Mathematical Theory of Quantum Teleportation Protocols

An Introduction for Scientists and Engineers

Binayak S. Choudhury and Soumen Samanta

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This book provides an in-depth exploration of the mathematical and theoretical foundations of quantum teleportation. Starting with the seminal 1993 work by C. H. Bennett and colleagues, this book delves into the intricate processes that enable the transfer of quantum information using non-local quantum entangled resources. It covers a broad spectrum of teleportation protocols for single and multi-qubit systems, explores the impact of quantum noise, and presents strategies to mitigate these effects.

Overall features of this book are the following:

- Detailed explanations of quantum mechanics concepts relevant to teleportation, such as qubits, entanglement, quantum evolution, noise, and measurement.
- In-depth analysis of teleportation through noisy quantum channels, with models for amplitude-damping, bit-flip, phase-flip, and phase-damping noise.
- Methods for minimizing the effects of noise, including weak and reversal measurements and environment-assisted techniques.
- Case studies that illustrate the application of various teleportation protocols.
- Comprehensive coverage of the necessary mathematics, including linear spaces, operators, tensor products, and partial trace operations.

Primarily aimed at theoretical physicists, applied mathematicians, computer scientists, and engineers belonging to all branches of electrical technology, this book is both an introduction and a comprehensive guide to the field of quantum teleportation. It will also be beneficial to the interested scientientists and professionls belonging to physical chemistry, material science and information technology.

Binayak S. Choudhury is a Professor of Mathematics at the Indian Institute of Engineering Science and Technology, Shibpur, India. He has more than three decades of research and teaching experience, in which more than two decades have been in the position of Full Professor. He has wide research interests in several topics of Pure and Applied Mathematics, Theoretical Physics and Decision Sciences. He has published more than 300 research articles and guided 26 research students for Ph.D. degree. He has handled several Research Projects as a Principal Investigator. Particularly, his contributions to Quantum Information Theory over about two decades have been commendable.

Soumen Samanta is currently an Assistant Professor of Mathematics in the Department of Applied Science at Haldia Institute of Technology, West Bengal, India. Before that, he has served as an Assistant Professor of Mathematics at IIEST, Shibpur, India, on a contractual basis. He has experience in teaching mathematics at Undergraduate and Postgraduate levels. He has obtained his Master of Science degree from the University of Calcutta, India and Ph.D. degree in the field of Quantum Information Theory from IIEST, Shibpur, India. Dr. Samanta has specialized in the Theory of Quantum Communication, particularly in Quantum Teleportation, Quantum Remote State Preparation, Quantum Walk, etc. Dr. Samanta has contributed significantly to the field of Quantum Information Theory through more than 30 research articles. He has also served as a reviewer for different journals.

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Dedication

From Dr Binayak S. Choudhury:
To Prof. J. N. Das who taught me quantum mechanics
From Dr. Soumen Samanta:
To my wife and son who are my inspiration

Contents

PART II   Teleportation Protocols in Ideal Environment

Preface

This book is written with an objective of providing a theoretical as well as mathematical introduction to the concept of teleportation in a self-contained manner. The required mathematics and physics is described at the beginning. The description starts from the elementary level and is restricted to the minimum required for serving the purpose of the book without making any compromise with rigor. This is done with a view to making the book accessible to scientists and engineers without having a specialized knowledge in physics, applied mathematics or computer science.

The book is divided into three parts. The contents of these three parts are concisely as follows.

In the first part the requisite background materials are given. It is a self-contained presentation. The mathematics is presented from the preliminary level and includes everything which is required for the understanding of the subject matter contained in the book. The requisite part of quantum mechanics is also described in a self-contained way. It is better that the reader has an introductory knowledge of quantum mechanics, but this is not necessary. The topics of quantum physics covered include qubits, multi-qubit systems, entanglement, quantum evolution, quantum noise, quantum measurement and a general description of quantum communication system. The mathematics discussed here includes linear space, Hilbert space, linear operators in general, Hermitian operators, unitary operators, projection

operators, tensor products and the partial trace operations. There are several illustrations on the topics included in this part.

In the second part the topics are teleportation protocols for different types of states using appropriate entangled quantum resources as channels. There are several versions of teleportation like controlled teleportation, multi-hop teleportation, probabilistic teleportation etc. whose final aim is to transfer quantum states with the help of entanglement resources. The above-mentioned variations are described in this section.

The third part of this book is dedicated to the analysis of teleportation protocols through noisy channels. Noise is an inalienable phenomena in every communication system. We consider quantum noise affecting the entangled communication channel. This noise is modeled through Kraus operators. Four types of noises are considered, namely, Amplitude-damping, Bit-flip, Phase-flip and Phase-damping noise. In this part we analyze the effects of these noises on the concerned teleportation protocols by calculating the fidelity of the process. Fidelity is a measure by which we understand the deviation of the quantum state actually obtained at the receiver's end from that which was originally intended for transfer. Further it is important to control the effect of noise as far as possible. We present in this part weak and reversal measurements and environment assisted measurements as methods for such control.

The book can be utilized by following the interdependency chart of the chapters given below.

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└───────┬───────┘                    └───────┬───────┘
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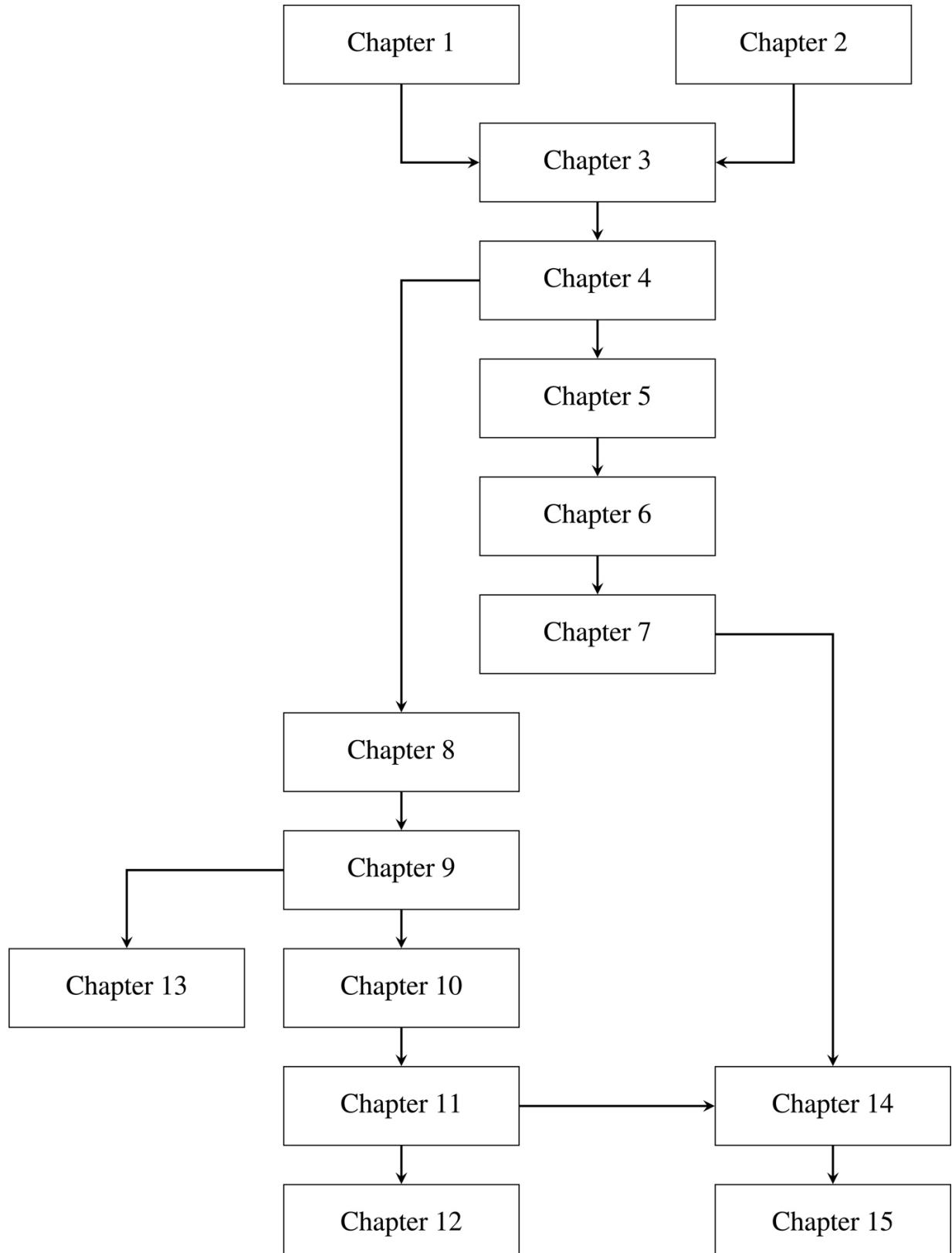
▶ Long Description for Figure

Since the literature on teleportation is vast, we discuss some representative protocols in order to present the basic ideas, quantum mechanical techniques and the methodologies which are in use in this study. The bibliography to a certain extent contains the prominent works on this subject. Particularly, the readers interested in the fundamental ideas of teleportation will find it in [Chapter 8](#) which is accessible after going through the prerequisites given in [Chapters 1](#), [3](#) and [4](#).

The primary readership of the book is for Theoretical Physicists, Applied Mathematicians, Computer Scientists, Telecommunication Engineers and Technologists belonging to all branches of Electrical Technology. Beyond the primary readership, interested Scientists and Engineers belonging to the disciplines of Chemistry, Chemical Engineering, Applied Physics, Space Science, Material Science and Information Technology professionals are supposed to be benefited through the book. Certain portions of the book can serve as parts of courses on Quantum technology/ Quantum information science.

We gratefully acknowledge all the authors whose works have been used in parts of the book. Also we express our gratitude to all who have helped directly or indirectly in making our project of writing the book into a reality.

Dr. Binayak S. Choudhury, Professor

Dr. Soumen Samanta, Assistant Professor

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Part I

Essential Concepts

1 Linear Spaces and Operators

1.1   INTRODUCTION

In this chapter, the mathematics that are used in the discussion of the teleportation protocols are provided. Hilbert spaces, which are linear spaces with appropriate geometries brought about by introducing inner products, form the mathematical framework of the above-mentioned discussion. These spaces along with the relevant operators for describing the required part of quantum physics are presented here. For extensive studies on the topics included in this chapter, the references [33, 34, 62, 78, 113, 136, 152, 156, 170, 189] are useful.

1.2   LINEAR SPACE

In this section we begin with the definition of linear space and introduce Hilbert space consequently. The mathematical framework for the discussion of our topics in this book is the Hilbert spaces. Moreover, we restrict our consideration only to finite-dimensional spaces as per our requirement. The notations used here are different from those which appear in the usual treatise on linear algebra and functional analysis. This is due to *Dirac* which is used in specific areas of quantum mechanics. Especially these notations (*ket* and *bra*

vectors as they are called) are standard notations in quantum information theory.

1.2.1 DEFINITIONS AND ILLUSTRATIONS

Let V be a non-empty set and $\backslash +'$ be a binary operation on V. Let $\mathbb{C}$ be the field of complex numbers, and let $\backslash \cdot$' be an external composition of $\mathbb{C}$ with V. Then V or more generally $(V, \mathbb{C}, +, .)$, is said to be a Linear space over the field $\mathbb{C}$ of complex numbers if the following conditions are satisfied:

1. $|\sigma_1\rangle + |\sigma_2\rangle = |\sigma_2\rangle + |\sigma_1\rangle$, $\forall |\sigma_1\rangle, |\sigma_2\rangle \in V$ (Commutativity);

2. $|\sigma_1\rangle + (|\sigma_2\rangle + |\sigma_3\rangle) = (|\sigma_1\rangle + |\sigma_2\rangle) + |\sigma_3\rangle$, $\forall |\sigma_1\rangle, |\sigma_2\rangle, |\sigma_3\rangle \in V$ (Associativity);

3. there exists an element $|\vartheta\rangle$ in V such that $|\sigma\rangle + |\vartheta\rangle = |\sigma\rangle$, $\forall |\sigma\rangle \in V$ (Existence of Identity);

4. for each $|\sigma_1\rangle \in V$ there exists an element $|\sigma_2\rangle \in V$ such that $|\sigma_1\rangle + |\sigma_2\rangle = |\vartheta\rangle$, (Existence of Inverse);

5. $(e.f).|\sigma\rangle = e.(f.|\sigma\rangle)$, $\forall e, f \in \mathbb{C}$ and $\forall |\sigma\rangle \in V$;

6. $e.(|\sigma_1\rangle + |\sigma_2\rangle) = (e.|\sigma_1\rangle) + (e.|\sigma_2\rangle)$, $\forall e \in \mathbb{C}$ and $\forall |\sigma_1\rangle, |\sigma_2\rangle \in V$;

7. $(e + f).|\sigma\rangle = (e.|\sigma\rangle) + (f.|\sigma\rangle)$, $\forall e, f \in \mathbb{C}$ and $\forall |\sigma\rangle \in V$;

8. $1.|\sigma\rangle = |\sigma\rangle$, 1 being the identity element in $\mathbb{C}$.

In mathematics, a linear space or a vector space is defined on an arbitrary field, which is an algebraic structure defined separately through certain algebraic operations. We do not require linear spaces except for those over the field of complex numbers. So we have noted the above definition with respect to the field of complex numbers only.

Let V be a linear space over the field $\mathbb{C}$ of complex numbers. Let $|\sigma_1\rangle, |\sigma_2\rangle, \ldots, |\sigma_k\rangle \in V$. A linear combination of a set of vectors $\{|\sigma_1\rangle, |\sigma_2\rangle, \ldots, |\sigma_k\rangle\}$ is any vector

$$|\Xi\rangle = e_1|\sigma_1\rangle + e_2|\sigma_2\rangle + \ldots + e_k|\sigma_k\rangle,$$

where $e_1, e_2, \ldots, e_k$ are complex numbers.

If $S = \{|\sigma_1\rangle, \ldots, |\sigma_n\rangle\}$, then the set of all linear combinations of the elements of S is denoted by $L(S)$ and is called the linear span or simply the span of S.

A finite set of vectors $\{|\sigma_1\rangle, |\sigma_2\rangle, \ldots, |\sigma_k\rangle\}$ is said to be linearly dependent if there exist scalars $e_1, e_2, \ldots, e_k$ not all zero in $\mathbb{C}$ such that

$$e_1|\sigma_1\rangle + e_2|\sigma_2\rangle + \ldots + e_k|\sigma_k\rangle = 0.$$

(1.1)

The set is said to be linearly independent in V if the equality given in Eq. (1.1) is satisfied only when $e_1 = e_2 = \cdots = e_k = 0$.

Hilbert spaces can be of infinite dimensions. Particularly, the L^2-space, the space of square-integrable functions, has extensive use in the quantum mechanics of continuous systems. As already noted, these spaces will not be used in the present context. For that reason, discussions on infinite-dimensional spaces are omitted.

1.2.2   INNER PRODUCT SPACE AND HILBERT SPACE

The inner product of two vectors $|\Omega\rangle$ and $|\Xi\rangle$ is given by

$$\langle \Omega | \Xi \rangle = f_1^* e_1 + f_2^* e_2 + \cdots + f_n^* e_n = \sum_{i=1}^{n} f_i^* e_i$$

(1.2)

where $|\Omega\rangle = \sum_{i=1}^{n} f_i |\sigma_i\rangle$ and $|\Xi\rangle = \sum_{i=1}^{n} e_i |\sigma_i\rangle$.

The inner product $\langle\Omega|\Xi\rangle$ is a complex number in general and independent of their representations.

If two vectors $|\Omega\rangle$ and $|\Xi\rangle$ are such that $\langle\Omega|\Xi\rangle = 0$, then $|\Omega\rangle$ and $|\Xi\rangle$ are orthogonal. If $\{|\Omega_1\rangle, |\Omega_2\rangle, \dots, |\Omega_p\rangle\}$ are p vectors such that $\langle\Omega_k|\Omega_l\rangle = \delta_{kl}$, $k, l = 1, 2, \dots, p$, then the set $\{|\Omega_1\rangle, |\Omega_2\rangle, \dots, |\Omega_p\rangle\}$ is called an orthonormal set.

Referring to the completeness relation $\sum_{i=1}^{n} |\sigma_i\rangle\langle\sigma_i| = I$, (described in) we have

$$|\Xi\rangle = I. |\Xi\rangle = \sum_{i=1}^{n} |\sigma_i\rangle\langle\sigma_i|\Xi\rangle = \sum_{i=1}^{n} e_i |\sigma_i\rangle,$$

where $e_i = \langle\sigma_i|\Xi\rangle$. The above formula provides with the determination of the coefficients for vectors $|\Xi\rangle$ in its expansion with respect to a given basis (see below Subsection 1.1.3).

The inner product has certain properties which are enumerated below:

1. $\langle\Omega|\Xi\rangle = \langle\Xi|\Omega\rangle^*$
2. $\langle\Omega|(e_1|\Xi_1\rangle + e_2|\Xi_2\rangle) = e_1\langle\Omega|\Xi_1\rangle + e_2\langle\Omega|\Xi_2\rangle$
3. $\langle\Xi|\Xi\rangle \geq 0$ and $\langle\Xi|\Xi\rangle = 0$ if and only if $|\Xi\rangle = 0$, the zero vector of the linear space.

NOTE: Actually the above properties are used for an axiomatic definition of Inner Product on a vector space to make it into an inner product space. For our special purpose, we have taken the definition as in Eq. (1.2).

A complete inner product space is called Hilbert space. The most elementary but very significant example of a Hilbert space is the space spanned by two elements $\{|0\rangle, |1\rangle\}$ which we denote by $\mathbb{H}_2$. Its elements are given as

$$|\Xi_2\rangle = e_0|0\rangle + e_1|1\rangle,$$

where e_0 and e_1 are complex numbers.

The inner product on this space is given by

$$\langle k|l\rangle = \delta_{kl}, \ \ k, l = 0, 1,$$

that is, explicitly

$$\langle 0|0\rangle = \langle 1|1\rangle = 1$$

and

$$\langle 0|1\rangle = \langle 1|0\rangle = 0.$$

This particular space $\mathbb{H}_2$ is used for the mathematical description of a qubit.

Another example is $\mathbb{H}_d$ (d is an integer) which describes the d-level quantum systems known as qudit.

A qudit is a generalization of a qubit. A qudit can be in any quantum state $|0\rangle, |1\rangle, \ldots |d-1\rangle$, and any superposition of these states, similar to a qubit being in a superposition of $|0\rangle$ and $|1\rangle$. For example, a qudit with $d = 3$, known as qutrit, could be in a superposition of three states:

$$|\Xi_3\rangle = e_1|0\rangle + e_2|1\rangle + e_3|2\rangle.$$

We will be concerned only with qubits.

1.2.3   BASIS AND DIMENSION

Let V be a linear space over $\mathbb{C}$ and $S = \{|\sigma_1\rangle, |\sigma_2\rangle, \ldots, |\sigma_k\rangle\}$ be a subset of V. We say that S is a spanning set of V if every vector $|\sigma\rangle \in V$ can be expressed as a linear combination of the elements in S. In such cases, we say that S spans V, that is, $V = L(S)$.

Let V be a linear space. A minimal set of elements in V that spans V is called a basis for V. Equivalently, a basis for V is a set of elements that is (i) linearly independent and (ii) spans V, that is, $V = L(S)$. The number of elements in a basis for V is called the dimension of V, denoted by $\dim V$.

From the above definition, any vector $|\Xi\rangle \in V$ can be written as

$$|\Xi\rangle = e_1|\sigma_1\rangle + e_2|\sigma_2\rangle + \cdots + e_n|\sigma_n\rangle = \sum_1^n e_i|\sigma_i\rangle,$$

where $\{|\sigma_1\rangle, |\sigma_2\rangle, \ldots, |\sigma_n\rangle\}$ forms a basis for the vector space V.

The above expression is unique insofar as the basis remains the same. With respect to the given basis mentioned above, we can represent the element $|\Xi\rangle$ as

a column vector $\begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}$.

As per the convention in quantum mechanics, we call $|\Xi\rangle$ a *'ket'* vector or simplify a ket. This is known as *Dirac*'s notation and is sometimes called *Dirac*'s ket vector. There is a corresponding concept of *'bra'* vector, which is written as $\langle\Xi|$ corresponding to the ket $|\Xi\rangle$.

The representation of $\langle\Xi|$ which is actually the complex conjugate of $|\Xi\rangle$ is given by a row vector $(e_1^*, e_2^*, \ldots, e_n^*)$ in the same basis.

A basis $\{|\sigma_1\rangle, |\sigma_2\rangle, \ldots, |\sigma_n\rangle\}$ is orthonormal if

$$\langle\sigma_k|\sigma_l\rangle = \delta_{kl}, \quad k, l = 1, 2, \ldots, n$$

where δ_{kl} is the Kronecker's delta, which is

$$\delta_{kl} = \begin{cases} 1, & \text{if } k = l. \\ 0, & \text{if } k \neq l \end{cases}$$

where $\langle \sigma_i | \sigma_j \rangle$ stands for the inner product.

In the Hilbert space $\mathbb{H}_2$, we have the basis $\{|0\rangle, |1\rangle\}$ which is a 2-dimensional Hilbert space.

As described above, we can describe $|0\rangle$ as $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle$ as $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then any $|\Xi_2\rangle = e_1|0\rangle + e_2|1\rangle$ will be written as

$$e_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

This is an alternative representation of $\mathbb{H}_2$.

1.2.4   CHANGE OF BASIS

There can be more than one basis of the same vector space (in fact, an infinite number of bases is possible). As an instance, for the space $\mathbb{H}_2$ described previously, two bases are noted in the following:

$$\{|0\rangle, |1\rangle\} \text{ and } \{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\}.$$

The choice of basis is important in quantum mechanics since a basis is related to a particular measurement performed on the system. The following is the mechanism by which we can bring about a change of basis. Since our consideration will be only orthonormal bases, we describe the corresponding change only for these types of bases.

It is possible to pass from an orthonormal basis $(|\gamma_i\rangle, i = 1, 2, \ldots, n)$ to another $(|\gamma_i'\rangle, i = 1, 2, \ldots, n)$ by means of a unitary transformation S:

$$|\gamma_i'\rangle = \sum_j S_{ji}|\gamma_j\rangle \quad (i = 1, 2, \ldots, n).$$

<div align="right">(1.3)</div>

Then $S_{ki} = \langle \gamma_k | \gamma_i' \rangle$.

A generic vector

$$|\alpha\rangle = \sum_i a_i |\gamma_i\rangle \quad (a_i \equiv \langle \gamma_i | \alpha \rangle),$$

(1.4)

can be expressed in the new basis as

$$|\alpha\rangle = \sum_j a_j' S_{ij} |\gamma_j\rangle \quad (a_j' \equiv \langle \gamma_j' | \alpha \rangle),$$

(1.5)

where we have used Eq. (1.2). Thus, the old and new vector components are linked by the relation

$$a_i = \sum_j S_{ij} a_j'.$$

(1.6)

As an illustration, if we consider $B_1 = \{|0\rangle, |1\rangle\}$ and $B_2 = \{|\xi_1\rangle, |\xi_2\rangle\}$ with $|\xi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |\xi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, then the matrix $T = (t_{ij})_{2\times 2}$ for the transformation from B_1 to B_2 is given by $T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$.

1.3 OPERATORS ON HILBERT SPACES

Our consideration is limited by our requirement of three kinds of operators on a linear space, namely self-adjoint operator, unitary operator, and projection operator. Since quantum mechanics is a linear theory, we only require linear operators.

1.3.1 LINEAR OPERATORS AND MATRIX REPRESENTATIONS

An *operator* is a mapping $L : \mathbb{H}_1 \to \mathbb{H}_2$ which is from a linear space $\mathbb{H}_1$ to another linear space $\mathbb{H}_2$.

The operator L is *linear* if

$$L(e|\tau_1\rangle + f|\tau_2\rangle) = (eL|\tau_1\rangle + fL|\tau_2\rangle)$$

for all $|\tau_1\rangle, |\tau_2\rangle \in \mathbb{H}_1$ and scalars (complex numbers in our consideration) e and f.

If L is a linear operator, then eL defined as $(eL)|\tau\rangle = e(L|\tau\rangle)$ is a linear operator and if L_1, L_2 are two linear operators, then $L_1 + L_2$ defined as $(L_1 + L_2)|\tau\rangle = L_1|\tau\rangle + L_2|\tau\rangle$ is also a linear operator.

Considering the above two statements, we conclude that $eL_1 + fL_2$ is a linear operator whenever L_1, L_2 are linear operators and e, f are scalars.

Given two bases of $\mathbb{H}_1$ and $\mathbb{H}_2$ of dimensions n and m, respectively, a linear operator $L : \mathbb{H}_1 \to \mathbb{H}_2$ can be represented by an $m \times n$ matrix that acts on the n-tuple corresponding to a vector in $\mathbb{H}_1$. This representation of a linear operator by a matrix is specific to the choice of bases.

If $L : \mathbb{H}_1 \to \mathbb{H}_2$ is a linear operator, $\{|\tau_1\rangle, \ldots, |\tau_n\rangle\}$ and $\{|\pi_1\rangle, \ldots, |\pi_m\rangle\}$ are bases for $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively, then its matrix representation is $(L)_{ij}$, given by

$$L_{ij} = \langle \pi_i | L | \tau_j \rangle.$$

Inverse Operator

Consider a linear operator L on a Hilbert space $\mathbb{H}$, that is, $L : \mathbb{H} \to \mathbb{H}$. If there exists an operator M such that

$$LM = ML = I,$$

we call M the *inverse* of L and write $M = L^{-1}$. If we have $|\pi\rangle = L|\tau\rangle$, then in that case $|\tau\rangle = L^{-1}|\pi\rangle$. It is possible to show that the inverse of an operator L exists if and only if the equation $L|\tau\rangle = 0$ (the zero ket $|0\rangle$) implies that $|\tau\rangle$ is the zero vector. Considering the matrix representation of L, it is immediate to conclude that the inverse of an operator L exists if and only if $det L \neq 0$.

1.3.2   OUTER PRODUCTS

Let $\mathbb{H}$ be any Hilbert space and $|\tau_1\rangle, |\tau_2\rangle \in \mathbb{H}$. Then the *outer product* of $|\tau_1\rangle$ and $|\tau_2\rangle$ is a linear operator $|\tau_1\rangle\langle\tau_2|$ on $\mathbb{H}$ defined by its action on an arbitrary ket $|\tau_3\rangle$ in $\mathbb{H}$ as

$$(|\tau_1\rangle\langle\tau_2|)|\tau_3\rangle = |\tau_1\rangle\langle\tau_2|\tau_3\rangle = (\langle\tau_2|\tau_3\rangle)|\tau_1\rangle.$$

The identity operator I is defined as $I|\tau\rangle = |\tau\rangle$, for all $|\tau\rangle \in \mathbb{H}$.

For an orthonormal basis $\{|\pi_1\rangle, |\pi_2\rangle, \ldots, |\pi_n\rangle\}$ of a (finite-dimensional) Hilbert space $\mathbb{H}$, we have for all $|\tau\rangle \in \mathbb{H}$,

$$
\begin{aligned}
&(|\pi_1\rangle\langle\pi_1| + |\pi_2\rangle\langle\pi_2| + \ldots + |\pi_n\rangle\langle\pi_n|)|\tau\rangle \\
&= |\pi_1\rangle\langle\pi_1||\tau\rangle + |\pi_2\rangle\langle\pi_2||\tau\rangle + \ldots + |\pi_n\rangle\langle\pi_n||\tau\rangle \\
&= \sum_{i=1}^{n}(\langle\pi_i||\tau\rangle)|\pi_i\rangle \\
&= |\tau\rangle.
\end{aligned}
$$

From the above, we have the completeness relation that for any orthonormal basis $\{|\pi_1\rangle, |\pi_2\rangle, \ldots, |\pi_n\rangle\}$, the following relation holds

$$\sum_{i=1}^{n}|\pi_i\rangle\langle\pi_i| = I.$$

(1.7)

The above relation is an extremely important result in the discussion of quantum information theory.

1.3.3   HERMITIAN OPERATORS

Given a linear operator $L : \mathbb{H} \rightarrow \mathbb{H}$ on a Hilbert space $\mathbb{H}$, $L^\dagger$ on $\mathbb{H}$, the adjoint or Hermitian conjugate of L is another linear operator $L^\dagger$ such that for all vectors $|\alpha\rangle, |\beta\rangle \in \mathbb{H}$,

$$\langle\alpha|L|\beta\rangle = \langle\beta|L^\dagger|\alpha\rangle^*.$$

(1.8)

We call an operator self-adjoint or Hermitian if $L^\dagger = L$.

For a linear operator L, if there exists a scalar (complex number) such that

$$L|\alpha\rangle = \lambda|\alpha\rangle$$

for some $|\alpha\rangle$, then λ is called the eigenvalue of L and $|\alpha\rangle$ is called eigenket corresponding to the eigenvalue λ.

The Hermitian operators have the following properties:

1. They have real eigenvalues.

2. The eigenkets corresponding to different eigenvalues are orthogonal.

3. There exists a complete set of orthogonal eigenkets corresponding to every Hermitian operator.

It follows from the above that there exist eigenkets $|\alpha_1\rangle, \dots, |\alpha_n\rangle$ for a self-adjoint operator L on a finite-dimensional Hilbert space $\mathbb{H}$ of dimension n such that

$$\langle\alpha_i|\alpha_j\rangle = \delta_{kl} \quad k, l = 1, 2, \dots, n$$

and for any $|\gamma\rangle \in \mathbb{H}$,

$$|\gamma\rangle = e_1|\alpha_1\rangle + \ldots + e_n|\alpha_n\rangle$$

for some scalars $e_1, \ldots, e_n$.

The above is equivalent to the fact that the eigenvalues $|\alpha_1\rangle, \ldots, |\alpha_n\rangle$ form an orthonormal basis of $\mathbb{H}$.

For a given basis $\{|\alpha_1\rangle, |\alpha_2\rangle, \ldots, |\alpha_n\rangle\}$, the matrix corresponding to L is

$$L_{ij} \equiv \langle \alpha_i | L | \alpha_j \rangle.$$

(1.9)

Then

$$
\begin{aligned}
(L^\dagger)_{ij} &= \langle \alpha_i | L^\dagger | \alpha_j \rangle \\
&= \langle \alpha_j | L | \alpha_i \rangle \\
&= \langle \alpha_i | L | \alpha_j \rangle^*
\end{aligned}
$$

$\langle L\gamma_i | \gamma_j \rangle = \langle \gamma_i | L^\dagger \gamma_j \rangle$ and this relation can be written as

$$(L_{ji})^* = (L^\dagger)_{ij}.$$

(1.10)

In matrix representation,

$$L^\dagger = (L^T)^*.$$

(1.11)

For a self-adjoint operator, we have

$$L = (L^T)^*.$$

(1.12)

1.3.4 UNITARY OPERATORS

An operator L on a Hilbert space $\mathbb{H}$ is said to be unitary if

$$LL^\dagger = L^\dagger L = I.$$

(1.13)

From this definition, we have that the adjoint of a unitary operator coincides with its inverse,

$$L^\dagger = L^{-1},$$

(1.14)

and that $L^\dagger$ is unitary. The product LM of two unitary operators is unitary, since

$$(LM)(LM)^\dagger = LMM^\dagger L^\dagger = I.$$

(1.15)

Unitary operators preserve the inner product between kets. For any two kets $|\sigma_1\rangle$ and $|\sigma_2\rangle$, if $|\tau\rangle = L|\sigma_1\rangle$ and $|\pi\rangle = L|\sigma_2\rangle$, then

$$\langle \tau | \pi \rangle = \langle L\sigma_1 | L\sigma_2 \rangle = \langle \sigma_1 | L^\dagger L | \sigma_2 \rangle = \langle \sigma_1 | \sigma_2 \rangle.$$

(1.16)

With $|\sigma_1\rangle = |\sigma_2\rangle$, we see that a unitary operator preserves the norm of a ket vector.

1.3.5 PROJECTION OPERATORS

A projection operator $P : \mathbb{H} \to \mathbb{H}$, where $\mathbb{H}$ is a Hilbert space, is an operator satisfying the following:

1. $P = P^\dagger$, that is, P is self-adjoint.

2. $P^2 = P$.

3. P is continuous.

When $\mathbb{H}$ is a finite-dimensional Hilbert space, as in our present consideration, P is automatically continuous.

Particularly, for a given vector $|\alpha\rangle$, the operator $P = |\alpha\rangle\langle\alpha|$ is a projection operator. It is proved in the theory of Hilbert spaces that a projection operator determines a subspace of a Hilbert space to which all elements of $\mathbb{H}$ are projected by the operator. This result has important consequences in problems of quantum measurements.

1.4 TENSOR PRODUCT

Tensor product of two or more Hilbert spaces is a method of combining these Hilbert spaces into a higher dimensional space. It is utilized to describe composite quantum systems. Tensor products of operators are also described in this section.

1.4.1 TENSOR PRODUCT OF HILBERT SPACES

Consider two Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$ of dimensions m and n, respectively. In the tensor product $\mathbb{H}$ of $\mathbb{H}_1$ and $\mathbb{H}_2$, written as $\mathbb{H} = \mathbb{H}_1 \otimes \mathbb{H}_2$, we can associate with each pair of vectors $|\mu\rangle \in \mathbb{H}_1$ and $|\nu\rangle \in \mathbb{H}_2$ a vector belonging to $\mathbb{H}$, denoted by $|\mu\rangle \otimes |\nu\rangle$ and call it the tensor product of $|\mu\rangle$ and $|\nu\rangle$. By definition, the vectors in $\mathbb{H}$ are linear superpositions of the above vectors $|\mu\rangle \otimes |\nu\rangle$ where the following properties are satisfied:

1. for any $|\mu\rangle \in \mathbb{H}_1$, $|\nu\rangle \in \mathbb{H}_2$ and $e \in \mathbb{C}$,
$e(|\mu\rangle \otimes |\nu\rangle) = (e|\mu\rangle) \otimes |\nu\rangle = |\mu\rangle \otimes (e|\nu\rangle)$;

2. for any $|\mu_1\rangle, |\mu_2\rangle \in \mathbb{H}_1$ and $|\nu\rangle \in \mathbb{H}_2$, $(|\mu_1\rangle + |\mu_2\rangle) \otimes |\nu\rangle = |\mu_1\rangle \otimes |\nu\rangle + |\mu_2\rangle \otimes |\nu\rangle$;

3. for any $|\mu\rangle \in \mathbb{H}_1$ and $|\nu_1\rangle, |\nu_2\rangle \in \mathbb{H}_2$, $|\mu\rangle \otimes (|\nu_1\rangle + |\nu_2\rangle) = |\mu\rangle \otimes |\nu_1\rangle + |\mu\rangle \otimes |\nu_2\rangle$;

In the following, instead of $|\mu\rangle \otimes |\nu\rangle$, we shall often use the notations $|\mu\rangle|\nu\rangle$, $|\mu, \nu\rangle$ or $|\mu\nu\rangle$.

Let $\{|\mu_1\rangle, |\mu_2\rangle, \ldots, |\mu_m\rangle\}$ and $\{|\nu_1\rangle, |\nu_2\rangle, \ldots, |\nu_n\rangle\}$ be the bases of the Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively. Let $|\mu\rangle = e_1|\mu_1\rangle + e_2|\mu_2\rangle + \ldots + e_m|\mu_m\rangle$ and $|\nu\rangle = f_1|\nu_1\rangle + f_2|\nu_2\rangle + \ldots + f_n|\nu_n\rangle$, respectively, be two elements of $\mathbb{H}_1$ and $\mathbb{H}_2$. Then with respect to the bases mentioned above, $|\mu\rangle$ and $|\nu\rangle$ are given by column vectors

$$|\mu\rangle \equiv \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{pmatrix} \quad \text{and} \quad |\nu\rangle \equiv \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

We consider the set $\{|\mu_1\nu_1\rangle, |\mu_2\nu_2\rangle, \ldots, |\mu_m\nu_n\rangle\}$, that is, $\{|\mu_i\nu_j\rangle : i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$. Then the linear space $\mathbb{H}$ consisting of vectors

$$\{|\Xi\rangle = \sum_{i,j} e_{ij}|\mu_i\nu_j\rangle, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\},$$

is the tensor product of $\mathbb{H}_1$ and $\mathbb{H}_2$ where the operations are

$$\begin{aligned} |\Xi\rangle &= \alpha|\Psi\rangle + \beta|\Omega\rangle \\ &= \alpha\left(\sum_{i,j} e_{ij}|\mu_i\nu_j\rangle\right) + \beta\left(\sum_{i,j} f_{ij}|\mu_i\nu_j\rangle\right) \\ &= \sum_{i,j} (\alpha e_{ij} + \beta f_{ij})|\mu_i\nu_j\rangle \end{aligned}$$

and the inner product is

$$\langle \Omega | \Psi \rangle \quad = \sum_{p,q} \sum_{i,j} f_{pq} e_{ij} \langle \nu_q u_p | \mu_i \nu_j \rangle$$

where

$$|\Psi\rangle = \sum_{i,j} e_{ij} |\mu_i \nu_j\rangle$$

and

$$|\Omega\rangle = \sum_{i,j} f_{ij} |\mu_i \nu_j\rangle.$$

The tensor product space is denoted by $H = \mathbb{H}_1 \otimes \mathbb{H}_2$.

Thus

$$|\mu v\rangle \quad \equiv \quad \begin{pmatrix} e_1 f_1 \\ e_1 f_2 \\ \vdots \\ e_m f_n \end{pmatrix}_{mn \times 1}.$$

It is computationally convenient to write this as

$$|\mu v\rangle \quad = \quad \begin{pmatrix} e_1 f \\ e_2 f \\ \vdots \\ e_m f \end{pmatrix}, \quad \text{where} \ f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

As an illustration, let us consider $\mathbb{H}_i = \mathbb{C}^2$, $i = 1, 2$ and specify the bases $(|0\rangle_1, |1\rangle_1)$ and $(|0\rangle_2, |1\rangle_2)$ for them, respectively.

Let $|\mu\rangle = \sqrt{\frac{2}{3}} |0\rangle_1 + \frac{1}{\sqrt{3}} |1\rangle_1$, and $|\nu\rangle = \frac{1}{\sqrt{2}} |0\rangle_2 - \frac{1}{\sqrt{2}} |1\rangle_2$.

Then

$$|\mu v\rangle = \sqrt{\tfrac{2}{3}}\,\tfrac{1}{\sqrt{2}}\,|00\rangle_{12} + \sqrt{\tfrac{2}{3}}\left(-\tfrac{1}{\sqrt{2}}\right)|01\rangle_{12} + \tfrac{1}{\sqrt{3}}\,\tfrac{1}{\sqrt{2}}\,|10\rangle_{12} + \tfrac{1}{\sqrt{3}}\left(-\tfrac{1}{\sqrt{3}}\right)|11\rangle_{12}$$

.

In this illustration,

$$|\mu v\rangle = \begin{pmatrix} \sqrt{\tfrac{2}{3}} \begin{pmatrix} \tfrac{1}{\sqrt{2}} \\ -\tfrac{1}{\sqrt{2}} \end{pmatrix} \\[2ex] \tfrac{1}{\sqrt{3}} \begin{pmatrix} \tfrac{1}{\sqrt{2}} \\ -\tfrac{1}{\sqrt{2}} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \tfrac{1}{\sqrt{3}} \\ -\tfrac{1}{\sqrt{3}} \\ \tfrac{1}{\sqrt{6}} \\ -\tfrac{1}{\sqrt{6}} \end{pmatrix} .$$

It is important to note that it is not always possible to express any vector $|\Xi\rangle \in \mathbb{H}_1 \otimes \mathbb{H}_2$ in the form $|\psi\rangle = |\mu\rangle \otimes |\nu\rangle$ for some $|\mu\rangle \in \mathbb{H}_1$ and $|\nu\rangle \in \mathbb{H}_2$. As an illustration, we can take $\mathbb{H}_1 = \mathbb{C}^2$, $\mathbb{H}_2 = \mathbb{C}^2$ and $\{|0\rangle_1, |1\rangle_1\}$ and $\{|0\rangle_2, |1\rangle_2\}$ as two bases of $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively. Then $|\Xi\rangle = |00\rangle + |11\rangle$ is such a vector of the above kind. These states are referred to as non-separable states. As we will see in the next chapter, such vectors describe 'quantum entanglement' which is the principal resource on which quantum communication theory stands.

The above concept of tensor product can be extended to any finite number of linear spaces like $\mathbb{H}_1 \otimes \mathbb{H}_2 \otimes \ldots \otimes \mathbb{H}_k$. If $n_1, n_2, \ldots, n_k$ are dimensions of $\mathbb{H}_1, \mathbb{H}_2, \ldots, \mathbb{H}_k$, respectively, then the dimension of their tensor product is $n_1.n_2 \ldots n_k$. As an illustration, let $\mathbb{H}_i = C^2, i = 1, 2, 3$. Then $\otimes_{i=1}^{3} \mathbb{H}_i = \mathbb{H}_1 \otimes \mathbb{H}_2 \otimes \mathbb{H}_3$ has dimension $2^3 = 8$. In this case $\otimes_{i=1}^{3} \mathbb{H}_i = (\mathbb{H}_1 \otimes \mathbb{H}_2) \otimes \mathbb{H}_3 = \mathbb{H}_1 \otimes (\mathbb{H}_2 \otimes \mathbb{H}_3)$, mathematically upto isomorphism. One (of several alternatives) basis consists of $\{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\}$. An example of a vector that is not separable is

$$|\mu\rangle = |000\rangle + |111\rangle$$

which is referred to as GHZ-state in the representation of a 3-qubit system.

1.4.2 TENSOR PRODUCTS OF OPERATORS

Here we investigate the action of operators on tensor products of Hilbert spaces. Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be two Hilbert spaces with bases $\{|\mu_1\rangle, |\mu_2\rangle, \ldots, |\mu_m\rangle\}$ and $\{|\nu_1\rangle, |\nu_2\rangle, \ldots, |\nu_n\rangle\}$, respectively. Then, any element of $\mathbb{H}_1 \otimes \mathbb{H}_2$ is expressible as

$$|\Xi\rangle = \sum_{ij} e_{ij}|i\rangle \otimes |j\rangle.$$

If L and M are two operators acting on $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively, then the tensor product $L \otimes M$ is defined by the equation

$$(L \otimes M)(\sum_{ij} e_{ij}|i\rangle \otimes |j\rangle) = \sum_{ij} e_{ij}L|i\rangle \otimes M|j\rangle.$$

It is possible to show that any linear operator O acting on $\mathbb{H}_1 \otimes \mathbb{H}_2$ can be written as a sum of tensor products of linear operators L_i acting on $\mathbb{H}_1$ and M_j acting on $\mathbb{H}_2$:

$$O = \sum_{ij} \gamma_{ij} L_i \otimes M_j.$$

The matrix representation of the operator $L \otimes M$ in the basis $|K\rangle \equiv |\mu_i\nu_j\rangle$, labeled by the single index $K = 1, 2, \ldots, mn$, is given by the $mn \times mn$ matrix.

$$L \otimes M = \begin{vmatrix} L_{11}M & L_{12}M & \cdots & AL1mM \\ L_{21}M & L_{22}M & \cdots & L_{2m}M \\ \vdots & \vdots & & \vdots \\ L_{m1}M & L_{m2}M & \cdots & L_{mm}M \end{vmatrix},$$

where the terms $L_{ij}M$ denote sub-matrices of size $n \times n$, with L and M being matrix representations of the operators L and M (L and M are $m \times m$ and $n \times n$ matrices, respectively).

1.5 TRACE

For any operator $L : \mathbb{H} \to \mathbb{H}$, where $\mathbb{H}$ is an n-dimensional Hilbert space,

$$Tr(L)(\equiv \text{Trace of L}) = \sum_{i=1}^{n} \langle \mu_i | L | \mu_i \rangle,$$

where $\{|\mu_i\rangle, i = 1, 2, \ldots, n\}$ is an orthonormal basis for $\mathbb{H}$.

The matrix representation of L indicates that the trace is $\sum_{i=1}^{n} L_{ii}$, which is the sum of all diagonal elements of the representation $L_{n \times n}$ of L.

1.5.1 INVARIANT OF TRACE

A result of profound influence in quantum mechanics is that for any operator $L : \mathbb{H} \to \mathbb{H}$, trace of L is independent of basis. This can be seen through the following argument.

Let $\{|\mu_i\rangle, i = 1, 2, \ldots, n\}$ and $\{|\nu_j\rangle, j = 1, 2, \ldots, n\}$ be two orthonormal bases for an n-dimensional Hilbert space $\mathbb{H}$.

Then in the basis $\{|\mu_i\rangle\}$, the trace of L is

$$Tr(L) = \sum_{i=1}^{n} \langle \mu_i | L | \mu_i \rangle$$

and in basis $\{|\nu_j\rangle\}$, the trace of L is

$$Tr(L) = \sum_{j=1}^{n} \langle \nu_j | L | \nu_j \rangle.$$

Now,

$$
\begin{aligned}
Tr(L) &= \sum_{j=1}^{n} \langle \nu_j | L | \nu_j \rangle \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} \langle \nu_j | \mu_i \rangle \langle \mu_i | L | \nu_j \rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \langle \mu_i | L | \nu_j \rangle \langle \nu_j | \mu_i \rangle \\
&= \sum_{i=1}^{n} \langle \mu_i | L | \mu_i \rangle.
\end{aligned}
$$

In the second step we recall the completeness relation

$$
\sum_{i=1}^{n} | \mu_i \rangle \langle \mu_i | = I.
$$

This indicates that the tr (L) is independent of basis.

We also have the following result:

$$
Tr(LM) = Tr(ML).
$$

Let L and M be two operator from an n-dimensional Hilbert space $\mathbb{H}$ to $\mathbb{H}$. Also let $\{|\mu_i\rangle, i = 1, \dots, n\}$ be an orthonormal basis for n-dimensional Hilbert space $\mathbb{H}$. Then

$$
Tr(L) = \sum_{i=1}^{n} \langle \mu_i | L | \mu_i \rangle.
$$

By applying the same relation for the basis, $\{|\mu_j\rangle\}$, we have

$$
\begin{aligned}
Tr(LM) &= \sum_i \langle \mu_i | LM | \mu_i \rangle \\
&= \sum_i \sum_j \langle \mu_i | L | \mu_j \rangle \langle \mu_j | M | \mu_i \rangle \\
&= \sum_j \sum_i \langle \mu_j | M | \mu_i \rangle \langle \mu_i | L | \mu_j \rangle \\
&= \sum_j \langle \mu_j | ML | \mu_j \rangle \\
&= Tr(ML)
\end{aligned}
$$

It is important to see that although $LM \neq ML$ in general, their traces have equal values.

1.5.2 PARTIAL TRACE

Let D be any operator acting on $\mathbb{H}_L \otimes \mathbb{H}_M$ where $\mathbb{H}_L$ and $\mathbb{H}_M$ are m and n dimensional Hilbert spaces, respectively. Then the partial trace of D over $\mathbb{H}_M$, denoted D_L, is given by

$$
D_L \equiv Tr_M D = \sum_j (I \otimes \langle j|) D (I \otimes |j\rangle),
$$

where $|j\rangle$ is any orthonormal basis for the Hilbert space $\mathbb{H}_M$.

From the above compactly written expression, if

$$
D = \sum_\alpha L_{1\alpha} \otimes L_{2\alpha}
$$

where $L_{1\alpha}$ and $L_{2\alpha}$ are operators on $\mathbb{H}_A$ and $\mathbb{H}_B$, respectively, we have,

$$
\begin{aligned}
D_L &= \sum_\alpha L_{1\alpha} \sum_J \langle j | L_{2\alpha} | j \rangle \\
&= \sum_\alpha L_{1\alpha} Tr(L_{2\alpha}).
\end{aligned}
$$

In particular, if $D = L_1 \otimes L_2$, then we have the partial trace $D_L = Tr(L_2). L_1$.

Similarly, the partial trace of D over $\mathbb{H}_L$, denoted D_M, is given by

$$D_M \equiv Tr_L D = \sum_i (\langle i| \otimes I) D(|i\rangle \otimes I),$$

where $|i\rangle$ is any orthonormal basis for the Hilbert space $\mathbb{H}_L$.

By taking a partial trace over $\mathbb{H}_M$, sometimes called 'tracing over $\mathbb{H}_M$' we exclude all the variables related to $\mathbb{H}_M$. It is an important operation, particularly in quantum communication protocols performed in noisy environments. We will be dealing with partial trace operations more in .

2 Classical Bits and Classical Gates

2.1 INTRODUCTION

In this chapter, a review of concepts of classical bits and gates. It is required for a correlation with the corresponding quantum concepts given in later chapters. References [54, 55, 147, 155] are useful for an in-depth study on the subjects of the chapter.

2.2 BITS AND BOOLEAN ALGEBRA

A bit or a classical bit as we call it here is the basic unit of classical information, which is realized in practice by different sorts of binary devices. We require classical bits in the communication protocols that we describe in the second part of the book for the purpose of classical assistance which is inevitably necessary in these protocols. A bit has two states that can be described by any two different symbols. In particular, they are often denoted by 0 and 1. The mathematical structure to describe and manipulate single bits is the Boolean Algebra. There are three basic operations on these two states which are given in the structure of a Boolean Algebra. Formally, a Boolean Algebra is an algebraic structure $(B, \vee, \wedge, ')$

where $B = \{0, 1\}$, $\vee$ and $\wedge$ are binary operations on B and '$'$' is a unitary operation on B described in Table 2.1.

Table 2.1

Truth tables for OR ($a \vee b$), AND ($a \wedge b$), and NOT (a') operations ⏎

a	b	$a \vee b$
0	0	0
0	1	1
1	0	1
1	1	1

$2.1\,(a)$

a	b	$a \wedge b$
0	0	0
0	1	0
1	0	0
1	1	1

$2.1\,(b)$

a	a'
0	1
1	0

$2.1\,(c)$

The general mathematical definition of Boolean Algebra is wider than the above. What is described above is to serve our purpose here.

2.3   CLASSICAL GATES

Logic gates are operations in which an n-bit input is provided to recover an m-bit output. This is given symbolically as $\{0,1\}^n \rightarrow \{0,1\}^m$. The above correspondence is also referred to as a Boolean operation. We describe some logical gates namely, NOT, AND, OR, and XOR gates, along with their symbols in the following figures.

1. **NOT gate:** A NOT gate inverts the input signal. It is a unary operator, which means that it operates on a single input. The truth table and circuit diagram for the NOT gate are shown in Figure 2.1.

2. **AND gate:** An AND gate performs the binary operation '∧' between two bits. The truth table and circuit diagram for the AND gate are shown in Figure 2.2.

3. **OR gate:** An OR gate performs the binary operation '∨' between two bits. The truth table and circuit representation for the OR gate are shown in Figure 2.3.

4. **XOR gate:** An XOR gate performs the operation $a \oplus b = a + b \,(mod\ 2)$. The corresponding truth table and circuit diagram for the 2-input XOR gate are given in Figure 2.4.

$$n = m = 1$$

a	a'
0	1
1	0
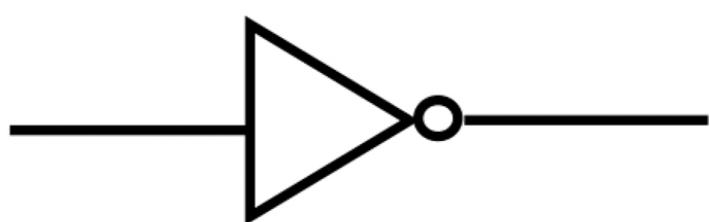
Figure 2.1   The truth table and symbol for NOT gate. ↵

$$n = 2, m = 1$$

a	b	a · b
0	0	0
0	1	0
1	0	0
1	1	1

Figure 2.2 The truth table and symbol for AND gate. ⏎

$$n = 2, m = 1$$

a	b	a ∨ b
0	0	0
0	1	1
1	0	1
1	1	1
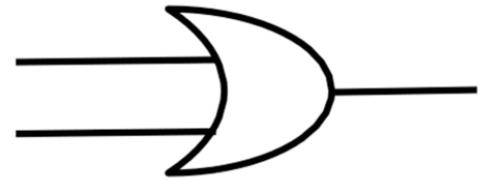
Figure 2.3 The truth table and symbol for OR gate. ⏎

$$n = 2, m = 1$$

a	b	$a \oplus b$ $= a + b (mod\ 2)$
0	0	0
0	1	1
1	0	1
1	1	0

Figure 2.4 The truth table and symbol for 2-input XOR gate. ⏎

2.4 UNIVERSAL GATES

A universal set of gates consists of those gates by which any Boolean operation $\{0,1\}^n \rightarrow \{0,1\}^m$ can be constructed. An example of such a set of universal gates is $\{NAND, NOR\}$. These two gates are described in the following two figures.

1. **NAND gate:** It is a combination of a NOT gate and an AND gate which performs the operation $(a \wedge b)'$ between two bits. The corresponding truth table and the circuit diagram for the NAND gate are given in Figure 2.5.

2. **NOR gate:** It is an combination of NOT gate and OR gate which performs the operation $(a \vee b)'$. The corresponding truth table and circuit diagram for the NOR gate are given in Figure 2.6.

$$n = 2, m = 1$$

a	b	$a \uparrow b$ $= (a \wedge b)'$
0	0	1
0	1	1
1	0	1
1	1	0
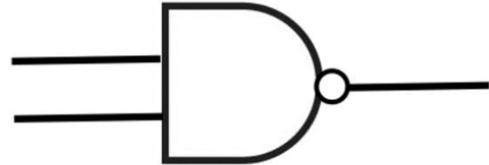
Figure 2.5 The truth table and symbol for NAND gate. ↵

$$n = 2, m = 1$$

a	b	$a \downarrow b = (a \vee b)'$
0	0	1
0	1	0
1	0	0
1	1	0
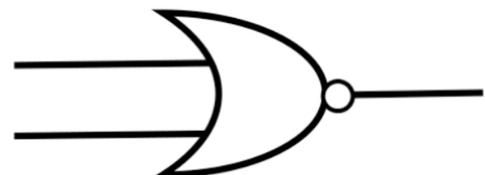
Figure 2.6 The truth table and symbol for NOR gate. ↵

It may be noted that the set of universal gates is not unique. There may be several collection of gates acting as set of universal gates.

3 Essentials From Quantum Physics

3.1 INTRODUCTION

This chapter contains concepts from quantum physics which are essential for the understanding of the teleportation protocols. Qubit system is discussed separately along with the laws of its evolution. Quantum gates and quantum circuits are described. The description is limited to the technical and conceptual requirements for understanding the materials in the book. For an extensive appraisal of the topics included in this chapter, [6, 12, 15, 32, 35, 55, 77, 81, 107, 113, 117, 134, 139, 140, 149, 163, 171] are helpful references.

3.2 POSTULATES OF QUANTUM MECHANICS

In this section we describe the basic principles of quantum mechanics in the form of postulates. This approach is conventional in the study of quantum mechanics.

Postulate 1:

The state of a quantum system is described by a ket vector which is an element $|\aleph\rangle$ of an appropriate Hilbert space $\mathbb{H}$. The space $\mathbb{H}$ is a mathematical description of the quantum system. A state of a quantum system is described by a ket $|\aleph\rangle$ up to a non-zero scalar multiple, that is, $|\aleph\rangle$ and $c|\aleph\rangle$ indicate the same quantum state. The zero ket does not represent any quantum state. If there are two quantum states $|\aleph_1\rangle$ and $|\aleph_2\rangle$, then ($\mathfrak{g}_1|\aleph_1\rangle + \mathfrak{g}_2|\aleph_2\rangle$)) is also a state for arbitrary choices of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ provided not both are zero. If a state $|\aleph\rangle$ is such that $\langle\aleph|\aleph\rangle = 1$, then we say that $|\aleph\rangle$ is normalized state.

In particular, $|\aleph\rangle$ and $e^{i\phi}|\aleph\rangle$ denote the same state which is that in ket representation the phase factor $e^{i\phi}$ is ignored. Physically it means that the phase has no observable consequences.

Postulate 2:

For every observable physical attribute pertaining to a physical system there exists a Hermitian operator on the Hilbert space describing that system whose eigenvalues are the possible observed values for the physical observable.

It follows that the eigenvalues of a Hermitian operator being always real, the observed values of an attribute of the physical system is also real.

Our consideration is limited to systems described by finite-dimensional Hilbert space in which case we have a set of distinct eigenvectors $\{|\varrho_1\rangle, |\varrho_2\rangle, \ldots, |\varrho_n\rangle\}$ of the Hermitian operator corresponding to the physical observable which forms an orthonormal basis of the Hilbert space.

Postulate 3:

Measurement in a quantum system described by a Hilbert space $\mathbb{H}$ is described by a set of operators $\{M_1, M_2, \ldots, M_k\}$ satisfying the completeness relation

$$\sum_{i=1}^{k} M_i^\dagger M_i = I,$$

where 'i' refers to the possible measurement outcome. The operators $M_i's$ are called measurement operators. If the system is given by $|\aleph\rangle$ assumed to be normalized, and the measurement is performed on $|\aleph\rangle$, then the outcome 'i' occurs with probability given by

$$p(i) = \langle\aleph|M_i^\dagger M_i|\aleph\rangle$$

in which case the state after the measurement reduces to $M_i|\aleph\rangle$.

The above is the most general description of quantum measurement. We will require mainly projective measurement in our protocols, which is elaborately described in the subsequent section. Occasionally, we will require Positive Operator Valued Measurement or POVM operators.

In a POVM measurement the corresponding set of operators $\{Q_1, \ldots, Q_n\}$ need not be idempotent as in the case of projective measurement described subsequently. They are supposed to satisfy the following assumptions for all i,

1. $Q_i = Q_i^{\dagger}$ (Hermitian)

2. $Q_i \geq 0$ (Positive semi-definite)

3. $\sum_{i=1}^{n} Q_i = I$

If $\mathbb{H}$ describes a quantum system, and $|\Xi\rangle, |\Omega\rangle$ are two arbitrary ket vectors in $\mathbb{H}$, then

$$Q_1 = (I - |\Xi\rangle\langle\Xi|)$$
$$Q_2 = (I - |\Omega\rangle\langle\Omega|)$$
$$Q_3 = I - Q_1 - Q_2$$

forms a specific example of the above type measurement.

If $\{|\varrho_1\rangle, |\varrho_2\rangle, \ldots, |\varrho_n\rangle\}$ forms an orthonormal basis of the Hilbert space $\mathbb{H}$ describing a system, then $\{|\varrho_1\rangle\langle\varrho_1|, |\varrho_2\rangle\langle\varrho_2|, \ldots, |\varrho_n\rangle\langle\varrho_n|\}$ describes a set of measurement operators. A corresponding measurement is also known as measurement in the basis $\{|\varrho_1\rangle, |\varrho_2\rangle, \ldots, |\varrho_n\rangle\}$. We will discuss more about it in the context of projective measurements in Section 3.4.

By a closed quantum system we mean a system which is free from interaction with the outside environment.

Postulate 4:

The time evolution of a closed quantum system is unitary which means that whenever a quantum system is specified by a ket $|\aleph(t_0)\rangle$ at time t_0 and by a ket $|\aleph(t_1)\rangle$ at time $t_1 > t_0$, both belonging to the Hilbert space $\mathbb{H}$ describing the system, there exists a unitary operator $U(t_1, t_0)$ on $\mathbb{H}$ such that

$$|\aleph(t_1)\rangle = U(t_1, t_0)|\aleph(t_0)\rangle.$$

From the above postulate it follows that we can only apply unitary operators to transform a closed quantum system state to some other state. If it is impossible to construct such an operator for a proposed transformation, then it is impossible to physically carry out that transformation. One important implication of the above rule is that the inner product of the

ket vectors are not altered, that is, under the unitary evaluation $|\aleph\rangle \rightarrow U|\aleph\rangle$ of the system, when $|\tau_1\rangle$ evolves to $|\tau_1{}'\rangle = U|\tau_1\rangle$ and $|\tau_2\rangle$ evolves to $|\tau_2{}'\rangle = U|\tau_2\rangle$. We have

$$\langle \tau_2{}'|\tau_1{}'\rangle = \langle \tau_2|U^\dagger U|\tau_1\rangle = \langle \tau_2|\tau_1\rangle.$$

The above observation has important consequences in quantum mechanics.

It is immediate from the postulates of quantum mechanics that the measurement of an observable inherently produces uncertain results having a probability distribution.

The expectation value (which is the average value in the probabilistic situation) of an observable A when the system is in the normalized state $|\aleph\rangle$ is given by

$$\langle A \rangle = \langle \aleph|A|\aleph\rangle.$$

This is demonstrated in the following case where A has eigenvalues ϱ_i with corresponding normalized eigenvectors $|\varrho_i\rangle, i = 1, 2, \ldots, n$ assumed for simplicity to be all distinct. Then

$$A = \sum_{i=1}^{n} \varrho_i |\varrho_i\rangle\langle\varrho_i|.$$

Let $|\aleph\rangle = \sum_{i=1}^{n} c_i |\varrho_i\rangle$ where $\sum_{i=1}^{n} |c_i|^2 = 1$. Therefore

$$\langle A \rangle = \sum_{i=1}^{n} \varrho_i |c_i|^2.$$

The uncertainty associated with A is given by the fact that the measurement of A is associated with the set of measurement operators $\{|\varrho_i\rangle\langle\varrho_i|, i = 1, \ldots, n\}$ which yields the value ϱ_i with probability.

The uncertainty in the measurement of the observable A is defined as

$$\triangle A = \langle (A - \langle A\rangle)^2 \rangle^{\frac{1}{2}},$$

where the expectation value is described above.

The uncertainties pertaining to the measured values of two operators A and B are related by the relation

$$\triangle A . \triangle B \geq \frac{|\langle\aleph|(AB - BA)|\aleph\rangle|}{2}.$$

The above result is the famous Heisenberg's uncertainty principle. We will not explicitly use it anywhere in this text. But it is to be kept in mind that this principle is present implicitly in the backdrop of every discussion on quantum mechanics.

3.3 THE QUBIT SYSTEM

A qubit is the simplest quantum system, which is described by a 2-dimensional Hilbert space. It is the quantum counterpart of the classical bit, which is popularly known as 'bit' amongst computer scientists. Qubits are structurally fundamental blocks of quantum information. The basic difference between a 'bit' and a 'qubit' is that whereas a bit can assume one of two given values, customarily written as 0 and 1, a qubit can be in a combination (superposition) of two states producing one of the two states with certain probabilities only when observed. A 'bit' can be described by a 'Boolean algebra' $\{\{0,1\}, +, .\}$ with two algebraic operations whereas a 'qubit' can be described by a linear space (of dimension 2) where superposition of the elements is allowable. This is why the mathematical treatment of 'bit' and 'qubit' are different. The physical realizations of these two concepts also differ accordingly. In the case of a 'bit', two-level devices like on-off switches, two-level voltage systems, etc. are sufficient for a physical realization. In the case of 'qubit' physical systems admitting of superposition are required. They include polarization states of a photon, spin states of a spin-$\frac{1}{2}$ particle, ground and first excited states of an atom, etc. From the standpoint of technology, physical realization and maneuvering of qubits are more complicated than bits.

3.3.1 SINGLE-QUBIT AND ITS REPRESENTATION

As an element of the 2-dimensional Hilbert space $\mathbb{H}_2$ or equivalently $\mathbb{C}^2$, a qubit is described by

$$|\aleph\rangle = \mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle$$

where $\{|0\rangle, |1\rangle\}$ is an orthonormal basis of $\mathbb{H}_2$, where $\mathfrak{g}_1, \mathfrak{g}_2$ are complex numbers.

Since the representation of a quantum system is given by a 'ket' vector up to a scalar multiplication (see [Section 3.1](#)), there is no loss of generality to assume that $|\mathfrak{g}_1|^2 + |\mathfrak{g}_2|^2 = 1$, in which case we say that the state $|\aleph\rangle$ is normalized.

Further, from the above consideration, $|\aleph\rangle$ is also independent of an overall (unobservable) phase factor. This allows us to represent a qubit as

$$|\aleph\rangle = cos(\frac{\kappa}{2})|0\rangle + e^{i\phi}sin(\frac{\kappa}{2})|1\rangle$$

when $0 \leq \kappa, \phi \leq 2\pi$.

The above expression of an arbitrary qubit state $|\aleph\rangle$ allows us to represent it on a sphere called the Bloch sphere. Figure 3.1 shows the Bloch sphere representation of a single-qubit pure state.
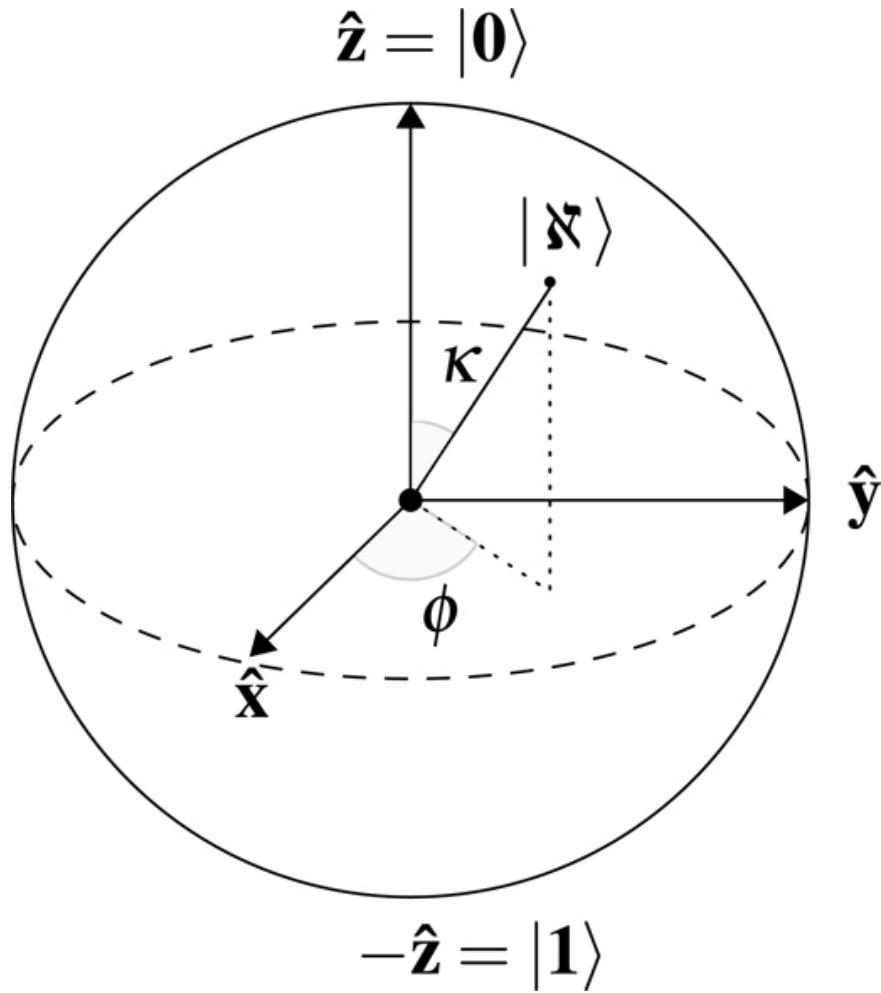
Figure 3.1 Single qubit state representation using Bloch sphere. ↵

It is important to observe that the Bloch sphere should not be confused with the usual 3-dimensional the sphere. For instance, the orthogonal states $|0\rangle$ and $|1\rangle$ are represented by

points along $z - axis$, but in opposite directions which is not the case with a 3-dimensional sphere.

3.3.2 SYSTEMS CONSISTING OF *N*-QUBITS

Let there be n qubits represented by n number of 2-dimensional Hilbert spaces $\mathbb{H}_1, \mathbb{H}_2, \ldots, \mathbb{H}_n$ having bases $\{|0\rangle_1, |1\rangle_1\}, \{|0\rangle_2, |1\rangle_2\}, \ldots, \{|0\rangle_n, |1\rangle_n\}$, respectively. Then the composite system of these n-qubits is described by $\mathbb{H}_1 \otimes \mathbb{H}_2 \otimes \cdots \otimes \mathbb{H}_n$.

It has an orthonormal basis consisting of 2^n elements given by

$$\{|j_1 \ldots j_n\rangle : j_i = 0, 1; \ i = 1, 2, \ldots, n\}.$$

The above basis is known as computational basis.

A state of the composite system is given by a linear combination of the 2^n states mentioned above. Thus an arbitrary $n-$ qubit state is given by

$$|\aleph\rangle = \sum_{j_1=1}^{2} \sum_{j_2=1}^{2} \cdots \sum_{j_n=1}^{2} a_{j_1 \ldots j_n} |j_1 \ldots j_n\rangle.$$

If $\sum_{j_1=1}^{2} \sum_{j_2=1}^{2} \cdots \sum_{j_n=1}^{2} |a_{j_1} \cdots j_n|^2 = 1$, then the state is said to be normalized. After performing a measurement on the basis mentioned above, the state $|j_{s_1} \cdots j_{s_n}\rangle$ is obtained with probability $|a_{j_{s_1} \cdots j_{s_n}}|^2$.

As an illustration, in the case of $n = 3$, the computational basis of a 3-qubit system is $\{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\}$. A state of the composite 3-qubit system, for instance, described by

$$\mathfrak{g}_1|000\rangle + \mathfrak{g}_2|101\rangle + \mathfrak{g}_3|110\rangle.$$

For a two-qubit system the most general two-qubit state is given by

$$|\aleph\rangle = \mathfrak{g}_1|00\rangle + \mathfrak{g}_2|01\rangle + \mathfrak{g}_3|10\rangle + \mathfrak{g}_4|11\rangle.$$

The state is normalized when the condition $|\mathfrak{g}_1|^2 + |\mathfrak{g}_2|^2 + |\mathfrak{g}_3|^2 + |\mathfrak{g}_4|^2 = 1$ is satisfied.

If we measure the system in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, we obtain the state $|00\rangle$ with probability $|\mathfrak{g}_1|^2$, $|01\rangle$ with probability $|\mathfrak{g}_2|^2$, $|10\rangle$ with probability $|\mathfrak{g}_3|^2$ and $|11\rangle$ with probability $|\mathfrak{g}_4|^2$.

Of particular importance are the following 2-qubit states and 3-qubit states known as Bell states given by

$$|\Upsilon_1\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle,$$

$$|\Upsilon_2\rangle = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle,$$

$$|\Upsilon_3\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle,$$

$$|\Upsilon_4\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle.$$

(3.1)

and 3-qubit states known as Greenberger–Horne–Zeilinger (GHZ)-states given by

$$|\varsigma_1\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}, \qquad |\varsigma_2\rangle = \frac{|000\rangle - |111\rangle}{\sqrt{2}},$$

$$|\varsigma_3\rangle = \frac{|001\rangle + |110\rangle}{\sqrt{2}}, \qquad |\varsigma_4\rangle = \frac{|001\rangle - |110\rangle}{\sqrt{2}},$$

$$|\varsigma_5\rangle = \frac{|010\rangle + |101\rangle}{\sqrt{2}}, \qquad |\varsigma_6\rangle = \frac{|010\rangle - |101\rangle}{\sqrt{2}},$$

$$|\varsigma_7\rangle = \frac{|011\rangle + |100\rangle}{\sqrt{2}}, \qquad |\varsigma_8\rangle = \frac{|011\rangle - |100\rangle}{\sqrt{2}}$$

(3.2)

The above-mentioned states (3.1) constitute the Bell basis which is an orthonormal basis for the 2-qubit system.

3.3.3   EVOLUTION OF A QUBIT SYSTEM

As in the general case of a quantum system, the evolution of an n-qubit system is realized by a unitary operator on the system. The evolution is given schematically as

$$|\aleph\rangle \rule{2cm}{0.4pt} \boxed{U} \rule{2cm}{0.4pt} U|\aleph\rangle$$

Since U^{-1} exists for a unitary operator, the evolution is always reversible. Thus any operation on an (isolated) n-qubit system is always reversible. We will discuss the other case where the system is non-isolated (noisy) in the next chapter.

Evolution of 1-qubit

Particularly we have the evolution of 1-qubit by application of three Pauli operators given by

$$\vartheta_x = |0\rangle\langle 1| + |1\rangle\langle 0|,$$
$$\vartheta_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|,$$
$$\vartheta_z = |0\rangle\langle 0| + |1\rangle\langle 1|.$$

In the matrix representation with respect to the basis $\{|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$, we have the usual form of Pauli matrices

$$X = \vartheta_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$Y = \vartheta_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$
$$Z = \vartheta_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These are referred to as Pauli gates or X-gate, Y-gate and Z-gate, respectively. Throughout the book we extensively use the above notations of Pauli operators and Pauli matrices.

The identity operator

$$I = |0\rangle\langle 0| + |1\rangle\langle 1| \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is trivially a unitary operator.

Also we have the Hadamard gate given by

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),$$
$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle),$$

which has the matrix representation in the computational basis as

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The phase-shift gate is defined as

$$R_z(\delta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{pmatrix}.$$

Considering the representation of the 1-qubit state in Section 3.2.1, we have

$$R_z(\delta) \begin{pmatrix} \cos\frac{\kappa}{2} \\ e^{i\phi}\sin\frac{\kappa}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\kappa}{2} \\ e^{i(\phi+\delta)}\sin\frac{\kappa}{2} \end{pmatrix}.$$

It indicates the change in the relative phase.

The above are some examples of operators which describe 1-qubit evolution.

Evolution of qubit systems:

We consider a system of n-qubits $q_1, q_2, \ldots, q_n$ individually represented by Hilbert spaces $\mathbb{H}_1, \mathbb{H}_2, \ldots, \mathbb{H}_n$. Then the composite system is given by $\mathbb{H}_1 \otimes \mathbb{H}_2 \otimes \cdots \otimes \mathbb{H}_n$. The evolution of the composite system is described by $\aleph \to U\aleph$ where U is a unitary operator. In particular, if $U_1, U_2, \ldots, U_n$ are unitary evolution operators for the qubits $q_1, q_2, \ldots, q_n$, respectively, then the evolution of the composite system is given by

$$\aleph \to (U_1 \otimes U_2 \otimes \cdots \otimes U_n)\aleph.$$

As an illustration, we consider the Bell-state $|\Upsilon_3\rangle_{q_1 q_2} = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)_{q_1 q_2}$. When the qubit q_1 is operated with ϑ_x and the qubit q_2 is operated with ϑ_z, then the Bell-state $|\Upsilon_3\rangle_{q_1 q_2}$ evolves into

$$\begin{aligned} (\vartheta_x \otimes \vartheta_z)(\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)) &= \frac{1}{\sqrt{2}}(\vartheta_x|1\rangle \otimes \vartheta_z|0\rangle + \vartheta_x|0\rangle \otimes \vartheta_z|1\rangle) \\ &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes (-)|1\rangle) \\ &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle). \end{aligned}$$

3.4   THE RELATION BETWEEN BITS AND QUBITS

Both bits and qubits when measured yield one of the two possible states usually denoted by 0 and 1 for bits and $|0\rangle$ and $|1\rangle$ for the case of qubits. The difference is that a bit is in a definite state 0 or 1 at any time (and hence prior to the measurement) and a measurement on it can be performed without affecting the state of the qubit, whereas a measurement on a qubit (generally) affects its state, yielding $|0\rangle$ or $|1\rangle$ with some probability. Prior to the measurement a qubit is in a superposed state of $|0\rangle$ and $|1\rangle$. The mathematics of superposition is required to describe a qubit which is supplied by a Hilbert space. From a mathematical point of view this is fundamental in understanding the difference between a bit and a qubit. Further, the dynamics of a qubit is subject to the laws of quantum mechanics. As a consequence, the changes (except by measurement) of qubit systems are unitary in general which are reversible. Along with qubits, classical bits are also necessary participants in teleportation processes.

3.5   PROJECTIVE MEASUREMENT

A measurement is a Projective measurement of $M_i's$ are projection operator with $M_i M_j = 0$ if $i \neq j$.

As an illustration, we consider a spin-$\frac{1}{2}$ system described by a 2-dimensional Hilbert space $\mathbb{H}_2$ having as basis elements $|\uparrow\rangle$ and $|\downarrow\rangle$, physically describing spin up and spin down states with respect to a fixed direction in space. Let us consider a projection measurement $\{M_1, M_2\}$ where $M_1 = |\uparrow\rangle\langle\uparrow|$ and $M_2 = |\downarrow\rangle\langle\downarrow|$ is performed on the state $|\aleph\rangle = \mathfrak{g}_1|\uparrow\rangle + \mathfrak{g}_2|\downarrow\rangle$. Then we obtain spin up as our measurement result with probability $\langle\aleph|(|\uparrow\rangle\langle\uparrow|)(|\uparrow\rangle\langle\uparrow|)|\aleph\rangle = |\mathfrak{g}_1|^2$. In this case the state $|\aleph\rangle$ of the system reduces to $\mathfrak{g}_1|\uparrow\rangle$ which, when normalized, is the same as $\frac{\mathfrak{g}_1}{|\mathfrak{g}_1|^2}|\uparrow\rangle$.

By a similar consideration we obtain spin down as the measurement result with probability $|\mathfrak{g}_2|^2$ with the state $|\aleph\rangle$ being reduced to $\frac{\mathfrak{g}_2}{|\mathfrak{g}_2|^2}|\downarrow\rangle$.

We sometimes talk of measuring in a basis of the Hilbert space corresponding to the quantum system. This practice is very often adopted in this book. The following is the explanation of the above.

If $\{|\varrho_1\rangle, |\varrho_2\rangle, \ldots, |\varrho_n\rangle\}$ is an orthonormal basis of $\mathbb{H}$, then $\{M_1, M_2, \ldots, M_n\}$ with $M_i = |\varrho_i\rangle\langle\varrho_i|$ constitutes a set of Projective measurement operators. Measuring with them

is referred to as measuring in the basis $\{|\varrho_1\rangle, |\varrho_2\rangle, \ldots, |\varrho_n\rangle\}$.

