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Quantum Theory and Fuzzy Systems: Traversing Uncertainty in Group Decision-Making and Social Networks

Quantum and Fuzzy Approaches to Social
Network Analysis and Group Decisions

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Editors

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This book is dedicated to all the pioneers in the fields of quantum theory, fuzzy systems, decision-making and social networks.

Tofiq Allahviranloo would like to dedicate the book to his mother.

Sovan Samanta would like to dedicate the book to his parents, wife, and kids, Adrita and Adrit.

Preface

The journey of writing this book has been both challenging and rewarding. The idea for this work stemmed from a deep curiosity about how quantum theory and fuzzy systems can be applied to real-world problems, particularly in group decision-making and graph theory/social networks.

This book aims to reduce the gap between abstract theoretical concepts of quantum theory and their practical implementations in decision-making and network theory. The chapters cover a wide range of topics, starting with the fundamentals of uncertainty in quantum theory and fuzzy systems, and the basics of quantum graphs. It then explores various types of quantum graphs, multi-attribute group decision-making methods integrating TOPSIS under linguistic information, and the selection of eco-friendly building materials using Pythagorean linguistic information. The book also includes the selection of wearable health technology devices, renewable energy options, and cell phones using different quantum decision-making methodologies. Further, it discusses green supplier selection, investment decision-making amidst uncertainty, and sustainable solar energy management. Advanced topics include dynamic centrality measures for brain networks, routing protocols, centrality metrics, link prediction, and applications of quantum planar graphs. The book also covers topological indices, domination theory, quantum computing applications, fuzzy and quantum social networks, competition scenarios, and optimization techniques in quantum graphs.

We hope that this book will serve as a valuable resource for students, researchers, and professionals who are keen to explore the ideas of applications of quantum theory in network sciences, quantum computing and decision making.

Istanbul, Turkey
Tamluk, India
November 2024

Tofiq Allahviranloo
Sovan Samanta

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The completion of this book would not have been possible without the support and encouragement of many individuals and organizations. We would like to express my deepest gratitude to:

- our colleagues and mentors, whose invaluable insights and guidance have significantly shaped the content and direction of this book.
- the research community for their continuous efforts and contributions to the fields of quantum theory and fuzzy systems, which have provided a rich foundation for this work.

We are thankful to all authors of the chapters including Dr. Prasenjit Mandal, Dr. Rupkumar Mahapatra.

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Uncertainty in Quantum Theory and Fuzzy Systems



Sovan Samanta^{ID} and Tofigh Allahviranloo^{ID}

Abstract This chapter investigates the uncertainties in both quantum theory and fuzzy systems. Quantum theory includes uncertainty as a core aspect of particle behavior at the atomic and subatomic levels in the form of superposition and entanglement. On the other hand, fuzzy systems offer a tool for dealing with uncertainty in more formal contexts, such as control systems and decision-making processes, where binary logic is not applicable. This chapter aims to provide an understanding of how these two fields can complement each other. We illustrate the representation and management of uncertainty in both domains. This combination of quantum and fuzzy uncertainty not only enhances theoretically but also opens new areas for practical implementations for future research and development in this interdisciplinary area.

Keywords Quantum theory · Fuzzy systems · Uncertainty · Superposition · Entanglement · Fuzzy logic

1 Introduction

Quantum theory, developed in the early 20th century, revolutionized the characteristics of microscopic world. It introduced concepts such as wave-particle duality, superposition, and entanglement. The uncertainty principle, formulated by Werner Heisenberg, is a popular theory of quantum mechanics, stating that certain pairs

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of physical properties, like position and momentum, cannot be simultaneously measured [1]. Fuzzy systems, introduced by Lotfi Zadeh in 1965, provide a mathematical tool for dealing with imprecise and vague information. Unlike binary logic, which operates on true or false values, fuzzy logic allows for degrees of truth, making it well-suited for modeling complex systems where uncertainty and ambiguity are present [2]. In [35], authors studied fuzzy variable precision rough sets, in [36, 37], authors introduced hesitant bipolar-valued fuzzy sets and bipolar-valued hesitant fuzzy sets, and studied multi-granulation bipolar-valued fuzzy probabilistic rough sets and their corresponding three-way decisions. Fuzzy systems have found applications in various fields, including control systems, decision-making, and artificial intelligence. Quantum theory addresses uncertainty at the primary level of physical reality, fuzzy systems offer practical tools for handling uncertainty in real-world applications. The motivation behind this study is to combine quantum uncertainty and fuzzy logic to make a powerful tool of capturing of uncertainties.

The capturing of fuzziness in various graph structures is a significant theme across multiple studies such as intuitionistic fuzzy competition graphs in complex systems [38], applications in scenarios where tolerance levels between entities are crucial [39], medical diagnosis to handle uncertainty and imprecision in patient data [40], complex Pythagorean fuzzy graphs, applications in various fields [41].

Uncertainty is a fundamental aspect of both quantum theory and fuzzy systems, which questions the completeness of quantum descriptions of reality [4, 16–18]. Fuzzy systems introduce uncertainty differently, linguistic Z-numbers enable modeling of uncertainty in dynamic systems [24, 26, 34].

The exploration of uncertainty in quantum theory through various studies reveals a rich interplay between quantum mechanics and fuzzy logic such as quantum mechanical approach to fuzzy theory, [42], the evolution and interconnections of these paradigms [43], in managing quantum uncertainties [44], in artificial intelligence and unconventional computing [45], advancements in quantum computing [46], emphasizing their theoretical and practical implications [47]. Ishikawa and Kikuchi (2021) analysed quantum fuzzy logic and its temporal aspects on quantum uncertainty [48]. Kaaz (1987) examined a quantum-like uncertainty relation for complementary fuzzy sets, contributing to the theoretical understanding of quantum fuzziness [49]. Other ongoing advancements in this field can be found in [50] and practical benefits of combining quantum theory and fuzzy logic [51].

The primary objectives of this chapter are:

- to explain the concept of uncertainty in quantum theory, focusing on superposition and entanglement.
- to discuss uncertainties in fuzzy systems and their applications.
- to combine quantum theory and fuzzy systems in practical applications.

2 Quantum Theory and Uncertainty

Quantum theory revolutionizes our understanding of the universe. The principle of superposition states that a quantum system can exist in multiple states simultaneously until it is observed. This leads to wave-particle duality, where particles like electrons exhibit both wave-like and particle-like properties. Measurement collapses the superposition into a single state. This principal includes randomness and uncertainty [3].

Mathematically, Superposition is represented by a state vector Φ , including all possible states. For a system with states $|\phi_1\rangle$ and $|\phi_2\rangle$, the superposition is:

$$|\Phi\rangle = \alpha|\phi_1\rangle + \beta|\phi_2\rangle$$

where α and β are complex coefficients satisfying:

$$|\alpha|^2 + |\beta|^2 = 1$$

Upon measurement, the state vector collapses to one of the basis states, with probabilities $|\alpha|^2$ and $|\beta|^2$. This collapse introduces randomness and uncertainty [6].

Wave-particle duality further complicates our concept of quantum systems. Particles such as electrons exhibit both wave-like and particle-like properties, depending on the experimental setup. This duality was famously demonstrated in the double-slit experiment, where electrons create an interference pattern when not observed, indicating wave-like behavior, but act as particles when observed [7].

Entanglement is a unique quantum phenomenon where the quantum states of two or more particles become interconnected. When particles are entangled, the state of one particle is directly related to the state of another, no matter the distance between them. This non-locality challenges the notions of separability and locality [8].

For two entangled particles, the combined state is:

$$|\Phi\rangle = \frac{1}{\sqrt{2}} (|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B)$$

where $|0\rangle$ and $|1\rangle$ represent possible states. Measurement of one particle immediately determines the state of the other. Experimental verification of entanglement has been achieved through tests of Bell's inequalities, confirming the non-local nature of quantum entanglement [9].

2.1 Uncertainty in Quantum Theory

Heisenberg's uncertainty principle limits the precision with which pairs of physical properties, such as position and momentum, can be known simultaneously. This principle is expressed as:

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$$

where Δx is the uncertainty in position, Δp is the uncertainty in momentum, and $\hbar$ is the reduced Planck constant [1].

The uncertainty principle implies that nature is probabilistic rather than deterministic, distinguishing quantum mechanics from classical physics [10].

Quantum measurement plays a crucial role in uncertainty. Measurement collapses the state vector to a definite state, introducing randomness. This collapse is fundamental to quantum mechanics, leading to concepts about reality and the observer's role [11].

Decoherence describes how a quantum system loses coherence and behaves classically due to environmental interactions. It provides a mechanism for state vector collapse and explains the transition from quantum to classical behavior [12].

Mathematically, decoherence is described using the density matrix ρ . For a pure state $|\phi\rangle$, the density matrix is:

$$\rho = |\phi\rangle\langle\phi|$$

Interaction with the environment causes the off-diagonal elements of the density matrix to decay, leading to a mixed state:

$$\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i|$$

where p_i are probabilities and $|\phi_i\rangle$ are possible states.

Quantum uncertainty is discussed as like the Copenhagen interpretation, which states that physical systems lack definite properties until measured, and the many-worlds interpretation, suggesting all possible outcomes occur in parallel universes [13–15].

The Copenhagen interpretation, developed by Bohr and Heisenberg, emphasizes the observer's role in measurement. Quantum systems exist in superposition until observed, collapsing to a definite state upon measurement [10].

The many-worlds interpretation, proposed by Everett, suggests all possible outcomes of a quantum measurement occur in separate universes, eliminating state vector collapse and providing a deterministic framework [15].

3 Fuzzy Systems

Fuzzy systems provide a tool for dealing with uncertainty introduced by Zadeh to handle the concept of partial truth. In fuzzy logic, truth values range between 0 and 1 [2].

Fuzzy set theory is used to represent vague information. For instance, the development of fuzzy fractional differential equations using fuzzy Laplace transforms

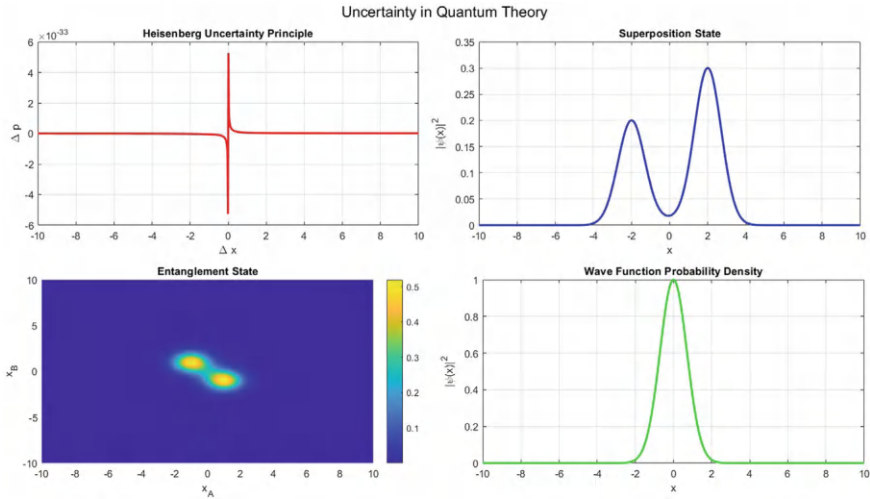


Fig. 1 Top Left Plot: This graph illustrates the Heisenberg Uncertainty Principle, which states that the product of the uncertainties in position (Δx) and momentum (Δp). Top Right Plot: This graph represents the probability density function $|\psi(x)|^2$ for a superposition state. Bottom Left Plot: This surface plot depicts the probability density function $|\psi(x_A, x_B)|^2$ for an entangled state. Bottom Right Plot: This graph shows the probability density function $|\psi(x)|^2$ for a single Gaussian wave function. The plot visualizes the spread of the probability distribution, which is centered around $x = 0$

is fit for modeling systems where the influence of individuals or groups evolves fractionally over time under uncertainty [16, 21] (Fig. 1).

The introduction of predictor-corrector methods (in solving fuzzy differential equations) has direct applications in social groups [17, 19, 20]. These methods allow for a more accurate simulation of how consensus forms over time, particularly in environments where group dynamics are shaped by uncertainty and imprecision. The bipolar fuzzy systems describes the concepts of positivity and negativity properties.

Fuzzy graph theory such as fuzzy threshold graphs, fuzzy competition graphs, fuzzy planar graphs and applications of edge coloring in fuzzy graphs is used to solve some real world problems.

Samanta et al. proposed generalized fuzzy graph theory connecting relationships of edges and end vertices. Generalised fuzzy graphs are further extended to quantum graphs with dynamic weights of vertices and edges changing over times [34].

Authors [5] have studied the use of fuzzy logic in the optimization of industrial processes. In 2016, Samanta et al. introduced on the completeness and regularity of generalized fuzzy graphs offered significant insights into the structural properties of these graphs [22]. In 2021, they introduced a measure of influences in social networks, which has practical implications for understanding and analyzing social dynamics [23]. The RSM index, proposed in 2019, presented a novel approach to link prediction in social networks, enhancing the accuracy of predicting future connections

[24]. Their work on radio fuzzy graphs and the assignment of frequency in radio stations in 2019 provided innovative solutions for frequency allocation problems [25]. Additionally, their research on the applications of edge coloring of fuzzy graphs in 2020 demonstrated the utility of fuzzy graph theory in solving coloring problems [26]. The study on generalized neutrosophic planar graphs and its applications in 2021 further expanded the scope of fuzzy graph theory [27]. Their contributions also include the development of methods for link prediction in social networks using neutrosophic graphs [28]. Other notable works include their research on vague graphs and strengths [29]. More works can be found in generalised fuzzy graph representation [30] and competition cluster hypergraphs [31]. Their research on generalized fuzzy trees [32] and coepetition bunch graphs [33] further demonstrate their innovative contributions to fuzzy graph theory and its applications in various fields. In 2015, they introduced a full fuzzy method for solving differential equations based on Taylor expansion for handling uncertainties in differential equations [18]. Other notable works include their research on fuzzy integer linear programming problems [17]. Their contributions also extend to the fuzzy solution of LR fuzzy linear systems [16], fuzzy linear fractional differential equations [18].

The first subplot of the Fig. 2 illustrates the membership functions for the input variable “Speed Error,” showing how different levels of error are categorized into negative, zero, and positive ranges using trapezoidal and triangular membership functions. The second subplot displays the membership functions for the “Change in Speed Error,” similarly categorized into negative, zero, and positive ranges. The third subplot presents the rule surface visualized as a 3D surface plot. This surface plot helps in understanding how the fuzzy rules interact to produce the control output.

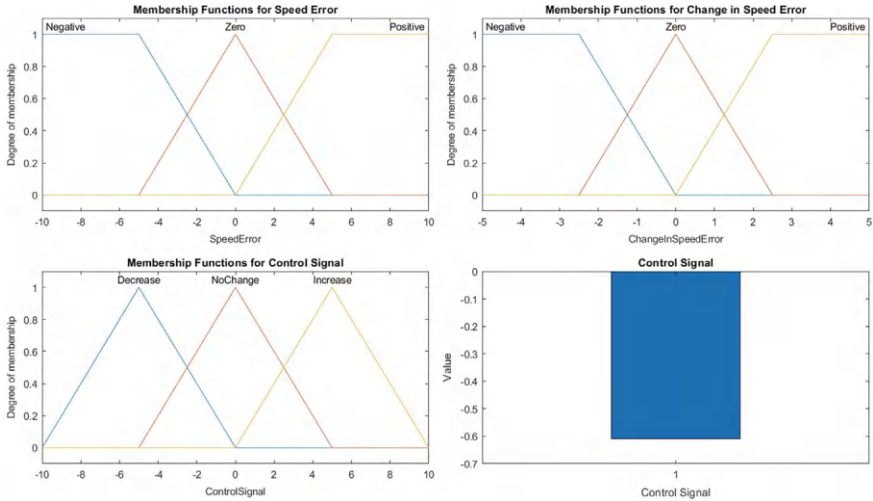


Fig. 2 Visualization of the fuzzy logic controller’s membership functions

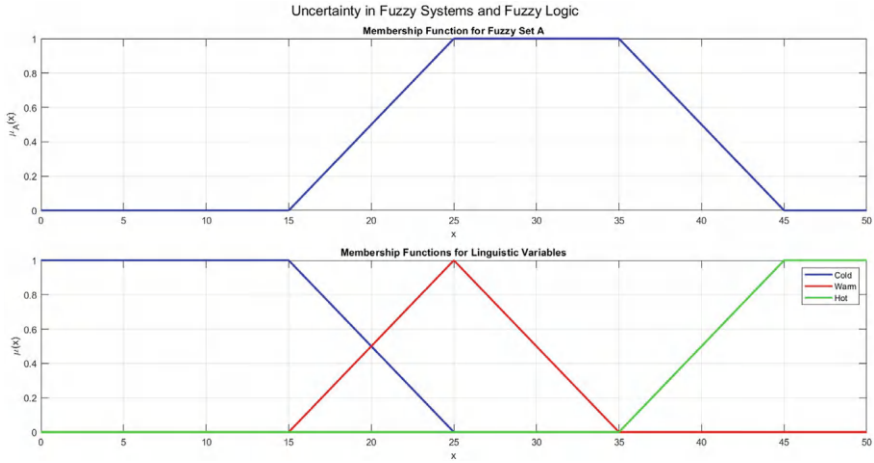


Fig. 3 Top Plot: This graph represents the membership function for a fuzzy set A within a universe of discourse x . Bottom Plot: This graph shows the membership functions for the linguistic variables “cold,” “warm,” and “hot,” which are used in fuzzy logic to handle imprecise concepts. This plot demonstrates how fuzzy logic uses linguistic variables and their membership functions to model and reason with imprecise concepts, enabling decision-making in complex and uncertain environments

Finally, the fourth subplot shows the output membership function for the “Control Signal,” indicating the degree of membership for the resulting control action (Fig. 3).

3.1 Uncertainty in Fuzzy Systems

Fuzzy systems handle uncertainty by using fuzzy sets and membership functions. A fuzzy set A in a universe of discourse X is characterized by a membership function $\mu_A : X \rightarrow [0, 1]$.

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \text{ is completely in } A \\ 0 & \text{if } x \text{ is not in } A \\ \text{a value between 0 and 1} & \text{if } x \text{ is partially in } A \end{cases} \quad (1)$$

3.2 Fuzzy Logic and Linguistic Variables

In fuzzy logic, linguistic variables are assumed to handle imprecise concepts. For example, the variable “temperature” may have linguistic values such as “cold,” “warm,” and “hot,” each represented by a fuzzy set.

$$\text{Temperature} = \{\text{cold, warm, hot}\} \quad (2)$$

Each linguistic value is associated with a membership function that defines its degree of truth. For instance, the membership function for “warm” might be defined as:

$$\mu_{\text{warm}}(x) = \begin{cases} 0 & \text{if } x \leq 15 \\ \frac{x-15}{10} & \text{if } 15 < x < 25 \\ 1 & \text{if } 25 \leq x \leq 35 \\ \frac{45-x}{10} & \text{if } 35 < x < 45 \\ 0 & \text{if } x \geq 45 \end{cases} \quad (3)$$

4 Combining Quantum Theory and Fuzzy Systems

We consider a quantum algorithm that uses a fuzzy inference system to determine the terms of quantum states. Mathematically, this can be represented as:

$$|\psi\rangle = \sum_{i=1}^n \mu_i |i\rangle$$

where μ_i are the membership values determined by the fuzzy inference system, and $|i\rangle$ are the basis states of the quantum system.

Another application is in the modeling of complex systems, where the probabilistic nature of quantum mechanics can be combined with the approximate reasoning of fuzzy logic to develop more accurate models. For example, in fluid dynamics, the behavior of particles can be modeled using quantum graphs, while the uncertainties in the system parameters can be managed using fuzzy logic. The state of a particle in a quantum graph can be described by a wave function $\psi(x, t)$, which changes according to the Schrödinger equation:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(x)\psi$$

where $V(x)$ is the potential energy function. By incorporating fuzzy logic, the potential energy function $V(x)$ can be represented as a fuzzy set, allowing for the modeling of uncertainties in the system. This results in a fuzzy Schrödinger equation:

$$i\hbar \frac{\partial \tilde{\psi}}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \tilde{\psi} + \tilde{V}(x) \tilde{\psi}$$

where $\tilde{\psi}$ and $\tilde{V}(x)$ are fuzzy wave functions and fuzzy potential energy functions, respectively.

The overall decision can then be made by combining the membership values using a quantum superposition:

$$|\Psi\rangle = \sum_{i=1}^n \mu_i |\psi_i\rangle$$

where μ_i are the membership values determined by the fuzzy inference system.

4.1 Fuzzy Quantum Sets

A Fuzzy Quantum Set $\tilde{Q}$ in a universe of discourse X is characterized by a membership function $\mu_{\tilde{Q}} : X \rightarrow [0, 1]$ and a quantum state function $\psi : X \rightarrow \mathbb{C}$ such that:

$$\mu_{\tilde{Q}}(x) = |\psi(x)|^2 \quad (4)$$

where $\psi(x)$ is the wave function representing the quantum state, and $|\psi(x)|^2$ is the probability density function.

4.2 Superposition in Fuzzy Quantum Sets

In a Fuzzy Quantum Set, elements can exist in a superposition of states. For example, if ψ_1 and ψ_2 are two possible states, then any linear combination $\psi = c_1\psi_1 + c_2\psi_2$ is also a valid state:

$$\mu_{\tilde{Q}}(x) = |c_1\psi_1(x) + c_2\psi_2(x)|^2 \quad (5)$$

4.2.1 Example: Superposition of States

Consider a particle that can be in two states $\psi_1(x) = \sqrt{0.6}$ and $\psi_2(x) = \sqrt{0.4}$. The superposition state is:

$$\psi(x) = \frac{1}{\sqrt{2}}(\psi_1(x) + \psi_2(x)) \quad (6)$$

The membership function is:

$$\mu_{\tilde{Q}}(x) = \left| \frac{1}{\sqrt{2}}(\sqrt{0.6} + \sqrt{0.4}) \right|^2 = 0.5 \quad (7)$$

4.3 Entanglement in Fuzzy Quantum Sets

Elements in a Fuzzy Quantum Set can be entangled, meaning the membership function of one element depends on the state of another. For an entangled pair (x_A, x_B) :

$$\mu_{\tilde{Q}}(x_A, x_B) = \left| \frac{1}{\sqrt{2}}(\psi_A(x_A)\psi_B(x_B) + \psi_A(x_B)\psi_B(x_A)) \right|^2 \quad (8)$$

4.3.1 Example: Entangled Qubits

Consider two entangled qubits in the Bell state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B) \quad (9)$$

The membership function for the Fuzzy Quantum Set is:

$$\mu_{\tilde{Q}}(x_A, x_B) = \left| \frac{1}{\sqrt{2}}(\delta_{x_A,0}\delta_{x_B,1} + \delta_{x_A,1}\delta_{x_B,0}) \right|^2 \quad (10)$$

where δ is the Kronecker delta function.

4.4 Theorem 1: Linearity of Superposition

Statement: If $\tilde{Q}_1$ and $\tilde{Q}_2$ are Fuzzy Quantum Sets with wave functions ψ_1 and ψ_2 , respectively, then any linear combination $\alpha\psi_1 + \beta\psi_2$ (where $\alpha, \beta \in \mathbb{C}$) is also a valid wave function for a Fuzzy Quantum Set.

Proof: Let $\tilde{Q}_1$ and $\tilde{Q}_2$ be Fuzzy Quantum Sets with wave functions ψ_1 and ψ_2 . A linear combination $\psi = \alpha\psi_1 + \beta\psi_2$ is considered. The membership function for the combined state is:

$$\mu_{\tilde{Q}}(x) = |\psi(x)|^2 = |\alpha\psi_1(x) + \beta\psi_2(x)|^2$$

Using the properties of the modulus of complex numbers, we have:

$$\mu_{\tilde{Q}}(x) = |\alpha|^2 |\psi_1(x)|^2 + |\beta|^2 |\psi_2(x)|^2 + \alpha\beta^* \psi_1(x) \psi_2^*(x) + \alpha^* \beta \psi_1^*(x) \psi_2(x)$$

Since $\mu_{\tilde{Q}}(x)$ is a valid membership function (i.e., it maps X to $[0, 1]$), ψ is a valid wave function for a Fuzzy Quantum Set.

Thus, the linear combination of wave functions is a valid wave function for a Fuzzy Quantum Set.

4.5 Theorem 2: Orthogonality and Measurement

Statement: If ψ_1 and ψ_2 are orthogonal wave functions (i.e., $\langle \psi_1 | \psi_2 \rangle = 0$), then measuring a Fuzzy Quantum Set in the state $\psi = \alpha\psi_1 + \beta\psi_2$ will collapse it to ψ_1 with probability $|\alpha|^2$ and to ψ_2 with probability $|\beta|^2$.

Proof: Let ψ_1 and ψ_2 be orthogonal wave functions such that $\langle \psi_1 | \psi_2 \rangle = 0$. Consider the state $\psi = \alpha\psi_1 + \beta\psi_2$. The probability of measuring the state and finding it in ψ_1 is given by:

$$P(\psi_1) = |\langle \psi_1 | \psi \rangle|^2 = |\langle \psi_1 | (\alpha\psi_1 + \beta\psi_2) \rangle|^2$$

Since ψ_1 and ψ_2 are orthogonal, $\langle \psi_1 | \psi_2 \rangle = 0$, so:

$$P(\psi_1) = |\alpha \langle \psi_1 | \psi_1 \rangle + \beta \langle \psi_1 | \psi_2 \rangle|^2 = |\alpha \langle \psi_1 | \psi_1 \rangle|^2 = |\alpha|^2$$

Similarly, the probability of measuring the state and finding it in ψ_2 is:

$$P(\psi_2) = |\langle \psi_2 | \psi \rangle|^2 = |\beta|^2$$

Thus, the measurement probabilities are $|\alpha|^2$ and $|\beta|^2$ for ψ_1 and ψ_2 , respectively.

4.6 Theorem 3: Non-commutativity and Uncertainty

Statement: For two non-commuting observables $\hat{A}$ and $\hat{B}$ in a Fuzzy Quantum Set, the uncertainty relation is given by:

$$\Delta A \cdot \Delta B \geq \frac{1}{2} |\langle [\hat{A}, \hat{B}] \rangle|$$

Proof: Let $\hat{A}$ and $\hat{B}$ be two observables with the commutator $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$. The uncertainty in A and B is defined as:

$$\Delta A = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2}, \quad \Delta B = \sqrt{\langle \hat{B}^2 \rangle - \langle \hat{B} \rangle^2}$$

Using the Cauchy-Schwarz inequality in the context of quantum mechanics, we have:

$$\Delta A \cdot \Delta B \geq \frac{1}{2} | \langle [\hat{A}, \hat{B}] \rangle |$$

Since $[\hat{A}, \hat{B}]$ is an operator, its expectation value $\langle [\hat{A}, \hat{B}] \rangle$ is a complex number.

Thus, the uncertainty relation for non-commuting observables is established.

5 Conclusion

In this chapter, we conceptualized the combination of quantum mechanics and fuzzy logic. Despite the promising potential of combining quantum theory and fuzzy systems, there are several limitations that need to be addressed. Quantum algorithms and fuzzy inference systems both require significant computational resources. Additionally, the development of hybrid models that requires further research and refinement. This area of fuzzy quantum sets may be the backbone of future research area including control systems and decision making.

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Basics of Quantum Graphs



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Abstract This chapter discusses the concepts of quantum graphs. Quantum graphs extend graph theory by incorporating quantum mechanical effects such as superposition and entanglement, representing quantum phenomena. We describe the concept of dynamic vertices, where each vertex has a time-dependent weight, and entangled edges, which represent correlations between vertex weights. Additionally, we analyse various dweight functions and their variations, highlighting how different choices can impact the properties of quantum graphs. Several properties of strength of connectedness in quantum graphs have been proved.

Keywords Quantum graphs · Dynamic weights · Strength of connectedness

1 Introduction

Quantum networks are well researched topic in advancing quantum communication and computation [40], in quantum chaos and universal spectral statistics, providing a comprehensive understanding of their mathematical underpinnings [41],

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for financial fraud detection, as demonstrated [42], for signal processing, as presented [43], network models in materials search [44], significance in secure communications [46]. The quantum networks based on algebraic structure lacks the user friendly graph theoretical structure. Recently, quantum graphs are introduced by Samanta et al. [4]. These are mathematical structures that extend graph theory to dynamical systems such as biological networks. Superposition and entanglement are main role in quantum theory. The concept of quantum graphs emerged from the necessity to model and analyze systems where quantum effects cannot be ignored. In graph theory, vertices and edges are static. These can represent fixed relationships and properties. But, in quantum graphs, these relationships are dynamic. Quantum graphs introduce dynamic weights for vertices and entangled edges, which can change over time and exhibit correlations that are not present in graphs.

1.1 Objectives

The primary objectives of this chapter are to:

- characterise quantum graphs and their components.
- establish the properties and applications of quantum graphs.
- present area of applications of quantum graphs.

2 Literature Review

Fuzzy systems provide a tool for dealing with uncertainty introduced by Zadeh to handle the concept of partial truth. In fuzzy logic, truth values range between 0 and 1. Fuzzy set theory is used to represent vague information. For instance, the development of fuzzy fractional differential equations using fuzzy Laplace transforms is fit for modeling systems where the influence of individuals or groups evolves fractionally over time under uncertainty [14].

The introduction of predictor-corrector methods (in solving fuzzy differential equations) has direct applications in social groups [15, 17–22]. Fuzzy graph theory [28] such as fuzzy competition graphs [6], fuzzy planar graphs [5] is used to solve some real world problems. In 2015, they introduced a full fuzzy method for solving differential equations based on Taylor expansion for handling uncertainties in differential equations [16]. Other notable works include their research on fuzzy integer linear programming problems [21], the trapezoidal approximation of fuzzy numbers [22].

Fuzzy graphs [23–25, 27] are used in various fields, from physics to machine learning, for tasks involving quantum systems, such as quantum computing and network modelling [24, 26, 27, 29, 30, 45]. Researchers have advanced their works in fields like decision processes and neural networks [1, 2, 23–25]

Theoretical developments of fuzzy graphs [3, 12, 13] are studied in competitive and tolerant interactions [34, 35], fuzzy graphs in practical applications [36], applicability of fuzzy logic in graph theory [37], algebraic structure and syntactic recognition of fuzzy graphs, underscoring their significance in the broader context of graph research [38, 39]. dealing with vague information [31], in multi-attribute decision-making processes [32] the complex decision-making scenarios [33].

Samanta et al. [10] proposed generalized fuzzy graph theory connecting relationships of edges and end vertices [7–9, 11]. Generalised fuzzy graphs are further extended to quantum graphs with dynamic weights of vertices and edges changing over times [4].

3 Quantum Graphs

Reference [4] a quantum graph with dynamic vertices and entangled edges is defined as follows:

1. **Vertices with Dynamic Weights (dweight):** Each vertex v_i has a dynamic weight $\text{dweight}_i(t)$ that changes over time:

$$0 \leq \text{dweight}_i(t) \leq 1$$

2. **Edges and Entanglement:** Edges e_{ij} between vertices v_i and v_j can be entangled, represented by a correlation function $g_{ij}(t)$:

$$g_{ij}(t) = h(\text{dweight}_i(t), \text{dweight}_j(t))$$

3. **Graph Structure:** The graph G is represented as $G = (V, E)$, with an adjacency matrix $A(t)$:

$$A_{ij}(t) = \begin{cases} g_{ij}(t) & \text{if } e_{ij} \text{ exists and is entangled} \\ 0 & \text{otherwise} \end{cases}$$

3.1 Example: Quantum Sensor Network

A network of quantum sensors can be modeled as a quantum graph where each sensor's sensitivity $\text{dweight}_i(t)$ changes over time due to calibration and environmental factors. The edges represent entangled states between sensors, enhancing the overall sensitivity of the network. The correlation function $g_{ij}(t)$ quantifies the entanglement, and the adjacency matrix $A(t)$ is used to analyze and improve the network's performance in detecting weak signals.

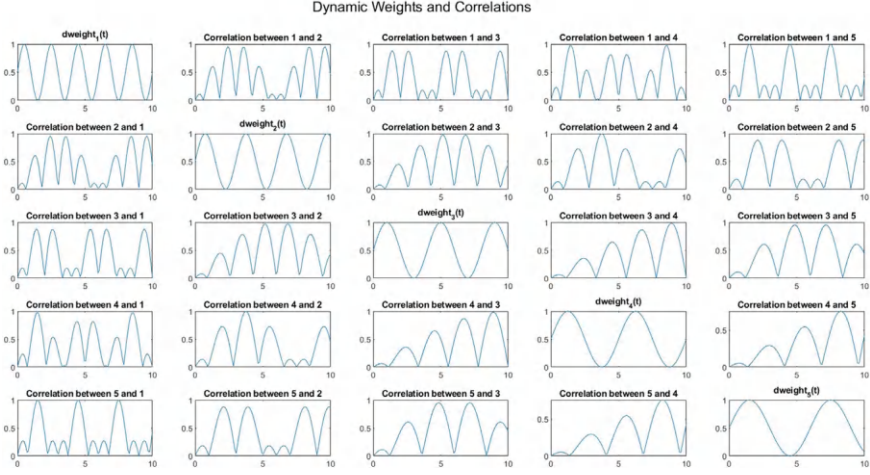


Fig. 1 Dynamic weights and correlations of quantum graph vertices

Let us construct a quantum graph of five vertices (see Fig. 1). We visualize the dynamic behavior of quantum graph vertices and their entanglement correlations over time. Each vertex v_i is assigned a dynamic weight $dweight_i(t)$, which varies sinusoidally to simulate over time. These dynamic weights are plotted along the diagonal subplots, showing how each vertex's weight changes over time. The off-diagonal subplots illustrate the correlations between pairs of vertices, defined by the function $h(dweight_i(t), dweight_j(t))$, which measures the absolute difference between the dynamic weights of two vertices. This setup allows us to observe both the individual behavior of each vertex and the entanglement relationships between them, providing a comprehensive view of the quantum graph's dynamic properties. We assume the following vertex dweights.

$$dweight_1(t) = 0.5 \left(1 + \sin \left(2\pi \frac{t}{2} \right) \right)$$

$$dweight_2(t) = 0.5 \left(1 + \sin \left(2\pi \frac{t}{3} \right) \right)$$

$$dweight_3(t) = 0.5 \left(1 + \sin \left(2\pi \frac{t}{4} \right) \right)$$

$$dweight_4(t) = 0.5 \left(1 + \sin \left(2\pi \frac{t}{5} \right) \right)$$

$$dweight_5(t) = 0.5 \left(1 + \sin \left(2\pi \frac{t}{6} \right) \right)$$

4 Choosing Dweight Functions and Variations

4.1 Examples of Dweight Functions

The choice of dweight function depends on the properties to model. Few examples include the following.

- **Sinusoidal Function:**

$$\text{dweight}_i(t) = \frac{1}{2} (1 + \sin(\omega_i t + \phi_i))$$

This function will be helpful for periodic behaviors/ oscillations in dynamical systems.

- **Exponential Decay:**

$$\text{dweight}_i(t) = e^{-\lambda_i t}$$

This function is useful for the vertices whose weights decay for example the cooling of a thermal system.

- **Gaussian Function:**

$$\text{dweight}_i(t) = e^{-\frac{(t-\mu_i)^2}{2\sigma_i^2}}$$

This function can represent phenomena that peak at a certain time and then decrease.

4.2 Correlations Among Nodes

Examples of correlation functions h include the following:

- **Linear Correlation:**

$$h(\text{dweight}_i(t), \text{dweight}_j(t)) = \text{dweight}_i(t) \cdot \text{dweight}_j(t)$$

This function models directly related to the product of their states.

- **Cosine Similarity:**

$$h(\text{dweight}_i(t), \text{dweight}_j(t)) = \cos(\pi \cdot (\text{dweight}_i(t) - \text{dweight}_j(t)))$$

This function measures the similarity between the weights of two nodes.

4.2.1 Example: Quantum Sensor Network with Gaussian Response

For a network of quantum sensors with a Gaussian response to a transient event:

$$\text{dweight}_i(t) = e^{-\frac{(t-\mu_i)^2}{2\sigma_i^2}}$$

The correlation between sensors could be modeled using cosine similarity:

$$h(\text{dweight}_i(t), \text{dweight}_j(t)) = \cos(\pi \cdot (\text{dweight}_i(t) - \text{dweight}_j(t)))$$

This setup can be used to analyze the collective response of sensors for a sudden change in the environment.

5 Properties of Quantum Graphs

Note 2.1 (Dynamic Behavior of Vertex Weights) In a quantum graph G , the dynamic weights $\text{dweight}_i(t)$ of the vertices are continuous functions of time t .

The dynamic weight $\text{dweight}_i(t)$ of a vertex v_i is a function that changes over time:

$$0 \leq \text{dweight}_i(t) \leq 1$$

Assume that the dynamic weights $\text{dweight}_i(t)$ are governed by a continuous process, such as a differential equation or a smooth function of time.

If $\text{dweight}_i(t)$ is a continuous function of time, then small changes in time t result in small changes in $\text{dweight}_i(t)$.

Therefore, the dynamic weights $\text{dweight}_i(t)$ of the vertices in a quantum graph G are continuous functions of time t .

Definition 2.1 (Strength of Connectedness) Let $P = (v_1, v_2, \dots, v_n)$ be a path in a quantum graph G with dynamic weights and entangled edges. The strength of connectedness $S(P, t)$ of the path P at time t is defined as the product of the average values of the local minima of the edges along the path:

$$S(P, t) = \prod_{i=1}^{n-1} \left(\frac{1}{k} \sum_{j=1}^k \min(g_{i,i+1}(t_j)) \right)$$

where $g_{i,i+1}(t_j)$ is the correlation function between vertices v_i and v_{i+1} at time t_j , and k is the number of local minima considered for each edge $e_{i,i+1}$.

Property 2.1 (Boundedness of Strength of Connectedness) *The strength of connectedness $S(P, t)$ is bounded by the product of the dynamic weights of the vertices along the path. Specifically:*

$$0 \leq S(P, t) \leq \prod_{i=1}^n \text{dweight}_i(t)$$

Proof The strength of connectedness $S(P, t)$ for a path $P = (v_1, v_2, \dots, v_n)$ is defined as:

$$S(P, t) = \prod_{i=1}^{n-1} \left(\frac{1}{k} \sum_{j=1}^k \min(g_{i,i+1}(t_j)) \right)$$

where $g_{i,i+1}(t_j)$ is the correlation function between vertices v_i and v_{i+1} at time t_j , and k is the number of local minima considered for each edge $e_{i,i+1}$.

By definition, the correlation functions $g_{i,i+1}(t)$ are bounded between 0 and 1:

$$0 \leq g_{i,i+1}(t) \leq 1$$

Therefore, the local minima of these correlation functions are also bounded:

$$0 \leq \min(g_{i,i+1}(t_j)) \leq 1$$

The average of the local minima for each edge $e_{i,i+1}$ is also bounded between 0 and 1:

$$0 \leq \frac{1}{k} \sum_{j=1}^k \min(g_{i,i+1}(t_j)) \leq 1$$

Since each term in the product $\prod_{i=1}^{n-1} \left(\frac{1}{k} \sum_{j=1}^k \min(g_{i,i+1}(t_j)) \right)$ is bounded between 0 and 1, the entire product is also bounded between 0 and 1:

$$0 \leq \prod_{i=1}^{n-1} \left(\frac{1}{k} \sum_{j=1}^k \min(g_{i,i+1}(t_j)) \right) \leq 1$$

The dynamic weights $\text{dweight}_i(t)$ of the vertices are bounded between 0 and 1:

$$0 \leq \text{dweight}_i(t) \leq 1$$

The strength of connectedness $S(P, t)$ is the product of the bounded average values of the local minima of the edges and the dynamic weights of the vertices:

$$S(P, t) = \prod_{i=1}^{n-1} \left(\frac{1}{k} \sum_{j=1}^k \min(g_{i,i+1}(t_j)) \right) \cdot \prod_{i=1}^n \text{dweight}_i(t)$$

Since both the average values of the local minima and the dynamic weights are bounded between 0 and 1, their product is also bounded:

$$0 \leq S(P, t) \leq \prod_{i=1}^n \text{dweight}_i(t)$$

Therefore, the strength of connectedness $S(P, t)$ is bounded by the product of the dynamic weights of the vertices along the path, completing the proof. ■ □

Definition 2.2 (*Maximum Possibility of Connectedness*) Let $P = (v_1, v_2, \dots, v_n)$ be a path in a quantum graph G with dynamic weights and entangled edges. The maximum possibility of connectedness $M(P, t)$ of the path P at time t is defined as the product of the average of the local maxima of the edges along the path:

$$M(P, t) = \prod_{i=1}^{n-1} \left(\frac{1}{k} \sum_{j=1}^k \max(g_{i,i+1}(t_j)) \right)$$

where $g_{i,i+1}(t_j)$ is the correlation function between vertices v_i and v_{i+1} at time t_j , and k is the number of local maxima considered for each edge $e_{i,i+1}$.

Theorem 2.1 (*Boundedness of Maximum Possibility of Connectedness*) *The maximum possibility of connectedness $M(P, t)$ of a path P in a quantum graph G is bounded by 1. Specifically:*

$$0 \leq M(P, t) \leq 1$$

Proof The maximum possibility of connectedness $M(P, t)$ for a path $P = (v_1, v_2, \dots, v_n)$ is defined as:

$$M(P, t) = \prod_{i=1}^{n-1} \left(\frac{1}{k} \sum_{j=1}^k \max(g_{i,i+1}(t_j)) \right)$$

where $g_{i,i+1}(t_j)$ is the correlation function between vertices v_i and v_{i+1} at time t_j , and k is the number of local maxima considered for each edge $e_{i,i+1}$.

By definition, the correlation functions $g_{i,i+1}(t)$ are bounded between 0 and 1:

$$0 \leq g_{i,i+1}(t) \leq 1$$

Therefore, the local maxima of these correlation functions are also bounded:

$$0 \leq \max(g_{i,i+1}(t)) \leq 1$$

Since each term in the product $\prod_{i=1}^{n-1} \left(\frac{1}{k} \sum_{j=1}^k \max(g_{i,i+1}(t_j)) \right)$ is bounded between 0 and 1, the entire product is also bounded between 0 and 1:

$$0 \leq \prod_{i=1}^{n-1} \left(\frac{1}{k} \sum_{j=1}^k \max(g_{i,i+1}(t_j)) \right) \leq 1$$

Therefore, the maximum possibility of connectedness $M(P, t)$ is bounded by 1, completing the proof. ■ □

Theorem 2.2 *The peak of the Gaussian dweight function $\text{dweight}_i(t) = e^{-\frac{(t-\mu_i)^2}{2\sigma_i^2}}$ occurs at $t = \mu_i$ and is time-invariant.*

Proof The Gaussian function $\text{dweight}_i(t) = e^{-\frac{(t-\mu_i)^2}{2\sigma_i^2}}$ reaches its maximum value when the exponent is zero, i.e., when $t = \mu_i$. At this point, $\text{dweight}_i(t) = e^0 = 1$. Since μ_i is a constant, the peak position is time-invariant. □

Theorem 2.3 *The sinusoidal dweight function $\text{dweight}_i(t) = \frac{1}{2} (1 + \sin(\omega_i t + \phi_i))$ is periodic with period $T = \frac{2\pi}{\omega_i}$.*

Proof The sine function $\sin(\omega_i t + \phi_i)$ is periodic with period $\frac{2\pi}{\omega_i}$. Therefore, the dweight function:

$$\text{dweight}_i(t) = \frac{1}{2} (1 + \sin(\omega_i t + \phi_i))$$

inherits this periodicity. Thus, $\text{dweight}_i(t + T) = \text{dweight}_i(t)$ for $T = \frac{2\pi}{\omega_i}$. □

Theorem 2.4 (Monotonicity of Strength of Connectedness) *If the dynamic weights of all vertices in a path P increase over time, then the strength of connectedness $S(P, t)$ is non-decreasing.*

Proof The strength of connectedness $S(P, t)$ for a path $P = (v_1, v_2, \dots, v_n)$ is defined as:

$$S(P, t) = \prod_{i=1}^{n-1} \left(\frac{1}{k} \sum_{j=1}^k \min(g_{i,i+1}(t_j)) \right) \cdot \prod_{i=1}^n \text{dweight}_i(t)$$

Assume that the dynamic weights $\text{dweight}_i(t)$ of all vertices v_i in the path P are non-decreasing over time:

$$\text{dweight}_i(t_1) \leq \text{dweight}_i(t_2) \quad \text{for } t_1 \leq t_2$$

Since the correlation functions $g_{i,i+1}(t)$ depend on the dynamic weights, and assuming $g_{i,i+1}(t)$ is a non-decreasing function of $\text{dweight}_i(t)$ and $\text{dweight}_{i+1}(t)$, we have:

$$g_{i,i+1}(t_1) \leq g_{i,i+1}(t_2) \quad \text{for } t_1 \leq t_2$$

The product of non-decreasing functions is also non-decreasing. Therefore, the strength of connectedness $S(P, t)$ is non-decreasing:

$$S(P, t_1) \leq S(P, t_2) \quad \text{for } t_1 \leq t_2$$

Therefore, if the dynamic weights of all vertices in a path P increase over time, then the strength of connectedness $S(P, t)$ is non-decreasing. ■ □

Theorem 2.5 (Maximum Possibility of Connectedness and Path Length) *The maximum possibility of connectedness $M(P, t)$ of a path P in a quantum graph G decreases as the length of the path increases.*

Proof The maximum possibility of connectedness $M(P, t)$ for a path $P = (v_1, v_2, \dots, v_n)$ is defined as:

$$M(P, t) = \prod_{i=1}^{n-1} \max(g_{i,i+1}(t))$$

The correlation functions $g_{i,i+1}(t)$ are bounded between 0 and 1:

$$0 \leq g_{i,i+1}(t) \leq 1$$

As the length of the path P increases, the number of terms in the product $\prod_{i=1}^{n-1} \max(g_{i,i+1}(t))$ increases. Since each term $\max(g_{i,i+1}(t))$ is less than or equal to 1, the product of more terms will generally result in a smaller value:

$$\prod_{i=1}^{n-1} \max(g_{i,i+1}(t)) \leq \prod_{i=1}^{m-1} \max(g_{i,i+1}(t)) \quad \text{for } n > m$$

Therefore, the maximum possibility of connectedness $M(P, t)$ of a path P decreases as the length of the path increases. ■ □

Note 2.2 (*Stability of Connectedness*) In a quantum graph G , if the dynamic weights $\text{dweight}_i(t)$ of the vertices are stable over time, then the strength of connectedness $S(P, t)$ of any path P is also stable.

The dynamic weights $\text{dweight}_i(t)$ are stable if they do not change significantly over time:

$$\frac{d}{dt} \text{dweight}_i(t) \approx 0$$

If $\text{dweight}_i(t)$ are stable, then the correlation functions $g_{i,i+1}(t)$ are also stable:

$$\frac{d}{dt} g_{i,i+1}(t) \approx 0$$

Since $S(P, t)$ is a product of stable functions, it is also stable:

$$\frac{d}{dt}S(P, t) \approx 0$$

Therefore, if the dynamic weights $\text{dweight}_i(t)$ of the vertices are stable over time, then the strength of connectedness $S(P, t)$ of any path P is also stable.

Theorem 2.6 (Connectivity and Path Length) *In a quantum graph G , if there exists a path P of length n with non-zero strength of connectedness $S(P, t)$, then the graph G is connected.*

Proof The strength of connectedness $S(P, t)$ for a path $P = (v_1, v_2, \dots, v_n)$ is defined as:

$$S(P, t) = \prod_{i=1}^{n-1} \left(\frac{1}{k} \sum_{j=1}^k \min(g_{i,i+1}(t_j)) \right) \cdot \prod_{i=1}^n \text{dweight}_i(t)$$

If $S(P, t) > 0$, then each term in the product must be greater than zero:

$$\frac{1}{k} \sum_{j=1}^k \min(g_{i,i+1}(t_j)) > 0 \quad \text{and} \quad \text{dweight}_i(t) > 0$$

Since $\min(g_{i,i+1}(t_j)) > 0$, there must exist edges $e_{i,i+1}$ between each pair of consecutive vertices v_i and v_{i+1} .

The existence of a path P with non-zero strength of connectedness implies that all vertices in P are connected through edges with non-zero correlation functions.

Therefore, if there exists a path P of length n with non-zero strength of connectedness $S(P, t)$, then the graph G is connected. ■ □

6 Conclusion

In this chapter, we have investigated properties of quantum graphs. We focused on the strength of connectedness and the maximum possibility of connectedness. We defined these concepts rigorously and established their boundedness, demonstrating that the strength of connectedness is influenced by the dynamic weights of the vertices and the correlation functions of the edges. We also proved that the maximum possibility of connectedness decreases with increasing path length. Additionally, we examined the continuity and stability of dynamic weights, showing their impact on the overall connectedness of the graph. In future, problems on dynamical systems such as brain networks can be solved with the help of quantum graphs.

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A Study on Quantum Graphs



Sovan Samanta 

Abstract This chapter investigates the structural properties of quantum graphs, including degree, completeness, planarity, and bipartiteness, and their dynamic behavior. The research identifies gaps in the current literature such as fuzzy graphs and generalised fuzzy graphs, with quantum graphs. Additionally, some properties have been established.

Keywords Quantum graphs · Fuzzy graphs · Generalized fuzzy graphs · Planarity · Bipartiteness · Dynamical systems

1 Introduction

Graph theory is very useful tool in representation of networks such as social networks, biological networks, etc. Fuzzy graph theory extends the concept of graph theory by including uncertainty in the systems. Quantum graphs [14] are used to represent advanced dynamical networks. This chapter introduces the classifications of quantum graphs. To capture uncertainty, fuzzy set is very popular technique. For instance, the development of fuzzy fractional differential equations using fuzzy Laplace transforms is fit for modeling systems where the influence of individuals or groups evolves fractionally over time under uncertainty [24].

Graph theory has been characterized in discrete applied mathematics [42], in neural computing [43], in theoretical framework of fuzzy graph theory [44], in the concept of divisor 2-equitable domination in fuzzy graphs [45], in classifications that serve as the foundation for advanced studies [46], in classifications in real-world problems [47].

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A significant area of research within graph theory are the studies of planar bipartite graphs [35], correctness of distributed systems [36], a bijective enumeration of planar bipartite maps with three tight boundaries [37], graph-theoretic concepts to describe and analyze these states [38], context of quantum information processing [39]. Their research demonstrates how graph-theoretic principles can be applied to the design and analysis of quantum systems.

Allahviranloo has also contributed to the field of fuzzy systems and their applications in differential equations and linear systems. Allahviranloo et al. [24] explored solving fuzzy fractional differential equations using fuzzy Laplace transforms, while Allahviranloo et al. [25] developed numerical solutions for fuzzy differential equations using the predictor-corrector method. These works provide foundational methods that can be extended to the analysis of quantum graphs.

The introduction of predictor-corrector methods (in solving fuzzy differential equations) has direct applications in social groups [25, 27–29, 31]. These methods allow for a more accurate simulation of how consensus forms over time, particularly in environments where group dynamics are shaped by uncertainty and imprecision. The bipolar fuzzy systems [48, 49] describes the concepts of positivity and negativity properties. Trapezoidal approximations [26, 30, 32, 50] provide methods in decision-making processes.

In the field of fuzzy graph theory, Somasundaram and Somasundaram (1998) discussed domination in fuzzy graphs, providing foundational concepts that have been expanded upon in subsequent research [40]. Mordeson, Mathew, and Malik (2018) further explored applications of fuzzy graph theory to human trafficking, illustrating the practical utility of these theoretical constructs [41].

Theoretical developments of fuzzy graphs are studied in competitive and tolerant interactions [7, 8], fuzzy graphs in practical applications [9], applicability of fuzzy logic in graph theory [10], algebraic structure and syntactic recognition of fuzzy graphs, underscoring their significance in the broader context of graph research [11, 12]. dealing with vague information [4], in multi-attribute decision-making processes [5] the complex decision-making scenarios [6].

Fuzzy graphs, introduced by Rosenfeld [1], extend classical graph theory by incorporating uncertainties. This foundational work has been expanded on fuzzy graphs and fuzzy hypergraphs by Mordeson and Nair [2] exploring their properties and applications. Authors [3] investigated edge coloring in fuzzy graphs, highlighting its significance in network design and optimization. Fuzzy graph theory such as fuzzy competition graphs [16], fuzzy planar graphs [15] is used to solve some real world problems [13, 22, 23].

Generalized fuzzy graphs extend the concept of fuzzy graphs by allowing more functional relationships between vertices. Samanta and Sarkar [17] introduced generalized fuzzy graphs and discussed their properties and applications. Further studies by Samanta and Sarkar [18, 19] explored the isomorphism and visualization of generalized fuzzy graphs, providing tools for comparing and representing these complex structures. Samanta and Sarkar [20] also discussed generalized fuzzy Euler and Hamiltonian graphs, while Sarkar and Samanta [21] analysed generalized fuzzy trees.

Recent reviews on quantum networks such as those by Wei et al. [33] and DiAdamo et al. [34] highlight the progress and challenges in developing scalable quantum networks. These networks require robust protocols and physical hardware to realize large-scale quantum communication systems. Quantum graphs, as introduced by Samanta and Allahviranloo [14], represent a significant advancement in the study of graph theory. These graphs incorporate quantum mechanical properties, making them suitable for modeling quantum networks and systems.

This study finds the following research gaps.

- Introduction of different types of quantum graphs.
- Detailed analysis of the time-averaged properties of dynamic quantum graphs.
- Investigation of the conditions under which quantum graphs transition between different states (e.g., planar to non-planar, bipartite to non-bipartite).

2 Definition of Quantum Graph

A Quantum Graph is defined mathematically as follows:

1. Vertices: Each vertex v_i has a dynamic weight $\text{dweight}_i(t)$ that changes over time. The weight is a real number between 0 and 1:

$$0 \leq \text{dweight}_i(t) \leq 1$$

The weight may follow a specific pattern, for example, a sinusoidal function as follows:

$$\text{dweight}_i(t) = \frac{1}{2} (1 + \sin(\omega_i t + \phi_i))$$

where ω_i is the frequency and ϕ_i is the phase shift.

2. Edges: Edges e_{ij} between vertices v_i and v_j can be either entangled or not. If entangled, the edge is represented by a correlation function $g_{ij}(t)$ that describes the relationship between the weights of the connected vertices:

$$g_{ij}(t) = h(\text{dweight}_i(t), \text{dweight}_j(t))$$

where h is a function or correlation.

3. Graph Structure: The graph G is represented as $G = (V, E)$, where V is the set of vertices and E is the set of edges. The adjacency matrix $A(t)$ of the graph is defined as:

$$A_{ij}(t) = \begin{cases} g_{ij}(t) & \text{if } e_{ij} \text{ exists and is entangled} \\ 0 & \text{otherwise} \end{cases}$$

2.1 Dynamic Degree

The degree of a vertex in a quantum graph is defined as the number of adjacent edges over a time period i.e., the degree can change over time based on the weights of the vertices and the nature of the entanglement.

In a dynamic quantum graph, the degree of a vertex v_i at time t , denoted as $\deg(v_i, t)$, is defined as:

$$\deg(v_i, t) = \sum_j A_{ij}(t)$$

where $A_{ij}(t)$ is the time-dependent adjacency matrix element. The adjacency matrix $A(t)$ is defined as:

$$A_{ij}(t) = \begin{cases} g_{ij}(t) & \text{if } e_{ij} \text{ exists and is entangled} \\ 0 & \text{otherwise} \end{cases}$$

Here, $g_{ij}(t)$ is the correlation function describing the entanglement between vertices v_i and v_j .

2.1.1 Example: Sinusoidal Weight Function

The following weight function is assumed

$$\text{dweight}_i(t) = \frac{1}{2} (1 + \sin(\omega_i t + \phi_i))$$

The correlation function $g_{ij}(t)$ can be defined as:

$$g_{ij}(t) = \text{dweight}_i(t) \cdot \text{dweight}_j(t)$$

Thus, the dynamic degree of vertex v_i at time t becomes:

$$\deg(v_i, t) = \sum_j \text{dweight}_i(t) \cdot \text{dweight}_j(t)$$

2.2 Time-Averaged Degree

The time-averaged degree of a vertex v_i over a period T is given by:

$$\langle \deg(v_i) \rangle_T = \frac{1}{T} \int_0^T \deg(v_i, t) dt$$

This provides a measure of the average connectivity of the vertex over time.

2.3 Graph Regularity

A quantum graph is considered regular if the degree of each vertex remains constant over a time period. Formally a graph G is regular if:

$$\deg(v_i, t) = k \quad \forall v_i \in V, \forall t$$

where k is a constant. If the degree varies over time period, the graph is not regular.

2.3.1 Example of Regular and Non-regular Quantum Graphs

We consider a quantum graph with three vertices v_1 , v_2 , and v_3 . If the degrees of these vertices are constant over time, say $\deg(v_1, t) = \deg(v_2, t) = \deg(v_3, t) = 2$, the graph is regular. However, if the degrees change over time, for example, $\deg(v_1, t) = 2 + \sin(t)$, $\deg(v_2, t) = 2$, and $\deg(v_3, t) = 2 - \sin(t)$, the graph is not regular though at $t = n \times \pi$, $n = 0, 1, 2 \dots$, the values are same.

3 Completeness of Quantum Graphs

A quantum graph is complete if there is an edge between every pair of vertices but in case dynamical nature, it can vary based on the edge correlation function. For instance, if the edge weight becomes zero, the graph may no longer be complete.

A quantum graph $G = (V, E)$ is said to be complete if every pair of distinct vertices is connected by an edge. Mathematically, this means:

$$\forall v_i, v_j \in V, i \neq j \implies e_{ij} \in E$$

where e_{ij} denotes the edge between vertices v_i and v_j .

3.1 Dynamic Completeness

In a dynamic quantum graph, the completeness can change over time based on the edge correlation function $g_{ij}(t)$. The adjacency matrix $A(t)$ of a complete quantum graph at time t is defined as:

$$A_{ij}(t) = \begin{cases} g_{ij}(t) & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

For the graph to remain complete at time t , the correlation function $g_{ij}(t)$ must be non-zero for all $i \neq j$:

$$g_{ij}(t) \neq 0 \quad \forall i \neq j$$

3.2 Example: Correlation Function with Zero Values

Let us consider a quantum graph where the edge correlation function is defined as:

$$g_{ij}(t) = \left(\frac{1}{2} (1 + \sin(\omega_i t + \phi_i)) \right) \cdot \left(\frac{1}{2} (1 + \cos(\omega_j t + \phi_j)) \right) \cdot f(t)$$

where $f(t)$ is a function that can introduce zero points independently of the dynamic weights. For example, let:

$$f(t) = \cos(\omega t)$$

Thus, the correlation function becomes:

$$g_{ij}(t) = \left(\frac{1}{2} (1 + \sin(\omega_i t + \phi_i)) \right) \cdot \left(\frac{1}{2} (1 + \cos(\omega_j t + \phi_j)) \right) \cdot \cos(\omega t)$$

In this case, the dynamic weights are always non-zero, but the edge weight $g_{ij}(t)$ can be zero when $\cos(\omega t) = 0$, i.e., at $t = \frac{(2k+1)\pi}{2\omega}$ for integer k .

3.3 Example: Time-Dependent Completeness

We consider a quantum graph with three vertices v_1 , v_2 , and v_3 . The edge correlation functions are given by:

$$g_{12}(t) = \left(\frac{1}{2} (1 + \sin(\omega_1 t + \phi_1)) \right) \cdot \left(\frac{1}{2} (1 + \cos(\omega_2 t + \phi_2)) \right) \cdot \cos(\omega t)$$

$$g_{13}(t) = \left(\frac{1}{2} (1 + \sin(\omega_1 t + \phi_1)) \right) \cdot \left(\frac{1}{2} (1 + \cos(\omega_3 t + \phi_3)) \right) \cdot \cos(\omega t)$$

$$g_{23}(t) = \left(\frac{1}{2} (1 + \sin(\omega_2 t + \phi_2)) \right) \cdot \left(\frac{1}{2} (1 + \cos(\omega_3 t + \phi_3)) \right) \cdot \cos(\omega t)$$

The graph is complete at time t if:

$$g_{12}(t) \neq 0, \quad g_{13}(t) \neq 0, \quad g_{23}(t) \neq 0$$

Analyzing these functions, we find that the graph is incomplete at times $t = \frac{(2k+1)\pi}{2\omega}$ for integer k , as $\cos(\omega t) = 0$ at these times.

A quantum graph becomes incomplete if any edge correlation function $g_{ij}(t)$ becomes zero at any time t , affecting the overall connectivity and functionality of the graph.

4 Planarity and Bipartiteness Properties of Quantum Graphs

A graph is planar if it can be embedded in the plane such that no edges intersect except at their endpoints. Mathematically, a graph $G = (V, E)$ is planar if there exists a mapping of G into the plane $\mathbb{R}^2$ such that no two edges cross.

A simple quantum graph G with five vertices v_1, v_2, v_3, v_4, v_5 and edges $e_{12}, e_{13}, e_{14}, e_{15}, e_{23}, e_{24}, e_{25}$ is considered. The adjacency matrix $A(t)$ is given by:

$$A(t) = \begin{pmatrix} 0 & g_{12}(t) & g_{13}(t) & g_{14}(t) & g_{15}(t) \\ g_{12}(t) & 0 & g_{23}(t) & g_{24}(t) & g_{25}(t) \\ g_{13}(t) & g_{23}(t) & 0 & 0 & 0 \\ g_{14}(t) & g_{24}(t) & 0 & 0 & 0 \\ g_{15}(t) & g_{25}(t) & 0 & 0 & 0 \end{pmatrix}$$

If the edge correlation functions $g_{ij}(t)$ are non-zero and do not cause any edges to intersect, the graph remains planar.

Now, the same graph G but with additional edges e_{34} and e_{45} is considered:

$$A(t) = \begin{pmatrix} 0 & g_{12}(t) & g_{13}(t) & g_{14}(t) & g_{15}(t) \\ g_{12}(t) & 0 & g_{23}(t) & g_{24}(t) & g_{25}(t) \\ g_{13}(t) & g_{23}(t) & 0 & g_{34}(t) & g_{35}(t) \\ g_{14}(t) & g_{24}(t) & g_{34}(t) & 0 & g_{45}(t) \\ g_{15}(t) & g_{25}(t) & g_{35}(t) & g_{45}(t) & 0 \end{pmatrix}$$

If $g_{34}(t) \neq 0$ and $g_{45}(t) \neq 0$, the graph may become non-planar depending on the values of $g_{ij}(t)$. For instance, if $g_{34}(t)$ and $g_{45}(t)$ cause e_{34} and e_{45} to intersect with other edges, the graph is no longer planar.

5 Properties of Quantum Graphs

Theorem 1 *In a dynamic quantum graph, the time-averaged degree of a vertex v_i over a period T is given by:*

$$\langle \deg(v_i) \rangle_T = \frac{1}{T} \int_0^T \deg(v_i, t) dt$$

Proof By definition, the time-averaged degree of a vertex v_i over a period T is:

$$\langle \deg(v_i) \rangle_T = \frac{1}{T} \int_0^T \deg(v_i, t) dt$$

Since $\deg(v_i, t) = \sum_j A_{ij}(t)$, we have:

$$\langle \deg(v_i) \rangle_T = \frac{1}{T} \int_0^T \sum_j A_{ij}(t) dt = \sum_j \frac{1}{T} \int_0^T A_{ij}(t) dt$$

Thus, the time-averaged degree is the sum of the time-averaged adjacency matrix elements. To further elaborate, consider the adjacency matrix $A(t)$, where each element $A_{ij}(t)$ represents the presence and weight of an edge between vertices v_i and v_j at time t . The degree of vertex v_i at time t is the sum of the weights of all edges connected to v_i :

$$\deg(v_i, t) = \sum_j A_{ij}(t)$$

Integrating this over the period T and dividing by T gives the time-averaged degree:

$$\langle \deg(v_i) \rangle_T = \frac{1}{T} \int_0^T \sum_j A_{ij}(t) dt = \sum_j \frac{1}{T} \int_0^T A_{ij}(t) dt$$

This shows that the time-averaged degree is the average of the adjacency matrix elements over time. $\square$

Theorem 2 *A quantum graph is regular if and only if the degree of each vertex remains constant over time.*

Proof To prove this theorem, we need to show the followings: if a quantum graph is regular, then the degree of each vertex remains constant over time, and if the degree of each vertex remains constant over time, then the graph is regular.

(1) If a quantum graph is regular, then the degree of each vertex remains constant over time.

By definition, a regular graph is one in which every vertex has the same degree. In a dynamic quantum graph, this means that for all vertices v_i and for all times t , the degree $\deg(v_i, t)$ is constant. Let k be the constant degree. Then:

$$\deg(v_i, t) = k \quad \forall v_i \in V, \forall t$$

This implies that the sum of the adjacency matrix elements for each vertex is constant over time:

$$\sum_j A_{ij}(t) = k \quad \forall v_i \in V, \forall t$$

Thus, the degree of each vertex remains constant over time.

(2) If the degree of each vertex remains constant over time, then the graph is regular.

Assume that the degree of each vertex remains constant over time. Let k_i be the constant degree of vertex v_i . Then:

$$\deg(v_i, t) = k_i \quad \forall v_i \in V, \forall t$$

Since the degree is constant for each vertex, we have:

$$\sum_j A_{ij}(t) = k_i \quad \forall v_i \in V, \forall t$$

If $k_i = k$ for all vertices v_i , then the graph is regular with degree k . Therefore, if the degree of each vertex remains constant over time, the graph is regular.

Hence, we have shown that a quantum graph is regular if and only if the degree of each vertex remains constant over time. $\square$

Theorem 3 *A quantum graph is complete if and only if the edge correlation function $g_{ij}(t) \neq 0$ for all $i \neq j$ and for all t .*

Proof To prove this theorem, we need to show both directions: if a quantum graph is complete, then the edge correlation function $g_{ij}(t) \neq 0$ for all $i \neq j$ and for all t , and if the edge correlation function $g_{ij}(t) \neq 0$ for all $i \neq j$ and for all t , then the graph is complete.

(1) If a quantum graph is complete, then the edge correlation function $g_{ij}(t) \neq 0$ for all $i \neq j$ and for all t .

By definition, a complete graph is one in which every pair of distinct vertices is connected by an edge. In a dynamic quantum graph, this means that for all pairs of vertices v_i and v_j with $i \neq j$, there is an edge e_{ij} at all times t . The presence of an edge is determined by the edge correlation function $g_{ij}(t)$. Therefore, for a complete graph:

$$g_{ij}(t) \neq 0 \quad \forall i \neq j, \forall t$$

(2) If the edge correlation function $g_{ij}(t) \neq 0$ for all $i \neq j$ and for all t , then the graph is complete.

Assume that the edge correlation function $g_{ij}(t) \neq 0$ for all $i \neq j$ and for all t . This implies that there is an edge between every pair of distinct vertices at all times t . Therefore, the graph is complete by definition.

Hence, we have shown that a quantum graph is complete if and only if the edge correlation function $g_{ij}(t) \neq 0$ for all $i \neq j$ and for all t . $\square$

Theorem 4 *A dynamic quantum graph can transition between planar and non-planar states depending on the edge correlation functions.*

Proof Let us consider a dynamic quantum graph with edge correlation functions $g_{ij}(t)$. The planarity of the graph depends on whether it can be embedded in the plane without edges crossing at any given time t .

(1) Planar State:

Assume that at time t_1 , the edge correlation functions $g_{ij}(t_1)$ are such that the graph can be embedded in the plane without any edges crossing. This means that the graph is planar at time t_1 .

(2) Non-Planar State:

Now, consider a later time t_2 where the edge correlation functions $g_{ij}(t_2)$ have changed such that some edges now intersect when the graph is embedded in the plane. This means that the graph is non-planar at time t_2 .

Example: Let us consider a quantum graph with six vertices $v_1, v_2, v_3, v_4, v_5, v_6$ and edges $e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{25}, e_{35}, e_{45}$. At time t_1 , the edge correlation functions are such that the graph is planar:

$$A(t_1) = \begin{pmatrix} 0 & g_{12}(t_1) & g_{13}(t_1) & g_{14}(t_1) & 0 & 0 \\ g_{12}(t_1) & 0 & g_{23}(t_1) & g_{24}(t_1) & g_{25}(t_1) & 0 \\ g_{13}(t_1) & g_{23}(t_1) & 0 & 0 & g_{35}(t_1) & 0 \\ g_{14}(t_1) & g_{24}(t_1) & 0 & 0 & g_{45}(t_1) & 0 \\ 0 & g_{25}(t_1) & g_{35}(t_1) & g_{45}(t_1) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

If at a later time t_2 , additional edges e_{34} and e_{56} appear with $g_{34}(t_2) \neq 0$ and $g_{56}(t_2) \neq 0$, the graph may become non-planar:

$$A(t_2) = \begin{pmatrix} 0 & g_{12}(t_2) & g_{13}(t_2) & g_{14}(t_2) & 0 & 0 \\ g_{12}(t_2) & 0 & g_{23}(t_2) & g_{24}(t_2) & g_{25}(t_2) & 0 \\ g_{13}(t_2) & g_{23}(t_2) & 0 & g_{34}(t_2) & g_{35}(t_2) & 0 \\ g_{14}(t_2) & g_{24}(t_2) & g_{34}(t_2) & 0 & g_{45}(t_2) & 0 \\ 0 & g_{25}(t_2) & g_{35}(t_2) & g_{45}(t_2) & 0 & g_{56}(t_2) \\ 0 & 0 & 0 & 0 & g_{56}(t_2) & 0 \end{pmatrix}$$

If $g_{34}(t_2)$ and $g_{56}(t_2)$ cause e_{34} and e_{56} to intersect with other edges, the graph is non-planar at time t_2 .

Hence, we have shown that a dynamic quantum graph can transition between planar and non-planar states depending on the edge correlation functions. $\square$

Note 1 A dynamic quantum graph can transition between bipartite and non-bipartite states depending on the edge correlation functions.

A quantum graph with four vertices v_1, v_2, v_3, v_4 and edges $e_{12}, e_{14}, e_{23}, e_{34}$ is considered. At time t_1 , the edge correlation functions are such that the graph is bipartite with sets $V_1 = \{v_1, v_3\}$ and $V_2 = \{v_2, v_4\}$:

$$A(t_1) = \begin{pmatrix} 0 & g_{12}(t_1) & 0 & g_{14}(t_1) \\ g_{12}(t_1) & 0 & g_{23}(t_1) & 0 \\ 0 & g_{23}(t_1) & 0 & g_{34}(t_1) \\ g_{14}(t_1) & 0 & g_{34}(t_1) & 0 \end{pmatrix}$$

If at a later time t_2 , an additional edge e_{13} appears with $g_{13}(t_2) \neq 0$, the graph may become non-bipartite:

$$A(t_2) = \begin{pmatrix} 0 & g_{12}(t_2) & g_{13}(t_2) & g_{14}(t_2) \\ g_{12}(t_2) & 0 & g_{23}(t_2) & 0 \\ g_{13}(t_2) & g_{23}(t_2) & 0 & g_{34}(t_2) \\ g_{14}(t_2) & 0 & g_{34}(t_2) & 0 \end{pmatrix}$$

If $g_{13}(t_2) \neq 0$, the graph is non-bipartite at time t_2 as v_1 and v_3 are now directly connected.

Hence, we have exemplified that a dynamic quantum graph can transition between bipartite and non-bipartite states depending on the edge correlation functions.

Theorem 5 *A quantum graph is time-averaged complete over a period T if the time-averaged edge correlation function $\langle g_{ij} \rangle_T \neq 0$ for all $i \neq j$.*

Proof A quantum graph with edge correlation functions $g_{ij}(t)$ is considered. The time-averaged completeness of the graph over a period T depends on the time-averaged edge correlation function $\langle g_{ij} \rangle_T$.

The time-averaged edge correlation function over a period T is given by:

$$\langle g_{ij} \rangle_T = \frac{1}{T} \int_0^T g_{ij}(t) dt$$

Assume that $\langle g_{ij} \rangle_T \neq 0$ for all $i \neq j$. This implies that the average value of the edge correlation function over the period T is non-zero for all pairs of vertices.

Since $\langle g_{ij} \rangle_T \neq 0$ for all $i \neq j$, the average presence of edges between all pairs of vertices over the period T is non-zero. This means that, on average, there is an edge between every pair of vertices over the period T . Therefore, the graph is time-averaged complete over the period T .

To generalize, if the time-averaged edge correlation function $\langle g_{ij} \rangle_T \neq 0$ for all $i \neq j$, then the graph is time-averaged complete over the period T . This is because the non-zero average value of the edge correlation function implies the presence of edges between all pairs of vertices on average over the period T .

Hence, we have shown that a quantum graph is time-averaged complete over a period T if the time-averaged edge correlation function $\langle g_{ij} \rangle_T \neq 0$ for all $i \neq j$. $\square$

Theorem 6 *If $\text{dweight}_i(t) \neq 0$ for all t , the graph can still be incomplete if $g_{ij}(t) = 0$ due to other factors.*

Proof A quantum graph with dynamic weights $\text{dweight}_i(t)$ for each vertex v_i is considered. The edge correlation function $g_{ij}(t)$ is given by the product of the dynamic weights of the connected vertices and an additional function $f(t)$.

Assume that $\text{dweight}_i(t) \neq 0$ for all t . This means that the dynamic weights of the vertices are always non-zero. However, the edge correlation function is given by:

$$g_{ij}(t) = \text{dweight}_i(t) \cdot \text{dweight}_j(t) \cdot f(t)$$

If $f(t) = 0$ for some t , then:

$$g_{ij}(t) = \text{dweight}_i(t) \cdot \text{dweight}_j(t) \cdot 0 = 0$$

Since $g_{ij}(t) = 0$, the adjacency matrix element $A_{ij}(t) = 0$. This implies that there is no edge between v_i and v_j at that time, making the graph incomplete.

To generalize, even if $\text{dweight}_i(t) \neq 0$ for all t , the graph can still be incomplete if $g_{ij}(t) = 0$ due to other factors, such as the function $f(t)$. If $f(t) = 0$ for some t , then $g_{ij}(t) = 0$, making the graph incomplete at that time.

Hence, we have shown that if $\text{dweight}_i(t) \neq 0$ for all t , the graph can still be incomplete if $g_{ij}(t) = 0$ due to other factors. $\square$

6 Conclusion

This chapter has provided an analysis of quantum graphs, focusing on the structural properties. In this study, the completeness, planarity and bipartiteness property of quantum graphs have been discussed. Some relevant properties have been proved. The main limitation is of capturing the dynamical nature in real data. Once this problem is resolved, the quantum graphs will be very useful in medical sciences such as analysing dynamic brain networks.

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Quantum Decision Scenario-Based MAGDM Integrating TOPSIS Under Linguistic Information



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Abstract In most of the current multi-criteria group decision-making (MCGDM) scenarios, experts are often viewed as autonomous, with their psychological behaviors seldom considered. Nonetheless, in several instances, the perspectives of experts are likely to affect one another, and these experts frequently exhibit subjective limited rationality during the decision-making process. A linguistic information (LI)-based MCGDM model using the TOPSIS technique and quantum decision theory is designed to address this issue. LI, informed by sample size data, are employed to compile group linguistic evaluations, effectively illustrating both quantitative dispersion and qualitative ambiguity. A comprehensive generalized linguistic TOPSIS technique is introduced to represent the limited rational behaviors of decision-makers. It is utilized to assess the dominance of each choice. A quantum possibilistic aggregation framework is developed to investigate the interference effects among experts' viewpoints. In this procedure, the views of experts are regarded as simultaneously occurring wave functions that interact with one another and affect the cumulative outcome. The comparisons with several current techniques validate the efficacy and stability of the new MCGDM approach.

Keywords Quantum decision theory · Group decision-making · TOPSIS

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1 Introduction

The research in fuzzy set theory and its applications plays a significant role in addressing uncertainty and complexity in social networks and group decision-making processes. The methods developed in this area offer valuable insights into how fuzzy logic can be used to model and analyze systems where imprecision and ambiguity are prevalent. Fuzzy set theory is particularly well-suited to modeling the uncertainty inherent in social networks, where relationships and interactions often involve incomplete or vague information. Traditional mathematical models struggle to capture this uncertainty, but fuzzy approaches provide a more nuanced framework for analyzing such dynamics. For instance, the development of fuzzy fractional differential equations using fuzzy Laplace transforms is essential for modeling systems where the influence of individuals or groups evolves fractionally over time under uncertainty (Salahshour et al., 2012; Allahviranloo et al., 2015; Chehlabi et al., 2016; Armand et al., 2018).

The introduction of predictor–corrector methods for solving fuzzy differential equations also has direct applications in the study of evolving opinions and preferences within social groups (Allahviranloo et al., 2007; 2010). These methods allow for a more accurate simulation of how consensus forms over time, particularly in environments where group dynamics are shaped by uncertainty and imprecision.

The development of methods for ranking fuzzy numbers using weighted distance (Allahviranloo et al., 2011) proves useful in group decision-making processes, where individuals often express preferences that are not crisply defined. The ability to rank such imprecise inputs is crucial in facilitating collective decisions, as it allows for the systematic comparison and prioritization of uncertain information.

A new centrality measure is introduced for neutrosophic and linguistic graphs (Mahapatra et al., 2019, 2020), including self-weight or individual merit within the network. Mahapatra et al. (Mahapatra et al., 2024a) present an innovative method for centrality measurement in fuzzy social networks where relationships between nodes are not binary but have different degrees of intensity.

Multi-criteria group decision-making (MCGDM) is an important area in decision science. In a typical MCGDM situation, several experts are asked to evaluate different options based on set criteria. Their individual opinions are combined using information fusion methods to get a group evaluation. For example, in (Hussain et al., 2024) studied TOPSIS approach-based energy supplier selection under complex fuzzy rough information, in (Senapati et al., 2024) introduced dual hesitant q-rung orthopair fuzzy information-based healthcare supply chain management, etc. However, as noted by Xu (2005) and Mandal et al. (2022a), experts tend to express their opinions more reliably using linguistic information, known as linguistic sets (LS), rather than fuzzy information. In real-life scenarios, experts often have limited knowledge and personal biases, making language-based information, combined with confidence and doubt levels (Mandal et al., 2021; 2022b; 2023a; 2023b; 2024a), a more flexible way for them to express their ideas. In this connection, Mandal et al. briefly studied MCGDM under LZN information with the MARCOS approach and

its application in the selection of logistic cold chain centers (Mandal et al., 2024b); TODIM-VIKRO approach (Mandal et al., 2024c) and its application in site selection of medical logistic center; ORESTE approach (Mandal et al., 2024d) and its application in assessment of regional energy; and MULTIMOORA approach (Mandal et al., 2024e) and its application in selection of software. To address the challenge of aggregating individual assessments, specific methods are needed. Although many aggregation strategies have been studied, most assume that decision-makers are independent and uninfluenced by one another. However, in practice, opinions are often influenced by surrounding perspectives. In decision-making, the views of multiple experts can interact, ultimately affecting the final decision, and introducing the concept of 'interference'.

To address MCGDM issues, we need to combine individual assessments using a specific method. While many aggregation strategies have been studied, most assume that decision-makers are independent and not influenced by each other. However, in reality, people's opinions can be influenced by their surroundings. In decision-making, we often take either a combined or separate view of different perspectives. Many experts' views can influence each other, affecting the final decision. This introduces the concept of "interference" in Quantum Probability Theory (QPT) (Busemeyer & Bruza, 2012; Moreira & Wichert, 2014; Yearsley & Busemeyer, 2016). QPT is a recent development in classical probability theory (CPT) and has been widely studied in fields like psychology (Busemeyer & Bruza, 2012), cognitive science (Yearsley & Busemeyer, 2016), and decision analysis (He et al., 2018a; 2018b; Wu et al., 2020; Xiao et al., 2022; Han & Liu, 2023; Mandal et al., 2024f; 2025) due to its strong descriptive capabilities and solid framework. It can explain paradoxes like the disjunction fallacy (He, Z., & Jiang, 2018a; 2018b), the Ellsberg paradox (Eichberger & Pirner, 2018), the order effect (Wang et al., 2014), and graph theory (Samanta & Allahviranloo, 2025), which CPT struggles with. QPT is particularly good at representing interference terms, making it very useful for studying how experts' views influence each other in the MCGDM process.

This work introduces a quantum-based behavioral linguistic information MCGDM model that considers the limited rationality of experts and how their opinions influence each other. The main contributions of this paper are:

1. An improved generalized linguistic information TOPSIS approach accounts for experts' limited rationality, simplifies the calculation process and resolves some paradoxes of the traditional TOPSIS method.
2. A quantum-based method for combining individual evaluation results, which helps us better understand human psychology through the effects of interference.
3. A case study demonstrating the effectiveness of our approach in choosing a medication logistics supplier. Comparative and sensitivity analyses show the flexibility and robustness of our method.

The structure of the paper is as follows: In Sect. 2, recall some basic definitions and the background of linguistic term sets (LTs) and quantum Decision Theory (QDT). The proposed MCGDM model is discussed in Sect. 3. A case study for the

selection of medication logistics suppliers with comparative analysis is provided in Sect. 4. Section 5 concludes our study.

2 Preliminaries

Here we recall some related definitions and results that are used in this paper.

2.1 LTS

Here we assume an LTS is: $H = \{h_\lambda | \lambda = -\zeta, \dots, -1, 0, 1, \dots, \zeta\}$ with cardinality $2\zeta + 1$ and holds the followings (Mandal et al., 2024a):

- (1) $h_{\lambda_1} \geq h_{\lambda_2}$ if and only if $\lambda_1 \geq \lambda_2$;
- (2) $\text{neg}(h_{\lambda_1}) = h_{-\lambda_1}$;
- (3) $\max(h_{\lambda_1}, h_{\lambda_2}) = h_{\lambda_1}$ if $h_{\lambda_1} \geq h_{\lambda_2}$;
- (4) $\min(h_{\lambda_1}, h_{\lambda_2}) = h_{\lambda_1}$ if $h_{\lambda_1} \leq h_{\lambda_2}$.

To preserve expert information, the continuous LTSs (Xu, 2005) denotes $\overline{H} = \{h_\lambda | \lambda \in [-t, t]\}$, $t > \zeta$.

The subscript index of any $h_\lambda \in \overline{H}$, represented by the following functions:

$$I : \overline{H} \rightarrow [-t, t], \text{ and } I(h_\lambda) = \lambda.$$

$$I^{-1} : [-t, t] \rightarrow \overline{H}, \text{ and } I^{-1}(\lambda) = h_\lambda.$$

By Ref. (Xu, 2005), the distance between two LTs h_{λ_1} and h_{λ_2} can be computed in the following way:

$$d(h_{\lambda_1}, h_{\lambda_2}) = \frac{|\lambda_1 - \lambda_2|}{2t}$$

Let $\mathcal{F}\Theta$ be the linguistic conversation function for any LT $G \in \overline{H}$, which is defined in the following way:

$$\mathcal{F} : \overline{H} \rightarrow [0, 1], \mathcal{F}(G) = \{\mathcal{F}(h_\lambda) = \frac{\lambda}{2t} + \frac{1}{2} | \lambda \in [-t, t]\} = \kappa.$$

The inverse function of $\mathcal{F}$ is denoted by $\mathcal{F}^{-1}$ and defined in the following way:

$$\mathcal{F}^{-1} : [0, 1] \rightarrow \overline{H}, \mathcal{F}^{-1}(\kappa) = \{h_{(2\kappa-1)t} | \kappa \in [0, 1]\} = h_\lambda.$$

2.2 Quantum Probability Theory (QPT)

A quantum decision (QD) model is based on the principles of quantum probability (QP) theory (QPT). To provide context, we will review some key concepts and notations from QPT. In quantum theory (QT), a state is represented as a unit-length vector within an N-dimensional vector space.

Definition 1. (Yearsley & Busemeyer, 2016) A finite-dimensional Hilbert space (HS), represented as $\mathfrak{H}$, is an abstract vector space over a complex field, spanned by a set of orthonormal basis vectors with complex components. In classical probability theory (CPT), events are fundamental (He et al., 2018), but in quantum probability theory (QPT), events are instead represented by subspaces of the Hilbert space $\mathfrak{H}$, rather than traditional sets.

Definition 2. (Busemeyer & Bruza, 2012) In QT, the superposition state $|S\rangle$, a unit-length vector in the N-dimensional Hilbert space $\mathfrak{H}$, defines all possible events within that space. The nonlinear combination of a set of orthogonal basis vectors $\{|\mathcal{V}\rangle_k, k = 1, 2, \dots, n\}$ are used to represent the superposition state $|S\rangle$, i.e.,

$$\begin{aligned} |S\rangle &= \langle \mathcal{V}_1 | S \mathcal{V}_1 \rangle + \langle \mathcal{V}_2 | S \mathcal{V}_2 \rangle + \dots + \langle \mathcal{V}_n | S \mathcal{V}_n \rangle \\ &= \xi_1 |\mathcal{V}_1\rangle + \xi_2 |\mathcal{V}_2\rangle + \dots + \xi_n |\mathcal{V}_n\rangle \\ &= \zeta_1 e^{i\alpha_1} |\mathcal{V}_1\rangle + \zeta_2 e^{i\alpha_2} |\mathcal{V}_2\rangle + \dots + \zeta_n e^{i\alpha_n} |\mathcal{V}_n\rangle \end{aligned} \quad (1)$$

where $\xi_k (= \zeta_k e^{i\alpha_k})$ is the amplitude of the probability, expressed as a complex number, characterizing the amplitude of the wave in the state $|\mathcal{V}_k\rangle$, and the amplitude of the phase angle represented by $e^{i\alpha_k}$.

Definition 3. (Busemeyer & Bruza, 2012) The CPT framework represents every event using a single set of components. In QPT, probabilities are assigned based on the overlap between cognitive states (shown as state vectors) and different probability measures (shown as subspaces). QPT extends CPT into the complex domain by including incompatible qualities and viewing events as vectors. Incompatible attributes are two observable features of an item that cannot be measured accurately at the same time. Both the incompatibility condition and the uncertainty principle can be described by the probability amplitude, also known as complex probability.

According to Born's rule, an event's probability amplitude and its classical probability, x_i are related to each other:

$$P(\mathcal{V}_k) = |\zeta_k e^{i\alpha_k}|^2 = (\zeta_k e^{i\alpha_k}) \times (\zeta_k e^{i\alpha_k})^* = \zeta_k e^{i\alpha_k} \times \zeta_k e^{-i\alpha_k} = (\zeta_k)^2 \quad (2)$$

For the total event probability to be 1, the sum of the squares of the probability amplitudes must also be 1.

$$\sum_{k=1}^n P(\mathcal{V}_k) = \sum_{k=1}^n (\zeta_k)^2 = 1. \quad (3)$$

An $N - 1$ mapping transforms complex probabilities into classical probabilities. For each given probability value $P(\mathcal{V}_k)$, countless complex probabilities correspond to it. Referencing Eq. (2) and Euler's formula,

$$e^{i\alpha} = \cos\alpha + i \times \sin\alpha \quad (4)$$

the complex probability satisfies:

$$\xi = \sqrt{P(\mathcal{V})}(\cos\alpha + i \times \sin\alpha) = \sqrt{P(\mathcal{V})}e^{i\alpha} \quad (5)$$

For example, in the following, we discussed the QPT in two-dimensional Hilbert space. Let $\{\mathcal{M}_1, \mathcal{M}_2\}$ be an event containing two basic states, where $|\mathcal{M}_1\rangle = (1, 0)$, and $|\mathcal{M}_2\rangle = (0, 1)$ as two basic orthogonal vectors in $\mathbf{H}$. The state $|\mathcal{S}\rangle$ is defined as a superposition and is graphically illustrated in Fig. 1. According to Eqs. (1) and (2), it can be expressed as.

$$|\mathcal{S}\rangle = \zeta_1 e^{i\alpha_1} |\mathcal{M}_1\rangle + \zeta_2 e^{i\alpha_2} |\mathcal{M}_2\rangle = \frac{e^{i\alpha_{\mathcal{M}_1}}}{\sqrt{2}} |\mathcal{M}_1\rangle + \frac{e^{i\alpha_{\mathcal{M}_2}}}{\sqrt{2}} |\mathcal{M}_2\rangle$$

and the probabilities of $|\mathcal{M}_1\rangle$ and $|\mathcal{M}_2\rangle$ are calculated as follows:

$$P(\mathcal{M}_1) = P(\mathcal{M}_2) = (e^{i\alpha})(e^{i\alpha})^* = \frac{1}{2}.$$

According to conventional Bayesian decision theory, decision-making can be mathematically represented as a Markov process (Moreira & Wichert, 2014). The state transition probability in a single route of QPT aligns with the CPT according to Feynman's first rule (Wu et al., 2020), which varies over several pathways. Consider the decision-making dilemma highlighted above as an illustration of the process

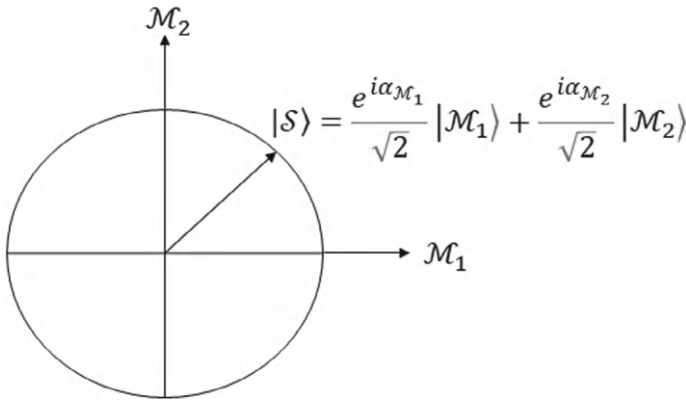


Fig. 1 Event in Hilbert space with two dimensions

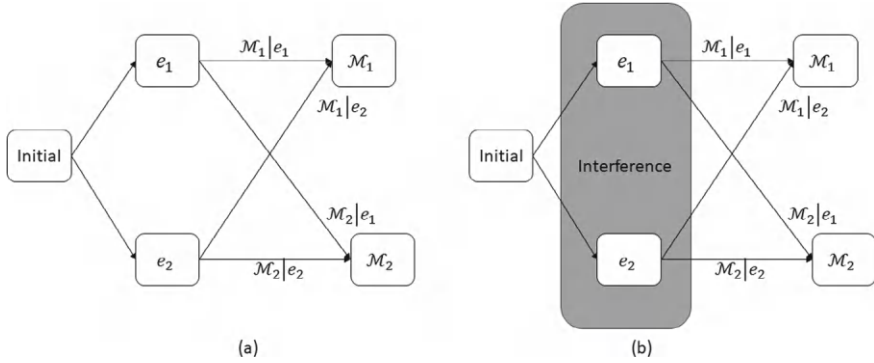


Fig. 2 DM in **a** Markov and **b** Quantum process

depicted in Fig. 2(a). Experts provide their assessments of $\mathcal{M}_1$ ('bad') or $\mathcal{M}_2$ ('good') as probability quantifications. State transition probabilities are determined by combining the individual probabilities for each route and using the total probability formula in CPT.

$$P(\mathcal{M}_1) = \sum_{k=1}^2 P(e_k)P(\mathcal{M}_1|e_k), \sum_{k=1}^2 P(e_k) = 1 \quad (6)$$

Since only one route is selected at a time, it is sometimes referred to as the single path principle. The conventional Markovian framework assumes that experts rely on one expert at a time and that both experts are completely autonomous. However, this approach is sometimes unsuitable for ordinary cognitive processes because many avenues are often considered when forming judgments. Such phenomena can result in illogical actions, including disjunction, conjunction errors, and interference effects. Thus, QPT is an excellent complement to CPT in this scenario.

Within the QPT framework (Fig. 2(b)), the state is formed by combining three fundamental states, which can be represented as different wave functions aligning according to Feynman's second rule. The amplitude of the final form is obtained by superimposing two individual routes running from I ('Initial') to $\mathcal{M}_1$. The amplitude is the sum of the individual pathways from source I to target $\mathcal{M}_1$, and the probability of reaching the target $\mathcal{M}_1$ is equal to the square of the amplitude.

$$P(\mathcal{M}_1) = |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 \quad (7)$$

Equation (7) can be simplified in the following way:

$$\begin{aligned} P(\mathcal{M}_1) &= |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 \\ &= |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}| |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^* \end{aligned}$$

$$\begin{aligned}
&= \left(\sqrt{P(e_1)P(\mathcal{M}_1|e_1)}e^{i\alpha_1} + \sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{i\alpha_2} \right) \\
&\quad \left(\sqrt{P(e_1)P(\mathcal{M}_1|e_1)}e^{-i\alpha_1} + \sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{-i\alpha_2} \right) \\
&= P(e_1)P(\mathcal{M}_1|e_1) + P(e_2)P(\mathcal{M}_1|e_2) \\
&\quad + \sqrt{P(e_1)P(\mathcal{M}_1|e_1)}\sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{i(\alpha_1-\alpha_2)} \\
&\quad + \sqrt{P(e_1)P(\mathcal{M}_1|e_1)}\sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{i(\alpha_2-\alpha_1)} \tag{8}
\end{aligned}$$

By Eq. (4), we get

$$\cos(\alpha_1 - \alpha_2) = \frac{e^{i(\alpha_2-\alpha_1)} + e^{i(\alpha_1-\alpha_2)}}{2} \tag{9}$$

Then Eq. (7) can be written in the following way:

$$\begin{aligned}
P(\mathcal{M}_1) &= P(e_1)P(\mathcal{M}_1|e_1) + P(e_2)P(\mathcal{M}_1|e_2) \\
&\quad + 2\sqrt{P(e_1)P(\mathcal{M}_1|e_1)}\sqrt{P(e_2)P(\mathcal{M}_1|e_2)}\cos(\alpha_1 - \alpha_2) \\
&= |\xi_{e_1}\xi_{\mathcal{M}_1|e_1}|^2 + |\xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 + 2\xi_{e_1}\xi_{\mathcal{M}_1|e_1}\xi_{e_2}\xi_{\mathcal{M}_1|e_2}\cos(\alpha_1 - \alpha_2) \tag{10}
\end{aligned}$$

Upon comparing Eqs. (6) and (10), it becomes evident that the quantum framework includes an interference component. If the pathways are observable, the quantum phase transition will reduce to the classical one, following Feynman's third rule.

$$P(\mathcal{M}_1) = |\xi_{e_1}\xi_{\mathcal{M}_1|e_1}|^2 + |\xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 = P(e_1)P(\mathcal{M}_1|e_1) + P(e_2)P(\mathcal{M}_1|e_2) \tag{11}$$

Additionally, the QPT can be expanded to a multi-path graph, resulting in an increased number of interference terms. The QPT enhances CPT by accurately representing cognitive processes in quantum terms, thereby more effectively modeling ambiguity, vagueness, and uncertainty in decision-making.

3 The Proposed Topsis-Based Magdm Under Linguistic Information and Quantum Decision Scenario

This section introduces quantum cognition-based MAGDM integrating the TOPSIS approach.

3.1 Background

Let us assume

$X = \{x_1, x_2, \dots, x_n\}$: Alternative set,

$C = \{c_1, c_2, \dots, c_m\}$: Attribute set,

$e = \{e_1, e_2, \dots, e_q\}$: Expert set,

$w^k = (w_1^k, w_2^k, \dots, w_m^k)$: Weight vector of attributes for expert e_k , where $w_j^k \in [0, 1]$, and $\sum_{j=1}^m w_j^k = 1$,

$\omega = (\omega_1, \omega_2, \dots, \omega_q)$: Weight vector of experts with $\omega_k \in [0, 1]$, $\sum_{k=1}^q \omega_k = 1$.

3.2 Decision Steps

Let $A^k = (a_{ij}^k)_{nm}$ ($i = 1, 2, \dots, n; j = 1, 2, \dots, m; k = 1, 2, \dots, q$) be the LN-based decision judgment provided by an expert e_k using pre-defined distinct LTSs $H = \{h_\lambda | \lambda = -\zeta, \dots, -1, 0, 1, \dots, \zeta\}$.

Step 1: Construction of decision matrix

Then, in the following, we represent the decision matrix of k -th expert:

$$A^k = (a_{ij}^k)_{nm} = \begin{matrix} & \begin{matrix} c_1 & c_2 & \dots & c_m \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} & \begin{bmatrix} a_{11}^k & a_{12}^k & \dots & a_{1m}^k \\ a_{21}^k & a_{22}^k & \dots & a_{2m}^k \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}^k & a_{n2}^k & \dots & a_{nm}^k \end{bmatrix} \end{matrix} \quad (12)$$

In a MAGDM or multi-attribute decision-making problem, the normalization of the decision matrix is vital for better evaluation. The above decision matrix (Eq. (12)) can be normalized to $L^k = (l_{ij}^k)_{nm}$ with

$$l_{ij}^k = \begin{cases} \frac{\mathcal{F}(A_{ij}^k)}{\sqrt{\sum_{i=1}^n (\mathcal{F}(A_{ij}^k))^2}}, & \text{for benefit criteria } c_j, \\ 1 - \frac{\mathcal{F}(A_{ij}^k)}{\sqrt{\sum_{i=1}^n (\mathcal{F}(A_{ij}^k))^2}}, & \text{for cost criteria } c_j. \end{cases} \quad (13)$$

Step 2: Expert weight computation process

Inspire by Ref. (Mandal et al., 2024a), the weight of each expert e_k ($k = 1, 2, \dots, q$) can be calculated in the following way:

$$\omega_k = \frac{\Omega(Z^k)}{\sum_{k=1}^q \Omega(Z^k)}, \text{ where } \Omega(Z^k) = \frac{\sum_{j=1}^m \sum_{i=1}^n l_{ij}^k}{mn}. \quad (14)$$

Step 3: Attribute weight computation process

- (1) The Shannon entropy $\Psi^k = (\Psi_1^k, \Psi_2^k, \dots, \Psi_m^k)$ can be calculated in the following way:

$$\Psi_j^k = -\frac{1}{\ln n} \sum_{i=1}^n l_{ij}^k \ln l_{ij}^k \quad (15)$$

$l_{ij}^k \ln l_{ij}^k$ is 0 if $l_{ij}^k = 0$.

- (2) The attribute weight $w^k = (w_1^k, w_2^k, \dots, w_m^k)$ concerning each expert e_k ($k = 1, 2, \dots, q$) opinion can be calculated in the following way:

$$w_j^k = \frac{1 - \Psi_j^k}{\sum_{j=1}^m (1 - \Psi_j^k)} \quad (16)$$

Step 4: Construction of Bayesian network

In this phase, we converted the pre-collected data into probabilities and developed a Bayesian Network for decision-making, as the quantum framework is a partial extension of the Bayesian Network. Figure 3 illustrates that, in a MAGDM situation, the decision-making process can be represented as a simple two-layer Bayesian Network. The experts' weights are converted into probabilities, which indicate the degree of belief in the experts, using the following equation:

$$P(e_k) = \omega_k, k = 1, 2, \dots, q. \quad (17)$$

In Eq. (17), the probabilities form the initial layer of the Bayesian Network (BN). The second layer depends on the assessment outcomes from each expert, which can be calculated using the TOPSIS approach as follows:

- (1) In the following, we define the LI-based positive ideal (LIPIS) and negative ideal (LINIS) concerning each expert e_k ($k = 1, 2, \dots, q$) opinion in the following way:

$$\begin{cases} LIPIS^k = (LIPIS_1^k, LIPIS_2^k, \dots, LIPIS_m^k) \\ LINIS^k = (LINIS_1^k, LINIS_2^k, \dots, LINIS_m^k) \end{cases} \quad (18)$$

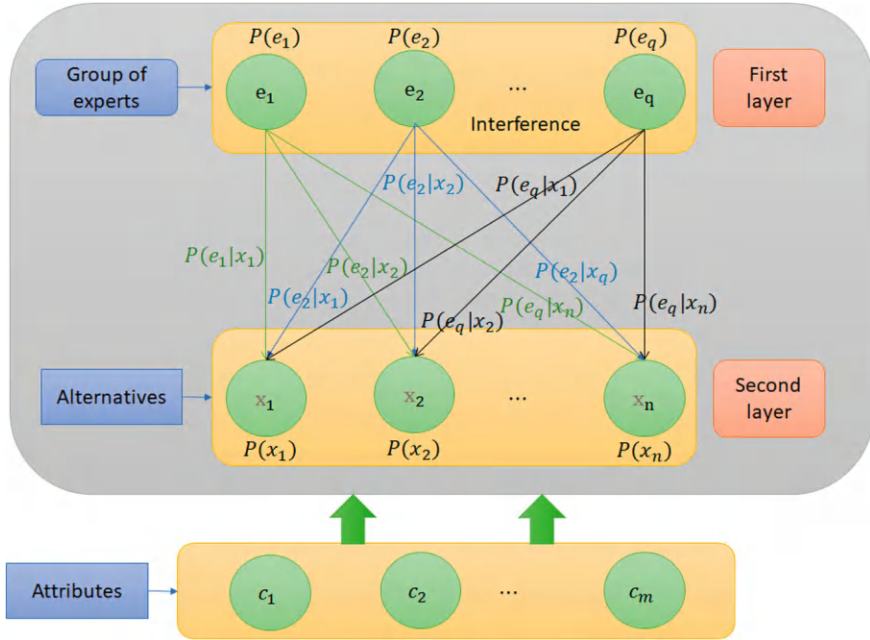


Fig. 3 MAGDM according to QSBN

where,

$$LIPIS_j^k = \left\{ c_j, \text{Max}_{i=1}^n \mathcal{F}^{-1} \left(l_{ij}^k \right) \middle| j = 1, 2, \dots, m \right\}$$

$$LINIS_j^k = \left\{ c_j, \text{Min}_{i=1}^n \mathcal{F}^{-1} \left(l_{ij}^k \right) \middle| j = 1, 2, \dots, m \right\}$$

- (2) Using attribute weight obtained by Eq. (16), the distance of each alternative from $LIPIS_j^k$ and $LINIS_j^k$ is obtained in the following way:

$$d_i^{k,+} = \sum_{j=1}^m w_j^k d \left(\mathcal{F}^{-1} \left(l_{ij}^k \right), LIPIS_j^k \right), \quad (19)$$

$$d_i^{k,-} = \sum_{j=1}^m w_j^k d \left(\mathcal{F}^{-1} \left(l_{ij}^k \right), LINIS_j^k \right). \quad (20)$$

- (3) Based on Eqs. (19) and (20), conditional probabilities which link the two layers in the BN as:

$$P(x_i|e_k) = \frac{CI_i^k}{\sum_{i=1}^n CI_i^k}, CI_i^k = \frac{d_i^{k,-}}{d_i^{k,+} + d_i^{k,-}}. \quad (21)$$

(4) The second layer consists of the probabilities of alternatives calculated as:

$$P(x_i) = \sum_{k=1}^q P(e_k)P(x_i|e_k) \quad (22)$$

Subsequently, we can decide based on the $P(x_i)$ values, as higher probabilities, indicate a superior ranking. The classical approach does not account for the interference effect, which can influence the final rankings. To address this issue, Bayesian networks are expanded into a quantum framework capable of properly modeling a practical approach to real interference in multi-attribute group decision-making (MAGDM).

