Innovation in Risk Analysis

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Risk Matrix

Rating Scheme Design and Risk Aggregation

Springer

Innovation in Risk Analysis

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Contents

Chapter 1
Risk Matrix: Foundations and Overview

At the beginning of 2020, the COVID-19 swept across China, and then spread to the whole world. This disease made a tremendous impact on human beings' lives, economics, and so on. So far, people are still suffering the serious effect of COVID-19, and Governments of different countries appeal to citizens to stay cautious to avoid the health risk. Besides, lots of enterprises sustained huge losses and even collapsed. This is the biggest event recently that makes so many people deeply feel the influence of risk, and consider the importance of risk management.

So, what is "**risk**?" When talking about risks, people usually refer to different issues. For example, risk may be referred to as the possibility of an adverse event's occurrence; risk is the combination of the uncertainty and severity of a consequence; risk is the deviation from a reference value (one may refer to Society for Risk Analysis Glossary for more definition of risk). Although people usually say we should avoid a certain risk, like the health risk, risk is not always referred to as the adverse aspect of an event (Park and Grant 2005; Lee et al. 2011; Orchowski et al. 2012). For example, market risk in the financial field may result in possible gain or loss. However, it has been shown that people care more about the negative side of an event with uncertainty. And thus adopting the right choices to make preferred consequences more probable is what we aim to do facing a risk. **Risk management** is such a process that helps minimize risk level through the main steps of risk identification, risk assessment, and risk mitigation (Zhang 2016; Johnson and Swedlow 2021; Rostamzadeh et al. 2018). For example, in the case where people are under the threat of COVID-19, she/he needs to identify all the possible sources that may cause the infection risk, assess the consequence and probability of the risk, and take appropriate actions to mitigate the risk according to her/his acceptance of the risk.

For a certain risk, a fairly important part is **risk assessment** when risk management process is conducted, which focuses on measuring the quantitative value of the risk, mainly answering whether the risk is tolerable (Aminbakhsh et al. 2013; Aven 2016; Hegde and Rokseth 2020; Lloyd-Jones et al. 2019). The accuracy of the risk assessment result affects the risk mitigation strategies and then the really-happened

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consequence on the stakeholders, and thus the risk assessment needs convincing methods and tools.

In general, risk assessment methods could be divided into quantitative, qualitative, and semi-quantitative ones. Quantitative methods heavily rely on sufficient data and thus are usually adopted in the fields with high-frequency data (Hong et al. 2020; Vatanpour et al. 2015). For example, when studying financial risks, we can explore probability distribution-based methods like VaR (Value at Risk) to assess the magnitude of the risk. However, in many cases, we usually suffer the flaw that risk assessment data is scarce. For example, at the very beginning of a certain major public health emergency, such as the COVID-19, we have little data to support a quantitative assessment of this risk. At this point, the qualitative or semi-quantitative risk assessment tools are the alternatives.

A typical semi-quantitative or quantitative risk assessment tool is the risk matrix, which is what we discuss in this book. The incentive of studying the risk matrix is driven by the wide application of this tool. For example, we may see a risk matrix in a hospital, on a building site, on a financial management manual, and so on. Despite the popularity of this tool, researchers have found that the risk matrix sssssss several theoretical flaws, which may prevent the accuracy of the risk assessment result. Therefore, in this book, we will focus on resolving some fundamental problems related to risk matrix, and the proposed solution will alleviate its shortages.

1.1 What Is Risk Matrix?

When we talk about the risk matrix, as we see in many cases, it is usually a colorful graph. A very simple risk matrix is shown in Fig. 1.1. The risk matrix measures a certain risk by two input dimensions which are usually presented by consequence and probability, and the cell corresponding to each combination of dimensions has a color, such as green, yellow, and red, to reflect the risk level (Cox 2008). For clearly understanding risk matrix, the following basic concepts should be clarified.

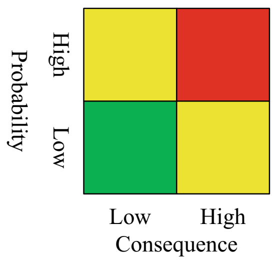

Fig. 1.1 A simple 2×2 risk matrix

1.1.1 What Is Risk?

As stated before, risk is differently defined in different cases. Although ISO (International Standardization Organization) defines risk as the effect of uncertainty on objectives, the definition is not that concrete. The Society for Risk Analysis Glossary summarized the following qualitative definitions of risk that are commonly adopted (SRA 2018):

(1) Risk is the possibility of an unfortunate occurrence.
(2) Risk is the potential for realization of unwanted, negative consequences of an event.
(3) Risk is exposure to a proposition (e.g., the occurrence of a loss) of which one is uncertain.
(4) Risk is the consequences of the activity and associated uncertainties.
(5) Risk is uncertainty about and severity of the consequences of an activity with respect to something that humans value.
(6) Risk is the occurrences of some specified consequences of the activity and associated uncertainties.
(7) Risk is the deviation from a reference value and associated uncertainties.

The diversification of risk definition is derived from our understanding of the core term, i.e., **uncertainty**, of the corresponding event in different situations. Risk essentially refers to future events, which is full of different kinds of uncertainty. For example, when we discuss (a) a possible future natural disaster, (b) a future financial investment, (c) a future production of uniformed goods, and so on, we may describe these future uncertainties in various ways. For case (a), definitions (1) and (2) are more suitable for the risk in a possible natural disaster since the uncertainty is always from the unwanted aspect. Definition (4) matches case (b) because in an investment one may get different returns with an uncertain probability. And when the uniformed goods are produced, definition (7) works for understanding risks, since one may care how much a produced good varies given its specification.

In general, the definition of risk is related to three elements, i.e., a certain objective (which may face an event that is uncertain), the consequence of an event on the objective, and the uncertainty of the consequence (uncertain about whether a consequence/event will happen or what is the consequence). We can briefly define risk in a triplet, namely $(objective, consequence|event, uncertainty|consequence)$.

1.1.2 What Is Risk Measure

Risk definition tells us what is risk, and for further risk management, we should know how big is the risk. That is what risk measure aims to do. Generally speaking, risk measure outputs the magnitude of a given risk considering some characteristics that the stakeholders value.

As we stated before, the definitions of risk are different as the risk assessment contexts change. Intuitively, the risk measure should correspond to risk definition. In other words, risk measures are various in different situations. For example, in a situation where the consequence is so severe that once the adverse event happens, the risk is unbearable, we need only to consider the uncertainty about the event's happening, and the probability of the event is a feasible risk measure (Aven 2012), namely,

$$risk = p = probability\ of\ the\ event's\ occurence, \tag{1.1}$$

where p is the abbreviation and $probability\ of\ the\ event$ is the detailed explanation (it is the same in the following equations in this subsection).

When we should consider both the consequence of an event and the uncertainty of the event's happening, the situations can be further divided as follows. For one case, where once an event occurs, the consequence is pre-assessed and nearly fixed, the risk could be measured as (Willis 2007):

$$risk = p \times c = probability\ of\ the\ event'\ occurence \times consequence\ the\ event\ brings. \tag{1.2}$$

For another case, where we know the event will happen for sure, but we don't know what are the possible consequences, the following risk measure is applicable (Ale et al. 2015):

$$risk = \sum_i p_i \times c_i = \sum_i probability\ of\ occurence\ i \times consequence\ i\ of\ the\ event. \tag{1.3}$$

If we have a reference value of the consequence, in the above case, the risk could also be measured by the variation of the consequence (Girardi and Ergun 2013), namely,

$$risk = Expectation(consequence - expected\ consequence). \tag{1.4}$$

We will not discuss other risk measures in this book, even though there are still lots of various risk measures in different fields, such as VaR (value at risk), CVaR (conditional value at risk), and so on, that are widely used in financial risk management and so on.

1.1.3 Objective Risk Measure Versus Subjective Risk Measure

One may find that to measure a risk using any appropriate risk measures, we need to obtain the estimation of the elements that the risk measure contains, such probability and consequence. The accuracy of the estimation highly relies on the sufficiency and reliability of the related data. However, in many risk assessment contexts, we do not have enough data supporting the adoption of data-relied risk measures.

Therefore, we can divide risk measures into two parts, one of which relies on sufficient objective data, and the other may use subjective judgment.

We focus on the subjective risk measure. Let's start with a very simple but widely used risk measure as shown in Eq. (1.2), namely $risk = p \times c$. Probability could be interpreted as the frequentist probability and be measured by statistical methods when data is sufficient, and it could also be interpreted as subjective probability and be measured just relying on the risk assessor's knowledge if data is scarce. When the measure $risk = p \times c$ is adopted, it is assumed that the consequence of a risk is fixed, namely, we should pre-estimate the effect that an adverse event will bring once it occurs. It could be measured or estimated according to historical data or expertise. Given the estimation of probability and consequence of the risk, the risk could be measured objectively or subjectively (the measure $risk = p \times c$ is also suitable for objective risk assessment).

As we can see in Fig. 1.1, in a risk matrix, the risk measure is not the same as we usually use like risk = p × c. Actually, the magnitude of a risk is a logical mapping of consequence and probability (in Fig. 1.1):

If consequence is "*Low*" *and probability is* "*Low,*" *then the risk is* "*Green.*"
If consequence is "*Low*" *and probability is* "*High,*" *then the risk is* "*Yellow.*"
If consequence is "*High*" *and probability is* "*Low,*" *then the risk is* "*Yellow.*"
If consequence is "*High*" *and probability is* "*High,*" *then the risk is* "*Red.*"

In a risk matrix, the estimation of consequence and probability is discrete, i.e., the stakeholders need to answer whether the consequence/probability is low or high. One may find that actually there is an additional mapping from the estimated value of consequence/probability to its rating. This mapping is different for different stakeholders. For example, given a certain value of consequence like 1, for stakeholder A, she/he may consider 1 as a "low" consequence, but for B, she/he may take it as "high."

The explanation for the above phenomenon is if an objective risk measure is adopted, it only outputs the objective value of the risk, which is not affected by the risk assessor's personal characteristic. We call the process of obtaining the risk using a risk measure as process p1, and the process of assessing whether the risk is acceptable as process p2. Ony finds that when we use objective risk measures, the two processes are sequential, namely, p1 goes first and then p2. While when we use subjective risk measures, the two processes overlap, namely, one may estimate the risk considering whether the consequence, the probability, or even the risk itself, is acceptable, namely, low or high, at the process of p1. This mixture of p1 and p2 reveals that the study of risk matrix should involve more issues that are related to human subjectivity.

1.2 Theoretical Flaws and the Topics in This Book

1.2.1 Some Theoretical Flaws of Risk Matrix

Considering the simplicity nature, risk matrix is widely used in various production activities including climate environment, public health, industrial production, commercial investment, and so on (Iverson et al. 2012; IEC 2009). In most cases, risk matrix plays the role of a specific graphical display, which involves risk communication between risk matrix designers and users. For example, for quick response to acute public health events, WHO (World Health Organization) published a quick risk assessment guideline where likelihood and consequence ratings of a disease are evaluated by answering several rigorously designed questions, and then a risk matrix is given to output the risk ratings (WHO 2012) (see Fig. 1.2).

Despite the wide usage of risk matrix, it suffers some fundamental flaws. Cox first proposed a systematic analysis of the risk matrix in the top journal Risk Analysis in the field of risk management in 2008 (Cox 2008). The analysis points out several shortcomings of the risk matrix:

(1) The risk matrix has false resolution. In the risk matrix, according to the definition of risk, there are some points whose value is greater than others, but they are assigned a lower risk level, which violates the most basic principle of monotonicity. For example, in Fig. 1.3, the point (0.3, 0.9) belongs to the green area, and the point (0.35, 0.7) belongs to the yellow area, but the value of the former (0.27) is greater than the latter (0.265).

(2) The risk ranking obtained according to the risk matrix violates the transfer invariance. For multiple risks, when their consequences all change a value at

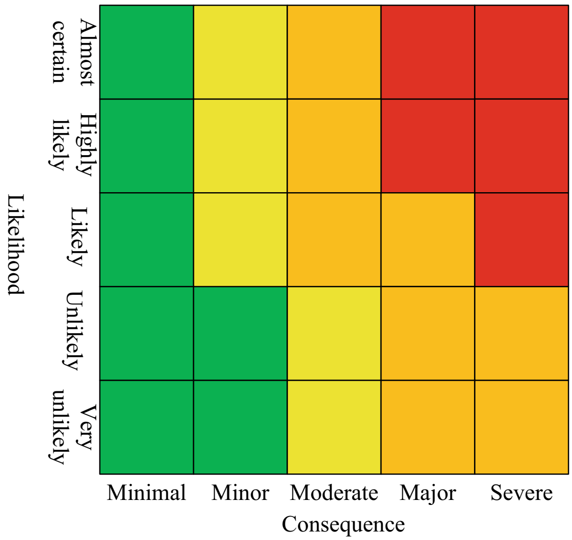

Fig. 1.2 A risk matrix used for rapid risk assessment of acute public health events

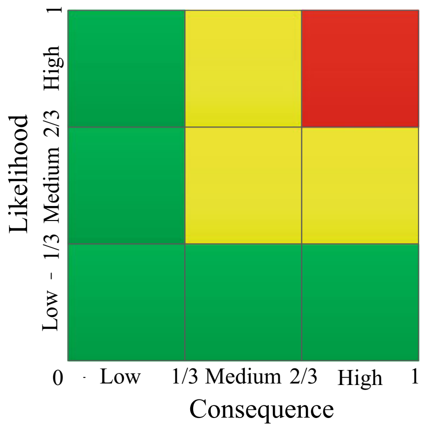

Fig. 1.3 A 3 × 3 risk matrix

the same time, the priority order of risks will change. For example, for three risks, the consequences and probabilities are (A) (0.95, 1), (B) (0.4, 0.5), (C) (0.15, 1). Obviously, according to the risk level in Fig. 1.3, (A) > (B) > (C), but when the consequences are reduced by 0.1 at the same time, according to the risk level in Fig. 1.3, (A) > (B) = (C), since (C) and (B) have the same degree of consequences, both of which are low.

(3) Relying only on the rating of the risk matrix does not necessarily make the correct decision. Different risk control may correspond to different control costs, and risk control measures depend on budget constraints. Therefore, the risk rating does not necessarily directly correspond to the risk mitigation priority. For example, (a) changing risk A from 100 to 80 requires a cost of 30, (b) changing risk B from 50 to 10 requires a cost of 40, and (c) changing risk C from 25 to 0 requires a cost of 20. The elimination priority of these three risks is constrained by the budget: when the budget is 20, risk C has the highest priority; when the budget is 40, risk B has the highest priority (the risk value of 40 can be eliminated); when the budget is 60, it is optimal to mitigate risks B and C at the same time.

(4) Different decision makers have different classifications of consequences (or probability), which leads to different assessments of risks. For example, for a certain consequence value like 0.5, some may consider it as “Medium,” which some may consider it as “High.”

(5) Risk attitudes will affect the assessment of risks. Obviously, for two risk assessors, one of which is risk-averse, and the other risk-neutral, the risk rating should not always be the same using the same risk matrix.

After Cox first proposed a systematic analysis of the shortcomings of the risk matrix itself, other scholars have also put forward discussions on the theory and application of the risk matrix.

Smith et al. pointed out the possible deviation of input data in the risk matrix (Smith et al. 2009). (1) People have subjective cognitive biases in the understanding of probability. (2) According to the prospect theory, people have different utilities at different consequences. (3) When people evaluate the consequences or probabilities, there will be a phenomenon that the evaluation is concentrated in the middle part. And (4) The consequences and probability data will appear linear relationship. All these phenomena show that when using the risk matrix, decision-makers cannot give accurate assessments of consequences and probabilities. Therefore, the final risk assessment results lack accuracy.

Ball and Watt asked some experimental participants to evaluate different risks based on a risk matrix. They found that even for the same risk assessor, his/her assessment of risk will change over time (Ball and Watt 2013). This shows that just using a risk matrix for risk assessment does not accurately reflect the magnitude of the risk.

In his review of the risk matrix, Duijm pointed out that (1) The difference in subjective judgment is due to people's cognitive bias. The use of qualitative descriptions of consequences and probabilities (such as the use of adjectives) will exacerbate this cognitive bias. Therefore, It is recommended to use a quantitative description (using an interval to describe an input category); (2) Regarding the integration of risk matrices, like ISO's point of view, Duijm believes that there are two difficulties affecting the integration of risk matrices: the inability to compare different vocabulary descriptions of the input categories, and different risk types cannot be compared. Therefore, it is considered that the risk matrix is difficult to be aggregated (Duijm 2015; IEC 2009).

1.2.2 The Topics in This Book

Main Topic 1: Rating scheme design of risk matrix

Given all the flaws of risk matrix that were stated before, we think the fundamental problem is that we lack a convincing scientific risk matrix design method to overcome the flaws that exist in the risk matrix designed using some traditional methods.

Designing the rating scheme of a risk matrix essentially answers two questions:

(1) how many risk ratings (denoted by colors as shown in Fig. 1.3) should the risk matrix have, and
(2) how to assign risk ratings to each cell (a cell is the combination of a consequence and a likelihood) that the risk matrix contains.

Figure 1.4 presents a graphic illustration of the risk matrix design. Although we may see so many risk matrices used in various situations, few of these applications explain clearly how the risk matrices are designed, and thus the risk assessment results are questionable.

Different decision makers may produce different mappings of the two inputs to risk categories based on their own methods to assess a risk. However, even if the designs of a risk matrix may be different from different designers, the rating

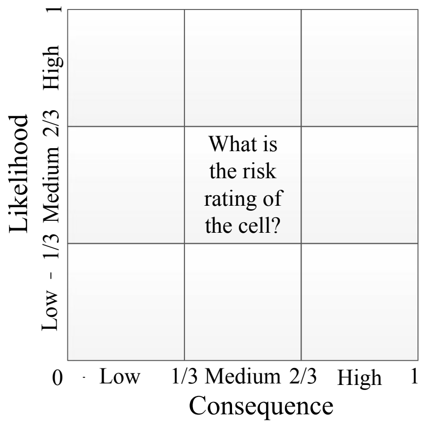

Fig. 1.4 The illustration of risk matrix's rating scheme design

schemes cannot be entirely arbitrary. At a minimum, risk should be designed as a monotonically increasing function of consequences and likelihood. Thus, we believe that while risk matrices may capture subjective assessments, they should be designed based on objective and recognized rules so that the users ensure they obtain relatively reliable information from the risk matrices.

We should avoid believing that because we are using qualitative risk management tools, the risk matrices should be designed in an entirely subjective way. What we will discuss in this book attempts to assist the designers in designing risk matrices as "reliably" as possible so that users will reach better evaluated and considered decisions. Although different designers may have different designs, we merely attempt to provide a more effective method to support the design process.

Main Topic 2: Risk aggregation of risk matrices.

Topic 1 focuses on one specific risk matrix. Another topic we concern is the aggregation of risk matrices, which is related to multiple risk matrices.

We discuss the topic of risk aggregation of risk matrices because, in practice, a risk assessment context usually contains several risks and decision makers need to consider the overall risk rather than individual risks (Bernard et al. 2014; Acharya et al. 2013; Kouvelis et al. 2012). Risk aggregation is a common topic in data-frequent fields like finance. However, the extant study on aggregation of risk matrices is sparse. Some issues that hinder aggregation of risk matrices are: (1) A risk matrix is essentially a qualitative tool and thus ratings of risks are often described only qualitatively, which makes it difficult to compare aggregated risks of different scenarios (a scenario means a case where each risk considered has a particular rating). For example, it is difficult to say how many low risks are equivalent to a medium one. Also, (2) consequences of different types of risks are usually different and cannot be compared, such as the consequences of economic loss, casualties, and so on (Duijm 2015). As a result, ISO (IEC 2009) states that 'risks cannot be aggregated.' To resolve the problem, in this book, another main topic is about how to aggregate risks assessed by risk matrices. Since a particular risk is measured by a predesigned risk matrix, the concept of "aggregating risk matrices" is substituted for "aggregating individual risks measured by risk matrices" for simplicity (Duijm 2015; IEC 2009).

How to understand the aggregation of risk matrices and the core problem of aggregation? In Fig. 1.3, consequence and likelihood are qualitatively described as 'Low', 'Medium' and 'High'. However, if ratings of inputs are described in the same way, mathematical operations of risk matrices may become very difficult. For example, if three 3×3 risk matrices are aggregated, 'Green (L + L) + Green (L + L) + Green (L + L)', where the output of consequence 'Low' and likelihood 'Low', denoted by 'L + L' and so on, is the lowest risk aggregation situation. (1) We care what is the risk rating of this combination of three individual risk matrices? And (2) which of the two scenarios, one with three risk ratings 'Green + Yellow + Red' and the other scenario with three risk ratings 'Yellow + Yellow + Yellow', should have a higher priority?

Rating scheme design and risk aggregation of risk matrices are discussed in detail in the remaining of this book. We hope the resolution of these two problems help the better usage of this popular risk assessment tool.

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Chapter 2
Different Types of Risk Matrices and Typical Applications

2.1 Origin and Fundamental Model

2.1.1 The Origin of the Risk Matrix

The term of risk has a long history, and the assessment and management of risk can be traced back to before Greek and Roman times. However, formal risk analysis has only begun to appear in modern times. In the 1870s, under the promotion of the U.S. Environmental Protection Agency, the role of risk assessment in management was enhanced, leading to the professionalization of risk analysis. During the 1990s, the concept of value-at-risk was widely used in the economic field, and research on risk also developed rapidly in various industries.

A significant prerequisite for risk assessment is a deep understanding of the connotation of risk. Some researchers summarize it as a collection of uncertain damage events and their probabilities and consequences (Kaplan 1997). And some scholars believe that the risk can be either the uncertain event itself that will cause loss, the probability of the uncertain event, or the expected value of the loss caused by the uncertain event. Generally speaking, the researchers' understanding of the connotation of risk is basically similar: the necessary factors that constitute risk include risk scenario, risk probability, and risk loss. The risk matrix assessment method directly and concisely reflects the understanding of the connotation of risk, which is one of the reasons why it is widely used.

In 1995, the US Air Force Electronic System Center (ESC) systematically proposed and widely used the risk matrix assessment method in the life cycle risk assessment of the acquisition project for the first time (Ni et al. 2010). Since 1996, a large number of ESC projects have adopted the risk matrix method to assess project risks. The risk matrix does not have a completely fixed form, and the specific form and content are also closely related to the subjectivity of decision-makers (Ayyub 2003; Franks and Maddison 2006). Even though, the risk matrix has its fundamental structure as shown in the following.

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2.1.2 Fundamental Mathematical Model of Risk Matrix

A risk matrix can be described in the form of a grading function:

$$R = f(p, c) = [R_{ij}],\ when \begin{cases} p_i \le p < p_{i+1} \\ c_i \le c < c_{i+1} \end{cases},$$

where R_{ij} represents the risk ranking corresponding to the *i*-th level of risk probability and the *j*-th level of risk consequence in the risk matrix; p_i and l_i represents the lower limit of the *i*-th level of risk probability and the *j*-th level of risk loss respectively; p_{i+1} and $c_{\mathrm{i+1}}$ respectively correspond to the upper limit.

Table 2.1 exhibits a typical risk matrix in the form of table, and we see that a risk matrix mainly includes three aspects: the category of consequence and probability, the number of ratings the risk matrix totally have, and the mapping of a risk rating with the combination of a consequence and a probability. The three aspects correspond to the use process of risk matrix. Firstly, we should know how to categorize the consequence and probability (how many categories there should be and what is the category of a certain consequence/probability given its estimated value). Then we should know how many risk ratings should there be, even though we are used to dividing the risk ratings into Low, Medium, and High. And lastly, we should know

Table 2.1 A typical risk matrix

Probability level	Consequences level				
	1	2	3	4	5
1	Negligible	Negligible	Receivability	Receivability	Reasonable control
2	Negligible	Negligible	receivability	Reasonable control	Strict control
3	Receivability	Receivability	Reasonable control	Strict control	Unacceptable
4	Receivability	Reasonable control	Strict control	Unacceptable	Unacceptable
5	Reasonable control	Strict control	Unacceptable	Unacceptable	Unacceptable

what the risk rating is (or what the mapping is, or what is the function f), given the category of consequence and probability of a risk.

2.2 Structure and the Design of Different Kinds of Risk Matrix

The risk matrix method is widely adopted as a convenient and efficient risk evaluation tool in many fields, such as engineering and software field, etc. (Ale et al. 2015; Smith et al. 2009; Hsu et al. 2016; Oliveira et al. 2018). The structure of the risk matrix is relatively simple, as shown in Fig. 2.1. There are two axes in the risk matrix named the horizontal axis and the vertical axis (Cox 2008; Levine 2012). The horizontal axis indicates the severity of risk consequences, and the vertical axis represents the probability of risk occurrence. When the horizontal and vertical axes are divided into intervals according to a certain ratio, the intersection of each row interval and column interval becomes the smallest unit of the risk matrix, i.e., the cell of the risk matrix. Generally, if it has M categories of consequence and N categories of likelihood, it has $M \times N$ cells.

In addition, in different application scenarios, risk matrices can be classified into qualitative risk matrices, semi-quantitative risk matrices, and quantitative risk matrices (Hong et al. 2020; Vatanpour et al. 2015). The structure and characteristics of the risk matrix will be further introduced accompanied by different types of risk matrices.

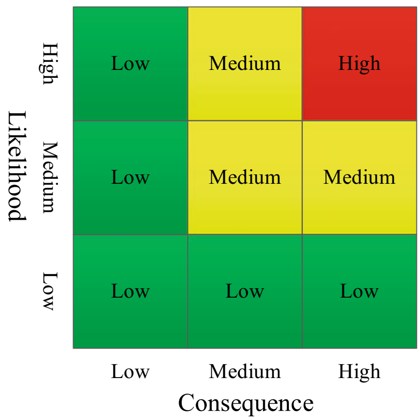

Fig. 2.1 A 3 × 3 risk matrix

2.2.1 Qualitative Risk Matrix

Qualitative risk matrix refers to a risk matrix where the categories of consequence and likelihood are described used adjectives like "High," "Medium," "Low." The risk matrices in Figs. 2.1 and 2.2 are typical qualitative ones.

When using a qualitative risk matrix, one should first judge the categories of the assessed risk's consequence and likelihood, and then the designed risk matrix outputs the risk rating. In a qualitative risk matrix, there is no explicit risk measure. For example, what is the risk if both the consequence and likelihood are categorized as "High."

For qualitative risk matrices, several scholars have summarized some design rules from practice: ***cells along a diagonal with the same slope have the same risk and adjacent risks are classified the same rating***. If the above rule works, it means some combinations of qualitative descriptions of risk criteria have the same degree of risk and it is more likely that risk is measured by the formula "*risk* = *consequence* + *likelihood*". For example, both consequences and likelihood are categorized as "high", "medium" or "low", then "high + low" = "medium + medium" = "low + high" or "high + medium" = "medium + high" or other alternative descriptions. By the rule, cells on adjacent diagonals can be classified together. However, there is no definitive answer to the question of which diagonals should be classified together. It is usually conducted based on decision-makers' knowledge (Duijm 2015). The risk matrix was rated by Pritchard et al. to assess the risk of drilling hazards according to the above rule (as shown in Fig. 2.2) and numerical representation of risk priorities (Pickering and Cowley 2010).

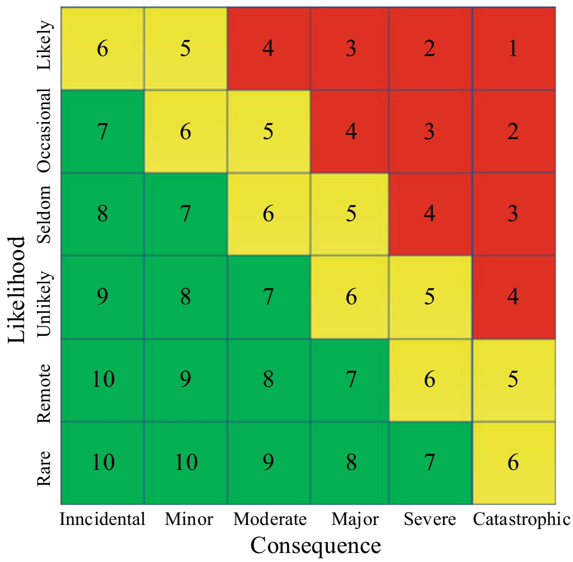

Fig. 2.2 The risk matrix designed by Pritchard et al.

Hewett et al. applied the rule in the design of risk matrices for managing agricultural pollution (Hewett et al. 2004). Other examples include Cook's risk matrix rating scheme for a security management system, Cox's design of the simplest 2×2 risk matrix, and so on (Cook 2008). The above rule is applied primarily when qualitative language is used to describe risk criteria, it is a crude qualitative category rule, which cannot be used when risk preference appears. In addition, in terms of rating the qualitative risk matrix, Holt et al. used a different utility function to determine a rank of a cell. Specifically, if there are still three categories for the two input criteria, then the utility function corresponding to the "minimum matrix" is Minimum, in other words, the risk rating depends on the lower of the consequence and likelihood ratings, so Minimum (very low, high) = very low (Holt et al. 2014). However, the flaw in this approach, is that the dimensions of consequence and likelihood must be equivalent.

2.2.2 Semi-quantitative Risk Matrices

In practice, the semi-quantitative expression is used to distinguish qualitative input categories more intuitively, whereby the categories of consequence and likelihood are expressed in discrete ordinal numbers, such as the risk matrix described by 1, 2, and 3, which means a severer consequence or a greater likelihood has a higher score (Soon and Baines 2015).

In a semi-quantitative risk matrix, the risk measure $risk = consequence \times likelihood$ is usually employed and each cell in a risk matrix gets a score. The decision-maker will be close to each other several scores are classified as the same rating. Accordingly, its rating rule can be summarized as: ***cells whose semi-quantitative scores are in the same interval have the same rating.*** Dethlefs and Chastain designed the risk matrix concerning the rating rules for semi-quantitative risk matrices, as shown in Fig. 2.3. The rule has been used in other literature, such as the rating of hazard risk assessment matrices (Donoghue 2001).

The semi-quantitative risk matrix gives a more reliable rating scheme based on quantitative inputs than the qualitative risk matrix rating rules, but there are still some problems that can't be ignored: (1) Threshold setting for both ratings is entirely subjective and there are no criteria for selecting thresholds. (2) Scores are assigned to categories of consequence and likelihood in the form of an arithmetic sequence, which means the importance of these categories grows evenly. This is not always the case, however, as Smithson et al. list a probability phrase numerical bootstrap Table 2.2, where probability expressions have different intervals, which are not even (Smithson et al. 2012). (3) The ranking of risks varies when different ordinal numbers are applied to the two input categories (Smithson et al. 2012).

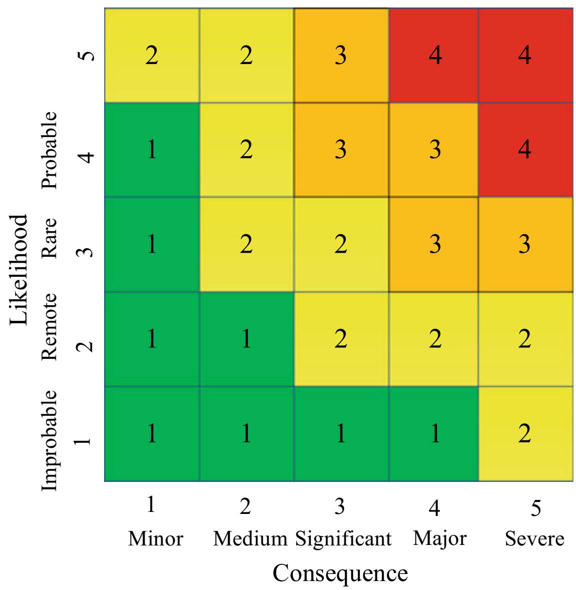

Fig. 2.3 Risk matrix designed by Dethlefs and Chastain

Table 2.2 IPCC probability phrases numerical guide

Phrase	IPCC range (%)
Virtually certain	>99
Extremely likely	>95
Very likely	>90
Likely	>66
More likely than not	>50
About as likely as not	33–66
Unlikely	<33
Very unlikely	<10
Extremely unlikely	<5
Exceptionally unlikely	<1

2.2.3 *Quantitative Risk Matrices*

As discussed earlier, one of the drawbacks of the semi-quantitative classification of inputs is that it does not reflect the true differences between the two categories. Thus, quantitative descriptions of inputs are employed in some practices and research (Pickering and Cowley 2010; Smith et al. 2009; Cox 2008). In quantitative descriptions, the category corresponds to a numerical interval of the consequence or likelihood instead of a specific score, for instance, “likely” corresponds to an interval [1/3, 2/3].

In the case of quantitative risk matrices, it is possible that assigning several thresholds of risk contours corresponding to numerical values, which may split a cell into

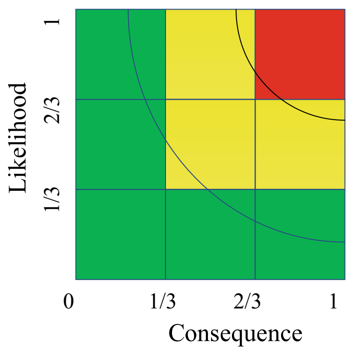

Fig. 2.4 Risk matrix as rated by Ruan et al.

two or even more areas, Ruan et al. adopted the area approach to determine the eventual rank of a cell: the rank associated with the largest area in a cell is the eventual rank of the cell (Ruan et al. 2015). As shown in Fig. 2.4, two risk contours separate a 3×3 risk matrix into three ranks, thus determining the eventual rank assigned to each cell. Other more systematic design approaches are also available. For example, Cox proposed three axioms to guide users in rating cells in risk matrices, namely weak consistency axiom, between-ness axiom, and consistent coloring axiom (Cox 2008). These three axioms express the relationships that should be fulfilled by cells of different colors. Although risk matrices rated according to Cox's axioms possess reliable mathematical logic, there are still two remarkable drawbacks. One is that the rating scheme of a risk matrix is not unique, the other is inadequate resolution. It is therefore difficult to apply the rule in practice to rating the risk matrix. More detailed risk matrix design methods will be introduced in Sect. 2.3.

2.3 Applications in Different Fields

Although the risk matrix has different types and can be differently used in different fields, as a mature risk assessment tool, the use of the risk matrix has its systematic process (Niyongabo et al. 2019). In the following, we will introduce in detail how to use the risk matrix to evaluate risks and the application examples of the risk matrix to further discuss the practical value of the risk matrix.

Risk assessment consists of three steps (or three sub-processes), which are risk identification, risk analysis, and risk evaluation sub-processes (ISO 2009). Risk identification is the first sub-process of the whole process of risk assessment. It provides input for risk analysis and is the process of discovering, acknowledging, and describing risks. The methods of risk identification mainly include brainstorming method, questionnaire method, decomposition method, fault tree method, and so on. Risk analysis is the second sub-process of the whole process of risk assessment. It is the process of understanding the characteristic of risk and determining the level of risk, which provides a basis for risk assessment and risk response decision-making.

Risk evaluation is the third sub-process of the whole process of risk assessment. It is the process of comparing the result with risk criteria to determine whether the risk and/or its magnitude is acceptable or tolerable. The correct risk assessment will help the organization to make decisions about risk response.

According to the process of risk assessment, risk matrix, as a tool of risk assessment, is used as follows:

(1) Determine the subject and needs of the assessment, conduct a systematic analysis on it, and give a risk definition;
(2) Carry out risk identification and determine risk factors that need to be assessed;
(3) Use specific methods such as expert judgment to conduct risk analysis to describe the probability and consequence level of each risk factor;
(4) Categorize the probability and consequence level of risk matrix, design the risk matrix, and carry out the risk evaluation of each risk situation according to the risk matrix.

As we all know risk matrix is widely used for rapid risk assessment in the fields of medical treatment and public health, food safety and public infrastructure operation, etc. Next, we will introduce some typical cases to explain the application of it in detail.

2.3.1 Food Safety Risk Assessment

Case: Rapid food safety risk assessment of compounds

The Food Safety Law of the People's Republic of China, which came into force in June, 2009, lays down that the State shall establish a food safety risk assessment system to assess the risks of chemical, biological and physical hazards in food and food additives. Over the past five years, China has launched multiple food safety risk assessment actions and risk assessment approaches are gradually applied to food safety field.

Codex Alimentarius defined risk as a function of the probability of an adverse health effect and the severity of that effect, consequential to a hazard(s) in food. Therefore, consequence and likelihood of adverse effects are key parameters used to evaluate a risk in this health risk classification system.

To establish a risk matrix and classify the health risks of chemicals in food, the following three steps are generally required: (1) determining the consequence severity; (2) determining the likelihood of adverse effects when exposure to a hazard; and (3) determining the risk level.

In the first step, when it comes to chemicals like regular environmental pollutants and food additives, such data as hospitalization rate and prevalence rate is normally not available. In this case, toxicity or severity of adverse effects can be used to measure consequence resulted from a chemical. By referring to relevant materials, acute toxicity (such as oral lethal dose in rats) and long-term toxicity (Carcinogenic

Likelihood (over score)	Consequence(over score)				
	Insignificant (1)	Minor (2)	Moderate (3)	Major (4)	Severe (5)
Almost certain(5)	Medium(5)	Medium(10)	High(15)	High(20)	High(25)
Likely(4)	Low(4)	Medium(8)	Medium(12)	High(16)	High(20)
Possible(3)	Low(3)	Medium(6)	Medium(9)	Medium(12)	High(15)
Unlikely(2)	Low(2)	Low(4)	Medium(6)	Medium(8)	Medium(10)
Rare(1)	Low(1)	Low(2)	Low(3)	Low(4)	Medium(5)

Fig. 2.5 Risk matrix of food safety

degree, mutagenic degree, etc.) were used to jointly define the severity of damage to human body caused by chemical substances.

In the second step, the probability of the adverse effect occurring should be considered. Due to lack of foodborne disease data, The exposure and dose–response data concerning a given chemical could be used to estimate its chance to cause health impacts.

In the final step, the food risk level of the evaluated compound is determined according to the following risk matrix (Fig. 2.5).

2.3.2 Risk Assessment of Public Infrastructure Operation

Case: The Risk matrix-Based assessment of Shenzhen metro operation

In recent years, with the rapid development of urbanization, the pressure of urban traffic is increasing rapidly. The rail transit industry shoulders the important task of building an urban public transport system. In the context of the great development of rail transit, Shenzhen Metro has been in the forefront of China. As of December 31, 2020, there are 11 metro lines in operation in Shenzhen, including line 1, line 2, line 3, line 4, line 5, line 6, line 7, line 8, line 9, line 10 and line 11. The total length of metro lines in the city is 411 km, with 283 stations.

Shenzhen Metro always regards safety as the foundation of its operation. However, despite the outstanding achievements in operational security, there are still potential operational risks. Inaccurate grasp of operational risks will affect the follow-up risk management process and seriously threaten the safety of people's lives. Metro operation risk refers to the impact of uncertainty in the process of metro operation and management on the production objectives of metro operation, and the impact can be comprehensively measured from two aspects: the severity of the consequences of the risk and the probability of the risk.

Based on the above analysis, the following risk matrix is constructed to evaluate the operation risk of Shenzhen Metro (Fig. 2.6). An expert group composed of ten experts that including station service safety engineer, Electromechanical Safety Engineer, automation safety engineer, maintenance center safety engineer, safety supervision engineer and metro public security, has carried out risk identification. Nine risks are identified at last. They are signal system risk, vehicle system risk, power supply system risk, civil construction risk, safety management risk, fire protection system risk, mechanical and electrical equipment risk, ticketing system risk and external environment risk.

After risk analysis and evaluation, the rating of each risk factor is obtained. Taking the fire protection system risk as an example, the risk level of these fire protection system risk factors are obtained by designing questionnaire, as shown in the Table 2.3. Among all the risk factors, the risk of arson, misoperation of gas fire extinguishing system and insufficient resistance of solenoid valve of gas fire extinguishing system whose consequence is more serious and the probability is higher is undoubtedly the highest. Although the probability of false fire alarm linkage factor in FAS system is relatively high, the risk level is not the highest. This is because the risk level does not entirely depend on the risk probability or consequence, but on the combinations of both, that is, the form of risk measurement.

2.3.3 Medical and Health Risk Assessment

Case: Rapid risk assessment of solid medical wastes: a case study in Burundi

Presence of infectious agents, toxic chemicals, radioactivity, used sharps, or biologically aggressive pharmaceuticals in solid medical waste (SMW) will have serious health effects when people are exposed to them. However, according to a survey conducted in Bujumbura, Burundi in 2019, current classification system of SMW in the national guidelines was not appropriate for safe collection and disposal. Pathological wastes, pharmaceutical wastes and discarded medical plastics, and absorbent cotton and placenta were main types of SMW, accounting for 84.4% from the healthcare facilities (HCFs). No HCFs followed the national guidelines completely, and most medical wastes have not been properly managed from the source separation stage. The generation rate per bed and the amounts of medical wastes per health care worker were 3.6 and 5.9 times higher in public HCFs than those in private HCFs, respectively, while the management practices of public HCFs were worse than those of private HCFs. Storage of medical wastes was the least managed step in the HCFs. All SMWs, HCFs, and people involved in SMW management were at very high risk or high risk. Hence, the assessment and specific description of risk in the SMW management process from generation to storage is imperative for reducing the unnecessary losses.

The risk from generation to storage in 12 HCFs in Bujumbura, Burundi, was assessed with the risk matrix below. Risk matrix a is used for assessing the risk of

risk matrix of Shenzhen metro			consequence level						
			1	2	3	4	5	6	7
			Very slightly	slight	general	larger	severity	serious	disaster
Probability level	9	several times a week	R4	R3	R2	R1	R1	R1	R1
	8	several times a month	R4	R3	R2	R1	R1	R1	R1
	7	several times a quarter	R4	R3	R2	R2	R1	R1	R1
	6	several times a year	R4	R3	R3	R2	R1	R1	R1
	5	several times a decade	R4	R3	R3	R2	R2	R1	R1
	4	several times a hundred years	R4	R4	R3	R3	R2	R2	R2
	3	improbable	R4	R4	R4	R3	R3	R3	R2
	2	very unlikely	R4	R4	R4	R4	R3	R3	R3
	1	incredible	R4	R4	R4	R4	R4	R4	R4

Fig. 2.6 Risk matrix used in the case

Table 2.3 Risk rating of public infrastructure operation

Sequence number	Fire system risk factors	Consequence score	Probability score	Risk rating
01	Arson	7	5	R1
02	The gas fire extinguishing system was misoperated	5	6	R1
03	Invalid fire protection facilities	4	4	R3
04	Short circuit of electrical equipment	2	4	R4
05	FAS system misalarms fire alarm linkage	3	7	R2
06	Bad function of fire escape passage	6	3	R3
07	The solenoid valve of gas fire extinguishing System has insufficient resistance ability	5	7	R1

SMW and HCFs, b is used for assessing the risk of people involved during SMW management in 12 HCFs.

Currently, the consequence of all SMWs and HCFs is severe and the likelihood of the severe consequence is almost certain. Undoubtedly, all SMWs and HCFs are of very high risks (the rating colored red), and improving the overall management practices is essential to reducing risk. Reducing exposure through segregation and safe storage may reduce the risk to some extent, but they are still of high risk as shown in the yellow region in Fig. 2.7a. To control and reduce risk to a safe low level (green), it is necessary to use additional measures such as disinfection of infectious wastes and medical sharps, and to implement safe and detailed guidelines for toxic chemicals and radioactive wastes.

As shown in matrix (b), most people including staffs and neighbors are at very high or high risk levels. Their potential risk from SMW depends on the management status and the frequency of contact with the SMW in the HCFs. Medical staffs, patients and waste workers are directly or indirectly involved from generation to storage and they should be exposed to SMW on a regular basis during SMW management. Because current overall SMW management practices were poor and those involved are not protected safely, they can be classified as very high risk (red color). Visitors and residents around HCFs are less likely to be exposed directly to SMW than hospital staff, but the risk level is at least high (orange or red) considering poor SMW

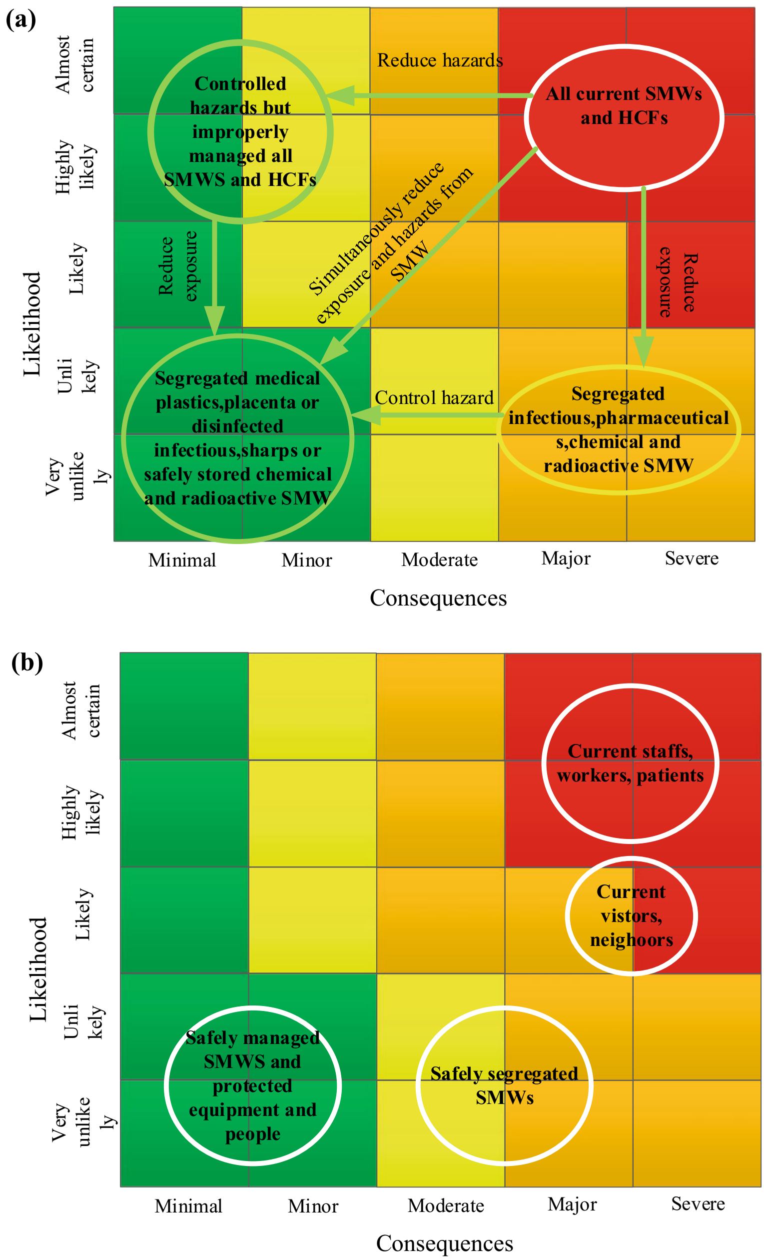

Fig. 2.7 Two risk matrices used in the case

management status, especially the storage stages in the HCFs. Safe classification and segregation can reduce the risk to a certain extent but reduce it to a low-risk level, there is a need for safe protection of staff and workers, proper use of equipment, and investment of infrastructure for the safe storage system.