A Hermitian operator A has associated with it an orthonormal basis $\{|\varrho_1\rangle, |\varrho_2\rangle, \ldots, |\varrho_n\rangle\}$ consisting of its distinct eigenvectors. Conversely, given an orthonormal basis, we can always associated A Hermitian operator, that is to say, a physical observable. Thus to measure an observable is the same as measuring in a basis.

3.6 QUANTUM GATES AND CIRCUITS

Quantum gates are unitary operators which act on the quantum states. One important difference between a quantum and a classical gate is that the action of most classical gates is irreversible, but for quantum gates, it is reversible. Quantum gates are represented by unitary operators on Hilbert spaces. Quantum gates are constituting elements of quantum circuits. They are sometimes (but not always) counterparts of Boolean gates in the Boolean circuit theory. Further discussion on their extremely important utility will drift us away from our objective in this book. We restrict ourselves to the extent to which they are used in teleportation protocols we discuss in the present context. Nevertheless they are relevant in the fabrication of the quantum entanglements we use as quantum channels in the protocols. We will mention in the passing how entanglement can be generated by the application of quantum circuits.

By their very definitions the quantum gates are reversible quantum operations. This reversibility is crucial in quantum information theory.

3.6.1 UNITARY GATES

In the matrix representation, a unitary gate U acting on a n-qubit system is a $2^n \times 2^n$ unitary matrix. Unitary matrices preserve the norm of the quantum state which is crucial for maintaining the probabilistic interpretation of quantum mechanics. When a unitary gate U acts on an arbitrary state $|\aleph\rangle$ of a qubit, the transformed quantum state $|\aleph'\rangle$ is given by

$$|\aleph'\rangle = U|\aleph\rangle.$$

Further, the existence of U ensures the existence of U^{-1}, that is,

$$|\aleph\rangle = U^{-1}|\aleph'\rangle.$$

It has the following visual representation:
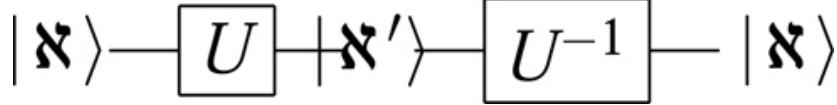
A general unitary gate U acting on a single qubit is capable of being represented as a 2×2 complex unitary matrix. Given the constraint $U^\dagger U = I$, such a matrix can always be written as:

$$U = \begin{pmatrix} g_1 & g_2 \\ -g_2^* & g_1^* \end{pmatrix},$$

where the condition $|g_1|^2 + |g_2|^2 = 1$ is satisfied.

The following are some types of quantum gates:

1. **Pauli Gates:** There are three Pauli gates which are Pauli-X gate, Pauli-Y gate, and Pauli-Z gate. These are single-qubit gates and perform rotations around the Bloch sphere's X, Y, and Z axes, respectively. The Pauli gates and their gate notations are the following:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \boxed{X},$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \boxed{Y},$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \boxed{Z}.$$

Any 2×2 unitary matrix can be expressed as a linear combination of Pauli gates and the identity gate. Also, any rotation on the Bloch sphere can be shown as composed of Pauli matrices.

2. **Hadamard Gate:** It is a single-qubit gate transforming the states $|0\rangle$ and $|1\rangle$ to $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, respectively. Its action creates an equal superposition of $|0\rangle$ and $|1\rangle$. The matrix representation and the gate notation are given respectively as

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \boxed{H}.$$

3. **Phase Gates:** It creates a phase shift of qubits. An instance of a phase gate is the S gate which adds a phase of $\frac{\pi}{2}$ to the quantum state $|1\rangle$ while keeping the state $|0\rangle$ unchanged. The matrix representation and notation of the S gate are

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad -\boxed{S}-.$$

4. **Controlled gates:** Controlled gates are two-qubit gates that act with a control qubit. A qubit is called target qubit whose state depends on the state of another qubit known as the control qubit.

One example of Controlled gates is the Controlled-NOT(CNOT) gate, which flips the state of the target qubit if the control qubit is in $|1\rangle$ state, and keeps it unchanged otherwise. Matrix representation and notation of CNOT gate are given as

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

Another example of a controlled gate is the controlled-Z gate, or (CZ) gate, which is the following

$$CZ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

5. **SWAP Gate:** The SWAP gate is a two-qubit gate whose action is to exchange the states of two qubits. Its matrix representation and notation are the following

$$SWAP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

3.6.2 QUANTUM CIRCUIT

A quantum circuit is a quantum analogue of a classical circuit. It represents a process performed on qubits. A circuit diagram represents operations sequentially performed on qubits. It consists of qubits, quantum gates, measurements, and wires for connecting qubits to other components.

Typically, an illustrative quantum circuit for entangled state $\frac{1}{2\sqrt{2}}(|00010\rangle + |00100\rangle + |11010\rangle + |11100\rangle + |00011\rangle - |00101\rangle - |11011\rangle + |11101\rangle)$ is given in Figure 3.2.
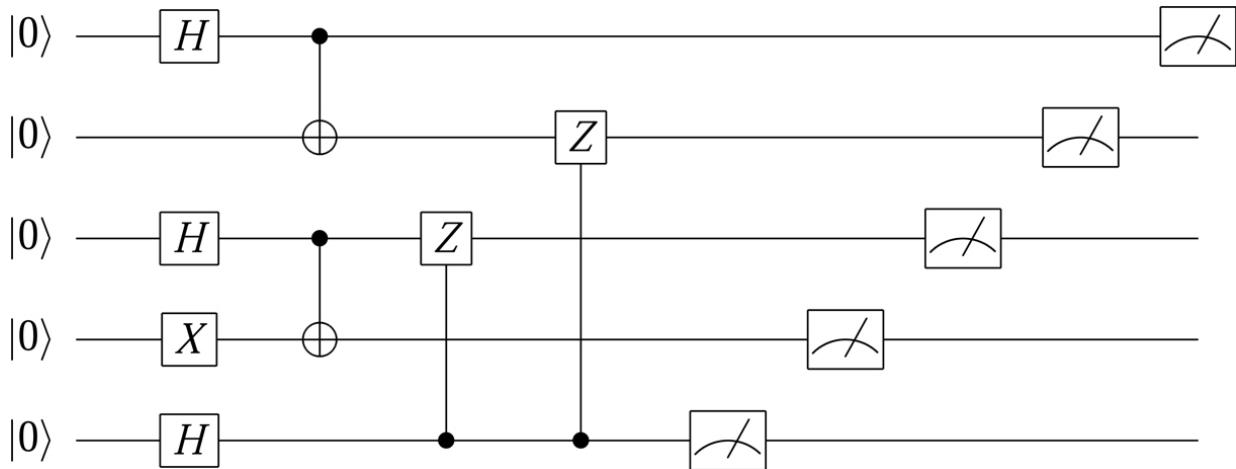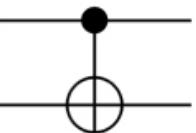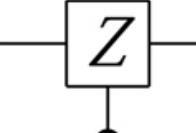
Figure 3.2 Circuit generation of entangled state. ↵

In the Figure 3.2,

$-\boxed{H}-$ stands for Hadamard gate

$-\boxed{X}-$ stands for Pauli X-gate

stands for controlled-NOT gate

stands for controlled-Z gate

stands for measurement operator

Quantum circuits for several purposes are constructed in the following chapters.

4 Entanglement

4.1 INTRODUCTION

This chapter deals with the concept of quantum entanglement which is the most precious resource in quantum communication science. Some topics on entanglement including construction of circuits for entanglement generation are presented. The materials presented here are limited by their requirements in the protocols presented here. Books and review articles [36, 37, 106, 113, 134, 139, 140] contain different aspects of quantum entanglement.

4.2 QUANTUM CORRELATION

Quantum correlations are fundamental concepts for the understanding of the phenomena of quantum entanglement. Two quantum systems can have correlation even if they are separated by arbitrarily large distances. This is completely quantum in nature having no corresponding classical counterpart. The concept of such correlation first appeared in the famous EPR paper by Einstein, Podolsky, and Rosen published in 1935, albeit in a different context. It is nonlocal in nature. It is the central theme in use in the domain of quantum technology. Particularly, entanglement forms the main quantum resource in quantum communication schemes like the teleportation protocol.

Entanglement is inseparability between two quantum systems. If two quantum systems A and B are represented by Hilbert spaces $\mathbb{H}_A$ and $\mathbb{H}_B$, then the state of

the composite system AB is given by $|\Xi\rangle \in \mathbb{H}_A \otimes \mathbb{H}_B$. If it is impossible to express

$$|\Xi\rangle = |\Xi_1\rangle_A \otimes |\Xi_2\rangle_B$$

for $|\Xi_1\rangle_A \in \mathbb{H}_A$ and $|\Xi_2\rangle_B \in \mathbb{H}_B$, then the system is entangled and the state $|\Xi\rangle$ is called an entangled state.

As an illustration, we consider two qubits A and B such that the composite 2-qubit system is given by one of the Bell states described in Section 3.2.2,

$$|\Upsilon_1\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B).$$

It is an entangled state of two qubits which can be seen in the following:

If possible, let $|\Upsilon_1\rangle_{AB} = |\Upsilon_{1A}\rangle \otimes |\Upsilon_{1B}\rangle$ where $|\Upsilon_{1A}\rangle = \mathfrak{g}_1|0\rangle_A + \mathfrak{g}_2|1\rangle_A$ and $|\Upsilon_{1B}\rangle = \mathfrak{g}_3|0\rangle_B + \mathfrak{g}_4|1\rangle_B$.

Then $|\Upsilon_1\rangle_{AB} = \mathfrak{g}_1\mathfrak{g}_3|0\rangle_A|0\rangle_B + \mathfrak{g}_1\mathfrak{g}_4|0\rangle_A|1\rangle_B + \mathfrak{g}_2\mathfrak{g}_3|1\rangle_A|0\rangle_B + \mathfrak{g}_2\mathfrak{g}_4|1\rangle_A|1\rangle_B$.

Comparing the two expressions of $|\Upsilon_1\rangle_{AB}$, we have $\mathfrak{g}_1\mathfrak{g}_3 = \frac{1}{\sqrt{2}}$, $\mathfrak{g}_1\mathfrak{g}_4 = 0$, $\mathfrak{g}_2\mathfrak{g}_3 = 0$ and $\mathfrak{g}_2\mathfrak{g}_4 = \frac{1}{\sqrt{2}}$. The above four equations are inconsistent, implying thereby that the state $|\Upsilon_1\rangle_{AB}$ is an entangled state. It can be similarly proved that all four Bell states are entangled.

4.3   MULTI-QUBIT ENTANGLED STATES

A multipartite system is a combination of more than two individual systems. If these are $p(> 2)$ systems described through Hilbert spaces $\mathbb{H}_1, \ldots, \mathbb{H}_p$, then the composite of these systems is a multipartite systems which is described by $H = \mathbb{H}_1 \otimes \ldots \otimes \mathbb{H}_p$. A state of the system $|\Xi\rangle$ is a member of H. If it is impossible to write (mathematically upto isomorphism)

$$|\Xi\rangle = |\Xi_1\rangle \otimes |\Xi_2\rangle$$

where $|\Xi_i\rangle$ belongs to the tensor product of n_i number of Hilbert spaces collected from $\mathbb{H}_1, \ldots, \mathbb{H}_n$, being all distinct, $i = 1, 2$ and $n_1 + n_2 = n$, then we have a multipartite entangled state $|\Xi\rangle$. It may be noted that some constituent states of $|\Xi\rangle$ may have entanglement amongst themselves. As an illustration, a 3-qubit state

$$|0\rangle_1 \otimes |0\rangle_2 \otimes |1\rangle_3 + |0\rangle_1 \otimes |1\rangle_2 \otimes |0\rangle_3 + |1\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3$$

customarily written as $|001\rangle + |010\rangle + |100\rangle$, is an entangled state.

On the contrary, the state of 3-qubits

$$|\Xi\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes |1\rangle_3 + |0\rangle_1 \otimes |1\rangle_2 \otimes |0\rangle_3 + |1\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3$$

is not entangled since we can write $|\Xi\rangle = |\Xi_1\rangle \otimes |\Xi_2\rangle$ where $|\Xi_1\rangle = |0\rangle_1 \otimes |1\rangle_2 + |1\rangle_1 \otimes |0\rangle_2$ and $|\Xi_2\rangle = |0\rangle_3$.

It may be noted that the part $|\Xi_1\rangle$ is an entangled state which is an unnormalized Bell state.

4.4 MAXIMALLY ENTANGLED STATES

There are several measures of entanglement on the basis of which we can say whether a state is more entangled than the other. In particular, we can speak of a maximally entangled state. In this section, we only discuss the issue of maximal entanglement in a bipartite system based on the Schmidt decomposition. For the concept of maximal entanglement in a multipartite system, we require density matrices. This part will be taken up in Chapter 5.

If there are two quantum systems 1 and 2 described mathematically by Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$ of dimensions d_1 and d_2, respectively, then for a state $|\Xi\rangle \in \mathbb{H}_1 \otimes \mathbb{H}_2$ it is possible to find bases $\{|\mu_1\rangle, \ldots, |\mu_{d_1}\rangle\}$ and $\{|\nu_1\rangle, \ldots, |\nu_{d_2}\rangle\}$ of $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively such that

$$|\Xi\rangle = \sum_{i=1}^{d} c_i |\mu_i\rangle \otimes |\nu_i\rangle$$

where $d = min\{d_1, d_2\}$.

The state is entangled only if more than one of c_i s are non-zero.

The state $|\Xi\rangle$ is maximally entangled if $c_1 = \ldots = c_d = \frac{1}{d}$ where $|\Xi\rangle$ is normalized.

It may be immediately seen that the Bell states are maximally entangled in view of the above consideration.

4.5   CIRCUITS FOR ENTANGLEMENT GENERATION

In this section, we present some quantum circuits for generating entangled states, as shown in Figures 4.1 to 4.4. The constructions of circuits are self-explanatory. We explain the generation through the circuits in Figure 4.4 as a representative case.
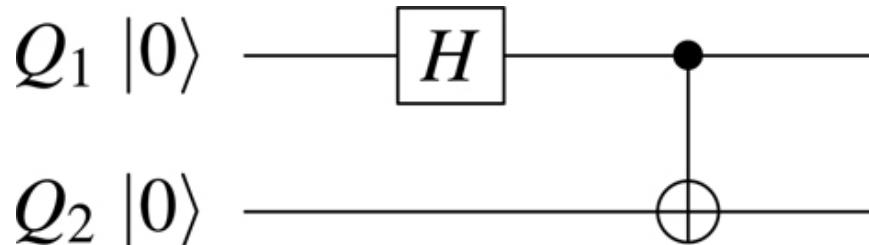
Figure 4.1   Circuit diagram of Bell-state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. ↵
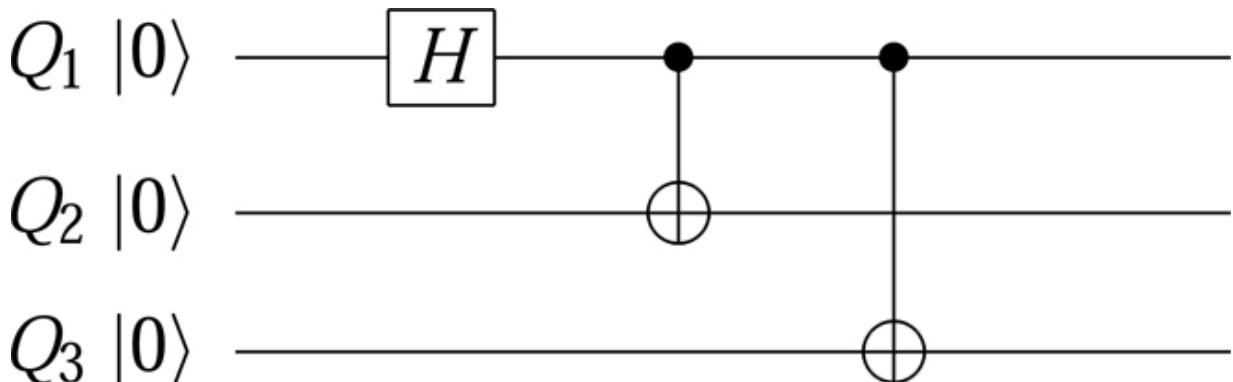
Figure 4.2   Circuit diagram of GHZ state $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$.

▶ Long Description for Figure 4.3
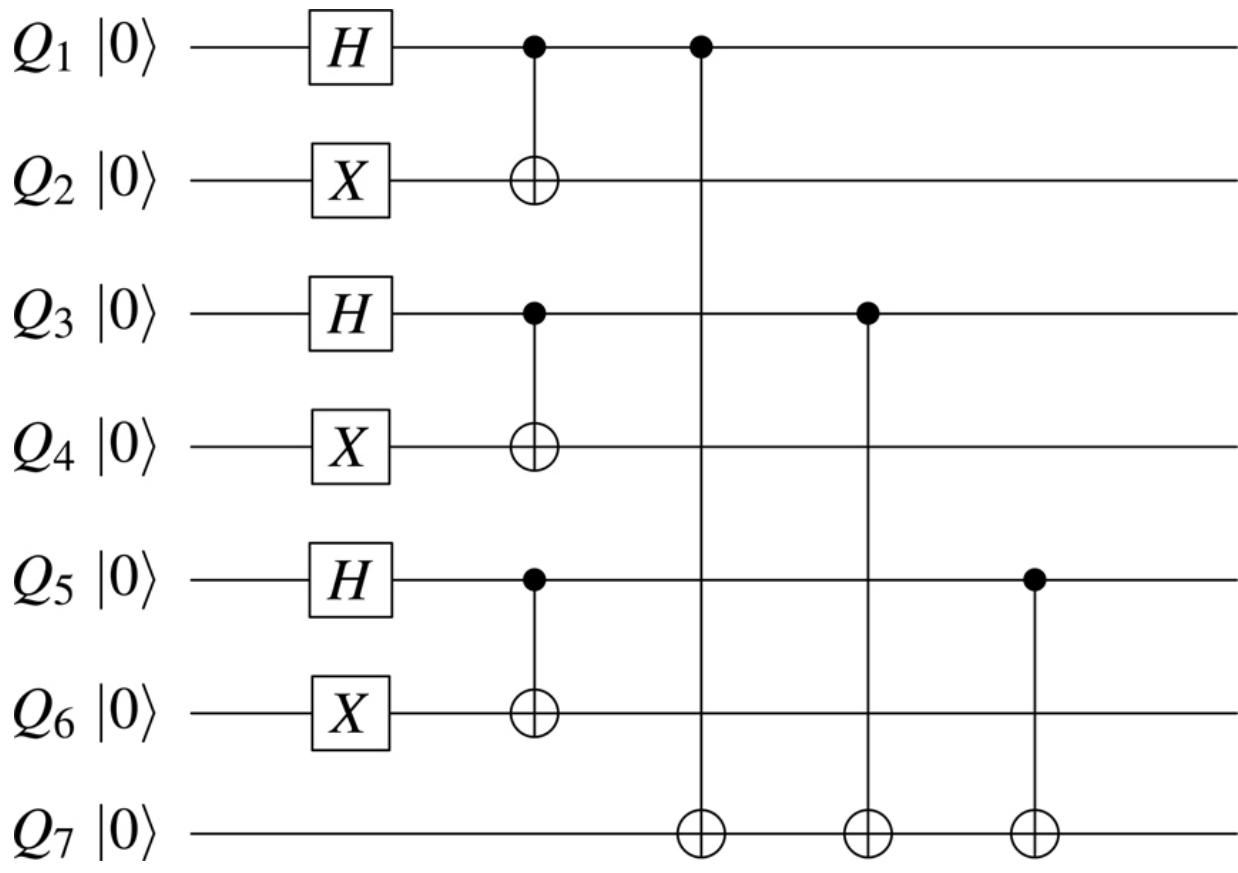
Figure 4.3 Circuit generation of entangled state $\frac{1}{2\sqrt{2}}(|0101010\rangle + |0101101\rangle + |0110011\rangle + |0110100\rangle + |1001011\rangle + |1001100\rangle + |1010010\rangle + |1010101\rangle)$.
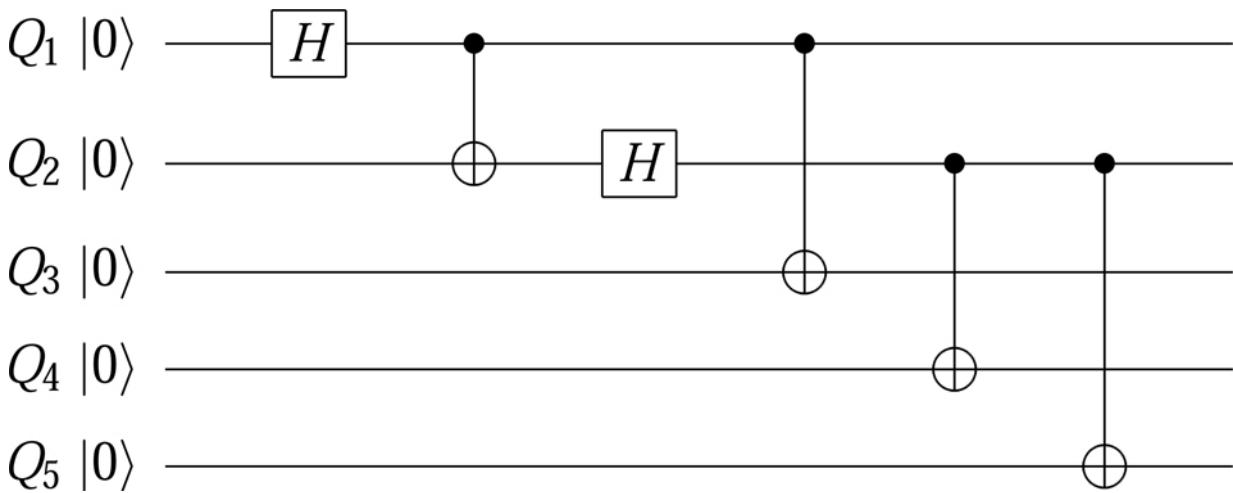
Figure 4.4 Quantum Circuit generation for the entangled state $|E\rangle_{Q_1Q_2Q_3Q_4Q_5}$ given in Eq. (4.1). ↵

Now we describe step-by-step circuit generation for an entangled five-qubit cluster state given in [Figure 4.4](#) in detail given by

$$|E\rangle_{Q_1Q_2Q_3Q_4Q_5} = \frac{1}{2}(|00000\rangle + |01011\rangle + |10100\rangle - |11111\rangle).$$

(4.1)

Step 0: A five-qubit state is prepared from a five ($|0\rangle$) zero initial state which is given by

$$|E_0\rangle_{Q_1Q_2Q_3Q_4Q_5} = |0\rangle_{Q_1} \otimes |0\rangle_{Q_2} \otimes |0\rangle_{Q_3} \otimes |0\rangle_{Q_4} \otimes |0\rangle_{Q_5}.$$

Step 1: Now, first a Hadamard gate is applied on qubit Q_1 and then a controlled-NOT gate is applied with qubit Q_1 as control qubit and qubit Q_2 as target qubit. Then the initial state $|E_0\rangle$ is transformed into the state

$$|E_1\rangle = \frac{1}{\sqrt{2}}\left(|00000\rangle + |11000\rangle\right)_{Q_1Q_2Q_3Q_4Q_5}.$$

Step 2: Again, a Hadamard gate is applied to the qubit Q_2 and then the state $|E_1\rangle$ evolves into the state

$$|E_2\rangle = \frac{1}{2}\left(|00000\rangle + |01000\rangle + |10000\rangle - |11000\rangle\right)_{Q_1Q_2Q_3Q_4Q_5}.$$

Step 3: Next, a controlled-NOT gate is applied with qubit Q_1 as the control qubit and qubit Q_3 as the target qubit. Then the state $|E_2\rangle$ becomes

$$|E_3\rangle = \frac{1}{2}\left(|00000\rangle + |01000\rangle + |10100\rangle - |11100\rangle\right)_{Q_1Q_2Q_3Q_4Q_5}.$$

Step 4: Lastly, two controlled-NOT gates are applied with qubit Q_2 as the control qubit for each of the qubits Q_4 and Q_5, respectively, as target qubits. Then the state $|E_3\rangle$ is transferred to

$$|E_4\rangle = \frac{1}{2}\Big(|00000\rangle + |01011\rangle + |10100\rangle - |11111\rangle\Big)_{Q_1Q_2Q_3Q_4Q_5},$$

which is the same as $|E\rangle_{Q_1Q_2Q_3Q_4Q_5}$ given in Eq. (4.1).

5 Density Matrix

5.1 INTRODUCTION

The chapter presents some aspects of the density matrix theory. The topics include the necessity of density matrix formalism and reduced density matrices by partial trace operation. References [10, 38, 113, 115, 125, 126, 145] contain several aspects of density matrix theory.

5.2 NECESSITY OF DENSITY OPERATOR

Density operator is a mathematical instrument that is a generalization of the idea of a ket vector. It combines classical information with quantum information and is suitable for the description of many physical systems for several practical purposes.

The idea originates from the observation that an element $|\Xi\rangle$ of a Hilbert space $\mathbb{H}$ representing a quantum system can be associated in a one-to-one correspondence with a Hermitian operator $|\Xi\rangle\langle\Xi|$ acting on $\mathbb{H}$. A ket $|\Xi\rangle$ describes the state of a quantum system which is a member of the Hilbert space $\mathbb{H}_n$ of dimension n describing the system. This can be described alternatively by a linear operator $|\Xi\rangle\langle\Xi|$ on the same Hilbert space $\mathbb{H}_n$ defined by

$$(|\Xi\rangle\langle\Xi|)|\aleph\rangle = \langle\Xi|\aleph\rangle|\Xi\rangle$$

Mathematically, the correspondence $|\Xi\rangle\langle\Xi| \leftrightarrow |\Xi\rangle$ is an isomorphism for the case of finite-dimensional Hilbert spaces. The operator $\varpi = |\Xi\rangle\langle\Xi|$ is called the density operator for the quantum state. But there can be other operators on $\mathbb{H}$ which also describe quantum systems. As an illustration, we consider the situation where there are quantum states $|\Xi_\alpha\rangle$ with probabilities p_α for $|\Xi_\alpha\rangle$ to be obtained in a random choice. The mixed quantum state is described by the density operator $\varpi = \sum_\alpha p_\alpha |\Xi_\alpha\rangle\langle\Xi_\alpha|$. There is no assumed correlation between two different quantum states $|\Xi_\alpha\rangle$ and $|\Xi_\beta\rangle$.

The specialty of the description is that it entails both classical and quantum uncertainties. Such physical situations are common in practice and experiments. This is the reason why density operators are important in quantum mechanics.

It is important to note that a density matrix may correspond to and describe more than one physical situation.

As an illustration, considering the 2-dimensional Hilbert space describing a qubit, a 50-50 mixture of $|0\rangle$ and $|1\rangle$ and a 50-50 mix up of the states $|\Xi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|\Xi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ are both described by the same density matrix. This can be verified as follows. In the former case the describing density matrix is

$$\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|,$$

while in the latter case the density matrix is

$$\frac{1}{2}|\Xi_1\rangle\langle\Xi_1| + \frac{1}{2}|\Xi_2\rangle\langle\Xi_2|$$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\cdot\frac{1}{\sqrt{2}}(\langle0|+\langle1|) + \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\cdot\frac{1}{\sqrt{2}}(\langle0|-\langle1|)\right)$$

$$= \frac{1}{2}|0\rangle\langle0| + \frac{1}{2}|1\rangle\langle1|.$$

It is impossible to describe the underlying real situation merely by looking as the density matrix.

5.3   PROPERTIES OF DENSITY OPERATOR OR MATRIX

In general, an operator ϖ describing a quantum system is a density operator if it satisfies the conditions:

(i) $\varpi^\dagger = \varpi$ *(Hermitian),*

(ii) ϖ *is positive semi-definite,*

(iii) $tr(\varpi) = 1.$

The matrix representation of the density operator is called the density matrix and is given by the same symbol ϖ.

Being a Hermitian operator, by virtue of the spectral decomposition theorem, it is possible to find a basis $\{|\tau_i\rangle : i = 1,\ldots,n\}$ (say) of the Hilbert space $\mathbb{H}$ in which the density matrix is diagonal. In that case we can write

$$\varpi = \sum_{j=1}^{n}\lambda_j|\tau_j\rangle\langle\tau_j|$$

$$(5.2)$$

Here, $\lambda_j \geq 0$ by the positive semi-definiteness of ϖ.

Then $tr(\varpi) = \sum_j \lambda_j$ and $tr(\varpi^2) = \sum_j \lambda_j^2$.

It then follows that $tr(\varpi^2) \leq tr(\varpi) = 1$ and that the equality follows only when $\lambda_k = 1$ for some k and $\lambda_j = 0$ for all $j \neq k$ in which case we have

$$\varpi = |\tau_k\rangle\langle\tau_k|,$$

that is, in this case, ϖ represents a pure state.:: noindent:: Conversely, it is immediate that for a pure state we have $tr(\varpi^2) = 1$.

From the above, it follows that ϖ describes a pure state if and only if $tr(\varpi) = tr(\varpi^2)$.

If A is an observable, then its expectation value when the quantum system is in the pure state $|\Xi\rangle$ is

$$
\begin{aligned}
\langle A \rangle_{|\Xi\rangle} &= \langle \Xi | A | \Xi \rangle \\
&= \langle \Xi | \Xi \rangle \langle \Xi | A | \Xi \rangle \\
&= \langle \Xi | \varpi A | \Xi \rangle \\
&= tr(\varpi A).
\end{aligned}
$$

(5.3)

Now in a situation where there are many quantum states, say, $|\Xi_1\rangle, |\Xi_2\rangle, \ldots, |\Xi_k\rangle$ mixed up (classically) in proportions $p_1, p_2, \ldots, p_k$, that is, there are totally m states with m_1 number of $|\Xi_1\rangle$ states, m_2 number of $|\Xi_2\rangle$ states, ..., m_k number of $|\Xi_k\rangle$ states, with $mp_i = m_i, i = 1, 2, \ldots, k$, the value of $\langle A \rangle$ depends on two factors. One is the drawing of the state $|\Xi_i\rangle$ from the above collection (ensemble), while the other is the finding of the value according to (5.3) with $|\Xi\rangle = |\Xi_i\rangle$.

In the first place the probability is classical, while in the latter consideration this is purely quantum. We can combine them into one formula by writing

$$\langle A \rangle = \sum_{i=1}^{k} p_i \langle A \rangle_{|\Xi_i\rangle}$$

$$= \sum_{i=1}^{k} p_i tr(A|\Xi_i\rangle\langle\Xi_i|)$$

$$= tr(A(\sum_{i=1}^{k} p_i|\Xi_i\rangle\langle\Xi_i|))$$

$$= tr(A\varpi),$$

where

$$\varpi = \sum_{i=1}^{k} p_i|\Xi_i\rangle\langle\Xi_i|$$

(5.4)

is the density operator describing the above situation.

5.4   DENSITY MATRIX OF COMPOSITE SYSTEMS

Let $\mathbb{H}_A$ and $\mathbb{H}_B$ be Hilbert spaces associated with two systems A and B, respectively, and ϖ is the density matrix describing the state of the composite system A and B. Occasionally, it is possible to write the density matrix of the composite system ϖ as $\varpi = \varpi_A \otimes \varpi_B$ for two density matrices ϖ_A and ϖ_B pertaining to the two systems A and B, respectively. If this is not the case, that is, if $\varpi \neq \varpi_A \otimes \varpi_B$ for some choices of density matrices ϖ_A and ϖ_B, then we have an entanglement existing between the two systems.

As an illustration, we take the density matrix of the Bell state $|\Upsilon_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)_{AB}$.

The density matrix of $|\Upsilon_2\rangle$ is $\varpi = |\Upsilon_2\rangle\langle\Upsilon_2|$ which is

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

It cannot be written as a tensor product of two density matrices ϖ_A and ϖ_B, which confirms the entangled character of $|\Upsilon_2\rangle$.

5.5 REDUCED DENSITY MATRIX

Let ϖ be the density matrix describing the composite quantum system as in Section 5.2. Then the reduced density matrices ϖ_A and ϖ_B, for the respective systems A and B, are given by $\varpi_A = tr_B\varpi$ and $\varpi_B = tr_A\varpi$ where tr_A and tr_B are the partial trace operations on the state ϖ of the composite system as described in Section 1.5.

If $\mathbb{H}_A$ admits of a basis $\{|\mu_i\rangle, i = 1, 2, \ldots, m\}$ and $\mathbb{H}_B$ admits of a basis, $\{|\nu_i\rangle, i = 1, 2, \ldots, n\}$ then the composite system of A and B corresponds to the Hilbert space $\mathbb{H}_A \otimes \mathbb{H}_B$ having dimension mn with a basis consisting of the element $|\mu_i\nu_j\rangle (= |\mu_i\rangle \otimes |\nu_j\rangle)$ $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$. The density matrix ϖ is described by its element $\varpi_{ik,lj}$ where $1 \leq i, l \leq m$ and $1 \leq k, j \leq n$.

Then ϖ_A and ϖ_B are given by

$$(\varpi_A)_{il} = \sum_{p=1}^{n} \varpi_{ip,lp} \quad i, l = 1, 2, \ldots, m$$

and

$$(\varpi_B)_{il} = \sum_{q=1}^{m} \varpi_{qk,qj} \quad k, j = 1, 2, \ldots, n.$$

The reduced density matrix describes a subsystem by eliminating the rest part from the total subsystem.

5.6 QUANTUM ENTROPY

The von Neuman entropy of a mixed state ϖ is given by the expression

$$E(\varpi) = -Tr(\varpi\, log\varpi).$$

If ϖ is a pure states, then $E(\varpi) = 0$.

If A is the subsystem of the system described by ϖ and ϖ_A is the reduced density matrix of the system A, the quantity $-Tr(\varpi_A\, log\varpi_A)$ is independent of the choice of the subsystem A.

We call the quantity $-Tr(\varpi_A\, log\varpi_A)$ is the entropy entanglement measure of ϖ. A state given by ϖ is a maximally entangled state if the above entropy entanglement measure is maximum. It may be observed that from the above viewpoint, the Bell-states are maximally entangled.

It should be mentioned that there are other entanglement measures like the widely used 'negativity', etc., which are different from the above one. The concept of the 'amount of entanglement' may be different for different choices of these measures.

6 Quantum Noise

6.1 INTRODUCTION

This chapter is on quantum noise. The origin of noise as a part of interaction with the environment is discussed. The Kraus operator formulation of noise along with various special types of noises are presented. References [14, 32, 44, 77, 109, 113, 115, 131] are helpful to understand different aspects of quantum noise.

6.2 ORIGIN OF QUANTUM NOISE

Noise is an unavoidable phenomena in any communication system regardless of whether it is classical or quantum in nature. The effect of noise we consider originates through the interaction of the quantum resource with the environment. It is a quantum decoherence phenomena [143, 144] by which the quantum resource shared by the different parties becomes less entangled and thereby the quality of output decreases at the receiver's end. The output becomes different from the input which is the desired output. The noise affects the resource when after its generation the qubits constituting the entangled resource are distributed to different parties. In the process of distribution, the qubits have to pass through the noisy environment and thereby become affected with noise.

6.3 KRAUS OPERATORS

Quantum noise we consider here is described by Kraus operators. Their origin lies in the quantum decoherence phenomena. An elaborate description of this origin is beyond the scope of the book. In the following we give a short deduction of this description of quantum noise under restricted assumptions.

Let the Hamiltonian function associated with the system and environment be H_S and H_E, respectively. Then the Hamiltonian for the system-Environment combination is given by $H_T = H_S \otimes H_E$. Let the space H_E have a basis $\{|e_1\rangle, \ldots, |e_n\rangle\}$ and the initial states of the system and the environment be given by $\varpi(0)$ and $|e\rangle\langle e|$, respectively, where we have assumed that the system is in a mixed state and the environment is in a pure state. We write

$$\varpi_T(0) = \varpi(0) \otimes |e\rangle\langle e|.$$

The System-Environment composition is assumed to form a closed system due to which it evolves unitarily. Thus $\varpi_T(t) = U^\dagger \varpi_T(0) U = U^\dagger \varpi(0) \otimes |e\rangle\langle e| U$ where U is the unitary operator on H_T. For finding the evolution of the system we take a partial trace over E. Then the evolution of the system is given by

$$
\begin{aligned}
\varpi(t) \quad &= tr_E(\varpi_T(t)) \\
&= \sum_{\mu=1}^{m} \langle e_\mu | U^\dagger \varpi(0) \otimes |e\rangle\langle e| U | e_\mu \rangle \\
&= \sum_{\mu=1}^{m} \langle e_\mu | U^\dagger |e\rangle \, \varpi(0) \langle e| U | e_\mu \rangle \\
&= \sum_{\mu=1}^{m} M_\mu^\dagger \varpi(0) M_\mu
\end{aligned}
$$

where $M_\mu = \langle e|U|e_n \rangle$ operates on the Hilbert space H_S and are referred to as Kraus operators. An important property of the Kraus operators is that they satisfy the condition $\sum_{\mu=1}^{m} M_\mu^\dagger M_\mu = I$.

It is important to note that the above derivation is obtained under certain restrictions. Also, it is noteworthy that the number of Kraus operators depends on the dimension of the Hilbert space describing the environment.

6.4 DIFFERENT TYPES OF NOISES

There are several types of noises that have different effects on the quantum system under consideration. They require different choices of Kraus operators. We note in the following some of these noises.

1. **Amplitude-damping Noise:** The Kraus operators of amplitude damping noise are expressed as:

$$K_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}, K_1 = \begin{bmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{bmatrix}$$

where p is the noise intensity parameter of amplitude damping noise.

2. **Bit-flip Noise:** The Kraus operators of bit-flip noise are expressed as:

$$K_0 = \begin{bmatrix} \sqrt{1-q} & 0 \\ 0 & \sqrt{1-q} \end{bmatrix}, K_1 = \begin{bmatrix} 0 & \sqrt{q} \\ \sqrt{q} & 0 \end{bmatrix}$$

where q is the noise intensity parameter of bit-flip noise.

3. **Phase-flip Noise:** The Kraus operators of phase-flip noise are expressed as:

$$K_0 = \begin{bmatrix} \sqrt{1-r} & 0 \\ 0 & \sqrt{1-r} \end{bmatrix}, K_1 = \begin{bmatrix} \sqrt{r} & 0 \\ 0 & -\sqrt{r} \end{bmatrix}$$

where r is the noise intensity parameter of phase-flip noise.

4. **Phase-damping Noise:** The Kraus operators of phase-damping noise are expressed as:

$$K_0 = \begin{bmatrix} \sqrt{1-s} & 0 \\ 0 & \sqrt{1-s} \end{bmatrix}, K_1 = \begin{bmatrix} \sqrt{s} & 0 \\ 0 & 0 \end{bmatrix}, K_2 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{s} \end{bmatrix}$$

where s is the noise intensity of phase-damping noise.

5. **Depolarizing noise:** Depolarizing noise is described by the following Kraus operators:

$$K_0 = \sqrt{1-l} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, K_1 = \sqrt{\frac{l}{3}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$K_2 = \sqrt{\frac{l}{3}} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, K_3 = \sqrt{\frac{l}{3}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

where l is the depolarizing probability.

In all the above cases the number of Kraus operators is determined by the dimension of the Hilbert space which describes the environment. They have different physical effects on the qubit. We do not enter into the details of these physical effects.

7 The Quantum Communication System

7.1 INTRODUCTION

In this chapter, the quantum communication system is described. The concept of fidelity quantifying the faithfulness of quantum state transfer is discussed. For detailed exposure on this topic, [71, 86, 113] is helpful references.

7.2 GENERAL DESCRIPTION

A general communication system can be described by the following diagram

```
┌─────────────────────────┐
│    Information source    │
└─────────────────────────┘
             │
             ▼
┌─────────────────────────┐
│        Encoding          │
└─────────────────────────┘
             │
             ▼
┌─────────────────────────┐
│      Transmission        │
└─────────────────────────┘
             │
             ▼
┌─────────────────────────┐
│        Decoding          │
└─────────────────────────┘
             │
             ▼
┌─────────────────────────┐
│       Destination        │
└─────────────────────────┘
```
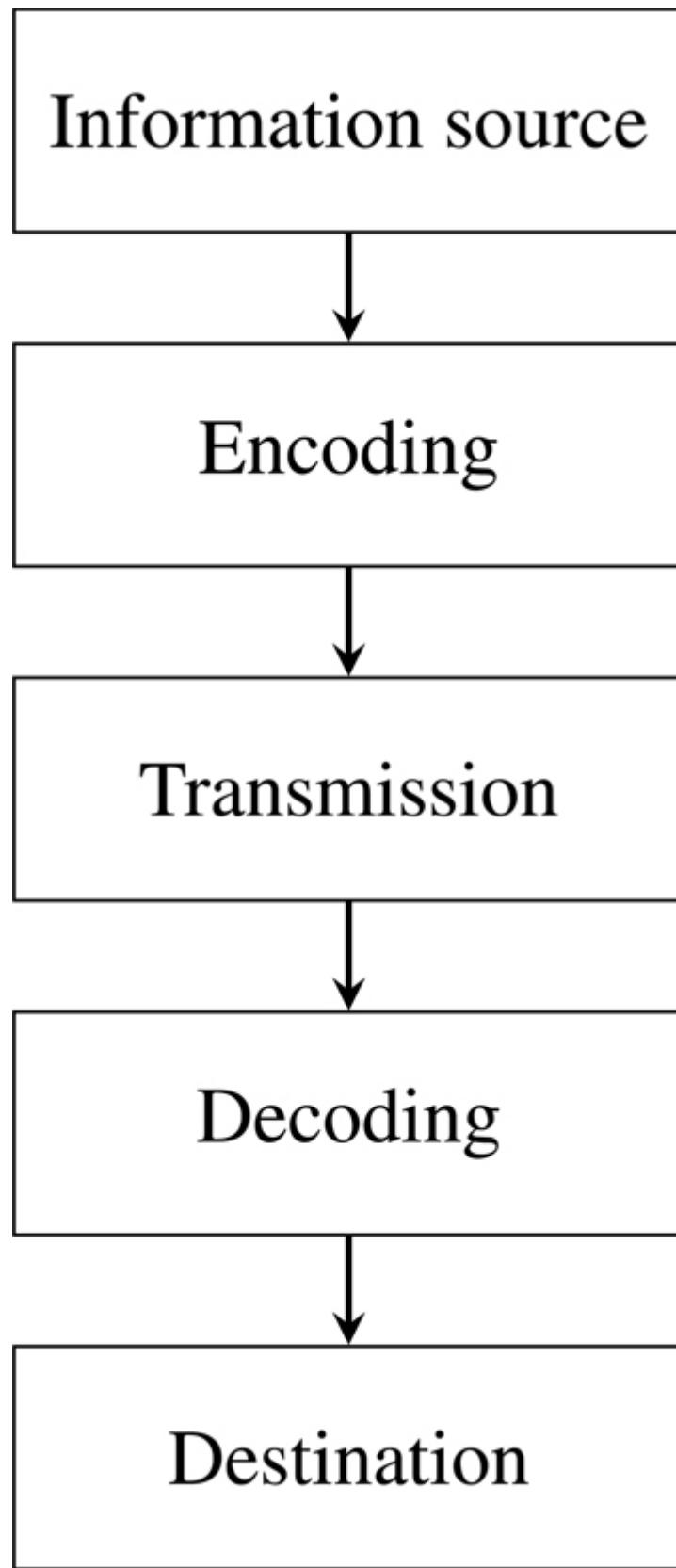
The system can be either classical or quantum or a combination of both. Quantum information is encoded in qubit systems which is the counterpart of classical information being encoded in bits. Teleportation is a process by which qubit systems are transmitted using quantum resources which are entangled states shared between the different parties participating in the process. In every teleportation protocol, the assistance of a classical information channel is unavoidable. Further, at the receiver's end decoding takes place which, in the case of quantum communications, is the decoding of information from qubit systems obtained after transmission. In this book we will be concerned with only the 'Transmission' part of the communication system.

7.3 QUANTUM CHANNEL

A quantum channel is an arrangement, mathematically some operations, that produces a density matrix ϖ' corresponding to an input density matrix ϖ.

$$\varpi \longrightarrow \boxed{\text{Quantum Channel}} \longrightarrow \varpi'.$$

In the teleportation protocols, entangled states act as resources in these quantum channels.

A formal mathematical description of a quantum channel is done by a superoperator which is trace-preserving and completely positive. We do not enter into such mathematical aspects of the theory.

7.4 FIDELITY

The degree of similarity between two quantum systems is mathematically quantified through the concept of fidelity. It is useful as well as necessary in

many practical situations. For instance, the preparation of a quantum state is generally limited by imperfections. It might be necessary to know in quantitative terms the amount of imperfection, that is, how much the prepared state has deviated from the state intended for preparation. As another example, it is known that exact cloning of a quantum state is impossible, but it is possible to create approximately cloned copies of a quantum state. In that situation, it may be necessary to determine the similarity of the cloned copy with the original one for the purpose of determining the quality of cloning and for possible measures toward optimizing the quality of the cloning. Formally, the fidelity between two mixed states ϖ_1 and ϖ_2 is given by

$$\mathscr{F}(\varpi_1, \varpi_2) = \left(tr\sqrt{\sqrt{\varpi_1}\varpi_2\sqrt{\varpi_1}} \right)^2.$$

(7.1)

In particular, if one of the states is a pure state, we have the following expression:

$$\mathscr{F}(\varpi, |\Xi\rangle\langle\Xi|) = tr\langle\Xi|\varpi|\Xi\rangle.$$

(7.2)

As an illustration, if both the states are pure states, that is, $\varpi_1 = |\Omega\rangle\langle\Omega|$ and $\varpi_2 = |\Xi\rangle\langle\Xi|$, then we have

$$\mathscr{F}(|\Omega\rangle\langle\Omega|, |\Xi\rangle\langle\Xi|) = |\langle\Omega|\Xi\rangle|^2.$$

(7.3)

When a quantum communication process is perfect, that is, the input is equal to the output, the fidelity takes unit value which can be seen from the

above expressions of fidelity.

Our use of the expression of Fidelity will be mainly based on Eq. (7.2). For our purpose, we will require the determination of fidelity when teleportation is imperfect which is when the process is executed in a noisy environment. The input state then differs from the output state due to the effect of noise which perturbs the otherwise perfect teleportation process. It is interesting to note the effects of variation of the noise parameter as well as other parameters in the protocol. If ϖ_1 is the density operator of the input state, the density matrix of the output state is ϖ_2, $\mu_1, \ldots, \mu_n$ are parameters in the protocol including noise parameters, then the fidelity $F(\varpi_1, \varpi_2, \mu_1, \ldots, \mu_n)$ varies with the parameters $\mu_1, \ldots, \mu_n$. If the noise parameters in the set $\{\mu_1, \ldots, \mu_n\}$ are made vanishingly small, then physical considerations show that the fidelity $\mathscr{F}$ will tend to the unit value. The above is a universal feature of fidelity analysis.

Part II

Teleportation Protocols in Ideal Environment

8 Teleportation of Single-qubit Quantum States

8.1 INTRODUCTION

In this chapter, the single qubit teleportation process is presented. The process is described elaborately. It forms the backbone of understanding of the rest of the protocols presented in the following chapters.

8.2 THE BASIC PROGRAM OF TELEPORTATION PROTOCOL

In this chapter, an overview of the basic proposal of teleportation process is provided. It was originally designed by C. H. Bennett and his coauthors in 1993 where an unknown one-qubit state is transferred from one party, namely Alice, to another party, namely Bob, by using Bell state as quantum resource and performing Bell state measurement (BSM) [7]. Additionally, description of the same protocol using measurement on computational basis is also included in this chapter. These names Alice and Bob are customary in information theory and are adopted from there in quantum communication research. The achievement of teleportation is through quantum mechanical operations although supported with a classical communication channel. This

support of classical communication precludes the possibility of super-luminal signaling, that is, the possibility of sending signals instantaneously or at least at a speed exceeding that of light. Also, such a necessary support of classical communication specifies that classical causality is not violated by the process of teleportation and, therefore, there is no violation of local realism. Incidentally, classical communication channels as assisting channels are indispensable in all types of teleportation which are described in the subsequent chapters. Therefore the above considerations are also applicable to all the following chapters. It is important to emphasize that teleportation is fundamentally a process of quantum state transfer with no physical object being transported.

The teleportation process has several versions and is applicable to the tasks of transferring various types of states. The basic methodology of teleportation is also applicable to other areas of quantum mechanics where remote action is warranted. They include protocols performing telecloning [29, 49, 51, 68, 108, 116, 181], remote implementation of operators [5, 66, 67, 70, 95, 122, 123, 164, 190, 201], etc. In the following chapters of part II several protocols are described whose understanding requires the basic ideas and methodologies presented in this chapter.

Since teleportation protocols are of various kinds, the types discussed here are not exhaustive. There are several other protocols which have not been addressed in this book although they are no less important. They include, for instance, teleportation by quantum walk [18, 40, 84, 148, 151, 167], mentor initiated teleportation [27], quantum conference by teleportation [26], multi-directional teleportation [72, 161, 193], short-distance teleportation [2, 3, 103, 157, 177], etc. Particularly teleportation of continuous variables [11, 13, 42, 47, 111, 127, 188] has been kept out of our discussion. The experimental verification of teleportation has been reported in several works [64, 100,

]. We have not discussed the experimental aspects since it is outside the scope of the book.

8.3   TELEPORTATION OF ARBITRARY SINGLE-QUBIT STATE

In this section a scheme for teleporting an unknown single-qubit quantum state from the sender Alice to the receiver Bob is described. Bell state measurement (BSM) is used in the protocol. The protocol was designed by Bennett et al. [7] through which the concept of teleportation was introduced.

Alice possesses a qubit without knowing any information about it. Let the single qubit in Alice's possession be given by

$$|\aleph\rangle_\mathfrak{a} = (\mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle),$$

(8.1)

with the parameters $\mathfrak{g}_1, \mathfrak{g}_2$ satisfy the normalization condition, that is,

$$|\mathfrak{g}_1|^2 + |\mathfrak{g}_2|^2 = 1.$$

It should be emphasized that no information on $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are available with either Alice or Bob except the above normalization condition.

For the purpose of teleportation, Alice and Bob share a Bell state that acts as quantum resource. Here we take the shared quantum state to be one of the four Bell state given by

$$|\Upsilon_1\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$

(8.2)

with Alice and Bob holding the first and second qubit, respectively.

The preparation of the entangled resource is shown in Figure 8.1.
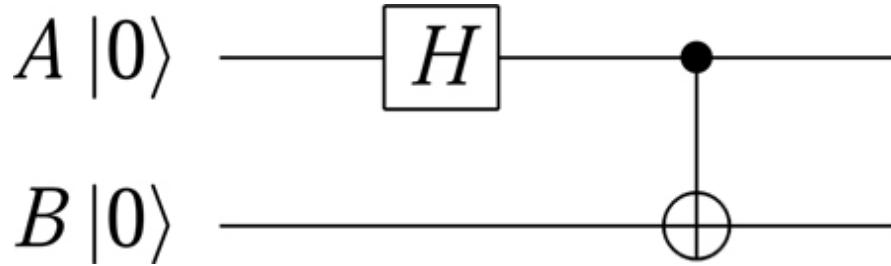
Figure 8.1   Circuit representation for the generation of entangled resource $|\Upsilon_1\rangle_{AB}$ given in Eq. (8.2).

A classical channel is assumed to exist between Alice and Bob capable of transmitting two classical bits from Alice to Bob. With the above setup, the state $|\aleph\rangle_a$ given in Eq. (8.1) is transmitted from Alice to Bob in the following way. The whole scenario is depicted in Figure 8.2.
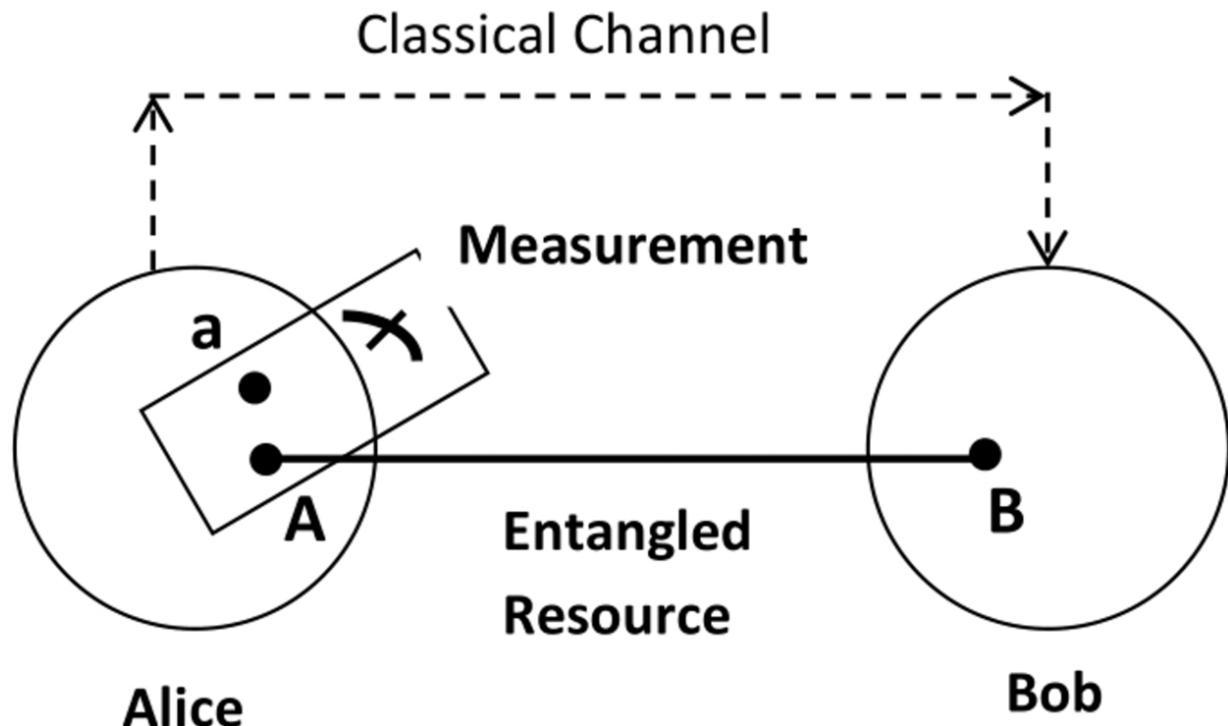
Figure 8.2   Diagram illustrating the single-qubit state transfer by teleportation.

First, Alice takes one particle from the pair constituting the Bell state, while the other is retained by Bob. In the entangled state, the indices A and B refer to the qubits in possessions of Alice and Bob, respectively. So Alice holds qubits $\mathfrak{a}$ (to be teleported, Eq. (8.1)), and A (one from the entangled pair which is given in Eq. (8.2)), and Bob holds the qubit B. The joint state of the composite three-qubit system is given by:

$$
\begin{aligned}
|\Gamma\rangle &= |\aleph\rangle_{\mathfrak{a}} \otimes |\Upsilon_1\rangle_{AB} \\
&= (\mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle)_{\mathfrak{a}} \otimes \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)_{AB}.
\end{aligned}
$$

$$(8.3)$$

Alice carries out a measurement on her qubits $(\mathfrak{a}, A)$, with respect to the Bell basis specified by

$$
\begin{aligned}
|\Upsilon_1\rangle_{\mathfrak{a}A} &= \frac{(|00\rangle + |11\rangle)}{\sqrt{2}}, \\
|\Upsilon_2\rangle_{\mathfrak{a}A} &= \frac{(|00\rangle - |11\rangle)}{\sqrt{2}}, \\
|\Upsilon_3\rangle_{\mathfrak{a}A} &= \frac{(|01\rangle + |10\rangle)}{\sqrt{2}}, \\
|\Upsilon_4\rangle_{\mathfrak{a}A} &= \frac{(|01\rangle - |10\rangle)}{\sqrt{2}}.
\end{aligned}
$$

$$(8.4)$$

That is, Alice performs a Bell State Measurement (BSM) using the four basis vectors $\{|\Upsilon_1\rangle, |\Upsilon_2\rangle, |\Upsilon_3\rangle, |\Upsilon_4\rangle\}$ described in Eq. (8.4).

To clearly express the outcome of her measurement, it is helpful to rewrite the joint state of Alice's qubits as a superposition of the above vectors. This can be achieved by employing the following general identities:

$$|0\rangle \otimes |0\rangle = \frac{1}{\sqrt{2}}(|\Upsilon_1\rangle + |\Upsilon_2\rangle),$$

$$|0\rangle \otimes |1\rangle = \frac{1}{\sqrt{2}}(|\Upsilon_3\rangle + |\Upsilon_4\rangle),$$

$$|1\rangle \otimes |0\rangle = \frac{1}{\sqrt{2}}(|\Upsilon_3\rangle - |\Upsilon_4\rangle),$$

$$|1\rangle \otimes |1\rangle = \frac{1}{\sqrt{2}}(|\Upsilon_1\rangle - |\Upsilon_2\rangle).$$

(8.5)

Using Eq. (8.5), the total system, that is, Eq. (8.3) can be rewritten as

$$
\begin{aligned}
|\Gamma\rangle = \ & \frac{1}{2}[|\Upsilon_1\rangle_{aA} \otimes (\mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle)_B + |\Upsilon_2\rangle_{aA} \otimes (\mathfrak{g}_1|0\rangle - \mathfrak{g}_2|1\rangle)_B \\
& + |\Upsilon_3\rangle_{aA} \otimes (\mathfrak{g}_1|1\rangle + \mathfrak{g}_2|0\rangle)_B + |\Upsilon_4\rangle_{aA} \otimes (\mathfrak{g}_1|1\rangle - \mathfrak{g}_2|0\rangle)_B] \\
= \ & \sum_{i=1}^{4} |\Upsilon_i\rangle_{aA} \otimes |v_i\rangle_B.
\end{aligned}
$$

(8.6)

So far, we have only performed a basis change on Alice's subsystem. No actual operations have been conducted and the overall state of the three qubits remains the same. Alice begins the teleportation procedure by performing a Bell state measurement (BSM) on her two qubits. This BSM projects the system onto one of the four possible outcomes, each occurring with equal likelihood which are

$$|\Upsilon_1\rangle_{aA} \otimes (\mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle)_B,$$

$$|\Upsilon_2\rangle_{aA} \otimes (\mathfrak{g}_1|0\rangle - \mathfrak{g}_2|1\rangle)_B,$$

$$|\Upsilon_3\rangle_{aA} \otimes (\mathfrak{g}_1|1\rangle + \mathfrak{g}_2|0\rangle)_B,$$

$$|\Upsilon_4\rangle_{aA} \otimes (\mathfrak{g}_1|1\rangle - \mathfrak{g}_2|0\rangle)_B.$$

The two qubits held by Alice are now in an entangled state, and the entanglement that initially existed between her and Bob's qubits is no longer present. By this breakage the entangled resource originally utilized in the teleportation process is lost forever. This indicates that the quantum resource can be used only once. There is no reuse of the resource. At this stage Bob is left with his qubit which is not entangled with any other qubit of the system.

Following the measurement, Alice informs her measurement result to Bob through the assisting classical channel. Due to the fact that there are four possible outcomes of Alice's BSM, two classical bits are necessary to convey the result. Once Bob receives the classical information, he can obtain the desired state to be teleported by the following procedure. Using this knowledge, Bob applies an appropriate unitary transformation, specified in Table 8.1, on his qubit to obtain the original state intended for teleportation. In the above Table 8.1 ϑ_x, ϑ_y and ϑ_z refer to the Pauli operators described in Section 3.2.3. This concludes the protocol.

Table 8.1

Local unitary operations for Bob's qubit are provided as determined by Alice's measurement ⏎

Alice's outcome	State of Bob's site	Bob's operation
$\lvert\Upsilon_1\rangle_{aA}$	$\lvert v_1\rangle = (\mathfrak{g}_1\lvert 0\rangle + \mathfrak{g}_2\lvert 1\rangle)_B$	$(I)_B$
$\lvert\Upsilon_2\rangle_{aA}$	$\lvert v_2\rangle = (\mathfrak{g}_1\lvert 0\rangle - \mathfrak{g}_2\lvert 1\rangle)_B$	$(\vartheta_z)_B$
$\lvert\Upsilon_3\rangle_{aA}$	$\lvert v_3\rangle = (\mathfrak{g}_1\lvert 1\rangle + \mathfrak{g}_2\lvert 0\rangle)_B$	$(\vartheta_x)_B$
$\lvert\Upsilon_4\rangle_{aA}$	$\lvert v_4\rangle = (\mathfrak{g}_1\lvert 1\rangle - \mathfrak{g}_2\lvert 0\rangle)_B$	$(\vartheta_z\vartheta_x)_B$

To illustrate, let us assume that Alice obtains the measurement result $\lvert\Upsilon_4\rangle_{aA}$, then the state of Bob's qubit becomes $(\mathfrak{g}_1\lvert 1\rangle - \mathfrak{g}_2\lvert 0\rangle)_B$. Now Alice conveys her measurement result to Bob using a classical channel. After receiving this

information, Bob executes the appropriate unitary operation given in [Table 8.1](#), which is $(\vartheta_z \vartheta_x)_B$ on his qubit, and thereby creates the desired state at his site. The teleportation is thereby accomplished. The other three cases arising out of Alice's measurement are similar.

The following are some special features of the teleportation process described above.

The state to be transmitted is arbitrary. It is unknown and remains so during the execution of the protocol.

The state is completely lost to Alice after the protocol is finished.

The assistance of a classical communication channel is indispensable.

Once used, the quantum resource (which is a Bell pair here) is lost. It cannot be reused.

The protocol is perfect by which it is meant that there is no case of failure.

There is no upper bound of the physical distance by which the sender and the receiver can be separated.

Remark: The Bell-state utilized as a quantum resource may well be any of the remaining three Bell-states $|\Upsilon_2\rangle_{AB}, |\Upsilon_3\rangle_{AB}$ and $|\Upsilon_4\rangle_{AB}$. The protocol will require slight modification with the main features remaining the same.

Alternatively, the same problem of transferring a single-qubit state between two communicating parties Alice (the sender) and Bob (the receiver) can be performed using a computational basis as the measurement basis. Alice aims

to transfer an unknown state, as specified in Eq. (8.1) to Bob, where $\mathfrak{g}_1$ and $\mathfrak{g}_1$ are complex numbers chosen such that

$$|\mathfrak{g}_1|^2 + |\mathfrak{g}_2|^2 = 1.$$

For this, we use the two-qubit Bell state given in Eq. (8.2) as a quantum resource between Alice and Bob. Thus, Alice possesses two qubits: the particle $\mathfrak{a}$, which is the one to be teleported and is described by Eq. (8.1), and particle A, that is the first qubit of the quantum resource defined in Eq. (8.2). Bob holds qubit B, the counterpart in the entangled pair shared with Alice. The entire three-particle system is expressed in Eq. (8.3).

In order to enable a successful measurement using the computational basis, Alice applies two quantum operations: a controlled-NOT gate is applied by Alice with '$\mathfrak{a}$' as control qubit and 'A' as target qubit after which she applies a Hadamard gate on her qubit '$\mathfrak{a}$'. These operations transform the state of the overall three-qubit system into a superposition of the states belonging to the computational basis of the qubits $\mathfrak{a}$ and A. After applying these transformations, the system evolves into the following state:

$$
\begin{aligned}
|\Gamma\rangle \quad &= |\aleph\rangle_{\mathfrak{a}} \otimes |\Upsilon_1\rangle_{AB} \\
&= (\mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle)_{\mathfrak{a}} \otimes \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)_{AB} \\
\underrightarrow{CNOT} \quad &= \frac{1}{\sqrt{2}}\left[\mathfrak{g}_1|000\rangle + \mathfrak{g}_1|011\rangle + \mathfrak{g}_2|110\rangle + \mathfrak{g}_2|101\rangle\right]_{\mathfrak{a}AB} \\
\underrightarrow{Hadamard\,Gate} \quad &= \frac{1}{2}\left[\mathfrak{g}_1(|0\rangle + |1\rangle)_{\mathfrak{a}}|00\rangle_{AB} + \mathfrak{g}_1(|0\rangle + |1\rangle)_{\mathfrak{a}}|11\rangle_{AB}\right. \\
&\quad + \mathfrak{g}_2(|0\rangle - |1\rangle)_{\mathfrak{a}}|10\rangle_{AB} + \mathfrak{g}_2(|0\rangle - |1\rangle)_{\mathfrak{a}}|01\rangle_{AB}\Big] \\
&= \frac{1}{2}\Big[(\mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle)_B|00\rangle_{\mathfrak{a}A} + (\mathfrak{g}_1|1\rangle + \mathfrak{g}_2|0\rangle)_B|01\rangle_{\mathfrak{a}A} \\
&\quad + (\mathfrak{g}_1|0\rangle - \mathfrak{g}_2|1\rangle)_B|10\rangle_{\mathfrak{a}A} + (\mathfrak{g}_1|1\rangle - \mathfrak{g}_2|0\rangle)_B|11\rangle_{\mathfrak{a}A}\Big].
\end{aligned}
$$

$$(8.7)$$

Alice now measures her two qubits $(\mathfrak{a}, A)$ in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. After the measurement, Alice sends the result of her measurement through the 2-bit classical channel to Bob. Depending on the two-bit outcome communicated by Alice, Bob applies a specific Pauli gate to recover the initial quantum state $|\aleph\rangle$, as outlined in Table 8.2. In the above Table 8.2 ϑ_x, ϑ_y and ϑ_z refer to the Pauli operators described in Section 3.2.3. This is the end of the protocol.

Table 8.2

State at Bob's location and corresponding unitary operations for Bob conditioned on Alice's results ↵

Alice's outcome	State of Bob's site	Bob's operation
$|00\rangle_{\mathfrak{a}A}$	$(\mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle)_B$	$(I)_B$
$|01\rangle_{\mathfrak{a}A}$	$(\mathfrak{g}_1|1\rangle + \mathfrak{g}_2|0\rangle)_B$	$(\vartheta_x)_B$
$|10\rangle_{\mathfrak{a}A}$	$(\mathfrak{g}_1|0\rangle - \mathfrak{g}_2|1\rangle)_B$	$(\vartheta_z)_B$
$|11\rangle_{\mathfrak{a}A}$	$(\mathfrak{g}_1|1\rangle - \mathfrak{g}_2|0\rangle)_B$	$(\vartheta_z\vartheta_x)_B$

Apart from teleportation protocols, there exists a separate class of similar quantum communication processes, which are designed for creating known quantum states at a distant place. Just the same as in the teleportation processes, these protocols use entanglement and classical communication for their accomplishment. They are called Remote State Preparation (RSP) protocols. In some cases, the information of the known states can be divided between two parties, where each party possesses only partial information about the state. This leads to the use of a specific class of protocols called Joint Remote State Preparation (JRSP) protocol. We discuss representative cases of these protocols in the APPENDIX.

9 Teleportation Protocol of Multi-qubit States

9.1 INTRODUCTION

This chapter explicitly deals with the transfer protocols of arbitrary unknown quantum states that involve two and three qubits. There is a large literature on the teleportation of multi-qubit states of various kinds which follow the basic techniques of the protocols described here [16, 22, 23, 50, 83, 94, 97, 99, 112, 130, 133, 158, 187, 191, 194].

9.2 TELEPORTATION OF ARBITRARY TWO-QUBIT STATES

This section outlines the scheme for teleporting unknown two-qubit general quantum states between two parties, Alice and Bob. Alice wishes to transmit the following general two-qubit state to Bob described as

$$|\aleph\rangle_{\mathfrak{a}_1\mathfrak{a}_2} = (\mathfrak{g}_1|00\rangle + \mathfrak{g}_2|01\rangle + \mathfrak{g}_3|10\rangle + \mathfrak{g}_4|11\rangle),$$

(9.1)

where the parameters $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3$, and $\mathfrak{g}_4$ meet the normalization condition, that is,

$$\sum_{k=1}^{4} |\mathfrak{g}_k|^2 = 1.$$

The state is unknown to both Alice and Bob which is equivalent to the fact that the coefficients $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3$, and $\mathfrak{g}_4$ are unknown except for the normalization relation. The

protocol described here is given by G. Rigolin [133].

The following are the sixteen generalized Bell states, also referred to as G states for simplicity (introduced by Rigolin [133]). These states are classified into four different groups.

Group 1:

$$
\begin{aligned}
|G_1\rangle &= \tfrac{1}{2}(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle), \\
|G_2\rangle &= \tfrac{1}{2}(|0000\rangle + |0101\rangle - |1010\rangle - |1111\rangle), \\
|G_3\rangle &= \tfrac{1}{2}(|0000\rangle - |0101\rangle + |1010\rangle - |1111\rangle), \\
|G_4\rangle &= \tfrac{1}{2}(|0000\rangle - |0101\rangle - |1010\rangle + |1111\rangle).
\end{aligned}
$$

(9.2)

Group 2:

$$
\begin{aligned}
|G_5\rangle &= \tfrac{1}{2}(|0001\rangle + |0100\rangle + |1011\rangle + |1110\rangle), \\
|G_6\rangle &= \tfrac{1}{2}(|0001\rangle + |0100\rangle - |1011\rangle - |1110\rangle), \\
|G_7\rangle &= \tfrac{1}{2}(|0001\rangle - |0100\rangle + |1011\rangle - |1110\rangle), \\
|G_8\rangle &= \tfrac{1}{2}(|0001\rangle - |0100\rangle - |1011\rangle + |1110\rangle).
\end{aligned}
$$

(9.3)

Group 3:

$$
\begin{aligned}
|G_9\rangle &= \tfrac{1}{2}(|0010\rangle + |0111\rangle + |1000\rangle + |1101\rangle), \\
|G_{10}\rangle &= \tfrac{1}{2}(|0010\rangle + |0111\rangle - |1000\rangle - |1101\rangle), \\
|G_{11}\rangle &= \tfrac{1}{2}(|0010\rangle - |0111\rangle + |1000\rangle - |1101\rangle), \\
|G_{12}\rangle &= \tfrac{1}{2}(|0010\rangle - |0111\rangle - |1000\rangle + |1101\rangle).
\end{aligned}
$$

(9.4)

Group 4:

$$|G_{13}\rangle = \tfrac{1}{2}(|0011\rangle + |0110\rangle + |1001\rangle + |1100\rangle),$$
$$|G_{14}\rangle = \tfrac{1}{2}(|0011\rangle + |0110\rangle - |1001\rangle - |1100\rangle),$$
$$|G_{15}\rangle = \tfrac{1}{2}(|0011\rangle - |0110\rangle + |1001\rangle - |1100\rangle),$$
$$|G_{16}\rangle = \tfrac{1}{2}(|0011\rangle - |0110\rangle - |1001\rangle + |1100\rangle).$$

(9.5)

The above G-states satisfy the condition given by

$$\sum_{k=1}^{16} |G_k\rangle\langle G_k| = I$$

and

$$\langle G_k|G_l\rangle = \delta_{kl}$$

and thus form an orthonormal basis, which is generally known as the G-basis.

Now, Alice and Bob share one of the sixteen G states to utilize it as the quantum resource in the teleportation process with the first two qubits held by Alice and the remaining two by Bob.

Assume that Alice and Bob share the state

$$|G_1\rangle_{A_1 A_2 B_1 B_2} = \frac{1}{2}(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle),$$

where the pair of qubits (A_1, A_2) and (B_1, B_2) belong to Alice and Bob, respectively. The circuit diagram corresponding to the generation of quantum resource $|G_1\rangle_{A_1 A_2 B_1 B_2}$ is given in Figure 9.1. Also there is a classical channel between Alice and Bob.
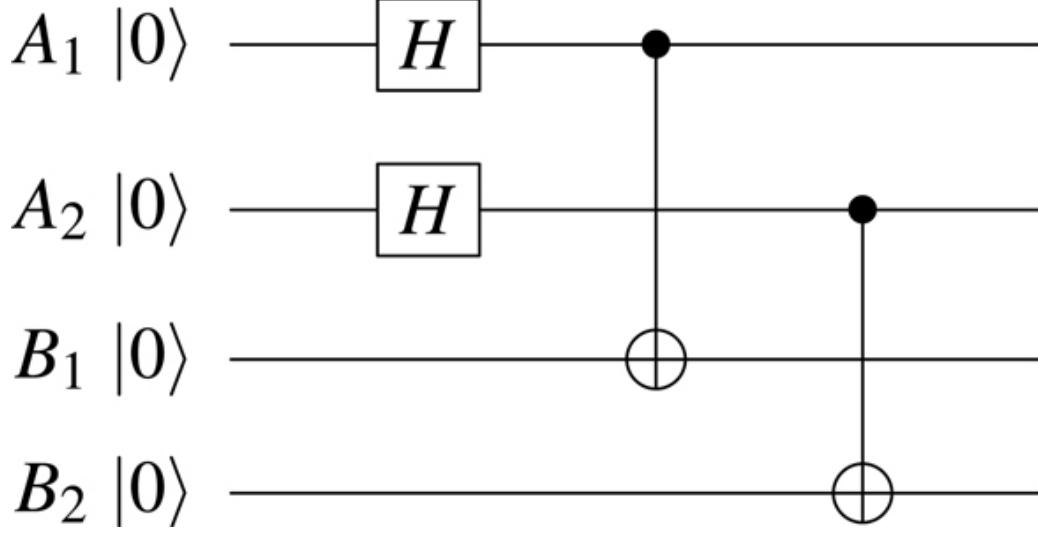
Figure 9.1 Circuit diagram for generation of the quantum resource $|G_1\rangle$. ↵

The initial combined state of the system is described as

$$
\begin{aligned}
|\Gamma\rangle &= |\aleph\rangle_{a_1 a_2} \otimes |G_1\rangle_{A_1 A_2 B_1 B_2} \\
&= (g_1|00\rangle + g_2|01\rangle + g_3|10\rangle + g_4|11\rangle)_{a_1 a_2} \\
&\otimes \frac{1}{2}(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle)_{A_1 A_2 B_1 B_2} \\
&= \frac{g_1}{2}(|000000\rangle + |000101\rangle + |001010\rangle + |001111\rangle)_{a_1 a_2 A_1 A_2 B_1 B_2} \\
&+ \frac{g_2}{2}(|010000\rangle + |010101\rangle + |011010\rangle + |011111\rangle)_{a_1 a_2 A_1 A_2 B_1 B_2} \\
&+ \frac{g_3}{2}(|100000\rangle + |100101\rangle + |101010\rangle + |101111\rangle)_{a_1 a_2 A_1 A_2 B_1 B_2} \\
&+ \frac{g_4}{2}(|110000\rangle + |110101\rangle + |111010\rangle + |111111\rangle)_{a_1 a_2 A_1 A_2 B_1 B_2}.
\end{aligned}
$$

(9.6)

No measurement has taken place yet, so the state of the qubits remains unchanged. Applying Eqs. (9.2) – (9.5), the combined state $|\Gamma\rangle$ given in Eq. (9.6) can be written as

$$
|\Gamma\rangle = \frac{1}{4} \sum_{j=1}^{16} |G_j\rangle_{a_1 a_2 A_1 A_2} |v_j\rangle_{B_1 B_2},
$$

(9.7)

where the states $|v_j\rangle$ are given in [Table 9.1](#).