Step 5: Construction of quantum scenario-based Bayesian network (QSBN)

The facilitator often consolidates opinions from several specialists. The interconnected decision-making processes in real-world situations, as demonstrated in Sect. 2.2, remain unobservable. The square of the sum of the amplitude probabilities of all possible pathways collectively constitutes the probability of the alternative x_i from QPT (He et al., 2018), as described.

$$\begin{aligned} P(x_i) &= \left| \sum_{k=1}^q \xi_{e_k} \xi_{x_i|e_k} e^{i\alpha_k} \right|^2 \\ &= \left| \sum_{k=1}^q \sqrt{P(e_k)} \sqrt{P(x_i|e_k)} e^{i\alpha_k} \right|^2, i = 1, 2, \dots, n \end{aligned} \quad (23)$$

By Eqs. (7) – (11), Eq. (23) can be written as

$$\begin{aligned} P(x_i) &= \sum_{k=1}^q P(e_k)P(x_i|e_k) + 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \cos(\alpha_k - \alpha_s) \\ &= \sum_{k=1}^q P(e_k)P(x_i|e_k) + INT_i \end{aligned} \quad (24)$$

In a quantum-based Bayesian network, a normalization factor is incorporated into Eq. (24) to standardize the probability of each alternative, ensuring that the overall estimated probability does not exceed 1. Then

$$Prob(x_i) = \sigma P(x_i) \quad (25)$$

where,

$$\sigma = \frac{1}{\sum_{i=1}^n P(x_i)} = \frac{1}{\sum_{i=1}^n (\sum_{k=1}^q P(e_k)P(x_i|e_k) + INT_i)} \quad (26)$$

3.3 Analysis of Interference Effect Among Experts' Opinions

Equation (26) uses the term INT_i , which represent the ambiguity and uncertainty of expert opinions, to explain the superposition of decision information. In Eq. (24), $\cos(\alpha_k - \alpha_s)$ represents the phase angle between two distinct experts e_k and e_s , which represents the connection vector between two experts. $\frac{q(q-1)}{2}$ is the total count of pairwise correspondences across q experts, excluding duplicates. The final interference value is derived by summing the computed $\frac{q(q-1)}{2}$ interference values for each expert pair, rather than assessing the overall uncertainty across experts. This approach enhances the accuracy of the estimated probability value.

In real-world situations, it is hard to measure the phase angle. Experts' opinions can differ or be influenced by outside factors, causing a lot of variation and mistakes. Belief entropy helps a lot in handling and combining unclear information, making it easier to make good decisions in group settings with many factors. It is used as a standard in areas like risk assessment, defect diagnosis, and pattern recognition. Belief entropy is the latest way to measure interference in QSNB. Unlike the old method of manually estimating the phase angle range, this new method is more reliable because it reduces errors from personal judgment, which can affect the final decision and the expert's probability calculation. Belief entropy is used in QSNB to manage the interference among different experts' beliefs in GDM.

Then the belief distance about alternative x_i according to the average probability $\theta = \frac{1}{nq}$ can be calculated in the following way:

$$D_{iu} = \left| \delta_{iu} + \frac{|\delta_{iu} - \theta| - |\varrho_{iu} - \theta|}{|\delta_{iu} - \theta| + |\varrho_{iu} - \theta|} \right|, i = 1, 2, \dots, n; u = 1, 2, \dots, \frac{q(q-1)}{2},$$

$$\begin{cases} \delta_{iu} = P(e_k)P(x_i|e_k) \\ \varrho_{iu} = P(e_s)P(x_i|e_s) \end{cases}, \quad (27)$$

$$i = 1, 2, \dots, n; k = 1, 2, \dots, q-1; s = k+1, k+2, \dots, \frac{q(q-1)}{2}.$$

In Eq. (27), u means the value reallocated to the correlation among q experts; $\theta = \frac{1}{nq}$ represent each expert probability value of each alternative; D_{iu} means deviation degree from θ ; $|\delta_{iu} - \theta|$ and $|\varrho_{iu} - \theta|$ represents two experts' magnitude of the deviation degree from θ about alternative x_i . If $|\delta_{iu} - \theta| < |\varrho_{iu} - \theta|$, then the position of δ_{iu} and ϱ_{iu} are interchangeable.

There are many ways to measure uncertainty. Deng entropy (Zhao et al., 2024), also known as belief entropy, is better at including the confidence function and the uncertainty of BPA because it measures both total non-specificity and disagreement.

Thus, by Deng's entropy, we get

$$\vartheta_{iu}(D_{iu}) = -D_{iu} \log_2 \frac{D_{iu}}{2^m - 1}, i = 1, 2, \dots, n; u = 1, 2, \dots, \frac{q(q-1)}{2}. \quad (28)$$

To ensure that the values of Deng entropy align with the range of $\cos(\alpha_k - \alpha_s)$, which is between $[-1, 1]$, Deng entropy is standardized.

$$\begin{aligned} \mu_{iu}(D_{iu}) &= 2 \frac{\vartheta_{iu}(D_{iu}) - \min\{\vartheta_{iu}(D_{iu})\}}{\max\{\vartheta_{iu}(D_{iu})\} - \min\{\vartheta_{iu}(D_{iu})\}} - 1, \\ i &= 1, 2, \dots, n; u = 1, 2, \dots, \frac{q(q-1)}{2}. \end{aligned} \quad (29)$$

Thus, the phase angle $\cos(\alpha_k - \alpha_s)$ can be determined by the following waw:

$$\cos(\alpha_k - \alpha_s) = \mu_{iu}(D_{iu}), i = 1, 2, \dots, n; u = 1, 2, \dots, \frac{q(q-1)}{2}. \quad (30)$$

Therefore, by Eq. (30), in Eq. (24), INT_i can be written as

$$\begin{aligned} INT_i &= 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \cos(\alpha_k - \alpha_s) \\ &= 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \mu_{iu}(D_{iu}) \end{aligned} \quad (31)$$

By Eq. (31), the Eq. (24) can be written as

$$P(x_i) = \sum_{k=1}^q P(e_k)P(x_i|e_k) + 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \mu_{iu}(D_{iu}) \quad (32)$$

Finally, the probability of each alternative x_i can be obtained in the following way:

$$Prob(x_i) = \sigma P(x_i) \quad (33)$$

where,

$$\sigma = \frac{1}{\sum_{i=1}^n P(x_i)} = \frac{1}{\sum_{i=1}^n (\sum_{k=1}^q P(e_k)P(x_i|e_k) + INT_i)} \quad (34)$$

Step 6: Ranking of alternative

According to Eq. (33), the higher probability of an alternative means the best alternative.

4 A Practical Example with Comparative Analysis

This section employs the QDS-based MAGDM approach to identify the ideal third-party cold chain medical logistics firm among several options. A comparative analysis is conducted to validate its efficacy.

4.1 A Case Study

The execution of the Healthy India policy has presented significant development potential for pharmaceutical logistics, a fundamental element in enhancing health protection. Conventional pharmaceutical logistics is frequently dominated by huge pharmaceutical corporations, which excel in pharmacology but exhibit deficiencies in professionalism and efficiency in logistics. Medical transportation is distinct from ordinary transportation due to its specific characteristics. The distribution of pharmaceutical products must adhere to pertinent standards, including Good Supply Practice (GSP) and Good Manufacturing Practice (GMP), while transportation conditions such as temperature and humidity must be meticulously regulated to avert the degradation of medications. The need for medical logistics is abrupt, resulting from the unexpected emergence and rapid transmission of epidemic illnesses. This necessitates an elevated capacity for swift reaction in pharmaceutical logistics (Forghani et al., 2018; Singh et al., 2018). Furthermore, medication safety must be prioritized during the transportation of drugs. The assurance of medication safety necessitates the ability to monitor pharmaceuticals throughout the transportation process, as well as the timely recall or destruction of expired or defective drugs. Alongside the standardization of medical logistics and the establishment of logistical technological infrastructure, the quality of staff is a significant element influencing safety.

Third-party professional cold chain logistics solutions provide outstanding performance. They can autonomously document, monitor, and transmit comprehensive cold chain temperature data, precisely locate via GPS (Jaberidoost et al., 2013), and oversee transportation vehicle information in real-time. Furthermore, they incorporate a professional logistics information management platform and an EDI (Electronic Data Interchange) center, capable of automatically querying items via label tracing. This feature facilitates the prompt detection, recall, or disposal of expired and defective pharmaceuticals, thereby considerably decreasing the prevalence of substandard medications and ensuring their quality (Jaberidoost et al., 2013). As a

result, several pharmaceutical companies choose third-party cold chain logistics to minimize expenses and improve their core competitiveness (Wen et al., 2019).

This research examines a major pharmaceutical company's strategy to reduce expenses and strengthen its core competitiveness by outsourcing logistical operations while focusing on its principal business activities. Due to the specific demands of drug logistics, a superior level of service is necessary. In the preliminary screening phase, many third-party logistic (3PL) businesses that fail to meet the pharmaceutical logistics requirements are excluded, resulting in a limited number of third-party cold chain logistics firms being considered.

The multinational pharmaceutical company is evaluating four third-party cold-chain logistics firms (x_1, x_2, x_3, x_4) for the delivery and distribution of its medicinal supplies. Four experts (e_1, e_2, e_3, e_4) from the quality management, production, material control departments, and a consulting firm are requested to assess the options against four criteria (c_1, c_2, c_3, c_4) using suitable LTSs. The four essential factors are infrastructure, cost, human quality, and service performance. The infrastructure equipment c_1 include logistical infrastructure and storage facilities, as well as specialized facilities and equipment, including an information system for tracking, freezers, and refrigerated vehicles that maintain low-temperature control throughout the process. The cost c_2 denotes the aggregate expense of logistical services. The personnel quality c_3 pertains to the professional caliber of employees, including their service awareness and the degree of their professional and ethical standards. The service performance c_4 encompasses timeliness rate, product qualification ratio, reaction speed, and emergency management capability. The second criterion is a cost criterion, whereas the others are benefit criteria.

Here we offer the LTSs, $H_1 = \{h_{-3}: \text{very low (VL)}, h_{-2}: \text{low (L)}, h_{-1}: \text{slightly low (SL)}, h_0: \text{medium (M)}, h_1: \text{slightly high (SH)}, h_2: \text{high (H)}, h_3: \text{very high (VH)}\}$ for the evaluation of logistics firms (x_1, x_2, x_3, x_4) about criteria c_2 and the LTSs, $H_2 = \{h_{-3}: \text{very bad (VB)}, h_{-2}: \text{bad (B)}, h_{-1}: \text{slightly bad (SB)}, h_0: \text{medium (M)}, h_1: \text{slightly good (SG)}, h_2: \text{good (G)}, h_3: \text{very good (VG)}\}$ for evaluation of logistics firms (x_1, x_2, x_3, x_4) about the rest criteria of the experts (e_1, e_2, e_3, e_4). The evaluation information is shown in the Eqs. (35)–(38).

$$A^1 = \begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} & \begin{bmatrix} h_{-2} & h_0 & h_{-1} & h_{-1} \\ h_1 & h_0 & h_1 & h_1 \\ h_2 & h_{-1} & h_2 & h_2 \\ h_3 & h_2 & h_1 & h_1 \end{bmatrix} \end{matrix} \quad (35)$$

$$A^2 = \begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} & \begin{bmatrix} h_{-2} & h_{-1} & h_1 & h_2 \\ h_{-3} & h_1 & h_2 & h_1 \\ h_2 & h_0 & h_3 & h_{-1} \\ h_2 & h_1 & h_1 & h_1 \end{bmatrix} \end{matrix} \quad (36)$$

$$A^3 = \begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} & \begin{bmatrix} h_{-3} & h_{-2} & h_{-1} & h_0 \\ h_{-2} & h_1 & h_3 & h_1 \\ h_1 & h_{-2} & h_1 & h_1 \\ h_2 & h_1 & h_2 & h_2 \end{bmatrix} \end{matrix} \quad (37)$$

$$A^4 = \begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} & \begin{bmatrix} h_{-2} & h_0 & h_{-1} & h_2 \\ h_{-1} & h_1 & h_2 & h_{-3} \\ h_2 & h_1 & h_2 & h_1 \\ h_3 & h_0 & h_2 & h_1 \end{bmatrix} \end{matrix} \quad (38)$$

Step 1: By Eq. (13), each expert evaluation can be normalized and the results are shown in the following:

$$L^1 = \begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} & \begin{bmatrix} 0.113 & 0.562 & 0.256 & 0.256 \\ 0.453 & 0.562 & 0.512 & 0.512 \\ 0.566 & 0.708 & 0.64 & 0.64 \\ 0.679 & 0.271 & 0.512 & 0.512 \end{bmatrix} \end{matrix}$$

$$L^2 = \begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} & \begin{bmatrix} 0.14 & 0.702 & 0.415 & 0.64 \\ 0 & 0.404 & 0.518 & 0.512 \\ 0.7 & 0.553 & 0.622 & 0.256 \\ 0.7 & 0.404 & 0.415 & 0.512 \end{bmatrix} \end{matrix}$$

$$L^3 = \begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} & \begin{bmatrix} 0 & 0.829 & 0.222 & 0.369 \\ 0.154 & 0.314 & 0.667 & 0.492 \\ 0.617 & 0.829 & 0.444 & 0.492 \\ 0.772 & 0.314 & 0.556 & 0.615 \end{bmatrix} \end{matrix}$$

$$L^4 = \begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} & \begin{bmatrix} 0.123 & 0.576 & 0.225 & 0.662 \\ 0.246 & 0.434 & 0.563 & 0 \\ 0.615 & 0.434 & 0.536 & 0.53 \\ 0.739 & 0.576 & 0.563 & 0.53 \end{bmatrix} \end{matrix}$$

Step 2: By Eq. (14), the weight of experts are $\omega_1 = 0.259$, $\omega_2 = 0.254$, $\omega_3 = 0.232$, and $\omega_4 = 0.254$.

Step 3: By Eqs. (15) and (16), the weight of attributes for all experts are:

$$w_1^1 = 0.0881, w_2^1 = 0.4302, w_3^1 = 0.2892, w_4^1 = 0.1925;$$

$$w_1^2 = 0.1452, w_2^2 = 0.4034, w_3^2 = 0.1062, w_4^2 = 0.3452;$$

$$w_1^3 = 0.0269, w_2^3 = 0.5844, w_3^3 = 0.3156, w_4^3 = 0.0731;$$

$$w_1^4 = 0.1135, w_2^4 = 0.1821, w_3^4 = 0.5979, w_4^4 = 0.1065.$$

Step 4: By Eq. (17), $P(e_1) = 0.259$, $P(e_2) = 0.254$, $P(e_3) = 0.232$, and $P(e_4) = 0.254$.

(1) From Eq. (18), we get

$$\begin{cases} LIPIS^1 = (\langle c_1, h_{1.076} \rangle, \langle c_2, h_{1.250} \rangle, \langle c_3, h_{0.841} \rangle, \langle c_4, h_{0.841} \rangle) \\ LINIS^1 = (\langle c_1, h_{-2.321} \rangle, \langle c_2, h_{-1.376} \rangle, \langle c_3, h_{-1.464} \rangle, \langle c_4, h_{-1.464} \rangle) \end{cases};$$

$$\begin{cases} LIPIS^2 = (\langle c_1, h_{1.201} \rangle, \langle c_2, h_{1.211} \rangle, \langle c_3, h_{0.733} \rangle, \langle c_4, h_{0.841} \rangle) \\ LINIS^2 = (\langle c_1, h_{-3} \rangle, \langle c_2, h_{-0.578} \rangle, \langle c_3, h_{-0.511} \rangle, \langle c_4, h_{-1.464} \rangle) \end{cases};$$

$$\begin{cases} LIPIS^3 = (\langle c_1, h_{1.629} \rangle, \langle c_2, h_{1.971} \rangle, \langle c_3, h_1 \rangle, \langle c_4, h_{0.693} \rangle) \\ LINIS^3 = (\langle c_1, h_{-3} \rangle, \langle c_2, h_{-1.116} \rangle, \langle c_3, h_{-1.667} \rangle, \langle c_4, h_{-0.784} \rangle) \end{cases};$$

$$\begin{cases} LIPIS^4 = (\langle c_1, h_{1.431} \rangle, \langle c_2, h_{0.454} \rangle, \langle c_3, h_{0.375} \rangle, \langle c_4, h_{0.394} \rangle) \\ LINIS^4 = (\langle c_1, h_{-2.261} \rangle, \langle c_2, h_{-0.394} \rangle, \langle c_3, h_{-1.65} \rangle, \langle c_4, h_{-3} \rangle) \end{cases}.$$

(2) By Eqs. (19) and (20), we get

$$\begin{cases} d_1^{1,+} = 0.2977, d_2^{1,+} = 0.1444, d_3^{1,+} = 0.010, d_4^{1,+} = 0.2499 \\ d_1^{1,-} = 0.1255, d_2^{1,-} = 0.2788, d_3^{1,-} = 0.4132, d_4^{1,-} = 0.1732 \end{cases};$$

$$\begin{cases} d_1^{2,+} = 0.1034, d_2^{2,+} = 0.2772, d_3^{2,+} = 0.1927, d_4^{2,+} = 0.1865 \\ d_1^{2,-} = 0.2732, d_2^{2,-} = 0.0994, d_3^{2,-} = 0.1838, d_4^{2,-} = 0.1901 \end{cases};$$

$$\begin{cases} d_1^{3,+} = 0.1790, d_2^{3,+} = 0.3263, d_3^{3,+} = 0.0833, d_4^{3,+} = 0.3358 \\ d_1^{3,-} = 0.3007, d_2^{3,-} = 0.1534, d_3^{3,-} = 0.3964, d_4^{3,-} = 0.1439 \end{cases};$$

$$\begin{cases} d_1^{4,+} = 0.2717, d_2^{4,+} = 0.1521, d_3^{4,+} = 0.0538, d_4^{4,+} = 0.0141 \\ d_1^{4,-} = 0.0963, d_2^{4,-} = 0.2158, d_3^{4,-} = 0.3141, d_4^{4,-} = 0.3538 \end{cases};$$

(3) By Eq. (21) we calculate the conditional probabilities of each alternative for each expert. The details results are shown in Table 1.

(4) By Eq. (22), the probabilities of alternatives are:

Table 1 The conditional probabilities of each alternative

$P(x_i e_k)$	e_1	e_2	e_3	e_4
x_1	0.1267	0.3660	0.3024	0.0982
x_2	0.2814	0.1331	0.1543	0.2202
x_3	0.4171	0.2463	0.3986	0.3205
x_4	0.1749	0.2546	0.1447	0.3611

$$P(x_1) = 0.2212, P(x_2) = 0.1986, P(x_3) = 0.3448, P(x_4) = 0.2355.$$

Step 5: By Eq. (27), the belief distance between two experts about each alternative is calculated, which is presented in Eq. (39).

$$D = \begin{bmatrix} 0.0182 & 0.6174 & 0.0831 & 0.6872 & 0.0084 & 0.5857 \\ 0.3973 & 0.3689 & 0.3010 & 0.0698 & 0.1004 & 0.6444 \\ 1.1015 & 0.3098 & 0.5173 & 0.9277 & 0.9220 & 0.3189 \\ 0.8128 & 0.2065 & 0.2144 & 0.7895 & 0.7917 & 0.0252 \end{bmatrix} \quad (39)$$

By Eq. (28), the Deng entropy between two experts about each alternative is shown in Eq. (40).

$$\vartheta = \begin{bmatrix} 0.1764 & 2.8415 & 0.6230 & 3.0567 & 0.0907 & 2.7404 \\ 2.0814 & 1.9718 & 1.6975 & 0.5409 & 0.7254 & 2.9262 \\ 4.1497 & 1.7341 & 2.5130 & 3.7247 & 3.7102 & 1.7717 \\ 3.4185 & 1.2769 & 1.3139 & 3.3536 & 3.3599 & 0.2326 \end{bmatrix} \quad (40)$$

By Eq. (29), the standardized Deng entropy between two experts about each alternative is shown in Eq. (41).

$$\mu = \begin{bmatrix} -0.9422 & 0.8542 & -0.6411 & 1 & -1 & 0.7867 \\ 0.2917 & 0.1998 & -0.0302 & -1 & -0.8453 & 1 \\ 1 & -1 & -0.3551 & 0.6481 & 0.6361 & -0.9689 \\ 1 & -0.3445 & -0.3212 & 0.9592 & 0.9632 & -1 \end{bmatrix} \quad (41)$$

By Eq. (31), the interface effect among experts for each alternative is:

$$INT_1 = 0.0725, INT_2 = -0.0082, INT_3 = -0.0809, INT_4 = 0.1669.$$

Finally, by Eq. (33), the probability of each alternative is:

$$Prob(x_1) = 0.2553, Prob(x_2) = 0.1655, Prob(x_3) = 0.2294, Prob(x_4) = 0.3498.$$

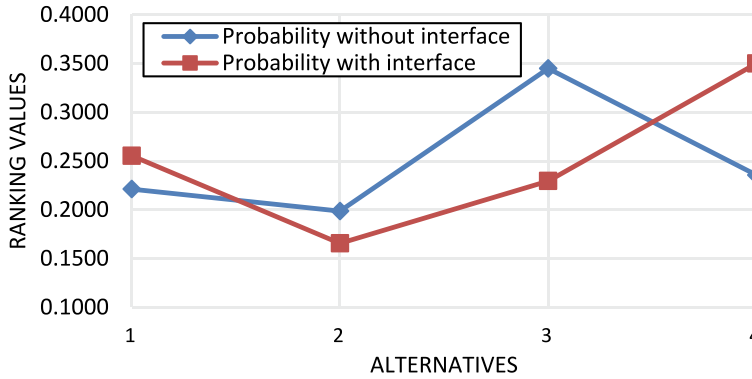


Fig. 4 The interface effect on ranking values

Step 6: According to the value of QP of each alternative, the final ranking is

$$x_4 \succ x_1 \succ x_3 \succ x_2.$$

4.2 Effect of the Quantum Interface on Decision Result

If we can decide based on the $P(x_i) (i = 1, 2, 3, 4)$ values according to Eq. (22), as higher probabilities indicate a superior ranking. The classical approach does not account for the interference effect, which can influence the final rankings, as shown in Fig. 4. The ranking result of alternatives according to Eq. (22) is $x_3 \succ x_4 \succ x_1 \succ x_2$. If we count the interface effect between two experts about an alternative the ranking result is $x_4 \succ x_1 \succ x_3 \succ x_2$.

4.3 Comparative Analysis

To test the feasibility and validity of the suggested method, we compare it with some existing methods for the example given above.

In the traditional decision-making model (Zhang et al., 2023; Wang et al., 2023; Büyüközkan et al., 2024) the ranking results are derived from the aggregate decision matrix of experts' opinions. Therefore, to compare our model and the traditional model, we first aggregate the experts' opinions. In this connection, by Ref. (Xu, 2005), the aggregation of four experts opinions is

$$Agg(A^1, A^2, A^3, A^4) = \begin{bmatrix} h_{-2.232} & h_{-0.719} & h_{-0.491} & h_{0.759} \\ h_{-1.224} & h_{0.741} & h_{1.974} & h_{-0.018} \\ h_{1.768} & h_{-0.469} & h_{2.022} & h_{0.75} \\ h_{2.513} & h_{1.004} & h_{1.487} & h_{1.232} \end{bmatrix}$$

(42)

Therefore, the ranking of alternatives according to the methods TOPSIS (Zhang et al., 2023), TODIM (Wang et al., 2023), VIKOR (Büyüközkan, et al., 2024) methods and our proposed method is presented in Table 2 and Fig. 5.

Table 2 Comparison of our model versus some existing models

Methods	Ranking values	Ranking orders
Zhang et al. (2023)	$C_1^* = 0.0817, C_2^* = 0.4231,$ $C_3^* = 0.6378, C_4^* = 0.4571$	$x_3 \succ x_4 \succ x_2 \succ x_1$
Wang et al. (2023)	$\phi(x_1) = -0.2382, \phi(x_2) =$ $-0.8277, \phi(x_3) = 0.6718, \phi(x_4) =$ $0.4066.$	$x_3 \succ x_4 \succ x_1 \succ x_2$
Büyüközkan, et al. (2024)	$Q_1 = 0.5232, Q_2 = 0.645,$ $Q_3 = 0.183, Q_4 = 0.3247$	$x_3 \succ x_4 \succ x_2 \succ x_1$
Proposed method without considering interface effect	$P(x_1) = 0.2212, P(x_2) = 0.1986,$ $P(x_3) = 0.3448,$ $P(x_4) = 0.2355$	$x_3 \succ x_4 \succ x_1 \succ x_2$
Our proposed method	$Prob(x_1) = 0.2553,$ $Prob(x_2) = 0.1655,$ $Prob(x_3) = 0.2294,$ $Prob(x_4) = 0.3498$	$x_4 \succ x_1 \succ x_3 \succ x_2$

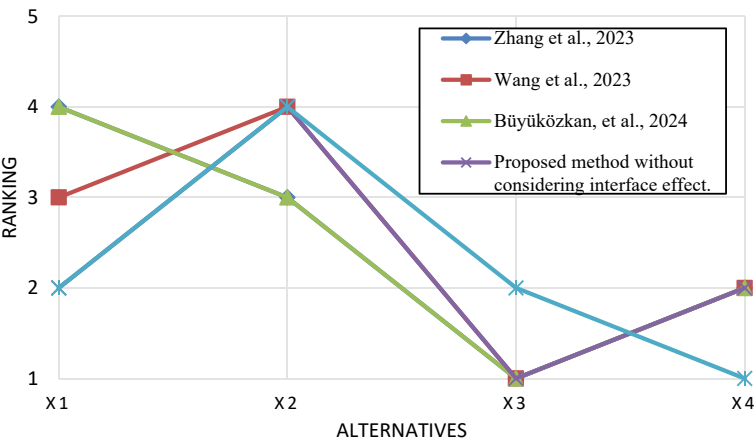


Fig. 5 Comparing ranking results among different models

The rankings of alternatives generated from Eq. (42) by the TOPSIS, TODIM, and VIKOR methods differ significantly from our proposed model. The optimal option identified by the TOPSIS, TODIM, and VIKOR methods is x_3 . When the proposed model is applied without considering the interface effect, the best plan is also x_3 . This suggests that the ranking results of the proposed approach align with those of traditional methods when the impact of decision-makers is disregarded, thereby validating the robustness of the suggested model. However, when evaluating the quantum interference among decision-makers, the outcomes of this model suggest that x_2 is the optimal strategy. This inconsistency arises from the conventional framework's exclusion of the interference effect among decision-makers, which leads to subjective biases during the evaluation of alternatives. As a result, the outcomes of this model better align with real-world complexities.

5 Conclusion

This research proposes a quantum group decision model to mitigate information uncertainty and opinion interference during the meteorological selection of medicine logistic providers. The model uses LIs to represent ambiguous information in complex decision-making scenarios. Furthermore, to replicate the reciprocal interference effects among decision-makers, opinion interferences throughout the MAGDM process are quantified using Deng entropy. This model effectively addresses the shortcomings of traditional methods, which treat decision-makers as isolated individuals. The proposed approach enhances existing processes by applying quantum decision-making techniques to tackle challenges such as determining interference impact values, assigning weights to decision-makers, and managing opinion interference.

Our findings demonstrate that the model significantly improves the decision-making process for the meteorological selection of medicine logistic providers by providing a comprehensive framework for managing information ambiguity and divergent viewpoints. However, as the number of decision-makers and attributes increases, the model's computational complexity grows substantially. This limitation highlights the need for advanced computational methods to maintain efficiency.

Future research will focus on integrating machine learning methodologies to address this issue, aiming to enhance the scalability and feasibility of the quantum group consensus decision-making process. This integration is intended to provide a more efficient approach to managing increasing complexity, ensuring the model remains effective in large-scale decision-making contexts.

Conflict of Interest The authors declare that they have no conflict of interest.

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Pythagorean Linguistic Information-Based Eco-Friendly Building Materials Selection Made from Recycled Plastic Using Quantum-Based Group Decision-Making Methods and the MARCOS Technique



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Abstract The recycling of waste plastics and agro-industrial byproducts for the creation of sustainable polymeric composites is acknowledged as an effective strategy to mitigate the harmful environmental impacts of plastic waste. Despite the considerable promise of sustainable composites in the Circular Economy (CE), their application remains minimal due to the absence of an appropriate material selection methodology. The presence of several contributing factors in material selection categorizes it as a multi-criteria group decision-making (MCGDM) issue. In contemporary MCGDM contexts, experts are frequently seen as independent agents, with their psychological behaviours rarely considered. However, in numerous cases, the viewpoints of experts are likely to influence each other, and these experts often demonstrate subjective bounded rationality in the decision-making process. The Pythagorean linguistic information (PLI)-based MCGDM model, utilizing the measurement, and ranking of alternatives via the compromise solution (MARCOS) approach and quantum decision theory, is designed to tackle this problem. The PLI consists of an ordered pair of a linguistic term (LT) and a Pythagorean fuzzy number (PFN), whereby the latter quantifies the degree of certainty regarding the former.

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PLIs, guided by sample size data, are utilized to aggregate group linguistic assessments, clearly demonstrating both quantitative variation and qualitative uncertainty. A thorough generalized Pythagorean linguistic MARCOS approach is presented to depict the constrained rational actions of decision-makers. It is employed to evaluate the superiority of each option. A quantum possibilistic aggregation framework is established to examine the interference effects among experts' perspectives. This technique considers expert opinions as concurrently existing wave functions that interact and influence the overall result. The comparisons with several contemporary methodologies confirm the effectiveness and reliability of the novel MCGDM strategy.

Keywords Quantum decision theory · Group decision-making · Pythagorean linguistic set · MARCOS

1 Introduction

The research in fuzzy set theory and its applications plays a significant role in addressing uncertainty and complexity in social networks and group decision-making processes. The methods developed in this area offer valuable insights into how fuzzy logic can be used to model and analyze systems where imprecision and ambiguity are prevalent. Fuzzy set theory is particularly well-suited to modeling the uncertainty inherent in social networks, where relationships and interactions often involve incomplete or vague information. Traditional mathematical models struggle to capture this uncertainty, but fuzzy approaches provide a more nuanced framework for analyzing such dynamics. For instance, the development of fuzzy fractional differential equations using fuzzy Laplace transforms is essential for modeling systems where the influence of individuals or groups evolves fractionally over time under uncertainty (Salahshour et al., 2012; Allahviranloo et al., 2015; Chehlabi et al., 2016; Armand et al., 2018). In this connection, a new centrality measure is introduced for neutrosophic and linguistic graphs (Mahapatra et al., 2019, 2020), including self-weight or individual merit within the network. Mahapatra et al. (Mahapatra et al., 2024a) present an innovative method for centrality measurement in fuzzy social networks where relationships between nodes are not binary but have different degrees of intensity.

Polymer-based composite material generated through this process consists of two components: fillers and matrices. Waste plastics, specifically low-density polyethylene (LDPE) and high-density polyethylene (HDPE) are used as matrices. The properties of LDPE and HDPE are described in detail in (Soni et al., 2024). Additionally, rice husk ash (RHA), with its chemical composition and physical properties are also described in (Soni et al., 2024). As per studies by (Soni et al., 2024), eco-friendly building materials selection made from recycled plastic is multi-criteria group decision-making (MCGDM). From studies in (Soni et al., 2024), we pick up the top five alternatives and their corresponding characteristic and corresponding

criteria, which are shown in Tables 1 & 2. The problem of eco-friendly building materials selection made from recycled plastic is effectively handled by (Soni et al., 2024) under q-rung ortho pair fuzzy information. However, we observe some drawbacks (Soni et al., 2024), which are listed in the following.

- (1)

Avoid expert confidence and douting level for providing their opinion judgment.
- (2)

Avoid experts' influence on each other opinion judgments, that is, does not count the observation of between two experts' opinions about criteria.

To resolve the above two drawbacks, in this study, we offer the Pythagorean linguistic information (PLI) (Mandal et al., 2019) to the experts to provide their opinion judgment for resolving the first drawback. Because in real-life situations, experts often have limited knowledge and personal biases, so using language-based information with expert confidence and douting levels (Mandal et al., 2021; 2022; 2023a; 2023b; 2024a) is more flexible for expressing their ideas. In this connection, Mandal et al. briefly studied MCGDM under LZN information with the MARCOS approach and its application in the selection of logistic cold chain centers (Mandal et al., 2024b); TODIM-VIKRO approach (Mandal et al., 2024c) and its application in site selection of medical logistic center; ORESTE approach (Mandal et al., 2024d) and its application in assessment of regional energy; and MULTIMOORA approach (Mandal et al., 2024e) and its application in selection of software. To address the second drawback we need to combine individual assessments using a specific method. While many aggregation strategies have been studied, most assume that decision-makers are independent and not influenced by each other. However, in reality, people's opinions can be influenced by their surroundings. In decision-making, we often take either a combined or separate view of different perspectives.

Table 1 Polymeric composites' chemical makeup

Alternatives	Mixture code	LDPE(Wt. %)	HDPE(Wt. %)	RHA(Wt. %)	Sand(Wt. %)
x_1	L70R15R15	70		15	15
x_2	L80R10S10	80		10	10
x_3	H50R25S25		50	25	25
x_4	H80R10S10		80	10	10
x_5	H90R5R5		90	5	5

Table 2 Description of criteria

Criteria	Characteristic	Behaviour
c_1	Compressive strength	Beneficial
c_2	Abrasive wear at low speed and load	Non-beneficial
c_3	Abrasive wear at high speed and load	Non-beneficial
c_4	Density	Beneficial

Many experts' views can influence each other, affecting the final decision. This introduces the concept of "interference" in Quantum Probability Theory (QPT) (Busemeyer & Bruza, 2012; Moreira & Wichert, 2014; Yearsley & Busemeyer, 2016). QPT is a recent development in classical probability theory (CPT) and has been widely studied in fields like psychology (Busemeyer & Bruza, 2012), cognitive science (Yearsley & Busemeyer, 2016), and decision analysis (He et al., 2018a; 2018b; Wu et al., 2020; Xiao et al., 2022; Han & Liu, 2023; Mandal et al., 2024f; 2025) due to its strong descriptive capabilities and solid framework. It can explain paradoxes like the disjunction fallacy (He et al., 2018a; 2018b), the Ellsberg paradox (Eichberger & Pirner, 2018), the order effect (Wang et al., 2014), and graph theory (Samanta & Allahviranloo, 2025), which CPT struggles with. QPT is particularly good at representing interference terms, making it very useful for studying how experts' views influence each other in the MCGDM process.

This work introduces a quantum-based behavioral PLI-based MCGDM model that considers the limited rationality of experts and how their opinions influence each other. The main contributions of this paper are:

1. An improved generalized PLI-based MARCOS approach accounts for experts' limited rationality, simplifies the calculation process and resolves some paradoxes of the traditional MARCOS method.
2. A quantum-based method for combining individual evaluation results, which helps us better understand human psychology through the effects of interference.
3. A case study demonstrating the effectiveness of our approach in eco-friendly building materials selection made from recycled plastic. Comparative and sensitivity analyses show the flexibility and robustness of our method.

The structure of the paper is as follows: In Sect. 2, recall some basic definitions and the background of linguistic term sets (LTs), the Pythagorean linguistic set (PLS), and quantum Decision Theory (QDT). The proposed MCGDM model is discussed in Sect. 3. A case study for the selection of eco-friendly building materials made from recycled plastic with comparative analysis is provided in Sect. 4. Section 5 concludes our study.

2 Preliminaries

Here we recall some related definitions and results that are used in this paper.

2.1 LTS

Here we assume an LTS is: $H = \{h_\lambda | \lambda = -\zeta, \dots, -1, 0, 1, \dots, \zeta\}$ with cardinality $2\zeta + 1$ and holds the followings (Mandal et al., 2024a):

- (1) $h_{\lambda_1} \geq h_{\lambda_2}$ if and only if $\lambda_1 \geq \lambda_2$;
- (2) $\text{neg}(h_{\lambda_1}) = h_{-\lambda_1}$;
- (3) $\max(h_{\lambda_1}, h_{\lambda_2}) = h_{\lambda_1}$ if $h_{\lambda_1} \geq h_{\lambda_2}$;
- (4) $\min(h_{\lambda_1}, h_{\lambda_2}) = h_{\lambda_1}$ if $h_{\lambda_1} \leq h_{\lambda_2}$.

To preserve expert information, the continuous LTSs (Xu, 2005) denotes $\bar{H} = \{h_\lambda | \lambda \in [-t, t]\}$, $t > \zeta$.

The subscript index of any $h_\lambda \in \bar{H}$, represented by the following functions:

$$I: \bar{H} \rightarrow [-t, t], \text{ and } I(h_\lambda) = \lambda.$$

$$I^{-1}: [-t, t] \rightarrow \bar{H}, \text{ and } I^{-1}(\lambda) = h_\lambda.$$

Let $\mathcal{F}\ominus$ be the linguistic conversation function for any $LT G \in \bar{H}$, which is defined in the following way:

$$\mathcal{F}: \bar{H} \rightarrow [0, 1], \mathcal{F}(G) = \{\mathcal{F}(h_\lambda) = \frac{\lambda}{2t} + \frac{1}{2} | \lambda \in [-t, t]\} = \kappa.$$

2.2 PTS

By Ref. (Mandal & Ranadive, 2019), a Pythagorean linguistic set (PLS) denoted by $\not\in$ in an alternative set X for continuous LTS $\bar{H} = \{h_{-\lambda} \leq h_\lambda \leq h_t | \lambda \in [-t, t]\}$ is represented in the following way:

$$\mathcal{R} = \{ \langle h_{\theta(x)}, (\mu_{\mathcal{R}}(x), \nu_{\mathcal{R}}(x)) \rangle | x \in X \},$$

where, $h_{\theta(x)} \in \bar{H}$, $\mu_{\mathcal{R}}: X \rightarrow [0, 1]$ and $\nu_{\mathcal{R}}: X \rightarrow [0, 1]$ are the membership and non-membership degrees of x to $h_{\theta(x)}$ such that $\mu_{\mathcal{R}}^2(x) + \nu_{\mathcal{R}}^2(x) \leq 1$. Here, the Pythagorean linguistic number (PLN) in $\mathcal{R}$ is denoted by $\not\in$ and represented by $\not\in = \langle h_{\theta(\not\in)}, (\mu(\not\in), \nu(\not\in)) \rangle$ with $h_{\theta(\not\in)} \in \bar{H}$ and $\mu(\not\in), \nu(\not\in) \in [0, 1]$, $\mu^2(\not\in) + \nu^2(\not\in) \leq 1$.

By Ref. (Mandal & Ranadive, 2019), the expected value of $\not\in = \langle h_{\theta(\not\in)}, (\mu(\not\in), \nu(\not\in)) \rangle$ is denoted by κ and defined in the following way:

$$\kappa(\not\in) = h_{\frac{\theta(\not\in) \times (\mu^2(\not\in) + 1 - \nu^2(\not\in))}{2}}.$$

2.3 Quantum Probability Theory (QPT)

A quantum decision (QD) model is based on the principles of quantum probability (QP) theory (QPT). To provide context, we will review some key concepts and notations from QPT. In quantum theory (QT), a state is represented as a unit-length vector within an N-dimensional vector space.

Definition 1. (Yearsley & Busemeyer, 2016) A finite-dimensional Hilbert space (HS), represented as $\mathfrak{H}$, is an abstract vector space over a complex field, spanned by a set of orthonormal basis vectors with complex components. In classical probability theory (CPT), events are fundamental (He et al., 2018), but in quantum probability theory (QPT), events are instead represented by subspaces of the Hilbert space $\mathfrak{H}$, rather than traditional sets.

Definition 2. (Busemeyer & Bruza, 2012) In QT, the superposition state $|S\rangle$, a unit-length vector in the N-dimensional Hilbert space $\mathfrak{H}$, defines all possible events within that space. The nonlinear combination of a set of orthogonal basis vectors $\{|\mathcal{V}\rangle_k, k = 1, 2, \dots, n\}$ are used to represent the superposition state $|S\rangle$, i.e.,

$$\begin{aligned} |S\rangle &= \langle \mathcal{V}_1 | S \mathcal{V}_1 \rangle + \langle \mathcal{V}_2 | S \mathcal{V}_2 \rangle + \dots + \langle \mathcal{V}_n | S \mathcal{V}_n \rangle \\ &= \xi_1 |\mathcal{V}_1\rangle + \xi_2 |\mathcal{V}_2\rangle + \dots + \xi_n |\mathcal{V}_n\rangle \\ &= \zeta_1 e^{i\alpha_1} |\mathcal{V}_1\rangle + \zeta_2 e^{i\alpha_2} |\mathcal{V}_2\rangle + \dots + \zeta_n e^{i\alpha_n} |\mathcal{V}_n\rangle \end{aligned} \quad (1)$$

where $\xi_k (= \zeta_k e^{i\alpha_k})$ is the amplitude of the probability, expressed as a complex number, characterizing the amplitude of the wave in the state $|\mathcal{V}_k\rangle$, and the amplitude of the phase angle represented by $e^{i\alpha_k}$.

Definition 3. (Busemeyer & Bruza, 2012) The CPT framework represents every event using a single set of components. In QPT, probabilities are assigned based on the overlap between cognitive states (shown as state vectors) and different probability measures (shown as subspaces). QPT extends CPT into the complex domain by including incompatible qualities and viewing events as vectors. Incompatible attributes are two observable features of an item that cannot be measured accurately at the same time. Both the incompatibility condition and the uncertainty principle can be described by the probability amplitude, also known as complex probability.

According to Born's rule, an event's probability amplitude and its classical probability, x_i are related to each other:

$$P(\mathcal{V}_k) = |\zeta_k e^{i\alpha_k}|^2 = (\zeta_k e^{i\alpha_k}) \times (\zeta_k e^{i\alpha_k})^* = \zeta_k e^{i\alpha_k} \times \zeta_k e^{-i\alpha_k} = (\zeta_k)^2 \quad (2)$$

For the total event probability to be 1, the sum of the squares of the probability amplitudes must also be 1.

$$\sum_{k=1}^n P(\mathcal{V}_k) = \sum_{k=1}^n (\zeta_k)^2 = 1. \quad (3)$$

An $N - 1$ mapping transforms complex probabilities into classical probabilities. For each given probability value $P(\mathcal{V}_k)$, countless complex probabilities correspond to it. Referencing Eq. (2) and Euler's formula,

$$e^{i\alpha} = \cos\alpha + i \times \sin\alpha \quad (4)$$

the complex probability satisfies:

$$\xi = \sqrt{P(\mathcal{V})}(\cos\alpha + i \times \sin\alpha) = \sqrt{P(\mathcal{V})}e^{i\alpha} \quad (5)$$

For example, in the following, we discussed the QPT in two-dimensional Hilbert space. Let $\{\mathcal{M}_1, \mathcal{M}_2\}$ be an event containing two basic states, where $|\mathcal{M}_1\rangle = (1, 0)$, and $|\mathcal{M}_2\rangle = (0, 1)$ as two basic orthogonal vectors in $\mathbf{H}$. The state $|\mathcal{S}\rangle$ is defined as a superposition and is graphically illustrated in Fig. 1. According to Eqs. (1) and (2), it can be expressed as.

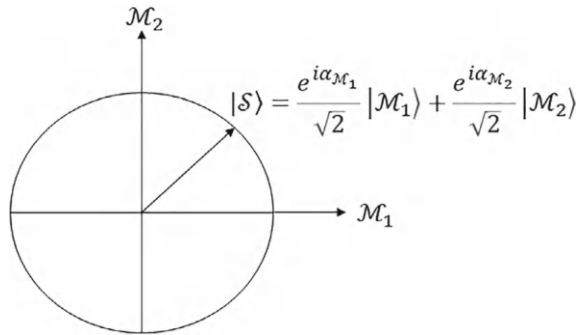
$$|\mathcal{S}\rangle = \zeta_1 e^{i\alpha_1} |\mathcal{M}_1\rangle + \zeta_2 e^{i\alpha_2} |\mathcal{M}_2\rangle = \frac{e^{i\alpha_{\mathcal{M}_1}}}{\sqrt{2}} |\mathcal{M}_1\rangle + \frac{e^{i\alpha_{\mathcal{M}_2}}}{\sqrt{2}} |\mathcal{M}_2\rangle$$

and the probabilities of $|\mathcal{M}_1\rangle$ and $|\mathcal{M}_2\rangle$ are calculated as follows:

$$P(\mathcal{M}_1) = P(\mathcal{M}_2) = (e^{i\alpha})(e^{i\alpha})^* = \frac{1}{2}.$$

According to conventional Bayesian decision theory, decision-making can be mathematically represented as a Markov process (Moreira & Wichert, 2014). The state transition probability in a single route of QPT aligns with the CPT according to Feynman's first rule (Wu et al., 2020), which varies over several pathways. Consider the decision-making dilemma highlighted above as an illustration of the process depicted in Fig. 2(a). Experts provide their assessments of $\mathcal{M}_1$ ('bad') or $\mathcal{M}_2$ ('good') as probability quantifications. State transition probabilities are determined by combining the individual probabilities for each route and using the total probability formula in CPT.

Fig. 1 Event in Hilbert space with two dimensions



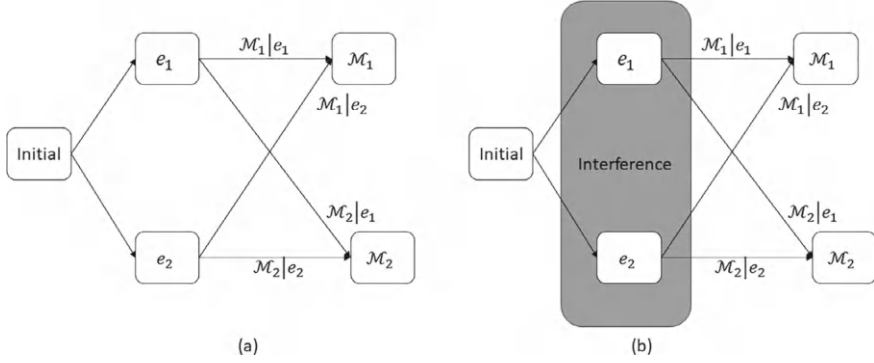


Fig. 2 DM in (a) Markov and (b) Quantum process

$$P(\mathcal{M}_1) = \sum_{k=1}^2 P(e_k)P(\mathcal{M}_1|e_k), \sum_{k=1}^2 P(e_k) = 1 \quad (6)$$

Since only one route is selected at a time, it is sometimes referred to as the single path principle. The conventional Markovian framework assumes that experts rely on one expert at a time and that both experts are completely autonomous. However, this approach is sometimes unsuitable for ordinary cognitive processes because many avenues are often considered when forming judgments. Such phenomena can result in illogical actions, including disjunction, conjunction errors, and interference effects. Thus, QPT is an excellent complement to CPT in this scenario.

Within the QPT framework (Fig. 2(b)), the state is formed by combining three fundamental states, which can be represented as different wave functions aligning according to Feynman's second rule. The amplitude of the final form is obtained by superimposing two individual routes running from I ('Initial') to $\mathcal{M}_1$. The amplitude is the sum of the individual pathways from source I to target $\mathcal{M}_1$, and the probability of reaching the target $\mathcal{M}_1$ is equal to the square of the amplitude.

$$P(\mathcal{M}_1) = |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 \quad (7)$$

Equation (7) can be simplified in the following way:

$$\begin{aligned} P(\mathcal{M}_1) &= |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 \\ &= |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}| |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^* \\ &= \left(\sqrt{P(e_1)P(\mathcal{M}_1|e_1)}e^{i\alpha_1} + \sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{i\alpha_2} \right) \\ &\quad \left(\sqrt{P(e_1)P(\mathcal{M}_1|e_1)}e^{-i\alpha_1} + \sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{-i\alpha_2} \right) \\ &= P(e_1)P(\mathcal{M}_1|e_1) + P(e_2)P(\mathcal{M}_1|e_2) \\ &\quad + \sqrt{P(e_1)P(\mathcal{M}_1|e_1)}\sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{i(\alpha_1-\alpha_2)} \end{aligned}$$

$$+ \sqrt{P(e_1)P(\mathcal{M}_1|e_1)}\sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{i(\alpha_2-\alpha_1)} \quad (8)$$

By Eq. (4), we get

$$\cos(\alpha_1 - \alpha_2) = \frac{e^{i(\alpha_2-\alpha_1)} + e^{i(\alpha_1-\alpha_2)}}{2} \quad (9)$$

Then Eq. (7) can be written in the following way:

$$\begin{aligned} P(\mathcal{M}_1) &= P(e_1)P(\mathcal{M}_1|e_1) + P(e_2)P(\mathcal{M}_1|e_2) \\ &+ 2\sqrt{P(e_1)P(\mathcal{M}_1|e_1)}\sqrt{P(e_2)P(\mathcal{M}_1|e_2)}\cos(\alpha_1 - \alpha_2) \\ &= |\xi_{e_1}\xi_{\mathcal{M}_1|e_1}|^2 + |\xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 + 2\xi_{e_1}\xi_{\mathcal{M}_1|e_1}\xi_{e_2}\xi_{\mathcal{M}_1|e_2}\cos(\alpha_1 - \alpha_2) \end{aligned} \quad (10)$$

Upon comparing Eqs. (6) and (10), it becomes evident that the quantum framework includes an interference component. If the pathways are observable, the quantum phase transition will reduce to the classical one, following Feynman's third rule.

$$P(\mathcal{M}_1) = |\xi_{e_1}\xi_{\mathcal{M}_1|e_1}|^2 + |\xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 = P(e_1)P(\mathcal{M}_1|e_1) + P(e_2)P(\mathcal{M}_1|e_2) \quad (11)$$

Additionally, the QPT can be expanded to a multi-path graph, resulting in an increased number of interference terms. The QPT enhances CPT by accurately representing cognitive processes in quantum terms, thereby more effectively modeling ambiguity, vagueness, and uncertainty in decision-making.

3 The Proposed Marcos-Based Magdm Under Pythagorean Linguistic Information and Quantum Decision Scenario

This section introduces quantum cognition-based MAGDM integrating the MARCOS approach.

3.1 Background

Let us assume.

$X = \{x_1, x_2, \dots, x_n\}$: Alternative set,

$C = \{c_1, c_2, \dots, c_m\}$: Attribute set,

$e = \{e_1, e_2, \dots, e_q\}$: Expert set,

$w^k = (w_1^k, w_2^k, \dots, w_m^k)$: Weight vector of attributes for expert e_k , where $w_j^k \in [0, 1]$, and $\sum_{j=1}^m w_j^k = 1$,

$\omega = (\omega_1, \omega_2, \dots, \omega_q)$: Weight vector of experts with $\omega_k \in [0, 1]$, $\sum_{k=1}^q \omega_k = 1$.

3.2 Decision Steps

Let $A^k = (a_{ij}^k)_{nm}$ ($i = 1, 2, \dots, n; j = 1, 2, \dots, m; k = 1, 2, \dots, q$) be the LN-based decision judgment provided by an expert e_k using pre-defined distinct LTSs $H = \{h_\lambda | \lambda = -\zeta, \dots, -1, 0, 1, \dots, \zeta\}$.

Step 1: Construction of decision matrix

Then, in the following, we represent the decision matrix of k-th expert:

$$A^k = (a_{ij}^k)_{nm} = \begin{matrix} & \begin{matrix} c_1 & c_2 & \dots & c_m \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} & \begin{bmatrix} a_{11}^k & a_{12}^k & \dots & a_{1m}^k \\ a_{21}^k & a_{22}^k & \dots & a_{2m}^k \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}^k & a_{n2}^k & \dots & a_{nm}^k \end{bmatrix} \end{matrix} \quad (12)$$

where, $a_{ij}^k = \langle h_{\theta_{ij}}^k, (\mu_{ij}^k, v_{ij}^k) \rangle$ is a PLN, called the Pythagorean linguistic information (PLI) of k-th expert for i-th alternative about j-th criteria.

Step 2: Expert weight computation process

Inspire by Ref. (Mandal et al., 2024a), the weight of each expert e_k ($k = 1, 2, \dots, q$) can be calculated in the following way:

$$\omega_k = \frac{\Omega(Z^k)}{\sum_{k=1}^q \Omega(Z^k)}, \text{ where } \Omega(Z^k) = \frac{\sum_{j=1}^m \sum_{i=1}^n \mathcal{F}(\kappa(a_{ij}^k))}{mn} \quad (13)$$

Step 3: Construct expert opinion according to expected value

Adding the ideal solution (IS) and anti-ideal solution (AIS) in the k-th expert opinion, the expected value-based extended (EVBE) k-th expert opinion is $L^k = (l_{ij}^k)_{nm}$ is as:

$$L^k = \left(l_{ij}^k \right)_{nm} = \begin{matrix} & & c_1 & c_2 & \cdots & c_m \\ \begin{matrix} AIS^k \\ x_1 \\ x_2 \\ \vdots \\ x_n \\ IS^k \end{matrix} & \left[\begin{array}{cccc} l_{AIS_1}^k & l_{AIS_2}^k & \cdots & l_{AIS_n}^k \\ l_{11}^k = \mathcal{F}(\kappa(a_{11}^k)) & l_{12}^k = \mathcal{F}(\kappa(a_{12}^k)) & \cdots & l_{1m}^k = \mathcal{F}(\kappa(a_{1m}^k)) \\ l_{21}^k = \mathcal{F}(\kappa(a_{21}^k)) & l_{22}^k = \mathcal{F}(\kappa(a_{22}^k)) & \cdots & l_{2m}^k = \mathcal{F}(\kappa(a_{2m}^k)) \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1}^k = \mathcal{F}(\kappa(a_{n1}^k)) & l_{n2}^k = \mathcal{F}(\kappa(a_{n2}^k)) & \cdots & l_{nm}^k = \mathcal{F}(\kappa(a_{nm}^k)) \\ l_{IS_1}^k & l_{IS_2}^k & \cdots & l_{IS_n}^k \end{array} \right] \end{matrix} \quad (14)$$

Here, the AIS means worst object and IS that of the best object. According to the characteristic of criteria, the AIS and IS can be determined in the following way:

$$AIS^k = \begin{cases} \min_i \{ l_{ij}^k \} \text{ iff } i \in \mathfrak{B} \\ \max_i \{ l_{ij}^k \} \text{ iff } i \in \mathfrak{C} \end{cases} \text{ and } IS^k = \begin{cases} \max_i \{ l_{ij}^k \} \text{ iff } i \in \mathfrak{B} \\ \min_i \{ l_{ij}^k \} \text{ iff } i \in \mathfrak{C} \end{cases}$$

where $\mathfrak{B}$ is the set of benefit criteria and $\mathfrak{C}$ that of the set of cost criteria.

Step 4: Normalization of process EVBE of each expert opinion

Let $\mathcal{N}^k = \left(n_{ij}^k \right)_{nm}$ be the normalized EVBE of $L^k = \left(l_{ij}^k \right)_{nm}$, written in the following way:

$$\mathcal{N}^k = \left(n_{ij}^k \right)_{nm} = \begin{matrix} & & c_1 & c_2 & \cdots & c_m \\ \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} & \left[\begin{array}{cccc} n_{11}^k & n_{12}^k & \cdots & n_{1m}^k \\ n_{21}^k & n_{22}^k & \cdots & n_{2m}^k \\ \vdots & \vdots & \ddots & \vdots \\ n_{n1}^k & n_{n2}^k & \cdots & n_{nm}^k \end{array} \right] \end{matrix} \quad (15)$$

where,

$$n_{ij}^k = \begin{cases} \frac{l_{IS_i}^k}{l_{ij}^k} \text{ iff } i \in C \\ \frac{l_{ij}^k}{l_{IS_i}^k} \text{ iff } i \in B \end{cases} \quad (16)$$

Step 5: Attribute weight computation process

(1) The Shannon entropy $\Psi^k = (\Psi_1^k, \Psi_2^k, \dots, \Psi_m^k)$ can be calculated from the normalized EVBE presented in Eq. (15) in the following way:

$$\Psi_j^k = -\frac{1}{\ln n} \sum_{i=1}^n n_{ij}^k \ln n_{ij}^k \quad (17)$$

$$n_{ij}^k \ln n_{ij}^k \text{ is } 0 \text{ if } n_{ij}^k = 0.$$

- (2) The attribute weight $w^k = (w_1^k, w_2^k, \dots, w_m^k)$ concerning each expert $e_k (k = 1, 2, \dots, q)$ opinion can be calculated in the following way:

$$w_j^k = \frac{1 - \Psi_j^k}{\sum_{j=1}^m (1 - \Psi_j^k)} \quad (18)$$

Step 6: Construction of Bayesian network

In this phase, we converted the pre-collected data into probabilities and developed a Bayesian Network for decision-making, as the quantum framework is a partial extension of the Bayesian Network. Figure 3 illustrates that, in a MAGDM situation, the decision-making process can be represented as a simple two-layer Bayesian Network. The experts' weights are converted into probabilities, which indicate the degree of belief in the experts, using the following equation:

$$P(e_k) = \omega_k, k = 1, 2, \dots, q. \quad (19)$$

In Eq. (19), the probabilities form the initial layer of the Bayesian Network (BN). The second layer depends on the assessment outcomes from each expert, which can be calculated using the MORCOS approach as follows:

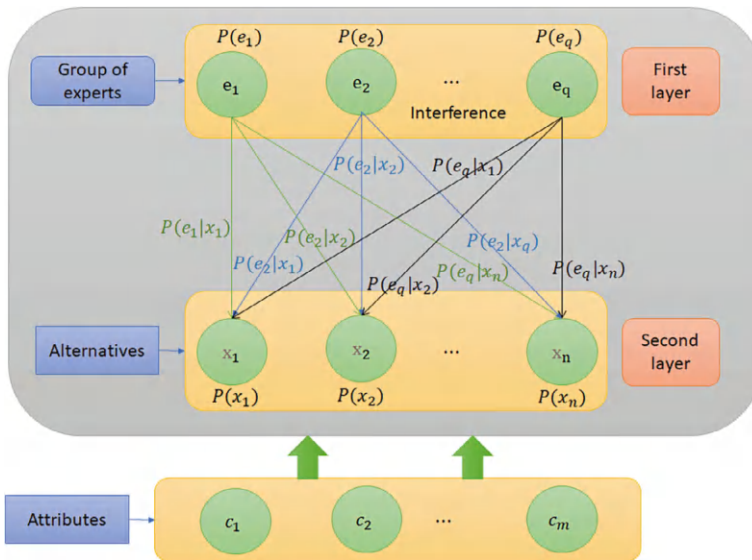


Fig. 3 MAGDM according to QSBN

- (1) Let $B^k = (b_{ij}^k)_{nm}$ be the weight-normalized EVBE, written in the following way:

$$B^k = (b_{ij}^k)_{nm} = \begin{matrix} & \begin{matrix} c_1 & c_2 & \cdots & c_m \end{matrix} \\ \begin{matrix} b_{AIS}^k \\ x_1 \\ x_2 \\ \vdots \\ x_n \\ b_{IS}^k \end{matrix} & \begin{bmatrix} b_{AIS_1}^k & b_{AIS_2}^k & \cdots & b_{AIS_n}^k \\ b_{11}^k & b_{12}^k & \cdots & b_{1m}^k \\ b_{21}^k & b_{22}^k & \cdots & b_{2m}^k \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}^k & b_{n2}^k & \cdots & b_{nm}^k \\ b_{IS_1}^k & b_{IS_2}^k & \cdots & b_{IS_n}^k \end{bmatrix} \end{matrix} \quad (20)$$

where the elements q_{ij}^k are calculated as: $b_{ij}^k = n_{ij}^k w_j^k$ and $b_{AIS}^k = \begin{cases} \min_i \{b_{ij}^k\} | i \in \mathfrak{B} \\ \max_i \{b_{ij}^k\} | i \in \mathfrak{C} \end{cases}$ and $b_{IS}^k = \begin{cases} \max_i \{b_{ij}^k\} | i \in \mathfrak{B} \\ \min_i \{b_{ij}^k\} | i \in \mathfrak{C} \end{cases}$.

- (2) With the help of IS and AIS, the utility degree of each alternative of each expert can be obtained in the following way:

$$U_i^{k,-} = \frac{\sum_{j=1}^m b_{ij}^k}{\sum_{i=1}^n b_{AIS_i}^k} \text{ and } U_i^{k,+} = \frac{\sum_{j=1}^m b_{ij}^k}{\sum_{i=1}^n b_{IS_i}^k} \quad (21)$$

- (3) The utility function of each alternative of each expert $e_k (k = 1, 2, \dots, q)$ opinion obtained in the following way:

$$\mathcal{T}(U_i^k) = \frac{U_i^{k,-} + U_i^{k,+}}{1 + \frac{1 - \mathcal{T}(U_i^{k,-})}{\mathcal{T}(U_i^{k,-})} + \frac{1 - \mathcal{T}(U_i^{k,+})}{\mathcal{T}(U_i^{k,+})}}, \quad (22)$$

where,

$$\mathcal{T}(U_i^{k,+}) = \frac{U_i^{k,+}}{U_i^{k,+} + U_i^{k,-}} \text{ and } \mathcal{T}(U_i^{k,-}) = \frac{U_i^{k,-}}{U_i^{k,+} + U_i^{k,-}}. \quad (23)$$

- (4) Based on Eq. (22), conditional probabilities which link the two layers in the BN as:

$$P(x_i | e_k) = \frac{\mathcal{T}(U_i^k)}{\sum_{i=1}^n \mathcal{T}(U_i^k)}. \quad (24)$$

- (5) The second layer consists of the probabilities of alternatives calculated as:

$$P(x_i) = \sum_{k=1}^q P(e_k)P(x_i|e_k) \quad (25)$$

Subsequently, we can decide based on the $P(x_i)$ values, as higher probabilities, indicate a superior ranking. The classical approach does not account for the interference effect, which can influence the final rankings. To address this issue, Bayesian networks are expanded into a quantum framework capable of properly modeling a practical approach to real interference in multi-attribute group decision-making (MAGDM).

Step 7: Construction of quantum scenario-based Bayesian network (QSBN)

The facilitator often consolidates opinions from several specialists. The interconnected decision-making processes in real-world situations, as demonstrated in Sect. 2.2, remain unobservable. The square of the sum of the amplitude probabilities of all possible pathways collectively constitutes the probability of the alternative x_i from QPT (He et al., 2018), as described.

$$\begin{aligned} P(x_i) &= \left| \sum_{k=1}^q \xi_{e_k} \xi_{x_i|e_k} e^{i\alpha_k} \right|^2 \\ &= \left| \sum_{k=1}^q \sqrt{P(e_k)} \sqrt{P(x_i|e_k)} e^{i\alpha_k} \right|^2, i = 1, 2, \dots, n \end{aligned} \quad (26)$$

By Eqs. (7) – (11), Eq. (26) can be written as

$$\begin{aligned} P(x_i) &= \sum_{k=1}^q P(e_k)P(x_i|e_k) + 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \cos(\alpha_k - \alpha_s) \\ &= \sum_{k=1}^q P(e_k)P(x_i|e_k) + INT_i \end{aligned} \quad (27)$$

In a quantum-based Bayesian network, a normalization factor is incorporated into Eq. (27) to standardize the probability of each alternative, ensuring that the overall estimated probability does not exceed 1. Then

$$Prob(x_i) = \sigma P(x_i) \quad (28)$$

where,

$$\sigma = \frac{1}{\sum_{i=1}^n P(x_i)} = \frac{1}{\sum_{i=1}^n (\sum_{k=1}^q P(e_k)P(x_i|e_k) + INT_i)} \quad (29)$$

3.3 Analysis of Interference Effect Among Experts' Opinions

Equation (29) uses the term INT_i , which represent the ambiguity and uncertainty of expert opinions, to explain the superposition of decision information. In Eq. (27), $\cos(\alpha_k - \alpha_s)$ represents the phase angle between two distinct experts e_k and e_s , which represents the connection vector between two experts. $\frac{q(q-1)}{2}$ is the total count of pairwise correspondences across q experts, excluding duplicates. The final interference value is derived by summing the computed $\frac{q(q-1)}{2}$ interference values for each expert pair, rather than assessing the overall uncertainty across experts. This approach enhances the accuracy of the estimated probability value.

In real-world situations, it is hard to measure the phase angle. Experts' opinions can differ or be influenced by outside factors, causing a lot of variation and mistakes. Belief entropy helps a lot in handling and combining unclear information, making it easier to make good decisions in group settings with many factors. It is used as a standard in areas like risk assessment, defect diagnosis, and pattern recognition. Belief entropy is the latest way to measure interference in QSBN. Unlike the old method of manually estimating the phase angle range, this new method is more reliable because it reduces errors from personal judgment, which can affect the final decision and the expert's probability calculation. Belief entropy is used in QSBN to manage the interference among different experts' beliefs in GDM.

Then the belief distance about alternative x_i according to the average probability $\nabla = \frac{1}{nq}$ can be calculated in the following way:

$$D_{iu} = \left| \delta_{iu} + \frac{|\delta_{iu} - \nabla| - |\varrho_{iu} - \nabla|}{|\delta_{iu} - \nabla| + |\varrho_{iu} - \nabla|} \right|, i = 1, 2, \dots, n; u = 1, 2, \dots, \frac{q(q-1)}{2},$$

$$\begin{cases} \delta_{iu} = P(e_k)P(x_i|e_k), i = 1, 2, \dots, n; k = 1, 2, \dots, q-1; s = k+1, k+2, \dots, \frac{q(q-1)}{2}. \\ \varrho_{iu} = P(e_s)P(x_i|e_s) \end{cases} \quad (30)$$

In Eq. (30), u means the value reallocated to the correlation among q experts; $\nabla = \frac{1}{nq}$ represent each expert probability value of each alternative; D_{iu} means deviation degree from ∇ ; $|\delta_{iu} - \nabla|$ and $|\varrho_{iu} - \nabla|$ represents two experts' magnitude of the deviation degree from ∇ about alternative x_i . If $|\delta_{iu} - \nabla| < |\varrho_{iu} - \nabla|$, then the position of δ_{iu} and ϱ_{iu} are interchangeable.

There are many ways to measure uncertainty. Deng entropy (Zhao et al., 2024), also known as belief entropy, is better at including the confidence function and the uncertainty of BPA because it measures both total non-specificity and disagreement.

Thus, by Deng's entropy, we get

$$\vartheta_{iu}(D_{iu}) = -D_{iu} \log_2 \frac{D_{iu}}{2^m - 1}, i = 1, 2, \dots, n; u = 1, 2, \dots, \frac{q(q-1)}{2}. \quad (31)$$

To ensure that the values of Deng entropy align with the range of $\cos(\alpha_k - \alpha_s)$, which is between $[-1, 1]$, Deng entropy is standardized.

$$\beta_{iu}(D_{iu}) = 2 \frac{\vartheta_{iu}(D_{iu}) - \min\{\vartheta_{iu}(D_{iu})\}}{\max\{\vartheta_{iu}(D_{iu})\} - \min\{\vartheta_{iu}(D_{iu})\}} - 1, i = 1, 2, \dots, n; u = 1, 2, \dots, \frac{q(q-1)}{2}. \quad (32)$$

Thus, the phase angle $\cos(\alpha_k - \alpha_s)$ can be determined by the following way:

$$\cos(\alpha_k - \alpha_s) = \beta_{iu}(D_{iu}), i = 1, 2, \dots, n; u = 1, 2, \dots, \frac{q(q-1)}{2}. \quad (33)$$

Therefore, by Eq. (33), in Eq. (27), INT_i can be written as

$$\begin{aligned} INT_i &= 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \cos(\alpha_k - \alpha_s) \\ &= 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \beta_{iu}(D_{iu}) \end{aligned} \quad (34)$$

By Eq. (34), the Eq. (27) can be written as

$$P(x_i) = \sum_{k=1}^q P(e_k)P(x_i|e_k) + 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \beta_{iu}(D_{iu}) \quad (35)$$

Finally, the probability of each alternative x_i can be obtained in the following way:

$$Prob(x_i) = \sigma P(x_i) \quad (36)$$

where,

$$\sigma = \frac{1}{\sum_{i=1}^n P(x_i)} = \frac{1}{\sum_{i=1}^n (\sum_{k=1}^q P(e_k)P(x_i|e_k) + INT_i)} \quad (37)$$

Step 8: Ranking of alternative

According to Eq. (36), the higher probability of an alternative means the best alternative.

4 A Practical Example with Comparative Analysis

Using a case study focused on the material selection problem of polymeric composites, this section demonstrates the viability of the proposed MARCOS-based MCGDM strategy, which integrates QDT under PLI. Through an availability test and parameter analysis, we highlight the numerous advantages and adaptability offered

by the technique. Additionally, we assess the effectiveness of the proposed method in comparison with several previously employed strategies.

4.1 A Case Study

The objective of this section is to examine the application of the proposed decision-making technique for solving the problem of selecting materials for polymeric composites. A unique QDT-based integrated MARCOS approach, illustrated in Fig. 4, is implemented to address this challenging material selection problem and expand the range of potential applications.

To evaluate the performance of the generated floor tile samples, five alternatives (x_1, x_2, x_3, x_4, x_5), representing different composite material compositions, are considered across four criteria (c_1, c_2, c_3, c_4). A panel of five experts (e_1, e_2, e_3, e_4, e_5) and researchers with extensive experience in composite materials and their use in construction have been formed to determine the most suitable composite material.

Step 1: Tables 1 and 2 present the chacterstic details of the criteria and alternatives. Here we offer the LTSs, $H_1 = \{h_{-3}: \text{very low (VL)}, h_{-2}: \text{low (L)}, h_{-1}: \text{slightly low (SL)}, h_0: \text{medium (M)}, h_1: \text{slightly high (SH)}, h_2: \text{high (H)}, h_3: \text{very high (VH)}\}$ for the evaluation of alternatives (x_1, x_2, x_3, x_4, x_5) about the criteria (c_1, c_2, c_3, c_4). The evaluation information is shown in the Eqs. (38–42).

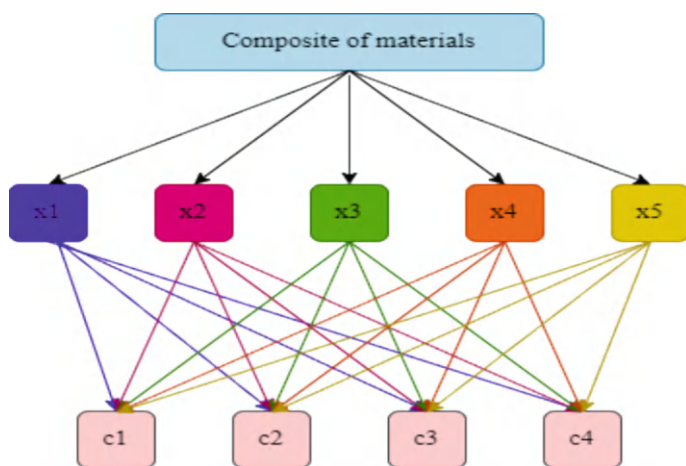


Fig. 4 Hierarchical diagram of material selection

$$A^1 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} \langle h_1, (0.82, 0.31) \rangle & \langle h_0, (0.8, 0.4) \rangle & \langle h_1, (0.25, 0.65) \rangle & \langle h_{-2}, (0.45, 0.2) \rangle \\ \langle h_2, (0.32, 0.75) \rangle & \langle h_2, (0.47, 0.75) \rangle & \langle h_2, (0.46, 0.5) \rangle & \langle h_1, (0.65, 0.6) \rangle \\ \langle h_{-3}, (0.45, 0.85) \rangle & \langle h_{-3}, (0.6, 0.5) \rangle & \langle h_2, (0.52, 0.6) \rangle & \langle h_2, (0.85, 0.2) \rangle \\ \langle h_3, (0.92, 0.12) \rangle & \langle h_1, (0.8, 0.3) \rangle & \langle h_{-1}, (0.9, 0.2) \rangle & \langle h_3, (0.9, 0.12) \rangle \\ \langle h_2, (0.78, 0.28) \rangle & \langle h_2, (0.6, 0.2) \rangle & \langle h_{-2}, (0.7, 0.6) \rangle & \langle h_{-2}, (0.8, 0.3) \rangle \end{bmatrix} \end{matrix} \quad (38)$$

$$A^2 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} \langle h_{-3}, (0.8, 0.41) \rangle & \langle h_0, (0.6, 0.2) \rangle & \langle h_2, (0.85, 0.2) \rangle & \langle h_{-2}, (0.3, 0.8) \rangle \\ \langle h_{-2}, (0.6, 0.3) \rangle & \langle h_{-2}, (0.2, 0.9) \rangle & \langle h_{-2}, (0.56, 0.25) \rangle & \langle h_{-1}, (0.8, 0.54) \rangle \\ \langle h_2, (0.5, 0.5) \rangle & \langle h_2, (0.6, 0.7) \rangle & \langle h_2, (0.82, 0.21) \rangle & \langle h_1, (0.6, 0.65) \rangle \\ \langle h_1, (0.8, 0.6) \rangle & \langle h_{-1}, (0.8, 0.4) \rangle & \langle h_{-1}, (0.87, 0.22) \rangle & \langle h_3, (0.9, 0.2) \rangle \\ \langle h_{-3}, (0.65, 0.74) \rangle & \langle h_{-2}, (0.7, 0.25) \rangle & \langle h_{-3}, (0.92, 0.15) \rangle & \langle h_{-3}, (0.7, 0.4) \rangle \end{bmatrix} \end{matrix} \quad (39)$$

$$A^3 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} \langle h_{-1}, (0.45, 0.65) \rangle & \langle h_2, (0.88, 0.2) \rangle & \langle h_0, (0.35, 0.2) \rangle & \langle h_1, (0.23, 0.8) \rangle \\ \langle h_{-2}, (0.25, 0.8) \rangle & \langle h_{-1}, (0.56, 0.65) \rangle & \langle h_1, (0.8, 0.3) \rangle & \langle h_2, (0.42, 0.6) \rangle \\ \langle h_3, (0.88, 0.25) \rangle & \langle h_{-2}, (0.8, 0.2) \rangle & \langle h_{-1}, (0.6, 0.5) \rangle & \langle h_{-3}, (0.35, 0.45) \rangle \\ \langle h_0, (0.65, 0.6) \rangle & \langle h_1, (0.6, 0.1) \rangle & \langle h_1, (0.78, 0.4) \rangle & \langle h_{-1}, (0.85, 0.3) \rangle \\ \langle h_{-3}, (0.8, 0.2) \rangle & \langle h_{-2}, (0.8, 0.3) \rangle & \langle h_2, (0.25, 0.8) \rangle & \langle h_2, (0.9, 0.2) \rangle \end{bmatrix} \end{matrix} \quad (40)$$

$$A^4 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} \langle h_3, (0.8, 0.2) \rangle & \langle h_{-2}, (0.5, 0.6) \rangle & \langle h_2, (0.2, 0.9) \rangle & \langle h_2, (0.9, 0.12) \rangle \\ \langle h_2, (0.7, 0.5) \rangle & \langle h_3, (0.6, 0.5) \rangle & \langle h_{-2}, (0.7, 0.5) \rangle & \langle h_1, (0.6, 0.35) \rangle \\ \langle h_3, (0.2, 0.6) \rangle & \langle h_2, (0.8, 0.5) \rangle & \langle h_3, (0.2, 0.6) \rangle & \langle h_1, (0.4, 0.85) \rangle \\ \langle h_1, (0.8, 0.1) \rangle & \langle h_{-1}, (0.2, 0.9) \rangle & \langle h_{-3}, (0.4, 0.4) \rangle & \langle h_{-3}, (0.8, 0.2) \rangle \\ \langle h_{-2}, (0.9, 0.2) \rangle & \langle h_1, (0.6, 0.7) \rangle & \langle h_{-3}, (0.8, 0.1) \rangle & \langle h_{-2}, (0.3, 0.8) \rangle \end{bmatrix} \end{matrix} \quad (41)$$

$$A^5 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} \langle h_2, (0.8, 0.3) \rangle & \langle h_2, (0.8, 0.2) \rangle & \langle h_3, (0.9, 0.2) \rangle & \langle h_{-1}, (0.8, 0.2) \rangle \\ \langle h_1, (0.6, 0.5) \rangle & \langle h_{-1}, (0.3, 0.1) \rangle & \langle h_{-3}, (0.8, 0.3) \rangle & \langle h_{-3}, (0.6, 0.6) \rangle \\ \langle h_2, (0.4, 0.6) \rangle & \langle h_{-3}, (0.8, 0.3) \rangle & \langle h_3, (0.9, 0.1) \rangle & \langle h_1, (0.7, 0.4) \rangle \\ \langle h_{-2}, (0.7, 0.5) \rangle & \langle h_2, (0.7, 0.25) \rangle & \langle h_{-1}, (0.85, 0.25) \rangle & \langle h_3, (0.6, 0.5) \rangle \\ \langle h_{-2}, (0.8, 0.3) \rangle & \langle h_3, (0.6, 0.3) \rangle & \langle h_{-2}, (0.92, 0.1) \rangle & \langle h_{-3}, (0.5, 0.2) \rangle \end{bmatrix} \end{matrix} \quad (42)$$

Step 2: By Eq. (13), the weight of experts are $\omega_1 = 0.2240$, $\omega_2 = 0.1763$, $\omega_3 = 0.1971$, $\omega_4 = 0.2034$, and $\omega_5 = 0.1992$.

Step 3: According to Eq. (14), we get the experts EVBE opinons, which are shown in Eqs. (43) – (47).

$$L^1 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} AIS^1 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ IS^1 \end{matrix} & \begin{bmatrix} 0.3800 & 0.7200 & 0.6603 & 0.2417 \\ 0.6314 & 0.5000 & 0.5533 & 0.3063 \\ 0.5900 & 0.6097 & 0.6603 & 0.5885 \\ 0.3800 & 0.2225 & 0.6517 & 0.7804 \\ 0.9580 & 0.6292 & 0.3525 & 0.9489 \\ 0.7550 & 0.7200 & 0.3117 & 0.2417 \\ 0.9580 & 0.2225 & 0.3117 & 0.9489 \end{bmatrix} \end{matrix} \quad (43)$$

$$L^2 = \begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ AIS^2 & \begin{bmatrix} 0.1320 & 0.6450 & 0.7804 & 0.1675 \\ 0.1320 & 0.5000 & 0.7804 & 0.4250 \\ 0.2883 & 0.4617 & 0.2915 & 0.3876 \\ 0.6667 & 0.6450 & 0.7714 & 0.5781 \\ 0.6067 & 0.3767 & 0.3576 & 0.9425 \\ 0.2813 & 0.2621 & 0.0440 & 0.1675 \end{bmatrix} \\ x_1 & \\ x_2 & \\ x_3 & \\ x_4 & \\ x_5 & \\ IS^2 & \begin{bmatrix} 0.6667 & 0.2621 & 0.0440 & 0.9425 \end{bmatrix} \end{matrix} \quad (44)$$

$$L^3 = \begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ AIS^3 & \begin{bmatrix} 0.1000 & 0.7891 & 0.6292 & 0.2700 \\ 0.4350 & 0.7891 & 0.5000 & 0.5344 \\ 0.4296 & 0.4257 & 0.6292 & 0.6361 \\ 0.9280 & 0.2333 & 0.4075 & 0.2700 \\ 0.5000 & 0.6125 & 0.6207 & 0.3640 \\ 0.1000 & 0.2417 & 0.5704 & 0.7950 \end{bmatrix} \\ x_1 & \\ x_2 & \\ x_3 & \\ x_4 & \\ x_5 & \\ IS^3 & \begin{bmatrix} 0.9280 & 0.2333 & 0.4075 & 0.7950 \end{bmatrix} \end{matrix} \quad (45)$$

$$L^4 = \begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ AIS^4 & \begin{bmatrix} 0.2050 & 0.7775 & 0.6700 & 0.1000 \\ 0.9000 & 0.3517 & 0.5383 & 0.7993 \\ 0.7067 & 0.7775 & 0.2933 & 0.6031 \\ 0.6700 & 0.7317 & 0.6700 & 0.5365 \\ 0.6358 & 0.4808 & 0.2500 & 0.1000 \\ 0.2050 & 0.5725 & 0.0925 & 0.4250 \end{bmatrix} \\ x_1 & \\ x_2 & \\ x_3 & \\ x_4 & \\ x_5 & \\ IS^4 & \begin{bmatrix} 0.9000 & 0.3517 & 0.0925 & 0.7993 \end{bmatrix} \end{matrix} \quad (46)$$

$$L^5 = \begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ AIS^5 & \begin{bmatrix} 0.2417 & 0.8175 & 0.9500 & 0.1975 \\ 0.7583 & 0.7667 & 0.9425 & 0.3667 \\ 0.5925 & 0.4100 & 0.1125 & 0.2500 \\ 0.6333 & 0.1125 & 0.9500 & 0.6108 \\ 0.2933 & 0.7379 & 0.3617 & 0.7775 \\ 0.2417 & 0.8175 & 0.1939 & 0.1975 \end{bmatrix} \\ x_1 & \\ x_2 & \\ x_3 & \\ x_4 & \\ x_5 & \\ IS^5 & \begin{bmatrix} 0.7583 & 0.1125 & 0.1125 & 0.7775 \end{bmatrix} \end{matrix} \quad (47)$$

Step 4: By Eq. (16), the the normalized EVBE experts opinions are shown in Eqs. (48–52).

$$\mathcal{N}^1 = \begin{array}{c} \begin{array}{ccccc} & c_1 & c_2 & c_3 & c_4 \\ x_1 & 0.6590 & 0.4450 & 0.5633 & 0.3227 \\ x_2 & 0.6158 & 0.3649 & 0.4720 & 0.6202 \\ x_3 & 0.3967 & 1 & 0.4782 & 0.8224 \\ x_4 & 1 & 0.3536 & 0.8842 & 1 \\ x_5 & 0.7881 & 0.3090 & 1 & 0.2547 \end{array} \end{array} \quad (48)$$

$$\mathcal{N}^2 = \begin{array}{c} \begin{array}{ccccc} & c_1 & c_2 & c_3 & c_4 \\ x_1 & 0.1980 & 0.5242 & 0.0564 & 0.4509 \\ x_2 & 0.4325 & 0.5677 & 0.1510 & 0.4113 \\ x_3 & 1 & 0.4063 & 0.0571 & 0.6134 \\ x_4 & 0.9100 & 0.6958 & 0.1231 & 1 \\ x_5 & 0.4219 & 1 & 1 & 0.1777 \end{array} \end{array} \quad (49)$$

$$\mathcal{N}^3 = \begin{array}{c} \begin{array}{ccccc} & c_1 & c_2 & c_3 & c_4 \\ x_1 & 0.4688 & 0.2957 & 0.8150 & 0.6722 \\ x_2 & 0.4629 & 0.5481 & 0.6477 & 0.8001 \\ x_3 & 1 & 1 & 1 & 0.3396 \\ x_4 & 0.5388 & 0.3810 & 0.6565 & 0.4578 \\ x_5 & 0.1078 & 0.9655 & 0.7144 & 1 \end{array} \end{array} \quad (50)$$

$$\mathcal{N}^4 = \begin{array}{c} \begin{array}{ccccc} & c_1 & c_2 & c_3 & c_4 \\ x_1 & 1 & 1 & 0.1718 & 1 \\ x_2 & 0.7852 & 0.4523 & 0.3153 & 0.7546 \\ x_3 & 0.7444 & 0.4806 & 0.1381 & 0.6712 \\ x_4 & 0.7065 & 0.7314 & 0.3700 & 0.1251 \\ x_5 & 0.2278 & 0.6143 & 1 & 0.5317 \end{array} \end{array} \quad (51)$$

$$\mathcal{N}^5 = \begin{array}{c} \begin{array}{ccccc} & c_1 & c_2 & c_3 & c_4 \\ x_1 & 1 & 0.1467 & 0.1194 & 0.4716 \\ x_2 & 0.7813 & 0.2744 & 1 & 0.3215 \\ x_3 & 0.8352 & 1 & 0.1184 & 0.7856 \\ x_4 & 0.3868 & 0.1525 & 0.3111 & 1 \\ x_5 & 0.3187 & 0.1376 & 0.5801 & 0.2540 \end{array} \end{array} \quad (52)$$

Step 5: By Eq. (18), the weight of attributes for all experts are:

$$w_1^1 = 0.3124, w_2^1 = 0.0978, w_3^1 = 0.3050, w_4^1 = 0.2848;$$

$$w_1^2 = 0.2609, w_2^2 = 0.1813, w_3^2 = 0.4055, w_4^2 = 0.1523;$$

$$w_1^3 = 0.1684, w_2^3 = 0.2688, w_3^3 = 0.3347, w_4^3 = 0.2281;$$

$$w_1^4 = 0.3387, w_2^4 = 0.2031, w_3^4 = 0.1655, w_4^4 = 0.2927;$$

$$w_1^5 = 0.3099, w_2^5 = 0.2397, w_3^5 = 0.2459, w_4^5 = 0.2045.$$

Step 6: According to Eq. (19), we get $P(e_1) = 0.2240$, $P(e_2) = 0.1763$, $P(e_3) = 0.1971$, $P(e_4) = 0.2034$, and $P(e_5) = 0.1992$.

(1) According to Eq. (20), the weight-normalized EVBE, experts opinions are shown in Eqs. (53) – (57).

$$B^1 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} b_{AIS}^1 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ b_{IS}^1 \end{matrix} & \begin{bmatrix} 0.1239 & 0.0978 & 0.3050 & 0.0725 \\ 0.2059 & 0.0435 & 0.1718 & 0.0919 \\ 0.1924 & 0.0357 & 0.1439 & 0.1767 \\ 0.1239 & 0.0978 & 0.1458 & 0.2342 \\ 0.3124 & 0.0346 & 0.2696 & 0.2848 \\ 0.2462 & 0.0302 & 0.3050 & 0.0725 \\ 0.3124 & 0.0302 & 0.1439 & 0.2848 \end{bmatrix} \end{matrix} \quad (53)$$

$$B^2 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} b_{AIS}^2 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ b_{IS}^2 \end{matrix} & \begin{bmatrix} 0.0517 & 0.1813 & 0.4055 & 0.0271 \\ 0.0517 & 0.0951 & 0.0229 & 0.0687 \\ 0.1128 & 0.1029 & 0.0612 & 0.0626 \\ 0.2609 & 0.0737 & 0.0231 & 0.0934 \\ 0.2374 & 0.1262 & 0.0499 & 0.1523 \\ 0.1101 & 0.1813 & 0.4055 & 0.0271 \\ 0.2609 & 0.0737 & 0.0229 & 0.1523 \end{bmatrix} \end{matrix} \quad (54)$$

$$B^3 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} b_{AIS}^3 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ b_{IS}^3 \end{matrix} & \begin{bmatrix} 0.0181 & 0.2688 & 0.3347 & 0.0775 \\ 0.0789 & 0.0795 & 0.2727 & 0.1533 \\ 0.0780 & 0.1473 & 0.2168 & 0.1825 \\ 0.1684 & 0.2688 & 0.3347 & 0.0775 \\ 0.0907 & 0.1024 & 0.2197 & 0.1044 \\ 0.0181 & 0.2596 & 0.2391 & 0.2281 \\ 0.1684 & 0.0795 & 0.2168 & 0.2281 \end{bmatrix} \end{matrix} \quad (55)$$

$$B^4 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} b_{AIS}^4 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ b_{IS}^4 \end{matrix} & \begin{bmatrix} 0.0772 & 0.2031 & 0.1655 & 0.4824 \\ 0.3387 & 0.2031 & 0.0284 & 0.8630 \\ 0.2660 & 0.0919 & 0.0522 & 0.6309 \\ 0.2522 & 0.0976 & 0.0228 & 0.5691 \\ 0.2393 & 0.1486 & 0.0612 & 0.4857 \\ 0.0772 & 0.1248 & 0.1655 & 0.5230 \\ 0.3387 & 0.0919 & 0.0228 & 0.7461 \end{bmatrix} \end{matrix} \quad (56)$$

$$B^5 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} b_{AIS}^5 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ b_{IS}^5 \end{matrix} & \begin{bmatrix} 0.0988 & 0.2397 & 0.2459 & 0.0519 \\ 0.3099 & 0.0352 & 0.0293 & 0.0964 \\ 0.2422 & 0.0658 & 0.2459 & 0.0657 \\ 0.2588 & 0.2397 & 0.0291 & 0.1606 \\ 0.1199 & 0.0365 & 0.0765 & 0.2045 \\ 0.0988 & 0.0330 & 0.1426 & 0.0519 \\ 0.3099 & 0.0330 & 0.0291 & 0.2045 \end{bmatrix} \end{matrix} \quad (57)$$

- (2) By Eq. (21), the utility degree of each alternative of each expert can be obtained and the results are shown in Table 3.
- (3) By Eqs. (22) and (23), the utility function of each alternative of each expert can be obtained and the results are shown in Table 4.
- (4) By Eq. (24) we calculate the conditional probabilities of each alternative for each expert. The details results are shown in Table 5.
- (5) By Eq. (25), the probabilities of alternatives are:

$$P(x_1) = 0.1825, P(x_2) = 0.1914, P(x_3) = 0.2182, P(x_4) = 0.2029, \text{ and } P(x_5) = 0.2050.$$

Step 7: By Eq. (30), the belief distance between two experts about each alternative is calculated, which is presented in Eq. (58).

$$D = \begin{bmatrix} 0.3453 & 0.0014 & 0.3699 & 0.0787 & 0.3694 & 0.6525 & 0.4352 & 0.3990 & 0.1118 & 0.2383 \\ 0.3052 & 0.0069 & 0.2167 & 0.8174 & 0.3289 & 0.5176 & 0.9115 & 0.2588 & 0.8334 & 0.7396 \\ 0.2764 & 0.9442 & 0.1640 & 0.3905 & 0.9826 & 0.1545 & 0.6346 & 0.8841 & 0.7573 & 0.5550 \\ 0.3552 & 0.1419 & 0.1042 & 0.0426 & 0.4260 & 0.3950 & 0.3428 & 0.0698 & 0.1322 & 0.0949 \\ 0.0141 & 0.0806 & 0.5025 & 0.6441 & 0.0119 & 0.4498 & 0.6020 & 0.4091 & 0.5730 & 0.1930 \end{bmatrix} \quad (58)$$

By Eq. (31), the Deng entropy between two experts about each alternative is shown in Eq. (59).

Table 3 The utility degree of each alternative of each expert

e_1	x_1	x_2	x_3	x_4	x_5
$U_i^{1,-}$	0.8563	0.9157	1.0043	1.5044	1.0913
$U_i^{1,+}$	0.6652	0.7113	0.7801	1.1686	0.8477
e_2					
$U_i^{2,-}$	0.3580	0.5103	0.6778	0.8500	1.0877
$U_i^{2,+}$	0.4674	0.6664	0.8850	1.1100	1.4204
e_3					
$U_i^{3,-}$	0.8361	0.8933	1.2149	0.7399	1.0655
$U_i^{3,+}$	0.8438	0.9016	1.2261	0.7467	1.0753
e_4					
$U_i^{4,-}$	1.7889	1.3078	1.1797	1.0069	1.0842
$U_i^{4,+}$	1.1566	0.8455	0.7627	0.6510	0.7010
e_5					
$U_i^{5,-}$	0.7400	0.9737	1.0817	0.6874	0.5128
$U_i^{5,+}$	0.8168	1.0747	1.1940	0.7587	0.5661

$$\vartheta = \begin{bmatrix} 1.8788 & 0.0191 & 1.9759 & 0.5959 & 1.9741 & 2.9513 & 2.2228 & 2.0877 & 0.7902 & 1.4243 \\ 1.7151 & 0.0762 & 1.3246 & 3.4313 & 1.8126 & 2.5140 & 3.6829 & 1.5159 & 3.4751 & 3.2113 \\ 1.5924 & 3.7670 & 1.0684 & 2.0554 & 3.8637 & 1.0201 & 2.8955 & 3.6113 & 3.2625 & 2.6399 \\ 1.9180 & 0.9543 & 0.7471 & 0.3603 & 2.1888 & 2.0724 & 1.8689 & 0.5406 & 0.9027 & 0.6931 \\ 0.1422 & 0.6075 & 2.4622 & 2.9253 & 0.1223 & 2.2759 & 2.7929 & 2.1257 & 2.6991 & 1.2122 \end{bmatrix} \quad (59)$$

By Eq. (32), the standardized Deng entropy between two experts about each alternative is shown in Eq. (60).

$$\beta = \begin{bmatrix} 0.2684 & -1 & 0.3347 & -0.6066 & 0.3335 & 1 & 0.5031 & 0.4110 & -0.4740 & -0.0415 \\ -0.0912 & -1 & -0.3077 & 0.8605 & -0.0371 & 0.3218 & 1 & -0.2016 & 0.8848 & 0.7385 \\ -0.5975 & 0.9320 & -0.9660 & -0.2718 & 1 & -1 & 0.3191 & 0.8225 & 0.5772 & 0.1393 \\ 0.7038 & -0.3502 & -0.5769 & -1 & 1 & 0.8727 & 0.6501 & -0.8028 & -0.4068 & -0.6360 \\ -0.9858 & -0.6538 & 0.6695 & 1 & -1 & 0.5366 & 0.9053 & 0.4295 & 0.8386 & -0.2223 \end{bmatrix} \quad (60)$$

By Eq. (34), the interface effect among experts for each alternative is:

$$INT_1 = 0.0354, INT_2 = 0.1858, INT_3 = 0.1209, INT_4 = -0.0337, \text{ and } INT_5 = 0.0569.$$

Finally, by Eq. (36), the probability of each alternative is:

$$\begin{aligned} Prob(x_1) &= 0.1596, Prob(x_2) = 0.2763, Prob(x_3) = 0.2484, Prob(x_4) \\ &= 0.1239, \text{ and } Prob(x_5) = 0.1918. \end{aligned}$$

Table 4 The utility function of each alternative of each expert

e_1	x_1	x_2	x_3	x_4	x_5
$\mathcal{T}(U_i^{1,-})$	0.4372	0.4372	0.4372	0.4372	0.4372
$\mathcal{T}(U_i^{1,+})$	0.5628	0.5628	0.5628	0.5628	0.5628
$\mathcal{T}(U_i^1)$	0.4965	0.5310	0.5824	0.8723	0.6328
e_2					
$\mathcal{T}(U_i^{2,-})$	0.5663	0.5663	0.5663	0.5663	0.5663
$\mathcal{T}(U_i^{2,+})$	0.2687	0.2687	0.2687	0.2687	0.2687
$\mathcal{T}(U_i^2)$	0.2687	0.3831	0.5088	0.6381	0.8165
e_3					
$\mathcal{T}(U_i^{3,-})$	0.5023	0.5023	0.5023	0.5023	0.5023
$\mathcal{T}(U_i^{3,+})$	0.4977	0.4977	0.4977	0.4977	0.4977
$\mathcal{T}(U_i^3)$	0.5599	0.5983	0.8136	0.4955	0.7136
e_4					
$\mathcal{T}(U_i^{4,-})$	0.3927	0.3927	0.3927	0.3927	0.3927
$\mathcal{T}(U_i^{4,+})$	0.6073	0.6073	0.6073	0.6073	0.6073
$\mathcal{T}(U_i^4)$	0.9224	0.6743	0.6083	0.5192	0.5591
e_5					
$\mathcal{T}(U_i^{5,-})$	0.5247	0.5247	0.5247	0.5247	0.5247
$\mathcal{T}(U_i^{5,+})$	0.4753	0.4753	0.4753	0.4753	0.4753
$\mathcal{T}(U_i^5)$	0.5173	0.6806	0.7561	0.4804	0.3585

Table 5 The conditional probabilities of each alternative

$P(x_i e_k)$	e_1	e_2	e_3	e_4	e_5
x_1	0.1594	0.1028	0.1760	0.2809	0.1852
x_2	0.1705	0.1465	0.1881	0.2054	0.2437
x_3	0.1870	0.1945	0.2558	0.1853	0.2707
x_4	0.2800	0.2440	0.1558	0.1581	0.1720
x_5	0.2031	0.3122	0.2243	0.1703	0.1284

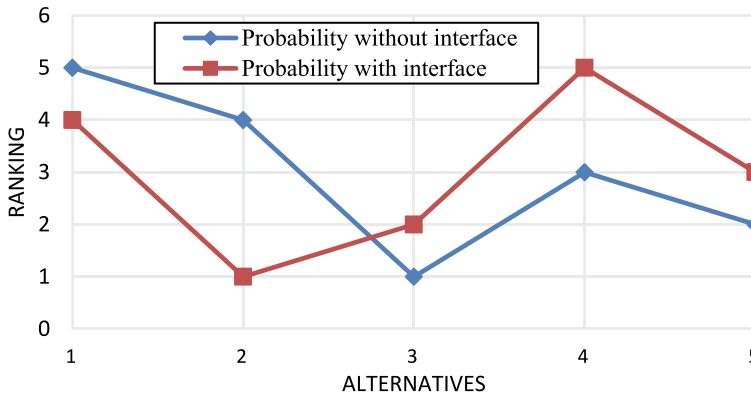


Fig. 5 The interface effect on ranking values

Step 8: According to the value of the QP of each alternative, the final ranking is.

$$x_2 \succ x_3 \succ x_5 \succ x_1 \succ x_4.$$

4.2 Effect of the Quantum Interface on Decision Result

If we can decide based on the $P(x_i) (i = 1, 2, 3, 4, 5)$ values according to Eq. (25), as higher probabilities indicate a superior ranking. The classical approach does not account for the interference effect, which can influence the final rankings, as shown in Fig. 5. The ranking result of alternatives according to Eq. (25) is $x_3 \succ x_5 \succ x_4 \succ x_2 \succ x_1$. If we count the interface effect between two experts about an alternative the ranking result is $x_2 \succ x_3 \succ x_5 \succ x_1 \succ x_4$.

4.3 Effect of the Expert Confidence and Doubting Levels on the Decision Result

If we can consider the expert to have full confidence in his/her opinion, then in Eq. (12), $a_{ij}^k = \left\langle h_{\alpha_{ij}}^k, \left(\mu_{ij}^k, v_{ij}^k \right) \right\rangle$ can be written as $a_{ij}^k = \left\langle h_{\alpha_{ij}}^k, (1, 0) \right\rangle$. In this case $Prob(x_1) = 0.1322$, $Prob(x_2) = 0.2551$, $Prob(x_3) = 0.1475$, $Prob(x_4) = 0.2620$, and $Prob(x_5) = 0.2031$. Therefore, the ranking of the alternative is $x_4 \succ x_2 \succ x_5 \succ x_3 \succ x_1$, which indicates that the expert's confidence levels are a great impact on the decision result. Therefore, in the MCGDM problem, the expert's confidence levels must be considered.

4.4 Comparative Analysis

To test the feasibility and validity of the suggested method, we compare it with some existing methods for the example given above.

In the traditional decision-making model (Mandal & Ranadive, 2019) the ranking results are derived from the aggregate decision matrix of experts' opinions. Therefore, to compare our model and the traditional model, we first aggregate the experts' opinions. In this connection, by Ref. (Mandal & Ranadive, 2019), using the Pythagorean linguistic weighted averaging aggregation operator, the aggregation of five experts opinions is

$$\text{Agg}(A^1, A^2, A^3, A^4, A^5) = \begin{matrix} & & c^1 & c^2 & c^3 & c^4 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \left[\begin{array}{cccc} \langle h_{0.51}, (0.77, 0.34) \rangle & \langle h_{0.39}, (0.76, 0.29) \rangle & \langle h_{1.58}, (0.67, 0.35) \rangle & \langle h_{-0.4}, (0.68, 0.3) \rangle \\ \langle h_{0.31}, (0.54, 0.55) \rangle & \langle h_{0.31}, (0.47, 0.46) \rangle & \langle h_{-0.71}, (0.7, 0.36) \rangle & \langle h_{0.05}, (0.64, 0.53) \rangle \\ \langle h_{1.28}, (0.60, 0.53) \rangle & \langle h_{-0.9}, (0.74, 0.4) \rangle & \langle h_{1.81}, (0.71, 0.34) \rangle & \langle h_{0.44}, (0.66, 0.45) \rangle \\ \langle h_{0.65}, (0.81, 0.28) \rangle & \langle h_{0.44}, (0.69, 0.31) \rangle & \langle h_{-1.01}, (0.82, 0.28) \rangle & \langle h_{0.99}, (0.84, 0.23) \rangle \\ \langle h_{-1.48}, (0.8, 0.29) \rangle & \langle h_{0.5}, (0.67, 0.32) \rangle & \langle h_{-1.59}, (0.81, 0.24) \rangle & \langle h_{-1.59}, (0.73, 0.33) \rangle \end{array} \right] \end{matrix}$$

Therefore, the ranking of alternatives according to the methods MARCOS method is $x_3 > x_4 > x_2 > x_1 > x_5$, i.e., the optimal option is x_3 . When the proposed model is applied without considering the interface effect, the optimal option is also x_3 . This suggests that the optimal option of the proposed approach aligns with those of traditional methods when the impact of decision-makers is disregarded, thereby validating the robustness of the suggested model. However, when evaluating the quantum interference among decision-makers, the outcomes of this model suggest that x_2 is the optimal strategy. This inconsistency arises from the conventional framework's exclusion of the interference effect among decision-makers, which leads to subjective biases during the evaluation of alternatives. As a result, the outcomes of this model better align with real-world complexities.

5 Conclusion

The purpose of this study is to propose a quantum group decision model to mitigate the effects of information uncertainty and opinion interference during the meteorological selection of environmentally friendly construction materials made from recycled plastic. The model employs PLIs to represent ambiguous information in complex decision-making scenarios. The Deng entropy is used to quantify opinion interference, simulating the reciprocal influence among decision-makers in the MAGDM process.

Unlike traditional approaches, which treat decision-makers as isolated individuals, this model addresses their limitations by managing interference impacts, assigning appropriate weights to decision-makers, and controlling opinion interference through quantum decision-making techniques. The proposed framework enhances existing

methods by providing more effective solutions to challenges related to opinion interference and decision weight allocation.