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Chapter 3
Rating Scheme Design Methods

3.1 Summary of Some Unwritten Rules

The accuracy of risk matrix design directly affects the accuracy of risk assessment by decision-makers using risk matrix, but there is no uniform design standard yet. Therefore, the lack of a standardized design mechanism of the risk matrix is one of the most important shortcomings of the risk matrix. we begin our review of the risk matrix assessment program by exploring three questions: first, how to classify two inputs (i.e., consequence and likelihood); Second, how to measure risk in a risk matrix; And last, how to categorize different risks.

3.1.1 Input Classification

Measuring a specific risk involves an assessment of the two input dimensions of the risk matrix consequence and likelihood. There are various measures of risk, which are different in the definition of risk due to variation, value at risk, etc. In the risk matrix, this old-fashioned risk metric is usually used with the expected loss (the product of the consequence and the likelihood). Therefore, in the risk matrix, we use the two input criteria of consequence and likelihood to assess risk. According to the classification of these two input criteria, the risk matrix can be divided into three categories: qualitative, semi-quantitative, and quantitative. The qualitative risk matrix describes the consequence and likelihood in qualitative language as "high, medium, and low" (Garvey et al. 1998). For semi-quantitative risk matrices, to distinguish the categories of their qualitative inputs, the categories entered are represented by ascending scores, such as 1, 2, 3, etc. (higher scores indicate more serious or more likely results). In the quantitative risk matrix, the category no longer corresponds to a particular value, but a numerical interval corresponding to the consequence and likelihood, for example, "seldom" corresponds to the interval of [0.4, 0.5].

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For a qualitative risk matrix, the extent of the consequence or likelihood is primarily determined by the user based on subjective experience. The semi-quantitative risk matrix quantifies qualitative descriptions to match different scores for the categories of the consequence and likelihood input criteria. However, both types of risk matrices are based on subjective empirical judgments to classify input categories and are not classified according to quantitative criteria. So different users can divide the same risk consequence or likelihood into different categories. This is why many researchers recommend the use of quantitative risk matrices, which provide a quantitative description of each category of results or possibilities using numerical intervals (Pritchard et al. 2010; Smith et al. 2009). While the division of consequence and likelihood inputs may be approximate for a single user or designer, it provides an opportunity for them to agree on which division is most reasonable in a particular situation. Therefore, a recognized standardized input classification will make the classification of risks more widely accepted. This is why the ISO proposes to increase the intervals of consequence and likelihood (Hsu et al. 2016; Oliveira et al. 2015; Bao et al. 2016).

3.1.2 Measure of Risks

Since the magnitude of the consequence and likelihood are not clear due to subjective judgments, the risk measured by expected loss (the product of the consequence and the likelihood) cannot be well reflected in the qualitative risk matrix. The risk in a qualitative risk matrix can be approximated as a monotonically increasing function of consequence and likelihood, for example, risks with "medium" consequence and likelihood are more severe than risks with "low" consequence and likelihood. However, if the two input dimensions of the risk matrix are not quantitatively measured, such as one risk with "low" consequence and "high" likelihood, and the other risk with "high" consequence and low likelihood, the two theoretical risks are incorrectly ranked.

A semi-quantitative risk matrix can define risk as the product of the consequence and the likelihood (MacKenzie 2014; Ball et al. 2013). But this can be problematic if both the consequence and the likelihood input dimensions are scaled at the same. For example, the "likely" corresponding score is "3", and the "very likely" corresponding score is "4", and the interval between the "likely" and the "very likely" is "1", but the user thinks that "likely" corresponds to the interval (66%, 90%), "very likely" corresponds to the interval (90%, 95%), so the interval between the two intervals is not proportional to the score (Iverson et al. 2012). And using different ordinals to describe the level of consequence and the likelihood, the ranking of the risks will change. For example, using 1, 2, 3… to describe the categories of consequence and the likelihood and using 5, 4, 3… to describe the inputs, the resulting risk rankings are different, so the ranking according to the risk of the product is not accurate (Thomas et al. 2014).

In a semi-quantitative risk matrix, each cell is treated as an independent point, and this number of points is finite. For the quantitative risk matrix, there are infinite risk points in each cell, because the scale is continuous, and the risk value corresponding to each risk point in the cell is represented by the product of the consequence and the likelihood. When using a quantitative risk matrix, the decision-maker will classify the risk as one of the cells, because the risk will correspond to a quantitative interval or a point in the cell in the risk matrix. Therefore, it is a common setting in quantitative risk matrix research that risk corresponds to infinite quantitative risk values in the risk matrix.

3.1.3 Classification of Different Risks

For a qualitative risk matrix, that is, the categories of consequence and likelihood are matrices described in qualitative language, the classification rules are not clear, its design rules for along the same lines on the cell with the same risk level, adjacent cell with the same level of risk, risk matrix for the similar reference Hewett et al. (2004) and Holt et al. (2014).

Semi-quantitative risk matrices, namely, categories of consequence and likelihood are risk matrices described by discrete ordinals. The classification design rules are explicit, that is, cells with risk values in the same interval have the same risk level. However, the threshold setting in the semi-quantitative risk matrix is completely dependent on subjectivity, and the risk ranking will change when different ordinals are used to describe the consequence and likelihood (Ale et al. 2015; Goerlandt et al. 2016; Ruan et al. 2015).

Concerning quantitative risk research, Cox first presented a question of how a reasonable risk matrix should be designed in his review of risk matrices. He proposed three theorems for determining the category of risk matrices, namely weak consistency axioms, intermediate axioms, and consistent coloring axioms (In Cox's theorem, the green risk is the lowest, yellow is second, and red is the highest risk). These three theorems illustrate the relationship that should be satisfied between the colors of different cells. The weak consistency axioms require that the risk value of the points in the high-level cells be greater than the risk value in the low-level cells. The intermediate axioms propose that given a line segment with a positive slope, and that line segment passes through the red and green cells, then this line segment must also pass through the yellow. The consistent coloring axioms require that all cells below or through the green risk line be green, and all cells above or through the red risk line are red. If one cell contains both the risk in the green cell a point with the same value, which in turn contains the same point as the risk value in the red cell, the cell is divided into yellow.

Although the risk matrix designed according to the Cox axioms satisfies strict mathematical logic, there are two problems. First, the risk matrix rating scheme is not unique. Cox also pointed this out, and he offers two possible rating schemes for a 4×4 risk matrix. Second, the number of risk levels is too small, only two risk levels are comparable (the green cell level is lower than the red cell, but there are some points in the yellow cell that are lower than the green cells and some points higher than the red cell, which is not comparable with the green or red cell). Obviously, in practice, low resolution can lead to errors in certain decisions. It is therefore difficult to use this rule in practice for risk matrix design.

3.2 Usage of Utility Functions to Design Qualitative Risk Matrix Design

The criteria or risk factors were described by a set of discrete categories or ratings that had linguistic definitions but that also had a definite order on a five-point scale. For example, a particular risk factor might be described as very low, low, moderate, high, or very high. The linguistic definitions are frequently supplemented by notes and examples. They are essentially relative or comparative so that while it is not usually possible to give the rating a quantitative interpretation.

It is based on a hierarchical decomposition of the problem into subconcepts and finally to a finite set of basic attributes, allowing (discrete) distributions of ratings to be used to describe the basic attributes. Thus, rating uncertainty associated with the criteria is expressed as a frequency distribution. The rules for integrating the attributes are described by small sets of utility functions, which are presented as tables or matrices that can be readily defined. The outputs of a utility function are the marginal frequencies from the joint rating frequency distribution of the two criteria, calculated according to the particular utility function used. Each utility function has only two inputs.

A limited palette of five matrices is defined to describe the outcome of aggregating or combining criteria, two at a time. The outcome is described in the same linguistic terms as the original criteria: very low, low, etc. These five matrices are minimum, round-down, round-out, round-up, maximum can be considered to express a decreasing degree of constraint by the criterion with the lower rating over the other. The output and application scenarios of the five tables are shown in the Table 3.1.

At one extreme, a minimum matrix defines the outcome as the lower of the two ratings, so the lower value imposes a complete constraint over the higher. This expresses the idea of a necessary condition so that both criteria must achieve a particular rating for the outcome to reach that rating. At the other extreme, a Maximum matrix defines the outcome as the higher of the two, so the lower rating is not a constraint on the outcome. This expresses the idea of a sufficient condition, so that if either criterion achieves a particular rating, then the outcome also reaches that rating, such as

$$\text{Minimum(very low, high)} = \text{very low},$$
$$\text{Maximum(very low, high)} = \text{high}.$$

Table 3.1 Descriptions of the five utility functions

Matrix name	Outcome	Applicability
Minimum	The lower of the two ratings; both are necessary conditions	The lower rating constrains the outcome
Round-down	The intermediate between the two ratings but where the intermediate falls between two categories, the lower	The outcome lies between the two ratings but tends to be constrained by the lower
Round-out	The intermediate between two ratings but where the intermediate falls between two ratings and is lower than moderate, the higher	The outcome lies between the two ratings but is more influenced by a higher or lower rating than a moderate rating
Round-up	The intermediate between the two ratings but where the intermediate falls between two categories, the higher	The outcome lies between the two ratings but tends to be more influenced by the higher
Maximum	The higher of the two ratings; either is a sufficient condition	A lower rating of one component does not constrain the outcome

For the other three, the outcome is related to the intermediate of the two ratings but, being a discrete model, if this falls on the boundary between two categories, the result is rounded up or down according to the matrix type. For example:

$$\text{Round - out(very low, high)} = \text{low},$$
$$\text{Round - out(low, very high)} = \text{high}.$$

The five matrices obtained according to the corresponding rules are shown in Fig. 3.1.

3.3 Cox's Risk Matrix Design Axioms

The risk matrix provides a potential quantitative relationship with the following formula: risk = likelihood × consequence. The two axes of the risk matrix are the likelihood and the consequence, and the risk value is the name of the product. For example, it can be assumed that the likelihood axis is divided into five ordered qualitative categories (e.g., from rare to likely) roughly corresponding to dividing the quantitative probability axis into intervals [0, 0.2), [0.2, 0.4), [04, 0.6), [0.6, 0.8) and [0.8, 1]. Similarly, the five ordered categories of consequence axes (e.g., from minor to severe) correspond to numerical intervals [0, 0.2), [0.2, 0.4), [04, 0.6), [0.6, 0.8), and [0.8, 1]. Where 0 is no adverse effect, 1 is the worst possible adverse result, and a value between 0 and 1 indicates that the value is between no adverse effects and the worst possible adverse effects.

To make a reliable rating scheme, Cox proposed three axioms to guide users in rating cells in risk matrices, namely weak consistency axiom, betweenness axiom,

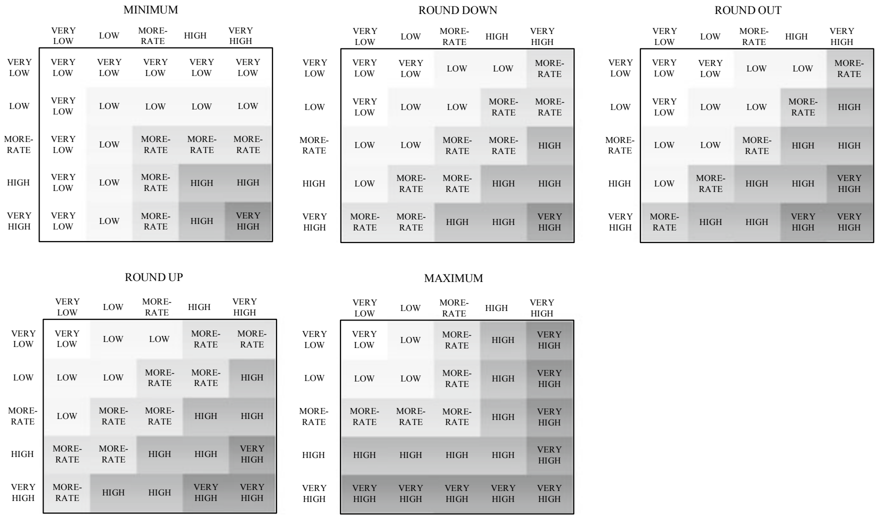


Fig. 3.1 The five symmetric matrices that express varying degrees of constraint of one criterion over the other

and consistent coloring axiom. (In Cox's axioms, the green risk is lowest, red is the highest, and yellow is in the middle.)

Weak consistency axiom: If a risk matrix contains multiple colors (levels), if it satisfies weak consistency, the risk value of the points in the high-level cells is greater than the risk value of the points in the low-level cells. (e.g., if a risk A is quantitatively higher than another risk B, we have risk (A) > risk (B)). The weak consistency of Lemma 1 indicates that if the risk matrix satisfies weak consistency, then the red cell and the green cell are not adjacent. And Lemma 2 describes if the risk matrix satisfies weak consistency and has at least two levels (green in the lower-left cell, red in the upper right cell, and the direction of the two axes shows increasing likelihood and consequence), then green cells do not appear in the right or top row of the risk matrix, and red cells do not appear in the left or lower rows.

Betweenness axiom: The hypothesis that the risk matrix provides an approximate qualitative representation of the underlying quantitative risk also means that any small increase in likelihood and consequence will not result in a discontinuous jump of the risk classification from the lowest level to the highest level. If a risk matrix satisfies the **betweenness axiom**, then given any line segment with a positive slope that goes through the red and green cells, the line segment must also go through the yellow cells.

Consistent Coloring axiom: Ideally, the same quantitative risk should have the same qualitative risk rating. However, this condition cannot be realized accurately in a discrete risk matrix. Therefore, we consider the axiom of uniform coloring. If a risk matrix satisfies consistent coloring, a cell will be red if it contains the same values of risk as the existing red cells and does not contain the same values of risk as the existing green cells. A cell is green when it contains the same risk points as the existing green cell and does not contain the same risk points as the existing red cell. A cell is yellow if it is between red and green, or if the cell contains both the same values of risk as the red cell and the same values of risk as the green cell. This axiom states that all sufficiently high risks should have the same color (red) and all sufficiently low risks should have the same color (green).

3.4 Sequential Updating Approach

The design of the risk matrix must conform to certain mathematical logic on the one hand and must meet the needs of practice on the other hand (Ni et al., 2010; Garvey et al. 1998; Hewett et al. 2004; Hong 2012; Pickering and Cowley 2010). In general, two requirements are important to practitioners: (Cox 2008; Levine 2012; Ruan et al. 2015; Organization 2009; Chen et al. 2020).

(a) The designed risk matrix can meet the user's requirements for resolution;
(b) Designed risks can help decision-makers make the right decisions.

In practice, there are two main purposes for using the risk matrix. One purpose is to determine the acceptability of a single risk. In this case, it is often only necessary to have three risk levels to meet the needs of decision-makers, such as "acceptable", "reasonable", and "unacceptable" (Duijm 2015). Another purpose is to rank multiple risks (Dethlefs and Chastain 2012). Obviously, in this case, the more the level of the risk matrix, the more different the risks can be distinguished. From the two purposes of the risk matrix, the risk matrix designed by Cox is not satisfied. Therefore, for demand (a), Cox's method needs to be improved.

According to Cox's weak consistency axiom, if the two risks are at different risk levels, then the magnitude of the two risks can be completely differentiated, that is, the decision-maker can make the right decision. However, in the risk matrix designed according to the Cox theorem, the number of risk levels is only three, which makes the two risks that are highly probable to be evaluated at the same risk level (this is called the risk knot). At this time, it is difficult to judge which risk is greater. In addition, the risk matrix designed according to Cox is not unique. As mentioned earlier, if different matrices are used to judge the magnitude of the two risks, the results may be different. Therefore, for demand (b), Cox's approach seems to be flawed.

Aiming at these two defects, a sequential update approach is proposed to design the risk matrix. The starting point of this approach is to be able to design a unique risk matrix that takes into account mathematical logic and practical needs.

3.4.1 Principles of the SUA to Design Rating Schemes for Risk Matrices

To prioritize cells in risk matrices, the primary problem is how to compare one cell with another. Intuitively, if cell A has a higher priority than B, A must be larger than B. "Larger" here is the result of the logical judgment between two cells. We propose to provide an appropriate criterion by which any two cells can be compared.

Cox stated in the axiom of weak consistency that "points in its top risk category represent higher quantitative risks than points in its bottom category." (1) Put differently, Cox's criterion of the logical judgment between two cells is if cell A has a higher priority than B, then any points in A should be quantitative larger than any points in B.

Although Cox's criterion is mathematically reasonable, there are some flaws. (1) Not any two cells can be compared. For example, in two adjacent cells, due to the multiplicative measure of risk, there must be some iso-risk contours passing through both of the cells, which means not all points in a cell are quantitatively larger than those in the other. As a result, we cannot tell which is larger. (2) An intermediate (unidentifiable) rating exists. The intermediate rating is denoted by the color yellow according to Cox. However, the rating "yellow" is not lower than the rating "red" and is not higher than the rating "green." Thus, the intermediate rating is not identifiable. (3) The designed risk matrix is of low resolution because only three colors are used. These flaws show that the criteria proposed by Cox should be improved to some extent.

Therefore, in the following, we first describe how to improve the criterion for comparing the size of two cells. Then, motivated by promising the consistency of logical judgment and improving the resolution of risk matrices, we provide another two principles.

Adjusted weak consistency

When we use a risk matrix to assess a risk, we should select a cell in the matrix to match the risk. However, in a cell of a quantitative risk matrix, there are infinite risk points, which means we allow risk to "vary within a cell, rather than considering risk within a single cell to be a single, discrete value." For a decision maker, he/she matches the risk with a particular cell because he/she considers the assessed risk is located at one point in the cell. Thus, from this sense, we may treat a point in a cell as a possible location of the risk to be assessed, which means the points in a cell may have a distribution, such as the uniform distribution.

We find that when using Cox's rule to compare two cells, only the quantitatively highest and lowest points of the two cells are used. However, since we believe the risk matched with a cell has a distribution, using only two points to determine the relationship between the two cells may lose some information. Therefore, a substitution of Cox's criterion to compare two cells is proposed, namely the criterion of logical comparison of two cells, defined in the following.

Definition of Logical Comparison of Two Cells

Cell A is (logically) larger than cell B if and only if the probability that a point in cell A is larger than a point in cell B is quantitatively larger than a predetermined threshold.

The criterion can be written mathematically as the following condition:

$$\Pr(a > b | a \in A, b \in B) \geq \alpha, \alpha > 0.5 \tag{3.1}$$

where a and b are two variables, representing points in cell A and B, respectively, and α is the threshold established by the decision maker.

According to condition (1), if more risk points in cell A are quantitatively larger than those in cell B, the corresponding probability will be larger. As a result, it reflects the difference between the two cells. The higher the corresponding probability, the larger the difference between the two cells.

Remark: Under the logical comparison of two cells, the possible relationships between two cells A and B are as follows: A is larger than B, A is equal to B, and A is smaller than B. At the risk point level, we use "quantitatively larger, equal, or smaller" to describe the relationship between two points. At the cell level, we use "larger, equal, or smaller" to describe the relationship between two cells. "Larger" means the difference between two cells is prominent, namely, the corresponding probability is larger than α. "Smaller" means the difference is prominent with the corresponding probability smaller than $1 - \alpha$. "Equal" occurs only when $\alpha = 0.5$. When the probability is during the interval of $(1 - \alpha, 0.5)$ or $(0.5, \alpha)$, we state there is a difference between the two cells, but the difference is not prominent. Thus, we cannot state that a cell is larger or smaller than another. In summary, the logical comparison criterion help us compare any two cells at the cell level considering all the information the single risk points in the cell bring.

We now explain why condition (1) is adopted. First, intuitively if we want to obtain the probability that a point in cell A is larger than a point in cell B, we must first generate two points randomly from A and B, respectively, and then compare them. After randomly sampling several times, we obtain the frequency of samples in which a point in cell A is larger than a point in cell B, which can be treated as an approximation of the corresponding probability. When the frequency remains unchanged (or the change is small) as the number of sampling increases, we consider that an acceptable approximation of the corresponding probability for the comparison of two cells has been obtained. As a result, in this process, distribution of risk values of a cell is embedded and theoretically every single point in a cell participates in the comparison process. Therefore, no information in the cell is ignored. Intuitively, if cell A has a higher rating than cell B, the probability that a point in cell A is larger than a point in cell B must be larger than 0.5. Thus, α should be set larger than 0.5.

We further describe the motivation for the logical comparison criterion. Intuitively, we may treat two cells as two populations and single risk points as individuals. The two cells (populations) are different because the risk points (individuals) have different characteristics (different distributions). Assume that we now want to compare the age of the two populations, A and B. Obviously, if the youngest person in population A is older than the oldest in population B, we state that population A is older than B. But, this condition is too strict. We may use the average age of the two populations. However, if we take the average risk values of the cells to compare the two cells, each cell will have a particular value and the quantitative risk matrix will effectively be treated as semi-quantitative, which will still suffer the flaws we discussed previously. Another concept is if most individuals in population A are older than individuals in population B, we may conclude that population A is higher than population B. This idea is normalized by condition (1). This measure will allow some cells with similar characteristics to be classified into the same rating, and this classification is not arbitrary (actually, the classification depends on the result of the corresponding probability).

We next state the reasonability of the logical comparison criterion. Remember that a risk matrix is a qualitative tool, and decision makers thus classify consequence and likelihood using qualitative categories such as “very likely.” Thus, between two adjacent ratings of consequence or likelihood there may be an overlap, i.e., it may be possible that some points in a cell with a higher rating are quantitatively lower than some points in a cell with a lower rating (for a visual representation of the overlap, one may refer to a fuzzy set (Zadeh 1965)). In this case, decision makers are more prone to believe that they assign a higher rating to cell A of a risk matrix than cell B because most, but not necessarily all, points in A are quantitatively larger than those in B. Although this will result in the so-called spurious-resolution, we claim that since risk matrices are qualitative tools, it is reasonable that some risk values in a higher-rated cell are quantitatively lower than those in a lower-rated cell if the proportion of the area with “spurious-resolution” is acceptable.

After explaining the new proposed criterion to compare two cells, we present the first principle that a risk matrix must satisfy, namely *adjusted weak consistency* (AWC).

Definition of Adjusted Weak Consistency

If cell A is assigned a higher rating (or higher priority) than cell B, cell A must be larger than cell B, according to the logical comparison criterion.

AWC is an intuitive principle. If cell A has a higher rating, it must show something different than cell B. The comparison of “logically larger” is the difference. Obviously, it is not reasonable to derive that A is equal to or smaller than B.

The criterion to compare two cells is different from that of Cox, and this is why “adjusted” is used. AWC can be employed to compare any two cells, irrespective of whether they are adjacent or not. Another automatic advantage of the principle of AWC is that the intermediate rating is no longer needed since any two cells are comparable. In other words, by using adjusted weak consistency, there will be no unidentifiable ratings.

One finds that if $\alpha = 1$, condition (1) becomes the condition Cox employed, namely, all points in A are larger than any in B (in this case, only the quantitatively highest and lowest points are needed). Thus, mathematically, if α is established with a higher value, the logical comparison between the two cells will be more accurate.

However, this does not mean that a higher α is a better choice (see the case where $\alpha = 1$ as Cox provided and the analysis in Sect. 4.1.4). For the decision maker, α represents the least confidence level at which they think A is larger than B. Once α is given, the criterion to compare any two cells in a risk matrix is determined.

Our final claim is that we do not require the measure of risk to be $risk = consequence \times likelihood$. The method to compare two cells applies to any form of risk measure, such as $risk = consequence^n \times likelihood$, where risk aversion can be modeled. One may refer to Appendix 1 to obtain the analytical result when the measure $risk = consequence \times likelihood$ is adopted. However, we believe the Monte Carlo simulation method is more convenient, especially in a more complicated setting (for example, non-uniform distributions of consequence and likelihood are used). The details of the Monte Carlo simulation are presented in Sect. 3.3.

Consistent internality

Adjusted weak consistency requires that if A has a higher priority than B, A must be larger than B. However, we question that although condition (1) is a necessary condition of A's rating being higher than B's, is it sufficient? Namely, if A is larger than B, can we conclude that A should have a higher rating than B? For example, in Fig. 3.2, assume that cell 1 and 3 have the same rating according to condition (1), cell 6 is larger than cell 1, but cell 6 is not larger than cell 3. In this case, should cell 6 have a higher priority (yellow or Y) than cell 1 and 3? The question can be normalized as the following:

Question: *If a cell is larger than a portion of the cells that have the same rating, should it have a higher rating than those cells?*

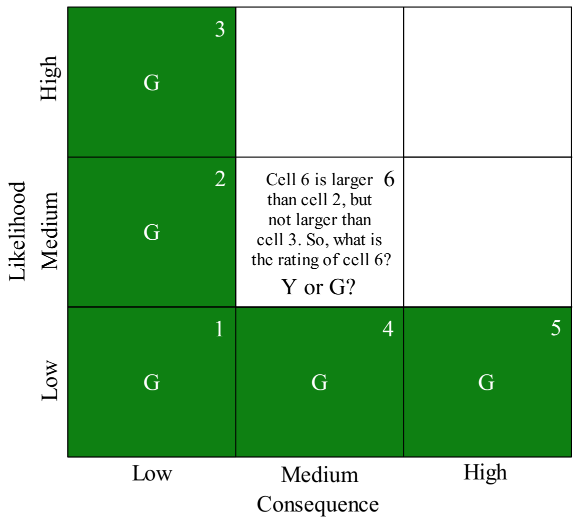


Fig. 3.2 Illustration of consistent internality

The answer is no. If the cell has a higher rating, it will violate the AWC principle because the cell has a higher rating than some of the cells with a lower rating, but the cell is not larger than them.

This analysis tells us if a cell has a higher rating, it should be larger than any cells in the lower rating, which forms the principle of *consistent internality* (CI).

Definition of Consistent Internality

A risk matrix satisfies consistent internality if a cell in a higher rating is larger than any cells in a lower rating, according to condition (1).

Remark: In a semi-quantitative risk matrix, we classify the cells with different ratings according to their risk scores. By using AWC and CI, only logical judgment is needed in the categorization of cells. The difference is: in semi-quantitative risk matrix, there is no unified standard based on which the thresholds of the scores are chosen and as a result, a risk matrix with *n* ratings should have *n-1* thresholds; while by using AWC and CI in a quantitative risk matrix, once the criterion to compare two cells is determined (namely, α is determined), it works throughout the design process and there will be no need to determine the thresholds between any two adjacent ratings.

Continuous screening

The last principle is proposed to address the following problem. Consider the situation: a risk matrix has 3 risk ratings, namely, 1, 2 and 3. The rating scheme is designed according to AWC and CI, and thus cells rated 3 are larger than cells rated 2 and cells rated 2 are larger than those rated 1. If we compress the risk matrix into 2 ratings, e.g., A (containing previous ratings 1 and 2) and B (containing previous rating 3), the latter rating scheme does not violate AWC and CI since cells rated B are larger than cells rated A. Of course, another rating scheme is feasible: rating A contains previous rating 1 and B contains 2 and 3. Although these two rating schemes are correct based on the last two principles, we can further subdivide the ratings because some cells in a rating are markedly different from the others. This motivates the following principle of *continuous screening* (CS).

Definition of Continuous Screening

If cells rated A are determined, any other cell X satisfying the condition that X is larger than any cells rated A updates the rating of X to a higher level.

We use CS to identify all the ratings whose cells can be distinguished according to AWC and CI to avoid blocking too many cells in a rating. This is why we call it the principle of continuous screening. The connection between CS and Cox's consistent coloring is, according to Cox, that cells larger than "red" cells are still red, while we subdivide those red cells. In this sense, CS is a principle maximizing the number of ratings to make the risk matrix as high-resolution as possible.

Remark: We believe the combination of these three principles is an improved version of Cox's rules. AWC and CI constitute the necessary and sufficient condition for a cell to have a higher priority than another. In some sense, these principles replace Cox's weak-consistency and between-ness axioms. The difference between AWC &

CI and weak consistency axiom on the one hand, and the between-ness axiom on the other is that the former are reasonable and milder criteria which can be used to compare the size of any two cells, and thus the intermediate (unidentifiable) rating is not needed. Obviously, due to the weak consistency and between-ness axioms, there are only three colors. However, according to AWC and CI, more colors may be created, and CS is used to maximize the number of colors to provide a higher resolution.

3.4.2 *The Uniqueness Principle of the Sequential Updating Approach*

Uniqueness Theorem: The risk matrix that satisfies weak consistency, consistent internality, and continuous escalation has a unique design.

In the remainder of this section, positive integers 1, 2, 3, etc. are used to indicate the magnitude of the risk level, where 1 indicates the lowest risk level.

Proposition 1 If a cell with a risk level m is determined, then if cells with a risk level greater than or equal to $m + 1$ exist, then they are also determined.

Proof: Assume that the number of cells rated m is S_m, and the total number of cells is N. For the remaining N-S_m cells, if they are larger than all the S_m cells rated m according to AWC and CI, their ratings will temporarily increase by 1 to $m + 1$ (this is required by CS). Record the number of cells temporarily rated $m + 1$ and denote it by T_{m+1}.

Of all the cells temporarily rated $m + 1$, find the smallest by following these steps: First, choose any cell x rated m satisfying the condition that the corresponding probabilities in condition (1) of cells temporarily rated $m + 1$ compared with x are not all 1. Then, select the smallest probability of the probabilities mentioned in last step (if there is more than one, select any of them) and the corresponding cell temporarily rated $m + 1$ is defined as the smallest one we need, which is denoted by $x_{0,m+1}$.

Next, we compare the remaining $T_{m+1}-1$ cells with $x_{0,m+1}$; if they are larger than $x_{0,m+1}$ according to AWC, their ratings temporarily increase by 1 to $m + 2$; otherwise, their ratings remain $m + 1$. Denote the set containing all cells temporarily rated $m + 1$ until now as L_{m+1} and the set containing all cells temporarily rated $m + 2$ as H_{m+2}. The members of the two sets remain unchanged if all cells in set H_{m+2} are larger than those in L_{m+1} according to AWC; otherwise, the cells in set H_{m+2} that do not match the criterion are moved into L_{m+1}.

Based on the above operations, the cells that ultimately remain in L_{m+1} are the cells rated $m + 1$ in the final rating scheme. This is because compared with cells rated m, cells with rating $m + 1$ satisfy all three principles and no other cells can be added into the rating $m + 1$ since the other cells are either smaller than cells rated m or larger than cells rated $m + 1$. Additionally, cells whose ratings are larger than $m + 1$ are in the ultimate H_{m+2}. ■

Proof of the uniqueness theorem

The lower left-most cell that has the lowest categories of consequence and likelihood should have the lowest rating. The lower left-most cell is the smallest cell among cells rated 1 and we easily find the cells ultimately rated 1 as we did in the proof of Proposition 1. According Proposition 1, if the cells rated 1 are determined, cells rated 2 will be determined. Similarly, we will determine the cells rated 3 given cells rated 2. Thus, we can now determine the ratings from low to high which are 1, 2, 3, and so on. The cells in the set $H_{highest}$ where the upper right-most cell is and any cell is not larger than any other one has the highest rating. Ultimately, every cell has a unique rating, which means the rating scheme is unique. ■

From the above process, we see that based on the three principles, the rating scheme can be designed by assigning ratings 1, 2, 3 and so on to cells in turn. Therefore, we call our method the ***sequential updating approach*** (SUA) to design the rating scheme of the risk matrix.

3.4.3 A Global Rating Algorithm

The proof process of the uniqueness theorem provides an algorithm for updating the cell rank for the sequential update method. This algorithm first compares the cells, then determines a subset of the cells, compares the other cells, determines their rank, and loops accordingly. This algorithm, referred to herein as the local rating algorithm, because it requires constant iteration of loops between local cells. This section will present a more convenient algorithm called the global rating algorithm for hierarchical design based on the sequential update approach. The global rating algorithm classifies the characteristics of each cell from a global perspective.

The global rating algorithm consists of three parts:

Part 1: Establishing the relationship between pairs of units;

Part 2: Find the characteristics of any level of cells;

Part 3: Divide the rank according to the corresponding cell characteristics.

Figure 3.3 above shows the schematic flow of the global rating algorithm. Establish a confidence level in advance. According to the above figure, the probability comparison matrix is first obtained. Then, based on the PCM, the determined rules for each unit are given.

Step 1: Establish a probability comparison matrix (PCM). An element in the PCM indicates the probability that any point in cell i is greater than any point in cell j. As mentioned earlier, there are two ways to get the probability. One is the result of mathematical analysis, and the other is the use of computer-assisted Monte Carlo simulation. Obviously, the mathematical analysis results are more accurate. However, when there are many cells in the risk matrix, it is not easy to calculate all the probabilities. It would be more convenient to cycle through all the probabilistic simulation methods within a computer program. Here are the details of the Monte Carlo simulation. Considering a risk matrix, the axes of the consequences are divided

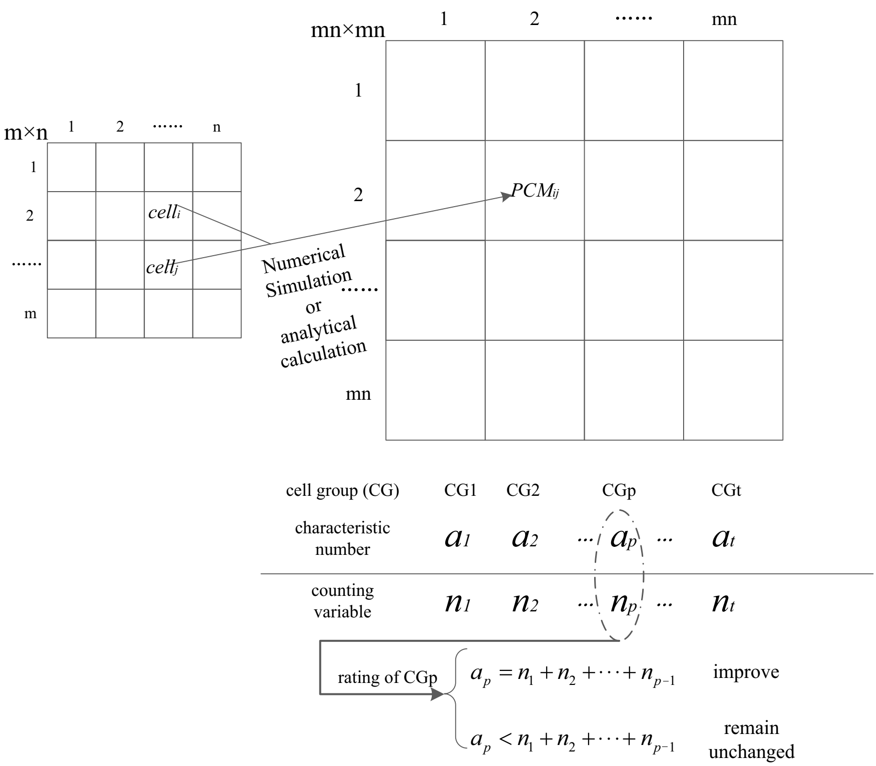


Fig. 3.3 The graphical explanation of the global rating algorithm

into *m* intervals on average, and the axes of the probabilities are equally divided into n intervals. First, *mn* cells are numbered 1 to *mn* in order from lower probability to higher probability, from lower consequence to higher consequence. The specific process is as follows:

Randomly generate two points located in *i* and *j* according to the consequence probability interval of cells *i* and *j*; (2) define the variable counter (initial value is 0), if the risk value of *j* is greater than *i*, Counter is incremented by 1, otherwise, it remains unchanged; (3) steps (1) and (2) are repeated K times to obtain the final size of the counter; (4) counter/K can be regarded as an approximation of the probability of any point in cell *i* is larger than cell *j* (the number of simulations K should be large enough so that when K becomes large, counter/K is almost constant). By repeating the above process, all items of the PCM can be obtained. PCM is a $mn \times mn$ matrix of one.

In the sub-steps below, as previously mentioned, for the sake of simplicity, the numbers 1, 2, … are used to indicate the cell level, and the larger numbers indicate the higher level.

Find out the characteristics of each cell. In each column of the PCM, the number of values satisfying $PCM_{ij} \geq \alpha$ is found, and is represented by "feature number". Count the number of features from small to large without repeating, denoted as $a_1, a_2, \cdots, a_t$. Then, use the "count variable" N_i to indicate the number of each a_i repetition (marked as 1 if there is no repetition).

Step 3: According to the characteristics of step 2, find out the rules contained in the level of the cell. Each unit level contains the following rule: First, the risk level of all feature numbers a_1 is recorded as 1. When the feature number is a_p, the rank is R, if $a_{p+1} < N_1 + N_2 + \cdots + N_p$, then the rank of the cell whose feature number is a_{p+1} is also R; otherwise, it is $R + 1$. This is because, when $a_{p+1} < N_1 + N_2 + \cdots + N_p$, the cell with the feature number a_{p+1} is only larger than the portion of the cell with the rank R, and if the level is raised, the consistent internality principle is violated.

By assigning each cell a level (color) based on the above steps, the risk matrix design is complete.

Like the LRA, GRA rates each cell in order. The two algorithms give the same results. The difference between them is as follows: with the use of LRA, the logical comparison and rating of cells are carried out alternately, and the determination of the level of each cell can be finally determined after several rounds of iteration. GRA separates logical comparison from rating. The GRA rates cells based on their global characteristics, the number of features, and counting variables, so that each cell is rated one round. The key point of GRA is to summarize the characteristics contained in the cell level. Although GRA and LRA have nearly the same computational cost (in theory, each cell pair needs to be compared), the GRA approach makes the process of updating the level of each cell clearer and easier to use.

3.4.4 Application: Design the Rating Scheme of a 4×4 *Risk Matrix Using SUA*

In this section, we provide a hypothetical case where the decision makers must assess some project risks with the same type of consequence (if the risks are of different types of consequence, we should normalize the consequences). They do not have sufficient data of the risks. Thus, they decide to employ a risk matrix to prioritize these risks. We are entrusted by these decision makers to design a reasonable risk matrix.

Preparation

To design the rating scheme of the risk matrix according the SUA, we should first investigate to identify the decision makers' needs for the risk matrix. So we designed some questions for the decision makers. We asked the decision makers to give well-thought-out answers to these questions. The questions and answers are supplied as follows:

Question 1. How many categories do you expect the consequence and likelihood of the risk matrix to have?
Answer. We believe it is appropriate if the consequence is divided into 4 categories and the same for the likelihood, which are "insignificant", "significant", "serious" and "major" for consequence, and "very improbable", "improbable", "probable" and "frequent" for likelihood.
Question 2. Based on your knowledge and experience, what are the intervals corresponding to the partitions of the consequence and likelihood you would like to provide?
Answer. For the consequence, we believe the following intervals are reasonable: [0, 2.5), [2.5, 5), [5, 7.5), and [7.5, 10] (unit: millions of dollars). We divide the likelihood into [0, 0.25), [0.25, 0.5), [0.5, 0.75), and [0.75, 1]. The larger values represent a more severe consequence or likelihood.
Question 3. Do you have any knowledge or experience concerning the distribution of the consequence and likelihood? If not, we assume that the consequence and likelihood are evenly distributed.
Answer. We currently have no idea concerning the distribution of the consequence and likelihood.
(For simplicity, we assume the consequence and likelihood are evenly distributed. However, if there is any additional information concerning the distribution of consequence and likelihood, it can be embedded into the design process.)
Question 4. In the design process, we must compare two cells, A and B. We compare A and B as follows: Randomly select one point each from A and B and compare them. If the probability that a point in A is quantitatively larger than a point in B is larger than a predefined threshold, we declare A is larger than B. So, what should be the minimum threshold?
(If necessary, we may need to supply a more detailed explanation of the method to compare two cells.)
Answer. It seems 90% is an appropriate choice.
Question 5. How many ratings do you expect the risk matrix to have?
Answer. 4 ratings at least.

The above is the minimum information we require. After normalizing the consequence axis from 0 to 1, we present the 4×4 risk matrix whose rating scheme must be designed in Fig. 3.4.

Get the probability comparison matrix

Probability comparison matrix (PCM) is the base for the further process. To get PCM, we need the information obtained from questions 1, 2, and 3, namely, the numerical intervals corresponding to the categories of consequence and likelihood, and the distribution of consequence and likelihood.

First, all the 16 cells are numbered "1–16" from lower likelihood to higher likelihood and lower consequence to higher consequence in turn (see Fig. 3.5).

Then, entries of the *PCM* can be obtained according to the steps we stated in Sect. 3.3. Table 3.2 gives the *PCM*. We provide two values of PCM_{ij}. The upper value

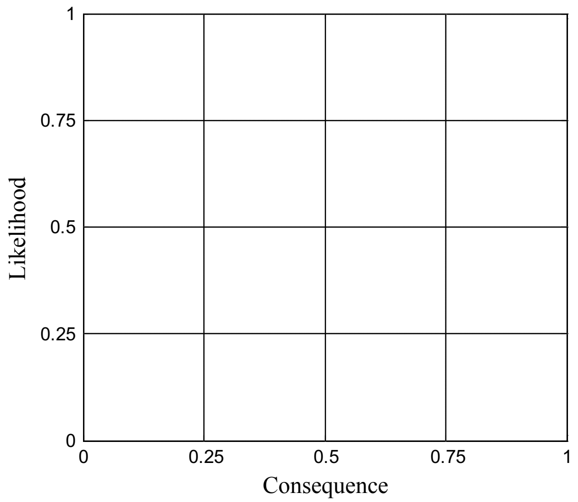


Fig. 3.4 A 4 × 4 risk matrix which needs coloring

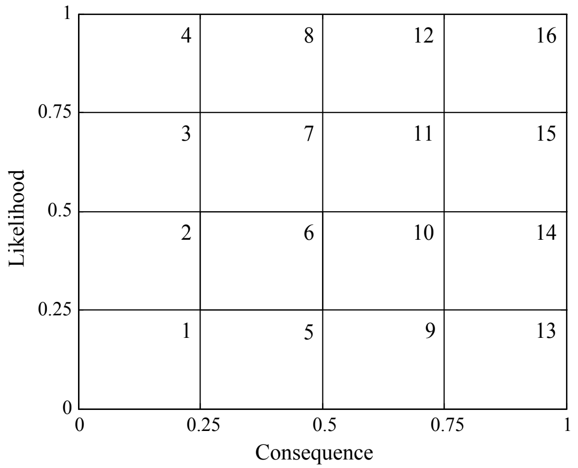


Fig. 3.5 A 4 × 4 risk matrix with cells numbered from 1 to 16

is the simulation result with simulation times $K = 10{,}000$ and the lower value is the analytical result. We find that there is very little difference between the true and estimated results. Therefore, we consider the simulation method is applicable especially when the distributions of consequence and likelihood are more complicated.

3.4.4.1 Rate the Cells According to the PCM

We have get the value of α from the decision makers, which is 0.9. To rate the cells, the following steps are necessary:

Table 3.2 Probability comparison matrix of a 4 × 4 risk matrix

Probability comparison		Cell number in Fig. 3.6									
		1	2	3	4	5	6	7	8	9	10
Cell number in Fig. 3.6	1	0.503/0.5	0.824/0.827	0.898/0.899	0.928/0.928	0.827/0.827	1	1	1	0.899/0.899	1
	2	0.172/0.173	0.501/0.5	0.697/0.696	0.783/0.784	0.497/0.5	0.979/0.980	1	1	0.696/0.696	1
	3	0.100/0.101	0.304/0.304	0.503/0.5	0.641/0.640	0.304/0.304	0.833/0.836	0.993/0.993	1	0.503/0.5	0.993/0.993
	4	0.071/0.072	0.217/0.216	0.361/0.360	0.505/0.5	0.218/0.216	0.646/0.644	0.929/0.931	0.997/0.997	0.356/0.360	0.929/0.931
	5	0.170/0.173	0.495/0.5	0.696/0.696	0.783/0.784	0.505/0.5	0.981/0.980	1	1	0.700/0.696	1
	6	0	0.020/0.020	0.167/0.164	0.352/0.356	0.020/0.020	0.501/0.5	0.924/0.924	0.997/0.997	0.162/0.164	0.924/0.924
	7	0	0	0.008/0.007	0.068/0.069	0	0.080/0.076	0.504/0.5	0.847/0.845	0.007/0.007	0.492/0.5
	8	0	0	0	0.003/0.003	0	0.003/0.003	0.146/0.145	0.499/0.5	0	0.147/0.145
	9	0.100/0.101	0.307/0.304	0.503/0.5	0.637/0.640	0.305/0.304	0.839/0.836	0.994/0.993	1	0.505/0.5	0.993/0.007
	10	0	0	0.007/0.007	0.071/0.069	0	0.080/0.076	0.503/0.5	0.854/0.845	0.007/0.007	0.497/0.5
	11	0	0	0	0	0	0	0.026/0.028	0.263/0.264	0	0.028/0.028
	12	0	0	0	0	0	0	0	0.014/0.014	0	0
	13	0.073/0.072	0.213/0.216	0.362/0.360	0.496/0.5	0.218/0.216	0.639/0.644	0.935/0.931	0.997/0.997	0.364/0.360	0.930/0.931
	14	0	0	0	0.004/0.003	0	0.003/0.003	0.148/0.145	0.499/0.5	0	0.147/0.146
	15	0	0	0	0	0	0	0	0.015/0.014	0	0
	16	0	0	0	0	0	0	0	0	0	0

(continued)

Table 3.2 (continued)

Probability comparison		Cell number in Fig. 3.6					
		11	12	13	14	15	16
Cell number in Fig. 3.6	1	1	1	0.928/0.928	1	1	1
	2	1	1	0.781/0.784	1	1	1
	3	1	1	0.639/0.640	1	1	1
	4	1	1	0.499/0.5	0.996/0.997	1	1
	5	1	1	0.787/0.784	1	1	1
	6	1	1	0.359/0.356	0.997/0.997	1	1
	7	0.971/0.972	1	0.066/0.069	0.854/0.855	1	1
	8	0.735/0.736	0.986/0.986	0.003/0.003	0.501/0.5	0.986/0.986	1
	9	1	1	0.642/0.640	1	1	1
	10	0.972/0.972	1	0.071/0.069	0.853/0.854	1	1
	11	0.496/0.5	0.939/0.939	0	0.264/0.264	0.938/0.939	1
	12	0.061/0.061	0.503/0.5	0	0.014/0.014	0.496/0.5	0.967/0.969
	13	1	1	0.499/0.5	0.997/0.997	1	1
	14	0.739/0.736	0.984/	0.004/0.003	0.506/0.5	0.986/0.986	1
	15	0.065/0.061	0.501/0.5	0	0.014/0.014	0.503/0.5	0.970/0.969
	16	0	0.032/0.031	0	0	0.030/0.031	0.499/0.5

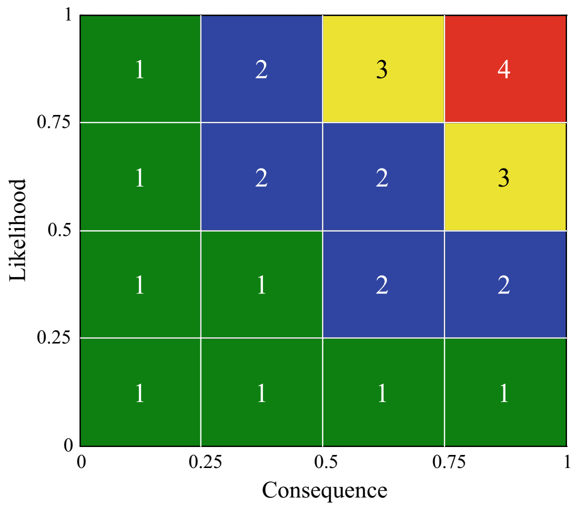


Fig. 3.6 Rating scheme of the 4 × 4 risk matrix

Table 3.3 Description of all the cells

Description	Cell group						
	1	2	3	4	5	6	7
Corresponding cells	1, 2, 3, 5, 9	4,13	6	7, 8, 10, 14	11	12, 15	16
Characteristic number	0	1	3	8	10	13	15
Counting variable	5	2	1	4	1	2	1
Rating	1	1	1	2	2	3	4

Step a1: Obtain the characteristic numbers of each cell and list the numbers without repetition with their corresponding counting variables (Table 3.3).