Table 9.1

Reduced state and Bob's unitary operations conditioned on Alice's outcomes ⏎

Alice's result	Reduced state with Bob	Bob's perfect operation
$|G_1\rangle$	$|v_1\rangle = (\mathfrak{g}_1|00\rangle + \mathfrak{g}_2|01\rangle + \mathfrak{g}_3|10\rangle + \mathfrak{g}_4|11\rangle)_{B_1 B_2}$	I
$|G_2\rangle$	$|v_2\rangle = (\mathfrak{g}_1|00\rangle + \mathfrak{g}_2|01\rangle - \mathfrak{g}_3|10\rangle - \mathfrak{g}_4|11\rangle)_{B_1 B_2}$	$(\vartheta_z)_{B_1}$
$|G_3\rangle$	$|v_3\rangle = (\mathfrak{g}_1|00\rangle - \mathfrak{g}_2|01\rangle + \mathfrak{g}_3|10\rangle - \mathfrak{g}_4|11\rangle)_{B_1 B_2}$	$(\vartheta_z)_{B_2}$
$|G_4\rangle$	$|v_4\rangle = (\mathfrak{g}_1|00\rangle - \mathfrak{g}_2|01\rangle - \mathfrak{g}_3|10\rangle + \mathfrak{g}_4|11\rangle)_{B_1 B_2}$	$(\vartheta_z)_{B_2} \otimes (\vartheta_z)_{B_1}$
$|G_5\rangle$	$|v_5\rangle = (\mathfrak{g}_1|01\rangle + \mathfrak{g}_2|00\rangle + \mathfrak{g}_3|11\rangle + \mathfrak{g}_4|10\rangle)_{B_1 B_2}$	$(\vartheta_x)_{B_2}$
$|G_6\rangle$	$|v_6\rangle = (\mathfrak{g}_1|01\rangle + \mathfrak{g}_2|00\rangle - \mathfrak{g}_3|11\rangle - \mathfrak{g}_4|10\rangle)_{B_1 B_2}$	$(\vartheta_z)_{B_1} \otimes (\vartheta_x)_{B_2}$
$|G_7\rangle$	$|v_7\rangle = (\mathfrak{g}_1|01\rangle - \mathfrak{g}_2|00\rangle + \mathfrak{g}_3|11\rangle - \mathfrak{g}_4|10\rangle)_{B_1 B_2}$	$(\vartheta_z \vartheta_x)_{B_2}$
$|G_8\rangle$	$|v_8\rangle = (\mathfrak{g}_1|01\rangle - \mathfrak{g}_2|00\rangle - \mathfrak{g}_3|11\rangle + \mathfrak{g}_4|10\rangle)_{B_1 B_2}$	$(\vartheta_z)_{B_1} \otimes (\vartheta_z \vartheta_x)_{B_2}$
$|G_9\rangle$	$|v_9\rangle = (\mathfrak{g}_1|10\rangle + \mathfrak{g}_2|11\rangle + \mathfrak{g}_3|00\rangle + \mathfrak{g}_4|01\rangle)_{B_1 B_2}$	$(\vartheta_x)_{B_1}$
$|G_{10}\rangle$	$|v_{10}\rangle = (\mathfrak{g}_1|10\rangle + \mathfrak{g}_2|11\rangle - \mathfrak{g}_3|00\rangle - \mathfrak{g}_4|01\rangle)_{B_1 B_2}$	$(\vartheta_z \vartheta_x)_{B_1}$
$|G_{11}\rangle$	$|v_{11}\rangle = (\mathfrak{g}_1|10\rangle - \mathfrak{g}_2|11\rangle + \mathfrak{g}_3|00\rangle - \mathfrak{g}_4|01\rangle)_{B_1 B_2}$	$(\vartheta_z)_{B_2} \otimes (\vartheta_x)_{B_1}$
$|G_{12}\rangle$	$|v_{12}\rangle = (\mathfrak{g}_1|10\rangle - \mathfrak{g}_2|11\rangle - \mathfrak{g}_3|00\rangle + \mathfrak{g}_4|01\rangle)_{B_1 B_2}$	$(\vartheta_z)_{B_2} \otimes (\vartheta_z \vartheta_x)_{B_1}$
$|G_{13}\rangle$	$|v_{13}\rangle = (\mathfrak{g}_1|11\rangle + \mathfrak{g}_2|10\rangle + \mathfrak{g}_3|01\rangle + \mathfrak{g}_4|00\rangle)_{B_1 B_2}$	$(\vartheta_x)_{B_2} \otimes (\vartheta_x)_{B_1}$
$|G_{14}\rangle$	$|v_{14}\rangle = (\mathfrak{g}_1|11\rangle + \mathfrak{g}_2|10\rangle - \mathfrak{g}_3|01\rangle - \mathfrak{g}_4|00\rangle)_{B_1 B_2}$	$(\vartheta_z)_{B_1} \otimes (\vartheta_x)_{B_2} \otimes (\vartheta_x)_{B_1}$
$|G_{15}\rangle$	$|v_{15}\rangle = (\mathfrak{g}_1|11\rangle - \mathfrak{g}_2|10\rangle + \mathfrak{g}_3|01\rangle - \mathfrak{g}_4|00\rangle)_{B_1 B_2}$	$(\vartheta_z \vartheta_x)_{B_2} \otimes (\vartheta_x)_{B_1}$
$|G_{16}\rangle$	$|v_{16}\rangle = (\mathfrak{g}_1|11\rangle - \mathfrak{g}_2|10\rangle - \mathfrak{g}_3|01\rangle + \mathfrak{g}_4|00\rangle)_{B_1 B_2}$	$(\vartheta_z)_{B_2} \otimes (\vartheta_z)_{B_1} \otimes (\vartheta_x)_{B_2} \otimes (\vartheta_x)_{B_1}$

Alice then executes a measurement on her four qubits in the *G*-basis mentioned above. Following the measurement, she transmits her outcomes classically to Bob, indicating the specific *G* state that was observed. With this information, Bob is able to identify and apply the correct unitary transformation to his two qubits, allowing him to faithfully reconstruct the general two-qubit quantum state originally in the possession of Alice. The corresponding Pauli operations against Alice's possible results are

provided in [Table 9.1](#). A visual representation of the entire protocol is presented in [Figure 9.2](#). The following is an illustration of the above.
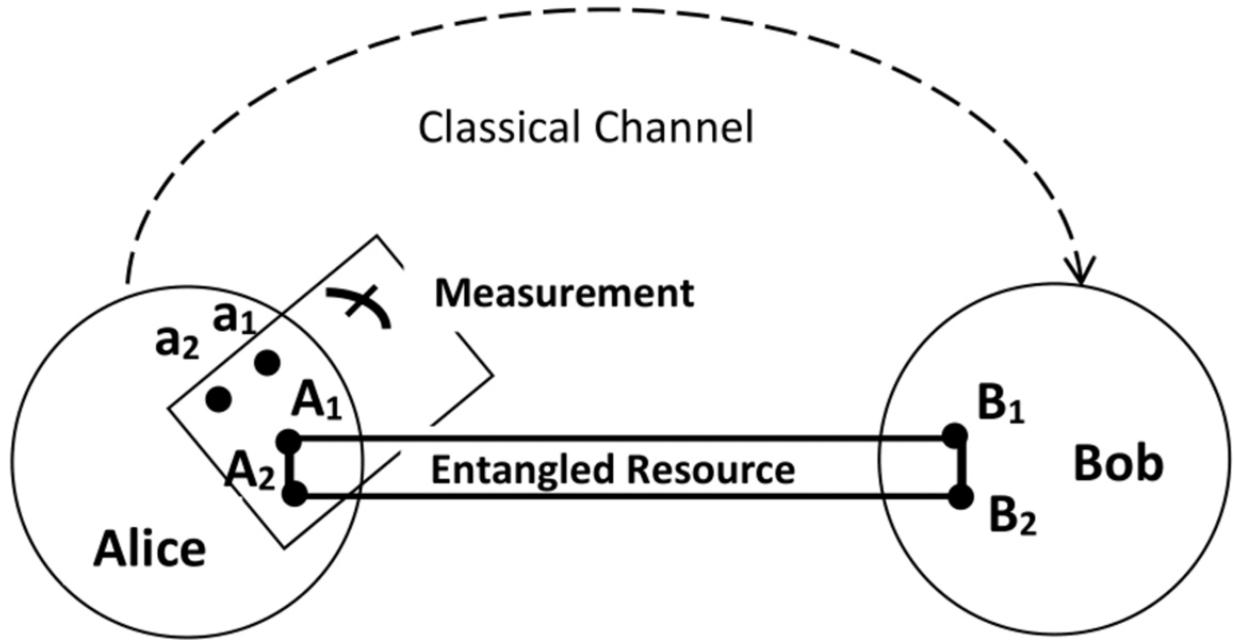
Figure 9.2 Schematic diagram of transferring a general 2-qubit state through quantum teleportation. ⏎

Suppose, for instance, that Alice obtains the measurement result $|G_{11}\rangle_{a_1 a_2 A_1 A_2}$, then the reduced state at Bob's site becomes

$$(\mathfrak{g}_1|10\rangle - \mathfrak{g}_2|11\rangle + \mathfrak{g}_3|00\rangle - \mathfrak{g}_4|01\rangle)_{B_1 B_2}.$$

Alice now uses the classical communication channel to transfer her results to Bob. Upon receiving this information, Bob performs the corresponding unitary operation $(\vartheta_z)_{B_2} \otimes (\vartheta_x)_{B_1}$ on his qubits as shown in [Table 9.1](#). This completes the teleportation of the general two-qubit state.

9.3   TELEPORTATION OF ARBITRARY THREE-QUBIT STATES

We now describe a teleportation protocol that enables the transfer of a general three-qubit state from Alice to Bob. There is no limit to the physical distance by which they can be separated. The protocol described here is designed by Yi-you et al. [112]. The unknown quantum state to be teleported is represented as follows:

$$|\aleph\rangle_{\mathfrak{a}_1 \mathfrak{a}_2 \mathfrak{a}_3} = (\mathfrak{g}_1|000\rangle \quad + \mathfrak{g}_2|001\rangle + \mathfrak{g}_3|010\rangle + \mathfrak{g}_4|011\rangle + \mathfrak{g}_5|100\rangle$$
$$+ \mathfrak{g}_6|101\rangle + \mathfrak{g}_7|110\rangle + \mathfrak{g}_8|111\rangle),$$

<div align="right">(9.8)</div>

where Alice possesses the qubits $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3$ and the parameters obey the criteria for the normalization condition, that is,

$$\sum_{k=1}^{8} |\mathfrak{g}_k|^2 = 1.$$

At the outset Alice and Bob share three W-class states which are given by

$$|E_1\rangle_{A_1 A_2 B_1} = \tfrac{1}{2}(|100\rangle + |010\rangle + \sqrt{2}|001\rangle)_{A_1 A_2 B_1},$$
$$|E_2\rangle_{A_3 A_4 B_2} = \tfrac{1}{2}(|100\rangle + |010\rangle + \sqrt{2}|001\rangle)_{A_3 A_4 B_2},$$
$$|E_3\rangle_{A_5 A_6 B_3} = \tfrac{1}{2}(|100\rangle + |010\rangle + \sqrt{2}|001\rangle)_{A_5 A_6 B_3}.$$

<div align="right">(9.9)</div>

Alice holds the qubits $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, A_1, A_2, A_3, A_4, A_5, A_6$, whereas qubits B_1, B_2, B_3 belong to Bob.

Also there is a classical communication channel between Alice and Bob.

The total quantum system, composed of the twelve qubits, can be written as

$$|\Gamma\rangle = |\aleph\rangle_{\mathfrak{a}_1 \mathfrak{a}_2 \mathfrak{a}_3} \otimes |E_1\rangle_{A_1 A_2 B_1} \otimes |E_2\rangle_{A_3 A_4 B_2} \otimes |E_3\rangle_{A_5 A_6 B_3}.$$

<div align="right">(9.10)</div>

In order to transfer the original state given in Eq. (9.8) at the site of Bob, first Alice executes three 3-qubit measurements on the basis containing a set of linearly independent vectors $\{|\varepsilon^{\pm}\rangle, |\omega^{\pm}\rangle\}$, respectively. These are given by

$$|\varepsilon^{\pm}\rangle_{(\mathfrak{a}_1 A_1 A_2)/(\mathfrak{a}_2 A_3 A_4)/(\mathfrak{a}_3 A_5 A_6)} = \tfrac{1}{2}(|010\rangle + |001\rangle \pm \sqrt{2}|100\rangle),$$
$$|\omega^{\pm}\rangle_{(\mathfrak{a}_1 A_1 A_2)/(\mathfrak{a}_2 A_3 A_4)/(\mathfrak{a}_3 A_5 A_6)} = \tfrac{1}{2}(|110\rangle + |101\rangle \pm \sqrt{2}|000\rangle).$$

<div align="right">(9.11)</div>

By a result of linear algebra, it is always possible for Alice to augment this set into a basis, that is, to find a basis containing the above set of four ket vectors and then perform the measurement.

After completing the measurement, Alice gets one of the 64 possible outcomes with equal probability described in the following cases. The state of the remaining qubits are detailed in Eqs. (9.12) - (9.19).

Case I: Measurement result of Alice: $|\varepsilon^{\pm}\rangle_{\mathfrak{a}_1 A_1 A_2} |\varepsilon^{\pm}\rangle_{\mathfrak{a}_2 A_3 A_4} |\varepsilon^{\pm}\rangle_{\mathfrak{a}_3 A_5 A_6}$

Then the state of the qubits remaining with Bob becomes

$$
\begin{aligned}
_{\mathfrak{a}_3 A_5 A_6}\langle \varepsilon^{\pm}|_{\mathfrak{a}_2 A_3 A_4}\langle \varepsilon^{\pm}|_{\mathfrak{a}_1 A_1 A_2}\langle \varepsilon^{\pm}||\Gamma\rangle \; &= \; \frac{1}{8}(\mathfrak{g}_1|000\rangle \pm ++ \mathfrak{g}_2|001\rangle +\pm+ \mathfrak{g}_3|010\rangle \\
&\quad \pm\pm+ \mathfrak{g}_4|011\rangle +++ \mathfrak{g}_5|100\rangle \pm+\pm \mathfrak{g}_6|101\rangle \\
&\quad +\pm\pm \mathfrak{g}_7|110\rangle \pm\pm\pm \mathfrak{g}_8|111\rangle)_{B_1 B_2 B_3}.
\end{aligned}
$$

$$(9.12)$$

Case-II: Measurement result of Alice: $|\varepsilon^{\pm}\rangle_{\mathfrak{a}_1 A_1 A_2} |\varepsilon^{\pm}\rangle_{\mathfrak{a}_2 A_3 A_4} |\omega^{\pm}\rangle_{\mathfrak{a}_3 A_5 A_6}$

Then the state of the qubits remaining with Bob becomes

$$
\begin{aligned}
_{\mathfrak{a}_3 A_5 A_6}\langle \omega^{\pm}|_{\mathfrak{a}_2 A_3 A_4}\langle \varepsilon^{\pm}|_{\mathfrak{a}_1 A_1 A_2}\langle \varepsilon^{\pm}||\Gamma\rangle \; &= \; \frac{1}{8}(\pm++\mathfrak{g}_1|001\rangle + \mathfrak{g}_2|000\rangle \pm\pm+ \mathfrak{g}_3|011\rangle \\
&\quad +\pm+ \mathfrak{g}_4|010\rangle \pm+\pm \mathfrak{g}_5|101\rangle +++ \mathfrak{g}_6|100\rangle \\
&\quad \pm\pm\pm \mathfrak{g}_7|111\rangle +\pm\pm \mathfrak{g}_8|110\rangle)_{B_1 B_2 B_3}.
\end{aligned}
$$

$$(9.13)$$

Case-III: Measurement result of Alice: $|\varepsilon^{\pm}\rangle_{\mathfrak{a}_1 A_1 A_2} |\omega^{\pm}\rangle_{\mathfrak{a}_2 A_3 A_4} |\varepsilon^{\pm}\rangle_{\mathfrak{a}_3 A_5 A_6}$

Then the state of the qubits remaining with Bob becomes

$$
\begin{aligned}
_{\mathfrak{a}_3 A_5 A_6}\langle \varepsilon^{\pm}|_{\mathfrak{a}_2 A_3 A_4}\langle \omega^{\pm}|_{\mathfrak{a}_1 A_1 A_2}\langle \varepsilon^{\pm}||\Gamma\rangle \; &= \; \frac{1}{8}(+\pm+\mathfrak{g}_1|010\rangle \pm\pm+ \mathfrak{g}_2|011\rangle + \mathfrak{g}_3|000\rangle \\
&\quad \pm++ \mathfrak{g}_4|001\rangle +\pm\pm \mathfrak{g}_5|110\rangle \pm\pm\pm \mathfrak{g}_6|111\rangle \\
&\quad ++\pm \mathfrak{g}_7|100\rangle \pm+\pm \mathfrak{g}_8|101\rangle)_{B_1 B_2 B_3}.
\end{aligned}
$$

$$(9.14)$$

Case-IV: Measurement result of Alice: $|\varepsilon^\pm\rangle_{\mathfrak{a}_1 A_1 A_2} |\omega^\pm\rangle_{\mathfrak{a}_2 A_3 A_4} |\omega^\pm\rangle_{\mathfrak{a}_3 A_5 A_6}$

Then the state of the qubits remaining with Bob becomes

$$
\begin{aligned}
{}_{\mathfrak{a}_3 A_5 A_6}\langle\omega^\pm|_{\mathfrak{a}_2 A_3 A_4}\langle\omega^\pm|_{\mathfrak{a}_1 A_1 A_2}\langle\varepsilon^\pm||\Gamma\rangle \ = \ &\frac{1}{8}(\pm\pm+\mathfrak{g}_1|011\rangle+\pm+\ \mathfrak{g}_2|010\rangle \pm++\ \mathfrak{g}_3|001\rangle \\
&+\ \mathfrak{g}_4|000\rangle \pm\pm\pm\ \mathfrak{g}_5|111\rangle + \pm\pm\pm\ \mathfrak{g}_6|110\rangle \\
&\pm+\pm\ \mathfrak{g}_7|101\rangle ++\pm\ \mathfrak{g}_8|100\rangle)_{B_1 B_2 B_3}.
\end{aligned}
$$

$$(9.15)$$

Case-V: Measurement result of Alice: $|\omega^\pm\rangle_{\mathfrak{a}_1 A_1 A_2} |\varepsilon^\pm\rangle_{\mathfrak{a}_2 A_3 A_4} |\varepsilon^\pm\rangle_{\mathfrak{a}_3 A_5 A_6}$

Then the state of the qubits remaining with Bob becomes

$$
\begin{aligned}
{}_{\mathfrak{a}_3 A_5 A_6}\langle\varepsilon^\pm|_{\mathfrak{a}_2 A_3 A_4}\langle\varepsilon^\pm|_{\mathfrak{a}_1 A_1 A_2}\langle\omega^\pm||\Gamma\rangle \ = \ &\frac{1}{8}(++\pm\mathfrak{g}_1|100\rangle\pm+\pm\mathfrak{g}_2|101\rangle+\pm\pm\mathfrak{g}_3|110\rangle \\
&\pm\pm\pm\ \mathfrak{g}_4|111\rangle + \mathfrak{g}_5|000\rangle \pm++\ \mathfrak{g}_6|001\rangle \\
&+\pm+\ \mathfrak{g}_7|010\rangle \pm\pm+\ \mathfrak{g}_8|011\rangle)_{B_1 B_2 B_3}.
\end{aligned}
$$

$$(9.16)$$

Case-VI: Measurement result of Alice: $|\omega^\pm\rangle_{\mathfrak{a}_1 A_1 A_2} |\varepsilon^\pm\rangle_{\mathfrak{a}_2 A_3 A_4} |\omega^\pm\rangle_{\mathfrak{a}_3 A_5 A_6}$

Then the state of the qubits remaining with Bob becomes

$$
\begin{aligned}
{}_{\mathfrak{a}_3 A_5 A_6}\langle\omega^\pm|_{\mathfrak{a}_2 A_3 A_4}\langle\varepsilon^\pm|_{\mathfrak{a}_1 A_1 A_2}\langle\omega^\pm||\Gamma\rangle \ = \ &\frac{1}{8}(\pm+\pm\mathfrak{g}_1|101\rangle++\pm\mathfrak{g}_2|100\rangle\pm\pm\pm\mathfrak{g}_3|111\rangle \\
&+\pm\pm\ \mathfrak{g}_4|110\rangle \pm++\ \mathfrak{g}_5|001\rangle + \mathfrak{g}_6|000\rangle \\
&\pm\pm+\ \mathfrak{g}_7|011\rangle +\pm+\ \mathfrak{g}_8|010\rangle)_{B_1 B_2 B_3}.
\end{aligned}
$$

$$(9.17)$$

Case-VII: Measurement result of Alice: $|\omega^\pm\rangle_{\mathfrak{a}_1 A_1 A_2} |\omega^\pm\rangle_{\mathfrak{a}_2 A_3 A_4} |\varepsilon^\pm\rangle_{\mathfrak{a}_3 A_5 A_6}$

Then the state of the qubits remaining with Bob becomes

$$
\begin{aligned}
{}_{\mathfrak{a}_3 A_5 A_6}\langle\varepsilon^\pm|_{\mathfrak{a}_2 A_3 A_4}\langle\omega^\pm|_{\mathfrak{a}_1 A_1 A_2}\langle\omega^\pm||\Gamma\rangle \ = \ &\frac{1}{8}(+\pm\pm\mathfrak{g}_1|110\rangle\pm\pm\pm\mathfrak{g}_2|111\rangle++\pm\mathfrak{g}_3|100\rangle \\
&\pm+\pm\ \mathfrak{g}_4|101\rangle + \pm+\ \mathfrak{g}_5|010\rangle \pm\pm+\ \mathfrak{g}_6|011\rangle \\
&+\ \mathfrak{g}_7|000\rangle \pm++\ \mathfrak{g}_8|001\rangle)_{B_1 B_2 B_3}.
\end{aligned}
$$

Case-VIII: Measurement result of Alice: $|\omega^{\pm}\rangle_{\mathfrak{a}_1 A_1 A_2} |\omega^{\pm}\rangle_{\mathfrak{a}_2 A_3 A_4} |\omega^{\pm}\rangle_{\mathfrak{a}_3 A_5 A_6}$

Then the state of the qubits remaining with Bob becomes

$$\begin{aligned}
_{\mathfrak{a}_3 A_5 A_6}\langle \omega^{\pm}|_{\mathfrak{a}_2 A_3 A_4}\langle \omega^{\pm}|_{\mathfrak{a}_1 A_1 A_2}\langle \omega^{\pm}||\Gamma\rangle \ &= \frac{1}{8}(\pm\pm\pm\mathfrak{g}_1|111\rangle +\pm\pm\,\mathfrak{g}_2|110\rangle \pm+\pm\,\mathfrak{g}_3|101\rangle \\
&\quad + +\pm\,\mathfrak{g}_4|100\rangle \pm\pm+\mathfrak{g}_5|011\rangle + \pm+\mathfrak{g}_6|010\rangle \\
&\quad \pm++\mathfrak{g}_7|001\rangle + \mathfrak{g}_8|000\rangle)_{B_1 B_2 B_3}.
\end{aligned}$$

The notation above is explained as follows. The signs '+' or '$\pm$' from right to left in the right hand side of Eqs. (9.12) - (9.19) reflect to Alice's measurements of qubits $(\mathfrak{a}_1 A_1 A_2), (\mathfrak{a}_2 A_3 A_4), (\mathfrak{a}_3 A_5 A_6)$, respectively. Although the notations '$\pm$' of the qubits B_1, B_2 and B_3 in the second column of Table 9.2, Table 9.3, Table 9.4, Table 9.5, Table 9.6, Table 9.7, Table 9.8, Table 9.9 correspond to the measurement of qubits $(\mathfrak{a}_1 A_1 A_2), (\mathfrak{a}_2 A_3 A_4), (\mathfrak{a}_3 A_5 A_6)$, respectively. If the three-qubit measurement is '+', the notation '$\pm$' will be '+' and while in the other case it will be '−'.

Table 9.2

Bob's required unitary operations determined by Alice's outcomes (first column), are presented in the second column for Case-I ↵

Alice's results	Bob's operations
$|\varepsilon^+\rangle_{\mathfrak{a}_1 A_1 A_2}|\varepsilon^+\rangle_{\mathfrak{a}_2 A_3 A_4}|\varepsilon^+\rangle_{\mathfrak{a}_3 A_5 A_6}$	$(I)_{B_1} \otimes (I)_{B_2} \otimes (I)_{B_3}$
$|\varepsilon^+\rangle_{\mathfrak{a}_1 A_1 A_2}|\varepsilon^+\rangle_{\mathfrak{a}_2 A_3 A_4}|\varepsilon^-\rangle_{\mathfrak{a}_3 A_5 A_6}$	$(I)_{B_1} \otimes (I)_{B_2} \otimes (\vartheta_z)_{B_3}$
$|\varepsilon^+\rangle_{\mathfrak{a}_1 A_1 A_2}|\varepsilon^-\rangle_{\mathfrak{a}_2 A_3 A_4}|\varepsilon^+\rangle_{\mathfrak{a}_3 A_5 A_6}$	$(I)_{B_1} \otimes (\vartheta_z)_{B_2} \otimes (I)_{B_3}$
$|\varepsilon^+\rangle_{\mathfrak{a}_1 A_1 A_2}|\varepsilon^-\rangle_{\mathfrak{a}_2 A_3 A_4}|\varepsilon^-\rangle_{\mathfrak{a}_3 A_5 A_6}$	$(I)_{B_1} \otimes (\vartheta_z)_{B_2} \otimes (\vartheta_z)_{B_3}$
$|\varepsilon^-\rangle_{\mathfrak{a}_1 A_1 A_2}|\varepsilon^+\rangle_{\mathfrak{a}_2 A_3 A_4}|\varepsilon^+\rangle_{\mathfrak{a}_3 A_5 A_6}$	$(\vartheta_z)_{B_1} \otimes (I)_{B_2} \otimes (I)_{B_3}$
$|\varepsilon^-\rangle_{\mathfrak{a}_1 A_1 A_2}|\varepsilon^+\rangle_{\mathfrak{a}_2 A_3 A_4}|\varepsilon^-\rangle_{\mathfrak{a}_3 A_5 A_6}$	$(\vartheta_z)_{B_1} \otimes (I)_{B_2} \otimes (\vartheta_z)_{B_3}$
$|\varepsilon^-\rangle_{\mathfrak{a}_1 A_1 A_2}|\varepsilon^-\rangle_{\mathfrak{a}_2 A_3 A_4}|\varepsilon^+\rangle_{\mathfrak{a}_3 A_5 A_6}$	$(\vartheta_z)_{B_1} \otimes (\vartheta_z)_{B_2} \otimes (I)_{B_3}$
$|\varepsilon^-\rangle_{\mathfrak{a}_1 A_1 A_2}|\varepsilon^-\rangle_{\mathfrak{a}_2 A_3 A_4}|\varepsilon^-\rangle_{\mathfrak{a}_3 A_5 A_6}$	$(\vartheta_z)_{B_1} \otimes (\vartheta_z)_{B_2} \otimes (\vartheta_z)_{B_3}$

Table 9.3

Bob's required unitary operations determined by Alice's outcomes (first column), are presented in the second column for Case-II ⏎

Alice's results	Bob's operations
$\lvert\varepsilon^{+}\rangle_{a_1 A_1 A_2}\lvert\varepsilon^{+}\rangle_{a_2 A_3 A_4}\lvert\omega^{+}\rangle_{a_3 A_5 A_6}$	$(I)_{B_1}\otimes(I)_{B_2}\otimes(\vartheta_x)_{B_3}$
$\lvert\varepsilon^{+}\rangle_{a_1 A_1 A_2}\lvert\varepsilon^{+}\rangle_{a_2 A_3 A_4}\lvert\omega^{-}\rangle_{a_3 A_5 A_6}$	$(I)_{B_1}\otimes(I)_{B_2}\otimes(\vartheta_x\vartheta_z)_{B_3}$
$\lvert\varepsilon^{+}\rangle_{a_1 A_1 A_2}\lvert\varepsilon^{-}\rangle_{a_2 A_3 A_4}\lvert\omega^{+}\rangle_{a_3 A_5 A_6}$	$(I)_{B_1}\otimes(\vartheta_z)_{B_2}\otimes(\vartheta_x)_{B_3}$
$\lvert\varepsilon^{+}\rangle_{a_1 A_1 A_2}\lvert\varepsilon^{-}\rangle_{a_2 A_3 A_4}\lvert\omega^{-}\rangle_{a_3 A_5 A_6}$	$(I)_{B_1}\otimes(\vartheta_z)_{B_2}\otimes(\vartheta_x\vartheta_z)_{B_3}$
$\lvert\varepsilon^{-}\rangle_{a_1 A_1 A_2}\lvert\varepsilon^{+}\rangle_{a_2 A_3 A_4}\lvert\omega^{+}\rangle_{a_3 A_5 A_6}$	$(\vartheta_z)_{B_1}\otimes(I)_{B_2}\otimes(\vartheta_x)_{B_3}$
$\lvert\varepsilon^{-}\rangle_{a_1 A_1 A_2}\lvert\varepsilon^{+}\rangle_{a_2 A_3 A_4}\lvert\omega^{-}\rangle_{a_3 A_5 A_6}$	$(\vartheta_z)_{B_1}\otimes(I)_{B_2}\otimes(\vartheta_x\vartheta_z)_{B_3}$
$\lvert\varepsilon^{-}\rangle_{a_1 A_1 A_2}\lvert\varepsilon^{-}\rangle_{a_2 A_3 A_4}\lvert\omega^{+}\rangle_{a_3 A_5 A_6}$	$(\vartheta_z)_{B_1}\otimes(\vartheta_z)_{B_2}\otimes(\vartheta_x)_{B_3}$
$\lvert\varepsilon^{-}\rangle_{a_1 A_1 A_2}\lvert\varepsilon^{-}\rangle_{a_2 A_3 A_4}\lvert\omega^{-}\rangle_{a_3 A_5 A_6}$	$(\vartheta_z)_{B_1}\otimes(\vartheta_z)_{B_2}\otimes(\vartheta_x\vartheta_z)_{B_3}$

Table 9.4

Bob's required unitary operations determined by Alice's outcomes (first column), are presented in the second column for Case-III ⏎

Alice's results	Bob's operations
$\lvert\varepsilon^{+}\rangle_{a_1 A_1 A_2}\lvert\omega^{+}\rangle_{a_2 A_3 A_4}\lvert\varepsilon^{+}\rangle_{a_3 A_5 A_6}$	$(I)_{B_1}\otimes(\vartheta_x)_{B_2}\otimes(I)_{B_3}$
$\lvert\varepsilon^{+}\rangle_{a_1 A_1 A_2}\lvert\omega^{+}\rangle_{a_2 A_3 A_4}\lvert\varepsilon^{-}\rangle_{a_3 A_5 A_6}$	$(I)_{B_1}\otimes(\vartheta_x)_{B_2}\otimes(\vartheta_z)_{B_3}$
$\lvert\varepsilon^{+}\rangle_{a_1 A_1 A_2}\lvert\omega^{-}\rangle_{a_2 A_3 A_4}\lvert\varepsilon^{+}\rangle_{a_3 A_5 A_6}$	$(I)_{B_1}\otimes(\vartheta_x\vartheta_z)_{B_2}\otimes(I)_{B_3}$
$\lvert\varepsilon^{+}\rangle_{a_1 A_1 A_2}\lvert\omega^{-}\rangle_{a_2 A_3 A_4}\lvert\varepsilon^{-}\rangle_{a_3 A_5 A_6}$	$(I)_{B_1}\otimes(\vartheta_x\vartheta_z)_{B_2}\otimes(\vartheta_z)_{B_3}$
$\lvert\varepsilon^{-}\rangle_{a_1 A_1 A_2}\lvert\omega^{+}\rangle_{a_2 A_3 A_4}\lvert\varepsilon^{+}\rangle_{a_3 A_5 A_6}$	$(\vartheta_z)_{B_1}\otimes(\vartheta_x)_{B_2}\otimes(I)_{B_3}$
$\lvert\varepsilon^{-}\rangle_{a_1 A_1 A_2}\lvert\omega^{+}\rangle_{a_2 A_3 A_4}\lvert\varepsilon^{-}\rangle_{a_3 A_5 A_6}$	$(\vartheta_z)_{B_1}\otimes(\vartheta_x)_{B_2}\otimes(\vartheta_z)_{B_3}$
$\lvert\varepsilon^{-}\rangle_{a_1 A_1 A_2}\lvert\omega^{-}\rangle_{a_2 A_3 A_4}\lvert\varepsilon^{+}\rangle_{a_3 A_5 A_6}$	$(\vartheta_z)_{B_1}\otimes(\vartheta_x\vartheta_z)_{B_2}\otimes(I)_{B_3}$
$\lvert\varepsilon^{-}\rangle_{a_1 A_1 A_2}\lvert\omega^{-}\rangle_{a_2 A_3 A_4}\lvert\varepsilon^{-}\rangle_{a_3 A_5 A_6}$	$(\vartheta_z)_{B_1}\otimes(\vartheta_x\vartheta_z)_{B_2}\otimes(\vartheta_z)_{B_3}$

Table 9.5

Bob's required unitary operations determined by Alice's outcomes (first column), are presented in the second column for Case-IV ⏎

Alice's results	Bob's operations
$\|\varepsilon^+\rangle_{a_1 A_1 A_2} \|\omega^+\rangle_{a_2 A_3 A_4} \|\omega^+\rangle_{a_3 A_5 A_6}$	$(I)_{B_1} \otimes (\vartheta_x)_{B_2} \otimes (\vartheta_x)_{B_3}$
$\|\varepsilon^+\rangle_{a_1 A_1 A_2} \|\omega^+\rangle_{a_2 A_3 A_4} \|\omega^-\rangle_{a_3 A_5 A_6}$	$(I)_{B_1} \otimes (\vartheta_x)_{B_2} \otimes (\vartheta_x \vartheta_z)_{B_3}$
$\|\varepsilon^+\rangle_{a_1 A_1 A_2} \|\omega^-\rangle_{a_2 A_3 A_4} \|\omega^+\rangle_{a_3 A_5 A_6}$	$(I)_{B_1} \otimes (\vartheta_x \vartheta_z)_{B_2} \otimes (\vartheta_x)_{B_3}$
$\|\varepsilon^+\rangle_{a_1 A_1 A_2} \|\omega^-\rangle_{a_2 A_3 A_4} \|\omega^-\rangle_{a_3 A_5 A_6}$	$(I)_{B_1} \otimes (\vartheta_x \vartheta_z)_{B_2} \otimes (\vartheta_x \vartheta_z)_{B_3}$
$\|\varepsilon^-\rangle_{a_1 A_1 A_2} \|\omega^+\rangle_{a_2 A_3 A_4} \|\omega^+\rangle_{a_3 A_5 A_6}$	$(\vartheta_z)_{B_1} \otimes (\vartheta_x)_{B_2} \otimes (\vartheta_x)_{B_3}$
$\|\varepsilon^-\rangle_{a_1 A_1 A_2} \|\omega^+\rangle_{a_2 A_3 A_4} \|\omega^-\rangle_{a_3 A_5 A_6}$	$(\vartheta_z)_{B_1} \otimes (\vartheta_x)_{B_2} \otimes (\vartheta_x \vartheta_z)_{B_3}$
$\|\varepsilon^-\rangle_{a_1 A_1 A_2} \|\omega^-\rangle_{a_2 A_3 A_4} \|\omega^+\rangle_{a_3 A_5 A_6}$	$(\vartheta_z)_{B_1} \otimes (\vartheta_x \vartheta_z)_{B_2} \otimes (\vartheta_x)_{B_3}$
$\|\varepsilon^-\rangle_{a_1 A_1 A_2} \|\omega^-\rangle_{a_2 A_3 A_4} \|\omega^-\rangle_{a_3 A_5 A_6}$	$(\vartheta_z)_{B_1} \otimes (\vartheta_x \vartheta_z)_{B_2} \otimes (\vartheta_x \vartheta_z)_{B_3}$

Table 9.6

Bob's required unitary operations determined by Alice's outcomes (first column), are presented in the second column for Case-V ⏎

Alice's results	Bob's operations
$\|\omega^+\rangle_{a_1 A_1 A_2} \|\varepsilon^+\rangle_{a_2 A_3 A_4} \|\varepsilon^+\rangle_{a_3 A_5 A_6}$	$(\vartheta_x)_{B_1} \otimes (I)_{B_2} \otimes (I)_{B_3}$
$\|\omega^+\rangle_{a_1 A_1 A_2} \|\varepsilon^+\rangle_{a_2 A_3 A_4} \|\varepsilon^-\rangle_{a_3 A_5 A_6}$	$(\vartheta_x)_{B_1} \otimes (I)_{B_2} \otimes (\vartheta_z)_{B_3}$
$\|\omega^+\rangle_{a_1 A_1 A_2} \|\varepsilon^-\rangle_{a_2 A_3 A_4} \|\varepsilon^+\rangle_{a_3 A_5 A_6}$	$(\vartheta_x)_{B_1} \otimes (\vartheta_z)_{B_2} \otimes (I)_{B_3}$
$\|\omega^+\rangle_{a_1 A_1 A_2} \|\varepsilon^-\rangle_{a_2 A_3 A_4} \|\varepsilon^-\rangle_{a_3 A_5 A_6}$	$(\vartheta_x)_{B_1} \otimes (\vartheta_z)_{B_2} \otimes (\vartheta_z)_{B_3}$
$\|\omega^-\rangle_{a_1 A_1 A_2} \|\varepsilon^+\rangle_{a_2 A_3 A_4} \|\varepsilon^+\rangle_{a_3 A_5 A_6}$	$(\vartheta_x \vartheta_z)_{B_1} \otimes (I)_{B_2} \otimes (I)_{B_3}$
$\|\omega^-\rangle_{a_1 A_1 A_2} \|\varepsilon^+\rangle_{a_2 A_3 A_4} \|\varepsilon^-\rangle_{a_3 A_5 A_6}$	$(\vartheta_x \vartheta_z)_{B_1} \otimes (I)_{B_2} \otimes (\vartheta_z)_{B_3}$
$\|\omega^-\rangle_{a_1 A_1 A_2} \|\varepsilon^-\rangle_{a_2 A_3 A_4} \|\varepsilon^+\rangle_{a_3 A_5 A_6}$	$(\vartheta_x \vartheta_z)_{B_1} \otimes (\vartheta_z)_{B_2} \otimes (I)_{B_3}$
$\|\omega^-\rangle_{a_1 A_1 A_2} \|\varepsilon^-\rangle_{a_2 A_3 A_4} \|\varepsilon^-\rangle_{a_3 A_5 A_6}$	$(\vartheta_x \vartheta_z)_{B_1} \otimes (\vartheta_z)_{B_2} \otimes (\vartheta_z)_{B_3}$

Table 9.7

Bob's required unitary operations determined by Alice's outcomes (first column), are presented in the second

column for Case-VI

Alice's results	Bob's operations
$\lvert\omega^+\rangle_{a_1A_1A_2}\lvert\varepsilon^+\rangle_{a_2A_3A_4}\lvert\omega^+\rangle_{a_3A_5A_6}$	$(\vartheta_x)_{B_1}\otimes(I)_{B_2}\otimes(\vartheta_x)_{B_3}$
$\lvert\omega^+\rangle_{a_1A_1A_2}\lvert\varepsilon^+\rangle_{a_2A_3A_4}\lvert\omega^-\rangle_{a_3A_5A_6}$	$(\vartheta_x)_{B_1}\otimes(I)_{B_2}\otimes(\vartheta_x\vartheta_z)_{B_3}$
$\lvert\omega^+\rangle_{a_1A_1A_2}\lvert\varepsilon^-\rangle_{a_2A_3A_4}\lvert\omega^+\rangle_{a_3A_5A_6}$	$(\vartheta_x)_{B_1}\otimes(\vartheta_z)_{B_2}\otimes(\vartheta_x)_{B_3}$
$\lvert\omega^+\rangle_{a_1A_1A_2}\lvert\varepsilon^-\rangle_{a_2A_3A_4}\lvert\omega^-\rangle_{a_3A_5A_6}$	$(\vartheta_x)_{B_1}\otimes(\vartheta_z)_{B_2}\otimes(\vartheta_x\vartheta_z)_{B_3}$
$\lvert\omega^-\rangle_{a_1A_1A_2}\lvert\varepsilon^+\rangle_{a_2A_3A_4}\lvert\omega^+\rangle_{a_3A_5A_6}$	$(\vartheta_x\vartheta_z)_{B_1}\otimes(I)_{B_2}\otimes(\vartheta_x)_{B_3}$
$\lvert\omega^-\rangle_{a_1A_1A_2}\lvert\varepsilon^+\rangle_{a_2A_3A_4}\lvert\omega^-\rangle_{a_3A_5A_6}$	$(\vartheta_x\vartheta_z)_{B_1}\otimes(I)_{B_2}\otimes(\vartheta_x\vartheta_z)_{B_3}$
$\lvert\omega^-\rangle_{a_1A_1A_2}\lvert\varepsilon^-\rangle_{a_2A_3A_4}\lvert\omega^+\rangle_{a_3A_5A_6}$	$(\vartheta_x\vartheta_z)_{B_1}\otimes(\vartheta_z)_{B_2}\otimes(\vartheta_x)_{B_3}$
$\lvert\omega^-\rangle_{a_1A_1A_2}\lvert\varepsilon^-\rangle_{a_2A_3A_4}\lvert\omega^-\rangle_{a_3A_5A_6}$	$(\vartheta_x\vartheta_z)_{B_1}\otimes(\vartheta_z)_{B_2}\otimes(\vartheta_x\vartheta_z)_{B_3}$

Table 9.8

Bob's required unitary operations determined by Alice's outcomes (first column), are presented in the second column for Case-VII ⏎

Alice's results	Bob's operations
$\lvert\omega^+\rangle_{a_1A_1A_2}\lvert\omega^+\rangle_{a_2A_3A_4}\lvert\varepsilon^+\rangle_{a_3A_5A_6}$	$(\vartheta_x)_{B_1}\otimes(\vartheta_x)_{B_2}\otimes(I)_{B_3}$
$\lvert\omega^+\rangle_{a_1A_1A_2}\lvert\omega^+\rangle_{a_2A_3A_4}\lvert\varepsilon^-\rangle_{a_3A_5A_6}$	$(\vartheta_x)_{B_1}\otimes(\vartheta_x)_{B_2}\otimes(\vartheta_z)_{B_3}$
$\lvert\omega^+\rangle_{a_1A_1A_2}\lvert\omega^-\rangle_{a_2A_3A_4}\lvert\varepsilon^+\rangle_{a_3A_5A_6}$	$(\vartheta_x)_{B_1}\otimes(\vartheta_x\vartheta_z)_{B_2}\otimes(I)_{B_3}$
$\lvert\omega^+\rangle_{a_1A_1A_2}\lvert\omega^-\rangle_{a_2A_3A_4}\lvert\varepsilon^-\rangle_{a_3A_5A_6}$	$(\vartheta_x)_{B_1}\otimes(\vartheta_x\vartheta_z)_{B_2}\otimes(\vartheta_z)_{B_3}$
$\lvert\omega^-\rangle_{a_1A_1A_2}\lvert\omega^+\rangle_{a_2A_3A_4}\lvert\varepsilon^+\rangle_{a_3A_5A_6}$	$(\vartheta_x\vartheta_z)_{B_1}\otimes(\vartheta_x)_{B_2}\otimes(I)_{B_3}$
$\lvert\omega^-\rangle_{a_1A_1A_2}\lvert\omega^+\rangle_{a_2A_3A_4}\lvert\varepsilon^-\rangle_{a_3A_5A_6}$	$(\vartheta_x\vartheta_z)_{B_1}\otimes(\vartheta_x)_{B_2}\otimes(\vartheta_z)_{B_3}$
$\lvert\omega^-\rangle_{a_1A_1A_2}\lvert\omega^-\rangle_{a_2A_3A_4}\lvert\varepsilon^+\rangle_{a_3A_5A_6}$	$(\vartheta_x\vartheta_z)_{B_1}\otimes(\vartheta_x\vartheta_z)_{B_2}\otimes(I)_{B_3}$
$\lvert\omega^-\rangle_{a_1A_1A_2}\lvert\omega^-\rangle_{a_2A_3A_4}\lvert\varepsilon^-\rangle_{a_3A_5A_6}$	$(\vartheta_x\vartheta_z)_{B_1}\otimes(\vartheta_x\vartheta_z)_{B_2}\otimes(\vartheta_z)_{B_3}$

Table 9.9

Bob's required unitary operations determined by Alice's outcomes (first column), are presented in the second column for Case-VIII ⏎

Alice's results	Bob's operations

Alice's results	Bob's operations
$\lvert\omega^+\rangle_{a_1A_1A_2}\lvert\omega^+\rangle_{a_2A_3A_4}\lvert\omega^+\rangle_{a_3A_5A_6}$	$(\vartheta_x)_{B_1}\otimes(\vartheta_x)_{B_2}\otimes(\vartheta_x)_{B_3}$
$\lvert\omega^+\rangle_{a_1A_1A_2}\lvert\omega^+\rangle_{a_2A_3A_4}\lvert\omega^-\rangle_{a_3A_5A_6}$	$(\vartheta_x)_{B_1}\otimes(\vartheta_x)_{B_2}\otimes(\vartheta_x\vartheta_z)_{B_3}$
$\lvert\omega^+\rangle_{a_1A_1A_2}\lvert\omega^-\rangle_{a_2A_3A_4}\lvert\omega^+\rangle_{a_3A_5A_6}$	$(\vartheta_x)_{B_1}\otimes(\vartheta_x\vartheta_z)_{B_2}\otimes(\vartheta_x)_{B_3}$
$\lvert\omega^+\rangle_{a_1A_1A_2}\lvert\omega^-\rangle_{a_2A_3A_4}\lvert\omega^-\rangle_{a_3A_5A_6}$	$(\vartheta_x)_{B_1}\otimes(\vartheta_x\vartheta_z)_{B_2}\otimes(\vartheta_x\vartheta_z)_{B_3}$
$\lvert\omega^-\rangle_{a_1A_1A_2}\lvert\omega^+\rangle_{a_2A_3A_4}\lvert\omega^+\rangle_{a_3A_5A_6}$	$(\vartheta_x\vartheta_z)_{B_1}\otimes(\vartheta_x)_{B_2}\otimes(\vartheta_x)_{B_3}$
$\lvert\omega^-\rangle_{a_1A_1A_2}\lvert\omega^+\rangle_{a_2A_3A_4}\lvert\omega^-\rangle_{a_3A_5A_6}$	$(\vartheta_x\vartheta_z)_{B_1}\otimes(\vartheta_x)_{B_2}\otimes(\vartheta_x\vartheta_z)_{B_3}$
$\lvert\omega^-\rangle_{a_1A_1A_2}\lvert\omega^-\rangle_{a_2A_3A_4}\lvert\omega^+\rangle_{a_3A_5A_6}$	$(\vartheta_x\vartheta_z)_{B_1}\otimes(\vartheta_x\vartheta_z)_{B_2}\otimes(\vartheta_x)_{B_3}$
$\lvert\omega^-\rangle_{a_1A_1A_2}\lvert\omega^-\rangle_{a_2A_3A_4}\lvert\omega^-\rangle_{a_3A_5A_6}$	$(\vartheta_x\vartheta_z)_{B_1}\otimes(\vartheta_x\vartheta_z)_{B_2}\otimes(\vartheta_x\vartheta_z)_{B_3}$

Now Alice notifies her outcomes to Bob over 6-bit classical channel. With the classical information from Alice in hand, Bob finally executes appropriate Pauli operations on his qubits to reconstruct the original state which Alice desires to teleport. The following Table 9.2, Table 9.3, Table 9.4, Table 9.5, Table 9.6, Table 9.7, Table 9.8, Table 9.9 include all 64 measurement outcomes of Alice and associated Bob's operations. The protocol terminates at this point. The whole scenario is depicted in Figure 9.3.
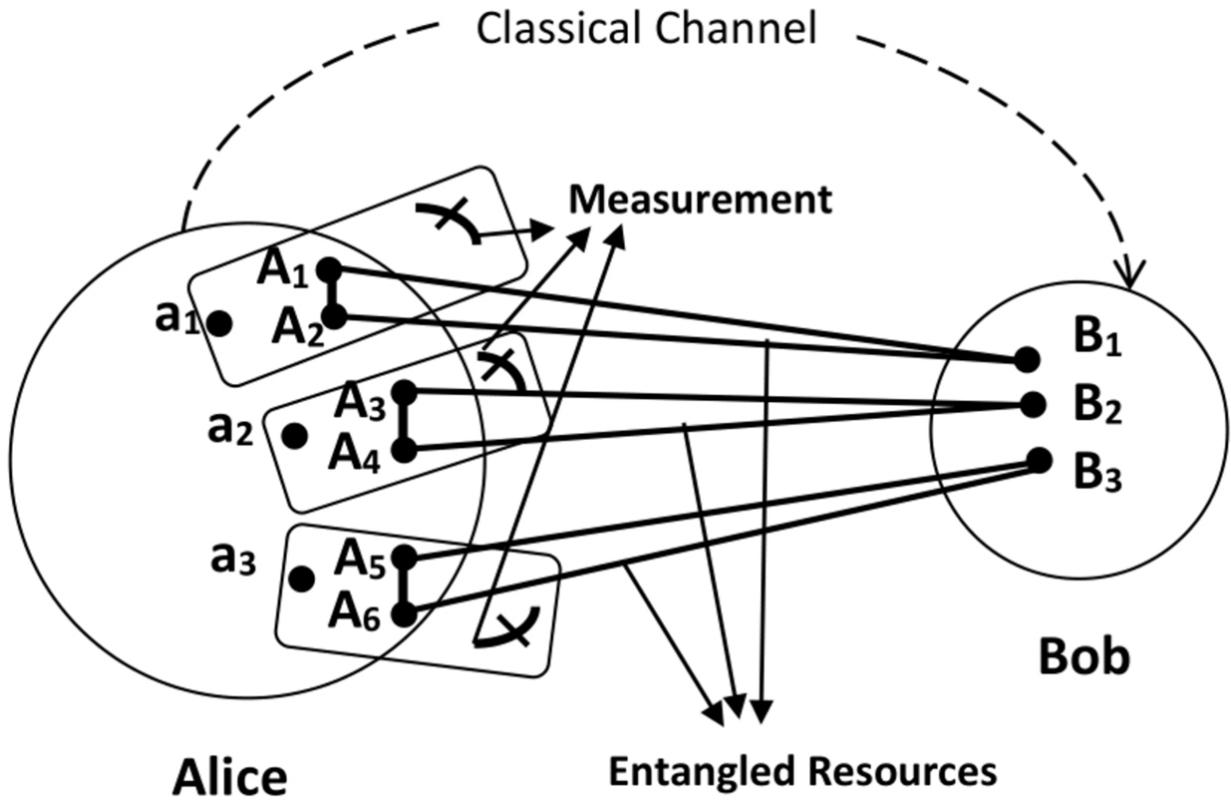
Figure 9.3 Schematic diagram for transfer of a general 3-qubit state using quantum teleportation. ↵

As an instance, consider the case where Alice's measurement yields

$$|\omega^-\rangle_{\mathfrak{a}_1 A_1 A_2}|\varepsilon^-\rangle_{\mathfrak{a}_2 A_3 A_4}|\varepsilon^+\rangle_{\mathfrak{a}_3 A_5 A_6}.$$

Then the state of the remaining qubits becomes

$$
\begin{aligned}
(\quad &+ + - \mathfrak{g}_1|100\rangle + + - \mathfrak{g}_2|101\rangle + - - \mathfrak{g}_3|110\rangle + - - \mathfrak{g}_4|111\rangle \\
&+ \mathfrak{g}_5|000\rangle + + + \mathfrak{g}_6|001\rangle + - + \mathfrak{g}_7|010\rangle + - + \mathfrak{g}_8|011\rangle)_{B_1 B_2 B_3} \\
&= (-\mathfrak{g}_1|100\rangle - \mathfrak{g}_2|101\rangle + \mathfrak{g}_3|110\rangle + \mathfrak{g}_4|111\rangle + \mathfrak{g}_5|000\rangle \\
&+ \mathfrak{g}_6|001\rangle - \mathfrak{g}_7|010\rangle - \mathfrak{g}_8|011\rangle)_{B_1 B_2 B_3}.
\end{aligned}
$$

Alice uses a classical channel to transmit her measurement result to Bob who then applies the relevant unitary operation from Table 9.6, which is

$$(\vartheta_x \vartheta_z)_{B_1} \otimes (\vartheta_z)_{B_2} \otimes (I)_{B_3},$$

by which the state transfer is achieved. That is the end of the protocol.

A special feature about the above protocol is that it uses three separate entangled resources. When more involved communication tasks are attempted, the use of multi-partite entanglement resources with large number of qubits become inevitable. In view of the fragile nature of the entangled states, and also due to the difficulties in the generation of such entangled states, it is sometimes recommended, if possible to use multiple quantum resources with relatively less number of involved qubits. The present teleportation scheme is an instance of that kind of protocol.

10 Bidirectional Teleportation Protocols

10.1 INTRODUCTION

In this chapter, the bidirectional teleportation process is described which is mutual exchange of quantum states between two distant parties connected through entanglement. Here, both parties act as sender and receiver. Bidirectional teleportation schemes of various kinds have been discussed in works like [19, 46, 59, 137, 159, 160, 162, 166, 203].

10.2 MUTUAL EXCHANGE OF SINGLE-QUBIT STATES

In this scenario, two individuals, Alice and Bob, each possesses a general unknown one-qubit state denoted $|\aleph_1\rangle$ and $|\aleph_2\rangle$, respectively, and given by

$$
\begin{aligned}
|\aleph_1\rangle_{\mathfrak{a}} &= (\mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle), \\
|\aleph_2\rangle_{\mathfrak{b}} &= (\mathfrak{h}_1|0\rangle + \mathfrak{h}_2|1\rangle),
\end{aligned}
$$

(10.1)

where coefficients $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{h}_1, \mathfrak{h}_2$ meet the normalization conditions, that is,

$$
\sum_{k=1}^{2} |\mathfrak{g}_k|^2 = 1 \quad \text{and} \quad \sum_{l=1}^{2} |\mathfrak{h}_l|^2 = 1.
$$

There is a classical communication channel connecting the two parties.

The objective is to exchange the two states between Alice and Bob which is performed by a bidirectional teleportation protocol given by Verma et al. [163].

Assume that Alice intends to communicate her single-qubit quantum state $|\aleph_1\rangle$ to Bob, whereas Bob intends to transfer his state $|\aleph_2\rangle$ to Alice simultaneously. To achieve this goal, a 4-qubit cluster state is used as a quantum resource, which is

$$|E\rangle_{A_1 B_1 A_2 B_2} = \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)_{A_1 B_1 A_2 B_2},$$

(10.2)

where the qubits (A_1, A_2) and (B_1, B_2) are held by Alice and Bob, respectively. Figure 10.1 shows the representation of the corresponding quantum circuit for generating the above quantum resource.
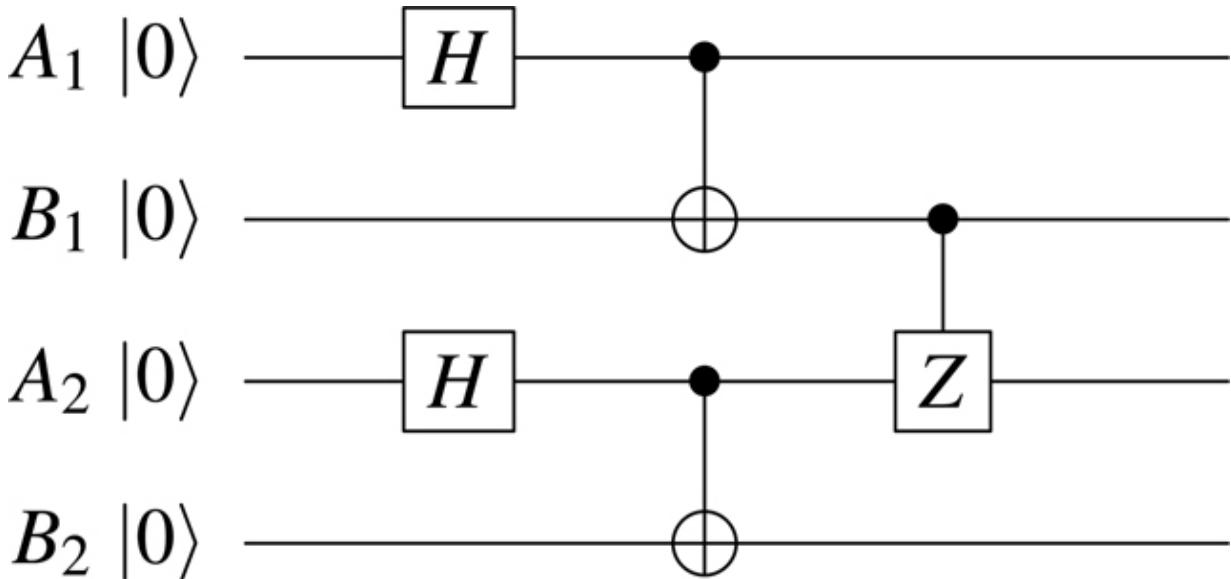
Figure 10.1 Circuit representation for generation of the entangled resource given in Eq. (10.2). ⏎

The system as a whole may be expressed as

$$
\begin{aligned}
|\Gamma\rangle &= |\aleph_1\rangle_\mathfrak{a} \otimes |\aleph_2\rangle_\mathfrak{b} \otimes |E\rangle_{A_1 B_1 A_2 B_2} \\
&= (\mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle)_\mathfrak{a} \otimes (\mathfrak{h}_1|0\rangle + \mathfrak{h}_2|1\rangle)_\mathfrak{b} \\
&\quad \otimes \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)_{A_1 B_1 A_2 B_2}.
\end{aligned}
$$

$$(10.3)$$

Alice possessed the qubits $(\mathfrak{a}, A_1, A_2)$ and the qubits $(\mathfrak{b}, B_1, B_2)$ are held by Bob, respectively. The teleportation process for exchange of qubit states is completed through the following steps:

Step I Alice measures her qubits $(\mathfrak{a}, A_1)$ on the basis given by

$$
|\Upsilon_1\rangle_{\mathfrak{a} A_1} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),
$$

$$
|\Upsilon_2\rangle_{\mathfrak{a} A_1} = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),
$$

$$
|\Upsilon_3\rangle_{\mathfrak{a} A_1} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),
$$

$$
|\Upsilon_4\rangle_{\mathfrak{a} A_1} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).
$$

$$(10.4)$$

Following the measurement, she uses a 2-bit classical channel to transmit her result to Bob.

Step II Bob performs a measurement on his qubits $(\mathfrak{b}, B_2)$ on the basis given by

$$|\Upsilon_1\rangle_{bB_2} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$

$$|\Upsilon_2\rangle_{bB_2} = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$

$$|\Upsilon_3\rangle_{bB_2} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$

$$|\Upsilon_4\rangle_{bB_2} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

(10.5)

After the measurement, he sends his result to Alice via a 2-bit classical channel.

Step III A controlled-Z (CZ) gate is applied to qubits (B_1, A_2), using A_2 as control qubit and B_1 as target qubit.

This operation requires that the party performing it will require to have access to both the qubits A_2 and B_1. It can be either Alice or Bob or a third party having access to A_2 and B_1. The above step is unavoidable in the protocol.

Step IV Finally, upon receiving the classical messages, both parties apply the appropriate unitary operations corresponding to these messages to recover the original quantum state. The protocol is thereby accomplished.

Now we discuss the protocol in details:

Using the Bell-basis given in Eq. (10.4), the whole system (10.3) can be rewritten as

$$
\begin{aligned}
|\Gamma\rangle \;=\; \frac{1}{4}\Big[&\,|\Upsilon_1\rangle_{\mathfrak{a}A_1}|\Upsilon_1\rangle_{\mathfrak{b}B_2}(\mathfrak{g}_1\mathfrak{h}_1|00\rangle + \mathfrak{g}_1\mathfrak{h}_2|01\rangle + \mathfrak{g}_2\mathfrak{h}_1|10\rangle - \mathfrak{g}_2\mathfrak{h}_2|11\rangle)_{B_1A_2} \\
+&\,|\Upsilon_1\rangle_{\mathfrak{a}A_1}|\Upsilon_2\rangle_{\mathfrak{b}B_2}(\mathfrak{g}_1\mathfrak{h}_1|00\rangle - \mathfrak{g}_1\mathfrak{h}_2|01\rangle + \mathfrak{g}_2\mathfrak{h}_1|10\rangle + \mathfrak{g}_2\mathfrak{h}_2|11\rangle)_{B_1A_2} \\
+&\,|\Upsilon_2\rangle_{\mathfrak{a}A_1}|\Upsilon_1\rangle_{\mathfrak{b}B_2}(\mathfrak{g}_1\mathfrak{h}_1|00\rangle + \mathfrak{g}_1\mathfrak{h}_2|01\rangle - \mathfrak{g}_2\mathfrak{h}_1|10\rangle + \mathfrak{g}_2\mathfrak{h}_2|11\rangle)_{B_1A_2} \\
+&\,|\Upsilon_2\rangle_{\mathfrak{a}A_1}|\Upsilon_2\rangle_{\mathfrak{b}B_2}(\mathfrak{g}_1\mathfrak{h}_1|00\rangle - \mathfrak{g}_1\mathfrak{h}_2|01\rangle - \mathfrak{g}_2\mathfrak{h}_1|10\rangle - \mathfrak{g}_2\mathfrak{h}_2|11\rangle)_{B_1A_2} \\
+&\,|\Upsilon_1\rangle_{\mathfrak{a}A_1}|\Upsilon_3\rangle_{\mathfrak{b}B_2}(\mathfrak{g}_1\mathfrak{h}_1|01\rangle + \mathfrak{g}_1\mathfrak{h}_2|00\rangle - \mathfrak{g}_2\mathfrak{h}_1|11\rangle + \mathfrak{g}_2\mathfrak{h}_2|10\rangle)_{B_1A_2} \\
+&\,|\Upsilon_1\rangle_{\mathfrak{a}A_1}|\Upsilon_4\rangle_{\mathfrak{b}B_2}(\mathfrak{g}_1\mathfrak{h}_1|01\rangle - \mathfrak{g}_1\mathfrak{h}_2|00\rangle - \mathfrak{g}_2\mathfrak{h}_1|11\rangle - \mathfrak{g}_2\mathfrak{h}_2|10\rangle)_{B_1A_2} \\
+&\,|\Upsilon_2\rangle_{\mathfrak{a}A_1}|\Upsilon_3\rangle_{\mathfrak{b}B_2}(\mathfrak{g}_1\mathfrak{h}_1|01\rangle + \mathfrak{g}_1\mathfrak{h}_2|00\rangle + \mathfrak{g}_2\mathfrak{h}_1|11\rangle - \mathfrak{g}_2\mathfrak{h}_2|10\rangle)_{B_1A_2} \\
+&\,|\Upsilon_2\rangle_{\mathfrak{a}A_1}|\Upsilon_4\rangle_{\mathfrak{b}B_2}(\mathfrak{g}_1\mathfrak{h}_1|01\rangle - \mathfrak{g}_1\mathfrak{h}_2|00\rangle + \mathfrak{g}_2\mathfrak{h}_1|11\rangle + \mathfrak{g}_2\mathfrak{h}_2|10\rangle)_{B_1A_2} \\
+&\,|\Upsilon_3\rangle_{\mathfrak{a}A_1}|\Upsilon_1\rangle_{\mathfrak{b}B_2}(\mathfrak{g}_1\mathfrak{h}_1|10\rangle - \mathfrak{g}_1\mathfrak{h}_2|11\rangle + \mathfrak{g}_2\mathfrak{h}_1|00\rangle + \mathfrak{g}_2\mathfrak{h}_2|01\rangle)_{B_1A_2} \\
+&\,|\Upsilon_3\rangle_{\mathfrak{a}A_1}|\Upsilon_2\rangle_{\mathfrak{b}B_2}(\mathfrak{g}_1\mathfrak{h}_1|10\rangle + \mathfrak{g}_1\mathfrak{h}_2|11\rangle + \mathfrak{g}_2\mathfrak{h}_1|00\rangle - \mathfrak{g}_2\mathfrak{h}_2|01\rangle)_{B_1A_2} \\
+&\,|\Upsilon_4\rangle_{\mathfrak{a}A_1}|\Upsilon_1\rangle_{\mathfrak{b}B_2}(\mathfrak{g}_1\mathfrak{h}_1|10\rangle - \mathfrak{g}_1\mathfrak{h}_2|11\rangle - \mathfrak{g}_2\mathfrak{h}_1|00\rangle - \mathfrak{g}_2\mathfrak{h}_2|01\rangle)_{B_1A_2} \\
+&\,|\Upsilon_4\rangle_{\mathfrak{a}A_1}|\Upsilon_2\rangle_{\mathfrak{b}B_2}(\mathfrak{g}_1\mathfrak{h}_1|10\rangle + \mathfrak{g}_1\mathfrak{h}_2|11\rangle - \mathfrak{g}_2\mathfrak{h}_1|00\rangle + \mathfrak{g}_2\mathfrak{h}_2|01\rangle)_{B_1A_2} \\
+&\,|\Upsilon_3\rangle_{\mathfrak{a}A_1}|\Upsilon_3\rangle_{\mathfrak{b}B_2}(-\mathfrak{g}_1\mathfrak{h}_1|11\rangle + \mathfrak{g}_1\mathfrak{h}_2|10\rangle + \mathfrak{g}_2\mathfrak{h}_1|01\rangle + \mathfrak{g}_2\mathfrak{h}_2|00\rangle)_{B_1A_2} \\
+&\,|\Upsilon_3\rangle_{\mathfrak{a}A_1}|\Upsilon_4\rangle_{\mathfrak{b}B_2}(-\mathfrak{g}_1\mathfrak{h}_1|11\rangle - \mathfrak{g}_1\mathfrak{h}_2|10\rangle + \mathfrak{g}_2\mathfrak{h}_1|01\rangle - \mathfrak{g}_2\mathfrak{h}_2|00\rangle)_{B_1A_2} \\
+&\,|\Upsilon_4\rangle_{\mathfrak{a}A_1}|\Upsilon_3\rangle_{\mathfrak{b}B_2}(-\mathfrak{g}_1\mathfrak{h}_1|11\rangle + \mathfrak{g}_1\mathfrak{h}_2|10\rangle - \mathfrak{g}_2\mathfrak{h}_1|01\rangle - \mathfrak{g}_2\mathfrak{h}_2|00\rangle)_{B_1A_2} \\
+&\,|\Upsilon_4\rangle_{\mathfrak{a}A_1}|\Upsilon_4\rangle_{\mathfrak{b}B_2}(-\mathfrak{g}_1\mathfrak{h}_1|11\rangle - \mathfrak{g}_1\mathfrak{h}_2|10\rangle - \mathfrak{g}_2\mathfrak{h}_1|01\rangle + \mathfrak{g}_2\mathfrak{h}_2|00\rangle)_{B_1A_2} \Big].
\end{aligned}
$$

(10.6)

Now, both parties make Bell-state measurement (BSM) on their respective pairs of qubits $(\mathfrak{a}, A_1)$ and $(\mathfrak{b}, B_2)$, and transmit the results of their measurements to each other via 2-bit classical channels. After BSM, the reduced states of qubits (B_1, A_2) are as follows:

$$|v_1\rangle_{B_1A_2} = (\mathfrak{g}_1\mathfrak{h}_1|00\rangle + \mathfrak{g}_1\mathfrak{h}_2|01\rangle + \mathfrak{g}_2\mathfrak{h}_1|10\rangle - \mathfrak{g}_2\mathfrak{h}_2|11\rangle)_{B_1A_2},$$
$$|v_2\rangle_{B_1A_2} = (\mathfrak{g}_1\mathfrak{h}_1|00\rangle - \mathfrak{g}_1\mathfrak{h}_2|01\rangle + \mathfrak{g}_2\mathfrak{h}_1|10\rangle + \mathfrak{g}_2\mathfrak{h}_2|11\rangle)_{B_1A_2},$$
$$|v_3\rangle_{B_1A_2} = (\mathfrak{g}_1\mathfrak{h}_1|00\rangle + \mathfrak{g}_1\mathfrak{h}_2|01\rangle - \mathfrak{g}_2\mathfrak{h}_1|10\rangle + \mathfrak{g}_2\mathfrak{h}_2|11\rangle)_{B_1A_2},$$
$$|v_4\rangle_{B_1A_2} = (\mathfrak{g}_1\mathfrak{h}_1|00\rangle - \mathfrak{g}_1\mathfrak{h}_2|01\rangle - \mathfrak{g}_2\mathfrak{h}_1|10\rangle - \mathfrak{g}_2\mathfrak{h}_2|11\rangle)_{B_1A_2},$$
$$|v_5\rangle_{B_1A_2} = (\mathfrak{g}_1\mathfrak{h}_1|01\rangle + \mathfrak{g}_1\mathfrak{h}_2|00\rangle - \mathfrak{g}_2\mathfrak{h}_1|11\rangle + \mathfrak{g}_2\mathfrak{h}_2|10\rangle)_{B_1A_2},$$
$$|v_6\rangle_{B_1A_2} = (\mathfrak{g}_1\mathfrak{h}_1|01\rangle - \mathfrak{g}_1\mathfrak{h}_2|00\rangle - \mathfrak{g}_2\mathfrak{h}_1|11\rangle - \mathfrak{g}_2\mathfrak{h}_2|10\rangle)_{B_1A_2},$$
$$|v_7\rangle_{B_1A_2} = (\mathfrak{g}_1\mathfrak{h}_1|01\rangle + \mathfrak{g}_1\mathfrak{h}_2|00\rangle + \mathfrak{g}_2\mathfrak{h}_1|11\rangle - \mathfrak{g}_2\mathfrak{h}_2|10\rangle)_{B_1A_2},$$
$$|v_8\rangle_{B_1A_2} = (\mathfrak{g}_1\mathfrak{h}_1|01\rangle - \mathfrak{g}_1\mathfrak{h}_2|00\rangle + \mathfrak{g}_2\mathfrak{h}_1|11\rangle + \mathfrak{g}_2\mathfrak{h}_2|10\rangle)_{B_1A_2},$$
$$|v_9\rangle_{B_1A_2} = (\mathfrak{g}_1\mathfrak{h}_1|10\rangle - \mathfrak{g}_1\mathfrak{h}_2|11\rangle + \mathfrak{g}_2\mathfrak{h}_1|00\rangle + \mathfrak{g}_2\mathfrak{h}_2|01\rangle)_{B_1A_2},$$
$$|v_{10}\rangle_{B_1A_2} = (\mathfrak{g}_1\mathfrak{h}_1|10\rangle + \mathfrak{g}_1\mathfrak{h}_2|11\rangle + \mathfrak{g}_2\mathfrak{h}_1|00\rangle - \mathfrak{g}_2\mathfrak{h}_2|01\rangle)_{B_1A_2},$$
$$|v_{11}\rangle_{B_1A_2} = (\mathfrak{g}_1\mathfrak{h}_1|10\rangle - \mathfrak{g}_1\mathfrak{h}_2|11\rangle - \mathfrak{g}_2\mathfrak{h}_1|00\rangle - \mathfrak{g}_2\mathfrak{h}_2|01\rangle)_{B_1A_2},$$
$$|v_{12}\rangle_{B_1A_2} = (\mathfrak{g}_1\mathfrak{h}_1|10\rangle + \mathfrak{g}_1\mathfrak{h}_2|11\rangle - \mathfrak{g}_2\mathfrak{h}_1|00\rangle + \mathfrak{g}_2\mathfrak{h}_2|01\rangle)_{B_1A_2},$$
$$|v_{13}\rangle_{B_1A_2} = (-\mathfrak{g}_1\mathfrak{h}_1|11\rangle + \mathfrak{g}_1\mathfrak{h}_2|10\rangle + \mathfrak{g}_2\mathfrak{h}_1|01\rangle + \mathfrak{g}_2\mathfrak{h}_2|00\rangle)_{B_1A_2},$$
$$|v_{14}\rangle_{B_1A_2} = (-\mathfrak{g}_1\mathfrak{h}_1|11\rangle - \mathfrak{g}_1\mathfrak{h}_2|10\rangle + \mathfrak{g}_2\mathfrak{h}_1|01\rangle - \mathfrak{g}_2\mathfrak{h}_2|00\rangle)_{B_1A_2},$$
$$|v_{15}\rangle_{B_1A_2} = (-\mathfrak{g}_1\mathfrak{h}_1|11\rangle + \mathfrak{g}_1\mathfrak{h}_2|10\rangle - \mathfrak{g}_2\mathfrak{h}_1|01\rangle - \mathfrak{g}_2\mathfrak{h}_2|00\rangle)_{B_1A_2},$$
$$|v_{16}\rangle_{B_1A_2} = (-\mathfrak{g}_1\mathfrak{h}_1|11\rangle - \mathfrak{g}_1\mathfrak{h}_2|10\rangle - \mathfrak{g}_2\mathfrak{h}_1|01\rangle + \mathfrak{g}_2\mathfrak{h}_2|00\rangle)_{B_1A_2}.$$

$$(10.7)$$

After that, a quantum phase gate operation given in [Chapter 3](#) on qubits A_2 and B_1 is needed to complete the bidirectional quantum teleportation. Here, the control qubit is A_2, whereas B_1 is the target qubit. The quantum states then turn into

$$|v_1'\rangle_{B_1 A_2} = (\mathfrak{g}_1\mathfrak{h}_1|00\rangle + \mathfrak{g}_1\mathfrak{h}_2|01\rangle + \mathfrak{g}_2\mathfrak{h}_1|10\rangle + \mathfrak{g}_2\mathfrak{h}_2|11\rangle)_{B_1 A_2},$$
$$|v_2'\rangle_{B_1 A_2} = (\mathfrak{g}_1\mathfrak{h}_1|00\rangle - \mathfrak{g}_1\mathfrak{h}_2|01\rangle + \mathfrak{g}_2\mathfrak{h}_1|10\rangle - \mathfrak{g}_2\mathfrak{h}_2|11\rangle)_{B_1 A_2},$$
$$|v_3'\rangle_{B_1 A_2} = (\mathfrak{g}_1\mathfrak{h}_1|00\rangle + \mathfrak{g}_1\mathfrak{h}_2|01\rangle - \mathfrak{g}_2\mathfrak{h}_1|10\rangle - \mathfrak{g}_2\mathfrak{h}_2|11\rangle)_{B_1 A_2},$$
$$|v_4'\rangle_{B_1 A_2} = (\mathfrak{g}_1\mathfrak{h}_1|00\rangle - \mathfrak{g}_1\mathfrak{h}_2|01\rangle - \mathfrak{g}_2\mathfrak{h}_1|10\rangle + \mathfrak{g}_2\mathfrak{h}_2|11\rangle)_{B_1 A_2},$$
$$|v_5'\rangle_{B_1 A_2} = (\mathfrak{g}_1\mathfrak{h}_1|01\rangle + \mathfrak{g}_1\mathfrak{h}_2|00\rangle + \mathfrak{g}_2\mathfrak{h}_1|11\rangle + \mathfrak{g}_2\mathfrak{h}_2|10\rangle)_{B_1 A_2},$$
$$|v_6'\rangle_{B_1 A_2} = (\mathfrak{g}_1\mathfrak{h}_1|01\rangle - \mathfrak{g}_1\mathfrak{h}_2|00\rangle + \mathfrak{g}_2\mathfrak{h}_1|11\rangle - \mathfrak{g}_2\mathfrak{h}_2|10\rangle)_{B_1 A_2},$$
$$|v_7'\rangle_{B_1 A_2} = (\mathfrak{g}_1\mathfrak{h}_1|01\rangle + \mathfrak{g}_1\mathfrak{h}_2|00\rangle - \mathfrak{g}_2\mathfrak{h}_1|11\rangle - \mathfrak{g}_2\mathfrak{h}_2|10\rangle)_{B_1 A_2},$$
$$|v_8'\rangle_{B_1 A_2} = (\mathfrak{g}_1\mathfrak{h}_1|01\rangle - \mathfrak{g}_1\mathfrak{h}_2|00\rangle - \mathfrak{g}_2\mathfrak{h}_1|11\rangle + \mathfrak{g}_2\mathfrak{h}_2|10\rangle)_{B_1 A_2},$$
$$|v_9'\rangle_{B_1 A_2} = (\mathfrak{g}_1\mathfrak{h}_1|10\rangle + \mathfrak{g}_1\mathfrak{h}_2|11\rangle + \mathfrak{g}_2\mathfrak{h}_1|00\rangle + \mathfrak{g}_2\mathfrak{h}_2|01\rangle)_{B_1 A_2},$$
$$|v_{10}'\rangle_{B_1 A_2} = (\mathfrak{g}_1\mathfrak{h}_1|10\rangle - \mathfrak{g}_1\mathfrak{h}_2|11\rangle + \mathfrak{g}_2\mathfrak{h}_1|00\rangle - \mathfrak{g}_2\mathfrak{h}_2|01\rangle)_{B_1 A_2},$$
$$|v_{11}'\rangle_{B_1 A_2} = (\mathfrak{g}_1\mathfrak{h}_1|10\rangle + \mathfrak{g}_1\mathfrak{h}_2|11\rangle - \mathfrak{g}_2\mathfrak{h}_1|00\rangle - \mathfrak{g}_2\mathfrak{h}_2|01\rangle)_{B_1 A_2},$$
$$|v_{12}'\rangle_{B_1 A_2} = (\mathfrak{g}_1\mathfrak{h}_1|10\rangle - \mathfrak{g}_1\mathfrak{h}_2|11\rangle - \mathfrak{g}_2\mathfrak{h}_1|00\rangle + \mathfrak{g}_2\mathfrak{h}_2|01\rangle)_{B_1 A_2},$$
$$|v_{13}'\rangle_{B_1 A_2} = (\mathfrak{g}_1\mathfrak{h}_1|11\rangle + \mathfrak{g}_1\mathfrak{h}_2|10\rangle + \mathfrak{g}_2\mathfrak{h}_1|01\rangle + \mathfrak{g}_2\mathfrak{h}_2|00\rangle)_{B_1 A_2},$$
$$|v_{14}'\rangle_{B_1 A_2} = (\mathfrak{g}_1\mathfrak{h}_1|11\rangle - \mathfrak{g}_1\mathfrak{h}_2|10\rangle + \mathfrak{g}_2\mathfrak{h}_1|01\rangle - \mathfrak{g}_2\mathfrak{h}_2|00\rangle)_{B_1 A_2},$$
$$|v_{15}'\rangle_{B_1 A_2} = (\mathfrak{g}_1\mathfrak{h}_1|11\rangle + \mathfrak{g}_1\mathfrak{h}_2|10\rangle - \mathfrak{g}_2\mathfrak{h}_1|01\rangle - \mathfrak{g}_2\mathfrak{h}_2|00\rangle)_{B_1 A_2},$$
$$|v_{16}'\rangle_{B_1 A_2} = (\mathfrak{g}_1\mathfrak{h}_1|11\rangle - \mathfrak{g}_1\mathfrak{h}_2|10\rangle - \mathfrak{g}_2\mathfrak{h}_1|01\rangle + \mathfrak{g}_2\mathfrak{h}_2|00\rangle)_{B_1 A_2}.$$

$$(10.8)$$

Following the exchange of the classical messages, both parties perform appropriate unitary operations, which are given in Table 10.1, to obtain the intended states. Through this procedure, Alice and Bob exchange the quantum state with each other. However, a successful reconstruction of the transmitted state is only possible if both participants cooperate. Figure 10.2 illustrates the complete scenario of the protocol.