Our findings demonstrate that the model significantly improves decision-making in the selection of eco-friendly construction materials derived from recycled plastic, particularly by offering a comprehensive framework for managing information ambiguity and conflicting opinions. However, the computational complexity of the model grows substantially with the increase in decision-makers and attributes. This highlights the need for advanced computational methods to maintain efficiency.

Future research will focus on integrating machine learning techniques to improve scalability and practicality, ensuring the model remains effective for large-scale decision-making tasks. This integration aims to address rising complexity, facilitating consensus within quantum group decision-making processes and enhancing the model's long-term usability.

Conflict of Interest: The authors declare that they have no conflict of interest.

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The Selection of Wearable Health Technology Devices is Based on the Pythagorean Linguistic Multi-attribute Group Quantum Decision Theory Technique Integrating the TODIM-VIKOR Methodology



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Abstract Determining the most effective wearable health technology devices for hospitals is crucial for emergency decision-making. The multi-attribute group decision-making (MAGDM) method is a widely used and effective technique for selecting wearable health devices. However, due to the numerous factors that must be considered when choosing equipment in emergencies, decision-makers often find it difficult to develop a comprehensive assessment methodology. This study presents a TODIM-VIKOR methodology that incorporates quantum decision theory within the framework of Pythagorean linguistic information, taking into account various decision-maker perspectives and demonstrating applicability across different research domains. We begin by explaining the fundamental concepts of Pythagorean linguistic numbers (PLNs) and quantum decision theory, followed by an introduction

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to the TODIM and VIKOR methodologies. Next, we propose an enhanced TODIM-VIKOR model for addressing MAGDM issues involving PLNs. Finally, we apply the suggested technique to a numerical case involving the site selection of a medical logistics facility to validate its effectiveness.

Keywords Quantum decision theory · Group decision-making · Pythagorean linguistic set · TODIM-VIKOR

1 Introduction

The methodologies for addressing multi-attribute group decision-making (MAGDM) challenges have evolved significantly, driven by the growing diversification of their application contexts in recent decades. Today, these techniques are essential for managing decision-making processes that require the evaluation of multiple, distinct criteria in real-world scenarios. Numerous MCGDM strategies have been introduced (Lei et al., 2023). Among them, TODIM and VIKOR are two prominent approaches that have attracted considerable attention. TODIM, an acronym in Portuguese for 'Interactive and Multicriteria Decision Making,' is a methodology rooted in prospect theory. It aims to capture the psychological behavior of experts when faced with uncertainty. The TODIM technique evaluates alternatives by measuring their dominance relative to others, considering both the aggregate value and the cognitive behavior of experts. It is particularly suited to situations where experts display irrational tendencies and must navigate ambiguous decision-making challenges. VIKOR, on the other hand, is a multi-criteria decision-making method designed to address problems with conflicting and non-commensurable criteria. It assumes that compromise is acceptable for resolving conflicts and aims to identify a solution that approximates the ideal by evaluating options and determining the best compromise. Originally developed by Serafim Opricovic, VIKOR is especially useful when decision-makers need to select the optimal alternative from a set of viable options evaluated across multiple attributes. Thus, each methodology offers distinct advantages.

The research in fuzzy set theory and its applications plays a significant role in addressing uncertainty and complexity in social networks and group decision-making processes. The methods developed in this area offer valuable insights into how fuzzy logic can be used to model and analyze systems where imprecision and ambiguity are prevalent. Fuzzy set theory is particularly well-suited to modeling the uncertainty inherent in social networks, where relationships and interactions often involve incomplete or vague information. Traditional mathematical models struggle to capture this uncertainty, but fuzzy approaches provide a more nuanced framework for analyzing such dynamics. For instance, the development of fuzzy fractional differential equations using fuzzy Laplace transforms is essential for modeling systems where the influence of individuals or groups evolves fractionally over time under uncertainty (Salahshour et al., 2012; Allahviranloo et al., 2015; Chehlabi et al., 2016; Armand et al., 2018).

A new centrality measure is introduced for neutrosophic and linguistic graphs (Mahapatra et al., 2019, 2020), including self-weight or individual merit within the network. Mahapatra et al. (Mahapatra et al., 2024a) present an innovative method for centrality measurement in fuzzy social networks where relationships between nodes are not binary but have different degrees of intensity.

According to the above discussion, fuzzy sets (FS) have become a popular approach for incorporating fuzzy information as part of an expert's evaluative measures. However, as noted by Xu (2005) and Mandal et al. (2022a), experts tend to express their opinions more reliably using linguistic information, known as linguistic sets (LS), rather than fuzzy information. In (Mandal et al. 2019), the LS concept was extended to the Pythagorean linguistic set (PLS), allowing experts to express both their confidence and doubt levels when providing judgments. In real-life scenarios, situations, experts often have limited knowledge and personal biases, so using language-based information with expert confidence and doubt levels (Mandal et al., 2021; 2022b; 2023a; 2023b; 2024a) is more flexible for expressing their ideas. In this connection, Mandal et al. briefly studied MCGDM under LZN information with the MARCOS approach and its application in the selection of logistic cold chain centers (Mandal et al., 2024b); TODIM-VIKRO approach (Mandal et al., 2024c) and its application in site selection of medical logistic center; ORESTE approach (Mandal et al., 2024d) and its application in assessment of regional energy; and MULTI-MOORA approach (Mandal et al., 2024e) and its application in selection of software. To address the challenge of aggregating individual assessments, specific methods are needed. Although many aggregation strategies have been studied, most assume that decision-makers are independent and uninfluenced by one another. However, in practice, opinions are often influenced by surrounding perspectives. In decision-making, the views of multiple experts can interact, ultimately affecting the final decision, and introducing the concept of 'interference.

Quantum Probability Theory (QPT) (Busemeyer & Bruza, 2012; Moreira & Wichert, 2014; Yearsley & Busemeyer, 2016) offers a framework for capturing these influences. QPT, a recent advancement over classical probability theory (CPT), has been extensively studied in fields such as psychology (Busemeyer & Bruza, 2012), cognitive science (Yearsley & Busemeyer, 2016), and decision analysis (He et al., 2018a; 2018b; Wu et al., 2020; Xiao et al., 2022; Han & Liu, 2023; Mandal et al., 2024f; 2025) due to its robust descriptive power. It effectively explains paradoxes, such as the disjunction fallacy (He et al., 2018a; 2018b), the Ellsberg paradox (Eichberger & Pirner, 2018), the order effect (Wang et al., 2014), and graph theory (Samanta & Allahviranloo, 2025), which are challenging for CPT to address. QPT's ability to represent interference terms makes it particularly valuable for understanding how expert opinions influence each other in the MCGDM process.

Building on the above discussion, this study proposes a TODIM-VIKOR-based MAGDM using the quantum decision theory (QDT) technique within the PLI context. This approach integrates the psychological aspects of experts' preferences with the optimization of group utility and the minimization of individual regret. The main contributions of this paper are:

1. An improved generalized PLI-based TODIM-VIKOR approach accounts for experts' limited rationality, simplifies the calculation process, and resolves some paradoxes of the traditional TODIM-VIKOR method.
2. A quantum-based method for combining individual evaluation results, which helps us better understand human psychology through the effects of interference.
3. A case study demonstrating the effectiveness of our approach in the selection of wearable health technology devices. Comparative and sensitivity analyses show the flexibility and robustness of our method.

The structure of the paper is as follows: In Sect. 2, recall some basic definitions and the background of linguistic term sets (LTSs), the Pythagorean linguistic set (PLS), and quantum Decision Theory (QDT). The proposed MCGDM model is discussed in Sect. 3. A case study for the selection of wearable health technology devices with comparative analysis is provided in Sect. 4. Section 5 concludes our study.

2 Preliminaries

Here we recall some related definitions and results that are used in this paper.

2.1 LTS

Here we assume an LTS is: $H = \{h_\lambda | \lambda = -\zeta, \dots, -1, 0, 1, \dots, \zeta\}$ with cardinality $2\zeta + 1$ and holds the followings (Mandal et al., 2024a):

- (1) $h_{\lambda_1} \geq h_{\lambda_2}$ if and only if $\lambda_1 \geq \lambda_2$;
- (2) $\text{neg}(h_{\lambda_1}) = h_{-\lambda_1}$;
- (3) $\max(h_{\lambda_1}, h_{\lambda_2}) = h_{\lambda_1}$ if $h_{\lambda_1} \geq h_{\lambda_2}$;
- (4) $\min(h_{\lambda_1}, h_{\lambda_2}) = h_{\lambda_1}$ if $h_{\lambda_1} \leq h_{\lambda_2}$.

To preserve expert information, the continuous LTSs (Xu, 2005) denotes $\bar{H} = \{h_\lambda | \lambda \in [-t, t]\}$, $t > \zeta$.

The subscript index of any $h_\lambda \in \bar{H}$, represented by the following functions:

$$I: \bar{H} \rightarrow [-t, t], \text{ and } I(h_\lambda) = \lambda.$$

$$I^{-1}: [-t, t] \rightarrow \bar{H}, \text{ and } I^{-1}(\lambda) = h_\lambda.$$

Let $\mathcal{F}\ominus$ be the linguistic conversation function for any LT $G \in \bar{H}$, which is defined in the following way:

$$\mathcal{F}: \bar{H} \rightarrow [0, 1], \mathcal{F}(G) = \{\mathcal{F}(h_\lambda) = \frac{\lambda}{2t} + \frac{1}{2} | \lambda \in [-t, t]\} = \kappa.$$

2.2 PTS

By Ref. (Mandal & Ranadive 2019), a Pythagorean linguistic set (PLS) denoted by $\mathcal{P}$ in an alternative set X for continuous LTS $\overline{H} = \{h_{-t} \leq h_\lambda \leq h_t | \lambda \in [-t, t]\}$ is represented in the following way:

$$\mathcal{R} = \{ h_{\theta(x)}, (\mu_{\mathcal{R}}(x), \nu_{\mathcal{R}}(x)) | x \in X \},$$

where, $h_{\theta(x)} \in \overline{H}$, $\mu_{\mathcal{R}} : X \rightarrow [0, 1]$ and $\nu_{\mathcal{R}} : X \rightarrow [0, 1]$ are the membership and non-membership degrees of x to $h_{\alpha(x)}$ such that $\mu_{\mathcal{R}}^2(x) + \nu_{\mathcal{R}}^2(x) \leq 1$. Here, the Pythagorean linguistic number (PLN) in $\mathcal{R}$ is denoted by $\not\approx$ and represented by $\not\approx = \langle h_{\theta(\not\approx)}, (\mu(\not\approx), \nu(\not\approx)) \rangle$ with $h_{\theta(\not\approx)} \in \overline{H}$ and $\mu(\not\approx), \nu(\not\approx) \in [0, 1]$, $\mu^2(\not\approx) + \nu^2(\not\approx) \leq 1$.

By Ref. (Mandal & Ranadive 2019), the expected value of $\not\approx = h_{\theta(\not\approx)}, (\mu(\not\approx), \nu(\not\approx))$ is denoted by κ and defined in the following way:

$$\kappa(\not\approx) = \frac{h_{\theta(\not\approx)} \times (\mu^2(\not\approx) + 1 - \nu^2(\not\approx))}{2}.$$

By Ref. (Wang et al. 2019), the distance between two PLNs $\not\approx_1 = \langle h_{\theta(\not\approx_1)}, (\mu(\not\approx_1), \nu(\not\approx_1)) \rangle$ and $\not\approx_2 = \langle h_{\theta(\not\approx_2)}, (\mu(\not\approx_2), \nu(\not\approx_2)) \rangle$ are defined in the following way:

$$d(\not\approx_1, \not\approx_2) = \frac{1}{2} \left| \left((1 + \mu^2(\not\approx_1) - \nu^2(\not\approx_1)) * \mathcal{F}(h_{\theta(\not\approx_1)}) \right) - \left((1 + \mu^2(\not\approx_2) - \nu^2(\not\approx_2)) * \mathcal{F}(h_{\theta(\not\approx_2)}) \right) \right|.$$

2.3 Quantum Probability Theory (QPT)

A quantum decision (QD) model is based on the principles of quantum probability (QP) theory (QPT). To provide context, we will review some key concepts and notations from QPT. In quantum theory (QT), a state is represented as a unit-length vector within an N-dimensional vector space.

Definition 1. (Yearsley & Busemeyer 2016) A finite-dimensional Hilbert space (HS), represented as $\mathfrak{H}$, is an abstract vector space over a complex field, spanned by a set of orthonormal basis vectors with complex components. In classical probability theory (CPT), events are fundamental (He et al. 2018), but in quantum probability theory (QPT), events are instead represented by subspaces of the Hilbert space $\mathfrak{H}$, rather than traditional sets.

Definition 2. (Busemeyer & Bruza, 2012) In QT, the superposition state $|\mathcal{S}\rangle$, a unit-length vector in the N-dimensional Hilbert space $\mathfrak{H}$, defines all possible events

within that space. The nonlinear combination of a set of orthogonal basis vectors $\{|\mathcal{V}\rangle_k, k = 1, 2, \dots, n\}$ are used to represent the superposition state $|\mathcal{S}\rangle$, i.e.,

$$\begin{aligned} |\mathcal{S}\rangle &= \langle \mathcal{V}_1 | \mathcal{S} \mathcal{V}_1 \rangle + \langle \mathcal{V}_2 | \mathcal{S} \mathcal{V}_2 \rangle + \dots + \langle \mathcal{V}_n | \mathcal{S} \mathcal{V}_n \rangle \\ &= \xi_1 |\mathcal{V}_1\rangle + \xi_2 |\mathcal{V}_2\rangle + \dots + \xi_n |\mathcal{V}_n\rangle \\ &= \zeta_1 e^{i\alpha_1} |\mathcal{V}_1\rangle + \zeta_2 e^{i\alpha_2} |\mathcal{V}_2\rangle + \dots + \zeta_n e^{i\alpha_n} |\mathcal{V}_n\rangle \end{aligned} \quad (1)$$

where $\xi_k (= \zeta_k e^{i\alpha_k})$ is the amplitude of the probability, expressed as a complex number, characterizing the amplitude of the wave in the state $|\mathcal{V}_k\rangle$, and the amplitude of the phase angle represented by $e^{i\alpha_k}$.

Definition 3. (Busemeyer & Bruza 2012) The CPT framework represents every event using a single set of components. In QPT, probabilities are assigned based on the overlap between cognitive states (shown as state vectors) and different probability measures (shown as subspaces). QPT extends CPT into the complex domain by including incompatible qualities and viewing events as vectors. Incompatible attributes are two observable features of an item that cannot be measured accurately at the same time. Both the incompatibility condition and the uncertainty principle can be described by the probability amplitude, also known as complex probability.

According to Born's rule, an event's probability amplitude and its classical probability, x_i are related to each other:

$$P(\mathcal{V}_k) = |\zeta_k e^{i\alpha_k}|^2 = (\zeta_k e^{i\alpha_k}) \times (\zeta_k e^{i\alpha_k})^* = \zeta_k e^{i\alpha_k} \times \zeta_k e^{-i\alpha_k} = (\zeta_k)^2 \quad (2)$$

For the total event probability to be 1, the sum of the squares of the probability amplitudes must also be 1.

$$\sum_{k=1}^n P(\mathcal{V}_k) = \sum_{k=1}^n (\zeta_k)^2 = 1. \quad (3)$$

An $N - 1$ mapping transforms complex probabilities into classical probabilities. For each given probability value $P(\mathcal{V}_k)$, countless complex probabilities correspond to it. Referencing Eq. (2) and Euler's formula,

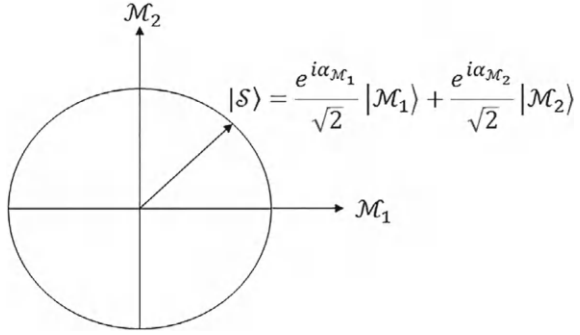
$$e^{i\alpha} = \cos\alpha + i \times \sin\alpha \quad (4)$$

the complex probability satisfies:

$$\xi = \sqrt{P(\mathcal{V})}(\cos\alpha + i \times \sin\alpha) = \sqrt{P(\mathcal{V})}e^{i\alpha} \quad (5)$$

For example, in the following, we discussed the QPT in two-dimensional Hilbert space. Let $\{\mathcal{M}_1, \mathcal{M}_2\}$ be an event containing two basic states, where $|\mathcal{M}_1\rangle = (1, 0)$, and $|\mathcal{M}_2\rangle = (0, 1)$ as two basic orthogonal vectors in $\mathbf{H}$. The state $|\mathcal{S}\rangle$ is

Fig. 1 Event in Hilbert space with two dimensions



defined as a superposition and is graphically illustrated in Fig. 1. According to Eqs. (1) and (2), it can be expressed as.

$$|S\rangle = \zeta_1 e^{i\alpha_1} |\mathcal{M}_1\rangle + \zeta_2 e^{i\alpha_2} |\mathcal{M}_2\rangle = \frac{e^{i\alpha_{\mathcal{M}_1}}}{\sqrt{2}} |\mathcal{M}_1\rangle + \frac{e^{i\alpha_{\mathcal{M}_2}}}{\sqrt{2}} |\mathcal{M}_2\rangle$$

and the probabilities of $|\mathcal{M}_1\rangle$ and $|\mathcal{M}_2\rangle$ are calculated as follows:

$$P(\mathcal{M}_1) = P(\mathcal{M}_2) = (e^{i\alpha})(e^{i\alpha})^* = \frac{1}{2}.$$

According to conventional Bayesian decision theory, decision-making can be mathematically represented as a Markov process (Moreira & Wichert 2014). The state transition probability in a single route of QPT aligns with the CPT according to Feynman's first rule (Wu et al. 2020), which varies over several pathways. Consider the decision-making dilemma highlighted above as an illustration of the process depicted in Fig. 2(a). Experts provide their assessments of $\mathcal{M}_1$ ('bad') or $\mathcal{M}_2$ ('good') as probability quantifications. State transition probabilities are determined by combining the individual probabilities for each route and using the total probability formula in CPT.

$$P(\mathcal{M}_1) = \sum_{k=1}^2 P(e_k) P(\mathcal{M}_1 | e_k), \sum_{k=1}^2 P(e_k) = 1 \quad (6)$$

Since only one route is selected at a time, it is sometimes referred to as the single path principle. The conventional Markovian framework assumes that experts rely on one expert at a time and that both experts are completely autonomous. However, this approach is sometimes unsuitable for ordinary cognitive processes because many avenues are often considered when forming judgments. Such phenomena can result in illogical actions, including disjunction, conjunction errors, and interference effects. Thus, QPT is an excellent complement to CPT in this scenario.

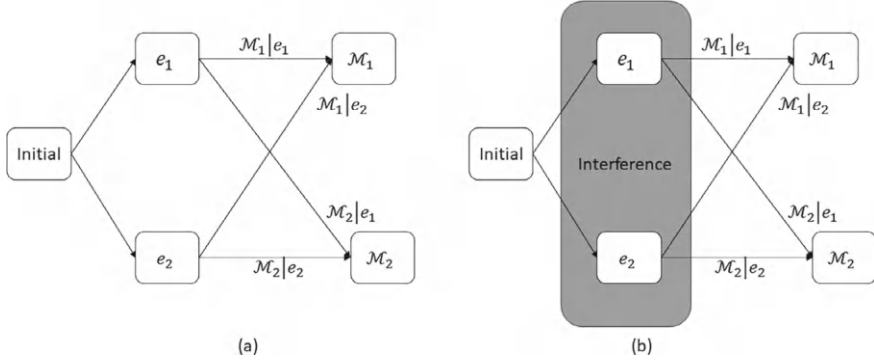


Fig. 2 DM in **a** Markov and **b** Quantum process

Within the QPT framework (Fig. 2b), the state is formed by combining three fundamental states, which can be represented as different wave functions aligning according to Feynman's second rule. The amplitude of the final form is obtained by superimposing two individual routes running from I ('Initial') to $\mathcal{M}_1$. The amplitude is the sum of the individual pathways from source I to target $\mathcal{M}_1$, and the probability of reaching the target $\mathcal{M}_1$ is equal to the square of the amplitude.

$$P(\mathcal{M}_1) = |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 \quad (7)$$

Equation (7) can be simplified in the following way:

$$\begin{aligned} P(\mathcal{M}_1) &= |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 \\ &= |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}| |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^* \\ &= \left(\sqrt{P(e_1)P(\mathcal{M}_1|e_1)}e^{i\alpha_1} + \sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{i\alpha_2} \right) \\ &\quad \left(\sqrt{P(e_1)P(\mathcal{M}_1|e_1)}e^{-i\alpha_1} + \sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{-i\alpha_2} \right) \\ &= P(e_1)P(\mathcal{M}_1|e_1) + P(e_2)P(\mathcal{M}_1|e_2) \\ &\quad + \sqrt{P(e_1)P(\mathcal{M}_1|e_1)}\sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{i(\alpha_1-\alpha_2)} \\ &\quad + \sqrt{P(e_1)P(\mathcal{M}_1|e_1)}\sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{i(\alpha_2-\alpha_1)} \end{aligned} \quad (8)$$

By Eq. (4), we get

$$\cos(\alpha_1 - \alpha_2) = \frac{e^{i(\alpha_2-\alpha_1)} + e^{i(\alpha_1-\alpha_2)}}{2} \quad (9)$$

Then Eq. (7) can be written in the following way:

$$P(\mathcal{M}_1) = P(e_1)P(\mathcal{M}_1|e_1) + P(e_2)P(\mathcal{M}_1|e_2)$$

$$\begin{aligned}
& + 2\sqrt{P(e_1)P(\mathcal{M}_1|e_1)}\sqrt{P(e_2)P(\mathcal{M}_1|e_2)}\cos(\alpha_1 - \alpha_2) \\
& = |\xi_{e_1}\xi_{\mathcal{M}_1|e_1}|^2 + |\xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 + 2\xi_{e_1}\xi_{\mathcal{M}_1|e_1}\xi_{e_2}\xi_{\mathcal{M}_1|e_2}\cos(\alpha_1 - \alpha_2) \quad (10)
\end{aligned}$$

Upon comparing Eqs. (6) and (10), it becomes evident that the quantum framework includes an interference component. If the pathways are observable, the quantum phase transition will reduce to the classical one, following Feynman's third rule.

$$P(\mathcal{M}_1) = |\xi_{e_1}\xi_{\mathcal{M}_1|e_1}|^2 + |\xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 = P(e_1)P(\mathcal{M}_1|e_1) + P(e_2)P(\mathcal{M}_1|e_2) \quad (11)$$

Additionally, the QPT can be expanded to a multi-path graph, resulting in an increased number of interference terms. The QPT enhances CPT by accurately representing cognitive processes in quantum terms, thereby more effectively modeling ambiguity, vagueness, and uncertainty in decision-making.

3 The Proposed Todim-Vikor-Based Magdm Under Pythagorean Linguistic Information and Quantum Decision Scenario

This section introduces quantum cognition-based MAGDM integrating the TODIM-VIKOR approach.

3.1 Background

Let us assume.

$X = \{x_1, x_2, \dots, x_n\}$: Alternative set,

$C = \{c_1, c_2, \dots, c_m\}$: Attribute set,

$e = \{e_1, e_2, \dots, e_q\}$: Expert set,

$w^k = (w_1^k, w_2^k, \dots, w_m^k)$: Weight vector of attributes for expert e_k , where $w_j^k \in [0, 1]$, and $\sum_{j=1}^m w_j^k = 1$,

$\omega = (\omega_1, \omega_2, \dots, \omega_q)$: Weight vector of experts with $\omega_k \in [0, 1]$, $\sum_{k=1}^q \omega_k = 1$.

3.2 Decision Steps

Let $A^k = (a_{ij}^k)_{nm}$ ($i = 1, 2, \dots, n; j = 1, 2, \dots, m; k = 1, 2, \dots, q$) be the PLI-based decision judgment matrix provided by an expert e_k using pre-defined distinct LTSs $H = \{h_\lambda | \lambda = -\zeta, \dots, -1, 0, 1, \dots, \zeta\}$.

Step 1: Construction of decision matrix

Then, in the following, we represent the decision matrix of k -th expert:

$$A^k = (a_{ij}^k)_{nm} = \begin{matrix} & \begin{matrix} c_1 & c_2 & \cdots & c_m \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} & \begin{bmatrix} a_{11}^k & a_{12}^k & \cdots & a_{1m}^k \\ a_{21}^k & a_{22}^k & \cdots & a_{2m}^k \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}^k & a_{n2}^k & \cdots & a_{nm}^k \end{bmatrix} \end{matrix}, \quad (12)$$

where, $a_{ij}^k = \langle h_{\theta_{ij}}^k, (\mu_{ij}^k, \nu_{ij}^k) \rangle$ is a PLN, called the Pythagorean linguistic information (PLI) of k -th expert for i -th alternative about j -th criteria.

Step 2: Expert weight computation process

Inspire by Ref. (Mandal et al., 2024), the weight of experts can be calculated in the following way:

$$\omega_k = \frac{\Omega(Z^k)}{\sum_{k=1}^q \Omega(Z^k)}, \text{ where } \Omega(Z^k) = \frac{\sum_{j=1}^m \sum_{i=1}^n \mathcal{F}(\kappa(a_{ij}^k))}{mn}. \quad (13)$$

Step 3: Attribute weight computation process

(1) The decision matrix (Eq. 12) can be normalized to $L^k = (l_{ij}^k)_{nm}$ with

$$l_{ij}^k = \begin{cases} \frac{\mathcal{F}(\kappa(a_{ij}^k))}{\sum_{i=1}^n \mathcal{F}(\kappa(a_{ij}^k))}, \text{ for benefit criteria } c_j, \\ 1 - \frac{\mathcal{F}(\kappa(a_{ij}^k))}{\sum_{i=1}^n \mathcal{F}(\kappa(a_{ij}^k))}, \text{ for cost criteria } c_j. \end{cases} \quad (14)$$

(2) The Shannon entropy $\Psi^k = (\Psi_1^k, \Psi_2^k, \dots, \Psi_m^k)$ concerning each expert e_k ($k = 1, 2, \dots, q$) can be calculated in the following way:

$$\Psi_j^k = -\frac{1}{\ln n} \sum_{i=1}^n l_{ij}^k \ln l_{ij}^k \quad (15)$$

$l_{ij}^k \ln l_{ij}^k$ is 0 if $l_{ij}^k = 0$.

- (3) The attribute weight $w^k = (w_1^k, w_2^k, \dots, w_m^k)$, concerning each expert $e_k (k = 1, 2, \dots, q)$ can be calculated in the following way:

$$w_j^k = \frac{1 - \Psi_j^k}{\sum_{j=1}^m (1 - \Psi_j^k)} \quad (16)$$

Step 4: Construction of Bayesian network

In this phase, we converted the pre-collected data into probabilities and developed a Bayesian Network for decision-making, as the quantum framework is a partial extension of the Bayesian Network. Figure 3 illustrates that, in a MAGDM situation, the decision-making process can be represented as a simple two-layer Bayesian Network. The experts' weights are converted into probabilities, which indicate the degree of belief in the experts, using the following equation:

$$P(e_k) = \omega_k, k = 1, 2, \dots, q. \quad (17)$$

In Eq. (17), the probabilities form the initial layer of the Bayesian Network (BN). The second layer depends on the assessment outcomes from each expert, which can be calculated using the TODIM-VIKOR approach as follows:

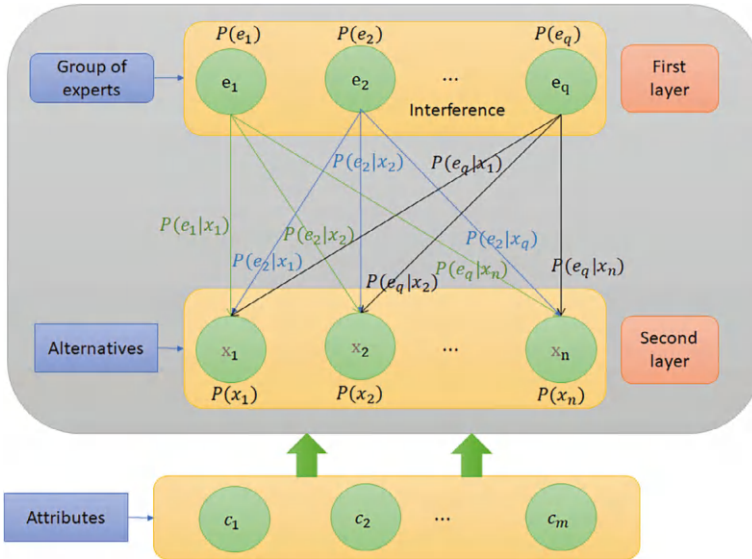


Fig. 3 MAGDM according to QSBN

(1) Attribute relative weight computation process

Let $w_j^{k,*}$ be the relative weight of the attribute c_j concerning each expert $e_k (k = 1, 2, \dots, q)$, where

$$w_j^{k,*} = \frac{w_j^k}{w_{max}^k}, w_{max}^k = \max_{j=1}^m w_j^k \quad (18)$$

(2) Dominance degree matrix construction process

The dominant degree of x_i over x_s for the attribute c_j concerning each expert $e_k (k = 1, 2, \dots, q)$ opinion is denoted by $\phi_j^k(x_i, x_s)$ and defined in the following way:

$$\phi_j^k(x_i, x_s) = \begin{cases} \sqrt{\frac{w_j^{k,*} \times d(a_{ij}^k, a_{sj}^k)}{\sum_{j=1}^m w_j^{k,*}}} & \text{if } \mathcal{F}(\kappa(a_{ij}^k)) > \mathcal{F}(\kappa(a_{sj}^k)) \\ 0 & \text{if } \mathcal{F}(\kappa(a_{ij}^k)) = \mathcal{F}(\kappa(a_{sj}^k)) \\ -\frac{1}{\theta} \sqrt{\frac{(\sum_{j=1}^m w_j^{k,*}) \times d(a_{ij}^k, a_{sj}^k)}{w_j^{k,*}}} & \text{if } \mathcal{F}(\kappa(a_{ij}^k)) < \mathcal{F}(\kappa(a_{sj}^k)) \end{cases} \quad (19)$$

where θ is a risk factor of gain and losses. $\mathcal{F}(\kappa(a_{ij}^k)) > \mathcal{F}(\kappa(a_{sj}^k))$ means $j(x_i, x_s)$ are gains and $\mathcal{F}(\kappa(a_{ij}^k)) < \mathcal{F}(\kappa(a_{sj}^k))$ means $\phi_j^k(x_i, x_s)$ are losers. Based on Eq. (19), the dominance degree matrix $\phi_j^k (j = 1, 2, \dots, m)$ for c_j concerning each expert $e_k (k = 1, 2, \dots, q)$ opinion is as follows:

$$\phi_j^k = (\phi_j^k(x_i, x_s))_{n \times n} = \begin{matrix} & \begin{matrix} x_1 & x_2 & \dots & x_n \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} & \begin{bmatrix} 0 & \phi_j^k(x_1, x_2) & \dots & \phi_j^k(x_1, x_n) \\ \phi_j^k(x_2, x_1) & 0 & \dots & \phi_j^k(x_2, x_n) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_j^k(x_n, x_1) & \phi_j^k(x_n, x_2) & \dots & 0 \end{bmatrix} \end{matrix} \quad (20)$$

(3) Overall dominance matrix construction process

The overall dominance degree of x_i concerning remaining alternatives for c_j with respect to each expert $e_k (k = 1, 2, \dots, q)$ opinion denoted by $\eta^k(x_i, c_j)$ and defined in the following way:

$$\eta^k(x_i, c_j) = \sum_{s=1}^n \phi_j^k(x_i, x_s) (i = 1, 2, \dots, n; j = 1, 2, \dots, m) \quad (21)$$

Based on Eq. (21), the overall dominance matrix η^k concerning each expert $e_k (k = 1, 2, \dots, q)$ opinion is as follows:

$$\eta^k = (\eta^k(x_i, c_j))_{n \times m}$$

$$= \begin{matrix} & \begin{matrix} c_1 & c_2 & \dots & c_m \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} & \begin{bmatrix} \eta^k(x_1, c_1) & \eta^k(x_1, c_2) & \dots & \eta^k(x_1, c_m) \\ \eta^k(x_2, c_1) & \eta^k(x_2, c_2) & \dots & \eta^k(x_2, c_m) \\ \vdots & \vdots & \ddots & \vdots \\ \eta^k(x_n, c_1) & \eta^k(x_n, c_2) & \dots & \eta^k(x_n, c_m) \end{bmatrix} \end{matrix} \quad (22)$$

(4) Positive and negative ideal solution process

With the help of max and min operations, in the following, we define the positive ideal (PI) and negative ideal (NI) solution in the following way:

$$\begin{cases} PIS^k = (PIS_1^k, PIS_2^k, \dots, PIS_m^k) \\ NIS^k = (NIS_1^k, NIS_2^k, \dots, NIS_m^k) \end{cases} \quad (23)$$

where,

$$PIS_j^k = \{(c_j, \max_{i=1}^n \eta^k(x_i, c_j)) | j = 1, 2, \dots, m\}$$

$$NIS_j^k = \{(c_j, \min_{i=1}^n \eta^k(x_i, c_j)) | j = 1, 2, \dots, m\}$$

(5) VIKOR-based conditional probability computation process

According to the VIKOR approach, the maximum group utility, $\mathbb{G}_i^k$, the regret value, $\mathbb{R}_i^k$, and the relationship degree between $\mathbb{G}_i^k$ and $\mathbb{R}_i^k$, $\mathbb{Q}_i^k$ of expert e_k are defined in the following way:

$$\mathbb{G}_i^k = \sum_{j=1}^m w_j^k \frac{|\eta^k(x_i, c_j) - PIS_j^k|}{|PIS_j^k - NIS_j^k|} \quad (24)$$

$$\mathbb{R}_i^k = \max_{j=1}^m w_j^k \frac{|\eta^k(x_i, c_j) - PIS_j^k|}{|PIS_j^k - NIS_j^k|} \quad (25)$$

$$\mathbb{Q}_i^k = \frac{\mathbb{G}_i^k - \mathbb{G}^{k,-}}{\mathbb{G}^{k,+} - \mathbb{G}^{k,-}} + (1 - \frac{\mathbb{R}_i^k - \mathbb{R}^{k,-}}{\mathbb{R}^{k,+} - \mathbb{R}^{k,-}}) \quad (26)$$

where $\mathbb{G}^{k,+} = \max_{i=1}^n \mathbb{G}_i^k$, $\mathbb{G}^{k,-} = \min_{i=1}^n \mathbb{G}_i^k$, $\mathbb{R}^{k,+} = \max_{i=1}^n \mathbb{R}_i^k$, and $\mathbb{R}^{k,-} = \min_{i=1}^n \mathbb{R}_i^k$.

- (1) Based on Eq. (26), conditional probabilities which link the two layers in the BN as:

$$P(x_i|e_k) = \frac{\mathcal{Q}_i^k}{\sum_{i=1}^n \mathcal{Q}_i^k}. \quad (27)$$

- (2) The second layer consists of the probabilities of alternatives calculated as:

$$P(x_i) = \sum_{k=1}^q P(e_k)P(x_i|e_k) \quad (28)$$

Subsequently, we can decide based on the $P(x_i)$ values, as higher probabilities, indicate a superior ranking. The classical approach does not account for the interference effect, which can influence the final rankings. To address this issue, Bayesian networks are expanded into a quantum framework capable of properly modeling a practical approach to real interference in MAGDM.

Step 5: Construction of quantum scenario-based Bayesian network (QSBN)

The facilitator often consolidates opinions from several specialists. The interconnected decision-making processes in real-world situations, as demonstrated in Sect. 2.2, remain unobservable. The square of the sum of the amplitude probabilities of all possible pathways collectively constitutes the probability of the alternative x_i from QPT (He et al., 2018), as described.

$$\begin{aligned} P(x_i) &= \left| \sum_{k=1}^q \xi_{e_k} \xi_{x_i|e_k} e^{i\alpha_k} \right|^2 \\ &= \left| \sum_{k=1}^q \sqrt{P(e_k)} \sqrt{P(x_i|e_k)} e^{i\alpha_k} \right|^2, i = 1, 2, \dots, n \end{aligned} \quad (29)$$

By Eqs. (7)–(11), (29) can be written as

$$\begin{aligned} P(x_i) &= \sum_{k=1}^q P(e_k)P(x_i|e_k) + 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \cos(\alpha_k - \alpha_s) \\ &= \sum_{k=1}^q P(e_k)P(x_i|e_k) + INT_i \end{aligned} \quad (30)$$

In a quantum-based Bayesian network, a normalization factor is incorporated into Eq. (30) to standardize the probability of each alternative, ensuring that the overall estimated probability does not exceed 1. Then

$$Prob(x_i) = \sigma P(x_i) \quad (31)$$

where,

$$\sigma = \frac{1}{\sum_{i=1}^n P(x_i)} = \frac{1}{\sum_{i=1}^n (\sum_{k=1}^q P(e_k)P(x_i|e_k) + INT_i)} \quad (32)$$

3.3 Analysis of Interference Effect Among Experts' Opinions

Equation (32) uses the term INT_i , which represent the ambiguity and uncertainty of expert opinions, to explain the superposition of decision information. In Eq. (30), $\cos(\alpha_k - \alpha_s)$ represents the phase angle between two distinct experts e_k and e_s , which represents the connection vector between two experts. $\frac{q(q-1)}{2}$ is the total count of pairwise correspondences across q experts, excluding duplicates. The final interference value is derived by summing the computed $\frac{q(q-1)}{2}$ interference values for each expert pair, rather than assessing the overall uncertainty across experts. This approach enhances the accuracy of the estimated probability value.

In real-world situations, it is hard to measure the phase angle. Experts' opinions can differ or be influenced by outside factors, causing a lot of variation and mistakes. Belief entropy helps a lot in handling and combining unclear information, making it easier to make good decisions in group settings with many factors. It is used as a standard in areas like risk assessment, defect diagnosis, and pattern recognition. Belief entropy is the latest way to measure interference in QSBN. Unlike the old method of manually estimating the phase angle range, this new method is more reliable because it reduces errors from personal judgment, which can affect the final decision and the expert's probability calculation. Belief entropy is used in QSBN to manage the interference among different experts' beliefs in GDM.

Then the belief distance about alternative x_i according to the average probability $\nabla = \frac{1}{nq}$ can be calculated in the following way:

$$D_{iu} = \left| \delta_{iu} + \frac{|\delta_{iu} - \nabla| - |\varrho_{iu} - \nabla|}{|\delta_{iu} - \nabla| + |\varrho_{iu} - \nabla|} \right|, i = 1, 2, \dots, n; u = 1, 2, \dots, \frac{q(q-1)}{2},$$

$$\begin{cases} \delta_{iu} = P(e_k)P(x_i|e_k) \\ \varrho_{iu} = P(e_s)P(x_i|e_s) \end{cases}, i = 1, 2, \dots, n; k = 1, 2, \dots, q-1; s = k+1, k+2, \dots, \frac{q(q-1)}{2}. \quad (33)$$

In Eq. (33), u means the value reallocated to the correlation among q experts; $\nabla = \frac{1}{nq}$ represent each expert probability value of each alternative; D_{iu} means deviation degree from ∇ ; $|\delta_{iu} - \nabla|$ and $|\varrho_{iu} - \nabla|$ represents two experts' magnitude of the deviation degree from ∇ about alternative x_i . If $|\delta_{iu} - \nabla| < |\varrho_{iu} - \nabla|$, then the position of δ_{iu} and ϱ_{iu} are interchangeable.

There are many ways to measure uncertainty. Deng entropy (Zhao et al. 2024), also known as belief entropy, is better at including the confidence function and the uncertainty of BPA because it measures both total non-specificity and disagreement.

Thus, by Deng's entropy, we get

$$\vartheta_{iu}(D_{iu}) = -D_{iu} \log_2 \frac{D_{iu}}{2^m - 1}, i = 1, 2, \dots, n; u = 1, 2, \dots, \frac{q(q-1)}{2}. \quad (34)$$

To ensure that the values of Deng entropy align with the range of $\cos(\alpha_k - \alpha_s)$, which is between $[-1, 1]$, Deng entropy is standardized.

$$\beta_{iu}(D_{iu}) = 2 \frac{\vartheta_{iu}(D_{iu}) - \min\{\vartheta_{iu}(D_{iu})\}}{\max\{\vartheta_{iu}(D_{iu})\} - \min\{\vartheta_{iu}(D_{iu})\}} - 1, i = 1, 2, \dots, n; u = 1, 2, \dots, \frac{q(q-1)}{2}. \quad (35)$$

Thus, the phase angle $\cos(\alpha_k - \alpha_s)$ can be determined by the following way:

$$\cos(\alpha_k - \alpha_s) = \beta_{iu}(D_{iu}), i = 1, 2, \dots, n; u = 1, 2, \dots, \frac{q(q-1)}{2}. \quad (36)$$

Therefore, by Eq. (36), in Eq. (30), INT_i can be written as

$$\begin{aligned} INT_i &= 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \cos(\alpha_k - \alpha_s) \\ &= 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \beta_{iu}(D_{iu}) \end{aligned} \quad (37)$$

By Eq. (37), the Eq. (30) can be written as

$$P(x_i) = \sum_{k=1}^q P(e_k)P(x_i|e_k) + 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \beta_{iu}(D_{iu}) \quad (38)$$

Finally, the probability of each alternative x_i can be obtained in the following way:

$$Prob(x_i) = \sigma P(x_i) \quad (39)$$

where,

$$\sigma = \frac{1}{\sum_{i=1}^n P(x_i)} = \frac{1}{\sum_{i=1}^n (\sum_{k=1}^q P(e_k)P(x_i|e_k) + INT_i)} \quad (40)$$

Step 6: Ranking of alternative

According to Eq. (39), the higher probability of an alternative means the best alternative.

4 A Practical Example with Comparative Analysis

This section describes a new approach to MAGDM based on the TODIM-VIKOR method integrating QDT with PLI. The TODIM and VIKOR methods are integrated to account for the psychological aspects of decision-maker satisfaction, while also maximizing overall group utility and minimizing individual regret. The detailed procedure for the combined TODIM-VIKOR approach is presented below:

4.1 A Case Study

This section will illustrate the use and versatility of the previously described technique through a MAGDM scenario involving wearable medical equipment. The market for wearable medical and healthcare products has grown significantly due to global industrialization. The rise of wearable technology has profoundly impacted the healthcare sector. Wearable devices, equipped with transmitters and target receptors, are worn on the body or integrated into clothing to monitor physiological changes and provide real-time feedback. This technology helps clinicians understand patient conditions and administer timely treatments. The primary goal of wearable technology in healthcare is to revolutionize the industry by offering patients extensive, actionable insights. Wearable devices monitor key indicators such as heart rate, blood pressure, and oxygen saturation, while also tracking health metrics like physical activity, diet, and sleep patterns. Individuals with chronic conditions—including diabetes, cardiovascular diseases, and neurological disorders—require continuous monitoring for appropriate medical intervention. Wearable technology plays a crucial role in meeting this need. By enabling individuals to collect and share their health data online, wearable medical devices reduce the need for in-person consultations. These technologies also help hospitals save time and money while improving patient convenience. Continuous patient monitoring is essential during rehabilitation, and wearable health devices enhance the efficiency of this process. Wearable fitness trackers and health-monitoring devices allow individuals to measure their physical activity, including steps, exercise, general mobility, and other vital health metrics. Additionally, these devices can assess psychological well-being using sensors to track mental health. Wearable technology in healthcare also provides educational benefits by offering access to key resources such as articles, case studies, and study materials via smartphones. Furthermore, wearable medical devices enhance electronic health record (EHR) systems. The integration of patient-generated fitness or

medical data with broader datasets—such as biological, genetic, and current health information—adds significant value and robustness to EHR systems.

The integration of wearable medical technology into healthcare systems presents several challenges. Financial constraints may limit organizations from adopting these devices, and patients may find them uncomfortable due to their weight. Although these devices contain multiple components designed for body attachment, there is insufficient evidence to confirm their unmodified accuracy due to data inconsistencies. Privacy concerns arise when individuals cannot control their personal information, as software installations require user consent for data access, posing risks such as forgery and data breaches. Integrating patient data from wearable devices is still in its early stages, with ongoing challenges related to compatibility and connectivity. Additionally, the vast amount of personal health data collected by these devices can cause anxiety, making users feel excessively monitored. The medical community also faces difficulties in managing the large volumes of data generated efficiently.

4.1.1 Decision Procedure

This section demonstrates the constructed model through a case study on selecting wearable health technology devices. Assume an employer has identified four potential firms globally to choose a suitable wearable health technology device. A team of global experts has been assembled to evaluate and select the most appropriate gadgets for the organization. Let $E = \{e_1, e_2, e_3, e_4, e_5\}$ represent the five decision-makers, and $X = \{x_1, x_2, x_3, x_4, x_5\}$ denote the five devices under consideration. Each device is evaluated based on four criteria: accuracy (c_1), intelligence (c_2), security (c_3), and constraints (c_4) with all features being beneficial. The following methods are applied to evaluate this MAGDM scenario using our proposed TODIM-VIKOR method integrating QDT with PLI for assessing the available options.

Step 1: Here we offer the LTSs, $H_1 = \{h_{-3}: \text{very bad (VB)}, h_{-2}: \text{bad (B)}, h_{-1}: \text{slightly bad (SB)}, h_0: \text{moderate (M)}, h_1: \text{slightly good (SG)}, h_2: \text{good (G)}, h_3: \text{very good (VG)}\}$ for the evaluation of alternatives (x_1, x_2, x_3, x_4, x_5) about the criteria (c_1, c_2, c_3, c_4). The evaluation information is shown in the Eqs. (41)–(45).

$$A^1 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} \langle h_2, (0.8, 0.4) \rangle & \langle h_3, (0.9, 0.2) \rangle & \langle h_{-2}, (0.4, 0.6) \rangle & \langle h_2, (0.6, 0.3) \rangle \\ \langle h_{-1}, (0.9, 0.3) \rangle & \langle h_{-2}, (0.7, 0.3) \rangle & \langle h_0, (0.5, 0.5) \rangle & \langle h_3, (0.7, 0.5) \rangle \\ \langle h_3, (0.7, 0.4) \rangle & \langle h_3, (0.6, 0.5) \rangle & \langle h_3, (0.6, 0.6) \rangle & \langle h_{-3}, (0.6, 0.4) \rangle \\ \langle h_{-2}, (0.6, 0.5) \rangle & \langle h_2, (0.5, 0.4) \rangle & \langle h_2, (0.8, 0.4) \rangle & \langle h_2, (0.8, 0.4) \rangle \\ \langle h_2, (0.5, 0.4) \rangle & \langle h_{-1}, (0.9, 0.3) \rangle & \langle h_1, (0.9, 0.1) \rangle & \langle h_2, (0.7, 0.2) \rangle \end{bmatrix} \end{matrix} \quad (41)$$

$$A^2 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} \langle h_1, (0.82, 0.3) \rangle & \langle h_{-2}, (0.8, 0.2) \rangle & \langle h_{-2}, (0.75, 0.45) \rangle & \langle h_{-3}, (0.9, 0.2) \rangle \\ \langle h_0, (0.78, 0.4) \rangle & \langle h_{-3}, (0.72, 0.32) \rangle & \langle h_1, (0.8, 0.35) \rangle & \langle h_2, (0.87, 0.35) \rangle \\ \langle h_{-3}, (0.65, 0.23) \rangle & \langle h_2, (0.8, 0.25) \rangle & \langle h_2, (0.23, 0.9) \rangle & \langle h_{-3}, (0.8, 0.4) \rangle \\ \langle h_2, (0.81, 0.32) \rangle & \langle h_1, (0.92, 0.12) \rangle & \langle h_3, (0.9, 0.25) \rangle & \langle h_{-1}, (0.75, 0.3) \rangle \\ \langle h_{-2}, (0.75, 0.45) \rangle & \langle h_0, (0.7, 0.3) \rangle & \langle h_{-3}, (0.8, 0.5) \rangle & \langle h_2, (0.6, 0.4) \rangle \end{bmatrix} \end{matrix} \quad (42)$$

$$A^3 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} \langle h_2, (0.9, 0.1) \rangle & \langle h_3, (0.6, 0.5) \rangle & \langle h_{-2}, (0.8, 0.3) \rangle & \langle h_3, (0.45, 0.7) \rangle \\ \langle h_{-1}, (0.45, 0.75) \rangle & \langle h_{-2}, (0.7, 0.65) \rangle & \langle h_0, (0.7, 0.6) \rangle & \langle h_{-2}, (0.68, 0.5) \rangle \\ \langle h_2, (0.75, 0.25) \rangle & \langle h_{-1}, (0.9, 0.2) \rangle & \langle h_3, (0.65, 0.45) \rangle & \langle h_3, (0.78, 0.25) \rangle \\ \langle h_3, (0.35, 0.45) \rangle & \langle h_2, (0.78, 0.25) \rangle & \langle h_2, (0.9, 0.7) \rangle & \langle h_2, (0.9, 0.3) \rangle \\ \langle h_{-1}, (0.9, 0.2) \rangle & \langle h_3, (0.65, 0.3) \rangle & \langle h_{-1}, (0.74, 0.3) \rangle & \langle h_{-2}, (0.7, 0.2) \rangle \end{bmatrix} \end{matrix} \quad (43)$$

$$A^4 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} \langle h_2, (0.8, 0.5) \rangle & \langle h_{-2}, (0.8, 0.5) \rangle & \langle h_2, (0.8, 0.3) \rangle & \langle h_0, (0.9, 0.3) \rangle \\ \langle h_{-2}, (0.9, 0.3) \rangle & \langle h_1, (0.9, 0.3) \rangle & \langle h_3, (0.92, 0.2) \rangle & \langle h_2, (0.6, 0.5) \rangle \\ \langle h_1, (0.75, 0.5) \rangle & \langle h_3, (0.7, 0.4) \rangle & \langle h_{-3}, (0.78, 0.25) \rangle & \langle h_{-2}, (0.85, 0.25) \rangle \\ \langle h_0, (0.6, 0.4) \rangle & \langle h_{-2}, (0.7, 0.3) \rangle & \langle h_2, (0.6, 0.5) \rangle & \langle h_1, (0.7, 0.6) \rangle \\ \langle h_3, (0.8, 0.3) \rangle & \langle h_0, (0.8, 0.25) \rangle & \langle h_1, (0.7, 0.3) \rangle & \langle h_3, (0.6, 0.4) \rangle \end{bmatrix} \end{matrix} \quad (44)$$

$$A^5 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} \langle h_2, (0.75, 0.4) \rangle & \langle h_{-2}, (0.83, 0.2) \rangle & \langle h_2, (0.8, 0.3) \rangle & \langle h_{-2}, (0.35, 0.6) \rangle \\ \langle h_1, (0.65, 0.5) \rangle & \langle h_{-3}, (0.84, 0.3) \rangle & \langle h_{-3}, (0.75, 0.4) \rangle & \langle h_{-1}, (0.8, 0.3) \rangle \\ \langle h_3, (0.75, 0.6) \rangle & \langle h_2, (0.92, 0.1) \rangle & \langle h_1, (0.68, 0.3) \rangle & \langle h_2, (0.7, 0.4) \rangle \\ \langle h_2, (0.85, 0.2) \rangle & \langle h_3, (0.9, 0.2) \rangle & \langle h_3, (0.56, 0.6) \rangle & \langle h_3, (0.65, 0.4) \rangle \\ \langle h_1, (0.65, 0.35) \rangle & \langle h_2, (0.7, 0.3) \rangle & \langle h_0, (0.7, 0.23) \rangle & \langle h_0, (0.8, 0.3) \rangle \end{bmatrix} \end{matrix} \quad (45)$$

Step 2: By Eq. (13), the weight of experts is $\omega_1 = 0.2170$, $\omega_2 = 0.1636$, $\omega_3 = 0.2054$, $\omega_4 = 0.2027$, and $\omega_5 = 0.2113$.

Step 3: By Eq. (16), the weight of attributes for all experts are:

$$w_1^1 = 0.2511, w_2^1 = 0.3475, w_3^1 = 0.1185, w_4^1 = 0.2829;$$

$$w_1^2 = 0.1748, w_2^2 = 0.2048, w_3^2 = 0.2193, w_4^2 = 0.4011;$$

$$w_1^3 = 0.1534, w_2^3 = 0.2426, w_3^3 = 0.2726, w_4^3 = 0.3315;$$

$$w_1^4 = 0.2218, w_2^4 = 0.2528, w_3^4 = 0.3432, w_4^4 = 0.1823;$$

$$w_1^5 = 0.0198, w_2^5 = 0.5731, w_3^5 = 0.2660, w_4^5 = 0.1410.$$

Step 4: According to Eq. (17), we get $P(e_1) = 0.2170$, $P(e_2) = 0.1636$, $P(e_3) = 0.2054$, $P(e_4) = 0.2027$, and $P(e_5) = 0.2113$.

(1) By Eq. (18), the relative weight of each attribute for all experts are:

$$w_{max}^1 = 0.3475, w_1^{1,*} = 0.7224, w_2^{1,*} = 1, w_3^{1,*} = 0.3410, w_4^{1,*} = 0.8141;$$

$$w_{max}^2 = 0.4011, w_1^{2,*} = 0.4356, w_2^{2,*} = 0.5105, w_3^{2,*} = 0.5467, w_4^{2,*} = 1;$$

$$w_{max}^3 = 0.3315, w_1^{3,*} = 0.4627, w_2^{3,*} = 0.7320, w_3^{3,*} = 0.8223, w_4^{3,*} = 1;$$

$$w_{max}^4 = 0.3432, w_1^{4,*} = 0.6462, w_2^{4,*} = 0.7365, w_3^{4,*} = 1, w_4^{4,*} = 0.5312;$$

$$w_{max}^5 = 0.5731, w_1^{5,*} = 0.0346, w_2^{5,*} = 1, w_3^{5,*} = 0.4642, w_4^{5,*} = 0.2460.$$

- (2) & (3) Let $\theta = 2.5$. Then by Eq. (22), using Eqs. (19) and (20), the overall dominance matrix η^k ($k = 1, 2, 3, 4, 5$) for all experts is shown in Eqs. (46) – (50).

$$\eta^1 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} 0.6771 & 1.6983 & -2.9727 & -0.2681 \\ -1.0556 & -1.7179 & -1.8246 & 0.6768 \\ 1.0275 & 0.4931 & 0.6252 & -2.3143 \\ -2.0139 & -0.0771 & 0.1113 & 0.5911 \\ -0.1820 & -0.9110 & -0.0624 & 0.3804 \end{bmatrix} \end{matrix} \quad (46)$$

$$\eta^2 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} 0.4113 & -1.4967 & -1.2163 & -1.2806 \\ -0.4401 & -2.2531 & 0.4063 & 1.7328 \\ -2.3627 & 1.0430 & -1.0935 & -1.2806 \\ 1.0099 & 0.7065 & 1.5418 & -0.1080 \\ -1.6515 & -0.4629 & -1.9651 & 0.9466 \end{bmatrix} \end{matrix} \quad (47)$$

$$\eta^3 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} 0.9276 & 0.4636 & -1.5868 & 0.3899 \\ -1.9919 & -2.1451 & -0.5265 & -1.3170 \\ 0.3048 & -1.1651 & 1.1050 & 1.5657 \\ -0.5731 & 0.2948 & 0.7103 & 0.1488 \\ -2.1340 & 0.9119 & -0.8651 & -1.4633 \end{bmatrix} \end{matrix} \quad (48)$$

$$\eta^4 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} 0.3611 & -1.5331 & 0.6232 & -0.5410 \\ -2.0327 & 0.6516 & 1.6299 & 0.0977 \\ -0.3831 & 1.1586 & -2.1284 & -2.1333 \\ -1.1628 & -1.5691 & -0.3094 & -0.4102 \\ 1.1808 & -0.2202 & -0.3840 & 0.8262 \end{bmatrix} \end{matrix} \quad (49)$$

$$\eta^5 = \begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} -1.2228 & -0.9477 & 1.1265 & -1.9894 \\ -4.7213 & -1.5586 & -2.1481 & -2.0830 \\ 0.1850 & 1.4020 & 0.0661 & 0.3248 \\ -0.6982 & 2.0447 & 0.2859 & 0.8018 \\ -3.7094 & 0.5683 & -0.6148 & -0.6122 \end{bmatrix} \end{matrix} \quad (50)$$

(3) By Eq. (23), we get

$$\begin{cases} \text{PIS}^1 = (\langle c_1, 1.0275 \rangle, \langle c_2, 1.6983 \rangle, \langle c_3, 0.6252 \rangle, \langle c_4, 0.6768 \rangle) \\ \text{NIS}^1 = (\langle c_1, -2.0139 \rangle, \langle c_2, -1.7179 \rangle, \langle c_3, -2.9727 \rangle, \langle c_4, -2.3143 \rangle) \end{cases};$$

$$\begin{cases} \text{PIS}^2 = (\langle c_1, 1.0099 \rangle, \langle c_2, 1.0430 \rangle, \langle c_3, 1.5418 \rangle, \langle c_4, 1.7328 \rangle) \\ \text{NIS}^2 = (\langle c_1, -2.3627 \rangle, \langle c_2, -2.2531 \rangle, \langle c_3, -1.9651 \rangle, \langle c_4, -1.2806 \rangle) \end{cases};$$

$$\begin{cases} \text{PIS}^3 = (\langle c_1, 0.9276 \rangle, \langle c_2, 0.9119 \rangle, \langle c_3, 1.1050 \rangle, \langle c_4, 1.5657 \rangle) \\ \text{NIS}^3 = (\langle c_1, -2.1340 \rangle, \langle c_2, -2.1451 \rangle, \langle c_3, -1.5868 \rangle, \langle c_4, -1.4633 \rangle) \end{cases};$$

$$\begin{cases} \text{PIS}^4 = (\langle c_1, 1.1808 \rangle, \langle c_2, 1.1586 \rangle, \langle c_3, 1.6299 \rangle, \langle c_4, 0.8262 \rangle) \\ \text{NIS}^4 = (\langle c_1, -2.0327 \rangle, \langle c_2, -1.5691 \rangle, \langle c_3, -2.1284 \rangle, \langle c_4, -2.1333 \rangle) \end{cases};$$

$$\begin{cases} \text{PIS}^5 = (\langle c_1, 0.1850 \rangle, \langle c_2, 2.0447 \rangle, \langle c_3, 1.1265 \rangle, \langle c_4, 0.8018 \rangle) \\ \text{NIS}^5 = (\langle c_1, -4.7213 \rangle, \langle c_2, -1.5586 \rangle, \langle c_3, -2.1481 \rangle, \langle c_4, -2.0830 \rangle) \end{cases}.$$

- (4) Let $\delta = 0.2$. Then, according to the VIKOR approach, the maximum group utility, $\mathbb{G}_i^k$, the regret value, $\mathbb{R}_i^k$, and the relationship degree between $\mathbb{G}_i^k$ and $\mathbb{R}_i^k$, $\mathcal{Q}_i^k$ for all experts are calculated by the Eqs. (24–26). The results are shown in Tables. 1, 2, 3, 4 and 5
- (5) By Eq. (27) we calculate the conditional probabilities of each alternative for each expert. The details results are shown in Table 6.
- (6) By Eq. (28), the probabilities of alternatives are:

$$P(x_1) = 0.2314, P(x_2) = 0.2670, P(x_3) = 0.2235, P(x_4) = 0.1102 \text{ and } P(x_5) = 0.1679.$$

Table 1 The output results of $\mathbb{G}_i^1$, $\mathbb{R}_i^1$, and $\mathcal{Q}_i^1$ for expert e_1

Alternatives	$\mathbb{G}_i^1$	$\mathbb{R}_i^1$	$\mathcal{Q}_i^1$
x_1	0.2368	0.1185	0.0000
x_2	0.6002	0.3475	1.0000
x_3	0.4055	0.2829	0.6671
x_4	0.4567	0.2511	0.5841
x_5	0.4160	0.2654	0.6119

Table 2 The output results of G_i^2 , R_i^2 , and Q_i^2 for expert e_2

Alternatives	G_i^2	R_i^2	Q_i^2
x_1	0.7624	0.4011	1.0000
x_2	0.3509	0.2048	0.0342
x_3	0.7407	0.4011	0.9912
x_4	0.2659	0.2450	0.1640
x_5	0.5554	0.2193	0.1759

Table 3 The output results of G_i^3 , R_i^3 , and Q_i^3 for expert e_3

Alternatives	G_i^3	R_i^3	Q_i^3
x_1	0.4368	0.2726	0.6044
x_2	0.8695	0.3155	0.9274
x_3	0.1960	0.1648	0.0444
x_4	0.3192	0.1551	0.0366
x_5	0.6843	0.3315	0.9450

Table 4 The output results of G_i^4 , R_i^4 , and Q_i^4 for expert e_4

Alternatives	G_i^4	R_i^4	Q_i^4
x_1	0.4821	0.2494	0.4249
x_2	0.3136	0.2218	0.1913
x_3	0.6334	0.3432	0.9807
x_4	0.6677	0.2528	0.5459
x_5	0.3117	0.1839	0.0000

Table 5 The output results of G_i^5 , R_i^4 , and Q_i^5 for expert e_5

Alternatives	G_i^5	R_i^5	Q_i^5
x_1	0.6181	0.4760	0.7637
x_2	1.0000	0.5731	1.0000
x_3	0.2117	0.1022	0.0839
x_4	0.0719	0.0683	0.0000
x_5	0.4612	0.2348	0.3478

Step 5: By Eq. (33), the belief distance between two experts about each alternative is calculated, which is presented in Eq. (51).

Table 6 The conditional probabilities of each alternative

$P(x_i e_k)$	e_1	e_2	e_3	e_4	e_5
x_1	0.0000	0.3493	0.2330	0.2040	0.2137
x_2	0.4228	0.0145	0.4191	0.0693	0.0743
x_3	0.2363	0.3626	0.0174	0.0143	0.3695
x_4	0.1983	0.0893	0.4577	0.2547	0.0000
x_5	0.3479	0.4555	0.0382	0.0000	0.1584

$$D = \begin{bmatrix} 0.4460 & 0.9430 & 0.6721 & 0.3601 & 0.9269 & 0.3921 & 0.0331 & 0.6912 & 0.8341 & 0.3714 \\ 0.2219 & 0.1018 & 0.0304 & 0.2086 & 0.3235 & 0.2001 & 0.0169 & 0.0574 & 0.2336 & 0.1656 \\ 0.8912 & 0.9258 & 0.9238 & 0.9232 & 0.3605 & 0.3265 & 0.3179 & 0.0445 & 0.0546 & 0.1029 \\ 0.6972 & 0.7384 & 0.5999 & 0.7498 & 0.0867 & 0.9300 & 0.1166 & 0.9356 & 0.0273 & 0.8849 \\ 0.7784 & 0.7075 & 0.8182 & 0.5931 & 0.1997 & 0.1263 & 0.4044 & 0.2419 & 0.2968 & 0.5034 \end{bmatrix} \quad (51)$$

By Eq. (34), the Deng entropy between two experts about each alternative is shown in Eq. (52).

$$\vartheta = \begin{bmatrix} 2.2620 & 3.7641 & 3.0112 & 1.9377 & 3.7229 & 2.0615 & 0.2923 & 3.0688 & 3.4770 & 1.9818 \\ 1.3487 & 0.7333 & 0.2719 & 1.2866 & 1.7908 & 1.2462 & 0.1653 & 0.4606 & 1.4029 & 3.0768 \\ 3.6299 & 3.7198 & 3.7147 & 3.7134 & 1.9391 & 1.8028 & 1.7678 & 0.3738 & 0.4423 & 0.7394 \\ 3.0867 & 3.2078 & 2.7861 & 3.2409 & 0.6447 & 3.7307 & 0.8169 & 3.7452 & 0.2486 & 3.6133 \\ 3.3226 & 3.1173 & 3.4334 & 2.7641 & 1.2443 & 0.8704 & 2.1018 & 1.4402 & 1.6798 & 2.4650 \end{bmatrix} \quad (52)$$

By Eq. (35), the standardized Deng entropy between two experts about each alternative is shown in Eq. (53).

$$\beta = \begin{bmatrix} 0.1347 & 1 & 0.5663 & -0.0521 & 0.9763 & 0.0192 & -1 & 0.5995 & 0.8346 & -0.0267 \\ 0.4561 & -0.3011 & -0.8688 & 0.3796 & 1 & 0.3299 & -1 & -0.6366 & 0.5227 & 0.1215 \\ 0.9462 & 1 & 0.9969 & 0.9961 & -0.0644 & -0.1458 & -0.1668 & -1 & -0.9590 & -0.7814 \\ 0.6234 & 0.6926 & 0.4514 & 0.7116 & -0.7734 & 0.9917 & -0.6749 & 1 & -1 & 0.9245 \\ 0.9135 & 0.7534 & 1 & 0.4777 & -0.7082 & -1 & -0.0341 & -0.5554 & -0.3684 & 0.2444 \end{bmatrix} \quad (53)$$

By Eq. (37), the interface effect among experts for each alternative is:

$$INT_1 = 0.1223, INT_2 = 0.0201, INT_3 = 0.1925, INT_4 = 0.1506, \text{ and } INT_5 = 0.0889.$$

Finally, by Eq. (38), the probability of each alternative is:

$$\begin{aligned} Prob(x_1) &= 0.2247, Prob(x_2) = 0.1823, Prob(x_3) = 0.2643, Prob(x_4) \\ &= 0.1656, \text{ and } Prob(x_5) = 0.1631. \end{aligned}$$

Step 6: According to the value of QP of each alternative, the final ranking is

$$x_3 \succ x_1 \succ x_2 \succ x_4 \succ x_5.$$

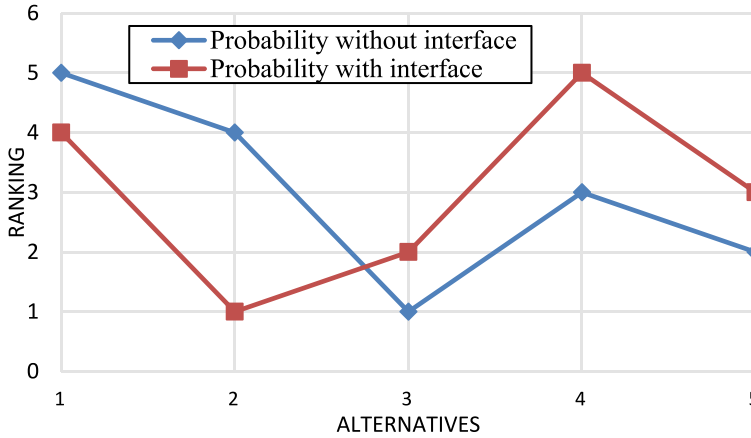


Fig. 4 The interface effect on ranking values

4.2 Effect of the Quantum Interface on Decision Result

If we can decide based on the $P(x_i) (i = 1, 2, 3, 4, 5)$ values according to Eq. (28), as higher probabilities indicate a superior ranking. The classical approach does not account for the interference effect, which can influence the final rankings, as shown in Fig. 4. The ranking result of alternatives according to Eq. (28) is $x_2 > x_1 > x_3 > x_5 > x_4$. If we count the interface effect between two experts about an alternative the ranking result is $x_3 > x_1 > x_2 > x_4 > x_5$.

4.3 Comparative Analysis

To test the feasibility and validity of the suggested method, we compare it with some existing methods for the example given above.

In the traditional decision-making model (Mandal & Ranadive 2019) the ranking results are derived from the aggregate decision matrix of experts' opinions. Therefore, to compare our model and the traditional model, we first aggregate the experts' opinions. In this connection, by Ref. (Mandal & Ranadive 2019), using the Pythagorean linguistic weighted averaging aggregation operator, the aggregation of five experts' opinions is

$$\text{Agg}(A^1, A^2, A^3, A^4, A^5) = \begin{matrix} & & c_1 & c_2 & c_3 & c_4 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \left[\begin{array}{cccc} \langle h_{1.84}, (0.82, 0.30) \rangle & \langle h_{0.11}, (0.81, 0.29) \rangle & \langle h_{-0.34}, (0.74, 0.37) \rangle & \langle h_{0.14}, (0.74, 0.39) \rangle \\ \langle h_{-0.62}, (0.80, 0.42) \rangle & \langle h_{-1.77}, (0.79, 0.36) \rangle & \langle h_{0.14}, (0.78, 0.39) \rangle & \langle h_{0.76}, (0.75, 0.42) \rangle \\ \langle h_{1.41}, (0.73, 0.38) \rangle & \langle h_{1.8}, (0.82, 0.25) \rangle & \langle h_{1.2}, (0.65, 0.44) \rangle & \langle h_{-0.51}, (0.76, 0.33) \rangle \\ \langle h_{0.93}, (0.69, 0.36) \rangle & \langle h_{1.24}, (0.81, 0.24) \rangle & \langle h_{2.37}, (0.8, 0.47) \rangle & \langle h_{1.31}, (0.78, 0.39) \rangle \\ \langle h_{0.72}, (0.76, 0.32) \rangle & \langle h_{0.82}, (0.78, 0.29) \rangle & \langle h_{-0.28}, (0.79, 0.24) \rangle & \langle h_{0.96}, (0.7, 0.28) \rangle \end{array} \right] \end{matrix}$$

Therefore, the ranking of alternatives according to the methods TODIM-VIKOR method is $x_2 > x_1 > x_5 > x_3 > x_4$, i.e., the optimal option is x_2 . When the proposed model is applied without considering the interface effect, the optimal option is also x_2 . This suggests that the optimal option of the proposed approach aligns with those of traditional methods when the impact of decision-makers is disregarded, thereby validating the robustness of the suggested model. However, when evaluating the quantum interference among decision-makers, the outcomes of this model suggest that x_3 is the optimal strategy. This inconsistency arises from the conventional framework's exclusion of the interference effect among decision-makers, which leads to subjective biases during the evaluation of alternatives. As a result, the outcomes of this model better align with real-world complexities.

5 Conclusion

This study presents a quantum group decision model designed to alleviate the effects of information ambiguity and opinion interference in the selection of wearable health technology devices. The model leverages PLIs to convey uncertain information in complex decision-making scenarios. Deng entropy measures opinion interference by modeling the reciprocal influence among decision-makers in the MAGDM process.

Unlike traditional methods that treat decision-makers as isolated entities, this framework addresses their limitations by managing interference effects, assigning appropriate weights to decision-makers, and controlling opinion interference through quantum decision-making techniques. The proposed model enhances current approaches by providing more effective solutions for issues related to opinion interference and weight distribution.

Our findings demonstrate that the model significantly improves decision-making in the selection of wearable health technology devices, particularly by offering a robust framework to handle information ambiguity and conflicting opinions. However, the model's computational complexity increases as more decision-makers and factors are introduced, highlighting the need for advanced computational methods to maintain efficiency.

Future research will focus on integrating machine learning techniques to enhance scalability and feasibility, ensuring the model's effectiveness in large-scale decision-making. This integration aims to manage growing complexity, foster consensus in quantum group decision-making, and ensure the model's long-term sustainability.

Conflict of Interest: The authors declare that they have no conflict of interest.

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The Multicriteria Quantum Decision Theory-Based Group Decision-Making Integrating TODIM-PROMETHEE II Approach Under Linguistic Z Number Information with Application to Selection of Renewable Energy



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Abstract Decision-making often relies on subjective experience; however, certain perspectives may lack clarity, making articulation difficult for policymakers. As a result, researchers need to provide scientific decision frameworks, such as the multicriteria group decision-making (MCGDM) approach within a linguistic context, which is becoming increasingly popular among academics. In modern MCGDM settings, experts are frequently viewed as independent, and their psychological behaviours are rarely considered. However, in many cases, experts' viewpoints are likely to influence one another, and they often exhibit subjective, bounded rationality during the decision-making process. In this chapter, we propose a multicriteria quantum decision theory-based group decision-making model that integrates the TODIM-PROMETHEE II method using linguistic Z-number information. A case study on renewable energy selection is conducted to evaluate the model's effectiveness, along with sensitivity analysis, comparative analysis, and discussion.

Keywords Quantum decision theory · Group decision-making · Linguistic Z number · TODIM-PROMETHEE II

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1 Introduction

In recent decades, techniques for addressing challenges in multi-criteria group decision-making (MCGDM) have advanced significantly, expanding their scope of application. These strategies are now essential for navigating complex decision-making processes involving multiple factors in real-world scenarios. Among various MCGDM methodologies, TODIM and PROMETHEE II have gained considerable attention.

TODIM, an acronym for “Interactive and Multicriteria Decision Making” in Portuguese, is based on prospect theory. It seeks to capture the psychological responses of decision-makers under uncertainty. The method evaluates alternatives by analyzing their relative superiority, considering both total value and the cognitive processes of the decision-makers. TODIM is particularly relevant in situations where decision-makers must navigate ambiguity without complete rational certainty.

PROMETHEE II is another MCGDM approach used to generate a comprehensive ranking of alternatives based on their multicriteria net flow. It integrates preferences and indifferences to create an overall ordering of options. This method is especially useful in complex decision-making scenarios involving numerous criteria, where human perception and judgment play critical roles. PROMETHEE II has been successfully applied across various fields, including business, transportation, health-care, and education, helping decision-makers identify the alternative that best aligns with their objectives and understanding of the problem (Wang et al. 2020).

The research in fuzzy set theory and its applications plays a significant role in addressing uncertainty and complexity in social networks and group decision-making processes. The methods developed in this area offer valuable insights into how fuzzy logic can be used to model and analyze systems where imprecision and ambiguity are prevalent. Fuzzy set theory is particularly well-suited to modeling the uncertainty inherent in social networks, where relationships and interactions often involve incomplete or vague information. Traditional mathematical models struggle to capture this uncertainty, but fuzzy approaches provide a more nuanced framework for analyzing such dynamics. For instance, the development of fuzzy fractional differential equations using fuzzy Laplace transforms is essential for modeling systems where the influence of individuals or groups evolves fractionally over time under uncertainty (Salahshour et al. 2012; Allahviranloo et al. 2015; Chehlabi and Allahviranloo 2016; Armand et al. 2018).

A new centrality measure is introduced for neutrosophic and linguistic graphs (Mahapatra et al. 2019, 2020), including self-weight or individual merit within the network. Mahapatra et al. (2024) present an innovative method for centrality measurement in fuzzy social networks where relationships between nodes are not binary but have different degrees of intensity.

According to the above discussion, fuzzy sets (FS) have become a popular approach for incorporating fuzzy information as part of an expert's evaluative measures. However, as noted by Xu (2005) and Mandal et al. (2022a), experts tend to express their opinions more reliably using linguistic information, known as linguistic sets (LS), rather than fuzzy information. In Mandal and Ranadive (2019), the LS concept was extended to the Pythagorean linguistic set (PLS), allowing experts to express both their confidence and doubt levels when providing judgments. In real-life scenarios, situations, experts often have limited knowledge and personal biases, so using language-based information with expert confidence and doubt levels (Mandal et al. 2021, 2022b, 2023a, b, 2024a) is more flexible for expressing their ideas. The field has evolved, leading to advanced fuzzy models such as the linguistic Z-number (LZN). Proposed by Wang and colleagues in 2017, the LZN combines Zadeh's concept of linguistic variables (1975) with his Z-number model (2011). In this connection, Mandal et al. briefly studied MCGDM under LZN information with the MARCOS approach and its application in the selection of logistic cold chain centers (Mandal et al. 2024b); TODIM-VIKRO approach (Mandal et al. 2024c) and its application in site selection of medical logistic center; ORESTE approach (Mandal et al. 2024d) and its application in assessment of regional energy; and MULTI-MOORA approach (Mandal et al. 2024e) and its application in selection of software. To address the challenge of aggregating individual assessments, specific methods are needed. Although many aggregation strategies have been studied, most assume that decision-makers are independent and uninfluenced by one another. However, in practice, opinions are often influenced by surrounding perspectives. In decision-making, the views of multiple experts can interact, ultimately affecting the final decision, and introducing the concept of 'interference'.

Quantum Probability Theory (QPT) (Busemeyer and Bruza 2012; Moreira and Wichert 2014; Yearsley and Busemeyer 2016) offers a framework for capturing these influences. QPT, a recent advancement over classical probability theory (CPT), has been extensively studied in fields such as psychology (Busemeyer and Bruza 2012), cognitive science (Yearsley and Busemeyer 2016), and decision analysis (He and Jiang 2018a, b; Wu et al. 2020; Xiao et al. 2022; Han and Liu 2023; Mandal et al., 2024f; 2025) due to its robust descriptive power. It effectively explains paradoxes, such as the disjunction fallacy (He and Jiang 2018a, b), the Ellsberg paradox (Eichberger and Pirner 2018), and the order effect (Wang et al. 2014), which are challenging for CPT to address. QPT's ability to represent interference terms makes it particularly valuable for understanding how expert opinions influence each other in the MCGDM process.

Reflecting on the discussed merits of TODIM, PROMETHEE II, and quantum decision theory (QDT) technique, this paper introduces a hybrid MCGDM approach that integrates TODIM with PROMETHEE II using QDT within an LZN framework. This novel method considers the psychological dimensions of expert preferences while aiming to maximize collective utility and minimize personal regret. The main contributions of this paper are:

1. An improved generalized LZNI-based TODIM-PROMETHEE II approach accounts for experts' limited rationality, simplifies the calculation process and resolves some paradoxes of the traditional TODIM-PROMETHEE II method.
2. A quantum-based method for combining individual evaluation results, which helps us better understand human psychology through the effects of interference.
3. A case study demonstrating the effectiveness of our approach in the site selection of renewable energy. Comparative and sensitivity analyses show the flexibility and robustness of our method.

The structure of the paper is as follows: In Sect. 2, recall some basic definitions and the background of linguistic term sets (LTSs), the linguistic Z number (LZN), and quantum Decision Theory (QDT). The proposed MCGDM model is discussed in Sect. 3. A case study for the site selection of renewable energy with comparative analysis is provided in Sect. 4. Section 5 concludes our study.

2 Preliminaries

Here we recall some related definitions and results that are used in this paper.

2.1 LTS

Here we assume two LTSs are: $H = \{h_\alpha | \alpha = -\zeta, \dots, -1, 0, 1, \dots, \zeta\}$ and $G = \{g_\beta | \beta = -\xi, \dots, -1, 0, 1, \dots, \xi\}$ with cardinality $2\zeta + 1$ and $2\xi + 1$ and holds the followings (Mandal et al. 2024a):

- (1) $h_{\alpha_1} \geq h_{\alpha_2}$ if and only if $\alpha_1 \geq \alpha_2$ ($g_{\beta_1} \geq g_{\beta_2}$ if and only if $\beta_1 \geq \beta_2$);
- (2) $neg(h_{\alpha_1}) = h_{-\alpha_1}$ ($neg(g_{\beta_1}) = g_{-\beta_1}$);
- (3) $max(h_{\alpha_1}, h_{\alpha_2}) = h_{\alpha_1}$ if $h_{\alpha_1} \geq h_{\alpha_2}$ ($max(g_{\beta_1}, g_{\beta_2}) = g_{\beta_1}$ if $g_{\beta_1} \geq g_{\beta_2}$);
- (4) $min(h_{\alpha_1}, h_{\alpha_2}) = h_{\alpha_1}$ if $h_{\alpha_1} \leq h_{\alpha_2}$ ($min(g_{\beta_1}, g_{\beta_2}) = g_{\beta_1}$ if $g_{\beta_1} \leq g_{\beta_2}$).

To preserve expert information, the continuous LTSs (Xu 2005) denotes $\bar{H} = \{h_{-t_1} \leq h_\alpha \leq h_{t_1} | \alpha \in [-t_1, t_1]\}$, $t_1 > \zeta$ and $\bar{G} = \{g_{-t_2} \leq g_\beta \leq g_{t_2} | \beta \in [-t_2, t_2]\}$, $t_2 > \xi$.

The subscript index of any $h_\alpha \in \bar{H}$ ($g_\beta \in \bar{G}$), represented by the following functions:

$$I : \bar{H} \rightarrow [-t_1, t_1], \text{ and } I(h_\alpha) = \alpha \left(I : \bar{G} \rightarrow [-t_2, t_2] \text{ and } I(g_\beta) = \beta \right).$$

$$I^{-1} : [-t_1, t_1] \rightarrow \bar{H}, \text{ and } I^{-1}(\alpha) = h_\alpha \left(I^{-1} : [-t_2, t_2] \rightarrow \bar{G}, \text{ and } I^{-1}(\beta) = g_\beta \right).$$

In this paper, we adopt the Sun et al. (2018) transformation formula from LTSs to real-valued $[0, 1]$:

$$T : \bar{H} \rightarrow [0, 1] \text{ and } T(h_\alpha) = \frac{|I(h_\alpha) - I(h_{-t_1})|}{2t_1}$$

$$\left(T : \bar{G}[0, 1] \text{ and } T(g_\beta) = \frac{|I(g_\beta) - I(g_{-t_2})|}{2t_2} \right)$$

The transformation formula for T also can be written in the following way:

$$T(h_\alpha) = 0.5 + \frac{I(h_\alpha)}{2t_1} \left(T(g_\beta) = 0.5 + \frac{I(g_\beta)}{2t_2} \right).$$

2.2 LZN

By Chai et al. (2023), Mandal et al. (2024a), an ordered pair $z = (A, B) = (h_\alpha, g_\beta)$ is called an LZN, where $A \in H$, and $B \in G$. Here B is the relative measure of A .

By Chai et al. (2023), $\kappa(z) = T(A) \times T(B)$ is called an expected value of any LZN $z = (A, B)$.

For any two LZNs, $z_1 = (A_1, B_1) = (h_{\alpha_1}, g_{\beta_1})$ and $z_2 = (A_2, B_2) = (h_{\alpha_2}, g_{\beta_2})$, $A_1, A_2 \in H$, $B_1, B_2 \in G$ we define the following max and min operations rule.

- (1) $Max(z_1, z_2) = \left(h_{\alpha_1}, g_{\frac{\beta_1 + \beta_2}{2}} \right)$ if $h_{\alpha_1} \geq h_{\alpha_2}$;
- (2) $Min(z_1, z_2) = \left(h_{\alpha_2}, g_{\frac{\beta_1 + \beta_2}{2}} \right)$ if $h_{\alpha_1} \geq h_{\alpha_2}$.

For any two $z_1 = (A_1, B_1)$ and $z_2 = (A_2, B_2)$, the distance between them is calculated in the following way (Wang et al. 2017):

$$d(z_1, z_2) = \frac{1}{2} \left(|T(A_1)T(B_1) - T(A_2)T(B_2)| + \max\{|T(A_1) - T(A_2)|, |T(B_1) - T(B_2)|\} \right)$$

2.3 Quantum Probability Theory (QPT)

A quantum decision (QD) model is based on the principles of quantum probability (QP) theory (QPT). To provide context, we will review some key concepts and notations from QPT. In quantum theory (QT), a state is represented as a unit-length vector within an N -dimensional vector space.

Definition 1 Yearsley and Busemeyer (2016) A finite-dimensional Hilbert space (HS), represented as $\mathcal{H}$, is an abstract vector space over a complex field, spanned by a set of orthonormal basis vectors with complex components. In classical probability theory (CPT), events are fundamental (He et al. 2018), but in quantum probability theory (QPT), events are instead represented by subspaces of the Hilbert space $\mathcal{H}$, rather than traditional sets.

Definition 2 Busemeyer and Bruza (2012) In QT, the superposition state $|S\rangle$, a unit-length vector in the N -dimensional Hilbert space $\mathcal{H}$, defines all possible events within that space. The nonlinear combination of a set of orthogonal basis vectors $\{|\mathcal{V}\rangle_k, \quad k = 1, 2, \dots, n\}$ are used to represent the superposition state $|S\rangle$, i.e.,

$$\begin{aligned} |S\rangle &= \langle \mathcal{V}_1 | S \rangle |\mathcal{V}_1\rangle + \langle \mathcal{V}_2 | S \rangle |\mathcal{V}_2\rangle + \dots + \langle \mathcal{V}_n | S \rangle |\mathcal{V}_n\rangle \\ &= \xi_1 |\mathcal{V}_1\rangle + \xi_2 |\mathcal{V}_2\rangle + \dots + \xi_n |\mathcal{V}_n\rangle \\ &= \zeta_1 e^{i\alpha_1} |\mathcal{V}_1\rangle + \zeta_2 e^{i\alpha_2} |\mathcal{V}_2\rangle + \dots + \zeta_n e^{i\alpha_n} |\mathcal{V}_n\rangle \end{aligned} \quad (1)$$

where $\xi_k (= \zeta_k e^{i\alpha_k})$ is the amplitude of the probability, expressed as a complex number, characterizing the amplitude of the wave in the state $|\mathcal{V}_k\rangle$, and the amplitude of the phase angle represented by $e^{i\alpha_k}$.

Definition 3 Busemeyer and Bruza (2012) The CPT framework represents every event using a single set of components. In QPT, probabilities are assigned based on the overlap between cognitive states (shown as state vectors) and different probability measures (shown as subspaces). QPT extends CPT into the complex domain by including incompatible qualities and viewing events as vectors. Incompatible attributes are two observable features of an item that cannot be measured accurately at the same time. Both the incompatibility condition and the uncertainty principle can be described by the probability amplitude, also known as complex probability.

According to Born's rule, an event's probability amplitude and its classical probability, x_i are related to each other:

$$P(\mathcal{V}_k) = |\zeta_k e^{i\alpha_k}|^2 = (\zeta_k e^{i\alpha_k}) \times (\zeta_k e^{i\alpha_k})^* = \zeta_k e^{i\alpha_k} \times \zeta_k e^{-i\alpha_k} = (\zeta_k)^2 \quad (2)$$

For the total event probability to be 1, the sum of the squares of the probability amplitudes must also be 1.

$$\sum_{k=1}^n P(\mathcal{V}_k) = \sum_{k=1}^n (\zeta_k)^2 = 1. \quad (3)$$

An $N - 1$ mapping transforms complex probabilities into classical probabilities. For each given probability value $P(\mathcal{V}_k)$, countless complex probabilities correspond to it. Referencing Eq. (2) and Euler's formula,

$$e^{i\alpha} = \cos\alpha + i \times \sin\alpha \quad (4)$$

the complex probability satisfies:

$$\xi = \sqrt{P(\mathcal{V})}(\cos\alpha + i \times \sin\alpha) = \sqrt{P(\mathcal{V})}e^{i\alpha} \quad (5)$$

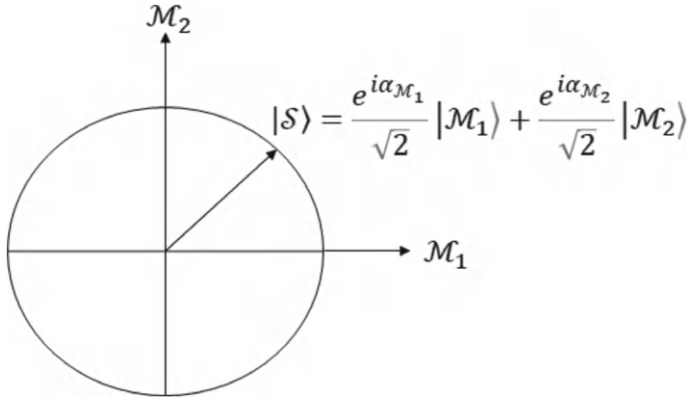


Fig. 1 Event in Hilbert space with two dimensions

For example, in the following, we discussed the QPT in two-dimensional Hilbert space. Let $\{\mathcal{M}_1, \mathcal{M}_2\}$ be an event containing two basic states, where $|\mathcal{M}_1\rangle = (1, 0)$, and $|\mathcal{M}_2\rangle = (0, 1)$ as two basic orthogonal vectors in $\mathbf{H}$. The state $|\mathcal{S}\rangle$ is defined as a superposition and is graphically illustrated in Fig. 1. According to Eqs. (1) and (2), it can be expressed as.

$$|\mathcal{S}\rangle = \zeta_1 e^{i\alpha_1} |\mathcal{M}_1\rangle + \zeta_2 e^{i\alpha_2} |\mathcal{M}_2\rangle = \frac{e^{i\alpha_{\mathcal{M}_1}}}{\sqrt{2}} |\mathcal{M}_1\rangle + \frac{e^{i\alpha_{\mathcal{M}_2}}}{\sqrt{2}} |\mathcal{M}_2\rangle$$

and the probabilities of $|\mathcal{M}_1\rangle$ and $|\mathcal{M}_2\rangle$ are calculated as follows:

$$P(\mathcal{M}_1) = P(\mathcal{M}_2) = (e^{i\alpha})(e^{i\alpha})^* = \frac{1}{2}.$$

According to conventional Bayesian decision theory, decision-making can be mathematically represented as a Markov process (Moreira and Wichert 2014). The state transition probability in a single route of QPT aligns with the CPT according to Feynman's first rule (Wu et al. 2020), which varies over several pathways. Consider the decision-making dilemma highlighted above as an illustration of the process depicted in Fig. 2a. Experts provide their assessments of $\mathcal{M}_1$ ('bad') or $\mathcal{M}_2$ ('good') as probability quantifications. State transition probabilities are determined by combining the individual probabilities for each route and using the total probability formula in CPT.

$$P(\mathcal{M}_1) = \sum_{k=1}^2 P(e_k)P(\mathcal{M}_1|e_k), \quad \sum_{k=1}^2 P(e_k) = 1 \quad (6)$$

Since only one route is selected at a time, it is sometimes referred to as the single path principle. The conventional Markovian framework assumes that experts rely

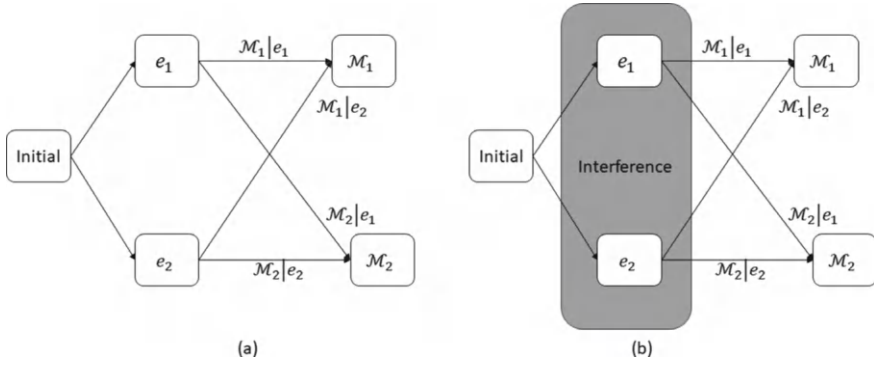


Fig. 2 DM in **a** Markov and **b** quantum process

on one expert at a time and that both experts are completely autonomous. However, this approach is sometimes unsuitable for ordinary cognitive processes because many avenues are often considered when forming judgments. Such phenomena can result in illogical actions, including disjunction, conjunction errors, and interference effects. Thus, QPT is an excellent complement to CPT in this scenario.

Within the QPT framework (Fig. 2b), the state is formed by combining three fundamental states, which can be represented as different wave functions aligning according to Feynman's second rule. The amplitude of the final form is obtained by superimposing two individual routes running from I ('Initial') to $\mathcal{M}_1$. The amplitude is the sum of the individual pathways from source I to target $\mathcal{M}_1$, and the probability of reaching the target $\mathcal{M}_1$ is equal to the square of the amplitude.

$$P(\mathcal{M}_1) = |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 \quad (7)$$

Equation (7) can be simplified in the following way:

$$\begin{aligned} P(\mathcal{M}_1) &= |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 \\ &= |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}| |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^* \\ &= \left(\sqrt{P(e_1)P(\mathcal{M}_1|e_1)}e^{i\alpha_1} + \sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{i\alpha_2} \right) \\ &\quad \left(\sqrt{P(e_1)P(\mathcal{M}_1|e_1)}e^{-i\alpha_1} + \sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{-i\alpha_2} \right) \\ &= P(e_1)P(\mathcal{M}_1|e_1) + P(e_2)P(\mathcal{M}_1|e_2) \\ &\quad + \sqrt{P(e_1)P(\mathcal{M}_1|e_1)}\sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{i(\alpha_1-\alpha_2)} \\ &\quad + \sqrt{P(e_1)P(\mathcal{M}_1|e_1)}\sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{i(\alpha_2-\alpha_1)} \end{aligned} \quad (8)$$

By Eq. (4), we get

$$\cos(\alpha_1 - \alpha_2) = \frac{e^{i(\alpha_2 - \alpha_1)} + e^{i(\alpha_1 - \alpha_2)}}{2} \quad (9)$$

Then Eq. (7) can be written in the following way:

$$\begin{aligned} P(\mathcal{M}_1) &= P(e_1)P(\mathcal{M}_1|e_1) + P(e_2)P(\mathcal{M}_1|e_2) \\ &\quad + 2\sqrt{P(e_1)P(\mathcal{M}_1|e_1)P(e_2)P(\mathcal{M}_1|e_2)} \cos(\alpha_1 - \alpha_2) \\ &= |\xi_{e_1}\xi_{\mathcal{M}_1|e_1}|^2 + |\xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 + 2\xi_{e_1}\xi_{\mathcal{M}_1|e_1}\xi_{e_2}\xi_{\mathcal{M}_1|e_2} \cos(\alpha_1 - \alpha_2) \end{aligned} \quad (10)$$

Upon comparing Eqs. (6) and (10), it becomes evident that the quantum framework includes an interference component. If the pathways are observable, the quantum phase transition will reduce to the classical one, following Feynman's third rule.

$$P(\mathcal{M}_1) = |\xi_{e_1}\xi_{\mathcal{M}_1|e_1}|^2 + |\xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 = P(e_1)P(\mathcal{M}_1|e_1) + P(e_2)P(\mathcal{M}_1|e_2) \quad (11)$$

Additionally, the QPT can be expanded to a multi-path graph, resulting in an increased number of interference terms. The QPT enhances CPT by accurately representing cognitive processes in quantum terms, thereby more effectively modeling ambiguity, vagueness, and uncertainty in decision-making.

3 The Proposed TODIM-PROMETHEE II-Based MAGDM Under Pythagorean Linguistic Information and Quantum Decision Scenario

This section introduces quantum cognition-based MAGDM integrating the TODIM-PROMETHEE II approach.

3.1 Background

Let us assume

$X = \{x_1, x_2, \dots, x_n\}$: Alternative set,

$C = \{c_1, c_2, \dots, c_m\}$: Attribute set,

$e = \{e_1, e_2, \dots, e_q\}$: Expert set,

$w^k = (w_1^k, w_2^k, \dots, w_m^k)$: Weight vector of attributes for expert e_k , where $w_j^k \in$

$[0, 1]$, and $\sum_{j=1}^m w_j^k = 1$,

$\omega = (\omega_1, \omega_2, \dots, \omega_q)$: Weight vector of experts with $\omega_k \in [0, 1]$, $\sum_{k=1}^q \omega_k = 1$.

3.2 Decision Steps

Let $z_{ij}^k = (A_{ij}^k, B_{ij}^k)$ ($i = 1, 2, \dots, n$; $j = 1, 2, \dots, m$; $k = 1, 2, \dots, q$) be a the LZN-based decision information provided by expert e_k using pre-defined distinct LTSs $H = \{h_\alpha | \alpha = -\zeta, \dots, -1, 0, 1, \dots, \zeta\}$ and $G = \{g_\beta | \beta = -\xi, \dots, -1, 0, 1, \dots, \xi\}$.

Step 1: Construction of decision matrix

Then, in the following, we represent the decision matrix of k-th expert:

$$Z^k = (z_{ij}^k)_{nm} = \begin{bmatrix} z_{11}^k = (A_{11}^k, B_{11}^k) & z_{12}^k = (A_{12}^k, B_{12}^k) & \dots & z_{1m}^k = (A_{1m}^k, B_{1m}^k) \\ z_{21}^k = (A_{21}^k, B_{21}^k) & z_{22}^k = (A_{22}^k, B_{22}^k) & \dots & z_{2m}^k = (A_{2m}^k, B_{2m}^k) \\ \vdots & \vdots & \ddots & \vdots \\ z_{n1}^k = (A_{n1}^k, B_{n1}^k) & z_{n2}^k = (A_{n2}^k, B_{n2}^k) & \dots & z_{nm}^k = (A_{nm}^k, B_{nm}^k) \end{bmatrix}. \quad (12)$$

Step 2: Expert weight computation process

Inspire by Chai et al. (2023), Mandal et al. (2024a, b, c, d, e), the weight of experts can be calculated in the following way:

$$\omega_k = \frac{\Omega(Z^k)}{\sum_{k=1}^q \Omega(Z^k)} \quad (13)$$

where $\Omega(Z^k) = \chi \frac{\sum_{j=1}^m \sum_{i=1}^n T(A_{ij}^k)}{mn} + (1 - \chi) \frac{\sum_{j=1}^m \sum_{i=1}^n T(B_{ij}^k)}{mn}$.

Step 3: Attribute weight computation process

(1) The decision matrix (Eq. (12)) can be normalized to $L^k = (l_{ij}^k)_{nm}$ with

$$l_{ij}^k = \begin{cases} \frac{\kappa(z_{ij}^k)}{\sum_{i=1}^n \kappa(z_{ij}^k)}, & \text{for benefit criteria } c_j, \\ 1 - \frac{\kappa(z_{ij}^k)}{\sum_{i=1}^n \kappa(z_{ij}^k)}, & \text{for cost criteria } c_j. \end{cases} \quad (14)$$

(2) The Shannon entropy $\Psi^k = (\Psi_1^k, \Psi_2^k, \dots, \Psi_m^k)$ concerning each expert e_k ($k = 1, 2, \dots, q$) can be calculated in the following way:

$$\Psi_j^k = -\frac{1}{\ln n} \sum_{i=1}^n l_{ij}^k \ln l_{ij}^k \quad (15)$$

$l_{ij}^k \ln l_{ij}^k$ is 0 if $l_{ij}^k = 0$.

- (3) The attribute weight $w^k = (w_1^k, w_2^k, \dots, w_m^k)$ concerning each expert $e_k (k = 1, 2, \dots, q)$ can be calculated in the following way:

$$w_j^k = \frac{1 - \Psi_j^k}{\sum_{j=1}^m (1 - \Psi_j^k)} \quad (16)$$

Step 4: Construction of Bayesian network

In this phase, we converted the pre-collected data into probabilities and developed a Bayesian Network for decision-making, as the quantum framework is a partial extension of the Bayesian Network. Figure 3 illustrates that, in a MAGDM situation, the decision-making process can be represented as a simple two-layer Bayesian Network. The experts' weights are converted into probabilities, which indicate the degree of belief in the experts, using the following equation:

$$P(e_k) = \omega_k, \quad k = 1, 2, \dots, q. \quad (17)$$

In Eq. (17), the probabilities form the initial layer of the Bayesian Network (BN). The second layer depends on the assessment outcomes from each expert, which can be calculated using the TODIM-PROMETHEE II approach as follows:

(1) Attribute relative weight computation process

Let $w_j^{k,*}$ be the relative weight of the attribute c_j concerning each expert $e_k (k = 1, 2, \dots, q)$, where

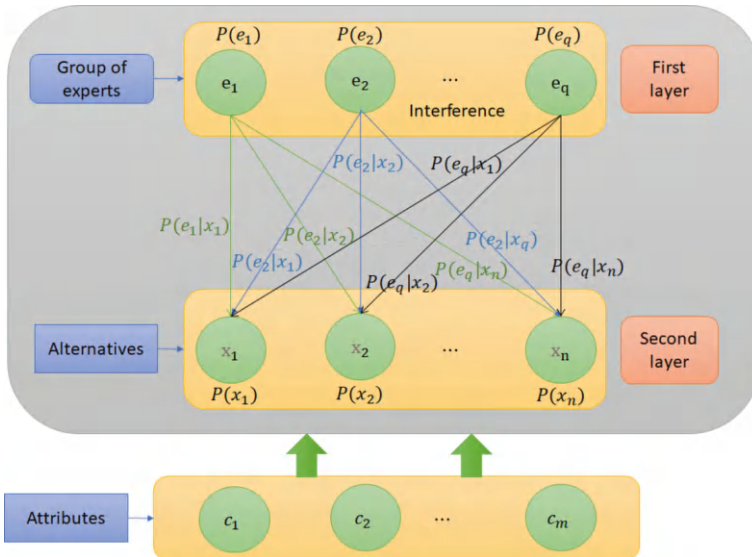


Fig. 3 MAGDM according to QSBN

$$w_j^{k,*} = \frac{w_j^k}{w_{max}^k}, \quad w_{max}^k = \max_{j=1}^m w_j^k \quad (18)$$

(2) Dominance degree matrix construction process

The dominant degree of x_i over x_s for the attribute c_j concerning each expert e_k ($k = 1, 2, \dots, q$) is denoted by $\phi_j^k(x_i, x_s)$ and defined in the following way:

$$\phi_j^k(x_i, x_s) = \begin{cases} \sqrt{\frac{w_j^{k,*} \times d(z_{ij}^k, z_{sj}^k)}{\sum_{j=1}^m w_j^{k,*}}} & \text{if } \kappa(z_{ij}^k) > \kappa(z_{sj}^k) \\ 0 & \text{if } \kappa(z_{ij}^k) = \kappa(z_{sj}^k) \\ -\frac{1}{\theta} \sqrt{\frac{\left(\sum_{j=1}^m w_j^{k,*}\right) \times d(z_{ij}^k, z_{sj}^k)}{w_j^*}} & \text{if } \kappa(z_{ij}^k) < \kappa(z_{sj}^k) \end{cases} \quad (19)$$

where θ is a risk factor of gain and losses. $\kappa(z_{ij}^k) > \kappa(z_{sj}^k)$ means $\phi_j^k(x_i, x_s)$ are gains and $\kappa(z_{ij}^k) < \kappa(z_{sj}^k)$ means $\phi_j^k(x_i, x_s)$ are losers. Based on Eq. (19), the dominance degree matrix ϕ_j^k ($j = 1, 2, \dots, m$) for c_j concerning each expert e_k ($k = 1, 2, \dots, q$) is as follows:

$$\phi_j^k = \left(\phi_j^k(x_i, x_s) \right)_{n \times n} = \begin{matrix} & \begin{matrix} x_1 & x_2 & \dots & x_n \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} & \begin{bmatrix} 0 & \phi_j^k(x_1, x_2) & \dots & \phi_j^k(x_1, x_n) \\ \phi_j^k(x_2, x_1) & 0 & \dots & \phi_j^k(x_2, x_n) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_j^k(x_n, x_1) & \phi_j^k(x_n, x_2) & \dots & 0 \end{bmatrix} \end{matrix} \quad (20)$$

(3) Overall dominance matrix construction process

The overall dominance degree of x_i concerning remaining alternatives for c_j of expert e^k is denoted by $\eta^k(x_i, x_s)$ and defined in the following way:

$$\phi^k(x_i, x_s) = \sum_{j=1}^m w_j^k \phi_j^k(x_i, x_s) \quad (i = 1, 2, \dots, n; j = 1, 2, \dots, m) \quad (21)$$

Based on Eq. (21), the overall dominance matrix η^k concerning each expert e_k ($k = 1, 2, \dots, q$) is as follows:

$$\eta^k = (\eta^k(x_i, x_s))_{n \times n} = \begin{matrix} & \begin{matrix} x_1 & x_2 & \cdots & x_n \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} & \begin{bmatrix} \phi^k(x_1, x_1) & \phi^k(x_1, x_2) & \cdots & \phi^k(x_1, x_n) \\ \phi^k(x_2, x_1) & \phi^k(x_2, x_2) & \cdots & \phi^k(x_2, x_n) \\ \vdots & \vdots & \ddots & \vdots \\ \phi^k(x_n, x_1) & \phi^k(x_n, x_2) & \cdots & \phi^k(x_n, x_n) \end{bmatrix} \end{matrix} \quad (22)$$

(4) Positive and negative flows of each alternative computation process

The positive and negative flows of each alternative for expert e^k are denoted by $^{k,+}(x_i)$ and $^{k,-}(x_i)$, ($i = 1, 2, \dots, n$) and calculated in the following way:

$$^{k,+}(x_i) = \sum_{s=1}^n \eta^k(x_i, x_s) \quad (23)$$

$$^{k,-}(x_i) = \sum_{s=1}^n \eta^k(x_s, x_i) \quad (24)$$

(5) Net flow of each alternative computation process

Based on Eqs. (23) and (24), we have the net flow of each alternative for expert e^k is as follows:

$$\eta_i^k = |^{k,+}(x_i) - ^{k,-}(x_i)| \quad (25)$$

(6) Based on Eq. (25), conditional probabilities which link the two layers in the BN as:

$$P(x_i|e_k) = \frac{\eta_i^k}{\sum_{i=1}^n \eta_i^k}. \quad (26)$$

(7) The second layer consists of the probabilities of alternatives calculated as:

$$P(x_i) = \sum_{k=1}^q P(e_k)P(x_i|e_k) \quad (27)$$

Subsequently, we can decide based on the $P(x_i)$ values, as higher probabilities, indicate a superior ranking. The classical approach does not account for the interference effect, which can influence the final rankings. To address this issue, Bayesian networks are expanded into a quantum framework capable of properly modeling a practical approach to real interference in MAGDM.