Step a2: Rate the cells according to the characteristic numbers of each cell. Cells 1, 2, 3, 5 and 9 are rated as "1." Cells 4, 13 and 6 are still rated "1" because $a_2 < N_1$ and $a_3 < N_1 + N_2$. Cells 7, 8, 10 and 14 are rated "2" because $a_4 = N_1 + N_2 + N_3$ and so on.

We get the rating of cell by the above operations. The rating scheme is presented in Fig. 3.6.

Sensitivity analysis of rating confidence

When we get the rating scheme of the risk matrix as shown in Fig. 3.6, we should turn back to see whether the designed risk matrix meets the decision makers' need. One may find that once α is determined, the rating scheme of the risk matrix is unique. Thus, it is possible that the resolution of the risk matrix is lower than what the decision makers require. We will show how the SUA deals with the problem next.

Since a specific α corresponds to a specific design of a risk matrix, a question of interest concerning the SUA is how rating schemes change as α changes. Intuitively, a

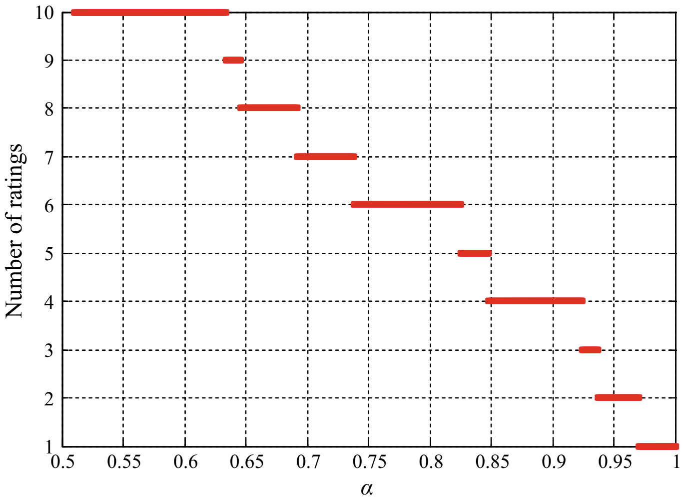


Fig. 3.7 Relationship between the rating number and α

higher α means the decision maker requires a stronger difference between two ratings, and there will thus be fewer ratings. To more intuitively reflect on the relationship between the number of ratings and α, a sensitivity analysis of α is performed.

As we presented, $\alpha \leq 0.5$ is meaningless, and α thus varies from 0.5 to 1. In particular, a $n \times n$ risk matrix with axes of consequence and likelihood evenly divided can be divided into $(n^2 + n)/2$ at most since it is symmetric around the lower-left-to-upper-right diagonal. Figure 3.7 shows the rating numbers of the 4×4 risk matrix mentioned above as α changes.

We notice that when $\alpha = 1$, there will be only one rating, which further explains why the transitional rating "yellow" is required according to Cox's method. This is because without transitional rating, there cannot be more than one rating. According to the result of Fig. 3.7, a smaller α corresponds to a higher resolution, but it is at the cost of the reliability of the ratings because a smaller α means the allowance of a larger difference between cells with the same rating. As a result, decision makers should perform trade-offs between higher resolution and higher reliability of ratings. From another perspective, the sensitivity analysis reported here supplies multiple choices for decision makers, based on the resolution they expect to have. In practice, an $m \times m$ risk matrix usually has m risk ratings and therefore choosing α to ensure that an $m \times m$ risk matrix is divided into m risk ratings is a reasonable choice.

We designed question 4 and 5 before to obtain the decision makers' need for accuracy and resolution of the risk matrix. If the designed risk matrix cannot satisfy both of the requirement on higher accuracy and higher resolution, we need to feedback to the decision makers, and give them other possible designs with different α and resolution.

In Fig. 3.8 we provide another four rating schemes of the same 4×4 risk matrix as α changes. When α becomes larger (representing that the decision maker provides

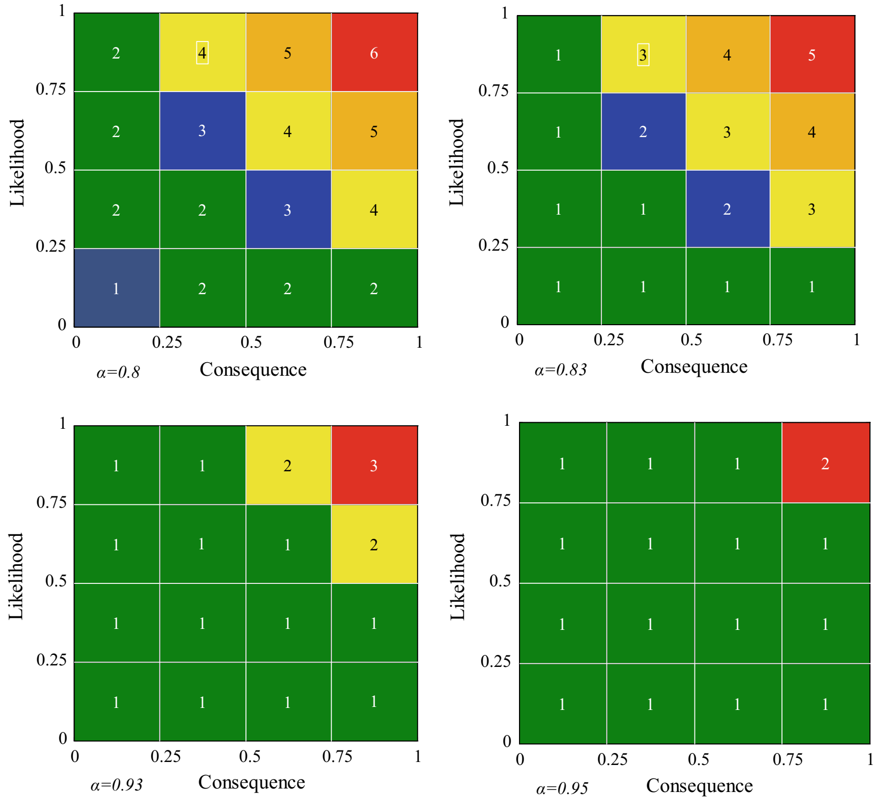


Fig. 3.8 Different rating schemes as α changes

stricter criteria based on which cell is larger than another), the number of the risk ratings becomes smaller. However, when α is set too large, for example, $\alpha = 0.95$, the rating scheme designed seems useless since there are only two ratings and only one cell belongs to the risk rating "2" (red). Therefore, decision makers must determine what resolution they expect to receive from the risk matrix.

Finally, we argue that the sensitivity analysis performed here is not contrary to the uniqueness of the rating scheme according to the SUA. α is predefined, reflecting the decision maker's acknowledgment of the accuracy of the criteria based on which two cells are compared. In other words, the choice of α is affected by the decision maker's requirement of resolution or accuracy. However, once α is determined, the rating scheme of a risk matrix is unique.

3.5 Summary and Comparison of the Rating Scheme Design Methods

3.5.1 Summary

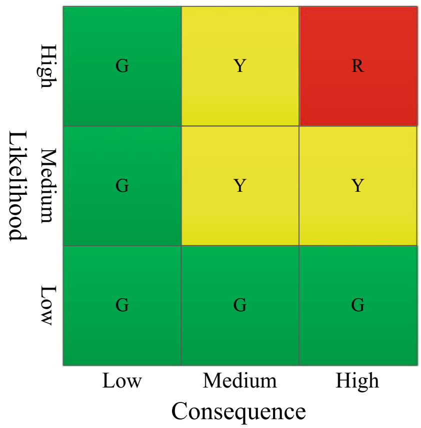
The accuracy of risk matrix design directly affects the accuracy of risk assessment by decision-makers using risk matrix, but there is no uniform design standard yet. The risk matrix design steps include: defining input variables, classifying input variables, and assigning a risk level to the cells corresponding to each set of input variables. The third step is the most critical.

For qualitative risk matrices, that is, the categories of consequence and likelihood are risk matrices described by qualitative language. The design rules are as follows: cells along the same line have the same risk, and adjacent cells have the same risk level.

Holt used different utility functions to determine the level of a cell when designing a qualitative risk matrix. For example, the consequences and probabilities are divided into three categories: "high", "medium", and "low". The utility function corresponding to the "minimum matrix" is Minimum (the level of the consequences, the level of the probability), such as Minimum (very low, high) = very low. A 3 × 3 risk matrix design result is shown in the Fig. 3.9. The drawback of this approach is that the dimensions of the consequences and probabilities must be the same, and the final number of risk levels must equal the number of levels of consequences or probabilities.

In the research of quantitative risk matrix, Cox first proposed three axioms that a reasonable risk matrix should satisfy in its review of the risk matrix, namely weak consistency axiom, **betweenness axiom**, **consistent coloring** axiom. These three theorems illustrate the relationship that should be satisfied between the colors of different cells.

Fig. 3.9 A 3 × 3 risk matrix



According to Cox's three theorems, a 3 × 3 risk matrix has a unique design result as shown in Fig. 3.10. Although Cox's theorem is a systematic approach to the design of risk matrices, there are several deficiencies in this approach. The first is that the risk matrix designed according to the Cox method can only have 3 colors, namely red, yellow and green. According to Cox's assertion, yellow is still an unrecognizable level, then only two colors are valid. Can be used to distinguish the size of the risk. Obviously, this feature does not meet the risk matrix itself needs sufficient resolution (sufficient resolution refers to the number of risk levels is sufficient) to give the risk rating requirements. The second is that the risk matrix designed according to the Cox method is not unique. For example, for a 4 × 4 risk matrix, both designs as shown Fig. 3.11 are possible. The non-uniqueness of this design leads to the inability to judge in practice which design should be adopted, for the same risk, for example, the

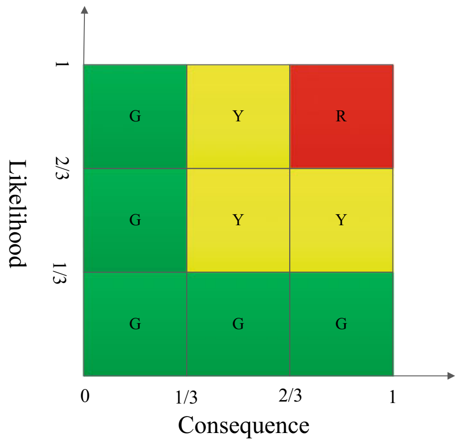


Fig. 3.10 Rating scheme of a 3 × 3 normal risk matrix guided by Cox's three axioms

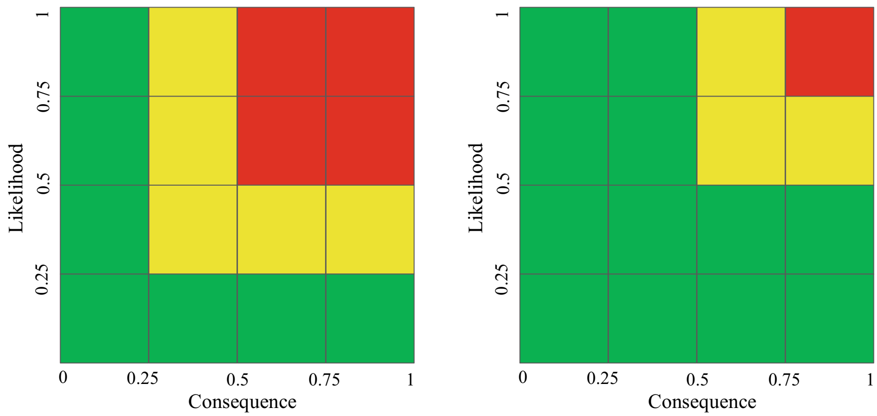


Fig. 3.11 Rating scheme of a 4 × 4 normal risk matrix guided by Cox's three axioms

Table 3.4 Description of three different types of risk matrices

Risk matrix type	Category 1	Category 2	Category 3	Category 4	Category 5
Qualitative risk matrix	Very low	Low	Medium	High	Very high
Cox risk matrix	[0, 20%]	[20%, 40%]	[40%, 60%]	[60%, 80%]	[80%, 100%]
SUA risk matrix	[0, 20%]	[20%, 40%]	[40%, 60%]	[60%, 80%]	[80%, 100%]

consequences and probability are [0.25, 0.5] risk, in the figure In the two matrices, one rank is "yellow" and one rank is "green".

Aiming at the deficiency of the Cox method, a sequential updating approach is proposed to design a reasonable risk matrix based on the improvement of Cox's three theorems. Based on the logical relation between the cells of the risk matrix, the sequential updating approach proposes the criterion for comparing the size relation between any two cells and then uses the three principles of weak consistency of adjustment, consistent internality, and continuous upgrade to regulate the requirements of the cell grade in the risk matrix. Then, a global rating algorithm is proposed to design the risk matrix on the operational level. The risk matrix designed by the sequential updating approach is unique and high resolution, and the decision-maker can also choose the resolution according to his own needs. Obviously, these two features of the sequential updating method are more in line with the requirements of the risk matrix in practice.

3.5.2 Comparison

Different risk matrix design methods use different interval descriptions, as shown in the following Table 3.4.

If we qualitatively describe the two inputs to the risk matrix, we can treat each cell as a point, which can set a score threshold for the qualitative risk matrix to classify risks. However, if the input uses quantitative description, there are infinite points in the cell. Therefore, if we compare two cells, cox's matrix design method only uses the highest or lowest quantitative points, while the others are not used, which means that most of the information in one cell will be ignored. Moreover, we believe that each cell shares a cross-section with other cells (meaning that two cells have the same quantitative high point), making it difficult to classify two adjacent cells, resulting in a lower resolution of the scoring scheme. Although SUA uses quantitative interval to describe the input categories, the probability comparison matrix in the steps of the global rating algorithm used by SUA covers most of the information of each cell and also satisfies the mathematical logic. The risk matrix designed by SUA can also choose the resolution according to the requirements.

The Table 3.5 lists some of the differences between the three design rules. First, according to our common sense, rating schemes that rely heavily on subjective judgments often lack scientific analysis and are therefore somewhat unreliable. By

Table 3.5 Comparison of different risk matrices design by different methods

Risk matrix type	Input category	Risk measure	Reliability of rating	Classification of scores	Resolution of rating scheme	Uniqueness of rating scheme
Qualitative risk matrix	Qualitative	Risk = consequence + likelihood	Low	High dependence on subjective judgment	High	Yes
Cox risk matrix	Quantitative	Risk = consequence × likelihood	High	Independence on subjective judgment	Low	No
SUA risk matrix	Quantitative	Risk = consequence × likelihood	High	Independence on subjective judgment	High	Yes

comparison, we find that the risk matrix of Cox design satisfies strong mathematical logic, but high reliability will lead to non-uniqueness of the rating scheme; secondly, more reliable rules are at the expense of low resolution. However, the risk matrix designed by SUA not only satisfies mathematical logic, but also has high reliability and resolution, and is unique.

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Chapter 4
Risk Perceptions in Risk Matrix: Sources and Impact to Risk Matrix Design

4.1 Introduction

In the field of risk management, risk matrices are popular tools due to their intuition and simplicity to handle subjective judgments. They are widely used in both the private sector and public organizations, especially when data is insufficient (Chen et al. 2020; Li et al. 2018). For example, during the recent COVID-19 pandemic, the risk matrix, which is a typical quick risk assessment tool, is widely used in assessing risks in major public health emergencies. A risk matrix assesses risk based on two inputs: typically, these are consequence and likelihood or other, similar factors. The two inputs are divided into several categories, which are textually described with words like “high”, “medium”, “low” and so on. The user of a risk matrix needs to provide estimations of the two inputs based on their experience and knowledge. Each combination of the two inputs corresponds to a particular cell in the risk matrix. The risk matrix outputs a risk rating for each cell, usually represented by a particular color, denoting the severity of a risk (Li et al. 2018; Bao et al. 2018).

The process of using risk matrices reveals that the designers of a risk matrix should have the same conception of the matrix as the matrix’s users; this conception includes factors such as (1) the categorization of the two criteria, (2) the measure of risks in the risk matrix, and (3) the rating mapping to consequence and likelihood (Li et al. 2018). Otherwise, the decisions made by users based on the risk matrix will be inaccurate. Some risk matrix critics consider risk matrices as defective for these tools rely too much on subjective judgments, and thus the users can not always reach an agreement on the rating of a risk (Duijm 2015; Ball and Watt 2013). Subjectivity is the nature of the risk matrix tool, accompanied by its simplicity and intuitive nature (Li et al. 2018), for it needs the estimates of consequence and likelihood of a risk as the inputs. When this criticism is proposed, it ignores the feedback between the design (or designer) and the use (or user) of a risk matrix (Fig. 4.1), namely, the user must tell the designer his/her perception of the risk, and designer embeds the user’s perception into the design process to establish a more accurate risk matrix, which forms the feedback. If the feedback is well handled, the assessment of a risk

C. Bao et al., *Risk Matrix*, Innovation in Risk Analysis,
https://doi.org/10.1007/978-981-19-1480-5_4

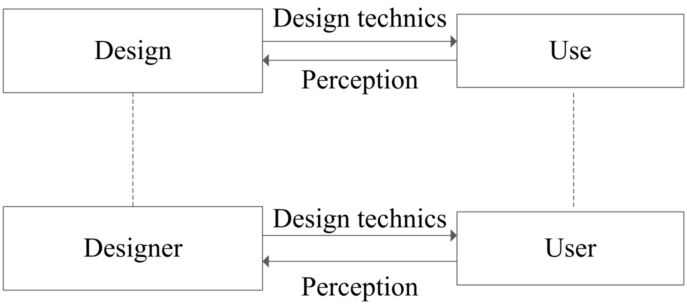


Fig. 4.1 Feedback between the design and use of a risk matrix

based on the corresponding risk matrix will have taken the users' subjectivity into consideration, and thus the assessment will be more accurate than that based on the risk matrix without considering the subjectivity. This raises the questions that how the perception of risks affects the design of a risk matrix, and how to embed the perception into the design process, which are novel issues in the risk matrix related researches.

Risk perception has been proven to exist in the risk management field (Burns and Slovic 2012; Slovic 1987; Rundmo and Nordfjærn 2017). According to a study by Fischhoff et al. (1978) that used psychometrics, nine general properties of the risk source are important: voluntariness of risk, immediacy of effect, knowledge about the risk by the person exposed to the risk source, knowledge about the risk in science, control over the risk, newness, chronic/catastrophic, common/dread, and severity of consequences. These nine properties cause perceptions of risk to vary in different fields. Thus, researchers have explored many kinds of perceptions, such as pedestrians' risk perception of traffic accidents (Rankavat and Tiwari 2016), the perception of risk in the fishing industry (Booth and Nelson 2014), and so on (Taylor and Snyder 2017; Zhao et al. 2016). Researchers underscore that risk perception should be added as a variable to occupational safety research models, such as the Health Belief Model (Rosenstock 1974).

Risk perception, in short, can be explained as how a stakeholder feels about a risk (Taylor and Snyder 2017; Rundmo and Nordfjærn 2017). This is a highly relevant concept in the risk matrix where risk is finally assessed as a particular level, and risk matrices are concrete tools that measure the perception of risks. Goerlandt and Reniers highlighted that in different approaches to risk analysis, risk perceptions are used in the risk ranking process. Obviously, risk ranking is the main goal of using risk matrices, and therefore risk perceptions must be highlighted in risk communication of risk matrices. As discussed before, in different fields, factors affecting risk perception are various. We care what should be the factors affecting the risk perception in risk matrices, namely, the risk rating of risk (Goerlandt and Reniers 2017). Obviously, as Ale et al. stated, "you cannot derive legitimate quantitative (QUANT) outputs from PCDS (probability consequence diagrams) when they are an essentially qualitative (QUAL) presentation" (Ale et al. 2015). Thus, the uncertainty, which is the reason for the qualitative presentation, about the quantitative settings in risk matrices makes the perception of risk different. These quantitative settings could be (Ale et al. 2015):

- What are the correct ordinates?—Probabilities, frequencies of events, outcomes, etc.?
- One or both linear scales, or logs, powers?
- Discrete points or area averages?
- Single points or distributions?
- Completeness?
- Uncertainties?
- "Level of Risk" (total, components).
- Criteria, acceptability, tolerance, appetite.
- Calibration with records, reality?

We contend that the perception of risk could be decomposed into the perceptions of different aspects, each of which may affect the final perception of risk. To construct the decomposed perception structure, it is started from the definition of risk in the risk matrix. Generally, the output of the risk matrix, namely, the rating of risk could be treated as the function of consequence and likelihood, e.g., $risk\, rating = f(consequence, likelihood)$ (Pickering and Cowley 2010). This function reveals that two main dimensions are important: (1) the inputs, and (2) the concrete function form. For the input dimension, users are not required to give precise estimations of the two inputs. The estimations are allowed to vary within an interval, which results in so-called input data biases (Smith et al. 2009). It is possible that for a particular category of likelihood, different users may give different quantitative intervals. Besides, researchers found that sometimes it is confused for risk matrix users to determine which category the consequence or likelihood of the assessed risk should belong to (Ball and Watt 2013). And for the function form dimension, though, in a risk matrix, the risk is usually measured by the product of consequence and likelihood, other forms are possible, such as the logarithmic form and so on (Levine 2012; Chen et al. 2020). Moreover, in risk management (only adverse events are considered here), decision-makers are possibly risk-averse besides risk-neutral (Bedford 2013; Thomas et al. 2014). One may refer to Li et al. (2018; Cox 2008; Iec 2009) for examples of risk matrices designed with and without risk aversion. Obviously, different people have different perceptions of these settings, and the variety of perceptions makes it difficult to design a satisfying risk matrix to assess risks.

Although risk matrix design is usually based on subjective judgment, Li et al. (2018) warn that "we should avoid believing that because we are using qualitative risk management tools, the risk matrices should be designed in an entirely subjective way." Several sets of rules have been proposed for designing risk matrices. Cox (2008) provided three axioms that a reasonable risk matrix should satisfy. Li et al. extended Cox's rules and proposed the sequential updating approach (SUA) to design unique risk matrices. In practice, the SUA seems more robust and effective in practice, and thus is adopted here.

4.2 Identifying Risk Perception in Risk Matrices

In this section, we explore the types of risk perception affecting the design of risk matrices instead of finding the factors that affect the perception of risk. We conclude based on the literature that perception is always object-oriented (Zhao et al. 2016; Booth and Nelson 2014). To clarify where risk perceptions derive in risk matrices, we must first have a thorough understanding of the design process.

As Li et al. stated, three steps are necessary for designing a risk matrix, namely, (s1) categorizing the inputs, (s2) defining the risk measure, and (s3) assigning each cell a particular risk rating. The first two steps are the preliminaries of the third step, and the third step is the core part of the design (Li et al. 2018). The types of perception identified in this section are based on the first two steps.

A risk matrix has a special structure as shown in Fig. 4.2. When assessing risk, users need to estimate and assign particular categories to the inputs, after which the risk matrix outputs a risk rating. The process of outputting a risk rating is complicated. The designers should first obtain the users' definition of the inputs and the measure of risks. Then the designers must use an effective set of rules to map the cells onto risk ratings. Throughout the process of designing risk matrices, most judgments (e.g., determining the categories of inputs) rely on the users' feelings or attitudes towards the perceived risk. As researchers have pointed out, uncertainty is a major factor affecting the perception of risks (Booth and Nelson 2014; Goerlandt and Reniers 2016). In this subsection, we identify various types of risk perception in risk matrices and show how they are affected by different kinds of uncertainty.

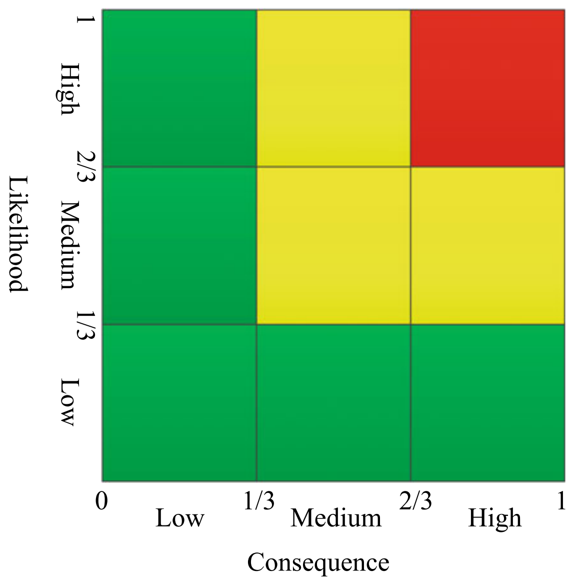


Fig. 4.2 A typical 3 × 3 risk matrix

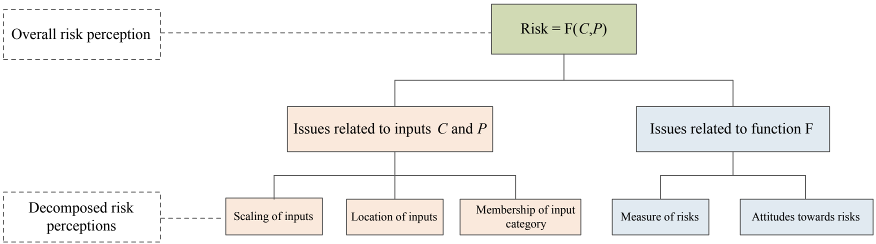


Fig. 4.3 Decomposition of risk perceptions in risk matrices

Figure 4.3 displays the five kinds of risk perceptions in risk matrices. Specifically, we start from the general measure of risks, which usually has the form like $risk = f(consequence, likelihood)$ (Aven 2012). In quantitative risk assessment, the risk is measured as a particular value, and the overall risk perception towards the perceived risk is based on this concrete value. While in risk matrices, even though the assessed risk corresponds to a cell instead of a single point, a risk matrix is a graphical representation of such risk measure (Ale et al. 2015). One may find that the elements contained in the risk measure correspond to the three steps for designing a risk matrix: inputs C and P correspond to step (s1), function F corresponds to step (s2), and the overall risk corresponds to step (s3). Obviously, the overall risk is determined by both the inputs and the risk measure. Therefore, we the overall risk perception into two parts which are related to the inputs and the risk measure. Then, these two parts are decomposed into more smaller parts as shown in Fig. 4.3. In the following, we will explain why these decomposed risk perceptions are proposed in detail.

4.2.1 *Perception of the Scaling of Inputs*

The first step of designing risk matrices is to define the intervals of inputs. The designers should ask the users to give well-thought-out definitions of the categories of consequence and likelihood. For example, if the consequence of a risk matrix needs to be divided into four categories—in this case, negligible, marginal, critical, and catastrophic—the designers and users should give quantitative descriptions of these four categories.

In risk matrices, the categories of inputs are textually described using adjectives. However, people may link the same adjective to different quantitative intervals. For example, in the field of drilling hazard management, Pritchard et al. thought the category "catastrophic" corresponded to the interval of $[\$20M, +\infty)$. While the U.S. Air Force described "catastrophic" as $[\$1M, +\infty)$. Even in the same category of the same risk matrix, individuals have different conceptions of risk. Payne asked respondents to provide estimates of the same consequence, and the scores varied among respondents (Payne 2014). This is because individuals have different knowledge regarding the risk, and uncertainty makes a uniform scaling of the inputs difficult. We call this kind of perception towards the scaling setting of the risk matrix "perception of the scaling of inputs".

Form of the perception of the scaling of inputs. For the consequence or likelihood of the risk matrix, the intervals of the categories can be normalized as $[c1, c2)$, $[c2, c3)$, …, $[cn, 1]$ or $[l1, l2)$, $[l2, l3)$, …, $[ln, 1]$. If decision-makers have differing perceptions of the scaling of the inputs, they may assign different values to the intervals, which will affect the design of the risk matrix.

In literature, consequence and likelihood axes are usually assumed to be evenly divided (Li et al. 2018). For example, in (a) of Fig. 4.4, evenly divided axes for the 4×4 risk matrix mean that the categories of consequence and likelihood are both partitioned with the quantitative intervals [0, 0.25), [0.25, 0.5), [0.5, 0.75), and [0.75, 1], where each category has the same length of 0.25. We provide two other types of possible settings for the input scaling (here we want to show two regular examples where the inputs are not evenly divided; in fact, any example can be presented and thus, whether the two examples are usually used in practice is trivial). One is the arithmetic progression form, which means that the length of the category increases with a fixed step. The setting of the arithmetic progression form is shown in (b) of Fig. 4.4 with the intervals [0, 0.1), [0.1, 0.3), [0.3, 0.6), and [0.6, 1], with fixed-step 0.1. The other possible kind of setting is the geometric progression form, where the length of the category increases geometrically with a fixed ratio. In Fig. 4.4, (c) presents a particular geometric progression form with intervals [0, 1/15), [1/15, 3/15), [3/15, 7/15), and [7/15, 1], with the ratio of 2. Obviously, given different perceptions towards the settings of input scaling, the structure of the risk matrix will change. From (a) to (c) in Fig. 4.4, the cells show more difference in area, and the cells with a low rating of inputs become more compressed. It is possible that the design of the risk matrix will change accordingly. Notice that, in Fig. 4.4, we show the same design under three different forms of input scaling. The accuracy of the designs in Fig. 4.4 will be discussed in Sect. 4.4.

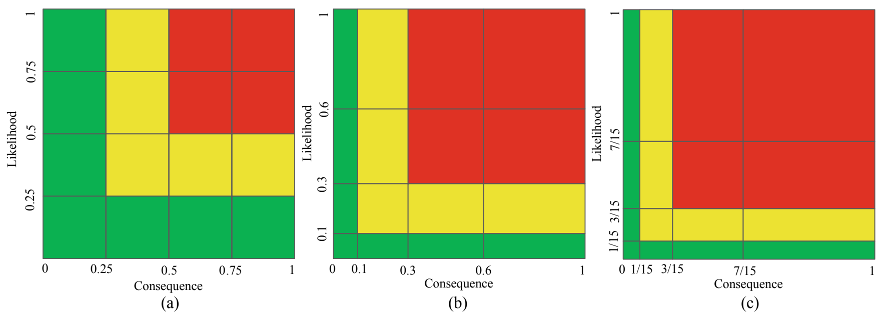


Fig. 4.4 Three different settings of scaling of inputs

4.2.2 *Perception of the Location of Inputs*

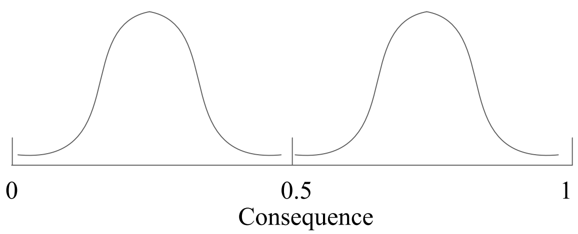
In 1951, Payne found that "Given a list of numbers [respondents] are prone to choose those near the middle of the list" (Pickering and Cowley 2010; Payne 2014). Similarly, in risk matrices, for a particular interval of a category of inputs, users may think the consequence or likelihood of the assessed risk will not be equiprobably located at any point of the interval based on their knowledge about the risk. In other words, the inputs may not be evenly distributed in an interval. For example, if the "medium" category of the consequence corresponds to the interval of $(0.4,\ 0.6]$, when using this risk matrix to assess risk, the user assigns the category "medium" to the consequence of the risk because he thinks the consequence is near 0.5 and 0.5 is in the interval of $(0.4,\ 0.6]$. In other words, for this risk assessor's perception, the consequence is not evenly distributed on the axis. We term this kind of perception in risk matrices "perception of the location of inputs".

In addition, the consequence may correlate to the likelihood. Smith et al. found that, given a collection of data of the estimation of consequence and likelihood, datum points tend to be located on the diagonal that is upward and rightward from the origin (Smith et al. 2009). Hong used a copula to obtain the joint distribution of consequence and likelihood (Cook 2008). This is another kind of perception of the location of inputs that considers their correlation.

Form of the perception of the location of inputs. Users may distribute the input in a particular category according to their knowledge. Generally, we denote the distribution of consequence by $f(C, L)$ and that of likelihood by $g(L, C)$, where their correlation is considered. For example, if users think the input is most probably located at the central part of the category, and the probability of its being located at the edge of the category gradually dwindles, a normal distribution may be reasonable. If the users think the input is equiprobably located at any point in an interval, a uniform distribution is suitable.

Of course, the perception of the location of inputs is also derived from the uncertainty regarding the cognition of the assessed risks. To visually present the uncertainty, we first consider one input at a time. We assume that, in each category of consequence, the consequence is normally distributed, namely, $N(0.5,\ 0.01)$ as shown in Fig. 4.5. This assumption is reasonable; for example, in practice, decision-makers tend to consider the consequence to be located in the center of the category with high probability as has been found by Payne.

Fig. 4.5 Normal distribution of inputs with 2 categories



If both the consequence and likelihood are normally distributed in each category, the points in a cell are distributed as Fig. 4.6.

Finally, we assume that the consequence is positively correlated to likelihood. Both of the inputs are normally distributed in each category of inputs. The distribution of the risk points in the risk matrix is shown in Fig. 4.7. In each cell, we assume that the binary variable ($consequence$, $likelihood$) obeys the normal, two-dimensional distribution with a variance of 0.01 and a correlation coefficient of 0.8. In this case, the points are distributed around the diagonal in each cell. Therefore, if the correlation between the two inputs is embedded, the information on the points in each cell changes.

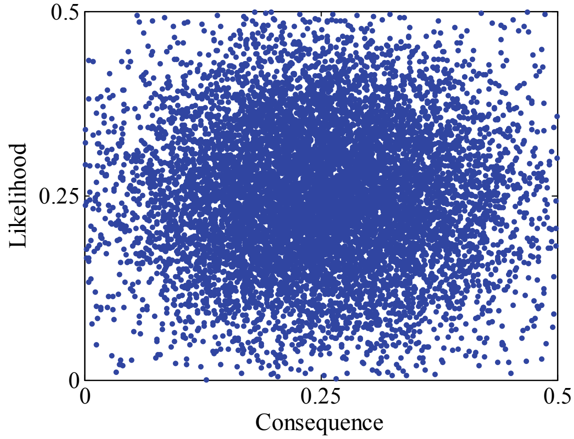


Fig. 4.6 Risk points in a cell with two normally distributed inputs

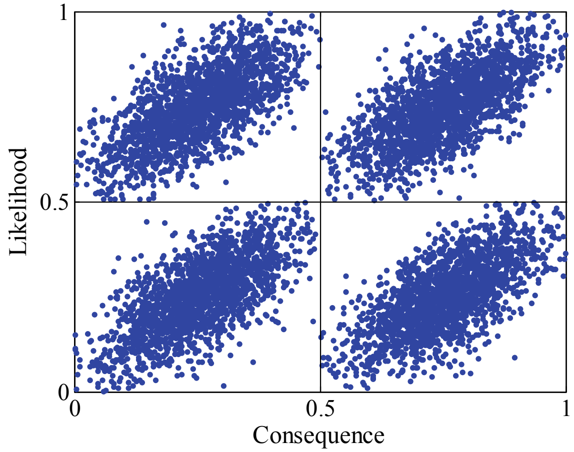


Fig. 4.7 Location of the risk points considering the correlation between the two inputs

4.2.3 *Perception of Input Category Membership*

In a risk matrix, it is common that decision-makers cannot assign each input of an event a correct category. This is because fuzzy information prevents the determination of the membership of the inputs to particular categories (Levine 2012). For example, Zhao et al. assessed the stakeholders' perceptions of risk in construction by asking the participants to judge the categories of consequence and likelihood of the same event (Zhao et al. 2016); the participants chose different categories. Ball and Watt found that, given different times to assess three kinds of risks, the same participants chose different categories for the inputs in risk matrices (Ball and Watt 2013). These studies reveal that the ability to determine the input category of an assessed risk is affected by many factors, such as the information possessed by the respondents, the mental processing related to the response time, the respondents' beliefs, and so on. All the uncertainty about these factors is the source of the fuzzy information. Fuzzy information leads to hesitation at deciding which category an input should belong to. Therefore, the perception of the proper categorization of the assessed risk may vary. We term this kind of perception "perception of input category membership".

For example, let us suppose that, in a risk matrix, two categories of consequence, "medium" and "high", are assigned the intervals (0.3, 0.6] and (0.6, 1]. The user of the risk matrix thinks the consequence of the assessed risk is around 0.6. In this case, he cannot determine the category to which the consequence belongs. The membership function which is usually used in fuzzy systems is adopted to describe the uncertainty of determining the membership of the input's category. This tool is introduced below.

Form of the perception of input category membership. The perception of input category membership can be expressed by the membership function, which is adapted to represent different grades of membership (Bao et al. 2018; Zadeh 1965). In applications, membership functions are usually assumed to be trapezoidal as shown in Fig. 4.8. When x is in the interval of $[a_2, a_3]$ and $[a_4, a_5]$, it 100% belongs to set 1 and set 2. When x is in the interval of $[a_1, a_2]$, the grade of x belonging to set 1 increases from 0 to 1. In a risk matrix, there may be an overlapping part

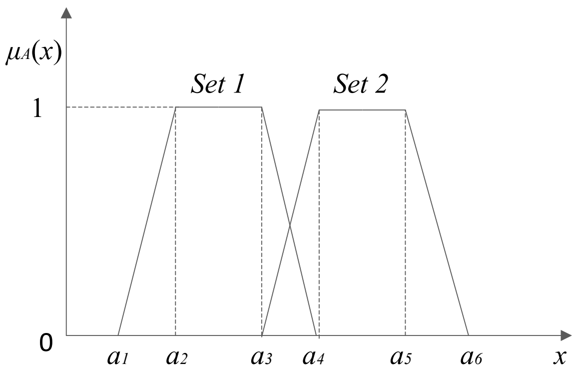


Fig. 4.8 An example of a trapezoidal membership function

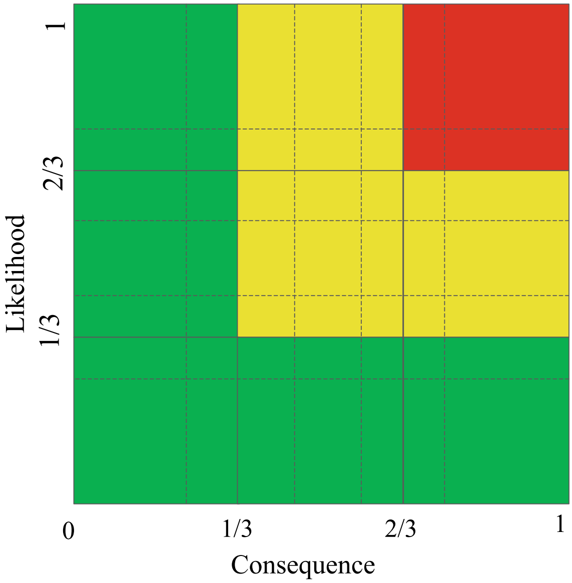


Fig. 4.9 Visualization of the fuzzy part of the inputs' adjacent categories

between the intervals of two categories, which is treated as a fuzzy part ($[a_3, a_4]$ in Fig. 4.8). For example, let us suppose that the designers of a risk matrix think that the category "low" corresponds to the interval of [0.3, 0.6] and the category "medium" corresponds to the interval of [0.5, 0.8]. This means [0.5, 0.6] is the overlapping of the two categories. The overlapping part exists because risk matrices are essentially qualitative tools and the boundary of two categories of the inputs is difficult to distinguish (Bao et al. 2018).

Figure 4.9 visualizes the fuzzy part of the intervals of the inputs' adjacent categories. If in each category, the assessed inputs of the risk belong to the category with the degree of 1, the boundaries of each cell are the original ones, which are the solid lines. When the degree of membership is not 1, the boundaries expand. The dashed lines outside a category are the new boundaries of each cell; this means risks that have the same consequence or likelihood may be assigned to different risk categories.

4.2.4 Perception of the Measure of Risks

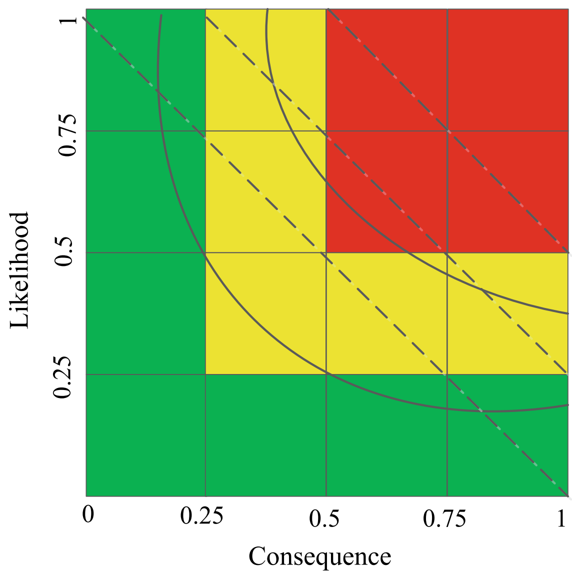
In quantitative risk matrices, the measure of risks is very important in the mapping of risk ratings to each cell. The difficulty of understanding the measure of risks in quantitative risk matrices is that the inputs are described by intervals and, thus, the cell cannot be denoted by a particular score. In a cell, there are infinite risk points. The risk measure works in giving the risk points different scores. Mapping a risk rating to a cell is based on the information that these infinite risk points provide (Bao et al. 2018). This is why risk measure is important in risk matrices.

Although the multiplication measure of risk, $risk = consequence \times likelihood$, is the most common one in the risk matrix, other forms of measure do exist. Li et al. (2018) found that, in some risk matrices, the risk is measured by the addition formula. Ni et al. (2010) provided different kinds of risk measures in risk matrices. We term this kind of perception "perception of the measure of risks".

Form of the perception of the measure of risks. The score of a risk point is the output of the function of consequence and likelihood, namely, $Risk = F(c, l)$. In risk matrices, the assessed risk corresponds to the cell instead of to a single point. The risk rating of a cell should be determined by calculating $Rating = G([c1, c2],\ [l1, l2],\ F\ (c, l))$. Obviously, the risk measure plays an important role in the risk matrix design.

To visually present how the perception of the risk measures affects the design of risk matrices, in Fig. 4.10, we draw some iso-risk contours based on the risk measures. The solid lines are drawn according to the multiplication measure: $risk = consequence \times likelihood$. The dashed lines are drawn according to the additional measure of $risk = consequence + likelihood$. Obviously, in this risk matrix, the additional measure seems not to be matched with the design. Generally speaking, the distribution of the risk ratings of the cells should be in accordance with the risk measure, specifically, the iso-risk contour. In Fig. 4.11, we provide a risk matrix used by Cook (2008) in the study of safety management. We see from the risk matrix that the rating distribution of the cells is well-matched with the iso-risk contours, which are of the addition form. This analysis reveals that risk measures may affect the design of risk matrices by changing the rating distribution of the cells.

Fig. 4.10 Explanation of how risk measures affect the design of risk matrices



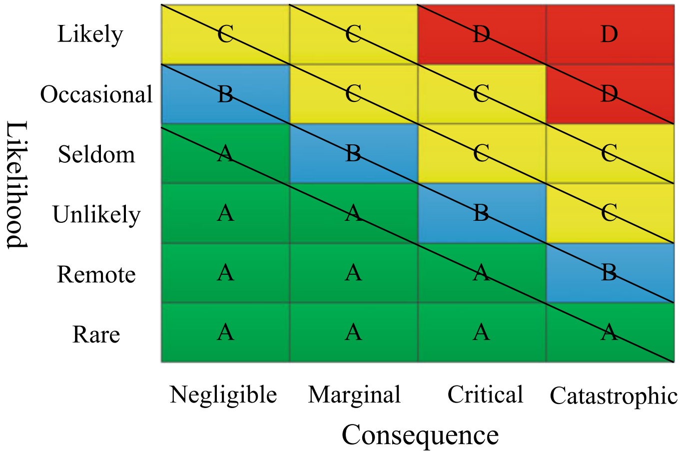


Fig. 4.11 A risk matrix with additional forms of risk measure

Although risk measures affect the designs of risk matrices, we see that the rating distribution of cells is not completely in accordance with the risk measure. In Fig. 4.11, the multiplication form of the risk measure seems to match the rating distribution of the cell. However, there must be an iso-risk contour passing through both yellow and red regions, which means according to the iso-contour, we can not find a region where all the cells with the same rating stay. Thus, the points with the same quantitative risk values may belong to different risk ratings. All this is because, in a risk matrix, the cells are square, and usually, the iso-risk contours are smooth, and thus, the iso-risk contours must pass through the cells. This phenomenon makes it more difficult for risk matrix users to have an accurate conception of the risk measures.