Table 10.1

Required unitary operations for Alice and Bob ⏎

Alice'soutcome	Bob'soutcome	Reducedstate	Alice's unitary operations	Bob's unitary operation
$\lvert \Upsilon_1 \rangle$	$\lvert \Upsilon_1 \rangle$	$\lvert v'_1 \rangle_{B_1 A_2}$	$(I)_{A_2}$	$(I)_{B_1}$
$\lvert \Upsilon_1 \rangle$	$\lvert \Upsilon_2 \rangle$	$\lvert v'_2 \rangle_{B_1 A_2}$	$(\vartheta_z)_{A_2}$	$(I)_{B_1}$
$\lvert \Upsilon_2 \rangle$	$\lvert \Upsilon_1 \rangle$	$\lvert v'_3 \rangle_{B_1 A_2}$	$(I)_{A_2}$	$(\vartheta_z)_{B_1}$
$\lvert \Upsilon_2 \rangle$	$\lvert \Upsilon_2 \rangle$	$\lvert v'_4 \rangle_{B_1 A_2}$	$(\vartheta_z)_{A_2}$	$(\vartheta_z)_{B_1}$
$\lvert \Upsilon_1 \rangle$	$\lvert \Upsilon_3 \rangle$	$\lvert v'_5 \rangle_{B_1 A_2}$	$(\vartheta_x)_{A_2}$	$(I)_{B_1}$
$\lvert \Upsilon_1 \rangle$	$\lvert \Upsilon_4 \rangle$	$\lvert v'_6 \rangle_{B_1 A_2}$	$(\vartheta_z \vartheta_x)_{A_2}$	$(I)_{B_1}$
$\lvert \Upsilon_2 \rangle$	$\lvert \Upsilon_3 \rangle$	$\lvert v'_7 \rangle_{B_1 A_2}$	$(\vartheta_x)_{A_2}$	$(\vartheta_z)_{B_1}$
$\lvert \Upsilon_2 \rangle$	$\lvert \Upsilon_4 \rangle$	$\lvert v'_8 \rangle_{B_1 A_2}$	$(\vartheta_z \vartheta_x)_{A_2}$	$(\vartheta_z)_{B_1}$
$\lvert \Upsilon_3 \rangle$	$\lvert \Upsilon_1 \rangle$	$\lvert v'_9 \rangle_{B_1 A_2}$	$(I)_{A_2}$	$(\vartheta_x)_{B_1}$
$\lvert \Upsilon_3 \rangle$	$\lvert \Upsilon_2 \rangle$	$\lvert v'_{10} \rangle_{B_1 A_2}$	$(\vartheta_z)_{A_2}$	$(\vartheta_x)_{B_1}$
$\lvert \Upsilon_4 \rangle$	$\lvert \Upsilon_1 \rangle$	$\lvert v'_{11} \rangle_{B_1 A_2}$	$(I)_{A_2}$	$(\vartheta_z \vartheta_x)_{B_1}$
$\lvert \Upsilon_4 \rangle$	$\lvert \Upsilon_2 \rangle$	$\lvert v'_{12} \rangle_{B_1 A_2}$	$(\vartheta_z)_{A_2}$	$(\vartheta_z \vartheta_x)_{B_1}$
$\lvert \Upsilon_3 \rangle$	$\lvert \Upsilon_3 \rangle$	$\lvert v'_{13} \rangle_{B_1 A_2}$	$(\vartheta_x)_{A_2}$	$(\vartheta_x)_{B_1}$
$\lvert \Upsilon_3 \rangle$	$\lvert \Upsilon_4 \rangle$	$\lvert v'_{14} \rangle_{B_1 A_2}$	$(\vartheta_z \vartheta_x)_{A_2}$	$(\vartheta_x)_{B_1}$
$\lvert \Upsilon_4 \rangle$	$\lvert \Upsilon_3 \rangle$	$\lvert v'_{15} \rangle_{B_1 A_2}$	$(\vartheta_x)_{A_2}$	$(\vartheta_z \vartheta_x)_{B_1}$
$\lvert \Upsilon_4 \rangle$	$\lvert \Upsilon_4 \rangle$	$\lvert v'_{16} \rangle_{B_1 A_2}$	$(\vartheta_z \vartheta_x)_{A_2}$	$(\vartheta_z \vartheta_x)_{B_1}$
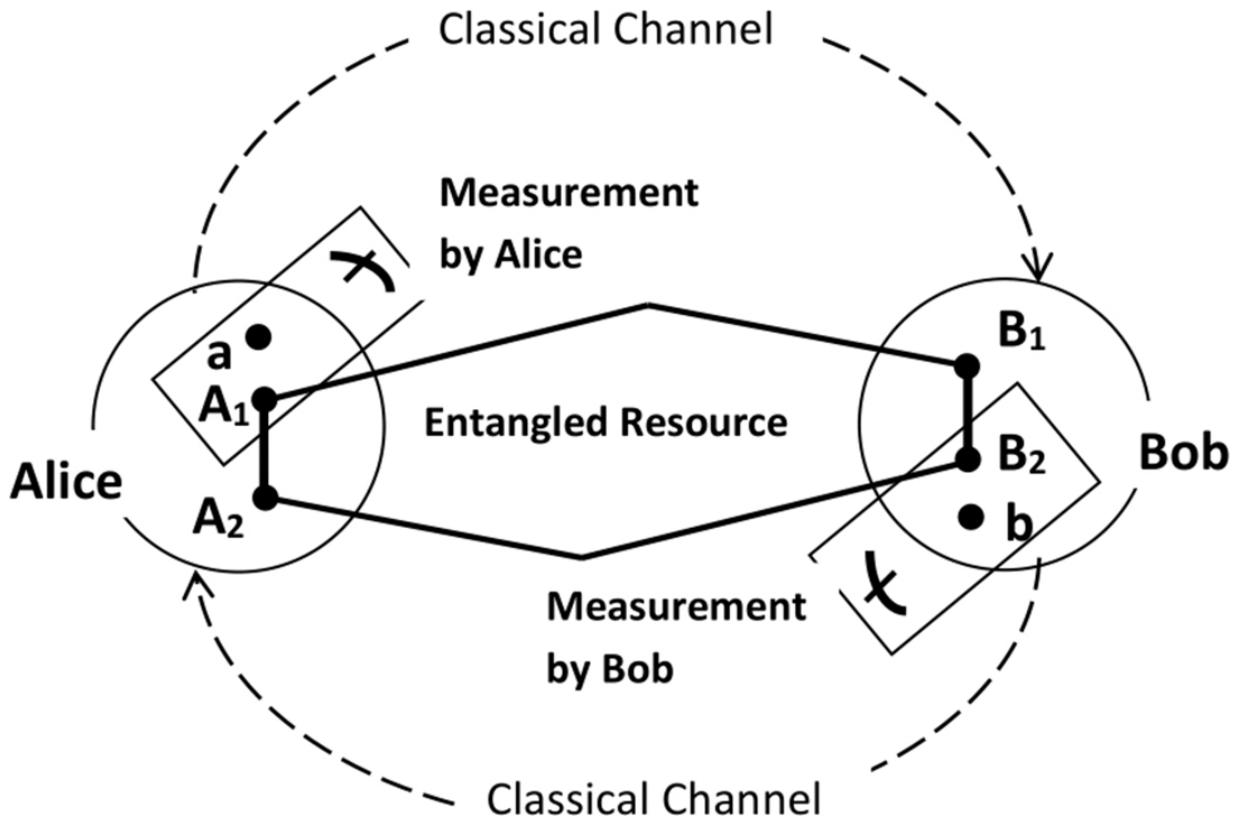
Figure 10.2 Bidirectional teleportation protocol for exchange of single-qubit states. ↵

To illustrate, assuming that Alice's and Bob's measurement outcomes are $|\Upsilon_3\rangle_{\mathfrak{a}A_1}$ and $|\Upsilon_2\rangle_{\mathfrak{b}B_2}$, respectively, the state of the remaining qubits becomes

$$(\mathfrak{g}_1\mathfrak{h}_1|10\rangle + \mathfrak{g}_1\mathfrak{h}_2|11\rangle + \mathfrak{g}_2\mathfrak{h}_1|00\rangle - \mathfrak{g}_2\mathfrak{h}_2|01\rangle)_{B_1A_2}.$$

After that, to complete the above bidirectional communication protocol, a quantum operation is applied on qubits B_1, A_2 where qubit A_2 plays the role of control qubit whereas qubit B_1 is the target qubit. Then the final reduced state becomes

$$(\mathfrak{g}_1\mathfrak{h}_1|10\rangle - \mathfrak{g}_1\mathfrak{h}_2|11\rangle + \mathfrak{g}_2\mathfrak{h}_1|00\rangle - \mathfrak{g}_2\mathfrak{h}_2|01\rangle)_{B_1A_2}$$
$$= (\mathfrak{g}_1|1\rangle + \mathfrak{g}_2|0\rangle)_{B_1} \otimes (\mathfrak{h}_1|0\rangle - \mathfrak{h}_2|1\rangle)_{A_2}.$$

Finally, Bob and Alice execute appropriate unitary operations on their respective qubits which are, from [Table 10.1](#), given by $(\vartheta_x)_{B_1}$ and $(\vartheta_z)_{A_2}$,

respectively. The mutual transfer of qubit states is thereby completed. That is the end of the protocol.

10.3 ASYMMETRIC BIDIRECTIONAL TELEPORTATION PROTOCOL OF $(2 \leftrightarrow 3)$-QUBIT STATES

In this section, we explore a bidirectional teleportation protocol characterized by its asymmetry. Specifically, one party, Alice, wishes to transfer a two-qubit state to Bob, while simultaneously Bob intends to transmits a three-qubit state to Alice, which are, respectively, given by

$$\begin{aligned}
|\aleph_1\rangle_{\mathfrak{a}_1\mathfrak{a}_2} &= (\mathfrak{g}_1|00\rangle + \mathfrak{g}_2|11\rangle), \\
|\aleph_2\rangle_{\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3} &= (\mathfrak{h}_1|000\rangle + \mathfrak{h}_2|111\rangle),
\end{aligned}$$

$$(10.9)$$

where the complex coefficients $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{h}_1, \mathfrak{h}_2$ meet the normalization conditions, that is,

$$\sum_{m=1}^{2} |\mathfrak{g}_m|^2 = 1, \quad \sum_{n=1}^{2} |\mathfrak{h}_n|^2 = 1.$$

Alice and Bob are connected through a classical communication channel.

The states $|\aleph_1\rangle$ and $|\aleph_2\rangle$ are unknown to both Alice and Bob except for their normalizations. The task is intended to be executed in an integrated manner using a single entanglement resource.

For this purpose, an 8-qubit state is utilized as quantum resource amongst the parties given by

$$\begin{aligned}
|E\rangle_{B_1A_1B_2A_2B_3A_3B_4A_4} = \frac{1}{2} \ [&|00000000\rangle + |10100001\rangle \\
&+ |01011110\rangle + |11111111\rangle],
\end{aligned}$$

where Alice possesses the qubits (A_1, A_2, A_3, A_4) and Bob holds the qubits (B_1, B_2, B_3, B_4).

The total system of 13 qubits is written as

$$
\begin{aligned}
|\Gamma\rangle = & \ |\aleph_1\rangle_{\mathfrak{a}_1\mathfrak{a}_2} \otimes |\aleph_2\rangle_{\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3} \otimes |E\rangle_{B_1A_1B_2A_2B_3A_3B_4A_4} \\
= & \ (\mathfrak{g}_1|00\rangle + \mathfrak{g}_2|11\rangle)_{\mathfrak{a}_1\mathfrak{a}_2} \otimes (\mathfrak{h}_1|000\rangle + \mathfrak{h}_2|111\rangle)_{\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3} \otimes \frac{1}{2}[|00000000\rangle \\
& + |10100001\rangle + |01011110\rangle + |11111111\rangle]_{B_1A_1B_2A_2B_3A_3B_4A_4}.
\end{aligned}
$$

$$(10.11)$$

Alice now performs a measurement on her three qubits $(\mathfrak{a}_1, \mathfrak{a}_2, A_4)$ using the basis defined by

$$
\begin{aligned}
|\varsigma_1\rangle &= \frac{|000\rangle + |111\rangle}{\sqrt{2}}, & |\varsigma_2\rangle &= \frac{|000\rangle - |111\rangle}{\sqrt{2}}, \\
|\varsigma_3\rangle &= \frac{|001\rangle + |110\rangle}{\sqrt{2}}, & |\varsigma_4\rangle &= \frac{|001\rangle - |110\rangle}{\sqrt{2}}, \\
|\varsigma_5\rangle &= \frac{|010\rangle + |101\rangle}{\sqrt{2}}, & |\varsigma_6\rangle &= \frac{|010\rangle - |101\rangle}{\sqrt{2}}, \\
|\varsigma_7\rangle &= \frac{|011\rangle + |100\rangle}{\sqrt{2}}, & |\varsigma_8\rangle &= \frac{|011\rangle - |100\rangle}{\sqrt{2}}.
\end{aligned}
$$

$$(10.12)$$

and Bob carries out on the basis given by

$$
\{|\varsigma_i\rangle_{\mathfrak{b}_1\mathfrak{b}_2B_3}|\Upsilon_j\rangle_{\mathfrak{b}_3B_4} : i = 1, 2, \ldots, 8; j = 1, 2, 3, 4\},
$$

where $\{|\Upsilon_j\rangle_{\mathfrak{b}_3B_4} : j = 1, 2, 3, 4\}$s are the Bell states given by

$$|\Upsilon_1\rangle_{b_3 B_4} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$

$$|\Upsilon_2\rangle_{b_3 B_4} = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$

$$|\Upsilon_3\rangle_{b_3 B_4} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$

$$|\Upsilon_4\rangle_{b_3 B_4} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

After the measurements both parties exchange their respective measurement results through classical channels. Based on their outcomes, the parties execute appropriate unitary operations to obtain the intended states. The complete process is shown schematically in Figure 10.3.
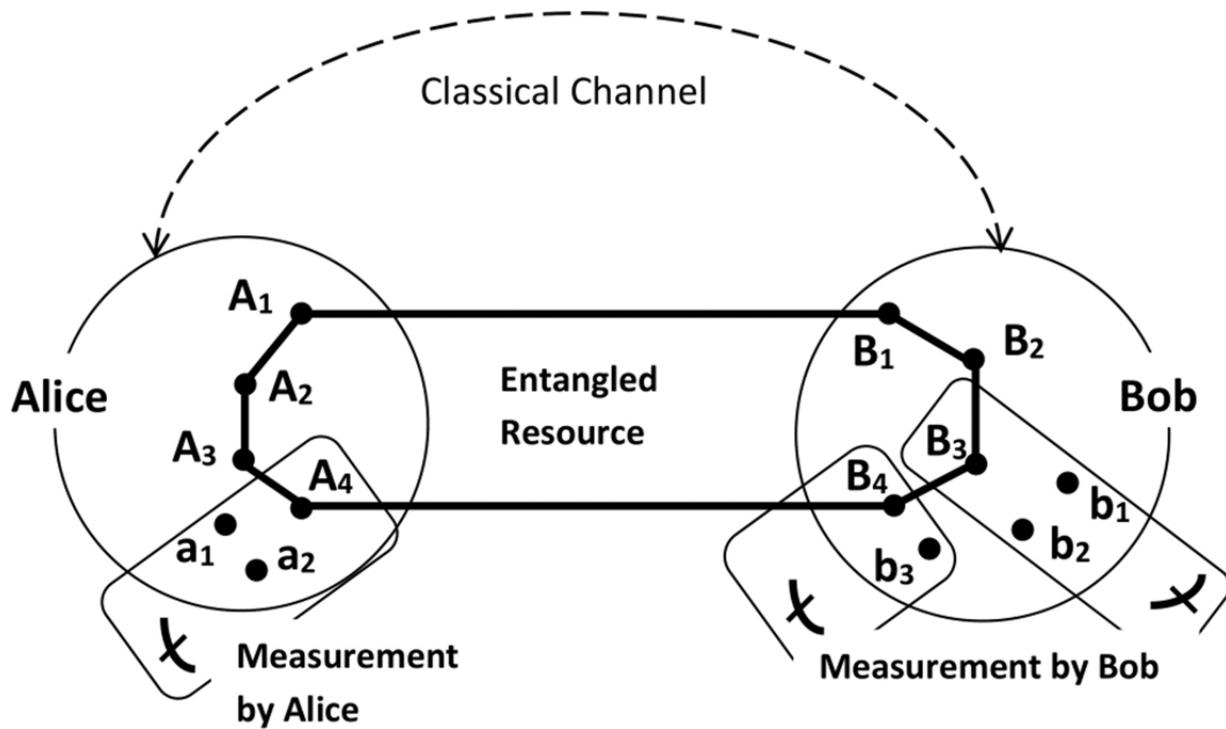
Figure 10.3 Bidirectional teleportation protocol for transferring two- and three-qubit state. ⏎

We discuss the protocol in detail in the following.

The composite state $|\Gamma\rangle$ in Eq. (10.11) can be written as

$$|\Gamma\rangle = (\mathfrak{h}_1|000\rangle + \mathfrak{h}_2|111\rangle)_{\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3} \otimes \sum_{i=1}^{4} |\varsigma_i\rangle_{\mathfrak{a}_1\mathfrak{a}_2 A_4} \otimes |M_i\rangle_{B_1 A_1 B_2 A_2 B_3 A_3 B_4},$$

where

$$
\begin{aligned}
|M_1\rangle_{B_1 A_1 B_2 A_2 B_3 A_3 B_4} =\ & \mathfrak{g}_1(|0000000\rangle + |0101111\rangle) \\
& + \mathfrak{g}_2(|1010000\rangle + |1111111\rangle), \\
|M_2\rangle_{B_1 A_1 B_2 A_2 B_3 A_3 B_4} =\ & \mathfrak{g}_1(|0000000\rangle + |0101111\rangle) \\
& - \mathfrak{g}_2(|1010000\rangle + |1111111\rangle), \\
|M_3\rangle_{B_1 A_1 B_2 A_2 B_3 A_3 B_4} =\ & \mathfrak{g}_1(|1010000\rangle + |1111111\rangle) \\
& + \mathfrak{g}_2(|0000000\rangle + |0101111\rangle), \\
|M_4\rangle_{B_1 A_1 B_2 A_2 B_3 A_3 B_4} =\ & \mathfrak{g}_1(|1010000\rangle + |1111111\rangle) \\
& - \mathfrak{g}_2(|0000000\rangle + |0101111\rangle).
\end{aligned}
$$

Now, Alice makes her measurement using the basis given in Eq. (10.12) and communicates the outcomes using a classical channel to Bob. There are four possible outcomes of Alice's measurement. We discuss their consequences in the following four cases.

Case I:

Suppose that the measurement result of Alice is $|\varsigma_1\rangle_{\mathfrak{a}_1\mathfrak{a}_2 A_4}$, then the state of the remaining qubits becomes

$$
\begin{aligned}
|\Gamma_1\rangle =\ & (\mathfrak{h}_1|000\rangle + \mathfrak{h}_2|111\rangle)_{\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3} \otimes [\mathfrak{g}_1(|0000000\rangle + |0101111\rangle) \\
& + \mathfrak{g}_2(|1010000\rangle + |1111111\rangle)]_{B_1 A_1 B_2 A_2 B_3 A_3 B_4}.
\end{aligned}
$$

$$(10.13)$$

The above Eq. (10.13) can be re-expressed as

$$
\begin{aligned}
|\Gamma_1\rangle = \ & |\varsigma_1\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_1\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|00000\rangle \\
& + \mathfrak{g}_1\mathfrak{h}_2|01011\rangle + \mathfrak{g}_2\mathfrak{h}_1|10100\rangle + \mathfrak{g}_2\mathfrak{h}_2|11111\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+ \ & |\varsigma_1\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_2\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|00000\rangle \\
& - \mathfrak{g}_1\mathfrak{h}_2|01011\rangle + \mathfrak{g}_2\mathfrak{h}_1|10100\rangle - \mathfrak{g}_2\mathfrak{h}_2|11111\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+ \ & |\varsigma_2\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_1\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|00000\rangle \\
& - \mathfrak{g}_1\mathfrak{h}_2|01011\rangle + \mathfrak{g}_2\mathfrak{h}_1|10100\rangle - \mathfrak{g}_2\mathfrak{h}_2|11111\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+ \ & |\varsigma_2\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_2\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|00000\rangle \\
& + \mathfrak{g}_1\mathfrak{h}_2|01011\rangle + \mathfrak{g}_2\mathfrak{h}_1|10100\rangle + \mathfrak{g}_2\mathfrak{h}_2|11111\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+ \ & |\varsigma_3\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_3\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|01011\rangle \\
& + \mathfrak{g}_1\mathfrak{h}_2|00000\rangle + \mathfrak{g}_2\mathfrak{h}_1|11111\rangle + \mathfrak{g}_2\mathfrak{h}_2|10100\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+ \ & |\varsigma_3\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_4\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|01011\rangle \\
& - \mathfrak{g}_1\mathfrak{h}_2|00000\rangle + \mathfrak{g}_2\mathfrak{h}_1|11111\rangle - \mathfrak{g}_2\mathfrak{h}_2|10100\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+ \ & |\varsigma_4\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_3\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|01011\rangle \\
& - \mathfrak{g}_1\mathfrak{h}_2|00000\rangle + \mathfrak{g}_2\mathfrak{h}_1|11111\rangle - \mathfrak{g}_2\mathfrak{h}_2|10100\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+ \ & |\varsigma_4\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_4\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|01011\rangle \\
& + \mathfrak{g}_1\mathfrak{h}_2|00000\rangle + \mathfrak{g}_2\mathfrak{h}_1|11111\rangle + \mathfrak{g}_2\mathfrak{h}_2|10100\rangle)_{B_1 A_1 B_2 A_2 A_3}.
\end{aligned}
$$

Now Bob performs his measurement on the corresponding basis and sends the measurement outcomes through a classical channel to Alice.

If Bob's measurement result is $|\varsigma_1\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3}|\Upsilon_1\rangle_{\mathfrak{b}_3 B_4}$, the resulting state of the remaining qubits is as follows:

$$
\begin{aligned}
(\mathfrak{g}_1\mathfrak{h}_1|00000\rangle & + \mathfrak{g}_1\mathfrak{h}_2|01011\rangle + \mathfrak{g}_2\mathfrak{h}_1|10100\rangle + \mathfrak{g}_2\mathfrak{h}_2|11111\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
& = (\mathfrak{g}_1|00\rangle + \mathfrak{g}_2|11\rangle)_{B_1 B_2} \otimes (\mathfrak{h}_1|000\rangle + \mathfrak{h}_2|111\rangle)_{A_1 A_2 A_3}.
\end{aligned}
$$

When Bob obtains $|\varsigma_1\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3}|\Upsilon_2\rangle_{\mathfrak{b}_3 B_4}$ as his measurement result, the state of the rest of the qubits is as follows:

$$
\begin{aligned}
(\mathfrak{g}_1\mathfrak{h}_1|00000\rangle & - \mathfrak{g}_1\mathfrak{h}_2|01011\rangle + \mathfrak{g}_2\mathfrak{h}_1|10100\rangle - \mathfrak{g}_2\mathfrak{h}_2|11111\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
& = (\mathfrak{g}_1|00\rangle + \mathfrak{g}_2|11\rangle)_{B_1 B_2} \otimes (\mathfrak{h}_1|000\rangle - \mathfrak{h}_2|111\rangle)_{A_1 A_2 A_3}.
\end{aligned}
$$

When Bob obtains $|\varsigma_2\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3}|\Upsilon_1\rangle_{\mathfrak{b}_3 B_4}$ as his measurement result, the state of the rest of the qubits is as follows:

$$(\mathfrak{g}_1\mathfrak{h}_1|00000\rangle - \mathfrak{g}_1\mathfrak{h}_2|01011\rangle + \mathfrak{g}_2\mathfrak{h}_1|10100\rangle - \mathfrak{g}_2\mathfrak{h}_2|11111\rangle)_{B_1 A_1 B_2 A_2 A_3}$$
$$= (\mathfrak{g}_1|00\rangle + \mathfrak{g}_2|11\rangle)_{B_1 B_2} \otimes (\mathfrak{h}_1|000\rangle - \mathfrak{h}_2|111\rangle)_{A_1 A_2 A_3}.$$

When Bob obtains $|\varsigma_2\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3}|\Upsilon_2\rangle_{\mathfrak{b}_3 B_4}$ as his measurement result, the state of the rest of the qubits is as follows:

$$(\mathfrak{g}_1\mathfrak{h}_1|00000\rangle + \mathfrak{g}_1\mathfrak{h}_2|01011\rangle + \mathfrak{g}_2\mathfrak{h}_1|10100\rangle + \mathfrak{g}_2\mathfrak{h}_2|11111\rangle)_{B_1 A_1 B_2 A_2 A_3}$$
$$= (\mathfrak{g}_1|00\rangle + \mathfrak{g}_2|11\rangle)_{B_1 B_2} \otimes (\mathfrak{h}_1|000\rangle + \mathfrak{h}_2|111\rangle)_{A_1 A_2 A_3}.$$

When Bob obtains $|\varsigma_3\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3}|\Upsilon_3\rangle_{\mathfrak{b}_3 B_4}$ as his measurement result, the state of the rest of the qubits is as follows:

$$(\mathfrak{g}_1\mathfrak{h}_1|01011\rangle + \mathfrak{g}_1\mathfrak{h}_2|00000\rangle + \mathfrak{g}_2\mathfrak{h}_1|11111\rangle + \mathfrak{g}_2\mathfrak{h}_2|10100\rangle)_{B_1 A_1 B_2 A_2 A_3}$$
$$= (\mathfrak{g}_1|00\rangle + \mathfrak{g}_2|11\rangle)_{B_1 B_2} \otimes (\mathfrak{h}_1|111\rangle + \mathfrak{h}_2|000\rangle)_{A_1 A_2 A_3}.$$

When Bob obtains $|\varsigma_3\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3}|\Upsilon_4\rangle_{\mathfrak{b}_3 B_4}$ as his measurement result, the state of the rest of the qubits is as follows:

$$(\mathfrak{g}_1\mathfrak{h}_1|01011\rangle - \mathfrak{g}_1\mathfrak{h}_2|00000\rangle + \mathfrak{g}_2\mathfrak{h}_1|11111\rangle - \mathfrak{g}_2\mathfrak{h}_2|10100\rangle)_{B_1 A_1 B_2 A_2 A_3}$$
$$= (\mathfrak{g}_1|00\rangle + \mathfrak{g}_2|11\rangle)_{B_1 B_2} \otimes (\mathfrak{h}_1|111\rangle - \mathfrak{h}_2|000\rangle)_{A_1 A_2 A_3}.$$

When Bob obtains $|\varsigma_4\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3}|\Upsilon_3\rangle_{\mathfrak{b}_3 B_4}$ as his measurement result, the state of the rest of the qubits is as follows:

$$(\mathfrak{g}_1\mathfrak{h}_1|01011\rangle - \mathfrak{g}_1\mathfrak{h}_2|00000\rangle + \mathfrak{g}_2\mathfrak{h}_1|11111\rangle - \mathfrak{g}_2\mathfrak{h}_2|10100\rangle)_{B_1 A_1 B_2 A_2 A_3}$$
$$= (\mathfrak{g}_1|00\rangle + \mathfrak{g}_2|11\rangle)_{B_1 B_2} \otimes (\mathfrak{h}_1|111\rangle - \mathfrak{h}_2|000\rangle)_{A_1 A_2 A_3}.$$

When Bob obtains $|\varsigma_4\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3}|\Upsilon_4\rangle_{\mathfrak{b}_3 B_4}$ as his measurement result, the state of the rest of the qubits is as follows:

$$(\mathfrak{g}_1\mathfrak{h}_1|01011\rangle + \mathfrak{g}_1\mathfrak{h}_2|00000\rangle + \mathfrak{g}_2\mathfrak{h}_1|11111\rangle + \mathfrak{g}_2\mathfrak{h}_2|10100\rangle)_{B_1 A_1 B_2 A_2 A_3}$$
$$= (\mathfrak{g}_1|00\rangle + \mathfrak{g}_2|11\rangle)_{B_1 B_2} \otimes (\mathfrak{h}_1|111\rangle + \mathfrak{h}_2|000\rangle)_{A_1 A_2 A_3}.$$

Finally, to reconstruct the original quantum state, Alice and Bob each applies a suitable unitary operation on their respective qubits. Details of the unitary operations for **Case I** are provided in Table 10.2.

Table 10.2

Required unitary operations for Alice and Bob for Case I ⏎

Bob's outcome	Alice's unitary operation	Bob's unitary operation
$\lvert \varsigma_1 \rangle_{\mathfrak{b}_1 \mathfrak{b}_2 B_3} \otimes \lvert \Upsilon_1 \rangle_{\mathfrak{b}_3 B_4}$	$(I \otimes I \otimes I)_{A_1 A_2 A_3}$	$(I \otimes I)_{B_1 B_2}$
$\lvert \varsigma_1 \rangle_{\mathfrak{b}_1 \mathfrak{b}_2 B_3} \otimes \lvert \Upsilon_2 \rangle_{\mathfrak{b}_3 B_4}$	$(\vartheta_z \otimes I \otimes I)_{A_1 A_2 A_3}$	$(I \otimes I)_{B_1 B_2}$
$\lvert \varsigma_2 \rangle_{\mathfrak{b}_1 \mathfrak{b}_2 B_3} \otimes \lvert \Upsilon_1 \rangle_{\mathfrak{b}_3 B_4}$	$(I \otimes \vartheta_z \otimes I)_{A_1 A_2 A_3}$	$(I \otimes I)_{B_1 B_2}$
$\lvert \varsigma_2 \rangle_{\mathfrak{b}_1 \mathfrak{b}_2 B_3} \otimes \lvert \Upsilon_2 \rangle_{\mathfrak{b}_3 B_4}$	$(I \otimes I \otimes I)_{A_1 A_2 A_3}$	$(I \otimes I)_{B_1 B_2}$
$\lvert \varsigma_3 \rangle_{\mathfrak{b}_1 \mathfrak{b}_2 B_3} \otimes \lvert \Upsilon_3 \rangle_{\mathfrak{b}_3 B_4}$	$(\vartheta_x \otimes \vartheta_x \otimes \vartheta_x)_{A_1 A_2 A_3}$	$(I \otimes I)_{B_1 B_2}$
$\lvert \varsigma_3 \rangle_{\mathfrak{b}_1 \mathfrak{b}_2 B_3} \otimes \lvert \Upsilon_4 \rangle_{\mathfrak{b}_3 B_4}$	$(\vartheta_z \vartheta_x \otimes \vartheta_x \otimes \vartheta_x)_{A_1 A_2 A_3}$	$(I \otimes I)_{B_1 B_2}$
$\lvert \varsigma_4 \rangle_{\mathfrak{b}_1 \mathfrak{b}_2 B_3} \otimes \lvert \Upsilon_3 \rangle_{\mathfrak{b}_3 B_4}$	$(\vartheta_x \otimes \vartheta_z \vartheta_x \otimes \vartheta_x)_{A_1 A_2 A_3}$	$(I \otimes I)_{B_1 B_2}$
$\lvert \varsigma_4 \rangle_{\mathfrak{b}_1 \mathfrak{b}_2 B_3} \otimes \lvert \Upsilon_4 \rangle_{\mathfrak{b}_3 B_4}$	$(\vartheta_x \otimes \vartheta_x \otimes \vartheta_x)_{A_1 A_2 A_3}$	$(I \otimes I)_{B_1 B_2}$

Case II:

Suppose that Alice's measurement result is $\lvert \varsigma_2 \rangle_{\mathfrak{a}_1 \mathfrak{a}_2 A_4}$, then the reduced state of the remaining qubits is

$$
\begin{aligned}
\lvert \Gamma_2 \rangle = \ & (\mathfrak{h}_1 \lvert 000 \rangle + \mathfrak{h}_2 \lvert 111 \rangle)_{\mathfrak{b}_1 \mathfrak{b}_2 \mathfrak{b}_3} \otimes [\mathfrak{g}_1 (\lvert 0000000 \rangle + \lvert 0101111 \rangle) \\
& - \mathfrak{g}_2 (\lvert 1010000 \rangle + \lvert 1111111 \rangle)]_{B_1 A_1 B_2 A_2 B_3 A_3 B_4}.
\end{aligned}
$$

(10.14)

The above reduced state Eq. (10.14) can be expressed as

$$\begin{aligned}
|\Gamma_2\rangle = \ &|\varsigma_1\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_1\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|00000\rangle \\
&+ \mathfrak{g}_1\mathfrak{h}_2|01011\rangle - \mathfrak{g}_2\mathfrak{h}_1|10100\rangle - \mathfrak{g}_2\mathfrak{h}_2|11111\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+\ &|\varsigma_1\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_2\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|00000\rangle \\
&- \mathfrak{g}_1\mathfrak{h}_2|01011\rangle - \mathfrak{g}_2\mathfrak{h}_1|10100\rangle + \mathfrak{g}_2\mathfrak{h}_2|11111\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+\ &|\varsigma_2\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_1\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|00000\rangle \\
&- \mathfrak{g}_1\mathfrak{h}_2|01011\rangle - \mathfrak{g}_2\mathfrak{h}_1|10100\rangle + \mathfrak{g}_2\mathfrak{h}_2|11111\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+\ &|\varsigma_2\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_2\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|00000\rangle \\
&+ \mathfrak{g}_1\mathfrak{h}_2|01011\rangle - \mathfrak{g}_2\mathfrak{h}_1|10100\rangle - \mathfrak{g}_2\mathfrak{h}_2|11111\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+\ &|\varsigma_3\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_3\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|01011\rangle \\
&+ \mathfrak{g}_1\mathfrak{h}_2|00000\rangle - \mathfrak{g}_2\mathfrak{h}_1|11111\rangle - \mathfrak{g}_2\mathfrak{h}_2|10100\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+\ &|\varsigma_3\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_4\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|01011\rangle \\
&- \mathfrak{g}_1\mathfrak{h}_2|00000\rangle - \mathfrak{g}_2\mathfrak{h}_1|11111\rangle + \mathfrak{g}_2\mathfrak{h}_2|10100\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+\ &|\varsigma_4\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_3\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|01011\rangle \\
&- \mathfrak{g}_1\mathfrak{h}_2|00000\rangle - \mathfrak{g}_2\mathfrak{h}_1|11111\rangle + \mathfrak{g}_2\mathfrak{h}_2|10100\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+\ &|\varsigma_4\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_4\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|01011\rangle \\
&+ \mathfrak{g}_1\mathfrak{h}_2|00000\rangle - \mathfrak{g}_2\mathfrak{h}_1|11111\rangle - \mathfrak{g}_2\mathfrak{h}_2|10100\rangle)_{B_1 A_1 B_2 A_2 A_3}.
\end{aligned}$$

Now Bob makes his measurement on the basis mentioned above and transmits it to Alice by the use of a classical channel.

If the measurement performed by Bob yields $|\varsigma_1\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3}|\Upsilon_1\rangle_{\mathfrak{b}_3 B_4}$, then the rest of the system is described by the state

$$\begin{aligned}
&(\mathfrak{g}_1\mathfrak{h}_1|00000\rangle + \mathfrak{g}_1\mathfrak{h}_2|01011\rangle - \mathfrak{g}_2\mathfrak{h}_1|10100\rangle - \mathfrak{g}_2\mathfrak{h}_2|11111\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
&= (\mathfrak{g}_1|00\rangle - \mathfrak{g}_2|11\rangle)_{B_1 B_2} \otimes (\mathfrak{h}_1|000\rangle + \mathfrak{h}_2|111\rangle)_{A_1 A_2 A_3}.
\end{aligned}$$

If the measurement performed by Bob yields $|\varsigma_1\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3}|\Upsilon_2\rangle_{\mathfrak{b}_3 B_4}$, then the rest of the system evolves into the state

$$\begin{aligned}
&(\mathfrak{g}_1\mathfrak{h}_1|00000\rangle - \mathfrak{g}_1\mathfrak{h}_2|01011\rangle - \mathfrak{g}_2\mathfrak{h}_1|10100\rangle + \mathfrak{g}_2\mathfrak{h}_2|11111\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
&= (\mathfrak{g}_1|00\rangle - \mathfrak{g}_2|11\rangle)_{B_1 B_2} \otimes (\mathfrak{h}_1|000\rangle - \mathfrak{h}_2|111\rangle)_{A_1 A_2 A_3}.
\end{aligned}$$

If the measurement performed by Bob yields $|\varsigma_2\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3}|\Upsilon_1\rangle_{\mathfrak{b}_3 B_4}$, then the rest of the system is described by the state

$$(\mathfrak{g}_1\mathfrak{h}_1|00000\rangle - \mathfrak{g}_1\mathfrak{h}_2|01011\rangle - \mathfrak{g}_2\mathfrak{h}_1|10100\rangle + \mathfrak{g}_2\mathfrak{h}_2|11111\rangle)_{B_1 A_1 B_2 A_2 A_3}$$
$$= (\mathfrak{g}_1|00\rangle - \mathfrak{g}_2|11\rangle)_{B_1 B_2} \otimes (\mathfrak{h}_1|000\rangle - \mathfrak{h}_2|111\rangle)_{A_1 A_2 A_3}.$$

If the measurement performed by Bob yields $|\varsigma_2\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3}|\Upsilon_2\rangle_{\mathfrak{b}_3 B_4}$, then the rest of the system is described by the state

$$(\mathfrak{g}_1\mathfrak{h}_1|00000\rangle + \mathfrak{g}_1\mathfrak{h}_2|01011\rangle - \mathfrak{g}_2\mathfrak{h}_1|10100\rangle - \mathfrak{g}_2\mathfrak{h}_2|11111\rangle)_{B_1 A_1 B_2 A_2 A_3}$$
$$= (\mathfrak{g}_1|00\rangle - \mathfrak{g}_2|11\rangle)_{B_1 B_2} \otimes (\mathfrak{h}_1|000\rangle + \mathfrak{h}_2|111\rangle)_{A_1 A_2 A_3}.$$

If the measurement performed by Bob yields $|\varsigma_3\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3}|\Upsilon_3\rangle_{\mathfrak{b}_3 B_4}$, then the rest of the system is described by the state

$$(\mathfrak{g}_1\mathfrak{h}_1|01011\rangle + \mathfrak{g}_1\mathfrak{h}_2|00000\rangle - \mathfrak{g}_2\mathfrak{h}_1|11111\rangle - \mathfrak{g}_2\mathfrak{h}_2|10100\rangle)_{B_1 A_1 B_2 A_2 A_3}$$
$$= (\mathfrak{g}_1|00\rangle - \mathfrak{g}_2|11\rangle)_{B_1 B_2} \otimes (\mathfrak{h}_1|111\rangle + \mathfrak{h}_2|000\rangle)_{A_1 A_2 A_3}.$$

If the measurement performed by Bob yields $|\varsigma_3\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3}|\Upsilon_4\rangle_{\mathfrak{b}_3 B_4}$, then the rest of the system is described by the state

$$(\mathfrak{g}_1\mathfrak{h}_1|01011\rangle - \mathfrak{g}_1\mathfrak{h}_2|00000\rangle - \mathfrak{g}_2\mathfrak{h}_1|11111\rangle + \mathfrak{g}_2\mathfrak{h}_2|10100\rangle)_{B_1 A_1 B_2 A_2 A_3}$$
$$= (\mathfrak{g}_1|00\rangle - \mathfrak{g}_2|11\rangle)_{B_1 B_2} \otimes (\mathfrak{h}_1|111\rangle - \mathfrak{h}_2|000\rangle)_{A_1 A_2 A_3}.$$

If the measurement performed by Bob yields $|\varsigma_4\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3}|\Upsilon_3\rangle_{\mathfrak{b}_3 B_4}$, then the rest of the system is described by the state

$$(\mathfrak{g}_1\mathfrak{h}_1|01011\rangle - \mathfrak{g}_1\mathfrak{h}_2|00000\rangle - \mathfrak{g}_2\mathfrak{h}_1|11111\rangle + \mathfrak{g}_2\mathfrak{h}_2|10100\rangle)_{B_1 A_1 B_2 A_2 A_3}$$
$$= (\mathfrak{g}_1|00\rangle - \mathfrak{g}_2|11\rangle)_{B_1 B_2} \otimes (\mathfrak{h}_1|111\rangle - \mathfrak{h}_2|000\rangle)_{A_1 A_2 A_3}.$$

If the measurement performed by Bob yields $|\varsigma_4\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3}|\Upsilon_4\rangle_{\mathfrak{b}_3 B_4}$, then the rest of the system is described by the state

$$(\mathfrak{g}_1\mathfrak{h}_1|01011\rangle + \mathfrak{g}_1\mathfrak{h}_2|00000\rangle - \mathfrak{g}_2\mathfrak{h}_1|11111\rangle - \mathfrak{g}_2\mathfrak{h}_2|10100\rangle)_{B_1 A_1 B_2 A_2 A_3}$$
$$= (\mathfrak{g}_1|00\rangle - \mathfrak{g}_2|11\rangle)_{B_1 B_2} \otimes (\mathfrak{h}_1|111\rangle + \mathfrak{h}_2|000\rangle)_{A_1 A_2 A_3}.$$

In the final step, Alice and Bob perform appropriate unitary transformations to retrieve the initial state. In **Case II**, the unitary operations are listed in Table 10.3.

Table 10.3

Appropriate unitary operation performed by Alice and Bob for Case II ↵

Bob's outcome	Alice's unitary operation	Bob's unitary operation
$\lvert\varsigma_1\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes \lvert\Upsilon_1\rangle_{\mathfrak{b}_3 B_4}$	$(I \otimes I \otimes I)_{A_1 A_2 A_3}$	$(I \otimes \vartheta_z)_{B_1 B_2}$
$\lvert\varsigma_1\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes \lvert\Upsilon_2\rangle_{\mathfrak{b}_3 B_4}$	$(I \otimes \vartheta_z \otimes I)_{A_1 A_2 A_3}$	$(\vartheta_z \otimes I)_{B_1 B_2}$
$\lvert\varsigma_2\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes \lvert\Upsilon_1\rangle_{\mathfrak{b}_3 B_4}$	$(\vartheta_z \otimes I \otimes I)_{A_1 A_2 A_3}$	$(\vartheta_z \otimes I)_{B_1 B_2}$
$\lvert\varsigma_2\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes \lvert\Upsilon_2\rangle_{\mathfrak{b}_3 B_4}$	$(I \otimes I \otimes I)_{A_1 A_2 A_3}$	$(I \otimes \vartheta_z)_{B_1 B_2}$
$\lvert\varsigma_3\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes \lvert\Upsilon_3\rangle_{\mathfrak{b}_3 B_4}$	$(\vartheta_x \otimes \vartheta_x \otimes \vartheta_x)_{A_1 A_2 A_3}$	$(\vartheta_z \otimes I)_{B_1 B_2}$
$\lvert\varsigma_3\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes \lvert\Upsilon_4\rangle_{\mathfrak{b}_3 B_4}$	$(\vartheta_x \otimes \vartheta_x \otimes \vartheta_z\vartheta_x)_{A_1 A_2 A_3}$	$(I \otimes \vartheta_z)_{B_1 B_2}$
$\lvert\varsigma_4\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes \lvert\Upsilon_3\rangle_{\mathfrak{b}_3 B_4}$	$(\vartheta_x \otimes \vartheta_z\vartheta_x \otimes \vartheta_x)_{A_1 A_2 A_3}$	$(\vartheta_z \otimes I)_{B_1 B_2}$
$\lvert\varsigma_4\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes \lvert\Upsilon_4\rangle_{\mathfrak{b}_3 B_4}$	$(\vartheta_x \otimes \vartheta_x \otimes \vartheta_x)_{A_1 A_2 A_3}$	$(I \otimes \vartheta_z)_{B_1 B_2}$

Case III:

Suppose that the measurement result of Alice is $\lvert\varsigma_3\rangle_{\mathfrak{a}_1\mathfrak{a}_2 A_4}$, then the state of the remaining qubits becomes

$$\lvert\Gamma_3\rangle = (\mathfrak{h}_1\lvert000\rangle + \mathfrak{h}_2\lvert111\rangle)_{\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3} \otimes [\mathfrak{g}_1(\lvert1010000\rangle + \lvert1111111\rangle) + \mathfrak{g}_2(\lvert0000000\rangle + \lvert0101111\rangle)]_{B_1 A_1 B_2 A_2 B_3 A_3 B_4}.$$

$$(10.15)$$

The above reduced state, given in Eq. (10.15), can be re-written as

$$
\begin{aligned}
|\Gamma_3\rangle = \ & |\varsigma_1\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_1\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|10100\rangle \\
& + \mathfrak{g}_1\mathfrak{h}_2|11111\rangle + \mathfrak{g}_2\mathfrak{h}_1|00000\rangle + \mathfrak{g}_2\mathfrak{h}_2|01011\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+ \ & |\varsigma_1\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_2\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|10100\rangle \\
& - \mathfrak{g}_1\mathfrak{h}_2|11111\rangle + \mathfrak{g}_2\mathfrak{h}_1|00000\rangle - \mathfrak{g}_2\mathfrak{h}_2|01011\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+ \ & |\varsigma_2\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_1\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|10100\rangle \\
& - \mathfrak{g}_1\mathfrak{h}_2|11111\rangle + \mathfrak{g}_2\mathfrak{h}_1|00000\rangle - \mathfrak{g}_2\mathfrak{h}_2|01011\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+ \ & |\varsigma_2\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_2\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|10100\rangle \\
& + \mathfrak{g}_1\mathfrak{h}_2|11111\rangle + \mathfrak{g}_2\mathfrak{h}_1|00000\rangle + \mathfrak{g}_2\mathfrak{h}_2|01011\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+ \ & |\varsigma_3\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_3\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|11111\rangle \\
& + \mathfrak{g}_1\mathfrak{h}_2|10100\rangle + \mathfrak{g}_2\mathfrak{h}_1|01011\rangle + \mathfrak{g}_2\mathfrak{h}_2|00000\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+ \ & |\varsigma_3\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_4\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|11111\rangle \\
& - \mathfrak{g}_1\mathfrak{h}_2|10100\rangle + \mathfrak{g}_2\mathfrak{h}_1|01011\rangle - \mathfrak{g}_2\mathfrak{h}_2|00000\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+ \ & |\varsigma_4\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_3\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|11111\rangle \\
& - \mathfrak{g}_1\mathfrak{h}_2|10100\rangle + \mathfrak{g}_2\mathfrak{h}_1|01011\rangle - \mathfrak{g}_2\mathfrak{h}_2|00000\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+ \ & |\varsigma_4\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_4\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|11111\rangle \\
& + \mathfrak{g}_1\mathfrak{h}_2|10100\rangle + \mathfrak{g}_2\mathfrak{h}_1|01011\rangle + \mathfrak{g}_2\mathfrak{h}_2|00000\rangle)_{B_1 A_1 B_2 A_2 A_3}.
\end{aligned}
$$

Now Bob performs his measurement on the corresponding basis and sends the measurement results to Alice via a classical channel.

After Bob measures his qubits and finds them in state $|\varsigma_1\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3}|\Upsilon_1\rangle_{\mathfrak{b}_3 B_4}$, the other qubits in the system are described by the state

$$
\begin{aligned}
& (\mathfrak{g}_1\mathfrak{h}_1|10100\rangle + \mathfrak{g}_1\mathfrak{h}_2|11111\rangle + \mathfrak{g}_2\mathfrak{h}_1|00000\rangle + \mathfrak{g}_2\mathfrak{h}_2|01011\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
& = (\mathfrak{g}_1|11\rangle + \mathfrak{g}_2|00\rangle)_{B_1 B_2} \otimes (\mathfrak{h}_1|000\rangle + \mathfrak{h}_2|111\rangle)_{A_1 A_2 A_3}.
\end{aligned}
$$

After Bob measures his qubits and finds them in state $|\varsigma_1\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3}|\Upsilon_2\rangle_{\mathfrak{b}_3 B_4}$, the other qubits in the system are described by the state

$$
\begin{aligned}
& (\mathfrak{g}_1\mathfrak{h}_1|10100\rangle - \mathfrak{g}_1\mathfrak{h}_2|11111\rangle + \mathfrak{g}_2\mathfrak{h}_1|00000\rangle - \mathfrak{g}_2\mathfrak{h}_2|01011\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
& = (\mathfrak{g}_1|11\rangle + \mathfrak{g}_2|00\rangle)_{B_1 B_2} \otimes (\mathfrak{h}_1|000\rangle - \mathfrak{h}_2|111\rangle)_{A_1 A_2 A_3}.
\end{aligned}
$$

After Bob measures his qubits and finds them in state $|\varsigma_2\rangle_{\mathfrak{b}_1\mathfrak{b}_2B_3}|\Upsilon_1\rangle_{\mathfrak{b}_3B_4}$, the other qubits in the system are described by the state

$$(\mathfrak{g}_1\mathfrak{h}_1|10100\rangle - \mathfrak{g}_1\mathfrak{h}_2|11111\rangle + \mathfrak{g}_2\mathfrak{h}_1|00000\rangle - \mathfrak{g}_2\mathfrak{h}_2|01011\rangle)_{B_1A_1B_2A_2A_3}$$
$$= (\mathfrak{g}_1|11\rangle + \mathfrak{g}_2|00\rangle)_{B_1B_2} \otimes (\mathfrak{h}_1|000\rangle - \mathfrak{h}_2|111\rangle)_{A_1A_2A_3}.$$

After Bob measures his qubits and finds them in state $|\varsigma_2\rangle_{\mathfrak{b}_1\mathfrak{b}_2B_3}|\Upsilon_2\rangle_{\mathfrak{b}_3B_4}$, the other qubits in the system are described by the state

$$(\mathfrak{g}_1\mathfrak{h}_1|10100\rangle + \mathfrak{g}_1\mathfrak{h}_2|11111\rangle + \mathfrak{g}_2\mathfrak{h}_1|00000\rangle + \mathfrak{g}_2\mathfrak{h}_2|01011\rangle)_{B_1A_1B_2A_2A_3}$$
$$= (\mathfrak{g}_1|11\rangle + \mathfrak{g}_2|00\rangle)_{B_1B_2} \otimes (\mathfrak{h}_1|000\rangle + \mathfrak{h}_2|111\rangle)_{A_1A_2A_3}.$$

After Bob measures his qubits and finds them in state $|\varsigma_3\rangle_{\mathfrak{b}_1\mathfrak{b}_2B_3}|\Upsilon_3\rangle_{\mathfrak{b}_3B_4}$, the other qubits in the system are described by the state

$$(\mathfrak{g}_1\mathfrak{h}_1|11111\rangle + \mathfrak{g}_1\mathfrak{h}_2|10100\rangle + \mathfrak{g}_2\mathfrak{h}_1|01011\rangle + \mathfrak{g}_2\mathfrak{h}_2|00000\rangle)_{B_1A_1B_2A_2A_3}$$
$$= (\mathfrak{g}_1|11\rangle + \mathfrak{g}_2|00\rangle)_{B_1B_2} \otimes (\mathfrak{h}_1|111\rangle + \mathfrak{h}_2|000\rangle)_{A_1A_2A_3}.$$

After Bob measures his qubits and finds them in state $|\varsigma_3\rangle_{\mathfrak{b}_1\mathfrak{b}_2B_3}|\Upsilon_4\rangle_{\mathfrak{b}_3B_4}$, the other qubits in the system are described by the state

$$(\mathfrak{g}_1\mathfrak{h}_1|11111\rangle - \mathfrak{g}_1\mathfrak{h}_2|10100\rangle + \mathfrak{g}_2\mathfrak{h}_1|01011\rangle - \mathfrak{g}_2\mathfrak{h}_2|00000\rangle)_{B_1A_1B_2A_2A_3}$$
$$= (\mathfrak{g}_1|11\rangle + \mathfrak{g}_2|00\rangle)_{B_1B_2} \otimes (\mathfrak{h}_1|111\rangle - \mathfrak{h}_2|000\rangle)_{A_1A_2A_3}.$$

After Bob measures his qubits and finds them in state $|\varsigma_4\rangle_{\mathfrak{b}_1\mathfrak{b}_2B_3}|\Upsilon_3\rangle_{\mathfrak{b}_3B_4}$, the other qubits in the system are described by the state

$$(\mathfrak{g}_1\mathfrak{h}_1|11111\rangle - \mathfrak{g}_1\mathfrak{h}_2|10100\rangle + \mathfrak{g}_2\mathfrak{h}_1|01011\rangle - \mathfrak{g}_2\mathfrak{h}_2|00000\rangle)_{B_1A_1B_2A_2A_3}$$
$$= (\mathfrak{g}_1|11\rangle + \mathfrak{g}_2|00\rangle)_{B_1B_2} \otimes (\mathfrak{h}_1|111\rangle - \mathfrak{h}_2|000\rangle)_{A_1A_2A_3}.$$

After Bob measures his qubits and finds them in state $|\varsigma_4\rangle_{\mathfrak{b}_1\mathfrak{b}_2B_3}|\Upsilon_4\rangle_{\mathfrak{b}_3B_4}$, the other qubits in the system are described by the state

$$(\mathfrak{g}_1\mathfrak{h}_1|11111\rangle + \mathfrak{g}_1\mathfrak{h}_2|10100\rangle + \mathfrak{g}_2\mathfrak{h}_1|01011\rangle + \mathfrak{g}_2\mathfrak{h}_2|00000\rangle)_{B_1A_1B_2A_2A_3}$$
$$= (\mathfrak{g}_1|11\rangle + \mathfrak{g}_2|00\rangle)_{B_1B_2} \otimes (\mathfrak{h}_1|111\rangle + \mathfrak{h}_2|000\rangle)_{A_1A_2A_3}.$$

Finally, to recover the intended state, Alice and Bob perform appropriate unitary operations on her/his qubits, respectively. The unitary operations corresponding to **Case III** are summarized in [Table 10.4](#).

Table 10.4

Appropriate unitary operation performed by Alice and Bob for Case III ⏎

Bob's outcome	Alice's unitary operation	Bob's unitary operation
$\lvert \varsigma_1 \rangle_{\mathfrak{b}_1 \mathfrak{b}_2 B_3} \otimes \lvert \Upsilon_1 \rangle_{\mathfrak{b}_3 B_4}$	$(I \otimes I \otimes I)_{A_1 A_2 A_3}$	$(\vartheta_x \otimes \vartheta_x)_{B_1 B_2}$
$\lvert \varsigma_1 \rangle_{\mathfrak{b}_1 \mathfrak{b}_2 B_3} \otimes \lvert \Upsilon_2 \rangle_{\mathfrak{b}_3 B_4}$	$(I \otimes \vartheta_z \otimes I)_{A_1 A_2 A_3}$	$(\vartheta_x \otimes \vartheta_x)_{B_1 B_2}$
$\lvert \varsigma_2 \rangle_{\mathfrak{b}_1 \mathfrak{b}_2 B_3} \otimes \lvert \Upsilon_1 \rangle_{\mathfrak{b}_3 B_4}$	$(\vartheta_z \otimes I \otimes I)_{A_1 A_2 A_3}$	$(\vartheta_x \otimes \vartheta_x)_{B_1 B_2}$
$\lvert \varsigma_2 \rangle_{\mathfrak{b}_1 \mathfrak{b}_2 B_3} \otimes \lvert \Upsilon_2 \rangle_{\mathfrak{b}_3 B_4}$	$(I \otimes I \otimes I)_{A_1 A_2 A_3}$	$(\vartheta_x \otimes \vartheta_x)_{B_1 B_2}$
$\lvert \varsigma_3 \rangle_{\mathfrak{b}_1 \mathfrak{b}_2 B_3} \otimes \lvert \Upsilon_3 \rangle_{\mathfrak{b}_3 B_4}$	$(\vartheta_x \otimes \vartheta_x \otimes \vartheta_x)_{A_1 A_2 A_3}$	$(\vartheta_x \otimes \vartheta_x)_{B_1 B_2}$
$\lvert \varsigma_3 \rangle_{\mathfrak{b}_1 \mathfrak{b}_2 B_3} \otimes \lvert \Upsilon_4 \rangle_{\mathfrak{b}_3 B_4}$	$(\vartheta_x \otimes \vartheta_x \otimes \vartheta_z \vartheta_x)_{A_1 A_2 A_3}$	$(\vartheta_x \otimes \vartheta_x)_{B_1 B_2}$
$\lvert \varsigma_4 \rangle_{\mathfrak{b}_1 \mathfrak{b}_2 B_3} \otimes \lvert \Upsilon_3 \rangle_{\mathfrak{b}_3 B_4}$	$(\vartheta_x \otimes \vartheta_x \otimes \vartheta_z \vartheta_x)_{A_1 A_2 A_3}$	$(\vartheta_x \otimes \vartheta_x)_{B_1 B_2}$
$\lvert \varsigma_4 \rangle_{\mathfrak{b}_1 \mathfrak{b}_2 B_3} \otimes \lvert \Upsilon_4 \rangle_{\mathfrak{b}_3 B_4}$	$(\vartheta_x \otimes \vartheta_x \otimes \vartheta_x)_{A_1 A_2 A_3}$	$(\vartheta_x \otimes \vartheta_x)_{B_1 B_2}$

Case IV:

Suppose that the measurement result of Alice is $\lvert \varsigma_4 \rangle_{\mathfrak{a}_1 \mathfrak{a}_2 A_4}$, then the state of the remaining qubits becomes

$$
\begin{aligned}
\lvert \Gamma_4 \rangle = \ & (\mathfrak{h}_1 \lvert 000 \rangle + \mathfrak{h}_2 \lvert 111 \rangle)_{\mathfrak{b}_1 \mathfrak{b}_2 \mathfrak{b}_3} \otimes [\mathfrak{g}_1 (\lvert 1010000 \rangle + \lvert 1111111 \rangle) \\
& - \mathfrak{g}_2 (\lvert 0000000 \rangle + \lvert 0101111 \rangle)]_{B_1 A_1 B_2 A_2 B_3 A_3 B_4}.
\end{aligned}
$$

(10.16)

The reduced state given in Eq. (10.16) can be written as

$$
\begin{aligned}
|\Gamma_4\rangle = \ & |\varsigma_1\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_1\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|10100\rangle \\
& + \mathfrak{g}_1\mathfrak{h}_2|11111\rangle - \mathfrak{g}_2\mathfrak{h}_1|00000\rangle - \mathfrak{g}_2\mathfrak{h}_2|01011\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+ \ & |\varsigma_1\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_2\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|10100\rangle \\
& - \mathfrak{g}_1\mathfrak{h}_2|11111\rangle - \mathfrak{g}_2\mathfrak{h}_1|00000\rangle + \mathfrak{g}_2\mathfrak{h}_2|01011\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+ \ & |\varsigma_2\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_1\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|10100\rangle \\
& - \mathfrak{g}_1\mathfrak{h}_2|11111\rangle - \mathfrak{g}_2\mathfrak{h}_1|00000\rangle + \mathfrak{g}_2\mathfrak{h}_2|01011\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+ \ & |\varsigma_2\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_2\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|10100\rangle \\
& + \mathfrak{g}_1\mathfrak{h}_2|11111\rangle - \mathfrak{g}_2\mathfrak{h}_1|00000\rangle - \mathfrak{g}_2\mathfrak{h}_2|01011\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+ \ & |\varsigma_3\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_3\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|11111\rangle \\
& + \mathfrak{g}_1\mathfrak{h}_2|10100\rangle - \mathfrak{g}_2\mathfrak{h}_1|01011\rangle - \mathfrak{g}_2\mathfrak{h}_2|00000\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+ \ & |\varsigma_3\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_4\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|11111\rangle \\
& - \mathfrak{g}_1\mathfrak{h}_2|10100\rangle - \mathfrak{g}_2\mathfrak{h}_1|01011\rangle + \mathfrak{g}_2\mathfrak{h}_2|00000\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+ \ & |\varsigma_4\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_3\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|11111\rangle \\
& - \mathfrak{g}_1\mathfrak{h}_2|10100\rangle - \mathfrak{g}_2\mathfrak{h}_1|01011\rangle + \mathfrak{g}_2\mathfrak{h}_2|00000\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
+ \ & |\varsigma_4\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3} \otimes |\Upsilon_4\rangle_{\mathfrak{b}_3 B_4} \otimes (\mathfrak{g}_1\mathfrak{h}_1|11111\rangle \\
& + \mathfrak{g}_1\mathfrak{h}_2|10100\rangle - \mathfrak{g}_2\mathfrak{h}_1|01011\rangle - \mathfrak{g}_2\mathfrak{h}_2|00000\rangle)_{B_1 A_1 B_2 A_2 A_3}.
\end{aligned}
$$

Now Bob executes his measurement on the corresponding basis and sends the measurement results to Alice via a classical channel.

Suppose that his measurement outcome is $|\varsigma_1\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3}|\Upsilon_1\rangle_{\mathfrak{b}_3 B_4}$, the resulting state of the remaining qubits is as follows:

$$
\begin{aligned}
& (\mathfrak{g}_1\mathfrak{h}_1|10100\rangle + \mathfrak{g}_1\mathfrak{h}_2|11111\rangle - \mathfrak{g}_2\mathfrak{h}_1|00000\rangle - \mathfrak{g}_2\mathfrak{h}_2|01011\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
& = (\mathfrak{g}_1|11\rangle - \mathfrak{g}_2|00\rangle)_{B_1 B_2} \otimes (\mathfrak{h}_1|000\rangle + \mathfrak{h}_2|111\rangle)_{A_1 A_2 A_3}.
\end{aligned}
$$

Suppose that his measurement outcome is $|\varsigma_1\rangle_{\mathfrak{b}_1\mathfrak{b}_2 B_3}|\Upsilon_2\rangle_{\mathfrak{b}_3 B_4}$, the resulting state of the remaining qubits is as follows:

$$
\begin{aligned}
& (\mathfrak{g}_1\mathfrak{h}_1|10100\rangle - \mathfrak{g}_1\mathfrak{h}_2|11111\rangle - \mathfrak{g}_2\mathfrak{h}_1|00000\rangle + \mathfrak{g}_2\mathfrak{h}_2|01011\rangle)_{B_1 A_1 B_2 A_2 A_3} \\
& = (\mathfrak{g}_1|11\rangle - \mathfrak{g}_2|00\rangle)_{B_1 B_2} \otimes (\mathfrak{h}_1|000\rangle - \mathfrak{h}_2|111\rangle)_{A_1 A_2 A_3}.
\end{aligned}
$$

Suppose that his measurement outcome is $|\varsigma_2\rangle_{\mathfrak{b}_1\mathfrak{b}_2B_3}|\Upsilon_1\rangle_{\mathfrak{b}_3B_4}$, the resulting state of the remaining qubits is as follows:

$$(\mathfrak{g}_1\mathfrak{h}_1|10100\rangle - \mathfrak{g}_1\mathfrak{h}_2|11111\rangle - \mathfrak{g}_2\mathfrak{h}_1|00000\rangle + \mathfrak{g}_2\mathfrak{h}_2|01011\rangle)_{B_1A_1B_2A_2A_3}$$
$$= (\mathfrak{g}_1|11\rangle - \mathfrak{g}_2|00\rangle)_{B_1B_2} \otimes (\mathfrak{h}_1|000\rangle - \mathfrak{h}_2|111\rangle)_{A_1A_2A_3}.$$

Suppose that his measurement outcome is $|\varsigma_2\rangle_{\mathfrak{b}_1\mathfrak{b}_2B_3}|\Upsilon_2\rangle_{\mathfrak{b}_3B_4}$, the resulting state of the remaining qubits is as follows:

$$(\mathfrak{g}_1\mathfrak{h}_1|10100\rangle + \mathfrak{g}_1\mathfrak{h}_2|11111\rangle - \mathfrak{g}_2\mathfrak{h}_1|00000\rangle - \mathfrak{g}_2\mathfrak{h}_2|01011\rangle)_{B_1A_1B_2A_2A_3}$$
$$= (\mathfrak{g}_1|11\rangle - \mathfrak{g}_2|00\rangle)_{B_1B_2} \otimes (\mathfrak{h}_1|000\rangle + \mathfrak{h}_2|111\rangle)_{A_1A_2A_3}.$$

Suppose that his measurement outcome is $|\varsigma_3\rangle_{\mathfrak{b}_1\mathfrak{b}_2B_3}|\Upsilon_3\rangle_{\mathfrak{b}_3B_4}$, the resulting state of the remaining qubits is as follows:

$$(\mathfrak{g}_1\mathfrak{h}_1|11111\rangle + \mathfrak{g}_1\mathfrak{h}_2|10100\rangle - \mathfrak{g}_2\mathfrak{h}_1|01011\rangle - \mathfrak{g}_2\mathfrak{h}_2|00000\rangle)_{B_1A_1B_2A_2A_3}$$
$$= (\mathfrak{g}_1|11\rangle - \mathfrak{g}_2|00\rangle)_{B_1B_2} \otimes (\mathfrak{h}_1|111\rangle + \mathfrak{h}_2|000\rangle)_{A_1A_2A_3}.$$

Suppose that his measurement outcome is $|\varsigma_3\rangle_{\mathfrak{b}_1\mathfrak{b}_2B_3}|\Upsilon_4\rangle_{\mathfrak{b}_3B_4}$, the resulting state of the remaining qubits is as follows:

$$(\mathfrak{g}_1\mathfrak{h}_1|11111\rangle - \mathfrak{g}_1\mathfrak{h}_2|10100\rangle - \mathfrak{g}_2\mathfrak{h}_1|01011\rangle + \mathfrak{g}_2\mathfrak{h}_2|00000\rangle)_{B_1A_1B_2A_2A_3}$$
$$= (\mathfrak{g}_1|11\rangle - \mathfrak{g}_2|00\rangle)_{B_1B_2} \otimes (\mathfrak{h}_1|111\rangle - \mathfrak{h}_2|000\rangle)_{A_1A_2A_3}.$$

Suppose that his measurement outcome is $|\varsigma_4\rangle_{\mathfrak{b}_1\mathfrak{b}_2B_3}|\Upsilon_3\rangle_{\mathfrak{b}_3B_4}$, the resulting state of the remaining qubits is as follows:

$$(\mathfrak{g}_1\mathfrak{h}_1|11111\rangle - \mathfrak{g}_1\mathfrak{h}_2|10100\rangle - \mathfrak{g}_2\mathfrak{h}_1|01011\rangle + \mathfrak{g}_2\mathfrak{h}_2|00000\rangle)_{B_1A_1B_2A_2A_3}$$
$$= (\mathfrak{g}_1|11\rangle - \mathfrak{g}_2|00\rangle)_{B_1B_2} \otimes (\mathfrak{h}_1|111\rangle - \mathfrak{h}_2|000\rangle)_{A_1A_2A_3}.$$

Suppose that his measurement outcome is $|\varsigma_4\rangle_{\mathfrak{b}_1\mathfrak{b}_2B_3}|\Upsilon_4\rangle_{\mathfrak{b}_3B_4}$, the resulting state of the remaining qubits is as follows:

$$(\mathfrak{g}_1\mathfrak{h}_1|11111\rangle + \mathfrak{g}_1\mathfrak{h}_2|10100\rangle - \mathfrak{g}_2\mathfrak{h}_1|01011\rangle - \mathfrak{g}_2\mathfrak{h}_2|00000\rangle)_{B_1A_1B_2A_2A_3}$$
$$= (\mathfrak{g}_1|11\rangle - \mathfrak{g}_2|00\rangle)_{B_1B_2} \otimes (\mathfrak{h}_1|111\rangle + \mathfrak{h}_2|000\rangle)_{A_1A_2A_3}.$$

Finally, to recover the intended states, Alice and Bob perform appropriate unitary operations on their respective qubits. A detailed overview of the unitary operations for **Case IV** is given in Table 10.5.

Table 10.5

Appropriate unitary operation performed by Alice and Bob for Case IV ⏎

Bob's outcome	Alice's unitary operation	Bob's unitary operation
$\lvert \varsigma_1 \rangle_{\mathfrak{b}_1 \mathfrak{b}_2 B_3} \otimes \lvert \Upsilon_1 \rangle_{\mathfrak{b}_3 B_4}$	$(I \otimes I \otimes I)_{A_1 A_2 A_3}$	$(\vartheta_z \vartheta_x \otimes \vartheta_x)_{B_1 B_2}$
$\lvert \varsigma_1 \rangle_{\mathfrak{b}_1 \mathfrak{b}_2 B_3} \otimes \lvert \Upsilon_2 \rangle_{\mathfrak{b}_3 B_4}$	$(I \otimes \vartheta_z \otimes I)_{A_1 A_2 A_3}$	$(\vartheta_x \otimes \vartheta_z \vartheta_x)_{B_1 B_2}$
$\lvert \varsigma_2 \rangle_{\mathfrak{b}_1 \mathfrak{b}_2 B_3} \otimes \lvert \Upsilon_1 \rangle_{\mathfrak{b}_3 B_4}$	$(\vartheta_z \otimes I \otimes I)_{A_1 A_2 A_3}$	$(\vartheta_z \vartheta_x \otimes \vartheta_x)_{B_1 B_2}$
$\lvert \varsigma_2 \rangle_{\mathfrak{b}_1 \mathfrak{b}_2 B_3} \otimes \lvert \Upsilon_2 \rangle_{\mathfrak{b}_3 B_4}$	$(I \otimes I \otimes I)_{A_1 A_2 A_3}$	$(\vartheta_x \otimes \vartheta_z \vartheta_x)_{B_1 B_2}$
$\lvert \varsigma_3 \rangle_{\mathfrak{b}_1 \mathfrak{b}_2 B_3} \otimes \lvert \Upsilon_3 \rangle_{\mathfrak{b}_3 B_4}$	$(\vartheta_x \otimes \vartheta_x \otimes \vartheta_x)_{A_1 A_2 A_3}$	$(\vartheta_x \otimes \vartheta_z \vartheta_x)_{B_1 B_2}$
$\lvert \varsigma_3 \rangle_{\mathfrak{b}_1 \mathfrak{b}_2 B_3} \otimes \lvert \Upsilon_4 \rangle_{\mathfrak{b}_3 B_4}$	$(\vartheta_x \otimes \vartheta_x \otimes \vartheta_z \vartheta_x)_{A_1 A_2 A_3}$	$(\vartheta_z \vartheta_x \otimes \vartheta_x)_{B_1 B_2}$
$\lvert \varsigma_4 \rangle_{\mathfrak{b}_1 \mathfrak{b}_2 B_3} \otimes \lvert \Upsilon_3 \rangle_{\mathfrak{b}_3 B_4}$	$(\vartheta_x \otimes \vartheta_x \otimes \vartheta_z \vartheta_x)_{A_1 A_2 A_3}$	$(\vartheta_x \otimes \vartheta_z \vartheta_x)_{B_1 B_2}$
$\lvert \varsigma_4 \rangle_{\mathfrak{b}_1 \mathfrak{b}_2 B_3} \otimes \lvert \Upsilon_4 \rangle_{\mathfrak{b}_3 B_4}$	$(\vartheta_x \otimes \vartheta_x \otimes \vartheta_x)_{A_1 A_2 A_3}$	$(\vartheta_z \vartheta_x \otimes \vartheta_x)_{B_1 B_2}$

The bidirectional exchange of state is thus completed.

11 Controlled Teleportation Protocols

11.1 INTRODUCTION

In this chapter, teleportation under supervision, that is, controlled teleportation process is described. Particularly, we describe controlled bidirectional and cyclic teleportation protocols. Some other works on teleportation under supervision are listed in [30, 31, 53, 73, 74, 87, 90, 91, 93, 124, 141, 142, 150, 154, 172, 195, 202]. Particularly, teleportation protocols under multiple and hierarchical control have appeared in works like [17, 65, 138, 182, 183, 184, 192].

11.2 BIDIRECTIONAL CONTROLLED TELEPORTATION PROTOCOL OF TWO SINGLE-QUBIT STATES

In this chapter, we explore the role of controller in teleportation protocol. These protocols which are performed under the supervision of a controller (sometimes called a supervisor) are known as controlled teleportation protocols. A controller is a party who acts toward the end of the protocol and signals for the ultimate steps to be executed for the completion of the process. If the controller is not satisfied by the performances of the other parties, then the controller can withhold his action in which case the teleportation process cannot be completed. In this section we describe a bi-directional teleportation scheme while in the following section a cyclic teleportation protocol is presented.

We consider the problem of exchange of two single-qubit quantum states between two parties Alice and Bob with the help of a third party Charlie, who acts as a controller in this protocol. Here, both Alice and Bob act as a sender as well as a receiver. The protocol we describe for performing the above task is a controlled bi-directional protocol given by Zha et al. [196].

Suppose that Alice wants to transfer to Bob the quantum state given by

$$|\aleph_1\rangle_\mathfrak{a} = (\mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle)$$

(11.1)

and that Bob wants to transfer to Alice the quantum state given by

$$|\aleph_2\rangle_\mathfrak{b} = (\mathfrak{h}_1|0\rangle + \mathfrak{h}_2|1\rangle),$$

(11.2)

where all the coefficients satisfy the normalization conditions, that is,

$$|\mathfrak{g}_1|^2 + |\mathfrak{g}_2|^2 = 1$$

and

$$|\mathfrak{h}_1|^2 + |\mathfrak{h}_2|^2 = 1.$$

There is another party, Charlie by name, who is the controller of the protocol.

A five-qubit cluster state is used as a quantum channel to achieve this job by connecting three mutually separated parties, which has the form:

$$|E\rangle = \frac{1}{2}(|00000\rangle + |00111\rangle + |11010\rangle + |11101\rangle)_{A_1 B_1 A_2 C B_2},$$

(11.3)

where the qubits (A_1, A_2), $(B_1 B_2)$ and C are in the possessions of Alice, Bob and Charlie, respectively. The entanglement generation process for the state (11.3) is shown in Figure 11.2. Further, all the parties are connected by classical channels amongst themselves.