Step 5: Construction of quantum scenario-based Bayesian network (QSBN)

The facilitator often consolidates opinions from several specialists. The interconnected decision-making processes in real-world situations, as demonstrated in Sect. 2.2, remain unobservable. The square of the sum of the amplitude probabilities of all possible pathways collectively constitutes the probability of the alternative x_i from QPT (He et al. 2018), as described.

$$P(x_i) = \left| \sum_{k=1}^q \xi_{e_k} \xi_{x_i|e_k} e^{i\alpha_k} \right|^2 = \left| \sum_{k=1}^q \sqrt{P(e_k)} \sqrt{P(x_i|e_k)} e^{i\alpha_k} \right|^2, \quad i = 1, 2, \dots, n \quad (28)$$

By Eqs. (7)–(11), Eq. (28) can be written as

$$\begin{aligned} P(x_i) &= \sum_{k=1}^q P(e_k)P(x_i|e_k) + 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \cos(\alpha_k - \alpha_s) \\ &= \sum_{k=1}^q P(e_k)P(x_i|e_k) + INT_i \end{aligned} \quad (29)$$

In a quantum-based Bayesian network, a normalization factor is incorporated into Eq. (29) to standardize the probability of each alternative, ensuring that the overall estimated probability does not exceed 1. Then

$$Prob(x_i) = \sigma P(x_i) \quad (30)$$

where,

$$\sigma = \frac{1}{\sum_{i=1}^n P(x_i)} = \frac{1}{\sum_{i=1}^n (\sum_{k=1}^q P(e_k)P(x_i|e_k) + INT_i)} \quad (31)$$

3.3 Analysis of Interference Effect Among Experts' Opinions

Equation (31) uses the term INT_i , which represent the ambiguity and uncertainty of expert opinions, to explain the superposition of decision information. In Eq. (29), $\cos(\alpha_k - \alpha_s)$ represents the phase angle between two distinct experts e_k and e_s , which represents the connection vector between two experts. $\frac{q(q-1)}{2}$ is the total count of pairwise correspondences across q experts, excluding duplicates. The final interference value is derived by summing the computed $\frac{q(q-1)}{2}$ interference values for each expert pair, rather than assessing the overall uncertainty across experts. This approach enhances the accuracy of the estimated probability value.

In real-world situations, it is hard to measure the phase angle. Experts' opinions can differ or be influenced by outside factors, causing a lot of variation and mistakes. Belief entropy helps a lot in handling and combining unclear information, making it easier to make good decisions in group settings with many factors. It is used as a standard in areas like risk assessment, defect diagnosis, and pattern recognition. Belief entropy is the latest way to measure interference in QSBN. Unlike the old method of manually estimating the phase angle range, this new method is more reliable because it reduces errors from personal judgment, which can affect the final decision and the expert's probability calculation. Belief entropy is used in QSBN to manage the interference among different experts' beliefs in GDM.

Then the belief distance about alternative x_i according to the average probability $\nabla = \frac{1}{nq}$ can be calculated in the following way:

$$D_{iu} = \left| \delta_{iu} + \frac{|\delta_{iu} - \nabla| - |\varrho_{iu} - \nabla|}{|\delta_{iu} - \nabla| + |\varrho_{iu} - \nabla|} \right|, \quad i = 1, 2, \dots, n; \quad u = 1, 2, \dots, \frac{q(q-1)}{2},$$

$$\begin{cases} \delta_{iu} = P(e_k)P(x_i|e_k) \\ \varrho_{iu} = P(e_s)P(x_i|e_s) \end{cases}, \quad i = 1, 2, \dots, n;$$

$$k = 1, 2, \dots, q-1; \quad s = k+1, k+2, \dots, \frac{q(q-1)}{2}.$$
(32)

In Eq. (32), u means the value reallocated to the correlation among q experts; $\nabla = \frac{1}{nq}$ represent each expert probability value of each alternative; D_{iu} means deviation degree from ∇ ; $|\delta_{iu} - \nabla|$ and $|\varrho_{iu} - \nabla|$ represents two experts' magnitude of the deviation degree from ∇ about alternative x_i . If $|\delta_{iu} - \nabla| < |\varrho_{iu} - \nabla|$, then the position of δ_{iu} and ϱ_{iu} are interchangeable.

There are many ways to measure uncertainty. Deng entropy (Zhao et al. 2024), also known as belief entropy, is better at including the confidence function and the uncertainty of BPA because it measures both total non-specificity and disagreement.

Thus, by Deng's entropy, we get

$$\vartheta_{iu}(D_{iu}) = -D_{iu} \log_2 \frac{D_{iu}}{2^m - 1}, \quad i = 1, 2, \dots, n; \quad u = 1, 2, \dots, \frac{q(q-1)}{2}. \quad (33)$$

To ensure that the values of Deng entropy align with the range of $\cos(\alpha_k - \alpha_s)$, which is between $[-1, 1]$, Deng entropy is standardized.

$$\beta_{iu}(D_{iu}) = 2 \frac{\vartheta_{iu}(D_{iu}) - \min\{\vartheta_{iu}(D_{iu})\}}{\max\{\vartheta_{iu}(D_{iu})\} - \min\{\vartheta_{iu}(D_{iu})\}} - 1,$$

$$i = 1, 2, \dots, n; \quad u = 1, 2, \dots, \frac{q(q-1)}{2}.$$
(34)

Thus, the phase angle $\cos(\alpha_k - \alpha_s)$ can be determined by the following was:

$$\cos(\alpha_k - \alpha_s) = \beta_{iu}(D_{iu}), i = 1, 2, \dots, n; \quad u = 1, 2, \dots, \frac{q(q-1)}{2}. \quad (35)$$

Therefore, by Eq. (35), in Eq. (29), INT_i can be written as

$$\begin{aligned} INT_i &= 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \cos(\alpha_k - \alpha_s) \\ &= 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \beta_{iu}(D_{iu}) \end{aligned} \quad (36)$$

By Eq. (36), the Eq. (29) can be written as

$$P(x_i) = \sum_{k=1}^q P(e_k)P(x_i|e_k) + 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \beta_{iu}(D_{iu}) \quad (37)$$

Finally, the probability of each alternative x_i can be obtained in the following way:

$$Prob(x_i) = \sigma P(x_i) \quad (38)$$

where,

$$\sigma = \frac{1}{\sum_{i=1}^n P(x_i)} = \frac{1}{\sum_{i=1}^n (\sum_{k=1}^q P(e_k)P(x_i|e_k) + INT_i)} \quad (39)$$

Step 6: Ranking of alternative

According to Eq. (38), the higher probability of an alternative means the best alternative.

4 A Practical Example with Comparative Analysis

This section demonstrates the viability of the proposed TODIM-PROMETHEE II-based MCGDM approach, which integrates QDT within LZNI, through a case study on renewable energy selection. The detailed procedure for the combined TODIM-PROMETHEE II approach is presented below.

4.1 A Case Study

Multiple scientific studies demonstrate that India is a premier global site for renewable energy production due to its favorable natural conditions. Our proposed methodology was applied to identify a location for a wind power facility. Five cities were prioritized for this purpose: x_1 : Madhya Pradesh, x_2 : Maharashtra, x_3 : Rajasthan, x_4 : Gujrat, and x_5 : Panjab. After a thorough review of previous studies, four criteria were defined: c_1 : environmental conditions, c_2 : economic factors, c_3 : technical possibilities, and c_4 site characteristics. A panel of five experts $e_1, e_2, \dots, e_5$ initially evaluated these criteria using the LZNs, which are derived from the LTSs $H = \{h_{-3}$: very low importance, h_{-2} : low importance, h_{-1} : slightly less importance, h_0 : equal importance, h_1 : slightly more importance, h_2 : high importance, h_3 : very high importance} and $G = \{g_{-2}$: not sure, g_{-1} : slightly not sure, g_0 : fair, g_1 : slightly sure, g_2 : sure}. Figure 4 illustrates the decision tree of the site selection of renewable energy. The following methods are applied to evaluate this MAGDM scenario using our proposed TODIM-PROMETHEE II method integrating QDT with LZNI for assessing the available options.

Step 1: The evaluation information is shown in the Eqs. (40)–(44).

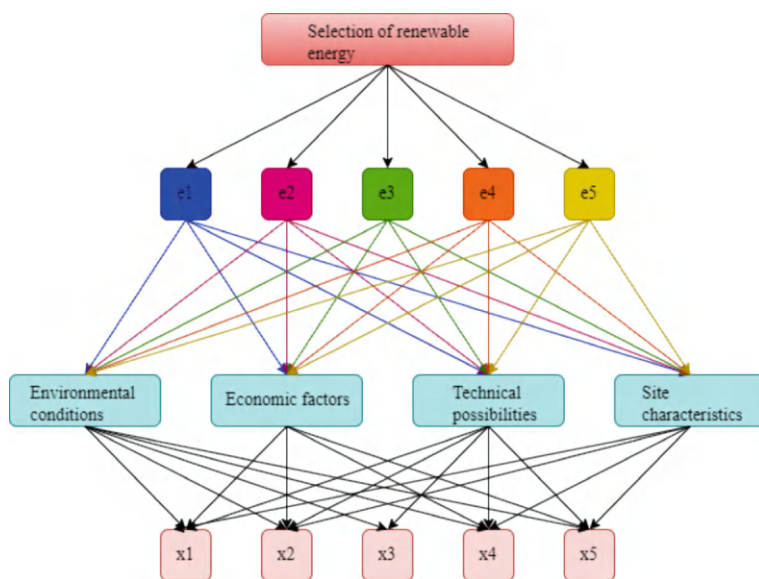


Fig. 4 The decision tree of the site selection of renewable energy

$$Z^1 = \begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} (h_0, g_1) & (h_2, g_1) & (h_3, g_1) & (h_{-1}, g_0) \\ (h_2, g_{-1}) & (h_1, g_{-1}) & (h_2, g_0) & (h_{-2}, g_1) \\ (h_3, g_2) & (h_2, g_2) & (h_3, g_2) & (h_1, g_{-1}) \\ (h_{-1}, g_0) & (h_2, g_0) & (h_{-2}, g_1) & (h_3, g_1) \\ (h_2, g_2) & (h_1, g_2) & (h_0, g_1) & (h_2, g_1) \end{bmatrix} \end{matrix} \quad (40)$$

$$Z^2 = \begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} (h_1, g_{-1}) & (h_1, g_2) & (h_1, g_0) & (h_1, g_1) \\ (h_{-2}, g_0) & (h_{-2}, g_1) & (h_{-2}, g_1) & (h_{-2}, g_{-1}) \\ (h_3, g_1) & (h_3, g_2) & (h_3, g_1) & (h_3, g_2) \\ (h_{-2}, g_2) & (h_1, g_0) & (h_{-1}, g_1) & (h_1, g_0) \\ (h_3, g_2) & (h_2, g_2) & (h_3, g_2) & (h_2, g_1) \end{bmatrix} \end{matrix} \quad (41)$$

$$Z^3 = \begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} (h_{-2}, g_0) & (h_2, g_2) & (h_1, g_1) & (h_1, g_2) \\ (h_1, g_0) & (h_{-2}, g_{-1}) & (h_{-2}, g_0) & (h_3, g_1) \\ (h_3, g_2) & (h_2, g_1) & (h_2, g_2) & (h_2, g_0) \\ (h_{-2}, g_2) & (h_1, g_0) & (h_{-1}, g_1) & (h_{-2}, g_{-2}) \\ (h_{-2}, g_1) & (h_0, g_2) & (h_{-2}, g_{-1}) & (h_1, g_2) \end{bmatrix} \end{matrix} \quad (42)$$

$$Z^4 = \begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} (h_{-2}, g_1) & (h_2, g_1) & (h_1, g_1) & (h_{-1}, g_1) \\ (h_{-2}, g_{-2}) & (h_{-2}, g_{-1}) & (h_0, g_1) & (h_{-1}, g_{-1}) \\ (h_2, g_3) & (h_3, g_1) & (h_2, g_1) & (h_2, g_1) \\ (h_{-1}, g_{-1}) & (h_2, g_1) & (h_3, g_0) & (h_3, g_{-1}) \\ (h_2, g_1) & (h_1, g_0) & (h_2, g_1) & (h_{-1}, g_1) \end{bmatrix} \end{matrix} \quad (43)$$

$$Z^5 = \begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} (h_1, g_2) & (h_2, g_{-1}) & (h_3, g_1) & (h_{-1}, g_2) \\ (h_{-2}, g_1) & (h_1, g_2) & (h_{-2}, g_2) & (h_{-2}, g_0) \\ (h_2, g_{-1}) & (h_{-2}, g_{-1}) & (h_3, g_1) & (h_1, g_{-1}) \\ (h_{-2}, g_2) & (h_1, g_2) & (h_1, g_{-1}) & (h_{-2}, g_1) \\ (h_2, g_1) & (h_3, g_1) & (h_{-1}, g_{-1}) & (h_{-2}, g_0) \end{bmatrix} \end{matrix} \quad (44)$$

Step 2: Suppose the expert risk parameter $\chi = 0.5$, then by Eq. (13), we get the expert weights as: $\omega_1 = 0.2112$, $\omega_2 = 0.2190$, $\omega_3 = 0.1923$, $\omega_4 = 0.1930$, and $\omega_5 = 0.1845$.

Step 3: By Eq. (16), the weight of attributes for all experts are:

$$w_1^1 = 0.2982, w_2^1 = 0.1065, w_3^1 = 0.2411, w_4^1 = 0.3542;$$

$$w_1^2 = 0.2880, w_2^2 = 0.1912, w_3^2 = 0.2542, w_4^2 = 0.2666;$$

$$w_1^3 = 0.3474, w_2^3 = 0.1764, w_3^3 = 0.3213, w_4^3 = 0.1549;$$

$$w_1^4 = 0.5433, w_2^4 = 0.2308, w_3^4 = 0.0192, w_4^4 = 0.2066;$$

$$w_1^5 = 0.2351, w_2^5 = 0.2609, w_3^5 = 0.3398, w_4^5 = 0.1642.$$

Step 4: According to Eq. (17), we get $P(e_1) = 0.2112$, $P(e_2) = 0.2190$, $P(e_3) = 0.1923$, $P(e_4) = 0.1930$, and $P(e_5) = 0.1845$.

(1) By Eq. (18), the relative weight of each attribute for all experts is:

$$w_{max}^1 = 0.3542, w_1^{1,*} = 0.8419, w_2^{1,*} = 0.3006, w_3^{1,*} = 0.6807, w_4^{1,*} = 1;$$

$$w_{max}^2 = 0.2880, w_1^{2,*} = 1, w_2^{2,*} = 0.6639, w_3^{2,*} = 0.8824, w_4^{2,*} = 0.9255;$$

$$w_{max}^3 = 0.3474, w_1^{3,*} = 1, w_2^{3,*} = 0.5078, w_3^{3,*} = 0.9250, w_4^{3,*} = 0.4458;$$

$$w_{max}^4 = 0.5433, w_1^{4,*} = 1, w_2^{4,*} = 0.4247, w_3^{4,*} = 0.0354, w_4^{4,*} = 0.3803;$$

$$w_{max}^5 = 0.3398, w_1^{5,*} = 0.6920, w_2^{5,*} = 0.7679, w_3^{5,*} = 1, w_4^{5,*} = 0.4833.$$

(2) and (3) Let $\theta = 2$. Then by Eq. (22), using Eqs. (19) and (20), the overall dominance matrix η^k ($k = 1, 2, 3, 4, 5$) for all experts is shown in Eqs. (45)–(49).

$$\eta^1 = \begin{matrix} & \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} 0 & 0.2625 & -0.3275 & -0.0400 & -0.3908 \\ -0.5168 & 0 & -0.6882 & -0.1690 & -0.8817 \\ 0.1813 & 0.3639 & 0 & 0.0481 & 0.5558 \\ -0.2518 & -0.1147 & -0.3863 & 0 & -1.0800 \\ 0.0237 & 0.2122 & -0.2120 & 0.0889 & 0 \end{bmatrix} \end{matrix} \quad (45)$$

$$\eta^2 = \begin{matrix} & \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} 0 & 0.3154 & -0.6447 & -0.0370 & -0.6191 \\ -0.6314 & 0 & -0.8941 & -0.5860 & -0.8694 \\ 0.3294 & 0.4508 & 0 & 0.3374 & -0.1491 \\ -0.2518 & 0.2991 & -0.6792 & 0 & -0.6560 \\ 0.3221 & 0.4412 & -0.0922 & 0.3278 & 0 \end{bmatrix} \end{matrix} \tag{46}$$

$$\eta^3 = \begin{matrix} & \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} 0 & -0.0819 & -0.3559 & 0.0421 & 0.0563 \\ -0.2231 & 0 & -0.5967 & -0.0900 & 0.0560 \\ 0.0689 & 0.2620 & 0 & 0.4035 & 0.2641 \\ -0.3541 & -0.2132 & -0.7289 & 0 & -0.0978 \\ -0.2392 & -0.3158 & -0.5884 & -0.1880 & 0 \end{bmatrix} \end{matrix} \tag{47}$$

$$\eta^4 = \begin{matrix} & \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} 0 & 0.3592 & -0.5741 & 0.2084 & -0.2503 \\ -0.5466 & 0 & -0.7360 & -0.4940 & -0.6112 \\ 0.4376 & 0.5182 & 0 & 0.4525 & 0.3217 \\ -0.1918 & 0.2790 & -0.5913 & 0 & -0.2383 \\ 0.1819 & 0.4421 & -0.4739 & 0.1652 & 0 \end{bmatrix} \end{matrix} \tag{48}$$

$$\eta^5 = \begin{matrix} & \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} 0 & 0.0912 & 0.2197 & 0.0597 & 0.0617 \\ -0.4407 & 0 & -0.3927 & -0.1700 & -0.1739 \\ -0.4905 & 0.0607 & 0 & 0.0403 & -0.1907 \\ -0.3793 & 0.0689 & -0.3620 & 0 & -0.1826 \\ -0.3584 & -0.0402 & -0.1532 & -0.0640 & 0 \end{bmatrix} \end{matrix} \tag{49}$$

(4) and (5) By Eqs. (23), (24), and (25), we calculate the positive, negative, and net flow of each alternative for all experts. The details results are shown in Tables 1, 2, 3, 4 and 5.

(6) By Eq. (26) we calculate the conditional probabilities of each alternative for each expert. The details results are shown in Table 6.

Table 1 The output results of $\eta^{1,+}$, $\eta^{1,-}$, and η_i^1

Alternatives	$\eta^{1,+}(x_i)$	$\eta^{1,-}(x_i)$	η_i^1
x_1	-0.3908	-0.5636	0.1728
x_2	-1.8817	0.7239	2.6056
x_3	0.5558	-1.6141	2.1699
x_4	-1.0800	-0.0716	1.0084
x_5	0.1127	-1.1586	1.2713

Table 2 The output results of $\eta^{2,+}$, $\eta^{2,-}$, and η_i^2

Alternatives	$\eta^{2,+}(x_i)$	$\eta^{2,-}(x_i)$	η_i^2
x_1	-0.9854	-0.2318	0.7536
x_2	-2.9812	1.5064	4.4876
x_3	0.9685	-2.3102	3.2787
x_4	-1.2880	0.0421	1.3301
x_5	0.9989	-2.2937	3.2926

Table 3 The output results of $\eta^{3,+}$, $\eta^{3,-}$, and η_i^3

Alternatives	$\eta^{3,+}(x_i)$	$\eta^{3,-}(x_i)$	η_i^3
x_1	-0.3394	-0.7476	0.4082
x_2	-0.8535	-0.3489	0.5046
x_3	0.9985	-2.2660	3.2645
x_4	-1.3940	0.1680	1.5620
x_5	-1.3274	0.2786	1.6060

Table 4 The output results of $\eta^{4,+}$, $\eta^{4,-}$, and η_i^4

Alternatives	$\eta^{4,+}(x_i)$	$\eta^{4,-}(x_i)$	η_i^4
x_1	-0.2568	-0.1190	0.1379
x_2	-2.3881	1.5985	3.9866
x_3	1.7300	-2.3752	4.1052
x_4	-0.7424	0.3318	1.0742
x_5	0.3153	-0.7781	1.0934

Table 5 The output results of $\eta^{5,+}$, $\eta^{5,-}$, and η_i^5

Alternatives	$\eta^{5,+}(x_i)$	$\eta^{5,-}(x_i)$	η_i^5
x_1	0.4323	-1.6690	2.1013
x_2	-1.1773	0.1806	1.3578
x_3	-0.5802	-0.6882	0.1080
x_4	-0.8550	-0.1340	0.7210
x_5	-0.6160	-0.4855	0.1304

(7) By Eq. (27), the probabilities of alternatives are:

$P(x_1) = 0.1186$, $P(x_2) = 0.2948$, $P(x_3) = 0.2842$, $P(x_4) = 0.1426$ and $P(x_5) = 0.1598$.

Table 6 The conditional probabilities of each alternative

$P(x_i e_k)$	e_1	e_2	e_3	e_4	e_5
x_1	0.0239	0.3605	0.3002	0.1395	0.1759
x_2	0.0573	0.3415	0.2495	0.1012	0.2505
x_3	0.0556	0.0687	0.4444	0.2127	0.2186
x_4	0.0133	0.3834	0.3948	0.1033	0.1052
x_5	0.4756	0.3073	0.0244	0.1632	0.0295

Step 5: By Eq. (32), the belief distance between two experts about each alternative is calculated, which is presented in Table 7.

By Eq. (33), the Deng entropy between two experts about each alternative is shown in Table 8.

Table 7 The belief distance between two experts about each alternative

D_{ui}	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
$u = 1(e_1, e_2)$	0.0962	0.1028	0.5499	0.1214	0.4886
$u = 2(e_1, e_3)$	0.0720	0.0933	0.3884	0.4154	0.2731
$u = 3(e_1, e_4)$	0.0219	0.1187	0.2599	0.1588	0.3587
$u = 4(e_1, e_5)$	0.0920	0.6675	0.4816	0.0455	0.5150
$u = 5(e_2, e_3)$	0.0118	0.1204	0.7146	0.5295	0.1863
$u = 6(e_2, e_4)$	0.1052	0.0907	0.6451	0.0162	0.6640
$u = 7(e_2, e_5)$	0.0084	0.6485	0.7607	0.1887	0.7485
$u = 8(e_3, e_4)$	0.6224	1.3571	1.4135	2.4340	2.5757
$u = 9(e_3, e_5)$	0.2754	3.1090	0.3290	1.8224	2.9594
$u = 10(e_4, e_5)$	0.8149	2.9030	1.2117	1.3568	1.1468

Table 8 The Deng entropy between two experts about each alternative

$\vartheta_{ui}(D_{ui})$	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
$u = 1(e_1, e_2)$	0.7005	0.7389	2.6228	0.8439	2.4137
$u = 2(e_1, e_3)$	0.5544	0.6839	2.0474	2.1495	1.5784
$u = 3(e_1, e_4)$	0.2063	0.8284	1.5207	1.0418	1.9321
$u = 4(e_1, e_5)$	0.6763	2.9970	2.3893	0.3808	2.5050
$u = 5(e_2, e_3)$	0.1215	0.8383	3.1383	2.5543	1.1795
$u = 6(e_2, e_4)$	0.7529	0.6682	2.9284	0.1600	2.9864
$u = 7(e_2, e_5)$	0.0906	2.9388	3.2722	1.1914	3.2370
$u = 8(e_3, e_4)$	0.6224	1.3571	1.4135	2.4340	2.5757
$u = 9(e_3, e_5)$	0.2754	3.1090	0.3290	1.8224	2.9594
$u = 10(e_4, e_5)$	0.8149	2.9030	1.2117	1.3568	1.1468

Table 9 The standardized Deng entropy between two experts about each alternative

$\beta_{ui}(D_{ui})$	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
$u = 1(e_1, e_2)$	0.6842	-0.9421	0.5588	-0.4287	0.2122
$u = 2(e_1, e_3)$	0.2807	-0.9872	0.1678	0.6618	-0.5870
$u = 3(e_1, e_4)$	-0.6806	-0.8687	-0.1902	-0.2635	-0.2486
$u = 4(e_1, e_5)$	0.6171	0.9082	0.4001	-0.8156	0.2996
$u = 5(e_2, e_3)$	-0.9149	-0.8606	0.9090	1.0000	-0.9687
$u = 6(e_2, e_4)$	0.8286	-1.0000	0.7664	-1.0000	0.7602
$u = 7(e_2, e_5)$	-1.0000	0.8605	1.0000	-0.1385	1.0000
$u = 8(e_3, e_4)$	0.4684	-0.4355	-0.2630	0.8995	0.3672
$u = 9(e_3, e_5)$	-0.4898	1.0000	-1.0000	0.3886	0.7344
$u = 10(e_4, e_5)$	1.0000	0.8312	-0.4001	-0.0003	-1.0000

By Eq. (34), the standardized Deng entropy between two experts about each alternative is shown in Table 9.

By Eq. (36), the interface effect among experts for each alternative is:

$INT_1 = -0.0372$, $INT_2 = -0.1714$, $INT_3 = 0.2355$, $INT_4 = 0.0523$, and $INT_5 = -0.0175$.

Finally, by Eq. (38), the probability of each alternative is:

$Prob(x_1) = 0.0767$, $Prob(x_2) = 0.1162$, $Prob(x_3) = 0.4895$, $Prob(x_4) = 0.1836$, and $Prob(x_5) = 0.1341$.

Step 6: According to the value of QP of each alternative, the final ranking is

$$x_3 \succ x_4 \succ x_5 \succ x_2 \succ x_1.$$

4.2 Result Discussion

In this section, we discussed the validation of our numerical results through sensitive analysis and some comparative analysis.

4.2.1 Effect of the Quantum Interface on Decision Result

If we can decide based on the $P(x_i)$ ($i = 1, 2, 3, 4, 5$) values according to Eq. (27), as higher probabilities indicate a superior ranking. The classical approach does not account for the interference effect, which can influence the final rankings, as shown in Fig. 5. The ranking result of alternatives according to Eq. (27) is $x_2 \succ x_3 \succ x_5 \succ x_4 \succ x_1$. If we count the interface effect between two experts about an alternative the ranking result is $x_3 \succ x_4 \succ x_5 \succ x_2 \succ x_1$.

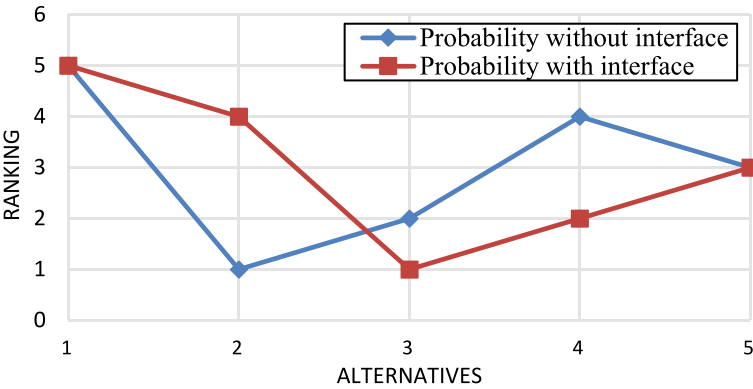


Fig. 5 The interface effect on ranking values

4.2.2 Effect of the Relative Information on Decision Result

In this section, first, we examine how much effect the relative measure of primary information has on decision results. By Chai et al. (2023), if the relative measure is g_2 of all experts, LZNI of all experts are degenerates to linguistic information (LI). Then we rank the alternatives of our proposed method for LZNI and LI, the details of the comparison results are shown in Fig. 6.

From 6, we can observe that if we avoid the relative measure of primary information, the ranking order of the site selection of reenable energy is different from the ranking order of the site selection of reenable energy if we consider the relative

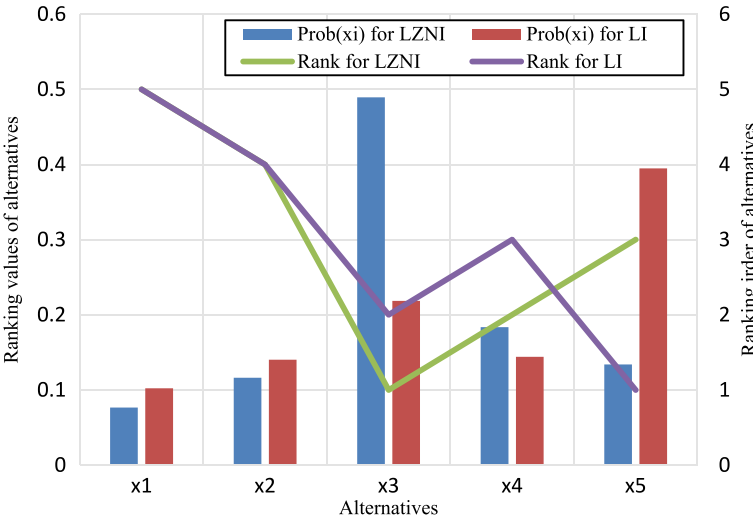


Fig. 6 Output of decision results for LZNI and LI

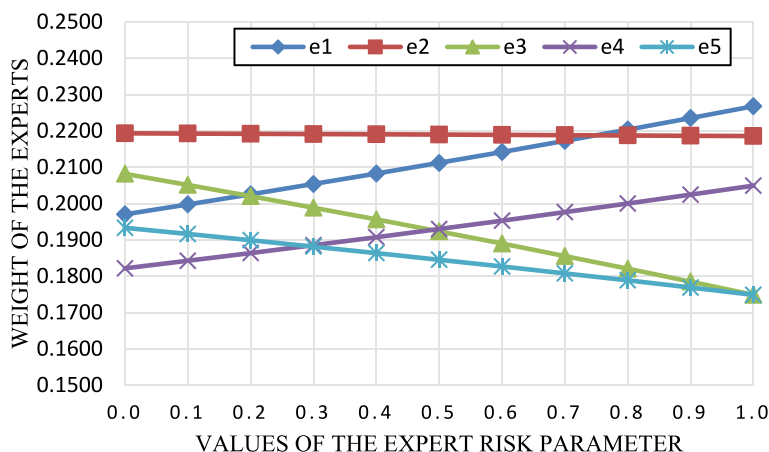


Fig. 7 The impact of χ on the weight of experts

measure of primary information. It demonstrates that the LZNI significant effect on decision results. Thus, we cannot neglect the relative measure of primary information.

4.2.3 Sensitive Analysis

Our proposed quantum TODIM-PROMETHEE II-based MCGDM approach under the LZN environment involved expert risk parameter $\chi \in [0, 1]$. This parameter is mainly used to obtain the weight of experts from their judgment decision matrix. In Fig. 7 we display the impact of χ on the weight of experts. In Fig. 7, we can observe that the increase of expert risk parameters χ , the weight of experts e_1 and e_4 are increases and the weight of expert e_2 , e_3 and e_5 are decreased. Mandal et al. (2021), pointed out that experts' weight has a great impact on decision results. In Fig. 8, we display how much effect the changes of the weight of experts for changes of χ on decision results. From Figs. 7 and 8, we can observe that the expert risk parameter χ has a moderate effect on the weight of the expert as well as decision results.

4.3 Comparative Analysis

To test the feasibility and validity of the suggested method, we compare it with some existing methods for the example given above.

In the traditional decision-making model (Chen et al. 2023) the ranking results are derived from the aggregate decision matrix of experts' opinions. Therefore, to compare our model and the traditional model, we first aggregate the experts' opinions. In this connection, by Mandal et al. (2024a), using the LZN weighted averaging aggregation operator, the aggregation of five experts' opinions is

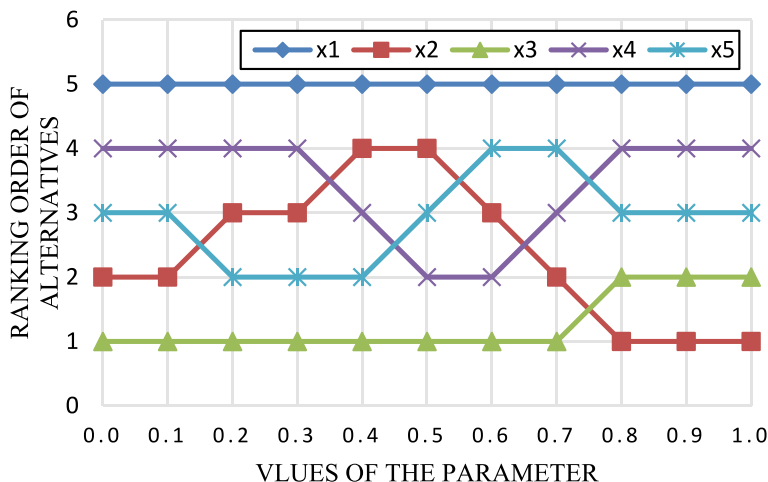


Fig. 8 The impact of χ on the decision result

$$\text{Agg}(Z^1, Z^2, Z^3, Z^4, Z^5) =$$

$$\begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} (h_{-0.3671}, g_{0.6}) & (h_{1.7810}, g_1) & (h_{1.7914}, g_{0.8}) & (h_{-0.1774}, g_{1.2}) \\ (h_{-0.5783}, g_{-0.2}) & (h_{-0.8129}, g_0) & (h_{-0.7693}, g_{0.8}) & (h_{-0.8454}, g_0) \\ (h_{2.6225}, g_{1.2}) & (h_{1.6738}, g_{0.8}) & (h_{2.6147}, g_{1.4}) & (h_{1.8233}, g_{0.2}) \\ (h_{-0.9389}, g_1) & (h_{1.4042}, g_{0.6}) & (h_{-0.0702}, g_{0.4}) & (h_{0.6777}, g_0) \\ (h_{1.4496}, g_{1.4}) & (h_{1.3957}, g_{1.2}) & (h_{0.4737}, g_{0.4}) & (h_{0.4906}, g_1) \end{bmatrix} \end{matrix}$$

Therefore, the ranking of alternatives according to the TODIM-VIKOR method is $x_2 \succ x_3 \succ x_4 \succ x_1 \succ x_5$, i.e., the optimal option is x_2 . When the proposed model is applied without considering the interface effect, the optimal option is also x_2 . This suggests that the optimal option of the proposed approach align with those of traditional methods when the impact of decision-makers is disregarded, thereby validating the robustness of the suggested model. However, when evaluating the quantum interference among decision-makers, the outcomes of this model suggest that x_3 is the optimal strategy. This inconsistency arises from the conventional framework's exclusion of the interference effect among decision-makers, which leads to subjective biases during the evaluation of alternatives. As a result, the outcomes of this model better align with real-world complexities.

5 Conclusion

This paper presents a quantum group decision model designed to reduce information ambiguity and opinion interference in site selection of renewable energy. The model employs LZNIs and QDT to convey ambiguous information in complex decision-making contexts and uses Deng entropy to assess opinion interference by modeling the reciprocal effects among decision-makers in the MAGDM process.

Unlike conventional methods that treat decision-makers as independent entities, this approach addresses interference effects, assigns appropriate weights to decision-makers, and regulates opinion interference using quantum decision-making techniques. Our methodology enhances existing approaches by offering more effective solutions to challenges related to opinion interference and weight distribution.

The findings demonstrate significant improvements in decision-making, particularly in site selection of renewable energy, by providing a comprehensive framework to manage information ambiguity and divergent viewpoints. However, the model's computational complexity increases with the addition of decision-makers and variables, highlighting the need for advanced computational techniques to maintain efficiency.

Future research will integrate machine learning methods to improve scalability and practicality, ensuring the model's effectiveness in large-scale decision-making processes. This integration aims to address growing complexity, foster consensus in quantum group decision-making, and ensure the model's long-term sustainability.

Conflict of Interest The authors declare that they have no conflict of interest.

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An Innovative Method Utilizing an Extended ORESTE-Based Pythagorean Linguistic Multicriteria Quantum Group Decision-Making Approach for Selecting Cell Phones



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Abstract Several studies have proposed ranking methodologies for cell phones, as new models with enhanced capabilities offer greater benefits to users, making them increasingly important in assisting daily decision-making. However, most research has revealed deficiencies related to the confidence levels of decision-makers in providing judgments, which significantly affect the final ranking outcomes. Moreover, expert opinions often influence one another, and these experts frequently demonstrate subjective, bounded rationality during the decision-making process. This research presents a novel approach utilizing an extended ORESTE-based Pythagorean linguistic multicriteria quantum group decision-making methodology to address this issue. A numerical example is provided to illustrate the proposed methodology, focusing on five prominent smartphone brands: Apple, Samsung, Huawei, OPPO, and Asus. The results show that Huawei is the most preferred brand, followed by OPPO and Apple, in the selection of the optimal smartphone.

Keywords Quantum decision theory · Group decision-making · Pythagorean linguistic set · ORESTE

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1 Introduction

The ORESTE method, which stands for “Organisation, Rangement Et Synthèse De Données Relationnelles” (Organization, Sorting and Synthesis of Relational Data), is a multi-criteria group decision-making (MCGDM) approach. It’s particularly useful when decision-makers face complex choices involving multiple, often conflicting criteria. The method was developed to handle both qualitative and quantitative data and is known for its flexibility in expressing the preferences of decision-makers. In the ORESTE method, alternatives are evaluated based on a set of criteria. The process involves:

- (1) Defining a set of alternatives and criteria.
- (2) Assigning importance to each criterion.
- (3) Evaluating each alternative according to the criteria.
- (4) Aggregating the evaluations to provide a global ranking of the alternatives.

The research in fuzzy set theory and its applications plays a significant role in addressing uncertainty and complexity in social networks and group decision-making processes. The methods developed in this area offer valuable insights into how fuzzy logic can be used to model and analyze systems where imprecision and ambiguity are prevalent. Fuzzy set theory is particularly well-suited to modeling the uncertainty inherent in social networks, where relationships and interactions often involve incomplete or vague information. Traditional mathematical models struggle to capture this uncertainty, but fuzzy approaches provide a more nuanced framework for analyzing such dynamics. For instance, the development of fuzzy fractional differential equations using fuzzy Laplace transforms is essential for modeling systems where the influence of individuals or groups evolves fractionally over time under uncertainty (Salahshour et al. 2012; Allahviranloo et al. 2015; Chehlabi and Allahviranloo 2016; Armand et al. 2018).

A new centrality measure is introduced for neutrosophic and linguistic graphs (Mahapatra et al. 2019, 2020), including self-weight or individual merit within the network. Mahapatra et al. (2024) present an innovative method for centrality measurement in fuzzy social networks where relationships between nodes are not binary but have different degrees of intensity.

According to the above discussion, Fuzzy sets (FS) have become a popular approach for incorporating fuzzy information as part of an expert’s evaluative measures. However, as noted by Xu (2005) and Mandal et al. (2022a), experts tend to express their opinions more reliably using linguistic information, known as linguistic sets (LS), rather than fuzzy information. In Mandal and Ranadive (2019), the LS concept was extended to the Pythagorean linguistic set (PLS), allowing experts to express both their confidence and doubt levels when providing judgments. In real-life scenarios, situations, experts often have limited knowledge and personal biases, so using language-based information with expert confidence and doubt levels (Mandal et al. 2021, 2022b, 2023a, b, 2024a) is more flexible for expressing their ideas. In this connection, Mandal et al. briefly studied MCGDM under LZN information with the

MARCOS approach and its application in the selection of logistic cold chain centers (Mandal et al. 2024b); TODIM-VIKRO approach (Mandal et al. 2024c) and its application in site selection of medical logistic center; ORESTE approach (Mandal et al. 2024d) and its application in assessment of regional energy; and MULTI-MOORA approach (Mandal et al. 2024e) and its application in selection of software. To address the challenge of aggregating individual assessments, specific methods are needed. Although many aggregation strategies have been studied, most assume that decision-makers are independent and uninfluenced by one another. However, in practice, opinions are often influenced by surrounding perspectives. In decision-making, the views of multiple experts can interact, ultimately affecting the final decision, and introducing the concept of 'interference'.

Quantum Probability Theory (QPT) (Busemeyer and Bruza 2012; Moreira and Wichert 2014; Yearsley and Busemeyer 2016) offers a framework for capturing these influences. QPT, a recent advancement over classical probability theory (CPT), has been extensively studied in fields such as psychology (Busemeyer and Bruza 2012), cognitive science (Yearsley and Busemeyer 2016), and decision analysis (He and Jiang 2018a, b; Wu et al. 2020; Xiao et al. 2022; Han and Liu 2023; Mandal et al., 2024f; 2025) due to its robust descriptive power. It effectively explains paradoxes, such as the disjunction fallacy (He et al. 2018a, b), the Ellsberg paradox (Eichberger and Pirner 2018), the order effect (Wang et al. 2014), and graph theory (Samanta and Allahviranloo 2025), which are challenging for CPT to address. QPT's ability to represent interference terms makes it particularly valuable for understanding how expert opinions influence each other in the MCGDM process.

Building on the above discussion, this study proposes an ORESTE-based MAGDM using the quantum decision theory (QDT) technique within the PLI context. This approach integrates the psychological aspects of experts' preferences with the optimization of group utility and the minimization of individual regret. The main contributions of this paper are:

1. An improved generalized PLI-based ORESTE approach accounts for experts' limited rationality, simplifies the calculation process, and resolves some paradoxes of the traditional ORESTE method.
2. A quantum-based method for combining individual evaluation results, which helps us better understand human psychology through the effects of interference.
3. A case study demonstrating the effectiveness of our approach in the selection of cell phones. Comparative and sensitivity analyses show the flexibility and robustness of our method.

The structure of the paper is as follows: In Sect. 2, recall some basic definitions and the background of linguistic term sets (LTSs), the Pythagorean linguistic set (PLS), and quantum Decision Theory (QDT). The proposed MCGDM model is discussed in Sect. 3. A case study for the selection of cell phones with comparative analysis is provided in Sect. 4. Section 5 concludes our study.

2 Preliminaries

Here we recall some related definitions and results that are used in this paper.

2.1 LTS

Here we assume two LTSs are: $H = \{h_\lambda | \lambda = -\zeta, \dots, -1, 0, 1, \dots, \zeta\}$ with cardinality $2\zeta + 1$ and holds the followings (Mandal et al. 2024a):

- (1) $h_{\lambda_1} \geq h_{\lambda_2}$ if and only if $\lambda_1 \geq \lambda_2$;
- (2) $neg(h_{\lambda_1}) = h_{-\lambda_1}$;
- (3) $max(h_{\lambda_1}, h_{\lambda_2}) = h_{\lambda_1}$ if $h_{\lambda_1} \geq h_{\lambda_2}$;
- (4) $min(h_{\lambda_1}, h_{\lambda_2}) = h_{\lambda_1}$ if $h_{\lambda_1} \leq h_{\lambda_2}$.

To preserve expert information, the continuous LTSs (Xu 2005) denotes $\overline{H} = \{h_{-t} \leq h_\lambda \leq h_t | \lambda \in [-t, t]\}$, $t > \zeta$.

The subscript index of any $h_\lambda \in \overline{H}$, represented by the following functions:

$$I : \overline{H} \rightarrow [-t, t], \text{ and } I(h_\lambda) = \lambda.$$

$$I^{-1} : [-t, t] \rightarrow \overline{H}, \text{ and } I^{-1}(\lambda) = h_\lambda.$$

Let $\mathcal{F}$ be the linguistic conversation function for any LT $G \in \overline{H}$, which is defined in the following way:

$$\mathcal{F} : \overline{H} \rightarrow [0, 1], \mathcal{F}(G) = \{\mathcal{F}(h_\lambda) = \frac{\lambda}{2t} + \frac{1}{2} | \lambda \in [-t, t]\} = \kappa.$$

2.2 PTS

By Mandal and Ranadive (2019), a Pythagorean linguistic set (PLS) denoted by $\mathcal{P}$ in an alternative set X for continuous LTS $\overline{H} = \{h_{-t} \leq h_\lambda \leq h_t | \lambda \in [-t, t]\}$ is represented in the following way:

$$\mathcal{R} = \{ \langle h_{\theta(x)}, (\mu_{\mathcal{R}}(x), \nu_{\mathcal{R}}(x)) \rangle | x \in X \},$$

where, $h_{\theta(x)} \in \overline{H}$, $\mu_{\mathcal{R}} : X \rightarrow [0, 1]$ and $\nu_{\mathcal{R}} : X \rightarrow [0, 1]$ are the membership and non-membership degrees of x to $h_{\theta(x)}$ such that $\mu_{\mathcal{R}}^2(x) + \nu_{\mathcal{R}}^2(x) \leq 1$. Here, the Pythagorean linguistic number (PLN) in $\mathcal{R}$ is denoted by $\wp$ and represented by $\wp = \langle h_{\theta(\wp)}, (\mu(\wp), \nu(\wp)) \rangle$ with $h_{\theta(\wp)} \in \overline{H}$ and $\mu(\wp), \nu(\wp) \in [0, 1]$, $\mu^2(\wp) + \nu^2(\wp) \leq 1$.

By Mandal and Ranadive (2019), the expected value of $\wp = h_{\theta(\wp)}, (\mu(\wp), \nu(\wp))$ is denoted by κ and defined in the following way:

$$\kappa(\wp) = h_{\frac{\theta(\wp) \times (\mu^2(\wp) + 1 - \nu^2(\wp))}{2}}.$$

By Wang et al. (2019), the distance between two PLNs $\wp_1 = \langle h_{\theta(\wp_1)}, (\mu(\wp_1), \nu(\wp_1)) \rangle$ and $\wp_2 = \langle h_{\theta(\wp_2)}, (\mu(\wp_2), \nu(\wp_2)) \rangle$ are defined in the following way:

$$d(\wp_1, \wp_2) = \frac{1}{2} \left| \begin{aligned} &(1 + \mu^2(\wp_1) - \mu^2(\wp_1)) * \mathcal{F}(h_{\theta(\wp_1)}) \\ &-(1 + \mu^2(\wp_2) - \mu^2(\wp_2)) * \mathcal{F}(h_{\theta(\wp_2)}) \end{aligned} \right|.$$

2.3 Quantum Probability Theory (QPT)

A quantum decision (QD) model is based on the principles of quantum probability (QP) theory (QPT). To provide context, we will review some key concepts and notations from QPT. In quantum theory (QT), a state is represented as a unit-length vector within an N-dimensional vector space.

Definition 1 Yearsley and Busemeyer (2016) A finite-dimensional Hilbert space (HS), represented as $\mathcal{H}$, is an abstract vector space over a complex field, spanned by a set of orthonormal basis vectors with complex components. In classical probability theory (CPT), events are fundamental (He et al. 2018), but in quantum probability theory (QPT), events are instead represented by subspaces of the Hilbert space $\mathcal{H}$, rather than traditional sets.

Definition 2 Busemeyer and Bruza (2012) In QT, the superposition state $|\mathcal{S}\rangle$, a unit-length vector in the N-dimensional Hilbert space $\mathcal{H}$, defines all possible events within that space. The nonlinear combination of a set of orthogonal basis vectors $\{|\mathcal{V}\rangle_k, k = 1, 2, \dots, n\}$ are used to represent the superposition state $|\mathcal{S}\rangle$, i.e.,

$$\begin{aligned} |\mathcal{S}\rangle &= \langle \mathcal{V}_1 | \mathcal{S} \rangle |\mathcal{V}_1\rangle + \langle \mathcal{V}_2 | \mathcal{S} \rangle |\mathcal{V}_2\rangle + \dots + \langle \mathcal{V}_n | \mathcal{S} \rangle |\mathcal{V}_n\rangle \\ &= \xi_1 |\mathcal{V}_1\rangle + \xi_2 |\mathcal{V}_2\rangle + \dots + \xi_n |\mathcal{V}_n\rangle \\ &= \zeta_1 e^{i\alpha_1} |\mathcal{V}_1\rangle + \zeta_2 e^{i\alpha_2} |\mathcal{V}_2\rangle + \dots + \zeta_n e^{i\alpha_n} |\mathcal{V}_n\rangle \end{aligned} \quad (1)$$

where $\xi_k (= \zeta_k e^{i\alpha_k})$ is the amplitude of the probability, expressed as a complex number, characterizing the amplitude of the wave in the state $|\mathcal{V}_k\rangle$, and the amplitude of the phase angle represented by $e^{i\alpha_k}$.

Definition 3 Busemeyer and Bruza (2012) The CPT framework represents every event using a single set of components. In QPT, probabilities are assigned based on

the overlap between cognitive states (shown as state vectors) and different probability measures (shown as subspaces). QPT extends CPT into the complex domain by including incompatible qualities and viewing events as vectors. Incompatible attributes are two observable features of an item that cannot be measured accurately at the same time. Both the incompatibility condition and the uncertainty principle can be described by the probability amplitude, also known as complex probability.

According to Born's rule, an event's probability amplitude and its classical probability, x_i are related to each other:

$$P(\mathcal{V}_k) = |\zeta_k e^{i\alpha_k}|^2 = (\zeta_k e^{i\alpha_k}) \times (\zeta_k e^{i\alpha_k})^* = \zeta_k e^{i\alpha_k} \times \zeta_k e^{-i\alpha_k} = (\zeta_k)^2 \quad (2)$$

For the total event probability to be 1, the sum of the squares of the probability amplitudes must also be 1.

$$\sum_{k=1}^n P(\mathcal{V}_k) = \sum_{k=1}^n (\zeta_k)^2 = 1. \quad (3)$$

An $N - 1$ mapping transforms complex probabilities into classical probabilities. For each given probability value $P(\mathcal{V}_k)$, countless complex probabilities correspond to it. Referencing Eq. (2) and Euler's formula,

$$e^{i\alpha} = \cos\alpha + i \times \sin\alpha \quad (4)$$

the complex probability satisfies:

$$\xi = \sqrt{P(\mathcal{V})}(\cos\alpha + i \times \sin\alpha) = \sqrt{P(\mathcal{V})}e^{i\alpha} \quad (5)$$

For example, in the following, we discussed the QPT in two-dimensional Hilbert space. Let $\{\mathcal{M}_1, \mathcal{M}_2\}$ be an event containing two basic states, where $|\mathcal{M}_1\rangle = (1, 0)$, and $|\mathcal{M}_2\rangle = (0, 1)$ as two basic orthogonal vectors in $\mathbf{H}$. The state $|\mathcal{S}\rangle$ is defined as a superposition and is graphically illustrated in Fig. 1. According to Eqs. (1) and (2), it can be expressed as.

$$|\mathcal{S}\rangle = \zeta_1 e^{i\alpha_1} |\mathcal{M}_1\rangle + \zeta_2 e^{i\alpha_2} |\mathcal{M}_2\rangle = \frac{e^{i\alpha_{\mathcal{M}_1}}}{\sqrt{2}} |\mathcal{M}_1\rangle + \frac{e^{i\alpha_{\mathcal{M}_2}}}{\sqrt{2}} |\mathcal{M}_2\rangle$$

and the probabilities of $|\mathcal{M}_1\rangle$ and $|\mathcal{M}_2\rangle$ are calculated as follows:

$$P(\mathcal{M}_1) = P(\mathcal{M}_2) = (e^{i\alpha})(e^{i\alpha})^* = \frac{1}{2}.$$

According to conventional Bayesian decision theory, decision-making can be mathematically represented as a Markov process (Moreira and Wichert 2014). The state transition probability in a single route of QPT aligns with the CPT according

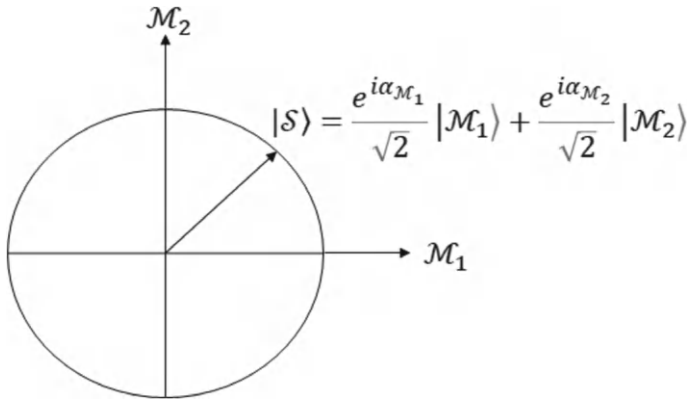


Fig. 1 Event in Hilbert space with two dimensions

to Feynman's first rule (Wu et al. 2020), which varies over several pathways. Consider the decision-making dilemma highlighted above as an illustration of the process depicted in Fig. 2a. Experts provide their assessments of $\mathcal{M}_1$ ('bad') or $\mathcal{M}_2$ ('good') as probability quantifications. State transition probabilities are determined by combining the individual probabilities for each route and using the total probability formula in CPT.

$$P(\mathcal{M}_1) = \sum_{k=1}^2 P(e_k)P(\mathcal{M}_1|e_k), \quad \sum_{k=1}^2 P(e_k) = 1 \quad (6)$$

Since only one route is selected at a time, it is sometimes referred to as the single path principle. The conventional Markovian framework assumes that experts rely on one expert at a time and that both experts are completely autonomous. However,

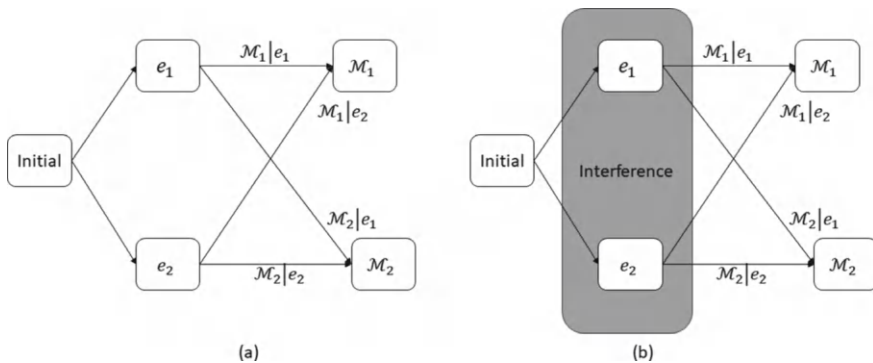


Fig. 2 DM in **a** Markov and **b** quantum process

this approach is sometimes unsuitable for ordinary cognitive processes because many avenues are often considered when forming judgments. Such phenomena can result in illogical actions, including disjunction, conjunction errors, and interference effects. Thus, QPT is an excellent complement to CPT in this scenario.

Within the QPT framework (Fig. 2b), the state is formed by combining three fundamental states, which can be represented as different wave functions aligning according to Feynman's second rule. The amplitude of the final form is obtained by superimposing two individual routes running from I ('Initial') to $\mathcal{M}_1$. The amplitude is the sum of the individual pathways from source I to target $\mathcal{M}_1$, and the probability of reaching the target $\mathcal{M}_1$ is equal to the square of the amplitude.

$$P(\mathcal{M}_1) = |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 \quad (7)$$

Equation (7) can be simplified in the following way:

$$\begin{aligned} P(\mathcal{M}_1) &= |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 \\ &= |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}| |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^* \\ &= \left(\sqrt{P(e_1)P(\mathcal{M}_1|e_1)}e^{i\alpha_1} + \sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{i\alpha_2} \right) \\ &\quad \left(\sqrt{P(e_1)P(\mathcal{M}_1|e_1)}e^{-i\alpha_1} + \sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{-i\alpha_2} \right) \\ &= P(e_1)P(\mathcal{M}_1|e_1) + P(e_2)P(\mathcal{M}_1|e_2) \\ &\quad + \sqrt{P(e_1)P(\mathcal{M}_1|e_1)}\sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{i(\alpha_1-\alpha_2)} \\ &\quad + \sqrt{P(e_1)P(\mathcal{M}_1|e_1)}\sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{i(\alpha_2-\alpha_1)} \end{aligned} \quad (8)$$

By Eq. (4), we get

$$\cos(\alpha_1 - \alpha_2) = \frac{e^{i(\alpha_2-\alpha_1)} + e^{i(\alpha_1-\alpha_2)}}{2} \quad (9)$$

Then Eq. (7) can be written in the following way:

$$\begin{aligned} P(\mathcal{M}_1) &= P(e_1)P(\mathcal{M}_1|e_1) + P(e_2)P(\mathcal{M}_1|e_2) \\ &\quad + 2\sqrt{P(e_1)P(\mathcal{M}_1|e_1)}\sqrt{P(e_2)P(\mathcal{M}_1|e_2)}\cos(\alpha_1 - \alpha_2) \\ &= |\xi_{e_1}\xi_{\mathcal{M}_1|e_1}|^2 + |\xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 + 2\xi_{e_1}\xi_{\mathcal{M}_1|e_1}\xi_{e_2}\xi_{\mathcal{M}_1|e_2}\cos(\alpha_1 - \alpha_2) \end{aligned} \quad (10)$$

Upon comparing Eqs. (6) and (10), it becomes evident that the quantum framework includes an interference component. If the pathways are observable, the quantum phase transition will reduce to the classical one, following Feynman's third rule.

$$P(\mathcal{M}_1) = |\xi_{e_1}\xi_{\mathcal{M}_1|e_1}|^2 + |\xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 = P(e_1)P(\mathcal{M}_1|e_1) + P(e_2)P(\mathcal{M}_1|e_2) \quad (11)$$

Additionally, the QPT can be expanded to a multi-path graph, resulting in an increased number of interference terms. The QPT enhances CPT by accurately representing cognitive processes in quantum terms, thereby more effectively modeling ambiguity, vagueness, and uncertainty in decision-making.

3 The Proposed Oreste-Based Magdm Under Pythagorean Linguistic Information and Quantum Decision Scenario

This section introduces quantum cognition-based MAGDM integrating the ORESTE approach.

3.1 Background

Let us assume

$X = \{x_1, x_2, \dots, x_n\}$: Alternative set,

$C = \{c_1, c_2, \dots, c_m\}$: Attribute set,

$e = \{e_1, e_2, \dots, e_q\}$: Expert set,

$w^k = (w_1^k, w_2^k, \dots, w_m^k)$: Weight vector of attributes for expert e_k , where $w_j^k \in [0, 1]$, and $\sum_{j=1}^m w_j^k = 1$,

$\omega = (\omega_1, \omega_2, \dots, \omega_q)$: Weight vector of experts with $\omega_k \in [0, 1]$, $\sum_{k=1}^q \omega_k = 1$.

3.2 Decision Steps

Let $A^k = (a_{ij}^k)_{nm}$ ($i = 1, 2, \dots, n$; $j = 1, 2, \dots, m$; $k = 1, 2, \dots, q$) be the PLI-based decision judgment matrix provided by an expert e_k using pre-defined distinct LTSs $H = \{h_\lambda | \lambda = -\zeta, \dots, -1, 0, 1, \dots, \zeta\}$.

Step 1: Construction of decision matrix

Then, in the following, we represent the decision matrix of k -th expert:

$$A^k = (a_{ij}^k)_{nm} = \begin{matrix} & \begin{matrix} c_1 & c_2 & \cdots & c_m \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} & \begin{bmatrix} a_{11}^k & a_{12}^k & \cdots & a_{1m}^k \\ a_{21}^k & a_{22}^k & \cdots & a_{2m}^k \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}^k & a_{n2}^k & \cdots & a_{nm}^k \end{bmatrix} \end{matrix}, \quad (12)$$

where, $a_{ij}^k = \left\langle h_{\theta_{ij}}^k, \left(\mu_{ij}^k, v_{ij}^k \right) \right\rangle$ is a PLN, called the Pythagorean linguistic information (PLI) of k -th expert for i -th alternative about j -th criteria.

Step 2: Expert weight computation process

Inspire by Mandal et al. (2024a), the weight of experts can be calculated in the following way:

$$\omega_k = \frac{\Omega(Z^k)}{\sum_{k=1}^q \Omega(Z^k)}, \text{ where } \Omega(Z^k) = \frac{\sum_{j=1}^m \sum_{i=1}^n \mathcal{F}(\kappa(a_{ij}^k))}{mn}. \quad (13)$$

Step 3: Attribute weight computation process

- (1) The decision matrix (Eq. (12)) can be normalized to $L^k = (l_{ij}^k)_{nm}$ with

$$l_{ij}^k = \begin{cases} \frac{\mathcal{F}(\kappa(a_{ij}^k))}{\sum_{i=1}^n \mathcal{F}(\kappa(a_{ij}^k))}, & \text{for benefit criteria } c_j, \\ 1 - \frac{\mathcal{F}(\kappa(a_{ij}^k))}{\sum_{i=1}^n \mathcal{F}(\kappa(a_{ij}^k))}, & \text{for cost criteria } c_j. \end{cases} \quad (14)$$

- (2) The Shannon entropy $\Psi^k = (\Psi_1^k, \Psi_2^k, \dots, \Psi_m^k)$ concerning each expert $e_k (k = 1, 2, \dots, q)$ can be calculated in the following way:

$$\Psi_j^k = -\frac{1}{\ln n} \sum_{i=1}^n l_{ij}^k \ln l_{ij}^k \quad (15)$$

$l_{ij}^k \ln l_{ij}^k$ is 0 if $l_{ij}^k = 0$.

- (3) The attribute weight $w^k = (w_1^k, w_2^k, \dots, w_m^k)$ concerning each expert $e_k (k = 1, 2, \dots, q)$ can be calculated in the following way:

$$w_j^k = \frac{1 - \Psi_j^k}{\sum_{j=1}^m (1 - \Psi_j^k)} \quad (16)$$

Step 4: Construction of Bayesian network

In this phase, we converted the pre-collected data into probabilities and developed a Bayesian Network for decision-making, as the quantum framework is a partial extension of the Bayesian Network. Figure 3 illustrates that, in a MAGDM situation, the decision-making process can be represented as a simple two-layer Bayesian Network. The experts' weights are converted into probabilities, which indicate the degree of belief in the experts, using the following equation:

$$P(e_k) = \omega_k, \quad k = 1, 2, \dots, q. \quad (17)$$

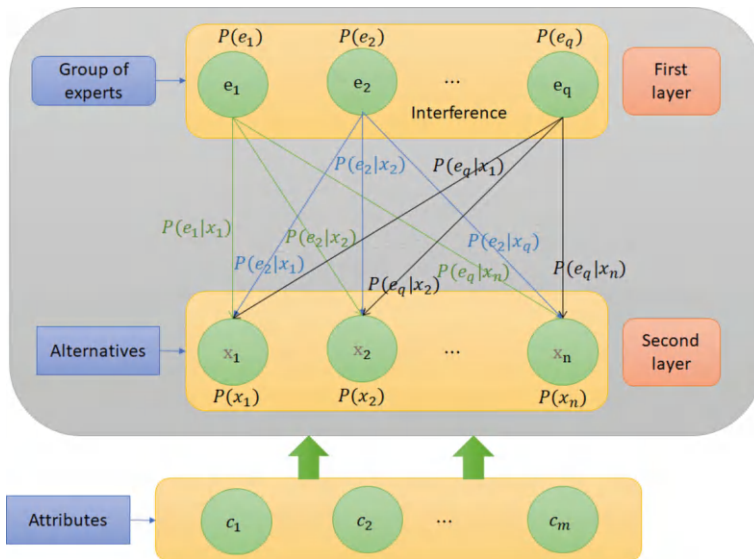


Fig. 3 MAGDM according to QSBN

In Eq. (17), the probabilities form the initial layer of the Bayesian Network (BN). The second layer depends on the assessment outcomes from each expert, which can be calculated using the extended ORESTE approach as follows:

(1) Identifying positive ideal solution under each criterion and the value of important criteria computation process

From the expert e_k opinion judgment matrix $A^k = (a_{ij}^k)_{nm}$, $a_{ij}^k = \langle h_{\alpha_{ij}}^k, (\mu_{ij}^k, \nu_{ij}^k) \rangle$, in the following, we obtain a positive ideal solution for each criterion c_j and the significance criteria value.

$$(a_j^k)^+ = \begin{cases} \max_{i=1}^n \{h_{\alpha_{ij}}^k\}, \left(\max_{i=1}^n \{\mu_{ij}^k\}, \min_{i=1}^n \{\nu_{ij}^k\} \right) & \text{if } c_j \text{ is a benefit criteria} \\ \min_{i=1}^n \{h_{\alpha_{ij}}^k\}, \left(\min_{i=1}^n \{\mu_{ij}^k\}, \max_{i=1}^n \{\nu_{ij}^k\} \right) & \text{if } c_j \text{ is a cost criteria} \end{cases} \quad (18)$$

$$w_{max}^k = \max_{j=1}^m \max_{j=1}^m w_j^k \quad (19)$$

(2) Global preference score (GPS) computation process

Using Eqs. (18) and (19), we obtain GPS of $x_i (i = 1, 2, \dots, n)$ over $c_j (j = 1, 2, \dots, m)$ of expert e_k in the following way:

$$GS(a_{ij}^k) = \sqrt{\kappa d(a_{ij}^k, (a_j^k)^+)^2 + (1 - \kappa)(w_j^k - w_{max}^k)^2} \quad (20)$$

where $\kappa \in (0, 1)$, be an adjustment parameter represents the significance between positive ideal solution under each criterion and the value of important criteria.

(3) Preference intensities computation process

The preference intensities of the alternative x_i over alternative x_s for expert e_k can be calculated in the following way:

$$\tau^k(x_i, x_s) = \frac{\sum_{j=1}^m \max\{GS(a_{sj}^k) - GS(a_{ij}^k), 0\}}{m} \quad (21)$$

where $GS(a_{sj}^k)$ and $GS(a_{ij}^k)$ are the GPS of alternatives x_i and x_s for criteria c_j , of expert e_k respectively.

Therefore, the net preference intensity of alternatives x_i over x_s of expert e_k can be calculated in the following way:

$$\Delta\tau^k(x_i, x_s) = \tau^k(x_i, x_s) - \tau^k(x_s, x_i) \quad (22)$$

(4) Positive and negative net preference intensity of each alternative computation process

The positive and negative net preference intensity of each alternative for expert e^k are denoted by $\eta^{k,+}(x_i)$ and $\eta^{k,-}(x_i)$, ($i = 1, 2, \dots, n$) and calculated in the following way:

$$\eta^{k,+}(x_i) = \sum_{s=1}^n \Delta\tau^k(x_i, x_s) \quad (23)$$

$$\eta^{k,-}(x_i) = \sum_{s=1}^n \Delta\tau^k(x_s, x_i) \quad (24)$$

(5) Net intensity of each alternative computation process

Based on Eqs. (14) and (15), we have the net intensity of each alternative for expert e^k is as follows:

$$\eta_i^k = |\eta^{k,+}(x_i) - \eta^{k,-}(x_i)| \quad (25)$$

(6) Based on Eq. (25), conditional probabilities which link the two layers in the BN as:

$$P(x_i|e_k) = \frac{\eta_i^k}{\sum_{i=1}^n \eta_i^k}. \quad (26)$$

(7) The second layer consists of the probabilities of alternatives calculated as:

$$P(x_i) = \sum_{k=1}^q P(e_k)P(x_i|e_k) \quad (27)$$

Subsequently, we can decide based on the $P(x_i)$ values, as higher probabilities, indicate a superior ranking. The classical approach does not account for the interference effect, which can influence the final rankings. To address this issue, Bayesian networks are expanded into a quantum framework capable of properly modeling a practical approach to real interference in MAGDM.

Step 5: Construction of quantum scenario-based Bayesian network (QSBN)

The facilitator often consolidates opinions from several specialists. The interconnected decision-making processes in real-world situations, as demonstrated in Sect. 2.2, remain unobservable. The square of the sum of the amplitude probabilities of all possible pathways collectively constitutes the probability of the alternative x_i from QPT (He et al. 2018), as described.

$$\begin{aligned} P(x_i) &= \left| \sum_{k=1}^q \xi_{e_k} \xi_{x_i|e_k} e^{i\alpha_k} \right|^2 \\ &= \left| \sum_{k=1}^q \sqrt{P(e_k)} \sqrt{P(x_i|e_k)} e^{i\alpha_k} \right|^2, \quad i = 1, 2, \dots, n \end{aligned} \quad (28)$$

By Eqs. (7)–(11), Eq. (28) can be written as

$$\begin{aligned} P(x_i) &= \sum_{k=1}^q P(e_k)P(x_i|e_k) + 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \cos(\alpha_k - \alpha_s) \\ &= \sum_{k=1}^q P(e_k)P(x_i|e_k) + INT_i \end{aligned} \quad (29)$$

In a quantum-based Bayesian network, a normalization factor is incorporated into Eq. (29) to standardize the probability of each alternative, ensuring that the overall estimated probability does not exceed 1. Then

$$Prob(x_i) = \sigma P(x_i) \quad (30)$$

where,

$$\sigma = \frac{1}{\sum_{i=1}^n P(x_i)} = \frac{1}{\sum_{i=1}^n (\sum_{k=1}^q P(e_k)P(x_i|e_k) + INT_i)} \quad (31)$$

3.3 Analysis of Interference Effect Among Experts' Opinions

Equation (31) uses the term INT_i , which represent the ambiguity and uncertainty of expert opinions, to explain the superposition of decision information. In Eq. (31), $\cos(\alpha_k - \alpha_s)$ represents the phase angle between two distinct experts e_k and e_s , which represents the connection vector between two experts. $\frac{q(q-1)}{2}$ is the total count of pairwise correspondences across q experts, excluding duplicates. The final interference value is derived by summing the computed $\frac{q(q-1)}{2}$ interference values for each expert pair, rather than assessing the overall uncertainty across experts. This approach enhances the accuracy of the estimated probability value.

In real-world situations, it is hard to measure the phase angle. Experts' opinions can differ or be influenced by outside factors, causing a lot of variation and mistakes. Belief entropy helps a lot in handling and combining unclear information, making it easier to make good decisions in group settings with many factors. It is used as a standard in areas like risk assessment, defect diagnosis, and pattern recognition. Belief entropy is the latest way to measure interference in QSNB. Unlike the old method of manually estimating the phase angle range, this new method is more reliable because it reduces errors from personal judgment, which can affect the final decision and the expert's probability calculation. Belief entropy is used in QSNB to manage the interference among different experts' beliefs in GDM.

Then the belief distance about alternative x_i according to the average probability $\chi = \frac{1}{nq}$ can be calculated in the following way:

$$D_{iu} = \left| \delta_{iu} + \frac{|\delta_{iu} - \chi| - |\varrho_{iu} - \chi|}{|\delta_{iu} - \chi| + |\varrho_{iu} - \chi|} \right|, i = 1, 2, \dots, n; \quad u = 1, 2, \dots, \frac{q(q-1)}{2},$$

$$\begin{cases} \delta_{iu} = P(e_k)P(x_i|e_k), & i = 1, 2, \dots, n; \quad k = 1, 2, \dots, q-1; \\ \varrho_{iu} = P(e_s)P(x_i|e_s), & s = k+1, k+2, \dots, \frac{q(q-1)}{2}. \end{cases} \quad (32)$$

In Eq. (32), u means the value reallocated to the correlation among q experts; $\chi = \frac{1}{nq}$ represent each expert probability value of each alternative; D_{iu} means deviation degree from χ ; $|\delta_{iu} - \chi|$ and $|\varrho_{iu} - \chi|$ represents two experts' magnitude of the deviation degree from χ about alternative x_i . If $|\delta_{iu} - \chi| < |\varrho_{iu} - \chi|$, then the position of δ_{iu} and ϱ_{iu} are interchangeable.

There are many ways to measure uncertainty. Deng entropy (Zhao et al. 2024), also known as belief entropy, is better at including the confidence function and the uncertainty of BPA because it measures both total non-specificity and disagreement.

Thus, by Deng's entropy, we get

$$\vartheta_{iu}(D_{iu}) = -D_{iu} \log_2 \frac{D_{iu}}{2^m - 1}, \quad i = 1, 2, \dots, n; \quad u = 1, 2, \dots, \frac{q(q-1)}{2}. \quad (33)$$

To ensure that the values of Deng entropy align with the range of $\cos(\alpha_k - \alpha_s)$, which is between $[-1, 1]$, Deng entropy is standardized.

$$\beta_{iu}(D_{iu}) = 2 \frac{\vartheta_{iu}(D_{iu}) - \min\{\vartheta_{iu}(D_{iu})\}}{\max\{\vartheta_{iu}(D_{iu})\} - \min\{\vartheta_{iu}(D_{iu})\}} - 1, \\ i = 1, 2, \dots, n; \quad u = 1, 2, \dots, \frac{q(q-1)}{2}. \quad (34)$$

Thus, the phase angle $\cos(\alpha_k - \alpha_s)$ can be determined by the following way:

$$\cos(\alpha_k - \alpha_s) = \beta_{iu}(D_{iu}), \quad i = 1, 2, \dots, n; \quad u = 1, 2, \dots, \frac{q(q-1)}{2}. \quad (35)$$

Therefore, by Eq. (35), in Eq. (29), INT_i can be written as

$$INT_i = 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \cos(\alpha_k - \alpha_s) \\ = 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \beta_{iu}(D_{iu}) \quad (36)$$

By Eq. (36), the Eq. (29) can be written as

$$P(x_i) = \sum_{k=1}^q P(e_k)P(x_i|e_k) + 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \beta_{iu}(D_{iu}) \quad (37)$$

Finally, the probability of each alternative x_i can be obtained in the following way:

$$Prob(x_i) = \sigma P(x_i) \quad (38)$$

where,

$$\sigma = \frac{1}{\sum_{i=1}^n P(x_i)} = \frac{1}{\sum_{i=1}^n (\sum_{k=1}^q P(e_k)P(x_i|e_k) + INT_i)} \quad (39)$$

Step 6: Ranking of alternative

According to Eq. (39), the higher probability of an alternative means the best alternative.

4 A Practical Example with Comparative Analysis

This section describes a new approach to MAGDM based on the ORESTE approach integrating QDT with PLI. The detailed procedure for the combined ORESTE approach is presented below.

4.1 A Case Study

This study analyzes smartphone selection in Malaysia, focusing on four attributes: Brand Name (c_1), Price (c_2), Features (c_3), and Social Influence (c_4). A questionnaire was administered to five experts e_1, e_2, e_3, e_4 , and e_5 , referred to as smartphone specialists. These professionals, as smartphone suppliers, bring substantial experience and expertise, having worked with prominent brands in Malaysia, including ASUS, Huawei, Apple, OPPO, and Samsung, the leading smartphone brands in Malaysia in 2017 (Al-Koliby and Rahman 2018). Each alternative smartphone is labeled as x_1, x_2, x_3, x_4 , and x_5 respectively.

4.1.1 Decision Procedure

The following methods are applied to evaluate this MAGDM scenario using our proposed ORESTE method integrating QDT with PLI for assessing the available options.

Step 1: Here we offer the LTSs, $H_1 = \{h_{-3}: \text{very bad (VB)}, h_{-2}: \text{bad (B)}, h_{-1}: \text{slightly bad (SB)}, h_0: \text{moderate (M)}, h_1: \text{slightly good (SG)}, h_2: \text{good (G)}, h_3: \text{very good (VG)}\}$ for the evaluation of alternatives (x_1, x_2, x_3, x_4, x_5) about the criteria (c_1, c_2, c_3, c_4). The evaluation information is shown in the Eqs. (40)–(44).

$$A^1 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} \langle h_{-1}, (0.87, 0.4) \rangle & \langle h_2, (0.85, 0.1) \rangle & \langle h_0, (0.9, 0.2) \rangle & \langle h_{-3}, (0.23, 0.7) \rangle \\ \langle h_2, (0.9, 0.21) \rangle & \langle h_1, (0.9, 0.12) \rangle & \langle h_{-2}, (0.7, 0.3) \rangle & \langle h_2, (0.45, 0.7) \rangle \\ \langle h_1, (0.45, 0.3) \rangle & \langle h_{-2}, (0.6, 0.5) \rangle & \langle h_3, (0.6, 0.4) \rangle & \langle h_1, (0.98, 0.1) \rangle \\ \langle h_{-2}, (0.36, 0.8) \rangle & \langle h_{-1}, (0.7, 0.5) \rangle & \langle h_1, (0.5, 0.6) \rangle & \langle h_{-1}, (0.8, 0.2) \rangle \\ \langle h_3, (0.2, 0.9) \rangle & \langle h_0, (0.65, 0.2) \rangle & \langle h_2, (0.9, 0.2) \rangle & \langle h_2, (0.9, 0.1) \rangle \end{bmatrix} \end{matrix} \quad (40)$$

$$A^2 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} \langle h_2, (0.9, 0.3) \rangle & \langle h_3, (0.45, 0.6) \rangle & \langle h_{-2}, (0.8, 0.25) \rangle & \langle h_3, (0.8, 0.1) \rangle \\ \langle h_3, (0.8, 0.4) \rangle & \langle h_{-3}, (0.5, 0.7) \rangle & \langle h_0, (0.65, 0.6) \rangle & \langle h_{-2}, (0.23, 0.5) \rangle \\ \langle h_1, (0.7, 0.23) \rangle & \langle h_0, (0.6, 0.6) \rangle & \langle h_2, (0.78, 0.3) \rangle & \langle h_{-1}, (0.65, 0.3) \rangle \\ \langle h_2, (0.8, 0.32) \rangle & \langle h_2, (0.8, 0.2) \rangle & \langle h_{-1}, (0.9, 0.2) \rangle & \langle h_{-3}, (0.9, 0.2) \rangle \\ \langle h_1, (0.9, 0.15) \rangle & \langle h_{-1}, (0.2, 0.9) \rangle & \langle h_2, (0.6, 0.1) \rangle & \langle h_2, (0.7, 0.2) \rangle \end{bmatrix} \end{matrix} \quad (41)$$

$$A^3 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} \langle h_{-3}, (0.9, 0.1) \rangle & \langle h_2, (0.9, 0.1) \rangle & \langle h_{-2}, (0.8, 0.2) \rangle & \langle h_2, (0.3, 0.5) \rangle \\ \langle h_0, (0.9, 0.2) \rangle & \langle h_3, (0.8, 0.1) \rangle & \langle h_3, (0.7, 0.35) \rangle & \langle h_3, (0.56, 0.4) \rangle \\ \langle h_2, (0.8, 0.3) \rangle & \langle h_1, (0.6, 0.3) \rangle & \langle h_2, (0.8, 0.4) \rangle & \langle h_1, (0.87, 0.3) \rangle \\ \langle h_{-3}, (0.7, 0.3) \rangle & \langle h_0, (0.7, 0.2) \rangle & \langle h_{-1}, (0.9, 0.2) \rangle & \langle h_0, (0.9, 0.2) \rangle \\ \langle h_3, (0.8, 0.2) \rangle & \langle h_2, (0.8, 0.1) \rangle & \langle h_2, (0.8, 0.3) \rangle & \langle h_{-3}, (0.5, 0.5) \rangle \end{bmatrix} \end{matrix} \quad (42)$$

$$A^4 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} \langle h_2, (0.8, 0.1) \rangle & \langle h_2, (0.8, 0.25) \rangle & \langle h_3, (0.56, 0.5) \rangle & \langle h_{-1}, (0.9, 0.2) \rangle \\ \langle h_0, (0.87, 0.25) \rangle & \langle h_1, (0.9, 0.1) \rangle & \langle h_{-3}, (0.5, 0.3) \rangle & \langle h_{-2}, (0.8, 0.3) \rangle \\ \langle h_1, (0.85, 0.2) \rangle & \langle h_0, (0.7, 0.2) \rangle & \langle h_2, (0.6, 0.4) \rangle & \langle h_0, (0.7, 0.1) \rangle \\ \langle h_{-2}, (0.9, 0.1) \rangle & \langle h_{-3}, (0.8, 0.1) \rangle & \langle h_{-1}, (0.7, 0.3) \rangle & \langle h_1, (0.9, 0.2) \rangle \\ \langle h_2, (0.6, 0.2) \rangle & \langle h_2, (0.9, 0.2) \rangle & \langle h_{-3}, (0.56, 0.2) \rangle & \langle h_2, (0.2, 0.9) \rangle \end{bmatrix} \end{matrix} \quad (43)$$

$$A^5 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} \langle h_{-3}, (0.8, 0.2) \rangle & \langle h_{-3}, (0.9, 0.2) \rangle & \langle h_3, (0.8, 0.2) \rangle & \langle h_2, (0.9, 0.25) \rangle \\ \langle h_{-2}, (0.2, 0.1) \rangle & \langle h_{-2}, (0.8, 0.4) \rangle & \langle h_{-2}, (0.3, 0.9) \rangle & \langle h_1, (0.8, 0.3) \rangle \\ \langle h_3, (0.9, 0.4) \rangle & \langle h_2, (0.7, 0.3) \rangle & \langle h_{-1}, (0.2, 0.7) \rangle & \langle h_3, (0.7, 0.5) \rangle \\ \langle h_{-3}, (0.8, 0.3) \rangle & \langle h_{-2}, (0.6, 0.1) \rangle & \langle h_{-3}, (0.9, 0.2) \rangle & \langle h_{-1}, (0.9, 0.4) \rangle \\ \langle h_{-2}, (0.6, 0.2) \rangle & \langle h_3, (0.7, 0.2) \rangle & \langle h_1, (0.3, 0.8) \rangle & \langle h_{-3}, (0.5, 0.5) \rangle \end{bmatrix} \end{matrix} \quad (44)$$

Step 2: By Eq. (13), the weight of experts is $\omega_1 = 0.2071$, $\omega_2 = 0.2150$, $\omega_3 = 0.2181$, $\omega_4 = 0.1891$, and $\omega_5 = 0.1706$.

Step 3: By Eq. (16), the weight of attributes for all experts are:

$$w_1^1 = 0.1849, w_2^1 = 0.2577, w_3^1 = 0.3144, w_4^1 = 0.2430;$$

$$w_1^2 = 0.0269, w_2^2 = 0.1308, w_3^2 = 0.22953, w_4^2 = 0.6127;$$

$$w_1^3 = 0.6237, w_2^3 = 0.0452, w_3^3 = 0.2072, w_4^3 = 0.1240;$$

$$w_1^4 = 0.1848, w_2^4 = 0.3302, w_3^4 = 0.3613, w_4^4 = 0.1237;$$

$$w_1^5 = 0.3710, w_2^5 = 0.3121, w_3^5 = 0.2177, w_4^5 = 0.0992.$$

Step 4: According to Eq. (17), we get $P(e_1) = 0.2071$, $P(e_2) = 0.2150$, $P(e_3) = 0.2181$, $P(e_4) = 0.1891$, and $P(e_5) = 0.1706$.

- (1) By Eq. (18), we get the positive ideal solution for each criterion c_j of all experts, which are shown in Table 1.

By Eq. (19), we get

$$w_{max}^1 = 0.3144, w_{max}^2 = 0.6127, w_{max}^3 = 0.6237, w_{max}^4 = 0.3613, w_{max}^5 = 0.3710.$$

- (2) Let $\alpha = 0.5$. Then by Eq. (20), we calculate the GPS of $x_i (i = 1, 2, \dots, n)$ over $c_j (j = 1, 2, \dots, m)$ of expert e_k , which are shown in Tables 2, 3, 4, 5 and 6.
- (3) By Eq. (22), using Eq. (21), we calculate the net preference intensity of alternatives x_i over x_s of expert e_k , which are shown in Tables 7, 8, 9, 10 and 11.

Table 1 The positive ideal solution for each criterion c_j of all experts

$(a_j^k)^+$	$j = 1$	$j = 2$	$j = 3$	$j = 4$
$k = 1$	$\langle h_3, (0.9, 0.21) \rangle$	$\langle h_2, (0.9, 0.1) \rangle$	$\langle h_3, (0.9, 0.2) \rangle$	$\langle h_2, (0.98, 0.1) \rangle$
$k = 2$	$\langle h_3, (0.9, 0.1) \rangle$	$\langle h_3, (0.8, 0.2) \rangle$	$\langle h_2, (0.9, 0.1) \rangle$	$\langle h_3, (0.9, 0.1) \rangle$
$k = 3$	$\langle h_3, (0.9, 0.1) \rangle$	$\langle h_3, (0.9, 0.1) \rangle$	$\langle h_3, (0.9, 0.2) \rangle$	$\langle h_3, (0.9, 0.2) \rangle$
$k = 4$	$\langle h_2, (0.9, 0.1) \rangle$	$\langle h_2, (0.9, 0.1) \rangle$	$\langle h_3, (0.7, 0.2) \rangle$	$\langle h_2, (0.9, 0.1) \rangle$
$k = 5$	$\langle h_3, (0.9, 0.1) \rangle$	$\langle h_3, (0.9, 0.1) \rangle$	$\langle h_3, (0.9, 0.2) \rangle$	$\langle h_3, (0.9, 0.25) \rangle$

Table 2 The GPS of the expert e_1

$GS(a_{ij}^1)$	c_1	c_2	c_3	c_4
x_1	0.4457	0.0477	0.3129	0.5769
x_2	0.1386	0.1144	0.5433	0.3682
x_3	0.3735	0.4666	0.2015	0.1255
x_4	0.6025	0.3863	0.4160	0.3894
x_5	0.5507	0.2887	0.1043	0.0672

Table 3 The GPS of the expert e_2

$GS(a_{ij}^2)$	c_1	c_2	c_3	c_4
x_1	0.4276	0.4334	0.5145	0.0601
x_2	0.4192	0.6604	0.4367	0.5891
x_3	0.5083	0.5170	0.2834	0.4794
x_4	0.4427	0.3535	0.4206	0.6364
x_5	0.4667	0.6373	0.3016	0.2092

Table 4 The GPS of the expert e_3

$GS(a_{ij}^3)$	c_1	c_2	c_3	c_4
x_1	0.6364	0.4226	0.6077	0.5176
x_2	0.3235	0.4135	0.3271	0.4151
x_3	0.1562	0.5300	0.3503	0.4232
x_4	0.6364	0.5584	0.5107	0.4719
x_5	0.0707	0.4379	0.3396	0.7186

Table 5 The GPS of the expert e_4

$GS(a_{ij}^4)$	c_1	c_2	c_3	c_4
x_1	0.1345	0.0691	0.1366	0.3630
x_2	0.2624	0.1083	0.5127	0.4700
x_3	0.1829	0.2749	0.1591	0.3169
x_4	0.4422	0.5308	0.3477	0.2025
x_5	0.1886	0.0237	0.5127	0.4921

Table 6 The GPS of the expert e_5

$GS(a_{ij}^5)$	c_1	c_2	c_3	c_4
x_1	0.6364	0.6378	0.1240	0.2180
x_2	0.5757	0.5508	0.6189	0.3173
x_3	0.0530	0.2278	0.5714	0.2629
x_4	0.6364	0.5584	0.6351	0.4650
x_5	0.5586	0.1306	0.5309	0.6470

Table 7 The net preference intensity of x_i over x_s of expert e_1

$\Delta\tau^1(x_i, x_s)$	x_1	x_2	x_3	x_4	x_5
x_1	0.0000	-0.1243	-0.0540	0.1028	-0.0930
x_2	0.1243	0.0000	0.0007	0.1574	-0.0384
x_3	0.0540	-0.0007	0.0000	0.1567	-0.0391
x_4	-0.1028	-0.1574	-0.1567	0.0000	-0.1958
x_5	0.0930	0.0384	0.0391	0.1958	0.0000

(4) and (5) By Eqs. (23), (24) and (25), we calculate the positive, negative, and net flow of each alternative for all experts. The details results are shown in Tables 12, 13, 14, 15 and 16.

Table 8 The net preference intensity of x_i over x_s of expert e_2

$\Delta\tau^2(x_i, x_s)$	x_1	x_2	x_3	x_4	x_5
x_1	0.0000	-0.0097	0.0881	0.2485	0.0448
x_2	0.0097	0.0000	-0.0793	-0.0630	-0.1226
x_3	-0.0881	0.0793	0.0000	0.0163	-0.0433
x_4	-0.2485	0.0630	-0.0163	0.0000	-0.0596
x_5	-0.0448	0.1226	0.0433	0.0596	0.0000

Table 9 The net preference intensity of x_i over x_s of expert e_3

$\Delta\tau^3(x_i, x_s)$	x_1	x_2	x_3	x_4	x_5
x_1	0.0000	-0.1763	-0.1811	-0.0017	-0.1544
x_2	0.1763	0.0000	-0.0049	0.1746	0.0978
x_3	0.1811	0.0049	0.0000	0.1794	0.0268
x_4	0.0017	-0.1746	-0.1794	0.0000	-0.1526
x_5	0.1544	-0.0978	-0.0268	0.1526	0.0000

Table 10 The net preference intensity of x_i over x_s of expert e_4

$\Delta\tau^4(x_i, x_s)$	x_1	x_2	x_3	x_4	x_5
x_1	0.0000	0.0102	0.0577	0.2050	0.1285
x_2	-0.0102	0.0000	-0.1049	0.0425	-0.0286
x_3	-0.0577	0.1049	0.0000	0.1474	0.0708
x_4	-0.2050	-0.0425	-0.1474	0.0000	-0.0765
x_5	-0.1285	0.0286	-0.0708	0.0765	0.0000

Table 11 The net preference intensity of x_i over x_s of expert e_5

$\Delta\tau^5(x_i, x_s)$	x_1	x_2	x_3	x_4	x_5
x_1	0.0000	-0.0276	-0.1253	0.2314	0.0627
x_2	0.0276	0.0000	-0.2369	0.0581	0.0336
x_3	0.1253	0.2369	0.0000	0.2950	0.1880
x_4	-0.2314	-0.0581	-0.2950	0.0000	-0.1069
x_5	-0.0627	-0.0336	-0.1880	0.1069	0.0000

Table 12 The output results of $\eta^{1,+}(x_i)$, $\eta^{1,-}(x_i)$, and η_i^1

Alternatives	$\eta^{1,+}(x_i)$	$\eta^{1,-}(x_i)$	η_i^1
x_1	-0.1685	0.1685	0.3370
x_2	0.2440	-0.2440	0.4880
x_3	0.1709	-0.1709	0.3418
x_4	-0.6127	0.6127	1.2255
x_5	0.3663	-0.3663	0.7327

Table 13 The output results of $\eta^{2,+}(x_i)$, $\eta^{2,-}(x_i)$, and η_i^2

Alternatives	$\eta^{2,+}(x_i)$	$\eta^{2,-}(x_i)$	η_i^2
x_1	0.3717	-0.3717	0.7434
x_2	-0.2553	0.2553	0.5105
x_3	-0.0358	0.0358	0.0717
x_4	-0.2613	0.2613	0.5226
x_5	0.1807	-0.1807	0.3614

Table 14 The output results of $\eta^{3,+}(x_i)$, $\eta^{3,-}(x_i)$, and η_i^3

Alternatives	$\eta^{3,+}(x_i)$	$\eta^{3,-}(x_i)$	η_i^3
x_1	-0.5135	0.5135	1.0270
x_2	0.4438	-0.4438	0.8875
x_3	0.3922	-0.3922	0.7844
x_4	-0.5049	0.5049	1.0098
x_5	0.1824	-0.1824	0.3649

Table 15 The output results of $\eta^{4,+}(x_i)$, $\eta^{4,-}(x_i)$, and η_i^4

Alternatives	$\eta^{4,+}(x_i)$	$\eta^{4,-}(x_i)$	η_i^4
x_1	0.4013	-0.4013	0.8026
x_2	-0.1012	0.1012	0.2023
x_3	0.2654	-0.2654	0.5308
x_4	-0.4714	0.4714	0.9427
x_5	-0.0942	0.0942	0.1884

Table 16 The output results of $\eta^{5,+}(x_i)$, $\eta^{5,-}(x_i)$, and η_i^5

Alternatives	$\eta^{5,+}(x_i)$	$\eta^{5,-}(x_i)$	η_i^5
x_1	0.1413	-0.1413	0.2825
x_2	-0.1177	0.1177	0.2353
x_3	0.8451	-0.8451	1.6903
x_4	-0.6914	0.6914	1.3827
x_5	-0.1774	0.1774	0.3547

(5) By Eq. (26) we calculate the conditional probabilities of each alternative for each expert. The details results are shown in Table 17.

(6) By Eq. (27), the probabilities of alternatives are:

$$P(x_1) = 0.2188, P(x_2) = 0.1541, P(x_3) = 0.1824, \\ P(x_4) = 0.3128 \text{ and } P(x_5) = 0.1320.$$

Table 17 The conditional probabilities of each alternative

$P(x_i e_k)$	e_1	e_2	e_3	e_4	e_5
x_1	0.1079	0.1562	0.1094	0.3921	0.2345
x_2	0.3364	0.2311	0.0324	0.2365	0.1636
x_3	0.2521	0.2179	0.1926	0.2479	0.0896
x_4	0.3010	0.0759	0.1990	0.3535	0.0706
x_5	0.0716	0.0596	0.4284	0.3505	0.0899

Table 18 The belief distance between two experts about each alternative

D_{ui}	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
$u = 1(e_1, e_2)$	0.1286	0.9970	0.2002	1.0277	0.7748
$u = 2(e_1, e_3)$	0.7173	0.7852	0.5696	0.8513	0.8614
$u = 3(e_1, e_4)$	0.6221	0.3053	0.4002	0.3801	0.8760
$u = 4(e_1, e_5)$	0.1323	0.3533	0.1067	0.6035	0.8718
$u = 5(e_2, e_3)$	0.7080	0.7241	0.6932	0.5999	0.3100
$u = 6(e_2, e_4)$	0.5994	0.9327	0.5608	0.8522	0.3885
$u = 7(e_2, e_5)$	0.1844	0.9345	0.3083	0.7881	0.3654
$u = 8(e_3, e_4)$	0.1083	0.8220	0.1715	0.5580	0.0724
$u = 9(e_3, e_5)$	0.7122	0.8348	0.4432	0.3604	0.0448
$u = 10(e_4, e_5)$	0.6335	0.0409	0.2654	0.3316	0.0411

Step 5: By Eq. (32), the belief distance between two experts about each alternative is calculated, which is presented in Table 18.

By Eq. (33), the Deng entropy between two experts about each alternative is shown in Table 19.

Table 19 The Deng entropy between two experts about each alternative

$\vartheta_{ui}(D_{ui})$	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
$u = 1(e_1, e_2)$	0.8831	3.8995	1.2469	3.9747	3.3124
$u = 2(e_1, e_3)$	3.1463	3.3416	2.6879	3.5236	3.5508
$u = 3(e_1, e_4)$	2.8565	1.7152	2.0922	2.0154	3.5897
$u = 4(e_1, e_5)$	0.9031	1.9108	0.7614	2.7974	3.5787
$u = 5(e_2, e_3)$	3.1188	3.1662	3.0746	2.7860	1.735
$u = 6(e_2, e_4)$	2.7844	3.7376	2.6589	3.5262	2.0478
$u = 7(e_2, e_5)$	1.1701	3.7423	1.7278	3.3497	1.9585
$u = 8(e_3, e_4)$	0.7705	3.4438	1.1064	2.6495	0.5572
$u = 9(e_3, e_5)$	3.1312	3.4790	2.2518	1.9386	0.3757
$u = 10(e_4, e_5)$	2.8921	0.3485	1.5449	1.8234	0.3502

Table 20 The standardized Deng entropy between two experts about each alternative

$\beta_{ui}(D_{ui})$	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
$u = 1(e_1, e_2)$	-0.9052	1.0000	-0.5802	1.0000	0.8288
$u = 2(e_1, e_3)$	1.0000	0.6858	0.6656	0.5806	0.9760
$u = 3(e_1, e_4)$	0.7560	-0.2302	0.1506	-0.8215	1.0000
$u = 4(e_1, e_5)$	-0.8884	-0.1201	-1.0000	-0.0944	0.9932
$u = 5(e_2, e_3)$	0.9768	0.5870	1.0000	-0.1051	-0.1450
$u = 6(e_2, e_4)$	0.6953	0.9088	0.6406	0.5830	0.0481
$u = 7(e_2, e_5)$	-0.6636	0.9115	-0.1644	0.4190	-0.0071
$u = 8(e_3, e_4)$	-1.0000	0.7433	-0.7018	-0.2320	-0.8722
$u = 9(e_3, e_5)$	0.9873	0.7632	0.2886	-0.8929	-0.9842
$u = 10(e_4, e_5)$	0.7860	-1.0000	-0.3226	-1.0000	-1.0000

By Eq. (34), the standardized Deng entropy between two experts about each alternative is shown in Table 20.

By Eq. (36), the interface effect among experts for each alternative is:

$$\begin{aligned} INT_1 &= 0.1756, INT_2 = 0.3144, INT_3 = -0.0562, \\ INT_4 &= -0.0814, \text{ and } INT_5 = 0.1371. \end{aligned}$$

Finally, by Eq. (38), the probability of each alternative is:

$$\begin{aligned} Prob(x_1) &= 0.2648, Prob(x_2) = 0.3145, Prob(x_3) = 0.0847, \\ Prob(x_4) &= 0.1553, \text{ and } Prob(x_5) = 0.1806. \end{aligned}$$

Step 6: According to the value of QP of each alternative, the final ranking is

$$x_2 \succ x_1 \succ x_5 \succ x_4 \succ x_3.$$

4.1.2 Effect of the Quantum Interface on Decision Result

If we can decide based on the $P(x_i) (i = 1, 2, 3, 4, 5)$ values according to Eq. (27), as higher probabilities indicate a superior ranking. The classical approach does not account for the interference effect, which can influence the final rankings, as shown in Fig. 4. The ranking result of alternatives according to Eq. (27) is $x_4 \succ x_1 \succ x_3 \succ x_2 \succ x_5$. If we count the interface effect between two experts about an alternative the ranking result is $x_2 \succ x_1 \succ x_5 \succ x_4 \succ x_3$.

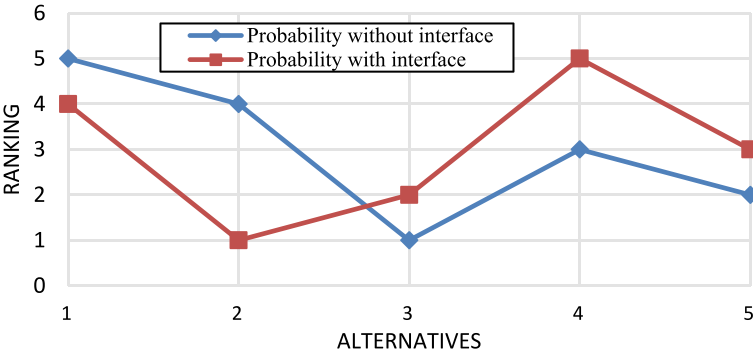


Fig. 4 The interface effect on ranking values

4.1.3 Sensitive Analysis

Our proposed quantum ORESTE-based MCGDM approach under the PLI environment involved an adjustment parameter $\kappa \in (0, 1)$ to represent the significance between the positive ideal solution under each criterion and the value of important criteria. In Fig. 5 we display the impact of κ on the QP of each alternative. In Fig. 6, we display how much effect the changes of the κ on decision results.

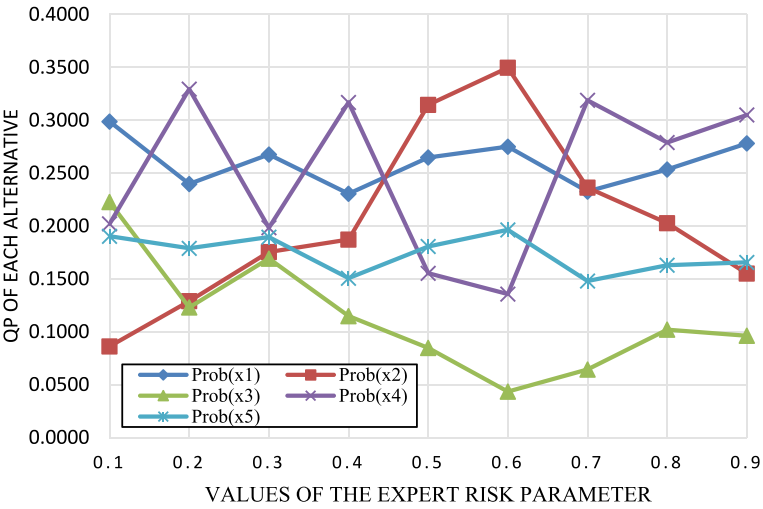


Fig. 5 The impact of κ on the QP of each alternative

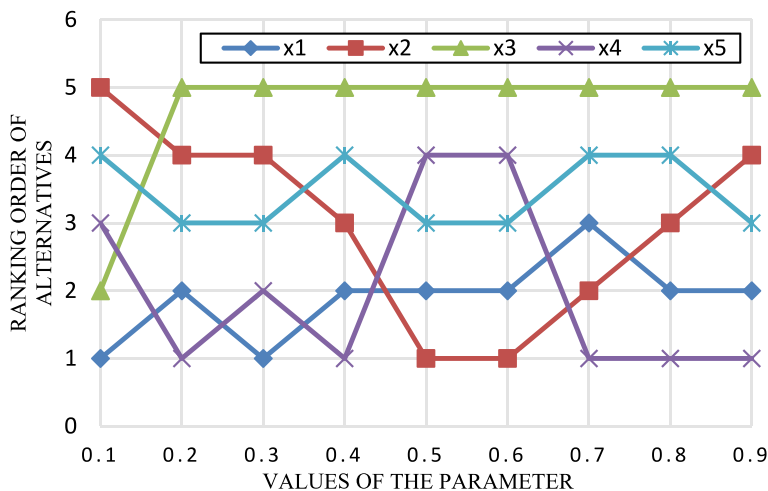


Fig. 6 The impact of κ on the decision result

4.1.4 Comparative Analysis

To test the feasibility and validity of the suggested method, we compare it with some existing methods for the example given above.