4.2.5 Perception of Attitudes Towards Risks

Risk attitudes are a common topic in risk management. Researchers have found that, when using risk matrices, decision-makers are not always risk-neutral. They show risk aversion in some cases (Goerlandt and Reniers 2016; Bedford 2013). Risk aversion in risk matrices is understood as the perception that consequence plays an important part in risk scoring. For example, let us suppose that we are assessing two risks; the consequence and likelihood of one risk are “medium” and “low”, respectively, and the consequence and likelihood of the other are “low” and “medium”, respectively. If the decision-maker is risk-neutral, the rating of the two risks is the same (e.g., “green”). But, if the decision-maker is risk-averse, the rating of the risk with “medium” consequence and “low” likelihood may be “yellow” while the rating of the other risk is “green” (“yellow” is a more severe rating than “green”)

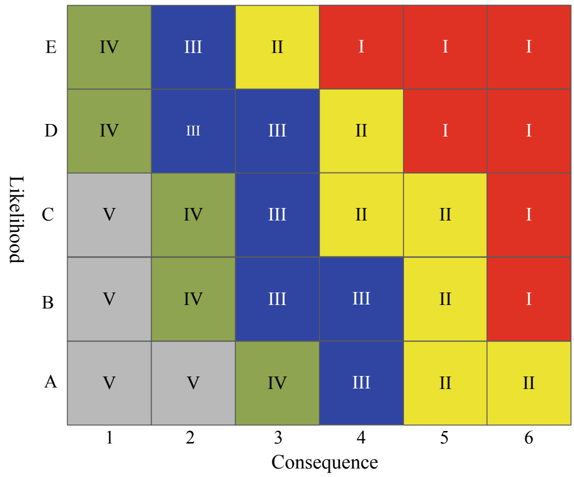


Fig. 4.12 A risk matrix given by ISO (2009) with risk aversion

because the consequence of the former risk is larger and, thus, the score of the risk is increased. In addition, even people who are aware that they are risk-averse display varying degrees of aversion. We term this kind of perception "perception of attitudes towards risks".

In the theoretical study of risk matrices, decision-makers are usually assumed to be risk-neutral. Figure 4.10 shows a typical risk matrix with a neutral attitude toward risk. This kind of risk matrix is diagonally symmetrical from (0,0) to (1,1). This is because a risk-neutral attitude means that the two inputs are given equal weight. In practice, people tend to display risk aversion. For example, Fig. 4.12 supplies a risk matrix created by ISO (International Standardization Organization). Obviously, in this risk matrix, consequence plays a more important role than likelihood; this proves the existence of risk aversion. However, ISO did not tell what the creator's degree of aversion is and how risk aversion is embedded in the design of the risk matrix. To describe the degree of risk aversion, the following form of this perception is provided.

Form of the perception of attitudes towards risks. The measure $risk = consequence \times likelihood$ is the most commonly used of all the possible forms, and thus, is adopted here to show to resent the risk attitude (Other forms of risk measure can be given here, and the analysis process is similar). One problem with this measure is that it cannot reflect decision makers' risk attitudes. In practice, many decision-makers are risk-averse ("risk" here means "adverse event") and, thus, a risk measure considering risk aversion is adopted by some risk matrix practitioners as shown in Eq. (4.1).

$$risk = likelihood \times consequence^n \quad (4.1)$$

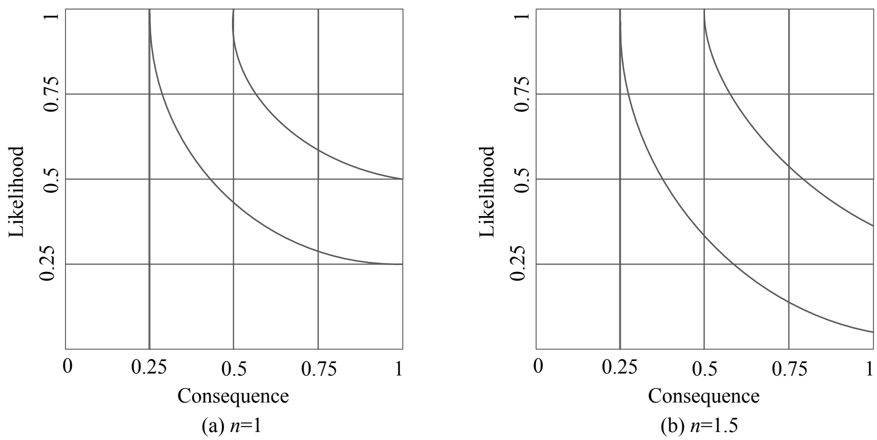


Fig. 4.13 Two risk matrices with the aversion coefficients set with 1 and 1.5

where $n \geq 1$ is a risk aversion coefficient. This means that even if consequence and likelihood are the same, consequence plays a more important role when risk attitude is risk-averse ($n \geq 1$). When $n \geq 1$, it is easy to prove that the risk attitude in Eq. (4.2) is risk-averse by using the Arrow–Pratt measure of absolute risk aversion:

$$A(risk) = -\frac{risk''(consequence)}{risk'(consequence)} = -\frac{n-1}{l} < 0 \tag{4.2}$$

Similar to the analysis of the risk measures, the perception of attitudes towards risks will affect the iso-risk contours and the distribution of the ratings of the cells. For example, in Fig. 4.13, we show two risk matrices where the aversion coefficients are set with 1 and 1.5, respectively; obviously, given different aversion coefficients, the iso-risk contours are different, and the distribution of the risk ratings may change. Detailed discussion will be given in Sect. 4.4.

4.3 A Sequential Updating Approach Used to Integrate Different Perception

The mapping of the cells to different risk ratings is the most important part of the design process because, if the mapping is not accurate, the assessment of the risks will not be reliable. Therefore, the design should not be arbitrary. As we stated above, the key to solving the mapping problem is to provide a function, namely $Rating = G([c1, c2], [l1, l2], F(c, l))$. This function is related to the definition of inputs and the risk measure.

Cox (2008) first discussed the mapping problem and found that the potential rules for guiding mapping in practice were not theoretically accurate. To solve this problem, Cox proposed three axioms that a reasonable risk matrix should satisfy. To our best knowledge, Cox is the first one who proposed a systematic method to give a reasonable design of a risk matrix. However, it seems Cox's axioms are too strict that the design risk matrix can still be improved. For example, according to Cox, only two risk ratings are effective, which leads to low resolution; besides, the designs of the same risk matrix are not unique. These two flaws may prevent the widespread use of the axioms in practice. Besides, we find that in some cases, Cox's axioms fail to output a risk matrix design (see the discussion in Sects. 4.1 and 4.5).

Li et al. (2018) extended Cox's method and recommended the use of the SUA, which allows decision-makers to choose the number of risk ratings they need and then output a unique design from the risk matrix. To the best of our knowledge, Cox's axioms and the SUA are the only two methods that have been proposed in a formative way in the literature. Compared with Cox's axioms, the SUA is a more robust method, and thus, it has been adopted here. In the rest of this section, we first briefly introduce the SUA, and then tell how to integrate different types of risk perception in the risk matrices.

4.3.1 Review of the SUA

Given the description of the inputs and the number of risk ratings, the key step is to assign the ratings to each cell. The SUA provides a systematic theory with which to undertake this process:

(a) The SUA assigns risk ratings to different cells, from low to high sequentially, mainly because it proposes a rule to compare any two cells. This rule is called the logical comparison of two cells, and it can be stated as follows: *Cell A is (logically) larger than cell B if, and only if, the probability that a point in cell A is quantitatively larger than a point in cell B exceeds a predetermined threshold*.

Mathematically, the criterion can be written thusly:

$$\Pr(a > b | a \in A, b \in B) \geq \alpha, \alpha > 0.5 \tag{4.3}$$

where a and b represent two possible points in cells A and B, respectively, and α is the predetermined threshold.

(b) Based on the cell comparison rule, three principles are proposed. They are explained as follows.

Adjusted weak consistency (AWC). AWC states that, if cell A has a higher rating than cell B, A must logically be larger than B.

Consistent internality (CI). CI holds that a higher-rated cell should be larger than all cells of a lower rating. It states that "A must logically be larger than B" is a necessary condition (instead of a sufficient condition) of "cell A has a higher rating than cell B". If, and only if, cell A is larger than all the cells that have the same risk rating as B will cell A have a higher rating than B.

Continuous screening (CS). CS states that "If cells rated A are determined, any other cell X satisfying the condition that X is larger than any cells rated A updates the rating of X to a higher level". If CS is not satisfied, the rating of a cell which should otherwise have a higher rating will remain unchanged. Thus, CS is intended to maximize the number of risk ratings.

(c) After the above three principles are provided, a global rating algorithm is proposed to complete the design that satisfies the three principles. Given a $m \times n$ risk matrix, the cells of which are numbered as $1 - mn$ from lower likelihood to higher likelihood and from lower consequence to higher consequence, the risk matrix can be designed according to the following steps.

c1. Obtain the probability comparison matrix (PCM). The element PCM_{ij} in the PCM represents the probability that a point in cell i is larger than a point in cell j, namely, $\Pr(a > b \middle| a \in cell_i, b \in cell_j)$. The PCM can be obtained analytically or by the simulation method.

c2. Give two characteristic variables of the risk matrix. In each column of the PCM, record the number of values satisfying $PCM_{ij} \geq \alpha$. This number is the first variable, which is called the *characteristic number*. Then rank these characteristic numbers without repetition in ascending order as $a_1, a_2, \ldots, a_t$. The other variable, the *counting variable* N_i, is used to record the number of repetitions of each characteristic number a_i.

c3. Discover the priorities of the cells based on the two characteristic variables. The rule is that the lowest risk rating is assigned to the cells whose characteristic number is a_1; For cells rated X with the characteristic number a_p, if $a_{p+1} < N_1 + N_2 + \cdots + N_p$, the rating of the cells with the characteristic number a_{p+1} remains unchanged; otherwise, the ratings of these cells will be upgraded to the next level. This is because, when $a_{p+1} < N_1 + N_2 + \cdots + N_p$, the cells with the characteristic number a_{p+1} are logically larger than only a portion of cells with ratings the same as or lower than X; this violates CI.

Figure 4.14 presents a simplified version of the process of the SUA. After the above steps, all of the cells will have a unique risk rating. The cell with the lowest consequence and likelihood has the lowest risk rating, and the risk ratings of other cells will be determined sequentially according to the principles stated above. Figure 4.15 shows a risk matrix created according to the SUA.

4.3.2 Integrating Different Types of Risk Perception

An important characteristic of the SUA is that it does not impose any special assumption on the description of the elements of risk matrices. Therefore, it can

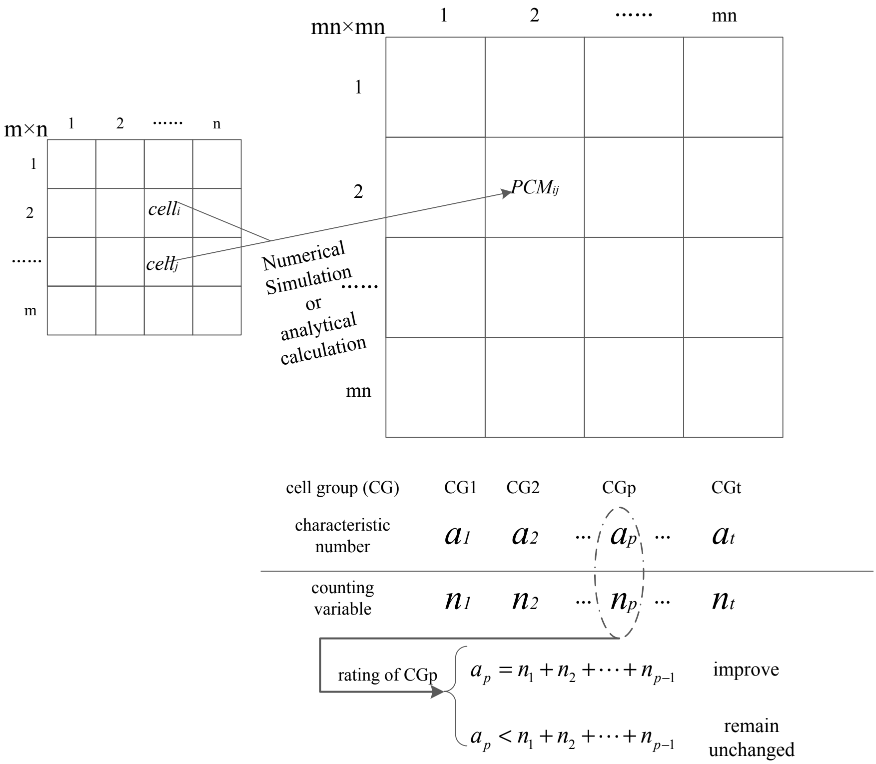


Fig. 4.14 A simple version of the SUA

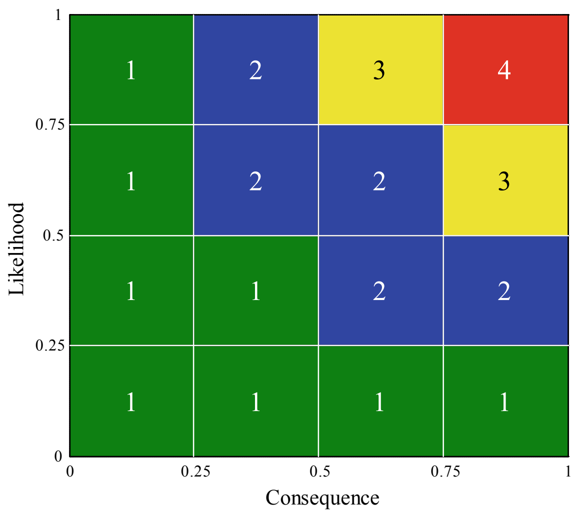


Fig. 4.15 The design of a 4×4 risk matrix with $\alpha = 0.9$, according to the SUA

still output the design of a risk matrix when different perceptions on these elements are integrated.

For possible scaling of the consequence and likelihood, $[c1, c2),\ [c2, c3), \ldots,\ [cn,\ 1]$ and $[l1, l2),\ [l2, l3), \ldots,\ [ln,\ 1]$, one just needs to replace the original intervals with the new ones. Then in the process to obtain the PCM, the boundaries of a cell are determined by the supplied intervals.

For the perception of the location of inputs, given a particular distribution of two inputs, the changes of the design process are also reflected in the way to obtain PCM. For example, when using the simulation method to calculate the PCM, if consequence and likelihood are evenly distributed on the axes, then risk points will be generated with the same probability; but if the new distribution is replaced, the probability differs.

To integrate different perceptions on the input category membership, one should first give the membership function. Then each risk point in the cell should multiply a weight corresponding to the membership value. For example, in the simulation method, if the membership of a risk of 0.5 to the category of "yellow" is 0.9, then the final value of the risk is 0.45. Other steps remain unchanged during the design.

Integrating different perceptions of risk measures is easy to understand. For example, given consequence of 0.5 and likelihood of 0.5, if the risk measure is considered as the addition form, the risk value is 1; while in the multiplication form, the risk value is 0.25. Under different risk measures, the values of the risk points will be different.

Integrating different perceptions of risk attitudes is similar to integrating the perception of risk measures. Since here risk attitude is represented by adding an aversion coefficient to the risk measure, the change of risk attitude will also affect the risk value, and further affect the comparison of two risk points.

Remark: The SUA extends Cox's method mainly because it allows any two cells comparable, which is the criterion to assign the two cells different risk ratings. Integrating different types of risk perception to the design of risk matrices is to give different settings, which are preferred by the users, to the elements of the risk matrices. The reason why the SUA can be used to study the influence of risk perceptions on the risk matrix design is the design created according to the SUA will always exist. This is formed as the property of the SUA as stated in the following.

Property of the SUA: The design according to the SUA will always exist.

Proof. The design of a risk matrix exists if there are at least two risk ratings (if there is only one risk rating, the risk matrix can not be used to prioritize risks). For a risk matrix, the upper right-most cell is larger than the adjacent two cells since one of the two inputs is the same and the other must be larger. If we set an α that will differentiate the cells into two risk ratings, the upper right-most cell will have a higher rating than all of the other cells. Therefore, there are at least two risk ratings, meaning that the design of a risk matrix will always exist.

4.4 Risk Matrices Integrating Different Risk Perceptions

Before involving different types of perception, the normal risk matrices are present. This kind of risk matrices is used by many researchers, and no particular perceptions are considered in its use (Li et al. 2018; Cox 2008). Its characteristics are as follows: the axes of the two inputs are evenly divided into several categories, the two inputs are evenly distributed, there is no fuzziness between different categories, the risks are measured by $risk = consequence \times likelihood$, and the attitude towards risks is neutral. According to the SUA, a normal 4×4 risk matrix is the same as the one in Fig. 4.15. When we study the effect of one perception on the design of risk matrices, we assume that other perceptions are normal. In the analysis, the cells are numbered as $1 - mn$ from lower likelihood to higher likelihood and from lower consequence to higher consequence.

When analyzing the performance of the risk matrix designs integrating risk perceptions, we must answer the following two questions:

Q1. Should the design under the normal setting change as risk perceptions are added?

Q2. If the answer to question 1 is yes, how does the design change?

Obviously, if the answer to Q1 is yes and the design does not change according to a method, the method fails to integrate the risk perceptions. Below, we give a more detailed analysis on integrating the perceptions.

4.4.1 Performance of Integrating Risk Perception on the Scaling of Inputs

To visualize the influence of this perception, here we illustrate a 4×4 risk matrix the quantitative category lengths of which consist of an increasing geometric sequence. Specifically, the intervals of consequence and likelihood are $[0, x)$, $[x, cx + x)$,$\left[cx + x, c^2x + cx + x\right)$, and $\left[c^2x + cx + x,\ 1\right]$ where $c > 1$(the lengths of these intervals are x, cx, c^2x, and c^3x where $x + cx + c^2x + c^3x = 1$). Notice that, here, c is a parameter that we set to present different degrees of perception on the scaling of inputs, and if $c = 0$, the risk matrix becomes normal.

In Fig. 4.16, we present two different settings of the scaling of consequence and likelihood. The first is of the parameters with $x = 8/65$, $c = 1.5$, and the second is of the parameters with $x = 1/15$, $c = 2$. The designs of the two risk matrices—namely, the color setting of each cell—are based on Cox's three axioms, and the two designs are the same (in fact, no matter how c changes, the design remains unchanged; see the proof in Appendix). Obviously, with different settings of the parameters, the structure of the risk matrix changes. The cells in the risk matrix with larger $c(c = 2)$ are of larger differences in area than those in the risk matrix with smaller $c(c = 1.5)$. The question is this: should the risk matrices with different scaling of inputs have the same design?

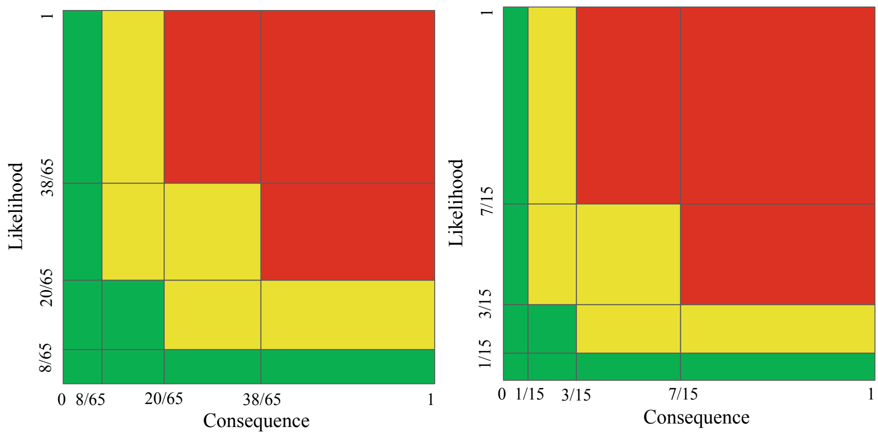


Fig. 4.16 Two different settings of the scaling of inputs

To answer this question, we first explore how the ratios of the three ratings in the risk matrix change as c vary. (Here, the ratio is measured between the area of the region of a rating and the total area of the risk matrix.) Fig. 4.17 presents their variation trend.

Apparently, when c gets larger, the red region occupies most of the risk matrix. In these cases, there are three red cells in the risk matrix, namely, "12", "15", and "16". Cell "16" is significantly larger than cells "12" and "15". (The areas of cells "16" and

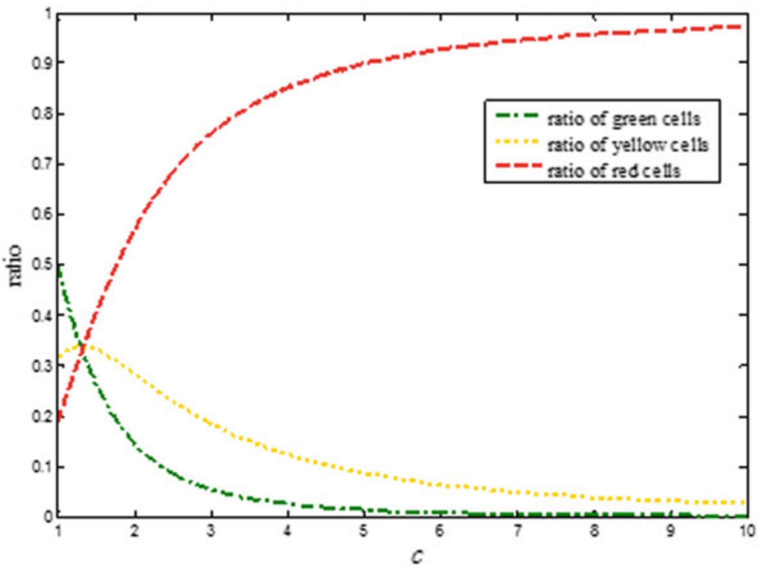


Fig. 4.17 Ratios of the three ratings as c changes

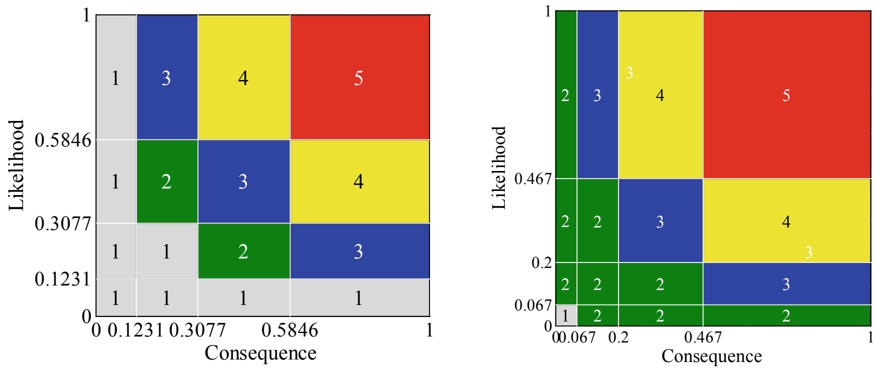


Fig. 4.18 Rating schemes of two 4×4 risk matrices the inputs of which are increasingly described with $c = 1.5$ and 2 according to the SUA

"12" are c^6x and c^5x, respectively, and area difference is quite large when c is very large.) As a result, we think cell "16" should be differently rated than cells "12" and "15." Thus, it is not reasonable that the risk matrix design in which c is close to 1 and the one in which c is far larger than 1 are the same. This example demonstrates that, as c changes, the information regarding each cell changes. Therefore, the invariability of the rating scheme according to Cox's axioms seems defective.

Next, we exhibit the designs of the two 4×4 risk matrices in Fig. 4.18, in which $c = 1.5$ and 2. The designs are given based on the SUA. To obtain the detailed process, one may refer to Li et al. (2018). We carry out the SUA by giving different scaling of inputs.

Comparing the two matrices in Fig. 4.18 with the normal one (the one in Fig. 4.15), we find that the cells with the same rating in the normal risk matrix are divided into more ratings. For example, the rating "2" in the original matrix is divided into the ratings "2" and "3". This is because, as c gets larger, the left-bottom cells become more compressed, causing these cells' information to change, and thus, the cells which have the same rating in the normal risk matrix, become different from each other.

To more clearly explain the change of design due to the increase of c, we first give the variation of the corresponding probability of cell "2" is larger than "1" on the right risk matrix of Fig. 4.18 as c changes (the probability is defined in formula (2)). Theoretically, the mapping of the probability of c is continuous. In Fig. 4.19, the points of probability are generated by simulation, where c increases in fixed steps of 0.02 and, due to simulation error, the simulated points can not be connected by a smooth line. Thus, we use the fit line (fitting of a polynomial) to reflect the trend of the change of probability. We see that the line increases from 0.82 to 0.94 steadily. In this case, it is set between 0.82 and 0.94, 0.85 for example, as c increases, the relationship between the two cells changes, namely, from being equal to each other (when the probability is smaller than) to cell "2" is larger than cell "1" (when the probability is larger than). We next give the variation of the corresponding probability

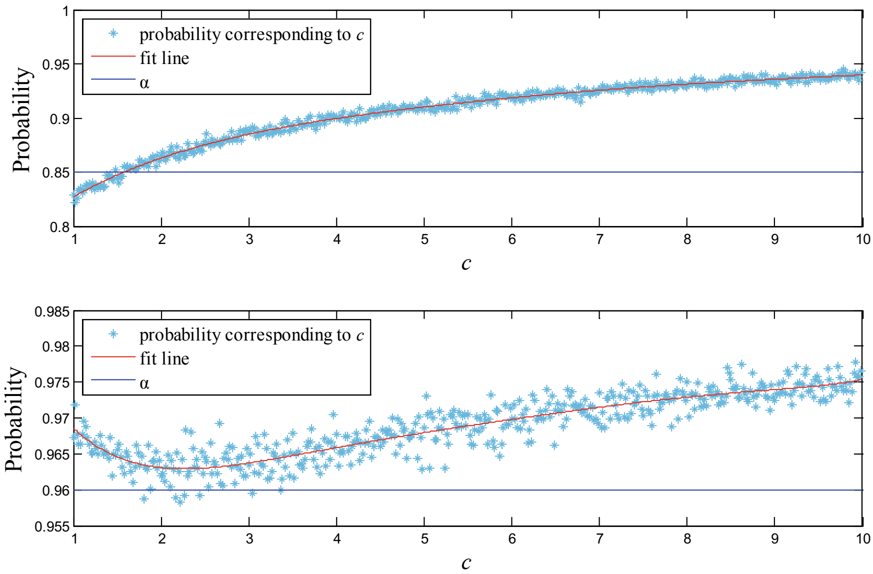


Fig. 4.19 Probability variation as c changes

of cell "16" is larger than "15" on the bottom of Fig. 4.18. As the figure shows, the lowest point is around 0.96, which means that it is set to be smaller than 0.96, cell "16" will always be larger than "15" and, thus, even if c increases, their relation will not change.

We learn from Fig. 4.19 that whether the rating scheme changes as c increases depends on the corresponding variation of the probability in the PCM and the setting of α. Since the correlation between any cell pair changes at different values of c, the rating scheme may be different at different values of c. The above analysis applies to any form of category definition.

4.4.2 Performance of Integrating Risk Perception on the Location of Inputs

Given different distributions of the inputs, the distribution of risk points in a cell varies. In Sect. 4.2.2, we show some examples of the risk point's distribution. If two cells are differently rated, they must show the difference they contain, and the distribution of the points in the cell is the source of the difference. For example, if the points in a cell are evenly distributed, the assessed risk corresponding to the cell may locate at any point in the cell with the same probability; while if the points are normally distributed, the assessed risk will be located at the center of the cell with high probability. What the probability directly affects is the comparison of two cells.

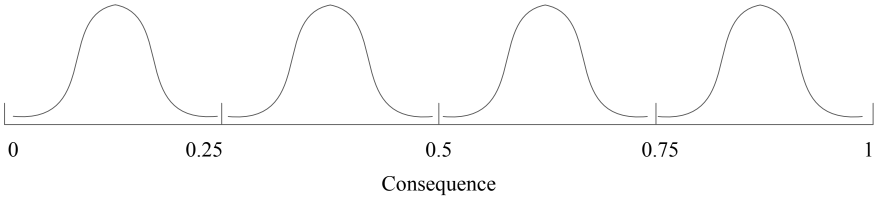


Fig. 4.20 Normal distribution of inputs with 4 categories

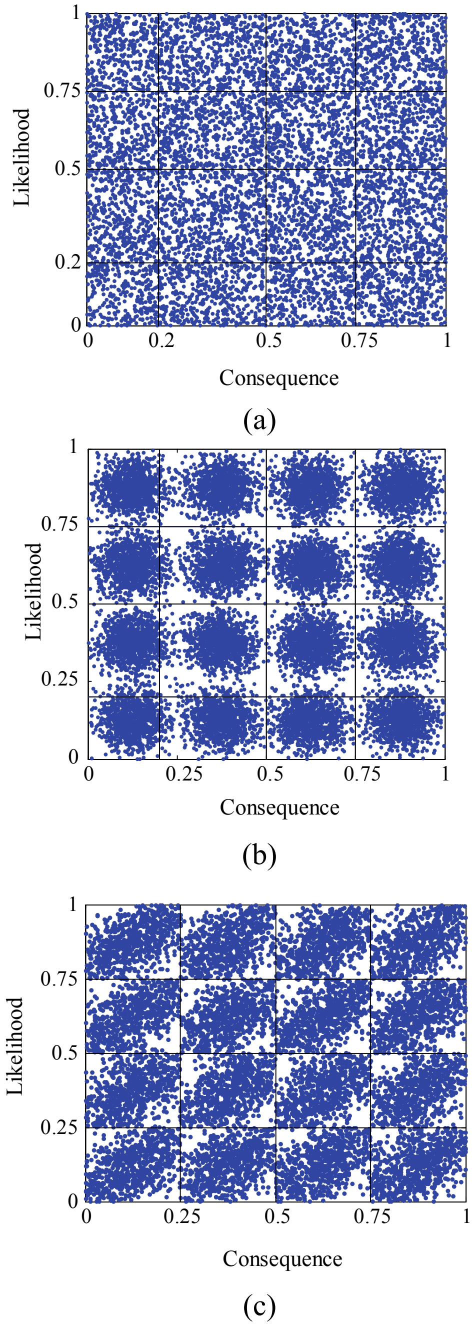
Obviously, when comparing two cells using the SUA, different distributions of the points in a cell will result in different relationships (one cell is larger than, equal to, or smaller than the other) between the two cells (Li et al. 2018). If the relationship between the two cells changes, it is not difficult to understand that their ratings may be different. The following are examples of how different locations of inputs affect the risk matrix design in detail.

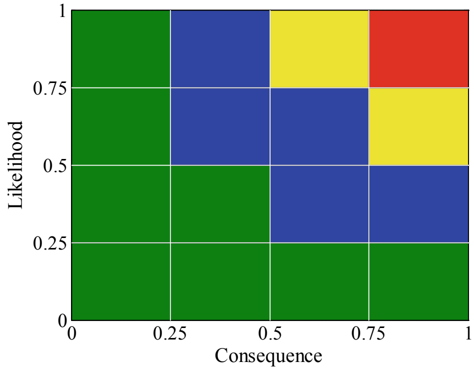
To show the design integrating perception of the location of inputs, we consider two possible forms of the location of inputs. In the first one, the two inputs are independent, and in each category of inputs, the inputs are normally distributed, namely, $input \sim N$ (the central value of the category, 0.01), as shown in Fig. 4.20. In the second one, consequence and likelihood are also normal distribution, and besides, they are correlated with the correlation coefficient 0.8.

We next explore whether the distribution of inputs affects the design of risk matrices. First of all, if the inputs are evenly distributed in each cell, the risk points are evenly distributed as shown in (a) of Fig. 4.21. But if the distribution of the inputs is changed (e.g., the normal distribution in Fig. 4.5), the distribution of the points in the cells will change. In Fig. 4.21, (b) shows the corresponding point distribution in the cells with the two settings of inputs described in Fig. 4.6. In Fig. 4.21, (c) is the point distribution integrating correlation between consequence and likelihood. If in each category of inputs, the inputs are normally distributed, most of the points in the cells are located in the center or on the edges of the cells, there are few points. When the correlation between two inputs is added, most points are located along the diagonals in each cell.

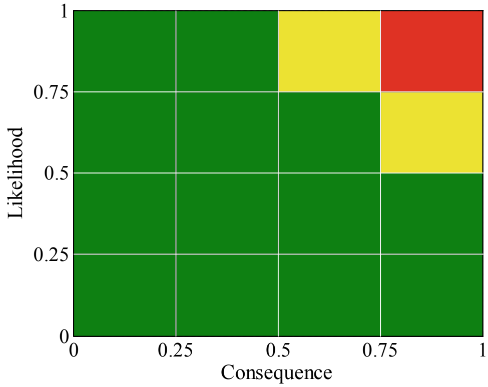
Figure 4.22 presents two designs with different distributions of inputs based on the SUA. We see that, if the inputs are normally distributed in each category, the design of the risk matrix is the same as that of the normal one because the information in each cell is similar to that of the normal one. But under the second setting of the distribution of inputs, the information in the risk matrix is apparently different, which leads to variation in the design. According to fundamentals of the SUA, it can be inferred that compared with the case under the first setting, in the second setting, the correlation between the inputs makes the difference between green and blue cells in (a) of Fig. 4.22 so small that they should not be divided as different ones.

Fig. 4.21 Distribution of points in the risk matrix under differing input distribution





(a) Design of the risk matrix where the consequence is normally distributed in each cell



(b) Design of the risk matrix with the correlation coefficient 0.8 based on (a)

Fig. 4.22 Two designs of the risk matrix with different distribution of inputs

4.4.3 Performance of Integrating Risk Perception on Input Category Membership

In the SUA, membership is embedded as follows: when obtaining the PCM, in the Monte Carlo simulation, the generated sample points will be multiplied by a weight, namely, the grade of membership. Since the weights are not the same for the input in an input category, it is easy to understand that under different settings of input category membership, the comparison results of two cells may change. Therefore, the designs of the risk matrices given different perceptions on input category membership may vary.

To reflect the change of the perception on the membership of inputs, we assume that both of the inputs have the membership given in Fig. 4.23. In the process of obtaining the PCM, if the inputs are multiplied by the weights which correspond to the degree of membership, the distribution of the points in a cell will change. In Fig. 4.24, we give the point distribution in the cell with the consequence interval $[0.25,\ 0.5]$ and the likelihood interval $[0.25,\ 0.5]$. We see that compared with the normal case ((a) in Fig. 4.22), due to the multiplying of the weights, the largest and the smallest points are no longer the original ones (in the normal case, in this cell, the smallest point is $[0.25,\ 0.25]$ and the largest is $[0.5,\ 0.5]$). Also, some points are located in the cell that are not there in the normal case (see the points in the region a, b, and c in Fig. 4.24).

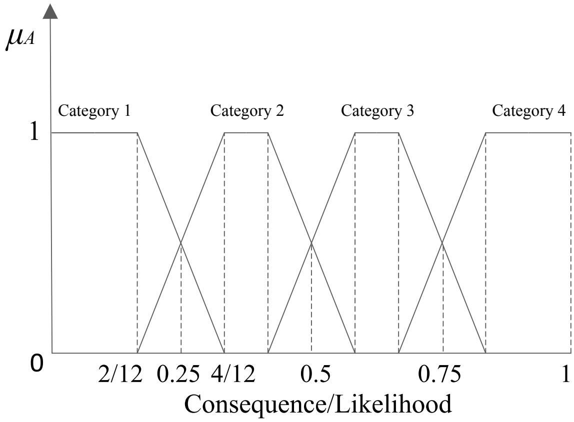


Fig. 4.23 Membership of the consequence and likelihood

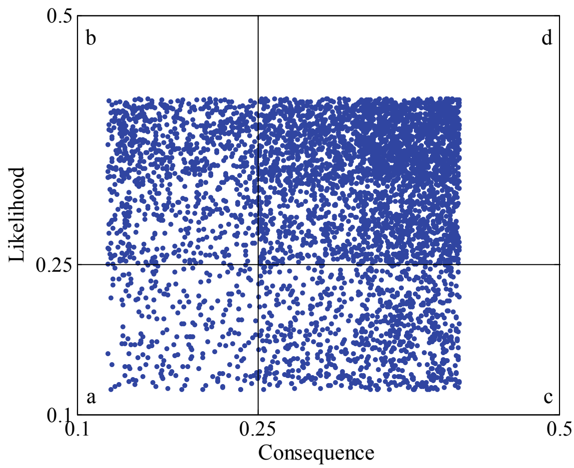


Fig. 4.24 Illustration of the point distribution in a cell considering input category membership

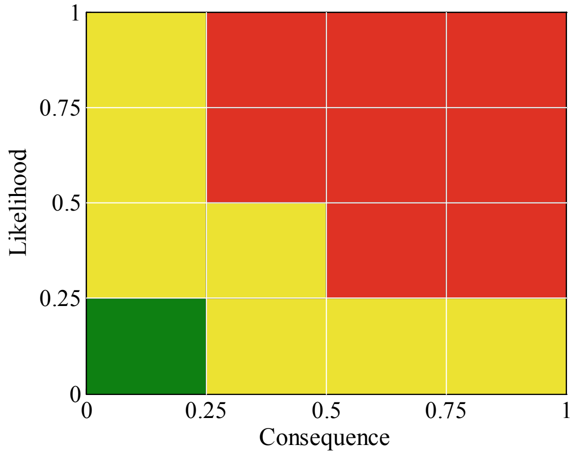


Fig. 4.25 Design of the risk matrix considering the perception of input category membership

The above analysis explains how the change of input category membership affects the risk matrix design by affecting each cell's information. In Fig. 4.25, we present the corresponding design of the risk matrix.

4.4.4 *Performance of Integrating Risk Perception on the Measure of Risks*

Intuitively, different perceptions on the risk measure will result in different risk matrix designs. The explanation is straightforward. As shown in Sect. 4.4, with different risk measures, the shapes of the iso-risk contour are different, which will result in the different distributions of the cells' ratings. Therefore, we would give an affirmative answer to Q1.

To show how different risk measures affect the design of risk matrices, we assume an additional form of risk measure is applied in the risk matrix, namely, $risk = consequence + likelihood$. Figure 4.26 reports the corresponding design. As we would expect, since the risk is measured by the sum of consequence and likelihood, the cells along the line with slope-1 have the same risk rating. This is because the members of the cells have the same rating. For example, if the risk measure is of the additional form, the points in the cells along a diagonal with slope-1 will have similar quantitative values; while if the risk measure is changed to a multiplication form, the points in the cells along a hyperbola will have the similar values. It is obvious that the cells with similar values should be categorized as the same rating. That is how risk measure works in a risk matrix design, and this mechanism is applied in the SUA.

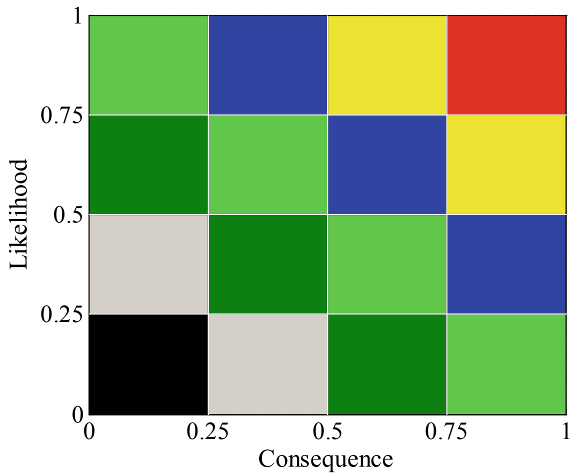


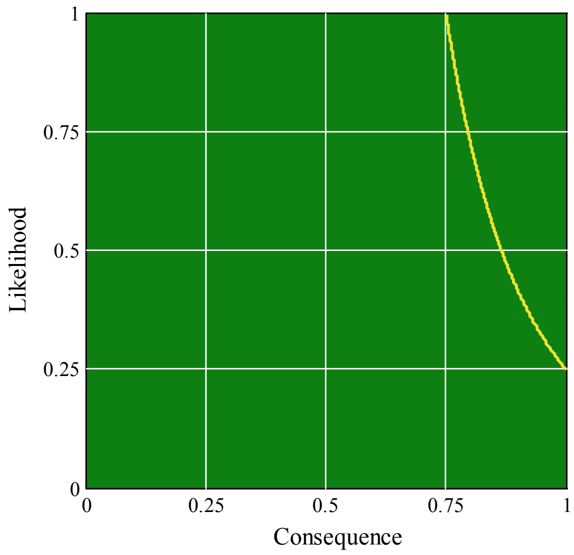
Fig. 4.26 The design of the risk matrix when risk is measured by the addition form

4.4.5 Performance of Integrating Risk Perception on Attitudes Towards Risks

Although Cox claimed regarding his method, "risk could be any smooth increasing function of frequency and severity, not necessarily their product", we find that, in some cases, we will fail to design a rating scheme for the risk matrix according to Cox's axioms. Let us take a 4×4 risk matrix as an example. If the risk is *measured by* $risk = likelihood \times consequence^n$*, where* $n \geq \log(0.25)/\log(0.75)$, then the risk *matrix will have only one color according to Cox's axioms*. We will explain this conclusion with the aid of Fig. 4.27. First of all, the cells at the bottom of the matrix should be green according to Cox's axioms. When $n = log\,(0.25)/log\,(0.75)$, the iso-risk contour with quantitative risk equal to 0.25 passes though points $(0.75,\ 1)$ and $(1,\ 0.25)$, which means the two cells should have the same rating. When $n > log\,(0.25)/log\,(0.75)$, $(1,\ 0.25)$ is larger than $(0.75,\ 1)$. Therefore, there is no red cell in the rating scheme governed by Cox's weak consistency and betweenness axioms [3]. In a normal risk matrix, the above case will not occur since such a matrix is symmetrical and the point $(1,\ 0.25)$ is always smaller than any point in cell "16", whose smallest point is $(0.75,\ 0.75)$. But when adapted to deal with risk aversion, Cox's method exposes its defects as shown above.

Next, we give the rating scheme ($\alpha = 0.85$) designed according to SUA to $n = log\,(0.25)/log\,(0.75)$ on the left in Fig. 4.26. Four iso-risk contours pass through (0.25, 0.25), (0.5, 0.25), (0.75, 0.25), and (1, 0.25) are drawn in the matrix. Compare the iso-risk contours on the left with the ones on the right, and one finds it is obvious that the influence of risk aversion on risk matrix designs essentially results from the change of iso-risk contours. For example, the rating distributions in both of the risk matrices in Fig. 4.27 are in line with the corresponding tendency of the iso-risk contours. In the left risk matrix with risk aversion, the contours are

Fig. 4.27 Rating scheme of a 4×4 risk matrix according to Cox's method with $n = log\,(0.25)/log\,(0.75)$



more vertical than those in the right matrix and, thus, cells with the same category of consequence tend to have the same rating; while in the right risk matrix with neutral risk attitude, the iso-risk contours are of the shape of hyperbola, and therefore, the distribution of the risk ratings is symmetric along the diagonal from (0,0) to (1,1).

In fact, as the risk aversion coefficient n changes, for any two horizontally adjacent cells, for example, the cell with consequence [0, 0.25] and likelihood [0, 0.25], and the one with inputs of [0, 0.25] and [0.25, 0.5], the probability of the right cell being larger than the left one has the trend shown on the top of Fig. 4.28; and for any two vertically adjacent cells, for example, the cell with consequence [0, 0.25]

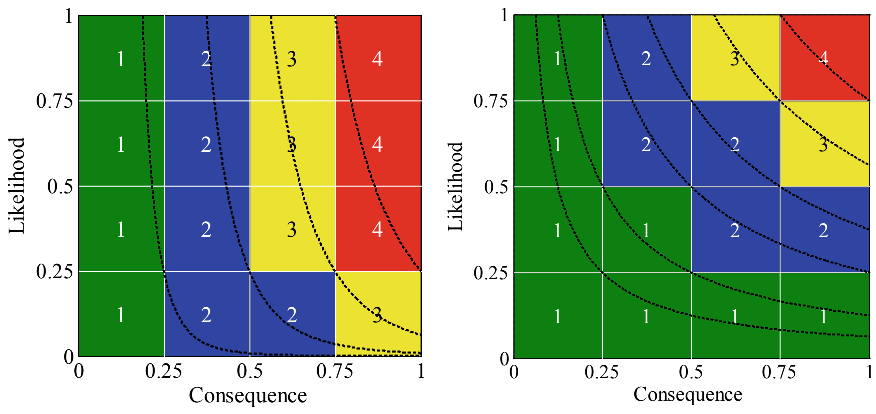


Fig. 4.28 Rating scheme according to SUA with $n = log(0.25)/log(0.75)$ and $n = 1$, respectively

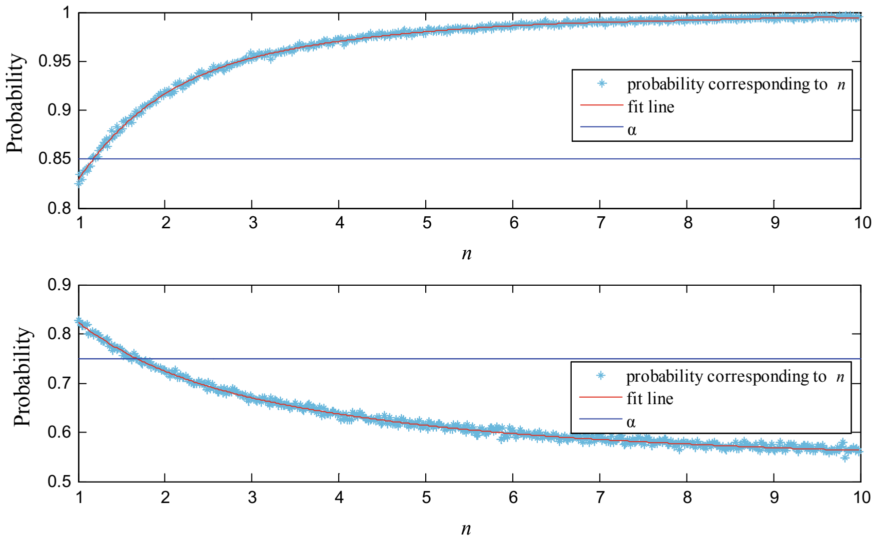


Fig. 4.29 Probability variation as n changes

and likelihood [0, 0.25], and the one with inputs of [0.25, 0.5] and [0, 0.25], the probability of the upper cell is larger than the left one has the trend shown on the bottom of Fig. 4.29.