The total system of seven qubits can be written as:

$$
\begin{aligned}
|\Gamma\rangle &= |\aleph_1\rangle_\mathfrak{a} \otimes |\aleph_2\rangle_\mathfrak{b} \otimes |E\rangle_{A_1 B_1 A_2 C B_2} \\
&= (\mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle)_\mathfrak{a} \otimes (\mathfrak{h}_1|0\rangle + \mathfrak{h}_2|1\rangle)_\mathfrak{b} \otimes \frac{1}{2}(|00000\rangle + |00111\rangle \\
&\quad + |11010\rangle + |11101\rangle)_{A_1 B_1 A_2 C B_2}.
\end{aligned}
$$

(11.4)

For this communication process Alice and Bob both execute measurements with Bell basis described as

$$|\Upsilon_1\rangle_{\mathfrak{a}A_1/\mathfrak{b}B_2} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$

$$|\Upsilon_2\rangle_{\mathfrak{a}A_1/\mathfrak{b}B_2} = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$

$$|\Upsilon_3\rangle_{\mathfrak{a}A_1/\mathfrak{b}B_2} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$

$$|\Upsilon_4\rangle_{\mathfrak{a}A_1/\mathfrak{b}B_2} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle),$$

$$(11.5)$$

on their pairs of qubits $(\mathfrak{a}, A_1)$ and $(\mathfrak{b}, B_2)$, respectively, and finally, the controller Charlie makes a von Neumann measurement on his qubit C on the basis given by

$$|\zeta_1\rangle_C = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),$$

$$|\zeta_2\rangle_C = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

$$(11.6)$$

Considering the basis discussed in Eq. (11.5) and the basis given by Eq. (11.6), the composite state $|\Gamma\rangle$ in Eq. (11.4) can be written as (ignoring the constant factor)

$$
\begin{aligned}
|\Gamma\rangle = \ & (\mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle)_{\mathfrak{a}} \otimes (\mathfrak{h}_1|0\rangle + \mathfrak{h}_2|1\rangle)_{\mathfrak{b}} \otimes \frac{1}{2}(|00000\rangle + |00111\rangle \\
& + |11010\rangle + |11101\rangle)_{A_1 B_1 A_2 C B_2} \\
= \ & \sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{k=1}^{2} |\Upsilon_i\rangle_{\mathfrak{a}A_1} \otimes |\Upsilon_j\rangle_{\mathfrak{b}B_2} \otimes |\zeta_k\rangle_C \otimes |\upsilon_{ijk}\rangle_{B_1 A_2}
\end{aligned}
$$

where $|\upsilon_{ijk}\rangle_{B_1 A_2}$'s are given by

$$|v_{111}\rangle_{B_1A_2} = \mathfrak{g}_1\mathfrak{h}_1|00\rangle + \mathfrak{g}_1\mathfrak{h}_2|01\rangle + \mathfrak{g}_2\mathfrak{h}_1|10\rangle + \mathfrak{g}_2\mathfrak{h}_2|11\rangle,$$
$$|v_{112}\rangle_{B_1A_2} = \mathfrak{g}_1\mathfrak{h}_1|00\rangle - \mathfrak{g}_1\mathfrak{h}_2|01\rangle - \mathfrak{g}_2\mathfrak{h}_1|10\rangle + \mathfrak{g}_2\mathfrak{h}_2|11\rangle,$$
$$|v_{121}\rangle_{B_1A_2} = \mathfrak{g}_1\mathfrak{h}_1|00\rangle - \mathfrak{g}_1\mathfrak{h}_2|01\rangle + \mathfrak{g}_2\mathfrak{h}_1|10\rangle - \mathfrak{g}_2\mathfrak{h}_2|11\rangle,$$
$$|v_{122}\rangle_{B_1A_2} = \mathfrak{g}_1\mathfrak{h}_1|00\rangle + \mathfrak{g}_1\mathfrak{h}_2|01\rangle - \mathfrak{g}_2\mathfrak{h}_1|10\rangle - \mathfrak{g}_2\mathfrak{h}_2|11\rangle,$$
$$|v_{211}\rangle_{B_1A_2} = \mathfrak{g}_1\mathfrak{h}_1|00\rangle + \mathfrak{g}_1\mathfrak{h}_2|01\rangle - \mathfrak{g}_2\mathfrak{h}_1|10\rangle - \mathfrak{g}_2\mathfrak{h}_2|11\rangle,$$
$$|v_{212}\rangle_{B_1A_2} = \mathfrak{g}_1\mathfrak{h}_1|00\rangle - \mathfrak{g}_1\mathfrak{h}_2|01\rangle + \mathfrak{g}_2\mathfrak{h}_1|10\rangle - \mathfrak{g}_2\mathfrak{h}_2|11\rangle,$$
$$|v_{221}\rangle_{B_1A_2} = \mathfrak{g}_1\mathfrak{h}_1|00\rangle - \mathfrak{g}_1\mathfrak{h}_2|01\rangle - \mathfrak{g}_2\mathfrak{h}_1|10\rangle + \mathfrak{g}_2\mathfrak{h}_2|11\rangle,$$
$$|v_{222}\rangle_{B_1A_2} = \mathfrak{g}_1\mathfrak{h}_1|00\rangle + \mathfrak{g}_1\mathfrak{h}_2|01\rangle + \mathfrak{g}_2\mathfrak{h}_1|10\rangle + \mathfrak{g}_2\mathfrak{h}_2|11\rangle,$$
$$|v_{131}\rangle_{B_1A_2} = \mathfrak{g}_1\mathfrak{h}_1|01\rangle + \mathfrak{g}_1\mathfrak{h}_2|00\rangle + \mathfrak{g}_2\mathfrak{h}_1|11\rangle + \mathfrak{g}_2\mathfrak{h}_2|10\rangle,$$
$$|v_{132}\rangle_{B_1A_2} = -\mathfrak{g}_1\mathfrak{h}_1|01\rangle + \mathfrak{g}_1\mathfrak{h}_2|00\rangle + \mathfrak{g}_2\mathfrak{h}_1|11\rangle - \mathfrak{g}_2\mathfrak{h}_2|10\rangle,$$
$$|v_{141}\rangle_{B_1A_2} = \mathfrak{g}_1\mathfrak{h}_1|01\rangle - \mathfrak{g}_1\mathfrak{h}_2|00\rangle + \mathfrak{g}_2\mathfrak{h}_1|11\rangle - \mathfrak{g}_2\mathfrak{h}_2|10\rangle,$$
$$|v_{142}\rangle_{B_1A_2} = -\mathfrak{g}_1\mathfrak{h}_1|01\rangle - \mathfrak{g}_1\mathfrak{h}_2|00\rangle + \mathfrak{g}_2\mathfrak{h}_1|11\rangle + \mathfrak{g}_2\mathfrak{h}_2|10\rangle,$$
$$|v_{231}\rangle_{B_1A_2} = \mathfrak{g}_1\mathfrak{h}_1|01\rangle + \mathfrak{g}_1\mathfrak{h}_2|00\rangle - \mathfrak{g}_2\mathfrak{h}_1|11\rangle - \mathfrak{g}_2\mathfrak{h}_2|10\rangle,$$
$$|v_{232}\rangle_{B_1A_2} = -\mathfrak{g}_1\mathfrak{h}_1|01\rangle + \mathfrak{g}_1\mathfrak{h}_2|00\rangle - \mathfrak{g}_2\mathfrak{h}_1|11\rangle + \mathfrak{g}_2\mathfrak{h}_2|10\rangle,$$
$$|v_{241}\rangle_{B_1A_2} = \mathfrak{g}_1\mathfrak{h}_1|01\rangle - \mathfrak{g}_1\mathfrak{h}_2|00\rangle - \mathfrak{g}_2\mathfrak{h}_1|11\rangle + \mathfrak{g}_2\mathfrak{h}_2|10\rangle,$$
$$|v_{242}\rangle_{B_1A_2} = -\mathfrak{g}_1\mathfrak{h}_1|01\rangle - \mathfrak{g}_1\mathfrak{h}_2|00\rangle - \mathfrak{g}_2\mathfrak{h}_1|11\rangle - \mathfrak{g}_2\mathfrak{h}_2|10\rangle,$$
$$|v_{311}\rangle_{B_1A_2} = \mathfrak{g}_1\mathfrak{h}_1|10\rangle + \mathfrak{g}_1\mathfrak{h}_2|11\rangle + \mathfrak{g}_2\mathfrak{h}_1|00\rangle + \mathfrak{g}_2\mathfrak{h}_2|01\rangle,$$
$$|v_{312}\rangle_{B_1A_2} = -\mathfrak{g}_1\mathfrak{h}_1|10\rangle + \mathfrak{g}_1\mathfrak{h}_2|11\rangle + \mathfrak{g}_2\mathfrak{h}_1|00\rangle - \mathfrak{g}_2\mathfrak{h}_2|01\rangle,$$
$$|v_{321}\rangle_{B_1A_2} = \mathfrak{g}_1\mathfrak{h}_1|10\rangle - \mathfrak{g}_1\mathfrak{h}_2|11\rangle + \mathfrak{g}_2\mathfrak{h}_1|00\rangle - \mathfrak{g}_2\mathfrak{h}_2|01\rangle,$$
$$|v_{322}\rangle_{B_1A_2} = -\mathfrak{g}_1\mathfrak{h}_1|10\rangle - \mathfrak{g}_1\mathfrak{h}_2|11\rangle + \mathfrak{g}_2\mathfrak{h}_1|00\rangle + \mathfrak{g}_2\mathfrak{h}_2|01\rangle,$$
$$|v_{411}\rangle_{B_1A_2} = \mathfrak{g}_1\mathfrak{h}_1|10\rangle + \mathfrak{g}_1\mathfrak{h}_2|11\rangle - \mathfrak{g}_2\mathfrak{h}_1|00\rangle - \mathfrak{g}_2\mathfrak{h}_2|01\rangle,$$
$$|v_{412}\rangle_{B_1A_2} = -\mathfrak{g}_1\mathfrak{h}_1|10\rangle + \mathfrak{g}_1\mathfrak{h}_2|11\rangle - \mathfrak{g}_2\mathfrak{h}_1|00\rangle + \mathfrak{g}_2\mathfrak{h}_2|01\rangle,$$
$$|v_{421}\rangle_{B_1A_2} = \mathfrak{g}_1\mathfrak{h}_1|10\rangle - \mathfrak{g}_1\mathfrak{h}_2|11\rangle - \mathfrak{g}_2\mathfrak{h}_1|00\rangle + \mathfrak{g}_2\mathfrak{h}_2|01\rangle,$$
$$|v_{422}\rangle_{B_1A_2} = -\mathfrak{g}_1\mathfrak{h}_1|10\rangle - \mathfrak{g}_1\mathfrak{h}_2|11\rangle - \mathfrak{g}_2\mathfrak{h}_1|00\rangle - \mathfrak{g}_2\mathfrak{h}_2|01\rangle,$$

$$|v_{331}\rangle_{B_1A_2} = \mathfrak{g}_1\mathfrak{h}_1|11\rangle + \mathfrak{g}_1\mathfrak{h}_2|10\rangle + \mathfrak{g}_2\mathfrak{h}_1|01\rangle + \mathfrak{g}_2\mathfrak{h}_2|00\rangle,$$
$$|v_{332}\rangle_{B_1A_2} = \mathfrak{g}_1\mathfrak{h}_1|11\rangle - \mathfrak{g}_1\mathfrak{h}_2|10\rangle - \mathfrak{g}_2\mathfrak{h}_1|01\rangle + \mathfrak{g}_2\mathfrak{h}_2|00\rangle,$$
$$|v_{341}\rangle_{B_1A_2} = \mathfrak{g}_1\mathfrak{h}_1|11\rangle - \mathfrak{g}_1\mathfrak{h}_2|10\rangle + \mathfrak{g}_2\mathfrak{h}_1|01\rangle - \mathfrak{g}_2\mathfrak{h}_2|00\rangle,$$
$$|v_{342}\rangle_{B_1A_2} = \mathfrak{g}_1\mathfrak{h}_1|11\rangle + \mathfrak{g}_1\mathfrak{h}_2|10\rangle - \mathfrak{g}_2\mathfrak{h}_1|01\rangle - \mathfrak{g}_2\mathfrak{h}_2|00\rangle,$$
$$|v_{431}\rangle_{B_1A_2} = \mathfrak{g}_1\mathfrak{h}_1|11\rangle + \mathfrak{g}_1\mathfrak{h}_2|10\rangle - \mathfrak{g}_2\mathfrak{h}_1|01\rangle - \mathfrak{g}_2\mathfrak{h}_2|00\rangle,$$
$$|v_{432}\rangle_{B_1A_2} = \mathfrak{g}_1\mathfrak{h}_1|11\rangle - \mathfrak{g}_1\mathfrak{h}_2|10\rangle + \mathfrak{g}_2\mathfrak{h}_1|01\rangle - \mathfrak{g}_2\mathfrak{h}_2|00\rangle,$$
$$|v_{441}\rangle_{B_1A_2} = \mathfrak{g}_1\mathfrak{h}_1|11\rangle - \mathfrak{g}_1\mathfrak{h}_2|10\rangle - \mathfrak{g}_2\mathfrak{h}_1|01\rangle + \mathfrak{g}_2\mathfrak{h}_2|00\rangle,$$
$$|v_{442}\rangle_{B_1A_2} = \mathfrak{g}_1\mathfrak{h}_1|11\rangle + \mathfrak{g}_1\mathfrak{h}_2|10\rangle + \mathfrak{g}_2\mathfrak{h}_1|01\rangle + \mathfrak{g}_2\mathfrak{h}_2|00\rangle.$$

Now Alice and Bob make Bell basis measurements on their respective qubits. After the completion of the measurements, Alice (Bob) sends her (his) measurement outcomes to Bob (Alice) and Charlie through some classical channels. At this stage, Charlie plays an

important role for completion of the protocol. Before making a decision Charlie carefully examines every concerned circumstances. If, even at the very last moment, Charlie observes something wrong, he will stop the protocol by doing nothing! Otherwise, he performs measurement on his single qubit C in the basis given in Eq. (11.6). After the measurement, Charlie sends his outcome to Alice and Bob through classical channels. By getting these information from Charlie, Alice and Bob make appropriate unitary operations on their remaining qubits which are given in the following [Table 11.1](#) and [Table 11.2](#) and thereby complete the exchange of qubit states. This is end of the protocol. The whole scenario is depicted in [Figure 11.1](#).

Table 11.1

Alice's and Bob's unitary operation conditioned on Bob's, Alice's and Charlie's measurement results ⏎

Alice's result	Bob's result	Charlie's result	Reduced state	Alice's unitary operation	Bob's unitary operation
$\|\Upsilon_1\rangle_{aA_1}$	$\|\Upsilon_1\rangle_{bB_2}$	$\|\zeta_1\rangle_C$	$\|\upsilon_{111}\rangle_{B_1A_2}$	$(I)_{A_2}$	$(I)_{B_1}$
$\|\Upsilon_1\rangle_{aA_1}$	$\|\Upsilon_1\rangle_{bB_2}$	$\|\zeta_2\rangle_C$	$\|\upsilon_{112}\rangle_{B_1A_2}$	$(\vartheta_z)_{A_2}$	$(\vartheta_z)_{B_1}$
$\|\Upsilon_1\rangle_{aA_1}$	$\|\Upsilon_2\rangle_{bB_2}$	$\|\zeta_1\rangle_C$	$\|\upsilon_{121}\rangle_{B_1A_2}$	$(\vartheta_z)_{A_2}$	$(I)_{B_1}$
$\|\Upsilon_1\rangle_{aA_1}$	$\|\Upsilon_2\rangle_{bB_2}$	$\|\zeta_2\rangle_C$	$\|\upsilon_{122}\rangle_{B_1A_2}$	$(I)_{A_2}$	$(\vartheta_z)_{B_1}$
$\|\Upsilon_2\rangle_{aA_1}$	$\|\Upsilon_1\rangle_{bB_2}$	$\|\zeta_1\rangle_C$	$\|\upsilon_{211}\rangle_{B_1A_2}$	$(I)_{A_2}$	$(\vartheta_z)_{B_1}$
$\|\Upsilon_2\rangle_{aA_1}$	$\|\Upsilon_1\rangle_{bB_2}$	$\|\zeta_2\rangle_C$	$\|\upsilon_{212}\rangle_{B_1A_2}$	$(\vartheta_z)_{A_2}$	$(I)_{B_1}$
$\|\Upsilon_2\rangle_{aA_1}$	$\|\Upsilon_2\rangle_{bB_2}$	$\|\zeta_1\rangle_C$	$\|\upsilon_{221}\rangle_{B_1A_2}$	$(\vartheta_z)_{A_2}$	$(\vartheta_z)_{B_1}$
$\|\Upsilon_2\rangle_{aA_1}$	$\|\Upsilon_2\rangle_{bB_2}$	$\|\zeta_2\rangle_C$	$\|\upsilon_{222}\rangle_{B_1A_2}$	$(I)_{A_2}$	$(I)_{B_1}$
$\|\Upsilon_1\rangle_{aA_1}$	$\|\Upsilon_3\rangle_{bB_2}$	$\|\zeta_1\rangle_C$	$\|\upsilon_{131}\rangle_{B_1A_2}$	$(\vartheta_x)_{A_2}$	$(I)_{B_1}$
$\|\Upsilon_1\rangle_{aA_1}$	$\|\Upsilon_3\rangle_{bB_2}$	$\|\zeta_2\rangle_C$	$\|\upsilon_{132}\rangle_{B_1A_2}$	$(\vartheta_z\vartheta_x)_{A_2}$	$(\vartheta_x\vartheta_z\vartheta_x)_{B_1}$
$\|\Upsilon_1\rangle_{aA_1}$	$\|\Upsilon_4\rangle_{bB_2}$	$\|\zeta_1\rangle_C$	$\|\upsilon_{141}\rangle_{B_1A_2}$	$(\vartheta_z\vartheta_x)_{A_2}$	$(I)_{B_1}$
$\|\Upsilon_1\rangle_{aA_1}$	$\|\Upsilon_4\rangle_{bB_2}$	$\|\zeta_2\rangle_C$	$\|\upsilon_{142}\rangle_{B_1A_2}$	$(\vartheta_x)_{A_2}$	$(\vartheta_x\vartheta_z\vartheta_x)_{B_1}$
$\|\Upsilon_2\rangle_{aA_1}$	$\|\Upsilon_3\rangle_{bB_2}$	$\|\zeta_1\rangle_C$	$\|\upsilon_{231}\rangle_{B_1A_2}$	$(\vartheta_x)_{A_2}$	$(\vartheta_z)_{B_1}$
$\|\Upsilon_2\rangle_{aA_1}$	$\|\Upsilon_3\rangle_{bB_2}$	$\|\zeta_2\rangle_C$	$\|\upsilon_{232}\rangle_{B_1A_2}$	$(\vartheta_z\vartheta_x)_{A_2}$	$(\vartheta_z\vartheta_x\vartheta_z\vartheta_x)_{B_1}$
$\|\Upsilon_2\rangle_{aA_1}$	$\|\Upsilon_4\rangle_{bB_2}$	$\|\zeta_1\rangle_C$	$\|\upsilon_{241}\rangle_{B_1A_2}$	$(\vartheta_z\vartheta_x)_{A_2}$	$(\vartheta_z)_{B_1}$
$\|\Upsilon_2\rangle_{aA_1}$	$\|\Upsilon_4\rangle_{bB_2}$	$\|\zeta_2\rangle_C$	$\|\upsilon_{242}\rangle_{B_1A_2}$	$(\vartheta_x)_{A_2}$	$(\vartheta_z\vartheta_x\vartheta_z\vartheta_x)_{B_1}$

Table 11.2

Continued: Alice's and Bob's unitary operation conditioned on Bob's, Alice's and Charlie's measurement results ⏎

Alice's result	Bob's result	Charlie's result	Reduced state	Alice's unitary operation	Bob's unitary operation
$\lvert\Upsilon_3\rangle_{aA_1}$	$\lvert\Upsilon_1\rangle_{bB_2}$	$\lvert\zeta_1\rangle_C$	$\lvert\upsilon_{311}\rangle_{B_1A_2}$	$(I)_{A_2}$	$(\vartheta_x)_{B_1}$
$\lvert\Upsilon_3\rangle_{aA_1}$	$\lvert\Upsilon_1\rangle_{bB_2}$	$\lvert\zeta_2\rangle_C$	$\lvert\upsilon_{312}\rangle_{B_1A_2}$	$(\vartheta_z)_{A_2}$	$(\vartheta_x\vartheta_z)_{B_1}$
$\lvert\Upsilon_3\rangle_{aA_1}$	$\lvert\Upsilon_2\rangle_{bB_2}$	$\lvert\zeta_1\rangle_C$	$\lvert\upsilon_{321}\rangle_{B_1A_2}$	$(\vartheta_z)_{A_2}$	$(\vartheta_x)_{B_1}$
$\lvert\Upsilon_3\rangle_{aA_1}$	$\lvert\Upsilon_2\rangle_{bB_2}$	$\lvert\zeta_2\rangle_C$	$\lvert\upsilon_{322}\rangle_{B_1A_2}$	$(I)_{A_2}$	$(\vartheta_x\vartheta_z)_{B_1}$
$\lvert\Upsilon_4\rangle_{aA_1}$	$\lvert\Upsilon_1\rangle_{bB_2}$	$\lvert\zeta_1\rangle_C$	$\lvert\upsilon_{411}\rangle_{B_1A_2}$	$(I)_{A_2}$	$(\vartheta_z\vartheta_x)_{B_1}$
$\lvert\Upsilon_4\rangle_{aA_1}$	$\lvert\Upsilon_1\rangle_{bB_2}$	$\lvert\zeta_2\rangle_C$	$\lvert\upsilon_{412}\rangle_{B_1A_2}$	$(\vartheta_z)_{A_2}$	$(\vartheta_z\vartheta_x\vartheta_z)_{B_1}$
$\lvert\Upsilon_4\rangle_{aA_1}$	$\lvert\Upsilon_2\rangle_{bB_2}$	$\lvert\zeta_1\rangle_C$	$\lvert\upsilon_{421}\rangle_{B_1A_2}$	$(\vartheta_z)_{A_2}$	$(\vartheta_z\vartheta_x)_{B_1}$
$\lvert\Upsilon_4\rangle_{aA_1}$	$\lvert\Upsilon_2\rangle_{bB_2}$	$\lvert\zeta_2\rangle_C$	$\lvert\upsilon_{422}\rangle_{B_1A_2}$	$(I)_{A_2}$	$(\vartheta_z\vartheta_x\vartheta_z)_{B_1}$
$\lvert\Upsilon_3\rangle_{aA_1}$	$\lvert\Upsilon_3\rangle_{bB_2}$	$\lvert\zeta_1\rangle_C$	$\lvert\upsilon_{331}\rangle_{B_1A_2}$	$(\vartheta_x)_{A_2}$	$(\vartheta_x)_{B_1}$
$\lvert\Upsilon_3\rangle_{aA_1}$	$\lvert\Upsilon_3\rangle_{bB_2}$	$\lvert\zeta_2\rangle_C$	$\lvert\upsilon_{332}\rangle_{B_1A_2}$	$(\vartheta_z\vartheta_x)_{A_2}$	$(\vartheta_z\vartheta_x)_{B_1}$
$\lvert\Upsilon_3\rangle_{aA_1}$	$\lvert\Upsilon_4\rangle_{bB_2}$	$\lvert\zeta_1\rangle_C$	$\lvert\upsilon_{341}\rangle_{B_1A_2}$	$(\vartheta_z\vartheta_x)_{A_2}$	$(\vartheta_x)_{B_1}$
$\lvert\Upsilon_3\rangle_{aA_1}$	$\lvert\Upsilon_4\rangle_{bB_2}$	$\lvert\zeta_2\rangle_C$	$\lvert\upsilon_{342}\rangle_{B_1A_2}$	$(\vartheta_x)_{A_2}$	$(\vartheta_z\vartheta_x)_{B_1}$
$\lvert\Upsilon_4\rangle_{aA_1}$	$\lvert\Upsilon_3\rangle_{bB_2}$	$\lvert\zeta_1\rangle_C$	$\lvert\upsilon_{431}\rangle_{B_1A_2}$	$(\vartheta_x)_{A_2}$	$(\vartheta_z\vartheta_x)_{B_1}$
$\lvert\Upsilon_4\rangle_{aA_1}$	$\lvert\Upsilon_3\rangle_{bB_2}$	$\lvert\zeta_2\rangle_C$	$\lvert\upsilon_{432}\rangle_{B_1A_2}$	$(\vartheta_z\vartheta_x)_{A_2}$	$(\vartheta_x)_{B_1}$
$\lvert\Upsilon_4\rangle_{aA_1}$	$\lvert\Upsilon_4\rangle_{bB_2}$	$\lvert\zeta_1\rangle_C$	$\lvert\upsilon_{441}\rangle_{B_1A_2}$	$(\vartheta_z\vartheta_x)_{A_2}$	$(\vartheta_z\vartheta_x)_{B_1}$
$\lvert\Upsilon_4\rangle_{aA_1}$	$\lvert\Upsilon_4\rangle_{bB_2}$	$\lvert\zeta_2\rangle_C$	$\lvert\upsilon_{442}\rangle_{B_1A_2}$	$(\vartheta_x)_{A_2}$	$(\vartheta_x)_{B_1}$
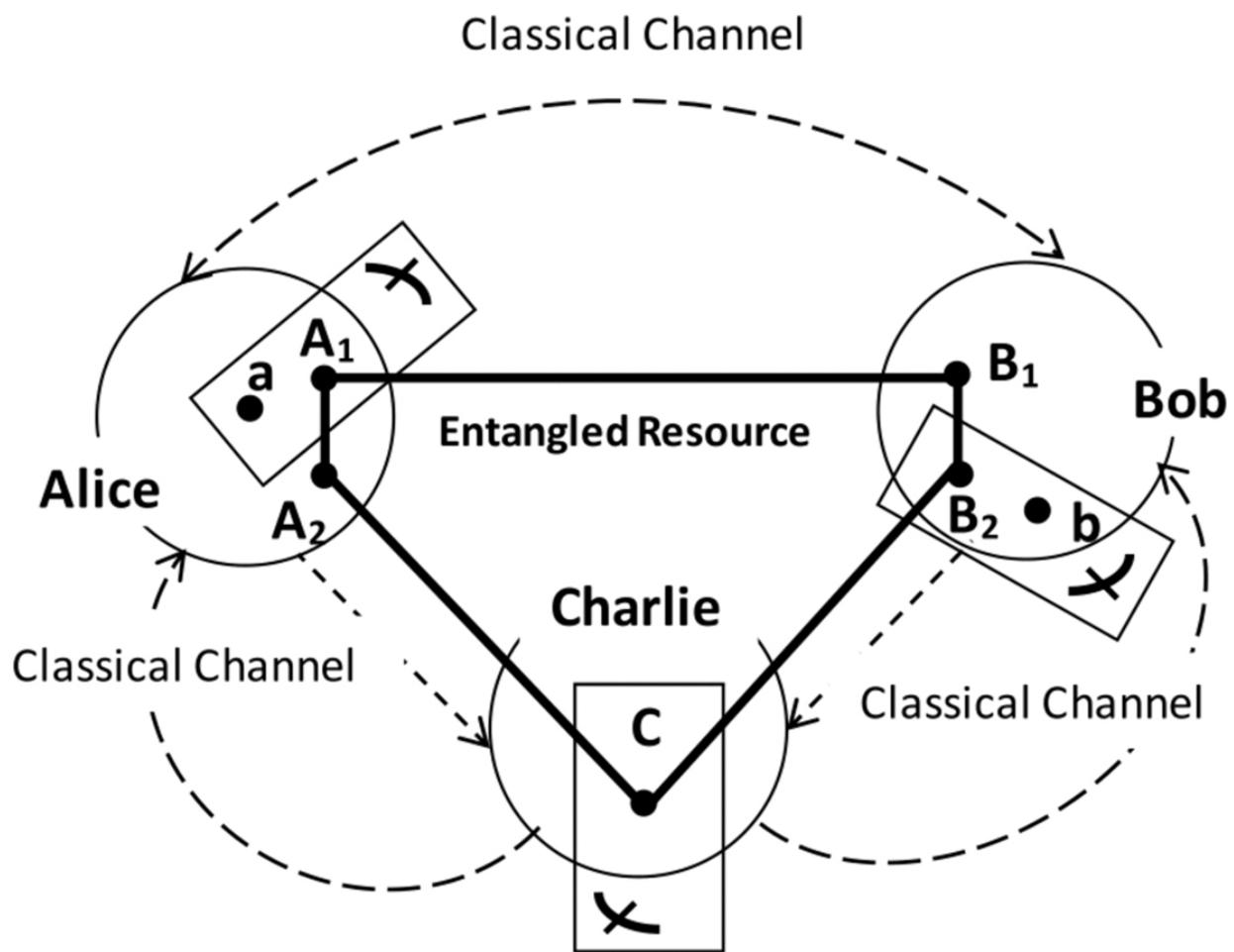
Figure 11.1 Schematic diagram of controlled bi-directional single-qubit quantum teleportation process. ⏎
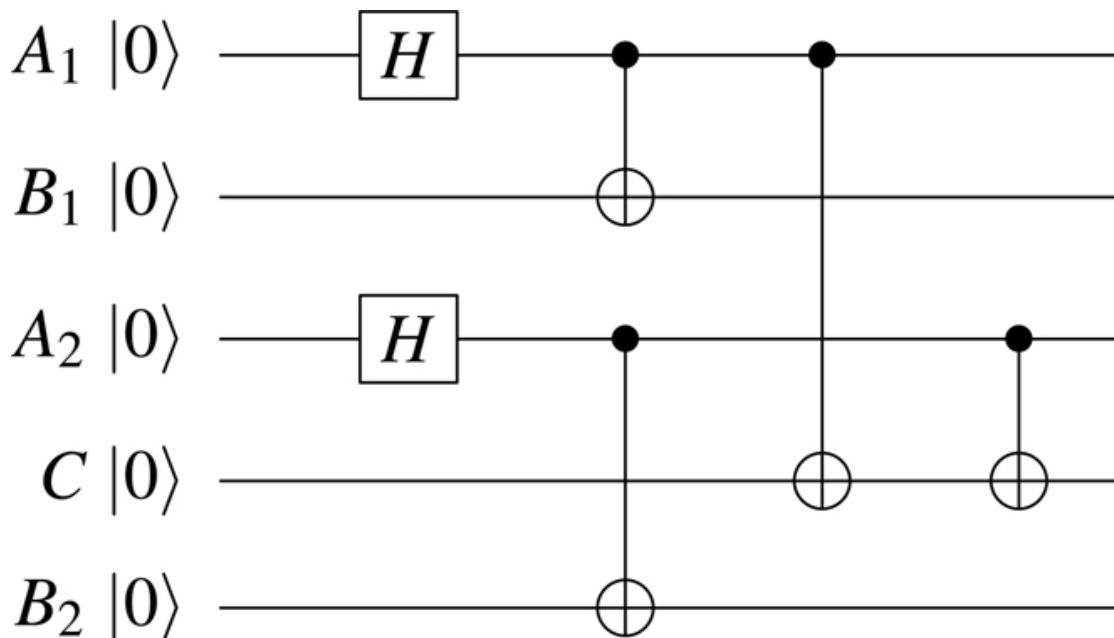
As an illustration, suppose Alice's, Bob's and Charlie's measurement outcomes are $|\Upsilon_4\rangle_{\mathfrak{a}A_1}$, $|\Upsilon_3\rangle_{\mathfrak{b}B_2}$ and $|\zeta_2\rangle_C$, respectively, then the state of the remaining qubits becomes

$$
\begin{aligned}
|v_{432}\rangle_{B_1A_2} =& \ \mathfrak{g}_1\mathfrak{h}_1|11\rangle - \mathfrak{g}_1\mathfrak{h}_2|10\rangle + \mathfrak{g}_2\mathfrak{h}_1|01\rangle - \mathfrak{g}_2\mathfrak{h}_2|00\rangle \\
=& \ (\mathfrak{g}_1|1\rangle + \mathfrak{g}_2|0\rangle)_{B_1} \otimes (\mathfrak{h}_1|1\rangle - \mathfrak{h}_2|0\rangle)_{A_2}.
\end{aligned}
$$

After receiving the classical information from the controller Charlie, Alice and Bob perform appropriate unitary operations, which are from [Table 11.2](#) respectively given by, $(\vartheta_z\vartheta_x)_{A_2}$ and $(\vartheta_x)_{B_1}$, on their respective qubits to recover the original quantum state. The goal of the protocol is thereby achieved.

11.3 CYCLIC CONTROLLED TELEPORTATION PROTOCOL AMONGST THREE PARTIES

We present here a cyclic teleportation process under a controller. The protocol has been developed by Zhi-wen Sang [143]. Here, we consider a scheme where three parties Alice, Bob and Charlie situated far apart from each other and each of them possesses an arbitrary single-qubit state without knowing any information of the state. These states in the possessions of Alice, Bob and Charlie are, respectively, given by

$$
\begin{aligned}
|\aleph_1\rangle_{\mathfrak{a}} &= (\mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle), \\
|\aleph_2\rangle_{\mathfrak{b}} &= (\mathfrak{h}_1|0\rangle + \mathfrak{h}_2|1\rangle), \\
|\aleph_3\rangle_{\mathfrak{c}} &= (\mathfrak{f}_1|0\rangle + \mathfrak{f}_2|1\rangle),
\end{aligned}
$$

(11.7)

where the coefficients $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{f}_1, \mathfrak{f}_2$ satisfy the normalization condition, that is,

$$
|\mathfrak{g}_1|^2 + |\mathfrak{g}_2|^2 = 1,
$$

$$
|\mathfrak{h}_1|^2 + |\mathfrak{h}_2|^2 = 1,
$$

$$
|\mathfrak{f}_1|^2 + |\mathfrak{f}_2|^2 = 1.
$$

Now Alice wants to transfer her single-qubit state $|\aleph_1\rangle_{\mathfrak{a}}$ to Bob, Bob wants to transfer his single-qubit state $|\aleph_2\rangle_{\mathfrak{b}}$ to Charlie and Charlie wants to transfer his single-qubit state $|\aleph_3\rangle_{\mathfrak{c}}$ to Alice. There is another party, namely David, whose role in the scheme is of a controller

from beginning to end of the scheme and without his action the scheme cannot be completed. To initiate the scheme, suppose that Alice, Bob, Charlie and David share a seven-qubit entangled state, which is given by

$$|E\rangle_{A_1 A_2 B_1 B_2 C_1 C_2 D} = \frac{1}{2\sqrt{2}} \; (|0101010\rangle + |0001111\rangle + |0111001\rangle + |0011100\rangle$$
$$+ \, |1100011\rangle + |1000110\rangle + |1110000\rangle + |1010101\rangle),$$

$$(11.8)$$
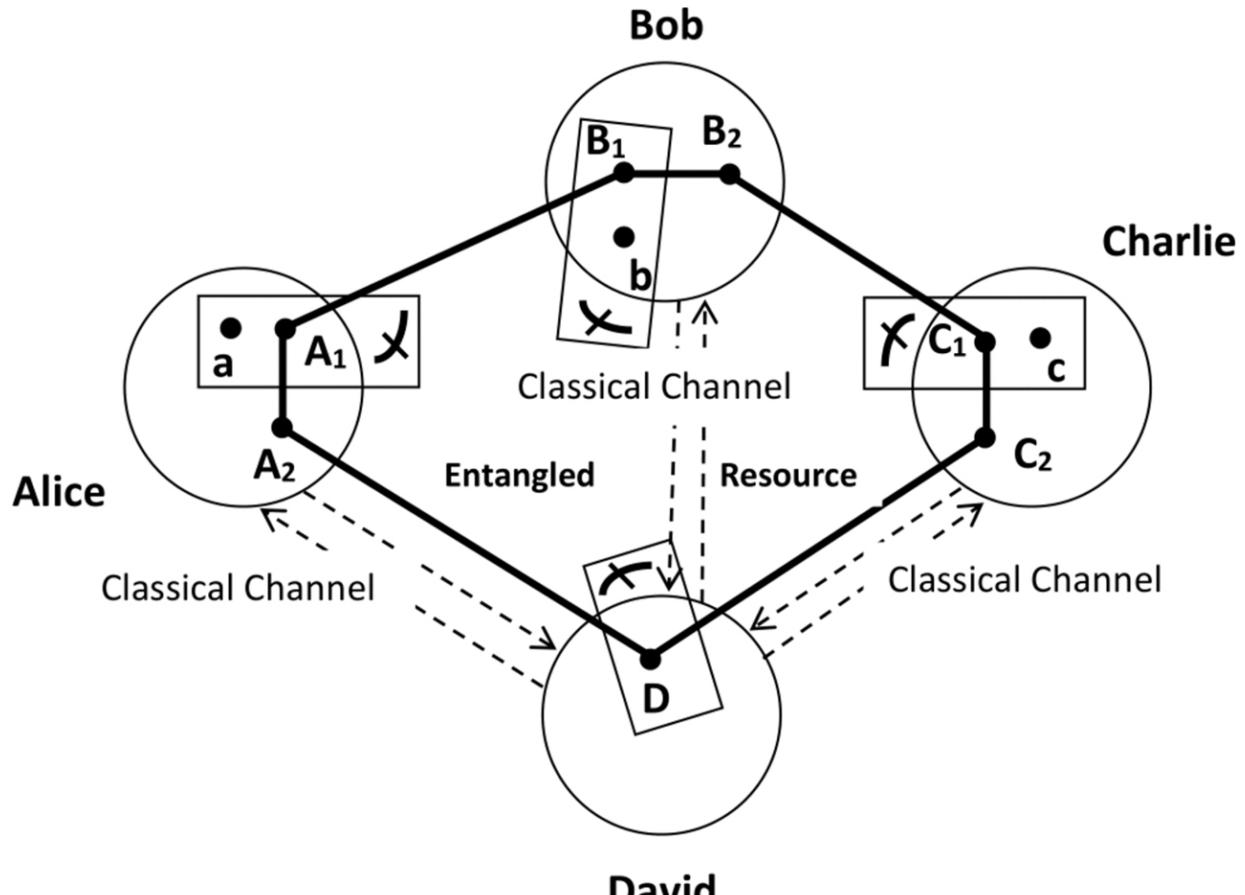
Figure 11.3 Schematic diagram for cyclic controlled teleportation protocol. ↵

where Alice possesses the qubits (A_1, A_2), Bob possesses the qubits (B_1, B_2), Charlie possesses the qubits (C_1, C_2) and the qubit D belongs to the controller David. The generation of quantum resource is shown in Figure 11.4. The state of the total quantum system can be written as

$$|\Gamma\rangle = |\aleph_1\rangle_{\mathfrak{a}} \otimes |\aleph_2\rangle_{\mathfrak{b}} \otimes |\aleph_3\rangle_{\mathfrak{c}} \otimes |E\rangle_{A_1 A_2 B_1 B_2 C_1 C_2 D}.$$

(11.9)
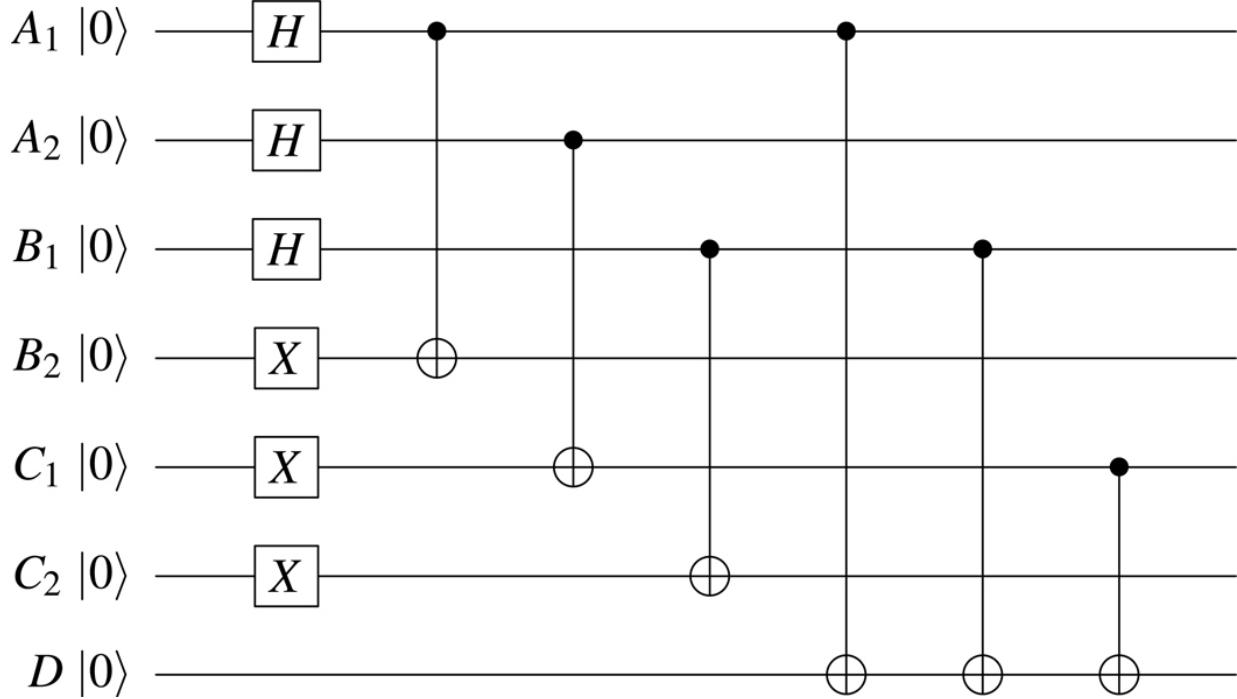
Figure 11.4 Circuit diagram for generation of entangled state $|E\rangle$ given in Eq. (11.8). ⏎

Further there are classical channels connecting all the four parties with one another.

In order to realize the quantum controlled cyclic teleportation, Alice applies a complete Bell-basis measurement on her qubits $(\mathfrak{a}, A_1)$ which are given by

$$|\Upsilon_1\rangle_{\mathfrak{a}A_1} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$

$$|\Upsilon_2\rangle_{\mathfrak{a}A_1} = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$

$$|\Upsilon_3\rangle_{\mathfrak{a}A_1} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$

$$|\Upsilon_4\rangle_{\mathfrak{a}A_1} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

Using this basis, the above combined state can be written as (ignoring the constant factor)

$$|\Gamma\rangle_{\mathfrak{a}\mathfrak{b}\mathfrak{c}A_1A_2B_1B_2C_1C_2D} = (\mathfrak{h}_1|0\rangle + \mathfrak{h}_2|1\rangle)_\mathfrak{b} \otimes (\mathfrak{f}_1|0\rangle + \mathfrak{f}_2|1\rangle)_\mathfrak{c}$$

$$\otimes \Big[|\Upsilon_1\rangle_{\mathfrak{a}A_1} \otimes \Big\{\mathfrak{g}_1\Big(|101010\rangle + |001111\rangle + |111001\rangle + |011100\rangle\Big)$$

$$+ \mathfrak{g}_2\Big(|100011\rangle + |000110\rangle + |110000\rangle + |010101\rangle\Big)\Big\}$$

$$+ |\Upsilon_2\rangle_{\mathfrak{a}A_1} \otimes \Big\{\mathfrak{g}_1\Big(|101010\rangle + |001111\rangle + |111001\rangle + |011100\rangle\Big)$$

$$- \mathfrak{g}_2\Big(|100011\rangle + |000110\rangle + |110000\rangle + |010101\rangle\Big)\Big\}$$

$$+ |\Upsilon_3\rangle_{\mathfrak{a}A_1} \otimes \Big\{\mathfrak{g}_1\Big(|100011\rangle + |000110\rangle + |110000\rangle + |010101\rangle\Big)$$

$$+ \mathfrak{g}_2\Big(|101010\rangle + |001111\rangle + |111001\rangle + |011100\rangle\Big)\Big\}$$

$$+ |\Upsilon_4\rangle_{\mathfrak{a}A_1} \otimes \Big\{\mathfrak{g}_1\Big(|100011\rangle + |000110\rangle + |110000\rangle + |010101\rangle\Big)$$

$$- \mathfrak{g}_2\Big(|101010\rangle + |001111\rangle + |111001\rangle + |011100\rangle\Big)\Big\}\Big].$$

(11.10)

After the measurement of Alice, she publicly announces her measurement outcomes through a 2-bit of classical message. Suppose Alice's measurement outcome is $|\Upsilon_2\rangle_{\mathfrak{a}A_1}$, then the state of the remaining qubits is reduced to the state

$$|\Gamma_1\rangle_{\mathfrak{b}\mathfrak{c}A_2B_1B_2C_1C_2D} = (\mathfrak{h}_1|0\rangle + \mathfrak{h}_2|1\rangle)_\mathfrak{b} \otimes (\mathfrak{f}_1|0\rangle + \mathfrak{f}_2|1\rangle)_\mathfrak{c}$$

$$\otimes \Big[\mathfrak{g}_1\Big(|101010\rangle + |001111\rangle + |111001\rangle + |011100\rangle\Big)$$

$$- \mathfrak{g}_2\Big(|100011\rangle + |000110\rangle + |110000\rangle + |010101\rangle\Big)\Big].$$

(11.11)

Subsequently, Bob performs a measurement on his own qubits $(\mathfrak{b}, B_1)$ in the basis given by

$$|\Upsilon_1\rangle_{\mathfrak{b}B_1} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad |\Upsilon_2\rangle_{\mathfrak{b}B_1} = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$

$$|\Upsilon_3\rangle_{\flat B_1} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$

$$|\Upsilon_4\rangle_{\flat B_1} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

Using this basis the above reduced state $|\Gamma_1\rangle_{\flat c A_2 B_1 B_2 C_1 C_2 D}$ can be written as (ignoring the constant factor)

$$
\begin{aligned}
|\Gamma_1\rangle_{\flat c A_2 B_1 B_2 C_1 C_2 D} \;=\; & (\mathfrak{f}_1|0\rangle + \mathfrak{f}_2|1\rangle)_c \\
& \otimes \Big[|\Upsilon_1\rangle_{\flat B_1} \otimes \Big\{ \mathfrak{g}_1\mathfrak{h}_1(|11010\rangle + |01111\rangle) + \mathfrak{g}_1\mathfrak{h}_2(|11001\rangle + |01100\rangle) \\
& \qquad - \mathfrak{g}_2\mathfrak{h}_1(|10011\rangle + |00110\rangle) - \mathfrak{g}_2\mathfrak{h}_2(|10000\rangle + |00101\rangle) \Big\} \\
& + |\Upsilon_2\rangle_{\flat B_1} \otimes \Big\{ \mathfrak{g}_1\mathfrak{h}_1(|11010\rangle + |01111\rangle) - \mathfrak{g}_1\mathfrak{h}_2(|11001\rangle + |01100\rangle) \\
& \qquad - \mathfrak{g}_2\mathfrak{h}_1(|10011\rangle + |00110\rangle) + \mathfrak{g}_2\mathfrak{h}_2(|10000\rangle + |00101\rangle) \Big\} \\
& + |\Upsilon_3\rangle_{\flat B_1} \otimes \Big\{ \mathfrak{g}_1\mathfrak{h}_1(|11001\rangle + |01100\rangle) + \mathfrak{g}_1\mathfrak{h}_2(|11010\rangle + |01111\rangle) \\
& \qquad - \mathfrak{g}_2\mathfrak{h}_1(|10000\rangle|00101\rangle) - \mathfrak{g}_2\mathfrak{h}_2(|10011\rangle + |00110\rangle) \Big\} \\
& + |\Upsilon_4\rangle_{\flat B_1} \otimes \Big\{ \mathfrak{g}_1\mathfrak{h}_1(|11001\rangle + |01100\rangle) - \mathfrak{g}_1\mathfrak{h}_2(|11010\rangle + |01111\rangle) \\
& \qquad - \mathfrak{g}_2\mathfrak{h}_1(|10000\rangle|00101\rangle) + \mathfrak{g}_2\mathfrak{h}_2(|10011\rangle + |00110\rangle) \Big\} \Big].
\end{aligned}
$$

$$(11.12)$$

After the measurement of Bob, he publicly announces his measurement results through a 2-bit classical message. Suppose Bob's measurement outcome is $|\Upsilon_3\rangle_{\flat B_1}$, then the state of the remaining qubits is reduced to the state

$$
\begin{aligned}
|\Gamma_2\rangle_{c A_2 B_2 C_1 C_2 D} \;=\; & (\mathfrak{f}_1|0\rangle + \mathfrak{f}_2|1\rangle)_c \\
& \otimes \Big[\mathfrak{g}_1\mathfrak{h}_1(|11001\rangle + |01100\rangle) + \mathfrak{g}_1\mathfrak{h}_2(|11010\rangle + |01111\rangle) \\
& \qquad - \mathfrak{g}_2\mathfrak{h}_1(|10000\rangle + |00101\rangle) - \mathfrak{g}_2\mathfrak{h}_2(|10011\rangle + |00110\rangle) \Big].
\end{aligned}
$$

Thirdly, Charlie makes a Bell-basis measurement on his own qubits $(\mathfrak{c}, C_1)$ given by

$$|\Upsilon_1\rangle_{\mathfrak{c}C_1} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$

$$|\Upsilon_2\rangle_{\mathfrak{c}C_1} = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$

$$|\Upsilon_3\rangle_{\mathfrak{c}C_1} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$

$$|\Upsilon_4\rangle_{\mathfrak{c}C_1} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

Using this basis the above reduced state $|\Gamma_2\rangle_{\mathfrak{c}A_2B_2C_1C_2D}$ can be written as (ignoring the constant factor)

$$
\begin{aligned}
|\Gamma_2\rangle_{\mathfrak{c}A_2B_2C_1C_2D} \;=\; &|\Upsilon_1\rangle_{\mathfrak{c}C_1} \otimes \Big[\mathfrak{g}_1\mathfrak{h}_1\mathfrak{f}_1|1101\rangle + \mathfrak{g}_1\mathfrak{h}_1\mathfrak{f}_2|0100\rangle + \mathfrak{g}_1\mathfrak{h}_2\mathfrak{f}_1|1110\rangle \\
&+ \mathfrak{g}_1\mathfrak{h}_2\mathfrak{f}_2|0111\rangle - \mathfrak{g}_2\mathfrak{h}_1\mathfrak{f}_1|1000\rangle - \mathfrak{g}_2\mathfrak{h}_1\mathfrak{f}_2|0001\rangle \\
&- \mathfrak{g}_2\mathfrak{h}_2\mathfrak{f}_1|1011\rangle - \mathfrak{g}_2\mathfrak{h}_2\mathfrak{f}_2|0010\rangle \Big] \\
&+ |\Upsilon_2\rangle_{\mathfrak{c}C_1} \otimes \Big[\mathfrak{g}_1\mathfrak{h}_1\mathfrak{f}_1|1101\rangle - \mathfrak{g}_1\mathfrak{h}_1\mathfrak{f}_2|0100\rangle + \mathfrak{g}_1\mathfrak{h}_2\mathfrak{f}_1|1110\rangle \\
&- \mathfrak{g}_1\mathfrak{h}_2\mathfrak{f}_2|0111\rangle - \mathfrak{g}_2\mathfrak{h}_1\mathfrak{f}_1|1000\rangle + \mathfrak{g}_2\mathfrak{h}_1\mathfrak{f}_2|0001\rangle \\
&- \mathfrak{g}_2\mathfrak{h}_2\mathfrak{f}_1|1011\rangle + \mathfrak{g}_2\mathfrak{h}_2\mathfrak{f}_2|0010\rangle \Big] \\
&+ |\Upsilon_3\rangle_{\mathfrak{c}C_1} \otimes \Big[\mathfrak{g}_1\mathfrak{h}_1\mathfrak{f}_2|1101\rangle + \mathfrak{g}_1\mathfrak{h}_1\mathfrak{f}_1|0100\rangle + \mathfrak{g}_1\mathfrak{h}_2\mathfrak{f}_2|1110\rangle \\
&+ \mathfrak{g}_1\mathfrak{h}_2\mathfrak{f}_1|0111\rangle - \mathfrak{g}_2\mathfrak{h}_1\mathfrak{f}_2|1000\rangle - \mathfrak{g}_2\mathfrak{h}_1\mathfrak{f}_1|0001\rangle \\
&- \mathfrak{g}_2\mathfrak{h}_2\mathfrak{f}_2|1011\rangle - \mathfrak{g}_2\mathfrak{h}_2\mathfrak{f}_1|0010\rangle \Big] \\
&+ |\Upsilon_4\rangle_{\mathfrak{c}C_1} \otimes \Big[-\mathfrak{g}_1\mathfrak{h}_1\mathfrak{f}_2|1101\rangle + \mathfrak{g}_1\mathfrak{h}_1\mathfrak{f}_1|0100\rangle - \mathfrak{g}_1\mathfrak{h}_2\mathfrak{f}_2|1110\rangle \\
&+ \mathfrak{g}_1\mathfrak{h}_2\mathfrak{f}_1|0111\rangle + \mathfrak{g}_2\mathfrak{h}_1\mathfrak{f}_2|1000\rangle - \mathfrak{g}_2\mathfrak{h}_1\mathfrak{f}_1|0001\rangle \\
&+ \mathfrak{g}_2\mathfrak{h}_2\mathfrak{f}_2|1011\rangle - \mathfrak{g}_2\mathfrak{h}_2\mathfrak{f}_1|0010\rangle \Big].
\end{aligned}
$$

After the measurement of Charlie, he publicly announces his measurement outcomes through a 2-bit classical message. Suppose that Charlie's measurement outcome is $|\Upsilon_1\rangle_{cC_1}$, then the state of the remaining qubits is reduced to the state

$$
\begin{aligned}
|\Gamma_3\rangle_{A_2B_2C_2D} \quad &= \mathfrak{g}_1\mathfrak{h}_1\mathfrak{f}_1|1101\rangle + \mathfrak{g}_1\mathfrak{h}_1\mathfrak{f}_2|0100\rangle + \mathfrak{g}_1\mathfrak{h}_2\mathfrak{f}_1|1110\rangle \\
&+ \mathfrak{g}_1\mathfrak{h}_2\mathfrak{f}_2|0111\rangle - \mathfrak{g}_2\mathfrak{h}_1\mathfrak{f}_1|1000\rangle - \mathfrak{g}_2\mathfrak{h}_1\mathfrak{f}_2|0001\rangle \\
&- \mathfrak{g}_2\mathfrak{h}_2\mathfrak{f}_1|1011\rangle - \mathfrak{g}_2\mathfrak{h}_2\mathfrak{f}_2|0010\rangle.
\end{aligned}
$$

(11.15)

Until now, the controller, David, has been inactive in the protocol. After receiving all the classical information from all the remaining parties, he scrutinizes the overall scenario. Once he is satisfied that everything is in order, only then he performs his measurement on his single-qubit D and announces the result classically via 1-bit messages. If David observes that something went wrong, he remains inactive by doing nothing, in which case the protocol cannot be completed.

Otherwise, David executes a single-qubit measurement on the basis given by

$$
\begin{aligned}
|\zeta_1\rangle_D &= \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle)_D, \\
|\zeta_2\rangle_D &= \tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle)_D.
\end{aligned}
$$

(11.16)

Using the basis (11.16), the above reduced state $|\Gamma_3\rangle_{A_2B_2C_2D}$ can be written as (ignoring the constant term)

$$
\begin{aligned}
|\Gamma_3\rangle_{A_2B_2C_2D} \quad = \; &|\zeta_1\rangle_D \otimes \Big(\mathfrak{g}_1\mathfrak{h}_1\mathfrak{f}_1|110\rangle + \mathfrak{g}_1\mathfrak{h}_1\mathfrak{f}_2|010\rangle + \mathfrak{g}_1\mathfrak{h}_2\mathfrak{f}_1|111\rangle + \mathfrak{g}_1\mathfrak{h}_2\mathfrak{f}_2|011\rangle \\
&- \mathfrak{g}_2\mathfrak{h}_1\mathfrak{f}_1|100\rangle - \mathfrak{g}_2\mathfrak{h}_1\mathfrak{f}_2|000\rangle - \mathfrak{g}_2\mathfrak{h}_2\mathfrak{f}_1|101\rangle - \mathfrak{g}_2\mathfrak{h}_2\mathfrak{f}_2|001\rangle \Big) \\
&+ |\zeta_2\rangle_D \otimes \Big(-\mathfrak{g}_1\mathfrak{h}_1\mathfrak{f}_1|110\rangle + \mathfrak{g}_1\mathfrak{h}_1\mathfrak{f}_2|010\rangle + \mathfrak{g}_1\mathfrak{h}_2\mathfrak{f}_1|111\rangle - \mathfrak{g}_1\mathfrak{h}_2\mathfrak{f}_2|011\rangle \\
&- \mathfrak{g}_2\mathfrak{h}_1\mathfrak{f}_1|100\rangle + \mathfrak{g}_2\mathfrak{h}_1\mathfrak{f}_2|000\rangle + \mathfrak{g}_2\mathfrak{h}_2\mathfrak{f}_1|101\rangle - \mathfrak{g}_2\mathfrak{h}_2\mathfrak{f}_2|001\rangle \Big).
\end{aligned}
$$

(11.17)

After completing the measurement, David announces his outcome via 1-bit classical message. Suppose David's measurement result is $|\zeta_1\rangle_D$, then the state of remaining qubits is reduced to the state

$$
\begin{aligned}
|\Gamma_4\rangle_{A_2 B_2 C_2} = \ & \Big(\mathfrak{g}_1 \mathfrak{h}_1 \mathfrak{f}_1 |110\rangle + \mathfrak{g}_1 \mathfrak{h}_1 \mathfrak{f}_2 |010\rangle + \mathfrak{g}_1 \mathfrak{h}_2 \mathfrak{f}_1 |111\rangle + \mathfrak{g}_1 \mathfrak{h}_2 \mathfrak{f}_2 |011\rangle \\
& - \mathfrak{g}_2 \mathfrak{h}_1 \mathfrak{f}_1 |100\rangle - \mathfrak{g}_2 \mathfrak{h}_1 \mathfrak{f}_2 |000\rangle - \mathfrak{g}_2 \mathfrak{h}_2 \mathfrak{f}_1 |101\rangle - \mathfrak{g}_2 \mathfrak{h}_2 \mathfrak{f}_2 |001\rangle \Big) \\
= \ & (\mathfrak{f}_1 |1\rangle + \mathfrak{f}_2 |0\rangle)_{A_2} \otimes (\mathfrak{g}_1 |1\rangle - \mathfrak{g}_2 |0\rangle)_{B_2} \otimes (\mathfrak{h}_1 |0\rangle + \mathfrak{h}_2 |1\rangle)_{C_2}.
\end{aligned}
$$

(11.18)

Lastly, Alice, Bob and Charlie perform local unitary operations $(\vartheta_x)_{A_2}$, $(\vartheta_z \vartheta_x)_{B_2}$ and I_{C_2} on their respective qubits to reconstruct the intended state. Thereby cyclic controlled teleportation is successfully realized. The protocol is described schematically in [Figure 11.3]. There are 128 number of possible cases in the protocol. Here, we illustrate only one such case.

12 Multi-hop Teleportation Schemes

12.1 INTRODUCTION

In this chapter, the multi-hop teleportation process is described in which there are intermediate nodes between the sender and the receiver. Protocols with and without controller are presented. These protocols with intermediate nodes are needed in cases of long distance teleportation where the shared entangled resources become vulnerable to environment disturbances.

So far in the previous chapters some direct communication protocols between the sender and receiver with and without the help of the controller are discussed. Practical challenges in implementing direct communication protocols across large distances are expected since the quantum resource may be distorted due to the interaction with nature making it unusable for teleportation. For that, intermediate nodes are introduced and the teleportation protocols are executed across the nodes in series over relatively short distances. The chances of being affected by noise are thus minimized. This is the concept of hop-by-hop teleportation, which is actually a combination of a number of teleportation processes performed sequentially. In this chapter, we discuss three different such protocols. Multi-hop teleportation protocols have been treated in works like [20, 24, 25, 39, 48, 101, 121, 173, 186, 196, 200, 205, 206].

12.2 MULTI-HOP TELEPORTATION PROTOCOL OF ARBITRARY SINGLE-QUBIT STATES

In this section, we discuss a protocol for transferring an arbitrary single-qubit state from a sender Alice to a receiver Bob who are situated far apart and are not directly connected by any kind of entangled resource. We considered the same problem of state transfer in

where the two parties were directly sharing a quantum resource. Here the problem of communication is approached by introducing intermediate nodes between the sender and the receiver. The scheme presented in this section is developed by Wang et al. [165].

One-hop quantum teleportation

This is the same as ordinary teleportation. Suppose that Alice wants to transmit an arbitrary single-qubit quantum state $|\aleph\rangle_\mathfrak{a}$ described in Eq. (8.1) to a distant receiver Bob. Also, suppose that the two parties shares a two-qubit maximally entangled Bell state in the form of Eq. (8.2). We recall briefly in the following the teleportation protocol described in .

The composite state of the whole system is described in Eq. (8.3). Using the four Bell states $\{|\Upsilon_1\rangle, |\Upsilon_2\rangle, |\Upsilon_3\rangle, |\Upsilon_4\rangle\}$ given in Eq. (8.4), the above composite state can be written as

$$
\begin{aligned}
|\Gamma\rangle = \ & \frac{1}{2}[|\Upsilon_1\rangle_{\mathfrak{a}A} \otimes (\mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle)_B + |\Upsilon_2\rangle_{\mathfrak{a}A} \otimes (\mathfrak{g}_1|0\rangle - \mathfrak{g}_2|1\rangle)_B \\
& + |\Upsilon_3\rangle_{\mathfrak{a}A} \otimes (\mathfrak{g}_1|1\rangle + \mathfrak{g}_2|0\rangle)_B + |\Upsilon_4\rangle_{\mathfrak{a}A} \otimes (\mathfrak{g}_1|1\rangle - \mathfrak{g}_2|0\rangle)_B] \\
= \ & \sum_{i=1}^{4} |\Upsilon_i\rangle_{\mathfrak{a}A} \otimes |v_i\rangle_B \\
= \ & \sum_{i=1}^{4} |\Upsilon_i\rangle_{\mathfrak{a}A} \otimes (U_i^{-1}|v_1\rangle_B)
\end{aligned}
$$

(12.1)

where $|v_i\rangle$'s are the reduced state and U_i's are the recovery operators, all of which are given in .

Now Alice executes on her two qubits using the Bell bases and transmits the outcome through a classical channel to Bob. Accordingly, Bob acts by performing the corresponding unitary operation to recover the original quantum state.

Two-hop quantum teleportation

In this case, we assume that the sender Alice wants to send the quantum state given in Eq. (8.1) to Bob, but initially there is no shared entangled quantum resource between them. In

this situation, quantum communication is feasible in the multi-hop way, where entangled swapping is used to distribute the entangled qubits to the sender as well as receiver. For two-hop cases, there is an intermediate node, say X_1, which can share one Bell pair with Alice and another with Bob. When the intermediate node X_1 makes a measurement on the two qubits and transmits the outcome to Alice and Bob, the remaining qubits at the sites of Alice and Bob get entangled. In this way, a quantum channel is created between the sender and receiver. We assume that the sender Alice and the receiver Bob share the entangled state $|\Upsilon_1\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ with the intermediate node X_1 separately.

The receiver Bob is connected to Alice and X_1 through a classical communication channel.

The total state of the system can be written as

$$
\begin{aligned}
&|\aleph\rangle_\mathfrak{a} \otimes |\Upsilon_1\rangle_{AX_1^1} \otimes |\Upsilon_1\rangle_{X_1^2 B} \\
=~ &(\mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle)_\mathfrak{a} \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{AX_1^1} \otimes |\Upsilon_1\rangle_{X_1^2 B} \\
=~ &\sum_{i=1}^{4} |\Upsilon_i\rangle_{\mathfrak{a}A} \otimes (U_i^{-1}|v_1\rangle_{X_1^1}) \otimes |\Upsilon_1\rangle_{X_1^2 B} \\
=~ &\sum_{i=1}^{4} |\Upsilon_i\rangle_{\mathfrak{a}A} \otimes \sum_{j=1}^{4} |\Upsilon_j\rangle_{X_1^1 X_1^2} \otimes (U_j^{-1} U_i^{-1}|v_1\rangle_B)
\end{aligned}
$$

(12.2)

where U_i's are unitary operations given in [Table 8.1](#) and $|\Upsilon_i\rangle$, are the Bell states described in Section 8 in Eq. 8.4.

Now Alice and the intermediate node X_1 execute measurements on their respective two qubits using Bell bases and transmit the measurement result to Bob via classical channels. Depending on the measurement results, Bob performs a unitary operation $U_i U_j$ to recover the intended state. That is the end of the two-hop teleportation protocol.

As an illustration, suppose that the measurement results of Alice and the party X_1 are $|\Upsilon_4\rangle_{\mathfrak{a}A}$ and $|\Upsilon_2\rangle_{X_1^1 X_1^2}$, respectively. Then the reduced state becomes

$$
U_2^{-1} U_4^{-1}|v_1\rangle_B = \vartheta_z \vartheta_x \vartheta_z (\mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle)_B = (-\mathfrak{g}_1|1\rangle - \mathfrak{g}_2|0\rangle)_B.
$$

Finally, after receiving the classical information from Alice and the party X_1, Bob accordingly acts by performing a unitary operation which is $U_4 U_2 = \vartheta_z \vartheta_x \vartheta_z$ on his qubit to recover the original quantum state.

N-hop quantum teleportation

Now, the above two cases (one-hop and two-hop) can be generalized to N-hop quantum teleportation where we assume in between Alice (source node) and Bob (destination node), $(N-1)$ intermediate nodes are present. There is no direct quantum entanglement between Alice and Bob, whereas each of consecutive pairs of parties are entangled through a sharing of the Bell state $|\Upsilon_1\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$. We denote the intermediate nodes by $X_1, X_2, X_3, \ldots, X_{N-1}$.

Further, Alice and all the intermediate nodes $X_1, X_2, X_3, \ldots, X_{N-1}$ are connected to Bob by classical communication channels.
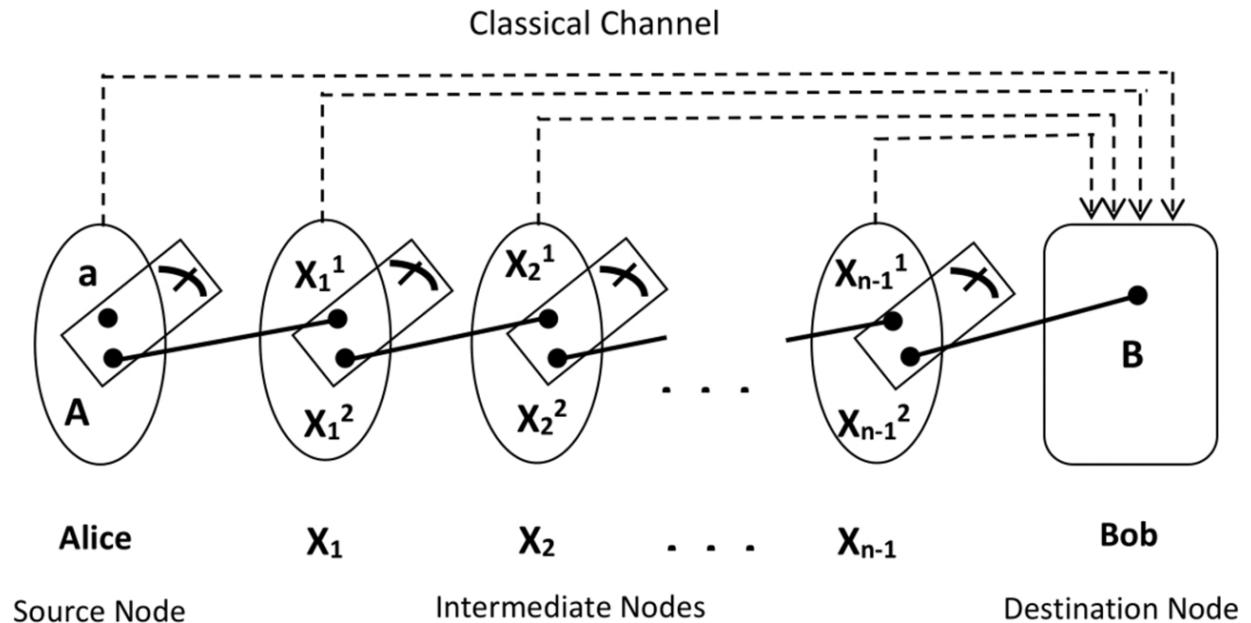

Figure 12.1 Multi-hop teleportation protocol for transferring single-qubit state.

The composite state of the whole system can be written as

$$|\aleph\rangle_\mathfrak{a} \otimes |\Upsilon_1\rangle_{AX_1^1} \otimes |\Upsilon_1\rangle_{X_1^2 X_2^1} \otimes \ldots \otimes |\Upsilon_1\rangle_{X_{N-1}^2 B}$$

$$= \ (\mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle)_\mathfrak{a} \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{AX_1^1} \otimes |\Upsilon_1\rangle_{X_1^2 X_2^1} \otimes \ldots \otimes |\Upsilon_1\rangle_{X_{N-1}^2 B}$$

$$= \ \sum_{i=1}^{4} |\Upsilon_i\rangle_{\mathfrak{a}A} \otimes (U_i^{-1}|v_1\rangle_{X_1^1}) \otimes |\Upsilon_1\rangle_{X_1^2 X_2^1} \otimes \ldots \otimes |\Upsilon_1\rangle_{X_{N-1}^2 B}$$

$$= \ \sum_{i=1}^{4} |\Upsilon_i\rangle_{\mathfrak{a}A} \otimes \sum_{j=1}^{4} |\Upsilon_j\rangle_{X_1^1 X_1^2} \otimes (U_j^{-1} U_i^{-1}|v_1\rangle_{X_2^1}) \otimes \ldots \otimes |\Upsilon_1\rangle_{X_{N-1}^2 B}$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$= \ \sum_{i=1}^{4} |\Upsilon_i\rangle_{\mathfrak{a}A} \otimes \sum_{j=1}^{4} |\Upsilon_j\rangle_{X_1^1 X_1^2} \otimes \cdots \otimes \sum_{k=1}^{4} |\Upsilon_k\rangle_{X_{N-1}^1 X_{N-1}^2} \otimes (U_k^{-1} \cdots U_j^{-1} U_i^{-1}|v_1\rangle_B).$$

<div align="right">(12.3)</div>

Now Alice and all intermediate nodes $X_1, X_2, \ldots, X_{N-1}$ make measurement on their respective two qubits using Bell bases and transmit the measurement result to Bob independently via the classical channels. Depending on the measurement results, Bob performs a unitary operation $U_i U_j \cdots U_k$ to recover the original quantum state by which the teleportation is successfully achieved. That is the end of the N-hop teleportation protocol.

12.3 MULTI-HOP TELEPORTATION PROTOCOL OF ARBITRARY TWO-QUBIT STATES

Let us assume that the sender Alice wants to transmit an unknown general two-qubit quantum state to Bob, who is situated far away from Alice. This problem is already discussed in Section 9.1, where the task of state transfer is accomplished through a teleportation protocol in which the sender and the receiver share an entangled resource. In our consideration it is a one-hop case which we briefly describe. The multi-hop protocol for the above problem is developed by Zou et al. [206]

One-hop quantum teleportation

We refer to the protocol presented in Section 9.1. Here the two parties share the entangled state

$$|G_1\rangle_{A_1 A_2 B_1 B_2} = \frac{1}{2}(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle)$$

For simplicity the total system (Eq. 9.7) can be written as

$$|\Gamma\rangle = \frac{1}{4} \sum_{j=1}^{16} |G_j\rangle_{\mathfrak{a}_1 \mathfrak{a}_2 A_1 A_2} \otimes |v_j\rangle_{B_1 B_2}$$

$$= \sum_{j=1}^{16} |G_j\rangle_{\mathfrak{a}_1 \mathfrak{a}_2 A_1 A_2} \otimes (U_j^{-1} |v_1\rangle_{B_1 B_2}),$$

(12.4)

where $|v_j\rangle$'s are the reduced state and U_j's are the recovery operators, all of which are given in Table 9.1.

Now Alice executes measurement on her four qubits using the basis given in Eqs. (9.2)--(9.5) and transmits her measurement results through a classical channel to Bob. Finally, Bob applies the corresponding unitary operation to recover the original quantum state.

Two-hop quantum teleportation

In this situation there is no direct quantum entanglement between the source party and destination party, rather an intermediate party, say X_1, is introduced who shares entanglement with the two parties. Let us assume an entangled state $|G_1\rangle = \frac{1}{2}(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle)$, as described in Eq. (9.2), is shared between the intermediate node X_1 and the sender Alice and also between X_1 and the receiver Bob. Also both Alice and X_1 are connected to Bob by classical communication channels.

Therefore, the total quantum system can be written as

$$|\aleph\rangle_{\mathfrak{a}_1\mathfrak{a}_2} \otimes |G_1\rangle_{A_1A_2X_1^1X_1^2} \otimes |G_1\rangle_{X_1^3X_1^4B_1B_2}$$

$$= (\mathfrak{g}_1|00\rangle + \mathfrak{g}_2|01\rangle + \mathfrak{g}_3|10\rangle + \mathfrak{g}_4|11\rangle)_{\mathfrak{a}_1\mathfrak{a}_2}$$

$$\otimes \frac{1}{2}(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle)_{A_1A_2X_1^1X_1^2} \otimes |G_1\rangle_{X_1^3X_1^4B_1B_2}$$

$$= \frac{1}{4}\sum_{j=1}^{16} |G_j\rangle_{\mathfrak{a}_1\mathfrak{a}_2A_1A_2} \otimes |v_j\rangle_{X_1^1X_1^2} \otimes |G_1\rangle_{X_1^3X_1^4B_1B_2}$$

$$= \sum_{j=1}^{16} |G_j\rangle_{\mathfrak{a}_1\mathfrak{a}_2A_1A_2} \otimes (U_j^{-1}|v_1\rangle_{X_1^1X_1^2}) \otimes |G_1\rangle_{X_1^3X_1^4B_1B_2}$$

$$= \sum_{j=1}^{16} |G_j\rangle_{\mathfrak{a}_1\mathfrak{a}_2A_1A_2} \otimes \sum_{k=1}^{16} |G_k\rangle_{X_1^1X_1^2X_1^3X_1^4} \otimes (U_k^{-1}U_j^{-1}|v_1\rangle_{B_1B_2})$$

$$(12.5)$$

where $|G_j\rangle$ s are described in Eqs. (9.2)–(9.5) and U_j s are given in Table 9.1.

Now both parties, sender Alice and intermediate node X_1, make measurements on their respective qubits on the basis given in Eqs. (9.2)--(9.5) and send the measurement results to Bob with the help of a 4 bit classical channel. After receiving the measurement result, Bob performs a unitary operation to recover the original state. If Alice's measurement result is $|G_j\rangle$ and the measurement result of X_1 is $|G_k\rangle$, then the unitary operation to be applied by Bob is U_jU_k. That is the end of the two-hop teleportation protocol.


▶ Long Description for Figure 12.2
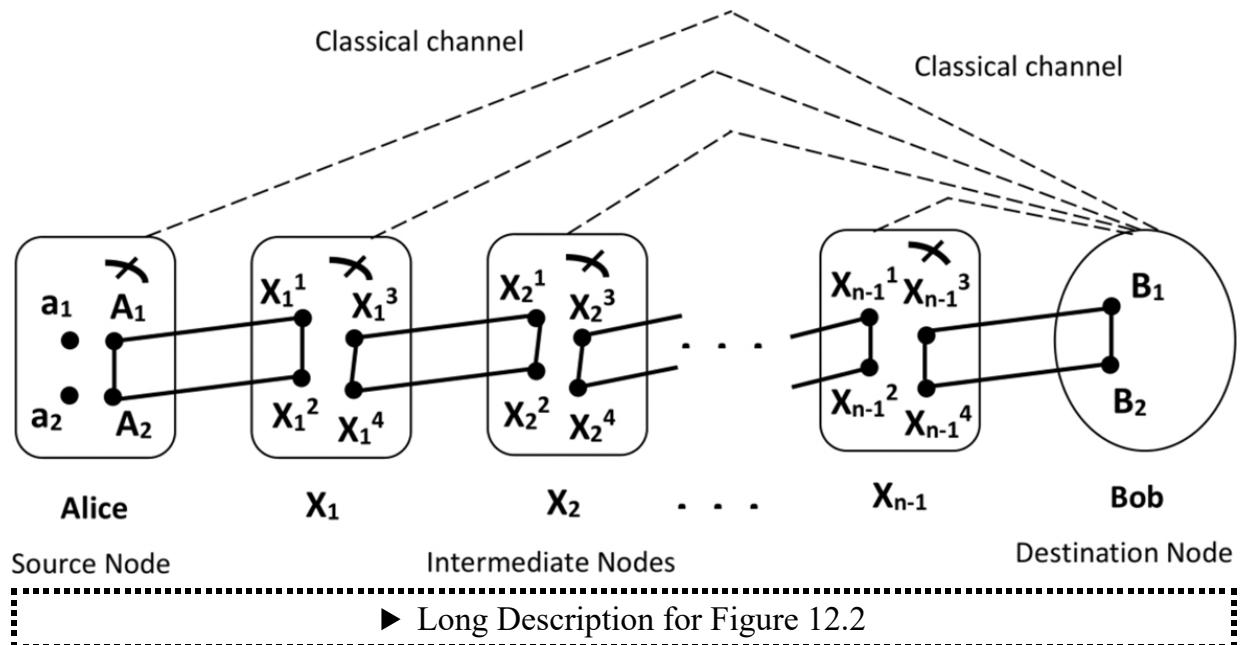
Figure 12.2 Multi-hop teleportation protocol for 2-qubit state.

As an illustration, suppose Bob receives the measurement results $|G_7\rangle_{\mathfrak{a}_1\mathfrak{a}_2 A_1 A_2}$ and $|G_3\rangle_{X_1^1 X_1^2 X_1^3 X_1^4}$ from Alice and the intermediate party X_1, respectively. The reduced state of the remaining qubits becomes

$$(-\mathfrak{g}_1|01\rangle - \mathfrak{g}_2|00\rangle - \mathfrak{g}_3|11\rangle - \mathfrak{g}_4|10\rangle)_{B_1 B_2}.$$

Then Bob applies a unitary operation $U_7 U_3 = (\vartheta_z \vartheta_x \vartheta_z)_{B_2}$ to obtain the intended state.

N-hop quantum teleportation

Now, the above two cases (one-hop and two-hop) can be generalized to N-hop quantum teleportation where we assume in between Alice (source node) and Bob (destination node), $(N-1)$ intermediate nodes are present. There is no direct quantum entanglement between Alice and Bob, whereas each of consecutive pairs of nodes are entangled. We assume that the intermediate nodes are $X_1, X_2, X_3, \ldots, X_{N-1}$.

The sender Alice and all the intermediate nodes $X_1, X_2, X_3, \ldots, X_{N-1}$ are individually connected to Bob through classical communication channels.

The composite state of the whole system can be written as

$$|\aleph\rangle_{\mathfrak{a}_1\mathfrak{a}_2} \otimes |G_1\rangle_{A_1 A_2 X_1^1 X_1^2} \otimes |G_1\rangle_{X_1^3 X_1^4 X_2^1 X_2^2} \otimes \ldots \otimes |G_1\rangle_{X_{N-1}^3 X_{N-1}^4 B_1 B_2}$$

$$= (\mathfrak{g}_1|00\rangle + \mathfrak{g}_2|01\rangle + \mathfrak{g}_3|10\rangle + \mathfrak{g}_4|11\rangle)_{\mathfrak{a}_1\mathfrak{a}_2} \otimes \frac{1}{2}(|0000\rangle + |0101\rangle + |1010\rangle$$
$$+ |1111\rangle)_{A_1 A_2 X_1^1 X_1^2} \otimes |G_1\rangle_{X_1^3 X_1^4 X_2^1 X_2^2} \otimes \ldots \otimes |G_1\rangle_{X_{N-1}^3 X_{N-1}^4 B_1 B_2}$$

$$= \sum_{j=1}^{16} |G_j\rangle_{\mathfrak{a}_1\mathfrak{a}_2 A_1 A_2} \otimes (U_j^{-1}|v_1\rangle_{X_1^1 X_1^2}) \otimes |G_1\rangle_{X_1^3 X_1^4 X_2^1 X_2^2} \otimes \ldots \otimes |G_1\rangle_{X_{N-1}^3 X_{N-1}^4 B_1 B_2}$$

$$= \sum_{j=1}^{16} |G_j\rangle_{\mathfrak{a}_1\mathfrak{a}_2 A_1 A_2} \otimes \sum_{k=1}^{16} |G_k\rangle_{X_1^1 X_1^2 X_1^3 X_1^4} \otimes (U_k^{-1} U_j^{-1}|v_1\rangle_{X_2^1 X_2^2}) \otimes \ldots \otimes |G_1\rangle_{X_{N-1}^3 X_{N-1}^4 B_1 B_2}$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$= \sum_{j=1}^{16} |G_j\rangle_{\mathfrak{a}_1\mathfrak{a}_2 A_1 A_2} \otimes \sum_{k=1}^{16} |G_k\rangle_{X_1^1 X_1^2 X_1^3 X_1^4} \otimes \cdots \otimes \sum_{l=1}^{16} |G_l\rangle_{X_{N-1}^1 X_{N-1}^2 X_{N-1}^3 X_{N-1}^4}$$
$$\otimes (U_l^{-1} \cdots U_k^{-1} U_j^{-1}|v_1\rangle_{B_1 B_2}).$$

$$(12.6)$$

Now, all the intermediate parties $X_1, X_2, \ldots, X_{N-1}$ and the sender Alice perform measurements on their respective four qubits using the basis given in Eq. (9.2)–(9.5) and transmit the measurement results to Bob independently through classical channels. After receiving the results of the measurement, Bob finally executes a unitary operation $U_j U_k \cdots U_l$ to recover the intended quantum state and that is the end of the N-hop teleportation protocol.

12.4 MULTI-HOP CONTROLLED TELEPORTATION PROTOCOL OF ARBITRARY SINGLE-QUBIT STATE

In this section the problem is that of transfer of a single qubit state to a distant party under the supervision of a controller. It is performed by introducing intermediate nodes in order to avoid the effect of long distances on entangled connections. The protocol is designed by Peng et al. [121].