In the traditional decision-making model (Mandal and Ranadive 2019) the ranking results are derived from the aggregate decision matrix of experts' opinions. Therefore, to compare our model and the traditional model, we first aggregate the experts' opinions. In this connection, by Mandal and Ranadive (2019), using the Pythagorean linguistic weighted averaging aggregation operator, the aggregation of five experts' opinions is

$$\text{Agg}(A^1, A^2, A^3, A^4, A^5) =$$

$$\begin{matrix} & & c_1 & c_2 & c_3 & c_4 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \left[\begin{array}{cccc} \left\langle h_{-0.56}, (0.87, 0.15) \right\rangle & \left\langle h_{1.36}, (0.83, 0.2) \right\rangle & \left\langle h_{0.21}, (0.8, 0.25) \right\rangle & \left\langle h_{0.61}, (0.76, 0.28) \right\rangle \\ \left\langle h_{0.72}, (0.84, 0.18) \right\rangle & \left\langle h_{0.06}, (0.82, 0.2) \right\rangle & \left\langle h_{-0.67}, (0.61, 0.43) \right\rangle & \left\langle h_{0.043}, (0.63, 0.42) \right\rangle \\ \left\langle h_{1.56}, (0.78, 0.21) \right\rangle & \left\langle h_{0.15}, (0.64, 0.36) \right\rangle & \left\langle h_{1.7}, (0.68, 0.41) \right\rangle & \left\langle h_{0.72}, (0.86, 0.21) \right\rangle \\ \left\langle h_{-1.53}, (0.77, 0.24) \right\rangle & \left\langle h_{-0.69}, (0.73, 0.19) \right\rangle & \left\langle h_{-0.93}, (0.83, 0.27) \right\rangle & \left\langle h_{-0.83}, (0.88, 0.23) \right\rangle \\ \left\langle h_{1.53}, (0.73, 0.26) \right\rangle & \left\langle h_{1.11}, (0.73, 0.24) \right\rangle & \left\langle h_{0.88}, (0.73, 0.24) \right\rangle & \left\langle h_{0.06}, (0.68, 0.33) \right\rangle \end{array} \right] \end{matrix}$$

Therefore, the ranking of alternatives according to the methods ORESTE method is $x_4 > x_3 > x_5 > x_1 > x_2$, i.e., the optimal option is x_4 . When the proposed

model is applied without considering the interface effect, the optimal option is also x_4 . This suggests that the optimal option of the proposed approach aligns with those of traditional methods when the impact of decision-makers is disregarded, thereby validating the robustness of the suggested model. However, when evaluating the quantum interference among decision-makers, the outcomes of this model suggest that x_2 is the optimal strategy. This inconsistency arises from the conventional framework's exclusion of the interference effect among decision-makers, which leads to subjective biases during the evaluation of alternatives. As a result, the outcomes of this model better align with real-world complexities.

5 Conclusion

This paper presents a quantum group decision model designed to mitigate information ambiguity and opinion interference in selecting cell phones. The approach uses PLIs to convey ambiguous information in complex decision-making contexts, employing Deng entropy to assess opinion interference by modeling the reciprocal effects among decision-makers in the MAGDM process. Unlike conventional methods that treat decision-makers as independent entities, this paradigm addresses interference by assigning appropriate weights to decision-makers and regulating opinion interference through quantum decision-making procedures.

The proposed model enhances existing methodologies, offering more efficient solutions to challenges related to opinion interference and weight distribution. Our findings show that the model significantly improves decision-making in the selection of cell phones by providing a comprehensive framework to address information ambiguity and conflicting perspectives. However, the computational complexity increases with the inclusion of additional decision-makers and variables, highlighting the need for advanced computational techniques to maintain efficiency.

Future research will focus on integrating machine learning methods to enhance scalability and practicality, ensuring the model's effectiveness in large-scale decision-making processes. This integration aims to manage rising complexity, foster consensus in quantum group decision-making, and ensure the model's long-term sustainability.

Conflict of Interest The authors declare that they have no conflict of interest.

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Pythagorean Linguistic Information-Based Green Supplier Using Quantum-Based Group Decision-Making Methodology and the MULTIMOORA Approach



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Abstract The selection of environmentally sustainable suppliers has historically posed a significant challenge in management decision-making (DM). Organizations must adopt a comprehensive strategy by evaluating prospective suppliers based on their competencies, financial sustainability, and ecological impact. Several studies have proposed ranking methodologies for selecting environmentally sustainable suppliers, as new models with enhanced capabilities offer greater environmental benefits. However, much research has revealed deficiencies in the confidence levels of decision-makers when providing judgments, which significantly affect the final ranking outcomes. The Pythagorean linguistic number (PLN) concept has proven to be a valuable tool for addressing multi-attribute decision-making (MAGDM) by incorporating experts' confidence levels and addressing challenges in complex real-world scenarios. Furthermore, in modern multicriteria group decision-making (MCGDM), expert opinions often influence one another, and experts frequently exhibit subjective, bounded rationality during the decision-making process. This research presents a novel approach that utilizes an extended MULTIMOORA-based Pythagorean linguistic multicriteria quantum group decision-making methodology to address these issues. A numerical example is provided to illustrate the proposed methodology, focusing on the selection of environmentally sustainable suppliers. Finally, we demonstrate the applicability and effectiveness of the method.

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Keywords Quantum decision theory · Group decision-making · Pythagorean linguistic set · MULTIMOORA

1 Introduction

Multi-criteria group decision making (DM) (MCGDM) is extensively utilized to assess or choose one or more options from a set of potential alternatives, even when faced with numerous and often conflicting criteria (Yeh 2002). With the rapid advancement of societies and economies, the complexity of DM has escalated, further complicated by the influx of data in daily activities. The presence of incomplete or inaccurate information, the ambiguity of the subject, and the cognitive limitations of experts often make existing MAGDM methods insufficient for real-world applications. Consequently, significant attempts have been directed at developing more compelling strategies to reduce the encounter of ambiguity and subjectivity in DM. Ideally, optimal outcomes are expressed as precise numerical values or clear directives. However, in practice, decisions often fall within a range of ambiguous outcomes, involving multiple possible rankings. To address this, Zadeh introduced the notion of fuzzy sets (FS). Subsequent research has refined FS theory, providing useful methods for handling complex decision problems under uncertainty (Salahshour et al. 2012; Allahviranloo et al. 2015; Chehlabi and Allahviranloo 2016; Armand et al. 2018). After that the research in fuzzy set theory and its applications plays a significant role in addressing uncertainty and complexity in social networks and group decision-making processes. The methods developed in this area offer valuable insights into how fuzzy logic can be used to model and analyze systems where imprecision and ambiguity are prevalent. Fuzzy set theory is particularly well-suited to modeling the uncertainty inherent in social networks (Mahapatra et al. 2019, 2020), where relationships and interactions often involve incomplete or vague information. Traditional mathematical models struggle to capture this uncertainty, but fuzzy approaches provide a more nuanced framework for analyzing such dynamics.

Recognizing the limitations of FS in representing information, Atanassov introduced the intuitionistic fuzzy set (IFS), incorporating the degree of opposition to a given attribute named non-membership degree. Compared to FS, IFS offers a more comprehensive depiction of decision information, capturing levels of support as well as opposition, and neutrality among experts. Numerous studies have highlighted IFS's ability to handle uncertainty and ambiguity through aggregation operators, distance measures, correlation coefficients, and MAGDM techniques (Senapati et al. 2022). However, the growing complexity of real-world decision problems has exposed limitations in IFS, particularly the requirement that the sum of membership and non-membership degrees (MANMDs) must not exceed 1. In response, Yager (Yager 2013) proposed the Pythagorean fuzzy set (PFS), which relaxes this constraint by allowing the addition of MANMDs to exceed 1, as long as the addition of their squares remains less than or equal to 1. Numerous studies have highlighted IFS's

ability to handle uncertainty and ambiguity through MCGDM techniques (Senapati et al. 2023). This extension enhances the ability to model fuzzy phenomena more effectively compared to FS and IFS. However, as noted by Xu (2005) and Mandal et al. (2022a), experts tend to express their opinions more reliably using linguistic information, known as linguistic sets (LS), rather than fuzzy information. In this connection, Mandal and Ranadive (2019) the LS concept was extended to the Pythagorean linguistic set (PLS), allowing experts to express both their confidence and doubt levels when providing judgments. But in real-life situations, experts often have limited knowledge and personal biases, so using language-based information with expert confidence and doubt levels (Mandal et al. 2021, 2022b, 2023a, b, 2024a) is more flexible for expressing their ideas. In this connection, Mandal et al. briefly studied MCGDM under LZN information with the MARCOS approach and its application in the selection of logistic cold chain centers (Mandal et al. 2024b); TODIM-VIKRO approach (Mandal et al. 2024c) and its application in site selection of medical logistic center; ORESTE approach (Mandal et al. 2024d) and its application in assessment of regional energy; and MULTIMOORA approach (Mandal et al. 2024e) and its application in selection of software. To address the challenge of aggregating individual assessments, specific methods are needed. Although many aggregation strategies have been studied, most assume that decision-makers are independent and uninfluenced by one another. However, in practice, opinions are often influenced by surrounding perspectives. In decision-making, the views of multiple experts can interact, ultimately affecting the final decision, and introducing the concept of 'interference'.

In practical scenarios, decision-makers may express their preferences based on personal perspectives, which can vary across issues. Therefore, it is crucial to consider the risk attitudes of decision-makers in MCGDM. The MULTIMOORA method stands for less time, high stability, simple calculation, and more concise by comparing TODIM, ORESTE, TOPSIS, and VIKOR according to rigorous studies by Hafezalkotob et al. (2019). Therefore, the MULTIMOORA method is a comprehensive MCGDM tool. It is designed to assess a range of options against a backdrop of varied and often conflicting criteria. The approach is particularly noted for:

- (1) **Ratio System (RS):** This part of the approach ranks alternatives based on the ratio of the beneficial criteria to the non-beneficial criteria. Each criterion is given equal weight, and the alternative with the highest ratio is considered the best.
- (2) **Reference Point (RP) Approach:** This involves comparing alternatives to a reference point, which usually represents the most desirable performance levels across all criteria. The alternative closest to this reference point is preferred.
- (3) **Full Multiplicative Form (FMF):** This component multiplies the performance ratings of each alternative across all criteria. The alternative with the highest product is deemed the most suitable.

The MULTIMOORA approach is appreciated for its simplicity, comprehensiveness, and ability to provide a more balanced decision-making process by combining

these three methods. It's used in engineering, economics, and healthcare, to facilitate complex decision-making scenarios.

To address the challenge of aggregating individual assessments, specific methods are needed. Although many aggregation strategies have been studied, most assume that decision-makers are independent and uninfluenced by one another. However, in practice, opinions are often influenced by surrounding perspectives. In decision-making, the views of multiple experts can interact, ultimately affecting the final decision, and introducing the concept of 'interference'. Quantum Probability Theory (QPT) (Busemeyer and Bruza 2012; Moreira and Wichert 2014; Yearsley and Busemeyer 2016) offers a framework for capturing these influences. QPT, a recent advancement over classical probability theory (CPT), has been extensively studied in fields such as psychology (Busemeyer and Bruza 2012), cognitive science (Yearsley and Busemeyer 2016), and decision analysis (He and Jiang 2018a, b; Wu et al. 2020; Xiao et al. 2022; Han and Liu 2023; Mandal et al., 2024f; 2025) due to its robust descriptive power. It effectively explains paradoxes, such as the disjunction fallacy (He and Jiang 2018a, b), the Ellsberg paradox (Eichberger and Pirner 2018), the order effect (Wang et al. 2014), and graph theory (Samanta and Allahviranloo 2025) which are challenging for CPT to address. QPT's ability to represent interference terms makes it particularly valuable for understanding how expert opinions influence each other in the MCGDM process.

In this chapter, a quantum-based Pythagorean linguistic MCGDM model is proposed. This model takes into account the constrained rationality of decision-makers as well as the impact of interference that occurs among different viewpoints. The following is a list of the key contributions that this paper makes:

- (1) An improved Pythagorean linguistic information-based MULTIMOORA approach accounts for experts' limited rationality, less time, high stability, simple calculation, and more concise, and resolves some paradoxes of the traditional MULTIMOORA approach.
- (2) A quantum decision theory (QDT)-based method for combining expert opinion results, which helps us better understand human psychology through the effects of interference.
- (3) A case study demonstrating the effectiveness of our approach in the GSCM. Comparative and sensitivity analyses show the flexibility and robustness of our method.

The structure of the paper is as follows: In Sect. 2, recall some basic definitions and the background of linguistic term sets (LTSs), the Pythagorean linguistic set (PLS), and quantum decision theory (QDT). The proposed MCGDM model is discussed in Sect. 3. A case study for the GSS with comparative analysis is provided in Sect. 4. Section 5 concludes our study.

2 Preliminaries

Here we recall some related definitions and results that are used in this paper.

2.1 Linguistic Term Set (LTS)

Here we assume an LTS is: $H = \{h_\lambda | \lambda = -\zeta, \dots, -1, 0, 1, \dots, \zeta\}$ with cardinality $2\zeta + 1$ and holds the followings (Mandal et al. 2024a):

- (1) $h_{\lambda_1} \geq h_{\lambda_2}$ if and only if $\lambda_1 \geq \lambda_2$;
- (2) $neg(h_{\lambda_1}) = h_{-\lambda_1}$;
- (3) $max(h_{\lambda_1}, h_{\lambda_2}) = h_{\lambda_1}$ if $h_{\lambda_1} \geq h_{\lambda_2}$;
- (4) $min(h_{\lambda_1}, h_{\lambda_2}) = h_{\lambda_1}$ if $h_{\lambda_1} \leq h_{\lambda_2}$.

To preserve expert information, the continuous LTSs (Xu 2005) denotes $\bar{H} = \{h_{-t} \leq h_\lambda h_t | \lambda \in [-t, t]\}$, $t > \zeta$.

The subscript index of any $h_\lambda \in \bar{H}$, represented by the following functions:

$$I : \bar{H} \rightarrow [-t, t], \text{ and } I(h_\lambda) = \lambda.$$

$$I^{-1} : [-t, t] \rightarrow \bar{H}, \text{ and } I^{-1}(\lambda) = h_\lambda.$$

Let $\mathcal{F}$ be the linguistic conversation function for any LT $G \in \bar{H}$, which is defined in the following way:

$$\mathcal{F} : \bar{H} \rightarrow [0, 1], \mathcal{F}(G) = \{\mathcal{F}(h_\lambda) = \frac{\lambda}{2t} + \frac{1}{2} | \lambda \in [-t, t]\} = \kappa.$$

2.2 Pythagorean Linguistic Set (PTS)

By Mandal and Ranadive (2019), a Pythagorean linguistic set (PLS) denoted by $\mathcal{P}$ in an alternative set X for continuous LTS $\bar{H} = \{h_{-t} h_\lambda h_t | \lambda \in [-t, t]\}$ is represented in the following way:

$$\mathcal{R} = \{\langle h_{\theta(x)}, (\mu_{\mathcal{R}}(x), \nu_{\mathcal{R}}(x)) \rangle | x \in X\},$$

where, $h_{\theta(x)} \in \bar{H}$, $\mu_{\mathcal{R}} : X \rightarrow [0, 1]$ and $\nu_{\mathcal{R}} : X \rightarrow [0, 1]$ are the membership and non-membership degrees of x to $h_{\theta(x)}$ such that $\mu_{\mathcal{R}}^2(x) + \nu_{\mathcal{R}}^2(x) \leq 1$. Here, the Pythagorean linguistic number (PLN) in $\mathcal{R}$ is denoted by $\wp$ and represented by $\wp = h_{\theta(\wp)}, (\mu(\wp), \nu(\wp))$ with $h_{\theta(\wp)} \in \bar{H}$ and $\mu(\wp), \nu(\wp) \in [0, 1]$, $\mu^2(\wp) + \nu^2(\wp) \leq 1$.

By Mandal and Ranadive (2019), the expected value of $\wp = h_{\theta(\wp)}, (\mu(\wp), \nu(\wp))$ is denoted by $\varkappa$ and defined in the following way:

$$\chi(\wp) = h_{\frac{\theta(\wp) \times (\mu^2(\wp) + 1 - \nu^2(\wp))}{2}}.$$

By Wang et al. (2019), the distance between two PLNs $\wp_1 = h_{\theta(\wp_1)}, (\mu(\wp_1), \nu(\wp_1))$ and $\wp_2 = h_{\theta(\wp_2)}, (\mu(\wp_2), \nu(\wp_2))$ are defined in the following way:

$$d(\wp_1, \wp_2) = \frac{1}{2} \left| \frac{(1 + \mu^2(\wp_1) - \nu^2(\wp_1)) * \mathcal{F}(h_{\theta(\wp_1)})}{-(1 + \mu^2(\wp_2) - \nu^2(\wp_2)) * \mathcal{F}(h_{\theta(\wp_2)})} \right|.$$

2.3 Quantum Probability Theory (QPT)

A QDT model is based on the principles of quantum probability (QP) theory (QPT). To provide context, we will review some key concepts and notations of QPT. In QDT, a state is represented as a unit-length vector within an N-dimensional vector space.

Definition 1 Yearsley and Busemeyer (2016) A Hilbert space (HS) with finite-dimensional is represented by $\mathcal{H}$, where $\mathcal{H}$ is defined in abstract vector space over a complex field, spanned by a set of orthonormal basis vectors with complex components. In classical probability theory (CPT), events are fundamental (He et al. 2018), but in QPT, events are instead represented by subspaces of the HS $\mathcal{H}$, rather than traditional sets.

Definition 2 Busemeyer and Bruza (2012) In QDT, the superposition state $|\mathcal{S}\rangle$, a unit-length vector in the N-dimensional HS $\mathcal{H}$, defines all possible events within that space. The nonlinear combination of a set of orthogonal basis vectors $\{|\mathcal{V}_k, k = 1, 2, \dots, n\}$ are used to represent the superposition state $|\mathcal{S}\rangle$, i.e.,

$$\begin{aligned} |\mathcal{S}\rangle &= \langle \mathcal{V}_1 | \mathcal{S} \rangle |\mathcal{V}_1\rangle + \langle \mathcal{V}_2 | \mathcal{S} \rangle |\mathcal{V}_2\rangle + \dots + \langle \mathcal{V}_n | \mathcal{S} \rangle |\mathcal{V}_n\rangle \\ &= \xi_1 |\mathcal{V}_1\rangle + \xi_2 |\mathcal{V}_2\rangle + \dots + \xi_n |\mathcal{V}_n\rangle \\ &= \zeta_1 e^{i\alpha_1} |\mathcal{V}_1\rangle + \zeta_2 e^{i\alpha_2} |\mathcal{V}_2\rangle + \dots + \zeta_n e^{i\alpha_n} |\mathcal{V}_n\rangle \end{aligned} \quad (1)$$

where $\xi_k (= \zeta_k e^{i\alpha_k})$ is the amplitude of the probability, expressed as a complex number, characterizing the amplitude of the wave in the state $|\mathcal{V}_k\rangle$, and the amplitude of the phase angle represented by $e^{i\alpha_k}$.

Definition 3 Busemeyer and Bruza (2012) The CPT framework represents every event using a single set of components. In QPT, probabilities are assigned based on the overlap between cognitive states (shown as state vectors) and different probability measures (shown as subspaces). QPT extends CPT into the complex domain by including incompatible qualities and viewing events as vectors. Incompatible attributes are two observable features of an item that cannot be measured accurately at the same time. Both the incompatibility condition and the uncertainty principle can be described by the probability amplitude, also known as complex probability.

According to Born's rule, an event's probability amplitude and its classical probability, x_i are related to each other:

$$P(\mathcal{V}_k) = |\zeta_k e^{i\alpha_k}|^2 = (\zeta_k e^{i\alpha_k}) \times (\zeta_k e^{i\alpha_k})^* = \zeta_k e^{i\alpha_k} \times \zeta_k e^{-i\alpha_k} = (\zeta_k)^2 \quad (2)$$

For the total probability of event to be 1, the sum of the squares of the probability amplitudes must also be 1.

$$\sum_{k=1}^n P(\mathcal{V}_k) = \sum_{k=1}^n (\zeta_k)^2 = 1. \quad (3)$$

An $N - 1$ mapping transforms complex probabilities into classical probabilities. For each given probability value $P(\mathcal{V}_k)$, countless complex probabilities correspond to it. Referencing Eq. (2) and Euler's formula,

$$e^{i\alpha} = \cos\alpha + i \times \sin\alpha \quad (4)$$

the complex probability satisfies:

$$\xi = \sqrt{P(\mathcal{V})}(\cos\alpha + i \times \sin\alpha) = \sqrt{P(\mathcal{V})}e^{i\alpha} \quad (5)$$

For example, in the following, we discussed the QPT in two-dimensional Hilbert space. Let $\{\mathcal{M}_1, \mathcal{M}_2\}$ be an event containing two basic states, where $|\mathcal{M}_1\rangle = (1,0)$, and $|\mathcal{M}_2\rangle = (0,1)$ are defined in $\mathbf{H}$. In Fig. 1, we graphically show the state $|\mathcal{S}\rangle$, which is defined as a superposition. According to Eqs. (1) and (2), we get.

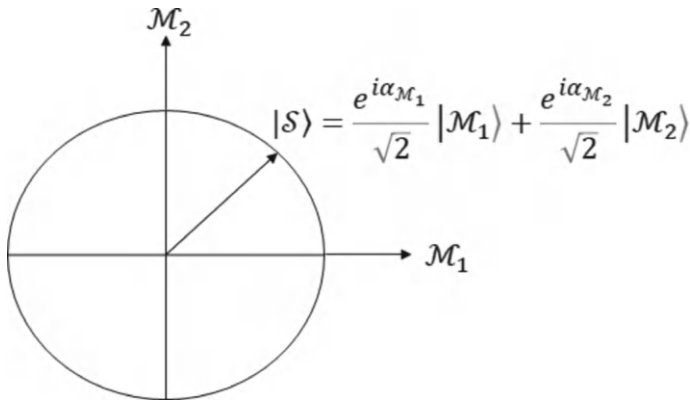


Fig. 1 Event in Hilbert space with two dimensions

$$|S\rangle = \zeta_1 e^{i\alpha_1} |\mathcal{M}_1\rangle + \zeta_2 e^{i\alpha_2} |\mathcal{M}_2\rangle = \frac{e^{i\alpha_{\mathcal{M}_1}}}{\sqrt{2}} |\mathcal{M}_1\rangle + \frac{e^{i\alpha_{\mathcal{M}_2}}}{\sqrt{2}} |\mathcal{M}_2\rangle.$$

and the probabilities of $|\mathcal{M}_1$ and $|\mathcal{M}_2$ are computed in the following way:

$$P(\mathcal{M}_1) = P(\mathcal{M}_2) = (e^{i\alpha})(e^{i\alpha})^* = \frac{1}{2}.$$

According to conventional Bayesian decision theory (BDT), DM can be mathematically represented as a Markov process (MP) (Moreira and Wichert 2014). The state transition probability (STP) in a single route of QPT aligns with the CPT according to Feynman's first rule (Wu et al. 2020), which varies over several pathways. Consider the decision-making dilemma highlighted above as an illustration of the process depicted in Fig. 2a. Experts provide their assessments of $\mathcal{M}_1$ ('bad') or $\mathcal{M}_2$ ('good') as probability quantifications. STPs are determined by combining the individual probabilities for each route and using the total probability formula in CPT.

$$P(\mathcal{M}_1) = \sum_{k=1}^2 P(e_k)P(\mathcal{M}_1|e_k), \sum_{k=1}^2 P(e_k) = 1 \quad (6)$$

Since only one route is selected at a time, it is sometimes referred to as the single path principle. The conventional MP assumes that experts rely on one expert at a time and that both experts are completely autonomous. However, this approach is sometimes unsuitable for ordinary cognitive processes because many avenues are often considered when forming judgments. Such phenomena can result in illogical actions, including conjunction errors as well as disjunction, and impacts of interference. Thus, QPT is an excellent complement to CPT in this scenario.

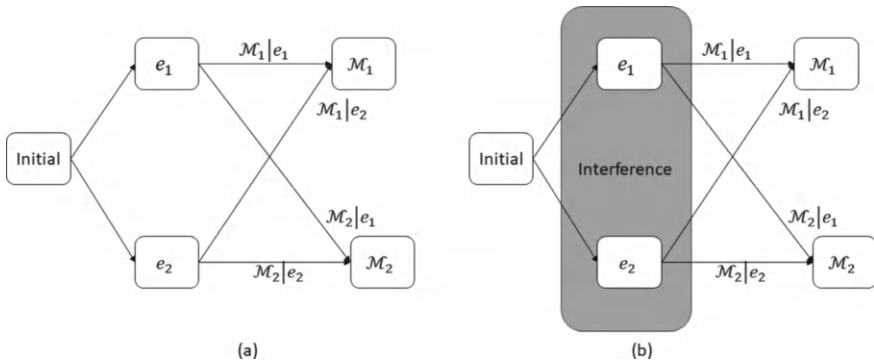


Fig. 2 DM in **a** MP and **b** QDT process

Within the QPT framework (Fig. 2b), the state is formed by combining three fundamental states, which can be represented as different wave functions aligning according to Feynman's second rule. The amplitude of the final form is obtained by superimposing two individual routes running from I ('Initial') to $\mathcal{M}_1$. The amplitude is the sum of the individual pathways from source I to target $\mathcal{M}_1$, and the probability of reaching the

$$P(\mathcal{M}_1) = |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 \quad (7)$$

Equation (7) can be simplified in the following way:

$$\begin{aligned} P(\mathcal{M}_1) &= |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 \\ &= |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}| |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^* \\ &= \left(\sqrt{P(e_1)P(\mathcal{M}_1|e_1)}e^{i\alpha_1} + \sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{i\alpha_2} \right) \\ &\quad \left(\sqrt{P(e_1)P(\mathcal{M}_1|e_1)}e^{-i\alpha_1} + \sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{-i\alpha_2} \right) \\ &= P(e_1)P(\mathcal{M}_1|e_1) + P(e_2)P(\mathcal{M}_1|e_2) \\ &\quad + \sqrt{P(e_1)P(\mathcal{M}_1|e_1)}\sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{i(\alpha_1-\alpha_2)} \\ &\quad + \sqrt{P(e_1)P(\mathcal{M}_1|e_1)}\sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{i(\alpha_2-\alpha_1)} \end{aligned} \quad (8)$$

By Eq. (4), we get

$$\cos(\alpha_1 - \alpha_2) = \frac{e^{i(\alpha_2-\alpha_1)} + e^{i(\alpha_1-\alpha_2)}}{2} \quad (9)$$

Then Eq. (7) can be written in the following way:

$$\begin{aligned} P(\mathcal{M}_1) &= P(e_1)P(\mathcal{M}_1|e_1) + P(e_2)P(\mathcal{M}_1|e_2) \\ &\quad + 2\sqrt{P(e_1)P(\mathcal{M}_1|e_1)}\sqrt{P(e_2)P(\mathcal{M}_1|e_2)}\cos(\alpha_1 - \alpha_2) \\ &= |\xi_{e_1}\xi_{\mathcal{M}_1|e_1}|^2 + |\xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 + 2\xi_{e_1}\xi_{\mathcal{M}_1|e_1}\xi_{e_2}\xi_{\mathcal{M}_1|e_2}\cos(\alpha_1 - \alpha_2) \end{aligned} \quad (10)$$

Upon comparing Eqs. (6) and (10), it becomes evident that the QDT framework includes an interference component. If the pathways are observable, the quantum phase transition will reduce to the classical one, following Feynman's third rule.

$$P(\mathcal{M}_1) = |\xi_{e_1}\xi_{\mathcal{M}_1|e_1}|^2 + |\xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 = P(e_1)P(\mathcal{M}_1|e_1) + P(e_2)P(\mathcal{M}_1|e_2) \quad (11)$$

Additionally, the QPT can be expanded to a multi-path graph, resulting in an increased number of interference terms. The QPT enhances CPT by accurately representing

cognitive processes in quantum terms, thereby more effectively modeling ambiguity, uncertainty, and vagueness in DM.

3 The Proposed MULTIMOORA-Based Magdm Under Pythagorean Linguistic Information and Quantum Decision Scenario

This section describes the new approach of quantum cognition MCGDM based on the MULTIMOORA method with PLN. The method evaluates multiple competing alternatives in the presence of diverse and conflicting criteria according to RS, RP, and FMF approaches. In the following, we described the details steps of the quantum cognition PLN-based MULTIMOORA approach.

3.1 Background

Let us assume

$X = \{x_1, x_2, \dots, x_n\}$: Alternative set,

$C = \{c_1, c_2, \dots, c_m\}$: Attribute set,

$e = \{e_1, e_2, \dots, e_q\}$: Expert set,

$w^k = (w_1^k, w_2^k, \dots, w_m^k)$: Weight vector of attributes for expert e_k , where $w_j^k \in [0, 1]$, and $\sum_{j=1}^m w_j^k = 1$,

$\omega = (\omega_1, \omega_2, \dots, \omega_q)$: Weight vector of experts with $\omega_k \in [0, 1]$, $\sum_{k=1}^q \omega_k = 1$.

3.2 Decision Steps

Let $A^k = (a_{ij}^k)_{nm}$ ($i = 1, 2, \dots, n; j = 1, 2, \dots, m; k = 1, 2, \dots, q$) be the PLI-based decision judgment matrix provided by an expert e_k using pre-defined distinct LTSs $H = \{h_\lambda | \lambda = -\varsigma, \dots, -1, 0, 1, \dots, \varsigma\}$.

Step 1: Construction of Decision Matrix

Then, in the following, we represent the decision matrix of k -th expert:

$$A^k = \left(a_{ij}^k \right)_{nm} = \begin{matrix} & \begin{matrix} c_1 & c_2 & \cdots & c_m \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} & \begin{bmatrix} a_{11}^k & a_{12}^k & \cdots & a_{1m}^k \\ a_{21}^k & a_{22}^k & \cdots & a_{2m}^k \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}^k & a_{n2}^k & \cdots & a_{nm}^k \end{bmatrix} \end{matrix}, \quad (12)$$

where, $a_{ij}^k = \left(h_{\theta_{ij}}^k, \left(\mu_{ij}^k, v_{ij}^k \right) \right)$ is a PLN, called the Pythagorean linguistic information (PLI) of k -th expert for i -th alternative about j -th criteria.

Step 2: Expert Weight Computation Process

Inspire by Mandal et al. (2024a), the weight of each expert $e_k (k = 1, 2, \dots, q)$ can be calculated in the following way:

$$\begin{aligned} \omega_k &= \frac{\Omega(Z^k)}{\sum_{k=1}^q (\Omega Z^k)}, \text{ where } \Omega(Z^k) \\ &= \frac{\sum_{j=1}^m \sum_{i=1}^n \mathcal{F}(\kappa(a_{ij}^k))}{mn} \end{aligned} \quad (13)$$

Step 3: Attribute Weight Computation Process

(1) The decision matrix (Eq. (12)) can be normalized to $L^k = \left(l_{ij}^k \right)_{nm}$ with

$$l_{ij}^k = \begin{cases} \frac{\mathcal{F}(\kappa(a_{ij}^k))}{\sum_{i=1}^n \mathcal{F}(\kappa(a_{ij}^k))}, & \text{for benefit criteria } c_j, \\ 1 - \frac{\mathcal{F}(\kappa(a_{ij}^k))}{\sum_{i=1}^n \mathcal{F}(\kappa(a_{ij}^k))}, & \text{for cost criteria } c_j. \end{cases} \quad (14)$$

(2) The Shannon entropy $\Psi^k = (\Psi_1^k, \Psi_2^k, \dots, \Psi_m^k)$ of each expert $e_k (k = 1, 2, \dots, q)$ can be calculated in the following way:

$$\Psi_j^k = -\frac{1}{\ln n} \sum_{i=1}^n l_{ij}^k \ln l_{ij}^k \quad (15)$$

$l_{ij}^k \ln l_{ij}^k$ is 0 if $l_{ij}^k = 0$.

(3) The attribute weight $w^k = (w_1^k, w_2^k, \dots, w_m^k)$ for expert $e_k (k = 1, 2, \dots, q)$ can be calculated in the following way:

$$w_j^k = \frac{1 - \Psi_j^k}{\sum_{j=1}^m (1 - \Psi_j^k)} \quad (16)$$

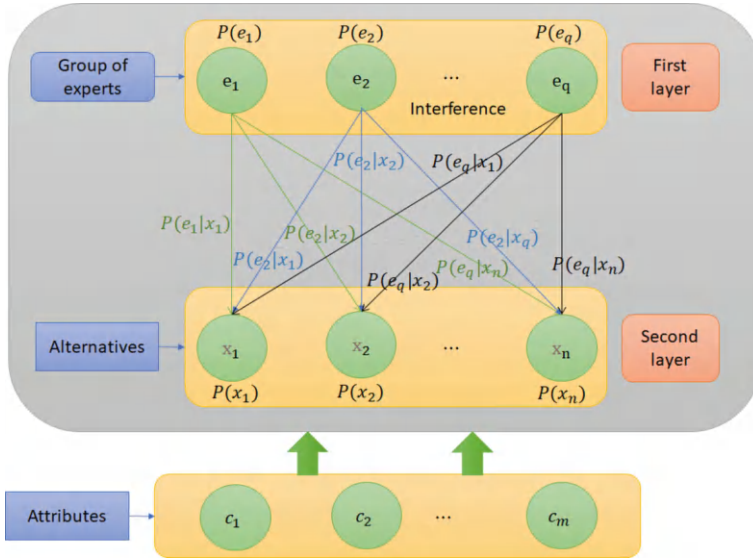


Fig. 3 MAGDM according to QSBN

Step 4: Construction of Bayesian Network

In this phase, we converted the pre-collected data into probabilities and developed a Bayesian Network for DM, as the QDT framework is a partial extension of the Bayesian Network. Figure 3 illustrates that, in a MAGDM situation, the DM process can be represented as a simple two-layer Bayesian Network. The experts' weights are converted into probabilities, which indicate the degree of belief in the experts, using the following equation:

$$P(e_k) = \omega_k, k = 1, 2, \dots, q. \quad (17)$$

In Eq. (17), the probabilities form the initial layer of the Bayesian Network (BN). The second layer depends on the assessment outcomes from each expert, which can be calculated using the MULTIMOORA approach as follows:

(1) The PLN Ratio Systems (PLN-RS) Approach

Consider $\mathcal{B}$ as the set of indices for benefit criteria and $\mathcal{C}$ for cost criteria. The overall utility is derived from the normalized decision matrix $L^k = (l_{ij}^k)_{nm}$ of each expert e_k for the i th alternative x_i , as per the PLN-RS, which is expressed as follows:

$$\dot{R}_1^k(x_i) = \left| \sum_{j \in \mathcal{B}} w_j^k l_{ij}^k - \sum_{j \in \mathcal{C}} w_j^k l_{ij}^k \right| \quad (18)$$

According to the greater value of $R_1^k(x_i)$ assigned a higher rank of alternative, which is denoted by $e_1^k(x_i)$ for expert e_k .

(2) The PLN Reference Point (PLN-RP) Approach

Let $\phi^k = (\phi_1^k, \phi_2^k, \dots, \phi_j^k, \dots, \phi_m^k)$ be the Maximal Objective Reference Vector (MORV) found from the normalized decision matrix $L^k = (l_{ij}^k)_{nm}$ of each expert e_k according to two following principles:

- (I) For the benefit criteria type, we have $\phi_j^k = \max_{i=1}^n l_{ij}^k, j \in \mathcal{B}$.
- (II) For cost criteria type we have $\phi_j^k = \min_{i=1}^n l_{ij}^k, j \in \mathcal{C}$.

Now from the MORV, the collective decision of the i th alternative x_i of each expert e_k calculate in the following way:

$$R_2^k(x_i) = \max_j |w_j^k l_{ij}^k - w_j^k \phi_j^k| \quad (19)$$

According to the lower value of $R_2^k(x_i)$ assigned a higher rank of alternative, which is denoted by $e_2^k(x_i)$ for expert e_k .

(3) The PLN Full Multiplicative Form (PLN-FMF) Approach

The overall utility is derived from the normalized decision matrix $L^k = (l_{ij}^k)_{nm}$ of each expert e_k for the i th alternative x_i , as per the PLN-FMF, which is expressed as follows:

$$R_3^k(x_i) = \frac{\mathbb{W}_i^{k,+}}{\mathbb{W}_i^{k,-}} \quad (20)$$

Where, $\mathbb{W}_i^{k,+} = \prod_{j \in \mathcal{B}} w_j^k l_{ij}^k$ and $\mathbb{W}_i^{k,-} = \prod_{j \in \mathcal{C}} w_j^k l_{ij}^k$.

According to the greater value of $R_3^k(x_i)$ assigned a higher rank of alternative, which is denoted by $e_3^k(x_i)$ for expert e_k .

(4) Borda Rule-Based Final Ranking Approach

- (I) By Wu et al. (2018), we assume the PLN-RS approach, PLN-RP approach, and PLN-FMF approach of each expert e_k are three attributes denoted by PLN – RS(ς_1^k), PLN – RP(ς_2^k) and PLN – FMF(ς_3^k). Therefore, each alternative for each expert e_k has three types of ranking values denoted by $R_\tau^k(x_i)$ ($\tau = 1, 2, 3$) and three types of ranking orders denoted by $e_\tau^k(x_i)$ ($\tau = 1, 2, 3$) concerning the attributes ς_1^k , ς_2^k and ς_3^k . Thus, the final ranking values of each alternative as multi-attribute decision-making contain a utility matrix $U^k = (R_\tau^k(x_i))_{n \times 3}$ (presented in Table 1) and ranking matrix $R^k = (e_\tau^k(x_i))_{n \times 3}$ (presented in Table 2).

Table 1 The utility matrix

$$\dot{U}^k = (R_\tau^k(x_i))_{n \times 3}$$

	PLN – RS(ζ_1^k)	PLN – RP(ζ_2^k)	PLN – FMF(ζ_3^k)
x_1	$R_1^k(x_1)$	$R_2^k(x_1)$	$R_3^k(x_1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_i	$R_1^k(x_i)$	$R_2^k(x_i)$	$R_3^k(x_i)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_n	$R_1^k(x_n)$	$R_2^k(x_n)$	$R_3^k(x_n)$

Table 2 The ranking matrix

$$R^k = (e_\tau^k(x_i))_{n \times 3}$$

	PLN – RS(ζ_1^k)	PLN – RP(ζ_2^k)	PLN – FMF(ζ_3^k)
x_1	$e_1^k(x_1)$	$e_2^k(x_1)$	$e_3^k(x_1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_i	$e_1^k(x_i)$	$e_2^k(x_i)$	$e_3^k(x_i)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_n	$e_1^k(x_n)$	$e_2^k(x_n)$	$e_3^k(x_n)$

(II) The normalized utility matrix of $\dot{U}^k = (R_\tau^k(x_i))_{n \times 3}$ is denoted by $N(\dot{U}^k) = (N(R_\tau^k(x_i)))_{n \times 3}$, where

$$N(R_\tau^k(x_i)) = \frac{R_\tau^k(x_i)}{\sum_{i=1}^n R_\tau^k(x_i)} \quad (21)$$

(III) The index of alternative x_i for expert e_k is denoted by β_i^k and obtain in the following way

$$\begin{aligned} \beta_i^k = & N((R_1^k(x_i))) \times \frac{n - e_1^k(x_i)}{n(n+1)/2} - N((R_2^k(x_i))) \times \frac{e_2^k(x_i)}{n(n+1)/2} \\ & + N((R_3^k(x_i))) \times \frac{n + e_3^k(x_i)}{n(n+1)/2}, \quad (i = 1, 2, \dots, n) \end{aligned} \quad (22)$$

(5) Based on Eq. (22), conditional probabilities which link the two layers in the BN as:

$$P(x_i|e_k) = \frac{\beta_i^k}{\sum_{i=1}^n \beta_i^k} \quad (23)$$

(6) The second layer consists of the probabilities of alternatives calculated as:

$$P(x_i) = \sum_{k=1}^q P(e_k)P(x_i|e_k) \quad (24)$$

Subsequently, we can decide based on the $P(x_i)$ values, as higher probabilities, indicate a superior ranking. The classical approach does not account for the interference effect, which can influence the final rankings. To address this issue, Bayesian networks are expanded into a quantum framework capable of properly modeling a practical approach to real interference in MAGDM.

Step 5: Construction of Quantum Scenario-Based Bayesian Network (QSBN)

The facilitator often consolidates opinions from several specialists. The interconnected decision-making processes in real-world situations, as demonstrated in Sect. 2.2, remain unobservable. The square of the sum of the amplitude probabilities of all possible pathways collectively constitutes the probability of the alternative x_i from QPT (He et al. 2018), as described.

$$\begin{aligned} P(x_i) &= \left| \sum_{k=1}^q \xi_{e_k} \xi_{x_i|e_k} e^{i\alpha_k} \right|^2 \\ &= \left| \sum_{k=1}^q \sqrt{P(e_k)} \sqrt{P(x_i|e_k)} e^{i\alpha_k} \right|^2, i = 1, 2, \dots, n \end{aligned} \quad (25)$$

By Eqs. (7)–(11), Eq. (25) can be written as

$$\begin{aligned} P(x_i) &= \sum_{k=1}^q P(e_k)P(x_i|e_k) \\ &+ 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \cos(\alpha_k - \alpha_s) \\ &= \sum_{k=1}^q P(e_k)P(x_i|e_k) + INT_i \end{aligned} \quad (26)$$

In a quantum-based Bayesian network, a normalization factor is incorporated into Eq. (26) to standardize the probability of each alternative, ensuring that the overall estimated probability does not exceed 1. Then

$$Prob(x_i) = \sigma P(x_i) \quad (27)$$

where,

$$\sigma = \frac{1}{\sum_{i=1}^n P(x_i)} = \frac{1}{\sum_{i=1}^n (\sum_{k=1}^q P(e_k)P(x_i|e_k) + INT_i)} \quad (28)$$

3.3 Analysis of Interference Effect Among Experts' Opinions

Equation (28) uses the term INT_i , which represent the ambiguity and uncertainty of expert opinions, to explain the superposition of decision information. In Eq. (28), $\cos(\alpha_k - \alpha_s)$ represents the phase angle between two distinct experts e_k and e_s , which represents the connection vector between two experts. $\frac{q(q-1)}{2}$ is the total count of pairwise correspondences across q experts, excluding duplicates. The final interference value is derived by summing the computed $\frac{q(q-1)}{2}$ interference values for each expert pair, rather than assessing the overall uncertainty across experts. This approach enhances the accuracy of the estimated probability value.

In real-world situations, it is hard to measure the phase angle. Experts' opinions can differ or be influenced by outside factors, causing a lot of variation and mistakes. Belief entropy helps a lot in handling and combining unclear information, making it easier to make good decisions in group settings with many factors. It is used as a standard in areas like risk assessment, defect diagnosis, and pattern recognition. Belief entropy is the latest way to measure interference in QSBN. Unlike the old method of manually estimating the phase angle range, this new method is more reliable because it reduces errors from personal judgment, which can affect the final decision and the expert's probability calculation. Belief entropy is used in QSBN to manage the interference among different experts' beliefs in GDM.

Then the belief distance about alternative x_i according to the average probability $\chi = \frac{1}{nq}$ can be calculated in the following way:

$$D_{iu} = \left| \delta_{iu} + \frac{|\delta_{iu} - \chi| - |\varrho_{iu} - \chi|}{|\delta_{iu} - \chi| + |\varrho_{iu} - \chi|} \right|, i = 1, 2, \dots, n; u = 1, 2, \dots, \frac{q(q-1)}{2},$$

$$\begin{cases} \delta_{iu} = P(e_k)P(x_i|e_k), & i = 1, 2, \dots, n; k = 1, \\ \varrho_{iu} = P(e_s)P(x_i|e_s), & 2, \dots, q-1; s = k+1, k+2, \dots, \frac{q(q-1)}{2}. \end{cases} \quad (29)$$

In Eq. (29), u means the value reallocated to the correlation among q experts; $\chi = \frac{1}{nq}$ represent each expert probability value of each alternative; D_{iu} means deviation degree from χ ; $|\delta_{iu} - \chi|$ and $|\varrho_{iu} - \chi|$ represents two experts' magnitude of the deviation degree from χ about alternative x_i . If $|\delta_{iu} - \chi| < |\varrho_{iu} - \chi|$, then the position of δ_{iu} and ϱ_{iu} are interchangeable.

There are many ways to measure uncertainty. Deng entropy (Zhao et al. 2024), also known as belief entropy, is better at including the confidence function and the uncertainty of BPA because it measures both total non-specificity and disagreement.

Thus, by Deng's entropy, we get

$$\begin{aligned}\vartheta_{iu}(D_{iu}) &= -D_{iu} \log_2 \frac{D_{iu}}{2^m - 1}, i = 1, 2, \dots, n; u \\ &= 1, 2, \dots, \frac{q(q-1)}{2}\end{aligned}\quad (30)$$

To ensure that the values of Deng entropy align with the range of $\cos(\alpha_k - \alpha_s)$, which is between $[-1, 1]$, Deng entropy is standardized.

$$\begin{aligned}\beta_{iu}(D_{iu}) &= 2 \frac{\vartheta_{iu}(D_{iu}) - \min\{\vartheta_{iu}(D_{iu})\}}{\max\{\vartheta_{iu}(D_{iu})\} - \min\{\vartheta_{iu}(D_{iu})\}} - 1, \\ i &= 1, 2, \dots, n; u = 1, 2, \dots, \frac{q(q-1)}{2}\end{aligned}\quad (31)$$

Thus, the phase angle $\cos(\alpha_k - \alpha_s)$ can be determined by the following waw:

$$\begin{aligned}\cos(\alpha_k - \alpha_s) &= \beta_{iu}(D_{iu}), i = 1, 2, \dots, n; u \\ &= 1, 2, \dots, \frac{q(q-1)}{2}\end{aligned}\quad (32)$$

Therefore, by Eq. (32), in Eq. (26), INT_i can be written as

$$\begin{aligned}INT_i &= 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \cos(\alpha_k - \alpha_s) \\ &= 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \beta_{iu}(D_{iu})\end{aligned}\quad (33)$$

By Eq. (33), the Eq. (26) can be written as

$$\begin{aligned}P(x_i) &= \sum_{k=1}^q P(e_k)P(x_i|e_k) \\ &+ 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \beta_{iu}(D_{iu})\end{aligned}\quad (34)$$

Finally, the probability of each alternative x_i can be obtained in the following way:

$$Prob(x_i) = \sigma P(x_i) \quad (35)$$

where,

$$\sigma = \frac{1}{\sum_{i=1}^n P(x_i)} = \frac{1}{\sum_{i=1}^n (\sum_{k=1}^q P(e_k)P(x_i|e_k) + INT_i)} \quad (36)$$

Step 6: Ranking of Alternative

According to Eq. (36), the higher probability of an alternative means the best alternative.

4 A Practical Example with Comparative Analysis

This section describes a new approach to MAGDM based on the MULTIMOORA approach integrating QDT with PLI. The detailed procedure for the combined MULTIMOORA approach is presented below.

4.1 A Case Study

Envision a new energy vehicle manufacturer seeking optimal auto components suppliers to meet its developmental needs and sustainable manufacturing standards. After investigating and evaluating market sources, the manufacturer selects five contenders: $X = \{x_1, x_2, x_3, x_4, x_5\}$. To identify the most suitable candidates, the manufacturer assembles an expert group for the evaluation of the supplier. After defining the assessment objectives and criteria for selecting green suppliers, the firm assembles a panel of five experts (e_1, e_2, e_3, e_4, e_5) with professional relevant backgrounds and expertise. This panel, comprising esteemed academics and industry professionals, brings substantial knowledge in technology innovation and supply chain management. They gather relevant information and conduct a detailed analysis of the prospective organizations' status of development and assessment goals, through in-depth discussions and the exchange of viewpoints, the panel reaches an agreement on four key evaluation indicators, guided by their collective judgment. In this connection, the four criteria are c_1 : Performance of price; c_2 : Quality of product; c_3 : Social influence and c_4 : Capability of service.

4.1.1 Decision Procedure

After the evaluation indicators are defined, the following methods are applied to evaluate the GSCM MAGDM scenario using our proposed MULTIMOORA method integrating QDT with PLI for assessing the available options.

Step 1: Here we offer the LTSs, $H_1 = \{h_{-3}: \text{very bad (VB)}, h_{-2}: \text{bad (B)}, h_{-1}: \text{slightly bad (SB)}, h_0: \text{moderate (M)}, h_1: \text{slightly good (SG)}, h_2: \text{good (G)}, h_3: \text{very good (VG)}\}$ for the evaluation of alternatives $(x_1, x_2, x_3, x_4, x_5)$ about the criteria (c_1, c_2, c_3, c_4) . The evaluation information is shown in the Eqs. (37)–(41).

$$A^1 =$$

$$\begin{array}{c} c_1 \quad c_2 \quad c_3 \quad c_4 \\ \begin{array}{l} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \left[\begin{array}{cccc} \langle h_1, (0.9, 0.1) \rangle & \langle h_2, (0.6, 0.6) \rangle & \langle h_2, (0.7, 0.2) \rangle & \langle h_0, (0.8, 0.3) \rangle \\ \langle h_3, (0.8, 0.3) \rangle & \langle h_{-1}, (0.7, 0.45) \rangle & \langle h_{-2}, (0.6, 0.7) \rangle & \langle h_2, (0.7, 0.4) \rangle \\ \langle h_2, (0.7, 0.5) \rangle & \langle h_0, (0.8, 0.35) \rangle & \langle h_1, (0.5, 0.6) \rangle & \langle h_1, (0.6, 0.6) \rangle \\ \langle h_{-1}, (0.8, 0.4) \rangle & \langle h_{-3}, (0.9, 0.25) \rangle & \langle h_1, (0.8, 0.4) \rangle & \langle h_{-2}, (0.5, 0.7) \rangle \\ \langle h_2, (0.9, 0.2) \rangle & \langle h_1, (0.7, 0.1) \rangle & \langle h_2, (0.7, 0.3) \rangle & \langle h_1, (0.6, 0.5) \rangle \end{array} \right] \end{array} \quad (37)$$

$$A^2 =$$

$$\begin{array}{c} c_1 \quad c_2 \quad c_3 \quad c_4 \\ \begin{array}{l} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \left[\begin{array}{cccc} \langle h_2, (0.8, 0.3) \rangle & \langle h_3, (0.6, 0.6) \rangle & \langle h_{-1}, (0.9, 0.1) \rangle & \langle h_3, (0.6, 0.5) \rangle \\ \langle h_{-2}, (0.7, 0.4) \rangle & \langle h_2, (0.7, 0.7) \rangle & \langle h_2, (0.8, 0.3) \rangle & \langle h_{-3}, (0.5, 0.7) \rangle \\ \langle h_3, (0.9, 0.5) \rangle & \langle h_1, (0.5, 0.8) \rangle & \langle h_2, (0.7, 0.4) \rangle & \langle h_2, (0.8, 0.4) \rangle \\ \langle h_1, (0.7, 0.3) \rangle & \langle h_{-2}, (0.6, 0.7) \rangle & \langle h_{-1}, (0.8, 0.2) \rangle & \langle h_1, (0.9, 0.4) \rangle \\ \langle h_2, (0.8, 0.2) \rangle & \langle h_1, (0.8, 0.4) \rangle & \langle h_3, (0.9, 0.3) \rangle & \langle h_2, (0.7, 0.6) \rangle \end{array} \right] \end{array} \quad (38)$$

$$A^3 =$$

$$\begin{array}{c} c_1 \quad c_2 \quad c_3 \quad c_4 \\ \begin{array}{l} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \left[\begin{array}{cccc} \langle h_{-2}, (0.6, 0.6) \rangle & \langle h_3, (0.8, 0.1) \rangle & \langle h_2, (0.9, 0.1) \rangle & \langle h_{-1}, (0.8, 0.2) \rangle \\ \langle h_2, (0.8, 0.4) \rangle & \langle h_2, (0.7, 0.4) \rangle & \langle h_3, (0.6, 0.2) \rangle & \langle h_2, (0.7, 0.6) \rangle \\ \langle h_1, (0.7, 0.6) \rangle & \langle h_{-2}, (0.6, 0.5) \rangle & \langle h_1, (0.4, 0.7) \rangle & \langle h_3, (0.6, 0.4) \rangle \\ \langle h_2, (0.5, 0.7) \rangle & \langle h_1, (0.9, 0.1) \rangle & \langle h_2, (0.8, 0.4) \rangle & \langle h_0, (0.4, 0.5) \rangle \\ \langle h_3, (0.6, 0.1) \rangle & \langle h_0, (0.8, 0.2) \rangle & \langle h_{-2}, (0.9, 0.2) \rangle & \langle h_1, (0.3, 0.1) \rangle \end{array} \right] \end{array} \quad (39)$$

$$A^4 =$$

$$\begin{array}{c} c_1 \quad c_2 \quad c_3 \quad c_4 \\ \begin{array}{l} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \left[\begin{array}{cccc} \langle h_{-2}, (0.3, 0.7) \rangle & \langle h_{-2}, (0.8, 0.4) \rangle & \langle h_0, (0.6, 0.7) \rangle & \langle h_2, (0.9, 0.25) \rangle \\ \langle h_3, (0.8, 0.4) \rangle & \langle h_2, (0.9, 0.3) \rangle & \langle h_{-2}, (0.7, 0.4) \rangle & \langle h_{-3}, (0.7, 0.1) \rangle \\ \langle h_2, (0.9, 0.1) \rangle & \langle h_3, (0.7, 0.2) \rangle & \langle h_3, (0.3, 0.6) \rangle & \langle h_1, (0.6, 0.7) \rangle \\ \langle h_{-1}, (0.7, 0.2) \rangle & \langle h_{-2}, (0.8, 0.2) \rangle & \langle h_{-1}, (0.7, 0.5) \rangle & \langle h_{-2}, (0.8, 0.4) \rangle \\ \langle h_{-2}, (0.9, 0.3) \rangle & \langle h_{-1}, (0.9, 0.1) \rangle & \langle h_3, (0.9, 0.2) \rangle & \langle h_3, (0.9, 0.1) \rangle \end{array} \right] \end{array} \quad (40)$$

$$A^5 = \begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} \langle h_{-3}, (0.3, 0.9) \rangle & \langle h_{-1}, (0.9, 0.1) \rangle & \langle h_{-3}, (0.5, 0.7) \rangle & \langle h_2, (0.8, 0.3) \rangle \\ \langle h_2, (0.8, 0.5) \rangle & \langle h_{-2}, (0.8, 0.4) \rangle & \langle h_2, (0.7, 0.3) \rangle & \langle h_3, (0.9, 0.1) \rangle \\ \langle h_3, (0.9, 0.12) \rangle & \langle h_0, (0.6, 0.5) \rangle & \langle h_1, (0.8, 0.5) \rangle & \langle h_1, (0.5, 0.6) \rangle \\ \langle h_{-1}, (0.7, 0.6) \rangle & \langle h_2, (0.9, 0.1) \rangle & \langle h_{-2}, (0.9, 0.4) \rangle & \langle h_2, (0.7, 0.4) \rangle \\ \langle h_2, (0.6, 0.4) \rangle & \langle h_1, (0.7, 0.3) \rangle & \langle h_3, (0.4, 0.2) \rangle & \langle h_2, (0.6, 0.2) \rangle \end{bmatrix} \end{matrix} \quad (41)$$

Step 2: By Eq. (13), the weight of experts is $\omega_1 = 0.1939$, $\omega_2 = 0.2123$, $\omega_3 = 0.2084$, $\omega_4 = 0.1802$, and $\omega_5 = 0.2052$.

Step 3: By Eq. (16), the weight of attributes for all experts are:

$$\begin{aligned} w_1^1 &= 0.0376, w_2^1 = 0.1642, w_3^1 = 0.7745, w_4^1 = 0.0237; \\ w_1^2 &= 0.0681, w_2^2 = 0.0322, w_3^2 = 0.8522, w_4^2 = 0.0474; \\ w_1^3 &= 0.0468, w_2^3 = 0.0641, w_3^3 = 0.8507, w_4^3 = 0.0384; \\ w_1^4 &= 0.0993, w_2^4 = 0.1238, w_3^4 = 0.6214, w_4^4 = 0.1554; \\ w_1^5 &= 0.0575, w_2^5 = 0.0852, w_3^5 = 0.8427, w_4^5 = 0.0147. \end{aligned}$$

Step 4: According to Eq. (17), we get $P(e_1) = 0.2071$, $P(e_2) = 0.2150$, $P(e_3) = 0.2181$, $P(e_4) = 0.1891$, and $P(e_5) = 0.1706$.

(1)–(4) (I) By Eqs. (18)–(20), we construct the utility matrix $\dot{U}^k = (R_\tau^k(x_i))_{5 \times 3}$ (shown in Tables 3, 4, 5, 6 and 7) and the ranking matrix $R^k = (e_\tau^k(x_i))_{5 \times 3}$ (shown in Tables 8, 9, 10, 11 and 12) of each expert e_k .

(II) By Eq. (21), the normalized utility matrix $N(\dot{U}^k) = (N(R_\tau^k(x_i)))_{5 \times 3}$ of each expert e_k are shown in Tables 13, 14, 15, 16 and 17.

(III) By Eq. (22), the index of alternative x_i of expert e_k is shown in Table 18.

(1) By Eq. (23) we calculate the conditional probabilities of each alternative for each expert. The details results are shown in Table 19.

Table 3 The utility matrix $\dot{U}^1 = (R_\tau^1(x_i))_{5 \times 3}$ of e_1

	PLN – RS(ζ_1^1)	PLN – RP(ζ_2^1)	PLN – FMF(ζ_3^1)
x_1	$R_1^1(x_1) = 0.5246$	$R_2^1(x_1) = 0.0026$	$R_3^1(x_1) = 2.55E - 06$
x_2	$R_1^1(x_2) = 0.6390$	$R_2^1(x_2) = 0.0989$	$R_3^1(x_2) = 2.53E - 06$
x_3	$R_1^1(x_3) = 0.5783$	$R_2^1(x_3) = 0.0429$	$R_3^1(x_3) = 2.26E - 06$
x_4	$R_1^1(x_4) = 0.6031$	$R_2^1(x_4) = 0.0441$	$R_3^1(x_4) = 9.93E - 08$
x_5	$R_1^1(x_5) = 0.5275$	$R_2^1(x_5) = 0.0032$	$R_3^1(x_5) = 3.44E - 06$

Table 4 The utility matrix $\dot{U}^2 = (R_\tau^2(x_i))_{5 \times 3}$ of e_2

	PLN – RS(ς_1^2)	PLN – RP(ς_2^2)	PLN – FMF(ς_3^2)
x_1	$R_1^2(x_1) = 0.7214$	$R_2^2(x_1) = 0.1581$	$R_3^2(x_1) = 1.97E - 06$
x_2	$R_1^2(x_2) = 0.6279$	$R_2^2(x_2) = 0.0468$	$R_3^2(x_2) = 3.01E - 07$
x_3	$R_1^2(x_3) = 0.6200$	$R_2^2(x_3) = 0.0568$	$R_3^2(x_3) = 1.89E - 06$
x_4	$R_1^2(x_4) = 0.7262$	$R_2^2(x_4) = 0.1532$	$R_3^2(x_4) = 6.26E - 07$
x_5	$R_1^2(x_5) = 0.5658$	$R_2^2(x_5) = 0.0025$	$R_3^2(x_5) = 1.86E - 06$

Table 5 The utility matrix $\dot{U}^3 = (R_\tau^3(x_i))_{5 \times 3}$ of e_3

	PLN – RS(ς_1^3)	PLN – RP(ς_2^3)	PLN – FMF(ς_3^3)
x_1	$R_1^3(x_1) = 0.6052$	$R_2^3(x_1) = 0.0081$	$R_3^3(x_1) = 7.05E - 07$
x_2	$R_1^3(x_2) = 0.5906$	$R_2^3(x_2) = 0.0038$	$R_3^3(x_2) = 2.39E - 06$
x_3	$R_1^3(x_3) = 0.6742$	$R_2^3(x_3) = 0.0743$	$R_3^3(x_3) = 8.62E - 07$
x_4	$R_1^3(x_4) = 0.6189$	$R_2^3(x_4) = 0.0226$	$R_3^3(x_4) = 2.27E - 06$
x_5	$R_1^3(x_5) = 0.7646$	$R_2^3(x_5) = 0.1695$	$R_3^3(x_5) = 1.25E - 06$

Table 6 The utility matrix $\dot{U}^4 = (R_\tau^4(x_i))_{5 \times 3}$ of e_4

	PLN – RS(ς_1^4)	PLN – RP(ς_2^4)	PLN – FMF(ς_3^4)
x_1	$R_1^4(x_1) = 0.4373$	$R_2^4(x_1) = 0.0982$	$R_3^4(x_1) = 1.68E - 05$
x_2	$R_1^4(x_2) = 0.4805$	$R_2^4(x_2) = 0.1474$	$R_3^4(x_2) = 1.70E - 05$
x_3	$R_1^4(x_3) = 0.3642$	$R_2^4(x_3) = 0.0577$	$R_3^4(x_3) = 9.00E - 05$
x_4	$R_1^4(x_4) = 0.4930$	$R_2^4(x_4) = 0.1211$	$R_3^4(x_4) = 4.50E - 06$
x_5	$R_1^4(x_5) = 0.3321$	$R_2^4(x_5) = 0.0255$	$R_3^4(x_5) = 1.84E - 05$

Table 7 The utility matrix $U^5 = (R_\tau^5(x_i))_{5 \times 3}$ of e_5

	PLN – RS(ς_1^5)	PLN – RP(ς_2^5)	PLN – FMF(ς_3^5)
x_1	$R_1^5(x_1) = 0.7221$	$R_2^5(x_1) = 0.1487$	$R_3^5(x_1) = 3.66E - 07$
x_2	$R_1^5(x_2) = 0.5853$	$R_2^5(x_2) = 0.0185$	$R_3^5(x_2) = 6.89E - 07$
x_3	$R_1^5(x_3) = 0.6118$	$R_2^5(x_3) = 0.0519$	$R_3^5(x_3) = 1.00E - 06$
x_4	$R_1^5(x_4) = 0.7343$	$R_2^5(x_4) = 0.1755$	$R_3^5(x_4) = 7.25E - 07$
x_5	$R_1^5(x_5) = 0.5598$	$R_2^5(x_5) = 0.0062$	$R_3^5(x_5) = 1.25E - 06$

Table 8 The ranking matrix

$R^1 = (\epsilon^1_\tau(x_i))_{5 \times 3}$

	PLN – RS(ζ^1_1)	PLN – RP(ζ^1_2)	PLN – FMF(ζ^1_3)
x_1	$\epsilon^1_1(x_1) = 5$	$\epsilon^1_2(x_1) = 1$	$\epsilon^1_3(x_1) = 2$
x_2	$\epsilon^1_1(x_2) = 1$	$\epsilon^1_2(x_2) = 5$	$\epsilon^1_3(x_2) = 3$
x_3	$\epsilon^1_1(x_3) = 3$	$\epsilon^1_2(x_3) = 3$	$\epsilon^1_3(x_3) = 4$
x_4	$\epsilon^1_1(x_4) = 2$	$\epsilon^1_2(x_4) = 4$	$\epsilon^1_3(x_4) = 5$
x_5	$\epsilon^1_1(x_5) = 4$	$\epsilon^1_2(x_5) = 2$	$\epsilon^1_3(x_5) = 1$

Table 9 The ranking matrix

$R^2 = (\epsilon^2_\tau(x_i))_{5 \times 3}$

	PLN – RS(ζ^2_1)	PLN – RP(ζ^2_2)	PLN – FMF(ζ^2_3)
x_1	$\epsilon^2_1(x_1) = 2$	$\epsilon^2_2(x_1) = 5$	$\epsilon^2_3(x_1) = 1$
x_2	$\epsilon^2_1(x_2) = 3$	$\epsilon^2_2(x_2) = 2$	$\epsilon^2_3(x_2) = 5$
x_3	$\epsilon^2_1(x_3) = 4$	$\epsilon^2_2(x_3) = 3$	$\epsilon^2_3(x_3) = 2$
x_4	$\epsilon^2_1(x_4) = 1$	$\epsilon^2_2(x_4) = 4$	$\epsilon^2_3(x_4) = 4$
x_5	$\epsilon^2_1(x_5) = 5$	$\epsilon^2_2(x_5) = 1$	$\epsilon^2_3(x_5) = 3$

Table 10 The ranking matrix

$R^3 = (\epsilon^3_\tau(x_i))_{5 \times 3}$

	PLN – RS(ζ^3_1)	PLN – RP(ζ^3_2)	PLN – FMF(ζ^3_3)
x_1	$\epsilon^3_1(x_1) = 4$	$\epsilon^3_2(x_1) = 2$	$\epsilon^3_3(x_1) = 5$
x_2	$\epsilon^3_1(x_2) = 5$	$\epsilon^3_2(x_2) = 1$	$\epsilon^3_3(x_2) = 1$
x_3	$\epsilon^3_1(x_3) = 2$	$\epsilon^3_2(x_3) = 4$	$\epsilon^3_3(x_3) = 4$
x_4	$\epsilon^3_1(x_4) = 3$	$\epsilon^3_2(x_4) = 3$	$\epsilon^3_3(x_4) = 2$
x_5	$\epsilon^3_1(x_5) = 1$	$\epsilon^3_2(x_5) = 5$	$\epsilon^3_3(x_5) = 3$

Table 11 The ranking matrix

$R^4 = (\epsilon^4_\tau(x_i))_{5 \times 3}$

	PLN – RS(ζ^4_1)	PLN – RP(ζ^4_2)	PLN – FMF(ζ^4_3)
x_1	$\epsilon^4_1(x_1) = 3$	$\epsilon^4_2(x_1) = 3$	$\epsilon^4_3(x_1) = 4$
x_2	$\epsilon^4_1(x_2) = 2$	$\epsilon^4_2(x_2) = 5$	$\epsilon^4_3(x_2) = 3$
x_3	$\epsilon^4_1(x_3) = 4$	$\epsilon^4_2(x_3) = 2$	$\epsilon^4_3(x_3) = 1$
x_4	$\epsilon^4_1(x_4) = 1$	$\epsilon^4_2(x_4) = 4$	$\epsilon^4_3(x_4) = 5$
x_5	$\epsilon^4_1(x_5) = 5$	$\epsilon^4_2(x_5) = 1$	$\epsilon^4_3(x_5) = 2$

Table 12 The ranking matrix

$R^5 = (\epsilon^5_\tau(x_i))_{5 \times 3}$

	PLN – RS(ζ^5_1)	PLN – RP(ζ^5_2)	PLN – FMF(ζ^5_3)
x_1	$\epsilon^5_1(x_1) = 2$	$\epsilon^5_2(x_1) = 4$	$\epsilon^5_3(x_1) = 5$
x_2	$\epsilon^5_1(x_2) = 4$	$\epsilon^5_2(x_2) = 2$	$\epsilon^5_3(x_2) = 4$
x_3	$\epsilon^5_1(x_3) = 3$	$\epsilon^5_2(x_3) = 3$	$\epsilon^5_3(x_3) = 2$
x_4	$\epsilon^5_1(x_4) = 1$	$\epsilon^5_2(x_4) = 5$	$\epsilon^5_3(x_4) = 3$
x_5	$\epsilon^5_1(x_5) = 5$	$\epsilon^5_2(x_5) = 1$	$\epsilon^5_3(x_5) = 1$

Table 13 The normalized utility matrix $N(\dot{U}^1) = (N(R_\tau^1(x_i)))_{5 \times 3}$ of e_1

$N(\dot{U}^1)$	PLN – RS(ς_1^1)	PLN – RP(ς_2^1)	PLN – FMF(ς_3^1)
x_1	0.1826	0.0136	0.2343
x_2	0.2225	0.5161	0.2326
x_3	0.2013	0.2235	0.2076
x_4	0.2100	0.2302	0.0091
x_5	0.1836	0.0165	0.3163

Table 14 The normalized utility matrix $N(\dot{U}^2) = (N(R_\tau^2(x_i)))_{5 \times 3}$ of e_2

$N(\dot{U}^2)$	PLN – RS(ς_1^2)	PLN – RP(ς_2^2)	PLN – FMF(ς_3^2)
x_1	0.2212	0.3784	0.2971
x_2	0.1925	0.1120	0.0453
x_3	0.1901	0.1359	0.2839
x_4	0.2227	0.3676	0.0943
x_5	0.1735	0.0061	0.2793

Table 15 The normalized utility matrix $N(\dot{U}^3) = (N(R_\tau^3(x_i)))_{5 \times 3}$ of e_3

$N(\dot{U}^3)$	PLN – RS(ς_1^3)	PLN – RP(ς_2^3)	PLN – FMF(ς_3^3)
x_1	0.1860	0.0292	0.1089
x_2	0.1815	0.0138	0.3690
x_3	0.2072	0.2670	0.1331
x_4	0.1902	0.0812	0.1956
x_5	0.2350	0.6087	0.1934

Table 16 The normalized utility matrix $N(\dot{U}^4) = (N(R_\tau^4(x_i)))_{5 \times 3}$ of e_4

$N(\dot{U}^4)$	PLN – RS(ς_1^4)	PLN – RP(ς_2^4)	PLN – FMF(ς_3^4)
x_1	0.2075	0.2182	0.1146
x_2	0.2280	0.3276	0.1160
x_3	0.1728	0.1282	0.6132
x_4	0.2340	0.2692	0.0307
x_5	0.1576	0.0567	0.1255

Table 17 The normalized utility matrix $N\left(\dot{U}^5\right)=N\left(\left(R_t^5\left(x_i\right)\right)\right)_{5 \times 3}$ of e_5

$N\left(\dot{U}^5\right)$	PLN – RS(ς_1^5)	PLN – RP(ς_2^5)	PLN – FMF(ς_3^5)
x_1	0.2247	0.3709	0.0908
x_2	0.1822	0.0461	0.1709
x_3	0.1904	0.1296	0.2492
x_4	0.2285	0.4380	0.1800
x_5	0.1742	0.0155	0.3091

Table 18 The output result of $\mathfrak{B}_i^k$

$\mathfrak{B}_i^k$	e_1	e_2	e_3	e_4	e_5
x_1	0.0616	0.0171	0.0158	0.0007	0.0479
x_2	0.0662	0.0138	0.1221	0.0404	0.0288
x_3	0.0098	0.0612	0.0120	0.1988	0.0659
x_4	0.0188	0.0261	0.0613	0.0074	0.0491
x_5	0.1155	0.0555	0.1016	0.0297	0.1020

Table 19 The conditional probabilities of each alternative

$P\left(x_i\left e_k\right.\right)$	e_1	e_2	e_3	e_4	e_5
x_1	0.2265	0.2434	0.0361	0.0691	0.4248
x_2	0.0986	0.0792	0.3525	0.1501	0.3195
x_3	0.0504	0.3904	0.0384	0.1960	0.3248
x_4	0.0025	0.1458	0.7179	0.0265	0.1072
x_5	0.1631	0.0980	0.2245	0.1670	0.3473

(2) By Eq. (24), the probabilities of alternatives are:

$P\left(x_1\right)=0.1093, P\left(x_2\right)=0.1918, P\left(x_3\right)=0.2653, P\left(x_4\right)=0.1252$ and $P\left(x_5\right)=0.3085$.

Step 5: By Eq. (29), the belief distance between two experts about each alternative is calculated, which is presented in Table 20.

By Eq. (30), the Deng entropy between two experts about each alternative is shown in Table 21.

By Eq. (31), the standardized Deng entropy between two experts about each alternative is shown in Table 22.

Table 20 The belief distance between two experts about each alternative

D_{ui}	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
$u = 1(e_1, e_2)$	0.6103	0.7978	0.2748	0.3506	0.3727
$u = 2(e_1, e_3)$	0.6895	0.7895	0.0188	0.6131	0.3758
$u = 3(e_1, e_4)$	0.7377	0.7425	0.2901	0.0920	0.1095
$u = 4(e_1, e_5)$	0.4185	0.7821	0.8393	0.4123	0.2897
$u = 5(e_2, e_3)$	0.1312	0.0450	0.1820	0.3608	0.0712
$u = 6(e_2, e_4)$	0.2396	0.1829	0.4484	0.3956	0.1975
$u = 7(e_2, e_5)$	0.2960	0.0690	0.8013	0.1031	0.0205
$u = 8(e_3, e_4)$	0.1024	0.2199	0.2998	0.6223	0.2008
$u = 9(e_3, e_5)$	0.4205	0.1054	0.8367	0.2230	0.0240
$u = 10(e_4, e_5)$	0.5002	0.0886	1.0348	0.4888	0.2005

Table 21 The Deng entropy between two experts about each alternative

$\vartheta_{ui}(D_{ui})$	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
$u = 1(e_1, e_2)$	2.8193	3.3768	1.5857	1.9000	1.987
$u = 2(e_1, e_3)$	3.0635	3.3537	0.1812	2.8279	1.999
$u = 3(e_1, e_4)$	3.2058	3.2198	1.6514	0.6764	0.7773
$u = 4(e_1, e_5)$	2.1610	3.3330	3.4912	2.1378	1.6496
$u = 5(e_2, e_3)$	0.8970	0.3769	1.1583	1.9403	0.5495
$u = 6(e_2, e_4)$	1.4302	1.1628	2.2709	2.0749	1.2338
$u = 7(e_2, e_5)$	1.6762	0.5358	3.3866	0.7408	0.1954
$u = 8(e_3, e_4)$	0.7369	1.3397	1.6924	2.8572	1.2494
$u = 9(e_3, e_5)$	2.1684	0.7541	3.4840	1.3542	0.223
$u = 10(e_4, e_5)$	2.4543	0.6559	3.9919	2.4143	1.2483

Table 22 The standardized Deng entropy between two experts about each alternative

$\beta_{ui}(D_{ui})$	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
$u = 1(e_1, e_2)$	0.6869	1.0000	-0.2629	0.1222	0.9867
$u = 2(e_1, e_3)$	0.8848	0.9846	-1.0000	0.9731	1.0000
$u = 3(e_1, e_4)$	1.0000	0.8953	-0.2284	-1.0000	-0.3548
$u = 4(e_1, e_5)$	0.1537	0.9708	0.7372	0.3402	0.6126
$u = 5(e_2, e_3)$	-0.8703	-1.0000	-0.4872	0.1591	-0.6074
$u = 6(e_2, e_4)$	-0.4384	-0.4761	0.0967	0.2825	0.1515
$u = 7(e_2, e_5)$	-0.2391	-0.8941	0.6823	-0.9409	-1.0000
$u = 8(e_3, e_4)$	-1.0000	-0.3581	-0.2069	1.0000	0.1688
$u = 9(e_3, e_5)$	0.1597	-0.7485	0.7334	-0.3784	-0.9694
$u = 10(e_4, e_5)$	0.3912	-0.8140	1.0000	0.5939	0.1676

By Eq. (33), the interface effect among experts for each alternative is:

$INT_1 = 0.0639$, $INT_2 = 0.0433$, $INT_3 = 0.2303$, $INT_4 = 0.0203$, and $INT_5 = 0.0421$.

Finally, by Eq. (35), the probability of each alternative is:

$Prob(x_1) = 0.1237$, $Prob(x_2) = 0.1679$, $Prob(x_3) = 0.3540$, $Prob(x_4) = 0.1039$, and $Prob(x_5) = 0.2504$.

Step 6: According to the value of QP of each alternative, the final ranking is $x_3 \succ x_5 \succ x_2 \succ x_1 \succ x_4$.

4.2 Result Discussion

In this section, we discussed the validation of our numerical results through sensitive analysis and some comparative analysis.

4.2.1 Effect of the Quantum Interface on Decision Result

If we can decide based on the $P(x_i) (i = 1, 2, 3, 4, 5)$ values according to Eq. (24), as higher probabilities indicate a superior ranking. The classical approach does not account for the interference effect, which can influence the final rankings, as shown in Fig. 4. The ranking result of alternatives according to Eq. (24) is $x_5 \succ x_3 \succ x_2 \succ x_4 \succ x_1$. If we count the interface effect between two experts about an alternative the ranking result is $x_3 \succ x_5 \succ x_2 \succ x_1 \succ x_4$.

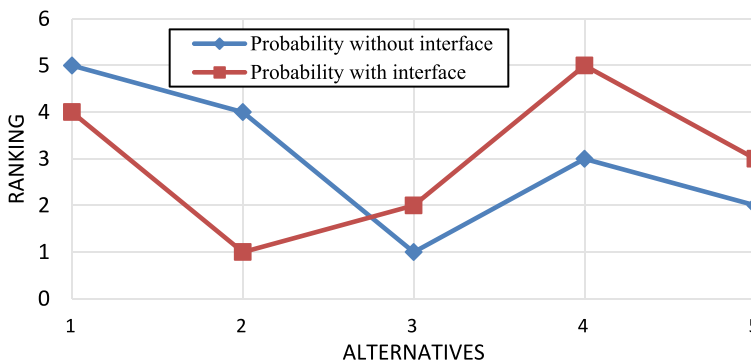


Fig. 4 The interface effect on ranking values

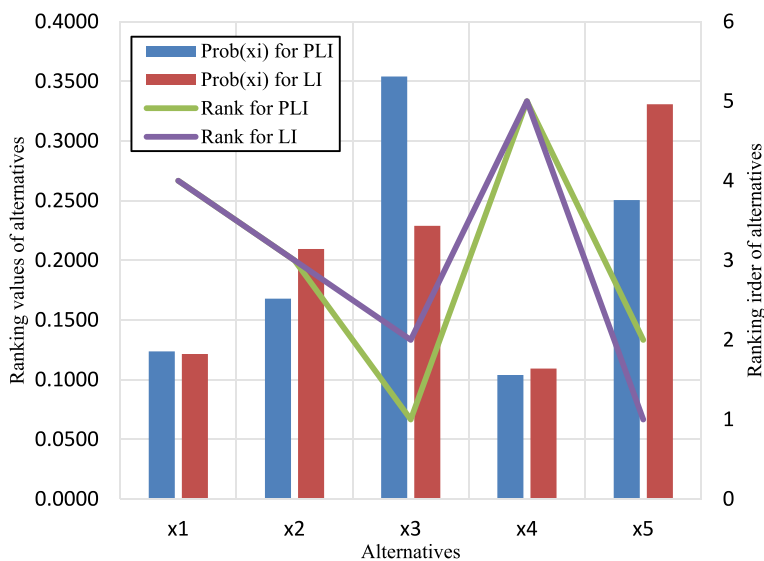


Fig. 5 Output of decision results for PLI and LI

4.2.2 Effect of the Expert Confidence Levels on Decision Result

In this section, first, we examine how much effect the expert confidence levels have on decision results. By Mandal and Ranadive (2019), if $\mu_{ij}^k = 1$ and $\nu_{ij}^k = 0$, the expert decision matrix shown in Eq. (12), can be degenerate into a linguistic information (LI)-based decision matrix. Then we rank the alternatives of our proposed method for PLI and LI, the details of the comparison results are shown in Fig. 5.

From 5, we can observe that if we avoid the expert confidence levels, the ranking order of the GSS is different from the ranking order of the GSS if we consider the expert confidence levels. It demonstrates that the PLI significant effect on decision results. Thus, we cannot neglect the expert confidence levels.

4.2.3 Comparative with Validity Analysis

To test the feasibility and validity of the suggested method, we compare it with some existing methods for the example given above.

In the traditional decision-making model (Mandal and Ranadive 2019) the ranking results are derived from the aggregate decision matrix of experts' opinions. Therefore, to compare our model and the traditional model, we first aggregate the experts' opinions. In this connection, by Mandal and Ranadive (2019), using the Pythagorean linguistic weighted averaging aggregation operator, the aggregation of five experts' opinions is

$$Agg(A^1, A^2, A^3, A^4, A^5) =$$

	$\left\langle \begin{matrix} h_{-0.77}, \\ (0.7, 0.41) \end{matrix} \right\rangle$	$\left\langle \begin{matrix} h_{1.08}, \\ (0.78, 0.27) \end{matrix} \right\rangle$	$\left\langle \begin{matrix} h_{-0.02}, \\ (0.79, 0.24) \end{matrix} \right\rangle$	$\left\langle \begin{matrix} h_{1.2}, \\ (0.8, 0.30) \end{matrix} \right\rangle$
x_1	$\left\langle \begin{matrix} h_{1.53}, \\ (0.78, 0.4) \end{matrix} \right\rangle$	$\left\langle \begin{matrix} h_{0.6}, \\ (0.78, 0.44) \end{matrix} \right\rangle$	$\left\langle \begin{matrix} h_{0.71}, \\ (0.69, 0.34) \end{matrix} \right\rangle$	$\left\langle \begin{matrix} h_{0.24}, \\ (0.74, 0.29) \end{matrix} \right\rangle$
x_2	$\left\langle \begin{matrix} h_{2.21}, \\ (0.85, 0.29) \end{matrix} \right\rangle$	$\left\langle \begin{matrix} h_{0.34}, \\ (0.66, 0.44) \end{matrix} \right\rangle$	$\left\langle \begin{matrix} h_{1.57}, \\ (0.61, 0.55) \end{matrix} \right\rangle$	$\left\langle \begin{matrix} h_{1.63}, \\ (0.64, 0.52) \end{matrix} \right\rangle$
x_3	$\left\langle \begin{matrix} h_{0.05}, \\ (0.7, 0.41) \end{matrix} \right\rangle$	$\left\langle \begin{matrix} h_{-0.75}, \\ (0.85, 0.2) \end{matrix} \right\rangle$	$\left\langle \begin{matrix} h_{-0.19}, \\ (0.81, 0.36) \end{matrix} \right\rangle$	$\left\langle \begin{matrix} h_{-0.13}, \\ (0.73, 0.47) \end{matrix} \right\rangle$
x_4	$\left\langle \begin{matrix} h_{1.49}, \\ (0.8, 0.21) \end{matrix} \right\rangle$	$\left\langle \begin{matrix} h_{0.43}, \\ (0.79, 0.19) \end{matrix} \right\rangle$	$\left\langle \begin{matrix} h_{1.76}, \\ (0.83, 0.24) \end{matrix} \right\rangle$	$\left\langle \begin{matrix} h_{1.78}, \\ (0.69, 0.23) \end{matrix} \right\rangle$
x_5				

Therefore, the ranking of alternatives according to the methods MULTIMOORA method is $x_5 > x_3 > x_4 > x_2 > x_1$, i.e., the optimal option is x_5 . When the proposed model is applied without considering the interface effect, the optimal option is also x_5 . This suggests that the optimal option of the proposed approach align with those of traditional methods when the impact of decision-makers is disregarded, thereby validating the robustness of the suggested model. However, when evaluating the quantum interference among decision-makers, the outcomes of this model suggest that x_3 is the optimal strategy. This inconsistency arises from the conventional framework's exclusion of the interference effect among decision-makers, which leads to subjective biases during the evaluation of alternatives. As a result, the outcomes of this model better align with real-world complexities.

5 Conclusion

This paper presents a quantum group decision model designed to mitigate information ambiguity and opinion interference in GSS. The approach uses PLIs to convey ambiguous information in complex decision-making contexts, employing Deng entropy to assess opinion interference by modeling the reciprocal effects among decision-makers in the MAGDM process. Unlike conventional methods that treat decision-makers as independent entities, this paradigm addresses interference by assigning appropriate weights to decision-makers and regulating opinion interference through quantum decision-making procedures.

The proposed model enhances existing methodologies, offering more efficient solutions to challenges related to opinion interference and weight distribution. Our findings show that the model significantly improves decision-making in GSS by providing a comprehensive framework to address information ambiguity and conflicting perspectives. However, the computational complexity increases with the inclusion of additional decision-makers and variables, highlighting the need for advanced computational techniques to maintain efficiency.

Future research will focus on integrating machine learning methods to enhance scalability and practicality, ensuring the model's effectiveness in large-scale decision-making processes. This integration aims to manage rising complexity, foster consensus in quantum group decision-making, and ensure the model's long-term sustainability.

Conflict of Interest The authors declare that they have no conflict of interest.

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Pythagorean Linguistic Quantum Scenario-Based Group Decision-Making Methodology Integrating the LogTODIM-TOPSIS Approach and Its Application to Navigating Investment Decisions Amidst Uncertainty for Growth



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Abstract Decision-making often relies on subjective experience; however, certain viewpoints may lack clarity, complicating their articulation for policymakers. Consequently, researchers must develop scientific decision frameworks, such as the multi-criteria group decision-making (MCGDM) approach in a linguistic setting, which is gaining popularity among scholars. In linguistic contexts, the degree of membership is typically expressed, but the degrees of non-membership and hesitation from experts often remain unarticulated. Pythagorean Linguistic Numbers/Values (PLNs/PLVs) offer innovative solutions to this problem. In modern MCGDM scenarios,

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experts are frequently seen as autonomous, and their psychological factors are rarely considered. Nevertheless, in many cases, experts' perspectives tend to influence each other, and they often exhibit subjective, bounded rationality throughout the decision-making process. This chapter introduces a quantum scenario-based multicriteria group decision-making framework that incorporates the Log-TODIM-TOPSIS method, utilizing Pythagorean linguistic information to address these challenges. Finally, we apply our proposed model to a numerical problem in an investment context, performing sensitivity and comparative analyses. The approach is grounded in the recommended sensitivity and comparative evaluation.

Keywords Quantum decision theory · Group decision-making · Pythagorean linguistic set · LogTODIM-TOPSIS

1 Introduction

The research in fuzzy set theory and its applications plays a significant role in addressing uncertainty and complexity in social networks and group decision-making processes. The methods developed in this area offer valuable insights into how fuzzy logic can be used to model and analyze systems where imprecision and ambiguity are prevalent. Fuzzy set theory is particularly well-suited to modeling the uncertainty inherent in social networks, where relationships and interactions often involve incomplete or vague information. Traditional mathematical models struggle to capture this uncertainty, but fuzzy approaches provide a more nuanced framework for analyzing such dynamics. For instance, the development of fuzzy fractional differential equations using fuzzy Laplace transforms is essential for modeling systems where the influence of individuals or groups evolves fractionally over time under uncertainty (Salahshour et al., 2012; Allahviranloo et al., 2015; Chehlabi et al., 2016; Armand et al., 2018).

A new centrality measure is introduced for neutrosophic and linguistic graphs (Mahapatra et al., 2019, 2020), including self-weight or individual merit within the network. Mahapatra et al. (Mahapatra et al., 2024a) present an innovative method for centrality measurement in fuzzy social networks where relationships between nodes are not binary but have different degrees of intensity.

According to the above discussion, fuzzy sets (FS) have become a popular approach for incorporating fuzzy information as part of an expert's evaluative measures. However, as noted by Xu (2005) and Mandal et al. (2022a), experts tend to express their opinions more reliably using linguistic information, known as linguistic sets (LS), rather than fuzzy information. In (Mandal et al. 2019), the LS concept was extended to the Pythagorean linguistic set (PLS), allowing experts to express both their confidence and doubt levels when providing judgments. In real-life scenarios, situations, experts often have limited knowledge and personal biases, so using language-based information with expert confidence and doubting levels (Mandal et al., 2021; 2022b; 2023a; 2023b; 2024a) is more flexible for expressing their ideas.

In this connection, Mandal et al. briefly studied MCGDM under LZN information with the MARCOS approach and its application in the selection of logistic cold chain centers (Mandal et al., 2024b); TODIM-VIKRO approach (Mandal et al., 2024c) and its application in site selection of medical logistic center; ORESTE approach (Mandal et al., 2024d) and its application in assessment of regional energy; and MULTI-MOORA approach (Mandal et al., 2024e) and its application in selection of software. To address the challenge of aggregating individual assessments, specific methods are needed. Although many aggregation strategies have been studied, most assume that decision-makers are independent and uninfluenced by one another. However, in practice, opinions are often influenced by surrounding perspectives. In decision-making, the views of multiple experts can interact, ultimately affecting the final decision, and introducing the concept of 'interference.

Quantum Probability Theory (QPT) (Busemeyer & Bruza, 2012; Moreira & Wichert, 2014; Yearsley & Busemeyer, 2016) offers a framework for capturing these influences. QPT, a recent advancement over classical probability theory (CPT), has been extensively studied in fields such as psychology (Busemeyer & Bruza, 2012), cognitive science (Yearsley & Busemeyer, 2016), and decision analysis (He et al., 2018a; 2018b; Wu et al., 2020; Xiao et al., 2022; Han & Liu, 2023; Mandal et al., 2024f; 2025) due to its robust descriptive power. It effectively explains paradoxes, such as the disjunction fallacy (He et al., 2018a; 2018b), the Ellsberg paradox (Eichberger & Pirner, 2018), and the order effect (Wang et al., 2014), which are challenging for CPT to address. QPT's ability to represent interference terms makes it particularly valuable for understanding how expert opinions influence each other in the MCGDM process.

This study addresses the MCGDM challenge in evaluating navigating investment decisions using Pythagorean linguistic information (PLI) integrating quantum decision theory (QDT) to manage uncertain and fuzzy data. It also introduces the quantum-based PLI-LogTODIM-TOPSIS model, which combines traditional LogTODIM (Leoneti and Gomes, 2021), TOPSIS (Lai et al., 1994) techniques, and QDT under the PLI environment. This novel approach is validated through a numerical example and comparative analyses, aiming to:

(1) extend the LogTODIM-TOPSIS technique to the quantum integrating PLI environment; (2) entropy-based criteria weight determination; and (3) apply the quantum-based PLI-LogTODIM-TOPSIS model to MCGDM in the context of PLI environment, with a case study on navigating investment decisions evaluation to demonstrate its efficacy.

The structure of the paper is as follows: In Sect. 2, recall some basic definitions and the background of linguistic term sets (LTSs), the Pythagorean linguistic set (PLS), and quantum decision theory (QDT). The proposed MCGDM model is discussed in Sect. 3. A case study for navigating investment decisions evaluation with comparative analysis is provided in Sect. 4. Section 5 concludes our study.

2 Preliminaries

Here we recall some related definitions and results that are used in this paper.

2.1 Linguistic Term Set (LTS)

Here we assume an LTS is: $H = \{h_\lambda | \lambda = -\zeta, \dots, -1, 0, 1, \dots, \zeta\}$ with cardinality $2\zeta + 1$ and holds the followings (Mandal et al., 2024a):

- (1) $h_{\lambda_1} \geq h_{\lambda_2}$ if and only if $\lambda_1 \geq \lambda_2$;
- (2) $neg(h_{\lambda_1}) = h_{-\lambda_1}$;
- (3) $max(h_{\lambda_1}, h_{\lambda_2}) = h_{\lambda_1}$ if $h_{\lambda_1} \geq h_{\lambda_2}$;
- (4) $min(h_{\lambda_1}, h_{\lambda_2}) = h_{\lambda_1}$ if $h_{\lambda_1} \leq h_{\lambda_2}$.

To preserve expert information, the continuous LTSs (Xu, 2005) denotes $\overline{H} = \{h_{-t} \leq h_\lambda \leq h_t | \lambda \in [-t, t]\}$, $t > \varsigma$.

The subscript index of any $h_\lambda \overline{H}$, represented by the following functions:

$$I: \overline{H} \rightarrow [-t, t], \text{ and } I(h_\lambda) = \lambda.$$

$$I^{-1}: [-t, t] \rightarrow \overline{H}, \text{ and } I^{-1}(\lambda) = h_\lambda.$$

Let $\mathcal{F}$ be the linguistic conversation function for any LT $G \in \overline{H}$, which is defined in the following way:

$$\mathcal{F}: \overline{H} \rightarrow [0, 1], \mathcal{F}(G) = \{\mathcal{F}(h_\lambda) = \frac{\lambda}{2t} + \frac{1}{2} | \lambda \in [-t, t]\} = \kappa.$$

2.2 Pythagorean Linguistic Set (PTS)

By Mandal and Ranadive (2019), a Pythagorean linguistic set (PLS) denoted by $\mathcal{P}$ in an alternative set X for continuous LTS $\overline{H} = \{h_{-t} \leq h_\lambda \leq h_t | \lambda \in [-t, t]\}$ is represented in the following way:

$$\mathcal{R} = \{\langle h_{\theta(x)}, (\mu_{\mathcal{R}}(x), \nu_{\mathcal{R}}(x)) \rangle | x \in X\}$$

where, $h_{\theta(x)} \in \overline{H}$, $\mu_{\mathcal{R}}: X \rightarrow [0, 1]$ and $\nu_{\mathcal{R}}: X \rightarrow [0, 1]$ are the membership and non-membership degrees of x to $h_{\theta(x)}$ such that $\mu_{\mathcal{R}}^2(x) + \nu_{\mathcal{R}}^2(x) \leq 1$. Here, the Pythagorean linguistic number (PLN) in $\mathcal{R}$ is denoted by $\wp$ and represented by $\langle h_{\theta(\wp)}, (\mu(\wp), \nu(\wp)) \rangle$ with $h_{\theta(\wp)} \in \overline{H}$ and $\mu(\wp), \nu(\wp) \in [0, 1]$, $\mu^2(\wp) + \nu^2(\wp) \leq 1$.

By Mandal and Ranadive (2019), the expected value of $\wp = h_{\theta(\wp)}, (\mu(\wp), \nu(\wp))$ is denoted by $\varkappa$ and defined in the following way:

$$\kappa(\wp) = \frac{h_{\theta(\wp)} \times (\mu^2 + 1 - \nu^2(\wp))}{2}$$

By Wang et al. (2019), the distance between two PLNs $\wp_1 = \langle h_{\theta(\wp_1)}, (\mu(\wp_1), \nu(\wp_1)) \rangle$ and $\wp_2 = \langle h_{\theta(\wp_2)}, (\mu(\wp_2), \nu(\wp_2)) \rangle$ are defined in the following way:

$$d(\wp_1, \wp_2) = \frac{1}{2} \left| (1 + \mu^2(\wp_1) - \nu^2(\wp_1)) * \mathcal{F}(h_{\theta(\wp_1)}) \right. \\ \left. - (1 + \mu^2(\wp_2) - \nu^2(\wp_2)) * \mathcal{F}(h_{\theta(\wp_2)}) \right|$$

2.3 Quantum Probability Theory (QPT)

A QDT model is based on the principles of quantum probability (QP) theory (QPT). To provide context, we will review some key concepts and notations of QPT. In QDT, a state is represented as a unit-length vector within an N-dimensional vector space.

Definition 1 (Yearsley & Busemeyer, 2016) A Hilbert space (HS) with finite-dimensional is represented by $\mathcal{H}$, where $\mathcal{H}$ is defined in abstract vector space over a complex field, spanned by a set of orthonormal basis vectors with complex components. In classical probability theory (CPT), events are fundamental (He et al., 2018), but in QPT, events are instead represented by subspaces of the HS $\mathcal{H}$, rather than traditional sets.

Definition 2 (Busemeyer & Bruza, 2012) In QDT, the superposition state $|S\rangle$, a unit-length vector in the N-dimensional HS $\mathcal{H}$, defines all possible events within that space. The nonlinear combination of a set of orthogonal basis vectors $\{|\mathcal{V}_k\rangle, k = 1, 2, \dots, n\}$ are used to represent the superposition state $|S\rangle$, i.e.,

$$\begin{aligned} |S\rangle &= \langle \mathcal{V}_1 | \langle S | \langle \mathcal{V}_1 + \mathcal{V}_2 | \langle S | \langle \mathcal{V}_2 + \dots + \mathcal{V}_n | \langle S | \langle \mathcal{V}_n \rangle \\ &= \xi_1 \langle \mathcal{V}_1 | + \xi_2 \langle \mathcal{V}_2 | + \dots + \xi_n \langle \mathcal{V}_n | \\ &= \zeta_1 e^{i\alpha_1} | \langle \mathcal{V}_1 + \zeta_2 e^{i\alpha_2} | \langle \mathcal{V}_2 + \dots + \zeta_n e^{i\alpha_n} | \langle \mathcal{V}_n \rangle \end{aligned} \quad (1)$$

where $\xi_k (= \zeta_k e^{i\alpha_k})$ is the amplitude of the probability, expressed as a complex number, characterizing the amplitude of the wave in the state $|\mathcal{V}_k\rangle$, and the amplitude of the phase angle represented by $e^{i\alpha_k}$.

Definition 3 (Busemeyer & Bruza, 2012) The CPT framework represents every event using a single set of components. In QPT, probabilities are assigned based on the overlap between cognitive states (shown as state vectors) and different probability measures (shown as subspaces). QPT extends CPT into the complex domain by including incompatible qualities and viewing events as vectors. Incompatible attributes are two observable features of an item that cannot be measured accurately

at the same time. Both the incompatibility condition and the uncertainty principle can be described by the probability amplitude, also known as complex probability.

According to Born's rule, an event's probability amplitude and its classical probability, x_i are related to each other:

$$P(\mathcal{V}_k) = |\zeta_k e^{i\alpha_k}|^2 = (\zeta_k e^{i\alpha_k}) \times (\zeta_k e^{i\alpha_k})^* = \zeta_k e^{i\alpha_k} \times \zeta_k e^{-i\alpha_k} = (\zeta_k)^2 \quad (2)$$

For the total probability of event to be 1, the sum of the squares of the probability amplitudes must also be 1.

$$\sum_{k=1}^n P(\mathcal{V}_k) = \sum_{k=1}^n (\zeta_k)^2 = 1. \quad (3)$$

An $N - 1$ mapping transforms complex probabilities into classical probabilities. For each given probability value $P(\mathcal{V}_k)$, countless complex probabilities correspond to it. Referencing Eq. (2) and Euler's formula,

$$e^{i\alpha} = \cos\alpha + i \times \sin\alpha \quad (4)$$

the complex probability satisfies:

$$\xi = \sqrt{P(\mathcal{V})}(\cos\alpha + i \times \sin\alpha) = \sqrt{P(\mathcal{V})}e^{i\alpha} \quad (5)$$

For example, in the following, we discussed the QPT in two-dimensional Hilbert space. Let $\{\mathcal{M}_1, \mathcal{M}_2\}$ be an event containing two basic states, where $|\mathcal{M}_1\rangle = (1, 0)$, and $|\mathcal{M}_2\rangle = (0, 1)$ are defined in $\mathbf{H}$. In Fig. 1, we graphically show the state $|\mathcal{S}\rangle$, which is defined as a superposition. According to Eqs. (1) and (2), we get

$$|\mathcal{S}\rangle = \zeta_1 e^{i\alpha_1} |\mathcal{M}_1\rangle + \zeta_2 e^{i\alpha_2} |\mathcal{M}_2\rangle = \frac{e^{i\alpha_{\mathcal{M}_1}}}{\sqrt{2}} |\mathcal{M}_1\rangle + \frac{e^{i\alpha_{\mathcal{M}_2}}}{\sqrt{2}} |\mathcal{M}_2\rangle$$

and the probabilities of $|\mathcal{M}_1\rangle$ and $|\mathcal{M}_2\rangle$ are computed in the following way:

$$P(\mathcal{M}_1) = P(\mathcal{M}_2) = (e^{i\alpha})(e^{i\alpha})^* = \frac{1}{2}.$$

According to conventional Bayesian decision theory (BDT), DM can be mathematically represented as a Markov process (MP) (Moreira & Wichert, 2014). The state transition probability (STP) in a single route of QPT aligns with the CPT according to Feynman's first rule (Wu et al., 2020), which varies over several pathways. Consider the decision-making dilemma highlighted above as an illustration of the process depicted in Fig. 2a. Experts provide their assessments of $\mathcal{M}_1$ ('bad') or $\mathcal{M}_2$ ('good') as probability quantifications. STPs are determined by combining the individual probabilities for each route and using the total probability formula in

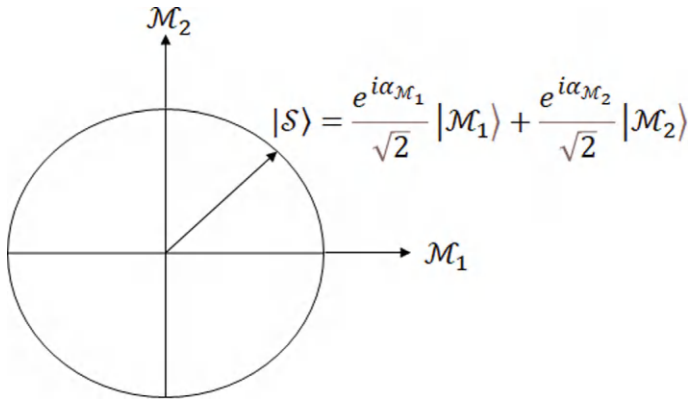


Fig. 1 Event in Hilbert space with two dimensions

CPT.

$$P(\mathcal{M}_1) = \sum_{k=1}^2 P(e_k)P(\mathcal{M}_1|e_k), \sum_{k=1}^2 P(e_k) = 1 \quad (6)$$

Since only one route is selected at a time, it is sometimes referred to as the single path principle. The conventional MP assumes that experts rely on one expert at a time and that both experts are completely autonomous. However, this approach is sometimes unsuitable for ordinary cognitive processes because many avenues are often considered when forming judgments. Such phenomena can result in illogical actions, including conjunction errors as well as disjunction, and impacts of interference. Thus, QPT is an excellent complement to CPT in this scenario.