As n gets larger, the difference between two horizontally adjacent cells increases to the maximum (the probability increases to 1), and the difference between two vertically adjacent cells decreases to the minimum (the probability decreases to 0.5). This reveals that, given a fixed α, as risk aversion increases, the horizontally adjacent cells tend to have different ratings, and the vertically adjacent cells tend to have the same rating. We can infer by these two facts that in the risk matrix with risk aversion, right-bottom cells tend to have a higher rating than those in a risk matrix with a neutral risk attitude.

4.5 Conclusion and Discussions

Risk matrices are risk management tools based on subjective judgments. Therefore, risk perceptions are inevitable factors in the design/use process of risk matrices. Risk matrices used in the literature and practice reveal that different user conceptions of some settings of risk matrices do exist and that the settings will affect the assessment of the risks. Therefore, such conceptions have been termed "risk perception" in this section. We have discussed how different types of risk perception affect the design of risk matrices, accordingly affecting risk assessment.

We first identified five different types of risk perception in risk matrices from the point of the risk matrix design process and the general definition of risks. Each perception, specifically, is a perception of the setting of a particular element in the risk matrix. We explained the five types of risk perception in detail, emphasizing how perception affects the information given by the points in each cell. Then we provided a brief introduction to the use of the SUA to design risk matrices, and introduce how to integrate different risk perceptions. Our main reason for selecting the SUA was that it can output a unique risk matrix design given the need for resolution of the risk matrix and the design will always exist. Finally, we comprehensively explored how risk perception affects the designs of risk matrices.

To show how each kind of perception affects the design, we integrate the perception one by one to the risk matrix design. In practice, different kinds of perception can be integrated at the same time.

We have emphasized that, for a risk matrix, the designers and the users should have the same conception of the risks. Differing conceptions will result in asymmetric information, which will make the result of the risk matrix inaccurate. Thus, in practice, risk matrices should be designed separately according to each field's particular risk perception.

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Chapter 5
Risk Matrix Design Assessment: Criteria and Quantitative Indicators

5.1 Criteria Used to Assess Risk Matrix

In the risk matrix, the risk is defined as outcome and possibility (Cox 2008; Duijm 2015; Albery et al. 2016; Ball and Watt 2013). Thus, The decision-maker goes through the following steps in designing the risk matrix: (1) Define the ratio of consequences to probabilities (2) Map the risk values and inputs to the risk matrix (3) Assign a corresponding level to each cell. Usually, decision-makers determine the magnitude of the consequences and probabilities of each event based on their experience and knowledge, so it's not clear which definition is better. In the study of risk matrices, the scaling of inputs is usually based on linear or logarithmic axes. And later We'll talk about quantifying how to come up with a standard in both cases. Accurate mapping of risks and inputs is the core technique for designing risk matrices and assigning a unique rating to each cell (Cox 2008; Li et al. 2018; Wall 2011; Skorupski 2016). Therefore, given the scale of the input, the next two steps affect the accuracy of the design and thus, the evaluation criteria to evaluate the operation of these two steps is particularly important. Figure 5.1 exhibits the relationship between the risk matrix design steps and the proposed criteria.

We provide a more explicit explanation of why a standard corresponding to a step should be proposed. Let's start with risk measurement. Risk is usually defined in the following form like $risk = f(consequence, likelihood)$. In essence, the risk matrix is a graphical representation of this risk measure: (1) First, the user of the risk matrix needs to give the input, and then (2) the risk matrix designer then employs a particular function to output a quantitative result of the risk (this function may be implicit in the qualitative risk matrix), and lastly (3) the risk matrix based on this design is used to determine the level of risk of the event being evaluated (Fig. 5.2).

During the design of the risk matrix, there are three design-related issues, namely (1) whether the risk measure function is reasonable, (2) whether the risk measure function is consistent for each point (there are infinite risk points in a risk matrix), and (3) whether the rating of cells is reasonable. These three questions correspond to the three criteria presented below.

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https://doi.org/10.1007/978-981-19-1480-5_5

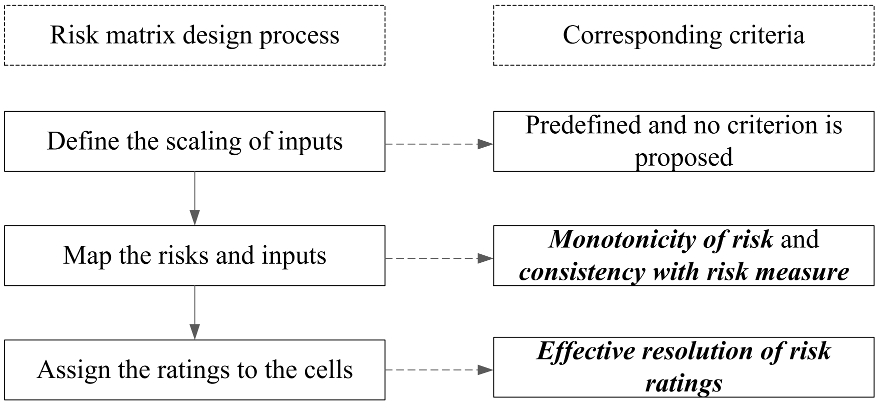


Fig. 5.1 Relationship between the risk matrix design process and the proposed criteria

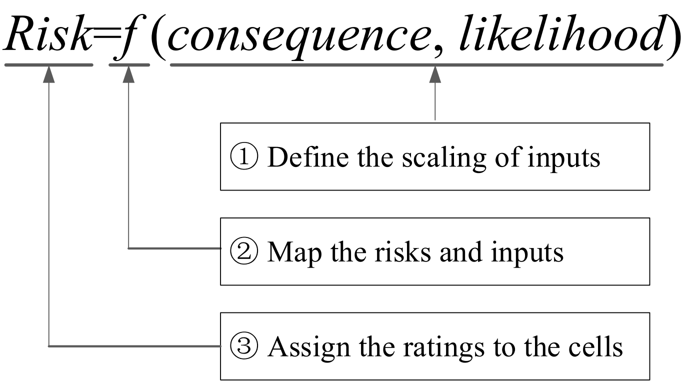


Fig. 5.2 Explanation of the risk matrix design process from the measure of risk

5.1.1 Monotonicity of Risks

The first guideline presented here is the mapping capability that the decision-maker needs to provide.

In the risk matrix, there is no fixed form to match the outcome and probability with the risk value. If there is no quantitative description of the given consequences and probabilities, then some scholars regard risks as the sum of the consequences and probabilities (Pritchard et al. 2010; Hewett et al. 2004; Holt et al. 2014; Cook 2008). Figure 5.3 shows a 6 × 6 risk matrix with inputs textually described. Pritchard et al. used it to assess drilling hazard risk. The same number in each grid represents the same risk score. According to the matrix, the grid on the diagonal of the same slope has the same risk, that is, it is more appropriate to use the following addition formula as the mapping function: $risk = consequence + likelihood$.

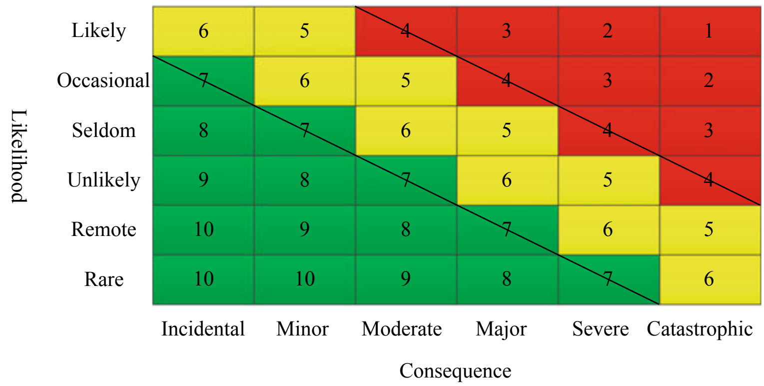
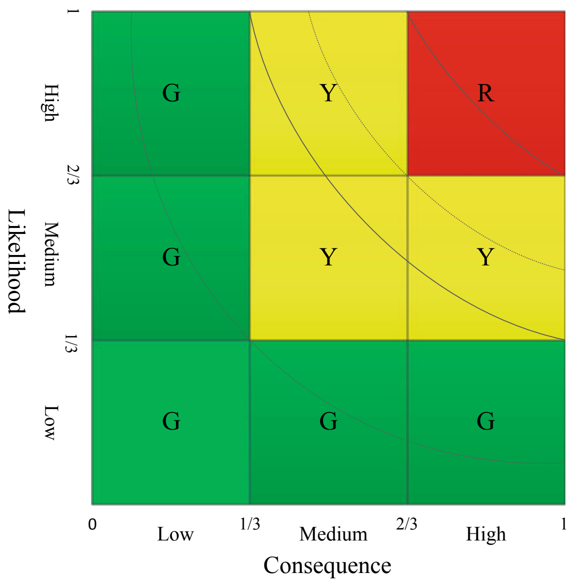


Fig. 5.3 A risk matrix with the risk being defined by the addition formula

Multiplication is another frequently used risk measure, $risk = consequence \times likelihood$. This is the usual way people define risk measures, which are expected consequences. See Fig. 5.4 for an example of a 3 × 3 risk matrix. We give 4 iso-risk contours, with quantitative risk values of 1/9, 1/3, 4/9, and 2/3. This formula is employed for quantitative analysis of risk matrices (Cox 2008; Bao et al. 2018; Ruan et al. 2015; Pickering and Cowley 2010). The iso-risk line where the risk value is larger usually has a higher risk level.

Regardless of which metric is used, the researcher needs to determine that the mapping function is monotonically increasing: an "increase in the outcome (the

Fig. 5.4 A risk matrix with the risk being defined by the multiplication formula



probability remains the same) or an increase in the probability (the outcome remains the same) does not result in a decrease in the specified risk" (Duijm 2015; Goerlandt and Reniers 2016).

Intuitively, it is easy to conclude that risk with higher outcomes and probabilities should be given a higher rating, that is, the risk should be monotonic. However, even in the quantitative risk matrix, if we evaluate risk, it does not correspond to a point, but a unit with an infinite number of points, which means that there will be different quantitative values of risk with the same risk rating. This is inherent in the risk matrix, so some point pairs violate monotony (Cox 2008; Engert and Lansdowne 1999). Since not all pairs of risk points in the risk matrix obey monotony, The degree to which they satisfy monotony is Worthy of attention.

5.1.2 Consistency with the Risk Measure

With a good understanding of the risk metrics in the risk matrix, designers need to assign risk ratings to different units. This distribution process will not be arbitrary. Intuitively, the distribution of cells with specific ratings should be consistent with the distribution of risk measures. For example, in the original risk matrix with linear axis characteristics, if the product of result and probability is defined as risk, the boundary of the yellow and green cells should be close to the hyperbola. A second criterion has thus emerged to characterize the consistency between the allocation of risk ratings and coordination of multilateral environment.

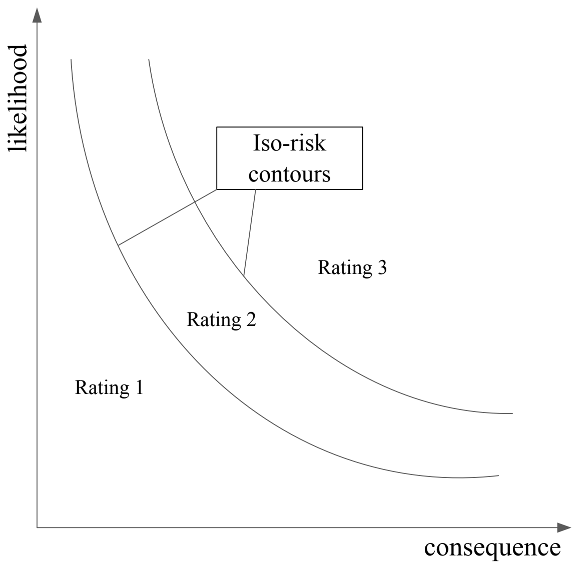


Fig. 5.5 Division of risk ratings according to iso-risk contours

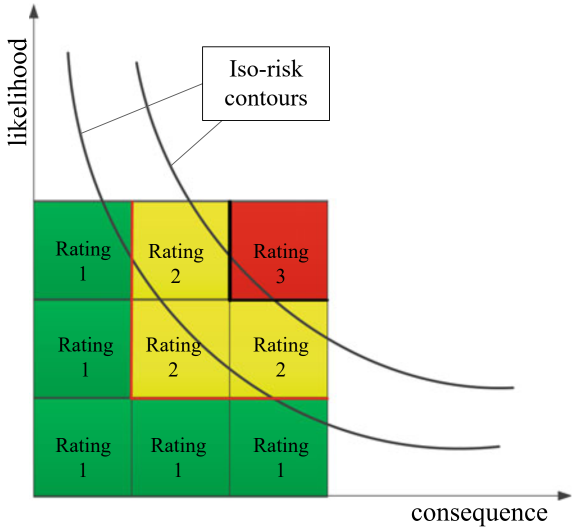


Fig. 5.6 Risk matrix with iso-risk contours

If the risk is measured by the product of outcomes and probabilities, and risk is graded with equal risk lines since the boundary is an equal risk line, the boundary of risk level is exactly the same as the risk measurement. (Refer to rating 2 in Fig. 5.5). Since the cells are square in the risk matrix, this means that an iso-risk contour must pass through two different cells (refer to Bao et al. 2018 for the proof). In this case, the boundaries of a risk rating are no longer the iso-risk contours. For example, in Fig. 5.6, the yellow cells are enveloped by the zig-zag broken lines (the bold red and black lines).

The above analysis does not ask the designer of the risk matrix to classify two risk levels according to the boundary. However, when designing the risk matrix, the risk level is classified according to several equal risk lines determined in advance. If a cell is between two iso-risk contours, the rating of the cell is determined. For example, in Fig. 5.6, if a cell is between the two iso-risk contours, the rating of the cell is rating 2. However, if an iso-risk contour line crosses a cell, part of the cell belongs to one rating, and part of the cell may belong to another rating. (Refer to the case where the iso-risk contour passes through the red and yellow cells in Fig. 5.6). In this case, the designer should be able to determine which rating the cell belongs to Ruan et al. gave their rule as follows: if most of the cell is located in a rating (its area proportion is the largest), for example, rating A. Finally, this cell belongs to the A rating (Ruan et al. 2015).

However, in the literature, risk matrix designers have not found a specific way to deal with this problem. But whether or not the designer explicitly proposes a specific approach, we assume that they follow a potential rule for determining the level of cells to design the risk matrix. Therefore, we explore the degree of consistency between cell ratings and risk measures based on potential rules. The greater the consistency, the better the risk measure is embedded in the rating allocation process. The design of a risk matrix is not acceptable if the consistency is quite small.

The standard requires that all boundaries of adjacent ratings be consistent with risk measures. The difficulty is that we don't know what a risk measure is unless the designer tells us. But there's another way to think about it. Based on a given risk measure, one boundary should be consistent with the other (if the risk measure is not given in advance). This inference can be proved by the risk matrix in practice. For example, in Fig. 5.3, for the "yellow" cells, Connect the smallest points in each unit with a score of "6", Also connect the largest points in each cell with a score of "5". When the minimum point and the maximum point are connected to the line respectively, a series of lines with the same slope are produced. As we discussed in Sect. 2.1, the addition formula is the appropriate mapping equation for the risk matrix. Risk measurement is consistent with design (if the risk measure is $risk = consequence + likelihood$, the slopes of lines given before should all be -1). When risk aversion attitude (risk aversion is understood as the perception that consequence plays a more important part in the risk score than likelihood; see Duijm (2015), Ale et al. (2015), is embedded, the design of risk matrix is obviously different from that when risk attitude is neutral. For example, in Fig. 5.8, If the user requires a risk-neutral risk matrix, the design of the risk matrix is symmetric from the bottom left to the top right (see the left figure in Fig. 5.8) (Dethlefs and Chastain 2012). If the user needs a risk aversion matrix, the risk matrix is designed asymmetrically (see the right figure of Fig. 5.7, available at www.npsa.nhs.uk).

From all the examples extracted from the literature, it can be found that the designer is trying to ensure that the risk measure is consistent in the whole risk matrix design. This also explains that the left risk matrix in Fig. 5.7 is symmetric with the line from the bottom left to the top right. It is easy to understand that a risk matrix with a neutral risk attitude satisfies this symmetry, and it is easy to guarantee that this consistency is 100%. How to ensure that risk measurements are consistent with risk aversion in the risk matrix is the real challenge (see the right risk matrix in

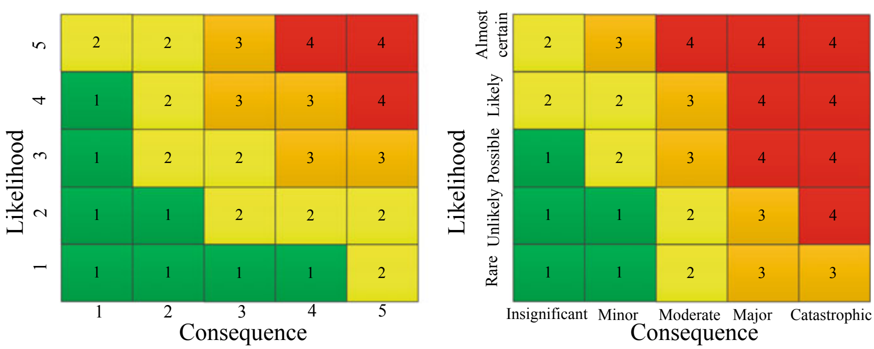


Fig. 5.7 Two 5 × 5 risk matrices with averse and neutral risk attitudes

Fig. 5.7). So it makes sense for us to come up with consistent standards. Obviously, the more reliable the risk matrix is, the more consistent the risk measure is.

The above analysis is based on the assumption that the risk matrix is designed with a systematic approach. If the risk matrix is designed solely on subjective factors, its reliability will be reduced and it will perform poorly in terms of consistency of risk measurement.

5.1.3 Effective Resolution of Risk Ratings

As mentioned earlier, assigning a risk level to each unit is the third step in designing a risk matrix. This is the key to the whole risk matrix. In this step, whether the allocation of risk rating meets the needs of decision-makers is an important criterion to be proposed (Li et al. 2018). This criterion is referred to as the validity of risk rating and is discussed in detail below.

The following methods are commonly used to rate the risk of a cell. First, the input is quantified with discrete Numbers such as “1” and “2”, each cell with have a score according to function $risk = f(consequence, likelihood)$. The cells are divided by several given thresholds (Dethlefs and Chastain 2012). For the risk matrix with continuous axes, several thresholds are given first also, For a risk matrix with a continuous axis, different rules are used to determine the rating of the unit by giving several thresholds (refer to Ruan et al. for example Ruan et al. 2015).

It is subjective to decide to divide the ratings where the decision is chosen from several thresholds of risks. Whether the selection of threshold is reasonable becomes the key. Obviously, risk matrices have several different risk levels (colors). Moreover, each rating has several cells. As a result, when different risks are prioritized using risk matrix tools, some risks may not be able to distinguish risk ratings. These same levels of risk are known as risk ties (Ni et al. 2010). Because risk ties have the same risk rating, they are difficult to distinguish. The number of cells with the same rating can be controlled through threshold tuning. Therefore, the selection of threshold will determine the number of risk ties in the risk matrix.

Clearly, a threshold that is not large enough leads to fewer risk rating categories, thus increasing the number of risk ties, which leads to what is known as low resolution (Duijm 2015). Users want to use the tool of a risk matrix to have enough resolution, to solve the scheduling problem with multiple risks so the risk of the low-resolution matrix is used to distinguish different risk priorities would be useless (Li et al. 2018).

Although given the same threshold, the number of risk relationships is also affected by the distance between adjacent thresholds. In this case, although the number of risk levels remains the same, the distribution of risk relationships varies. Therefore, in the process of threshold change, efficiency also changes with the priority of the risk matrix. Therefore, a series of relatively more efficient risk matrices should be designed. The resolution in risk matrices should not only contain the number of risk ratings but also the distribution of the ratings. Both changes affect the efficiency of the resolution.

Generally speaking, the purpose of using a risk matrix is to help decision-makers correctly evaluate risks or distinguish between different risks. Therefore, effective problem solving is the criterion of designing a risk matrix.

Remark on the three criteria:

(1) There are three criteria to be followed in the design of the risk matrix. As a tool for risk assessment, the first condition that should be followed as much as possible in risk matrix design is risk monotony. Although not all risk points in the risk matrix satisfy monotony, it is necessary to propose monotony as a specific criterion. The consistency of risk measure seems to be easily satisfied in a risk matrix with a neutral risk attitude. What risk matrix designers need to pay most attention to is what extent the designed matrix can satisfy the consistency of risk measurement and risk preference. Finally, we return to the purpose of using a risk matrix to properly evaluate the risks to be evaluated or to rank the strengths and weaknesses of different risks. Whether the risks are properly assessed depends on the validity of the matrix.

(2) People may think, If the first two standards perform well in design, the third standard will also perform well, so the three standards overlap. However, always exist in such a situation, namely in the design, even if the monotonicity of risk and risk measure consistency are well satisfied, but for the performance to solve the problem is likely to be different. There are two reasons. One is that the number of iso-hazard lines may differ, as may the quantitative values of the decision-makers. So the resolution is different. Another reason is that the methods used to assign levels to cells may also be different, and some methods may follow the steps shown in Fig. 5.1 exactly, but in step 3, the methods are different (Cox 2008; Ruan et al. 2015; Li et al. 2018). Although special methods have been proposed to solve the problem, these steps are not strictly followed (no risk measurement is given) (Holt et al. 2014; Thomas et al. 2014). So all three criteria need to be considered separately, they correspond to separate steps, but these steps are continuous. In Sect. 3, we will further discuss the differences between these three criteria.

(3) Different decision-makers may have different criteria to improve the accuracy of the decisions supported by the risk matrices. For example, Cox suggested that the priorities based on risk matrices should support the right decision, for example, in relation to resource allocation (Cox 2008). For the enterprise risk matrix, Duijm points out that the risk matrix at the company level needs to be different from that at the department level (Duijm 2015). We mainly studies the design process of the risk matrix. Other issues related to risk identification and risk reduction are generally not considered during the design process. Moreover, the risk matrix is essentially a qualitative risk management tool and thus, some mathematically rigorous approaches, such as the principle of translation invariance (Cox 2008), should not be imposed on them. They should not be imposed. We propose three standards according to the common requirements of risk matrix users. It also summarizes the theory and practice.

Table 5.1 Mapping of the three indicators to the three criteria

Number	Criterion	Indicator
1	Monotonicity of risks	Proportion of wrong risk pairs
2	Consistency with risk measure	Volatility of risk measures
3	Effective resolution of risk ratings	Probability of correct decision

5.2 Quantitative Indicators of the Criteria

In the previous section, three criteria were proposed to evaluate the design of a risk matrix. Next, we will describe how to quantify these standards.

Since the three criteria we are concerned about are all quantitative indicators, it is necessary to conduct a quantitative analysis on the risk matrix. In our approach, the possibility and outcome axes are defined as continuous, as most researchers do when they study risk matrices (Cox 2008; Ruan et al. 2015; Li et al. 2018). In particular, we make the following reasonable assumptions: without providing any additional information about the risk distribution, we assume that the points in the risk matrix are evenly distributed (Cox 2008), The most commonly used multiplication formula of results and likelihood is used for risk measure (Cox 2008; Li et al. 2018).[1] The following three quantitative indicators can be used to represent the three criteria proposed in Sect. 5.1. Table 5.1 depicts the mapping.

5.2.1 *Proportion of Wrong Risk Pairs (PWRP)*

(1) **Incentive of PWRP**

In the monotony criterion, the decision-maker wants events with high-risk values to have higher risk levels. However, in the risk matrix, due to the risk measure curve and the square shape of the cell, some of the points are in the higher-rated cell, but the risk value is lower, especially when the risk is defined as the product of results and possibilities. For example, in Fig. 5.8, a risk R_1 whose consequence and likelihood are 0.4 and 0.4 is classified with a rating of "yellow". Moreover, for another risk R_2, whose consequence and likelihood are 0.3 and 0.9, Because of the definition of the risk matrix, it should be classified as "green". In this case, R_2 has a higher quantitative value of 0.27, but the rating is lower. Obviously, this point doesn't satisfy monotony. To explain how to measure the monotony of the risk matrix, we first define the "wrong risk pair".

[1] The proposed criteria can be used in quantitative risk matrices regardless of the distribution of risks and the measure of risks in the risk matrix. To show how the criteria are achieved specifically, we offer this assumption.

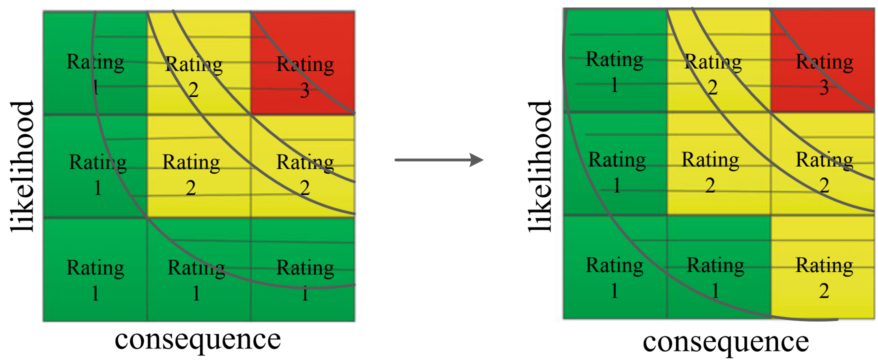


Fig. 5.8 The PWRP increases as the rating of a cell change irregularly

Definition of a wrong risk pair in a risk matrix. Wrong risk pairs are called if one of the two risks has a lower risk level but a higher risk value.

By randomly selecting two points in the risk matrix, three possible scenarios can be generated as follows: (1) One is that one point has a higher risk rating than the other and has a higher value at risk; (2) one is that they have the same risk rating as the other, but not the same value of risk; (3) One is that one has a higher risk rating than the other, but lower risk. Then, the risk pairs in scenarios 1 and 2 do not violate monotony and can be defined as normal risk pairs. But the risk pair in scenario 3 violates monotony and is defined as the wrong risk pair.

The more errors the risk matrix has, the less monotonic it is. Therefore, we propose the following monotonicity index, that is, the proportion of wrong risk pairs (PWRP).

$$Proportion\ of\ wrong\ risk\ pairs = Number\ of\ wrong\ risk\ pairs/total\ risk\ pairs. \tag{5.1}$$

Based on the definition of the wrong risk pair, we can infer that the value of PWRP will increase if the rating of any cell is changed irregularly. For example, in Fig. 5.8, if the risk is measured by the formula $risk = consequence \times likelihood$, the design on the left is obviously reasonable since the risk matrix after the rating should be symmetric with respect to the line with slope 1 and pass through the lower-left corner. But if changed the rating of the lower-right cell to 2, the design is no longer regular under the measure $risk = consequence \times likelihood$. PWRP will increase as the area of the two-level intersection increases. Therefore, we can capture cells with irregular levels by using PWRP during design.

(2) Algorithm of PWRP

Next, we provide the simulation method to calculate the indicator.

Step a1. Generate two points $r_{1,i}$ and $r_{2,i}$ of a risk matrix randomly, where i represents the ith simulation;

Step a2. If the rating of $r_{1,i}$ is higher than $r_{2,i}$ and $r_{2,i}$ is quantitatively higher than $r_{1,i}$, or, the rating of $r_{2,i}$ is higher than $r_{1,i}$ and $r_{1,i}$ is quantitatively higher than $r_{2,i}$, then the value of the variable m will increase by 1. The variable m is set to count the number of wrong risk pairs, with the initial value of 0; return to step a1 until $i = N$.

Step a3. Get the approximate value of the percentage of wrong risk pairs, namely, m/N.

N is a relatively large number so that m/N changes in a slight range as N increases.

Remark on the proportion of wrong risk pairs: The "wrong risk pair" here is only for monotony. However, this does not mean that the risk matrix with the best monotonicity is better. One may find that the wrong pairs of points occur at the intersection of two adjacent ratings, where the equal risk line crosses the two ratings. So, if the number of risk ratings goes down, the wrong risk pair goes away. If only one level is kept. At this time, PWRP is 0, and the risk matrix completely satisfies the monotonicity, but the risk matrix also loses its function. In other words, if only considering the monotony criterion of risk, it may lead to wrong judgment on the design of the risk matrix. For example, as shown in the previous example, a risk matrix with higher monotony may have a lower effective resolution. Therefore, the comparison between the two risk matrix designs based on monotony is based on the fact that the performance of the two risk matrix designs is almost the same under other conditions. This condition also applies to the other two cases.

(3) **PWRP with logarithmic axes**

In theoretical studies of risk matrices it is often necessary to assume that the input axis is linear, but using a logarithmic input axis is more common in practice (Iec 2009; Thomas et al. 2014). Therefore, discussing how to obtain the quantitative indicators with logarithmic axes is necessary.

Given the risk measure, namely, $risk = f(consequence, likelihood)$, in the logarithmic axes system, the risk measure was also changed to a logarithmic form, i.e., $\log(risk) = \log(f(consequence, likelihood))$. In this case, each risk point in the risk matrix with a logarithm gets a value based on the risk measure. Therefore, PWRP can be easily obtained by using Eq. (5.1).

5.2.2 The Volatility of Risk Measures (VRM)

(1) **Incentive of VRM**

What we have stated is that we should measure the consistency between the risk measure and the boundaries. However, we, as the assessors who design the risk matrix usually do not know the risk measure for a given risk matrix by the designer. Therefore, measuring consistency in another way is a good way. Before designing a risk matrix, assuming that the designers should have a predetermined risk measure is reasonable. If we give the boundary of two ratings, then the risk measure of this boundary will be determined. It is noticed that if all the boundaries are consistent with the predetermined risk measure, there will be no differences between the risk measures of any two boundaries. Otherwise, each boundary has a particular risk measure, which shows the difference between these risk measures. Thus, we use the volatility of risk measures (VRM) of all the boundaries to indicate the consistency between the risk measure of the risk matrix displayed and the predetermined risk measure by the designer.

(2) **Algorithm of VRM**

When the VRM is used, we need to obtain the risk measure of each boundary separately. Although the contours dividing the two ratings are not zigzag boundaries of the two ratings, some contour information is included. Therefore, we try to use the information contained in the zigzag boundary to fit the equal risk line.

Risk preference is a key factor affecting risk matrix design. It is common practice for researchers to add risk aversion coefficient into the standard multiplication formula to express the degree of risk aversion, $risk = consequence^n \times likelihood$, $n \geq 1$ (Ale et al. 2015; Thomas et al. 2014). When $n = 1$, the designer is risk-neutral. Under different risk levels of the risk matrix, designers may use different risk aversion coefficients, so different boundary measures will be inconsistent.

The risk measures of boundaries have the unified form as follows:

$$risk_i = consequence^{n_i} \times likelihood, \tag{5.2}$$

where $risk_i$ and n_i denote the risk and risk aversion of the ith iso-risk contour, respectively.

The logarithm form is as follows:

$$\log(risk_i) = n_i \times \log(consequence) + \log(likelihood). \tag{5.3}$$

Next, we need to estimate $\log(risk_i)$ and n_i. To provide symmetry in the estimated iso-risk contour when $n = 1$, Using the symmetrical least square method to estimate the $\log(risk_i)$ and n_i[2] (Wu et al. 2013). Sample data are provided below.

Step b1. Get the edge Shared by any two adjacent levels.[3]

Step b2. Get the median points of the shared edges.

Any point on the edge can be used theoretically, but it is better to use the median edge point to represent the average information. Figure 5.9 shows the estimated iso-risk contours of the 3 × 3 risk matrix. For example, in order to obtain the iso-risk contours between the green and yellow ratings respectively, four Shared edges need to be found first, and four median points of the four sides need to be obtained, $\log(risk_i)$ and n_i can be estimated by symmetric least square method.

If $n_i = n_j, i \neq j$, it is believed that the two iso-contours have the same risk measure ($risk_i$ and $risk_j$ must be different, representing different risk values), and the two iso-contours can be superimposed by translation. Therefore, for the same n_i, Even if we change the value of $risk_i$, the risk measure remains unchanged. Now for

[2] When the sample data have linear symmetry, if we adopt the least square method, the estimated iso-risk contour has no linear symmetry. The estimation formula of the simple linear regression model is given in Appendix 1.

[3] If there is only one edge, the iso-risk contour between the two adjacent ratings cannot be determined.

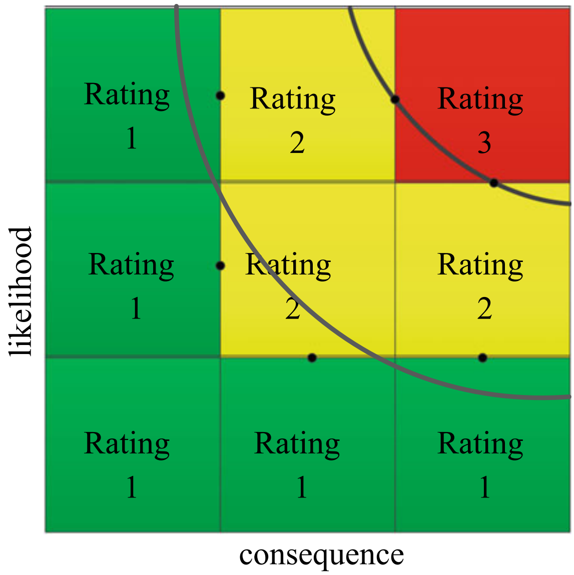


Fig. 5.9 Two estimated iso-risk contours of a 3 × 3 risk matrix

different iso-risk contours, we establish that $risk_p = 1$, where $p = 1, 2, \ldots$. In this case, the risk measure of contour i has not changed. Even the n of the contours are different, the point $(1,\ 1)$ is going to be passed by all the iso-risk contours (because $1_{risk} = 1^{n_i}_{consequence} \times 1_{likelihood}$).

Moreover, the slopes of the iso-risk contours with different n at $(1,\ 1)$ are different, which distinguish the contours (see Fig. 5.10). Therefore, the variance of the slope can be used to measure the volatility of the risk measure, which is expressed as follows:

$$volitility\ of\ risk\ measures = \sum_{i=1}^{N} (slope_i - \overline{slope})^2/(N-1), \tag{5.4}$$

The number of iso-risk contours is equal to N, $slope_i$ represents the slope at $(1,\ 1)$ of the iso-risk contour of $1 = consequence^{n_i} \times likelihood$, and the average of $slope_i$ is $\overline{slope}$.

Since the slope of the contour at $(1,\ 1)$ $likelihood = consequence^{-n_i}$(namely, $1 = consequence^{n_i} \times likelihood$) is $-n_i$, we can rewrite Eq. (5.4) as:

$$volitility\ of\ risk\ measures = \sum_{i=1}^{N} \left(\frac{\sum_{i=1}^{N} n_i}{N} - n_i \right)^2 /(N-1) \tag{5.5}$$

Obviously, higher volatility means a lower consistency with a predetermined risk measure, which means the volatility is a good indicator.

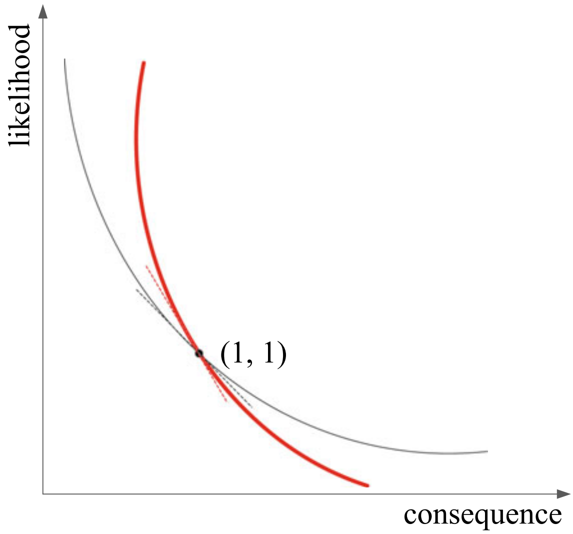


Fig. 5.10 Slopes of two different iso-risk contours at (1, 1)

(3) **VRM with logarithmic axes**

The VRM algorithm is complex to use in linear axes. But it's much easier on the logarithm axes.

Under the condition of the logarithmic axes, the risk measure with risk aversion coefficient is transformed from $risk = consequence^n \times likelihood$ to $\log(risk) = n\log(consequence) + log(likelihood)$. Therefore, the contours become a series of sloping lines as shown in Fig. 5.10. The advantages of the logarithmic scale axis are illustrated by Levine in Fig. 5.11 (Levine 2012). In this case, the volatility of the risk measure can be obtained simply by calculating the variance of these slopes.

Assume that all the diagonal lines have the form of $\log(risk_i) = n_i\log(consequence) + \log(likelihood)$. The volatility of the risk measure, as the variance of the slope, namely, $volitility\ of\ risk\ measures = \sum_{i=1}^{N}\left(\frac{\sum_{i=1}^{N} n_i}{N} - n_i\right)^2/(N-1)$. According to the algorithm, it can be found that the performance of VRM in the logarithmic and linear axes is the same. This seems to be a good result, indicating that the VRM is available on the most commonly used linear and logarithmic axes and that the VRM is independent of the form of the axis.

5.2.3 Probability of a Correct Decision (PCD)

(1) **Incentive of PCD**

For the criteria of effective resolution of risk ratings, we put forward the PCD method. What is the problem with a low-resolution risk matrix? Essentially, the risk matrix

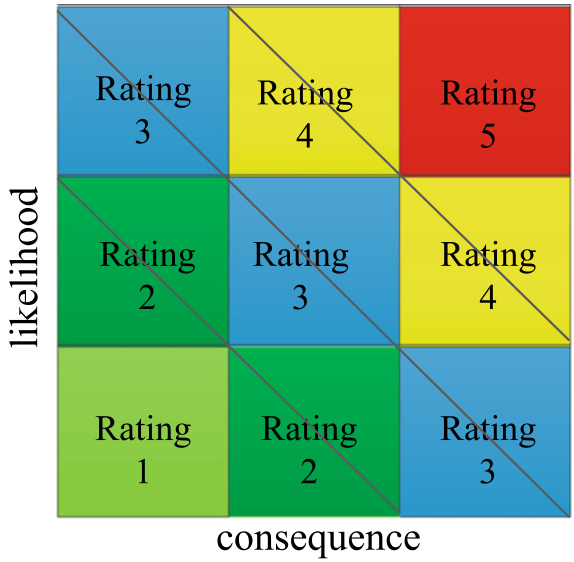


Fig. 5.11 Iso-risk contours in a risk matrix with logarithmic axes

tool is used to prioritize different risks[4] (Duijm 2015). Obviously, a good risk matrix needs to have the function to help the decision-maker make the right decision; That is, multiple risks can be distinguished using this risk matrix. However, for those point at risk ties, it's hard to find the difference between their value at risk, because they are in the same rating, which may cause the decision-maker to make the wrong priority ranking. Therefore, it is necessary for us to use the probability of the correct decision to represent the effective solution.

Any two risks may exist two kinds of risk relationship: (1) different ratings or (2) the same rating.

We first consider the first. If two risks have different risk ratings, the probability exists that the decision-makers will assign a lower rating to a quantitatively higher risk. This corresponds to the wrong pairs described before. In this case, the probability is calculated as follows.

$$\begin{aligned} Probability\ of\ a\ correct\ decision\ in\ case\ 1 &= 1 - Probability\ of\ decision\ error\ in\ case\ 1 \\ &= 1 - \frac{Number\ of\ wrong\ risk\ pairs}{total\ risk\ pairs}. \end{aligned} \tag{5.6}$$

In general, if there are fewer types of risk matrix ratings, there will be fewer wrong risk pairs. However, when the risk matrix has fewer ratings it also has a lower resolution, there will be more risk ties.

[4] Risk matrices are used to determine the acceptance of risks or prioritize risks. In this section, we focus on the latter.

Let's talk about the second case. When the two risks have the same level, the two risks may have the following five positions:

(a) The two risks are in the same cell;
(b) The two risks are in different cells, and a risk is to the right of the other risk;
(c) The two risks are in different cells, and a risk is above the other risk;
(d) The two risks are in different cells, and a risk is to the upper right of the other risk;
(e) The two risks are in different cells, and a risk is to the lower right of the other risk.

For case (a), decision makers cannot judge which risk level is higher. In this case, the probability that they make the right or wrong ranking is 0.5. For case (d), decision makers would give a higher rating to the risk in the upper-right cell. The occurrence probability of wrong risk pairs in case (d) is 0 since the upper-right cell is strictly larger. For cases (b) and (c), the decision makers will choose the right or the upper one because one of the inputs is the same and the other input is larger. In this case, calculate the corresponding probability. For case (e) decision makers could not tell which one would have a higher rating, so they picked a scheme at random and assumed that the probability of it being wrong was 0.5. Therefore, we should summarize all five different cases and give a total PCD estimate.

Comment on the difference between PWRP and PCD:

(1) Monotony is a basic criterion that both researchers and practitioners of risk matrix care about and attach great importance to. Therefore, it is reflected by a single standard: PWRP. But PCD is used to calculate the misclassification probability of two different risk ratings. PWRP focuses on the function when the risk matrix is designed, while PCD focuses on the efficiency when the risk matrix is used. Clearly, a risk matrix with both high PWRP and PCD is better designed.
(2) When calculating PCD, the wrong risk pairs used in calculating PWRP are also used. However, these wrong risk pairs have a small influence on PCD. Figure 5.12 shows the possible areas of wrong risk pairs. It's easy to see that these areas are near the border (the bold segments in Fig. 5.12). PCD is used to measure the probability of the correct ranking of two kinds of risks. On the one hand, we should have fewer boundaries or fewer risk ratings, thus reducing the proportion of wrong risk pairs. However, on the other hand, cutting the number of risk ratings will produce more risk ties, which makes it difficult to make the right decision. According to the data in the section, the more types of risk levels (the input is unchanged), the more points violating monotony in the risk matrix, that is, the higher PWRP is. However, PCD will be lower because of the decrease of risk ties. In other words, the risk ties are the main factor affecting PCD. Thus, the two indicators are indeed facing different standards.

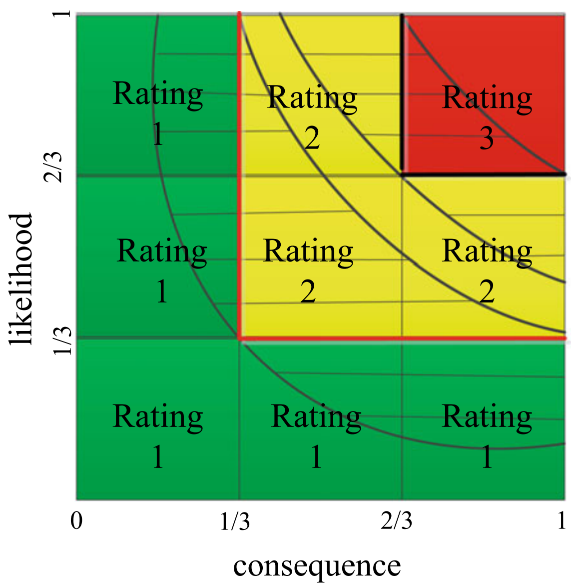


Fig. 5.12 Regions where wrong risk pairs may exist

(2) **Algorithm of PCD**

As discussed, there are many scenarios to consider in the PCD approach. So it's very complicated to compute PCD analytically. Therefore, We use the risk matrix PCD simulation method as follows:

Step c1. Generate two points $r_{1,i}(c_{1,i}, l_{1,i})$ and $r_{2,i}(c_{2,i}, l_{2,i})$ of a risk matrix randomly, where c represent consequence and l likelihood, and i represents the ith simulation;

Step c2. If $c_{1,i} \times l_{1,i}$ is quantitatively higher than $c_{2,i} \times l_{2,i}$ and the rating of $r_{1,i}$ is higher than $r_{2,i}$, or, $c_{2,i} \times l_{2,i}$ is quantitatively higher than $c_{1,i} \times l_{1,i}$ and the rating of $r_{2,i}$ is higher than $r_{1,i}$, then the value of the variable m will increase by 1. If $c_{1,i}$ equals $c_{2,i}$, $l_{1,i}$ is larger than $l_{2,i}$ and the rating of $r_{2,i}$ is higher than $r_{1,i}$, or, $c_{2,i}$ equals to $c_{1,i}$, $l_{2,i}$ is larger than $l_{1,i}$ and the rating of $r_{1,i}$ is higher than $r_{2,i}$, then the value of the variable m will increase by 1. If $l_{1,i}$ equals $l_{2,i}$, $c_{1,i}$ is larger than $c_{2,i}$ and the rating of $r_{2,i}$ is higher than $r_{1,i}$, or, $l_{2,i}$ equals to $l_{1,i}$, $c_{2,i}$ is larger than $c_{1,i}$ and the rating of $r_{1,i}$ is higher than $r_{2,i}$, then the value of the variable m will increase by 1. Otherwise, the value of the variable n will increase by 1. The variable m with the initial value 0 is set to count the number of risk pairs in the case where the probability cannot be given straightway. The variable n with the initial value 0 is set to count the number of risk pairs in the case where the probability is 0.5. Return to step c1 until $i = N$.

Step c3. The approximation of the correct decision percentage can be obtained as follows, namely,$m/N + n/N \times 0.5$. N is a relatively large number so that $m/N + n/N \times 0.5$ changes in a very small range as N increases.

(3) **PCD with logarithmic axes**

The analysis method for the wrong decision is the same in the logarithmic axis system as in the linear axis system.

Remark on the three quantitative indicators: From the perspective of theoretical analysis, we put forward the wrong point pairs rate and the volatility of risk measurement, and from the perspective of the application, we put forward the correct decision rate. Although these three criteria are presented from different perspectives, the good performance of one criterion does not mean that the overall design of the risk matrix is satisfactory. For example, the lower the risk level of the risk matrix, the fewer wrong pairs there will be and the smaller the proportion of error-prone regions on the boundary. However, the lower the resolution will increase the probability of decision-makers making mistakes. Moreover, a trade-off between the dimension of theoretical accuracy and practical applicability should be made. If the criteria are used to optimize the design of a particular risk matrix, Pareto optimality should be considered. However, if these criteria are used to compare different designs in the same risk matrix when the design does not reach Pareto optimality, maybe one design will perform better than another in all three criteria.