One-hop Quantum Controlled Teleportation

There are two nodes; Alice is the source node whereas Bob is the destination node, and Candy is the controller. The sender (Alice) intends to transmit a single-qubit state to the receiver (Bob) which is given by

$$|\aleph\rangle_\mathfrak{a} = (\mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle).$$

(12.7)

Here, the parameters $\mathfrak{g}_1$ and $\mathfrak{g}_2$ meet the normalization condition, that is,

$$|\mathfrak{g}_1|^2 + |\mathfrak{g}_2|^2 = 1.$$

There is a 3-qubit quantum resource connecting Alice, Bob and Candy given by

$$|E\rangle_{ABC} = \frac{1}{\sqrt{2}}(|000\rangle + cos\kappa|110\rangle + sin\kappa|111\rangle),$$

(12.8)

where qubits A,B,C are held by Alice, Bob and Candy, respectively. The corresponding circuit diagram for its generation is given in Figure 12.3, where $R_y(\kappa)$ is described by the matrix

$$R_y(\kappa) = \begin{pmatrix} \cos\frac{\kappa}{2} & -\sin\frac{\kappa}{2} \\ \sin\frac{\kappa}{2} & \cos\frac{\kappa}{2} \end{pmatrix}.$$
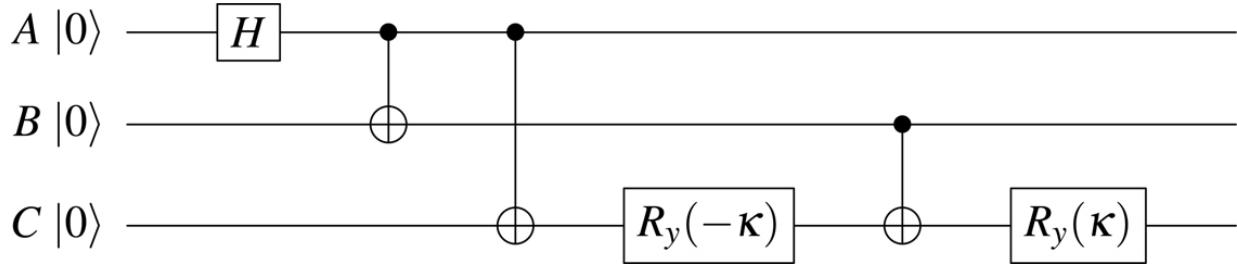
Figure 12.3 Circuit diagram for the generation of quantum resource given in Eq. (12.8). ⏎

The composite system is given by

$$
\begin{aligned}
|\Gamma\rangle &= |\aleph\rangle_{\mathfrak{a}} \otimes |E\rangle_{ABC} \\
&= (\mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle)_{\mathfrak{a}} \otimes \frac{1}{\sqrt{2}}(|000\rangle + \cos\kappa|110\rangle + \sin\kappa|111\rangle)_{ABC}.
\end{aligned}
$$

$$(12.9)$$

To complete the process, Alice first performs measurements on her qubits $\mathfrak{a}$ and A on the Bell basis (vide Eq. (8.4)). After the measurement, she sends the measurement outcomes to Bob and Candy through classical channels. After receiving the classical information, Candy makes a single-qubit rotation on his qubit C and then executes a projective measurement on the same qubit C and transmits the results to the receiver Bob. Depending on the classical information from Alice and Candy, an appropriate unitary operation is implemented to recover the intended quantum state. These operations are summarized in Table 12.1. That is the end of the protocol.

Table 12.1

Possible local unitary operation performed by Bob in one-hop controlled teleportation according to the measurement outcomes of Alice and Candy ⏎

Alice's results	Candy's results	State at Bob's site	Bob's unitary operation
$|\Upsilon_1\rangle_{\mathfrak{a}A}$	$|0\rangle_C$	$(\mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle)_B$	$(I)_B$

Alice's results	Candy's results	State at Bob's site	Bob's unitary operation
$\lvert\Upsilon_1\rangle_{\mathfrak{a}A}$	$\lvert 1\rangle_C$	$(\mathfrak{g}_1\lvert 0\rangle - \mathfrak{g}_2\lvert 1\rangle)_B$	$(\vartheta_z)_B$
$\lvert\Upsilon_2\rangle_{\mathfrak{a}A}$	$\lvert 0\rangle_C$	$(\mathfrak{g}_1\lvert 0\rangle - \mathfrak{g}_2\lvert 1\rangle)_B$	$(\vartheta_z)_B$
$\lvert\Upsilon_2\rangle_{\mathfrak{a}A}$	$\lvert 1\rangle_C$	$(\mathfrak{g}_1\lvert 0\rangle + \mathfrak{g}_2\lvert 1\rangle)_B$	$(I)_B$
$\lvert\Upsilon_3\rangle_{\mathfrak{a}A}$	$\lvert 0\rangle_C$	$(\mathfrak{g}_1\lvert 1\rangle + \mathfrak{g}_2\lvert 0\rangle)_B$	$(\vartheta_x)_B$
$\lvert\Upsilon_3\rangle_{\mathfrak{a}A}$	$\lvert 1\rangle_C$	$(\mathfrak{g}_1\lvert 1\rangle - \mathfrak{g}_2\lvert 0\rangle)_B$	$(\vartheta_z\vartheta_x)_B$
$\lvert\Upsilon_4\rangle_{\mathfrak{a}A}$	$\lvert 0\rangle_C$	$(\mathfrak{g}_1\lvert 1\rangle - \mathfrak{g}_2\lvert 0\rangle)_B$	$(\vartheta_z\vartheta_x)_B$
$\lvert\Upsilon_4\rangle_{\mathfrak{a}A}$	$\lvert 1\rangle_C$	$(\mathfrak{g}_1\lvert 1\rangle + \mathfrak{g}_2\lvert 0\rangle)_B$	$(\vartheta_x)_B$

We illustrate the whole process in the following way.

The entire quantum system (12.9) can be written using the Bell basis $\{\lvert\Upsilon_1\rangle, \lvert\Upsilon_2\rangle, \lvert\Upsilon_3\rangle, \lvert\Upsilon_4\rangle\}$ as

$$
\begin{aligned}
\lvert\Gamma\rangle \quad =\ & \frac{1}{2}\Big[\lvert\Upsilon_1\rangle_{\mathfrak{a}A} \otimes (\mathfrak{g}_1\lvert 00\rangle + \mathfrak{g}_2 cos\kappa\lvert 10\rangle + \mathfrak{g}_2 sin\kappa\lvert 11\rangle)_{BC} \\
& + \lvert\Upsilon_2\rangle_{\mathfrak{a}A} \otimes (\mathfrak{g}_1\lvert 00\rangle - \mathfrak{g}_2 cos\kappa\lvert 10\rangle - \mathfrak{g}_2 sin\kappa\lvert 11\rangle)_{BC} \\
& + \lvert\Upsilon_3\rangle_{\mathfrak{a}A} \otimes (\mathfrak{g}_1 cos\kappa\lvert 10\rangle + \mathfrak{g}_1 sin\kappa\lvert 11\rangle + \mathfrak{g}_2\lvert 00\rangle)_{BC} \\
& + \lvert\Upsilon_4\rangle_{\mathfrak{a}A} \otimes (\mathfrak{g}_1 cos\kappa\lvert 10\rangle + \mathfrak{g}_1 sin\kappa\lvert 11\rangle - \mathfrak{g}_2\lvert 00\rangle)_{BC}\Big].
\end{aligned}
$$

(12.10)

After measuring qubits $(\mathfrak{a}, A)$, Alice gets the outcomes $\lvert\Upsilon_1\rangle_{\mathfrak{a}A}$ or $\lvert\Upsilon_2\rangle_{\mathfrak{a}A}$ with probability

$$
\frac{\lvert\mathfrak{g}_1\rvert^2 + \lvert\mathfrak{g}_2\rvert^2 cos^2\kappa + \lvert\mathfrak{g}_2\rvert^2 sin^2\kappa}{2[(\lvert\mathfrak{g}_1\rvert^2 + \lvert\mathfrak{g}_2\rvert^2 cos^2\kappa + \lvert\mathfrak{g}_2\rvert^2 sin^2\kappa) + (\lvert\mathfrak{g}_1\rvert^2 cos^2\kappa + \lvert\mathfrak{g}_2\rvert^2 + \lvert\mathfrak{g}_1\rvert^2 sin^2\kappa)]} = \frac{1}{4}
$$

(12.11)

and $\lvert\Upsilon_3\rangle_{\mathfrak{a}A}$ or $\lvert\Upsilon_4\rangle_{\mathfrak{a}A}$ with probability

$$
\frac{\lvert\mathfrak{g}_1\rvert^2 cos^2\kappa + \lvert\mathfrak{g}_2\rvert^2 + \lvert\mathfrak{g}_1\rvert^2 sin^2\kappa}{2[(\lvert\mathfrak{g}_1\rvert^2 + \lvert\mathfrak{g}_2\rvert^2 cos^2\kappa + \lvert\mathfrak{g}_2\rvert^2 sin^2\kappa) + (\lvert\mathfrak{g}_1\rvert^2 cos^2\kappa + \lvert\mathfrak{g}_2\rvert^2 + \lvert\mathfrak{g}_1\rvert^2 sin^2\kappa)]} = \frac{1}{4}.
$$

(12.12)

Suppose that the measurement outcome of Alice's measurement is $\lvert\Upsilon_2\rangle_{\mathfrak{a}A}$, then the state of the remaining qubits becomes

$$(\mathfrak{g}_1|00\rangle - \mathfrak{g}_2 cos\kappa|10\rangle - \mathfrak{g}_2 sin\kappa|11\rangle)_{BC}.$$

$$(12.13)$$

Upon accepting Alice's classical message, the controller, Candy, applies the rotation operator $R_y(-\kappa)$ on his particle C.

Now,

$$R_y(-\kappa) = \begin{pmatrix} cos\frac{\kappa}{2} & sin\frac{\kappa}{2} \\ -sin\frac{\kappa}{2} & cos\frac{\kappa}{2} \end{pmatrix},$$

and hence,

$$\begin{aligned} R_y(-\kappa)|0\rangle &= \begin{pmatrix} cos\frac{\kappa}{2} & sin\frac{\kappa}{2} \\ -sin\frac{\kappa}{2} & cos\frac{\kappa}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} cos\frac{\kappa}{2} \\ -sin\frac{\kappa}{2} \end{pmatrix} \\ &= cos\frac{\kappa}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - sin\frac{\kappa}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= cos\frac{\kappa}{2}|0\rangle - sin\frac{\kappa}{2}|1\rangle, \end{aligned}$$

$$(12.14)$$

$$\begin{aligned} R_y(-\kappa)|1\rangle &= \begin{pmatrix} cos\frac{\kappa}{2} & sin\frac{\kappa}{2} \\ -sin\frac{\kappa}{2} & cos\frac{\kappa}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} sin\frac{\kappa}{2} \\ cos\frac{\kappa}{2} \end{pmatrix} \\ &= sin\frac{\kappa}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + cos\frac{\kappa}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= sin\frac{\kappa}{2}|0\rangle + cos\frac{\kappa}{2}|1\rangle, \end{aligned}$$

$$(12.15)$$

Using the Eqs. (12.14)-(12.15), the state (12.13) can be expressed as

$$R_y(-\kappa)\big(\mathfrak{g}_1|00\rangle - \mathfrak{g}_2 cos\kappa|10\rangle - \mathfrak{g}_2 sin\kappa|11\rangle\big)_{BC}$$

$$= R_y(-\kappa)\big(\mathfrak{g}_1|00\rangle - \mathfrak{g}_2(2cos^2\frac{\kappa}{2} - 1)|10\rangle - 2\mathfrak{g}_2 sin\frac{\kappa}{2}cos\frac{\kappa}{2}|11\rangle\big)_{BC}$$

$$= \big(\mathfrak{g}_1 cos\frac{\kappa}{2}|00\rangle - \mathfrak{g}_1 sin\frac{\kappa}{2}|01\rangle - \mathfrak{g}_2 cos\frac{\kappa}{2}|10\rangle - \mathfrak{g}_2 sin\frac{\kappa}{2}|11\rangle\big)_{BC}$$

$$= cos\frac{\kappa}{2}(\mathfrak{g}_1|0\rangle - \mathfrak{g}_2|1\rangle)_B|0\rangle_C - sin\frac{\kappa}{2}(\mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle)_B|1\rangle_C.$$

(12.16)

Candy then performs measurement on his particle C on the computational basis $\{|0\rangle, |1\rangle\}$ and communicates the outcome to Bob through a classical channel.

If the outcome is $|0\rangle_C$, which occurs with probability $cos^2(\kappa/2)$, the state of particle becomes $(\mathfrak{g}_1|0\rangle - \mathfrak{g}_2|1\rangle)_B$. The joint probability for this outcome (including Alice's result) is $\frac{1}{4}cos^2(\kappa/2)$. Based on the measurement results from Alice and Candy, Bob applies a unitary operation ϑ_z to his particle to recover the intended state $|\aleph\rangle$.

If the result is $|1\rangle_C$, which occurs with probability $sin^2(\kappa/2)$, the state of the particle becomes $(\mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle)_B$. The joint probability for this result (including Alice's result) is $\frac{1}{4}sin^2(\kappa/2)$. Based on the measurement results from Alice and Candy, Bob applies identity operation (I) to his particle to recover the intended state $|\aleph\rangle$.

Since Bob can always recover the target state using a suitable unitary operation in all possible scenarios, this controlled teleportation scheme is perfect. The total success probability is

$$4 \times \frac{1}{4} \times \left[cos^2(\frac{\kappa}{2}) + sin^2(\frac{\kappa}{2})\right] = 1.$$

The one-hop teleportation is thereby achieved in the case where the measurement of Alice yields $|\Upsilon_2\rangle_{\mathfrak{a}A}$. The other three cases are similar to the above.

Two-hop controlled teleportation

In this scenario, Alice serves as the source (sender) node, Bob as the destination node, with X_1 functioning as the intermediate node. Candy and David act as controllers at the intermediate and destination nodes, respectively. Alice intends to transmit the quantum state $|\aleph\rangle_{\mathfrak{a}}$ as defined in Eq. (12.7) to Bob. However, there is no direct quantum channel between the source (Alice) and the destination (Bob). Instead, two quantum resources are

available: one shared between Alice and the intermediate node X_1 and another between X_1 and Bob, which are as in the following.

$$|E\rangle_{AX_1^1C} = \frac{1}{\sqrt{2}}(|000\rangle + cos\kappa|110\rangle + sin\kappa|111\rangle),$$

$$|E\rangle_{X_1^2BD} = \frac{1}{\sqrt{2}}(|000\rangle + cos\kappa|110\rangle + sin\kappa|111\rangle).$$

Based on the outcome of the one-hop controlled teleportation, the qubit X_1^1 at the intermediate node X_1 can be reduced to one of the four possible states: $(\mathfrak{g}_1|0\rangle \pm \mathfrak{g}_2|1\rangle)_{X_1^1}$ or $(\mathfrak{g}_1|1\rangle \pm \mathfrak{g}_2|0\rangle)_{X_1^1}$. Suppose the state $(\mathfrak{g}_1|1\rangle - \mathfrak{g}_2|0\rangle)_{X_1^1}$ is obtained at node X_1 after the first controlled teleportation, then the resulting system state can be expressed as

$$|\Gamma_1\rangle = (\mathfrak{g}_1|1\rangle - \mathfrak{g}_2|0\rangle)_{X_1^1} \otimes \frac{1}{\sqrt{2}}(|000\rangle + cos\kappa|110\rangle + sin\kappa|111\rangle)_{X_1^2BD}.$$

(12.17)

Using the Bell basis, the above system state can be rewritten as

$$
\begin{aligned}
|\Gamma_1\rangle = \ & (\mathfrak{g}_1|1\rangle - \mathfrak{g}_2|0\rangle)_{X_1^1} \otimes \frac{1}{\sqrt{2}}(|000\rangle + cos\kappa|110\rangle + sin\kappa|111\rangle)_{X_1^2BD} \\
= \ & \frac{1}{2}\big[|\Upsilon_1\rangle_{X_1^1X_1^2} \otimes (\mathfrak{g}_1cos\kappa|10\rangle + \mathfrak{g}_1sin\kappa|11\rangle - \mathfrak{g}_2|00\rangle)_{BD} \\
& - |\Upsilon_2\rangle_{X_1^1X_1^2} \otimes (\mathfrak{g}_1cos\kappa|10\rangle + \mathfrak{g}_1sin\kappa|11\rangle + \mathfrak{g}_2|00\rangle)_{BD} \\
& + |\Upsilon_3\rangle_{X_1^1X_1^2} \otimes (\mathfrak{g}_1|00\rangle - \mathfrak{g}_2cos\kappa|10\rangle - \mathfrak{g}_2sin\kappa|11\rangle)_{BD} \\
& - |\Upsilon_4\rangle_{X_1^1X_1^2} \otimes (\mathfrak{g}_1|00\rangle + \mathfrak{g}_2cos\kappa|10\rangle + \mathfrak{g}_2sin\kappa|11\rangle)_{BD} \big].
\end{aligned}
$$

(12.18)

Following a similar approach as in the one-hop controlled teleportation, one of the four possible states- $(\mathfrak{g}_1|0\rangle \pm \mathfrak{g}_2|1\rangle)_B$ or $(\mathfrak{g}_1|1\rangle \pm \mathfrak{g}_2|0\rangle)_B$ -is obtained at the destination node Bob, with the assistance of the controller David.

Bob can always recover the target state $|\aleph\rangle$ by applying a suitable unitary operation on the particle B. The other three possible cases of the state relating to the qubit X_1^1 is similarly treated. In this way Bob obtains the intended state with certainty.

Multi-hop controlled teleportation

By extending the idea of 2-hop controlled teleportation, it is possible to induct $(N-1)$ intermediate nodes $X_1, X_2, \ldots, X_{N-1}$ between Alice and Bob and also controllers corresponding to each node where a quantum resource described in Eq. (12.8) is shared by the consecutive nodes including Alice and the controller. The steps in the 2-hop case can be repeated at each of the intermediate nodes with the classical information obtained from Alice and the intermediate nodes, the protocol can be completed by Bob through an application of appropriate unitary operation.

13 Probabilistic Teleportation Protocols

13.1 INTRODUCTION

The protocols presented in this chapter are probabilistic teleportation protocols in which there are cases of failures with certain probabilities. Probabilistic teleportation schemes of various kinds have been discussed in works like [1, 41, 45, 76, 82, 85, 98, 118, 135, 168, 169, 180]. Generally the quantum resources used in these protocols are not maximally entangled. This is one explanation for the probabilistic nature of these protocols. The justification of studying these protocols is that the generation and preservation of resources for these protocols are less difficult compared to that of those where maximally entangled states are used. Also, there is a class of resumable protocols in which the probabilistic teleportation process can be repeated in the case where the attempt fails in the first place [21, 52, 104, 105].

13.2 PROBABILISTIC TELEPORTATION PROTOCOL OF ARBITRARY SINGLE-QUBIT STATE

Assume that Alice possesses an unknown single-qubit quantum state given by

$$|\aleph\rangle_{\mathfrak{a}} = (\mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle)$$

$$(13.1)$$

with normalization condition $|\mathfrak{g}_1|^2 + |\mathfrak{g}_2|^2 = 1$ that she wants to transmit to the distant receiver Bob.

There is a classical communication channel between Alice and Bob.

A pure entangled quantum state is shared between the parties which acts as a quantum channel given by

$$|E\rangle_{AB} = \frac{1}{\sqrt{1+|m|^2}}(|00\rangle + m|11\rangle),$$

(13.2)

where m is a known complex number. The qubits 'A' and 'B' are with Alice and Bob, respectively.

The circuit diagram for generation of (13.2) is shown in Figure 13.1, where U is given by

$$U = \begin{pmatrix} \frac{1}{\sqrt{1+|m|^2}} & -\frac{m}{\sqrt{1+|m|^2}} \\ \frac{m}{\sqrt{1+|m|^2}} & \frac{1}{\sqrt{1+|m|^2}} \end{pmatrix}.$$
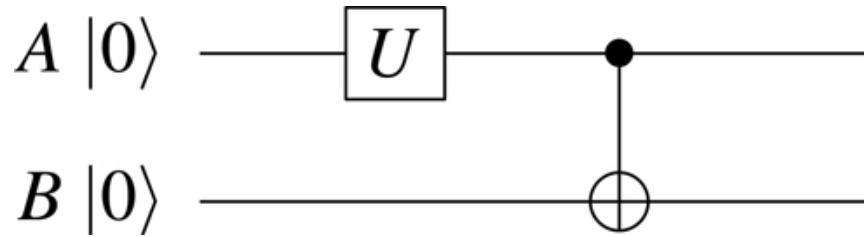
Figure 13.1   Circuit diagram for the generation of non-maximally entangled state given in Eq. (13.2).

The aforesaid task of state transfer using the above resource is done probabilistically with certain success probability by the following teleportation process designed by Agrawal et al. [1].

The total system can be written as

$$|\Gamma\rangle = |\aleph\rangle_a \otimes |E\rangle_{AB}.$$

(13.3)

Alice first makes a measurement on her qubits (a, A). If Alice performs measurement in the Bell basis given in Eq. (8.4), then the teleportation process cannot be completed

with unit fidelity and unit probability. However, if the measurement is performed in a non-maximally entangled Bell basis then it is possible for Alice to transfer the state $|\aleph\rangle_{\mathfrak{a}}$ with unit fidelity, though not with unit probability.

For this Alice uses a set of non-maximally entangled orthogonal Bell-states as basis states which are given by

$$|\varkappa_1\rangle_{\mathfrak{a}A} = \frac{1}{\sqrt{1+|\mathfrak{p}|^2}}(|00\rangle + \mathfrak{p}|11\rangle),$$
$$|\varkappa_2\rangle_{\mathfrak{a}A} = \frac{1}{\sqrt{1+|\mathfrak{p}|^2}}(\mathfrak{p}^*|00\rangle - |11\rangle),$$
$$|\varkappa_3\rangle_{\mathfrak{a}A} = \frac{1}{\sqrt{1+|\mathfrak{q}|^2}}(|01\rangle + \mathfrak{q}|10\rangle),$$
$$|\varkappa_4\rangle_{\mathfrak{a}A} = \frac{1}{\sqrt{1+|\mathfrak{q}|^2}}(\mathfrak{q}^*|01\rangle - |10\rangle),$$

(13.4)

where $\mathfrak{p}$ and $\mathfrak{q}$ are complex numbers. When $\mathfrak{p} = \mathfrak{q} = 0$, the above basis reduces to the computational basis which is not entangled, and when $\mathfrak{p} = \mathfrak{q} = 1$, it reduces to the maximally entangled Bell basis.

We have the following relations.

$$|00\rangle = \frac{1}{\sqrt{1+|\mathfrak{p}|^2}}(|\varkappa_1\rangle + \mathfrak{p}|\varkappa_2\rangle),$$
$$|11\rangle = \frac{1}{\sqrt{1+|\mathfrak{p}|^2}}(\mathfrak{p}^*|\varkappa_1\rangle - |\varkappa_2\rangle),$$
$$|01\rangle = \frac{1}{\sqrt{1+|\mathfrak{q}|^2}}(|\varkappa_3\rangle + \mathfrak{q}|\varkappa_4\rangle),$$
$$|10\rangle = \frac{1}{\sqrt{1+|\mathfrak{q}|^2}}(\mathfrak{q}^*|\varkappa_3\rangle - |\varkappa_4\rangle).$$

(13.5)

Using the above relations the total system can be rewritten in the following form:

$$
\begin{aligned}
|\Gamma\rangle \quad &= |\aleph\rangle_{\mathfrak{a}} \otimes |E\rangle_{AB} \\
&= M(\mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle)_{\mathfrak{a}} \otimes (|00\rangle + m|11\rangle)_{AB} \\
&= M(\mathfrak{g}_1|00\rangle_{\mathfrak{a}A}|0\rangle_B + \mathfrak{g}_1 m|01\rangle_{\mathfrak{a}A}|1\rangle_B + \mathfrak{g}_2|10\rangle_{\mathfrak{a}A}|0\rangle_B + \mathfrak{g}_2 m|11\rangle_{\mathfrak{a}A}|1\rangle_B) \\
&= M\left[|\varkappa_1\rangle_{\mathfrak{a}A}\left(P\mathfrak{g}_1|0\rangle + Pm\mathfrak{g}_2\mathfrak{p}^*|1\rangle \right)_B + |\varkappa_2\rangle_{\mathfrak{a}A}\left(P\mathfrak{p}\mathfrak{g}_1|0\rangle - Pm\mathfrak{g}_2|1\rangle \right)_B \right. \\
&\quad \left. + |\varkappa_3\rangle_{\mathfrak{a}A}\left(Q\mathfrak{g}_2\mathfrak{q}^*|0\rangle + Q\mathfrak{g}_1 m|1\rangle \right)_B + |\varkappa_4\rangle_{\mathfrak{a}A}\left(-Q\mathfrak{g}_2|0\rangle + Q\mathfrak{g}_1 m\mathfrak{q}|1\rangle \right)_B \right].
\end{aligned}
$$

$$(13.6)$$

Here, $M = \frac{1}{\sqrt{1+|m|^2}}$, $P = \frac{1}{\sqrt{1+|\mathfrak{p}|^2}}$ and $Q = \frac{1}{\sqrt{1+|\mathfrak{q}|^2}}$ are real numbers. After that Alice performs a measurement on the basis given in Eq. (13.4) and communicates this result to Bob by a classical channel.

In general, this classical information will be of no use for Bob in obtaining the state intended for teleportation with the exception of the following cases.

Case I: If $\mathfrak{p} = \frac{1}{\mathfrak{p}^*} = \mathfrak{q}^* = \frac{1}{\mathfrak{q}} = m$, that is, the parameters $m, \mathfrak{p}, \mathfrak{q}$ are complex numbers with unit modulus and are related as above, then it is possible for Bob to apply appropriate unitary operations on his qubit to produce the state $|\aleph\rangle_{\mathfrak{a}}$ at his end.

Following the above relations we have $|\mathfrak{p}|^2 = \mathfrak{p}\mathfrak{p}^* = 1$, $|\mathfrak{q}|^2 = \mathfrak{q}\mathfrak{q}^* = 1$ which implies that $P = Q = \frac{1}{\sqrt{2}}$. Putting in (13.6) and rewriting we have

$$
\begin{aligned}
|\Gamma\rangle = \quad &\frac{M}{\sqrt{2}}\left[|\varkappa_1\rangle_{\mathfrak{a}A}\left(\mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle \right)_B + m|\varkappa_2\rangle_{\mathfrak{a}A}\left(\mathfrak{g}_1|0\rangle - \mathfrak{g}_2|1\rangle \right)_B \right. \\
&\left. + m|\varkappa_3\rangle_{\mathfrak{a}A}\left(\mathfrak{g}_2|0\rangle + \mathfrak{g}_1|1\rangle \right)_B + |\varkappa_4\rangle_{\mathfrak{a}A}\left(-\mathfrak{g}_2|0\rangle + \mathfrak{g}_1|1\rangle \right)_B \right].
\end{aligned}
$$

$$(13.7)$$

Now Alice executes her measurement with the basis given in Eq. (13.4) and sends the results to Bob through a classical channel. Depending on the outcomes of Alice, Bob applies the corresponding appropriate unitary operation to obtain the intended state. Details of unitary operations and reduced states obtained by Bob are given in Table 13.1.

Table 13.1

State at Bob's location and corresponding unitary operations for Bob conditioned on Alice's results ↵

Alice's outcome	State of Bob's site	unitary operation
$\vert\varkappa_1\rangle_{aA}$	$(\mathfrak{g}_1\vert0\rangle + \mathfrak{g}_2\vert1\rangle)_B$	$(I)_B$
$\vert\varkappa_2\rangle_{aA}$	$(\mathfrak{g}_1\vert0\rangle - \mathfrak{g}_2\vert1\rangle)_B$	$(\vartheta_z)_B$
$\vert\varkappa_3\rangle_{aA}$	$(\mathfrak{g}_2\vert0\rangle + \mathfrak{g}_1\vert1\rangle)_B$	$(\vartheta_x)_B$
$\vert\varkappa_4\rangle_{aA}$	$(-\mathfrak{g}_2\vert0\rangle + \mathfrak{g}_1\vert1\rangle)_B$	$(\vartheta_z\vartheta_x)_B$

Case II: If we consider $\mathfrak{p} = \mathfrak{m} = \mathfrak{q}^*$, or $\mathfrak{p} = \mathfrak{m} = \frac{1}{\mathfrak{q}}$, or $\mathfrak{p}^* = \frac{1}{\mathfrak{m}} = \mathfrak{q}$, or $\mathfrak{p}^* = \frac{1}{\mathfrak{m}} = \frac{1}{\mathfrak{q}^*}$, then the teleportation of the state $\vert\aleph\rangle_a$ is possible only in two cases of measurement outcomes, that is,

(i) if the condition $\mathfrak{p} = \mathfrak{m} = \mathfrak{q}^*$ holds, then teleportation is possible only when Alice obtains the measurement result $\vert\varkappa_2\rangle_{aA}$ and $\vert\varkappa_3\rangle_{aA}$. This is immediate from the expression (13.6). The other three case also described below follow similarly.

(ii) if the condition $\mathfrak{p} = \mathfrak{m} = \frac{1}{\mathfrak{q}}$ holds, then teleportation is possible only when Alice obtains the measurement result $\vert\varkappa_2\rangle_{aA}$ and $\vert\varkappa_4\rangle_{aA}$.

(iii) if the condition $\mathfrak{p}^* = \frac{1}{\mathfrak{m}} = \mathfrak{q}$ holds, then teleportation is possible only when Alice obtains the measurement result $\vert\varkappa_1\rangle_{aA}$ and $\vert\varkappa_4\rangle_{aA}$.

(iv) if the condition $\mathfrak{p}^* = \frac{1}{\mathfrak{m}} = \frac{1}{\mathfrak{q}^*}$ holds, then teleportation is possible only when Alice obtains the measurement result $\vert\varkappa_1\rangle_{aA}$ and $\vert\varkappa_3\rangle_{aA}$.

The process fails in the other two cases of measurement outcomes in each condition. It also follows from Eq. (13.6) that the total probability of success corresponding to each condition is

$$\mathscr{P} = \frac{2\vert\mathfrak{m}\vert^2}{(1 + \vert\mathfrak{m}\vert^2)^2}.$$

It follows from the above that in order to perform probabilistic teleportation the knowledge of the entanglement resource is necessary on the part of the sender Alice in

the choice of her basis of measurement while the receiver Bob need not have to possess such information.

13.3 PROBABILISTIC TELEPORTATION PROTOCOL OF AN UNKNOWN TWO-QUBIT STATE

In this section, it is shown that an unknown two-qubit quantum state can be transferred from one party to another party with certain probability by the use of two Bell state measurements, a POVM measurement and an appropriate unitary operation. A 4-qubit entanglement resource is utilized in the protocol which is not maximally entangled. The protocol has been designed by Yan et al. [41].

Suppose that two parties, namely Alice and Bob, are situated at distant places. Alice plays the role of a sender and Bob is the receiver. Alice wants to transport her two qubit quantum state given by

$$|\aleph\rangle_{\mathfrak{a}_1\mathfrak{a}_2} = (\mathfrak{g}_1|00\rangle + \mathfrak{g}_2|01\rangle + \mathfrak{g}_3|10\rangle + \mathfrak{g}_4|11\rangle),$$

(13.8)

to the receiver Bob where the coefficients $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3, \mathfrak{g}_4$ satisfy normalization condition, that is,

$$|\mathfrak{g}_1|^2 + |\mathfrak{g}_2|^2 + |\mathfrak{g}_3|^2 + |\mathfrak{g}_4|^2 = 1.$$

For this purpose a four qubit entangled state, shared between the parties and generally not maximally entangled, is used as quantum resource which is given as

$$|E\rangle_{A_1 A_2 B_1 B_2} = (x|0000\rangle + y|1001\rangle + z|0110\rangle + w|1111\rangle),$$

(13.9)

where the coefficients are non-zero real numbers and meet the normalization condition, that is, $x^2 + y^2 + z^2 + w^2 = 1$. The qubits A_1 and A_2, and qubit pair $(\mathfrak{a}_1, \mathfrak{a}_2)$ are in Alice's possession, and other two qubits B_1 and B_2 are in Bob's possession. The circuit for the generation of (13.9) is given in Figure 13.2 where the operator U is described as

$$U = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & w \end{pmatrix}.$$
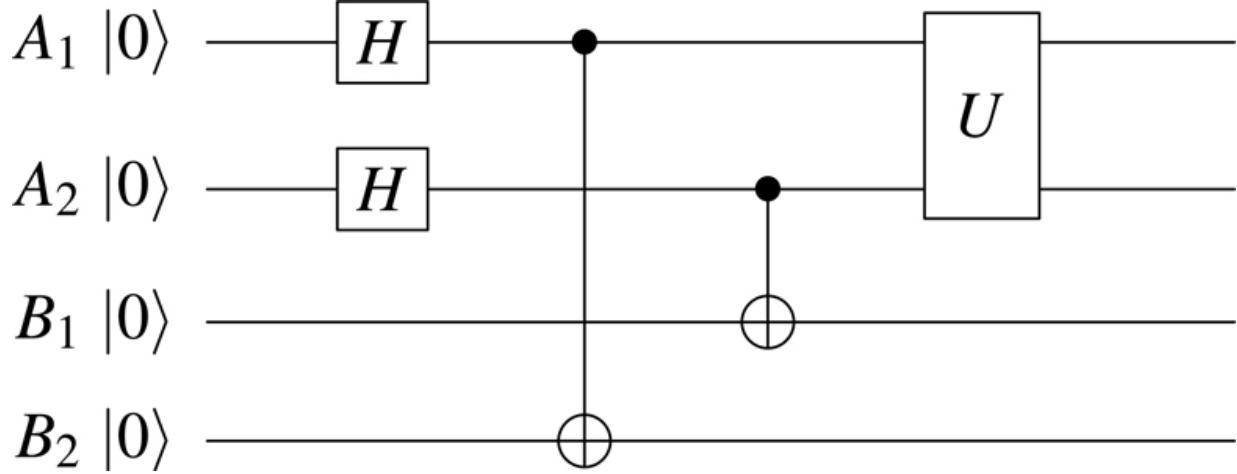
Figure 13.2 Circuit diagram for the generation of the non-maximally entangled resource given in Eq. (13.9). ⏎

Also the two parties are connected amongst themselves by a classical communication channel.

The composite system of six qubits is given by

$$|\Gamma\rangle = |\aleph\rangle_{\mathfrak{a}_1 \mathfrak{a}_2} \otimes |E\rangle_{A_1 A_2 B_1 B_2}.$$

(13.10)

The above state can be written as

$$
\begin{aligned}
|\Gamma\rangle &= \sum_{i=1}^{4} |\Upsilon_i\rangle_{\mathfrak{aa}1 A_2} \otimes |\Upsilon_1\rangle_{\mathfrak{a}_2 A_1} \otimes |R_{1i}\rangle_{B_1 B_2} + \sum_{i=1}^{4} |\Upsilon_i\rangle_{\mathfrak{a}_1 A_2} \otimes |\Upsilon_2\rangle_{\mathfrak{a}_2 A_1} \otimes |R_{2i}\rangle_{B_1 B_2} \\
&+ \sum_{i=1}^{4} |\Upsilon_i\rangle_{\mathfrak{a}_1 A_2} \otimes |\Upsilon_3\rangle_{\mathfrak{a}_2 A_1} \otimes |R_{3i}\rangle_{B_1 B_2} + \sum_{i=1}^{4} |\Upsilon_i\rangle_{\mathfrak{a}_1 A_2} \otimes |\Upsilon_4\rangle_{\mathfrak{a}_2 A_1} \otimes |R_{4i}\rangle_{B_1 B_2}
\end{aligned}
$$

(13.11)

where

$$|R_{11}\rangle_{B_1B_2} = \frac{1}{2}(x\mathfrak{g}_1|00\rangle + y\mathfrak{g}_2|01\rangle + z\mathfrak{g}_3|10\rangle + w\mathfrak{g}_4|11\rangle),$$

$$|R_{12}\rangle_{B_1B_2} = \frac{1}{2}(x\mathfrak{g}_1|00\rangle + y\mathfrak{g}_2|01\rangle - z\mathfrak{g}_3|10\rangle - w\mathfrak{g}_4|11\rangle),$$

$$|R_{13}\rangle_{B_1B_2} = \frac{1}{2}(z\mathfrak{g}_1|10\rangle + w\mathfrak{g}_2|11\rangle + x\mathfrak{g}_3|00\rangle + y\mathfrak{g}_4|01\rangle),$$

$$|R_{14}\rangle_{B_1B_2} = \frac{1}{2}(z\mathfrak{g}_1|10\rangle + w\mathfrak{g}_2|11\rangle - x\mathfrak{g}_3|00\rangle - y\mathfrak{g}_4|01\rangle),$$

$$(13.12)$$

$$|R_{21}\rangle_{B_1B_2} = \frac{1}{2}(x\mathfrak{g}_1|00\rangle - y\mathfrak{g}_2|01\rangle + z\mathfrak{g}_3|10\rangle - w\mathfrak{g}_4|11\rangle),$$

$$|R_{22}\rangle_{B_1B_2} = \frac{1}{2}(x\mathfrak{g}_1|00\rangle - y\mathfrak{g}_2|01\rangle - z\mathfrak{g}_3|10\rangle + w\mathfrak{g}_4|11\rangle),$$

$$|R_{23}\rangle_{B_1B_2} = \frac{1}{2}(z\mathfrak{g}_1|10\rangle - w\mathfrak{g}_2|11\rangle + x\mathfrak{g}_3|00\rangle - y\mathfrak{g}_4|01\rangle),$$

$$|R_{24}\rangle_{B_1B_2} = \frac{1}{2}(z\mathfrak{g}_1|10\rangle - w\mathfrak{g}_2|11\rangle - x\mathfrak{g}_3|00\rangle + y\mathfrak{g}_4|01\rangle),$$

$$(13.13)$$

$$|R_{31}\rangle_{B_1B_2} = \frac{1}{2}(y\mathfrak{g}_1|01\rangle + x\mathfrak{g}_2|00\rangle + w\mathfrak{g}_3|11\rangle + z\mathfrak{g}_4|10\rangle),$$

$$|R_{32}\rangle_{B_1B_2} = \frac{1}{2}(y\mathfrak{g}_1|01\rangle + x\mathfrak{g}_2|00\rangle - w\mathfrak{g}_3|11\rangle - z\mathfrak{g}_4|10\rangle),$$

$$|R_{33}\rangle_{B_1B_2} = \frac{1}{2}(w\mathfrak{g}_1|11\rangle + z\mathfrak{g}_2|10\rangle + y\mathfrak{g}_3|01\rangle + x\mathfrak{g}_4|00\rangle),$$

$$|R_{34}\rangle_{B_1B_2} = \frac{1}{2}(w\mathfrak{g}_1|11\rangle + z\mathfrak{g}_2|10\rangle - y\mathfrak{g}_3|01\rangle - x\mathfrak{g}_4|00\rangle),$$

$$(13.14)$$

$$|R_{41}\rangle_{B_1B_2} = \frac{1}{2}(y\mathfrak{g}_1|01\rangle - x\mathfrak{g}_2|00\rangle + w\mathfrak{g}_3|11\rangle - z\mathfrak{g}_4|10\rangle),$$

$$|R_{42}\rangle_{B_1B_2} = \frac{1}{2}(y\mathfrak{g}_1|01\rangle - x\mathfrak{g}_2|00\rangle - w\mathfrak{g}_3|11\rangle + z\mathfrak{g}_4|10\rangle),$$

$$|R_{43}\rangle_{B_1B_2} = \frac{1}{2}(w\mathfrak{g}_1|11\rangle - z\mathfrak{g}_2|10\rangle + y\mathfrak{g}_3|01\rangle - x\mathfrak{g}_4|00\rangle),$$

$$|R_{44}\rangle_{B_1B_2} = \frac{1}{2}(w\mathfrak{g}_1|11\rangle - z\mathfrak{g}_2|10\rangle - y\mathfrak{g}_3|01\rangle + x\mathfrak{g}_4|00\rangle).$$

To complete the teleportation processes, Alice first executes two Bell basis measurements on her qubit pairs $(\mathfrak{a}_2, A_1)$ and $(\mathfrak{a}_1, A_2)$ given by

$$|\Upsilon_1\rangle_{(\mathfrak{a}_2,A_1)/(\mathfrak{a}_1,A_2)} = \frac{(|00\rangle + |11\rangle)}{\sqrt{2}},$$

$$|\Upsilon_2\rangle_{(\mathfrak{a}_2,A_1)/(\mathfrak{a}_1,A_2)} = \frac{(|00\rangle - |11\rangle)}{\sqrt{2}},$$

$$|\Upsilon_3\rangle_{(\mathfrak{a}_2,A_1)/(\mathfrak{a}_1,A_2)} = \frac{(|01\rangle + |10\rangle)}{\sqrt{2}},$$

$$|\Upsilon_4\rangle_{(\mathfrak{a}_2,A_1)/(\mathfrak{a}_1,A_2)} = \frac{(|01\rangle - |10\rangle)}{\sqrt{2}}.$$

(13.16)

Suppose the outcomes of the Alice's measurement are $|\Upsilon_1\rangle_{\mathfrak{a}_1 A_2}$ and $|\Upsilon_1\rangle_{\mathfrak{a}_2 A_1}$, then Bob's qubits B_1, B_2 are in the state

$$|R_{11}\rangle'_{B_1 B_2} = \frac{(x\mathfrak{g}_1|00\rangle + y\mathfrak{g}_2|01\rangle + z\mathfrak{g}_3|10\rangle + w\mathfrak{g}_4|11\rangle)}{\sqrt{|x\mathfrak{g}_1|^2 + |y\mathfrak{g}_2|^2 + |z\mathfrak{g}_3|^2 + |w\mathfrak{g}_4|^2}}.$$

(13.17)

After the measurements, Alice sends her outcomes to Bob with the help of a 4-bit classical channel. After receiving the classical messages from Alice, Bob introduces two auxiliary qubits q_1, q_2 with the initial state $|0\rangle_{q_1}|0\rangle_{q_2}$. Then the state of Bob's qubits becomes

$$|R_{11}\rangle'_{B_1 B_2}|00\rangle_{q_1 q_2} = \frac{(x\mathfrak{g}_1|0000\rangle + y\mathfrak{g}_2|0100\rangle + z\mathfrak{g}_3|1000\rangle + w\mathfrak{g}_4|1100\rangle)_{B_1 B_2 q_1 q_2}}{\sqrt{|x\mathfrak{g}_1|^2 + |y\mathfrak{g}_2|^2 + |z\mathfrak{g}_3|^2 + |w\mathfrak{g}_4|^2}}.$$

(13.18)

Now Bob executes two CNOT operations on his qubits with B_1, B_2 as control qubits and q_1, q_2 as the corresponding target qubits. After completion of this operation, the above state of the qubits becomes the following.

$$|R_{11}\rangle''_{B_1B_2q_1q_2} = \frac{1}{N}(x\mathfrak{g}_1|0000\rangle + y\mathfrak{g}_2|0101\rangle + z\mathfrak{g}_3|1010\rangle + w\mathfrak{g}_4|1111\rangle)_{B_1B_2q_1q_2},$$

(13.19)

where $N = \sqrt{|x\mathfrak{g}_1|^2 + |y\mathfrak{g}_2|^2 + |z\mathfrak{g}_3|^2 + |w\mathfrak{g}_4|^2}$.

We can rewrite the above state as

$$
\begin{aligned}
|R_{11}\rangle''_{B_1B_2q_1q_2} = \frac{1}{4N}\Big[\ &(\mathfrak{g}_1|00\rangle + \mathfrak{g}_2|01\rangle + \mathfrak{g}_3|10\rangle + \mathfrak{g}_4|11\rangle)_{B_1B_2} \\
&\otimes (x|00\rangle + y|01\rangle + z|10\rangle + w|11\rangle)_{q_1q_2} \\
&+ (\mathfrak{g}_1|00\rangle + \mathfrak{g}_2|01\rangle - \mathfrak{g}_3|10\rangle - \mathfrak{g}_4|11\rangle)_{B_1B_2} \\
&\otimes (x|00\rangle + y|01\rangle - z|10\rangle - w|11\rangle)_{q_1q_2} \\
&+ (\mathfrak{g}_1|00\rangle - \mathfrak{g}_2|01\rangle + \mathfrak{g}_3|10\rangle - \mathfrak{g}_4|11\rangle)_{B_1B_2} \\
&\otimes (x|00\rangle - y|01\rangle + z|10\rangle - w|11\rangle)_{q_1q_2} \\
&+ (\mathfrak{g}_1|00\rangle - \mathfrak{g}_2|01\rangle - \mathfrak{g}_3|10\rangle + \mathfrak{g}_4|11\rangle)_{B_1B_2} \\
&\otimes (x|00\rangle - y|01\rangle - z|10\rangle + w|11\rangle)_{q_1q_2}\Big].
\end{aligned}
$$

(13.20)

Now Bob executes on his auxiliary qubits q_1, q_2 with a POVM given by

$$
\begin{aligned}
F_i &= \frac{1}{n}|\chi_i\rangle\langle\chi_i|; \ \ i = 1, 2, 3, 4 \\
F_5 &= I - (F_1 + F_2 + F_3 + F_4),
\end{aligned}
$$

(13.21)

where

$$|\chi_1\rangle = M(\frac{1}{x}|00\rangle + \frac{1}{y}|01\rangle + \frac{1}{z}|10\rangle + \frac{1}{w}|11\rangle),$$

$$|\chi_2\rangle = M(\frac{1}{x}|00\rangle + \frac{1}{y}|01\rangle - \frac{1}{z}|10\rangle - \frac{1}{w}|11\rangle),$$

$$|\chi_3\rangle = M(\frac{1}{x}|00\rangle - \frac{1}{y}|01\rangle + \frac{1}{z}|10\rangle - \frac{1}{w}|11\rangle),$$

$$|\chi_4\rangle = M(\frac{1}{x}|00\rangle - \frac{1}{y}|01\rangle - \frac{1}{z}|10\rangle + \frac{1}{w}|11\rangle),$$

$$M = \frac{1}{\sqrt{\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} + \frac{1}{w^2}}};$$

(13.22)

and I is an identity operator, n is a coefficient related with the coefficients x,y,z,w such that $1 \le n \le 4$, and makes F_5 into a positive operator. For simplicity of the calculation, we can write the above five operators F_1, F_2, F_3, F_4, F_5 in the matrix form respectively as

$$F_1 = \frac{M^2}{n} \begin{pmatrix} \frac{1}{x^2} & \frac{1}{xy} & \frac{1}{xz} & \frac{1}{xw} \\ \frac{1}{xy} & \frac{1}{y^2} & \frac{1}{yz} & \frac{1}{yw} \\ \frac{1}{xz} & \frac{1}{yz} & \frac{1}{z^2} & \frac{1}{zw} \\ \frac{1}{xw} & \frac{1}{yw} & \frac{1}{zw} & \frac{1}{w^2} \end{pmatrix}, \quad F_2 = \frac{M^2}{n} \begin{pmatrix} \frac{1}{x^2} & \frac{1}{xy} & -\frac{1}{xz} & -\frac{1}{xw} \\ \frac{1}{xy} & \frac{1}{y^2} & -\frac{1}{yz} & -\frac{1}{yw} \\ -\frac{1}{xz} & -\frac{1}{yz} & \frac{1}{z^2} & \frac{1}{zw} \\ -\frac{1}{xw} & -\frac{1}{yw} & \frac{1}{zw} & \frac{1}{w^2} \end{pmatrix},$$

$$F_3 = \frac{M^2}{n} \begin{pmatrix} \frac{1}{x^2} & -\frac{1}{xy} & \frac{1}{xz} & -\frac{1}{xw} \\ -\frac{1}{xy} & \frac{1}{y^2} & -\frac{1}{yz} & \frac{1}{yw} \\ \frac{1}{xz} & -\frac{1}{yz} & \frac{1}{z^2} & -\frac{1}{zw} \\ -\frac{1}{xw} & \frac{1}{yw} & -\frac{1}{zw} & \frac{1}{w^2} \end{pmatrix}, \quad F_4 = \frac{M^2}{n} \begin{pmatrix} \frac{1}{x^2} & -\frac{1}{xy} & -\frac{1}{xz} & \frac{1}{xw} \\ -\frac{1}{xy} & \frac{1}{y^2} & \frac{1}{yz} & -\frac{1}{yw} \\ -\frac{1}{xz} & \frac{1}{yz} & \frac{1}{z^2} & -\frac{1}{zw} \\ \frac{1}{xw} & -\frac{1}{yw} & -\frac{1}{zw} & \frac{1}{w^2} \end{pmatrix},$$

$$F_5 = M^2 \begin{pmatrix} D_1 & 0 & 0 & 0 \\ 0 & D_2 & 0 & 0 \\ 0 & 0 & D_3 & 0 \\ 0 & 0 & 0 & D_4 \end{pmatrix},$$

where

$$D_1 = \left(1 - \frac{4}{n}\right)\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} + \frac{1}{w^2},$$

$$D_2 = \left(1 - \frac{4}{n}\right)\frac{1}{y^2} + \frac{1}{x^2} + \frac{1}{z^2} + \frac{1}{w^2},$$

$$D_3 = \left(1 - \frac{4}{n}\right)\frac{1}{z^2} + \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{w^2},$$

$$D_4 = \left(1 - \frac{4}{n}\right)\frac{1}{w^2} + \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}.$$

(13.23)

If Bob's POVM outcome is F_1, then he obtains the original state which Alice wanted to transfer, that is,

$$|\aleph\rangle_{B_1 B_2} = (\mathfrak{g}_1|00\rangle + \mathfrak{g}_2|01\rangle + \mathfrak{g}_3|10\rangle + \mathfrak{g}_4|11\rangle).$$

If Bob's POVM outcome is F_2, then he gets the original state $|\aleph\rangle_{B_1 B_2}$ by performing the appropriate unitary operation $\sigma_z \otimes I$ on the qubits B_1, B_2.

If Bob's POVM outcome is F_3, then he recovers the original state $|\aleph\rangle_{B_1 B_2}$ by performing the appropriate unitary operation $I \otimes \sigma_z$ on the qubits B_1, B_2.

If Bob's POVM outcome is F_4, then he obtains the original state $|\aleph\rangle_{B_1 B_2}$ by performing the appropriate unitary operation $\sigma_z \otimes \sigma_z$ on the qubits B_1, B_2.

In all the above four cases mentioned above, we see that the teleportation processes is successfully realized. However, if Bob's measurement result is F_5, he gets no information about the state of the qubits B_1, B_2. In this case, the teleportation process fails.

The other cases arising out of Alice's measurement are similarly treated.

In all cases we see that there are possibilities of failure of the protocol. Thus the state can be transferred only with partial success.

13.4 PROBABILISTIC RESUMABLE TELEPORTATION SCHEME

In this section we discuss a probabilistic teleportation protocol in which, unlike in the usual teleportations, the state to be transferred is not destroyed. Rather, the state can be recovered by the sender in the case where the teleportation fails. The process can then be repeated till the success is achieved. The protocol has been designed by Meng et al. [105].

In this scheme there are two mutually separated parties, namely Alice and Bob, playing the role of sender and receiver, respectively. The sender Alice has two qubits in an arbitrary two-qubit state given by

$$|\aleph\rangle_{a_1 a_2} = (\mathfrak{g}_1|00\rangle + \mathfrak{g}_2|01\rangle + \mathfrak{g}_3|10\rangle + \mathfrak{g}_4|11\rangle),$$

(13.24)

where the coefficients $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3, \mathfrak{g}_4$ satisfy normalization condition, that is,

$$\sum_{i=1}^{4} |\mathfrak{g}_i|^2 = 1.$$

Alice wishes to transmit this two-qubit state to Bob through a pre-shared quantum channel between the parties.

There is also a classical communication channel between Alice and Bob.

For this purpose, two 2-qubit entangled states are shared between Alice and Bob which are given by

$$|E_1\rangle_{A_1 B_1} = (\tau_1|00\rangle + \tau_2|11\rangle),$$

(13.25)

$$|E_2\rangle_{A_2 B_2} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

(13.26)

where $|\tau_1| \geq |\tau_2|, |\tau_1|^2 + |\tau_2|^2 = 1$. Qubits A_1, A_2 belong to Alice, and qubits B_1, B_2 belong to Bob. The combined system becomes

$$
\begin{aligned}
|\Gamma\rangle =\ & |\aleph\rangle_{\mathfrak{a}_1\mathfrak{a}_2} \otimes |E_1\rangle_{A_1 B_1} \otimes |E_2\rangle_{A_2 B_2} \\
=\ & \frac{1}{\sqrt{2}} \Big(\mathfrak{g}_1\tau_1|000001\rangle + \mathfrak{g}_1\tau_1|000010\rangle + \mathfrak{g}_1\tau_2|001101\rangle \\
& + \mathfrak{g}_1\tau_2|001110\rangle + \mathfrak{g}_2\tau_1|010001\rangle + \mathfrak{g}_2\tau_1|010010\rangle \\
& + \mathfrak{g}_2\tau_2|011101\rangle + \mathfrak{g}_2\tau_2|011110\rangle + \mathfrak{g}_3\tau_1|100001\rangle \\
& + \mathfrak{g}_3\tau_1|100010\rangle + \mathfrak{g}_3\tau_2|101101\rangle + \mathfrak{g}_3\tau_2|101110\rangle \\
& + \mathfrak{g}_4\tau_1|110001\rangle + \mathfrak{g}_4\tau_1|110010\rangle + \mathfrak{g}_4\tau_2|111101\rangle \\
& + \mathfrak{g}_4\tau_2|111110\rangle \Big)_{\mathfrak{a}_1\mathfrak{a}_2 A_1 B_1 A_2 B_2}.
\end{aligned}
$$

$$(13.27)$$

To achieve the goal of state transfer, Alice initially makes two CNOT operations on the qubit pairs $(\mathfrak{a}_1, A_1)$ and $(\mathfrak{a}_2, A_2)$, where qubits A_1 and A_2 are the control qubits, and qubits $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are the target qubits. Then the state of the whole quantum system evolves into the state

$$
\begin{aligned}
|\Gamma_1\rangle =\ & \frac{1}{\sqrt{2}} \Big(\mathfrak{g}_1\tau_1|0000\rangle_{\mathfrak{a}_1\mathfrak{a}_2 A_1 A_2}|01\rangle_{B_1 B_2} + \mathfrak{g}_1\tau_1|0101\rangle_{\mathfrak{a}_1\mathfrak{a}_2 A_1 A_2}|00\rangle_{B_1 B_2} \\
& + \mathfrak{g}_1\tau_2|1010\rangle_{\mathfrak{a}_1\mathfrak{a}_2 A_1 A_2}|11\rangle_{B_1 B_2} + \mathfrak{g}_1\tau_2|1111\rangle_{\mathfrak{a}_1\mathfrak{a}_2 A_1 A_2}|10\rangle_{B_1 B_2} \\
& + \mathfrak{g}_2\tau_1|0100\rangle_{\mathfrak{a}_1\mathfrak{a}_2 A_1 A_2}|01\rangle_{B_1 B_2} + \mathfrak{g}_2\tau_1|0001\rangle_{\mathfrak{a}_1\mathfrak{a}_2 A_1 A_2}|00\rangle_{B_1 B_2} \\
& + \mathfrak{g}_2\tau_2|1110\rangle_{\mathfrak{a}_1\mathfrak{a}_2 A_1 A_2}|11\rangle_{B_1 B_2} + \mathfrak{g}_2\tau_2|1011\rangle_{\mathfrak{a}_1\mathfrak{a}_2 A_1 A_2}|10\rangle_{B_1 B_2} \\
& + \mathfrak{g}_3\tau_1|1000\rangle_{\mathfrak{a}_1\mathfrak{a}_2 A_1 A_2}|01\rangle_{B_1 B_2} + \mathfrak{g}_3\tau_1|1101\rangle_{\mathfrak{a}_1\mathfrak{a}_2 A_1 A_2}|00\rangle_{B_1 B_2} \\
& + \mathfrak{g}_3\tau_2|0010\rangle_{\mathfrak{a}_1\mathfrak{a}_2 A_1 A_2}|11\rangle_{B_1 B_2} + \mathfrak{g}_3\tau_2|0111\rangle_{\mathfrak{a}_1\mathfrak{a}_2 A_1 A_2}|10\rangle_{B_1 B_2} \\
& + \mathfrak{g}_4\tau_1|1100\rangle_{\mathfrak{a}_1\mathfrak{a}_2 A_1 A_2}|01\rangle_{B_1 B_2} + \mathfrak{g}_4\tau_1|1001\rangle_{\mathfrak{a}_1\mathfrak{a}_2 A_1 A_2}|00\rangle_{B_1 B_2} \\
& + \mathfrak{g}_4\tau_2|0110\rangle_{\mathfrak{a}_1\mathfrak{a}_2 A_1 A_2}|11\rangle_{B_1 B_2} + \mathfrak{g}_4\tau_2|0011\rangle_{\mathfrak{a}_1\mathfrak{a}_2 A_1 A_2}|10\rangle_{B_1 B_2} \Big).
\end{aligned}
$$

$$(13.28)$$

Now, Alice introduces two auxiliary qubits q_1 and q_2 with initial state $|00\rangle_{q_1 q_2}$ and executes two CNOT operation on her qubit pairs $(\mathfrak{a}_1, q_1)$ and $(\mathfrak{a}_2, q_2)$, where qubits $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are control qubits and the qubits q_1 and q_2 are the respective target qubits. After performing these operation, the quantum state of the system evolves into the state

$$|\Gamma_2\rangle = \frac{1}{\sqrt{2}}\Big(\mathfrak{g}_1\tau_1|00\rangle_{q_1q_2}|0000\rangle_{\mathfrak{a}_1\mathfrak{a}_2A_1A_2}|01\rangle_{B_1B_2} + \mathfrak{g}_1\tau_1|01\rangle_{q_1q_2}$$

$$|0101\rangle_{\mathfrak{a}_1\mathfrak{a}_2A_1A_2}|00\rangle_{B_1B_2} + \mathfrak{g}_1\tau_2|10\rangle_{q_1q_2}|1010\rangle_{\mathfrak{a}_1\mathfrak{a}_2A_1A_2}|11\rangle_{B_1B_2}$$

$$+ \mathfrak{g}_1\tau_2|11\rangle_{q_1q_2}|1111\rangle_{\mathfrak{a}_1\mathfrak{a}_2A_1A_2}|10\rangle_{B_1B_2} + \mathfrak{g}_2\tau_1|01\rangle_{q_1q_2}$$

$$|0100\rangle_{\mathfrak{a}_1\mathfrak{a}_2A_1A_2}|01\rangle_{B_1B_2} + \mathfrak{g}_2\tau_1|00\rangle_{q_1q_2}|0001\rangle_{\mathfrak{a}_1\mathfrak{a}_2A_1A_2}|00\rangle_{B_1B_2}$$

$$+ \mathfrak{g}_2\tau_2|11\rangle_{q_1q_2}|1110\rangle_{\mathfrak{a}_1\mathfrak{a}_2A_1A_2}|11\rangle_{B_1B_2} + \mathfrak{g}_2\tau_2|10\rangle_{q_1q_2}$$

$$|1011\rangle_{\mathfrak{a}_1\mathfrak{a}_2A_1A_2}|10\rangle_{B_1B_2} + \mathfrak{g}_3\tau_1|10\rangle_{q_1q_2}|1000\rangle_{\mathfrak{a}_1\mathfrak{a}_2A_1A_2}|01\rangle_{B_1B_2}$$

$$+ \mathfrak{g}_3\tau_1|11\rangle_{q_1q_2}|1101\rangle_{\mathfrak{a}_1\mathfrak{a}_2A_1A_2}|00\rangle_{B_1B_2} + \mathfrak{g}_3\tau_2|00\rangle_{q_1q_2}$$

$$|0010\rangle_{\mathfrak{a}_1\mathfrak{a}_2A_1A_2}|11\rangle_{B_1B_2} + \mathfrak{g}_3\tau_2|01\rangle_{q_1q_2}|0111\rangle_{\mathfrak{a}_1\mathfrak{a}_2A_1A_2}|10\rangle_{B_1B_2}$$

$$+ \mathfrak{g}_4\tau_1|11\rangle_{q_1q_2}|1100\rangle_{\mathfrak{a}_1\mathfrak{a}_2A_1A_2}|01\rangle_{B_1B_2} + \mathfrak{g}_4\tau_1|10\rangle_{q_1q_2}$$

$$|1001\rangle_{\mathfrak{a}_1\mathfrak{a}_2A_1A_2}|00\rangle_{B_1B_2} + \mathfrak{g}_4\tau_2|01\rangle_{q_1q_2}|0110\rangle_{\mathfrak{a}_1\mathfrak{a}_2A_1A_2}|11\rangle_{B_1B_2}$$

$$+ \mathfrak{g}_4\tau_2|00\rangle_{q_1q_2}|0011\rangle_{\mathfrak{a}_1\mathfrak{a}_2A_1A_2}|10\rangle_{B_1B_2}\Big).$$

(13.29)

Next, to accomplish resumable quantum teleportation of the two-qubit entangled state, Alice makes the following controlled unitary transform under the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ on qubits $\mathfrak{a}_1, \mathfrak{a}_2, A_1, A_2, q_1$ and q_2

$$
\begin{aligned}
U_{\mathfrak{a}_1\mathfrak{a}_2}^{A_1A_2q_1q_2} &= |00\rangle_{A_1A_2}\langle00| \otimes |00\rangle_{q_1q_2}\langle00| \otimes u_1^{\mathfrak{a}_1\mathfrak{a}_2} + |00\rangle_{A_1A_2}\langle00| \\
&\otimes |01\rangle_{q_1q_2}\langle01| \otimes u_2^{\mathfrak{a}_1\mathfrak{a}_2} + |00\rangle_{A_1A_2}\langle00| \otimes |10\rangle_{q_1q_2}\langle10| \\
&\otimes u_2^{\mathfrak{a}_1\mathfrak{a}_2} + |00\rangle_{A_1A_2}\langle00| \otimes |11\rangle_{q_1q_2}\langle11| \otimes u_1^{\mathfrak{a}_1\mathfrak{a}_2} \\
&+ |01\rangle_{A_1A_2}\langle01| \otimes |00\rangle_{q_1q_2}\langle00| \otimes u_1^{\mathfrak{a}_1\mathfrak{a}_2} + |01\rangle_{A_1A_2}\langle01| \\
&\otimes |01\rangle_{q_1q_2}\langle01| \otimes u_2^{\mathfrak{a}_1\mathfrak{a}_2} + |01\rangle_{A_1A_2}\langle01| \otimes |10\rangle_{q_1q_2}\langle10| \\
&\otimes u_2^{\mathfrak{a}_1\mathfrak{a}_2} + |01\rangle_{A_1A_2}\langle01| \otimes |11\rangle_{q_1q_2}\langle11| \otimes u_1^{\mathfrak{a}_1\mathfrak{a}_2} \\
&+ |10\rangle_{A_1A_2}\langle10| \otimes |00\rangle_{q_1q_2}\langle00| \otimes I + |10\rangle_{A_1A_2}\langle10| \\
&\otimes |01\rangle_{q_1q_2}\langle01| \otimes I + |10\rangle_{A_1A_2}\langle10| \otimes |10\rangle_{q_1q_2}\langle10| \otimes I \\
&+ |10\rangle_{A_1A_2}\langle10| \otimes |11\rangle_{q_1q_2}\langle11| \otimes I + |11\rangle_{A_1A_2}\langle11| \\
&\otimes |00\rangle_{q_1q_2}\langle00| \otimes I + |11\rangle_{A_1A_2}\langle11| \otimes |01\rangle_{q_1q_2}\langle01| \otimes I \\
&+ |11\rangle_{A_1A_2}\langle11| \otimes |10\rangle_{q_1q_2}\langle10| \otimes I \\
&+ |11\rangle_{A_1A_2}\langle11| \otimes |11\rangle_{q_1q_2}\langle11| \otimes I,
\end{aligned}
$$

(13.30)

where I stands for 4×4 identity matrix, and the unitary operators $u_1^{a_1 a_2}$ and $u_2^{a_1 a_2}$ are given by

$$
u_1^{a_1 a_2} = \begin{bmatrix} \frac{\tau_2}{\tau_1} & 0 & 0 & \sqrt{1 - (\frac{\tau_2}{\tau_1})^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sqrt{1 - (\frac{\tau_2}{\tau_1})^2} & 0 & 0 & -\frac{\tau_2}{\tau_1} \end{bmatrix},
$$

$$
u_2^{a_1 a_2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\tau_2}{\tau_1} & \sqrt{1 - (\frac{\tau_2}{\tau_1})^2} & 0 \\ 0 & \sqrt{1 - (\frac{\tau_2}{\tau_1})^2} & -\frac{\tau_2}{\tau_1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

After the operation $U_{a_1 a_2}^{A_1 A_2 q_1 q_2}$ on the quantum state $|\Gamma_2\rangle$, Alice executes $U_{Not}^{a_1 q_1}$ and $U_{Not}^{a_2 q_2}$ again, the above state evolves into the state

$$|\Gamma_3\rangle = \frac{1}{\sqrt{2}}\Big(\mathfrak{g}_1\tau_2|00\rangle_{q_1q_2}|00\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|00\rangle_{A_1A_2}|01\rangle_{B_1B_2} + \mathfrak{g}_1\sqrt{\tau_1^2-\tau_2^2}|11\rangle_{q_1q_2}$$

$$|11\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|00\rangle_{A_1A_2}|01\rangle_{B_1B_2} + \mathfrak{g}_1\tau_2|00\rangle_{q_1q_2}|01\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|01\rangle_{A_1A_2}|00\rangle_{B_1B_2}$$

$$+ \mathfrak{g}_1\sqrt{\tau_1^2-\tau_2^2}|11\rangle_{q_1q_2}|10\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|01\rangle_{A_1A_2}|00\rangle_{B_1B_2} + \mathfrak{g}_1\tau_2|00\rangle_{q_1q_2}$$

$$|10\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|10\rangle_{A_1A_2}|11\rangle_{B_1B_2} + \mathfrak{g}_1\tau_2|00\rangle_{q_1q_2}|11\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|11\rangle_{A_1A_2}|10\rangle_{B_1B_2}$$

$$+ \mathfrak{g}_2\tau_2|00\rangle_{q_1q_2}|00\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|01\rangle_{A_1A_2}|00\rangle_{B_1B_2} + \mathfrak{g}_2\sqrt{\tau_1^2-\tau_2^2}|11\rangle_{q_1q_2}$$

$$|11\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|01\rangle_{A_1A_2}|00\rangle_{B_1B_2} + \mathfrak{g}_2\tau_2|00\rangle_{q_1q_2}|01\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|00\rangle_{A_1A_2}|01\rangle_{B_1B_2}$$

$$+ \mathfrak{g}_2\sqrt{\tau_1^2-\tau_2^2}|11\rangle_{q_1q_2}|10\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|00\rangle_{A_1A_2}|01\rangle_{B_1B_2} + \mathfrak{g}_2\tau_2|00\rangle_{q_1q_2}$$

$$|10\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|11\rangle_{A_1A_2}|10\rangle_{B_1B_2} + \mathfrak{g}_2\tau_2|00\rangle_{q_1q_2}|11\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|10\rangle_{A_1A_2}|11\rangle_{B_1B_2}$$

$$+ \mathfrak{g}_3\tau_2|00\rangle_{q_1q_2}|00\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|10\rangle_{A_1A_2}|11\rangle_{B_1B_2} + \mathfrak{g}_3\tau_2|00\rangle_{q_1q_2}|01\rangle_{\mathfrak{a}_1\mathfrak{a}_2}$$

$$|11\rangle_{A_1A_2}|10\rangle_{B_1B_2} - \mathfrak{g}_3\tau_2|00\rangle_{q_1q_2}|10\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|00\rangle_{A_1A_2}|01\rangle_{B_1B_2}$$

$$+ \mathfrak{g}_3\sqrt{\tau_1^2-\tau_1^2}|11\rangle_{q_1q_2}|01\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|00\rangle_{A_1A_2}|01\rangle_{B_1B_2} - \mathfrak{g}_3\tau_2|00\rangle_{q_1q_2}$$

$$|11\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|01\rangle_{A_1A_2}|00\rangle_{B_1B_2} + \mathfrak{g}_3\sqrt{\tau_1^2-\tau_2^2}|11\rangle_{q_1q_2}|00\rangle_{\mathfrak{a}_1\mathfrak{a}_2}$$

$$|01\rangle_{A_1A_2}|00\rangle_{B_1B_2} + \mathfrak{g}_4\tau_2|00\rangle_{q_1q_2}|00\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|11\rangle_{A_1A_2}|10\rangle_{B_1B_2}$$

$$+ \mathfrak{g}_4\tau_2|00\rangle_{q_1q_2}|01\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|10\rangle_{A_1A_2}|11\rangle_{B_1B_2} - \mathfrak{g}_4\tau_2|00\rangle_{q_1q_2}|10\rangle_{\mathfrak{a}_1\mathfrak{a}_2}$$

$$|01\rangle_{A_1A_2}|00\rangle_{B_1B_2} + \mathfrak{g}_4\sqrt{\tau_1^2-\tau_2^2}|11\rangle_{q_1q_2}|01\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|01\rangle_{A_1A_2}$$

$$|00\rangle_{B_1B_2} - \mathfrak{g}_4\tau_2|00\rangle_{q_1q_2}|11\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|00\rangle_{A_1A_2}|01\rangle_{B_1B_2}$$

$$+ \mathfrak{g}_4\sqrt{\tau_1^2-\tau_2^2}|11\rangle_{q_1q_2}|00\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|00\rangle_{A_1A_2}|01\rangle_{B_1B_2}\Big).$$

<div align="right">(13.31)</div>

Again, Alice performs two CNOT operation $U_{Not}^{A_1\mathfrak{a}_1}$ and $U_{Not}^{A_2\mathfrak{a}_2}$ on qubits $(\mathfrak{a}_1, A_1)$ and $(\mathfrak{a}_2, A_2)$, and after that the above state $|\Gamma_3\rangle$ evolve into the state

$$
\begin{aligned}
|\Gamma_4\rangle = \ \ &\frac{1}{\sqrt{2}}\Bigg(\mathfrak{g}_1\tau_2|00\rangle_{q_1q_2}|00\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|00\rangle_{A_1A_2}|01\rangle_{B_1B_2} + \mathfrak{g}_1\sqrt{\tau_1^2-\tau_2^2}|11\rangle_{q_1q_2}\\
&|11\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|00\rangle_{A_1A_2}|01\rangle_{B_1B_2} + \mathfrak{g}_1\tau_2|00\rangle_{q_1q_2}|00\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|01\rangle_{A_1A_2}|00\rangle_{B_1B_2}\\
&+ \mathfrak{g}_1\sqrt{\tau_1^2-\tau_2^2}|11\rangle_{q_1q_2}|11\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|01\rangle_{A_1A_2}|00\rangle_{B_1B_2} + \mathfrak{g}_1\tau_2|00\rangle_{q_1q_2}\\
&|00\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|10\rangle_{A_1A_2}|11\rangle_{B_1B_2} + \mathfrak{g}_1\tau_2|00\rangle_{q_1q_2}|00\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|11\rangle_{A_1A_2}|10\rangle_{B_1B_2}\\
&+ \mathfrak{g}_2\tau_2|00\rangle_{q_1q_2}|01\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|01\rangle_{A_1A_2}|00\rangle_{B_1B_2} + \mathfrak{g}_2\sqrt{\tau_1^2-\tau_2^2}|11\rangle_{q_1q_2}\\
&|10\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|01\rangle_{A_1A_2}|00\rangle_{B_1B_2} + \mathfrak{g}_2\tau_2|00\rangle_{q_1q_2}|01\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|00\rangle_{A_1A_2}\\
&|01\rangle_{B_1B_2} + \mathfrak{g}_2\sqrt{\tau_1^2-\tau_2^2}|11\rangle_{q_1q_2}|10\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|00\rangle_{A_1A_2}|01\rangle_{B_1B_2}\\
&+ \mathfrak{g}_2\tau_2|00\rangle_{q_1q_2}|01\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|11\rangle_{A_1A_2}|10\rangle_{B_1B_2} + \mathfrak{g}_2\tau_2|00\rangle_{q_1q_2}|01\rangle_{\mathfrak{a}_1\mathfrak{a}_2}\\
&|10\rangle_{A_1A_2}|11\rangle_{B_1B_2} + \mathfrak{g}_3\tau_2|00\rangle_{q_1q_2}|10\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|10\rangle_{A_1A_2}|11\rangle_{B_1B_2}\\
&+ \mathfrak{g}_3\tau_2|00\rangle_{q_1q_2}|10\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|11\rangle_{A_1A_2}|10\rangle_{B_1B_2} - \mathfrak{g}_3\tau_2|00\rangle_{q_1q_2}|10\rangle_{\mathfrak{a}_1\mathfrak{a}_2}\\
&|00\rangle_{A_1A_2}|01\rangle_{B_1B_2} + \mathfrak{g}_3\sqrt{\tau_1^2-\tau_2^2}|11\rangle_{q_1q_2}|01\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|00\rangle_{A_1A_2}|01\rangle_{B_1B_2}\\
&- \mathfrak{g}_3\tau_2|00\rangle_{q_1q_2}|10\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|01\rangle_{A_1A_2}|00\rangle_{B_1B_2} + \mathfrak{g}_3\sqrt{\tau_1^2-\tau_2^2}|11\rangle_{q_1q_2}\\
&|01\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|01\rangle_{A_1A_2}|00\rangle_{B_1B_2} + \mathfrak{g}_4\tau_2|00\rangle_{q_1q_2}|11\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|11\rangle_{A_1A_2}|10\rangle_{B_1B_2}\\
&+ \mathfrak{g}_4\tau_2|00\rangle_{q_1q_2}|11\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|10\rangle_{A_1A_2}|11\rangle_{B_1B_2} - \mathfrak{g}_4\tau_2|00\rangle_{q_1q_2}\\
&|11\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|01\rangle_{A_1A_2}|00\rangle_{B_1B_2} + \mathfrak{g}_4\sqrt{\tau_1^2-\tau_2^2}|11\rangle_{q_1q_2}|00\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|01\rangle_{A_1A_2}\\
&|00\rangle_{B_1B_2} - \mathfrak{g}_4\tau_2|00\rangle_{q_1q_2}|11\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|00\rangle_{A_1A_2}|01\rangle_{B_1B_2}\\
&+ \mathfrak{g}_4\sqrt{\tau_1^2-\tau_2^2}|11\rangle_{q_1q_2}|00\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|00\rangle_{A_1A_2}|01\rangle_{B_1B_2}\Bigg).
\end{aligned}
$$

$$(13.32)$$

The state $|\Gamma_4\rangle$ given in Eq. (13.32) can be written in a simplified form as

$$|\Gamma_4'\rangle = \sqrt{2}\tau_2 |00\rangle_{q_1 q_2} \otimes \frac{1}{2}\Big(\mathfrak{g}_1|00\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|00\rangle_{A_1 A_2}|01\rangle_{B_1 B_2}$$

$$+ \mathfrak{g}_1|00\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|01\rangle_{A_1 A_2}|00\rangle_{B_1 B_2} + \mathfrak{g}_1|00\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|10\rangle_{A_1 A_2}|11\rangle_{B_1 B_2}$$

$$+ \mathfrak{g}_1|00\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|11\rangle_{A_1 A_2}|10\rangle_{B_1 B_2} + \mathfrak{g}_2|01\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|01\rangle_{A_1 A_2}|00\rangle_{B_1 B_2}$$

$$+ \mathfrak{g}_2|01\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|00\rangle_{A_1 A_2}|01\rangle_{B_1 B_2} + \mathfrak{g}_2|01\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|11\rangle_{A_1 A_2}|10\rangle_{B_1 B_2}$$

$$+ \mathfrak{g}_2|01\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|10\rangle_{A_1 A_2}|11\rangle_{B_1 B_2} + \mathfrak{g}_3|10\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|10\rangle_{A_1 A_2}|11\rangle_{B_1 B_2}$$

$$+ \mathfrak{g}_3|10\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|11\rangle_{A_1 A_2}|10\rangle_{B_1 B_2} - \mathfrak{g}_3|10\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|00\rangle_{A_1 A_2}|01\rangle_{B_1 B_2}$$

$$- \mathfrak{g}_3|10\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|01\rangle_{A_1 A_2}|00\rangle_{B_1 B_2} + \mathfrak{g}_4|11\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|11\rangle_{A_1 A_2}|10\rangle_{B_1 B_2}$$

$$+ \mathfrak{g}_4|11\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|10\rangle_{A_1 A_2}|11\rangle_{B_1 B_2} - \mathfrak{g}_4|11\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|01\rangle_{A_1 A_2}|00\rangle_{B_1 B_2}$$

$$- \mathfrak{g}_4|11\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|00\rangle_{A_1 A_2}|01\rangle_{B_1 B_2} \Big) + \sqrt{\tau_1^2 - \tau_2^2}\,|11\rangle_{q_1 q_2} \otimes$$

$$\frac{1}{\sqrt{2}}\Big(\mathfrak{g}_1|11\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|00\rangle_{A_1 A_2}|01\rangle_{B_1 B_2} + \mathfrak{g}_1|11\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|01\rangle_{A_1 A_2}|00\rangle_{B_1 B_2}$$

$$+ \mathfrak{g}_2|10\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|01\rangle_{A_1 A_2}|00\rangle_{B_1 B_2} + \mathfrak{g}_2|10\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|00\rangle_{A_1 A_2}|01\rangle_{B_1 B_2}$$

$$+ \mathfrak{g}_3|01\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|00\rangle_{A_1 A_2}|01\rangle_{B_1 B_2} + \mathfrak{g}_3|01\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|01\rangle_{A_1 A_2}|00\rangle_{B_1 B_2}$$

$$+ \mathfrak{g}_4|00\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|01\rangle_{A_1 A_2}|00\rangle_{B_1 B_2} + \mathfrak{g}_4|00\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|00\rangle_{A_1 A_2}|01\rangle_{B_1 B_2} \Big).$$

$$(13.33)$$

Now Alice makes two-qubit projection measurements on her qubits q_1 and q_2.