Within the QPT framework Fig. 2b, the state is formed by combining three fundamental states, which can be represented as different wave functions aligning

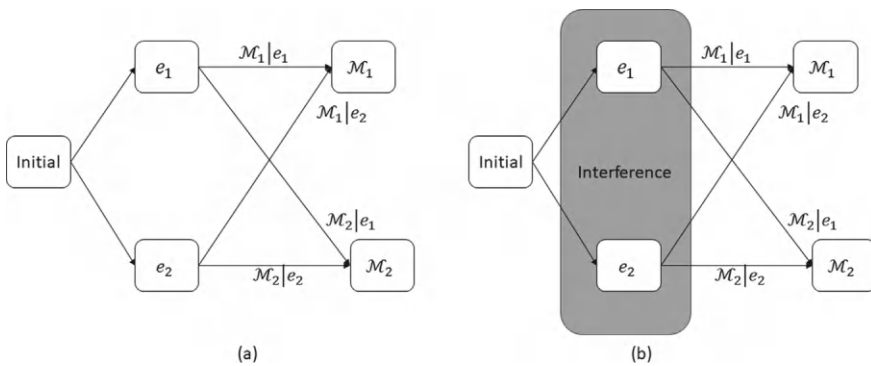


Fig. 2 DM in **a** MP and **b** QDT process

according to Feynman's second rule. The amplitude of the final form is obtained by superimposing two individual routes running from I ('Initial') to $\mathcal{M}_1$. The amplitude is the sum of the individual pathways from source I to target $\mathcal{M}_1$, and the probability of reaching the target $\mathcal{M}_1$ is equal to the square of the amplitude.

$$P(\mathcal{M}_1) = |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 \quad (7)$$

Equation (7) can be simplified in the following way:

$$\begin{aligned} P(\mathcal{M}_1) &= |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 \\ &= |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^* \\ &= \left(\sqrt{P(e_1)P(\mathcal{M}_1|e_1)}e^{i\alpha_1} + \sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{i\alpha_2} \right) \\ &\quad \left(\sqrt{P(e_1)P(\mathcal{M}_1|e_1)}e^{-i\alpha_1} + \sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{-i\alpha_2} \right) \\ &= P(e_1)P(\mathcal{M}_1|e_1) + P(e_2)P(\mathcal{M}_1|e_2) \\ &\quad + \sqrt{P(e_1)P(\mathcal{M}_1|e_1)}\sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{i(\alpha_1-\alpha_2)} \\ &\quad + \sqrt{P(e_1)P(\mathcal{M}_1|e_1)}\sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{i(\alpha_2-\alpha_1)} \end{aligned} \quad (8)$$

By Eq. (4), we get

$$\cos(\alpha_1 - \alpha_2) = \frac{e^{i(\alpha_2-\alpha_1)} + e^{i(\alpha_1-\alpha_2)}}{2} \quad (9)$$

Then Eq. (7) can be written in the following way:

$$\begin{aligned} P(\mathcal{M}_1) &= P(e_1)P(\mathcal{M}_1|e_1) + P(e_2)P(\mathcal{M}_1|e_2) \\ &\quad + 2\sqrt{P(e_1)P(\mathcal{M}_1|e_1)}\sqrt{P(e_2)P(\mathcal{M}_1|e_2)}\cos(\alpha_1 - \alpha_2) \\ &= |\xi_{e_1}\xi_{\mathcal{M}_1|e_1}|^2 + |\xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 + 2\xi_{e_1}\xi_{\mathcal{M}_1|e_1}\xi_{e_2}\xi_{\mathcal{M}_1|e_2}\cos(\alpha_1 - \alpha_2) \end{aligned} \quad (10)$$

Upon comparing Eqs. (6) and (10), it becomes evident that the QDT framework includes an interference component. If the pathways are observable, the quantum phase transition will reduce to the classical one, following Feynman's third rule.

$$P(\mathcal{M}_1) = |\xi_{e_1}\xi_{\mathcal{M}_1|e_1}|^2 + |\xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 = P(e_1)P(\mathcal{M}_1|e_1) + P(e_2)P(\mathcal{M}_1|e_2) \quad (11)$$

Additionally, the QPT can be expanded to a multi-path graph, resulting in an increased number of interference terms. The QPT enhances CPT by accurately representing cognitive processes in quantum terms, thereby more effectively modeling ambiguity, uncertainty, and vagueness in decision-making.

3 The Proposed Logtodim Topsis-Based Magdm Under Pythagorean Linguistic Information and Quantum Decision Scenario

This section describes the new approach of quantum cognition MAGDM based on the LogTODIM TOPSIS method with PLN. The LogTODIM and TOPSIS methods are integrated to take into account the psychological aspects of decision-maker satisfaction, as well as to maximize the overall utility for the group and minimize the regret of the individual. The detailed procedure for the combined quantum cognition PLN-based LogTODIM-TOPSIS approach is presented below.

3.1 Background

Let us assume.

$X = \{x_1, x_2, \dots, x_n\}$: Alternative set,

$C = \{c_1, c_2, \dots, c_m\}$: Attribute set,

$e = \{e_1, e_2, \dots, e_q\}$: Expert set,

$w^k = (w_1^k, w_2^k, \dots, w_m^k)$: Weight vector of attributes for expert e_k , where $w_j^k \in [0, 1]$, and $\sum_{j=1}^m w_j^k = 1$,

$\omega = (\omega_1, \omega_2, \dots, \omega_q)$: Weight vector of experts with $\omega_k \in [0, 1]$, $\sum_{k=1}^q \omega_k = 1$.

3.2 Decision Steps

Let $A^k = (a_{ij}^k)_{nm}$ ($i = 1, 2, \dots, n; j = 1, 2, \dots, m; k = 1, 2, \dots, q$) be the PLI-based decision judgment matrix provided by an expert e_k using pre-defined distinct LTSs $H = \{h_\lambda | \lambda = -, \dots, -1, 0, 1, \dots, \}$.

Step 1: Construction of decision matrix

Then, in the following, we represent the decision matrix of k -th expert:

$$A^k = (a_{ij}^k)_{nm} = \begin{matrix} & \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} & \begin{bmatrix} c_1 & c_2 & \cdots & c_m \\ a_{11}^k & a_{12}^k & \cdots & a_{1m}^k \\ a_{21}^k & a_{22}^k & \cdots & a_{2m}^k \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}^k & a_{n2}^k & \cdots & a_{nm}^k \end{bmatrix} \end{matrix} \quad (12)$$

where, $a_{ij}^k = h_{\theta_{ij}}^k, (\mu_{ij}^k, v_{ij}^k)$ is a PLN, called the Pythagorean linguistic information (PLI) of k -th expert for i -th alternative about j -th criteria.

Step 2: Expert weight computation process

Inspire by Mandal et al. (2024a), the weight of each expert $e_k (k = 1, 2, \dots, q)$ can be calculated in the following way:

$$\omega_k = \frac{\Omega(Z^k)}{\sum_{k=1}^q \Omega(Z^k)}, \text{ where } \Omega(Z^k) = \frac{\sum_{j=1}^m \sum_{i=1}^n \mathcal{F}(\kappa(a_{ij}^k))}{mn} \quad (13)$$

Step 3: Attribute weight computation process

- (1) The decision matrix (Eq. (12)) can be normalized to $L^k = (l_{ij}^k)_{nm}$ with

$$l_{ij}^k = \begin{cases} \frac{\mathcal{F}(\kappa(a_{ij}^k))}{\sum_{i=1}^n \mathcal{F}(\kappa(a_{ij}^k))}, & \text{for benefit criteria } c_j, \\ 1 - \frac{\mathcal{F}(\kappa(a_{ij}^k))}{\sum_{i=1}^n \mathcal{F}(\kappa(a_{ij}^k))}, & \text{for cost criteria } c_j. \end{cases} \quad (14)$$

- (2) The Shannon entropy $\Psi^k = (\Psi_1^k, \Psi_2^k, \dots, \Psi_m^k)$ can be calculated in the following way:

$$\Psi_j^k = -\frac{1}{\ln n} \sum_{i=1}^n l_{ij}^k \ln l_{ij}^k \quad (15)$$

is 0 if $l_{ij}^k = 0$.

- (3) The attribute weight $w^k = (w_1^k, w_2^k, \dots, w_m^k)$ for each expert $e_k (k = 1, 2, \dots, q)$ can be calculated in the following way:

$$w_j^k = \frac{1 - \Psi_j^k}{\sum_{j=1}^m (1 - \Psi_j^k)} \quad (16)$$

Step 4: Construction of Bayesian network

In this phase, we converted the pre-collected data into probabilities and developed a Bayesian Network for DM, as the QDT framework is a partial extension of the Bayesian Network. Figure 3 illustrates that, in a MAGDM situation, the decision-making process can be represented as a simple two-layer Bayesian Network. The experts' weights are converted into probabilities, which indicate the degree of belief in the experts, using the following equation:

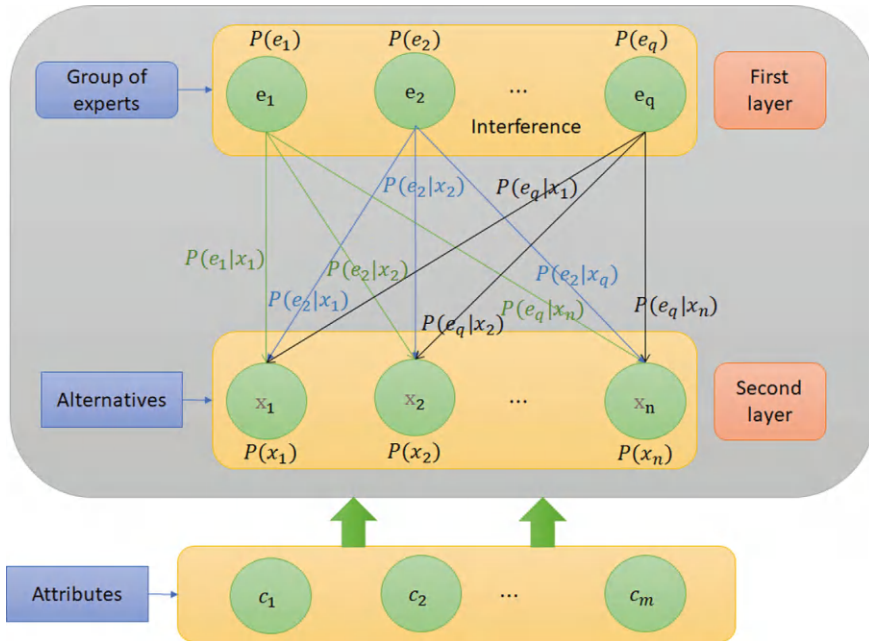


Fig. 3 MAGDM according to QSBN

$$P(e_k) = \omega_k, k = 1, 2, \dots, q. \quad (17)$$

In Eq. (17), the probabilities form the initial layer of the Bayesian Network (BN). The second layer depends on the assessment outcomes from each expert, which can be calculated using the LogTODIM-TOPSIS approach as follows:

(1) **Criteria relative weight computation process**

Let $w_j^{k,*}$ be the relative weight of the criteria c_j determinate from each expert opinion e_k ($k = 1, 2, \dots, q$), where

$$w_j^{k,*} = \frac{w_j^k}{w_{max}^k}, w_{max}^k = \max_{j=1}^m w_j^k \quad (18)$$

(2) **Dominance degree matrix construction process**

The dominant degree of x_i over x_s for the attribute c_j with respect to expert e_k ($k = 1, 2, \dots, q$) is denoted by $\phi_j^k(x_i, x_s)$ and defined in the following way:

$$\phi_j^k(x_i, x_s) = \begin{cases} w_j^{k,*} \log(1 + 10\rho d(a_{ij}^k, a_{sj}^k)) & \text{if } \mathcal{F}(\kappa(a_{ij}^k)) > \mathcal{F}(\kappa(a_{sj}^k)) \\ 0 & \text{if } \mathcal{F}(\kappa(a_{ij}^k)) = \mathcal{F}(\kappa(a_{sj}^k)) \\ -w_j^{k,*} \theta \log(1 + 10\rho d(a_{ij}^k, a_{sj}^k)) & \mathcal{F}(\kappa(a_{ij}^k)) < \mathcal{F}(\kappa(a_{sj}^k)) \end{cases} \quad (19)$$

where θ is a risk factor of gain and losses, and ρ indicates how significant the decision is according to the agent's perception. $\mathcal{F}(\kappa(a_{ij}^k)) > \mathcal{F}(\kappa(a_{sj}^k))$ means $\phi_j^k(x_i, x_s)$ are gains and $\mathcal{F}(\kappa(a_{ij}^k)) < \mathcal{F}(\kappa(a_{sj}^k))$ means $\phi_j^k(x_i, x_s)$ are losers. Based on Eq. (19), the dominance degree matrix ϕ_j^k ($j = 1, 2, \dots, m$) for c_j with respect to each expert e_k ($k = 1, 2, \dots, q$) is as follows:

$$\phi_j^k = (\phi_j^k(x_i, x_k))_{n \times n} = \begin{matrix} & \begin{matrix} x_1 & x_2 & \cdots & x_n \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} & \begin{bmatrix} 0 & \phi_j^k(x_1, x_2) & \cdots & \phi_j^k(x_1, x_n) \\ \phi_j^k(x_2, x_1) & 0 & \cdots & \phi_j^k(x_2, x_n) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_j^k(x_n, x_1) & \phi_j^k(x_n, x_2) & \cdots & 0 \end{bmatrix} \end{matrix} \quad (20)$$

(3) Overall dominance matrix construction process

The overall dominance degree of x_i concerning remaining alternatives for c_j with respect to each expert e_k ($k = 1, 2, \dots, q$) is denoted by $\eta^k(x_i, c_j)$ and defined in the following way:

$$\eta^k(x_i, c_j) = \sum_{k=1}^n \phi_j^k(x_i, x_k) \quad (i = 1, 2, \dots, n; j = 1, 2, \dots, m) \quad (21)$$

Based on Eq. (21), the overall dominance matrix η^k with respect to each expert e_k ($k = 1, 2, \dots, q$) is as follows:

$$\phi^k = (\eta^k(x_i, c_j))_{n \times m} = \begin{matrix} & \begin{matrix} c_1 & c_2 & \cdots & c_m \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} & \begin{bmatrix} \eta^k(x_1, c_1) & \eta^k(x_1, c_2) & \cdots & \eta^k(x_1, c_m) \\ \eta^k(x_2, c_1) & \eta^k(x_2, c_2) & \cdots & \eta^k(x_2, c_m) \\ \vdots & \vdots & \ddots & \vdots \\ \eta^k(x_n, c_1) & \eta^k(x_n, c_2) & \cdots & \eta^k(x_n, c_m) \end{bmatrix} \end{matrix} \quad (22)$$

(4) Positive and negative ideal solution process

With the help of max and min operations, in the following, we define the positive ideal (PI) and negative ideal (NI) solution with respect to each expert e_k ($k = 1, 2, \dots, q$) in the following way:

$$\begin{cases} PIS^k = (PIS_1^k, PIS_2^k, \dots, PIS_m^k) \\ NIS^k = (NIS_1^k, NIS_2^k, \dots, NIS_m^k) \end{cases} \quad (23)$$

where,

$$PIS_j^k = \{(c_j, \max_{i=1}^n \eta^k(x_i, c_j)) | j = 1, 2, \dots, m\}$$

$$NIS_j^k = \{(c_j, \min_{i=1}^n \eta^k(x_i, c_j)) | j = 1, 2, \dots, m\}$$

(5) Distance computation process of each alternative from PIS and NIS

Using attribute weight determinate from each expert opinion, the distance of each alternative from PIS^k and PIS^k with respect to each expert $e_k (k = 1, 2, \dots, q)$ is obtained in the following way:

$$\vartheta_i^{k,+} = \sum_{j=1}^m w_j^k \left| \eta^k(x_i, c_j) - PIS_j^k \right| \quad (24)$$

$$\vartheta_i^{k,-} = \sum_{j=1}^m w_j^k \left| \eta^k(x_i, c_j) - NIS_j^k \right| \quad (25)$$

(6) Closeness index computation process

Determined the relative closeness index (RCI) for each alternative with respect to each expert $e_k (k = 1, 2, \dots, q)$ in the following way:

$$RCI_i^k = \frac{\vartheta_i^{k,-}}{\vartheta_i^{k,+} + \vartheta_i^{k,-}} \quad (26)$$

(7) Based on Eq. (26), conditional probabilities which link the two layers in the BN as:

$$P(x_i|e_k) = \frac{RCI_i^k}{\sum_{i=1}^n RCI_i^k} \quad (27)$$

1. The second layer consists of the probabilities of alternatives calculated as:

$$P(x_i) = \sum_{k=1}^q P(e_k) P(x_i|e_k) \quad (28)$$

Subsequently, we can decide based on the $P(x_i)$ values, as higher probabilities, indicate a superior ranking. The classical approach does not account for the interference effect, which can influence the final rankings. To address this issue, Bayesian

networks are expanded into a quantum framework capable of properly modeling a practical approach to real interference in MAGDM.

Step 5: Construction of quantum scenario-based Bayesian network (QSBN)

The facilitator often consolidates opinions from several specialists. The interconnected decision-making processes in real-world situations, as demonstrated in Sect. 2.2, remain unobservable. The square of the sum of the amplitude probabilities of all possible pathways collectively constitutes the probability of the alternative x_i from QPT (He et al., 2018), as described.

$$\begin{aligned} P(x_i) &= \left| \sum_{k=1}^q \xi_{e_k} \xi_{x_i|e_k} e^{i\alpha_k} \right|^2 \\ &= \left| \sum_{k=1}^q \sqrt{P(e_k)} \sqrt{P(x_i|e_k)} e^{i\alpha_k} \right|^2, \quad i = 1, 2, \dots, n \end{aligned} \quad (29)$$

By Eqs. (7–11), Eq. (29) can be written as

$$\begin{aligned} P(x_i) &= \sum_{k=1}^q P(e_k)P(x_i|e_k) \\ &\quad + 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \cos(\alpha_k - \alpha_s) \\ &= \sum_{k=1}^q P(e_k)P(x_i|e_k) + INT_i \end{aligned} \quad (30)$$

In a quantum-based Bayesian network, a normalization factor is incorporated into Eq. (30) to standardize the probability of each alternative, ensuring that the overall estimated probability does not exceed 1. Then

$$Prob(x_i) = \sigma P(x_i) \quad (31)$$

where,

$$\sigma = \frac{1}{\sum_{i=1}^n P(x_i)} = \frac{1}{\sum_{i=1}^n (\sum_{k=1}^q P(e_k)P(x_i|e_k) + INT_i)} \quad (32)$$

3.3 Analysis of Interference Effect Among Experts' Opinions

Equation (32) uses the term INT_i , which represent the ambiguity and uncertainty of expert opinions, to explain the superposition of decision information. In Eq. (32), $\cos(\alpha_k - \alpha_s)$ represents the phase angle between two distinct experts e_k and e_s , which represents the connection vector between two experts. $\frac{q(q-1)}{2}$ is the total count of pairwise correspondences across q experts, excluding duplicates. The final interference value is derived by summing the computed $\frac{q(q-1)}{2}$ interference values for each expert pair, rather than assessing the overall uncertainty across experts. This approach enhances the accuracy of the estimated probability value.

In real-world situations, it is hard to measure the phase angle. Experts' opinions can differ or be influenced by outside factors, causing a lot of variation and mistakes. Belief entropy helps a lot in handling and combining unclear information, making it easier to make good decisions in group settings with many factors. It is used as a standard in areas like risk assessment, defect diagnosis, and pattern recognition. Belief entropy is the latest way to measure interference in QSBN. Unlike the old method of manually estimating the phase angle range, this new method is more reliable because it reduces errors from personal judgment, which can affect the final decision and the expert's probability calculation. Belief entropy is used in QSBN to manage the interference among different experts' beliefs in GDM.

Then the belief distance about alternative x_i according to the average probability $\chi = \frac{1}{nq}$ can be calculated in the following way:

$$D_{iu} = \left| \delta_{iu} + \frac{|\delta_{iu} - \chi| - |\varrho_{iu} - \chi|}{|\delta_{iu} - \chi| + |\varrho_{iu} - \chi|} \right|, i = 1, 2, \dots, n; u = 1, 2, \dots, \frac{q(q-1)}{2},$$

$$\begin{cases} \delta_{iu} = P(e_k)P(x_i|e_k), & i = 1, 2, \dots, n; k = 1, 2, \dots, q-1; \\ \varrho_{iu} = P(e_s)P(x_i|e_s) \end{cases}$$

$$s = k+1, k+2, \dots, \frac{q(q-1)}{2} \quad (33)$$

In Eq. (33), u means the value reallocated to the correlation among q experts; $\chi = \frac{1}{nq}$ represent each expert probability value of each alternative; D_{iu} means deviation degree from χ ; $|\delta_{iu} - \chi|$ and $|\varrho_{iu} - \chi|$ represents two experts' magnitude of the deviation degree from χ about alternative x_i . If $|\delta_{iu} - \chi| < |\varrho_{iu} - \chi|$, then the position of δ_{iu} and ϱ_{iu} are interchangeable.

There are many ways to measure uncertainty. Deng entropy (Zhao et al., 2024), also known as belief entropy, is better at including the confidence function and the uncertainty of BPA because it measures both total non-specificity and disagreement.

Thus, by Deng's entropy, we get

$$\vartheta_{iu}(D_{iu}) = -D_{iu} \log_2 \frac{D_{iu}}{2^m - 1}, i = 1, 2, \dots, n; u = 1, 2, \dots, \frac{q(q-1)}{2} \quad (34)$$

To ensure that the values of Deng entropy align with the range of $\cos(\alpha_k - \alpha_s)$, which is between $[-1, 1]$, Deng entropy is standardized.

$$\beta_{iu}(D_{iu}) = 2 \frac{\vartheta_{iu}(D_{iu}) - \min\{\vartheta_{iu}(D_{iu})\}}{\max\{\vartheta_{iu}(D_{iu})\} - \min\{\vartheta_{iu}(D_{iu})\}} - 1, \quad (35)$$

$$i = 1, 2, \dots, n; u = 1, 2, \dots, \frac{q(q-1)}{2}$$

Thus, the phase angle $\cos(\alpha_k - \alpha_s)$ can be determined by the following way:

$$\cos(\alpha_k - \alpha_s) = \beta_{iu}(D_{iu}), i = 1, 2, \dots, n; u = 1, 2, \dots, \frac{q(q-1)}{2}. \quad (36)$$

Therefore, by Eq. (36), in Eq. (30), INT_i can be written as

$$\begin{aligned} INT_i &= 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \cos(\alpha_k - \alpha_s) \\ &= 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \beta_{iu}(D_{iu}) \end{aligned} \quad (37)$$

By Eq. (37), the Eq. (30) can be written as

$$\begin{aligned} P(x_i) &= \sum_{k=1}^q P(e_k)P(x_i|e_k) \\ &+ 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \beta_{iu}(D_{iu}) \end{aligned} \quad (38)$$

Finally, the probability of each alternative x_i can be obtained in the following way:

$$Prob(x_i) = \sigma P(x_i) \quad (39)$$

where,

$$\sigma = \frac{1}{\sum_{i=1}^n P(x_i)} = \frac{1}{\sum_{i=1}^n (\sum_{k=1}^q P(e_k)P(x_i|e_k) + INT_i)} \quad (40)$$

Step 6: Ranking of alternative

According to Eq. (39), the higher probability of an alternative means the best alternative.

4 A Practical Example with Comparative Analysis

This section describes a new approach to MAGDM based on the LogTODIM-TOPSIS approach integrating QDT with PLI. The detailed procedure for the combined LogTODIM-TOPSIS approach is presented below.

4.1 A Case Study

The economic decision-making process of a firm is complex and involves a variety of variables, including but not limited to pricing, investments, market analysis, regulatory compliance, risk management, resource allocation, production, cost control, and sustainability. To be successful in a complicated economic climate and to ensure their continued success over the long term, businesses need to adopt a decision-making style that is both well-informed and strategic.

According to (Khan et al., 2024), a business must choose the most advantageous nation to invest in from among five different possibilities in order to boost the amount of money it invests overseas $X = \{x_1, x_2, x_3, x_4, x_5\}$. The four key variables that need to be taken into consideration are infrastructure, economy, resources, and policy. The equation $C = \{c_1, c_2, c_3, c_4\}$ represents these variables. After identifying the assessment objectives and criteria, the firm assembles a panel of five experts (e_1, e_2, e_3, e_4, e_5) with professional relevant backgrounds and expertise.

4.1.1 Decision Procedure

After the evaluation indicators are defined, the following methods are applied to evaluate the GSCM MAGDM scenario using our proposed LogTODIM-TOPSIS method integrating QDT with PLI for assessing the available options.

Step 1: Here we offer the LTSs, $H_1 = \{h_{-3}: \text{very bad (VB)}, h_{-2}: \text{bad (B)}, h_{-1}: \text{slightly bad (SB)}, h_0: \text{moderate (M)}, h_1: \text{slightly good (SG)}, h_2: \text{good (G)}, h_3: \text{very good (VG)}\}$ for the evaluation of alternatives (x_1, x_2, x_3, x_4, x_5) about the criteria (c_1, c_2, c_3, c_4). The evaluation information is shown in the Eqs. (41–45).

$$A^1 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \left[\begin{array}{cccc} \langle h_2, (0.9, 0.1) \rangle & \langle h_3, (0.7, 0.3) \rangle & \langle h_{-2}, (0.7, 0.2) \rangle & \langle h_2, (0.9, 0.1) \rangle \\ \langle h_{-1}, (0.7, 0.25) \rangle & \langle h_{-2}, (0.8, 0.2) \rangle & \langle h_3, (0.6, 0.5) \rangle & \langle h_{-2}, (0.7, 0.25) \rangle \\ \langle h_{-3}, (0.8, 0.1) \rangle & \langle h_1, (0.9, 0.1) \rangle & \langle h_1, (0.8, 0.6) \rangle & \langle h_2, (0.9, 0.3) \rangle \\ \langle h_1, (0.6, 0.2) \rangle & \langle h_0, (0.65, 0.3) \rangle & \langle h_2, (0.7, 0.2) \rangle & \langle h_1, (0.7, 0.2) \rangle \\ \langle h_{-1}, (0.9, 0.3) \rangle & \langle h_2, (0.8, 0.4) \rangle & \langle h_{-1}, (0.6, 0.4) \rangle & \langle h_{-2}, (0.9, 0.2) \rangle \end{array} \right] \end{matrix} \quad (41)$$

$$A^2 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} \langle h_2, (0.8, 0.4) \rangle & \langle h_{-3}, (0.8, 0.2) \rangle & \langle h_{-2}, (0.6, 0.6) \rangle & \langle h_{-1}, (0.8, 0.4) \rangle \\ \langle h_3, (0.65, 0.6) \rangle & \langle h_2, (0.7, 0.4) \rangle & \langle h_3, (0.7, 0.5) \rangle & \langle h_0, (0.7, 0.5) \rangle \\ \langle h_{-2}, (0.72, 0.2) \rangle & \langle h_2, (0.6, 0.6) \rangle & \langle h_2, (0.8, 0.3) \rangle & \langle h_2, (0.6, 0.4) \rangle \\ \langle h_0, (0.5, 0.2) \rangle & \langle h_{-1}, (0.8, 0.24) \rangle & \langle h_{-1}, (0.7, 0.4) \rangle & \langle h_{-1}, (0.5, 0.6) \rangle \\ \langle h_1, (0.8, 0.1) \rangle & \langle h_2, (0.9, 0.1) \rangle & \langle h_{-2}, (0.9, 0.2) \rangle & \langle h_3, (0.9, 0.2) \rangle \end{bmatrix} \end{matrix} \quad (42)$$

$$A^3 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} \langle h_{-2}, (0.8, 0.3) \rangle & \langle h_2, (0.9, 0.5) \rangle & \langle h_{-2}, (0.7, 0.6) \rangle & \langle h_3, (0.5, 0.3) \rangle \\ \langle h_3, (0.7, 0.1) \rangle & \langle h_{-1}, (0.6, 0.5) \rangle & \langle h_0, (0.6, 0.4) \rangle & \langle h_3, (0.6, 0.5) \rangle \\ \langle h_2, (0.6, 0.5) \rangle & \langle h_0, (0.8, 0.2) \rangle & \langle h_3, (0.9, 0.2) \rangle & \langle h_2, (0.8, 0.3) \rangle \\ \langle h_0, (0.5, 0.4) \rangle & \langle h_2, (0.7, 0.1) \rangle & \langle h_1, (0.7, 0.3) \rangle & \langle h_{-2}, (0.9, 0.1) \rangle \\ \langle h_1, (0.7, 0.3) \rangle & \langle h_{-3}, (0.6, 0.5) \rangle & \langle h_2, (0.6, 0.5) \rangle & \langle h_1, (0.7, 0.2) \rangle \end{bmatrix} \end{matrix} \quad (43)$$

$$A^4 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} \langle h_2, (0.4, 0.8) \rangle & \langle h_{-2}, (0.8, 0.2) \rangle & \langle h_2, (0.7, 0.5) \rangle & \langle h_{-1}, (0.8, 0.3) \rangle \\ \langle h_0, (0.5, 0.7) \rangle & \langle h_3, (0.7, 0.6) \rangle & \langle h_1, (0.8, 0.6) \rangle & \langle h_3, (0.7, 0.2) \rangle \\ \langle h_{-1}, (0.6, 0.5) \rangle & \langle h_2, (0.9, 0.1) \rangle & \langle h_{-3}, (0.7, 0.3) \rangle & \langle h_1, (0.9, 0.1) \rangle \\ \langle h_2, (0.9, 0.1) \rangle & \langle h_{-2}, (0.7, 0.4) \rangle & \langle h_3, (0.9, 0.2) \rangle & \langle h_0, (0.6, 0.3) \rangle \\ \langle h_1, (0.4, 0.2) \rangle & \langle h_1, (0.7, 0.3) \rangle & \langle h_{-3}, (0.7, 0.3) \rangle & \langle h_1, (0.8, 0.25) \rangle \end{bmatrix} \end{matrix} \quad (44)$$

$$A^5 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} \langle h_2, (0.7, 0.3) \rangle & \langle h_{-3}, (0.8, 0.4) \rangle & \langle h_2, (0.5, 0.7) \rangle & \langle h_0, (0.7, 0.5) \rangle \\ \langle h_3, (0.6, 0.6) \rangle & \langle h_2, (0.7, 0.3) \rangle & \langle h_3, (0.7, 0.3) \rangle & \langle h_2, (0.6, 0.3) \rangle \\ \langle h_{-2}, (0.8, 0.4) \rangle & \langle h_0, (0.9, 0.1) \rangle & \langle h_{-1}, (0.8, 0.5) \rangle & \langle h_{-1}, (0.8, 0.4) \rangle \\ \langle h_0, (0.9, 0.1) \rangle & \langle h_2, (0.2, 0.8) \rangle & \langle h_{-2}, (0.9, 0.4) \rangle & \langle h_2, (0.9, 0.1) \rangle \\ \langle h_1, (0.5, 0.6) \rangle & \langle h_{-2}, (0.9, 0.1) \rangle & \langle h_{-1}, (0.4, 0.2) \rangle & \langle h_{-3}, (0.7, 0.2) \rangle \end{bmatrix} \end{matrix} \quad (45)$$

Step 2: By Eq. (13), the weight of experts is $\omega_1 = 0.1952$, $\omega_2 = 0.2011$, $\omega_3 = 0.2148$, $\omega_4 = 0.2056$, and $\omega_5 = 0.1833$.

Step 3: By Eq. (16), the weight of attributes for all experts are:

$$\begin{aligned} w_1^1 &= 0.3766, w_2^1 = 0.1714, w_3^1 = 0.1615, w_4^1 = 0.2905; \\ w_1^2 &= 0.1546, w_2^2 = 0.3863, w_3^2 = 0.3053, w_4^2 = 0.1538; \\ w_1^3 &= 0.2433, w_2^3 = 0.2798, w_3^3 = 0.2052, w_4^3 = 0.2717; \\ w_1^4 &= 0.0892, w_2^4 = 0.3219, w_3^4 = 0.4783, w_4^4 = 0.1106; \\ w_1^5 &= 0.1341, w_2^5 = 0.3639, w_3^5 = 0.2111, w_4^5 = 0.2908. \end{aligned}$$

Step 4: According to Eq. (17), we get $P(e_1) = 0.1952$, $P(e_2) = 0.2011$, $P(e_3) = 0.2148$, $P(e_4) = 0.2056$, and $P(e_5) = 0.1833$.

(1) By Eq. (18), the relative weight of each attribute for all experts is:

$$w_{max}^1 = 0.3766, w_1^{1,*} = 1, w_2^{1,*} = 0.4551, w_3^{1,*} = 0.4287, w_4^{1,*} = 0.7713;$$
$$w_{max}^2 = 0.3863, w_1^{2,*} = 0.4003, w_2^{2,*} = 1, w_3^{2,*} = 0.7904, w_4^{2,*} = 0.3981;$$
$$w_{max}^3 = 0.2798, w_1^{3,*} = 0.8695, w_2^{3,*} = 1, w_3^{3,*} = 0.7331, w_4^{3,*} = 0.9708;$$
$$w_{max}^4 = 0.4783, w_1^{4,*} = 0.1866, w_2^{4,*} = 0.6731, w_3^{4,*} = 1, w_4^{4,*} = 0.2312;$$
$$w_{max}^5 = 0.3639, w_1^{5,*} = 0.3686, w_2^{5,*} = 1, w_3^{5,*} = 0.5800, w_4^{5,*} = 0.7991.$$

2 and (3) Let $\theta = 2.5$ and $\rho = 3$. Then by Eq. (22), using Eqs. (19) and (20), the overall dominance matrix $\eta^k(k = 1, 2, 3, 4, 5)$ for all experts is shown in Tables 1–5.

Table 1 The overall dominance matrix η^1 of expert e_1

η^1	c_1	c_2	c_3	c_4
x_1	5.1292	1.3670	−3.9550	2.4544
x_2	−3.8552	−5.0820	1.4115	−6.8378
x_3	−11.5733	0.2213	−1.1910	2.3234
x_4	0.2173	−3.0426	0.9112	−1.9455
x_5	−4.3223	−0.0458	−2.8285	−7.3800

Table 2 The overall dominance matrix η^2 of expert e_2

η^2	c_1	c_2	c_3	c_4
x_1	0.6098	−12.1947	−7.0168	−3.6181
x_2	1.0822	0.8500	2.9842	−1.5878
x_3	−4.6448	−2.0834	2.5202	0.1301
x_4	−2.5088	−6.3685	−3.4810	−2.9404
x_5	−0.2301	4.4425	−8.0920	2.0014

Table 3 The overall dominance matrix η^3 of expert e_3

η^3	c_1	c_2	c_3	c_4
x_1	−9.5972	4.3452	−8.6559	1.7291
x_2	3.8225	−7.1622	−4.5325	1.5346
x_3	−0.1180	−3.3812	3.7757	−0.4337
x_4	−5.5370	2.5571	−0.8021	−10.9314
x_5	−0.5712	−10.8354	−0.6881	−4.5923

Table 4 The overall dominance matrix η^4 of expert e_4

η^4	c_1	c_2	c_3	c_4
x_1	−0.4035	−8.0201	0.1148	−2.4476
x_2	−0.9912	0.5827	−2.1898	0.8828
x_3	−1.3427	3.0619	−8.4617	0.2623
x_4	0.8897	−5.8014	5.0710	−1.6077
x_5	0.1113	−1.2299	−9.7161	−0.3618

Table 5 The overall dominance matrix η^5 of expert e_5

η^5	c_1	c_2	c_3	c_4
x_1	0.7234	−10.8257	1.9422	−1.5018
x_2	1.0627	4.2426	1.9650	0.7013
x_3	−4.0075	−2.7540	−2.9233	−4.9326
x_4	−1.3192	−0.8939	−3.5214	3.6860
x_5	−0.9060	−7.1250	−1.7040	−8.8842

(4) By Eq. (23), we get

$$\begin{aligned} & \left\{ \begin{array}{l} \text{PIS}^1 = (\langle c_1, 5.1292 \rangle, \langle c_2, 1.3670 \rangle, \langle c_3, 0.4115 \rangle, \langle c_4, 2.4544 \rangle) \\ \text{NIS}^1 = (\langle c_1, -11.5733 \rangle, \langle c_2, -5.0820 \rangle, \langle c_3, -3.9550 \rangle, \langle c_4, -7.3800 \rangle); \end{array} \right. \\ & \left\{ \begin{array}{l} \text{PIS}^2 = (\langle c_1, 1.00822 \rangle, \langle c_2, 4.4425 \rangle, \langle c_3, 2.9842 \rangle, \langle c_4, 2.0014 \rangle) \\ \text{NIS}^2 = (\langle c_1, -4.6448 \rangle, \langle c_2, -12.1947 \rangle, \langle c_3, -8.0920 \rangle, \langle c_4, -3.6181 \rangle); \end{array} \right. \\ & \left\{ \begin{array}{l} \text{PIS}^3 = (\langle c_1, 3.8225 \rangle, \langle c_2, 4.3452 \rangle, \langle c_3, 3.7757 \rangle, \langle c_4, 1.7291 \rangle) \\ \text{NIS}^3 = (\langle c_1, -9.5972 \rangle, \langle c_2, -10.8354 \rangle, \langle c_3, -8.6559 \rangle, \langle c_4, -10.9314 \rangle); \end{array} \right. \\ & \left\{ \begin{array}{l} \text{PIS}^4 = (\langle c_1, 0.8897 \rangle, \langle c_2, 3.0619 \rangle, \langle c_3, 5.0710 \rangle, \langle c_4, 0.8828 \rangle) \\ \text{NIS}^4 = (\langle c_1, -1.3427 \rangle, \langle c_2, -8.0201 \rangle, \langle c_3, -9.7161 \rangle, \langle c_4, -2.4476 \rangle); \end{array} \right. \\ & \left\{ \begin{array}{l} \text{PIS}^5 = (\langle c_1, 1.0627 \rangle, \langle c_2, 4.2426 \rangle, \langle c_3, 1.9650 \rangle, \langle c_4, 3.6860 \rangle) \\ \text{NIS}^5 = (\langle c_1, -4.0075 \rangle, \langle c_2, -10.8257 \rangle, \langle c_3, -3.5214 \rangle, \langle c_4, -8.8842 \rangle) \end{array} \right. \end{aligned}$$

(5) and (6) By Eqs. (24)–(26), relative closeness index (RCI) for each alternative with respect to each expert e_k ($k = 1, 2, \dots, q$) are computed and shown in Tables 6–10.

(7) By Eq. (27) we calculate the conditional probabilities of each alternative for each expert. The details results are shown in Table 11.

(8) By Eq. (28), the probabilities of alternatives are:

$$P(x_1) = 0.1821, P(x_2) = 0.2561, P(x_3) = 0.1962, P(x_4) = 0.2228 \text{ and } P(x_5) = 0.1428.$$

Table 6 The output results of $\vartheta_i^{1,+}$, $\vartheta_i^{1,-}$, and RCI_i^1 of expert e_1

Alternatives	$\vartheta_i^{1,+}$	$\vartheta_i^{1,-}$	RCI_i^1
x_1	0.8665	10.2529	0.9221
x_2	7.1886	3.9308	0.3535
x_3	6.9452	4.1742	0.3754
x_4	3.9648	7.1546	0.6434
x_5	7.3434	3.7761	0.3396

Table 7 The output results of $\vartheta_i^{2,+}$, $\vartheta_i^{2,-}$, and RCI_i^2 of expert e_2

Alternatives	$\vartheta_i^{2,+}$	$\vartheta_i^{2,-}$	RCI_i^2
x_1	10.4173	1.1408	0.0987
x_2	1.9396	9.6185	0.8322
x_3	3.8359	7.7223	0.6681
x_4	7.4653	4.0929	0.3541
x_5	3.5847	7.9734	0.6899

Table 8 The output results of $\vartheta_i^{3,+}$, $\vartheta_i^{3,-}$, and RCI_i^3 of expert e_3

Alternatives	$\vartheta_i^{3,+}$	$\vartheta_i^{3,-}$	RCI_i^3
x_1	5.8157	7.6879	0.5693
x_2	4.9776	8.5259	0.6314
x_3	3.7086	9.7949	0.7254
x_4	7.1565	6.3470	0.4700
x_5	7.9505	5.5530	0.4112

Table 9 The output results of $\vartheta_i^{4,+}$, $\vartheta_i^{4,-}$, and RCI_i^4 of expert e_4

Alternatives	$\vartheta_i^{4,+}$	$\vartheta_i^{4,-}$	RCI_i^4
x_1	6.4215	4.7855	0.4270
x_2	4.4385	6.7685	0.6040
x_3	6.7399	4.4671	0.3986
x_4	3.1287	8.0783	0.7208
x_5	8.6607	2.5463	0.2272

Table 10 The output results of $\vartheta_i^{5,+}$, $\vartheta_i^{5,-}$, and RCI_i^5 of expert e_5

Alternatives	$\vartheta_i^{5,+}$	$\vartheta_i^{5,-}$	RCI_i^5
x_1	7.0431	3.9350	0.3584
x_2	0.8681	10.1100	0.9209
x_3	6.7649	4.2132	0.3838
x_4	3.3469	7.6311	0.6951
x_5	8.8316	2.1465	0.1955

Table 11 The conditional probabilities of each alternative

$P(x_i e_k)$	e_1	e_2	e_3	e_4	e_5
x_1	0.3501	0.1342	0.1425	0.2443	0.1289
x_2	0.0373	0.3149	0.2528	0.1340	0.2610
x_3	0.2028	0.2249	0.2584	0.1674	0.1465
x_4	0.1796	0.2540	0.1676	0.3032	0.0956
x_5	0.1404	0.3606	0.1503	0.2722	0.0766

Step 5: By Eq. (33), the belief distance between two experts about each alternative is calculated, which is presented in Table 12.

By Eq. (34), the Deng entropy between two experts about each alternative is shown in Table 13.

Table 12 The belief distance between two experts about each alternative

D_{ui}	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
$u = 1(e_1, e_2)$	0.3288	0.3091	0.9561	0.7698	0.8437
$u = 2(e_1, e_3)$	0.5488	0.8940	0.6300	0.6690	0.1705
$u = 3(e_1, e_4)$	0.2359	0.8548	0.2042	0.6362	0.0748
$u = 4(e_1, e_5)$	0.0711	0.2192	0.0217	0.9594	0.1579
$u = 5(e_2, e_3)$	0.7445	0.8385	0.6887	0.2704	0.7117
$u = 6(e_2, e_4)$	0.5370	0.7760	0.8483	0.3298	0.7966
$u = 7(e_2, e_5)$	0.2804	0.0318	0.8783	1.0177	0.8186
$u = 8(e_3, e_4)$	0.2967	0.0914	0.4209	0.1002	0.2103
$u = 9(e_3, e_5)$	0.5374	0.7622	0.5506	1.0208	0.2884
$u = 10(e_4, e_5)$	0.2630	0.7043	0.1479	1.0452	0.0650

Table 13 The Deng entropy between two experts about each alternative

$\vartheta_{ui}(D_{ui})$	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
$u = 1(e_1, e_2)$	1.8124	1.7312	3.7974	3.2981	3.5032
$u = 2(e_1, e_3)$	2.6192	3.6373	2.8814	3.0016	1.1012
$u = 3(e_1, e_4)$	1.4131	3.5330	1.2659	2.9008	0.5721
$u = 4(e_1, e_5)$	0.5488	1.3364	0.2046	3.8057	1.0375
$u = 5(e_2, e_3)$	3.2256	3.4891	3.0613	1.5666	3.1297
$u = 6(e_2, e_4)$	2.5797	3.3157	3.5157	1.8165	3.3735
$u = 7(e_2, e_5)$	1.6098	0.2823	3.5960	3.9503	3.4347
$u = 8(e_3, e_4)$	1.6795	0.6724	2.1699	0.7238	1.2948
$u = 9(e_3, e_5)$	2.5810	3.2765	2.6252	3.9579	1.6443
$u = 10(e_4, e_5)$	1.5344	3.1078	0.9855	4.0167	0.5104

Table 14 The standardized Deng entropy between two experts about each alternative

$\beta_{ui}(D_{ui})$	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
$u = 1(e_1, e_2)$	-0.0559	-0.1363	1.0000	0.5635	1.0000
$u = 2(e_1, e_3)$	0.5469	1.0000	0.4901	0.3834	-0.6052
$u = 3(e_1, e_4)$	-0.3542	0.9378	-0.4092	0.3222	-0.9588
$u = 4(e_1, e_5)$	-1.0000	-0.3716	-1.0000	0.8719	-0.6477
$u = 5(e_2, e_3)$	1.0000	0.9117	0.5903	-0.4881	0.7504
$u = 6(e_2, e_4)$	0.5174	0.8083	0.8432	-0.3364	0.9133
$u = 7(e_2, e_5)$	-0.2072	-1.0000	0.8879	0.9597	0.9542
$u = 8(e_3, e_4)$	-0.1552	-0.7675	0.0940	-1.0000	-0.4759
$u = 9(e_3, e_5)$	0.5184	0.7850	0.3474	0.9643	-0.2423
$u = 10(e_4, e_5)$	-0.2636	0.6844	-0.5653	1.0000	-1.0000

By Eq. (35), the standardized Deng entropy between two experts about each alternative is shown in Table 14.

By Eq. (37), the interface effect among experts for each alternative is:

$INT_1 = -0.0086$, $INT_2 = 0.2550$, $INT_3 = 0.2333$, $INT_4 = 0.0203$, and $INT_5 = 0.0760$.

Finally, by Eq. (39), the probability of each alternative is:

$Prob(x_1) = 0.0933$, $Prob(x_2) = 0.2750$, $Prob(x_3) = 0.2310$, $Prob(x_4) = 0.2830$, and $Prob(x_5) = 0.1177$.

Step 6: According to the value of QP of each alternative, the final ranking is $x_4 \succ x_2 \succ x_3 \succ x_5 \succ x_1$.

4.2 Result Discussion

In this section, we discussed the validation of our numerical results through sensitive analysis and some comparative analysis.

4.2.1 Effect of the Quantum Interface on Decision Result

If we can decide based on the $P(x_i)(i = 1, 2, 3, 4, 5)$ values according to Eq. (28), as higher probabilities indicate a superior ranking. The classical approach does not account for the interference effect, which can influence the final rankings, as shown in Fig. 4. The ranking result of alternatives according to Eq. (28) is $x_2 \succ x_4 \succ x_3 \succ x_1 \succ x_5$. If we count the interface effect between two experts about an alternative the ranking result is $x_4 \succ x_2 \succ x_3 \succ x_5 \succ x_1$.

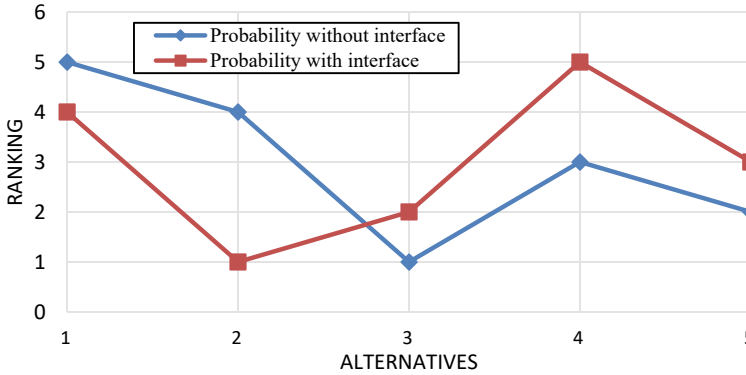


Fig. 4 The interface effect on ranking values

4.2.2 Comparative with Validity Analysis

To test the feasibility and validity of the suggested method, we compare it with some existing methods for the example given above.

In the traditional decision-making model (Mandal & Ranadive, 2019) the ranking results are derived from the aggregate decision matrix of experts' opinions. Therefore, to compare our model and the traditional model, we first aggregate the experts' opinions. In this connection, by Mandal and Ranadive (2019), using the Pythagorean linguistic weighted averaging aggregation operator, the aggregation of five experts' opinions is

$$Agg(A^1, A^2, A^3, A^4, A^5) = \begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} \langle h_{1.14}, (0.77, 0.31) \rangle \\ \langle h_{1.60}, (0.64, 0.36) \rangle \\ \langle h_{-1.13}, (0.75, 0.26) \rangle \\ \langle h_{0.61}, (0.76, 0.18) \rangle \\ \langle h_{0.61}, (0.73, 0.25) \rangle \end{bmatrix} & \begin{bmatrix} \langle h_{-0.55}, (0.81, 0.30) \rangle \\ \langle h_{0.78}, (0.71, 0.38) \rangle \\ \langle h_{1.01}, (0.85, 0.17) \rangle \\ \langle h_{0.18}, (0.67, 0.29) \rangle \\ \langle h_{-0.01}, (0.81, 0.23) \rangle \end{bmatrix} & \begin{bmatrix} \langle h_{-0.44}, (0.68, 0.43) \rangle \\ \langle h_{1.94}, (0.68, 0.49) \rangle \\ \langle h_{0.65}, (0.78, 0.36) \rangle \\ \langle h_{0.65}, (0.81, 0.24) \rangle \\ \langle h_{-0.97}, (0.70, 0.36) \rangle \end{bmatrix} & \begin{bmatrix} \langle h_{0.63}, (0.78, 0.28) \rangle \\ \langle h_{1.24}, (0.66, 0.33) \rangle \\ \langle h_{1.24}, (0.88, 0.27) \rangle \\ \langle h_{-0.07}, (0.78, 0.21) \rangle \\ \langle h_{0.08}, (0.82, 0.21) \rangle \end{bmatrix} \end{matrix}$$

Therefore, the ranking of alternatives according to the methods LogTODIM-TOPSIS method is $x_2 \succ x_4 \succ x_1 \succ x_3 \succ x_5$, i.e., the optimal option is x_2 . When the proposed model is applied without considering the interface effect, the optimal option is also x_2 . This suggests that the optimal option of the proposed approach aligns with those of traditional methods when the impact of decision-makers is disregarded, thereby validating the robustness of the suggested model. However, when evaluating the quantum interference among decision-makers, the outcomes of this model suggest that x_4 is the optimal strategy. This inconsistency arises from the conventional framework's exclusion of the interference effect among decision-makers, which leads to subjective biases during the evaluation of alternatives. As a result, the outcomes of this model better align with real-world complexities.

5 Conclusion

Within the context of investment decision-making, this study provides a quantum group decision model that was developed to reduce the impact of information ambiguity and opinion interference. In order to transmit ambiguous information in complex decision-making situations, this technique makes use of PLIs. Additionally, Deng entropy is utilized in order to evaluate opinion interference through the modeling of the reciprocal impacts among decision-makers in the MAGDM process. By assigning appropriate weights to decision-makers and managing opinion interference using quantum decision-making techniques, this paradigm tackles interference in a manner that is distinct from conventional approaches, which consider decision-makers as separate entities.

The model that has been provided improves upon the approaches that are already in place, providing solutions that are more effective for problems that are associated with opinion interference and weight distribution. By offering a complete framework to manage information ambiguity and opposing views, our findings demonstrate that the model considerably enhances decision-making in the process of navigating investment decisions. The introduction of new decision-makers and variables, on the other hand, has the effect of increasing computational complexity, which highlights the necessity of employing more advanced computing approaches in order to keep efficiency intact.

Research in the future will concentrate on incorporating machine learning techniques in order to improve scalability and practicality, so assuring that the model is successful in decision-making processes that include large-scale decision-making. With this integration, we want to control the increasing complexity, encourage consensus in the decision-making process of quantum groups, and guarantee the model's continued viability over the long run.

Conflict of Interest The authors declare that they have no conflict of interest

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Sustainable Solar Energy Management Driven by Pythagorean Linguistic Information Under Quantum Group Decision-Making Procedures and the CoCoSo Approach



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Abstract This research focuses on optimizing energy management in organizations to reduce costs and increase revenues. It emphasizes the importance of proactive, coordinated, and systematic energy utilization, highlighting the necessity of informed decision-making through building management, energy audits, and equipment retrofits. The Combined Compromise Solution (CoCoSo) method, a powerful technique for multi-attribute group decision-making (MAGDM), merges a compromise decision algorithm with an aggregation strategy to achieve a balanced solution. In MAGDM, the type of information is crucial, as effectively addressing uncertainty in the information type leads to better outcomes. The Pythagorean Linguistic Number (PLN) is an effective tool for managing uncertainty related to information. This research extends the original CoCoSo approach to operate within a PLN context, thus enhancing its applicability in addressing qualitative decision-making problems under quantum decision theory (QDT). We propose an innovative CoCoSo methodology using QDT within a PLN-based framework, demonstrating flexibility and scalability that extend beyond QDT. The method's superior performance and resilience are validated in the context of sustainable solar energy management.

Keywords Quantum decision theory · Group decision-making · Pythagorean linguistic set · CoCoSo

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1 Introduction

The global improvement in living circumstances, the growth of the population, and the expansion of the economy have all contributed to a steady increase in the demand for energy during the last few decades. On the other hand, fossil fuel supplies are continually being depleted, which is causing prices to gradually increase over time. The appropriate management of energy is very necessary in order to deal with the ever-increasing costs and regulatory duties. With the decreasing manufacturing costs of photovoltaic panels (solar cells) and accumulators, renewable energy sources are becoming increasingly desirable. They have a number of advantages, including the fact that they require less maintenance, do not have any moving parts built within, do not generate any noise, and do not contribute to pollution.

It was in the year 1839 that the French physicist Edmond Becquerel made the initial discovery of the photovoltaic phenomenon. At Bell Laboratories, the first solar cell that could be used in practical applications was shown to the public on April 25, 1954. Individual photovoltaic modules, which are more often referred to as solar panels, are frequently utilized as the structural components of photovoltaic systems. They are able to avoid fuel expenses and supply concerns by utilizing renewable energy, which ultimately results in a reduction in power prices. However, the intermittent and unpredictable nature of the electricity provided by solar cells, which is determined by the environment of the surrounding area, may be addressed by combining solar cells with other power sources, such as batteries for energy storage. The batteries are kept charged and the energy supply is maintained at a constant level by these systems, which effectively manage several sources to fulfill the demand and provide energy independent of fluctuations in load. The fact that efficient energy management provides several strategic advantages in addition to the reduction of costs is one of the reasons why decision-makers are becoming increasingly interested in it.

In the highly competitive business environment of today, it is essential to examine alternative solutions by employing a broad variety of criteria and a number of different possibilities in order to accomplish any objective. It is not sufficient to only choose the best choice; rather, it is necessary to precisely rank all of the alternatives in order to have a complete understanding of their classification and characterization. The ability to make decisions increases the possibility of selecting the best suitable alternative from among a number of different problems. It is possible that individuals who are confronted with the responsibility of forming judgments in both favorable and unfavourable circumstances might benefit from decision-making methods. When faced with a variety of sources of uncertainty or conflicting information, decision-makers frequently find it difficult to evaluate the available alternatives. In 1965, Zadeh presented the significant notion of fuzzy sets (FS) in order to address the problems that were being discussed.

The research in fuzzy set theory and its applications plays a significant role in addressing uncertainty and complexity in social networks and group decision-making processes. The methods developed in this area offer valuable insights into how fuzzy logic can be used to model and analyze systems where imprecision and ambiguity

are prevalent. Fuzzy set theory is particularly well-suited to modeling the uncertainty inherent in social networks, where relationships and interactions often involve incomplete or vague information. Traditional mathematical models struggle to capture this uncertainty, but fuzzy approaches provide a more nuanced framework for analyzing such dynamics. For instance, the development of fuzzy fractional differential equations using fuzzy Laplace transforms is essential for modeling systems where the influence of individuals or groups evolves fractionally over time under uncertainty (Salahshour et al. 2012; Allahviranloo et al. 2015; Chehlabi et al. 2016; Armand et al. 2018).

A new centrality measure is introduced for neutrosophic and linguistic graphs (Mahapatra et al. 2019, 2020), including self-weight or individual merit within the network. Mahapatra et al. (Mahapatra et al. 2024a) present an innovative method for centrality measurement in fuzzy social networks where relationships between nodes are not binary but have different degrees of intensity.

According to the above discussion, fuzzy sets (FS) have become a popular approach for incorporating fuzzy information as part of an expert's evaluative measures. However, as noted by Xu (2005) and Mandal et al. (2022a), experts tend to express their opinions more reliably using linguistic information, known as linguistic sets (LS), rather than fuzzy information. In (Mandal et al. 2019), the LS concept was extended to the Pythagorean linguistic set (PLS), allowing experts to express both their confidence and doubt levels when providing judgments. In real-life scenarios, situations, experts often have limited knowledge and personal biases, so using language-based information with expert confidence and doubt levels (Mandal et al., 2021; 2022b; 2023a, b; 2024a) is more flexible for expressing their ideas. In this connection, Mandal et al. briefly studied MCGDM under LZN information with the MARCOS approach and its application in the selection of logistic cold chain centers (Mandal et al. 2024b); TODIM-VIKRO approach (Mandal et al. 2024c) and its application in site selection of medical logistic center; ORESTE approach (Mandal et al. 2024d) and its application in assessment of regional energy; and MULTI-MOORA approach (Mandal et al. 2024e) and its application in selection of software. To address the challenge of aggregating individual assessments, specific methods are needed. Although many aggregation strategies have been studied, most assume that decision-makers are independent and uninfluenced by one another. However, in practice, opinions are often influenced by surrounding perspectives. In decision-making, the views of multiple experts can interact, ultimately affecting the final decision, and introducing the concept of 'interference.

Quantum Probability Theory (QPT) (Busemeyer and Bruza 2012; Moreira and Wichert 2014; Yearsley and Busemeyer 2016) offers a framework for capturing these influences. QPT, a recent advancement over classical probability theory (CPT), has been extensively studied in fields such as psychology (Busemeyer and Bruza 2012), cognitive science (Yearsley and Busemeyer 2016), and decision analysis (He et al. 2018a, b; Wu et al. 2020; Xiao et al. 2022; Han and Liu 2023; Mandal et al., 2024f; 2025) due to its robust descriptive power. It effectively explains paradoxes, such as the disjunction fallacy (He et al. 2018a, b), the Ellsberg paradox (Eichberger and Pirner 2018), the order effect (Wang et al. 2014), and graph theory (Samanta and

Allahviranloo 2025) which are challenging for CPT to address. QPT's ability to represent interference terms makes it particularly valuable for understanding how expert opinions influence each other in the MCGDM process.

This study addresses the MCGDM challenge in sustainable solar energy management decisions using Pythagorean linguistic information (PLI) integrating quantum decision theory (QDT) to manage uncertain and fuzzy data. It also introduces the quantum-based PLI-CoCoSo model, which combines which combines traditional CoCoSo (Yazdani et al. 2019), and QDT under the PLI (Mandal et al. 2019) environment. This novel approach is validated through a numerical example and comparative analyses, aiming to:

- (1) An improved PLI-based CoCoSo approach accounts for experts' limited rationality, less time, high stability, simple calculation, and more concise, and resolves some paradoxes of the traditional CoCoSo approach.
- (2) A QDT-based method for combining expert opinion results, helps us better understand human psychology through the effects of interference.
- (3) A case study demonstrating the effectiveness of our approach in sustainable solar energy management decisions. Comparative and sensitivity analyses show the flexibility and robustness of our method.

The structure of the paper is as follows: In Sect. 2, recall some basic definitions and the background of linguistic term sets (LTSs), the Pythagorean linguistic set (PLS), and quantum decision theory (QDT). The proposed MCGDM model is discussed in Sect. 3. A case study for sustainable solar energy management decisions evaluation with comparative analysis is provided in Sect. 4. Section 5 concludes our study.

2 Preliminaries

Here we recall some related definitions and results that are used in this paper.

2.1 Linguistic Term Set (LTS)

Here we assume an LTS is: $H = \{h_\lambda | \lambda = -\zeta, \dots, -1, 0, 1, \dots, \zeta\}$ with cardinality $2\zeta + 1$ and holds the followings (Mandal et al., 2024a):

- (1) $h_{\lambda_1} \geq h_{\lambda_2}$ if and only if $\lambda_1 \geq \lambda_2$;
- (2) $\text{neg}(h_{\lambda_1}) = h_{-\lambda_1}$;
- (3) $\max(h_{\lambda_1}, h_{\lambda_2}) = h_{\lambda_1}$ if $\lambda_1 \geq \lambda_2$;
- (4) $\min(h_{\lambda_1}, h_{\lambda_2}) = h_{\lambda_1}$ if $\lambda_1 \leq \lambda_2$.

To preserve expert information, the continuous LTSs (Xu 2005) denotes $\overline{H} = \{h_{-t} \leq h_\lambda \leq h_t | \lambda \in [-t, t], t > \zeta\}$.

The subscript index of any $h_\lambda \in \overline{H}$, represented by the following functions:

$$I : \overline{H}[-t, t], \text{ and } I(h_\lambda) = \lambda.$$

$$I^{-1} : [-t, t] \rightarrow \overline{H}, \text{ and } I^{-1}(\lambda) = h_\lambda.$$

Let $\mathcal{F}$ be the linguistic conversation function for any LT $G \in \overline{H}$, which is defined in the following way:

$$\mathcal{F} : \overline{H} \rightarrow [0, 1], \mathcal{F}(G) = \{\mathcal{F}(h_\lambda) = \frac{\lambda}{2t} + \frac{1}{2} | \lambda \in [-t, t]\} = \kappa.$$

2.2 Pythagorean Linguistic Set (PTS)

By Mandal and Ranadive (2019), a Pythagorean linguistic set (PLS) denoted by $\mathcal{P}$ in an alternative set X for continuous LTS $\overline{H} = \{h_{-t} \leq h_\lambda \leq h_t | \lambda \in [-t, t]\}$ is represented in the following way:

$$\mathcal{R} = \{h_{\theta(x)}, (\mu_{\mathcal{R}}(x), \nu_{\mathcal{R}}(x)) | x \in X\},$$

where, $h_{\theta(x)} \in \overline{H}$, $\mu_{\mathcal{R}} : X \rightarrow [0, 1]$ and $\nu_{\mathcal{R}} : X \rightarrow [0, 1]$ are the membership and non-membership degrees of x to $h_{\theta(x)}$ such that $\mu_{\mathcal{R}}^2(x) + \nu_{\mathcal{R}}^2(x) \leq 1$. Here, the Pythagorean linguistic number (PLN) in $\mathcal{R}$ is denoted by $\wp$ and represented by $\wp = h_{\theta(\wp)}, (\mu(\wp), \nu(\wp))$ with $h_{\theta(\wp)} \in \overline{H}$ and $\mu(\wp), \nu(\wp) \in [0, 1]$, $\mu^2(\wp) + \nu^2(\wp) \leq 1$.

By Mandal and Ranadive (2019), the expected value of $\wp = h_{\theta(\wp)}, (\mu(\wp), \nu(\wp))$ is denoted by $\varkappa$ and defined in the following way:

$$\varkappa(\wp) = h_{\frac{\theta(\wp) \times (\mu^2(\wp) + 1 - \nu^2(\wp))}{2}}.$$

By Wang et al. (2019), the distance between two PLNs $\wp_1 = \langle h_{\theta(\wp_1)}, (\mu(\wp_1), \nu(\wp_1)) \rangle$ and $\wp_2 = \langle h_{\theta(\wp_2)}, (\mu(\wp_2), \nu(\wp_2)) \rangle$ are defined in the following way:

$$d(\wp_1, \wp_2) = \frac{1}{2} \left| (1 + \mu^2(\wp_1) - \nu^2(\wp_1)) * \mathcal{F}(h_{\theta(\wp_1)}) \right. \\ \left. - (1 + \mu^2(\wp_2) - \nu^2(\wp_2)) * \mathcal{F}(h_{\theta(\wp_2)}) \right|$$

2.3 Quantum Probability Theory (QPT)

A QDT model is based on the principles of quantum probability (QP) theory (QPT). To provide context, we will review some key concepts and notations of QPT. In QDT, a state is represented as a unit-length vector within an N-dimensional vector space.

Definition 1 (Yearsley and Busemeyer 2016) A Hilbert space (HS) with finite-dimensional is represented by $\mathcal{H}$, where $\mathcal{H}$ is defined in abstract vector space over a complex field, spanned by a set of orthonormal basis vectors with complex components. In classical probability theory (CPT), events are fundamental (He et al. 2018), but in QPT, events are instead represented by subspaces of the HS, rather than traditional sets.

Definition 2 (Busemeyer and Bruza 2012) In QDT, the superposition state $|\mathcal{S}\rangle$, a unit-length vector in the N-dimensional HS $\mathcal{H}$, defines all possible events within that space. The nonlinear combination of a set of orthogonal basis vectors $\{|\mathcal{V}_k\rangle, k = 1, 2, \dots, n\}$ are used to represent the superposition state $|\mathcal{S}\rangle$, i.e.,

$$\begin{aligned} |\mathcal{S}\rangle &= \mathcal{V}_1|\mathcal{S}\rangle\mathcal{V}_1 + \mathcal{V}_2|\mathcal{S}\rangle\mathcal{V}_2 + \dots + \mathcal{V}_n|\mathcal{S}\rangle\mathcal{V}_n \\ &= \xi_1|\mathcal{V}_1\rangle + \xi_2|\mathcal{V}_2\rangle + \dots + \xi_n|\mathcal{V}_n\rangle \\ &= \zeta_1 e^{i\alpha_1}|\mathcal{V}_1\rangle + \zeta_2 e^{i\alpha_2}|\mathcal{V}_2\rangle + \dots + \zeta_n e^{i\alpha_n}|\mathcal{V}_n\rangle \end{aligned} \quad (1)$$

where $\xi_k (= \zeta_k e^{i\alpha_k})$ is the amplitude of the probability, expressed as a complex number, characterizing the amplitude of the wave in the state $|\mathcal{V}_k\rangle$, and the amplitude of the phase angle represented by $e^{i\alpha_k}$.

Definition 3 (Busemeyer and Bruza 2012) The CPT framework represents every event using a single set of components. In QPT, probabilities are assigned based on the overlap between cognitive states (shown as state vectors) and different probability measures (shown as subspaces). QPT extends CPT into the complex domain by including incompatible qualities and viewing events as vectors. Incompatible attributes are two observable features of an item that cannot be measured accurately at the same time. Both the incompatibility condition and the uncertainty principle can be described by the probability amplitude, also known as complex probability.

According to Born's rule, an event's probability amplitude and its classical probability, x_i are related to each other:

$$P(\mathcal{V}_k) = |\zeta_k e^{i\alpha_k}|^2 = (\zeta_k e^{i\alpha_k}) \times (\zeta_k e^{i\alpha_k})^* = \zeta_k e^{i\alpha_k} \times \zeta_k e^{-i\alpha_k} = (\zeta_k)^2 \quad (2)$$

For the total probability of event to be 1, the sum of the squares of the probability amplitudes must also be 1.

$$\sum_{k=1}^n P(\mathcal{V}_k) = \sum_{k=1}^n (\zeta_k)^2 = 1. \quad (3)$$

An $N - 1$ mapping transforms complex probabilities into classical probabilities. For each given probability value $P(\mathcal{V}_k)$, countless complex probabilities correspond to it. Referencing Eq. (2) and Euler's formula,

$$e^{i\alpha} = \cos\alpha + i \times \sin\alpha \quad (4)$$

the complex probability satisfies:

$$\xi = \sqrt{P(\mathcal{V})}(\cos\alpha + i \times \sin\alpha) = \sqrt{P(\mathcal{V})}e^{i\alpha} \quad (5)$$

For example, in the following, we discussed the QPT in two-dimensional Hilbert space. Let $\{\mathcal{M}_1, \mathcal{M}_2\}$ be an event containing two basic states, where $|\mathcal{M}_1\rangle = (1,0)$, and $|\mathcal{M}_2\rangle = (0,1)$ are defined in $\mathbf{H}$. In Fig. 1, we graphically show the state $|\mathcal{S}\rangle$, which is defined as a superposition. According to Eqs. (1) and (2), we get

$$|\mathcal{S}\rangle = \zeta_1 e^{i\alpha_1} |\mathcal{M}_1\rangle + \zeta_2 e^{i\alpha_2} |\mathcal{M}_2\rangle = \frac{e^{i\alpha_{\mathcal{M}_1}}}{\sqrt{2}} |\mathcal{M}_1\rangle + \frac{e^{i\alpha_{\mathcal{M}_2}}}{\sqrt{2}} |\mathcal{M}_2\rangle$$

and the probabilities of $|\mathcal{M}_1\rangle$ and $|\mathcal{M}_2\rangle$ are computed in the following way:

$$P(\mathcal{M}_1) = P(\mathcal{M}_2) = (e^{i\alpha})(e^{i\alpha})^* = \frac{1}{2}.$$

According to conventional Bayesian decision theory (BDT), DM can be mathematically represented as a Markov process (MP) (Moreira and Wichert 2014). The state transition probability (STP) in a single route of QPT aligns with the CPT according to Feynman's first rule (Wu et al. 2020), which varies over several pathways. Consider the decision-making dilemma highlighted above as an illustration of the process depicted in Fig. 2a. Experts provide their assessments of $\mathcal{M}_1$ ('bad')

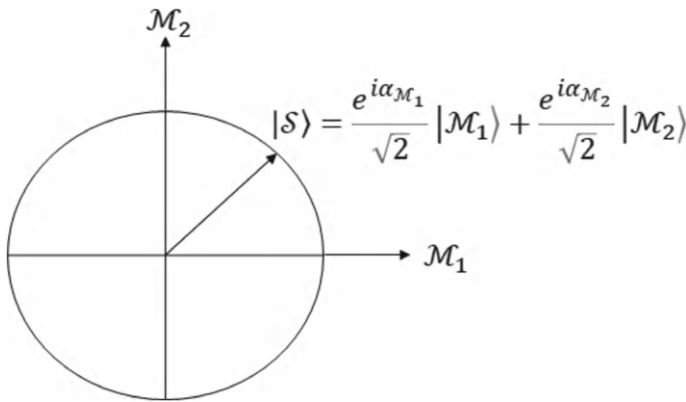


Fig. 1 Event in Hilbert space with two dimensions

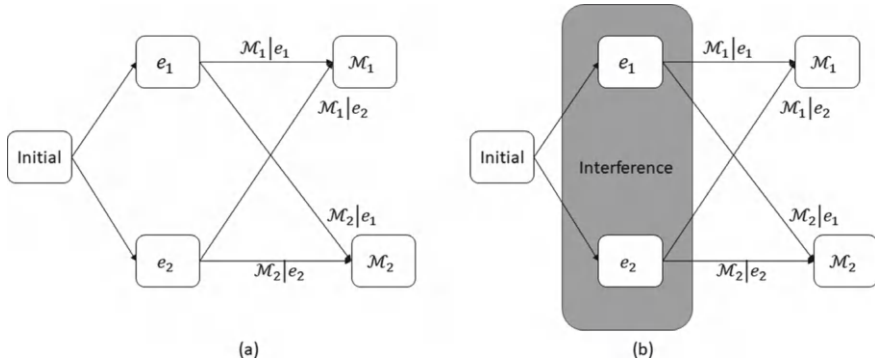


Fig. 2 DM in **a** MP and **b** QDT process

or $\mathcal{M}_2$ ('good') as probability quantifications. STPs are determined by combining the individual probabilities for each route and using the total probability formula in CPT.

$$P(\mathcal{M}_1) = \sum_{k=1}^2 P(e_k)P(\mathcal{M}_1|e_k), \sum_{k=1}^2 P(e_k) = 1 \quad (6)$$

Since only one route is selected at a time, it is sometimes referred to as the single path principle. The conventional MP assumes that experts rely on one expert at a time and that both experts are completely autonomous. However, this approach is sometimes unsuitable for ordinary cognitive processes because many avenues are often considered when forming judgments. Such phenomena can result in illogical actions, including conjunction errors as well as disjunction, and impacts of interference. Thus, QPT is an excellent complement to CPT in this scenario.

Within the QPT framework Fig. 2b, the state is formed by combining three fundamental states, which can be represented as different wave functions aligning according to Feynman's second rule. The amplitude of the final form is obtained by superimposing two individual routes running from I ('Initial') to $\mathcal{M}_1$. The amplitude is the sum of the individual pathways from source I to target $\mathcal{M}_1$, and the probability of reaching the target $\mathcal{M}_1$ is equal to the square of the amplitude.

$$P(\mathcal{M}_1) = |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 \quad (7)$$

Equation (7) can be simplified in the following way:

$$\begin{aligned} P(\mathcal{M}_1) &= |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 \\ &= |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}| |\xi_{e_1}\xi_{\mathcal{M}_1|e_1} + \xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^* \\ &= \left(\sqrt{P(e_1)P(\mathcal{M}_1|e_1)}e^{i\alpha_1} + \sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{i\alpha_2} \right) \end{aligned}$$

$$\begin{aligned}
& \left(\sqrt{P(e_1)P(\mathcal{M}_1|e_1)}e^{-i\alpha_1} + \sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{-i\alpha_2} \right) \\
&= P(e_1)P(\mathcal{M}_1|e_1) + P(e_2)P(\mathcal{M}_1|e_2) \\
& \quad + \sqrt{P(e_1)P(\mathcal{M}_1|e_1)}\sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{i(\alpha_1-\alpha_2)} \\
& \quad + \sqrt{P(e_1)P(\mathcal{M}_1|e_1)}\sqrt{P(e_2)P(\mathcal{M}_1|e_2)}e^{i(\alpha_2-\alpha_1)} \quad (8)
\end{aligned}$$

By Eq. (4), we get

$$\cos(\alpha_1 - \alpha_2) = \frac{e^{i(\alpha_2-\alpha_1)} + e^{i(\alpha_1-\alpha_2)}}{2} \quad (9)$$

Then Eq. (7) can be written in the following way:

$$\begin{aligned}
P(\mathcal{M}_1) &= P(e_1)P(\mathcal{M}_1|e_1) + P(e_2)P(\mathcal{M}_1|e_2) \\
& \quad + 2\sqrt{P(e_1)P(\mathcal{M}_1|e_1)}\sqrt{P(e_2)P(\mathcal{M}_1|e_2)}\cos(\alpha_1 - \alpha_2) \\
&= |\xi_{e_1}\xi_{\mathcal{M}_1|e_1}|^2 + |\xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 + 2\xi_{e_1}\xi_{\mathcal{M}_1|e_1}\xi_{e_2}\xi_{\mathcal{M}_1|e_2}\cos(\alpha_1 - \alpha_2) \quad (10)
\end{aligned}$$

Upon comparing Eqs. (6) and (10), it becomes evident that the QDT framework includes an interference component. If the pathways are observable, the quantum phase transition will reduce to the classical one, following Feynman's third rule.

$$P(\mathcal{M}_1) = |\xi_{e_1}\xi_{\mathcal{M}_1|e_1}|^2 + |\xi_{e_2}\xi_{\mathcal{M}_1|e_2}|^2 = P(e_1)P(\mathcal{M}_1|e_1) + P(e_2)P(\mathcal{M}_1|e_2) \quad (11)$$

Additionally, the QPT can be expanded to a multi-path graph, resulting in an increased number of interference terms. The QPT enhances CPT by accurately representing cognitive processes in quantum terms, thereby more effectively modeling ambiguity, uncertainty, and vagueness in decision-making.

3 The Proposed Cocoso-Based Magdm Under Pythagorean Linguistic Information and Quantum Decision Scenario

The CoCoSo method (Yazdani et al. 2019) serves as an effective MCGDM approach, leveraging both the weighted sum measure (WSM) and the weighted product measure (WPM) to evaluate each option. In this context, this section introduces the novel approach of quantum-based MAGDM based on the CoCoSo method with Pythagorean linguistic information (PLI). The detailed procedure for the combined quantum cognition PLN-based CoCoSo approach is presented below:

3.1 Background

Let us assume

$X = \{x_1, x_2, \dots, x_n\}$: Alternative set,

$C = \{c_1, c_2, \dots, c_m\}$: Attribute set,

$e = \{e_1, e_2, \dots, e_q\}$: Expert set,

$w^k = (w_1^k, w_2^k, \dots, w_m^k)$: Weight vector of attributes for expert e_k , where $w_j^k \in [0, 1]$, and $\sum_{j=1}^m w_j^k = 1$,

$\omega = (\omega_1, \omega_2, \dots, \omega_q)$: Weight vector of experts with $\omega_k \in [0, 1]$, $\sum_{k=1}^q \omega_k = 1$.

3.2 Decision Steps

Let $A^k = (a_{ij}^k)_{nm}$ ($i = 1, 2, \dots, n; j = 1, 2, \dots, m; k = 1, 2, \dots, q$) be the PLI-based decision judgment matrix provided by an expert e_k using pre-defined distinct LTSs $H = \{h_\lambda | \lambda = -\varsigma, \dots, -1, 0, 1, \dots, \varsigma\}$.

Step 1: Construction of decision matrix

Then, in the following, we represent the decision matrix of k -th expert:

$$A^k = (a_{ij}^k)_{nm} = \begin{matrix} & \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} & \begin{bmatrix} c_1 & c_2 & \cdots & c_m \\ a_{11}^k & a_{12}^k & \cdots & a_{1m}^k \\ a_{21}^k & a_{22}^k & \cdots & a_{2m}^k \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}^k & a_{n2}^k & \cdots & a_{nm}^k \end{bmatrix} \end{matrix}, \quad (12)$$

Where, $a_{ij}^k = \langle h_{\theta_{ij}}^k, (\mu_{ij}^k, \nu_{ij}^k) \rangle$ is a PLN, called the Pythagorean linguistic information (PLI) of k -th expert for i -th alternative about j -th criteria.

Step 2: Expert weight computation process

Inspire by Mandal et al. (2024a), the weight of each expert e_k ($k = 1, 2, \dots, q$) can be calculated in the following way:

$$\omega_k = \frac{\Omega(Z^k)}{\sum_{k=1}^q \Omega(Z^k)}, \text{ where } \Omega(Z^k) = \frac{\sum_{j=1}^m \sum_{i=1}^n \mathcal{F}(\chi(a_{ij}^k))}{mn}. \quad (13)$$

Step 3: Attribute weight computation process

(1) The decision matrix (Eq. (12)) can be normalized to $L^k = (l_{ij}^k)_{nm}$ with

$$l_{ij}^k = \begin{cases} \frac{\mathcal{F}(\kappa(a_{ij}^k))}{\sum_{i=1}^n \mathcal{F}(\kappa(a_{ij}^k))}, \text{for benefit criteria } c_j, \\ 1 - \frac{\mathcal{F}(\kappa(a_{ij}^k))}{\sum_{i=1}^n \mathcal{F}(\kappa(a_{ij}^k))}, \text{for cost criteria } c_j. \end{cases} \quad (14)$$

(2) The Shannon entropy $\Psi^k = (\Psi_1^k, \Psi_2^k, \dots, \Psi_m^k)$ can be calculated in the following way:

$$\Psi_j^k = -\frac{1}{\ln n} \sum_{i=1}^n l_{ij}^k \ln l_{ij}^k \quad (15)$$

$l_{ij}^k \ln l_{ij}^k$ is 0 if $l_{ij}^k = 0$.

(3) The attribute weight $w^k = (w_1^k, w_2^k, \dots, w_m^k)$ for each expert $e_k (k = 1, 2, \dots, q)$ can be calculated in the following way:

$$w_j^k = \frac{1 - \Psi_j^k}{\sum_{j=1}^m (1 - \Psi_j^k)} \quad (16)$$

Step 4: Construction of Bayesian network

In this phase, we converted the pre-collected data into probabilities and developed a Bayesian Network for DM, as the QDT framework is a partial extension of the Bayesian Network. Figure 3 illustrates that, in a MAGDM situation, the decision-making process can be represented as a simple two-layer Bayesian Network. The experts' weights are converted into probabilities, which indicate the degree of belief in the experts, using the following equation:

$$P(e_k) = \omega_k, k = 1, 2, \dots, q \quad (17)$$

In Eq. (17), the probabilities form the initial layer of the Bayesian Network (BN). The second layer depends on the assessment outcomes from each expert, which can be calculated using the CoCoSo approach as follows:

(1) WSM and WPM computation process

Using attributes weight obtained from each expert $e_k (k = 1, 2, \dots, q)$ opinion by Eq. (16), the WSM and WPM of each alternative are determined in the following way:

$$S_i^k = \sum_{j=1}^m w_j^k l_{ij}^k, i = 1, 2, \dots, n \quad (18)$$

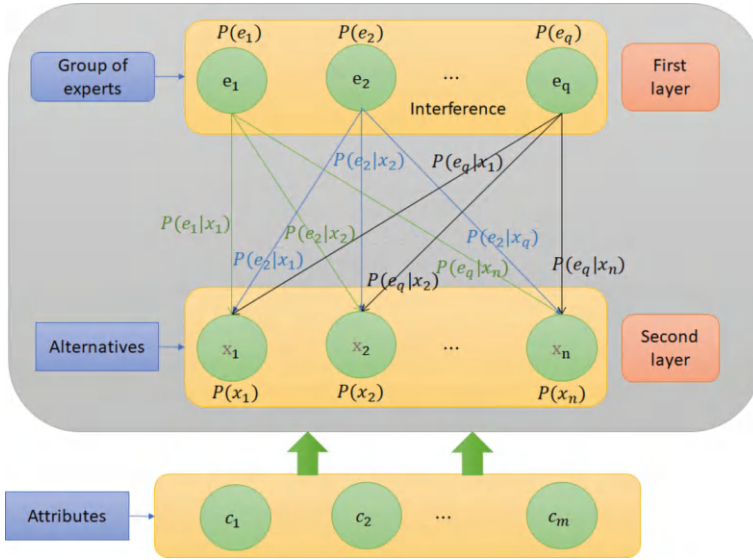


Fig. 3 MAGDM according to QSBN

$$\mathcal{P}_i^k = \prod_{j=1}^n \left(l_{ij}^k \right)^{w_j^k}, i = 1, 2, \dots, n \quad (19)$$

(2) **Each alternative relative importance computation process**

Using S_i^k and $\mathcal{P}_i^k$, three different functions are used for the computation of the relative importance of each alternative for each expert $e_k (k = 1, 2, \dots, q)$. There are as follows:

$$\left\{ \begin{array}{l} \mathcal{F}_i^{1,k} = \frac{S_i^k + \mathcal{P}_i^k}{\sum_{i=1}^n (S_i^k + \mathcal{P}_i^k)} \\ \mathcal{F}_i^{2,k} = \frac{S_i^k}{\min_{i=1}^n S_i^k} + \frac{\mathcal{P}_i^k}{\min_{i=1}^n \mathcal{P}_i^k} \\ \mathcal{F}_i^{3,k} = \frac{\epsilon S_i^k + (1-\epsilon) \mathcal{P}_i^k}{\epsilon \max_{i=1}^n S_i^k + (1-\epsilon) \max_{i=1}^n \mathcal{P}_i^k}, 0 \leq \epsilon \leq 1 \end{array} \right. \quad (20)$$

(3) **Each alternative total relative importance computation process**

By Eq. (20), the overall relative importance of each alternative for each expert $e_k (k = 1, 2, \dots, q)$ can be calculated in the following way:

$$\mathcal{F}_i^k = \left(\mathcal{F}_i^{1,k} \mathcal{F}_i^{2,k} \mathcal{F}_i^{3,k} \right)^{\frac{1}{3}} + \frac{1}{3} \left(\mathcal{F}_i^{1,k} + \mathcal{F}_i^{2,k} + \mathcal{F}_i^{3,k} \right) \quad (21)$$

(4) Based on Eq. (21), conditional probabilities which link the two layers in the BN as:

$$P(x_i|e_k) = \frac{\mathcal{F}_i^k}{\sum_{i=1}^n \mathcal{F}_i^k} \quad (22)$$

(5) The second layer consists of the probabilities of alternatives calculated as:

$$P(x_i) = \sum_{k=1}^q P(e_k)P(x_i|e_k) \quad (23)$$

Subsequently, we can decide based on the $P(x_i)$ values, as higher probabilities, indicate a superior ranking. The classical approach does not account for the interference effect, which can influence the final rankings. To address this issue, Bayesian networks are expanded into a quantum framework capable of properly modeling a practical approach to real interference in MAGDM.

Step 5: Construction of quantum scenario-based Bayesian network (QSBN)

The facilitator often consolidates opinions from several specialists. The interconnected decision-making processes in real-world situations, as demonstrated in Sect. 2.2, remain unobservable. The square of the sum of the amplitude probabilities of all possible pathways collectively constitutes the probability of the alternative x_i from QPT (He et al., 2018), as described.

$$\begin{aligned} P(x_i) &= \left| \sum_{k=1}^q \xi_{e_k} \xi_{x_i|e_k} e^{i\alpha_k} \right|^2 \\ &= \left| \sum_{k=1}^q \sqrt{P(e_k)} \sqrt{P(x_i|e_k)} e^{i\alpha_k} \right|^2, i = 1, 2, \dots, n \end{aligned} \quad (24)$$

By Eqs. (7–11), Eq. (24) can be written as

$$\begin{aligned} P(x_i) &= \sum_{k=1}^q P(e_k)P(x_i|e_k) \\ &\quad + 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \cos(\alpha_k - \alpha_s) \\ &= \sum_{k=1}^q P(e_k)P(x_i|e_k) + INT_i \end{aligned} \quad (25)$$

In a quantum-based Bayesian network, a normalization factor is incorporated into Eq. (25) to standardize the probability of each alternative, ensuring that the overall estimated probability does not exceed 1. Then

$$Prob(x_i) = \sigma P(x_i) \quad (26)$$

where,

$$\sigma = \frac{1}{\sum_{i=1}^n P(x_i)} = \frac{1}{\sum_{i=1}^n (\sum_{k=1}^q P(e_k)P(x_i|e_k) + INT_i)} \quad (27)$$

3.3 Analysis of Interference Effect Among Experts' Opinions

Equation (27) uses the term INT_i , which represent the ambiguity and uncertainty of expert opinions, to explain the superposition of decision information. In Eq. (27), $\cos(\alpha_k - \alpha_s)$ represents the phase angle between two distinct experts e_k and e_s , which represents the connection vector between two experts. $\frac{q(q-1)}{2}$ is the total count of pairwise correspondences across q experts, excluding duplicates. The final interference value is derived by summing the computed $\frac{q(q-1)}{2}$ interference values for each expert pair, rather than assessing the overall uncertainty across experts. This approach enhances the accuracy of the estimated probability value.

In real-world situations, it is hard to measure the phase angle. Experts' opinions can differ or be influenced by outside factors, causing a lot of variation and mistakes. Belief entropy helps a lot in handling and combining unclear information, making it easier to make good decisions in group settings with many factors. It is used as a standard in areas like risk assessment, defect diagnosis, and pattern recognition. Belief entropy is the latest way to measure interference in QSNB. Unlike the old method of manually estimating the phase angle range, this new method is more reliable because it reduces errors from personal judgment, which can affect the final decision and the expert's probability calculation. Belief entropy is used in QSNB to manage the interference among different experts' beliefs in GDM.

Then the belief distance about alternative x_i according to the average probability $\chi = \frac{1}{nq}$ can be calculated in the following way:

$$D_{iu} = \left| \delta_{iu} + \frac{|\delta_{iu} - \chi| - |\varrho_{iu} - \chi|}{|\delta_{iu} - \chi| + |\varrho_{iu} - \chi|} \right|, i = 1, 2, \dots, n; u = 1, 2, \dots, \frac{q(q-1)}{2}, \quad (27)$$

$$\begin{cases} \delta_{iu} = P(e_k)P(x_i|e_k), i = 1, 2, \dots, n; k = 1, 2, \dots, q-1; \\ \varrho_{iu} = P(e_s)P(x_i|e_s), \\ s = k+1, k+2, \dots, \frac{q(q-1)}{2} \end{cases} \quad (28)$$

In Eq. (28), u means the value reallocated to the correlation among q experts; $\chi = \frac{1}{nq}$ represent each expert probability value of each alternative; D_{iu} means deviation degree from χ ; $|\delta_{iu} - \chi|$ and $|\varrho_{iu} - \chi|$ represents two experts' magnitude of the deviation degree from χ about alternative x_i . If $|\delta_{iu} - \chi| < |\varrho_{iu} - \chi|$, then the position of δ_{iu} and ϱ_{iu} are interchangeable.

There are many ways to measure uncertainty. Deng entropy (Zhao et al., 2024), also known as belief entropy, is better at including the confidence function and the uncertainty of BPA because it measures both total non-specificity and disagreement.

Thus, by Deng's entropy, we get

$$\vartheta_{iu}(D_{iu}) = -D_{iu} \log_2 \frac{D_{iu}}{2^m - 1}, i = 1, 2, \dots, n; u = 1, 2, \dots, \frac{q(q-1)}{2} \quad (29)$$

To ensure that the values of Deng entropy align with the range of $\cos(\alpha_k - \alpha_s)$, which is between $[-1, 1]$, Deng entropy is standardized.

$$\beta_{iu}(D_{iu}) = 2 \frac{\vartheta_{iu}(D_{iu}) - \min\{\vartheta_{iu}(D_{iu})\}}{\max\{\vartheta_{iu}(D_{iu})\} - \min\{\vartheta_{iu}(D_{iu})\}} - 1, i = 1, 2, \dots, n; \\ u = 1, 2, \dots, \frac{q(q-1)}{2}. \quad (30)$$

Thus, the phase angle $\cos(\alpha_k - \alpha_s)$ can be determined by the following way:

$$\cos(\alpha_k - \alpha_s) = \beta_{iu}(D_{iu}), i = 1, 2, \dots, n; u = 1, 2, \dots, \frac{q(q-1)}{2}. \quad (31)$$

Therefore, by Eq. (31), in Eq. (25), INT_i can be written as

$$INT_i = 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \cos(\alpha_k - \alpha_s) \\ = 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \beta_{iu}(D_{iu}) \quad (32)$$

By Eq. (32), the Eq. (25) can be written as

$$P(x_i) = \sum_{k=1}^q P(e_k)P(x_i|e_k) + 2 \sum_{k=1}^{q-1} \sum_{s=k+1}^q \sqrt{P(e_k)P(x_i|e_k)P(e_s)P(x_i|e_s)} \beta_{iu}(D_{iu}) \quad (33)$$

Finally, the probability of each alternative x_i can be obtained in the following way:

$$\Pr ob(x_i) = \sigma P(x_i) \quad (34)$$

where,

$$\sigma = \frac{1}{\sum_{i=1}^n P(x_i)} = \frac{1}{\sum_{i=1}^n (\sum_{k=1}^q P(e_k)P(x_i|e_k) + INT_i)} \quad (35)$$

Step 6: Ranking of alternative

According to Eq. (34), the higher probability of an alternative means the best alternative.

4 A Practical Example with Comparative Analysis

This section describes a new approach to MAGDM based on the CoCoSo approach integrating QDT with PLI. The detailed procedure for the combined CoCoSo approach is presented below:

4.1 A Case Study

There are a lot of people who are under the incorrect impression that solar cells are easier to understand than they actually are, and it is not frequently understood that there are an assortment of distinct varieties. Taking a more in-depth look at the many different types of solar cells that are currently accessible makes it easier to comprehend and choose the one that is most appropriate. Photovoltaic systems typically consist of individual photovoltaic modules, which are more usually referred to as solar panels. These modules provide the core components of the system. Because they use renewable energy sources, they eliminate fuel charges and supply problems, which results in a reduction in the expenses associated with power generation. As a consequence of this, the board of directors of the firm made the decision to cut expenses in order to enhance profits, and they identified the cost of energy as another significant expense. It dawned on them that the use of solar energy for electricity generation may potentially reduce these expenses.

Suppose the board of directors invites four experts e_1 , e_2 , e_3 , and e_4 for evaluating the alternatives x_1 : Crystalline silicon cells, x_2 : Monocrystalline cells, x_3 : Polycrystalline cells, x_4 : Thin film solar cells concerning four attributes c_1 : Reliability, c_2 : Environment friendly, c_3 : Heat tolerance, and c_4 : cost.

4.1.1 Decision Procedure

After the evaluation indicators are defined, the following methods are applied to evaluate the solar cell of sustainable solar energy management MAGDM scenario using our proposed CoCoSo method integrating QDT with PLI for assessing the available options.

Step 1: Here we offer the LTSs, $H_1 = \{h_{-3}: \text{very bad (VB)}, h_{-2}: \text{bad (B)}, h_{-1}: \text{slightly bad (SB)}, h_0: \text{moderate (M)}, h_1: \text{slightly good (SG)}, h_2: \text{good (G)}, h_3: \text{very good (VG)}\}$ for the evaluation of alternatives (x_1, x_2, x_3, x_4) about the criteria (c_1, c_2, c_3, c_4) . The

evaluation information is shown in the Eqs. (36–39).

$$A^1 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} & \begin{bmatrix} \langle h_{-2}, (0.85, 0.2) \rangle & \langle h_0, (0.8, 0.3) \rangle & \langle h_2, (0.6, 0.5) \rangle & \langle h_{-2}, (0.9, 0.1) \rangle \\ \langle h_2, (0.35, 0.6) \rangle & \langle h_{-1}, (0.6, 0.6) \rangle & \langle h_{-1}, (0.9, 0.23) \rangle & \langle h_{-1}, (0.7, 0.4) \rangle \\ \langle h_3, (0.9, 0.1) \rangle & \langle h_2, (0.9, 0.25) \rangle & \langle h_1, (0.95, 0.1) \rangle & \langle h_3, (0.9, 0.3) \rangle \\ \langle h_{-1}, (0.2, 0.8) \rangle & \langle h_{-2}, (0.6, 0.5) \rangle & \langle h_0, (0.2, 0.7) \rangle & \langle h_1, (0.6, 0.6) \rangle \end{bmatrix} \end{matrix} \quad (36)$$

$$A^1 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} & \begin{bmatrix} \langle h_{-2}, (0.8, 0.1) \rangle & \langle h_1, (0.8, 0.3) \rangle & \langle h_1, (0.8, 0.4) \rangle & \langle h_1, (0.6, 0.5) \rangle \\ \langle h_2, (0.9, 0.2) \rangle & \langle h_2, (0.7, 0.2) \rangle & \langle h_0, (0.7, 0.3) \rangle & \langle h_{-1}, (0.7, 0.3) \rangle \\ \langle h_3, (0.7, 0.1) \rangle & \langle h_{-1}, (0.7, 0.5) \rangle & \langle h_2, (0.6, 0.2) \rangle & \langle h_2, (0.8, 0.2) \rangle \\ \langle h_1, (0.6, 0.2) \rangle & \langle h_2, (0.8, 0.1) \rangle & \langle h_{-1}, (0.9, 0.1) \rangle & \langle h_1, (0.9, 0.1) \rangle \end{bmatrix} \end{matrix} \quad (37)$$

$$A^1 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} & \begin{bmatrix} \langle h_{-2}, (0.7, 0.2) \rangle & \langle h_{-1}, (0.8, 0.4) \rangle & \langle h_1, (0.8, 0.1) \rangle & \langle h_0, (0.8, 0.4) \rangle \\ \langle h_1, (0.6, 0.3) \rangle & \langle h_2, (0.7, 0.3) \rangle & \langle h_{-2}, (0.7, 0.5) \rangle & \langle h_1, (0.7, 0.1) \rangle \\ \langle h_3, (0.6, 0.7) \rangle & \langle h_0, (0.6, 0.3) \rangle & \langle h_1, (0.6, 0.1) \rangle & \langle h_3, (0.9, 0.3) \rangle \\ \langle h_{-2}, (0.5, 0.2) \rangle & \langle h_{-2}, (0.7, 0.2) \rangle & \langle h_2, (0.7, 0.2) \rangle & \langle h_{-2}, (0.8, 0.2) \rangle \end{bmatrix} \end{matrix} \quad (38)$$

$$A^1 = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} & \begin{bmatrix} \langle h_1, (0.8, 0.3) \rangle & \langle h_1, (0.8, 0.1) \rangle & \langle h_{-3}, (0.6, 0.7) \rangle & \langle h_1, (0.7, 0.3) \rangle \\ \langle h_{-2}, (0.6, 0.7) \rangle & \langle h_{-3}, (0.7, 0.4) \rangle & \langle h_1, (0.9, 0.25) \rangle & \langle h_{-1}, (0.8, 0.4) \rangle \\ \langle h_1, (0.8, 0.3) \rangle & \langle h_3, (0.9, 0.1) \rangle & \langle h_2, (0.8, 0.3) \rangle & \langle h_2, (0.9, 0.2) \rangle \\ \langle h_2, (0.9, 0.2) \rangle & \langle h_1, (0.6, 0.2) \rangle & \langle h_{-1}, (0.7, 0.4) \rangle & \langle h_3, (0.7, 0.3) \rangle \end{bmatrix} \end{matrix} \quad (39)$$

Step 2: By Eq. (13), the weight of experts is $\omega_1 = 0.2384$, $\omega_2 = 0.2673$, $\omega_3 = 0.2310$, and $\omega_4 = 0.2634$.

Step 3: By Eq. (16), the weight of attributes for all experts are:

$$\begin{aligned} w_1^1 &= 0.3388, w_2^1 = 0.1737, w_3^1 = 0.0915, w_4^1 = 0.3961; \\ w_1^2 &= 0.5039, w_2^2 = 0.1591, w_3^2 = 0.1812, w_4^2 = 0.1558; \\ w_1^3 &= 0.2885, w_2^3 = 0.2272, w_3^3 = 0.1572, w_4^3 = 0.3271; \\ w_1^4 &= 0.1324, w_2^4 = 0.4651, w_3^4 = 0.2508, w_4^4 = 0.1517; \end{aligned}$$

Step 4: According to Eq. (17), we get $P(e_1) = 0.2384$, $P(e_2) = 0.2673$, $P(e_3) = 0.2310$, and $P(e_4) = 0.2634$.

- (1) By Eqs. (18) and (19), the obtained results of WSM and WPM for each alternative of each expert e_k ($k = 1, 2, 3, 4$) are shown in Tables 1, 2, 3 and 4.

Table 1 The output results of WSM and WPM for exp_rt e_1

Alternative	$\mathcal{S}_i^1$	$\mathcal{P}_i^1$
x_1	0.1420	0.1260
x_2	0.2177	0.2134
x_3	0.4127	0.4105
x_4	0.2276	0.2229

Table 2 The output results of WSM and WPM for exp_rt e_2

Alternative	$\mathcal{S}_i^2$	$\mathcal{P}_i^2$
x_1	0.1755	0.1535
x_2	0.2728	0.2653
x_3	0.3094	0.2991
x_4	0.2424	0.2378

Table 3 The output results of WSM and WPM for exp_rt e_3

Alternative	$\mathcal{S}_i^3$	$\mathcal{P}_i^3$
x_1	0.2007	0.1950
x_2	0.2915	0.2761
x_3	0.3461	0.3401
x_4	0.1616	0.1490

Table 4 The output results of WSM and WPM for exp_rt e_4

Alternative	$\mathcal{S}_i^4$	$\mathcal{P}_i^4$
x_1	0.2293	0.2211
x_2	0.1522	0.1258
x_3	0.3589	0.3548
x_4	0.2596	0.2545

(2) and (3) Let $\epsilon = 0.5$, then by Eqs. (20) and (21), the three types of relative importance $\mathcal{F}_i^{1,k}$, $\mathcal{F}_i^{2,k}$, and $\mathcal{F}_i^{3,k}$; and the overall relative importance $\mathcal{F}_i^k$ for each expert $e_k(k = 1, 2, 3, 4)$ is calculated and the details output results are shown in Tables 5, 6, 7 and 8.

Table 5 The output results of $\mathcal{F}_i^{1,1}$, $\mathcal{F}_i^{2,1}$, $\mathcal{F}_i^{3,1}$ and $\mathcal{F}_i^1$ of expert e_1

Alternative	$\mathcal{F}_i^{1,1}$	$\mathcal{F}_i^{2,1}$	$\mathcal{F}_i^{3,1}$	$\mathcal{F}_i^1$
x_1	0.1358	2.0000	0.3256	1.2660
x_2	0.2185	3.2269	0.5237	2.0405
x_3	0.4173	6.1640	1.0000	3.8972
x_4	0.2284	3.3718	0.5473	2.1322

Table 6 The output results of $\mathcal{F}_i^{1,2}$, $\mathcal{F}_i^{2,2}$, $\mathcal{F}_i^{3,2}$ and $\mathcal{F}_i^2$ of expert e_2

Alternative	$\mathcal{F}_i^{1,2}$	$\mathcal{F}_i^{2,2}$	$\mathcal{F}_i^{3,2}$	$\mathcal{F}_i^2$
x_1	0.1682	2.0000	0.5407	1.4696
x_2	0.2751	3.2827	0.8843	2.4085
x_3	0.3111	3.7115	1.0000	2.7234
x_4	0.2455	2.9303	0.7891	2.1497

Table 7 The output results of $\mathcal{F}_i^{1,3}$, $\mathcal{F}_i^{2,3}$, $\mathcal{F}_i^{3,3}$ and $\mathcal{F}_i^3$ of expert e_3

Alternative	$\mathcal{F}_i^{1,3}$	$\mathcal{F}_i^{2,3}$	$\mathcal{F}_i^{3,3}$	$\mathcal{F}_i^3$
x_1	0.2019	2.5509	0.5767	1.7771
x_2	0.2896	3.6569	0.8272	2.5481
x_3	0.3501	4.4241	1.0000	3.0817
x_4	0.1585	2.0000	0.4527	1.3939

Table 8 The output results of $\mathcal{F}_i^{1,4}$, $\mathcal{F}_i^{2,4}$, $\mathcal{F}_i^{3,4}$ and $\mathcal{F}_i^4$ of expert e_4

Alternative	$\mathcal{F}_i^{1,4}$	$\mathcal{F}_i^{2,4}$	$\mathcal{F}_i^{3,4}$	$\mathcal{F}_i^4$
x_1	0.2302	3.2640	0.6311	2.1550
x_2	0.1421	2.0000	0.3895	1.3240
x_3	0.3648	5.1782	1.0000	3.4172
x_4	0.2628	3.7289	0.7204	2.4612

(4) By Eq. (22) we calculate the conditional probabilities of each alternative for each expert. The details results are shown in Table 9.

(5) By Eq. (23), the probabilities of alternatives are:

$$P(x_1) = 0.1845, P(x_2) = 0.2298, P(x_3) = 0.3597, \text{ and } P(x_4) = 0.2260.$$

Step 5: By Eq. (28), the belief distance between two experts about each alternative is calculated, which is presented in Table 10.

By Eq. (29), the Deng entropy between two experts about each alternative is shown in Table 11.