5.3 Applications of the Criteria

5.3.1 Instruction for Practitioners Designing Risk Matrices

The audience of the criteria is practitioners of designing risk matrices. Several steps are particularly important in risk matrix design, e.g., defining scales of consequence and likelihood, classifying likelihood and consequence into different categories, assign a rating (color) to each cell. The most difficult step is setting a risk rating.

As far as we know, two rules have been formally proposed (Cox 2008; Li et al. 2018). We can't simply judge which rule is better, because both of these rules seem reasonable at some point. In practice or the literature, it is often entirely dependent on subjective judgment to determine the priority of risk in the risk matrix. However, subjectivity does not mean arbitrariness—at least, the cell with higher consequence and likelihood will have a higher risk priority. This subjectivity means that decision-makers have yet to find a uniform and correct way to help rate risk.

Therefore, our proposed standard is aimed at (1) Providing criteria for comparing risk matrix designs, (2) What should be noted for practitioners designing risk matrices with new rules, and (3) It provides solutions for practitioners to choose the best of several risk matrices designed.

We will introduce in the next section, how to apply these three standards in practice. Although the criteria are not proposed as a guide for designing risk matrices, The level of a particular cell can be determined by them. As mentioned before, the optimal values of these three criteria are difficult to obtain directly, but we may employ them to assess the current risk matrices in the literature.

5.3.2 Determine the Risk Rating of Some Cells

According to Cox's three axioms, the risk ratings of the nine cells are shown in Fig. 5.13: the top-right cell is rated 3, the bottom row and left column are rated 1, and the others are rated 2. First of all, Cox stated in this risk matrix that the risk is measured by the product of consequence and likelihood. Therefore, if only the risk matrix design is an asymmetric aspect, the design is reasonable. Then we focus on the cells that are rated 2. According to Cox's between-ness axiom, rating 2 is not a comparable rating because some of the points in cells rated 2 are larger than some points in cells rated 3, and some are smaller than some points in cells rated 1.

We drew a series of equal risk lines on the risk matrix. In addition, we found that most of the points in the central cell have the same value as the midpoint in the upper-left cell. Therefore, if we want all three ratings to be comparable (rating 3 higher than 2, rating 2 higher than 1), we want to know if we should change the central cell rating to 1. Figure 5.14 is the risk matrix designed according to the sequential updating approach proposed by Li et al. (2018). In Fig. 5.14 the area of the intersection part is smaller. According to our analysis, Fig. 5.14 is better designed based on monotony. The number of green risk ties increases by 1, while the number of yellow risk ties decreases by 1. Therefore, we cannot compare intuitively the performance of the two designs in effective resolution.

The two designs were evaluated using the three criteria and the results are reported in Table 5.2.

The results show that if the rating of the central cell is changed to 1, the risk measure is still consistent with the design. It reduces the likelihood of making the

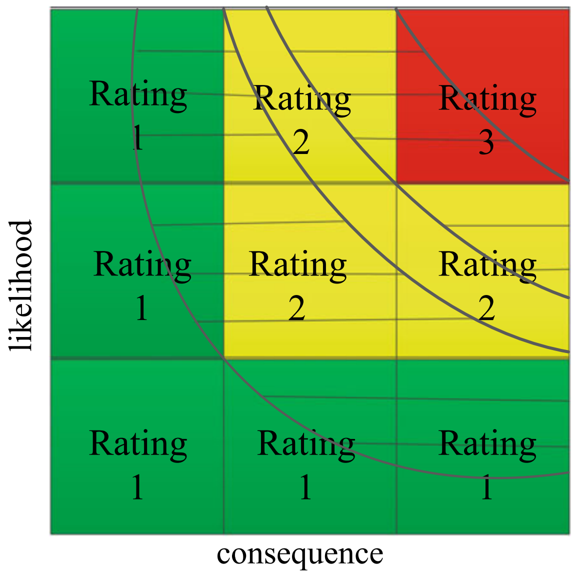


Fig. 5.13 The design of a 3×3 risk matrix according to Cox's axioms

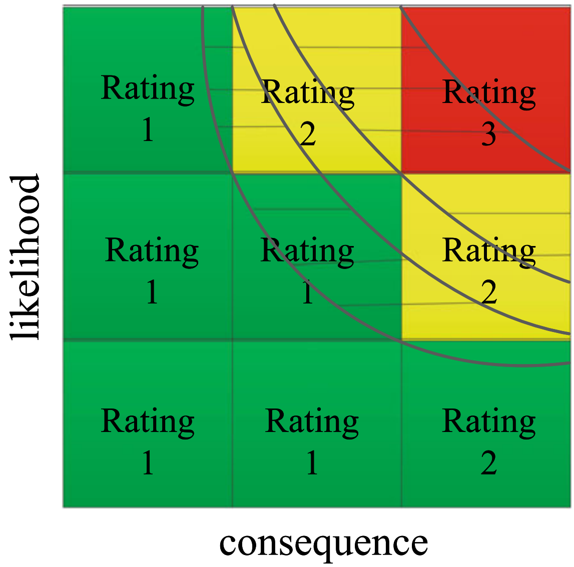


Fig. 5.14 The design of a 3 × 3 risk matrix when the rating of the central cell is 1

Table 5.2 Comparison of two designs of a 3 × 3 risk matrix according to the three criteria

Case	Proportion of wrong risk pairs	Volatility of risk measures	Probability of a correct decision
The central cell is rated 2	0.0111	0	0.8390
The central cell is rated 1	0.0058	0	0.8229

right decision a little bit. However, a significant decrease in the proportion of wrong risk pairs means that the theoretical accuracy of the design has increased. However, a significant decrease in the proportion of wrong risk pairs means that the theoretical accuracy of the design has increased. In this sense, changing the rating of the central unit from 2 to 1 theoretically makes the design more accurate, and the sequential updating method of designing the risk matrix is better in this case.

5.3.3 Assessing Some of the Risk Matrices Used in the Literature

Using these criteria to evaluate the design of risk matrices, risk matrix design does not have a standard to determine which quantitative metrics should be used. The purpose of this section is to calculate the quantitative indicators of risk matrices commonly used in the literature. This process will help us to recognize the normal range of indicators.

Next, we represent only some of the risk matrices smaller than 6 × 6 in size, which are the most common ones.

The 2 × 2 risk matrices are the smallest and are presented in Fig. 5.15. The bottom-left cell has a rating of "1", the top-right cell is rated "3", and the rest are rated "2"(Cox 2008).

The common 3 × 3 risk matrices were discussed in Sect. 5.3. A 3 × 4 risk matrix was used by Iverson et al. To evaluate climatic change responses of forested habitats. It is shown in Fig. 5.16 (Iverson et al. 2012).

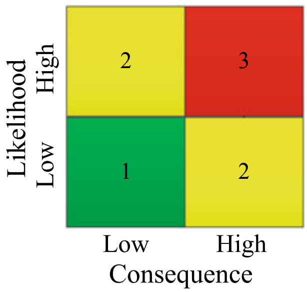


Fig. 5.15 A 2 × 2 risk matrix

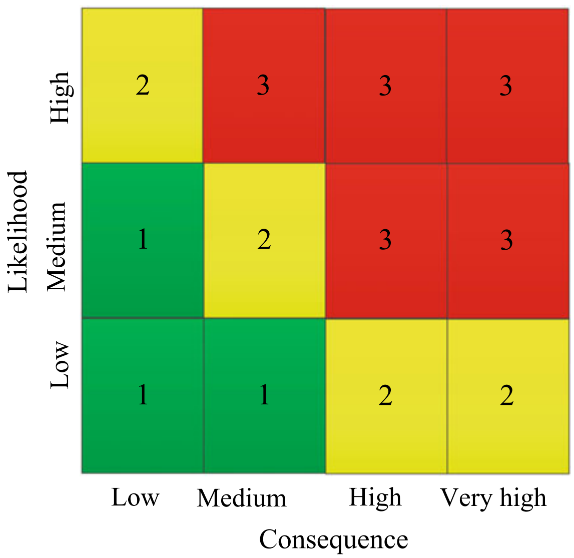


Fig. 5.16 A 3 × 4 risk matrix

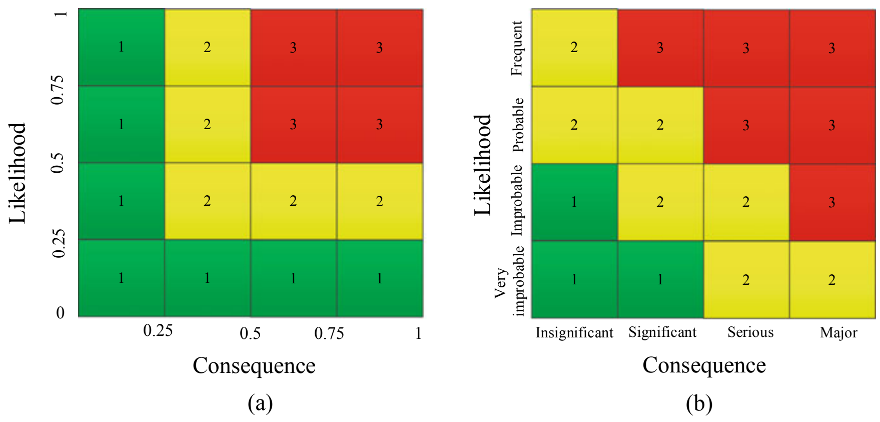


Fig. 5.17 Two 4 × 4 risk matrices

According to three axioms, Cox proposed the possible design of 4 × 4 risk matrix, which are shown in (a) of Fig. 5.17, in which the likelihood and consequence axes are evenly divided. A differently designed of 4 × 4 risk matrix Presented by Duijm, which is shown in (b) of Fig. 5.17 (Duijm 2015).

The most common is 5 × 5 risk matrices. Dethlefs and Chastian used a 5 × 5 risk matrix to assess well integrity risk. Discrete scores from 1 to 5 are used to describe the categories of results and possibilities (in Fig. 5.18) (Dethlefs and Chastain 2012). The National Health Service (NHS) in the United Kingdom also used a 5 × 5 risk matrix to rank risks (b in Fig. 5.18, available at www.npsa.nhs.uk). The federal highway administration uses a 5 × 5 risk matrix to prioritize risks (c in Fig. 5.18) (Cox 2008).

The ISO (2009) also provided an example of a 5 × 6 risk matrix (in Fig. 5.19) (Iec 2009). Moreover, Pritchard et al. used a 6 × 6 risk matrix to manage drilling hazards (b in Fig. 5.19).

Some of the risk matrices described above are qualitative or semi-quantitative, that is, they are input axes described in adjectives or discrete Numbers. However, the three evaluation criteria previously set are used for the quantitative matrix. Therefore,

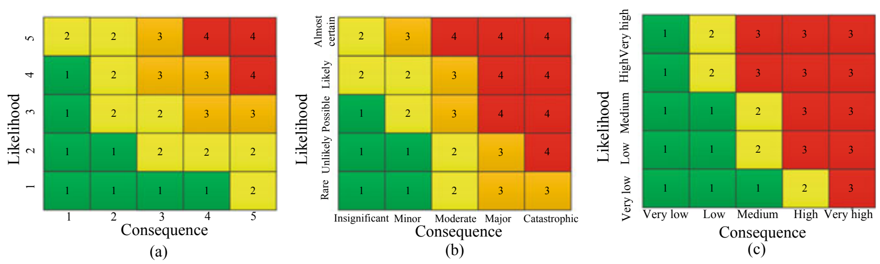


Fig. 5.18 Three 5 × 5 risk matrices

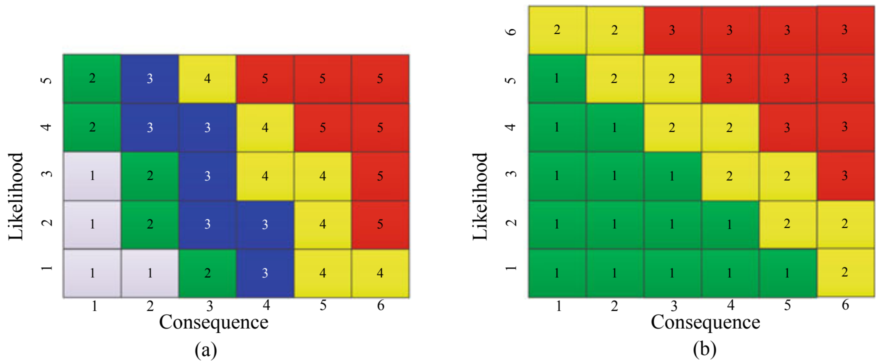


Fig. 5.19 A 5 × 6 risk matrix and a 6 × 6 one

it is necessary to convert qualitative or semi-quantitative risk matrix into the quantitative matrix. The operation is as follows: divide the input axes into equal intervals according to the number of categories. Such as, if the consequence of a risk matrix has five categories, the corresponding intervals are [0, 0.2), [0.2, 0.4), [0.4, 0.6), [0.6, 0.8), [0.8, 1]. We think such dividing is reasonable because it is derived from the distribution of the risk matrices ratings. For example, in the qualitative risk matrices, Cells with the same slash are rated the same, and in the semiquantitative risk matrices, the discrete numbers are ordered.

After preprocessing the risk matrix, we show the quantitative indicators of the three criteria in Table 5.3 according to the steps we introduced in Sect. 5.2.

As can be seen from Table 5.3, since the cell is square and the risk is defined as a multiplication measure, there is no risk matrix where all risk pairs meet monotonicity. First of all, different designs will present different results even for the same risk matrix, which indicates that the change of design method does affect the overall monotony of the risk matrix. In addition, because false risk pairs only exist in the intersection region of two adjacent ratings, the proportion of false risk pairs is less than 10%. When the risk matrix is larger, the intersection area will increase and the proportion of the wrong risk pairs will increase. However, when the proportion of the wrong risk pairs is larger, the higher the dimension, the better the design effect of the risk matrix. For example, in cases 9 and 10, because case 10 has a lower risk rating than case 9, case 10 has a lower proportion of design wrong risk pairs.

For the volatility of risk measures, we know that if the designers design a risk matrix with the same dimensions of likelihood and consequence (such as 3 × 3 or 4 × 4) are risk-neutral, the risk measure is completely consistent with the risk matrix, namely, the volatility of risk measures is 0. This is because the designer knows that the risk matrix should be designed to be symmetric with the line from the bottom left to the top right. However, if the number of categories of possibilities and results is different, it is difficult to design a risk matrix that is completely consistent with the risk measurement, even if the designer wants to be risk-neutral. (see case 3 for example). In addition, the consistency of risk measures decreases as the risk level

Table 5.3 Quantitative indicators of the three criteria under different cases

No	Case	Source	Proportion of wrong risk pairs	Volatility of risk measures	Probability of a correct decision
1	2 × 2 risk matrix in Fig. 5.15	Cox (2008)	0.0484	0 (risk neutral)	0.7645
2	3 × 3 risk matrix in Fig. 5.13	Cox (2008)	0.0111	0 (risk neutral)	0.8390
3	3 × 4 risk matrix in Fig. 5.16	Iverson et al. (2012)	0.0317	0.0112	0.8502
4	4 × 4 risk matrix in (a) of Fig. 5.17	Cox (2008)	0.0159	0 (risk neutral)	0.8738
5	4 × 4 risk matrix in (b) of Fig. 5.17	Duijm (2015)	0.0244	0 (risk neutral)	0.8469
6	5 × 5 risk matrix in (a) of Fig. 5.18	Dethlefs and Chastain (2012)	0.0189	0 (risk neutral)	0.8757
7	5 × 5 risk matrix in (b) of Fig. 5.18	The National Health Service (NHS) in the UK, available at www.npsa.nhs.uk	0.0651	0.2335	0.8488
8	5 × 5 risk matrix in (c) of Fig. 5.18	Federal Highway Administration, 2006 Cox (2008)	0.0523	0.6629	0.8439
9	5 × 6 risk matrix in (a) of Fig. 5.19	Iec (2009)	0.0686	0.8745	0.8325
10	6 × 6 risk matrix in (b) of Fig. 5.19	Pritchard et al. (2010)	0.0444	0 (risk neutral)	0.8376

increases, because the more risk levels there are, the harder it is for designers to ensure consistency.

Wrong decisions occur only when two risks are in the same rating unit or when they constitute the wrong risk pairs. However, the probability of both cases is small, so the probability of correct decision is relatively large, about 0.85, as shown in the table. Still, every different design has a different probability. But because the probability of making a correct decision is influenced by two factors—the proportion of wrong risk pairs and the number of risk ratings—the difference in probability is not significant. Both a smaller proportion of wrong risk pairs and a larger number of risk ratings will improve the probability of a correct decision. For example, in cases 6, 7, and 8, the design of the risk matrix in case 8 has the fewest risk ratings (the number of the average risk ties is the smallest) and thus has the smallest probability of a correct decision. In cases 9 and 10, the design of the risk matrix in case 10 has fewer risk ratings but has a lower proportion of wrong risk pairs than the matrix in case 9, and at last, it has a higher probability of a correct decision, which suggests that the proportion of wrong risk pairs has a greater impact than the number of risk ratings.

Finally, for the same risk matrix, there is no design in which all indicators reach the optimal value. However, for different designs of the same risk matrix, one design may perform better than another in all the indicators. For example, for the 4×4 risk matrices, the designs in cases 4 and 5 both are risk-neutral; however, the design in case 4 has a lower proportion of the wrong risk pairs and a higher probability of a correct decision. This further shows It is feasible to use these criteria to improve the value of one or more indicators to improve the risk matrix design.

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Chapter 6
Risk Matrix Aggregation: A General Framework

6.1 Term Explanation of Risk Matrix Aggregation

In practice, many risk scenarios are consisting of several individual risks. To make the right decisions at the whole organization level, decision-makers have to assess the overall risks rather than single risks (Kouvelis et al. 2011; Acharya et al. 2013; Bernard et al. 2014). Risk aggregation is the process to obtain the overall risk in a certain way based on the multiple individual risks. In the cases where data are sufficient, the overall risks can be assessed by quantitative methods (Binkowitz and Wartenberg 2001). For instance, in a bank, the overall risk should be the aggregated result of market risk, credit risk, operational risk, and so on; and the result can be obtained using copulas (Li et al. 2015).

However, there are many cases where data is not sufficient enough to assess overall risks through quantitative methods (Wu et al. 2019; Shao et al. 2012; Gao et al. 2013; John et al. 2014; Lyu et al. 2020). In a vague environment where data are sparse, risk matrices have been popular and effective tools to assess individual risks due to rrrrr simplicity and low reliance on data (Iec 2009; Iverson et al. 2012; Ruan et al. 2015; Ni et al. 2010; Garvey and Lansdowne 1998; Oliveira et al. 2018; Hsu et al. 2016). Figure 6.1 presents a typical 3×3 risk matrix. Therefore, obtaining overall risk by aggregating several individual risks measured with risk matrices is considered. Since a particular risk is measured by a pre-designed risk matrix, for simplification, the concept of 'aggregating risk matrices' is substituted for 'aggregating individual risks measured by risk matrices' (Duijm 2015; Iec 2009; Bao et al. 2018). The process that obtaining the overall risk through 'aggregating risk matrices' is defined as risk matrix aggregation. Figure 6.2 presents the notion of risk matrix aggregation.

It is imperative to do a further explanation about the aggregability of risk matrices. Few extant works of literature concerned about the aggregation of different risks measured with risk matrices and even ISO state that 'risks cannot be aggregated' (Iec 2009). The notions of non-aggregability of risk matrices are supported by two main pieces of evidence, namely, incomparability of different qualitative risk ratings and incomparability of different types of risk (Bao et al. 2018; Duijm 2015; Iec

C. Bao et al., *Risk Matrix*, Innovation in Risk Analysis,
https://doi.org/10.1007/978-981-19-1480-5_6

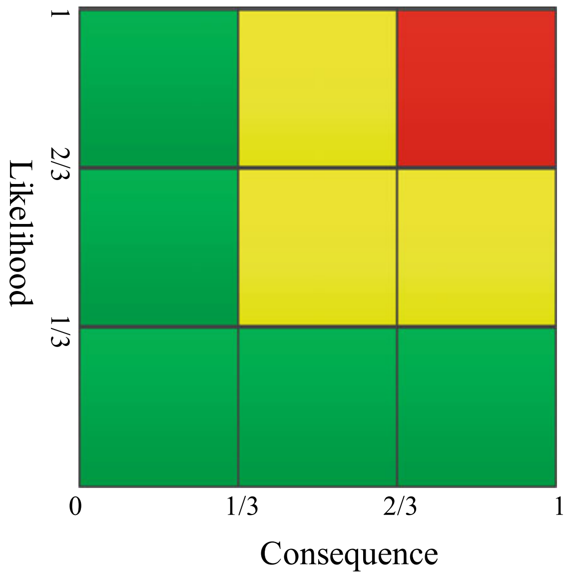


Fig. 6.1 A 3 × 3 risk matrix

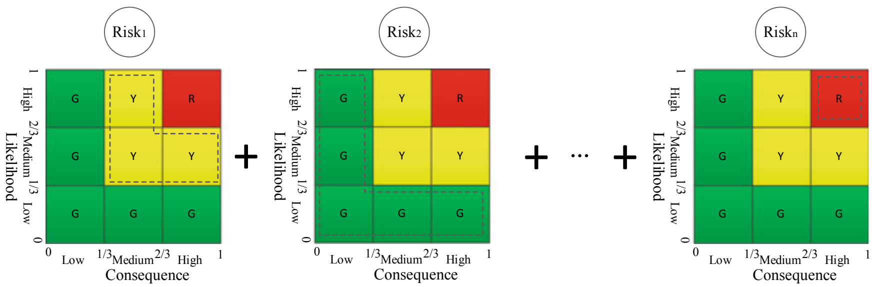


Fig. 6.2 The notion of risk matrix aggregation

2009). Using a risk matrix to assess the individual risks, some risk scenarios can be clearly compared by qualitative descriptions, for example, high risk is more severe than low risk and a scenario with three high risks is more severe than a scenario with three low risks. However, we seem unable to differentiate whether a scenario with high risk and a low risk implies greater risk than a scenario with two medium risks. The second piece of evidence is easy to understand. For different risks, the types of consequences could be economic loss, casualties, and so on. Due to different types of consequences, risks can't be aggregated.

The obstacles analyzed above will be conquered with the usage of the normalized quantitative risk matrices. Using normalized quantitative risk matrices is the precise and basic step of the aggregating risk matrix. Then, a general framework for risk matrix aggregation is proposed which is the major par. The basic idea is to transform the risk matrices into other equivalents to operate the aggregation process. Based on this framework, three methods, namely, the fuzzy set method, the interval number, and the probability density function methods, are introduced. The detailed analyses will be presented in the next sections.

6.2 Normal Framework to Aggregate Risk Matrices

In this section, a more general framework is extracted to aggregate individual risks. Under this normal framework, based on the different understanding of the points in risk matrices, we can develop different aggregation methods. The normal framework of risk matrix aggregation mainly has four steps, they are: (1) Assessing individual risks according to the normalized quantitative risk matrices and getting the rating of the risks, (2) Finding appropriate expressions of the risk ratings, (3) Aggregating individual risks by composition methods and getting the expression of the overall risk, and (4) Transforming the composition results to specific values. Figure 6.3 gives a more intuitive view of the risk matrix aggregation framework. The framework is described in detail next.

6.2.1 Assess Individual Risk According to the Normalized Quantitative Risk Matrices

As the researchers suggest, the input of the risk matrix, namely, consequence and likelihood, should be described with numerical intervals (Bao et al. 2018; Li et al. 2018; Cox 2008). For example, the category 'low' of consequence corresponds to

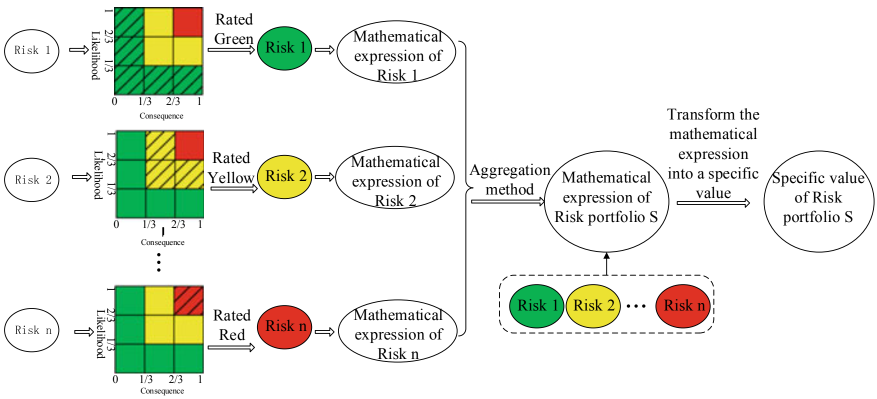


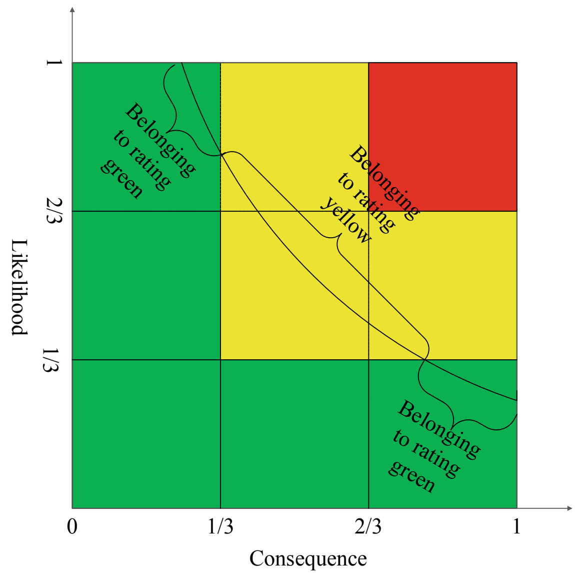
Fig. 6.3 A framework of risk aggregation based on risk matrices

the interval of ($0M, $30M), and the category ‘Medium’ of likelihood corresponds to the interval of (1/3, 2/3). Besides, the quantitative descriptions of consequences should be normalized from 0 to 1 as the qualitative ratings of consequences is the descriptions of the relative severity of a risk, instead of the magnitude (Cox 2008; Ni et al. 2010). This setting makes different types of consequences comparable and even aggregatable (Cokorilo et al. 2014). Figure 6.1 presents a normalized quantitative risk matrix. Using normalized quantitative risk matrices to assess individual risk can reduce the impact of subjective judgment and make mathematical operations on risk matrices possible, which makes risk matrix aggregation feasible.

After obtaining the normalized quantitative risk matrix, the ratings of risks consequence and likelihood are needed for decision-makers to acquire the rating of the risks. Due to the difference in knowledge and experience of decision-makers or experts, the categories of consequence and likelihood to the same risk may be different. However, to guarantee the uniqueness of the risk rating, decision-makers or experts must agree on the rating of risk consequence and likelihood.

The process of assessing the risks with the normalized quantitative risk matrix is the precise and the basic step to aggregating, and it is shown to the left of the brace in Fig. 6.4. The following steps are carried out based on these assessed risks and the corresponding risk matrices.

Fig. 6.4 The uncertainty of a risk value belonging to a risk rating



6.2.2 Find the Appropriate Expressions of the Ratings

Using a risk matrix to assess the risks, the outputs are the rating of these risks. Therefore, 'aggregating individual risks measured with risk matrices' is equivalent to 'aggregating ratings of individual risks measured with risk matrices. However, as the analyses above, we can't acquire the relative severity of the risk portfolio according to qualitative risk ratings. It is intuitive to consider finding appropriate mathematical expressions of the rating so that we could do further aggregation operations.

In a risk matrix, there are infinite points. So each rating of the risk matrix is made up of an infinite number of points. It is essential to understand the meaning of these points. From the process of assessing risks with a risk matrix, it can be easily understood that a point presents one possible location of the risk to be assessed, and the value of the point, which is the product of the consequence and the likelihood, is the relative severity of the risk. Based on the different understanding of points and their risk value, different expressions of rating may be obtained. It will be explained in detail next.

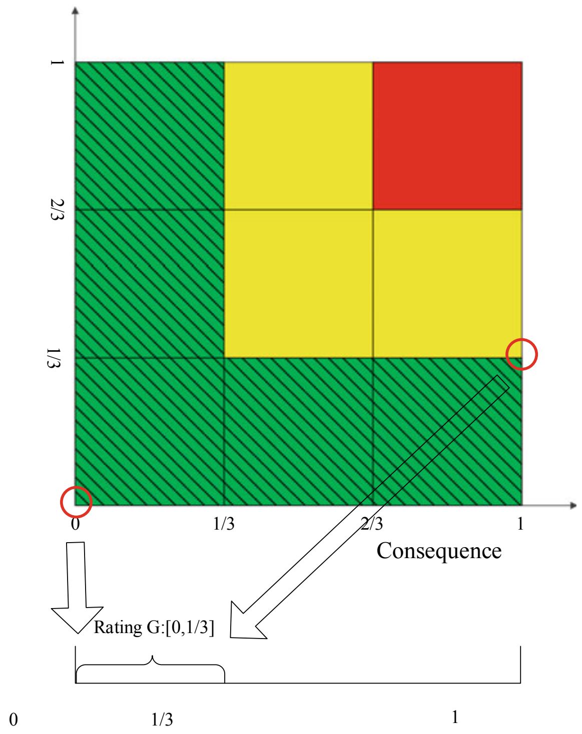
In the normalized quantitative risk matrix, as the risk value is measured by the product of the consequence and the likelihood (Cox 2008), the points with the same risk value may locate in different ratings, which results in the uncertainty of a risk value belonging to a risk rating. Figure 6.4 presents the uncertainty of a risk value belonging to a risk rating intuitively. In the figure, there is an ISO-risk contour, the points on the ISO-risk contour may have a different rating. Therefore, we need to measure the degree that a risk value belongs to a rating, which is an analogy to fuzzy set (Bao et al. 2018).

Supposed $X = \{x\}$ is a space of objectives, A belongs to X, that is, A is the subset of X. The membership function $\mu_A(x) \in [0, 1]$ represents the degree of x belonging to A. The larger the $\mu_A(x)$, the higher the degree of membership of x in A. As shown in Fig. 6.4, an ISO-risk contour passes through different cells which have different ratings. Thus, for a particular risk value in a different rating, it has a corresponding membership. Therefore, each risk rating could be seen as a membership function consisting of the risk values of the points and their degree of membership.

A fuzzy set is just one kind of possible expression of risk ratings. Next another two expressions, namely, the interval number and the probability density function will be introduced.

As analyzed above, each rating consists of an infinite number of points, and these points have different risk values. Therefore, it is intuitive to adopt interval numbers to express the risk ratings. An interval number of a rating contains all the possible risk values of the rating. The interval number can be obtained by finding the maximum and minimum boundary values. In general, the lower-left corner of the risk matrix has the lowest value of risk and the upper right corner has the highest value of risk. Therefore, interval numbers can be used to denote a certain rating. In Fig. 6.5, by determining the maximum and minimum risk values which are circled in the figure, the rating green can be described by interval number $[0, \frac{1}{3}]$.

Fig. 6.5 Rating expressed by interval number



The probability density function is considered as the expression of the rating of the risk matrix. Since each point in the rating is a possible location of the risk. To obtain the density function of the risk valve in a rating, we first obtain the cumulative distribution function $F_X(r) = P(X \le r)$; where r denotes the given risk value in rating G (green), Y (yellow), or R (red). As shown in, given a risk value r, the probability of $X \le r$ in rating G can be approximately equal to the proportion of shaded area in the whole area of G. The probability density function of r in a certain rating can be obtained by deriving the cumulative distribution function (Fig. 6.6).

Three possible expressions of risk rating, which reflect the information of the rating from three different perspectives, are introduced above. It's because the risk rating has infinite points, not a single point, that we can construct these expressions. Based on the expressions, aggregating of risk rating is operated. Next, we will introduce it detailedly.

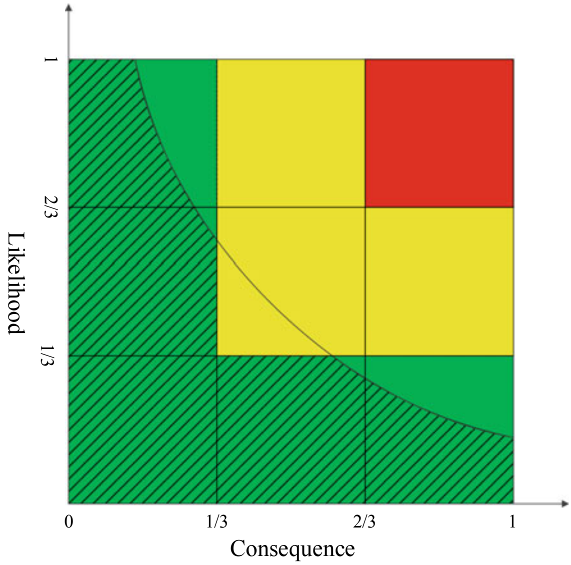


Fig. 6.6 Express the rating by the cumulative distribution function

6.2.3 Aggregate the Individual Risks by Composition Methods

Given the mathematical expressions of the rating, the severity of the overall risks can be obtained by using some kinds of composition methods. It should be noted that different aggregation methods are adopted when different expressions of risk rating are used.

When the ratings of risks are denoted by the fuzzy membership function, we make use of some mature techniques of fuzzy sets to aggregate risk matrices. For any two fuzzy sets, the most used composition method is the max–min rule shown in Zimmermann (2001):

$$\mu_{A*B}(z) = \max_{z=x+y} \min\{\mu_A(x), \mu_B(y)\} \tag{6.1}$$

Here, all risks are assumed to be independent of each other. Therefore, by repeatedly using the rule in Eq. (6.2), the results of operations of two or more risk ratings which can be seen as fuzzy sets including risk value and its degree of membership can be achieved by the following rule:

$$\begin{aligned}\mu_{A_1+A_2+\cdots+A_i+\cdots+A_n}(z) = \max_{z=x_1+x_2+\cdots x_i+\cdots+x_n} &\min\{\mu_{A_1}(x_1),\\ &\mu_{A_2}(x_2), \cdots, \mu_{A_i}(x_i), \ldots, \mu_{A_n}(x_n)\}\end{aligned} \tag{6.2}$$

where $\mu_{A_i}(x_i)$ denotes the degree of x_i in set A_i (or rating A_i) and $z = x_1 + x_2 ... + x_i ... + x_n$.

If the risk ratings are expressed by interval numbers, the aggregating of risks rating should be based on the addition principle of the interval number. As the sum of the two-interval numbers is an interval number, the result of the risk rating aggregating is an interval number too.

Similarly, when the expression of the risk ratings is the probability density function, to obtain the overall risks, we should acquire the probability density function of the sum of the individual risk value. It will be introduced in detail next section.

As analyzed above, the composition methods help aggregate several risks into the overall risks which have the same form of expression as the individual risks. If individual risk rating is expressed by the interval numbers, the expression of the aggregated risk still is interval number.

6.2.4 Transform the Result of Aggregation into a Specific Value

The above results are mathematical expressions, which means comparisons of any outcomes are difficult. As the particular expressions of the aggregated risk can't be compared directly, the relative severity of the aggregated risk is still not comparable by far. To compare the relative severity of the overall risks of the risk portfolios, the expressions of the aggregated risk need to be converted into a comparable value. By comparing the magnitude of these values, the ranking of a risk portfolio among all the risk portfolios can be obtained. The larger the specific value is, the severer the risk is.

In the method of fuzzy set, to convert the fuzzy number into a crisp value, defuzzification is applied (Zimmermann 2001). There are many defuzzification methods, such as max membership principle, centroid method, weighted average method, and so on (Ross 2004). The centroid method is the most prevalent, defined as:

$$z^* = \frac{\int \mu_C(z) \cdot z dz}{\int \mu_C(z) dz}. \tag{6.3}$$

In the method of interval number, the interval number can't be converted into a crisp value if there is only single interval number. Only there are several intervals to be compared, the interval number can be transformed into a specific value. The detailed process will be presented in the next section.

Lastly, the expectation of distribution is usually used to represent the distribution of the objective in the method of the probability density function. The expectation can be obtained with the usage of the Monte Carlo simulation method.

The four steps of the framework are necessary to aggregate the risk matrix. The framework extracted here is more general. Under the framework, different methods are allowed to resolve the aggregation problem. Furthermore, we can explore the difference between the methods and show which one is more reliable.

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Chapter 7
Risk Matrix Aggregation Methods: Introduction and Comparative Analysis

7.1 Fuzzy Set-Based Method

7.1.1 Similarities Between Risk Matrices and Fuzzy Sets

The fuzzy set theory, firstly introduced by Zadeh (1965), is used effectively in ambiguous environments, especially where exists many subjective judgments (Chen and Yu 2020; Goerlandt and Reniers 2016). Similar to the fuzzy set, the risk matrix is widely used in vague environments where are lack of precise data of consequence and likelihood. The descriptions of two inputs of the risk matrix depend on the subjective judgment of decision makers. For example, for a particular risk, the values of the consequence and likelihood, which are judged by the decision makers, are not exact values, but within the intervals (Kouvelis et al. 2011; Acharya et al. 2013; Bernard et al. 2014).

In addition, several unstructured linguistic descriptions can be translated into a structured one by fuzzy sets with fuzzy 'if–then' rule (Mamdani 1975; Rezaei et al. 2011):

$$\text{If } X_1 \text{ is } A_{1i}, \ldots \text{ and } X_n \text{ is } A_{ni}, \text{ then } Y \text{ is } B_i \text{ for } i = 1, 2, \ldots, K.$$

Similar to the application in fuzzy logic, mapping the two inputs to a risk rating in the risk matrix is also based on such an 'if–then' rule (Markowski and Mannan 2008; Duijm 2015). For example, in Fig. 7.2, there are nine rules and one of these rules is if the consequence is 'high' and the likelihood is 'low', then the risk rating is 'Green'. Generally, when a risk matrix is given, the 'if–then' rules are determined (Pickering and Cowley 2010; Ni et al. 2010; Levine 2012).

Last, as analyzed in Sect. 6.2.2, a specific risk value may belong to two or more different risk ratings and the degree that the risk value belongs to a rating, which is an analogy to fuzzy membership function, needs to be considered. In a fuzzy set, supposed $X = \{x\}$ is a space of objectives, A belongs to X, that is, A is the subset of X. The membership function $\mu_A(x) \in [0, 1]$ represents the degree of x belonging

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to A. The larger the $\mu_A(x)$, the higher the degree of membership of x in A (Zadeh 1965). In practical application, it is generally assumed that the membership function is trapezoidal (Zadeh 1975; Ferdous et al. 2011; Rezaei et al. 2011), as shown in Fig. 7.1.

The corresponding equation of trapezoidal membership function of fuzzy set 2 in Fig. 7.1 is:

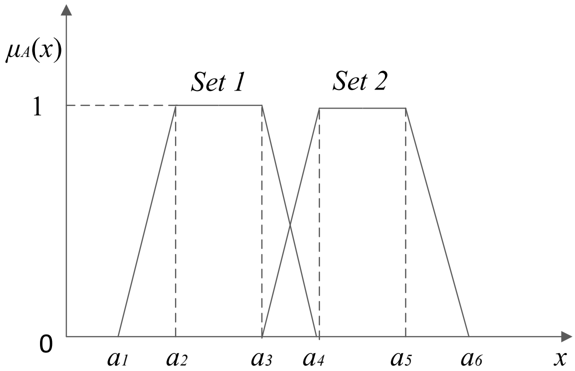


Fig. 7.1 A sample of trapezoidal membership function

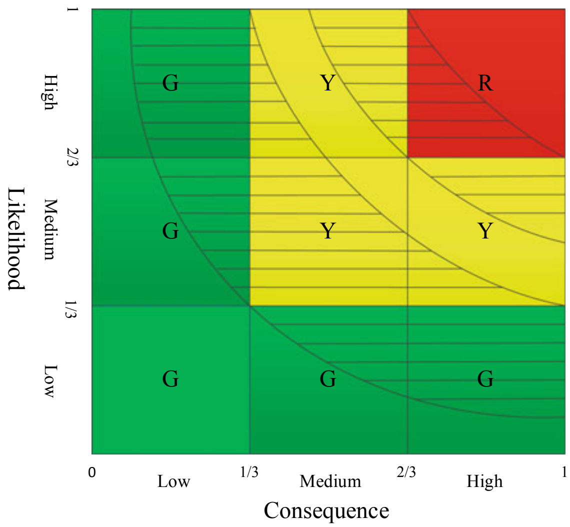


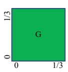
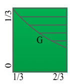
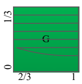
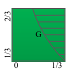
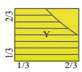
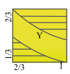
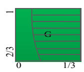
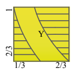
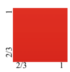
Fig. 7.2 A 3 × 3 risk matrix divided by four ISO-risk contours

$$\mu_{set1}(x) = \begin{cases} 0, & x < a_1, \\ \dfrac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2, \\ 1, & a_2 \leq x \leq a_3, \\ \dfrac{x - a_4}{a_3 - a_4}, & a_3 \leq x \leq a_4 \\ 0, & x > a_4. \end{cases} \tag{7.1}$$

To explain where the uncertainty of the risk value belonging to the risk rating comes from in detail, the risk matrix (this risk matrix is designed according to Cox's three axioms (Cox 2008), as shown in Fig. 7.2, is adopted. There are four ISO-risk (IEC 2009) contours in the risk matrix and these contours pass through the highest and lowest points of each risk rating. The dashed area in the figure represents the shared parts of two adjacent ratings, which means that in such an area, two risks with the same risk value may belong to two or more ratings. The points in the dashed area are called abnormal risk points, and the points in the remaining part are called normal risk points. Thus, there are five different area ratings, namely, explicitly rated green (I), intangibly rated green and yellow (II), explicitly rated yellow (III), intangibly rated yellow and red (IV), and explicitly rated red (V). Table 7.1 reports the area proportion (AP) of each cell.

The cells with ith rating of consequence and jth rating of likelihood in Fig. 7.2 are represented by C_{ij}. From the table we can see, in most cells, only a part of the risk values belongs to an explicit rating, while the rest are rated ambiguously. The

Table 7.1 Area proportion of each cell in a 3 × 3 risk matrix

Cells of Risk Matrix	C_{11}	C_{21}	C_{31}	C_{12}	C_{22}	C_{32}	C_{13}	C_{23}	C_{33}
Figure									
AP of I (%)	100%	69.31%	40.55%	69.31%	0%	0%	40.55%	0%	0%
AP of II (%)	0%	30.69%	59.45%	30.69%	86%	21.64%	59.45%	21.64%	0%
AP of III (%)	0%	0%	0%	0%	14%	40.55%	0%	40.55%	0%
AP of IV (%)	0%	0%	0%	0%	0%	37.81%	0%	37.81%	43.28%
AP of V (%)	0%	0%	0%	0%	0%	0%	0%	0%	56.72%

risk values with ambiguous ratings correspond to the overlap of the two fuzzy sets in Fig. 7.1 and it explains the source of uncertainty in risk ratings.

Based on the above analysis, the risk matrix can be transformed into other equivalents with the usage of a fuzzy set. Through this transformation, some mature fuzzy set aggregating techniques can be used to aggregate the risk matrix. Here, using a fuzzy set to express the risk matrix is the core step for risk aggregating.

7.1.2 Fuzzy Membership in Risk Matrices

(1) **Necessity of studying fuzzy membership**

As stated before, in the risk matrix, two kinds of risk points, namely normal and abnormal points, exist. The former refers to quantitatively equivalent points assembling in the same risk rating and the latter in different risk ratings. Since these abnormal points belong to two or more risk ratings, the degrees of membership in different ratings have to be considered. Membership coincidentally reflects a risk rating's vagueness. However, ISO-risk contour is determined in the risk matrix, thus the vagueness of a rating depends on which cells belong to the rating, namely, the color scheme of the risk matrix. Hence, whether the vagueness exists in all reasonable risk matrices needs to be established. But based on the following proposition, we can assert that vagueness is inevitable in the risk matrix (Bao et al. 2018).

Proposition 1 Degrees of membership of points in the same rated cells in a risk matrix cannot be all 1.

Proof First, it is clear that there must be two adjacent (upper-lower or left–right) cells that have different ratings. Otherwise, the risk matrix will only have one rating, which is meaningless for the risk matrix. More importantly, for any two adjacent cells, there are countless ISO-risk contours passing through these two cells. If the ISO-risk contour passes through two cells with different ratings, it means that there are some risk values belonging to two ratings simultaneously, which implies that the degree of the membership of the corresponding quantified risk in these ratings is not 1.

Proposition 1 illustrates the necessity of studying membership in the risk matrix. Based on the above analysis, the following definition is proposed:

Definition 1 *Risks are located in two kinds of intervals: overlapping and non-overlapping. The former interval* consists *of a series of continuous risks whose degrees of membership are less than 1 and the latter 1.*

As the consequences and likelihoods of cells are different, no two cells can be the same in theory. Thus, it is indispensable to argue whether we should consider cells with the same rating as a whole or as individual cells. Cells with the same rating are seen as a whole for the following reasons. First, the cells in the same rating have no significant difference. In addition, if each cell is regarded as an individual and all possible scenarios are considered to be aggregated, on the one hand, it violates the original intention of using a risk matrix where several different cells are approximately divided into the same rating, on the other hand, such operation will lead to the high resolution of risk matrix and the complexity of calculation will increase. As a result, we only consider the rating of a risk, for example, we use fuzzy membership function to express the Green rating rather than the single cell in the Green rating.

(2) **Measure and property of fuzzy membership function**

Another question about fuzzy membership is how does the degree of membership change between 0 and 1? This is determined by the membership function. As shown in Fig. 7.1, the membership function is assumed to be trapezoidal (Ferdous et al. 2011; John et al. 2014; Mentes et al. 2015). Therefore, under this assumption, the corresponding membership function of the risk matrix shown in Fig. 7.2 can be drawn as Fig. 7.3. Obviously, if the trapezoidal membership function is used, the degree of membership increases or decreases linearly.