Case I

If the measurement result is $|00\rangle_{q_1 q_2}$, then the above state $|\Gamma_4'\rangle$ becomes

$$|\Gamma_4''\rangle = \frac{1}{2}\Big(\mathfrak{g}_1|00\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|00\rangle_{A_1 A_2}|01\rangle_{B_1 B_2}$$

$$+ \mathfrak{g}_1|00\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|01\rangle_{A_1 A_2}|00\rangle_{B_1 B_2} + \mathfrak{g}_1|00\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|10\rangle_{A_1 A_2}|11\rangle_{B_1 B_2}$$

$$+ \mathfrak{g}_1|00\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|11\rangle_{A_1 A_2}|10\rangle_{B_1 B_2} + \mathfrak{g}_2|01\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|01\rangle_{A_1 A_2}|00\rangle_{B_1 B_2}$$

$$+ \mathfrak{g}_2|01\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|00\rangle_{A_1 A_2}|01\rangle_{B_1 B_2} + \mathfrak{g}_2|01\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|11\rangle_{A_1 A_2}|10\rangle_{B_1 B_2}$$

$$+ \mathfrak{g}_2|01\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|10\rangle_{A_1 A_2}|11\rangle_{B_1 B_2} + \mathfrak{g}_3|10\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|10\rangle_{A_1 A_2}|11\rangle_{B_1 B_2}$$

$$+ \mathfrak{g}_3|10\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|11\rangle_{A_1 A_2}|10\rangle_{B_1 B_2} - \mathfrak{g}_3|10\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|00\rangle_{A_1 A_2}|01\rangle_{B_1 B_2}$$

$$- \mathfrak{g}_3|10\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|01\rangle_{A_1 A_2}|00\rangle_{B_1 B_2} + \mathfrak{g}_4|11\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|11\rangle_{A_1 A_2}|10\rangle_{B_1 B_2}$$

$$+ \mathfrak{g}_4|11\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|10\rangle_{A_1 A_2}|11\rangle_{B_1 B_2} - \mathfrak{g}_4|11\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|01\rangle_{A_1 A_2}|00\rangle_{B_1 B_2}$$

$$- \mathfrak{g}_4|11\rangle_{\mathfrak{a}_1\mathfrak{a}_2}|00\rangle_{A_1 A_2}|01\rangle_{B_1 B_2} \Big).$$

Lastly, Alice executes two Bell-state measurements (BSM) on qubits $(\mathfrak{a}_1, A_1)$ and $(\mathfrak{a}_2, A_2)$. With the Bell basis the above state $|\Gamma_4''\rangle$ can be expressed as

$$
\begin{aligned}
|\Gamma_4''\rangle = \ & (|\Upsilon_1\rangle_{\mathfrak{a}_1 A_1} \otimes |\Upsilon_1\rangle_{\mathfrak{a}_2 A_2}) \otimes (\mathfrak{g}_1|01\rangle + \mathfrak{g}_2|00\rangle + \mathfrak{g}_3|11\rangle + \mathfrak{g}_4|10\rangle)_{B_1 B_2} \\
+ \ & (|\Upsilon_1\rangle_{\mathfrak{a}_1 A_1} \otimes |\Upsilon_2\rangle_{\mathfrak{a}_2 A_2}) \otimes (\mathfrak{g}_1|01\rangle - \mathfrak{g}_2|00\rangle + \mathfrak{g}_3|11\rangle - \mathfrak{g}_4|10\rangle)_{B_1 B_2} \\
+ \ & (|\Upsilon_1\rangle_{\mathfrak{a}_1 A_1} \otimes |\Upsilon_3\rangle_{\mathfrak{a}_2 A_2}) \otimes (\mathfrak{g}_1|00\rangle + \mathfrak{g}_2|01\rangle + \mathfrak{g}_3|10\rangle + \mathfrak{g}_4|11\rangle)_{B_1 B_2} \\
+ \ & (|\Upsilon_1\rangle_{\mathfrak{a}_1 A_1} \otimes |\Upsilon_4\rangle_{\mathfrak{a}_2 A_2}) \otimes (\mathfrak{g}_1|00\rangle - \mathfrak{g}_2|01\rangle + \mathfrak{g}_3|10\rangle - \mathfrak{g}_4|11\rangle)_{B_1 B_2} \\
+ \ & (|\Upsilon_2\rangle_{\mathfrak{a}_1 A_1} \otimes |\Upsilon_1\rangle_{\mathfrak{a}_2 A_2}) \otimes (\mathfrak{g}_1|01\rangle + \mathfrak{g}_2|00\rangle - \mathfrak{g}_3|11\rangle - \mathfrak{g}_4|10\rangle)_{B_1 B_2} \\
+ \ & (|\Upsilon_2\rangle_{\mathfrak{a}_1 A_1} \otimes |\Upsilon_2\rangle_{\mathfrak{a}_2 A_2}) \otimes (\mathfrak{g}_1|01\rangle - \mathfrak{g}_2|00\rangle - \mathfrak{g}_3|11\rangle + \mathfrak{g}_4|10\rangle)_{B_1 B_2} \\
+ \ & (|\Upsilon_2\rangle_{\mathfrak{a}_1 A_1} \otimes |\Upsilon_3\rangle_{\mathfrak{a}_2 A_2}) \otimes (\mathfrak{g}_1|00\rangle + \mathfrak{g}_2|01\rangle - \mathfrak{g}_3|10\rangle - \mathfrak{g}_4|11\rangle)_{B_1 B_2} \\
+ \ & (|\Upsilon_2\rangle_{\mathfrak{a}_1 A_1} \otimes |\Upsilon_4\rangle_{\mathfrak{a}_2 A_2}) \otimes (\mathfrak{g}_1|00\rangle - \mathfrak{g}_2|01\rangle - \mathfrak{g}_3|10\rangle + \mathfrak{g}_4|11\rangle)_{B_1 B_2} \\
+ \ & (|\Upsilon_3\rangle_{\mathfrak{a}_1 A_1} \otimes |\Upsilon_1\rangle_{\mathfrak{a}_2 A_2}) \otimes (\mathfrak{g}_1|11\rangle + \mathfrak{g}_2|10\rangle - \mathfrak{g}_3|01\rangle - \mathfrak{g}_4|00\rangle)_{B_1 B_2} \\
+ \ & (|\Upsilon_3\rangle_{\mathfrak{a}_1 A_1} \otimes |\Upsilon_2\rangle_{\mathfrak{a}_2 A_2}) \otimes (\mathfrak{g}_1|11\rangle - \mathfrak{g}_2|10\rangle - \mathfrak{g}_3|01\rangle + \mathfrak{g}_4|00\rangle)_{B_1 B_2} \\
+ \ & (|\Upsilon_3\rangle_{\mathfrak{a}_1 A_1} \otimes |\Upsilon_3\rangle_{\mathfrak{a}_2 A_2}) \otimes (\mathfrak{g}_1|10\rangle + \mathfrak{g}_2|11\rangle - \mathfrak{g}_3|00\rangle - \mathfrak{g}_4|01\rangle)_{B_1 B_2} \\
+ \ & (|\Upsilon_3\rangle_{\mathfrak{a}_1 A_1} \otimes |\Upsilon_4\rangle_{\mathfrak{a}_2 A_2}) \otimes (\mathfrak{g}_1|10\rangle - \mathfrak{g}_2|11\rangle - \mathfrak{g}_3|00\rangle + \mathfrak{g}_4|01\rangle)_{B_1 B_2} \\
+ \ & (|\Upsilon_4\rangle_{\mathfrak{a}_1 A_1} \otimes |\Upsilon_1\rangle_{\mathfrak{a}_2 A_2}) \otimes (\mathfrak{g}_1|11\rangle + \mathfrak{g}_2|10\rangle + \mathfrak{g}_3|01\rangle + \mathfrak{g}_4|00\rangle)_{B_1 B_2} \\
+ \ & (|\Upsilon_4\rangle_{\mathfrak{a}_1 A_1} \otimes |\Upsilon_2\rangle_{\mathfrak{a}_2 A_2}) \otimes (\mathfrak{g}_1|11\rangle - \mathfrak{g}_2|10\rangle + \mathfrak{g}_3|01\rangle - \mathfrak{g}_4|00\rangle)_{B_1 B_2} \\
+ \ & (|\Upsilon_4\rangle_{\mathfrak{a}_1 A_1} \otimes |\Upsilon_3\rangle_{\mathfrak{a}_2 A_2}) \otimes (\mathfrak{g}_1|10\rangle + \mathfrak{g}_2|11\rangle + \mathfrak{g}_3|00\rangle + \mathfrak{g}_4|01\rangle)_{B_1 B_2} \\
+ \ & (|\Upsilon_4\rangle_{\mathfrak{a}_1 A_1} \otimes |\Upsilon_4\rangle_{\mathfrak{a}_2 A_2}) \otimes (\mathfrak{g}_1|10\rangle - \mathfrak{g}_2|11\rangle + \mathfrak{g}_3|00\rangle - \mathfrak{g}_4|01\rangle)_{B_1 B_2}.
\end{aligned}
$$

$$(13.35)$$

After the execution of Alice's measurement, she sends her results to Bob via 4-bit of classical channel. Once Bob obtains classical messages from Alice, he performs an appropriate unitary operation on his qubits to recover the intended state. The details of the operations are given in Table 13.2.

Table 13.2

Bob's unitary operations conditioned on Alice's outcomes ⏎

Alice's outcome	Bob's operation
$|\Upsilon_1\rangle_{\mathfrak{a}_1 A_1}|\Upsilon_1\rangle_{\mathfrak{a}_2 A_2}$	$I_{B_1} \otimes (\vartheta_x)_{B_2}$
$|\Upsilon_1\rangle_{\mathfrak{a}_1 A_1}|\Upsilon_2\rangle_{\mathfrak{a}_2 A_2}$	$I_{B_1} \otimes (\vartheta_z \vartheta_x)_{B_2}$

Alice's outcome	Bob's operation
$\lvert \Upsilon_1 \rangle_{\mathfrak{a}_1 A_1} \lvert \Upsilon_3 \rangle_{\mathfrak{a}_2 A_2}$	$I_{B_1} \otimes (I)_{B_2}$
$\lvert \Upsilon_1 \rangle_{\mathfrak{a}_1 A_1} \lvert \Upsilon_4 \rangle_{\mathfrak{a}_2 A_2}$	$I_{B_1} \otimes (\vartheta_z)_{B_2}$
$\lvert \Upsilon_2 \rangle_{\mathfrak{a}_1 A_1} \lvert \Upsilon_1 \rangle_{\mathfrak{a}_2 A_2}$	$(\vartheta_z)_{B_1} \otimes (\vartheta_x)_{B_2}$
$\lvert \Upsilon_2 \rangle_{\mathfrak{a}_1 A_1} \lvert \Upsilon_2 \rangle_{\mathfrak{a}_2 A_2}$	$(\vartheta_z)_{B_1} \otimes (\vartheta_z \vartheta_x)_{B_2}$
$\lvert \Upsilon_2 \rangle_{\mathfrak{a}_1 A_1} \lvert \Upsilon_3 \rangle_{\mathfrak{a}_2 A_2}$	$(\vartheta_z)_{B_1} \otimes (I)_{B_2}$
$\lvert \Upsilon_2 \rangle_{\mathfrak{a}_1 A_1} \lvert \Upsilon_4 \rangle_{\mathfrak{a}_2 A_2}$	$(\vartheta_z)_{B_1} \otimes (\vartheta_z)_{B_2}$
$\lvert \Upsilon_3 \rangle_{\mathfrak{a}_1 A_1} \lvert \Upsilon_1 \rangle_{\mathfrak{a}_2 A_2}$	$(\vartheta_z \vartheta_x)_{B_1} \otimes (\vartheta_x)_{B_2}$
$\lvert \Upsilon_3 \rangle_{\mathfrak{a}_1 A_1} \lvert \Upsilon_2 \rangle_{\mathfrak{a}_2 A_2}$	$(\vartheta_z \vartheta_x)_{B_1} \otimes (\vartheta_z \vartheta_x)_{B_2}$
$\lvert \Upsilon_3 \rangle_{\mathfrak{a}_1 A_1} \lvert \Upsilon_3 \rangle_{\mathfrak{a}_2 A_2}$	$(\vartheta_z \vartheta_x)_{B_1} \otimes (I)_{B_2}$
$\lvert \Upsilon_3 \rangle_{\mathfrak{a}_1 A_1} \lvert \Upsilon_4 \rangle_{\mathfrak{a}_2 A_2}$	$(\vartheta_z \vartheta_x)_{B_1} \otimes (\vartheta_z)_{B_2}$
$\lvert \Upsilon_4 \rangle_{\mathfrak{a}_1 A_1} \lvert \Upsilon_1 \rangle_{\mathfrak{a}_2 A_2}$	$(\vartheta_x)_{B_1} \otimes (\vartheta_x)_{B_2}$
$\lvert \Upsilon_4 \rangle_{\mathfrak{a}_1 A_1} \lvert \Upsilon_2 \rangle_{\mathfrak{a}_2 A_2}$	$(\vartheta_x)_{B_1} \otimes (\vartheta_z \vartheta_x)_{B_2}$
$\lvert \Upsilon_4 \rangle_{\mathfrak{a}_1 A_1} \lvert \Upsilon_3 \rangle_{\mathfrak{a}_2 A_2}$	$(\vartheta_x)_{B_1} \otimes (I)_{B_2}$
$\lvert \Upsilon_4 \rangle_{\mathfrak{a}_1 A_1} \lvert \Upsilon_4 \rangle_{\mathfrak{a}_2 A_2}$	$(\vartheta_x)_{B_1} \otimes (\vartheta_z)_{B_2}$

As an illustration, assume that Alice's measurement yields $\lvert \Upsilon_3 \rangle_{\mathfrak{a}_1 A_1} \otimes \lvert \Upsilon_2 \rangle_{\mathfrak{a}_2 A_2}$. Then the state of the Bob's qubits becomes

$$(\mathfrak{g}_1 \lvert 11 \rangle - \mathfrak{g}_2 \lvert 10 \rangle - \mathfrak{g}_3 \lvert 01 \rangle + \mathfrak{g}_4 \lvert 00 \rangle)_{B_1 B_2}$$

with success probability $\frac{\lvert \tau_2 \rvert^2}{8}$ and finally Bob performs an appropriate unitary operation $(\vartheta_z \vartheta_x)_{B_1} \otimes (\vartheta_z \vartheta_x)_{B_2}$ given from Table 13.2 to recover the original state. The total success probability of the protocol for the case I is

$$\frac{\lvert \tau_2 \rvert^2}{8} \times 16 = 2 \lvert \tau_2 \rvert^2.$$

Case II

If the measurement result of Alice is $\lvert 11 \rangle_{q_1 q_2}$, the state $\lvert \Gamma_4' \rangle$ given in Eq. (13.33) becomes

$$\begin{aligned} |\Gamma_4'''\rangle = & \frac{1}{\sqrt{2}} \Big(\mathfrak{g}_1 |11\rangle_{\mathfrak{a}_1\mathfrak{a}_2} |00\rangle_{A_1A_2} |01\rangle_{B_1B_2} + \mathfrak{g}_1 |11\rangle_{\mathfrak{a}_1\mathfrak{a}_2} |01\rangle_{A_1A_2} |00\rangle_{B_1B_2} \\ & + \mathfrak{g}_2 |10\rangle_{\mathfrak{a}_1\mathfrak{a}_2} |01\rangle_{A_1A_2} |00\rangle_{B_1B_2} + \mathfrak{g}_2 |10\rangle_{\mathfrak{a}_1\mathfrak{a}_2} |00\rangle_{A_1A_2} |01\rangle_{B_1B_2} \\ & + \mathfrak{g}_3 |01\rangle_{\mathfrak{a}_1\mathfrak{a}_2} |00\rangle_{A_1A_2} |01\rangle_{B_1B_2} + \mathfrak{g}_3 |01\rangle_{\mathfrak{a}_1\mathfrak{a}_2} |01\rangle_{A_1A_2} |00\rangle_{B_1B_2} \\ & + \mathfrak{g}_4 |00\rangle_{\mathfrak{a}_1\mathfrak{a}_2} |01\rangle_{A_1A_2} |00\rangle_{B_1B_2} + \mathfrak{g}_4 |00\rangle_{\mathfrak{a}_1\mathfrak{a}_2} |00\rangle_{A_1A_2} |01\rangle_{B_1B_2} \Big) \\ = & \;\; (\mathfrak{g}_1 |11\rangle + \mathfrak{g}_2 |10\rangle + \mathfrak{g}_3 |01\rangle + \mathfrak{g}_4 |00\rangle)_{\mathfrak{a}_1\mathfrak{a}_2} |00\rangle_{A_1B_1} (|01\rangle + |10\rangle)_{A_2B_2}. \end{aligned}$$

$$(13.36)$$

The probability of getting this result is $(1 - 2|\tau_2|^2)$. We see from Eq. (13.36) that three pairs of qubits $(\mathfrak{a}_1, \mathfrak{a}_2), (A_1, B_1)$ and (A_2, B_2) are decoupled. Qubits (A_2, B_2) recover to the initial Bell state of Eq. (13.26). Qubits $(\mathfrak{a}_1, \mathfrak{a}_2)$ become a two-qubit arbitrary entangled state different from Eq. (13.24). Such a result means that our teleportation process fails. However, Alice can recover the initial state $|\aleph\rangle_{\mathfrak{a}_1\mathfrak{a}_2}$ by local operations $(\vartheta_x)_{\mathfrak{a}_1} \otimes (\vartheta_x)_{\mathfrak{a}_2}$ on her qubits $\mathfrak{a}_1$ and $\mathfrak{a}_2$. This is the specific meaning of probabilistic resumable quantum teleportation scheme, that is, the initial state to be teleported can be recovered by the sender when probabilistic teleportation fails. This ensures that the teleportation process can be repeatedly performed between the sender and the receiver until it succeeds. But at each time of its repetition, we need to have fresh quantum resource for use.

Part III

Teleportation in Noisy Environment

14 Teleportation Under Noisy Environments

14.1 INTRODUCTION

In the actual quantum communication process, there will inevitably be noise in the quantum channel. In this chapter, we consider teleportation protocols in noisy environment. There are different types of noises, which are amplitude-damping, bit-flip, phase-flip, phase-damping, depolarizing noise, etc. Throughout this chapter, four types of noises are considered, namely amplitude-damping, bit-flip, phase-flip, phase-damping, under which teleportation without and with control are presented. The analysis of fidelity of the process against the variation of noise parameters are also discussed. Teleportation through noisy environment have been discussed in a good number of papers like [43, 57, 60, 63, 69, 79, 92, 96, 114, 178, 198, 204].

14.2 TELEPORTATION OF AN ARBITRARY SINGLE-QUBIT STATE UNDER NOISY ENVIRONMENT

In this section, we consider the protocol which is discussed in Chapter 8 under Subsection 8.2.

Suppose that the sender Alice prepares the entangled resource given in Eq. (8.2) and distributes the qubit to the receiver Bob through a noisy environment. The particle A belonging to Alice is not affected by the noise of the environment whereas the particle B is affected by the environmental noise. Kraus operator is used here to characterize the different types of noises. The density matrix corresponding to the quantum state $|\aleph\rangle_{\mathfrak{a}}$ given in Eq. (8.1) can be written as $\varpi_{\mathfrak{a}} = |\aleph\rangle_{\mathfrak{a}}\langle\aleph|$ and that of the quantum resource as

$\varpi_{AB} = |\Upsilon_1\rangle_{AB}\langle\Upsilon_1|$. For different noises in the channel, the evolution of the quantum resource under the effects of quantum noise can be expressed as

$$\varepsilon(\varpi) = \sum_i M_i\,\varpi_{AB}\,M_i^{\dagger},$$

(14.1)

where $M_i = I^A \otimes K_i^B$, and K_i s are the Kraus operators corresponding to different noises. Here, the superscripts denote the respective qubits and '†' denotes the conjugate transpose.

The output state of the protocol can be expressed as

$$\varpi_i^{out} = Tr_{\mathfrak{a}A}\{U_i[\varpi_{\mathfrak{a}} \otimes \varepsilon(\varpi)]U_i^{\dagger}\},$$

(14.2)

where $Tr_{\mathfrak{a}A}$ is the partial trace over the pairs of qubits $(\mathfrak{a}, A)$ and U_i, $i \in \{1, 2, 3, 4\}$ is given by

$$U_i = \{I_{\mathfrak{a}A} \otimes (\vartheta^i)_B\}\{|\Upsilon_i\rangle_{\mathfrak{a}A}\langle\Upsilon_i| \otimes I_B\}$$

with $|\Upsilon_i\rangle_{\mathfrak{a}A}\langle\Upsilon_i|$ being Alice's measurement results, and $(\vartheta^i)_B$ being Bob's corresponding recovery operation.

The influence of noise on quantum teleportation can be measured by fidelity which, as discussed in [Chapter 7](), is given by

$$\mathscr{F} =_B \langle\aleph|\varpi_i^{out}|\aleph\rangle_B,$$

(14.3)

where $|\aleph\rangle_B$ represents the ideal output state. Here, the ideal output state is

$$|\aleph\rangle_B = (\mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle).$$

If the fidelity is close to zero, then it indicates that a significant amount of information is lost due to environmental noise. On the other hand, the value of fidelity close to one implies that the communication is highly efficient and the transmitted quantum state is well preserved. In the following we consider different types of noises separately.

14.2.1 TELEPORTATION IN AMPLITUDE-DAMPING NOISY ENVIRONMENT

The Kraus operators of amplitude damping noise are expressed as

$$K_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}, K_1 = \begin{bmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{bmatrix}$$

where p is the intensity of the noise from amplitude damping.

According to the formula given in Eq. (14.1), the quantum resource changes to

$$\varepsilon_{Aamp-Damp}(\varpi)_{AB} = \frac{1}{2}\Big[\{|00\rangle + \sqrt{1-p}|11\rangle\} \times \{\langle 00| + \sqrt{1-p}\langle 11|\} + p|10\rangle\langle 10|\Big].$$

(14.4)

Now, the combined state of the whole system is given by

$$\varpi' = |\aleph\rangle_{\mathfrak{a}}\langle\aleph| \otimes \varepsilon_{AD}(\varpi)_{AB}.$$

(14.5)

As an illustration, suppose that Alice's measurement result is $|\Upsilon_4\rangle_{\mathfrak{a}A}$. Then the reduced density matrix of the final output state is given by

$$\varpi_4^{out-Amp-Damp} = Tr_{\mathfrak{a}A}\{U_4[\varpi_{\mathfrak{a}} \otimes \varepsilon_{Amp-Damp}(\varpi)]U_4^{\dagger}\},$$

(14.6)

where U_4 is given by

$$U_4 = \{I_{\mathfrak{a}A} \otimes (\vartheta_z\vartheta_x)_B\}\{|\Upsilon_4\rangle_{\mathfrak{a}A}\langle\Upsilon_4| \otimes I_B\}.$$

Therefore, the final output state is given by

$$\varpi_4^{out-Amp-Damp} = \frac{1}{N_1}\Big[(\mathfrak{g}_2|1\rangle + \mathfrak{g}_1\sqrt{1-p}|0\rangle)$$
$$\times (\mathfrak{g}_2\langle 1| + \mathfrak{g}_1\sqrt{1-p}\langle 0|) + \mathfrak{g}_1^2 p|1\rangle\langle 1|\Big],$$

(14.7)

where N_1 is given as $N_1 = \mathfrak{g}_1^2(1-p) + \mathfrak{g}_2^2 + \mathfrak{g}_1^2 p = 1$.

Now, according to the formula described in Eq. (14.3), the fidelity $\mathscr{F}$ is

$$
\begin{aligned}
\mathscr{F}^{Amp-Damp} &= \left[\mathfrak{g}_1^2\sqrt{1-p}+\mathfrak{g}_2^2\right]^2 + \mathfrak{g}_1^2\mathfrak{g}_2^2 p \\
&= \left[\mathfrak{g}_1^2\sqrt{1-p}-\mathfrak{g}_1^2+1\right]^2 + \mathfrak{g}_1^2(1-\mathfrak{g}_1^2)p.
\end{aligned}
$$

(14.8)

The variation of fidelity is given in .
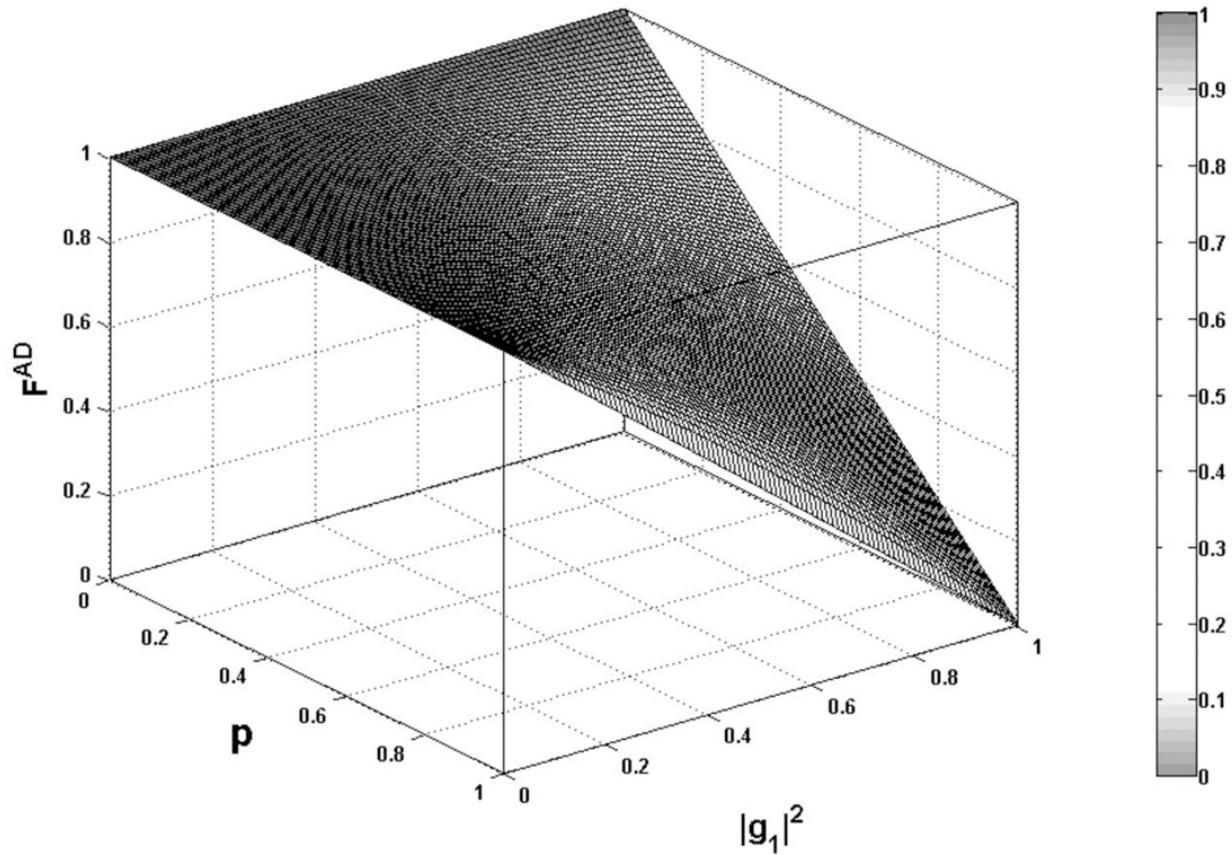
Figure 14.1 3-Dimensional surface plot for amplitude damping noise as a function of $|\mathfrak{g}_1|^2$ and noise intensity parameter p.

14.2.2 TELEPORTATION IN BIT-FLIP NOISY ENVIRONMENT

The Kraus operators of Bit-flip noise are expressed as

$$
K_0 = \begin{bmatrix} \sqrt{1-q} & 0 \\ 0 & \sqrt{1-q} \end{bmatrix}, K_1 = \begin{bmatrix} 0 & \sqrt{q} \\ \sqrt{q} & 0 \end{bmatrix}
$$

where q is the noise intensity parameter of bit-flip noise.

According to the formula in Eq. (14.1), the quantum resource becomes

$$\varepsilon_{Bit-Flip}(\varpi)_{AB} = \frac{1}{2}\Big[(1-q)(|00\rangle + |11\rangle) \times (\langle 00| + \langle 11|) + q(|01\rangle + |10\rangle) \times (\langle 01| + \langle 10|)\Big].$$

(14.9)

Now, the combined state of the whole system is given by

$$\varpi^{'} = |\aleph\rangle_{\mathfrak{a}}\langle\aleph| \otimes \varepsilon_{Bit-Flip}(\varpi)_{AB}.$$

(14.10)

As an illustration, suppose that Alice's measurement result is $|\Upsilon_4\rangle_{\mathfrak{a}A}$. Then the reduced density matrix of the final output state is given by

$$\varpi_4^{out-Bit-Flip} = Tr_{\mathfrak{a}A}\big\{U_4[\varpi_{\mathfrak{a}} \otimes (\varepsilon_{Bit-Flip}(\varpi))_{AB}]U_4^{\dagger}\big\}$$

(14.11)

where U_4 is given by

$$U_4 = \big\{I_{\mathfrak{a}A} \otimes (\vartheta_z\vartheta_x)_B\big\}\big\{|\Upsilon_4\rangle_{\mathfrak{a}A}\langle\Upsilon_4| \otimes I_B\big\}.$$

Then, the final output state is given by

$$\varpi_4^{out-Bit-Flip} = \frac{1}{N_2}\Big[(1-q)(\mathfrak{g}_2|1\rangle + \mathfrak{g}_1|0\rangle) \times (\mathfrak{g}_2\langle 1| + \mathfrak{g}_1\langle 0|) + q(-\mathfrak{g}_1|1\rangle - \mathfrak{g}_2|0\rangle) \times (-\mathfrak{g}_1\langle 1| - \mathfrak{g}_2\langle 0|)\Big],$$

(14.12)

where $N_2 = (\mathfrak{g}_1^2 + \mathfrak{g}_2^2)(1-q) + (\mathfrak{g}_2^2 + \mathfrak{g}_1^2)q = 1$.

Now, according to the formula in (14.3), the fidelity $\mathscr{F}$ is calculated as

$$\begin{aligned}\mathscr{F}^{Bit-Flip} &= \big[(\mathfrak{g}_1^2 + \mathfrak{g}_2^2)\sqrt{1-q}\big]^2 + 4\mathfrak{g}_1^2\mathfrak{g}_2^2 q \\ &= \big[(1-q) + 4\mathfrak{g}_1^2(1-\mathfrak{g}_1^2)q\big].\end{aligned}$$

The variation of fidelity is given in <u>Figure 14.2</u>.
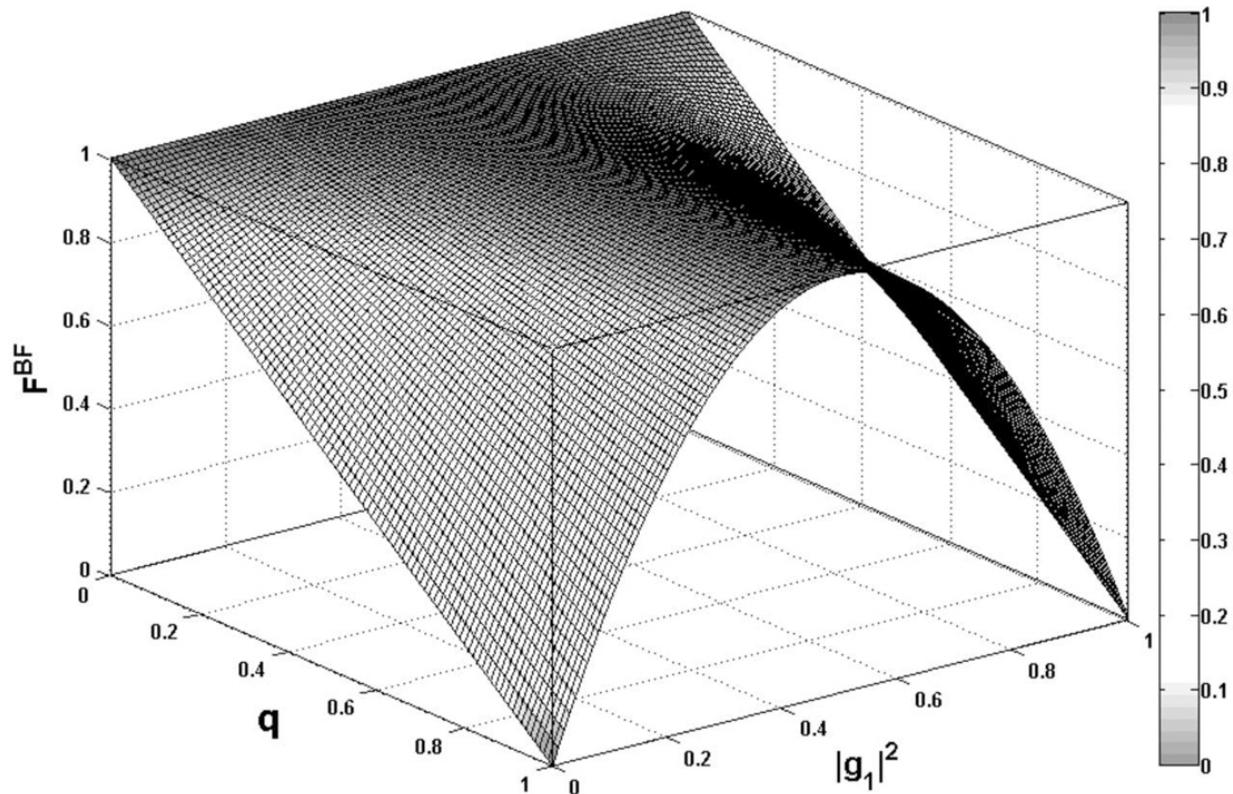
Figure 14.2 3-Dimensional surface plot for bit-flip noise as a function of $|g_1|^2$ and noise intensity parameter q.

14.2.3 TELEPORTATION IN PHASE-FLIP NOISY ENVIRONMENT

The Kraus operators of phase-flip noise are expressed as

$$K_0 = \begin{bmatrix} \sqrt{1-r} & 0 \\ 0 & \sqrt{1-r} \end{bmatrix}, K_1 = \begin{bmatrix} \sqrt{r} & 0 \\ 0 & -\sqrt{r} \end{bmatrix},$$

where r is the noise intensity of phase-flip noise.

According to formula (14.1), the quantum resource becomes

$$\varepsilon_{Phase-Flip}(\varpi)_{AB} = \frac{1}{2}\Big[(1-r)(|00\rangle + |11\rangle) \times (\langle 00| + \langle 11|)$$
$$+ r(|00\rangle - |11\rangle) \times (\langle 00| - \langle 11|)\Big]$$
$$= \frac{1}{2}\Big[(|00\rangle + |11\rangle) \times (\langle 00| + \langle 11|)\Big].$$

(14.14)

We see that for any value of $r \in [0\,1]$, the state of the quantum resource remains as in the noiseless case. So, the protocol remains unaffected by the phase-flip noise.

A remarkable feature with this case is that it demonstrates the fact that teleportation protocols can be unaffected by noise in some cases.

14.2.4 TELEPORTATION IN PHASE-DAMPING NOISY ENVIRONMENT

The Kraus operators of phase-damping noise are expressed as:

$$K_0 = \begin{bmatrix} \sqrt{1-s} & 0 \\ 0 & \sqrt{1-s} \end{bmatrix}, K_1 = \begin{bmatrix} \sqrt{s} & 0 \\ 0 & 0 \end{bmatrix}, K_2 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{s} \end{bmatrix},$$

where s is the noise intensity of phase-damping noise.

According to formula (14.1), the quantum resource changes to

$$\varepsilon_{Phase-Damp}(\varpi)_{AB} = \frac{1}{2}\Big[(1-s)(|00\rangle + |11\rangle) \times (\langle 00| + \langle 11|)$$
$$+ s(|00\rangle \times \langle 00| + |11\rangle \times \langle 11|)\Big].$$

(14.15)

Now, the combined state of the whole system is given by

$$\varpi' = |\aleph\rangle_\mathfrak{a}\langle\aleph| \otimes (\varepsilon_{Phase-Damp}(\varpi))_{AB}.$$

(14.16)

As an illustration, suppose that Alice's measurement result is $|\Upsilon_4\rangle_{\mathfrak{a}A}$. Then the reduced density matrix of the final output state is given by

$$\varpi_4^{out-Phase-Damp} = Tr_{\mathfrak{a}A}\{U_4[\varpi_a \otimes (\varepsilon_{Phase-Damp}(\varpi))_{AB}]U_4^\dagger\}$$

$$(14.17)$$

where U_4 is given by

$$U_4 = \left\{ I_{\mathfrak{a}A} \otimes (\vartheta_z \vartheta_x)_B \right\} \left\{ |\Upsilon_4\rangle_{\mathfrak{a}A} \langle \Upsilon_4| \otimes I_B \right\}.$$

Then, the final output state is given by

$$
\varpi_4^{out-Phase-Damp} = \frac{1}{N_3} \Big[(1-s)(\mathfrak{g}_2|1\rangle + \mathfrak{g}_1|0\rangle) \times (\mathfrak{g}_2\langle 1| + \mathfrak{g}_1\langle 0|) \\
+ s(\mathfrak{g}_2^2|1\rangle\langle 1| + \mathfrak{g}_1^2|0\rangle\langle 0|) \Big],
$$

$$(14.18)$$

where $N_3 = (\mathfrak{g}_1^2 + \mathfrak{g}_2^2)(1-s) + (\mathfrak{g}_2^2 + \mathfrak{g}_1^2)s = 1$.

Now, according to the formula provided in Eq. (14.3), the fidelity $\mathscr{F}$ is calculated as

$$
\begin{aligned}
\mathscr{F}^{Phase-Damp} &= \left[(\mathfrak{g}_1^2 + \mathfrak{g}_2^2)\sqrt{1-s} \right]^2 + (\sqrt{s}\mathfrak{g}_1^2)^2 + (\sqrt{s}\mathfrak{g}_2^2)^2 \\
&= \left[(1-s) + s(\mathfrak{g}_1^4 + (1-\mathfrak{g}_1^2)^2) \right].
\end{aligned}
$$

$$(14.19)$$

The variation of fidelity is given in Figure 14.3.
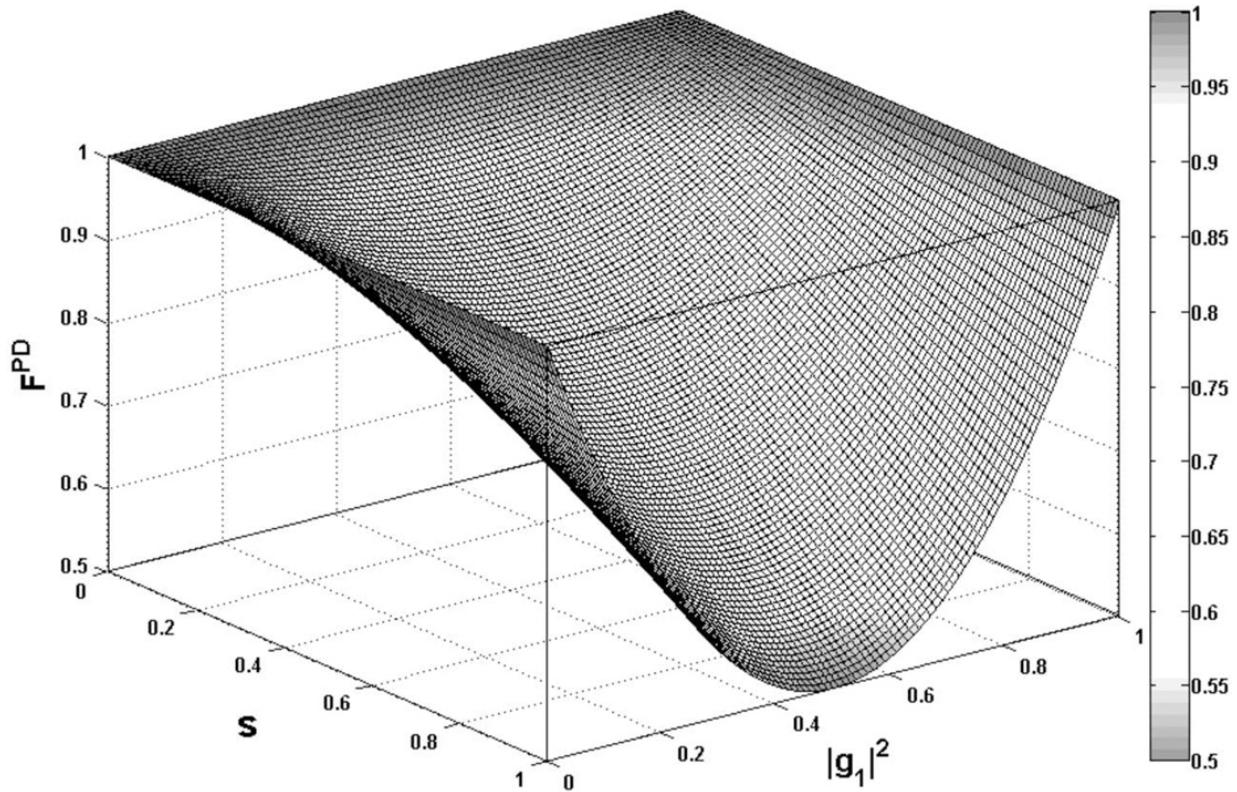
Figure 14.3 3-Dimensional surface plot for phase-damping noise as a function of $|g_1|^2$ and noise intensity parameter s.

A study of Figure 14.1, Figure 14.2, Figure 14.3 reveal the expected feature that the fidelity tends to unit value as the noise parameter tends to zero.

In the above the fidelity analysis is done only with respect to one of the four alternative results in Alice's measurement. The other three cases can be similarly investigated.

14.3 CONTROLLED TELEPORTATION PROTOCOL OF 2-QUBIT STATE UNDER NOISY ENVIRONMENT

In this section, we demonstrate a controlled teleportation protocol of 2-qubit state in a noisy environment. Three parties, namely Alice, Bob, and David, are located in three different places. Alice plays the role of sender and Bob acts as receiver, whereas David plays the role of controller. The protocol has two parts. In the first part, we discuss the protocol under ideal conditions without noise and in the second part, we discuss teleportation through noisy environment.

Part 1: Teleportation in an ideal environment

Alice wishes to send an unknown 2-qubit state to Bob given by

$$|\aleph\rangle_{a_1 a_2} = (\mathfrak{g}_1|01\rangle + \mathfrak{g}_2|10\rangle),$$

(14.20)

where $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are unknown coefficients for Alice and meets the normalization condition

$$|\mathfrak{g}_1|^2 + |\mathfrak{g}_2|^2 = 1.$$

For this purpose, a 4-qubit entangled state is shared amongst Alice, Bob and David which is given by

$$|E\rangle_{AB_1 B_2 D} = \frac{1}{\sqrt{2}}(|0100\rangle + |1011\rangle),$$

(14.21)

where Alice possesses the qubit A, Bob owns the qubits $\{B_1, B_2\}$ and the single qubit D belongs to the controller David. The circuit generation for the entangled state (14.21) is given in Figure 14.4.
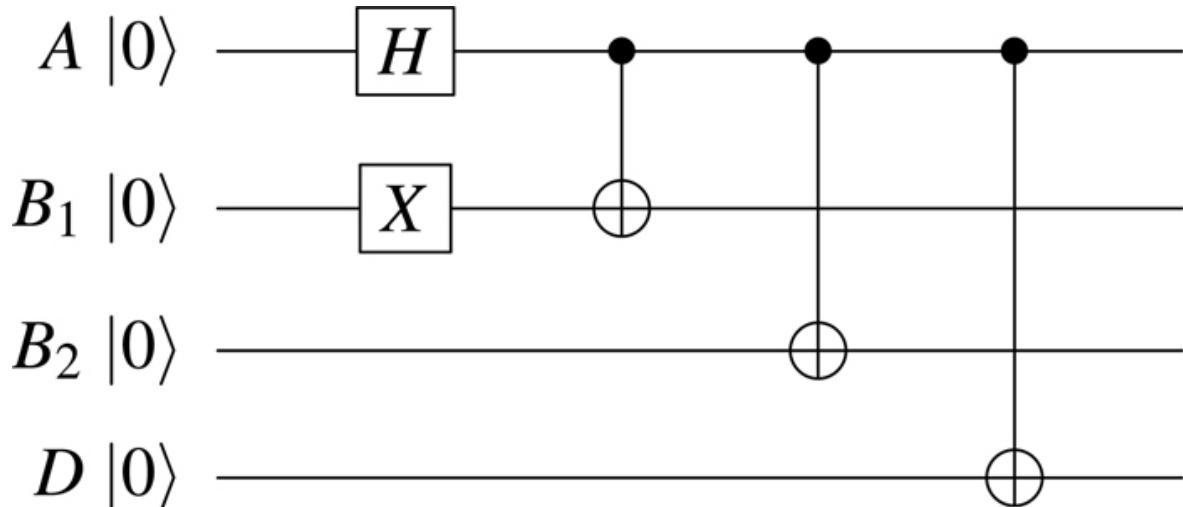
Figure 14.4 Circuit diagram for the generation of the entangled resource given in Eq. (14.21). ↵

Also it is assumed that all the parties are connected by classical communication channels.

The complete quantum system can be written as

$$
\begin{aligned}
|\Gamma\rangle &= |\aleph\rangle_{\mathfrak{a}_1 \mathfrak{a}_2} \otimes |E\rangle_{AB_1 B_2 D} \\
&= (\mathfrak{g}_1|01\rangle + \mathfrak{g}_2|10\rangle)_{\mathfrak{a}_1 \mathfrak{a}_2} \otimes \frac{1}{\sqrt{2}} (|0100\rangle + |1011\rangle)_{AB_1 B_2 D} \\
&= \frac{1}{\sqrt{2}} \Big(\mathfrak{g}_1|010100\rangle + \mathfrak{g}_1|011011\rangle + \mathfrak{g}_2|100100\rangle + \mathfrak{g}_2|101011\rangle \Big)_{\mathfrak{a}_1 \mathfrak{a}_2 AB_1 B_2 D}.
\end{aligned}
$$

$$(14.22)$$

Alice measures her qubits $(\mathfrak{a}_1, \mathfrak{a}_2, A)$ in the basis given by

$$
\begin{aligned}
|\varsigma_1\rangle_{\mathfrak{a}_1 \mathfrak{a}_2 A} &= \frac{|000\rangle + |111\rangle}{\sqrt{2}}, & |\varsigma_2\rangle_{\mathfrak{a}_1 \mathfrak{a}_2 A} &= \frac{|000\rangle - |111\rangle}{\sqrt{2}}, \\
|\varsigma_3\rangle_{\mathfrak{a}_1 \mathfrak{a}_2 A} &= \frac{|001\rangle + |110\rangle}{\sqrt{2}}, & |\varsigma_4\rangle_{\mathfrak{a}_1 \mathfrak{a}_2 A} &= \frac{|001\rangle - |110\rangle}{\sqrt{2}}, \\
|\varsigma_5\rangle_{\mathfrak{a}_1 \mathfrak{a}_2 A} &= \frac{|010\rangle + |101\rangle}{\sqrt{2}}, & |\varsigma_6\rangle_{\mathfrak{a}_1 \mathfrak{a}_2 A} &= \frac{|010\rangle - |101\rangle}{\sqrt{2}}, \\
|\varsigma_7\rangle_{\mathfrak{a}_1 \mathfrak{a}_2 A} &= \frac{|011\rangle + |100\rangle}{\sqrt{2}}, & |\varsigma_8\rangle_{\mathfrak{a}_1 \mathfrak{a}_2 A} &= \frac{|011\rangle - |100\rangle}{\sqrt{2}}.
\end{aligned}
$$

$$(14.23)$$

After the measurement, Alice sends her result to Bob and David using a classical channels. After receiving the classical information from Alice, David starts his job by checking the whole protocol. Once satisfied, he immediately executes a single-qubit measurement on the basis given by

$$
\begin{aligned}
|\zeta_1\rangle_D &= \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \\
|\zeta_2\rangle_D &= \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle).
\end{aligned}
$$

$$(14.24)$$

With the basis given in Eq. (14.23), the complete quantum system (14.22) can be rewritten as

$$|\Gamma\rangle = \frac{1}{\sqrt{2}}\Big(\mathfrak{g}_1|010100\rangle + \mathfrak{g}_1|011011\rangle + \mathfrak{g}_2|100100\rangle + \mathfrak{g}_2|101011\rangle\Big)_{\mathfrak{a}_1\mathfrak{a}_2AB_1B_2D}$$

$$= |\varsigma_5\rangle_{\mathfrak{a}_1\mathfrak{a}_2A} \otimes \Big(\mathfrak{g}_1|100\rangle + \mathfrak{g}_2|011\rangle\Big)_{B_1B_2D} + |\varsigma_6\rangle_{\mathfrak{a}_1\mathfrak{a}_2A} \otimes \Big(\mathfrak{g}_1|100\rangle - \mathfrak{g}_2|011\rangle\Big)_{B_1B_2D}$$

$$+ |\varsigma_7\rangle_{\mathfrak{a}_1\mathfrak{a}_2A} \otimes \Big(\mathfrak{g}_1|011\rangle + \mathfrak{g}_2|100\rangle\Big)_{B_1B_2D} + |\varsigma_8\rangle_{\mathfrak{a}_1\mathfrak{a}_2A} \otimes \Big(\mathfrak{g}_1|011\rangle - \mathfrak{g}_2|100\rangle\Big)_{B_1B_2D}.$$

$$(14.25)$$

Case I:

If the result of Alice's measurement yields $|\varsigma_5\rangle_{\mathfrak{a}_1\mathfrak{a}_2A}$, then the remaining qubits are in the state

$$|\Gamma_1\rangle = \Big(\mathfrak{g}_1|100\rangle + \mathfrak{g}_2|011\rangle\Big)_{B_1B_2D}.$$

Using the basis given in Eq. (14.24), the above state becomes

$$|\Gamma_1\rangle = |\zeta_1\rangle_D \otimes (\mathfrak{g}_1|10\rangle + \mathfrak{g}_2|01\rangle)_{B_1B_2} + |\zeta_2\rangle_D \otimes (\mathfrak{g}_1|10\rangle - \mathfrak{g}_2|01\rangle)_{B_1B_2}.$$

After the completion of David's measurement, he sends his outcomes to Bob via 1-bit classical channel.

If David's outcome is $|\zeta_1\rangle_D$, then the state at Bob's qubit becomes $(\mathfrak{g}_1|10\rangle + \mathfrak{g}_2|01\rangle)_{B_1B_2}$. To recover the original state, Bob executes the Pauli operation, which is $(\vartheta_x)_{B_1} \otimes (\vartheta_x)_{B_2}$.

If David's outcome is $|\zeta_2\rangle_D$, then the state at Bob's qubit becomes $(\mathfrak{g}_1|10\rangle - \mathfrak{g}_2|01\rangle)_{B_1B_2}$. To recover the original state, Bob executes the Pauli operation, which is $(\vartheta_x)_{B_1} \otimes (\vartheta_x\vartheta_z)_{B_2}$.

Case II:

If the result of Alice's measurement yields $|\varsigma_6\rangle_{\mathfrak{a}_1\mathfrak{a}_2A}$, then the remaining qubits are in the state

$$|\Gamma_2\rangle = \Big(\mathfrak{g}_1|100\rangle - \mathfrak{g}_2|011\rangle\Big)_{B_1B_2D}.$$

Using the basis given in Eq. (14.24), the above state becomes

$$|\Gamma_2\rangle = |\zeta_1\rangle_D \otimes (\mathfrak{g}_1|10\rangle - \mathfrak{g}_2|01\rangle)_{B_1B_2} + |\zeta_2\rangle_D \otimes (\mathfrak{g}_1|10\rangle + \mathfrak{g}_2|01\rangle)_{B_1B_2}.$$

After the completion of David's measurement, he sends his outcomes to Bob via 1-bit classical channel.

If David's outcome is $|\zeta_1\rangle_D$, then the state at Bob's qubit becomes $(\mathfrak{g}_1|10\rangle + \mathfrak{g}_2|01\rangle)_{B_1B_2}$. To recover the original state, Bob executes the Pauli operation, which is $(\vartheta_x)_{B_1} \otimes (\vartheta_x\vartheta_z)_{B_2}$.

If David's outcome is $|\zeta_2\rangle_D$, then the state at Bob's qubit becomes $(\mathfrak{g}_1|10\rangle + \mathfrak{g}_2|01\rangle)_{B_1B_2}$. To recover the original state, Bob executes the Pauli operation, which is $(\vartheta_x)_{B_1} \otimes (\vartheta_x)_{B_2}$.

Case III:

If the result of Alice's measurement yields $|\varsigma_7\rangle_{\mathfrak{a}_1\mathfrak{a}_2A}$, then the remaining qubits are in the state

$$|\Gamma_3\rangle = \Big(\mathfrak{g}_1|011\rangle + \mathfrak{g}_2|100\rangle\Big)_{B_1B_2D}.$$

Using the basis given in Eq. (14.24), the above state becomes

$$|\Gamma_3\rangle = |\zeta_1\rangle_D \otimes (\mathfrak{g}_1|01\rangle + \mathfrak{g}_2|10\rangle)_{B_1B_2} + |\zeta_2\rangle_D \otimes (-\mathfrak{g}_1|01\rangle + \mathfrak{g}_2|10\rangle)_{B_1B_2}.$$

After the completion of David's measurement, he sends his outcomes to Bob via 1-bit classical channel.

If David's outcome is $|\zeta_1\rangle_D$, then the state at Bob's qubit becomes $(\mathfrak{g}_1|01\rangle + \mathfrak{g}_2|10\rangle)_{B_1B_2}$. To recover the original state, Bob executes identity operation on his qubits, which is $(I)_{B_1} \otimes (I)_{B_2}$ which means that Bob need not act.

If David's outcome is $|\zeta_2\rangle_D$, then the state at Bob's qubit becomes $(-\mathfrak{g}_1|01\rangle + \mathfrak{g}_2|10\rangle)_{B_1B_2}$. To recover the original state, Bob executes the Pauli operation, which is $(I)_{B_1} \otimes (\vartheta_z)_{B_2}$.

Case IV:

If the result of Alice's measurement yields $|\varsigma_8\rangle_{\mathfrak{a}_1\mathfrak{a}_2A}$, then the remaining qubits are in the state

$$|\Gamma_4\rangle = \Big(\mathfrak{g}_1|011\rangle - \mathfrak{g}_2|100\rangle\Big)_{B_1B_2D}.$$

Using the basis given in Eq. (14.24), the above state becomes

$$|\Gamma_4\rangle = |\zeta_1\rangle_D \otimes (\mathfrak{g}_1|01\rangle - \mathfrak{g}_2|10\rangle)_{B_1B_2} + |\zeta_2\rangle_D \otimes (-\mathfrak{g}_1|01\rangle - \mathfrak{g}_2|10\rangle)_{B_1B_2}.$$

After the completion of David's measurement, he sends his outcomes to Bob via 1-bit classical channel.

If David's outcome is $|\zeta_1\rangle_D$, then the state at Bob's qubit becomes $(\mathfrak{g}_1|01\rangle - \mathfrak{g}_2|10\rangle)_{B_1 B_2}$. To recover the original state, Bob executes identity operation on his qubits, which is $(\vartheta_z)_{B_1} \otimes (I)_{B_2}$.

If David's outcome is $|\zeta_2\rangle_D$, then the state at Bob's qubit becomes $(-\mathfrak{g}_1|01\rangle - \mathfrak{g}_2|10\rangle)_{B_1 B_2}$. To recover the original state, Bob executes the Pauli operation, which is $(\vartheta_z)_{B_1} \otimes (\vartheta_z)_{B_2}$.

This is the description of the perfect protocol.

Part 2: Teleportation in a noisy environment

In this part, the same protocol discussed in Part 1 is analyzed in the presence of environmental noise. The 4-qubit entangled state given in Eq. (14.21) is used as the quantum resource. We suppose that the controller David produces the entangled resource in his laboratory and circulates the required particles to the other parties through noisy environment. The particle D belonging to David is not affected by environmental noise, whereas the particles A, B_1, and B_2 are affected by environmental noise. We consider four different types of noises which are given by Kraus operators discussed in Chapter 6. The density matrix of the quantum resource can be described as

$$\varpi_{AB_1B_2D} = |E\rangle_{AB_1B_2D}\langle E|$$

and that of the intended state given in Eq. (14.20) as $\varpi_{\mathfrak{a}_1\mathfrak{a}_2} = |\aleph\rangle_{\mathfrak{a}_1\mathfrak{a}_2}\langle\aleph|$.

The evolution of the quantum resource under the effects of quantum noise can be expressed as follows:

$$\varepsilon(\varpi) = \sum_{i,j,k}(K_i^A \otimes K_j^{B_1} \otimes K_k^{B_2} \otimes I^D)\,\varpi_{AB_1B_2D}\,(K_i^A \otimes K_j^{B_1} \otimes K_k^{B_2} \otimes I^D)^\dagger,$$

(14.26)

where K_is are the Kraus operators corresponding to the type of existing noise and meet the completeness criteria which is

$$\sum(K_i^A \otimes K_j^{B_1} \otimes K_k^{B_2} \otimes I^D)^\dagger(K_i^A \otimes K_j^{B_1} \otimes K_k^{B_2} \otimes I^D) = I.$$

The output state of the protocol can be written as

$$\varpi_{lm}^{out} = Tr_{\mathfrak{a}_1 \mathfrak{a}_2 AD} \{ U_{lm} [\varpi_{\mathfrak{a}_1 \mathfrak{a}_2 \mathfrak{a}_3} \otimes \varepsilon(\varpi)] U_{lm}^{\dagger} \}$$

(14.27)

where $Tr_{\mathfrak{a}_1 \mathfrak{a}_2 AD}$ is the partial trace over qubits $(\mathfrak{a}_1, \mathfrak{a}_2, A, D)$ and U_{lm} is given by

$$\begin{aligned} U_{lm} = \quad & \{ I_{\mathfrak{a}_1 \mathfrak{a}_2 A} \otimes (\vartheta^{lm})_{B_1 B_2} \otimes I_D \} \\ & \{ I_{\mathfrak{a}_1 \mathfrak{a}_2 A} \otimes \otimes I_{B_1 B_2} \otimes |\zeta_m\rangle_D \langle \zeta_m| \} \\ & \{ |\varsigma_l\rangle_{\mathfrak{a}_1 \mathfrak{a}_2 A} \langle \varsigma_l| \otimes I_{B_1 B_2} \otimes I_D \} \end{aligned}$$

(14.28)

where $l \in \{1, 2, \ldots, 8\}$ and $m \in \{1, 2\}$ with $|\varsigma_l\rangle_{\mathfrak{a}_1 \mathfrak{a}_2 A} \langle \varsigma_l|$ and $|\zeta_m\rangle_D \langle \zeta_m|$ representing the corresponding Alice's measurement results and the controller David's measurement result, respectively, and $(\vartheta^{lm})_{B_1 B_2}$ being Bob's appropriate unitary operation. In view of the expression (14.25) only four outcomes $|\varsigma_l\rangle_{\mathfrak{a}_1 \mathfrak{a}_2 A} \langle \varsigma_l|$ $(l = 5, 6, 7, 8)$ are possible from the measurement of Alice.

The influence of noise on quantum teleportation can be measured by fidelity $\mathscr{F}$. The definition of fidelity is based on the inner product between the output state and the ideal output state, which is given by

$$\mathscr{F} = _{B_1 B_2} \langle \aleph | \varpi_{lm}^{out} | \aleph \rangle_{B_1 B_2}$$

(14.29)

where $|\aleph\rangle_{B_1 B_2}$ represents the ideal output state. Here, the ideal output state is

$$|\aleph\rangle_{B_1 B_2} = (\mathfrak{g}_1 |01\rangle + \mathfrak{g}_2 |10\rangle).$$

14.3.1 CONTROLLED TELEPORTATION IN AMPLITUDE-DAMPING NOISY ENVIRONMENT

The Kraus operators of the amplitude damping noise are defined as:

$$K_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{bmatrix}, K_1 = \begin{bmatrix} 0 & \sqrt{\lambda} \\ 0 & 0 \end{bmatrix}$$

where λ is the strength of the amplitude damping noise.

According to formula given in Eq. (14.26), the quantum resource becomes

$$\varepsilon_{Amp-Damp}(\varpi) = \frac{1}{4}\left[\left(\sqrt{\frac{1-\lambda}{2}}|0100\rangle + \frac{1-\lambda}{\sqrt{2}}|1011\rangle\right) \times \left(\sqrt{\frac{1-\lambda}{2}}\langle 0100|\right.\right.$$

$$\left. + \frac{1-\lambda}{\sqrt{2}}\langle 1011|\right) + \left(\sqrt{\frac{\lambda(1-\lambda)}{2}}|1001\rangle\right) \times \left(\sqrt{\frac{\lambda(1-\lambda)}{2}}\langle 1001|\right)$$

$$+ \left(\sqrt{\frac{\lambda}{2}}|0000\rangle\right) \times \left(\sqrt{\frac{\lambda}{2}}\langle 0000|\right) + \left(\sqrt{\frac{\lambda(1-\lambda)}{2}}|0011\rangle\right)$$

$$\times \left(\sqrt{\frac{\lambda(1-\lambda)}{2}}\langle 0011|\right) + \left(\frac{\lambda}{\sqrt{2}}|0001\rangle\right) \times \left(\frac{\lambda}{\sqrt{2}}\langle 0001|\right)\right].$$

$$(14.30)$$

Now, the composite state of the whole system is given by

$$\varpi' = |\aleph\rangle_{\mathfrak{a}_1\mathfrak{a}_2}\langle \aleph| \otimes (\varepsilon_{Amp-Damp}(\varpi))_{AB_1B_2D}.$$

$$(14.31)$$

As in the illustration in [part 1](#) of this section, let us assume that Alice's measurement outcome yields $|\varsigma_7\rangle_{\mathfrak{a}_1\mathfrak{a}_2 A}$ and that David's measurement outcome is $|\zeta_2\rangle_D$. Then the density matrix of the final output state is given by

$$\varpi_{72}^{out-Amp-Damp} = Tr_{\mathfrak{a}_1\mathfrak{a}_2 AD}\left\{U_{72}[\varpi_{\mathfrak{a}_1\mathfrak{a}_2} \otimes \varepsilon_{Amp-Damp}(\varpi)]U_{72}^\dagger\right\}$$

$$(14.32)$$

where U_{72} is given by

$$U_{72} = \left\{I_{\mathfrak{a}_1\mathfrak{a}_2 A} \otimes [(I)_{B_1} \otimes (\vartheta_z)_{B_2}] \otimes I_D\right\}$$
$$\left\{I_{\mathfrak{a}_1\mathfrak{a}_2 A} \otimes \otimes I_{B_1B_2} \otimes |\zeta_2\rangle_D\langle \zeta_2|\right\}$$
$$\left\{|\varsigma_7\rangle_{\mathfrak{a}_1\mathfrak{a}_2 A}\langle \varsigma_7| \otimes I_{B_1B_2} \otimes I_D\right\}.$$

$$(14.33)$$

Therefore, the final output state becomes

$$\varpi_{72}^{out-Amp-Damp} = \sum_{p=1}^{5} |R_p\rangle_{B_1B_2}\langle R_p|,$$

$$(14.34)$$

where $\{|R_p\rangle_{B_1 B_2}, \ p = 1, 2, 3, 4, 5\}$ s are given by

$$\begin{aligned}
|R_1\rangle_{B_1 B_2} &= \mathfrak{g}_1(1-\lambda)|01\rangle + \mathfrak{g}_2\sqrt{1-\lambda}|10\rangle, \\
|R_2\rangle_{B_1 B_2} &= -\mathfrak{g}_1\sqrt{1-\lambda}\sqrt{\lambda}|00\rangle, \\
|R_3\rangle_{B_1 B_2} &= \mathfrak{g}_2\sqrt{\lambda}|00\rangle, \\
|R_4\rangle_{B_1 B_2} &= \mathfrak{g}_2\sqrt{1-\lambda}\sqrt{\lambda}|01\rangle, \\
|R_5\rangle_{B_1 B_2} &= -\mathfrak{g}_2\lambda|00\rangle.
\end{aligned}$$

Now, according to the formula in Eq. (14.29) the fidelity $\mathscr{F}$ is

$$\begin{aligned}
\mathscr{F}^{Amp-Damp} = \ & (\lambda - 1)\{2|\mathfrak{g}_2|^4(\lambda + \sqrt{1-\lambda} - 1) \\
& + |\mathfrak{g}_2|^2(2 - 2\sqrt{1-\lambda} - 3\lambda) + \lambda - 1\}.
\end{aligned}$$

(14.35)
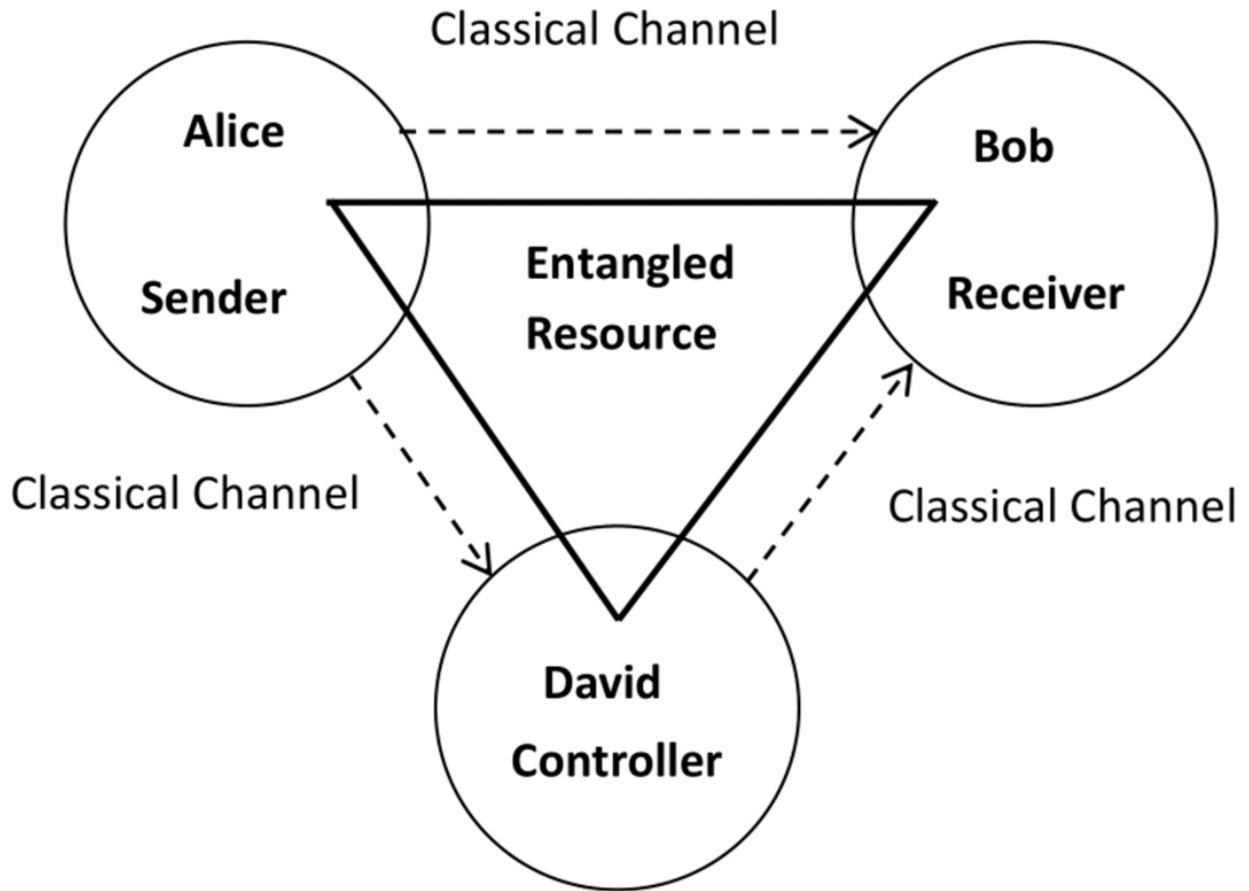
Figure 14.5 Schematic diagram for transfer of quantum states in ideal environment.

The variation of fidelity is shown in [Figure 14.6](). The other three cases corresponding to Alice's measurement results can be similarly analyzed.
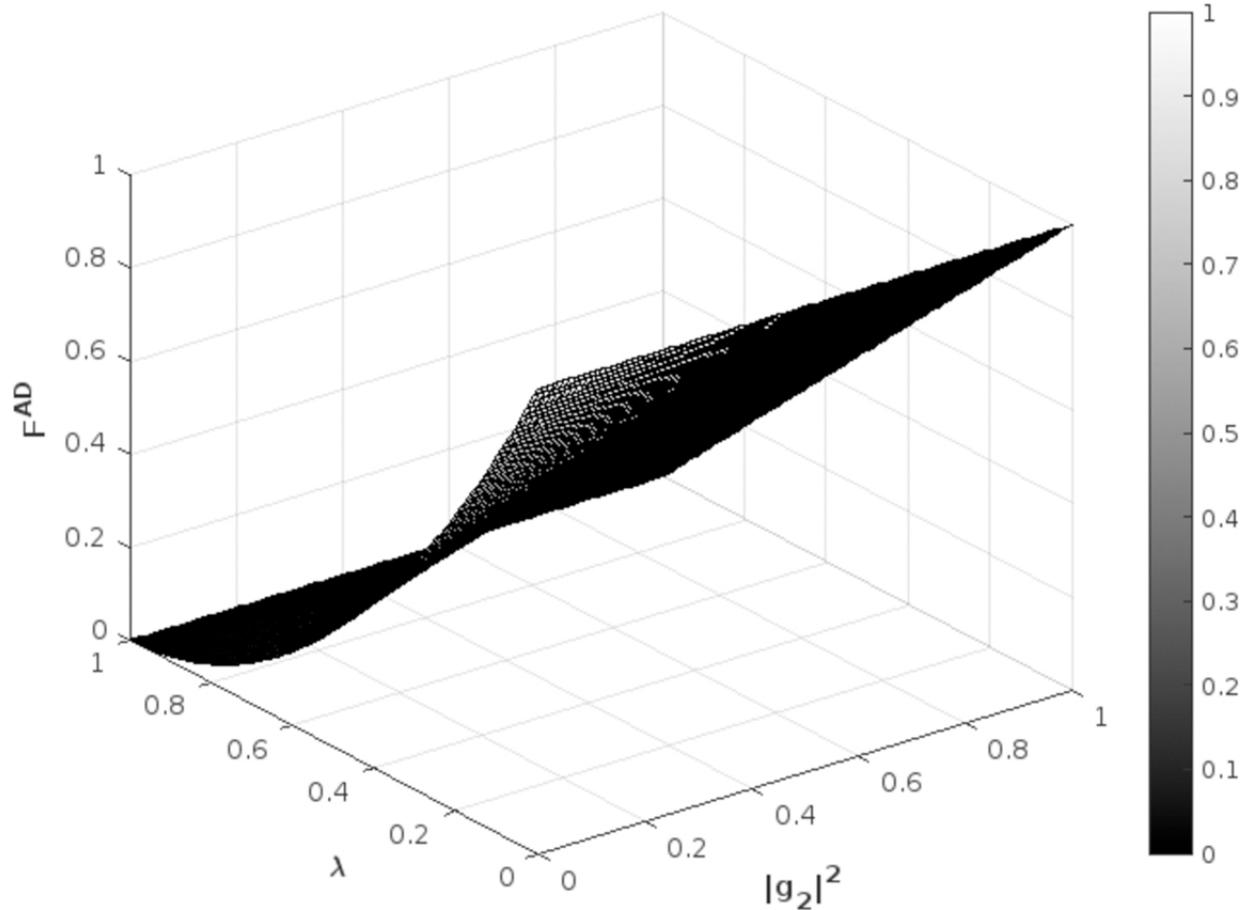
**Figure 14.6** 3-Dimensional surface plot of fidelity for amplitude damping noise as a function of $|g_2|^2$ and noise intensity parameter λ. ↵

14.3.2 CONTROLLED TELEPORTATION IN BIT-FLIP NOISY ENVIRONMENT

The Kraus operators for the Bit-flip noise are described as:

$$K_0 = \begin{bmatrix} \sqrt{1-\kappa} & 0 \\ 0 & \sqrt{1-\kappa} \end{bmatrix}, K_1 = \begin{bmatrix} 0 & \sqrt{\kappa} \\ \sqrt{\kappa} & 0 \end{bmatrix}$$

where κ is the noise intensity parameter of bit-flip noise.

According to the formula given in Eq. (14.26), the quantum resource becomes

$$\varepsilon_{Bit-Flip}(\varpi) = \frac{1}{4}\left[\left(\frac{(1-\kappa)^{\frac{3}{2}}}{\sqrt{2}}|0100\rangle + \frac{(1-\kappa)^{\frac{3}{2}}}{\sqrt{2}}|1011\rangle\right)\right.$$

$$\times \left(\frac{(1-\kappa)^{\frac{3}{2}}}{\sqrt{2}}\langle 0100| + \frac{(1-\kappa)^{\frac{3}{2}}}{\sqrt{2}}\langle 1011|\right)$$

$$+ \left(\frac{(1-\kappa)\sqrt{\kappa}}{\sqrt{2}}|0110\rangle + \frac{(1-\kappa)\sqrt{\kappa}}{\sqrt{2}}|1001\rangle\right)$$

$$\times \left(\frac{(1-\kappa)\sqrt{\kappa}}{\sqrt{2}}\langle 0110| + \frac{(1-\kappa)\sqrt{\kappa}}{\sqrt{2}}\langle 1001|\right)$$

$$+ \left(\frac{(1-\kappa)\sqrt{\kappa}}{\sqrt{2}}|0000\rangle + \frac{(1-\kappa)\sqrt{\kappa}}{\sqrt{2}}|1111\rangle\right)$$

$$\times \left(\frac{(1-\kappa)\sqrt{\kappa}}{\sqrt{2}}\langle 0000| + \frac{(1-\kappa)\sqrt{\kappa}}{\sqrt{2}}\langle 1111|\right)$$

$$+ \left(\frac{\kappa\sqrt{(1-\kappa)}}{\sqrt{2}}|0010\rangle + \frac{\kappa\sqrt{(1-\kappa)}}{\sqrt{2}}|1101\rangle\right)$$

$$\times \left(\frac{\kappa\sqrt{(1-\kappa)}}{\sqrt{2}}\langle 0010| + \frac{\kappa\sqrt{(1-\kappa)}}{\sqrt{2}}\langle 1101|\right)$$

$$+ \left(\frac{(1-\kappa)\sqrt{\kappa}}{\sqrt{2}}|0011\rangle + \frac{(1-\kappa)\sqrt{\kappa}}{\sqrt{2}}|1100\rangle\right)$$

$$\times \left(\frac{(1-\kappa)\sqrt{\kappa}}{\sqrt{2}}\langle 0011| + \frac{(1-\kappa)\sqrt{\kappa}}{\sqrt{2}}\langle 1100|\right)$$

$$+ \left(\frac{\kappa\sqrt{(1-\kappa)}}{\sqrt{2}}|0001\rangle + \frac{\kappa\sqrt{(1-\kappa)}}{\sqrt{2}}|1110\rangle\right)$$

$$\times \left(\frac{\kappa\sqrt{(1-\kappa)}}{\sqrt{2}}\langle 0001| + \frac{\kappa\sqrt{(1-\kappa)}}{\sqrt{2}}\langle 1110|\right)$$

$$+ \left(\frac{\kappa\sqrt{(1-\kappa)}}{\sqrt{2}}|0111\rangle + \frac{\kappa\sqrt{(1-\kappa)}}{\sqrt{2}}|1000\rangle\right)$$

$$\times \left(\frac{\kappa\sqrt{(1-\kappa)}}{\sqrt{2}}\langle 0111| + \frac{\kappa\sqrt{(1-\kappa)}}{\sqrt{2}}\langle 1000|\right)$$

$$+ \left(\frac{\kappa^{\frac{3}{2}}}{\sqrt{2}}|0101\rangle + \frac{\kappa^{\frac{3}{2}}}{\sqrt{2}}|1010\rangle\right)$$

$$\left.\times \left(\frac{\kappa^{\frac{3}{2}}}{\sqrt{2}}\langle 0101| + \frac{\kappa^{\frac{3}{2}}}{\sqrt{2}}\langle 1010|\right)\right].$$

(14.36)

Now, the composite state of the whole system can be written as

$$\varpi' = |\aleph\rangle_{\mathfrak{a}_1 \mathfrak{a}_2} \langle \aleph| \otimes \left(\varepsilon_{Bit-Flip}(\varpi) \right)_{AB_1 B_2 D}.$$

(14.37)

As in the illustration in <u>part 1</u> of this section, assume that the measurement result of Alice and David is $|\varsigma_7\rangle_{\mathfrak{a}_1 \mathfrak{a}_2 A}$ and $|\zeta_2\rangle_D$, respectively. Then the density matrix of the final output state is given by

$$\varpi_{72}^{out-Bit-Flip} = Tr_{\mathfrak{a}_1 \mathfrak{a}_2 AD} \left\{ U_{72}[\varpi_{\mathfrak{a}_1 \mathfrak{a}_2} \otimes \varepsilon_{Bit-Flip}(\varpi)] U_{72}^\dagger \right\},$$

(14.38)

where U_{72} is given in Eq. (14.33).