Table 9 The conditional probabilities of each alternative

$P(x_i e_k)$	e_1	e_2	e_3	e_4
x_1	0.1356	0.2186	0.4174	0.2284
x_2	0.1679	0.2752	0.3112	0.2456
x_3	0.2019	0.2895	0.3502	0.1584
x_4	0.2303	0.1415	0.3652	0.2630

Table 10 The belief distance between two experts about each alternative

D_{ui}	$i = 1$	$i = 2$	$i = 3$	$i = 4$
$u = 1(e_1, e_2)$	0.2953	0.0209	0.3822	0.4903
$u = 2(e_1, e_3)$	0.3432	0.4606	0.4359	0.4711
$u = 3(e_1, e_4)$	0.9169	0.3641	0.1465	0.1416
$u = 4(e_2, e_3)$	0.0970	0.5078	0.1426	0.7165
$u = 5(e_2, e_4)$	0.8549	0.7874	0.1556	0.2968
$u = 6(e_3, e_4)$	0.8379	0.6380	0.2132	0.6224

Table 11 The Deng entropy between two experts about each alternative

$\vartheta_{ui}(D_{ui})$	$i = 1$	$i = 2$	$i = 3$	$i = 4$
$u = 1(e_1, e_2)$	1.6735	0.1982	2.0235	2.4197
$u = 2(e_1, e_3)$	1.8702	2.3148	2.2253	2.3520
$u = 3(e_1, e_4)$	3.6969	1.9531	0.9785	0.9528
$u = 4(e_2, e_3)$	0.7053	2.4803	0.9578	3.1440
$u = 5(e_2, e_4)$	3.5332	3.3479	1.0258	1.6796
$u = 6(e_3, e_4)$	3.4874	2.9062	1.3083	2.8575

By Eq. (30), the standardized Deng entropy between two experts about each alternative is shown in Table 12.

By Eq. (32), the interface effect among experts for each alternative is:

$$INT_1 = 0.1374, INT_2 = 0.1664, INT_3 = -0.2885, \text{ and } INT_4 = 0.0694.$$

Finally, by Eq. (34), the probability of each alternative is:

$$Prob(x_1) = 0.2967, Prob(x_2) = 0.3653, Prob(x_3) = 0.0657, \text{ and } Prob(x_4) = 0.2723.$$

Step 6: According to the value of QP of each alternative, the final ranking is

$$x_2 \succ x_1 \succ x_4 \succ x_3$$

Table 12 The standardized Deng entropy between two experts about each alternative

$\beta_{ui}(D_{ui})$	$i = 1$	$i = 2$	$i = 3$	$i = 4$
$u = 1(e_1, e_2)$	−0.3527	−1.0000	0.6815	0.3389
$u = 2(e_1, e_3)$	−0.2212	0.3440	1.0000	0.2771
$u = 3(e_1, e_4)$	1.0000	0.1143	−0.9674	−1.0000
$u = 4(e_2, e_3)$	−1.0000	0.4491	−1.0000	1.0000
$u = 5(e_2, e_4)$	0.8906	1.0000	−0.8928	−0.3366
$u = 6(e_3, e_4)$	0.8599	0.7195	−0.4470	0.7385

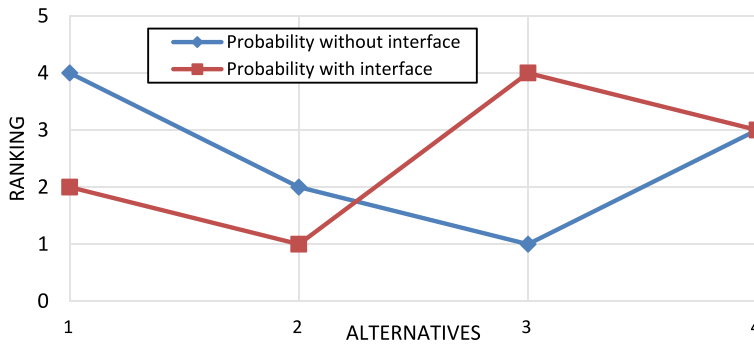


Fig. 4 The interface effect on ranking values

4.2 Result Discussion

In this section, we discussed the validation of our numerical results through sensitive analysis and some comparative analysis.

4.2.1 Effect of the Quantum Interface on Decision Result

If we can decide based on the $P(x_i) (i = 1, 2, 3, 4)$ values according to Eq. (23), as higher probabilities indicate a superior ranking. The classical approach does not account for the interference effect, which can influence the final rankings, as shown in Fig. 4. The ranking result of alternatives according to Eq. (23) is $x_3 > x_2 > x_4 > x_1$. If we count the interface effect between two experts about an alternative the ranking result is $x_2 > x_1 > x_4 > x_3$.

4.2.2 Comparative with Validity Analysis

To test the feasibility and validity of the suggested method, we compare it with some existing methods for the example given above.

In the traditional decision-making model (Mandal and Ranadive 2019) the ranking results are derived from the aggregate decision matrix of experts' opinions. Therefore, to compare our model and the traditional model, we first aggregate the experts' opinions. In this connection, by Mandal and Ranadive (2019), using the Pythagorean linguistic weighted averaging aggregation operator, the aggregation of five experts' opinions is

$$\text{Agg}(A^1, A^2, A^3, A^4) = \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ \langle h_{-1.21}, (0.80, 0.18) \rangle & \langle h_{0.3}, (0.80, 0.24) \rangle & \langle h_{0.185}, (0.72, 0.35) \rangle & \langle h_{0.054}, (0.78, 0.28) \rangle \\ \langle h_{0.716}, (0.71, 0.40) \rangle & \langle h_{-0.032}, (0.68, 0.34) \rangle & \langle h_{-0.437}, (0.83, 0.30) \rangle & \langle h_{-0.538}, (0.73, 0.27) \rangle \\ \langle h_{2.473}, (0.78, 0.21) \rangle & \langle h_{0.999}, (0.82, 0.25) \rangle & \langle h_{1.531}, (0.81, 0.16) \rangle & \langle h_{2.469}, (0.88, 0.24) \rangle \\ \langle h_{0.094}, (0.68, 0.28) \rangle & \langle h_{-0.141}, (0.69, 0.21) \rangle & \langle h_{-0.069}, (0.74, 0.27) \rangle & \langle h_{0.834}, (0.79, 0.24) \rangle \end{bmatrix}$$

Therefore, the ranking of alternatives according to the methods CoCoSo method is $x_3 > x_4 > x_2 > x_1$, i.e., the optimal option is x_3 . When the proposed model is applied without considering the interface effect, the optimal option is also x_3 . This suggests that the optimal option of the proposed approach aligns with those of traditional methods when the impact of decision-makers is disregarded, thereby validating the robustness of the suggested model. However, when evaluating the quantum interference among decision-makers, the outcomes of this model suggest that x_2 is the optimal strategy. This inconsistency arises from the conventional framework's exclusion of the interference effect among decision-makers, which leads to subjective biases during the evaluation of alternatives. As a result, the outcomes of this model better align with real-world complexities.

5 Conclusion

This paper presents a quantum group decision model that was built to lessen the influence of information ambiguity and opinion interference within the framework of sustainable solar energy management. This model intended to reduce the impact of these factors. This method calls for the utilization of PLIs in order to facilitate the transmission of ambiguous information in scenarios that involve complicated decision-making. In addition, Deng entropy is utilized in order to analyze opinion interference by modeling the reciprocal effects among decision-makers in the MAGDM process. This is done in order to determine whether or not there is interference. This paradigm addresses interference in a manner that is unique from conventional methods, which consider decision-makers to be independent entities. This is accomplished by assigning suitable weights to decision-makers and regulating opinion interference through the use of quantum decision-making procedures.

Solutions that are more effective for issues that are linked with opinion interference and weight distribution have been supplied by the model that has been presented. This model is an improvement upon the techniques that are presently in place. Our findings reveal that the model significantly improves decision-making in the process of sustainable solar energy management. This is demonstrated by the fact that the model provides a comprehensive framework to manage information ambiguity and conflicting opinions. The introduction of new decision-makers and variables, on the other hand, has the impact of increasing computational complexity, which underscores the requirement of adopting more modern computing technologies in order to maintain efficiency intact.

It is anticipated that future research will focus on the incorporation of machine learning techniques in order to enhance scalability and practicality. This will ensure that the model is effective in decision-making procedures that involve making decisions on a wide scale. Through the implementation of this integration, our goals are to maintain control over the ever-increasing complexity, foster consensus across quantum groups during the decision-making process, and ensure that the model will continue to be viable over the long term.

Conflict of Interest The authors declare that they have no conflict of interest

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Dynamic Centrality Measure for Bunch Nodes in Quantum Graph: Perspective on Brain Network



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Abstract Dynamic centrality measures are important for recognizing how different brain areas coordinate over time to support various cognitive functions. Conventional centrality metrics focusing on specific nodes have limitations when capturing the complex and ever-changing connections between groups of interconnected regions. In this chapter, we introduce a novel approach for evaluating dynamic centrality from the perspective of node bunches, which are clusters of functionally or structurally connected brain areas within the brain network. By moving from single-node centrality to group-level analysis we present a more comprehensive view of how brain modules interact and influence information across various cognitive states. Our analysis's findings show that node bunch centrality can capture more nuanced changes in the organization of brain networks, providing a promising new way to think about how diseases develop as well as how healthy brains function.

Keywords Dynamic centrality · Bunch nodes · Quantum graph

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1 Introduction

The term centrality in network theory describes the importance or influence of every node inside a network. There are various types of centralities, such as closeness centrality which measures how quickly a node can access all other nodes, betweenness centrality which highlights nodes that serve as bridges between others, and degree centrality which is based on the number of direct connections (Das et al. 2018). These centrality measures are more effective for static networks. But in a real-world system, there are many networks where nodes, edges, and their relationships change over time are called dynamic networks.

Dynamic centrality is a measure used to account for the changing roles and connections of nodes over time in a dynamic network. In contrast to static centrality, which evaluates a node's influence based on a single snapshot of a network, dynamic centrality captures how a node's position and significance change in response to shifting connections and network structures. Dynamic centrality measure is particularly helpful in networks where interactions are time-dependent, such as social networks, co-authorship networks, brain networks, etc.

In social networks, people's importance or centrality is not static. Because in social networks new connections are always formed, old ones fade, and patterns of interaction shift. Therefore, by using the concept of dynamic centrality emerging influencers can be identified, and how social groups or communities reorganize themselves in response to events can be recognized. Dynamic centrality in co-authorship networks shows how the influence or responsibilities of researchers change over time when new papers are released and partnerships change. It also helps to find the leaders in a particular domain and important collaborators that support diverse cooperation. Dynamic centrality is vital in understanding brain networks. As human beings are always engaged in different types of thinking, learning, or responding to stimuli therefore different brain regions change in their roles over time.

This paper introduces the concept of measuring the dynamic centrality of a group of nodes or "bunch nodes". A bunch graph may display one or more groups of linked persons as nodes, known as a bunch, with each individual being a simple node. In networks measuring the dynamic centrality of a cluster of nodes is important because it illustrates how several nodes connectivity and combined influence change over time. Clusters or groups of nodes collaborate in many real-world networks, and their total dynamic centrality can have a greater impact than that of individual nodes. For example, in brain networks, a set of regions may cooperate dynamically to perform a particular cognitive task and as a result, the performance will be faster and more accurate.

From the perspective of the quantum graph, brain networks can be viewed as graphs, where nodes represent neurons or brain areas and edges represent connections between neurons or functional relationships. The brain is extremely dynamic, with functional connections varying depending on tasks, resting states, or neurological conditions. Centrality measures need to reflect these changing patterns. The transfer of information in a quantum graph is not only traditional (by synaptic transmission),

but it may also involve quantum coherence or entanglement, especially at small scales.

The rest of the chapter is organised as follows. Section 2 presents the literature review in this area. Section 3 introduces some preliminaries of centrality, bunch graph, quantum graph etc. Section 4 defines the quantum bunch graph. In Sect. 5 we define the procedure for finding the dynamic centrality value for the bunch nodes. Section 6 presents the application of dynamic centrality measures for bunch nodes in the brain network. Section 7 concludes the paper and outline some limitation also.

2 Literature Review

Centrality measure in network analysis evaluates the importance or influence of a node within a network. Depending on the various factors like connectivity, position, or information flow throughout the network it helps to identify the key nodes. Das et al. (2020) provide a complete evaluation of centrality metrics used in network analysis. They explore several centrality metrics like degree, closeness, betweenness, eigenvector, PageRank, etc. for understanding the structural dynamics in a network. They suggest that, while no one centrality measure can fully capture all aspects of node importance, the measure used should be adjusted to the network's specific characteristics. Fuzzy set theory is particularly well-suited to modeling the uncertainty inherent in social networks, where relationships and interactions often involve incomplete or vague information. Traditional mathematical models struggle to capture this uncertainty, but fuzzy approaches provide a more nuanced framework for analyzing such dynamics. For instance, the development of fuzzy fractional differential equations using fuzzy Laplace transforms is essential for modeling systems where the influence of individuals or groups evolves fractionally over time under uncertainty (Allahviranloo 2020; Salahshour et al. 2012; Sahoo et al. 2024). Pythagorean fuzzy systems (Akram et al. 2021b) and trapezoidal approximations (Allahviranloo and Firozja 2007) provide additional methods for handling high levels of uncertainty in decision-making processes. These approaches offer flexible tools for analyzing complex decision problems, ensuring that a wide range of uncertain factors are considered when making group decisions.

Mahapatra et al. (2024) present an innovative method for centrality measurement in fuzzy social networks where relationships between nodes are not binary but have different degrees of intensity. Their research work highlights the limitation of traditional centrality measures in fuzzy environments and proposes new approaches that account for the nuanced relationships observed in real-world social networks. Saxena and Iyengar (2020) survey centrality measures in static complex networks to address the complexities in real-world networks. Ghanem et al. (2018) focus on centrality metrics in dynamic networks and compare their effectiveness in capturing temporal variations in node importance. They demonstrate the need for time-sensitive measures in accurately reflecting developing network architectures. Zareie and Sakellariou (2023) explore the centrality measure in fuzzy social

networks where the connection between nodes is characterized by varying degrees of uncertainty. They suggest modifications to classic centrality metrics to account for the fuzziness inherent in these networks. Their findings highlight the need for more adaptable and context-sensitive methods to centrality, particularly in real-world social systems with ambiguous or imprecise interactions. Huang et al. (2017) focused on how the centrality measures are applicable in temporal networks and how time series analysis can be used to understand the dynamics of these networks. Their proposed method may not apply to very large networks due to computational complexity. To measure influences in social networks (Samanta et al. 2021) proposed several algorithms like influence propagation algorithm, community detection algorithm, and weighted centrality measures algorithm. Community detection algorithms identify the groups within the network where influence might be more concentrated. Das et al. (2020a) explored how the different traditional centrality measures can be used in both online and offline social networks to find out the key individuals who can influence others. Muhiuddin et al. (2023) proposed a new methodology to predict novel gene candidates for unresolved rare diseases. Their research also explores a comprehensive understanding of genetic influence on disease phenotypes. A new centrality measure is introduced for neutrosophic and linguistic graphs (Guchhait et al. 2024a, b; Mahapatra et al. 2023), including self-weight or individual merit within the network.

Samanta et al. (2024) presented an innovative concept to find out the dynamic centrality in quantum graphs. Quantum graphs have two main features. These are superposition and entanglement. By using the features of quantum graphs, they explicitly represent how to compute the dynamic centrality of every node individual. Sahoo and Guchhait (2024) proposed a new meaning of dynamic centrality. By the word dynamic, we understand the value changes over time. However, they defined that depending on different network characteristics how the centrality value is changing. The authors performed their experience on a co-authorship network. Time based analysis in social network is very important to get the result properly (Namin et al. 2024).

Bhadoria et al. (2022) introduced some dimensionality reduction methods for character recognition which is bunch graph-based. Those nodes have similar types of dimensions they form a bunch of nodes. They also used auto-encoder architecture which is a type of neural network used for unsupervised learning. By using the bunch graph auto-encoder can focus on the most relevant features for character recognition. Samata et al. (2022) analyzed the bunch graph to identify the patterns of cooperation and competition. They found that while there is significant cooperation among researchers and institutions, competition also plays a crucial role. They look at metrics like the number of joint publications, co-authorships, and citation networks to understand the balance between collaborative and competitive behaviors.

Multi-criteria group decision-making (MCGDM) is an important area in decision science. In a typical MCGDM situation, several experts are asked to evaluate different options based on set criteria. Their individual opinions are combined using information fusion methods to get a group evaluation. For example, in (Mandal and Ranadive 2017) studied fuzzy variable precision rough sets, in (Mandal and Ranadive

2018a, b) introduced hesitant bipolar-valued fuzzy sets and bipolar-valued hesitant fuzzy sets, and studied multi-granulation bipolar-valued fuzzy probabilistic rough sets and their corresponding three-way decisions. The decision-theoretic rough sets and their applications are studied in (Mandal and Ranadive 2018c, 2019a, b, c, d). However, the experts tend to express their opinions more reliably using linguistic information, known as linguistic sets (LS), rather than fuzzy information. In (Mandal et al. 2019e), the LS concept was extended to the Pythagorean linguistic set (PLS), allowing experts to express both their confidence and doubt levels when providing judgments. A quantum framework for modelling subjectivity in multi-attribute group decision-making is important for MCGDM (He et al. 2018; He and Jiang 2018a, b).

Perseguers et al. (2010) discussed how quantum entanglement and superposition principles can lead to distinct behaviors in network formation and critical probabilities for the appearance of subgraphs. Samanta and Allahviranloo (2025) focus on the quantum graph and their stability analysis. They clearly explained the concept of the superposition of nodes and edge entanglement of the quantum graph.

3 Preliminaries

In this section, we provide the necessary definitions of the quantum graph, bunch graph, dynamic centrality, etc. which are essential to understanding the dynamic centrality of bunch nodes in the graph.

Definition 1 (Quantum graph) A quantum graph Q_G (Samanta and Allahviranloo 2025) is a dynamic graph with a set of dynamic vertices $Q_V = \{Q_{v_1}, Q_{v_2}, Q_{v_3}, \dots, Q_{v_n}\}$ and set of edges $Q_E = \{Q_{e_{ij}}\}$ where $Q_{e_{ij}}$ links to the vertices Q_{v_i} and Q_{v_j} . An edge can be either entangled or not. Every vertex Q_{v_i} has a dynamic weight $DW_i(t)$ represents the time-dependent influence of the vertex and a real number bounded between 0 and 1. Mathematically it is defined as follows:

$$0 \leq DW_i(t) \leq 1$$

The weight changes over time, often following a sinusoidal function of time t , described as:

$$DW_i(t) = \frac{1}{2}(1 + \sin(f_i t + \emptyset_i))$$

where, f_i is the frequency of the dynamic weight change for the vertex Q_{v_i} and $\emptyset_i$ is the phase shift.

When two vertices Q_{v_i} and Q_{v_j} are joined by an edge the relationship between their dynamic weights is determined by a correlation function $CF_{ij}(t)$ which specifies the nature of the entanglement.

$$CF_{ij}(t) = h(DW_i(t), DW_j(t))$$

The function h describes how dynamic weights are associated and models entanglement between vertices. The adjacency matrix $QA(t)$ for the quantum graph is expressed as follows:

$$QA_{ij}(t) = \begin{cases} CF_{ij}(t), & \text{if } Q_{e_{ij}} \text{ exist and entangled} \\ 0, & \text{otherwise} \end{cases}$$

Definition 2 (Bunch graph) A graph $G = (V_s, E)$ is called a bunch graph (Samanta et al. 2022) where V_s is the set of vertices and E be a multi-edge set that connects the vertices. S is the non-empty set and $V_s \subset P(S)$ such that $V_s \notin \emptyset$, $S \subset V_s$, $V_s - S \neq \emptyset$ and $E \subset V_s \times V_s$. $P(S)$ is the power set on the set S . Each node in the bunch is connected to the bunch itself, but the nodes might not be directly connected to one another.

Example 1 In the following Fig. 1, $S = \{V_1, V_2, V_3, V_4, V_5, V_6, V_7\}$, $V_s = \{\{V_1\}, \{V_2\}, \{V_3\}, \{V_4\}, \{V_5\}, \{V_6\}, \{V_7\}, \{V_1, V_2, V_3\}\}$ and $E = \{(V_1, V_4), (V_2, V_3), (V_3, V_4), (V_2, V_5), (V_2, V_6), (V_1 V_2 V_3, V_7)\}$ and $V_s - S \neq \emptyset$.

In the bunch graph, a bunch is also considered as a node, and within the bunch, the connection between nodes is shown in the following Fig. 2.

Definition 3 (Dynamic centrality) Dynamic centrality (Samanta et al. 2024) measures the importance of a vertex in a graph that changes over time. The dynamic centrality of a vertex V_i is calculated as the integral of its time-varying centrality measure $DC_i(t)$ over a specific time period T . This can be mathematically stated as follows:

$$DC_i(t) = \int_0^T dc_i(t)dt$$

Fig. 1 Bunch graph

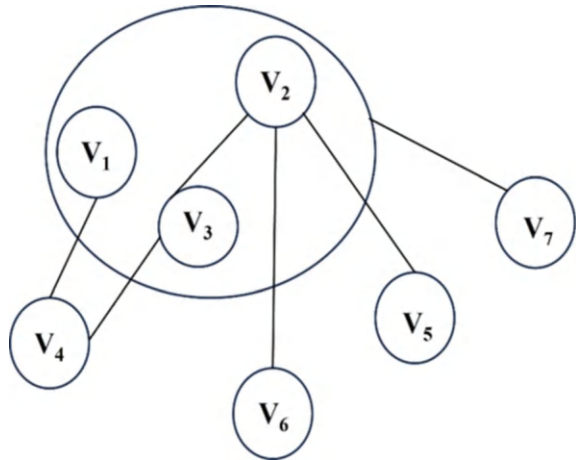
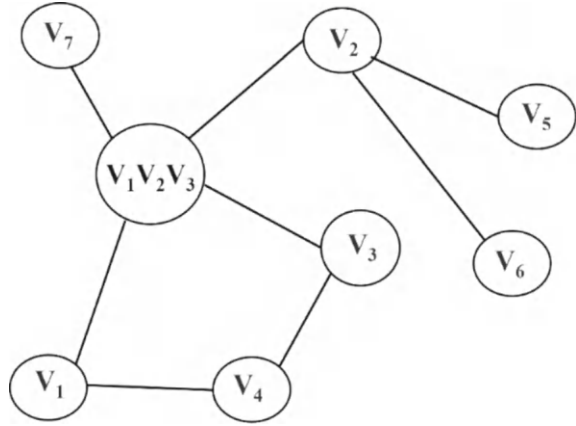


Fig. 2 Virtual depiction of bunch graph which is shown in Fig. 1



The integral captures the cumulative importance of the vertex V_i across the entire time period T , reflecting how its centrality fluctuates and evolves as the graph structure changes dynamically.

Common centrality measures are degree centrality, betweenness centrality, closeness centrality, and eigenvector centrality. We have only considered the degree centrality for simplicity. Degree centrality $d_i(t)$ is the total weight of the edges linked to the vertex V_i at time t .

$$d_i(t) = \sum_j QA_{ij}(t)$$

To compute the dynamic centrality $DC_i(t)$ for each vertex, we calculate the integral of its degree centrality over the time period T :

$$DC_i(t) = \int_0^T d_i(t)dt = \int_0^T QA_{ij}(t)dt$$

To get this integral numerically, we use the trapezoidal rule method. Mathematically it is expressed as:

$$DC_i(t) \approx \sum_{k=0}^{N-1} \frac{d_i(t_k) + d_i(t_{k+1})}{2} \Delta t$$

where $d_i(t_k)$ is the sum of the pairwise correlations at the time t_k , Δt represents the time interval and N is the total number of time steps.

Example 2 We compute the dynamic centrality for the bunch graph given in Fig. 1.

Let's assume the dynamic weight for the vertex V_1 to V_7 respectively

$$DW_1(t) = t, DW_2(t) = 0.5t, DW_3(t) = t + 1, DW_4(t) = 2t, \\ DW_5(t) = t + 2, DW_6(t) = t - 1, DW_7(t) = t - 2$$

Also, consider the time interval $[0,10]$ which is divided into $N = 5$ intervals for simplicity. Therefore, the time step is $\Delta t = \frac{10-0}{2} = 5$.

By substituting the dynamic weight, we get the degree centrality for each vertex.

The degree centrality for vertex V_2 is computed as follows:

$$\begin{aligned} d_2(t) &= CF_{23}(t) + CF_{25}(t) + CF_{26}(t) \\ &= DW_2(t) \times DW_3(t) + DW_2(t) \times DW_5(t) + DW_2(t) \\ &\quad \times DW_6 \\ &= 0.5t \times (t + 1) + 0.5t \times (t + 2) + 0.5t \\ &\quad \times (t - 1) \\ &= 0.5t(t + 1 + t + 2 + t - 1) \\ &= 0.5t(3t + 2) = 1.5t^2 + t \end{aligned}$$

Next, we calculate the $d_2(t)$ for each t_k .

$$t_0 = 0, d_2(0) = 1.5 \times 0 + 0 = 0$$

$$t_1 = 2, d_2(2) = 1.5 \times 2^2 + 2 = 8$$

$$t_1 = 4, d_2(4) = 1.5 \times 4^2 + 4 = 28$$

$$t_1 = 6, d_2(6) = 1.5 \times 6^2 + 6 = 60$$

$$t_1 = 8, d_2(8) = 1.5 \times 8^2 + 8 = 104$$

$$t_1 = 10, d_2(10) = 1.5 \times (10)^2 + 10 = 160$$

Now, using the trapezoidal rule we compute the dynamic centrality for the vertex V_2 is

$$\begin{aligned} DC_2 &\approx \frac{0+8}{2} \times 2 + \frac{8+28}{2} \times 2 + \frac{28+60}{2} \times 2 + \frac{60+104}{2} \times 2 + \frac{104+160}{2} \\ &\approx 8 + 36 + 88 + 164 + 264 \approx 560 \end{aligned}$$

In this way, we compute the dynamic centrality for all vertex in the graph.

4 Quantum Bunch Graph

A quantum bunch graph Q_{BG} is a graph where each vertex Q_{BV_i} represents a quantum entity with a dynamic weight $dweight_i(t)$ that changes over time. dynamic weight $dweight_i(t)$ represents the time-dependent influence of the vertex and a real number bounded between 0 and 1. Mathematically it is defined as follows:

$$0 \leq dweight_i(t) \leq 1$$

Vertices are grouped into bunches based on similarity and the dynamic weight of a bunch is defined as a combination of the dynamic weights of the nodes within the bunch. The bunch's dynamic weight could be an average or another function of the individual dynamic weights.

$$dweight_B(t) = f(Q_{BV_1}, Q_{BV_2}, \dots, Q_{BV_n})$$

Edges between vertices can be entangled, and this entanglement is represented by a correlation function dependent on the dynamic weights of the connected vertices. When two vertices Q_{BV_i} and Q_{BV_j} are joined by an edge $Q_{Be_{ij}}$ then the relationship between their dynamic weights is determined by a correlation function $BCF_{ij}(t)$ which specifies the nature of the entanglement.

$$BCF_{ij}(t) = h(dweight_i(t), dweight_j(t))$$

The function h describes how dynamic weights are associated and models entanglement between vertices. The adjacency matrix $QBA(t)$ for the quantum graph is expressed as follows:

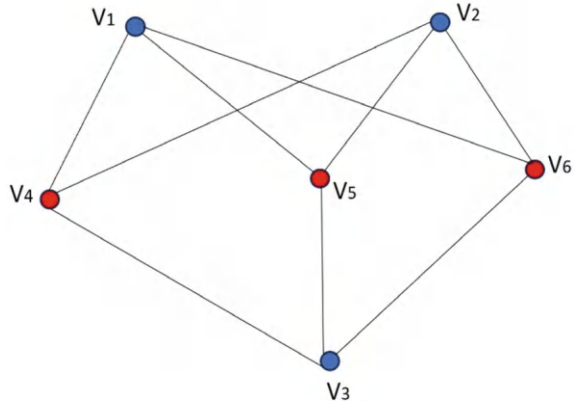
$$QBA_{ij}(t) = \begin{cases} BCF_{ij}(t), & \text{if } Q_{Be_{ij}} \text{ exist and entangled} \\ 0, & \text{otherwise} \end{cases}$$

Example 3 We consider a quantum bunch graph shown in Fig. 3 with simple nodes V_1, V_2, V_3 and a bunch node B_1 containing vertices V_4, V_5, V_6 . The dynamic weight of the vertices $V_1, V_2, V_3, V_4, V_5, V_6$ are $dweight_1(t), dweight_2(t), dweight_3(t), dweight_4(t), dweight_5(t), dweight_6(t)$ respectively. The dynamic weight of the bunch node B_1 is

$$dweight_{B_1}(t) = \frac{dweight_4(t) + dweight_5(t) + dweight_6(t)}{3}$$

Note 1 Basically, vertices are grouped into bunches based on similarity. But depending on different perspective nodes may form a bunch where nodes are not similar. In the perspective of brain network a bunch may have dissimilar nodes.

Fig. 3 Quantum bunch graph



5 Dynamic Centrality Measure for Bunch Nodes in Quantum Graph

The dynamic centrality value for bunch nodes in the quantum graph is defined as the maximum centrality value among the nodes that form the bunch node. These centrality measures can be helpful in quantum computing networks where connectivity or influence may change over time and the detection of a central node in the bunch can shift over time. The dynamic centrality for bunch nodes in the quantum graph can be mathematically represented as follows:

$$DC_{bunch}(B, t) = \max_{V_i \in B} DC(V_i, t)$$

where, $DC_{bunch}(B, t)$ is the centrality value of the bunch nodes B at time t and $B \subseteq V$ (V is the set of nodes in the graph). $DC(V_i, t)$ shows the centrality value of the node V_i at time t and $V_i \in B$ is the set of vertices that are in the bunch node B .

Example 4 For the Fig. 1 to find out the dynamic centrality value of the bunch node first we compute the dynamic centrality of the vertices V_1, V_2, V_3 using the above-expanded formula. In example 2 we have just computed the dynamic centrality value of V_2 which is approximately 560. Similarly, we also compute the centrality value for the vertices V_1 and V_3 are approximately 580 and 975 respectively.

So according to the formula for finding dynamic centrality for the bunch nodes we get the value for Fig. 1 is $\max(580, 560, 975) = 975$.

6 Application of Dynamic Centrality Measure for Bunch Nodes in Brain Network

The human brain is a vital organ that controls all types of thought, emotion, behavior, and psychological control. It comprises nearly 86 billion neurons communicating via synapses to form a complex neural network. The brain is divided into several regions each of which specializes in separate functions. Our brain performs different types of thinking such as positive, negative, realistic, etc., and takes decisions every moment in a frequency of time. Different regions or different nodes in the brain network perform separate functions. A pictorial representation of the brain network is shown in Fig. 4, where the nodes indicate the performance of their thinking. As the brain network is dynamic therefore the connection of nodes within the network changes over time. Dynamic centrality measures can be used to find the more influential or important node in the brain network. To make an effective decision in a particular situation several regions of the brain network work combinedly. Realistic optimism arises from the productive interaction of positive and negative thinking, where a person maintains an optimistic outlook but grounds it in reality by also considering potential challenges. In problem-solving, positive thinking encourages humans to believe that solutions are possible, while negative thinking helps identify flaws and potential pitfalls, refining the process. Therefore, several nodes make a bunch of nodes. Among the nodes in bunch nodes whose dynamic centrality is maximum is considered the dynamic centrality value of that bunch node and depending on the result of that particular node decision is made (Fig. 5).

Where the node's name indicates P, N, R, P nodes indicate that they perform positive thinking, N nodes perform negative thinking and R nodes perform realistic thinking (Fig. 4). Bunch node representation of brain network is shown in Fig. 5.

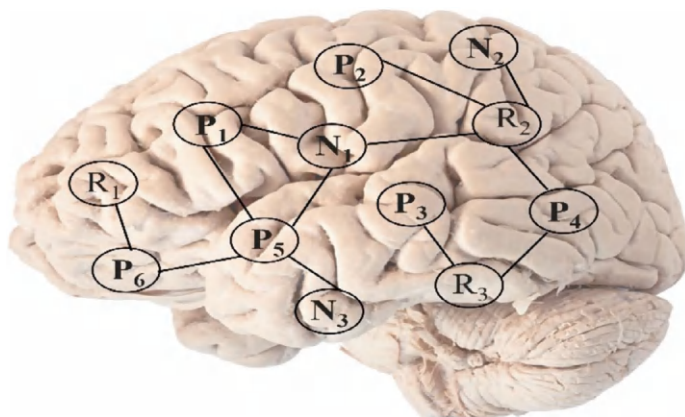


Fig. 4 Representation of a brain network (background image is collected from open source of google image)

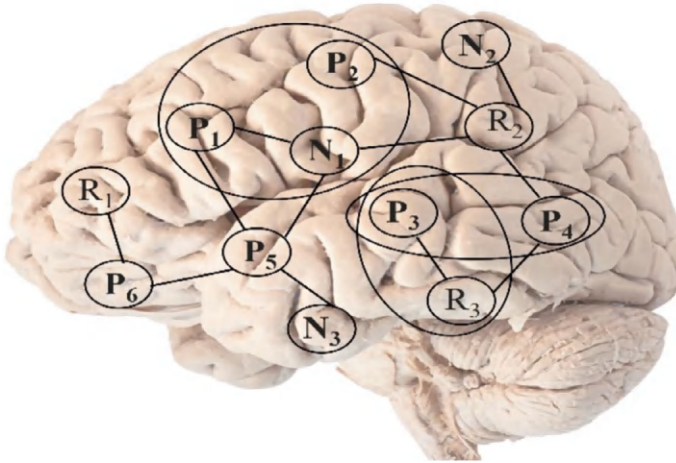


Fig. 5 Bunch node representation in the brain network

Individual nodes in the brain network can form a bunch of nodes due to several reasons. Suppose we are considering some positive thinking in a certain situation. In that time which nodes are performing positive thinking they may form a bunch of positive nodes. The dynamic centrality value for bunch nodes in the brain network is computed in the previously defined formula. The maximum centrality value among the nodes in the bunch is considered as the dynamic centrality value for bunch nodes in the brain network. One bunch can overlap another bunch also. Some of the application of dynamic centrality measures for bunch nodes in the brain network is presented as follows.

The link between the different regions in the brain is dynamic. Its performance changes with tasks, learning and different states of awareness. Dynamic centrality monitors these changes to understand how brain connections alter and adapt.

Identifying the central nodes in a bunch graph helps to evaluate the capacity of information transfer in the brain. High centrality nodes in bunch nodes could identify the regions that perform quick communication throughout the network which is essential for effective cognitive functioning.

By calculating the dynamic centrality of bunch nodes which region of the brain is more influential at different time can be identified. For example, during different cognitive task (e.g., resting vs. Active), certain regions might exhibit higher centrality, indicating their crucial role in processing information.

7 Conclusion

Centrality measure plays a major role in identifying the most important or influential node in the network. However, the usual centrality measures are not sufficient for capturing the time-varying interaction between the nodes or in a complex network. Therefore, the dynamic centrality measure for every node in the network is vital for understanding the role of every node that changes over time. Traditional centrality measures only focus on individual nodes, often missing the collective influence of a bunch of nodes that work together. In this chapter, we introduce the concept of dynamic centrality measure for bunch nodes in quantum graphs. For the brain network, this idea is more useful as the human brain works all the time and has to make decisions frequently. We have clearly explained in this chapter how the dynamic centrality of bunch nodes affects in brain network.

While the approach attempts to link dynamic centrality to brain functions like decision-making and cognitive processes, the direct interpretation of these values in the context of brain behavior could be challenging. Neural networks are complex and multidimensional, and a simplistic representation of centrality might not reflect the true nature of cognitive processes, such as how different brain regions coordinate specific tasks like problem-solving or decision-making. Besides that for computing the centrality value we have only considered the degree centrality. Other centrality measure methods are not included in the experiment. But those methods may yield better results than degree centrality.

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Routing Protocols Based on Quantum Bunch Graph



Nivedita Kuity, Laxminarayan Sahoo, Swasti Hazra, and Kalyani Maity Das

Abstract Routing Protocols based on Quantum bunch graphs acquaint an advanced approach to implementing routing in multifarious networks, commercialising principles of quantum computing and graph theory. This chapter discusses quantum bunch graphs, focusing on entanglement-based, degree-based, spectral, path-based, and indirect neighbour-based irregularity measures. These measures are essential for understanding the complexity and behavior of quantum bunch graphs, which are characterized by dynamic weights and entangled edges. The degree-based measure extends the concept of vertex degrees to dynamic quantum graphs by considering the time-averaged degree of each vertex, while the entanglement-based measure leverages the correlation functions of entangled edges. We used the Fuzzy Bunch Graph (FBG) concept to find the uncertain route. Both quantum bunch graphs and fuzzy bunch graph work on uncertainty, but they belong to different framework. The time-averaged adjacency matrix, and the path-based measure considers the variability in the shortest path lengths between all pairs of vertices. The indirect neighbor-based measure examines the influence of indirect neighbors by considering the variability in their weights. These irregularity measures are crucial in the analysis of dynamical networks, such as communication networks, where maintaining consistent performance and reliability is essential. For instance, in a quantum communication network, the degree-based irregularity measure can help identify nodes with fluctuating connectivity, which may affect the overall network stability and efficiency. By applying these measures, network designers can optimize the network structure to minimize irregularities and enhance performance. The result indicates that

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quantum-based routing guarantees achieving outstanding procure in network performance, flooring the way for a more reliable and efficient communication system in the quantum period.

Keywords Routing protocol · Dynamical systems · Fuzzy bunch graph · Bunch graph · Quantum networks

1 Introduction

A network may be defined as two or more communicating devices that are linked together to share common resources. Traditional Graph Theory is tool to represent a network in terms of interconnected nodes (vertices) and edges. Using classical graph theory, we can classify there are four types of networks. Those are edges have either direction or not and edges have either weights (distance, capacity) or not.

Quantum Graph Theory is a more powerful tool to represent a network; it is a mathematical structure that combines the concept of classical graph theory and power of quantum mechanics. Quantum computing enhance new possibilities to process and transmit information. In this paper we introduce shortest path algorithm to find the suitable path for forming route in the computer network routing protocol. Based on the concept of quantum graph theory and bunch graph we proposed a new concept quantum bunch graph. It is a quantum-based methods that improves communication, routing efficiency or data transmission in networks. It proposed an emerging concept that integrates quantum mechanics, graph theory, and network communication. Quantum computing, specifically through **quantum algorithms** (Strubell 2011; Adedoyin et al. 2018; Dalzell et al. 2023; Ugwuishiwu et al. 2020), we can improve routing protocols by allowing more methodical searches and pathfinding in complex networks. Quantum bunch graphs might represent a network where multiple quantum states (or bunches of states) propagate through the network to find optimal routes. Fuzzy bunch graph in the relationship between nodes can consider to represent uncertainty. It allows edges to have degrees of membership or weights that represent the strength of the relationship.

1.1 Motivation for Defining Quantum Bunch Graph

The motivation for the defining of quantum bunch graphs arises from the need to model and analyze system in which quantum mechanics effects play a crucial role. Classical graph theory concepts have been proven to be a vital role in the study of classical networks, such as communication networks, and social networks. Nevertheless, these traditional approaches are deficient for quantum systems, where entanglement and superposition must be considered. It is a dynamical approach. Vertices will be defining with dynamic weights and edges will be entangled. QBG

provide a more accurate mechanism for modeling quantum systems. This approach provides a new opportunity for research in quantum communication. Our main moto is to find the appropriate path for create a route in network system. It motivates us to improve search efficiency by exploring multiple routes simultaneously through quantum superposition. Routing protocol must be allowed by a quantum bunch graph to strength this parallelism, reducing the time and resources required to find the most efficient routes in a large, complex network.

2 Literature Review

The article (Murthy et al. 1996) addresses the development of an optimized routing protocol. The authors designed to improve the performance and reliability of wireless networks. It is able to handle dynamic topologies, reduce control overhead, and conserve energy makes it a valuable solution for enhancing wireless communication systems, especially in applications where stability and energy conservation are critical. The paper (Chakeres et al. 2004) motivates the researcher to develop their own on-demand ad hoc routing protocols and assist users in determining the implementation design that best fits their needs. Social media networks are implemented using semi-directed graph. The directed and undirected edges combines both mutual and non-mutual interactions between uses, providing an ideal tool for studying the complexity of social network (Samanta et al. 2021). This numerical method introduces a new definition of the derivative for fuzzy functions to solve fuzzy linear partial differential equations (Allahviranloo et al. 2010). The article (Mahapatra et al. 2020c) approached a graph network-based quality control model designed for the smart traceability and communication of packaged food by combining graph theory for IoT devices, and machine learning. The model addresses critical challenges in modern food supply chains, including product quality assurance, transparency and safety. The integration of IoT sensors enables real-time data collection on key environmental parameters (e.g., temperature, humidity) throughout the food's journey, ensuring continuous quality monitoring.

The article (Samanta et al. 2022) generalized the structure of graphs using bunch graph. The bunch graph illustrates both cooperative partnerships, such as joint ventures and shared scientific data, and competitive rivalries over patents, market share, and funding. The study explores how coopetition has accelerated the development of vaccines and therapies, demonstrating the crucial role of strategic collaboration amongst competition in addressing global crises). The article (Mahapatra et al. 2024a, c) introduces the linguistic Z-graph. This framework is designed within social networks to analyze linguistic structure. In online communities it provides a detailed understanding of language dynamics. A fuzzy routing algorithm proposed by the researchers to find shortest routing path. They update the Dijkstra's single-source shortest path algorithm by using fuzzy-logic membership function (Boominath et al. 2024). The article (Baranidharan and Shanthi 2011) reviewed the different clustering algorithms and proposed clustering technique using Minimum Spanning Tree. They

used the concept of shortest path with its strength and limitations. (Mahapatra et al. 2020a, b, 2022, 2023) provided several applications and developed the colouring idea of fuzzy graphs.

The Aggressive Multi-Path routing protocol proposed by the researcher. The AMA routing protocol resolve the single shortest path and multi-paths depletion problems of the Vertex-Transitive (VT) and Class-level Vertex-Transitive (CVT) routing protocols. It updates the routing table to reflect topology changes in the network system (Kim et al. 2013). The article (Das et al. 2024) enhances the link prediction within co-authorship networks using fuzzy-set theory. Researcher uses Common Neighbor index, Jaccard coefficient and Adamic-Adar Index to maintain the uncertainty and partial memberships which is useful in scientific research. The researcher mainly focused on associated network in fuzzy environment. A new centrality measure is introduced for neutrosophic and linguistic graphs (Mahapatra et al. 2024b) including self-weight or individual merit within the network. It plays a vital role to determine influence in social network, highlighting the importance of understanding how entities affect the behavior of other (Samanta et al. 2021). The techniques of edge coloring and radio k coloring in fuzzy graphs were first presented by Mahapatra et al. (2019a, b, 2020c).

The article explores the principles of quantum computing without a deep background in quantum mechanics or physics. It introduces the concepts of qubits, entanglement, superposition and quantum gates. It C includes common **CNOT gate** (for entanglement) (Zajac et al. 2018) (Abdessaied et al. 2014), **Hadamard gate** (for superposition) and provide the concept of **Grover's Algorithm** (for database search) (Shrivastava et al. 2019), **Shor's Algorithm** (for factoring large numbers) (Rieffel et al. 2000). The researchers survey an important issue in quantum communications and computing. It tried to correct error, secure quantum hardware and scalable of quantum hardware (Yang et al. 2023). Researchers introduced a reliable logical operation to implement advanced quantum algorithm at large scale network. They suggest a technique to overcome noise and error by integrating fault tolerance into quantum gates and quantum algorithms using error correction technique (Zhou et al. 2024).

Pandey et al. (2024) proposed methodologies for representing collaboration among entities. They introduced bipolar fuzzy graph (BFBG). It captures the positive and negative effects of both the terms competition and collaboration. They used communication potential effect (CPE) to form a clique in network to analyze bipolar fuzzy coopetition. Allahviranloo et al. (2011) proposed a method for solving fuzzy linear differential equations based on the assumption of differentiability that is strongly generalized.

Allahviranloo and Salahshour (2013) modeled the uncertainty inherently in real-life problem. The relations and interactions often involve incomplete or vague information. Traditional mathematical model cannot capture this uncertainty. The development of fuzzy fractional differential equations using fuzzy Laplace transforms is essential for modeling where the influence of individuals or group evolves fractionally over time. Mandal and Ranadive (2017) studied fuzzy variable precision rough sets, in Mandal and Ranadive (2018a, b) introduced hesitant bipolar-valued fuzzy

sets and bipolar-valued hesitant fuzzy sets, and studied multi-granulation bipolar-valued fuzzy probabilistic rough sets and their corresponding three-way decisions. The decision-theoretic rough sets and their applications are studied in Mandal and Ranadive (2018c, 2019a, b). In real-life scenarios, experts often have limited knowledge and personal biases, making language-based information, combined with confidence and doubt levels, a more flexible way for them to express their ideas. Samanta et al. (2016) introduced a regular and complete generalized fuzzy graph to overcome the limitation of fuzzy graphs. It modeled complex relationships under uncertainty. Sumitha (2013) analyzed routing algorithms such as the adoptive routing algorithms and the non-adoptive routing algorithms. The results are better to the adoptive routing algorithms with compared to non-adoptive routing algorithms. They got the better result because of the dynamic routing table. For instance, the development of fuzzy fractional differential equations using fuzzy Laplace transforms is essential for modeling systems where the influence of individuals or groups evolves fractionally over time under uncertainty (Allahviranloo 2020). Pythagorean fuzzy systems (Akram et al. 2021) and trapezoidal approximations (Allahviranloo and Firozja 2007) provide additional methods for handling high levels of uncertainty in decision-making processes. These approaches offer flexible tools for analyzing complex decision problems, ensuring that a wide range of uncertain factors are considered when making group decisions.

There are many limitations in classical routing algorithms. These algorithms struggled with complex networks due to high time complexity. They are mainly designed for static graphs. When network conditions change frequently over time, they don't work efficiently without re-calculating paths. This article addresses this gap by introducing a model based on the concept of quantum bunch graphs and fuzzy bunch graph with dynamic vertices, entangled edges and uncertainty. It can consider the certain route and also uncertain route for finding the correct shortest route.

3 Fuzzy Bunch Graph

Let X be a non-empty set and $V_X \subset P(X)$ where $P(X)$ is the power set of X 's, such that $\phi \notin V_X, X \subset V_X$ i.e., $V_X - X \neq \phi$. Now a non-empty set V_X , together with a pair of functions $\sigma_X : V_X \rightarrow [0, 1]$ and $\mu_X : V_X \times V_X \rightarrow [0, 1]$ such that for all $A, B \in V_X, \mu_X(A, B) \leq \min\{\sigma_X(A), \sigma_X(B)\}$, then the graph $G = (V_X, \sigma_X, \mu_X)$ is called a FBG (Fig. 1).

3.1 Shortest Path of FBG

If $G = (V_X, \sigma_X, \mu_X)$ be a FBG, where $V_X \subset P(X)$ [$P(X)$ is the power set of X , such that $\phi \notin V_X, X \subset V_X$], $\sigma_X : V_X \rightarrow [0, 1]$ and $\mu_X : V_X \times V_X \rightarrow [0, 1]$ such that

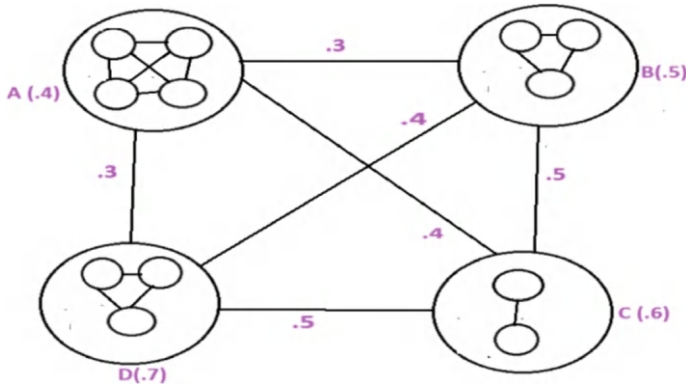


Fig. 1 Fuzzy Bunch Graph for finding shortest path

for all $A, B \in V_X$, the shortest path of the FBG is the minimum of $(\frac{\sigma_A + \sigma_B}{2} \times \mu_i)$ for $A \neq B$ and $\mu_i \in V_A \times V_B$.

If $G = (V_X, \sigma_X, \mu_X)$ be a FBG, where $V_X \subset P(X)$ [$P(X)$ is the power set of X , such that $\phi \notin V_X, X \subset V_X$], $\sigma_X : V_X \rightarrow [0, 1]$ and $\mu_X : V_X \times V_X \rightarrow [0, 1]$ such that for all $A, B \in V_X$, shortest path from A to B in different route is the minimum of $\sum_A^B \frac{\sigma_A + \sigma_B}{2} \times \mu_i$ for $A \neq B$ and $\mu_i \in V_A \times V_B$.

Example Let us consider a FBG below.

To find out the shortest path of the FBG, we have to calculate the values of $\frac{\sigma_i + \sigma_j}{2} \times \mu_k$, where $i \neq j$ and $\mu_k : V_i \times V_j$ for every edge from the above figure,

For $AB \rightarrow \frac{0.4 + 0.5}{2} \times 0.3 = 0.135$, $AC \rightarrow \frac{0.4 + 0.6}{2} \times 0.4 = 0.2$, $AD \rightarrow \frac{0.4 + 0.7}{2} \times 0.3 = 0.165$, $BD \rightarrow \frac{0.5 + 0.7}{2} \times 0.4 = 0.24$, $BC \rightarrow \frac{0.5 + 0.6}{2} \times 0.5 = 0.275$, $CD \rightarrow \frac{0.7 + 0.6}{2} \times 0.5 = 0.325$.

This shows that AB is the shortest path of the above FBG.

Now we want to find out the shortest path from any one of the vertices to other in different route from the above figure, by calculating the values $\sum_j^i \frac{\sigma_i + \sigma_j}{2} \times \mu_k$, where $i \neq j$ and $\mu_k : V_i \times V_j$ for A to C

$$AC \rightarrow 0.2$$

$$ABC \rightarrow AB + BC = 0.135 + 0.275 = 0.41$$

$$ADC \rightarrow AD + DC = 0.165 + 0.325 = 0.49$$

$$ABDC \rightarrow AB + BD + DC = 0.135 + 0.24 + 0.325 = 0.7$$

$$ADBC \rightarrow AD + DB + BC = 0.165 + 0.24 + 0.275 = 0.68$$

This shows that AC is the shortest path from the vertex A to the vertex C.

Theorem 1 The shortest path of a FBG having different membership value of each bunch node is unique.

Proof If $G = (V_X, \sigma_X, \mu_X)$ be a FBG whose any three nodes are V_1, V_2 and V_3 having membership values a_1, a_2 and a_3 where $a_1 > a_2 > a_3$ and E_1 and E_2 are their connecting edges having membership values e_1 and e_2 where $E_1 = V_2 \times V_3$.

Since

$$\begin{aligned} e_1 &= \min(a_1, a_2) \\ &= a_2 \text{ (If possible)} \\ e_2 &= \min(a_2, a_3) \\ &= a_3 \text{ (If possible)} \end{aligned}$$

Since $e_1 > e_2$.

Now the path value of $V_1 V_2$

$$\begin{aligned} &= \frac{a_1 + a_2}{2} \times e_1 \\ &> \frac{a_2 + a_2}{2} \times e_1 [a_1 > a_2] \\ &> a_2 \times e_1 \end{aligned}$$

The path value of $V_2 V_3$

$$\begin{aligned} &= \frac{a_2 + a_3}{2} \times e_2 \\ &< \frac{a_2 + a_2}{2} \times e_2 [a_2 > a_3] \\ &< a_2 \times e_2 \end{aligned}$$

Since $e_1 > e_2$.

$\therefore$ The path values of different edges are unequal.

Hence the condition is necessary.

Now let the path values of any two edges $V_1 V_2$ and $V_2 V_3$ be equal,

Which gives us

$$\frac{a_1 + a_2}{2} \times e_1 = \frac{a_2 + a_3}{2} \times e_2$$

$$\Rightarrow a_1 e_1 + a_2 e_1 = a_2 e_2 + a_3 e_2$$

$$\Rightarrow a_2(e_1 - e_2) = a_3 e_2 - a_1 e_1$$

Since the membership values of both vertices and edges are lies between 0 and 1, $e_1 > e_2$, $a_1 > a_2 > a_3$, then the value of L.H.S is always positive and the value of R.H.S is negative.

Which contradict our assumption.

Hence the condition is sufficient.

Therefore, the shortest path of a FBG having different membership value of any node is unique.

4 Quantum Graphs

A Quantum Graph is defined mathematically as follows.

4.1 Dynamic Weights of Vertices

Weight of each vertex v_i changes over time. The Weight is between 0 and 1:

$$0 \leq dweight_i(t) \leq 1$$

The weight follows a sinusoidal function:

$$dweight_i(t) = \frac{1}{2}(1 + \sin(w_i t + \emptyset_i))$$

where $\emptyset_i$ is the phase shift and w_i is the frequency.

4.2 Edges and Entanglement

Edges e_{ij} between vertices v_i and v_j can be either entangled or not. The edges are represented by a correlation function $g_{ij}(t)$, if it is entangled. The relationship between the weights of the connected vertices represented by:

$$g_{ij}(t) = h(dweight_i(t), dweight_j(t))$$

where the nature of the entanglement is represented by a function h .

4.3 Structure of the Graph

The graph G is represented as $G = (V, E)$, where V is represented set of vertices and E is represented set of edges. The adjacency matrix is defined as:

$$A_{ij}(t) = \begin{cases} g_{ij}(t) & \text{if } e_{ij} \text{ exists and is entangled} \\ 0 & \text{otherwise} \end{cases}$$

5 Quantum Bunch Graphs

A Quantum Bunch Graph (Samanta and Allaviranloo 2024) with dynamic vertices and entangled edges is defined mathematically as follows.

5.1 Bunch Nodes with Combined Weights

Each bunch node B_k combines the weights of simple nodes v_i within the bunch. The combined weight $bweight_k(t)$ is defined as the average of the dynamic weights of the simple nodes in the bunch:

$$bweight_k(t) = \frac{1}{n_k} \sum_{i=1}^{n_k} dweight_i(t)$$

where n_k is the number of simple nodes in bunch B_k .

5.2 Vertices with Dynamic Weights (dweight)

Each simple node v_i has a dynamic weights $dweight_i(t)$ that changes over time. The weight is a real number between 0 and 1:

$$0 \leq dweight_i(t) \leq 1$$

The weight follows a specific pattern, for example a sinusoidal function as follows:

$$dweight_i(t) = \frac{1}{2}(1 + \sin(w_i + \emptyset_i))$$

where w_i is the frequency and $\emptyset_i$ is the phase shift.

5.3 Edges and Entanglement

Edges e_{kl} between bunch nodes B_k and B_l can be either entangled or not. If entangled the edges is represented by a correlation function $g_{kl}(t)$ that describes the relationship between the combined weights of the connected bunch nodes:

$$g_{kl}(t) = h(bweight_k(t), bweight_l(t))$$

5.4 Structure of Graph

The graph G is represented as $G = (B, E)$, where B is the set of bunch nodes and E is the set of edges. The adjacency matrix $A_{kl}(t)$ of the graph is defined as:

$$A_{kl}(t) = \begin{cases} g_{kl}(t) & \text{if } e_{kl} \text{ exists and is entangled} \\ 0 & \text{otherwise} \end{cases}$$

Note 1. Dynamic Nature (Superposition Behaviour)

Vertices have the dynamic nature, means weights of the vertices $dweight_i(t)$ change over time following a specific modeled, pattern using time-dependent function.

Note 2. The $\{\phi\}$ is phase shift in the dynamic weight function of a node, affects the timing of the oscillations of weight of the node. Typically, the dynamic weight function is represented by:

$$dweight_i(t) = \frac{1}{2}(1 + \sin(w_i + \phi_i))$$

The phase shift at $t = 0$ dictates the initial position of the sine wave. The sine waves of different nodes start at different points will cause the different phase shifts.

5.5 Example of Quantum Bunch Graph

1. Vertices with Dynamic Weights:

Simple Nodes: v_1 with $dweight_1(t)$, v_2 with $dweight_2(t)$, v_3 with $dweight_3(t)$.

Bunch Node: Bunch B_1 containing nodes v_4, v_5, v_6 . Dynamic weights of nodes in the bunch: $dweight_4(t), dweight_5(t), dweight_6(t)$.

Dynamic weight of the bunch:

$$dweightB_1(t) = f(dweight_4(t), dweight_5(t), dweight_6(t))$$

For example, f could be the average:

$$dweightB_1(t) = \frac{dweight_4(t) + dweight_5(t) + dweight_6(t)}{3}$$

2. Edges and Entanglement: Edges between simple nodes and bunch nodes: e_1B_1 between v_1 and B_1 with correlation function $g_1B_1(t)$, e_2B_1 between v_2 and B_1 with correlation function $g_2B_1(t)$, e_3B_1 between v_3 and B_1 with Correlation function $g_3B_1(t)$.
3. Graph Structure: The graph G is represented as $G = (V, E)$, with vertices $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edges E . Adjacency matrix $A(t)$ is given by:

$$A(t) = \begin{pmatrix} 0 & 0 & 0 & g_1B_1(t) & g_1B_1(t) & g_1B_1(t) \\ 0 & 0 & 0 & g_2B_1(t) & g_2B_1(t) & g_2B_1(t) \\ 0 & 0 & 0 & g_3B_1(t) & g_3B_1(t) & g_3B_1(t) \\ g_1B_1(t) & g_2B_1(t) & g_3B_1(t) & 0 & 0 & 0 \\ g_1B_1(t) & g_2B_1(t) & g_3B_1(t) & 0 & 0 & 0 \\ g_1B_1(t) & g_2B_1(t) & g_3B_1(t) & 0 & 0 & 0 \end{pmatrix}$$

A representational diagram of the Quantum Bunch Graph is shown in the following Fig. 2.

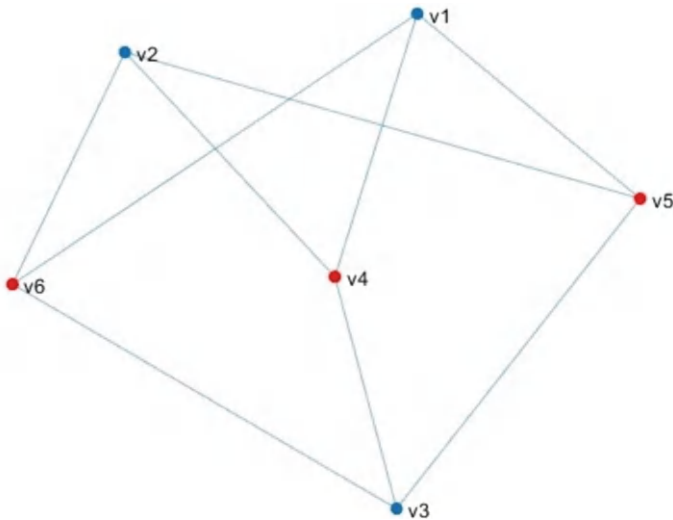


Fig. 2 Quantum bunch graph example

6 Routing Problem Using Quantum Bunch Graphs

6.1 Problem Statement

Given a network of Wi-Fi routers represented as a quantum bunch graph, find the shortest path from a source to destination router. Each bunch node represents a group of routers, and the edges between bunch nodes represent the communication links with dynamic weights.

7 Distance Definition in Quantum Bunch Graph Algorithm

In the algorithm for finding the shortest path in a quantum bunch graph, the distance is defined as the minimum cumulative weight from the source node in the graph. The steps to calculate and update the distance are as follows:

1. Initialization:

- Set the distance to all bunch nodes B_K to infinity: $\text{dist}[B_K] \leftarrow \infty$
- Set the distance to the source node s to 0: $\text{dist}[s] \leftarrow 0$.

2. Priority Queue:

- Use a priority queue Q to keep track of nodes to be processed, prioritized by their current known shortest distance.

3. Distance Calculation:

- For each node u extracted from the priority queue, examine its neighbors v .
- Calculate the weight $w_{uv}(t)$ of the edge between u and v using the function $g_{uv}(t)$, which depends on the dynamic weights of the edges.

4. Distance Update:

- If the newly calculated distance to a neighbor v through u ($\text{dist}[u] + w_{uv}(t)$) is less than the currently known distance to v ($\text{dist}[v]$), update the distance: $\text{dist}[v] \leftarrow \text{dist}[u] + w_{uv}(t)$.
- Insert the updated distance and node v back into the priority queue.

5. Termination:

- Continue the process until the priority queue is empty, ensuring that the shortest path distances to all nodes have been found.
- Return the final distance to the destination node d : $\text{dist}[d]$.

In summary, the distance in this algorithm represents the shortest path length from the source node to each other node, dynamically adjusted based on the weights of the edges at different time t .

7.1 Algorithm to Find Shortest Path for Routing Problem:

Algorithm: Shortest Path Search in Quantum Bunch Graphs

- 1: **Input:** Quantum Bunch Graph $G = \{B, E\}$, source node s , destination node d
- 2: **Output:** Shortest path from s to d
- 3: Initialize distance $\text{dist}[B_k] \leftarrow \infty$ for all bunch nodes $B_k \in B$
- 4: Set $\text{dist}[s] \leftarrow 0$
- 5: Initialize priority queue Q
- 6: Insert $(0, s)$ into Q **while** Q is not empty
do
- 7: Extract node u with minimum distance from Q for each neighbor v of u
Do
- 8: Calculate weight $w_{uv}(t)$ using $g_{uv}(t)$ **if** $\text{dist}[u] + w_{uv}(t) < \text{dist}[v]$ **then**
- 9: Update $\text{dist}[v] \leftarrow \text{dist}[u] + w_{uv}(t)$
- 10: Inset $(\text{dist}[v], v)$ into Q
- 11: **Return** $\text{dist}[d]$

7.2 Example: Wi-Fi Routing Problem

Consider a small network of Wi-Fi routers represented by the following quantum bunch graph:

- Bunch Nodes: B_1, B_2, B_3
- Simple Nodes in Bunches: $B_1 = \{v_1, v_2\}, B_2 = \{v_3, v_4\}, B_3 = \{v_5\}$
- Dynamic Weights:

$$dweight_1(t) = \frac{1}{2} = (1 + \sin(2t + \frac{\pi}{4}))$$

$$dweight_2(t) = \frac{1}{2}(1 + \sin(2t + \frac{\pi}{2}))$$

$$dweight_3(t) = \frac{1}{2}(1 + \sin(2t + \frac{\pi}{3}))$$

$$dweight_4(t) = \frac{1}{2}(1 + \sin(2t + \frac{\pi}{6}))$$

$$dweight_5(t) = \frac{1}{2}(1 + \sin(2t))$$

- Combined Weights:

$$bweight_1(t) = \frac{1}{2}(dweight_1(t) + dweight_2(t))$$

$$bweight_2(t) = \frac{1}{2}(dweight_3(t) + dweight_4(t))$$

$$bweight_3(t) = dweight_5(t)$$

- Edges and Weights:

$$g_{12}(t) = h(bweight_1(t), bweight_2(t))$$

$$g_{23}(t) = h(bweight_2(t), bweight_3(t))$$

Using the algorithm, we can find the shortest path from the source bunch node B_1 to the destination bunch node B_3 .

1. Initialize distance: $dist[B_1] = 0, dist[B_2] = \infty, dist[B_3] = \infty$
2. Insert $(0, B_1)$ into the priority queue
3. Extract B_1 and update distances for neighbors B_2
4. Insert $(dist[B_2], B_2)$ into the priority queue
5. Extract B_2 and update distances for neighbors B_3
6. Insert $(dist[B_3], B_3)$ into the priority queue
7. Extract B_3 and return $dist[B_3]$ as the shortest par.

A schematic diagram of network routing using bunches are shown in the Fig. 3.

The step-by-step diagram of a Wi-Fi routing is shown in the Figs. 4 and 5.

7.3 Comparison with Classical Routing Problem

Classical routing problem and routing problem using quantum bunch graphs significantly differ in structure and applications. In large network the classical routing algorithms like Dijkstra's and Bellman- Ford explore multiple paths sequentially based on distance, weights, or costs. On the other hand, quantum bunch graph-based algorithm explores multiple paths simultaneously. It can provide efficient, faster and potentially more secure routing- specifically in large-scale, complex or highly dynamic network. It considers uncertain route based on the concept of fuzzy bunch

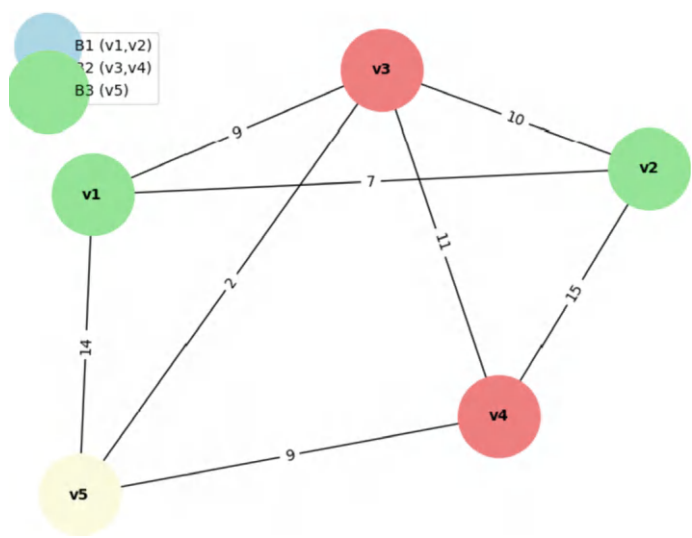


Fig. 3 Network routing based on bunch

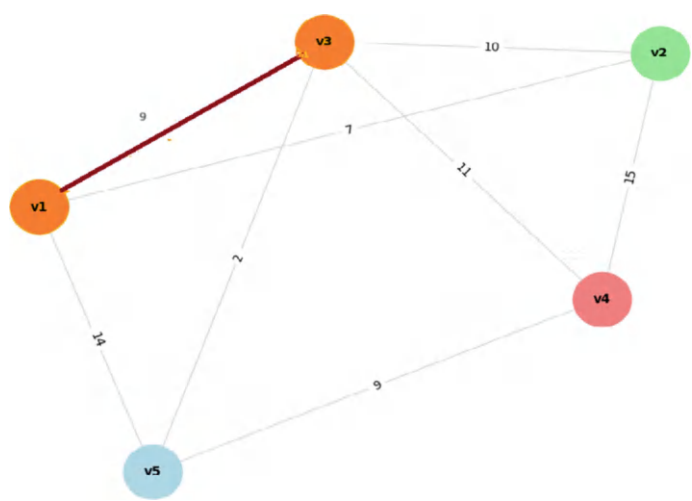


Fig. 4 Route the path from v_1 to v_3

graph. In quantum computing, dynamic and complex nature of quantum bunch graphs and fuzzy bunch graphs makes them applicable in advanced field.

A comparison table shown in the Table 1.

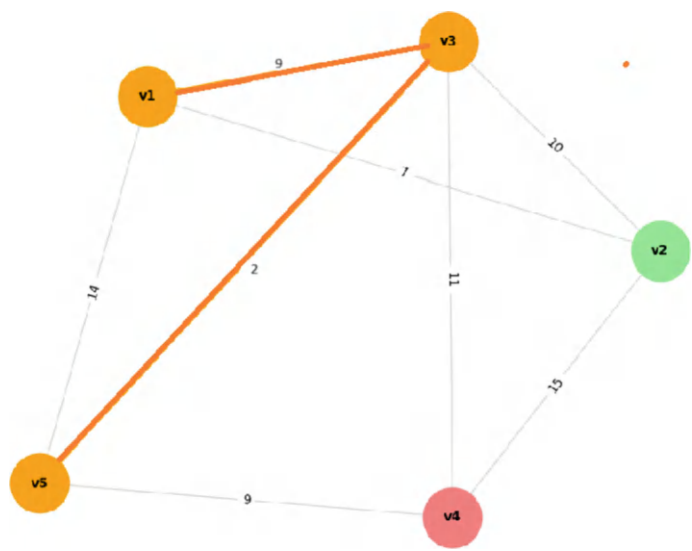


Fig. 5 Route from v_1 to v_5 through v_3

Table 1 Comparison between classical routing algorithms and quantum bunch graphs

Aspect	Classical routing algorithms	Quantum bunch graph routing algorithms
Path selection	Uses algorithms based on distances, weights, or costs	Uses superposition to explore multiple paths simultaneously, finding optimal path faster
Speed	Linearly or exponentially increases with network size	Quantum superposition allows for faster path computation
Structure	Fixed structure	Dynamic structure
Fault tolerance	To handle failures redundant paths and periodic updates are needed	Quantum concept could adapt dynamically to failures
Scalability	In complex networks performance can degrade due to memory requirements and increased computation time	Quantum parallelism can handle complex networks more efficiently and potentially improving scalability
Applications	Mobile networks, network routing, IoT, and traditional network infrastructure	Complex network environments

8 Conclusion

In this chapter, we have proposed the concept of a quantum bunch graph for solving routing problems. The quantum bunch graph extends classical theory methods into the quantum domain by introducing fundamental quantum properties such as entanglement and superposition. This approach provides a more accurate tool for solving routing problems in computer networks. Quantum bunch graph-based algorithm can

optimize path selection in routing problems by processing huge amounts of data in parallel. It also can improve the scalability security of routing in large networks with high connectivity. This method provides a comprehensive framework for transforming modern networks by supporting faster, more reliable, and more secure routing solutions.

Future research in the field of quantum bunch graphs is advancing both theoretical and practical applications. The practical implementation of quantum routing protocols is still in the experimental stage, constrained by current technological limitations in quantum computing infrastructure. Additionally, integrating these protocols with existing classical network architectures requires extensive research and development.

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Centrality Measure in Quantum Graphs and Its Application



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Abstract Centrality measures play a crucial role in identifying the most important nodes in a network, with applications ranging from social networks to biological systems. Traditional centrality on classical graph theory, which may struggle to capture complex interactions in large and interconnected systems. Quantum graphs, integrating quantum principles such as superposition and entanglement, offer a novel framework for evaluating centrality. By representing nodes and edges with quantum states, quantum graphs can account for both direct and indirect relationships with greater precision. This study introduces quantum centrality measures, showing how entangled edges and dynamic vertex states can reveal more nuanced insights into node importance. We apply these quantum centrality measures to various real-world networks, such as social, transportation, and communication networks.

Keywords Quantum graph · Centrality · Dynamic graph · Dynamic centrality

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1 Introduction

Key quantum features like superposition and entanglement are incorporated into quantum graphs, which expand classical graph theory into the quantum domain. The fundamental ideas of quantum graphs are presented in this chapter, with an emphasis on dynamical vertices and entangled edges. A more accurate framework for understanding quantum events is provided by quantum graphs, which incorporate entangled edges and dynamical weights. By suggesting a model that mimics the dynamic properties of actual quantum systems, this study fills the gap in the application of dynamical vertices and entangled edges. Numerous branches of research and engineering, including as physics, biology, economics, and engineering, use these systems. In today's fast-evolving digital landscape, communication networks are becoming increasingly complex, with vast amounts of data flowing through highly interconnected and dynamic systems. Traditional graph models often fall short in capturing the nuanced, multi-layered interactions and rapid fluctuations inherent in these networks. Quantum graphs, with their foundation in quantum mechanics, introduce a new approach to network modeling by leveraging principles like "superposition" and "entanglement". These quantum features allow for more flexible and precise representations of communication pathways, capturing the probabilistic nature of network connections and supporting the simultaneous evaluation of multiple states.

Centrality measures are critical in analyzing networks to identify influential nodes, optimize information flow, and ensure robust connectivity. However, traditional centrality measures often fall short in capturing the complex, multi-dimensional relationships present in many real-world networks, such as social networks, communication systems, and biological networks. These networks contain elements of uncertainty and non-linear interactions that are not easily modeled by classical approaches. Quantum graph theory introduces principles such as superposition and entanglement, which allow for the simultaneous evaluation of multiple potential relationships and interactions within the network. This quantum framework, the centrality measure capturing both direct and indirect influences in a single, unified model. By adopting quantum centrality measures, researchers can analyze networks more precisely, account for complex interdependencies, and better understand the hidden dynamics driving network behavior. This advanced approach has the potential to the network analysis, making it especially valuable in fields that more accurate understanding of network structures, from online social platforms to financial markets and health care systems.

1.1 Literature Review

In 1965, Zadeh presented the significant notion of fuzzy sets (FS) in order to address the problems that were being discussed. Fuzzy sets (FS) [33] have become a popular approach for incorporating fuzzy information as part of an expert's evaluative measures. For example, in [7] studied fuzzy variable precision rough sets, in [8]

introduced hesitant bipolar-valued fuzzy sets and bipolar-valued hesitant fuzzy sets, and studied multi-granulation bipolar-valued fuzzy probabilistic rough sets and their corresponding three-way decisions. The decision-theoretic rough sets and their applications are studied in [10–12]. However, as noted by Mandal et al. [13], experts tend to express their opinions more reliably using linguistic information, known as linguistic sets (LS), rather than fuzzy information. In the LS concept was extended to the Pythagorean linguistic set (PLS), allowing experts to express both their confidence and doubt levels when providing judgments.

The basis for the creation of quantum graphs has been laid by earlier efforts. As an example, Kottos and Smilansky [25] looked into the spectral characteristics of graphs with chaos and showed that graphs might be utilized to examine wave dynamics and quantum chaos in complicated systems. The promise of quantum graphs in quantum computing was highlighted more recently by Razavinia and Haghightadoost [26], who offered technical structures pertaining to quantum graphs and quantum computing concepts. A thorough overview of quantum graph computing and learning was given by Tang et al. [27], who emphasized the benefits of using quantum algorithms to solve graph-related issues [24]. Samanta and Allahviranloo [28] introduced a the quantum graph by the concept quantum theory.

The previous research in fuzzy set theory and its applications plays a significant role in addressing uncertainty and complexity in social networks and group decision-making processes. The methods developed in this area offer valuable insights into how fuzzy logic can be used to model and analyze systems where imprecision and ambiguity are prevalent.

Fuzzy set theory is particularly well-suited to modeling the uncertainty inherent in social networks, where relationships and interactions often involve incomplete or vague information. Traditional mathematical models struggle to capture this uncertainty, but fuzzy approaches provide a more nuanced framework for analyzing such dynamics. For instance, the development of fuzzy fractional differential equations using fuzzy Laplace transforms is essential for modeling systems where the influence of individuals or groups evolves fractionally over time under uncertainty [1].

The introduction of predictor-corrector methods for solving fuzzy differential equations also has direct applications in the study of evolving opinions and preferences within social groups [2]. These methods allow for a more accurate simulation of how consensus forms over time, particularly in environments where group dynamics are shaped by uncertainty and imprecision.

The development of methods for ranking fuzzy numbers using weighted distance [3] proves useful in group decision-making processes, where individuals often express preferences that are not crisply defined. The ability to rank such imprecise inputs is crucial in facilitating collective decisions, as it allows for the systematic comparison and prioritization of uncertain information.

Additionally, the use of fuzzy heat equations via the Adomian decomposition method [4] is highly relevant to understanding how influence or information spreads through social networks. This approach models the flow of ideas or behaviors in an uncertain environment, providing a framework for analyzing the propagation of influence within a network.

The work on fully fuzzy linear systems and the signed decomposition method [5] is crucial for modeling the complex interrelations within social networks, where interactions can be represented as linear systems with fuzzy parameters. This framework supports a more realistic analysis of decision-making processes, particularly when dealing with multiple interdependent and uncertain factors.

Fuzzy linear programming methods [6] are directly applicable to optimization problems in group decision-making, where the objectives and constraints are often imprecise. These techniques help navigate complex decision-making environments within social networks, where individual preferences and goals may conflict or remain undefined.

Mahapatra et al. [14] offers a new and improved definition, introducing the generalized neutrosophic planar graph (GNPG) by the concept of neutrosophic graph [32, 34, 35]. Here, planarity is evaluated more precisely by considering the true, false, and indeterminacy values for the degree of planarity, which gives a clearer measure of the graph's overall planarity. Link prediction is a crucial task in social networks, with various methods available to predict connections. The RSM index [18] is one such method, useful in fuzzy environments, but it lacks consideration for falsity and indeterminacy value common in uncertain settings in modified RSM index [15]. A COVID-19 infection example is provided to demonstrate how directed fuzzy graph colouring can model transmission influence and control by Mathapatra et al. [16]. A new concept of the chromatic number for neutrosophic graphs [37, 38] and discusses the differences between fuzzy colouring in fuzzy graphs [36] and colouring in NG [17, 19, 20]. Identifying central nodes in social networks is essential for tasks like business growth, news dissemination, and advertising. Traditional centrality methods focus on a node's direct reach but often overlook uncertainty in relationships. A new centrality measure is introduced for neutrosophic graphs and linguistic graph [21–23, 31], which includes self-weight or individual merit within the network. Kalampakas et al. [29, 30] introduced a syntactic recognition theory for graphs with fuzzy edges and nodes, using the algebraic structure of magmoids.

2 Preliminaries

Definition 1 (*Quantum Graphs*) The mathematical definition of a quantum graph with dynamic vertices and entangled edges is as

1. Vertices with Dynamic Weights (dweight): The dynamic weight $dweight_i(t)$ of each vertex v_i varies with time. A real number between 0 and 1 represents the weight:

$$0 \leq dweight_i(t) \leq 1$$

The weight exhibits a certain pattern, such as a sinusoidal function, as stated below.

$$dweight_i(t) = \frac{1}{2}(1 + \sin(\omega_i t + \phi_i))$$

2. Entanglement and Edges: There is an option for edges e_{ij} between vertices v_i and v_j to be entangled or not. A correlation function $g_{ij}(t)$ and $0 \leq g_{ij}(t) \leq 1$, which depicts the relationship between the weights of the connected vertices, is used to represent an edge if it is entangled.

$$g_{ij}(t) = h(dweight_i(t), dweight_j(t))$$

where h is a function defining the nature of the entanglement or correlation. 3. Graph Structure: The collection of vertices V , and the set of edges E are the components that make up the graph G , which is expressed as $G = (V, E)$. The graph's adjacency matrix $A(t)$ is described as follows:

$$\begin{aligned} A_{ij}(t) &= g_{ij}(t) \text{ if } e_{ij} \text{ exists and is entangled} \\ &= 0, \text{ otherwise} \end{aligned}$$

3 Centrality Measure of a Vertex in Quantum Graph

Centrality measures in classical graphs are used to determine the most important or influential nodes based on the structure of the graph. In quantum graphs, however, both nodes and edges have dynamic weight, which adds complexity to traditional centrality measures. To adapt centrality for quantum graphs, can define a centrality that accounts for the uncertainty and dynamic values of both edges and nodes.

Now, the centrality of a node of a quantum graph defined as follows-

Definition 2 $G = (V, E)$ be a quantum graph. A correlation function $g_{ij}(t)$ and $0 \leq g_{ij}(t) \leq 1$, which depicts the relationship between the weights of the connected vertices, is used to represent an edge if it is entangled.

$$g_{ij}(t) = h(dweight_i(t), dweight_j(t))$$

where h is a function defining the nature of the entanglement or correlation. Then the centrality of a vertex k is denoted by $D_k(t)$ and defined as

$$\begin{aligned} M_k(t) &= \frac{\sum_{i \neq j \& (i,j) \in G} \max_r \frac{\lambda_{(i,j)/k}^r(t)}{a}}{\sum_{i \neq j \& (i,j) \in G} \max_s \frac{\lambda_{(i,j)}^s(t)}{b}} \times dweight_k(t) \\ D_k &= \int_{t=0}^T M_k(t) dt \end{aligned}$$

The integral captures the cumulative importance of the vertex D_k across the entire time period T , showing how the importance of each node changes and adapts as the network structure shifts over time.

- a is the number of edges of shortest distance between the vertices i and j through the vertex k .
- b is the number of edges of shortest distance between the vertices i and j .
- $\lambda_{(i,j)/k}(t)$ is the sum of dynamic weight of a number of edges of shortest distance between every pair of vertices in G through the vertex k .
- $\lambda_{(i,j)}(t)$ is the sum of dynamic weight of b number of edges of shortest distance between every pair of vertices in G .
- $dweight_k(t)$ is the dynamic weight of the vertex k .

Example 1 We consider, the value of ω are in the range $[0, 2\pi]$ and the value of ϕ is $\phi = \frac{\pi}{4}$. Also, consider the value of t is $t = 4$ (here we consider a particular time) and the correlation value $h(a, b) = \min\{a, b\}$ with the value of $\omega_1 = \frac{\pi}{2}$, $\omega_2 = \frac{\pi}{4}$, $\omega_3 = \frac{\pi}{3}$, $\omega_4 = \frac{3\pi}{4}$.

We consider, $V = \{A, B, C, D\}$ denote vertex set, and $G = (V, E)$ represent a quantum graph. Now, the dynamic value of all vertices A, B, C, D with the ω value are $\omega_1 = \frac{\pi}{2}$, $\omega_2 = \frac{\pi}{4}$, $\omega_3 = \frac{3\pi}{2}$, $\omega_4 = \frac{3\pi}{4}$ respectively. The quantum graph depicted in Fig. 1 is considered. The dynamic value of all vertices in this quantum graph are detailed in Table 1. The dynamic correlation value of the quantum graph are presented in Table 2. Now, the Calculation of centrality of the quantum graph of Fig. 1 has been shown in Table 3. The centrality of the vertex A is $\frac{0.428}{0.51} \times 0.85 = 0.839 \times 0.85 = 0.987$. Centrality of all vertices of the quantum graph of Fig. 1 has

Fig. 1 A quantum graph

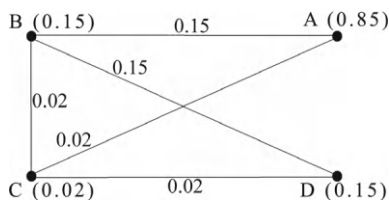


Table 1 Dynamic values of the quantum graph of Fig. 1

Vertex	Dynamic value
A	$0.5 \times (1 + \sin(\frac{\pi}{2} \times 4 + \frac{\pi}{4})) = 0.85$
B	$0.5 \times (1 + \sin(\frac{\pi}{4} \times 4 + \frac{\pi}{4})) = 0.15$
C	$0.5 \times (1 + \sin(\frac{\pi}{3} \times 4 + \frac{\pi}{4})) = 0.02$
D	$0.5 \times (1 + \sin(\frac{3\pi}{4} \times 4 + \frac{\pi}{4})) = 0.15$

Table 2 Dynamic correlation values of the quantum graph of Fig. 1

Edge	Dynamic correlation value
(A, B)	$\min\{0.85, 0.15\} = 0.15$
(A, C)	$\min\{0.85, 0.02\} = 0.02$
(B, C)	$\min\{0.15, 0.02\} = 0.02$
(B, D)	$\min\{0.15, 0.15\} = 0.15$
(D, C)	$\min\{0.15, 0.02\} = 0.02$

Table 3 Calculation of centrality of the quantum graph of Fig. 1

Pair of vertices	$\frac{\lambda_{(i,j)}(t)}{a}$	$\frac{\lambda_{(i,j)/A}(t)}{b}$	$\frac{\lambda_{(i,j)/B}(t)}{b}$	$\frac{\lambda_{(i,j)/C}(t)}{b}$	$\frac{\lambda_{(i,j)/D}(t)}{b}$
(A, B)	0.15	0.15	0.15	0.02	0.063
(A, C)	0.02	0.02	0.085	0.02	0.107
(A, D)	0.15	0.00	0.15	0.02	0.00
(B, C)	0.02	0.085	0.02	0.02	0.085
(B, D)	0.15	0.063	0.15	0.02	0.15
(C, D)	0.02	0.11	0.085	0.02	0.02
Sum	0.51	0.428	0.64	0.12	0.425

Table 4 Centrality of all vertices of the quantum graph of Fig. 1

Vertex	Centrality
A	$\frac{0.428}{0.51} \times 0.85 = 0.839 \times 0.85 = 0.987$
B	$\frac{0.64}{0.51} \times 0.15 = 1.255 \times 0.15 = 0.188$
C	$\frac{0.12}{0.51} \times 0.02 = 0.12 \times 0.02 = 0.002$
D	$\frac{0.425}{0.51} \times 0.15 = 0.833 \times 0.15 = 0.125$

been shown in Table 4. Now, the central node of the quantum graph of Fig. 1 is $\max\{D_A(4), D_B(4), D_C(4), D_D(4)\} = D_A(4)$ i.e. vertex A.

3.1 Algorithm to Calculate Centrality of All Vertices of a Quantum Graph

Input:- A quantum graph $G = (V, E)$.

Output:- Centrality of all vertices of G .

Step1:- All the vertices of the quantum graph G are labelled as $1, 2, \dots, p$.

Step 2:- Calculate

$$M_k(t) = \frac{\sum_{i \neq j \& (i,j) \in G} \max_r \frac{\lambda_{(i,j)/k}^r(t)}{a}}{\sum_{i \neq j \& (i,j) \in G} \max_s \frac{\lambda_{(i,j)}^s(t)}{b}} \times dweight_k(t)$$

Step 3:- Calculate the centrality of a vertex by the formula $D_k = \int_{t=0}^T M_k(t) dt$.

Step 4:- Continue Step 4 for all vertices of the quantum graph G .

4 Application in Quantum Centrality in Transportation Networks

Transportation networks often face dynamic values due to varying traffic conditions, weather changes, or construction. These factors affect the reliability of routes (roads, railways, etc.) and the efficiency of travel between destinations (cities, stations, etc.). To handle this uncertainty, traditional graph theory might fall short because it cannot effectively capture the vagueness of the real-world conditions. Quantum graph theory is a useful tools to address this problem. Quantum centrality measures provide a way to assess the importance of nodes (cities, stations, etc.) and edges (routes) in a transportation network, where the reliability or strength of connections is uncertain or imprecise. This helps in optimizing travel routes, planning infrastructure, and managing congestion or disruptions in the network.

Quantum centrality measures can be used to identify critical intersections or routes that are central to traffic flow. These critical nodes and routes can be prioritized for traffic management or infrastructure upgrades. In a city's road network, intersections with high quantum centrality can be optimized by adjusting traffic signals or adding lanes to reduce congestion.

Now, we consider 7 important of a city as a vertices of a quantum graph and there exists a edge if they connected by a rout. In a quantum graph, each node and edge has a dynamic value that reflects its reliability or strength, allowing a more nuanced representation of the transportation system. This represents the capacity of a transportation hub (such as a city, airport, or train station). A higher dynamic value means the hub is more reliable or has a higher capacity. Dynamic correlation value represents the strength, reliability, or capacity of the connection (such as roads, railways, or air routes) between two hubs. A higher dynamic value indicates a more reliable or stronger connection.

We consider, the value of ω are in the range $[0, 2\pi]$ and the value of ϕ is $\phi = \frac{\pi}{4}$. Also, consider the value of t is $t = 6$ and the correlation value $h(a, b) = \min\{a, b\}$ with the value of $\omega_1 = \frac{\pi}{2}, \omega_2 = \frac{4\pi}{5}, \omega_3 = \frac{3\pi}{2}, \omega_4 = \frac{3\pi}{4}, \omega_5 = \frac{\pi}{3}, \omega_6 = \frac{\pi}{4}, \omega_7 = \frac{2\pi}{5}$.

We consider, $V = \{A, B, C, D, E, F, G\}$ denote vertex set, and $G = (V, E)$ represent a quantum graph. Now, the dynamic value of all vertices A, B, C, D, E, F, G with the ω value are $\omega_1 = \frac{\pi}{2}, \omega_2 = \frac{4\pi}{5}, \omega_3 = \frac{3\pi}{2}, \omega_4 = \frac{3\pi}{4}, \omega_5 = \frac{\pi}{3}, \omega_6 = \frac{\pi}{4}, \omega_7 = \frac{2\pi}{5}$ respectively. The quantum graph depicted in Fig. 2 is considered. The dynamic value of all vertices in this quantum graph are detailed in Table 5. The dynamic correlation value of the quantum graph are presented in Table 6. Here, we consider a particular time $t = 6$ and the maximum value of centrality of a vertex is 1. So, the centrality of a vertex a of this application is $C_a = \min\{1, D_a\}$.

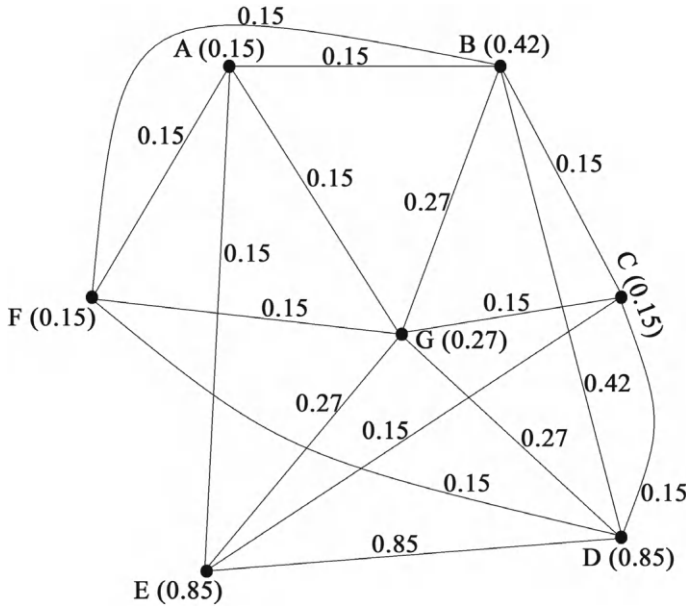


Fig. 2 A quantum graph of communication network

Table 5 Dynamic values of the quantum graph of Fig. 2

Vertex	Dynamic value
A	$0.5 \times (1 + \sin(\frac{\pi}{2} \times 6 + \frac{\pi}{4})) = 0.15$
B	$0.5 \times (1 + \sin(\frac{4\pi}{5} \times 6 + \frac{\pi}{4})) = 0.42$
C	$0.5 \times (1 + \sin(\frac{3\pi}{2} \times 6 + \frac{\pi}{4})) = 0.15$
D	$0.5 \times (1 + \sin(\frac{3\pi}{4} \times 6 + \frac{\pi}{4})) = 0.85$
E	$0.5 \times (1 + \sin(\frac{\pi}{3} \times 6 + \frac{\pi}{4})) = 0.85$
F	$0.5 \times (1 + \sin(\frac{\pi}{4} \times 6 + \frac{\pi}{4})) = 0.15$
G	$0.5 \times (1 + \sin(\frac{2\pi}{5} \times 6 + \frac{\pi}{4})) = 0.27$

The centrality of the vertex A is $\frac{3.755}{5.665} \times 0.15 = 0.663 \times 0.15 = 0.0995$ $\min\{1, 0.0995\} = 0.0995$. Centrality of all vertices of the quantum graph of Fig. 2 has been shown in Table 8. All the calculation are shown in Table 7. Now, the central nodes of the quantum graph of Fig. 2 are $\max\{C_A(6), C_B(6), C_C(6), C_D(6), C_E(6), C_F(6), C_G(6)\} = C_D(6) \text{ and } C_E(6)$ i.e. vertices D and E.

So, at time $t = 6$, the vertices D and E experience higher traffic congestion compared to others. Therefore, special attention is required for vertices D and E to manage this traffic. The quantum graph centrality helps in controlling the traffic, and the graph will change over time since the quantum graph depends on the time variable t . Additionally, this quantum centrality measure aids in other major areas, such as:

Table 6 Dynamic correlation values of the quantum graph of Fig. 2

Edge	Dynamic correlation value
(A, B)	$\min\{0.42, 0.15\} = 0.15$
(A, G)	$\min\{0.27, 0.15\} = 0.15$
(A, E)	$\min\{0.85, 0.15\} = 0.15$
(A, F)	$\min\{0.15, 0.15\} = 0.15$
(B, F)	$\min\{0.42, 0.15\} = 0.15$
(B, C)	$\min\{0.42, 0.15\} = 0.15$
(B, G)	$\min\{0.42, 0.27\} = 0.27$
(B, D)	$\min\{0.42, 0.85\} = 0.42$
(C, G)	$\min\{0.15, 0.27\} = 0.15$
(C, E)	$\min\{0.85, 0.15\} = 0.15$
(C, D)	$\min\{0.85, 0.15\} = 0.15$
(D, E)	$\min\{0.85, 0.85\} = 0.85$
(D, F)	$\min\{0.85, 0.15\} = 0.15$
(D, G)	$\min\{0.85, 0.27\} = 0.27$
(E, G)	$\min\{0.85, 0.27\} = 0.27$
(F, G)	$\min\{0.15, 0.27\} = 0.15$

Route Planning:

Transportation agencies can use quantum centrality to design more efficient routes. For identifying cities with high centrality values helps plan routes that minimize travel time and avoid unreliable roads. Example: In an airline network, airports with high centrality value can be chosen as central hubs to minimize flight delays and cancellations.

Infrastructure Investment:

Cities or hubs with high centrality might be prioritized for infrastructure investment because they are directly connected to many other hubs. Example: In a national highway system, highways connecting cities with high fuzzy degree centrality can be prioritized for expansion to improve overall network efficiency.

Resilience and Disaster Management:

Quantum centrality measures can also be used to improve the resilience of transportation networks. Nodes with high centrality values are critical for maintaining traffic flow during disruptions. Ensuring these nodes have redundant connections can improve network resilience. Example: In a railway network, if a city with high centrality values is affected by a natural disaster, alternative routes should be available to maintain network connectivity.

Quantum centrality provides a powerful way to analyze and optimize transportation networks under uncertainty. By incorporating dynamic values for both nodes and edges, quantum centrality measures help transportation planners make informed decisions regarding traffic management, route planning, infrastructure investment, and resilience. In a world where uncertainties like traffic congestion and road conditions

Table 7 Calculation of centrality of the quantum graph of Fig. 2

Pair of vertices	$\frac{\lambda_{(i,j)}(t)}{a}$	$\frac{\lambda_{(i,j)/A}(t)}{b}$	$\frac{\lambda_{(i,j)/B}(t)}{b}$	$\frac{\lambda_{(i,j)/C}(t)}{b}$	$\frac{\lambda_{(i,j)/D}(t)}{b}$	$\frac{\lambda_{(i,j)/E}(t)}{b}$	$\frac{\lambda_{(i,j)/F}(t)}{b}$	$\frac{\lambda_{(i,j)/G}(t)}{b}$
(A, B)	0.15	0.15	0.15	0.15	0.473	0.473	0.24	0.21
(A, C)	0.15	0	0.15	0	0.383	0.15	0.15	0.15
(A, D)	0.5	0	0.285	0.15	0	0.5	0.15	0.21
(A, E)	0.15	0.15	0.473	0.15	0.423	0.15	0.15	0.15
(A, F)	0.15	0.15	0.24	0.15	0.383	0.383	0.15	0.15
(A, G)	0.15	0.15	0.21	0.15	0.423	0.21	0.15	0.15
(B, C)	0.15	0.15	0.15	0.15	0.285	0.385	0.15	0.21
(B, D)	0.42	0.38	0.42	0.15	0.42	0.56	0.15	0.27
(B, E)	0.635	0.15	0	0.15	0.635	0	0.383	0.27
(B, F)	0.15	0.15	0.15	0.15	0.285	0.422	0.15	0.21
(B, G)	0.27	0.15	0.27	0.15	0.345	0.513	0.15	0.27
(C, D)	0.15	0.325	0.285	0.15	0.15	0.5	0.15	0.21
(C, E)	0.15	0.15	0.473	0.15	0.5	0.15	0.325	0.21
(C, F)	0.15	0.15	0.15	0	0.15	0.383	0	0.15
(C, G)	0.15	0.15	0.21	0.15	0.21	0.21	0.15	0.15
(D, E)	0.85	0.19	0.32	0.383	0.85	0.85	0.19	0.27
(D, F)	0.15	0.38	0.285	0.15	0.15	0.423	0.15	0.21
(D, G)	0.27	0.38	0.345	0.15	0.27	0.56	0.15	0.27
(E, F)	0.5	0.15	0.473	0.15	0.423	0	0	0.21
(E, G)	0.27	0.15	0.513	0.15	0.56	0.27	0.383	0.27
(F, G)	0.15	0.15	0.21	0.15	0.423	0.42	0.15	0.15
Sum	5.665	3.755	5.762	3.083	7.741	7.512	3.621	4.35

can significantly impact travel, quantum centrality offers a flexible and insightful approach to managing transportation systems efficiently.

Table 8 Centrality of all vertices of the quantum graph of Fig. 2

Vertex	$M_{k(t)}$	Centrality
A	$\frac{3.755}{5.665} \times 0.15 = 0.663 \times 0.15 = 0.0995$	$\min\{1, 0.0995\} = 0.0995$
B	$\frac{5.762}{5.665} \times 0.42 = 1.017 \times 0.42 = 0.4271$	$\min\{1, 0.4271\} = 0.4271$
C	$\frac{3.083}{5.665} \times 0.15 = 0.544 \times 0.15 = 0.0816$	$\min\{1, 0.0816\} = 0.0816$
D	$\frac{7.741}{5.665} \times 0.85 = 1.366 \times 0.85 = 1.161$	$\min\{1, 1.161\} = 1.0$
E	$\frac{7.512}{5.665} \times 0.85 = 1.326 \times 0.85 = 1.127$	$\min\{1, 1.127\} = 1.0$
F	$\frac{3.621}{5.665} \times 0.15 = 0.639 \times 0.15 = 0.0959$	$\min\{1, 0.0959\} = 0.0959$
G	$\frac{4.35}{5.665} \times 0.27 = 0.768 \times 0.27 = 0.207$	$\min\{1, 0.207\} = 0.207$

5 Conclusions

In this paper, explored the concept of centrality within quantum networks, where dynamic value in route reliability and node strength play a significant role. Traditional centrality measures fail to adequately capture the complexities and variability of real-world transportation systems, but quantum graph centrality extends these models by incorporating dynamic values that reflect the dynamic nature of connections. These measures help in identifying critical points in the network for route optimization, traffic flow management, infrastructure investment, and improving resilience during disruptions. By using quantum theory to handle uncertainty, transportation planners can better account for variations in traffic conditions, route reliability, and infrastructure strength. This leads to more flexible transportation systems capable of adapting to real-world challenges. Future research may delve further into applying quantum centrality across different types of transportation networks and integrating dynamic updates based on real-time data, enhancing the overall planning and management of these essential networks.

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Link Prediction by Quantum Graphs and Its Application



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Abstract Link prediction is a fundamental problem in network science, aiming to forecast missing and future connections within a graph. Traditional approaches on classical graph theory and often struggle with complex networks. Quantum graphs, incorporating quantum features such as superposition and entanglement, offer a more powerful framework for link prediction. By using quantum concept to represent graph edges and vertices. This study explores the application of quantum graphs to link prediction, focusing on how entangled edges and dynamic weights enhance predictive capabilities. Additionally, we discuss the practical implementation of quantum link prediction in real-world networks.

Keywords Quantum graph · Centrality · Dynamic graph · Dynamic link prediction

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1 Introduction

The quantum theory, like superposition and entanglement, are included in quantum graphs, which extend classical graph theory into the quantum world. This chapter explains the basic ideas of quantum graphs, focusing on changing (or “dynamical”) points (vertices) and connected, entangled links (edges). Quantum graphs give a more accurate way to understand quantum events by using these entangled links and changing weights. This study proposes a model that imitates the dynamic properties of real quantum systems, filling a gap in how these changing points and entangled links can be applied. These systems are used in many fields, such as physics, biology, economics, and engineering. In today’s fast-evolving digital landscape, communication networks are becoming increasingly complex, with vast amounts of data flowing through highly interconnected and dynamic systems. Traditional graph models often fall short in capturing the nuanced, multi-layered interactions and rapid fluctuations inherent in these networks. Quantum graphs, with their foundation in quantum mechanics, introduce a new approach to network modeling by leveraging principles like “superposition” and “entanglement”. These quantum features allow for more flexible and precise representations of communication pathways, capturing the probabilistic nature of network connections and supporting the simultaneous evaluation of multiple states.

Link prediction is fundamental in network analysis, aiding in identifying future or missing connections, which has applications across social networks, biological networks, and recommendation systems. Traditional link prediction methods are depends on patterns from historical data but often struggle with complex, uncertain relationships and multi-layered interactions. Quantum graph theory provides an innovative approach by using principles like superposition and entanglement to represent multiple potential relationships between nodes simultaneously. This quantum framework captures uncertainty more effectively than classical methods, allowing for the prediction of connections that may not be obvious or even visible in the current state of the network. Applying quantum based link prediction can uncover deeper patterns in network connectivity, such as hidden or probabilistic links, and improve the accuracy of predictions in rapidly evolving networks. This advancement is particularly valuable for dynamic networks, such as social media, where user behaviors and connections shift frequently, as well as in biology and finance, where understanding underlying connections can reveal critical insights for innovation and decision-making.

1.1 Literature Review

In 1965, Zadeh presented the significant notion of fuzzy sets (FS) in order to address the problems that were being discussed. Fuzzy sets (FS) [33] have become a popular approach for incorporating fuzzy information as part of an expert’s evaluative measures. For example, in [7] studied fuzzy variable precision rough sets, in [8]

introduced hesitant bipolar-valued fuzzy sets and bipolar-valued hesitant fuzzy sets, and studied multi-granulation bipolar-valued fuzzy probabilistic rough sets and their corresponding three-way decisions. The decision-theoretic rough sets and their applications are studied in [10–12]. However, as noted by Mandal et al. [13], experts tend to express their opinions more reliably using linguistic information, known as linguistic sets (LS), rather than fuzzy information. In the LS concept was extended to the Pythagorean linguistic set (PLS), allowing experts to express both their confidence and doubt levels when providing judgments.

The basis for the creation of quantum graphs has been laid by earlier efforts. As an example, Kottos and Smilansky [25] looked into the spectral characteristics of graphs with chaos and showed that graphs might be utilized to examine wave dynamics and quantum chaos in complicated systems. The promise of quantum graphs in quantum computing was highlighted more recently by Razavinia and Haghightadoost [26], who offered technical structures pertaining to quantum graphs and quantum computing concepts. A thorough overview of quantum graph computing and learning was given by Tang et al. [27], who emphasized the benefits of using quantum algorithms to solve graph-related issues [24]. Samanta and Allahviranloo [28] introduced a the quantum graph by the concept quantum theory.

The previous research in fuzzy set theory and its applications plays a significant role in addressing uncertainty and complexity in social networks and group decision-making processes. The methods developed in this area offer valuable insights into how fuzzy logic can be used to model and analyze systems where imprecision and ambiguity are prevalent.