To test the truth of this assumption, we will find a reasonable method to calculate the degree of membership. As shown in Fig. 7.2, the ISO-risk contour includes all the points having the specific same risk value, and these points may belong to different risk ratings. Intuitively, the length of the ISO-risk contour in the area of the particular rating represents the information that the corresponding risk value belongs to this rating. Hence, the ratio of the length of the partial contour to the overall length of this contour can be used to measure the degree of membership of a risk in a rating, namely:

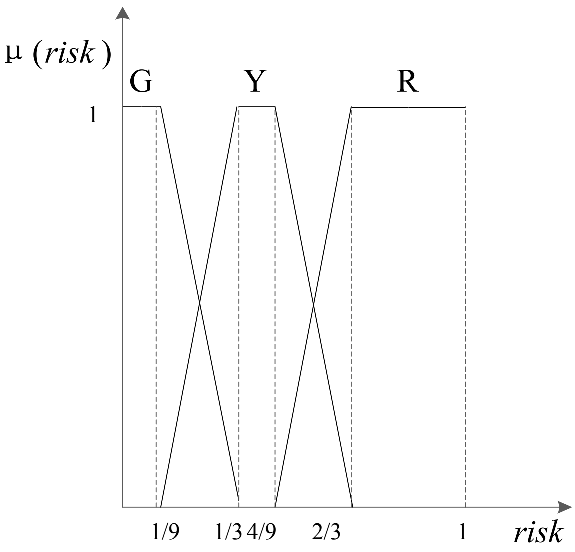


Fig. 7.3 Trapezoidal membership function of a 3 × 3 risk matrix

$$\mu_A(risk) = \frac{L_{risk,A}}{L_{risk}}. \tag{7.2}$$

In the above expression, $L_{risk,A}$ represents the length of the counter with the quantitative value *risk* in the region whose the risk rating is A. L_{risk} represents the overall length of this ISO-risk counter.

The length of the counter in the risk matrix can be calculated by the following expression:

$$\begin{aligned} L &= \int_{x_1}^{x_2} dl = \int_{x_1}^{x_2} \frac{dx}{\cos\theta} = \int_{x_1}^{x_2} \sqrt{1+\tan^2\theta}dx \\ &= \int_{x_1}^{x_2} \sqrt{1+f'^2(x)}dx = \int_{x_1}^{x_2} \sqrt{1+\frac{risk^2}{x^4}}dx \end{aligned} \tag{7.3}$$

where x is the input variable corresponding to the consequence in a risk matrix and $f(x)$ is the output corresponding to the likelihood, namely, $f(x) = \frac{risk}{x}$; θ is the slant angle of the curve in an infinite small triangle during the integration process.

Since there are no elementary antiderivatives of the integrand, formula (7.4) cannot be obtained by numerical calculation. In order to obtain the results, the approximate algorithm, namely, the trapezoidal rule, is used. Under this algorithm, the length of the curve can be approximately calculated as follows:

$$\int_a^b f(x)dx \approx \frac{b-a}{2n}(f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)) \tag{7.4}$$

where n is the number of subintervals of $[a, b]$ and $x_i = a + i\frac{b-a}{n}, (i = 0, 1, 2, \ldots, n)$.

It is notable that if the regions of ISO-risk contour in a specific rating are segmented, i.e. the cells that are crossed by the curve with the same rating are not adjacent, the degree of membership can be determined by the ratio of the length of these segments to the length of the overall length. Furthermore, in a risk matrix, a particular risk value may belong to several (more than 2) risk ratings, depending on the design of the risk matrix.

Based on the reasonable method above, the problem of whether membership is linearly increasing or decreasing is considered. The following proposition provides the answer.

Proposition 2 *If the default risk measure, namely, risk = probability × consequences is adopted to assess the risk, the membership function of risks in the overlapping part of two ratings is not linear.*

If the membership function within an interval increases or decreases linearly, the degree of membership of the beginning, middle and ending points of the interval will be on the same line. However, it is easy to find that the three membership degrees are not on the same line according to our method, which proves the truth of Proposition 2.

We claim that the membership functions of a risk in a rating are different in different risk matrices, which comprise the following proposition.

Proposition 3 *It is the design of* the *risk matrix, namely, how to assign a different rating to the different cells in a risk matrix, that determine the membership function of a risk in a rating.*

A risk matrix is essentially a qualitative risk assessment tool and decision-makers have their own perceptions of the rating of a risk. Their designs of the risk matrices determine their perceptions of the degree to which a risk belongs to a rating. We do not impose any assumption on the design of the risk matrices for the sake of universality of our method.

One may question why we should adopt the method we have introduced to achieve the degree of membership instead of accepting the assumption that the membership function is trapezoidal. We claim that there are two main reasons, as follows.

(1) Trapezoidal membership function results in a biased estimation of the aggregated risk and thus is not accurate;
(2) The assumption method is not applicable where the membership function does not change between 0 and 1, while our method for measuring the membership function is more universal.

To explain the reason (1), we first give two kinds of membership functions of a risk in risk rating Y (Fig. 7.4). The blue outline is drawn according to our method and the red outline is a trapezoid as the assumption described before. *One may see that when the membership function increases from 0 to 1, it is convex and when the membership function decreases from 1 to 0, it is concave.* Due to the complexity of

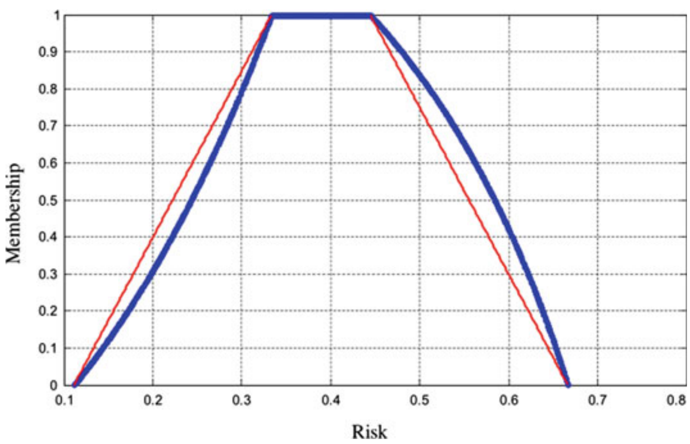


Fig. 7.4 Membership of risk in rating Y

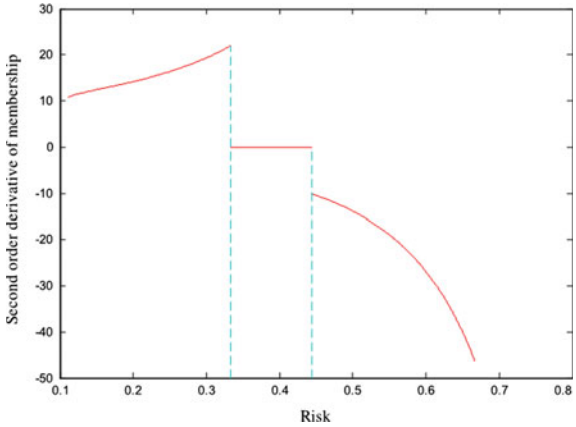


Fig. 7.5 The second derivative of membership with respect to risk

formula (1), without strict mathematical proof, with the aid of computer we report the second derivative of the membership function with respect to risk in the two overlapping parts in Fig. 7.5.

Figure 7.5 shows that the second derivative of the membership relationship in the increasing part is greater than 0, while the second derivative in the decreasing part is less than 0, which supports our conjecture. Our conjecture first describes the shape of the membership function and refutes the assumption that the risk membership function in the overlap interval is linear. Moreover, it has the following important property.

Property of the membership of risks in different ratings in a 3 × 3 risk matrix as shown in Fig.7.2**:**

The membership *functions of risks in different ratings G, Y, and R are asymmetric.*

Notice that when using a fuzzy set, defuzzification methods should be used to get a crisp value of the fuzzy set that represents its magnitude. The centroid method is a commonly used defuzzification method. When using the centroid method, the results of the trapezoidal membership function and the asymmetric membership function must be different. Compared with the centroid of the trapezoidal membership function, the asymmetric membership function's centroid moves to the right, which affects the accuracy of further operations. This is why the exact form of membership function should be determined first.

To illustrate the second reason, we construct an arbitrary risk matrix without considering the accuracy of its design, as shown on the left of Fig. 7.6. By using the method before, the outline of the membership function of risk in each rating is drawn on the right of Fig. 7.6. From the figure, we can clearly see that the maximum degree of membership of the risk-rated green and yellow is not 1, while the minimum degree of membership of the risk-rated red is not 0, which is contrary to the hypothesis that the shape of the risk membership function is trapezoidal. In addition, there are still shortcomings described in the first reason. For the above two reasons, the method proposed is recommended to output an accurate membership function.

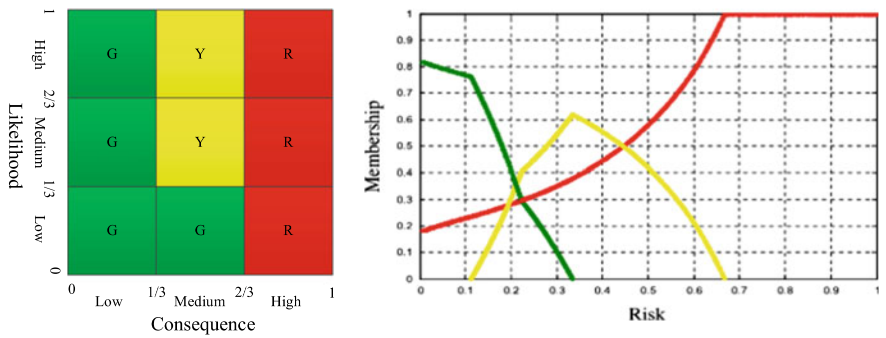


Fig. 7.6 A fictitious 3 × 3 risk matrix and the corresponding membership functions achieved by our method

(3) Fuzzy membership of aggregated risks

As analyzed in the Sect. 6.2.3, for any two fuzzy sets, the most used composition method is the max–min rule shown in Zimmermann (2001):

$$\mu_{A*B}(z) = \max_{z=x*y} \min\{\mu_A(x), \mu_B(y)\} \tag{7.5}$$

All risks are assumed to be independent of each other here. Therefore, by repeatedly using the rule in Eq. (7.5), the results of operations of two or more risk ratings which can be seen as fuzzy sets including risk value and its degree of membership can be achieved by the following rule:

$$\mu_{A_1+A_2+\cdots+A_i+\cdots+A_n}(z) = \max_{z=x_1+x_2+\cdots x_i+\cdots+x_n} \min\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots, \mu_{A_i}(x_i), \ldots, \mu_{A_n}(x_n)\} \tag{7.6}$$

where $\mu_{A_i}(x_i)$ denotes the degree of x_i in set A_i (or rating A_i) and $z = x_1 + x_2 ... + x_i ... + x_n$.

Next, we state how the above rules are applied to obtain the membership function of a set of n different risks (ratings of these risks are A_1, A_2, ..., A_n) with the help of Monte Carlo simulation in detail.

Step 1: Obtain the interval of the risk value in each rating of the evaluated risks. First, generate a risk value randomly belonging to the interval, and n risk values are generated in all. Second, calculate the corresponding n memberships of all the n risk values. Both risks are recorded in a two-dimensional array (the two dimensions are risk value and risk membership).

Step 2: Total risk of the n risk values is measured by the sum of all risk values, denoted by R_s, where s presents the sth round. And membership of R_s is the minimum of all memberships of single risks, denoted by $\mu(R_s)$.

Step 3: Repeat Steps 1–2 N times and we now have $R_1, R_2, ...R_N$, and $\mu(R_1), \mu(R_2), ..., \mu(R_n)$. Check if there are duplicate values of $R_p (p = 1, 2, ..., N)$.

If R_p is duplicate (in this case, there are several memberships of R_p), membership of R_p should be the maximum of all memberships of R_p. After the screening, there are in total N' overall risk values and memberships. If the number of simulations is large enough, we will obtain membership of each possible overall risk in theory.

(4) **Defuzzification of fuzzy membership**

Here, the centroid method is used to convert a fuzzy risk value into a crisp value. The centroid method is presented as follow:

$$z^* = \frac{\int \mu_C(z) \cdot z dz}{\int \mu_C(z) dz}. \tag{7.7}$$

However, in the simulation, due to the limited number of simulations, we cannot obtain the degree of membership of every possible overall risk. According to the definition of integral, if we choose m points of risk axis with the same step from the lowest risk to the highest one and obtain the corresponding memberships of these points, the defuzzification result according to the following formula will be comparatively accurate (Ross 2004).

$$R = \frac{\int \mu(r) r dr}{\int \mu(r) dr} \approx \frac{\sum \mu(r) r}{\sum \mu(r)} \tag{7.8}$$

However, the limited number of simulations will also result in few duplicate values, that is, generally, the true degree of membership of the risk value cannot be obtained (the membership of a risk value we obtain is smaller than the true one). For example, in Fig. 7.7, the degree of membership obtained is usually smaller than the membership on the boundary. The problem that the membership of selected risk may not be output in the simulation is solved with the following approximate method:

Step 1: Sort the two-dimensional array in ascending order according to risk.

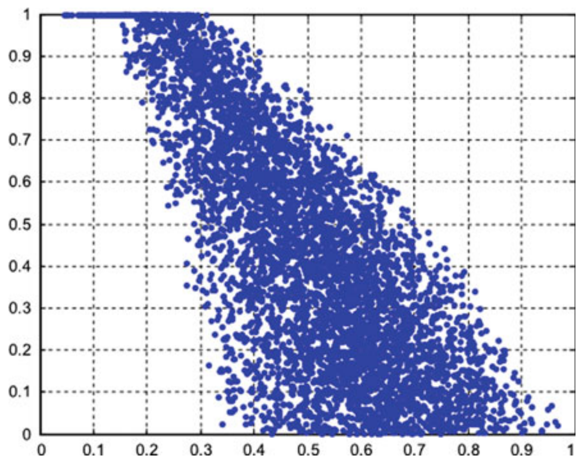


Fig. 7.7 Distribution of integrated risks with simulation method

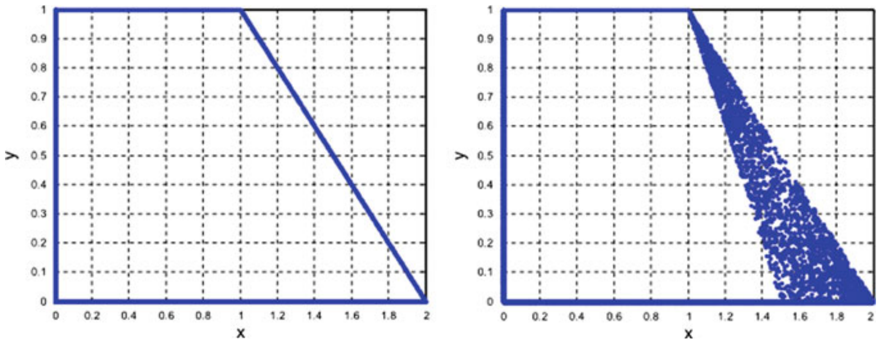


Fig. 7.8 Checking accuracy of the method to find the centroid of a trapezoid

Step 2: Divide N membership samples into m groups: elements of membership dimension in the array from 1 to N/m are in group 1; elements from N/m + 1 to 2 N/m are in group 2 and so on.

Step 3: Select the maximal element in each group and totally m elements are chosen.

Step 4: Select m risk values: $r_{\min,} r_{\min} + \frac{r_{\max} - r_{\min}}{m}$, ..., $r_{\max}$; where $r_{\min}$ and $r_{\max}$ present the minimum and maximum risks, respectively.

To check the accuracy of this method, we apply it to a simple problem: calculate the centroid of the left in Fig. 7.8. The exact value of the centroid is 7/9. After some pretreatment and simulation, some sample points as shown on the right of Fig. 7.8 are obtained. According to formula (7.8), the simulation result was 0.7760, close to 7/9. The variance of 100 simulations is $1.6230e^{-5}$, which is quite small. The above analysis shows the accuracy of our method in obtaining the centroid of the graph.

So far, based on the previous analysis, the aggregation process can be concluded as follows:

Step 1: Assess the n risks in their respective risk matrices to obtain the n risk ratings.

Step 2: Calculate the fuzzy membership function of each of the ratings using formula 7.5.

Step 3: Use Monte Carlo simulation to obtain the fuzzy membership of aggregated risks.

Step 3.1: Obtain the interval of the risk value in each rating of the evaluated risks. First, generate a risk value randomly belonging to the interval, and n risk values are generated in all. Second, calculate the corresponding n memberships of all the n risk values. Both risks are recorded in a two-dimensional array (the two dimensions are risk value and risk membership).

Step 3.2: Total risk of the n risk values is measured by the sum of all risk values, denoted by R_s, where s presents the sth round. And membership of R_s is the minimum of all memberships of single risks, denoted by $\mu(R_s)$.

Step 3.3: Repeat Steps $a1 - a2N$ times and we now have $R_1, R_2, \ldots R_N$, and $\mu(R_1), \mu(R_2), \ldots, \mu(R_n)$. Check if there are duplicate values of $R_p (p = 1, 2, \ldots, N)$.

If R_p is duplicate (in this case, there are several memberships of R_p), membership of R_p should be the maximum of all memberships of R_p. After the screening, there are in total N' overall risk values and memberships. If the number of simulations is large enough, we will obtain membership of each possible overall risk in theory.

Step 4: Apply the defuzzification method to obtain the crisp value of a risk portfolio, which can be compared in magnitude with the crisp values of other risk scenarios.

7.2 Interval Number-Based Method

An interval number is a set of real numbers on a closed interval, which is often denoted by [a, b], where a represents the lower bound of the interval number and b represents the higher bound. Interval number is a fundamental tool often used in a vague environment.

Let $\widetilde{a} = [a^L, a^U], \widetilde{b} = [b^L, b^U]$, where $\widetilde{a}, \widetilde{b}$ are both interval numbers. The addition operations are shown as follows (Sengupta and Pal 2000, 2009; Sun and Yao 2008):

$$\widetilde{a} + \widetilde{b} = [a^L + b^L, a^U + b^U]$$

$Let\, \widetilde{a} = [a^L, a^U], \widetilde{b} = [b^L, b^U], la = a^U - a^L, l\widetilde{b} = b^U - b^L,$
Then

$$p(\widetilde{a} \geq \widetilde{b}) = \frac{min\{l_{\tilde{a}} + l_{\tilde{b}}, max(a^U - b^L, 0)\}}{l_{\tilde{a}} + l_{\tilde{b}}} \tag{7.9}$$

which represents the probability of $\widetilde{a} \geq \widetilde{b}$. The relation can be denoted by $\widetilde{a} \geq_\rho \widetilde{b}$. Suppose there are N interval numbers, namely, $\underset{i}{\widetilde{a}} = [a_i^L, a_i^U], i \in N$, formula (7.9) can be used to calculate $p(\underset{i}{\widetilde{a}} \geq \underset{i}{\widetilde{b}}), i, j \in N$, denoted by p_{ij}, Furthermore,

$$v_i = \frac{1}{n(n-1)}(\sum_{j=1}^{n} p_{ij} + \frac{n}{2} - 1), i \in N \tag{7.10}$$

Formula (7.10) represents the magnitudes of interval numbers $\widetilde{a}_i$, v_i can be used to compare the magnitudes of different interval numbers.

Aggregating risk matrices by the aggregation method of interval number is fairly simple, which is achieved just by using the addition operation of the interval numbers. Specifically, each risk rating can be denoted by an interval number, by the addition of interval numbers, we can obtain the overall risk denoted by an interval number.

What should be paid attention to is an interval number itself does not have a v and thus there is not a crisp value to represent its magnitude. Therefore, to obtain the crisp value of a portfolio, we should collect all the interval numbers of all possible portfolios. The detailed process is as follows:

Step 1: Assess the n risks in their respective risk matrices to obtain the risk ratings.

Step 2: Denote the ratings of the evaluated risks by interval numbers, take the specific risk matrix in Fig. 7.2 for instance, $\widetilde{g} = [0, {}^1\!/_3]$, $\widetilde{y} = [{}^1\!/_9, {}^2\!/_3]$, $\widetilde{r} = [{}^4\!/_9, 1]$ ($\widetilde{g}, \widetilde{y}, \widetilde{r}$ represent interval numbers of rating green, rating yellow, and rating red, respectively).

Step 3: According to the calculation principle of interval numbers, we can aggregate the individual risks by the addition of interval numbers, the formula is $\widetilde{a} + \widetilde{b} = [a^L + b^L, a^U + b^U]$. The use of interval numbers should correspond to the ratings of risks in the portfolio.

Step 4: Use formula (7.10) to calculate v_i to compare the different interval numbers.

As analyzed before, notice only there are several intervals to be compared, the interval number can be transformed into a specific value.

7.3 Probability Density Function-Based Method

As analyzed in Chap. 6, in risk matrices, given a risk value x, as shown in Fig. 7.2 the probability of $X \le x$ equals to the proportion of the dashed area in the whole area of a rating. Therefore, the cumulative distribution function is an explicit formulation with respect to x. Then the probability density function is the derivation of the cumulative function, namely, $f(x) = F'(x)$.

The Monte Caro simulation method can be used to obtain the distribution of the overall (sum) risk of the risk portfolio. And the expectation of the distribution is taken as the crisp value of the distribution.

The aggregation process is as follows:

Step 1: Assess the n risks to be aggregated in their respective risk matrices to obtain the risk ratings of these risks.

Step 2: Obtain the probability density functions, such as $f_{yellow}(r)$ and so on, to express the risk ratings.

Step 3: Generate n risk values from the n distributions of the n risks according to their corresponding probability density functions. The sum of the n risk values is the overall risk of the portfolio in a simulation. Denote the sum by: r_i.

Step 4: Repeat step 3 N times and the mean of r_i is taken as the expectation of the distribution of the overall risk of the portfolio, namely, ${}^{r_1 + r_2 + \ldots r_N}\!/_N$.

7.4 Comparison of Different Methods to Aggregate Risk Matrices

In order to compare the three methods, we will apply these three methods to a specific example, and we will introduce the application in detail as the following.

7.4.1 An Illustrative Example

In this example, suppose there are four different risks, R_1, R_2, R_3, R_4, as shown in Table 7.2. Due to the complexity of the environment, the data needed to assess these risks is inadequate and decision makers want to know the relative severity of the overall risk. Therefore, the risk matrix (As shown in Fig. 7.9) is used to evaluate these risks. Notice that different risk matrices can be used to evaluate different risks, and for simplicity, the same risk matrix is used in this example.

First, policymakers need to estimate the consequences and probabilities of each risk. Table 7.2 provides a quantitative description of the consequences of each risk.

As is shown in the table above, R_1, R_2, R_3, R_4 are schedule delay, casualties, improper operation, and environmental pollution respectively. Risk loss is a discrete indicator in some of these four risks and thus needs to be converted into a continuous indicator. For example, the number of casualties is usually measured in the number of deaths or injuries, so it needs to be converted into the expenses spent. Any infinite intervals are not considered in risk aggregation because they have the highest priority. In addition, the probability interval corresponding to the three ratings of likelihood is [0, 1/3], (1/3, 2/3), and (2/3, 1), respectively. How the probability interval is divided,

Table 7.2 Definition of consequences of three risk matrices

Risk	Description of risk	Low	Medium	High	Extreme
R_1	Schedule delay: measures in days (unit: days)	[0, 10]	(10, 20]	(20, 30]	>30
R_2	Casualty: measured in expenses incurred for treatment of injured people (unit: $)	[0, 200 K]	(200 K, 400 K]	(400 K, 600 K]	>600 K
R_3	Misoperation: measured by the corresponding loss (unit: $)	[0, 50 K]	(50 K, 100 K]	(100 K, 150 K]	>150 K
R_4	Environmental pollution: measured by the fee curbing environmental pollution (unit: $)	[0, 100 K]	(100 K, 200 K]	(200 K, 300 K]	>300 K

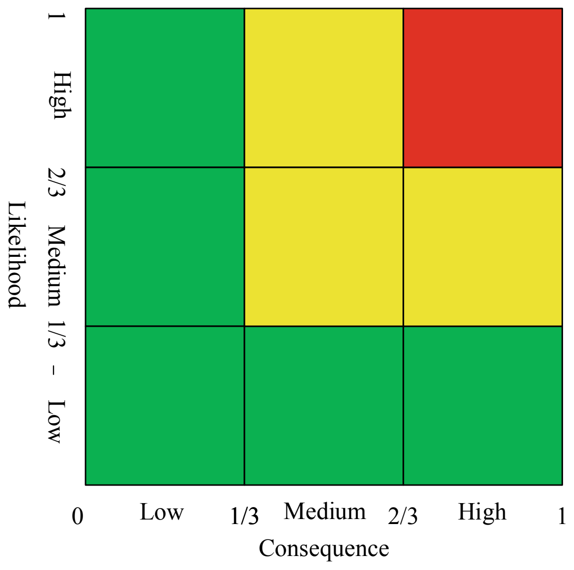


Fig. 7.9 A 3 × 3 risk matrix used to assess the 4 risks

namely the length and number of the probability interval depend on the subjective judgment of the decision maker.

After estimating the consequences and probabilities of the four risks, the decision maker can obtain the risk rating for each risk based on the risk matrix in Fig. 7.9. The risk rating of the four risks is shown in Table 7.3.

Based on the information above, three methods proposed are used to obtain the severity of the overall risk. Obviously, in order to obtain the severity of a risk portfolio, it is necessary to obtain the severity of all possible risk portfolios. Theoretically, there are 81 (= 3 × 3 × 3) possible risk combinations. But some risk combinations have the same risk value as the risk matrix used in this section is uniform. Therefore, we can compress the number of different portfolios. After this operation, totally, there are 15 different risk portfolios.

Based on the aforementioned methods, the results are exhibited in Table 7.4.

Firstly, we notice that some results are consistent with common sense, like the scenario of four 'red' risks has a higher priority than the one of four 'yellow' or 'green' risks. Furthermore, the above tables provide some results that cannot be estimated intuitively; for instance, the portfolio $(risk_{red}, risk_{green}, risk_{green}, risk_{green})$ has a

Table 7.3 Risk ratings of the 4 assessed risks

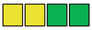
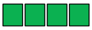
Risk	Consequence	Likelihood	Rating
R_1	Medium	Medium	Yellow
R_2	High	Low	Green
R_3	High	Low	Green
R_4	High	High	Red

Table 7.4 Ranking of the severity of different portfolios of the aggregated risks shown in Table 7.2 by different methods

Portfolios		Fuzzy set	Interval number	Cumulative distribution function	Ranking
	$risk_{red}, risk_{red}, risk_{red}, risk_{red}$	2.776/1[a]	17.056/1	2.776/1	1
	$risk_{red}, risk_{red}, risk_{red}, risk_{yellow}$	2.444/0.862	16.041/0.875	2.445/0.862	2
	$risk_{red}, risk_{red}, risk_{red}, risk_{green}$	2.175/0.750	15.453/0.803	2.182/0.752	3
	$risk_{red}, risk_{red} risk_{yellow}, risk_{yellow}$	2.108/0.722	14.940/0.740	2.115/0.724	4
	$risk_{red}, risk_{red}, risk_{yellow}, risk_{green}$	1.847/0.613	14.258/0.656	1.844/0.611	5
	$risk_{red}, risk_{yellow}, risk_{yellow}, risk_{yellow}$	1.779/0.585	13.810/0.601	1.777/0.584	6
	$risk_{red}, risk_{red}, risk_{green}, risk_{green}$	1.579/0.501	13.494/0.562	1.575/0.500	7
	$risk_{red}, risk_{yellow}, risk_{yellow}, risk_{green}$	1.510/0.473	13.063/0.510	1.511/0.472	8
	$risk_{yellow}, risk_{yellow}, risk_{yellow}, risk_{yellow}$	1.441/0.444	12.679/0.463	1.445/0.445	9
	$risk_{red}, risk_{yellow}, risk_{green}, risk_{green}$	1.247/0.363	12.226/0.407	1.247/0.363	10
	$risk_{yellow}, risk_{yellow}, risk_{yellow}, risk_{green}$	1.178/0.334	11.868/0.363	1.178/0.334	11
	$risk_{red}, risk_{green}, risk_{green}, risk_{green}$	0.981/0.253	11.283/0.291	0.978/0.251	12

(continued)

Table 7.4 (continued)

Portfolios		Fuzzy set	Interval number	Cumulative distribution function	Ranking
	$risk_{yellow}$, $risk_{yellow}$, $risk_{green}$, $risk_{green}$	0.910/0.223	10.959/0.251	0.910/0.223	13
	$risk_{yellow}$, $risk_{green}$, $risk_{green}$, $risk_{green}$	0.647/0.113	9.961/0.129	0.643/0.111	14
	$risk_{green}$, $risk_{green}$, $risk_{green}$, $risk_{green}$	0.375/0	8.913/0	0.376/0	15

[a] Two values of risk portfolios are given. One is the original value and the other is the normalized one

higher ranking than the portfolio ($risk_{yellow}$, $risk_{yellow}$, $risk_{green}$, $risk_{green}$) which is one of the main contributions of the methods.

Besides, we find that all three methods output the same ranking of the risk portfolios. However, the ranking is not the final result we are to obtain. In the next step, we will divide the 15 risk portfolios into several ratings for the purpose to obtain the severity of a risk portfolio.

Under the normalized risk results, all 15 risk portfolios were divided into three risk levels according to the threshold values of 1/3 and 2/3. Figures 7.10, 7.11, and 7.12 report the division of the risk portfolios calculated by three methods.

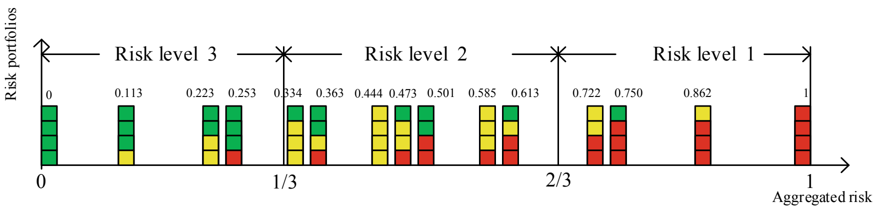


Fig. 7.10 Risk levels of different scenarios (calculated by fuzzy set)

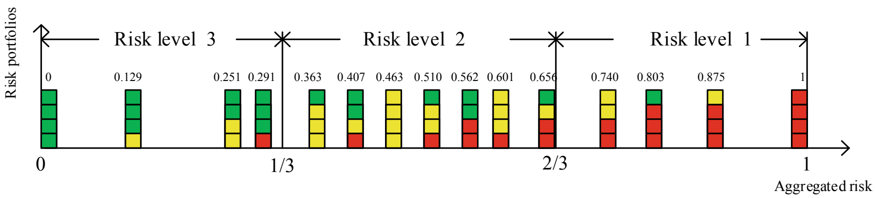


Fig. 7.11 Risk levels of different scenarios (calculated by interval number)

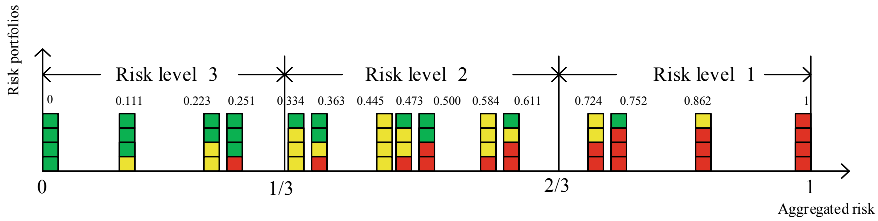


Fig. 7.12 Risk levels of different scenarios (calculated by cumulative distribution function)

It can be seen that, even if different methods are used to calculate, the classification of the risk portfolio is uniform under the same dividing criteria, that is, the lowest 4 risk portfolios are assigned the level of 3, the largest 4 are of risk level 1, and the remaining 2.

So far, by the aggregation methods, we reach our goal, namely, obtaining the severity (or risk rating) of the overall risk. Therefore, in this example, since the 4 risks are estimated as 'yellow', 'green', 'green', and 'red', respectively, the overall risk should have the risk rating of rating '2', which is at a lower–moderate level.

7.4.2 Robustness of Different Methods

In the previous section, the risk portfolios obtained by the three methods are divided uniformly. However, we find some differences between the risk value of aggregated risk assessed with different methods. The distributions of different risk scenarios obtained by a fuzzy set and cumulative distribution function are almost the same. Furthermore, compared with the result obtained by a fuzzy set and cumulative distribution function, the normalized values obtained by interval number are larger. Does it raise the question that whether the interval number method overestimates the risk of a portfolio? We wonder whether these methods are robust under other circumstances. This will be discussed in the following section.

Intuitively, in the risk matrix, the setting of consequence and likelihood is the major factor influencing the aggregating results. In order to explore the robustness of the three methods in different situations, another reasonable risk matrix different from the previous section is designed. As shown in Fig. 7.13, the only difference between the two risk matrices is that the intervals of inputs changed to [0, 1/4), [1/4, 3/4), and [3/4, 1], which will affect the membership of risk, the interval numbers of risk ratings, and the distribution of points, the risk matrix contains.

Table 7.5 reports the new ranking of the possible portfolios of the four aggregated risks.

Firstly, the results based on the methods of the fuzzy number and cumulative distribution function are the same, while the result of the method of interval number is different in the ranking of two scenarios $(risk_{red}, risk_{green}, risk_{green}, risk_{green})$

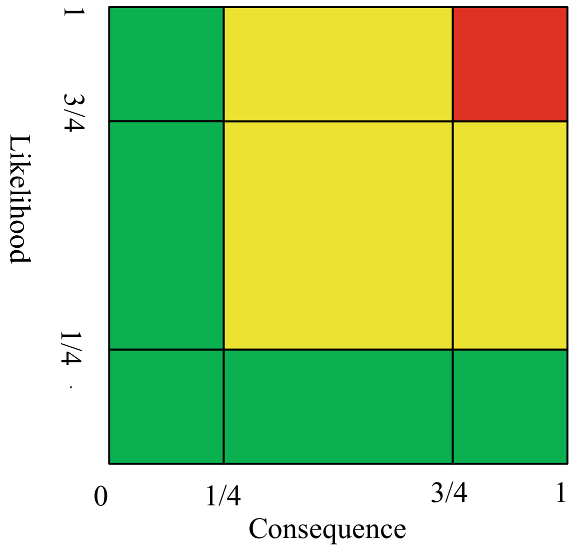


Fig. 7.13 The adjusted 3 × 3 risk matrix

and $(risk_{red}, risk_{green}, risk_{green}, risk_{green})$. The former two methods show that $(risk_{red}, risk_{green}, risk_{green}, risk_{green})$ ranks higher than $(risk_{red}, risk_{green}, risk_{green}, risk_{green})$, while the ranking judged by the method of interval numbers is opposite.

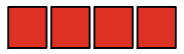
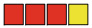
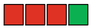
Secondly, we observe that the values of the results of the fuzzy number and cumulative distribution function are relatively close. In most cases, the values of the results of the fuzzy number and cumulative distribution function are the same until the percentile of the number, whereas the magnitude of v_i which is obtained by the method of interval number is far from the magnitude which the former two methods get. We regard this as an implication that the methods of fuzzy set and of the cumulative distribution function are more stable.

As shown above, the interval number-based method can't obtain well results in all cases. This is because interval numbers only take advantage of the maximum and minimum risk values in each cell. In Fig. 7.14, the design of two kinds of risk matrices is given. Although the design of these two risk matrices is different, the corresponding interval number of each rating is the same. This indicates that the interval number cannot describe the characteristics of each rating well, which also leads to inaccurate results in some cases.

7.4.3 Comparison of the Three Aggregation Methods

Based on the analyses above, three different risk matrix aggregation methods from the perspectives of comprehensibility, complexity, and explanation of risk matrix will be compared in this section. Table 7.6 gives the overall comparison results, which will be analyzed in detail next.

Table 7.5 Ranking of the severity of different portfolios of the aggregated risks shown in Table 7.2 by different methods based on the risk matrix in Fig. 7.13

Portfolios		Fuzzy set		Cumulative distribution function		Interval number	
		Value	Ranking	Value	Ranking	Value	Ranking
	$risk_{red}, risk_{red}, risk_{red}, risk_{red}$	3.275/1[a]	1	2.778/1	1	17.827/1	1
	$risk_{red}, risk_{red}, risk_{red}, risk_{yellow}$	2.894/0.867	2	2.448/0.863	2	16.612/0.875	2
	$risk_{red}, risk_{red}, risk_{red}, risk_{green}$	2.548/0.746	3	2.175/0.749	3	15.960/0.809	3
	$risk_{red}, risk_{red} risk_{yellow}, risk_{yellow}$	2.455/0.713	4	2.111/0.722	4	15.286/0.739	4
	$risk_{red}, risk_{red}, risk_{yellow}, risk_{green}$	2.163/0.611	5	1.843/0.611	5	14.501/0.659	5
	$risk_{red}, risk_{yellow}, risk_{yellow}, risk_{yellow}$	2.073/0.579	6	1.780/0.585	6	13.989/0.606	6
	$risk_{red}, risk_{red}, risk_{green}, risk_{green}$	1.818/0.490	7	1.577/0.500	7	13.501/0.556	7
	$risk_{red}, risk_{yellow}, risk_{yellow} risk_{green}$	1.743/0.464	8	1.511/0.473	8	13.085/0.514	8

(continued)

Table 7.5 (continued)

Portfolios	Fuzzy set		Cumulative distribution function		Interval number	
	Value	Ranking	Value	Ranking	Value	Ranking
$risk_{yellow}, risk_{yellow}, risk_{yellow}, risk_{yellow}$	1.553/0.398	9	1.444/0.445	9	12.773/0.482	9
$risk_{red}, risk_{yellow}, risk_{green}, risk_{green}$	1.415/0.349	10	1.242/0.361	10	11.950/0.397	10
$risk_{yellow}, risk_{yellow}, risk_{yellow}, risk_{green}$	1.329/0.319	11	1.177/0.333	11	11.792/0.381	11
$risk_{red}, risk_{green}, risk_{green}, risk_{green}$	1.083/0.233	12	0.978/0.250	12	10.639/0.263	12
$risk_{yellow}, risk_{yellow}, risk_{green}, risk_{green}$	0.993/0.202	13	0.910/0.222	13	10.640/0.263	13
$risk_{yellow}, risk_{green}, risk_{green}, risk_{green}$	0.699/0.099	14	0.645/0.112	14	9.372/0.133	14
$risk_{green}, risk_{green}, risk_{green}, risk_{green}$	0.417/0	15	0.377/0	15	8.074/0	15

[a] Two values of risk portfolios are given. One is the original value and the other is the normalized one

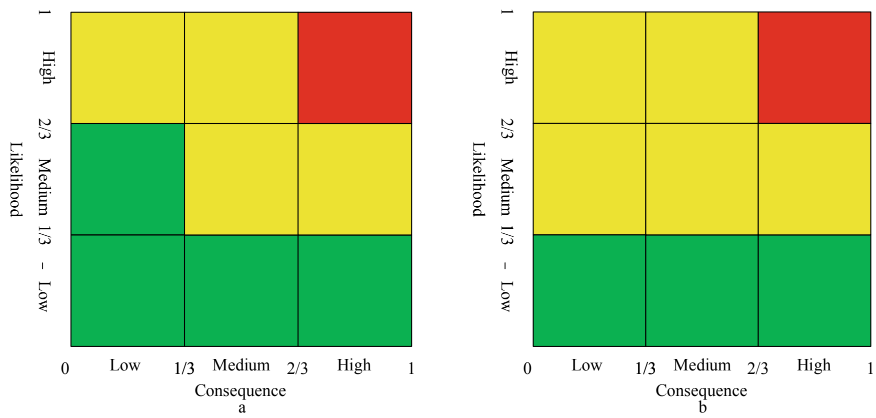


Fig. 7.14 Two different 3 × 3 risk matrices

Table 7.6 Different risk matrix integration methods of comparison

Methods	Comprehensibility	Complexity	Degree of explanation of risk matrix
Fuzzy set	Hard to comprehend	Most complex	High
Interval number	Easy to comprehend	Easiest	Low
Probability density function	Medium	Medium	Medium

Comprehensibility: Among the three methods, the interval number-based method is the easiest to understand because of its simplicity and feasibility. Decision makers only need to estimate the consequence and likelihood in their respective risk matrices to obtain the respective ratings and then get three interval numbers to add up. In addition, the comprehensibility of the probability density function-based method is between the fuzzy set method and interval number method. The fuzzy set-based method is the most difficult for users to understand since it has many unfrequent but significant concepts, such as membership function, defuzzification, and so on.

Complexity: Similar to the analysis of comprehensibility, the fuzzy set-based method is the most complicated, as the process of obtaining fuzzy membership function and crisp value is relatively complicated. The method based on the probability density function is in the middle of complexity, and the method of interval number is the simplest.

Degree of explanation of risk matrix: Although the interval number-based method is simple, it ignores a lot of information about the risk matrix. In other words, it conveys less information than the fuzzy set-based method as it only focuses on the maximum and minimum risk values in each cell and ignores the specific characteristics of different risk matrices, such as the relative position and the relative size of the cells. However, fuzzy sets and probability density function-based methods can contain this information.

As the above discussion shows, the interval number method is the least robust as it is the easiest method. Compared with the fuzzy set-based method, the probability density function-based method has a lower degree of explanation of risk matrix but can output similar results, thus it could be taken as a substitution as the fuzzy set method, thanks to its simplicity.

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Chapter 8
Three-Dimensional Risk Matrix: Theoretical Basis and Construction

8.1 Background of the Three-Dimensional Risk Matrix

Risk assessment is a comprehensive process for assessing the possible effect that an event or outcome will bring. Usually, the possible effect is quantified with the aid of a risk measure, which outputs the risk measurement of the risk, given several risk factors. For example, in the field of engineering, the most common measure of risk is $Risk = C \times P$, where C is the consequence of the events if they occur, and P is the corresponding probability (Aven 2012; Willis 2007; Nieto-Morote and Ruz-Vila 2011). The adopted risk measure relies heavily on the definition of risk. In the measure mentioned above, the risk is defined as the expected consequence of the event, which is one of the prevailing definitions of risk. Following are some common risk definitions (for historical and recent definitions of risk, see Aven 2012):

(1) Risk = Expected value. When we focus on the damage of an event, risk can be measured by the product of the possible loss and the corresponding probability (Ale et al. 2015; Willis 2007). Sometimes, researchers adopt the (dis)utility of the consequence as the effect on a decision maker, instead of the consequence; in this case, the risk is the product of the (dis)utility of the consequence and the probability (Ruan et al. 2015).

(2) Risk = Probability of an adverse event. This definition is usually employed when the consequence of the event is fixed, or the consequence is unacceptable no matter what the consequence is, and thus, only the probability is considered (Campbell 2005). An event tree is one of the typical tools in which this measure is embedded (Borgonovo and Smith 2011; Ferdous et al. 2011).

(3) Risk = Uncertainty. Understanding uncertainty is twofold. One is objective (aleatory) uncertainty, which is an inherent characteristic of an event but embodied in the event in the external world (Aven 2012). For example, in a coin toss, we know the obverse or the reverse of the coin will appear, but we are uncertain which one will appear. The other is subjective (epistemic) uncertainty (Winkler 1996). One does not know what will happen and what the consequence will be, depending on the risk assessor's knowledge.

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In brief, the risk is essentially uncertainty-oriented. Traditionally, the risk is related to the event (E), the consequence (C), and the probability (P), namely, a triplet (E, C, P). However, in this case, the probability is only a special measure of uncertainty, and thus, is not the foundation (Aven 2012). In practice, we estimate the probability in various ways. For example, we may perform sample testing many times to obtain the defect rate of a product; we may directly give the estimation of probability based on our experience. No matter which method we use, probability itself cannot reflect how we deal with the uncertainty based on our knowledge and what the degree of the uncertainty is. For example, when assessing the uncertainty of an event, a decision maker gives a probability of 0.8 while he or she thinks 0.9 is also an alternative, but 0.8 seems to be more accurate to him or her. Ball and Watt found that in different situations, the same risk assessor may give different estimates of the consequence and the probability (Ball and Watt 2013). It is revealed that uncertainty still exists in the subjective judgment of probability itself. Therefore, recently researchers suggested adopting uncertainty rather than probability in risk perspectives (Ho et al. 2010; Durbach and Stewart 2012). In this sense, risk should be defined by a more accurate triplet $(E,\ C,\ U)$, where U denotes the uncertainty of the consequence.

Uncertainty has been studied directly on the perception of risk. In 1978, Fischhoff et al. studied the factors influencing the perception of risk (Fischhoff et al. 1978). They explored nine factors: the voluntariness of risk, immediacy of effect, knowledge about the risk by the person exposed to the risk source, knowledge about the risk in science, control over the risk, newness, chronic/catastrophic, common/dread, and severity of consequences. Given a particular risk scenario, the perceived risk is different for the different participants, which reveals that in a broader sense, the risk of the same event may vary for diverse stakeholders in the context of a risk assessment based on their subjectivity, or more accurately, knowledge of the risk (e.g., knowledge of the risk in science). For measuring risk, knowledge supports the decision for determining the C and U of the risk. Therefore, let K be the knowledge of the risk of an event, and the risk triplet is further extended as $(E,\ C,\ U(K))$. The accuracy of this expression has been proved by several researchers. For example, Ball and Watt found that even for the same stakeholder, different ratings of the consequence and probability are assigned, and factors (such as information, psychosocial influences, mental processing, knowledge and beliefs, and so on) are analyzed to explain the difference (Ball and Watt 2013). Aven et al. provided the same suggestion for the risk measure (Aven and Eidesen 2007).

When additional risk factors are involved, the equation for the risk measure is more complicated and more difficult to express explicitly. For example, Risk $=$ Consequence $\times$ Probability is utilized when only the consequence and the probability of risk are considered. The problem is how to measure risk given additional factors. In other words, how to obtain the output of the tetrad $(E,\ C,\ U(K))$? There may be multiple dimensions of knowledge, and intuitively, no explicit function outputs the strength of knowledge incorporating all the dimensions.

8.2 Three-Dimensional Risk Measure Considering the Strength of Knowledge

The simplest form of risk measurement is the probability of an event, i.e., $P(E)$. A more complex measure that incorporates the consequences, complementing the results as a probabilistic model, the risk metric becomes a function of the overall estimate of the outcome and probability, i.e., risk = outcome and probability. In order to make this function explicit, scholars and practitioners usually measure the risk by taking the form of the product of the consequence and probability, the expected consequence. Usually, few models incorporate factors other than consequences and probabilities into risk measures because it is difficult to form an explicit expression to measure risk. Therefore, the definition of risk in the form of $(E, C, U(K))$, the first thing to solve is to form an explicit expression. In the field of risk management, the probability is the most commonly used to express uncertainty. To be precise, this uncertainty refers to objective (accidental) uncertainty (Aven 2017a). As mentioned earlier, using probability merely ignores how the evaluator gives an estimate of the probability, that is, the subjective (cognitive) uncertainty of the evaluator does not be reflected. The knowledge-based conditional probability $P(K)$ is used instead of $U(K)$ in the triples $(E, C, U(K))$. In so doing, the risk measure of adding subjective knowledge is still related to the concept of the most common probability, and also gives the risk measure an explicit expression.