Therefore, the final output state becomes

$$\varpi_{72}^{out-Bit-Flip} = \sum_{q=1}^{8} |S_q\rangle_{B_1 B_2} \langle S_q|,$$

(14.39)

where $\{|S_q\rangle_{B_1 B_2} \, q = 1, 2, \dots, 8\}$ s are given by

$$
\begin{aligned}
|S_1\rangle_{B_1 B_2} &= \mathfrak{g}_1 (1-\kappa)^{\frac{3}{2}} |01\rangle + \mathfrak{g}_2 (1-\kappa)^{\frac{3}{2}} |10\rangle, \\
|S_2\rangle_{B_1 B_2} &= -\mathfrak{g}_1 \sqrt{\kappa}(1-\kappa)|00\rangle - \mathfrak{g}_2 \sqrt{\kappa}(1-\kappa)|11\rangle, \\
|S_3\rangle_{B_1 B_2} &= \mathfrak{g}_2 \sqrt{\kappa}(1-\kappa)|00\rangle + \mathfrak{g}_1 \sqrt{\kappa}(1-\kappa)|11\rangle, \\
|S_4\rangle_{B_1 B_2} &= -\mathfrak{g}_2 \kappa \sqrt{1-\kappa}|01\rangle - \mathfrak{g}_1 \kappa \sqrt{1-\kappa}|10\rangle, \\
|S_5\rangle_{B_1 B_2} &= \mathfrak{g}_2 \sqrt{\kappa}(1-\kappa)|01\rangle + \mathfrak{g}_1 \sqrt{\kappa}(1-\kappa)|10\rangle, \\
|S_6\rangle_{B_1 B_2} &= -\mathfrak{g}_2 \kappa \sqrt{1-\kappa}|00\rangle - \mathfrak{g}_1 \kappa \sqrt{1-\kappa}|11\rangle, \\
|S_7\rangle_{B_1 B_2} &= \mathfrak{g}_1 \kappa \sqrt{1-\kappa}|00\rangle + \mathfrak{g}_2 \kappa \sqrt{1-\kappa}|11\rangle, \\
|S_8\rangle_{B_1 B_2} &= -\mathfrak{g}_1 \kappa^{\frac{3}{2}} |01\rangle - \mathfrak{g}_2 \kappa^{\frac{3}{2}} |10\rangle.
\end{aligned}
$$

Now, according to the formula in Eq. (14.29), the fidelity $\mathscr{F}$ is

$$\mathscr{F}^{Bit-Flip} = 1 + \kappa(\kappa - 1)(3 - 4|\mathfrak{g}_2|^2 + 4|\mathfrak{g}_2|^4).$$

(14.40)

The variation of fidelity $\mathscr{F}^{Bit-Flip}$ is shown in <u>Figure 14.7</u>. The other three cases corresponding to Alice's measurement results can be similarly analyzed.
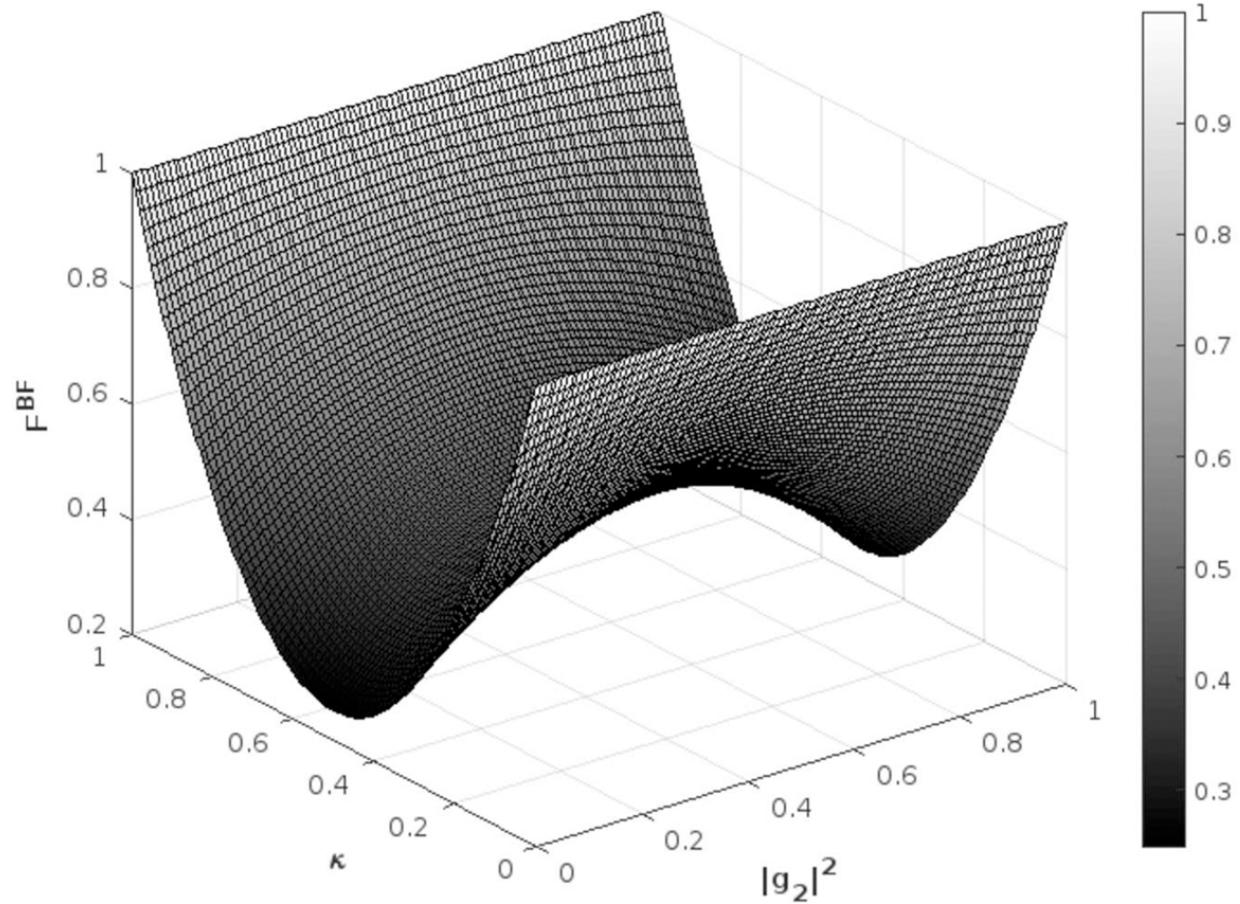

Figure 14.7 3-Dimensional surface plot of fidelity for bit-flip noise as a function of $|g_2|^2$ and noise intensity parameter κ.

14.3.3 CONTROLLED TELEPORTATION IN PHASE-FLIP NOISY ENVIRONMENT

The Kraus operators of phase-flip noise are defined as:

$$K_0 = \begin{bmatrix} \sqrt{1-\tau} & 0 \\ 0 & \sqrt{1-\tau} \end{bmatrix}, K_1 = \begin{bmatrix} \sqrt{\tau} & 0 \\ 0 & -\sqrt{\tau} \end{bmatrix}$$

where τ is the noise intensity parameter of phase-flip noise.

According to the formula given in Eq. (14.26), the quantum resource becomes

$$
\begin{aligned}
\varepsilon_{Phase-Flip}(\varpi) = \quad & \frac{1}{4}\Bigg[\left(\frac{(1-\tau)^{\frac{3}{2}}}{\sqrt{2}}\,|0100\rangle + \frac{(1-\tau)^{\frac{3}{2}}}{\sqrt{2}}\,|1011\rangle\right) \\
& \times \left(\frac{(1-\tau)^{\frac{3}{2}}}{\sqrt{2}}\,\langle0100| + \frac{(1-\tau)^{\frac{3}{2}}}{\sqrt{2}}\,\langle1011|\right) \\
& + \left(\frac{(1-\tau)\sqrt{\tau}}{\sqrt{2}}\,|0100\rangle - \frac{(1-\tau)\sqrt{\tau}}{\sqrt{2}}\,|1011\rangle\right) \\
& \times \left(\frac{(1-\tau)\sqrt{\tau}}{\sqrt{2}}\,\langle0100| - \frac{(1-\tau)\sqrt{\tau}}{\sqrt{2}}\,\langle1011|\right) \\
& + \left(-\frac{(1-\tau)\sqrt{\tau}}{\sqrt{2}}\,|0100\rangle + \frac{(1-\tau)\sqrt{\tau}}{\sqrt{2}}\,|1011\rangle\right) \\
& \times \left(-\frac{(1-\tau)\sqrt{\tau}}{\sqrt{2}}\,\langle0100| + \frac{(1-\tau)\sqrt{\tau}}{\sqrt{2}}\,\langle1011|\right) \\
& + \left(-\frac{\tau\sqrt{(1-\tau)}}{\sqrt{2}}\,|0100\rangle - \frac{\tau\sqrt{(1-\tau)}}{\sqrt{2}}\,|1011\rangle\right) \\
& \times \left(-\frac{\tau\sqrt{(1-\tau)}}{\sqrt{2}}\,\langle0100| - \frac{\tau\sqrt{(1-\tau)}}{\sqrt{2}}\,\langle1011|\right) \\
& + \left(\frac{(1-\tau)\sqrt{\tau}}{\sqrt{2}}\,|0100\rangle - \frac{(1-\tau)\sqrt{\tau}}{\sqrt{2}}\,|1011\rangle\right) \\
& \times \left(\frac{(1-\tau)\sqrt{\tau}}{\sqrt{2}}\,\langle0100| - \frac{(1-\tau)\sqrt{\tau}}{\sqrt{2}}\,\langle1011|\right) \\
& + \left(\frac{\tau\sqrt{(1-\tau)}}{\sqrt{2}}\,|0100\rangle + \frac{\tau\sqrt{(1-\tau)}}{\sqrt{2}}\,|1011\rangle\right) \\
& \times \left(\frac{\tau\sqrt{(1-\tau)}}{\sqrt{2}}\,\langle0100| + \frac{\tau\sqrt{(1-\tau)}}{\sqrt{2}}\,\langle1011|\right) \\
& + \left(-\frac{\tau\sqrt{(1-\tau)}}{\sqrt{2}}\,|0100\rangle - \frac{\tau\sqrt{(1-\tau)}}{\sqrt{2}}\,|1011\rangle\right) \\
& \times \left(-\frac{\tau\sqrt{(1-\tau)}}{\sqrt{2}}\,\langle0100| - \frac{\tau\sqrt{(1-\tau)}}{\sqrt{2}}\,\langle1011|\right) \\
& + \left(-\frac{\tau^{\frac{3}{2}}}{\sqrt{2}}\,|0100\rangle + \frac{\tau^{\frac{3}{2}}}{\sqrt{2}}\,|1011\rangle\right) \\
& \times \left(-\frac{\tau^{\frac{3}{2}}}{\sqrt{2}}\,\langle0100| + \frac{\tau^{\frac{3}{2}}}{\sqrt{2}}\,\langle1011|\right)\Bigg].
\end{aligned}
$$

$$(14.41)$$

The total state of the whole system can be expressed as

$$\varpi' = |\aleph\rangle_{\mathfrak{a}_1\mathfrak{a}_2}\langle\aleph| \otimes (\varepsilon_{Phase-Flip}(\varpi))_{AB_1B_2D}.$$

(14.42)

As discussed in the illustration of [part 1](#) of this section, assume that the outcome of Alice's measurement is $|\varsigma_7\rangle_{\mathfrak{a}_1\mathfrak{a}_2A}$ and that of the controller David is $|\zeta_2\rangle_D$. Then the density matrix of the final output state becomes

$$\varpi_{72}^{out-Phase-Flip} = Tr_{\mathfrak{a}_1\mathfrak{a}_2AD}\{U_{72}[\varpi_{\mathfrak{a}_1\mathfrak{a}_2} \otimes \varepsilon_{Phase-Flip}(\varpi)]U_{72}^{\dagger}\},$$

(14.43)

where U_{72} is given in Eq. (14.33).

Hence, the final output state is

$$\varpi_{72}^{out-Phase-Flip} = \sum_{r=1}^{8} |H_r\rangle_{B_1B_2}\langle H_r|,$$

(14.44)

where $\{|H_r\rangle_{B_1B_2}, \; r = 1, 2, \ldots, 8\}$ s are given by

$$
\begin{aligned}
|H_1\rangle_{B_1B_2} &= \; \mathfrak{g}_1(1-\tau)^{\frac{3}{2}}|01\rangle + \mathfrak{g}_2(1-\tau)^{\frac{3}{2}}|10\rangle, \\
|H_2\rangle_{B_1B_2} &= \; -\mathfrak{g}_1\sqrt{\tau}(1-\tau)|01\rangle + \mathfrak{g}_2\sqrt{\tau}(1-\tau)|10\rangle, \\
|H_3\rangle_{B_1B_2} &= \; \mathfrak{g}_1\sqrt{\tau}(1-\tau)|01\rangle - \mathfrak{g}_2\sqrt{\tau}(1-\tau)|10\rangle, \\
|H_4\rangle_{B_1B_2} &= \; -\mathfrak{g}_1\tau\sqrt{1-\tau}|01\rangle - \mathfrak{g}_2\tau\sqrt{1-\tau}|10\rangle, \\
|H_5\rangle_{B_1B_2} &= \; -\mathfrak{g}_1\sqrt{\tau}(1-\tau)|01\rangle + \mathfrak{g}_2\sqrt{\tau}(1-\tau)|10\rangle, \\
|H_6\rangle_{B_1B_2} &= \; \mathfrak{g}_1\tau\sqrt{1-\tau}|01\rangle + \mathfrak{g}_2\tau\sqrt{1-\tau}|10\rangle, \\
|H_7\rangle_{B_1B_2} &= \; -\mathfrak{g}_1\tau\sqrt{1-\tau}|01\rangle - \mathfrak{g}_2\tau\sqrt{1-\tau}|10\rangle, \\
|H_8\rangle_{B_1B_2} &= \; \mathfrak{g}_1\tau^{\frac{3}{2}}|01\rangle - \mathfrak{g}_2\tau^{\frac{3}{2}}|10\rangle.
\end{aligned}
$$

Now, according to the formula in Eq. (14.29), the fidelity $\mathscr{F}$ is given by

$$\mathscr{F}^{Phase-Flip} = 1 + 4\tau|\mathfrak{g}_2|^2(|\mathfrak{g}_2|^2 - 1)(3 - 6\tau + 4\tau^2).$$

(14.45)

The variation of fidelity $\mathscr{F}^{Phase-Flip}$ is shown in Figure 14.8. The other three cases corresponding to Alice's measurement results can be similarly analyzed.
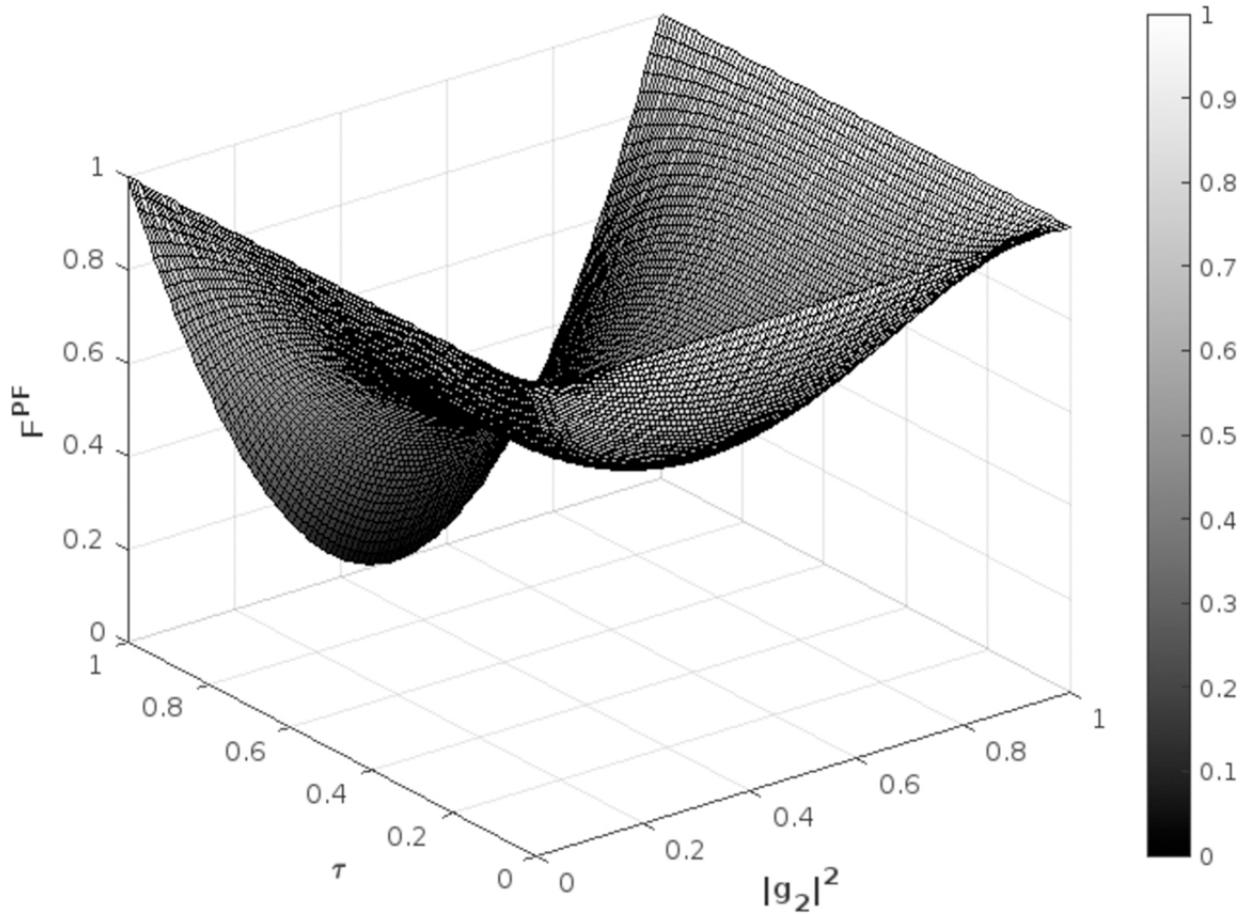

Figure 14.8 3-Dimensional surface plot of fidelity for phase-flip noise as a function of $|g_2|^2$ and noise intensity parameter τ.

14.3.4 CONTROLLED TELEPORTATION IN PHASE-DAMPING NOISY ENVIRONMENT

The Kraus operators of phase-damping noise are described as:

$$K_0 = \begin{bmatrix} \sqrt{1-\mu} & 0 \\ 0 & \sqrt{1-\mu} \end{bmatrix}, K_1 = \begin{bmatrix} \sqrt{\mu} & 0 \\ 0 & 0 \end{bmatrix}, K_2 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\mu} \end{bmatrix}$$

where μ is the noise strength of phase-damping noise.

According to the formula given in Eq. (14.26), the quantum resource becomes

$$\varepsilon_{Phase-Damp}(\varpi) = \frac{1}{4}\left[\left(\frac{(1-\mu)^{\frac{3}{2}}}{\sqrt{2}}|0100\rangle + \frac{(1-\mu)^{\frac{3}{2}}}{\sqrt{2}}|1011\rangle\right)\right.$$

$$\times \left(\frac{(1-\mu)^{\frac{3}{2}}}{\sqrt{2}}\langle 0100| + \frac{(1-\mu)^{\frac{3}{2}}}{\sqrt{2}}\langle 1011|\right)$$

$$+ \left(\frac{(1-\mu)\sqrt{\mu}}{\sqrt{2}}|0100\rangle\right) \times \left(\frac{(1-\mu)\sqrt{\mu}}{\sqrt{2}}\langle 0100|\right)$$

$$+ \left(\frac{(1-\mu)\sqrt{\mu}}{\sqrt{2}}|1011\rangle\right) \times \left(\frac{(1-\mu)\sqrt{\mu}}{\sqrt{2}}\langle 1011|\right)$$

$$+ \left(\frac{(1-\mu)\sqrt{\mu}}{\sqrt{2}}|1011\rangle\right) \times \left(\frac{(1-\mu)\sqrt{\mu}}{\sqrt{2}}\langle 1011|\right)$$

$$+ \left(\frac{\mu\sqrt{1-\mu}}{\sqrt{2}}|1011\rangle\right) \times \left(\frac{\mu\sqrt{1-\mu}}{\sqrt{2}}\langle 1011|\right)$$

$$+ \left(\frac{(1-\mu)\sqrt{\mu}}{\sqrt{2}}|0100\rangle\right) \times \left(\frac{(1-\mu)\sqrt{\mu}}{\sqrt{2}}\langle 0100|\right)$$

$$+ \left(\frac{\mu\sqrt{1-\mu}}{\sqrt{2}}|0100\rangle\right) \times \left(\frac{\mu\sqrt{1-\mu}}{\sqrt{2}}\langle 0100|\right)$$

$$+ \left(\frac{(1-\mu)\sqrt{\mu}}{\sqrt{2}}|0100\rangle\right) \times \left(\frac{(1-\mu)\sqrt{\mu}}{\sqrt{2}}\langle 0100|\right)$$

$$+ \left(\frac{\mu\sqrt{1-\mu}}{\sqrt{2}}|0100\rangle\right) \times \left(\frac{\mu\sqrt{1-\mu}}{\sqrt{2}}\langle 0100|\right)$$

$$+ \left(\frac{\mu\sqrt{1-\mu}}{\sqrt{2}}|0100\rangle\right) \times \left(\frac{\mu\sqrt{1-\mu}}{\sqrt{2}}\langle 0100|\right)$$

$$+ \left(\frac{\mu^{\frac{3}{2}}}{\sqrt{2}}|0100\rangle\right) \times \left(\frac{\mu^{\frac{3}{2}}}{\sqrt{2}}\langle 0100|\right)$$

$$+ \left(\frac{(1-\mu)\sqrt{\mu}}{\sqrt{2}}|1011\rangle\right) \times \left(\frac{(1-\mu)\sqrt{\mu}}{\sqrt{2}}\langle 1011|\right)$$

$$+ \left(\frac{\mu\sqrt{1-\mu}}{\sqrt{2}}|1011\rangle\right) \times \left(\frac{\mu\sqrt{1-\mu}}{\sqrt{2}}\langle 1011|\right)$$

$$+ \left(\frac{\mu\sqrt{1-\mu}}{\sqrt{2}}|1011\rangle\right) \times \left(\frac{\mu\sqrt{1-\mu}}{\sqrt{2}}\langle 1011|\right)$$

$$\left. + \left(\frac{\mu^{\frac{3}{2}}}{\sqrt{2}}|1011\rangle\right) \times \left(\frac{\mu^{\frac{3}{2}}}{\sqrt{2}}\langle 1011|\right)\right].$$

(14.46)

Therefore, the overall state of the system is given by

$$\varpi' = |\aleph\rangle_{\mathfrak{a}_1\mathfrak{a}_2}\langle\aleph| \otimes \left(\varepsilon_{Phase-Damp}(\varpi)\right)_{AB_1B_2D}.$$

(14.47)

As in the illustration in [part 1](#) of this section, let us assume that Alice's measurement outcome yields $|\varsigma_7\rangle_{\mathfrak{a}_1\mathfrak{a}_2A}$ and that David's measurement outcome is $|\zeta_1\rangle_D$. Then the density matrix of the final output state is as follows

$$\varpi_{72}^{out-Phase-Damp} = Tr_{\mathfrak{a}_1\mathfrak{a}_2AD}\left\{U_{72}[\varpi_{\mathfrak{a}_1\mathfrak{a}_2} \otimes \varepsilon_{Phase-Damp}(\varpi)]U_{72}^{\dagger}\right\},$$

(14.48)

where U_{72} is given in Eq. (14.33).

Therefore, the final output state is given by

$$\varpi_{72}^{out-Phase-Damp} = \sum_{s=1}^{15} |J_s\rangle_{B_1B_2}\langle J_s|,$$

(14.49)

where $\{|J_s\rangle_{B_1B_2,\ s=1,2,...,15}\}$ s are given by

$$
\begin{aligned}
|J_1\rangle_{B_1 B_2} &= \mathfrak{g}_1 (1-\mu)^{\frac{3}{2}} |01\rangle + \mathfrak{g}_2 (1-\mu)^{\frac{3}{2}} |10\rangle, \\
|J_2\rangle_{B_1 B_2} &= \mathfrak{g}_2 \sqrt{\mu}(1-\mu)|10\rangle, \\
|J_3\rangle_{B_1 B_2} &= \mathfrak{g}_1 \sqrt{\mu}(1-\mu)|01\rangle, \\
|J_4\rangle_{B_1 B_2} &= \mathfrak{g}_1 \sqrt{\mu}(1-\mu)|01\rangle, \\
|J_5\rangle_{B_1 B_2} &= \mathfrak{g}_1 \mu\sqrt{1-\mu}|01\rangle, \\
|J_6\rangle_{B_1 B_2} &= \mathfrak{g}_2 \sqrt{\mu}(1-\mu)|10\rangle, \\
|J_7\rangle_{B_1 B_2} &= \mathfrak{g}_2 \mu\sqrt{1-\mu}|10\rangle, \\
|J_8\rangle_{B_1 B_2} &= \mathfrak{g}_2 \sqrt{\mu}(1-\mu)|10\rangle, \\
|J_9\rangle_{B_1 B_2} &= \mathfrak{g}_2 \mu\sqrt{1-\mu}|10\rangle, \\
|J_{10}\rangle_{B_1 B_2} &= \mathfrak{g}_2 \mu\sqrt{1-\mu}|10\rangle, \\
|J_{11}\rangle_{B_1 B_2} &= \mathfrak{g}_2 \mu^{\frac{3}{2}}|10\rangle, \\
|J_{12}\rangle_{B_1 B_2} &= \mathfrak{g}_1 \sqrt{\mu}(1-\mu)|01\rangle, \\
|J_{13}\rangle_{B_1 B_2} &= \mathfrak{g}_1 \mu\sqrt{1-\mu}|01\rangle, \\
|J_{14}\rangle_{B_1 B_2} &= \mathfrak{g}_1 \mu\sqrt{1-\mu}|01\rangle, \\
|J_{15}\rangle_{B_1 B_2} &= \mathfrak{g}_1 \mu^{\frac{3}{2}}|01\rangle.
\end{aligned}
$$

According to the formula given in Eq. (14.29), the fidelity $\mathscr{F}$ is

$$
\mathscr{F}^{Phase-Damp} = 1 + 2|\mathfrak{g}_2|^2 \mu(|\mathfrak{g}_2|^2 - 1)\{3 + \mu(\mu - 3)\}.
$$

(14.50)

The variation of fidelity $\mathscr{F}^{Phase-Damp}$ is shown in Figure 14.9. The other three cases corresponding to Alice's measurement results can be similarly analyzed.
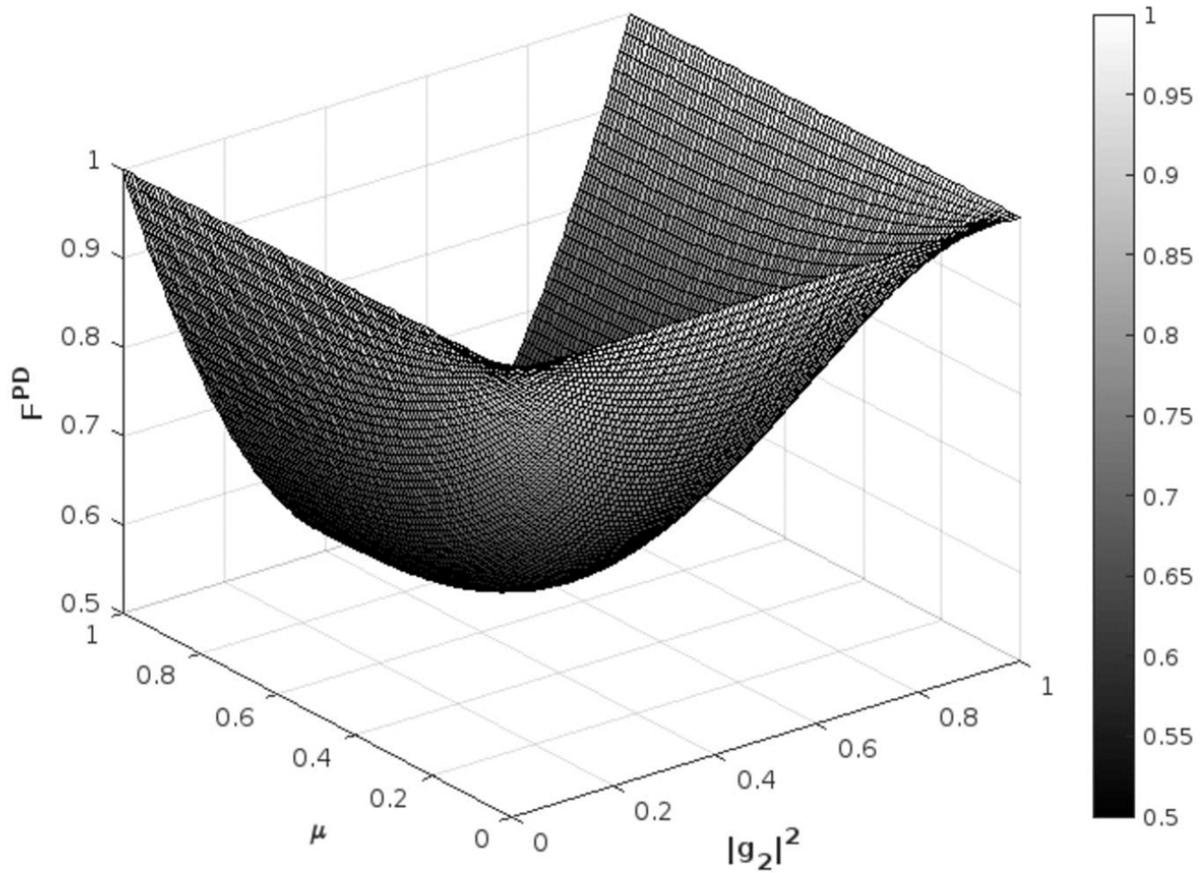
Figure 14.9 3-Dimensional surface plot of fidelity for phase-damping noise as a function of $|g_2|^2$ and noise intensity parameter μ.

In Figure 14.6, Figure 14.7, Figure 14.8, Figure 14.9, since the state is unknown to both Alice and Bob, the exact value of $|g_2|^2$ is not available with these two parties. An average fidelity can be calculated taking into account all four possible outcomes of Alice's measurement for a fixed parameter of the corresponding type of noise.

A scrutiny of Figure 14.6, Figure 14.7, Figure 14.8, Figure 14.9 shows that the fidelity tends to 1 as the noise parameter tends to zero. This is what is expected since in that situation the protocol becomes a perfect protocol where the fidelity is of unit value.

15 Control of Noise in Teleportation Processes

15.1 INTRODUCTION

In this chapter, two mechanisms for minimizing the effect of noise on the teleportation protocol are presented. The fidelity improvement is analyzed with respect to the variations of control parameters. There are several works on this topic in recent times [61, 75, 88, 89, 128, 129, 146, 175, 185].

15.2 PROTECTING TELEPORTATION PROTOCOL BY WEAK AND REVERSAL MEASUREMENTS

Teleportation of single qubit in ideal environment was discussed in Chapter 8 which was followed by the study of the effect of noise on the same protocol in Chapter 14.

In this section it is shown that fidelity can be improved by applications of weak measurement (WM) and weak measurement reversal (WMR). The protocol for such applications discussed in the following is given by Li et al. [89].

The WM and WMR operators on the single-qubit quantum system is described, respectively, as

$$W_m = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-k_w} \end{bmatrix}, \quad R_m = \begin{bmatrix} \sqrt{1-k_r} & 0 \\ 0 & 1 \end{bmatrix},$$

$$(15.1)$$

where the coefficients k_w and k_r are the strength of the weak and reversal measurements, respectively.

The quantum resource, which is a Bell state here, is created by Alice and is given by $|E\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. It is shared with Bob by sending one particle to him through a noisy environment whereby the entangled resource becomes affected with noise.

First, Alice makes a weak measurement on the qubit B before it is distributed. Then the quantum channel is reduced to the state

$$|E^W\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + \sqrt{1 - k_w}|11\rangle).$$

(15.2)

After that Alice distributes the particle B to Bob through the amplitude damping noisy channel. Then according to Eq. (14.1), the quantum channel becomes

$$\varepsilon_{AD}(\varpi_W) = \frac{1}{2}\Big[\{|00\rangle + \sqrt{(1 - k_w)(1 - p)}|11\rangle\}$$
$$\times \{\langle 00| + \sqrt{(1 - k_w)(1 - p)}\langle 11|\} + p(1 - k_w)|10\rangle\langle 10|\Big].$$

(15.3)

Upon receiving the particle B, Bob applies a WMR operator given in Eq. (15.1) on the particle B. The above quantum state subsequently reduces to the state

$$\varepsilon_{AD}(\varpi_{W-R}) = \frac{1}{2}\Big[\{\sqrt{1 - k_r}|00\rangle + \sqrt{(1 - k_w)(1 - p)}|11\rangle\} \times \{\sqrt{1 - k_r}\langle 00|$$
$$+ \sqrt{(1 - k_w)(1 - p)}\langle 11|\} + p(1 - k_w)(1 - k_r)|10\rangle\langle 10|\Big].$$

(15.4)

Now we have the combined state of the whole system which becomes

$$\varpi_{W-R}^{AD} = |\aleph\rangle_a\langle\aleph| \otimes \varepsilon_{AD}(\varpi)_{W-R}.$$

(15.5)

As an illustration, assuming Alice obtains the measurement result $|\Upsilon_4\rangle_{aA}$, the reduced density matrix is described as

$$\varpi_{W-R-4}^{out-AD} = Tr_{aA}\{U_4[\varpi_a \otimes \varepsilon_{AD}(\varpi)_{W-R}]U_4^\dagger\},$$

where U_4 is given by

$$U_4 = \{I_{aA} \otimes (\sigma_z \sigma_x)_B\}\{|\Upsilon_4\rangle_{aA}\langle\Upsilon_4| \otimes I_B\}.$$

Therefore, the final output state can be written as

$$\varpi_{W-R-4}^{out-AD} = \frac{1}{N_1}\Big[(\mathfrak{g}_2\sqrt{1-k_r}|1\rangle + \mathfrak{g}_1\sqrt{(1-k_w)(1-p)}|0\rangle) \times (\mathfrak{g}_2\sqrt{1-k_r}\langle 1|$$
$$+ \mathfrak{g}_1\sqrt{(1-k_w)(1-p)}\langle 0|) + \mathfrak{g}_1^2 p(1-k_w)(1-k_r)|1\rangle\langle 1|\Big],$$

(15.7)

where $N_1 = \mathfrak{g}_1^2(1-p)(1-k_w) + \mathfrak{g}_2^2(1-k_r) + \mathfrak{g}_1^2 p(1-k_w)(1-k_r)$.

Based on the formula in Eq. (14.3), the fidelity $\mathscr{F}$ is computed as

$$\mathscr{F}_{W-R}^{AD} = \frac{\left[\mathfrak{g}_1^2\sqrt{(1-k_w)(1-p)} + \mathfrak{g}_2^2\sqrt{1-k_r}\right]^2 + \mathfrak{g}_1^4\mathfrak{g}_2^2 p^2(1-k_w)^2(1-k_r)^2}{N_1}.$$

(15.8)

The optimal fidelity can be derived from Eq. (15.8) as

$$F_{OP}^{AD} = \frac{\left[\mathfrak{g}_1^2\sqrt{(1-k_w)(1-p)} + \mathfrak{g}_2^2\sqrt{(1-k_w)(1-p)}\right]^2 + \mathfrak{g}_1^4\mathfrak{g}_2^2 p^2(1-k_w)^4(1-p)^2}{\mathfrak{g}_1^2(1-p)(1-k_w) + \mathfrak{g}_2^2(1-k_w)(1-p) + \mathfrak{g}_1^2 p(1-k_w)^2(1-p)}$$
$$= \frac{1 + \mathfrak{g}_1^4(1-\mathfrak{g}_1^2)p^2(1-k_w)^3(1-p)}{1 + \mathfrak{g}_1^2 p(1-k_w)},$$

(15.9)

which is under the optimal reversal measurement condition $k_r = k_w + p(1-k_w)$.

The fidelity is plotted in the [Figure 15.1](#) for the process in the cases of amplitude damping noise (F^{AD}) and noise after protection by WM and WMR (F_{OP}^{AD}).
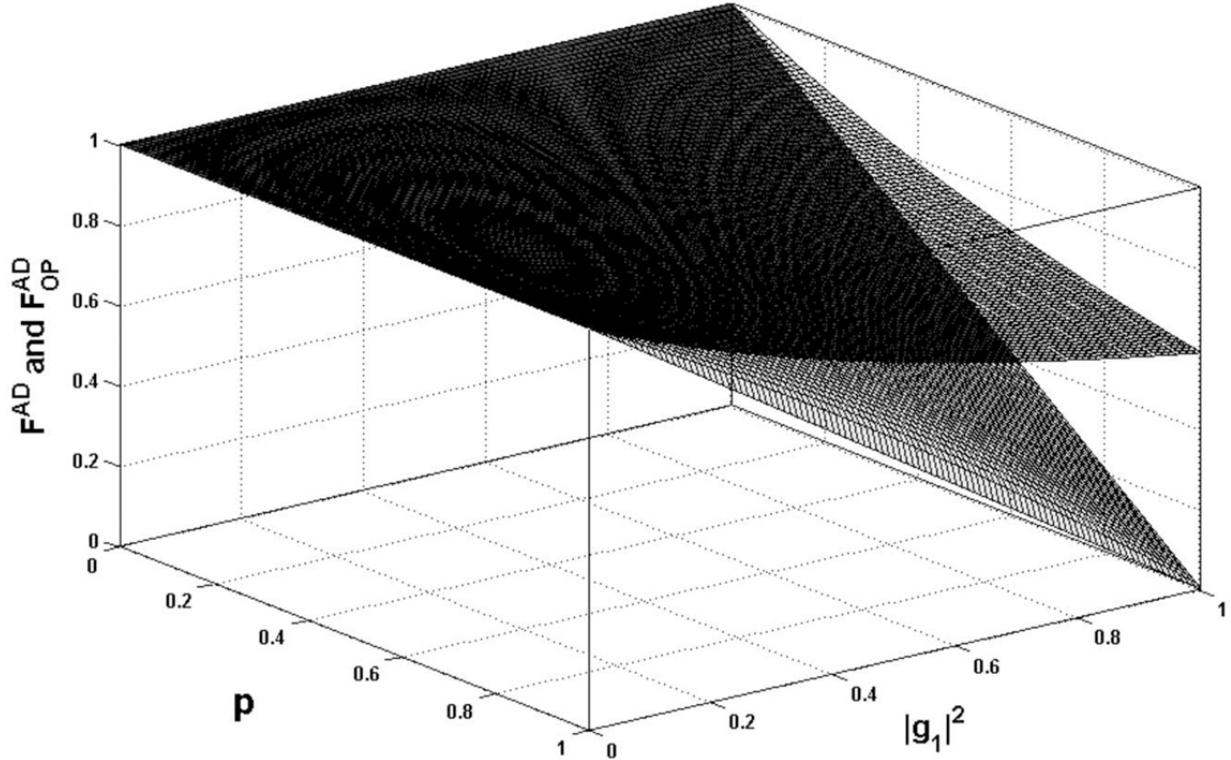
Figure 15.1 3-Dimensional surface comparison of the original amplitude damping fidelity F^{AD} and the optimized fidelity F_{OP}^{AD}, for a fixed weak measurement strength $k_w = 0.3$. ↵

From Figure 15.1 we see that the optimized fidelity demonstrates a significant improvement over the original for moderate to high damping levels illustrating the effectiveness of WM and WMR applications.

15.3 CONTROL BY ENVIRONMENT ASSISTED MEASUREMENTS (EAM)

Here we consider the same problem as in the previous section, but the control of noise is done in a different way by employing EAM along with WM and WMR. The protocol is given by Harraz et al. [56]. Other similar works employing EAM are discussed in [58, 176, 179].

Let us consider that Alice prepares a maximally entangled pair of particles and sends one of them to Bob via a noisy quantum resource. The shared entangled state used as the teleportation resource is represented as

$$|E\rangle_{AB} = \frac{1}{\sqrt{2}} (|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B),$$

<div align="right">(15.10)</div>

where qubit A remains with Alice and qubit B is received by Bob.

The entanglement protection scheme involves three stages: pre-decoherence operations, environment-assisted measurement (EAM) during decoherence, and post-decoherence recovery. Before transmitting the qubit B, Alice applies a weak measurement (WM) and a flip operation on it. She then sends the measurement outcome to Bob via the classical channel used for teleportation. During transmission, EAM is applied to select system states corresponding to invertible Kraus operators. Based on Alice's message, Bob performs a post-flip operation followed by weak measurement reversal (WMR) to recover his share of the entangled pair. The rest of the protocol is the same as the usual teleportation protocol. The complete protocol is outlined in the following five steps. The entire protocol is illustrated in [Figure 15.2](#).
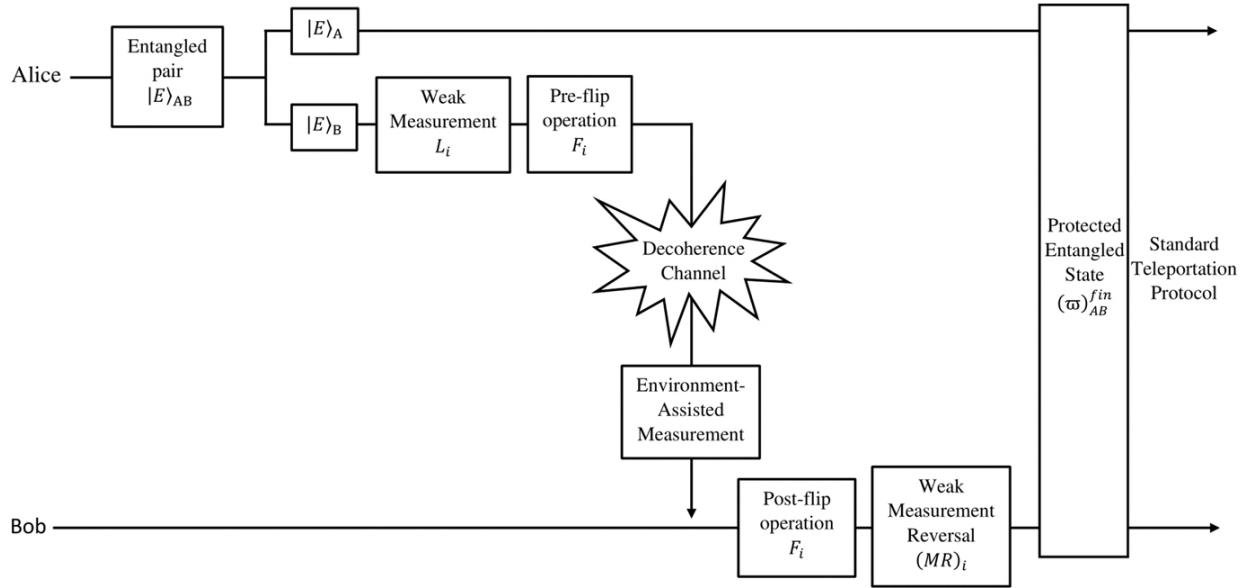
Figure 15.2 Schematic diagram for entanglement protection via EAM. ↵

Step 1. Alice performs a weak measurement (WM) on the qubit intended for Bob using a complete set of POVM operators $P_0 = L_0^\dagger L_0$, $P_1 = L_1^\dagger L_1$ with L_0 and L_1 being given by

$$L_0 = \begin{bmatrix} cos(\kappa/2) & 0 \\ 0 & sin(\kappa/2) \end{bmatrix}, L_1 = \begin{bmatrix} sin(\kappa/2) & 0 \\ 0 & cos(\kappa/2) \end{bmatrix}$$

(15.11)

where $\kappa \in [0, \pi/2]$ controls the measurement strength. For $\kappa = \pi/2$, there is no measurement, and for $\kappa = 0$, it becomes a projective measurement. The case $0 < \kappa < \pi/2$ corresponds to a weak measurement.

Step 2. Based on the measurement outcome, a pre-flip operation is applied to the qubit. The flip operators are defined as

$$F_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, F_1 = \vartheta_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(15.12)

where I is the identity and ϑ_x is the Pauli-X operator.

If the outcome corresponds to L_0, no action is needed ($F_0 = I$). If the outcome is L_1, ϑ_x (F_1) is applied to the state.

Step 3. The prepared state is then transmitted to Bob through a noisy quantum environment. The noise is assumed to be AD noise. The standard Kraus operators for AD noise are:

$$K_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{(1-p)} \end{bmatrix}, K_1 = \begin{bmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{bmatrix}$$

(15.13)

where p is the noise parameter.

We implement the Environment-Assisted Measurement (EAM) on the decoherence channel in the following way:

A measurement is performed on the channel, causing it to collapse into one of the eigenstates of the measured observable. As a result, the system is projected into a state conditioned on the corresponding outcome. If the channel collapses into the jth eigenstate $(j = 0, 1)$, the system evolves into the state $\varpi_S^j = K_j \varpi_S(0) K_j^\dagger$, up to

normalization. In our study, we consider only the outcome corresponding to the invertible Kraus operator K_0 and discard the measurement result of K_1.

Step 4. Bob then performs post-flip operations. The outcome of the weak measurement (WM) performed by Alice is communicated to Bob through a classical channel. Accordingly, Bob applies the same post-flip operations as those used by Alice, as given in Eq. (15.12).

Step 5. Finally, Bob applies the reversal measurement (WMR) to recover his part of the entangled state. The RM operators is given by:

$$(MR)_0 = \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix}, \ (MR)_1 = \begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix}$$

(15.14)

where $u \in (0, 1)$ is the strength of the WMR.

At this stage, the protected entangled pair shared between Alice and Bob is prepared for use in the standard quantum teleportation protocol.

The entire protection procedure can be represented by a control map denoted as C:

$$\varpi_{AB}^{fin} = C(\varpi_{AB}) = \sum_{j=0,1} (MR)_j F_j K_0 F_j L_j (\varpi_{AB}) \times L_j^\dagger F_j^\dagger K_0^\dagger F_j^\dagger (MR)_j^\dagger$$

(15.15)

where ϖ_{AB}^{fin} is the protected entangled pair. Thus, after the completion of the protection process, the entangled state shared between Alice and Bob transforms into the protected state given by

$$\varpi_{AB}^{fin} = \frac{1}{2} \begin{bmatrix} \varpi_{11}^{fin} & 0 & 0 & \varpi_{14}^{fin} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \varpi_{41}^{fin} & 0 & 0 & \varpi_{44}^{fin} \end{bmatrix}$$

(15.16)

where $\varpi_{11}^{fin} = \varpi_{44}^{fin} = cos^2 \frac{\kappa}{2} u^2 + (1-p) sin^2 \frac{\kappa}{2}$ and $\varpi_{14}^{fin} = \varpi_{41}^{fin\dagger} = u sin\kappa \sqrt{1-p}$.

Due to the partial nature of the RM and the exclusion of certain outcomes during the EAM process, the proposed scheme operates probabilistically, with an overall success probability given by

$$success\,probability = Trace(\varpi_{AB}^{fin}) = cos^2\frac{\kappa}{2}u^2 + (1-p)sin^2\frac{\kappa}{2}.$$

(15.17)

Using the protected teleportation channel given in Eq. (15.16), Alice begins the standard teleportation protocol. She does this by interacting with the unknown input state with her part of the entangled pair. The input state $|\aleph_{in}\rangle_a = \mathfrak{g}_1|0\rangle + \mathfrak{g}_2|1\rangle$ that Alice wants to send to Bob is given by

$$\varpi_{in} = |\aleph_{in}\rangle\langle\aleph_{in}| = \begin{bmatrix} |\mathfrak{g}_1|^2 & \mathfrak{g}_1\mathfrak{g}_2^* \\ \mathfrak{g}_1^*\mathfrak{g}_2 & |\mathfrak{g}_2|^2 \end{bmatrix}$$

(15.18)

where $|\mathfrak{g}_1|^2 + |\mathfrak{g}_2|^2 = 1$ and * denotes the complex conjugate.

At the end of the standard teleportation protocol, Bob obtains the output state ϖ_{out}, which can be expressed as

$$\varpi_{out} = \frac{1}{4}\begin{bmatrix} \varpi_{11}^{out} & \varpi_{14}^{out} \\ \varpi_{41}^{out} & \varpi_{44}^{out} \end{bmatrix}$$

(15.19)

where $\varpi_{11}^{out} = |\mathfrak{g}_1|^2, \varpi_{44}^{out} = |\mathfrak{g}_2|^2$ and $\varpi_{14}^{out} = \varpi_{41}^{out\dagger} = \frac{\mathfrak{g}_1^*\mathfrak{g}_2 usin\kappa\sqrt{1-p}}{cos^2\frac{\kappa}{2}u^2+(1-p)sin^2\frac{\kappa}{2}}.$

To assess the effectiveness of our protected quantum teleportation scheme, we calculate the average teleportation fidelity between the input state in Eq. (15.18) and the output state received by Bob in Eq. (15.19), averaged over all possible input states, as follows:

$$\mathscr{F}_{av}^{EAM} = \int\langle\aleph_{in}|\varpi_{out}|\aleph_{in}\rangle d\aleph$$

$$= \frac{11}{15} + \frac{4usin\kappa\sqrt{1-p}}{15[cos^2\frac{\kappa}{2}u^2 + (1-p)sin^2\frac{\kappa}{2}]}.$$

Similarly, the average teleportation fidelity for the standard teleportation protocol through an amplitude damping channel (AD) without any protection is calculated as

$$\mathscr{F}_{av}^{standard} = \frac{1}{15}\left(4\sqrt{1-p} - \frac{7}{2}p + 11\right).$$

(15.21)

To evaluate the performance of the proposed EAM-based protected teleportation scheme, we consider the average teleportation fidelity $\mathscr{F}_{av}^{EAM}$ from Eq. (15.20) along with the entanglement protection success probability provided in Eq. (15.21).

In Table. 15.1 fidelity and success probability are presented against WM strength κ and RM strength u under the assumption that the decoherence rate is $p = 0.5$.

Table 15.1

Average Teleportation Fidelity and Success Probability of Entanglement Protection in the EAM Scheme for Different Measurement Strengths with Fixed $p = 0.5$. ⏎

WM strength (κ)	WMR strength (u)	$\mathscr{F}_{av}^{EAM}$	success probability
$\pi/4$	0.3	1	0.15
$\pi/6$	0.5	0.84	0.25
$\pi/3$	0.7	0.96	0.5
0	1	0.73	1

There are some interesting outcomes from the data in Table 15.1.

The fidelity value 1 is attainable with proper choices of (κ, u) but with probability of success being 0.15.

The usual teleportation in the absence of the protection discussed above, has an average probability 0.80 with the decoherence parameter fixed at $p = 0.5$. It is possible to obtain fidelity higher than this value by fixing the parameters (κ, u) appropriately with some success probability. In the overall scenario the choices of the parameters (κ, u) determine the level of control on the fidelity.

A Remote State Preparation Scheme of Single-qubit State

Remote State Preparation (RSP) scheme is a quantum communication scheme that allows one party, namely Alice, to prepare a known quantum state at a remote location, namely in the location of Bob, using a previously shared entangled resource and classical communication. Unlike quantum teleportation, in RSP the quantum state to be prepared is known to the sender but need not be physically available beforehand. There are several works on this topic as, for instances, [8, 9, 80, 199].

We now describe a basic RSP protocol involving two parties: Alice (the sender) and Bob (the receiver). Alice wants to remotely prepare a specific single-qubit quantum state at Bob's site given as

$$|\aleph\rangle = \iota_1|0\rangle + \iota_2|1\rangle,$$

$$(A.1)$$

where the parameters are known to Alice and satisfy the normalization condition, that is,

$$|\iota_1|^2 + |\iota_2|^2 = 1.$$

To implement the protocol, Alice and Bob share a three-qubit Greenberger–Horne–Zeilinger (GHZ) state as a quantum resource which is

$$|E\rangle_{A_1 A_2 B} = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle),$$

(A.2)

where Alice holds the first two qubits A_1, A_2 and Bob holds the third qubit B.

Based on the known values of ι_1 and ι_2, Alice now defines a set of four mutually orthogonal two-qubit basis vectors for her qubits A_1, A_2, given by

$$
\begin{aligned}
|M_1\rangle_{A_1 A_2} &= (\iota_1|00\rangle + \iota_2|11\rangle), \\
|M_2\rangle_{A_1 A_2} &= (\iota_2|00\rangle - \iota_1|11\rangle), \\
|M_3\rangle_{A_1 A_2} &= (\iota_1|01\rangle + \iota_2|10\rangle), \\
|M_4\rangle_{A_1 A_2} &= (\iota_2|01\rangle - \iota_1|10\rangle).
\end{aligned}
$$

(A.3)

Using this basis, the shared quantum resource can be written as

$$
|E\rangle_{A_1 A_2 B} = \frac{1}{\sqrt{2}} \Big[|M_1\rangle_{A_1 A_2} \otimes (\iota_1|0\rangle + \iota_2|1\rangle)_B
$$
$$
+ |M_2\rangle_{A_1 A_2} \otimes (\iota_2|0\rangle - \iota_1|1\rangle)_B \Big].
$$

(A.4)

Alice then performs a measurement on her two qubits on the basis $\{|M_1\rangle, |M_2\rangle, |M_3\rangle, |M_4\rangle\}$. After the measurement, Alice communicates her

result to Bob using a classical channel. Based on this information, Bob applies the corresponding unitary operation given in [Table A.1](#) to recover the target state $|\aleph\rangle$.

Table A1

The appropriate unitary operations Bob needs to apply are summarized below: ⏎

Alice's result	State of Bob's site	Bob's operation
$|M_1\rangle_{A_1A_2}$	$(\iota_1|0\rangle + \iota_2|1\rangle)_B$	I
$|M_2\rangle_{A_1A_2}$	$(\iota_2|0\rangle - \iota_1|1\rangle)_B$	$\sigma_x\sigma_z$

If the outcome of Alice's measurement is $|M_1\rangle_{A_1A_2}$, then the state at Bob's site becomes $(\iota_1|0\rangle + \iota_2|1\rangle)_B$, which is the same as the intended state. So in this case Bob uses an identity operation, which is to say that Bob need not act in any way.

If the outcome of Alice's measurement is $|M_2\rangle_{A_1A_2}$, then the state at Bob's site becomes $(\iota_2|0\rangle - \iota_1|1\rangle)_B$. To complete the protocol and obtain the original quantum state, Bob applies a unitary operation $\sigma_x\sigma_z$.

Thus, Bob successfully reconstructs the desired state $|\aleph\rangle$ using the shared entangled channel and classical communication, although the state was never physically transmitted and was never possessed by the party intending to create the state at the site of the receiver Bob.

B Joint Remote State Preparation Protocol of Single-qubit State

In Remote State Preparation (RSP), a single sender, typically referred to as Alice, assists a receiver, Bob, in generating a quantum state known to her at Bob's location using a shared entangled state and exchanging classical information. In this process, the entire knowledge of the quantum state is possessed by a single party, which may not be ideal in scenarios involving multiple parties. In some scenarios, the complete information about the quantum state to be prepared may not be available to a single party due to technical constraints. Instead, the information is distributed between two separate parties. In that case, to enable the remote preparation of such a state at a distant location, a new type of protocol called Joint Remote State Preparation (JRSP) is introduced. Each sender only knows partial information about the quantum state, and none of them alone can perform the preparation. Joint remote state preparation protocols have been discussed in a good number of papers like [4, 110, 119, 120, 197].

Here, we discuss a basic JRSP protocol involving two senders: Alice and Candy and one receiver, namely Bob. Alice and Candy jointly want to remotely prepare an arbitrary single-qubit state

$$|\aleph\rangle_a = \iota_1|0\rangle + \iota_2 e^{i\kappa}|1\rangle,$$

in Bob's laboratory. The parameters ι_1, ι_2 satisfy the normalization condition, that is, $|\iota_1|^2 + |\iota_2|^2 = 1$ and the phase parameter $\kappa \in (0, 2\pi)$. Both senders know only partial information about the state, and the receiver does not know anything about the intended state. In this protocol, we assume that Alice knows ι_1, ι_2 and Candy knows the phase parameter κ.

To initiate the protocol, three parties share a three-qubit maximally entangled Greenberger–Horne–Zeilinger (GHZ) state as a quantum resource, which is given by

$$|E\rangle_{ACB} = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle),$$

(B.2)

where Alice holds the first qubit A, Candy holds the second qubit C and the third qubit B belongs to Bob.

Based on the known information of ι_1, ι_2, Alice makes a projective measurement of her single qubit on the basis given by

$$|M_1\rangle_A = (\iota_1|0\rangle + \iota_2|1\rangle),$$
$$|M_2\rangle_A = (\iota_2|0\rangle - \iota_1|1\rangle).$$

(B.3)

Using this basis, the shared quantum resource $|E\rangle_{ACB}$ can be expressed as

$$|E\rangle_{ACB} = \frac{1}{\sqrt{2}}\left[|M_1\rangle_A \otimes (\iota_1|00\rangle + \iota_2|11\rangle)_{CB} + |M_2\rangle_A \otimes (\iota_2|00\rangle - \iota_1|11\rangle)_{CB}\right].$$

After the measurement, Alice transmits her outcomes through 1-bit classical channels to Candy and the receiver Bob. Depending on the measurement results

received from Alice, Candy chooses two sets of basis for measurement of his qubit.

If Alice's outcome is $|M_1\rangle_A$, then Candy performs a projective measurement on the basis given by

$$|N_1^1\rangle_C = (|0\rangle + e^{-i\kappa}|1\rangle),$$
$$|N_2^1\rangle_C = (|0\rangle - e^{-i\kappa}|1\rangle).$$

(B.4)

If Alice's outcome is $|M_2\rangle_A$, then Candy performs a projective measurement on the basis given by

$$|N_1^2\rangle_C = (e^{-i\kappa}|0\rangle + |1\rangle),$$
$$|N_2^2\rangle_C = (e^{-i\kappa}|0\rangle - |1\rangle).$$

(B.5)

Case I:

If Alice's result is $|M_1\rangle_A$, then the state of the remaining particles becomes (ignoring the constant factor)

$$|E_1\rangle = (\iota_1|00\rangle + \iota_2|11\rangle)_{CB}.$$

Using the basis B.4, the above reduced state $|E_1\rangle_{CB}$ can be written as

$$|E_1\rangle = \frac{1}{2}\left[|N_1^1\rangle_C \otimes (\iota_1|0\rangle + \iota_2 e^{i\kappa}|1\rangle)_B + |N_2^1\rangle_C \otimes (\iota_1|0\rangle - \iota_2 e^{i\kappa}|1\rangle)_B\right]$$

Candy now performs his single-qubit projective measurement with the basis $\{|N_1^1\rangle, |N_2^1\rangle\}$. After the measurement, he sends his result classically to Bob. Finally, after receiving all the classical information from the senders, he applies an appropriate unitary operation given in Table B.1 to prepare the intended state $\aleph\rangle$.

Table B1

The appropriate unitary operations performed by Bob corresponding to Alice's outcome $|M_1\rangle_A$ ⏎

Candy's result	State of Bob's site	Unitary operation performed by Bob
$|N_1^1\rangle_C$	$(\iota_1|0\rangle + \iota_2 e^{i\kappa}|1\rangle)_B$	I
$|N_2^1\rangle_C$	$(\iota_1|0\rangle - \iota_2 e^{i\kappa}|1\rangle)_B$	σ_z

If the outcome of Candy's measurement is $|N_1^1\rangle_C$, then the reduced state at Bob's site becomes $(\iota_1|0\rangle + \iota_2 e^{i\kappa}|1\rangle)_B$, which is the same as the intended state. So in this case Bob uses an identity operation, that is, does not have to act.

If the outcome of Alice's measurement is $|N_2^1\rangle_C$, then the state at Bob's site becomes $(\iota_1|0\rangle - \iota_2 e^{i\kappa}|1\rangle)_B$. To complete the protocol and obtain the intended quantum state, Bob applies a unitary operation σ_z on his qubit. That is end of the protocol.

Case II:

If Alice's result is $|M_2\rangle_A$, then the state of the remaining particles becomes (ignoring the constant factor)

$$|E_2\rangle = (\iota_2|00\rangle - \iota_1|11\rangle)_{CB}.$$

Using the basis B.5, the above reduced state $|E_2\rangle_{CB}$ can be written as

$$|E_2\rangle = \frac{1}{2}\left[|N_1^2\rangle_C \otimes (e^{i\kappa}\iota_2|0\rangle + \iota_1|1\rangle)_B + |N_2^2\rangle_C \otimes (e^{i\kappa}\iota_2|0\rangle - \iota_1|1\rangle)_B\right]$$

Candy now performs his single-qubit projective measurement with the basis $\{|N_1^2\rangle, |N_2^2\rangle\}$. After the measurement, he sends his result classically to Bob. Finally, after receiving all the classical information from the senders, he applies an appropriate unitary operation given in Table B.2 to recover the intended state $|\aleph\rangle$.

Table B2

The appropriate unitary operations performed by Bob corresponding to Alice's outcome $|M_1\rangle_A$ ⏎

Candy's result	State of Bob's site	Unitary operation performed by Bob
$|N_1^2\rangle_C$	$(e^{i\kappa}\iota_2|0\rangle + \iota_1|1\rangle)_B$	σ_x
$|N_2^2\rangle_C$	$(e^{i\kappa}\iota_2|0\rangle - \iota_1|1\rangle)_B$	$\sigma_x\sigma_z$

If the outcome of Candy's measurement is $|N_1^2\rangle_C$, then the state at Bob's site becomes $(e^{i\kappa}\iota_2|0\rangle + \iota_1|1\rangle)_B$. To obtain the intended state, Bob uses a unitary operation σ_x on his particle.

If the outcome of Candy's measurement is $|N_2^2\rangle_C$, then the state at Bob's site becomes $(e^{i\kappa}\iota_2|0\rangle - \iota_1|1\rangle)_B$. To complete the protocol and obtain the original quantum state, Bob applies a unitary operation $\sigma_x\sigma_z$ on his qubit. That is the end of the protocol.

C Hybrid Bi-directional Communication Protocol

In the theory of quantum communications, teleportation enables the transfer of an unknown quantum state using shared entangled resource and classical communication while RSP allows the creation of a known quantum state at a distant location when the sender has knowledge of the state.

A hybrid bi-directional communication protocol integrates both QT and RSP in a single protocol, enabling simultaneous two-way quantum information transfer between two parties (namely Alice and Bob). In such protocols, one party (say Alice) teleports an unknown quantum state to Bob, while Bob simultaneously prepares a known quantum state at Alice's location using a prior shared entangled resource. The following works [28, 102, 153, 174] include hybrid protocols amongst others.

Here, we assume that Alice wants to transfer an unknown single-qubit state $|\aleph_1\rangle_a$ to Bob and simultaneously Bob wants to create a known single-qubit state $|\aleph_2\rangle$ to Alice. These states are given by

$$
\begin{aligned}
|\aleph_1\rangle_a &= \iota_1|0\rangle + \iota_2|1\rangle, \\
|\aleph_2\rangle &= \varphi_1|0\rangle + \varphi_2|1\rangle,
\end{aligned}
$$

$$(C.1)$$

where coefficients $\iota_1, \iota_2, \varphi_1, \varphi_2$ meet the normalization conditions, that is,

$$|\iota_1|^2 + |\iota_2|^2 = 1,$$

and

$$|\varphi_1|^2 + |\varphi_2|^2 = 1.$$

To complete the communication task, Alice and Bob share a 4-qubit entangled state as a quantum resource, which is given by

$$|E\rangle_{A_1B_1A_2B_2} = \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)_{A_1B_1A_2B_2},$$

(C.2)

where the qubits (a, A_1, A_2) and (B_1, B_2) are held by Alice and Bob, respectively.

The entire system can be expressed as

$$
\begin{aligned}
|\Gamma\rangle =& \ |\aleph_1\rangle_a \otimes |E\rangle_{A_1B_1A_2B_2} \\
=& \ (\iota_1|0\rangle + \iota_2|1\rangle)_a \otimes \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)_{A_1B_1A_2B_2}.
\end{aligned}
$$

(C.3)

Bob's measurement basis is given by

$$
\begin{aligned}
|M_1\rangle_{B_2} &= \varphi_1|0\rangle + \varphi_2|1\rangle, \\
|M_2\rangle_{B_2} &= \varphi_2|0\rangle - \varphi_1|1\rangle.
\end{aligned}
$$

(C.4)

Alice's measurement basis is given by

$$|\Upsilon_1\rangle_{aA_1} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$

$$|\Upsilon_2\rangle_{aA_1} = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$

$$|\Upsilon_3\rangle_{aA_1} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$

$$|\Upsilon_4\rangle_{aA_1} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

<div align="right">(C.5)</div>

The choice of such bases is possible since the coefficients φ_1, φ_2 are known to Bob.

Using the above basis $\{|M_1\rangle_{B_2}, |M_2\rangle_{B_2}\}$, the entire quantum system can be rewritten as

$$
\begin{aligned}
|\Gamma\rangle &= (\iota_1|0\rangle + \iota_2|1\rangle)_a \otimes \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)_{A_1B_1A_2B_2} \\
&= (\iota_1|0\rangle + \iota_2|1\rangle)_a \otimes \frac{1}{2}\Big[|M_1\rangle_{B_2} \otimes \Big(\varphi_1|000\rangle + \varphi_2|001\rangle + \varphi_1|110\rangle \\
&\quad - \varphi_2|111\rangle\Big)_{A_1B_1A_2} + |M_2\rangle_{B_2} \otimes \Big(\varphi_2|000\rangle - \varphi_1|001\rangle + \varphi_2|110\rangle \\
&\quad + \varphi_1|111\rangle\Big)_{A_1B_1A_2}\Big].
\end{aligned}
$$

<div align="right">(C.6)</div>

After the measurement, Bob classically sends his results to Alice. After that the following cases arise. A nonlocal (CZ) -operation is involved in each case. This operation can be performed by a third party who can have access to the two involved qubits.

Case 1:

If Bob's measurement result is $|M_1\rangle_{B_2}$, then the reduced state of the remaining particles becomes

$$
\begin{aligned}
|\Gamma_1\rangle = \; & (\iota_1|0\rangle + \iota_2|1\rangle)_a \otimes \frac{1}{2}\Big(\varphi_1|000\rangle + \varphi_2|001\rangle + \varphi_1|110\rangle - \varphi_2|111\rangle\Big)_{A_1 B_1 A_2} \\
= \; & \frac{1}{\sqrt{2}}\Big[|\Upsilon_1\rangle_{a A_1} \otimes (\iota_1\varphi_1|00\rangle + \iota_1\varphi_2|01\rangle + \iota_2\varphi_1|10\rangle - \iota_2\varphi_2|11\rangle)_{B_1 A_2} \\
& + |\Upsilon_2\rangle_{a A_1} \otimes (\iota_1\varphi_1|00\rangle + \iota_1\varphi_2|01\rangle - \iota_2\varphi_1|10\rangle + \iota_2\varphi_2|11\rangle)_{B_1 A_2} \\
& + |\Upsilon_3\rangle_{a A_1} \otimes (\iota_1\varphi_1|10\rangle - \iota_1\varphi_2|11\rangle + \iota_2\varphi_1|00\rangle + \iota_2\varphi_2|01\rangle)_{B_1 A_2} \\
& + |\Upsilon_4\rangle_{a A_1} \otimes (\iota_1\varphi_1|10\rangle - \iota_1\varphi_2|11\rangle - \iota_2\varphi_1|00\rangle - \iota_2\varphi_2|01\rangle)_{B_1 A_2} \Big].
\end{aligned}
$$

$$(C.7)$$

Now Alice makes her measurement on the basis $\{|\Upsilon_1\rangle_{a A_1}, |\Upsilon_2\rangle_{a A_1}, |\Upsilon_3\rangle_{a A_1}, |\Upsilon_4\rangle_{a A_1}\}$ and after completing the measurement then transmits the measurement result to Bob.

After that a quantum phase gate (CZ) operation is applied on qubit pairs (B_1, A_2) with qubit A_2 acting as a control qubit and qubit B_1 as the target qubit.

Sub-case I Suppose Alice's measurement outcome is $|\Upsilon_1\rangle_{a A_1}$, then the reduced state becomes

$$
\begin{aligned}
& (\iota_1\varphi_1|00\rangle + \iota_1\varphi_2|01\rangle + \iota_2\varphi_1|10\rangle - \iota_2\varphi_2|11\rangle)_{B_1 A_2} \\
\xrightarrow{CZ} \; & (\iota_1\varphi_1|00\rangle + \iota_1\varphi_2|01\rangle + \iota_2\varphi_1|10\rangle + \iota_2\varphi_2|11\rangle)_{B_1 A_2} \\
= \; & (\iota_1|0\rangle + \iota_2|1\rangle)_{B_1} \otimes (\varphi_1|0\rangle + \varphi_2|1\rangle)_{A_2}
\end{aligned}
$$

Finally, after receiving all the measurement results, Alice and Bob perform appropriate unitary operation to get the original state. In this case, both the parties perform identity operation on their respective particles, that is, they are not required to act in the situation under this sub-case. That is the end of the protocol.

Sub-case II If Alice's measurement outcome is $|\Upsilon_2\rangle_{a A_1}$, then the reduced state becomes

$$(\iota_1\varphi_1|00\rangle + \iota_1\varphi_2|01\rangle - \iota_2\varphi_1|10\rangle + \iota_2\varphi_2|11\rangle)_{B_1A_2}$$
$$\xrightarrow{CZ} (\iota_1\varphi_1|00\rangle + \iota_1\varphi_2|01\rangle - \iota_2\varphi_1|10\rangle - \iota_2\varphi_2|11\rangle)_{B_1A_2}$$
$$= (\iota_1|0\rangle - \iota_2|1\rangle)_{B_1} \otimes (\varphi_1|0\rangle + \varphi_2|1\rangle)_{A_2}$$

Finally, after receiving all the measurement results, Alice and Bob perform appropriate unitary operations $(I)_{A_2}$ and $(\sigma_z)_{B_1}$ to recover the original state. That is the end of the protocol.

Sub-case III If Alice's measurement outcome is $|\Upsilon_3\rangle_{aA_1}$, then the reduced state becomes

$$(\iota_1\varphi_1|10\rangle - \iota_1\varphi_2|11\rangle + \iota_2\varphi_1|00\rangle + \iota_2\varphi_2|01\rangle)_{B_1A_2}$$
$$\xrightarrow{CZ} (\iota_1\varphi_1|10\rangle + \iota_1\varphi_2|11\rangle + \iota_2\varphi_1|00\rangle + \iota_2\varphi_2|01\rangle)_{B_1A_2}$$
$$= (\iota_1|1\rangle + \iota_2|0\rangle)_{B_1} \otimes (\varphi_1|0\rangle + \varphi_2|1\rangle)_{A_2}$$

Finally, after receiving all the measurement results, Alice and Bob perform appropriate unitary operations $(I)_{A_2}$ and $(\sigma_x)_{B_1}$ to recover the original state. That is the end of the protocol.

Sub-case IV If Alice's measurement outcome is $|\Upsilon_4\rangle_{aA_1}$, then the reduced state becomes

$$(\iota_1\varphi_1|10\rangle - \iota_1\varphi_2|11\rangle - \iota_2\varphi_1|00\rangle - \iota_2\varphi_2|01\rangle)_{B_1A_2}$$
$$\xrightarrow{CZ} (\iota_1\varphi_1|10\rangle + \iota_1\varphi_2|11\rangle - \iota_2\varphi_1|00\rangle - \iota_2\varphi_2|01\rangle)_{B_1A_2}$$
$$= (\iota_1|1\rangle - \iota_2|0\rangle)_{B_1} \otimes (\varphi_1|0\rangle + \varphi_2|1\rangle)_{A_2}$$

Finally, after receiving all the measurement results, Alice and Bob perform appropriate unitary operations $(I)_{A_2}$ and $(\sigma_z\sigma_x)_{B_1}$ to recover the original state. That is the end of the protocol.

Case 2:

If Bob's measurement result is $|M_2\rangle_{B_2}$, then the reduced state of the remaining particles becomes

$$|\Gamma_2\rangle = (\iota_1|0\rangle + \iota_2|1\rangle)_a \otimes \frac{1}{2}\left(\varphi_2|000\rangle - \varphi_1|001\rangle + \varphi_2|110\rangle + \varphi_1|111\rangle\right)_{A_1B_1A_2}$$

$$= \frac{1}{\sqrt{2}}\left[|\Upsilon_1\rangle_{aA_1} \otimes (\iota_1\varphi_2|00\rangle - \iota_1\varphi_1|01\rangle + \iota_2\varphi_2|10\rangle + \iota_2\varphi_1|11\rangle)_{B_1A_2}\right.$$

$$+ |\Upsilon_2\rangle_{aA_1} \otimes (\iota_1\varphi_2|00\rangle - \iota_1\varphi_1|01\rangle - \iota_2\varphi_2|10\rangle - \iota_2\varphi_1|11\rangle)_{B_1A_2}$$

$$+ |\Upsilon_3\rangle_{aA_1} \otimes (\iota_1\varphi_2|10\rangle + \iota_1\varphi_1|11\rangle + \iota_2\varphi_2|00\rangle - \iota_2\varphi_1|01\rangle)_{B_1A_2}$$

$$\left. + |\Upsilon_4\rangle_{aA_1} \otimes (\iota_1\varphi_2|10\rangle + \iota_1\varphi_1|11\rangle - \iota_2\varphi_2|00\rangle + \iota_2\varphi_1|01\rangle)_{B_1A_2}\right].$$

$$(C.8)$$

Now Alice makes her measurement on the basis $\{|\Upsilon_1\rangle_{aA_1}, |\Upsilon_2\rangle_{aA_1}, |\Upsilon_3\rangle_{aA_1}, |\Upsilon_4\rangle_{aA_1}\}$ and after completing the measurement she transmits the outcome to Bob through a classical channel.

After that a quantum phase gate (CZ) operation is applied on qubit pairs (B_1, A_2) with qubit A_2 acting as a control qubit and qubit B_1 as the target qubit.

Sub-case I If Alice's measurement outcome is $|\Upsilon_1\rangle_{aA_1}$, then the reduced state becomes

$$(\iota_1\varphi_2|00\rangle - \iota_1\varphi_1|01\rangle + \iota_2\varphi_2|10\rangle + \iota_2\varphi_1|11\rangle)_{B_1A_2}$$

$$\xrightarrow{CZ} (\iota_1\varphi_2|00\rangle - \iota_1\varphi_1|01\rangle + \iota_2\varphi_2|10\rangle - \iota_2\varphi_1|11\rangle)_{B_1A_2}$$

$$= (\iota_1|0\rangle + \iota_2|1\rangle)_{B_1} \otimes (\varphi_2|0\rangle - \varphi_1|1\rangle)_{A_2}$$

Finally, after receiving the measurement results, Alice and Bob perform appropriate unitary operation $(\sigma_x\sigma_z)_{A_2}$ and $(I)_{B_1}$ on their respective qubits to get the original state. That is the end of the protocol.

Sub-case II If Alice's measurement outcome is $|\Upsilon_2\rangle_{aA_1}$, then the reduced state becomes

$$(\iota_1\varphi_2|00\rangle - \iota_1\varphi_1|01\rangle - \iota_2\varphi_2|10\rangle - \iota_2\varphi_1|11\rangle)_{B_1A_2}$$

$$\xrightarrow{CZ} (\iota_1\varphi_2|00\rangle - \iota_1\varphi_1|01\rangle - \iota_2\varphi_2|10\rangle + \iota_2\varphi_1|11\rangle)_{B_1A_2}$$

$$= (\iota_1|0\rangle - \iota_2|1\rangle)_{B_1} \otimes (\varphi_2|0\rangle - \varphi_1|1\rangle)_{A_2}$$

Finally, after receiving all the measurement results, Alice and Bob perform appropriate unitary operations $(\sigma_x\sigma_z)_{A_2}$ and $(\sigma_z)_{B_1}$ to recover the original states. That is the end of the protocol.

Sub-case III If Alice's measurement outcome is $|\Upsilon_3\rangle_{aA_1}$, then the reduced state becomes

$$
\begin{aligned}
&\quad (\iota_1\varphi_2|10\rangle + \iota_1\varphi_1|11\rangle + \iota_2\varphi_2|00\rangle - \iota_2\varphi_1|01\rangle)_{B_1A_2} \\
&\xrightarrow{CZ} (\iota_1\varphi_2|10\rangle - \iota_1\varphi_1|11\rangle + \iota_2\varphi_2|00\rangle - \iota_2\varphi_1|01\rangle)_{B_1A_2} \\
&= \quad (\iota_1|1\rangle + \iota_2|0\rangle)_{B_1} \otimes (\varphi_2|0\rangle - \varphi_1|1\rangle)_{A_2}
\end{aligned}
$$

Finally, after receiving all the measurement results, Alice and Bob perform appropriate unitary operations $(\sigma_x\sigma_z)_{A_2}$ and $(\sigma_x)_{B_1}$ to recover the original states. That is the end of the protocol.

Sub-case IV If Alice's measurement outcome is $|\Upsilon_4\rangle_{aA_1}$, then the reduced state becomes

$$
\begin{aligned}
&\quad (\iota_1\varphi_2|10\rangle + \iota_1\varphi_1|11\rangle - \iota_2\varphi_2|00\rangle + \iota_2\varphi_1|01\rangle)_{B_1A_2} \\
&\xrightarrow{CZ} (\iota_1\varphi_2|10\rangle - \iota_1\varphi_1|11\rangle - \iota_2\varphi_2|00\rangle + \iota_2\varphi_1|01\rangle)_{B_1A_2} \\
&= \quad (\iota_1|1\rangle - \iota_2|0\rangle)_{B_1} \otimes (\varphi_2|0\rangle - \varphi_1|1\rangle)_{A_2}
\end{aligned}
$$

Finally, after receiving all the measurement results, Alice and Bob perform appropriate unitary operations $(\sigma_x\sigma_z)_{A_2}$ and $(\sigma_z\sigma_x)_{B_1}$ to recover the original states. That concludes the protocol.

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