Definition 8.1 The risk measure with the dimension of knowledge. Given a risk, the magnitude of the risk assessed considering the assessor's knowledge is defined as:

$$Risk = [C - (C - C^l) \times (1 - M(K_C)), C + (C^u - C) \times (1 - M(K_C))]\times \\ [P - (P - P^l) \times (1 - M(K_P)), P + (P^u - P) \times (1 - M(K_P))] \quad (8.1)$$

where $M(K)$ is the measure of knowledge, C^l and C^u are the lower bound and upper bound of C respectively, and P^l and P^u are the lower bound and upper bound of P respectively. In practice, given a risk assessment scenario, the upper and lower bounds of the consequences are usually determined. For example, in an adverse event, the most serious consequence (upper bound) is the loss of all owned resources. The best consequence (lower bound) is that no resources are lost. For probability, the upper and lower bounds are 1 and 0, respectively.

To begin with, we explain the reasonability of the risk measure in Eq. (8.1).

(I) In addition to probability, knowledge related to the assessment of consequences have also been taken into account. This is because, in practice, the consequences are usually diversified rather than singular (Cox 2008). Tools used to analyze the consequences and probability of risk are not the same. For example, when analyzing the consequences, tools such as flow charts, cost analysis, and so on are used; when analyzing probability, Bayesian networks, event trees, and so on are often be employed. Therefore, we embed the knowledge dimensions of consequences and probabilities.

(II) $M(K)$ can be understood as the degree of belief that describes the risk assessor in his assessment of C and P. In general, the closer $M(K)$ is to 1, means that the evaluator has more knowledge to support his assessment of C and P; and the closer $M(K)$ is to 0, the more knowledge the evaluator can rely on when making an assessment. less. Stronger knowledge means that the subjective uncertainty of the evaluation of C or P is smaller, thus corresponding to a narrower interval of C or P. In particular, when $M(K) = 1$, it means that the evaluator fully understands C and P, or that the estimates of C and P reach a completely objective level. For example, if the evaluator has complete knowledge, the estimated probability (probability of subjective judgment) is the same as the frequency-based probability (objective probability). When $M(K) = 0$, this means that the evaluator knows very little about C and P, so the evaluator can only give the upper and lower bounds of C and P, that is, $[C^l, C^u]$ and $[P^l, P^u]$, this setting is in line with our common sense. The range of $M(K)$ should be from 0 to 1.

(III) When the consequence is a fixed value, for example, the consequence is a failure, and the definition of the risk at this time depends entirely on the probability. Therefore, at this time, the C and $M(K_C)$ both in the definition (8.1) are set to be 1, which is consistent with the definition of the second risk.

(IV) The expression in Eq. (8.1) is itself a representation of uncertainty. The uncertainty levels of C and P are:

$$d(C) = (C^u - C^l) \times (1 - M(K_C), \tag{8.2}$$

$$d(P) = (P^u - P^l) \times (1 - M(K_P). \tag{8.3}$$

Intuitively, stronger knowledge is in line with a lower degree of uncertainty.

(V) Although we embed the knowledge dimension for both consequences $M(K_C)$ and probabilities $M(K_P)$, the two are different. First, the knowledge that supports the evaluator to make judgments about the consequences and probabilities is different. Another more important difference is that, in general, there is no correlation between the consequences of the assessment and its knowledge strength $M(K_C)$, and there is a clear logical relationship between the probability and the knowledge of the probability $M(K_P)$ of assessment. In practice, even if an evaluator has no knowledge of the consequences, an arbitrary value can be given to assess the consequences. However, it is different for probability. Suppose an event has two possible consequences. If an evaluator knows little about the probability of these two consequences, then it is very reasonable to think that the probability of each result is 0.5; given that the probability of one of these consequences is 0.9, it is clear that the evaluator has additional knowledge to support it in making such an assessment. As

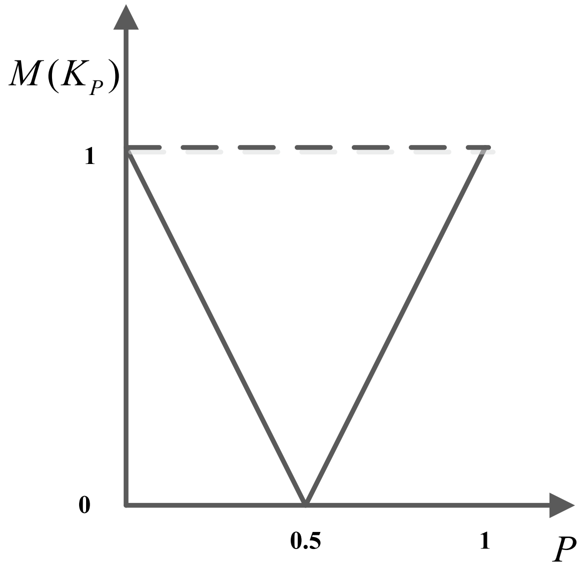

Fig. 8.1 An explanation of the relation between P and measure of the knowledge of P

shown in Fig. 8.1, although 0.5 may not correspond to the lowest probability in all cases, Fig. 8.1 shows a possible relationship between probability P and its corresponding knowledge strength $M(P)$. This means that P and K_P are not independent.

When $M(K) \neq 1$, according to the definition (8.1), the risk cannot be measured by a certain value because it is the product of the two-interval numbers. As $M(K)$ gets closer to 0, the interval between C and P becomes larger, because the lower the knowledge strength, the more choices C and P the evaluator faces. Therefore, the variance of the values of C and P appearing in the respective intervals can be used to reflect the magnitude of the uncertainty. Given an interval number [a, b], the decision maker does not have any distribution information about its corresponding variable X in the interval, so it can be assumed that the variable X is evenly distributed in [a, b]. Therefore, the variance of X is $\frac{(b-a)^2}{12}$. According to this, the variances of C and P are respectively $\frac{(d(C))^2}{12}$ and $\frac{(d(P))^2}{12}$. The final value of the risk measure is:

$$
\begin{aligned}
Risk &= C \times (1 + \frac{VarC}{Var_{\max}}) \times P \times (1 + \frac{VarP}{Var_{\max}}) \\
&= C \times (1 + \frac{((C^u - C^l) \times (1 - M(K_C))^2}{(C^u - C^l)^2}) \times P \\
&\times (1 + \frac{((P^u - P^l) \times (1 - M(K_P))^2}{(P^u - P^l)^2}) \\
&= C \times (1 + (1 - M(K_C))^2) \times P \times (1 + (1 - M(K_P))^2).
\end{aligned}
\tag{8.4}
$$

In Eq. (8.4), $\frac{VarC}{Var_{\max}}$ and $\frac{VarP}{Var_{\max}}$ are used to normalize the uncertainty caused by different knowledge strengths, so that they vary from 0 to 1. It can be seen from the derivation that the normalized form makes its uncertainty not affected by the boundaries of C and P, but only related to the strength of knowledge, and this is a reasonable setting.

According to Eq. (8.4), in terms of our common sense, the risk of higher consequences and possibility will have greater risks. In addition, repairing C and P may reveal that if the risk assessor has a lower level of belief in the estimates of C and P, in other words, his/her knowledge is less, the risk assessed will have a greater magnitude. This is the same as the analysis of (Aven 2017b), which proves the proposed measures are reasonable.

8.3 Impact Factors of the Strength of Knowledge

The risk measure considering the knowledge dimension is given in the previous section, where the embedding of knowledge is achieved by adding a knowledge strength variable. This section describes which factors should be considered when making a measure of knowledge strength.

Intuitively, in different risk assessment environments, knowledge strength can be given from a global or local perspective. For example, when exploring factors that influence risk perception, Fischhoff et al. overall perspective summarize two categories of individual perceptions of risk and knowledge of existing scientific knowledge (Fischhoff et al. 1978). When studying the risk matrix for risk assessment, Ball and Watt explored the role of matrix technology from a local perspective, such as how to include risk exposure and scaling of inputs (Ball and Watt 2013).

Based on Aven's work, we argue that although specific local knowledge is needed in a specific context, a general framework that influences the strength of knowledge from a global perspective is also necessary (Aven 2017a). According to Aven, knowledge strength can be considered "weak" when the following conditions are met:

(w1) The assumption is too simplistic.

(w2) data/information is lacking or unreliable.

(w3) Expert opinions are not uniform.

(w4) The phenomenon involved is misunderstood, lacking a predictive model or the model results are unreliable.

(w5) The assessment team lacks expertise in the risk (this knowledge may be known to others, but the assessment team does not know).

The setting here is based on Aven's condition that the strength of knowledge is "weak" (Aven 2017a), and also take the views of Fischhoff et al. (1978), Ball and Watt (2013) as a reference. We summarized the factors that should be considered in assessing the strength of knowledge in 5: (k1) public knowledge of risk, (k2) personal knowledge of risk, (k3) model, (k4) data, and (k5) expert opinion. Their detailed explanation is as follows:

(k1) Regardless of whether the risk assessor has specific expertise on the risk, public knowledge (e.g., reporting of related events, broadness of risk, and so on) will affect the accuracy of the assessment of the consequences and probabilities. At the same time, the accuracy of public knowledge will also have an impact. For example, the instructions for a product may not be reliable, so the risk of using such a product is miscalculated. Therefore, when considering public knowledge of risk, its circumstance and reliability should be considered.

(k2) k2 refers to the knowledge acquired by individuals in the risk assessment environment. It includes lessons learned from similar risks, individual professional skills, and so on. Therefore, when you pay attention to your personal knowledge, you need to consider its familiarity and professionalism.

(k3) The model is a key tool for supporting decision making. Regardless of whether the risk assessor knows the relevant model in advance, it is necessary to find some models to assist in the assessment of the consequences and probabilities. For model knowledge, it is important to consider whether there are enough models to predict the consequences and possibilities of the risks, the reliability of the model itself, and the consistency of the results.

(k4) Data is required for the model. If the original data is not available, the assumed data is necessary to run the model. Obviously, knowledge about data should consider the availability, authority, and adequacy of the data.

(k5) Expert opinions are sometimes more reliable than model predictions, which reveals the importance of expert opinion. Risk assessment, which evaluates its consequences and probabilities, can depend on subjective judgments, as well as other experts' opinions. The authority and consistency of expert opinions need to be considered.

Based on the above analysis, in addition to the five categories of global knowledge factors, the corresponding sub-factors are given. See Table 8.1 for details.

The factors influencing the strength of knowledge are summarized according to some opinions of scholars. It is a general knowledge factor framework. In practice, other factors can be included in the specific risk assessment scenario.

8.4 Measure of the Strength of Knowledge Based on Fuzzy MCDM

In the previous section, we explored the impact factors that influence the strength of knowledge. This section will explore how to gain knowledge based on these factors.

As analyzed above, knowledge strength is affected by multiple knowledge dimensions. Each factor affecting the strength of knowledge can be regarded as a decision criterion. Each risk assessed can be regarded as an action in multi-attribute decision making. Therefore, this section will use the relevant tools of multi-attribute decision making to obtain the knowledge strength. Multi-attribute decision making methods are often used in situations where multiple criteria cannot be used to connect these

Table 8.1 Global factors affecting knowledge strength and its sub-factors

Global factors	Sub-factors
k1. Public knowledge of risk	k11. Spread of public knowledge k12. Reliability of public knowledge
k2. Personal knowledge of risk	k21. Familiarity with risk k22. Professionalism of risk
k3. Model	k31. Adequacy of the model k32. Reliability of the model k33. Consistency of model results
k4. Data	k41. Availability of data k42. Authoritativeness of data k43. Adequacy of data
k5. Experts' opinions	k51. Authoritativeness of experts k52. Consistency of experts' opinions

criteria with an explicit function to obtain a final score. This is also in line with the factor characteristics of knowledge strength: it is not possible to connect all the factors with an explicit function to obtain the final knowledge strength.

In a multi-attribute decision-making problem, the action is sorted according to the weighted score of all indicators, or by using TOPSIS (Triantaphyllou 2000; Ho et al. 2010). Obviously, when gaining the strength of knowledge, the purpose is to obtain a measure. Therefore, only the method of weighted score can be adopted. The method of comparing the advantages and disadvantages of TOPSIS is not desirable. When using multi-attribute decision making methods to obtain knowledge strength, it is necessary to obtain the score of each knowledge factor. Therefore, it is first necessary to normalize the factors identified.

Normalized settings in the measure of knowledge strength: For all factors affecting the strength of knowledge, the concept of "degree" needs to be used to reflect their size. For example, for "public knowledge of risk", when assessing its size, you should ask "how much is the public knowledge of risk". Without loss of generality, we assume that all factors have a positive impact on knowledge strength (negative effects can be converted to positive factors by adding negative words), and the degree of influence ranges from 0 to 1.

In a multi-attribute decision problem, different criteria are usually non-independent (or there is an interaction between the criteria). Therefore, the results obtained by simply weighting the scores of all criteria are biased (Angilella et al. 2004). In general, two criteria are given, such as A and B, if they are related, then where f is the measure of the criterion. When measuring knowledge factors, it is difficult to ensure that all factors are independent of each other. For example, public knowledge of risk (k1) clearly affects personal knowledge of risk (k2). Therefore, when measuring the strength of knowledge, it is obviously not possible to use the traditional method of weighted scores, and the correlation between the criteria should be taken into account.

The remainder will describe how to use the method of multi-attribute decision making to gain knowledge strength.

8.4.1 Multi-Attribute Decision-Making Method Based on Fuzzy Measure and Choquet Integral

Figure 8.2 shows the logical framework of the method presented in this section. The main steps to obtain the strength of knowledge include: (1) obtaining the overall weight of each knowledge factor, (2) obtaining the interaction between any two knowledge factors, (3) obtaining the fuzzy measure of all indicators by using the entropy value optimization method, (4) using Choquet integral to get the strength of knowledge. The specific steps will be stated below.

(1) Overall weight of individual knowledge factors.

Given the factors that measure the strength of knowledge, the next step is to determine the weight of these factors. If these factors are independent, that is, they have no first inertia with each other, then the scores of each factor multiplied by its weight can be found first, and then these scores are added together to obtain a score of the overall knowledge strength. However, it is difficult to find mutually independent knowledge factors. For example, in the five global factors given in Sect. 8.3, public knowledge affects individual knowledge, the accuracy of the data affects the accuracy of the model, and the experts' opinions may conflict with the evaluator's own judgment. Therefore, the biggest challenge in determining the weight of each factor is the correlation between the processing factors.

To express the correlation between factors, scholars recommend the use of non-additive measures or fuzzy measures (Angilella et al. 2004; Grabisch et al. 2008). A fuzzy measure is defined as follows:

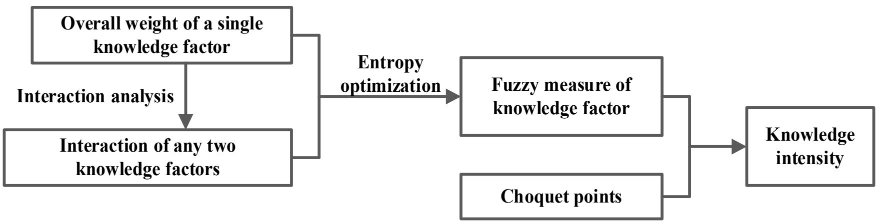

Fig. 8.2 Framework for obtaining knowledge strength using fuzzy multi-attribute decision-making

Definition 8.2 Fuzzy measure. Given a subset X of the set S and S, the function $\mu : X \to [0, 1]$ is called the fuzzy measure for X if the properties of the following are satisfied (Grabisch 1997):

(1) $\mu(\varnothing) = 0$ and $\mu(S) = 1$, and
(2) $\forall M \subseteq N \subseteq S,\ \mu(M) \leq \mu(N)$.

Definition 8.3 Definition of Möbius transform of fuzzy measure. Given a non-empty set S and any subset X of S, for any function $\mu : X \to \mathbf{R}$, its Mobius transform is defined as (Mikenina and Zimmermann 1999):

$$a(T) = \sum_{K \subset T} (-1)^{|T-K|}\mu(K),\ \forall T \subset S. \tag{8.5}$$

Möbius transform is reversible. When a is given, the arbitrary fuzzy measure has the following relationship with its Mobius transform (Mikenina and Zimmermann 1999):

$$\mu(T) = \sum_{M \subset T} a(M),\ \forall T \subset S. \tag{8.6}$$

Theorem 8.1 If $a(T)$ is a fuzzy measure of the Mobius transformation, if and only if the following conditions are true (Grabisch 1997):

(1) $a(\varnothing) = 0,\ \sum_{T \subset S} a(T) = 1;$
(2) $\sum_{i \in B \subset T} a(B) \geq 0,\ \forall T \subset S,\ \forall i \in T.$

According to the definition of fuzzy measures, the value of $2^n - 2$ arguments needs to be determined when using fuzzy measures (Grabisch 1996). Therefore, when n is large, determining the fuzzy measure requires a huge amount of calculation. In practice, in order to save computational effort, a 2-additive measure is typically used, that is, more than two sets are not considered. The 2-additive measure is defined as follows:

Definition 8.4 2-additive fuzzy measure. A fuzzy measure μ is referred to as 2-addable fuzzy measures. If for all T satisfies $|T| > 2$, $a(T) = 0$, and there is at least one subset T of S, it contains two elements and satisfies $a(T) \neq 0$ (Mikenina and Zimmermann 1999).

Therefore, according to Eq. (8.6), for $K \subseteq S$, $|K| > 2$, the 2-addittive fuzzy measure is defined as:

$$\mu(K) = \sum_{i \in K} a_i + \sum_{\{i,j\} \subset K} a_{ij} \tag{8.7}$$

Corollary 8.1 If $a(T)$ is a 2-additive fuzzy measure $\mu(T)$ of measurable Möbius transformation, the following conditions must be established (Grabisch 1997):

(1) $a(\varnothing) = 0$;
(2) $a_i \geq 0, \forall i \in S$;
(3) $\sum_{i \in S} a_i + \sum_{\{i,j\} \subset S} a_{ij} = 1$;
(4) $a_i + \sum_{j \in T} a_{ij} \geq 0, \forall i \in S, \forall T \subset S \backslash \{i\}$.

Under the fuzzy measure, the interaction of any two factors, such as A and B, has the following relationship: (1) There is no interaction between the two, that is, the two are independent, at this time $\mu(A + B) = \mu(A) + \mu(B)$, (2) There is a positive interaction between the two, that is, there is positive synergy or complementarity between the two, at this time $\mu(A + B) > \mu(A) + \mu(B)$, (3) there is a negative interaction between the two, that is, both There is a negative synergy or redundancy between them, at this time $\mu(A + B) < \mu(A) + \mu(B)$. For example, k3 and k4 are clearly complementary factors, so it can be inferred $\mu(k3 + k4) > \mu(k3) + \mu(k4)$.

As mentioned above, in a fuzzy measure, each contained subset f_i has an effect on its weight. Therefore, you cannot just use $\mu(f_i)$ to express the weight of the factor i. Shapley proposes an indicator of importance to reflect the overall contribution of a member of the system, known as the Shaply value (Shapley 1952). It is defined as follows (Mikenina and Zimmermann 1999; Grabisch 1997).

Definition 8.5 A fuzzy measure representing. A set S of n elements, the Shapley value element $x_i \in S$ is defined by:

$$v_i = \sum_{k=0}^{n-1} \frac{(n-k-1!K!)}{n!} \sum_{T \subset S \backslash x_i,\, T=k} (\mu_{iT} - \mu_T) \tag{8.8}$$

The interaction of any two elements is defined as:

Definition 8.6 In fuzzy measures μ, the interaction of any two elements is defined as (Grabisch 1997):

$$I_{ij} = \sum_{k=0}^{n-2} \frac{(n-k-2!k!)}{(n-1)!} \sum_{T \subset S \backslash \{x_i, x_j\},\, T=k} (\mu_{ijT} - \mu_{iT} - \mu_{jT} + \mu_T) \tag{8.9}$$

It has been proven that the Shapley values of all elements satisfy the following relationship (Shapley 1952):

$$\sum_{i=1}^{n} v_i = 1 \tag{8.10}$$

In a multi-attribute decision problem, the weights of all indicators must satisfy $\sum_{i=1}^{n} w_i = 1$, which w_i represents the weight of criterion i. v_i analogies w_i in Eq. (8.10), and the difference is that the former considers the interaction between the criteria, while the latter is based on the assumption that all criteria are independent. When interactions are included between the criteria, it is obvious that w_i cannot be weights, but only v_i represents the importance of each criterion. The results of directly multiplying them by the index scores have no theoretical explanation for the rationality of the results. In fact, the analysis below will show that take $\sum_{i=1}^{n} f_i v_i$ as the final score is wrong, that is, when there is an interaction, even if the overall weight of the interaction is considered, the weighted average cannot be directly made.

For the assignment of v_i, it can be obtained through the subjective judgment of the decision maker, and only needs to satisfy Eq. (8.10). In order to reflect the uncertainty of the decision makers in the assignment of v_i, it can be allowed to fluctuate within a certain range, that is, v_i can correspond to the interval of $[v_i^d, v_i^u]$.

Definition 8.7 If each factor corresponds to the interval of $[v_i^d, v_i^u]$ in the knowledge strength measure, then at least exist one set of $V = (v_1, v_2, \cdots, v_n)$ is feasible for the assignment in the interval of $[v_i^d, v_i^u]$ when the following conditions are met:

$$\begin{cases} v_t^d \le v_t \le v_t^u, \ \forall t \in \{1, 2, \cdots, n\}, \\ \sum_{i=1}^{n} v_i = 1. \end{cases} \tag{8.11}$$

(2) Analysis of interaction between any two knowledge factors

Grabisch demonstrated the following relationship between interaction and its corresponding Möbius transformation (Grabisch 1997):

$$I_p(T) = \sum_{K \subset S \backslash T} \frac{1}{|K| + 1} a_p(T \cup K), \forall T \subset S, \ \forall p \subset S. \tag{8.12}$$

Therefore, for the 2-additive measure, the interaction of any order can be written as:

$$\begin{cases} I(\varnothing) = a(\varnothing) + \frac{1}{2} \sum_{i \in S} a(i) + \frac{1}{3} \sum_{\{i,j\} \subset S} a(ij) \\ I(i) = a(i) + \frac{1}{2} \sum_{j \in S \backslash i} a(ij) \\ I(ij) = a(ij) \\ I(A) = 0, \ \forall |A| > 2. \end{cases} \tag{8.13}$$

By transforming the above equation group, equations can be obtained:

$$\begin{cases} a(\varnothing) = I(\varnothing) - \frac{1}{2}\sum_{i \in S} I(i) + \frac{1}{6}\sum_{\{i,j\} \subset S} I(ij) \\ a(i) = I(i) - \frac{1}{2}\sum_{j \in S \setminus i} I(ij) \\ a(ij) = I(ij) \\ a(A) = 0, \ \forall |A| > 2. \end{cases} \tag{8.14}$$

According to Corollary 8.1, the following conditions related to $I(ij)$ must be met:

$$\begin{cases} (a) \quad \sum_{i \in S} a_i + \sum_{\{i,j\} \subset S} a_{ij} = 1 \\ (b) \quad a_i + \sum_{j \in T} a_{ij} \geq 0, \ \forall i \in S, \ \forall T \subset S \setminus \{i\}. \end{cases} \tag{8.15}$$

For condition (a), we have:

$$\begin{aligned} \sum_{i \in S} a_i + \sum_{\{i,j\} \subset S} a_{ij} &= \sum_{i=1}^{n} a(i) + \sum_{\{i,j\} \subset S} I_{ij} \\ &= \sum_{i=1}^{n} (I(i) - \frac{1}{2}\sum_{j \in S \setminus i} I(ij)) + \sum_{\{i,j\} \subset S} I_{ij} \\ &= \sum_{i=1}^{n} I(i) - \frac{1}{2}\sum_{i=1}^{n}\sum_{j \in S \setminus i} I(ij) + \sum_{\{i,j\} \subset S} I_{ij} \\ &= \sum_{i=1}^{n} I(i) = 1. \end{aligned} \tag{8.16}$$

Therefore, condition (a) is always true. For condition (b):

$$\begin{aligned} a_i + \sum_{j \in T} a_{ij} &= I(i) - \frac{1}{2}\sum_{j \in S \setminus i} I(ij) + \sum_{j \in T} I_{ij} \\ &= I(i) - \frac{1}{2}\sum_{j \in S \setminus T} I(ij) + \frac{1}{2}\sum_{j \in T} I_{ij} \\ &\geq I(i) - \frac{1}{2}\sum_{j \in S \setminus i} |I(ij)| \end{aligned} \tag{8.17}$$

Proposition 8.1 Condition (b) will be established if the following conditions are true:

$$\forall T \subset S,\ \forall i \in T,\ \forall j \in T\backslash i,\ I(i) \geq \frac{1}{2}(n-1)|I(ij)|. \tag{8.18}$$

Proof According to the condition (8.18), we have:

$$\forall j \in S\backslash i,\ I(i) \geq \frac{1}{2}(n-1)|I(ij)| \Rightarrow (n-1)I(i) \geq \frac{1}{2}(n-1)\sum_{j\in S\backslash i}|I(ij)|$$
$$\Rightarrow I(i) \geq \frac{1}{2}\sum_{j\in S\backslash i}|I(ij)| \tag{8.19}$$

And thus:

$$a_i + \sum_{j\in T} a_{ij} \geq I(i) - \frac{1}{2}\sum_{j\in S\backslash i}|I(ij)| \geq 0 \tag{8.20}$$

According to Eq. (8.18), we can get the boundary of any two factor interactions:

$$\forall T \subset S,\ \forall i \in T,\ \forall j \in T\backslash i,\ |I(ij)| \leq \min\{\ 2I(i)/(n-1), 2I(j)/(n-1)\} \tag{8.21}$$

As mentioned above, interaction can be understood as a complementary or redundant relationship between factors. In practice, it is difficult for decision makers to give accurate estimates of interactions. Therefore, based on the boundary of interaction (8.21), the estimation of interaction is obtained in the following way.

Step 1. According to the boundary of the interaction, the interaction is divided into N parts, $[-B, -B+\frac{2B}{N})$, $[-B+\frac{2B}{N}, -B+\frac{4B}{N})$, …, $[B-\frac{2B}{N}, B)$, among them $B = \min\{\ 2I(i)/(n-1), 2I(j)/(n-1)\}$.

Step 2. Give each interval a textual explanation of the degree of interaction;

Step 3. The evaluator selects an extent of interaction according to the description of the text in step S2, corresponding to one of the intervals, $[-B, -B+\frac{2B}{N})$, $[-B+\frac{2B}{N}, -B+\frac{4B}{N}), \cdots, [B-\frac{2B}{N}, B)$, and is recorded as $B_{ij} = [B_{ij}^d, B_{ij}^u)$.

An example of an interaction estimate is shown below:

Example 8.1 Divide the interval $[-B, B]$ into 7 subintervals, namely, $[-B, -\frac{5}{7}B)$,$[-\frac{5}{7}B, -\frac{3}{7}B)$, $[-\frac{3}{7}B, -\frac{1}{7}B)$, $[-\frac{1}{7}B, \frac{1}{7}B)$, $[\frac{1}{7}B, \frac{3}{7}B)$, $[\frac{3}{7}B, \frac{5}{7}B)$, and $[\frac{5}{7}B, B)$. As shown in Fig. 8.3, the interaction of these seven intervals is explained as follows: (1) high redundancy, (2) medium redundancy, (3) lower redundancy, and (4) almost no interaction between the two factors. The role, (5) lower complementarity, (6) medium complementarity, and (7) highly complementary. If an evaluator believes that there is a medium complementarity between the two factors, then the interaction corresponds to the interval $[\frac{3}{7}B, \frac{5}{7}B)$.

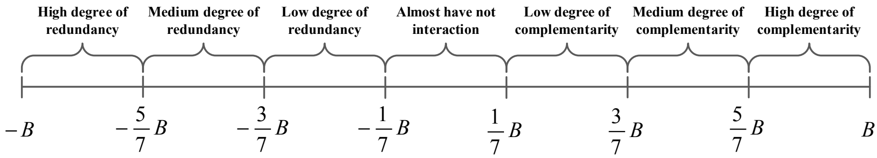

Fig. 8.3 Explanation of interaction in different intervals

(3) Solving Fuzzy Measures by Entropy Optimization

In the previous two subsections, we explore how to express the overall weight of a single knowledge factor and the interaction between any two factors. The purpose is to solve the fuzzy measure by transforming the abstracted fuzzy measure into the overall weight of the image and the interaction, and then by the values of the overall weight and interaction. Therefore, this section will first introduce how to use the entropy optimization method to obtain the overall weight and the specific value of the interaction, rather than the interval between the two.

The Shannon entropy of a variable x can be defined as:

$$H(x) = \sum_{i=1}^{n} h(x_i) \tag{8.22}$$

where $h(x) = -x_i \ln x_i$ is called the Shannon entropy.

Shannon entropy is a concept proposed by Shannon in 1948 to indicate the degree of uncertainty of a random variable (Shannon 1948). When a certain probability distribution has only a limited amount of information given, there will be many distributions that satisfy the constraints. Jaynes found that in all distributions, only one entropy was the largest. In the distribution corresponding to the maximum entropy, there is no information that is not known (Jaynes 1957). Therefore, the principle of maximum entropy can be used to determine the value of an unknown variable given some constraints. According to Marichal's research, the entropy value of a fuzzy measure μ is defined as follows (Marichal 2002):

$$H(\mu) = \sum_{i=1}^{n} \sum_{T \subset S \backslash i} \frac{(n - |T| - 1)!|T|}{n!} h(\mu(iT) - \mu(T)) \tag{8.23}$$

For the 2-additive fuzzy measure,

$$\begin{aligned} H(\mu) &= \sum_{i=1}^{n} \sum_{T \subset S \backslash i} \frac{(n - |T| - 1)!|T|}{n!} h(a(i) + \sum_{j \in T} a(ij)) \\ &= \sum_{i=1}^{n} \sum_{T \subset S \backslash i} \frac{(n - |T| - 1)!|T|}{n!} h(I(i) - \frac{1}{2} \sum_{j \in S \backslash i} I(ij) + \sum_{j \in T} I_{ij}). \end{aligned} \tag{8.24}$$

Therefore, the overall weight and the interaction of risk factors can be obtained by through the following optimization problem:

$$\max H(\mu) = \sum_{i=1}^{n} \sum_{T \subset S \backslash i} \frac{(n - |T| - 1)!|T|}{n!} h(I(i) - \frac{1}{2} \sum_{j \in S \backslash i} I(ij) + \sum_{j \in T} I_{ij})$$

$$s.t. \begin{cases} v_i = I(i), \\ v_t^d \leq v_t \leq v_t^u, \ \forall t \in \{1, 2, \cdots, n\}, \\ \sum_{i=1}^{n} v_i = 1, \\ \forall T \subset S, \ \forall i \in T, \ \forall j \in T \backslash i, \ B_{ij}^d \leq I(ij) \leq B_{ij}^u. \end{cases} \tag{8.25}$$

The exact value of the sum can be obtained by solving the above optimization problem, and then the corresponding Mobius transform is obtained by the Eq. (8.18), and then the corresponding fuzzy measure can be obtained according to the Eq. (8.8).

(4) Using Choquet Points to Gain Knowledge Strength

In the multi-attribute decision problem, when there is an interaction between the criteria, the score of an action cannot be obtained by a simple weighted average method. In this case, the most representative fuzzy integral, the Choquet integral, is usually used to integrate the final score (Marichal 2002; Marichal and Roubens 2000).

Definition 8.8 Definition of discrete Choquet integral. Given a set *S*, containing elements $(x_1, x_2, \cdots, x_n)$, the discrete Choquet integrals of functions $f : S \rightarrow \mathrm{R}^+$ for fuzzy measures μ are:

$$C_\mu(f(x_1),\ f(x_2),\ \cdots,\ f(x_n)) = \sum_{i=1}^{n} (f(x_{(i)}) - f(x_{(i-1)}))\mu(A_{(i)}) \tag{8.26}$$

where the subscript (i) represents a permutation of all elements in the set, such that so that $f(x_{(1)}) \leq f(x_{(2)}) \leq \cdots \leq f(x_{(n)})$, and $A_{(i)} = \{x_{(i)}, \cdots, x_{(n)}\}$.

The following is an example of a specific algorithm for Choquet integrals.

Example 8.2 Assume that knowledge strength requires consideration of three knowledge factors *A*, *B*, and *C*. The fuzzy measures corresponding to all of their possible subsets are: $\mu(A) = 0.3$, $\mu(B) = 0.4$, $\mu(C) = 0.5$, $\mu(A, B) = 0.6$, $\mu(A, C) = 0.6$, $\mu(B, C) = 0.6$, and $\mu(A, B, C) = 1$. In addition, the scores of the three factors *A*, *B*, and *C* are respectively $x_A = 0.9$, $x_B = 0.6$, and $x_C = 0.3$, and then, according to the Choquet score, the final knowledge strength is $(x_C - x_0)\mu(A, B, C) + (x_B - x_C)\mu(A, B) + (x_A - x_B)\mu(A) = 0.489$.

In the normalized setting in the knowledge strength measure, the knowledge factor scores from 0 to 1, and according to the Eq. (8.23), it is easy to prove:

$$0 < C_\mu \le \sum_{i=1}^{n} (f(x_{(i)}) - f(x_{(i-1)})) = \max\{f(x_1), \cdots, f(x_n)\} \le 1 \tag{8.27}$$

Example 8.3 In a risk measure, three risk factors A, B, C are included. The fuzzy measures of the subsets are: $\mu(A) = 0.3$, $\mu(B) = 0.4$, $\mu(C) = 0.5$, $\mu(A, B) = 0.6$, $\mu(A, C) = 0.6$, $\mu(B, C) = 0.6$, and $\mu(A, B, C) = 1$. Given a risk, the scores assigned to each factors are: $x_A = 0.9$, $x_B = 0.6$, and $x_C = 0.3$. Then the Choquet integral of the risk is $(x_C - x_0)\mu(A, B, C) + (x_B - x_C)\mu(A, B) + (x_A - x_B)\mu(A)$ $= 0.489$.

Therefore, the final knowledge strength is also from 0 to 1, which is why the knowledge factor's score range is set to 0 to 1 in the normalized setting in the knowledge strength measure.

The fuzzy measure of any subset of knowledge factors can be obtained from the work of the previous three subsections, and then the final knowledge strength can be obtained by Eq. (8.26).

8.4.2 Knowledge Strength Under Multi-Layer Factors

Section 8.4.1 describes how to measure knowledge strength using fuzzy multi-attribute methods, and the method is based on the setting of single-layer factors, that is k1, k2, k3, k4, and k5. But each global factor contains several local factors, which means that the factors are hierarchical. In practical applications, there may be cases where each factor contains multiple layers.

This section describes how to handle this multi-level factor. Given the multi-level factor shown in Fig. 8.4, if f_l is at layer l and has factors of k sub-factors, then the score S_l is:

$$S_l = \sum_{i=1}^{k} (f(x_{(i)}^{l-1}) - f(x_{(i-1)}^{l-1}))\mu(A_{(i)}^{l-1}) \tag{8.28}$$

where, $f(x_{(i)}^{l-1})$ represents the score of l-1 layer's sub-factor, $\mu(A_{(i)}^{l-1})$ indicates the corresponding fuzzy measure.

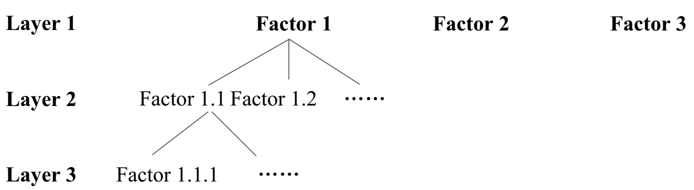

Fig. 8.4 Illustration of the knowledge factors in multiple layers

Equation (8.24) is also a Choquet score. Regardless of the factor of which layer, you can get the score of the factor from the lowest factor integration.

8.4.3 *Knowledge Strength Under Multiple Decision Makers*

When acquiring the strength of knowledge, it is common for multiple decision makers to exist at the same time. Suppose that m decision makers simultaneously evaluate the strength of knowledge. For the decision maker k, the overall weight is evaluated as $(v_1^{(k)}, v_2^{(k)}, \ldots, v_2^{(m)})$. Then for the x factor, the score is:

$$v_x = \sum_{i=1}^{m} t_i v_x^{(i)} \tag{8.29}$$

where t_i represents the weight of the i decision maker.

For the decision maker k, the interaction effect is evaluated as $B_{xy}^{(k)} = \left[B_{xy}^{d(k)}, B_{xy}^{u(k)}\right)$, and then, the integration interaction is:

$$B_{xy} = \left[\sum_{i=1}^{m} t_i B_{xy}^{d(i)}, \sum_{i=1}^{m} t_i B_{xy}^{u(i)}\right) \tag{8.30}$$

According to Eqs. (8.29) and (8.30), if a decision maker gives a score for each factor, then the strength of knowledge can be derived from Eq. (8.26). If multiple decision makers participate in the assessment, and for the decision maker k, whose Choquet score is $S^{(k)}$, then the ultimate integrated knowledge strength is:

$$S = \sum_{i=1}^{m} t_i S^{(i)} \tag{8.31}$$

8.5 Construction of Three-Dimensional Risk Matrices

Based on the work before, this section will explore how to build a three-dimensional risk matrix of consequences, probabilities, and knowledge strength based on new risk measures and use it for risk assessment.

The construction of the risk matrix depends on the specific application scenarios and the decision makers' requirements for the risk matrix.

In the traditional risk matrix, the criteria for risk assessment are composed of consequences and probabilities. When constructing a risk matrix, the inherent risk measure is $Risk = C \times P$. We showed how to use the sequential update method to obtain a two-dimensional matrix in Chap. 3. When adding the knowledge dimension, although the risk matrix changes from two-dimensional to three-dimensional, and the risk measure becomes $Risk = C \times (1+(1-M(K_C))^2) \times P \times (1+(1-M(K_P))^2)$, the sequential update method still is applied. This section will show how to use the sequential update method to design a three-dimensional risk matrix.

As with the two-dimensional risk matrix, as for three-dimensional risk matrix, the following basic information should also be obtained first:

(1) The number of categories of consequences, losses, and knowledge strength.
(2) The interval corresponding to the consequences, losses, and knowledge strength.
(3) Distribution of consequences, losses, and knowledge strength.
(4) The number of risk levels of the risk matrix or α.

It should be noted here that we integrate the knowledge strength of the consequences and the probability into the knowledge strength, that is, the integrated knowledge strength is $(1 + (1 - M(K_C))^2)(1 + (1 - M(K_P))^2)$.

It is assumed that the decision makers based on this basic information give the following requirements for the risk matrix of the required construction:

(1) The consequences, losses, and knowledge strength are divided into three categories, namely “high”, “medium”, and “low”.
(2) After the normalization of the consequences, losses and knowledge strength, the intervals corresponding to “high”, “medium” and “low” are (0, 1/3], (1/3, 2/3], (2/3, 1].
(3) There is no special information about the distribution of consequences, losses, and knowledge strength, so it is assumed that they are evenly distributed on their respective axes.
(4) The number of risk levels is four.

Based on the above information, the following two steps are required:

Step 1: Obtain a pairwise comparison matrix. In the three-dimensional risk matrix, the cells become unit blocks. The effect of the new risk measure is that the numerical comparison of the points is based on $Risk = C \times (1 + (1 - M(K_C))^2) \times P \times (1 + (1 - M(K_P))^2)$, other than $Risk = C \times P$. While the principle is still $\Pr(a > b | a \in A, b \in B) \geq \alpha,\ \alpha > 0.5$ when comparing the two-unit blocks, where a and b are two variables, which represent any point of the unit blocks A and B respectively.

Step 2: Rating according to the pairwise comparison matrix. The rules are consistent with the design of the two-dimensional risk matrix. When $a_{p+1} < N_1 + N_2 + \cdots + N_p$, the level of the unit block is unchanged. When $a_{p+1} = N_1 + N_2 + \cdots + N_p$, the risk level is increased by one level.

According to the above steps, a 3 × 3 × 3 risk matrix can be finally formed, and its design is shown in Fig. 8.5. In the risk matrix, the unit blocks with the consequences, probability, and knowledge strength (high, high, low) have the highest risk (red), which is consistent with the conclusion of Aven (2017a), while red The three blocks adjacent to the block are ranked second (yellow), and the consequences, probability,

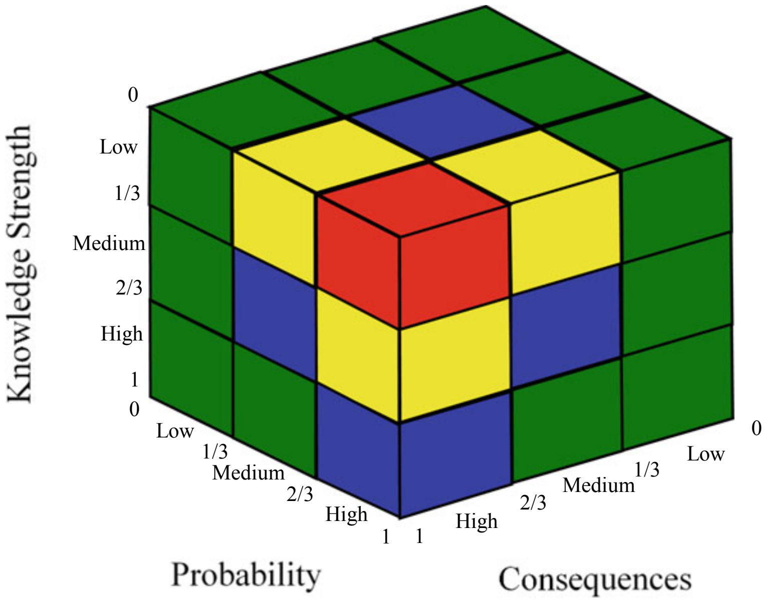

Fig. 8.5 3 × 3 × 3 risk matrix based on consequences, probability, and knowledge strength

and knowledge strength are (high, medium, medium), (medium, high, medium), (medium, medium, high), and (high, high, low). The four-unit blocks of the high and low have the third highest level (blue) and the remaining unit blocks have the lowest level (green).

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Chapter 9
Conclusions and Future Research

9.1 Conclusions

The risk matrix has been widely used in various fields mainly because it is a risk assessment tool that does not rely too much on sufficient data, and it helps provide quick risk assessment without heavy computational cost. Essentially, the risk matrix is a qualitative risk assessment tool and highly related to the stakeholder's subjectivity.

This book provides several methods for resolving problems related to two important topics of the risk matrix, namely, the rating scheme design and the aggregation. The former issue is derived from the phenomenon that in various risk assessment scenarios using risk matrices, they usually do not tell how the adopted risk is designed and current researches on this issue are spare. Another issue, risk aggregation, focuses on the overall risk of a risk assessment context that contains multiple risks. The aggregation of risk matrices is claimed to be impossible by ISO, given several technical obstacles. In this book, we mainly discussed the following issues in connection with risk matrix design and aggregation.

- **A sequential updating approach for the rating scheme design of risk matrix**. Although currently several apparent but very simple rules can be extracted from the frequently used risk matrices, there is a lack of systematic, logical, scientific, and convincing methods that can be used to guide the risk matrix design. The sequential updating approach is the first proposed method that combines principles from the perspective of decision and the algorithm for implementing the design, overcoming some obvious disadvantages of the traditionally designed risk matrices. The detail is presented in Chap. 3 together with some other existing methods.
- **The effect of different kinds of risk perception on the risk matrix design**. The risk matrix is a qualitative risk assessment tool that different perceptions related to the scaling of inputs, the location of inputs, the input category membership, the measure of risks, and the attitudes towards risks may affect the rating scheme. We explore how the perceptions work in the process of designing a risk matrix. This is explained in Chap. 4.

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- **The criterion that can be used to assess the design of the risk matrix**. We consider that the lack of appropriate rules to design risk matrices is mainly because designers tend to think that designing risk matrices is entirely a subjective process. Moreover, these scholars did not have quantitative criteria to assess the performance of the risk matrices they designed. In turn, reasonable criteria improve the accuracy of risk matrix design. The criteria and corresponding quantitative indicators are exhibited in Chap. 5.
- **A general framework for risk aggregation of risk matrices**. Although risk matrices are declared unable to be aggregated by ISO, we overcome some obstacles by using quantitative risk matrices and normalizing different types of consequences. Four steps are the general framework contained are illustrated in Chap. 6.
- **Several detailed methods for risk aggregation of risk matrices**. The methods of fuzzy set, interval number, and probability density function are proposed to aggregate risk matrices for their conformance with risk matrices in some properties. The technical steps and comparison of these methods are given in Chap. 7.
- **Extension from the two-dimensional risk matrices to three-dimensional ones**. The traditional risk matrices are two-dimensional since the risk measure is based on $risk = consequence \times likelihood$, which have two dimensions. Considering that this risk measure ignores the strength of knowledge to support the estimation of consequence and likelihood, we propose a risk measure containing three dimensions, i.e., consequence, likelihood, and the strength of knowledge. The measure of strength and the construction of three-dimensional risk matrices are shown in Chap. 8.

These issues help a better understanding of how the flaws of traditional risk matrices generate, how to deal with the subjectivity of the qualitative tools, and how to improve the accuracy of risk assessment and further decision making.

9.2 Future Research

The risk matrices are frequently used based on personal subjective judgement of risk related elements like the category of consequence and likelihood, the risk measure, the risk acceptance, and so on. Despite these qualitative nature, the design and use of risk matrices needs quantitative guidance. Therefore, in the future research of risk matrix, the qualitative and quantitative methods should be combined. To be specific, the following points are worthy of further research.

- Risk matrix design integrating subjective and objective methods. Subjectivity here mainly refers to the fact that the boundaries of two adjacent ratings are not so determined for a decision maker. Objectivity here mainly refers to the basic requirement that the matrix design should obey a convincing logic. And thus given some certain subjective boundaries, a risk matrix design method should work based on its given framework.

- Risk matrix aggretion considering risk dependence. The methods introduced in this book assume that the aggregated risks are independent for simplicity. However, this assumption is not always ture since risks usually mutually affect. The aggregation of risk matrices considering risk dependence should start with the risk measure of the aggregated risk of the risk assessment context with several risks that interact.
- Risk measure combing objectivity and subjectivity. As stated in this book, a objective risk measure like $risk = consequence \times probability$, can not refelect the perceived risk of a decision maker. More convincing risk measures that both obey some objective rules and handle the decision makers' subjectivity are necessary for the risk matrix research in the future.

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