Fuzzy set theory is particularly well-suited to modeling the uncertainty inherent in social networks, where relationships and interactions often involve incomplete or vague information. Traditional mathematical models struggle to capture this uncertainty, but fuzzy approaches provide a more nuanced framework for analyzing such dynamics. For instance, the development of fuzzy fractional differential equations using fuzzy Laplace transforms is essential for modeling systems where the influence of individuals or groups evolves fractionally over time under uncertainty [1].

The introduction of predictor-corrector methods for solving fuzzy differential equations also has direct applications in the study of evolving opinions and preferences within social groups [2]. These methods allow for a more accurate simulation of how consensus forms over time, particularly in environments where group dynamics are shaped by uncertainty and imprecision.

The development of methods for ranking fuzzy numbers using weighted distance [3] proves useful in group decision-making processes, where individuals often express preferences that are not crisply defined. The ability to rank such imprecise inputs is crucial in facilitating collective decisions, as it allows for the systematic comparison and prioritization of uncertain information.

Additionally, the use of fuzzy heat equations via the Adomian decomposition method [4] is highly relevant to understanding how influence or information spreads through social networks. This approach models the flow of ideas or behaviors in an uncertain environment, providing a framework for analyzing the propagation of influence within a network.

The work on fully fuzzy linear systems and the signed decomposition method [5] is crucial for modeling the complex interrelations within social networks, where interactions can be represented as linear systems with fuzzy parameters. This framework supports a more realistic analysis of decision-making processes, particularly when dealing with multiple interdependent and uncertain factors.

Fuzzy linear programming methods [6] are directly applicable to optimization problems in group decision-making, where the objectives and constraints are often imprecise. These techniques help navigate complex decision-making environments within social networks, where individual preferences and goals may conflict or remain undefined.

Mahapatra et al. [14] offers a new and improved definition, introducing the generalized neutrosophic planar graph (GNPG) by the concept of neutrosophic graph [32, 34, 35]. Here, planarity is evaluated more precisely by considering the true, false, and indeterminacy values for the degree of planarity, which gives a clearer measure of the graph's overall planarity. Link prediction is a crucial task in social networks, with various methods available to predict connections. The RSM index [18] is one such method, useful in fuzzy environments, but it lacks consideration for falsity and indeterminacy value common in uncertain settings in modified RSM index [15]. A COVID-19 infection example is provided to demonstrate how directed fuzzy graph colouring can model transmission influence and control by Mathapatra et al. [16]. A new concept of the chromatic number for neutrosophic graphs [37, 38] and discusses the differences between fuzzy colouring in fuzzy graphs [36] and colouring in NG [17, 19, 20]. Identifying central nodes in social networks is essential for tasks like business growth, news dissemination, and advertising. Traditional centrality methods focus on a node's direct reach but often overlook uncertainty in relationships. A new centrality measure is introduced for neutrosophic graphs and linguistic graph [21–23, 31], which includes self-weight or individual merit within the network. Kalampakas et al. [29, 30] introduced a syntactic recognition theory for graphs with fuzzy edges and nodes, using the algebraic structure of magmoids.

2 Preliminaries

Definition 1 (*Quantum Graphs*) The mathematical definition of a quantum graph with dynamic vertices and entangled edges is as

1. Vertices with Dynamic Weights (dweight): The dynamic weight $dweight_i(t)$ of each vertex v_i varies with time. A real number between 0 and 1 represents the weight:

$$0 \leq dweight_i(t) \leq 1$$

The weight exhibits a certain pattern, such as a sinusoidal function, as stated below.

$$dweight_i(t) = \frac{1}{2}(1 + \sin(\omega_i t + \phi_i))$$

2. Entanglement and Edges: There is an option for edges e_{ij} between vertices v_i and v_j to be entangled or not. A correlation function $g_{ij}(t)$ and $0 \leq g_{ij}(t) \leq 1$, which depicts the relationship between the weights of the connected vertices, is used to represent an edge if it is entangled.

$$g_{ij}(t) = h(dweight_i(t), dweight_j(t))$$

where h is a function defining the nature of the entanglement or correlation. 3. Graph Structure: The collection of vertices V , and the set of edges E are the components that make up the graph G , which is expressed as $G = (V, E)$. The graph's adjacency matrix $A(t)$ is described as follows:

$$\begin{aligned} A_{ij}(t) &= g_{ij}(t) \text{ if } e_{ij} \text{ exists and is entangled} \\ &= 0, \text{ otherwise} \end{aligned}$$

3 Link Prediction by Quantum Graph

Link prediction is a critical problem in network science, where the goal is to predict the likelihood of the existence of a connection (link) between two nodes in a graph based on the current structure of the network. In quantum graphs, which extend classical graph theory to the quantum domain by incorporating quantum features like superposition and entanglement, link prediction becomes even more powerful and complex.

One key approach to predicting links in a quantum graph is by using centrality measures. Centrality measures quantify the importance or influence of individual nodes within the graph. By leveraging these measures, one can predict new or missing links in the quantum graph, aiding in optimizing network structure and understanding its dynamics.

Now, the degree centrality of a node of a quantum graph defined as follows.

Definition 2 $G = (V, E)$ be a quantum graph. A correlation function $g_{ij}(t)$ and $0 \leq g_{ij}(t) \leq 1$, which depicts the relationship between the weights of the connected vertices, is used to represent an edge if it is entangled.

$$g_{ij}(t) = h(dweight_i(t), dweight_j(t))$$

where h is a function defining the nature of the entanglement or correlation. Then the degree centrality of a vertex is denoted by $D(t)$ and defined as

$$D(t) = \sum_{r=1}^n \frac{g_{ij}^r(t)}{n}$$

where the vertex connected with n number of vertices of this quantum graph G .

Table 1 dynamic values of the quantum multigraph of Fig. 1

Vertex	Dynamic value
A	$0.5 \times (1 + \sin(\frac{\pi}{2} \times 4 + \frac{\pi}{4})) = 0.85$
B	$0.5 \times (1 + \sin(\frac{\pi}{4} \times 4 + \frac{\pi}{4})) = 0.15$
C	$0.5 \times (1 + \sin(\frac{\pi}{3} \times 4 + \frac{\pi}{4})) = 0.02$
D	$0.5 \times (1 + \sin(\frac{3\pi}{4} \times 4 + \frac{\pi}{4})) = 0.15$

Now, the link prediction between two vertices of a quantum graph defined as follows-

Definition 3 $G = (V, E)$ be a quantum graph. A correlation function $g_{ij}(t)$ and $0 \leq g_{ij}(t) \leq 1$, which depicts the relationship between the weights of the connected vertices, is used to represent an edge if it is entangled.

$$g_{ij}(t) = h(dweight_i(t), dweight_j(t))$$

where h is a function defining the nature of the entanglement or correlation. Then the link prediction between the vertices i and j is denoted by $L_{ij}(t)$ and defined by

$$L_{ij}(t) = \sum_{r=1}^n \frac{D_{ij}^r(t)}{n}$$

where, $D_{ij}^r(t)$ are the degree centrality of n number of common neighbours between vertices i and j .

Now, the total link prediction between the vertices i and j is

$$S_{ij} = \int_{t=0}^T L_{ij}(t) dt$$

The integral captures the cumulative importance of link prediction between the vertices i and j across the entire time period T , showing how the importance of each node changes and adapts as the network structure shifts over time.

Fig. 1 A quantum graph

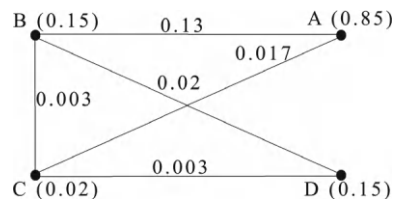


Table 2 Dynamic correlation values of the quantum multi-graph of Fig. 1

Edge	Dynamic correlation value
(A, B)	$0.85 \times 0.15 = 0.13$
(A, C)	$0.85 \times 0.02 = 0.017$
(B, C)	$0.15 \times 0.02 = 0.003$
(B, D)	$0.15 \times 0.15 = 0.02$
(D, C)	$0.15 \times 0.02 = 0.003$

Example 1 We consider, the value of ω are in the range $[0, 2\pi]$ and the value of ϕ is $\phi = \frac{\pi}{4}$. Also, consider the value of t is $t = 4$ (here we consider a particular time) and the correlation value $h(a, b) = a \times b$ with the value of $\omega_1 = \frac{\pi}{2}$, $\omega_2 = \frac{\pi}{4}$, $\omega_3 = \frac{\pi}{3}$, $\omega_4 = \frac{3\pi}{4}$.

We consider, $V = \{A, B, C, D\}$ denote vertex set, and $G = (V, E)$ represent a quantum graph. Now, the dynamic value of all vertices A, B, C, D with the ω value are $\omega_1 = \frac{\pi}{2}$, $\omega_2 = \frac{\pi}{4}$, $\omega_3 = \frac{3\pi}{2}$, $\omega_4 = \frac{3\pi}{4}$ respectively. The quantum graph depicted in Fig. 1 is considered. The dynamic value of all vertices in this quantum graph are detailed in Table 1. The dynamic correlation value of the quantum graph are presented in Table 2. Now, the centrality of the vertex B at $t = 4$ is $D_B(4) = \frac{0.13+0.02+0.003}{3} = \frac{0.153}{3} = 0.051$ and C is $D_C(4) = \frac{0.017+0.003+0.003}{3} = \frac{0.023}{3} = 0.008$. So, the link prediction between the vertices A and D is $L_{AD} = \frac{0.051+0.008}{2} = \frac{0.059}{2} = 0.0295$.

3.1 Algorithm to Calculate the Link Prediction Between Two Vertices of a Quantum Graph

Input:- A quantum graph $G = (V, E)$.

Output:- Link prediction between two non connected vertices of G .

Step1:- All the vertices of the quantum graph G are labelled as $v^1, v^2, \dots, v^p$.

Step 2:- Calculate the degree centrality of the vertex v^1 the formula

$$D_{v^1}(t) = \sum_{r=1}^n \frac{g_{ij}^r(t)}{n}$$

where the vertex v^1 is connected with n number of vertices.

Step 3:- Continue Step 2 for all vertices of the quantum graph G .

Step 4:- Calculate the link prediction between two not connected vertices by the formula $L_{ij}(t) = \sum_{r=1}^n \frac{D_{ij}^r(t)}{n}$.

Step 5:- Calculate total link prediction between the vertices i and j is $S_{ij} = \int_{t=0}^T L_{ij}(t) dt$.

Step 6:- Continue Step 4 and Step 5 for all non connected vertices of the quantum graph G .

4 Application

In network science, link prediction aims to predict future or missing links between nodes in a network based on current data. A quantum graph extends fuzzy graph theory by incorporating uncertainty. Link prediction in quantum graphs can play a crucial role in various applications where uncertainty exists, such as social networks. In social networks, connections between people are not always binary (either they exist or they don't). The strength of relationships between individuals can vary, represented by quantum values (dynamic correlation values). By analyzing existing connections and the strength of these dynamic correlation values, the system can predict future friendships or interactions between users. Predicting who is likely to influence or connect with whom based on the dynamic influence of users. For example, if person A and person B have a mutual friend C with high dynamic connection strengths to both A and B, it's likely that A and B may also form a connection in the future. In recommendation systems (like Netflix or Amazon), quantum graphs can model user-item interactions, where the dynamic strength of the edge between a user and an item indicates the likelihood or degree of interest the user has in that item. Link prediction in quantum graphs can recommend new products, services, or content to users based on their past interactions.

In a quantum recommendation graph, nodes represent users and items (movies, books, etc.), and the edges between them represent the degree of interest or preference, expressed as a dynamic membership value. Link prediction can predict new items that users are likely to be interested in by analyzing dynamic relationships in the network. We consider, the value of ω are in the range $[0, 2\pi]$ and the value of ϕ is $\phi = \frac{\pi}{4}$. Also, consider the value of t is $t = 6$ and the correlation value $h(a, b) = a \times b$ with the value of $\omega_1 = \frac{\pi}{2}, \omega_2 = \frac{4\pi}{5}, \omega_3 = \frac{3\pi}{2}, \omega_4 = \frac{3\pi}{4}, \omega_5 = \frac{\pi}{3}, \omega_6 = \frac{\pi}{4}, \omega_7 = \frac{2\pi}{5}$.

We consider, $V = \{P_1, P_2, P_3, P_4, M_1, M_2, M_3\}$ denote vertex set, and $G = (V, E)$ represent a quantum graph. Here, P_1, P_2, P_3, P_4 are four friends of a social networks and M_1, M_2, M_3 are three movies. Now, the dynamic value of all vertices $P_1, P_2, P_3, P_4, M_1, M_2, M_3$ with the ω value are $\omega_1 = \frac{\pi}{2}, \omega_2 = \frac{4\pi}{5}, \omega_3 = \frac{3\pi}{2}, \omega_4 = \frac{3\pi}{4}, \omega_5 = \frac{\pi}{3}, \omega_6 = \frac{\pi}{4}, \omega_7 = \frac{2\pi}{5}$ respectively. The quantum graph depicted in Fig. 2 is considered. The dynamic value of all vertices in this quantum graph are detailed in Table 3. The dynamic correlation value of the quantum graph are presented in Table 4. The strength of all intersecting edges of the quantum graph are presented in Table 5.

Now, the centrality of the vertex P_1 at $t = 6$ is $D_{P_1}(6) = \frac{0.128+0.063+0.023+0.128}{4} = \frac{0.342}{4} = 0.0855$. Similarly the centrality of all users has been shown in Table 5. So, the link prediction between the vertices P_1 and M_2 is $L_{P_1 M_2} = \frac{0.1942}{1} = 0.1942$. Similarly, link prediction of all predicted link of this quantum graph has been shown Table 6. So, the system will predict that P_1 will be recommended for watch the movie M_2 . Similarly, other online platforms like Netflix and Spotify use link prediction to recommend new movies, TV shows, or songs to users. By analyzing user behavior, such as viewing or listening history, these systems predict which new movies or songs users might enjoy. If a user has watched many action movies, link prediction algorithms can suggest other action movies that the user has not yet watched.

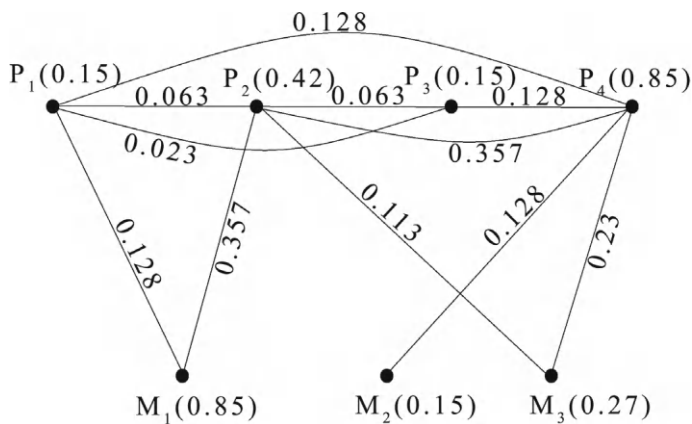


Fig. 2 A quantum graph of communication network

Table 3 Dynamic values of the quantum graph of Fig. 2

Vertex	Dynamic value
P_1	$0.5 \times (1 + \sin(\frac{\pi}{2} \times 6 + \frac{\pi}{4})) = 0.15$
P_2	$0.5 \times (1 + \sin(\frac{4\pi}{5} \times 6 + \frac{\pi}{4})) = 0.42$
P_3	$0.5 \times (1 + \sin(\frac{3\pi}{2} \times 6 + \frac{\pi}{4})) = 0.15$
P_4	$0.5 \times (1 + \sin(\frac{3\pi}{4} \times 6 + \frac{\pi}{4})) = 0.85$
M_1	$0.5 \times (1 + \sin(\frac{\pi}{3} \times 6 + \frac{\pi}{4})) = 0.85$
M_2	$0.5 \times (1 + \sin(\frac{\pi}{4} \times 6 + \frac{\pi}{4})) = 0.15$
M_3	$0.5 \times (1 + \sin(\frac{2\pi}{5} \times 6 + \frac{\pi}{4})) = 0.27$

Table 4 Dynamic correlation values of the quantum graph of Fig. 2

Edge	Dynamic correlation value
(P_1, P_2)	$0.15 \times 0.42 = 0.063$
(P_1, P_3)	$0.15 \times 0.15 = 0.023$
(P_1, P_4)	$0.15 \times 0.85 = 0.128$
(P_2, P_3)	$0.42 \times 0.15 = 0.063$
(P_2, P_4)	$0.42 \times 0.85 = 0.357$
(P_3, P_4)	$0.15 \times 0.85 = 0.128$
(P_1, M_1)	$0.15 \times 0.85 = 0.128$
(P_2, M_1)	$0.42 \times 0.85 = 0.357$
(P_2, M_3)	$0.42 \times 0.27 = 0.113$
(P_4, M_2)	$0.85 \times 0.15 = 0.128$
(P_4, M_3)	$0.85 \times 0.27 = 0.230$

Table 5 Centrality of vertices of the quantum graph of Fig. 2

Vertex	Centrality
P_1	$\frac{0.128+0.063+0.023+0.128}{4} = \frac{0.342}{4} = 0.0855$
P_2	$\frac{0.063+0.063+0.357+0.113+0.357}{5} = \frac{0.953}{5} = 0.1906$
P_3	$\frac{0.128+0.063+0.023}{3} = \frac{0.214}{3} = 0.0713$
P_4	$\frac{0.128+0.128+0.357+0.128+0.230}{5} = \frac{0.971}{5} = 0.1942$

Table 6 Score of link prediction of all predicted link of the quantum graph of Fig. 2

Predicted link	Score of link prediction
(P_1, M_2)	$\frac{0.1942}{1} = 0.1942$
(P_1, M_3)	$\frac{0.1906+0.1942}{2} = \frac{0.3848}{2} = 0.1924$
(P_2, M_2)	$\frac{1942}{1} = 1942$
(P_4, M_1)	$\frac{0.855+0.1906}{2} = \frac{0.2761}{2} = 0.1381$
(P_3, M_1)	$\frac{0.0855+0.1906}{2} = \frac{0.2761}{2} = 1381$
(P_3, M_2)	$\frac{0.1942}{1} = 0.1942$
(P_3, M_3)	$\frac{0.1906+0.1942}{2} = \frac{0.3848}{2} = 0.1924$

5 Conclusions

Link prediction in quantum graphs offers a cutting-edge approach to enhancing recommendation systems, especially in online platforms for movies and songs. By leveraging quantum graph features like superposition and entanglement, this method captures complex relationships between users and items, allowing for more accurate predictions of user preferences. In traditional recommendation systems, link prediction based on classical graph techniques to infer future interactions based on existing data. However, quantum graphs can provide deeper insights into user behavior by accounting for probabilistic states and dynamic interactions, representing both direct and indirect connections more effectively. This allows the system to better understand the subtle patterns in users’ evolving tastes, improving recommendation accuracy. When applied to online platforms such as Netflix or Spotify, quantum graph-based link prediction can recommend movies and songs that users are more likely to enjoy, even in cases where traditional methods struggle, such as with new users (cold start

problem) or rare items. Quantum link prediction enhances personalization, offering recommendations that are not only based on historical interactions but also consider the quantum-structured connections across the user-item graph, resulting in more nuanced and adaptive suggestions.

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Quantum Planar Graphs and Its Application



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Abstract This chapter introduces quantum planar graphs, which build on classical graph theory by adding key quantum features like superposition and entanglement. The focus is on how these quantum graphs have dynamic vertices and possibly entangled edges. By combining entangled edges and changing weights, quantum graphs become a better tool for studying quantum behavior. The study also explains the how quantum graphs can be used is in quantum planar graphs. Through theoretical analysis and practical application examples, this chapter demonstrates how quantum planar graphs can effectively identify critical nodes, manage data flow, and reduce congestion. Our findings reveal that this quantum-based approach has significant

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potential to enhance network resilience, optimize resource utilization, and support real-time adaptability in fast-evolving communication environments. These insights highlight the promise of quantum planar graphs as a next-generation framework for advanced communication network design and management.

Keywords Quantum graph · Planar graph · Dynamic graph

1 Introduction

The fundamental quantum concepts like superposition and entanglement are built into quantum graphs, which extend traditional graph theory into the quantum world. This chapter introduces the basics of quantum graphs, focusing on active vertices and connected edges that show entanglement. Quantum graphs provide a better way to understand quantum events by using these connected edges and dynamic weights. This study offers a model that simulates the changing nature of real quantum systems, bridging the gap in applying dynamic vertices and entangled edges. Many fields, such as physics, biology, economics, and engineering, use these systems. In today's fast-evolving digital landscape, communication networks are becoming increasingly complex, with vast amounts of data flowing through highly interconnected and dynamic systems. Traditional graph models often fall short in capturing the nuanced, multi-layered interactions and rapid fluctuations inherent in these networks. Quantum planar graphs, with their foundation in quantum mechanics, introduce a new approach to network modeling by leveraging principles like "superposition" and "entanglement". These quantum features allow for more flexible and precise representations of communication pathways, capturing the probabilistic nature of network connections and supporting the simultaneous evaluation of multiple states.

The quantum planar graphs in communication networks lies in their potential to address key challenges such as "network congestion", "information loss", and "optimization of data routes". By understanding and optimizing the structure and centrality of critical nodes within this quantum framework, network managers can achieve enhanced efficiency, improved resilience to failures, and better adaptability to changing demands. This advancement could be the way for faster, more reliable, and future-ready communication infrastructures, essential for supporting emerging technologies like IoT, 5G, and beyond.

1.1 Literature Review

In 1965, Zadeh presented the significant notion of fuzzy sets (FS) in order to address the problems that were being discussed. Fuzzy sets (FS) [32] have become a popular approach for incorporating fuzzy information as part of an expert's evaluative measures. For example, in [7] studied fuzzy variable precision rough sets, in [8]

introduced hesitant bipolar-valued fuzzy sets and bipolar-valued hesitant fuzzy sets, and studied multi-granulation bipolar-valued fuzzy probabilistic rough sets and their corresponding three-way decisions. The decision-theoretic rough sets and their applications are studied in [9–11]. However, as noted by Mandal et al. [12], experts tend to express their opinions more reliably using linguistic information, known as linguistic sets (LS), rather than fuzzy information. In the LS concept was extended to the Pythagorean linguistic set (PLS), allowing experts to express both their confidence and doubt levels when providing judgments.

The basis for the creation of quantum graphs has been laid by earlier efforts. As an example, Kottos and Smilansky [24] looked into the spectral characteristics of graphs with chaos and showed that graphs might be utilized to examine wave dynamics and quantum chaos in complicated systems. The promise of quantum graphs in quantum computing was highlighted more recently by Razavinia and Haghightadoost [25], who offered technical structures pertaining to quantum graphs and quantum computing concepts. A thorough overview of quantum graph computing and learning was given by Tang et al. [26], who emphasized the benefits of using quantum algorithms to solve graph-related issues [23]. Samanta and Allahviranloo [27] introduced a the quantum graph by the concept quantum theory.

The previous research in fuzzy set theory and its applications plays a significant role in addressing uncertainty and complexity in social networks and group decision-making processes. The methods developed in this area offer valuable insights into how fuzzy logic can be used to model and analyze systems where imprecision and ambiguity are prevalent.

Fuzzy set theory is particularly well-suited to modeling the uncertainty inherent in social networks, where relationships and interactions often involve incomplete or vague information. Traditional mathematical models struggle to capture this uncertainty, but fuzzy approaches provide a more nuanced framework for analyzing such dynamics. For instance, the development of fuzzy fractional differential equations using fuzzy Laplace transforms is essential for modeling systems where the influence of individuals or groups evolves fractionally over time under uncertainty [1].

The introduction of predictor-corrector methods for solving fuzzy differential equations also has direct applications in the study of evolving opinions and preferences within social groups [2]. These methods allow for a more accurate simulation of how consensus forms over time, particularly in environments where group dynamics are shaped by uncertainty and imprecision.

The development of methods for ranking fuzzy numbers using weighted distance [3] proves useful in group decision-making processes, where individuals often express preferences that are not crisply defined. The ability to rank such imprecise inputs is crucial in facilitating collective decisions, as it allows for the systematic comparison and prioritization of uncertain information.

Additionally, the use of fuzzy heat equations via the Adomian decomposition method [4] is highly relevant to understanding how influence or information spreads through social networks. This approach models the flow of ideas or behaviors in an uncertain environment, providing a framework for analyzing the propagation of influence within a network.

The work on fully fuzzy linear systems and the signed decomposition method [5] is crucial for modeling the complex interrelations within social networks, where interactions can be represented as linear systems with fuzzy parameters. This framework supports a more realistic analysis of decision-making processes, particularly when dealing with multiple interdependent and uncertain factors.

Fuzzy linear programming methods [6] are directly applicable to optimization problems in group decision-making, where the objectives and constraints are often imprecise. These techniques help navigate complex decision-making environments within social networks, where individual preferences and goals may conflict or remain undefined.

Mahapatra et al. [13] offers a new and improved definition, introducing the generalized neutrosophic planar graph (GNPG) by the concept of neutrosophic graph [31, 33, 34]. Here, planarity is evaluated more precisely by considering the true, false, and indeterminacy values for the degree of planarity, which gives a clearer measure of the graph's overall planarity. Link prediction is a crucial task in social networks, with various methods available to predict connections. The RSM index [17] is one such method, useful in fuzzy environments, but it lacks consideration for falsity and indeterminacy value common in uncertain settings in modified RSM index [14]. A COVID-19 infection example is provided to demonstrate how directed fuzzy graph colouring can model transmission influence and control by Mathapatra et al. [15]. A new concept of the chromatic number for neutrosophic graphs [36, 37] and discusses the differences between fuzzy colouring in fuzzy graphs [35] and colouring in NG [16, 18, 19]. Identifying central nodes in social networks is essential for tasks like business growth, news dissemination, and advertising. Traditional centrality methods focus on a node's direct reach but often overlook uncertainty in relationships. A new centrality measure is introduced for neutrosophic graphs and linguistic graph [20–22, 30], which includes self-weight or individual merit within the network. Kalampakas et al. [28, 29] introduced a syntactic recognition theory for graphs with fuzzy edges and nodes, using the algebraic structure of magmoids.

2 Preliminaries

Definition 1 (*Quantum Graphs*) The mathematical definition of a quantum graph with dynamic vertices and entangled edges is as

1. Vertices with Dynamic Weights (dweight): The dynamic weight $dweight_i(t)$ of each vertex v_i varies with time. A real number between 0 and 1 represents the weight:

$$0 \leq dweight_i(t) \leq 1$$

The weight exhibits a certain pattern, such as a sinusoidal function, as stated below.

$$dweight_i(t) = \frac{1}{2}(1 + \sin(\omega_i t + \phi_i))$$

2. Entanglement and Edges: There is an option for edges e_{ij} between vertices v_i and v_j to be entangled or not. A correlation function $g_{ij}(t)$ and $0 \leq g_{ij}(t) \leq 1$, which depicts the relationship between the weights of the connected vertices, is used to represent an edge if it is entangled.

$$g_{ij}(t) = h(dweight_i(t), dweight_j(t))$$

where h is a function defining the nature of the entanglement or correlation. 3. Graph Structure: The collection of vertices V , and the set of edges E are the components that make up the graph G , which is expressed as $G = (V, E)$. The graph's adjacency matrix $A(t)$ is described as follows:

$$\begin{aligned} A_{ij}(t) &= g_{ij}(t) \text{ if } e_{ij} \text{ exists and is entangled} \\ &= 0, \text{ otherwise} \end{aligned}$$

3 Quantum Multigraph

Definition 2 $G = (V, E)$ be a quantum graph. A correlation function $g_{ij}^i(t)$ and $0 \leq g_{ij}^i(t) \leq 1$, which depicts the relationship between the weights of the connected vertices, is used to represent an edge if it is entangled.

$$g_{ij}^i(t) = h(dweight_i(t), dweight_j(t))$$

where h is a function defining the nature of the entanglement or correlation. Thus, G is said as quantum multigraph.

Example 1 We consider, the value of ω are in the range $[0, 2\pi]$ and the value of ϕ is $\phi = \frac{\pi}{4}$. Also, consider the value of t is $t = 4$ and the correlation value $h(a, b) = a \times b$ with the value of $\omega_1 = \frac{\pi}{2}, \omega_2 = \frac{\pi}{4}, \omega_3 = \frac{\pi}{3}, \omega_4 = \frac{3\pi}{4}$.

We consider, $V = \{A, B, C, D\}$ denote vertex set, and $G = (V, E)$ represent a quantum multigraph. Now, the dynamic value of all vertices A, B, C, D with the ω value are $\omega_1 = \frac{\pi}{2}, \omega_2 = \frac{\pi}{4}, \omega_3 = \frac{3\pi}{2}, \omega_4 = \frac{3\pi}{4}$ respectively. The quantum multigraph depicted in Fig. 1 is considered. The dynamic value of all vertices in this quantum multigraph are detailed in Table 1. The dynamic correlation value of the quantum multigraph are presented in Table 2.

3.1 Strength of an Edge of a Quantum Graph

Definition 3 $G = (V, E)$ be a quantum graph. A correlation function $g_{ij}(t)$ and $0 \leq g_{ij}(t) \leq 1$, which depicts the relationship between the weights of the connected

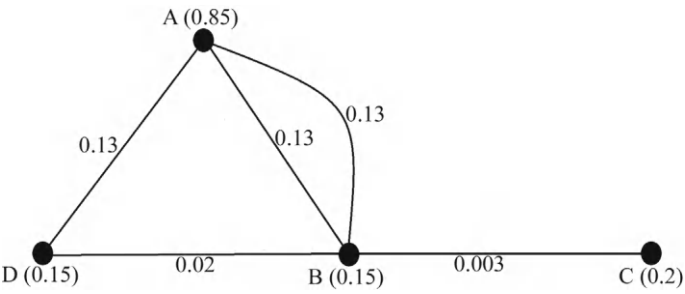


Fig. 1 A quantum multigraph

Table 1 Dynamic values of the quantum multigraph of Fig. 1

Vertex	Dynamic value
A	$0.5 \times (1 + \sin(\frac{\pi}{2} \times 4 + \frac{\pi}{4})) = 0.85$
B	$0.5 \times (1 + \sin(\frac{\pi}{4} \times 4 + \frac{\pi}{4})) = 0.15$
C	$0.5 \times (1 + \sin(\frac{\pi}{3} \times 4 + \frac{\pi}{4})) = 0.02$
D	$0.5 \times (1 + \sin(\frac{3\pi}{4} \times 4 + \frac{\pi}{4})) = 0.15$

Table 2 Dynamic correlation values of the quantum multigraph of Fig. 1

Edge	Dynamic correlation value
(A, B)	$0.85 \times 0.15 = 0.13$
(A, B)	$0.85 \times 0.15 = 0.13$
(A, D)	$0.85 \times 0.15 = 0.13$
(B, D)	$0.15 \times 0.15 = 0.02$
(B, C)	$0.15 \times 0.02 = 0.003$

vertices, is used to represent an edge if it is entangled and $g_{ij}(t) = h(dweight_i(t), dweight_j(t))$, where h is a function defining the nature of the entanglement or correlation.

Then the strength of an edge is denoted by $S_{(i,j)}(t)$ and

$$S_{(i,j)}(t) = \frac{g_{ij}(t)}{\min\{dweight_i(t), dweight_j(t)\}}$$

Example 2 We consider, $V = \{A, B, C, D\}$ denote vertex set, and $G = (V, E)$ represent a quantum multigraph depicted in Fig. 1. The strength of all edges of the quantum multigraph are presented in Table 3.

Table 3 Strength of all edges of the quantum multigraph of Fig. 1

Edge	Strength
(A, B)	$\frac{0.13}{0.15} = 0.87$
(A, B)	$\frac{0.13}{0.15} = 0.87$
(A, D)	$\frac{0.13}{0.15} = 0.87$
(B, D)	$\frac{0.02}{0.15} = 0.13$
(B, C)	$\frac{0.003}{0.02} = 0.15$

4 Quantum Planar Graphs

A quantum planar graph is a concept that merges ideas from graph theory and quantum theory. In graph theory, a planar graph is one that can be embedded in the plane, meaning it can be drawn on a flat surface without any of its edges crossing. A quantum planar graph could refer to a quantum network modeled by such a planar structure. Now the quantum planar graph defined as follows.

Now, the score of intersecting value between two edges are defined below.

Definition 4 $G = (V, E)$ be a quantum graph. A correlation function $g_{ij}(t)$ and $0 \leq g_{ij}(t) \leq 1$, which depicts the relationship between the weights of the connected vertices, is used to represent an edge if it is entangled and $g_{ij}(t) = h(dweight_i(t), dweight_j(t))$, where h is a function defining the nature of the entanglement or correlation.

Then the score of intersecting value between two edges $g_{ij}(t)$ and $g_{kl}(t)$ of quantum graph is denoted by $I(t)$ and

$$I(t) = \frac{g_{ij}(t) + g_{kl}(t)}{2}.$$

Now, the score of planarity value of a quantum graph is defined below.

Definition 5 $G = (V, E)$ be a quantum graph. Let G contains n number of points of intersections and the intersection values be $I^i(t)$, $i = 1, 2, \dots, n$. The degree of planarity of this quantum graph is denoted by $P(t)$ and is defined by

$$P(t) = \frac{1}{1 + \{I^1(t) + I^2(t) + \dots + I^n(t)\}}.$$

Now, the score of planarity value of a quantum graph

$$P = \int_{t=0}^T P(t) dt$$

The integral captures the cumulative importance of the intersection points i.e. the planarity value P across the entire time period T , showing how the importance of each intersection of edges changes and adapts as the network structure shifts over time.

4.1 Algorithm to Calculate the Score of Planarity Value of a Quantum Graph

Input:- A quantum graph $G = (V, E)$.

Output:- Score of planarity of G .

Step 1:- Calculate the strength of an edge by the formula

$$S_{(i,j)}(t) = \frac{g_{ij}(t)}{\min\{dweight_i(t), dweight_j(t)\}}$$

Step 2:- All the n number of intersecting points are labelled as $I^1, I^2, \dots, I^n$.

Step 3:- Calculate the score of intersecting value by the formula.

Step 4:- Calculate the degree of planarity of the graph

$$P(t) = \frac{1}{1 + \{I^1(t) + I^2(t) + \dots + I^n(t)\}}.$$

Step 5:- Calculate the score of planarity of the graph

$$P = \int_{t=0}^T P(t)dt$$

Example 3 We consider, the value of ω are in the range $[0, 2\pi]$ and the value of ϕ is $\phi = \frac{\pi}{4}$. Also, consider the value of t is $t = 5$ (here we consider a particular time) and the correlation value $h(a, b) = a \times b$ with the value of $\omega_1 = \frac{\pi}{2}, \omega_2 = \frac{\pi}{5}, \omega_3 = \frac{3\pi}{2}, \omega_4 = \frac{3\pi}{4}, \omega_5 = \frac{\pi}{3}$.

We consider, $V = \{A, B, C, D, E\}$ denote vertex set, and $G = (V, E)$ represent a quantum graph. Now, the dynamic value of all vertices A, B, C, D, E with the ω value are $\omega_1 = \frac{\pi}{2}, \omega_2 = \frac{\pi}{5}, \omega_3 = \frac{3\pi}{2}, \omega_4 = \frac{3\pi}{4}, \omega_5 = \frac{\pi}{3}$ respectively. The quantum graph depicted in Fig.2 is considered. The dynamic value of all vertices in this quantum graph are detailed in Table 4. The dynamic correlation value of the quantum graph are presented in Table 5. The strength of all intersecting edges of the quantum graph are presented in Table 6. Then the score of intersecting value at the points I^1 and I^2 of quantum graph are $I^1(5) = \frac{0.84+0.47}{2} = \frac{1.31}{2} = 0.655$ and $I^2(5) = \frac{0.33+0.47}{2} = \frac{0.8}{2} = 0.4$. So, the degree of planarity of the quantum graph $P = \frac{1}{1+\{I^1(t)+I^2(t)\}} = \frac{1}{1+0.655+0.4} = \frac{1}{2.055} = 0.49$.

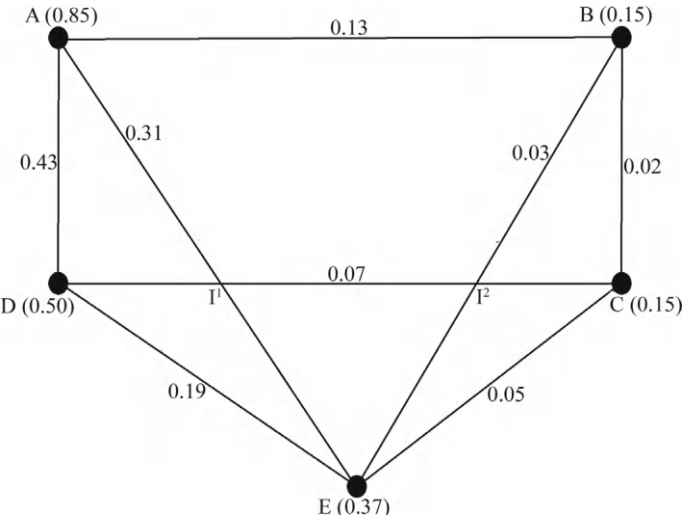


Fig. 2 A quantum graph

Table 4 Dynamic values of the quantum graph of Fig. 2

Vertex	Dynamic value
A	$0.5 \times (1 + \sin(\frac{\pi}{2} \times 5 + \frac{\pi}{4})) = 0.85$
B	$0.5 \times (1 + \sin(\frac{\pi}{3} \times 5 + \frac{\pi}{4})) = 0.15$
C	$0.5 \times (1 + \sin(\frac{3\pi}{2} \times 5 + \frac{\pi}{4})) = 0.15$
D	$0.5 \times (1 + \sin(\frac{3\pi}{4} \times 5 + \frac{\pi}{4})) = 0.50$
E	$0.5 \times (1 + \sin(\frac{\pi}{3} \times 5 + \frac{\pi}{4})) = 0.37$

Table 5 Dynamic correlation values of the quantum graph of Fig. 2

Edge	Dynamic correlation value
(A, B)	$0.85 \times 0.15 = 0.13$
(A, D)	$0.85 \times 0.50 = 0.43$
(A, E)	$0.85 \times 0.37 = 0.31$
(B, C)	$0.15 \times 0.15 = 0.02$
(B, E)	$0.15 \times 0.37 = 0.05$
(C, D)	$0.15 \times 0.50 = 0.07$
(C, E)	$0.15 \times 0.37 = 0.05$
(D, E)	$0.50 \times 0.37 = 0.19$

Table 6 Strength of all intersecting edges of the quantum graph of Fig. 2

Edge	Strength
(A, E)	$\frac{0.31}{0.37} = 0.84$
(E, B)	$\frac{0.05}{0.15} = 0.33$
(C, D)	$\frac{0.07}{0.15} = 0.47$

Table 7 Dynamic values of the quantum graph of Fig. 3

Vertex	Dynamic value
A	$0.5 \times (1 + \sin(\frac{\pi}{2} \times 6 + \frac{\pi}{4})) = 0.15$
B	$0.5 \times (1 + \sin(\frac{4\pi}{5} \times 6 + \frac{\pi}{4})) = 0.42$
C	$0.5 \times (1 + \sin(\frac{3\pi}{2} \times 6 + \frac{\pi}{4})) = 0.15$
D	$0.5 \times (1 + \sin(\frac{3\pi}{4} \times 6 + \frac{\pi}{4})) = 0.85$
E	$0.5 \times (1 + \sin(\frac{\pi}{3} \times 6 + \frac{\pi}{4})) = 0.85$
F	$0.5 \times (1 + \sin(\frac{\pi}{4} \times 6 + \frac{\pi}{4})) = 0.15$

5 Application

Quantum planar graphs are a novel concept that combines the principles of quantum mechanics with classical planar graphs, where nodes and edges are arranged on a two-dimensional plane without any edges crossing. This structure has practical applications in quantum communication, where quantum planar graphs are used to optimize the transfer of information between nodes while minimizing interference. The entangled edges in these graphs can facilitate faster and more secure communication channels.

In sensor or communication networks, optimizing the transfer of information between nodes is essential for enhancing the performance, reliability, and efficiency of the network. Quantum planar graphs provide a robust framework to model and optimize information flow, especially when the connections (edges) between nodes (sensors or communication devices) are uncertain.

Problem Statement: Given a network of nodes (e.g., sensors or routers) connected by communication links, the goal is to optimize the transfer of information by identifying the most reliable paths, considering both distance and the quality (reliability) of connections. In real-world networks, communication links are not always perfect, and their reliability can be represented by quantum membership values.

Quantum Planar Graph Representation: Nodes (Vertices): Represent devices (e.g., sensors, routers) in the network. Edges: Represent the communication links between the devices. Quantum Weights: Each edge and vertex have a dynamic weight. A value close to 1 indicates a highly reliable link and sensor, while a value close to 0 indicates a weak or unreliable link.

We consider, the value of ω are in the range $[0, 2\pi]$ and the value of ϕ is $\phi = \frac{\pi}{4}$. Also, consider the value of t is $t = 6$ and the correlation value $h(a, b) = a \times b$ with the value of $\omega_1 = \frac{\pi}{2}$, $\omega_2 = \frac{4\pi}{5}$, $\omega_3 = \frac{3\pi}{2}$, $\omega_4 = \frac{3\pi}{4}$, $\omega_5 = \frac{\pi}{3}$, $\omega_6 = \frac{\pi}{4}$.

We consider, $V = \{A, B, C, D, E, F\}$ denote vertex set, and $G = (V, E)$ represent a quantum graph. Now, the dynamic value of all vertices A, B, C, D, E, F with the ω value are $\omega_1 = \frac{\pi}{2}$, $\omega_2 = \frac{4\pi}{5}$, $\omega_3 = \frac{3\pi}{2}$, $\omega_4 = \frac{3\pi}{4}$, $\omega_5 = \frac{\pi}{3}$, $\omega_6 = \frac{\pi}{4}$ respectively. The quantum graph depicted in Fig. 3 is considered. The dynamic value of all vertices in this quantum graph are detailed in Table 7. The dynamic correlation value of the quantum graph are presented in Table 8. The strength of all intersecting edges of the

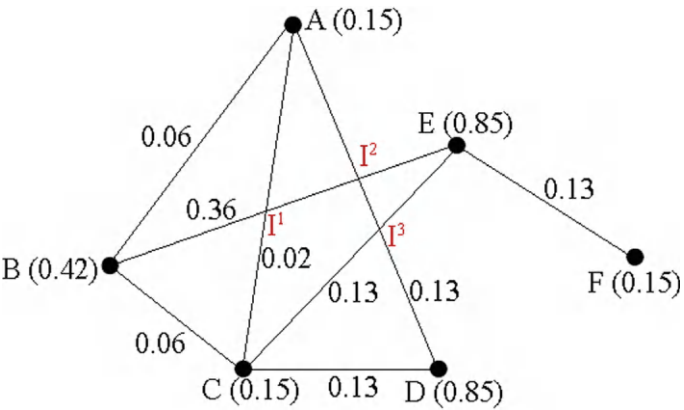


Fig. 3 A quantum graph of communication network

Table 8 Dynamic correlation values of the quantum graph of Fig. 3

Edge	Dynamic correlation value
(A, B)	$0.15 \times 0.42 = 0.06$
(A, D)	$0.15 \times 0.85 = 0.13$
(B, C)	$0.42 \times 0.15 = 0.06$
(B, E)	$0.42 \times 0.85 = 0.36$
(C, D)	$0.15 \times 0.85 = 0.13$
(C, E)	$0.15 \times 0.85 = 0.13$
(E, F)	$0.85 \times 0.15 = 0.13$
(A, C)	$0.15 \times 0.15 = 0.02$

Table 9 Strength of all intersecting edges of the quantum graph of Fig. 3

Edge	Strength
(A, C)	$\frac{0.02}{0.15} = 0.13$
(B, E)	$\frac{0.36}{0.42} = 0.86$
(C, E)	$\frac{0.13}{0.15} = 0.87$
(A, D)	$\frac{0.13}{0.15} = 0.87$

quantum graph are presented in Table 9. Then the score of intersecting value at the points I^1 , I^2 and I^3 of quantum graph are $I^1(6) = \frac{0.13+0.86}{2} = \frac{0.99}{2} = 0.49$, $I^2(6) = \frac{0.87+0.86}{2} = \frac{1.73}{2} = 0.87$ and $I^3(6) = \frac{0.87+0.87}{2} = \frac{1.74}{2} = 0.87$. So, the degree of planarity of the quantum graph $P = \frac{1}{1+[I^1(t)+I^2(t)+I^3(t)]} = \frac{1}{1+0.49+0.87+0.87} = \frac{1}{3.23} = 0.31$.

A high planarity value suggests that the graph is nearly planar or can be made planar with minimal modifications. A low planarity value means the graph is far from planar and has many edge crossings when drawn on a plane. This suggests the graph is more complex and tangled, with dense connections between the nodes. Here, the

planarity value is 0.31, which is compare to low. So, this network need some minimal modifications.

6 Conclusion

In summary, quantum planar graphs provide a powerful framework for analyzing and optimizing communication networks. By incorporating quantum principles like superposition and entanglement, quantum planar graphs can represent complex, multi-dimensional interactions more effectively than classical models. This approach allows for a more accurate assessment of centrality and connectivity, which are essential for identifying critical nodes, optimizing data flow, and reducing congestion in communication networks. Quantum planar graphs also support dynamic adaptability, making them ideal for applications where network demands and interactions fluctuate over time. This capability can significantly improve the performance and resilience of modern communication systems, highlighting the potential of quantum planar graphs as an innovative tool for advancing network design and management.

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Topological Indices in Quantum Graphs



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Abstract Graph theory plays a significant role in understanding the structure of complex systems in many areas, including chemistry, biology, computer science, and network theory. One key part of graph theory is the study of topological indices, which are numbers that describe important features of graph structures. In quantum mechanics, chemistry, and material science, quantum graphs have emerged as a significant topic as they demonstrate quantum systems using graphs. The collaboration of quantum mechanics and graph theory has been key to introducing quantum graphs, which offer unique ways to model and understand quantum systems. The ideas of traditional graph theory have been extended to develop quantum graphs with properties like superposition and entanglement. This chapter discusses quantum graphs and two of their main properties, such as superposition and entanglement. We have introduced some topological indices for quantum graphs and proved a few theorems. At last, we have concluded the book chapter with potential future works.

Keywords Quantum graphs · Topological indices · Zagreb index · Sombor index

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1 Introduction and Motivation

Quantum graphs have emerged as a prominent discipline of study in recent years. In quantum graphs, the vertices indicate quantum particles or quantum states, and the edges represent the paths through which quantum interaction occurs. The two most important components of quantum graphs are superposition and entanglement. Unlike classical graphs, in quantum graphs, a particle can exist in several states at a time, which is demonstrated through the superposition property. A renowned example is Schrödinger's cat, which is assumed to be alive and dead simultaneously. A quantum system may exist in any combination, such as $\gamma|0\rangle + \delta|1\rangle$, where γ and δ are complex numbers, provided that it can exist in the states $|0\rangle$ and $|1\rangle$. Another significant property of quantum graphs is entanglement. When two or more particles are deeply connected, and the state of one particle cannot be determined separately from the others, then this is referred to as entanglement. No matter how far away particles are, measuring the state of one particle indicates the state of the others when they are entangled. Experimental evidence for this phenomenon, which Einstein (Brang et al., 2024) termed "spooky action at a distance," has been gathered, and it is fundamental to the concept of quantum theory. We present topological indices in quantum graphs because we need to understand how a graph's structure influences its quantum behavior. Important quantum characteristics observed in graphs include superposition, entanglement, and wave function behavior, all of which interact in complex ways with the graph's structure.

The huge real-life application area of topological indices and quantum mechanics motivated us to study quantum graphs. Conventional topological indices are useful in classical graph theory, but their special quantum properties require modifications or expansions. We aim to present tools that can assess the structural characteristics of quantum systems and improve our understanding of their physical behaviors. To achieve this, we propose topological indices for quantum graphs. These indices offer a link between graph structure and quantum mechanics by assisting in the forecasting and analyzing quantum information flow and energy levels involved in quantum systems. These indices can also help with the modeling and design of quantum computing systems, quantum networks, and other advanced quantum technologies. In conclusion, developing topological indices in quantum graphs mostly translates traditional graph theory to the quantum domain. This offers a means of improving comprehension and forecasting the behavior of intricate quantum systems depending on their structure.

2 Literature Review

The notion of fuzzy graphs was first proposed by Rosenfeld (1975) in 1975. In 2015, Samanta and Pal (2015) came up with the idea of a fuzzy planar graph. In the next year, Samanta et al. (2016) presented a new concept called fuzzy coloring for fuzzy

graphs. In fuzzy graphs, the degree of membership of an edge must not exceed the lowest membership value of the two adjacent vertices. To address this, Samanta et al. (2016) proposed the idea of generalized fuzzy graphs in 2016, which explored properties like regularity and completeness. In 2020, the concept of fuzzy tolerance graphs was introduced by the authors of (Pramanik et al., 2017). Samanta and Sarkar (2020) also explored graph isomorphism in generalized fuzzy graphs. That same year, Akram et al. (2020) extended the concept of Pythagorean fuzzy planar graphs to more complex versions and compared them. In 2022, Mahapatra et al. (2022) suggested using edge coloring for neurotropic graphs to detect phishing websites. In 2024, Kalampakas et al. (2024) introduced a fuzzy logic model to study how external factors influence the spread of diseases. Abdullah et al. (2024) developed social networks based on linguistic Z-numbers and compared different centrality metric in 2024. Additionally, fuzzy graphs and sets have been applied in differential equations, numerical analysis, linear algebra and LPP by various authors from 2014 to 2023 in the articles (Allahviranloo & Taheri, 2009; Allahviranloo et al., 2011; Salahshour et al., 2012; Behzadi et al., 2014; Allahviranloo et al., 2015; Abbasi et al., 2015; Chehlabi & Allahviranloo, 2016; Alinezhad & Allahviranloo, 2017; Seifi et al., 2019; Akram et al., 2019, 2021; Alade et al., 2021; Saqib et al., 2021; De & Abdullah, 2023).

Quantum properties have led to the creation of quantum graphs, which build on classical graph theory by incorporating quantum elements. These graphs feature dynamic vertices and potentially entangled edges, allowing for better modeling of quantum systems (Cornish et al., 2024; Kottos & Smilansky, 1997). Over recent decades, the study of dynamic graphs has advanced significantly. Kottos and Smilansky's (1997) early research explored how graphs can reveal spectral properties related to quantum chaos and wave behavior in complex systems. Dynes et al. (2019) showcased a major development in quantum key distribution, presenting a high-speed network that operates using existing fiber optics and combines quantum key delivery with regular data traffic. Graphs have numerous applications, especially in quantum computing, where they help model qubit interactions and improve quantum algorithms and error correction. In quantum communication, they assist in understanding entanglement and information transfer in networks, leading to the development of secure communication protocols (Lloyd, 1996). One exciting application of graphs is in studying brain networks. The brain's complex and dynamic neural connections can be modelled using graphs, although traditional methods often miss quantum-like interactions. Quantum graphs can better represent these connections by including dynamic weights that reflect changes in neural activity over time, influenced by factors like synaptic input (Bennett & Brassard, 1985; Sporns, 2011). Additionally, the entangled edges in quantum graphs can show how different brain regions work together (Bassett & Bullmore, 2006). Neural connections are not fixed; they can strengthen or weaken, and their activity is often correlated (Bullmore & Sporns, 2009). This innovative approach may reveal new insights into brain function and connectivity, helping researchers understand changes during learning, memory formation, and the impact of neurological disorders like Alzheimer's disease and schizophrenia (Stam, 2014). By using quantum graphs, scientists can explore

the complex dynamics of neural interactions and improve diagnosis and treatment strategies.

The Zagreb Indices (1st and 2nd) are among the earliest topological indices, introduced by Gutman and Trinajstić (1972). Furtula and Gutman (2015) proposed the F-index, also known as the Forgotten Index, demonstrating that it improves the psycho-chemical relevance of the 1st Zagreb Index. Alameri et al. (2020) introduced a new topological measure called the Y-index and applied it to nano-torus and nano-tubes. Shirdel et al. (2013) developed the Hyper-Zagreb Index, while Gutman (2021) introduced the Sombor Index, highlighting its geometric basis. Kale and Mini-rani (2021) extended the 1st, 2nd, and Hyper-Zagreb indices to fuzzy graphs, and Hosamani (2023) applied the Sombor Index to fuzzy graphs. Islam and Pal (2023) developed the F-index for fuzzy graphs and explored its applications. Abdullah et al. (2024) proposed topological indices for linguistic generalized Z-graphs in 2024. In the same year, Abdullah et al. (2024) presented the Forgotten Index for bipolar fuzzy graphs and discussed applications. Some other significant related articles are (Kalathian et al., 2020; Lotfi et al., 2024; Poulik et al., 2022; Poulik & Ghorai, 2021).

3 Preliminaries

1st and 2nd Zagreb Indices for a graph $\Psi = (T, P)$ is defined, respectively, as,

$$I^{FZ}(\Psi) = \sum_{fk \in P} [\deg(f) + \deg(k)]; \text{ and}$$

$$I^{SZ}(\Psi) = \sum_{fk \in P} [\deg(f) \cdot \deg(k)].$$

The Sombor Index for the graph $\Psi = (T, P)$ is provided as,

$$I^{SO}(\Psi) = \sum_{fk \in P} \sqrt{\{\deg(f)\}^2 + \{\deg(k)\}^2}.$$

The F-index, Hyper-Zagreb Index and Y-index for a graph $\Psi = (T, P)$ are respectively defined as,

$$I^{FO}(\Psi) = \sum_{f \in T} \{\deg(f)\}^3;$$

$$I^{HZ}(\Psi) = \sum_{fk \in P} [\deg(f) + \deg(k)]^2; \text{ and}$$

Table 1 Some notations and their meanings

Notations	Meanings
Ψ	Graph
T	Vertex set of Ψ
P	Edge set of Ψ
Θ	Quantum graph
Υ	Vertex set of Θ
Ω	Edge set of Θ
I^{FZ}	1st Zagreb index
I^{SZ}	2nd Zagreb index
I^{SO}	Sombor index
I^{HZ}	Hyper-Zagreb index
I^{FO}	Forgotten index
I^Y	Y-index

$$I^Y(\Psi) = \sum_{fk \in P} [\{\deg(f)\}^3 + \{\deg(k)\}^3].$$

Notations used in this chapter and their meanings are provided in Table 1.

4 Quantum Graph and Topological Indices Defined on It

In a Quantum graph $\Theta = (\Upsilon, \Omega)$, the dynamic weight of a vertex λ_i in time t is given by, $w_t(\lambda_i) = \frac{a_i + b_i t}{c_i}$, (Samanta & Allahviranloo, 2025)

where a_i , b_i and c_i are constant chosen in such a way that $0 \leq w_t(\lambda_i) \leq 1$.

The time-independent weight of the vertex λ_i is defined as,

$$w(\lambda_i) = \frac{1}{t} \int_0^t w_t(\lambda_i) dt$$

The edge e_{ij} between the vertices λ_i and λ_j is either entangled or not. If the edge e_{ij} is entangled then the weight of the edge at time t is provided as,

$$w_t(e_{ij}) = w_t(\lambda_i) \cdot w_t(\lambda_j).$$

The adjacency matrix $A_t(\Theta)$ of a quantum graph $\Theta = (\Upsilon, \Omega)$ at time t is defined as,

$$A_t^{ij} = \begin{cases} w_t(e_{ij}), & \text{if } e_{ij} \text{ exists and is entangled} \\ 0, & \text{otherwise} \end{cases}.$$

The degree of λ_i at time t is given as, $d_t(\lambda_i) = \sum_j w_t(e_{ij})$.

The time-independent degree of the vertex λ_i is provided as,

$$d(\lambda_i) = \frac{1}{t} \int_0^t d_t(\lambda_i) dt.$$

Here, we introduce some topological indices for the quantum graph

$$\Theta = (\Upsilon, \Omega) :$$

$$I^{FZ}(\Theta) = \sum_{\lambda_i \lambda_j \in \Omega} [\{w(\lambda_i).d(\lambda_i)\} + \{w(\lambda_j).d(\lambda_j)\}];$$

$$I^{SZ}(\Theta) = \sum_{\lambda_i \lambda_j \in \Omega} [\{w(\lambda_i).d(\lambda_i)\} \cdot \{w(\lambda_j).d(\lambda_j)\}];$$

$$I^{SO}(\Theta) = \sum_{\lambda_i \lambda_j \in \Omega} \sqrt{\{w(\lambda_i).d(\lambda_i)\}^2 + \{w(\lambda_j).d(\lambda_j)\}^2};$$

$$I^{HZ}(\Theta) = \sum_{\lambda_i \lambda_j \in \Omega} [\{w(\lambda_i).d(\lambda_i)\} + \{w(\lambda_j).d(\lambda_j)\}]^2;$$

$$I^{FO}(\Theta) = \sum_{\lambda_i \in \Upsilon} \{w(\lambda_i).d(\lambda_i)\}^3; \text{ and}$$

$$I^Y(\Theta) = \sum_{\lambda_i \lambda_j \in \Omega} [\{w(\lambda_i).d(\lambda_i)\}^3 + \{w(\lambda_j).d(\lambda_j)\}^3].$$

Example Let us consider a Quantum graph $\Theta = (\Upsilon, \Omega)$, where the no. of vertices is 4 and the number of entangled edges is 5 at any time t . Let the vertices be $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and the edges be $e_{12}, e_{13}, e_{23}, e_{24}, e_{34}$.

Then, at time t , the adjacency matrix is,

$$A_t^{ij} = \begin{pmatrix} 0 & w_t(e_{12}) & w_t(e_{13}) & 0 \\ w_t(e_{21}) & 0 & w_t(e_{23}) & w_t(e_{24}) \\ w_t(e_{31}) & w_t(e_{32}) & 0 & w_t(e_{34}) \\ 0 & w_t(e_{42}) & w_t(e_{43}) & 0 \end{pmatrix}.$$

Let, t ranges from 1 to 10 with step size 0.05. We also assume that $a_1 = 0, b_1 = 1, c_1 = 10; a_2 = 10, b_2 = -1, c_2 = 10; a_3 = 5, b_3 = 0.5, c_3 = 10; a_4 = 3, b_4 = 0.5, c_4 = 10$.

At $t = 5$, let us calculate the dynamic weights of the vertices:

$$w_5(\lambda_1) = \frac{a_1 + 5b_1}{c_1} = 0.50;$$

$$w_5(\lambda_2) = \frac{a_2 + 5b_2}{c_2} = 0.50;$$

$$w_5(\lambda_3) = \frac{a_3 + 5b_3}{c_3} = 0.75;$$

$$w_5(\lambda_4) = \frac{a_4 + 5b_4}{c_4} = 0.55.$$

Now,

$$w_5(e_{12}) = 0.5 \times 0.5 = 0.25;$$

$$w_5(e_{13}) = 0.5 \times 0.75 = 0.38;$$

$$w_5(e_{23}) = 0.5 \times 0.75 = 0.38;$$

$$w_5(e_{24}) = 0.5 \times 0.55 = 0.28;$$

$$w_5(e_{34}) = 0.75 \times 0.55 = 0.41.$$

The adjacency matrix at $t = 5$ is given by,

$$A_5^{ij} = \begin{pmatrix} 0 & 0.25 & 0.38 & 0 \\ 0.25 & 0 & 0.38 & 0.28 \\ 0.38 & 0.38 & 0 & 0.41 \\ 0 & 0.28 & 0.41 & 0 \end{pmatrix}$$

The quantum graph at $t = 5$ is demonstrated in Fig. 1.

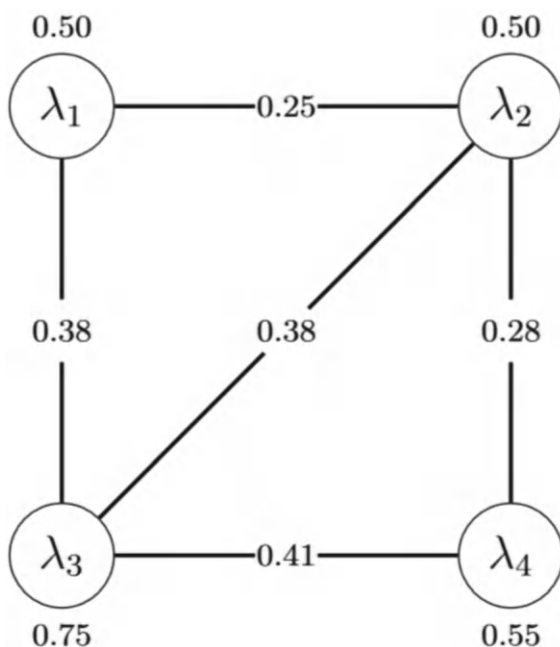
We can easily calculate the time-independent weights and degrees of the vertices of the quantum graph $\Theta = (\Upsilon, \Omega)$, and calculate the topological indices.

5 Some Theorems On Topological Indices On Quantum Graphs

Theorem 1 Let, $\Theta = (\Upsilon, \Omega)$ be a quantum graph having ν vertices. Then,

- (i) $I^{FZ}(\Theta) \leq \nu(\nu - 1)^2$;
- (ii) $I^{SZ}(\Theta) \leq \frac{\nu(\nu-1)^3}{2}$;

Fig. 1 The quantum graph at $t = 5$



- (iii) $I^{SO}(\Theta) \leq \frac{v(v-1)^2}{\sqrt{2}}$;
- (iv) $I^{HZ}(\Theta) \leq 2v(v-1)^3$;
- (v) $I^{FO}(\Theta) \leq v(v-1)^3$; and
- (vi) $I^Y(\Theta) \leq v(v-1)^4$.

Proof As the no. of vertices in Θ is v , the no. of edges cannot exceed $\frac{v(v-1)}{2}$, and a particular vertex can have at most $(v-1)$ incident edges. At any time t we have, $d_t(\lambda_i) = \sum_j w_t(e_{ij}) \leq (v-1).1 = (v-1), \forall \lambda_i \in \Upsilon$.

$$d_t(\lambda_i) \leq (v-1)$$

$$\Rightarrow \int_0^t d_t(\lambda_i) dt \leq \int_0^t (v-1) dt = (v-1)t$$

$$\Rightarrow \frac{1}{t} \int_0^t d_t(\lambda_i) dt \leq \frac{1}{t} \cdot (v-1) \cdot t = (v-1)$$

$$\Rightarrow d(\lambda_i) \leq (v-1), \forall \lambda_i \in \Upsilon$$

Now,

(i) $I^{FZ}(\Theta)$

$$\begin{aligned}
&= \sum_{(\lambda_i, \lambda_j) \in \Omega} [w(\lambda_i).d(\lambda_i) + w(\lambda_j).d(\lambda_j)] \\
&\leq \sum_{(\lambda_i, \lambda_j) \in \Omega} [1.(v-1) + 1.(v-1)] \\
&[\because w(\lambda_i) \leq 1 \text{ and } d(\lambda_i) \leq (v-1), \forall \lambda_i \in \Upsilon] \\
&\leq \frac{v(v-1)}{2}.2(v-1) \\
&= v(v-1)^2 \\
&\therefore I^{FZ}(\Theta) \leq v(v-1)^2
\end{aligned}$$

(ii) $I^{SZ}(\Theta)$

$$\begin{aligned}
&= \sum_{(\lambda_i, \lambda_j) \in \Omega} [\{w(\lambda_i).d(\lambda_i)\}.\{w(\lambda_j).d(\lambda_j)\}] \\
&\leq \sum_{(\lambda_i, \lambda_j) \in \Omega} [\{1.(v-1)\}.\{1.(v-1)\}] \\
&[\because w(\lambda_i) \leq 1 \text{ and } d(\lambda_i) \leq (v-1), \forall \lambda_i \in \Upsilon] \\
&\leq \frac{v(v-1)}{2}.(v-1)^2 \\
&= \frac{v(v-1)^3}{2} \\
&\therefore I^{SZ}(\Theta) \leq \frac{v(v-1)^3}{2}
\end{aligned}$$

(iii) $I^{SO}(\Theta)$

$$\begin{aligned}
&= \sum_{(\lambda_i, \lambda_j) \in \Omega} \left[\sqrt{\{w(\lambda_i).d(\lambda_i)\}^2 + \{w(\lambda_j).d(\lambda_j)\}^2} \right] \\
&\leq \sum_{(\lambda_i, \lambda_j) \in \Omega} \left[\sqrt{\{1.(v-1)\}^2 + \{1.(v-1)\}^2} \right] \\
&[\because w(\lambda_i) \leq 1 \text{ and } d(\lambda_i) \leq (v-1), \forall \lambda_i \in \Upsilon] \\
&\leq \frac{v(v-1)}{2}.\sqrt{2}(v-1) \\
&= \frac{v(v-1)^2}{\sqrt{2}} \\
&\therefore I^{SO}(\Theta) \leq \frac{v(v-1)^2}{\sqrt{2}}
\end{aligned}$$

(iv) $I^{HZ}(\Theta)$

$$\begin{aligned}
&= \sum_{(\lambda_i, \lambda_j) \in \Omega} [w(\lambda_i)d(\lambda_i) + w(\lambda_j)d(\lambda_j)]^2 \\
&\leq \sum_{(\lambda_i, \lambda_j) \in \Omega} [1.(v-1) + 1.(v-1)]^2 \\
&[\because w(\lambda_i) \leq 1 \text{ and } d(\lambda_i) \leq (v-1), \forall \lambda_i \in \Upsilon] \\
&\leq \frac{v(v-1)}{2}.4(v-1)^2 \\
&= 2v(v-1)^3 \\
&\therefore I^{HZ}(\Theta) \leq 2v(v-1)^3
\end{aligned}$$

(v) $I^{FO}(\Theta)$

$$\begin{aligned}
&= \sum_{\lambda_i \in \Upsilon} \{w(\lambda_i)d(\lambda_i)\}^3 \\
&\leq \sum_{\lambda_i \in \Upsilon} [1.(v-1)]^3 \\
&[\because w(\lambda_i) \leq 1 \text{ and } d(\lambda_i) \leq (v-1), \forall \lambda_i \in \Upsilon] \\
&= v.(v-1)^3. \\
&\therefore I^{FO}(\Theta) \leq v(v-1)^3.
\end{aligned}$$

(vi) $I^Y(\Theta)$

$$\begin{aligned}
&= \sum_{(\lambda_i, \lambda_j) \in \Omega} [\{w(\lambda_i).d(\lambda_i)\}^3 + \{w(\lambda_j).d(\lambda_j)\}^3] \\
&\leq \sum_{(\lambda_i, \lambda_j) \in \Omega} [\{1.(v-1)\}^3 + \{1.(v-1)\}^3] \\
&[\because w(\lambda_i) \leq 1 \text{ and } d(\lambda_i) \leq (v-1), \forall \lambda_i \in \Upsilon] \\
&\leq \frac{v(v-1)}{2}.2(v-1)^3 \\
&= v(v-1)^4 \\
&\therefore I^Y(\Theta) \leq v(v-1)^4. \quad \blacksquare
\end{aligned}$$

Theorem 2 Let, $\Theta = (\Upsilon, \Omega)$ be a quantum graph having v vertices. Also let, at any time t , Θ be a star.

Then,

(i) $I^{FZ}(\Theta) \leq v(v-1)$;

- (ii) $I^{SZ}(\Theta) \leq (v-1)^2$;
- (iii) $I^{SO}(\Theta) \leq (v-1)\sqrt{v^2 - 2v + 2}$;
- (iv) $I^{HZ}(\Theta) \leq v^2(v-1)$;
- (v) $I^{FO}(\Theta) \leq (v-1)\{v^2 - 2v + 2\}$; and
- (vi) $I^Y(\Theta) \leq v(v-1)(v^2 - 3v + 3)$.

Proof Let, $\Upsilon = \{\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_v\}$ and λ_1 be the center of the star. As λ_1 is the center, so it is adjacent to exactly $(v-1)$ vertices, and every other vertex except λ_1 is a pendant vertex. Clearly, the number of edges of Θ is $(v-1)$.

We have, $w(\lambda_i) \leq 1, \forall \lambda_i \in \Upsilon$.

Clearly, $d_t(\lambda_1) \leq (v-1)$

$$\Rightarrow \int_0^t d_t(\lambda_1) dt \leq \int_0^t (v-1) dt = (v-1)t$$

$$\Rightarrow \frac{1}{t} \int_0^t d_t(\lambda_1) dt \leq \frac{1}{t} \cdot (v-1) \cdot t = (v-1)$$

$$\Rightarrow d(\lambda_1) \leq (v-1).$$

and

$$d_t(\lambda_i) \leq 1, \forall \lambda_i \in \{\lambda_2, \lambda_3, \dots, \lambda_v\}$$

$$\Rightarrow \int_0^t d_t(\lambda_i) dt \leq \int_0^t 1 dt = t$$

$$\Rightarrow \frac{1}{t} \int_0^t d_t(\lambda_i) dt \leq \frac{1}{t} \cdot t = 1$$

$$\Rightarrow d(\lambda_i) \leq 1, \forall \lambda_i \in \{\lambda_2, \lambda_3, \dots, \lambda_v\}.$$

Now,

$$(i) \quad I^{FZ}(\Theta)$$

$$\begin{aligned}
 &= \sum_{(\lambda_i, \lambda_j) \in \Omega} [w(\lambda_i)d(\lambda_i) + w(\lambda_j)d(\lambda_j)] \\
 &= [w(\lambda_1)d(\lambda_1) + w(\lambda_2)d(\lambda_2)] + [w(\lambda_1)d(\lambda_1) + w(\lambda_3)d(\lambda_3)] \\
 &\quad + [w(\lambda_1)d(\lambda_1) + w(\lambda_4)d(\lambda_4)] + \dots + [w(\lambda_1)d(\lambda_1) + w(\lambda_v)d(\lambda_v)] \\
 &[(v-1) \text{ terms}] \\
 &\leq (v-1)[1 \cdot (v-1) + 1 \cdot 1] \\
 &= v(v-1) \\
 &\therefore I^{FZ}(\Theta) \leq v(v-1).
 \end{aligned}$$

(ii) $I^{SZ}(\Theta)$

$$\begin{aligned}
&= \sum_{(\lambda_i, \lambda_j) \in \Omega} [\{w(\lambda_i).d(\lambda_i)\}.\{w(\lambda_j).d(\lambda_j)\}] \\
&= [\{w(\lambda_1).d(\lambda_1)\}.\{w(\lambda_2).d(\lambda_2)\}] + [\{w(\lambda_1).d(\lambda_1)\}.\{w(\lambda_3).d(\lambda_3)\}] \\
&\quad + [\{w(\lambda_1).d(\lambda_1)\}.\{w(\lambda_4).d(\lambda_4)\}] \\
&\quad + \dots + [\{w(\lambda_1).d(\lambda_1)\}.\{w(\lambda_v).d(\lambda_v)\}] \\
&[(v-1) \text{ terms}] \\
&\leq (v-1)[\{1.(v-1)\}.\{1.1\}] \\
&= (v-1)^2 \\
&\therefore I^{SZ}(\Theta) \leq (v-1)^2
\end{aligned}$$

(iii) $I^{SO}(\Theta)$

$$\begin{aligned}
&= \sum_{(\lambda_i, \lambda_j) \in \Omega} \left[\sqrt{\{w(\lambda_i).d(\lambda_i)\}^2 + \{w(\lambda_j).d(\lambda_j)\}^2} \right] \\
&= \left[\sqrt{\{w(\lambda_1).d(\lambda_1)\}^2 + \{w(\lambda_2).d(\lambda_2)\}^2} \right] \\
&\quad + \left[\sqrt{\{w(\lambda_1).d(\lambda_1)\}^2 + \{w(\lambda_3).d(\lambda_3)\}^2} \right] \\
&\quad + \left[\sqrt{\{w(\lambda_1).d(\lambda_1)\}^2 + \{w(\lambda_4).d(\lambda_4)\}^2} \right] \\
&\quad + \dots + \left[\sqrt{\{w(\lambda_1).d(\lambda_1)\}^2 + \{w(\lambda_v).d(\lambda_v)\}^2} \right] \\
&[(v-1) \text{ terms}] \\
&\leq (v-1) \sqrt{\{1.(v-1)\}^2 + \{1.1\}^2} \\
&= (v-1) \sqrt{v^2 - 2v + 2} \\
&\therefore I^{SO}(\Theta) \leq (v-1) \sqrt{v^2 - 2v + 2}
\end{aligned}$$

(iv) $I^{HZ}(\Theta)$

$$\begin{aligned}
&= \sum_{(\lambda_i, \lambda_j) \in \Omega} [\{w(\lambda_i).d(\lambda_i)\} + \{w(\lambda_j).d(\lambda_j)\}]^2 \\
&= [\{w(\lambda_1).d(\lambda_1)\} + \{w(\lambda_2).d(\lambda_2)\}]^2 \\
&\quad + [\{w(\lambda_1).d(\lambda_1)\} + \{w(\lambda_3).d(\lambda_3)\}]^2
\end{aligned}$$

$$\begin{aligned}
& + [\{w(\lambda_1).d(\lambda_1)\} + \{w(\lambda_4).d(\lambda_4)\}]^2 \\
& + \dots + [\{w(\lambda_1).d(\lambda_1)\} + \{w(\lambda_v).d(\lambda_v)\}]^2 \\
& [(v-1)\text{terms}] \\
& \leq (v-1)[\{1.(v-1)\} + \{1.1\}]^2 \\
& = v^2(v-1) \\
& \therefore I^{HZ}(\Theta) \leq v^2(v-1)
\end{aligned}$$

(v) $I^{FO}(\Theta)$

$$\begin{aligned}
& = \sum_{\lambda_i \in \Upsilon} \{w(\lambda_i).d(\lambda_i)\}^3 \\
& = \{w(\lambda_1).d(\lambda_1)\}^3 + \{w(\lambda_2).d(\lambda_2)\}^3 + \{w(\lambda_3).d(\lambda_3)\}^3 \\
& \quad + \dots + \{w(\lambda_v).d(\lambda_v)\}^3 \\
& \leq \{1.(v-1)\}^3 + (v-1)(1)^3 \\
& = (v-1)\{(v-1)^2 + 1\} \\
& = (v-1)\{v^2 - 2v + 2\} \\
& \therefore I^{FO}(\Theta) \leq (v-1)\{v^2 - 2v + 2\}
\end{aligned}$$

(vi) $I^Y(\Theta)$

$$\begin{aligned}
& = \sum_{(\lambda_i, \lambda_j) \in \Omega} [\{w(\lambda_i).d(\lambda_i)\}^3 + \{w(\lambda_j).d(\lambda_j)\}^3] \\
& = [\{w(\lambda_1).d(\lambda_1)\}^3 + \{w(\lambda_2).d(\lambda_2)\}^3] \\
& \quad + [\{w(\lambda_1).d(\lambda_1)\}^3 + \{w(\lambda_3).d(\lambda_3)\}^3] \\
& \quad + [\{w(\lambda_1).d(\lambda_1)\}^3 + \{w(\lambda_4).d(\lambda_4)\}^3] \\
& \quad + \dots + [\{w(\lambda_1).d(\lambda_1)\}^3 + \{w(\lambda_v).d(\lambda_v)\}^3] \\
& [(v-1)\text{ terms}] \\
& \leq (v-1)[\{1.(v-1)\}^3 + \{1.1\}^3] \\
& = (v-1)(v^3 - 3v^2 + 3v) \\
& = v(v-1)(v^2 - 3v + 3) \\
& \therefore I^Y(\Theta) \leq v(v-1)(v^2 - 3v + 3). \quad \blacksquare
\end{aligned}$$

Theorem 3 Let, $\Theta = (\Upsilon, \Omega)$ be a quantum graph. Then,

(i) $I^{SO}(\Theta) \geq \frac{1}{\sqrt{2}}I^{FZ}(\Theta)$; and

$$(ii) \quad I^{HZ}(\Theta) \geq 2\{I^{SZ}(\Theta)\}.$$

Proof

(i) For any two non-negative real numbers h and p , we have,

$$(h - p)^2 \geq 0$$

$$\Rightarrow h^2 + p^2 \geq 2hp$$

$$\Rightarrow 2(h^2 + p^2) \geq (h + p)^2$$

$$\Rightarrow \sqrt{2}\sqrt{h^2 + p^2} \geq (h + p)$$

Now, putting $h = w(\lambda_i).d(\lambda_i)$ and $p = w(\lambda_j).d(\lambda_j)$, we get

$$\begin{aligned} & \sqrt{2}\sqrt{\{w(\lambda_i).d(\lambda_i)\}^2 + \{w(\lambda_j).d(\lambda_j)\}^2} \\ & \geq [\{w(\lambda_i).d(\lambda_i)\} + \{w(\lambda_j).d(\lambda_j)\}] \\ \Rightarrow & \sqrt{2} \sum_{(\lambda_i, \lambda_j) \in \Omega} \sqrt{\{w(\lambda_i).d(\lambda_i)\}^2 + \{w(\lambda_j).d(\lambda_j)\}^2} \\ & \geq \sum_{(\lambda_i, \lambda_j) \in \Omega} [\{w(\lambda_i).d(\lambda_i)\} + \{w(\lambda_j).d(\lambda_j)\}] \\ \Rightarrow & \sqrt{2}I^{SO}(\Theta) \geq I^{FZ}(\Theta) \\ \Rightarrow & I^{SO}(\Theta) \geq \frac{1}{\sqrt{2}}I^{FZ}(\Theta) \\ \therefore & I^{SO}(\Theta) \geq \frac{1}{\sqrt{2}}I^{FZ}(\Theta). \end{aligned}$$

(ii) For any two non-negative real numbers h and p , we have,

$$(h + p)^2 = h^2 + p^2 + 2hp$$

$$\Rightarrow (h + p)^2 \geq 2hp$$

Now putting $h = w(\lambda_i).d(\lambda_i)$ and $p = w(\lambda_j).d(\lambda_j)$, we get

$$\begin{aligned}
& [w(\lambda_i).d(\lambda_i) + w(\lambda_j).d(\lambda_j)]^2 \\
& \geq 2[\{w(\lambda_i).d(\lambda_i)\}.\{w(\lambda_j).d(\lambda_j)\}] \\
& \Rightarrow \sum_{(\lambda_i, \lambda_j) \in \Omega} [w(\lambda_i).d(\lambda_i) + w(\lambda_j).d(\lambda_j)]^2 \\
& \geq 2 \sum_{(\lambda_i, \lambda_j) \in \Omega} [\{w(\lambda_i).d(\lambda_i)\}.\{w(\lambda_j).d(\lambda_j)\}] \\
& \Rightarrow I^{HZ}(\Theta) \geq 2\{I^{SZ}(\Theta)\} \\
& \therefore I^{HZ}(\Theta) \geq 2\{I^{SZ}(\Theta)\}. \quad \blacksquare
\end{aligned}$$

6 Area of Application

The application area of topological indices is huge. Topological indices were first introduced for application in molecular chemistry. Later on, different topological indices were developed to apply in chemistry, molecular biology, network theory, etc. In this chapter, we propose topological indices for quantum graphs. These topological indices can be applied where the weights of the vertices and the edges are dynamic. Topological indices can be applied to the co-authorship network to find the most influential researcher. These indices can be applied to analyze the structure of chemical compounds. Another application area would be in gene regulatory networks to observe how genes behave there. We can apply topological indices in network theory to analyze the resilience and dynamics of quantum networks. Topological indices can be applied to quantum chemistry to describe the behavior of electrons in quantum chemistry. Over the years, numerous researchers have applied topological indices in many real-life scenarios. As the application area is so big, it is a hot topic among researchers.

7 Conclusion

Topological indices are a very hot topic in research. Their application area in real life is enormous. Recently, quantum graphs have attracted the attention of researchers. In this book chapter, we have discussed quantum graphs and different topological indices and shown examples. We have proved some theorems regarding the topological indices' bounds and relationships. In the future, we would like to introduce new topological indices for quantum graphs and prove some exciting theorems. We

would also like to explore the application areas of the topological indices for quantum graphs.

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Domination in Quantum Graphs



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Abstract This chapter investigates the concept of domination in quantum graphs. The dynamic weights of vertices represent time-dependent values, while the entangled edges capture the quantum correlations between vertices. We begin by discussing quantum graphs and introducing the concept of domination and threshold domination in the graph. Then we discuss the properties of domination in quantum graphs.

Keywords Quantum graphs · Domination · Threshold · Dynamic weights

1 Introduction

The initial example of a quantum graph describes that each vertex represents a quantum state or a quantum particle, and each edge represents a quantum interaction or entanglement between these states [24, 45, 46]. The vertices are characterized by dynamic weights which can change over time. These dynamic weights, denoted as $dweight_i(t)$, range between 0 and 1. The edges, on the other hand, can be entangled, are correlated. This entanglement is represented by a correlation function $g_{ij}(t)$,

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which depends on the dynamic weights of the connected vertices. Thus the definition of quantum graphs can be used for any dynamical systems/networks where weights of the vertices and edges may change in little change in time [43, 44].

The motivation behind studying domination in quantum graphs is to understand and control dynamical systems in better way. Domination is a well-known concept in classical graph theory, where a dominating set is a subset of vertices such that every vertex in the graph is either in the dominating set or adjacent to a vertex in the dominating set [1, 2, 4, 5]. This concept has numerous applications in network design, resource allocation, and social network analysis [3, 6, 8, 9].

In the context of quantum graphs, domination nature of the dynamic and entangled nature of the vertices and edges is useful in quantum computing. For example, in quantum computing, identifying a dominating set of qubits could help optimize quantum algorithms and error correction protocols. In quantum communication networks, a dominating set of quantum nodes could enhance the security of communication protocols.

The primary objectives of this chapter are to:

1. define the concept of domination in quantum graphs.
2. establish the properties of domination in quantum graphs.
3. identify applications of quantum graphs in various fields.

2 Literature Review

Domination in graph theory is a well-established area of research with numerous applications in network design, social network analysis, and optimization problems. The authors [1, 2, 4, 5] discuss the algorithms of minimal dominating sets, disjoint dominating sets and the domination number and its applications in different domains such as wireless networking.

Domination in graphs extends traditional domination concepts in complex networks (competitive in nature) [11–14]. In classical graph theory, a dominating set is a subset of vertices such that each vertex is either in the set or adjacent to a vertex in the set. In quantum networks, domination expands to incorporate quantum states and principles, probabilistic relationships [17–22]. Researchers [31–33] worked on domination can improve graph/networks for applications in quantum information processing.

Graph theory with uncertainty, named fuzzy graphs are also useful in domination techniques. Fuzzy sets (FS) have become a popular approach for incorporating fuzzy information as part of an expert's evaluative measures [34–37]. For example, the development of fuzzy fractional differential equations using fuzzy Laplace transforms is fit for modeling systems [38–41] where the influence of individuals or groups evolves fractionally over time under uncertainty [34, 42]. In [15, 16] studied fuzzy variable precision rough sets, in [16] introduced hesitant bipolar-valued fuzzy sets.

Fuzzy graph theory [6, 23] such as fuzzy competition graphs [26], fuzzy planar graphs [25] is used to solve some real world problems. Mordeson, Mathew, and Malik

discuss the application of fuzzy graph theory to human trafficking networks. The authors define domination in fuzzy graphs and explore its implications for modeling and analyzing trafficking flows [7]. The authors introduce the concept of dominating sets, domination integrity, equitable domination in fuzzy graphs [8–10]. Few recent studies on domination can be found in optimizing tasks [47], the structural properties and algorithmic challenges associated with these graph classes [48], the interplay between graph theory and quantum physics [49]. Samanta et al. proposed generalized fuzzy graph theory connecting relationships of edges and end vertices [27–29, 31]. Generalised fuzzy graphs [30] are further extended to quantum graphs with dynamic weights of vertices and edges changing over times [24].

3 Domination Techniques in Graph Theory

Domination in graph theory is a fundamental concept with lots of applications in network design, social network analysis, and optimization problems. A dominating set in a graph is a subset of vertices such that every vertex in the graph is either in the dominating set or adjacent to a vertex in the dominating set.

3.1 Dominating Set

Let $G = (V, E)$ be a graph with vertex set V and edge set E .

A subset $D \subseteq V$ is called a *dominating set* if every vertex $v \in V \setminus D$ is adjacent to at least one vertex in D . Formally,

$$D \text{ is a dominating set if } \forall v \in V \setminus D, \exists u \in D \text{ such that } (u, v) \in E.$$

3.2 Domination Number

The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in G . That is,

$$\gamma(G) = \min\{|D| : D \text{ is a dominating set of } G\}.$$

3.2.1 Examples of Dominating Sets

We consider the graph G with vertex set $V = \{1, 2, 3, 4, 5\}$ and edge set $E = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\}$. A possible dominating set is $D = \{1, 3\}$ because every vertex in $V \setminus D = \{2, 4, 5\}$ is adjacent to at least one vertex in D .

We consider a 3×3 grid graph. The vertices can be labeled as follows:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}$$

A possible dominating set is $D = \{1, 5, 9\}$. Each vertex in the grid is either in D or adjacent to a vertex in D .

3.3 Types of Domination

3.3.1 Connected Domination

A subset $D \subseteq V$ is a *connected dominating set* if D is a dominating set and the subgraph induced by D is connected. Formally,

D is a connected dominating set if D is a dominating set and $G[D]$ is connected.

3.3.2 Independent Domination

A subset $D \subseteq V$ is an *independent dominating set* if D is a dominating set and the subgraph induced by D has no edges. Formally,

D is an independent dominating set if D is a dominating set and $G[D]$ has no edges.

3.4 Mathematical Modeling of Domination in Graph Theory

3.4.1 Integer Linear Programming (ILP) Formulation

The domination problem can be formulated as an integer linear programming problem. Let x_i be a binary variable indicating whether vertex i is in the dominating set ($x_i = 1$) or not ($x_i = 0$).

The ILP formulation is:

$$\min \sum_{i \in V} x_i$$

subject to

$$x_i + \sum_{j \in N(i)} x_j \geq 1 \quad \forall i \in V,$$

where $N(i)$ is the set of neighbors of vertex i .

3.4.2 Example: ILP for a Simple Graph

We consider the graph G with vertex set $V = \{1, 2, 3, 4\}$ and edge set $E = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$.

The ILP formulation is:

$$\min x_1 + x_2 + x_3 + x_4$$

subject to

$$x_1 + x_2 \geq 1,$$

$$x_2 + x_3 \geq 1,$$

$$x_3 + x_4 \geq 1,$$

$$x_4 + x_1 \geq 1.$$

Solving this ILP, we find that $x_1 = 1$, $x_3 = 1$ and $x_2 = 0$, $x_4 = 0$, so the minimum dominating set is $\{1, 3\}$.

4 Domination in Quantum Graphs

A quantum graph with dynamic vertices and entangled edges is defined as follows:

1. **Vertices with Dynamic Weights (dweight):** Each vertex v_i has a dynamic weight $\text{dweight}_i(t)$ that changes over time:

$$0 \leq \text{dweight}_i(t) \leq 1$$

2. **Edges and Entanglement:** Edges e_{ij} between vertices v_i and v_j can be entangled, represented by a correlation function $g_{ij}(t)$:

$$g_{ij}(t) = h(\text{dweight}_i(t), \text{dweight}_j(t))$$

3. **Graph Structure:** The graph G is represented as $G = (V, E)$, with an adjacency matrix $A(t)$:

$$A_{ij}(t) = \begin{cases} g_{ij}(t) & \text{if } e_{ij} \text{ exists and is entangled} \\ 0 & \text{otherwise} \end{cases}$$

4.1 Domination Techniques in Quantum Graphs

4.1.1 Basic Domination

A subset $D \subseteq V$ is a *dominating set* in a quantum graph if every vertex $v \in V \setminus D$ is adjacent to at least one vertex in D . But, due to the dynamic nature of the vertices and the entanglement of the edges, a dominating set D may sometimes fail to dominate all vertices at certain times.

Example We consider a quantum graph with vertices $V = \{v_1, v_2, v_3, v_4\}$ and edges $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1)\}$. Let the dynamic weights be $\text{dweight}_i(t)$ for $i = 1, 2, 3, 4$. A possible dominating set could be $D = \{v_1, v_3\}$, assuming the entanglement between these vertices is strong enough to influence the entire graph. However, if $\text{dweight}_2(t)$ is significantly low at a particular time, vertex v_2 may not be dominated.

4.1.2 Threshold Domination

A subset $D \subseteq V$ is a *threshold dominating set* if every vertex $v \in V \setminus D$ is adjacent to at least one vertex in D such that the correlation value $g_{ij}(t)$ exceeds a predefined threshold τ :

$$g_{ij}(t) > \tau$$

This means that the domination of a vertex depends on the strength of the correlation with the dominating set.

Example In the quantum graph, let $\tau = 0.5$. A possible threshold dominating set could be $D = \{v_1, v_3\}$ if $g_{12}(t) > \tau$ and $g_{32}(t) > \tau$ at certain times. If $g_{12}(t)$ and $g_{32}(t)$ are below τ , v_2 may not be dominated.

4.1.3 Total Domination

A subset $D \subseteq V$ is a *total dominating set* if every vertex $v \in V$ is adjacent to at least one vertex in D . This means that even the vertices in D must be adjacent to other vertices in D .

Example In the same quantum graph, a total dominating set could be $D = \{v_1, v_2, v_3\}$, ensuring that each vertex, including those in D , is adjacent to at least one other vertex in D .

4.1.4 Connected Domination

A subset $D \subseteq V$ is a *connected dominating set* if D is a dominating set and the subgraph induced by D is connected. This ensures that the dominating set forms a connected subgraph.

Example For the quantum graph, $D = \{v_1, v_2, v_3\}$ could also serve as a connected dominating set if the entanglement between these vertices ensures connectivity. Time-dependent changes in weights could lead to disconnections among the vertices in D .

4.1.5 Independent Domination

A subset $D \subseteq V$ is an *independent dominating set* if D is a dominating set and the subgraph induced by D has no edges. This means that no two vertices in D are adjacent.

Example In the quantum graph, $D = \{v_1, v_3\}$ could be an independent dominating set if v_1 and v_3 are not directly entangled but still dominate the graph through their connections.

4.1.6 Weighted Domination

In quantum graphs, the concept of weighted domination can be extended to account for the dynamic weights of the vertices. A weighted dominating set D minimizes the total weight while ensuring domination.

Example We consider the same quantum graph with dynamic weights $\text{dweight}_i(t)$. The goal is to find a set D that minimizes $\sum_{v_i \in D} \text{dweight}_i(t)$ while dominating the graph. If $\text{dweight}_1(t) = 0.2$, $\text{dweight}_2(t) = 0.5$, $\text{dweight}_3(t) = 0.3$, and $\text{dweight}_4(t) = 0.4$, a possible weighted dominating set could be $D = \{v_1, v_3\}$ with a total weight of 0.5. But, changes in $\text{dweight}_2(t)$ may necessitate reconsideration of this choice.

4.1.7 Entangled Domination

In quantum graphs, entangled domination considers the entanglement between vertices. A set D is an entangled dominating set if the entanglement between vertices in D and those outside D ensures domination.

Example If $g_{ij}(t)$ represents the entanglement between vertices v_i and v_j , a set $D = \{v_1, v_3\}$ could be an entangled dominating set if $g_{13}(t)$ is sufficiently strong to influence the entire graph.

5 Properties of Domination in Quantum Graphs

The properties of domination in quantum graphs include dynamic adaptability, enhanced connectivity through entanglement, and time-dependent behavior.

Theorem 1 *In a quantum graph $G = (V, E)$ with dynamic weights $dweight_i(t)$ for each vertex $v_i \in V$, a dominating set D can adapt dynamically over time such that $D(t)$ remains a threshold dominating set for all $t \geq 0$.*

Proof Let $G = (V, E)$ be a quantum graph with dynamic weights $dweight_i(t)$ for each vertex $v_i \in V$. Assume $D(t_0)$ is a threshold dominating set at time t_0 . For $D(t)$ to remain a threshold dominating set for all $t \geq 0$, the following condition must hold:

$$\forall v \in V \setminus D(t), \exists u \in D(t) \text{ such that } (u, v) \in E \text{ and } g_{uv}(t) > \tau,$$

where τ is the predefined threshold.

Since the dynamic weights $dweight_i(t)$ are continuous functions of time, small changes in t will result in small changes in $dweight_i(t)$. Therefore, the adjacency matrix $A(t)$ will also change continuously. By continuously updating $D(t)$ to include vertices with higher dynamic weights while ensuring that the entanglement $g_{uv}(t)$ remains above the threshold τ , $D(t)$ can adapt dynamically to remain a threshold dominating set for all $t \geq 0$. $\square$

Theorem 2 *In a quantum graph $G = (V, E)$ with entangled edges, the connectivity of the threshold dominating set D is enhanced by the entanglement between vertices.*

Proof Let $G = (V, E)$ be a quantum graph with entangled edges represented by the correlation function $g_{ij}(t)$. Assume $D \subseteq V$ is a threshold dominating set. The connectivity of D is determined by the entanglement between vertices in D . For D to be connected, the subgraph induced by D must be connected, i.e., there must be a path between any two vertices in D .

We consider two vertices $u, v \in D$. If $(u, v) \in E$ and $g_{uv}(t) > \tau$, then u and v are directly connected. If $(u, v) \notin E$, there must exist a sequence of vertices $u = v_0, v_1, \dots, v_k = v$ such that $(v_i, v_{i+1}) \in E$ and $g_{v_i v_{i+1}}(t) > \tau$ for all $i = 0, 1, \dots, k-1$.

Since the entanglement $g_{ij}(t)$ enhances the probability of quantum interactions between vertices, the connectivity of D is enhanced by the entanglement between vertices. Therefore, the threshold dominating set D in a quantum graph with entangled edges has enhanced connectivity. $\square$

Theorem 3 *In a quantum graph $G = (V, E)$ with time-dependent dynamic weights $dweight_i(t)$ and entanglement $g_{ij}(t)$, the threshold domination number $\gamma_\tau(G, t)$ is a time-dependent function.*

Proof Let $G = (V, E)$ be a quantum graph with time-dependent dynamic weights $dweight_i(t)$ and entanglement $g_{ij}(t)$. The threshold domination number $\gamma_\tau(G, t)$ is defined as the minimum cardinality of a threshold dominating set at time t .

Consider the set of all possible threshold dominating sets $\mathcal{D}_\tau(t)$ at time t . The threshold domination number $\gamma_\tau(G, t)$ is given by:

$$\gamma_\tau(G, t) = \min_{D \in \mathcal{D}_\tau(t)} |D|.$$

Since $dweight_i(t)$ and $g_{ij}(t)$ are continuous functions of time, the set of threshold dominating sets $\mathcal{D}_\tau(t)$ may change as t varies. Consequently, the cardinality of the minimum threshold dominating set, and hence the threshold domination number $\gamma_\tau(G, t)$, is a time-dependent function.

For example, if at time t_1 , the dynamic weights and entanglement favor a threshold dominating set D_1 with cardinality $|D_1|$, and at time t_2 , the changes in $dweight_i(t)$ and $g_{ij}(t)$ favor a different threshold dominating set D_2 with cardinality $|D_2|$, then $\gamma_\tau(G, t_1) \neq \gamma_\tau(G, t_2)$.

Therefore, the threshold domination number $\gamma_\tau(G, t)$ in a quantum graph is a time-dependent function. $\square$

6 Conclusion

In this chapter, we introduced threshold domination in quantum graph and discussed several properties of it. The concept of threshold domination introduces a new layer of complexity, as it requires not only adjacency but also a measure of entanglement that must surpass a predefined threshold. Moreover, the findings suggest that as the weights and entanglement of vertices evolve over time, so too does the structure of the dominating sets. This dynamic nature poses interesting challenges for researchers, for further investigation.

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Quantum Computing and GAN Aspect



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Abstract This chapter aims to explore the convergence of two emerging fields of computer science: quantum computing and Generative Adversarial Networks. Though capable of individually performing heavy tasks themselves, we would try to understand how their interlinking can prove to be a beneficial path that may lead to the birth of different breakthroughs in coming times. Quantum computing, which can process large amounts of information simultaneously in high-dimensional data spaces, offers a promising solution to some of the limitations faced by classical learning models. We begin by reviewing the fundamentals of quantum computing, followed by an overview of GANs and their applications in generative tasks. Then, we dive into the field of QuantumGANs, examining key algorithms and architectures. Current challenges are also discussed, along with future developments.

Keywords Quantum computing · Generative adversarial networks · Quantum machine learning · Quantum GANs · Quantum algorithms · Quantum circuits · Quantum speed up · Quantum generators · Discriminators

1 Introduction

Quantum computing (Preskill, 2018) and GAN technologies are some of the recent breakthroughs that potentially may act as torch-bearers for upcoming innovations (Aaronson, 2013) in the fields of advanced computing. Quantum computing (Biamonte et al., 2018) leverages phenomena such as superposition, entanglement

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and parallelism that would not be feasible with classical systems. With recent developments in hardware, it promises to eradicate the huge computational requirements of learning models, particularly machine learning.

On the other hand, GANs have emerged as a powerful breakthrough in the domain of deep learning for tasks such as image synthesis and data augmentation. By framing the problem as a two-player game (generator and discriminator), GANs have been able to produce high-quality results even in the absence of abundant training samples. However, their high computational costs for high dimensional data still hinder their broad development.

In recent years, the intersection of these two fields for exploiting the capabilities of quantum computing to enhance GANs training for high dimensional data has been a topic of interest. Quantum algorithms (Rudolf & Howland, 2019), with their inherent parallelism and potential for exponential speedup, offer promising solutions to such training challenges.

Mahapatra et al. (Mahapatra et al., 2020b) offers a new and improved definition, introducing the generalized neutrosophic planar graph (GNPG). Here, planarity is evaluated more precisely by considering the true, false, and indeterminacy values for the degree of planarity, which gives a clearer measure of the graph's overall planarity. Link prediction is a crucial task in social networks, with various methods available to predict connections. The RSM index (Mahapatra et al., 2019b) is one such method, useful in fuzzy environments, but it lacks consideration for falsity and indeterminacy value common in uncertain settings in modified RSM index (Mahapatra et al., 2020a). A COVID-19 infection example is provided to demonstrate how directed fuzzy graph colouring can model transmission influence and control by Mathapatra et al. (Mahapatra et al., 2020d, 2021). A new concept of the chromatic number for neutrosophic graphs and discusses the differences between fuzzy colouring in fuzzy graphs and colouring in NG (Mahapatra et al., 2019a, 2020c, 2022). Identifying central nodes in social networks is essential for tasks like business growth, news dissemination, and advertising.

Traditional centrality methods focus on a node's direct reach but often overlook uncertainty in relationships. A new centrality measure is introduced for neutrosophic graphs and linguistic graph (Mahapatra et al., 2023, 2024a, 2024b), which includes self-weight or individual merit within the network.

The results are better to the adoptive routing algorithms with compared to non-adoptive routing algorithms. They got the better result because of the dynamic routing table. For instance, the development of fuzzy fractional differential equations using fuzzy Laplace transforms is essential for modeling systems where the influence of individuals or groups evolves fractionally over time under uncertainty (Allahviranloo & Afshar Kermani, 2010; Allahviranloo et al., 2011; Allahviranloo & Salahshour, 2013; Allahviranloo, 2020). Pythagorean fuzzy systems (Akram et al. 2021) and trapezoidal approximations (Allahviranloo & Firozja 2007) provide additional methods for handling high levels of uncertainty in decision-making processes. These approaches offer flexible tools for analyzing complex decision problems, ensuring that a wide range of uncertain factors are considered when making group decisions.

2 Quantum Computing: Fundamentals and Applications

Quantum computing (Biamonte et al., 2018; Preskill, 2018) has proven to be a major shift in learning models' computational paradigms (Aaronson, 2013), leveraging quantum mechanics (Lloyd et al., 2014) to solve problems outside the scope of classical models. Quantum computers use quantum bits (Farhi et al., 2014), or **qubits**, which exist in multiple states thanks to superposition and entanglement. This enables them to perform parallelism, hence speeding up the operation.

2.1 *Quantum Bits and Superposition*

Computer science has taught us all about the classical bit, which can take only two states, 0 or 1. A quantum bit, however, may exist as 0 and 1 simultaneously, as a superposition of both. This means that a quantum computer with n qubits can represent 2^n possible states at the same time, allowing vast parallelism opportunities.

Such simultaneous processing allows quantum computers to explore multiple problems like optimisation and simulations in parallel. The ability of qubits to interfere (constructively or destructively) with one another also plays a critical role in finding optimal solutions.

2.2 *Quantum Entanglement and Parallelism*

Quantum entanglement occurs between qubits, directly linking the state of one qubit to the state of another qubit, ignoring the distance between them. This property helps in complex calculations across large datasets with minimal resources.

Superposition and entanglement result in quantum parallelism to speed up processes.

2.3 *Quantum Gates and Circuits*

Quantum gates work similarly to logic gates, with the difference being that they manipulate quantum bits instead of classical ones through unitary operations. Some basic operations are the Hadamard gate (for superposition), the Pauli-X gate (analogue to NOT gate) and the CNOT gate (for entanglement). In particular, quantum gates are reversible to maintain the coherence of qubits during computations.

Quantum circuit is the sequence of quantum gates and forms the building blocks of complex algorithms.

2.4 Quantum Algorithms

To exploit the unique features of quantum systems (Zurek, 2003), the following algorithms (Cai & Liu, 2018) are designed:

- **Shor's Algorithm** (Shor, 1994): Used in cryptography to provide an exponential speedup for factoring large integers, an expensive process in classical computers. Quantum computers do this in polynomial time.
- **Grover's Algorithm**: Used in unsorted database searches to reduce the steps required to find a specific item in it, providing a quadratic speedup and demonstrating quantum computing's potential in optimisation problems.

2.5 Applications of Quantum Computing

Since large amounts of data can be handled in parallel using quantum computing, it is a promising candidate for learning models. It can accelerate tasks such as training large-scale models, optimisation of architectures and solving NP problems.

In the field of Generative Adversarial Networks, quantum computing can be a potential game-changer by providing faster convergence rates and improving output quality. Such advancements are possible due to the inherent parallelism and optimisation capabilities of quantum systems.

3 Generative Adversarial Networks

GANs (Goodfellow et al., 2014) have proven to be a revolution in the generative modelling field (Arjovsky et al., 2017) by allowing machines to generate synthetic data. Introduced by Ian Goodfellow in 2014, GANs have since found their involvement in data synthesis, image generation and content production. Its core is the process between two neural networks: the generator and the discriminator (Kingma & Welling, 2013).

3.1 GAN Architecture Overview

Two main components:

- **Generator (G)**: The generator receives random noise and is responsible for producing synthetic data from it. The created samples must resemble the training data in such a way that the discriminator cannot distinguish between them and real ones.

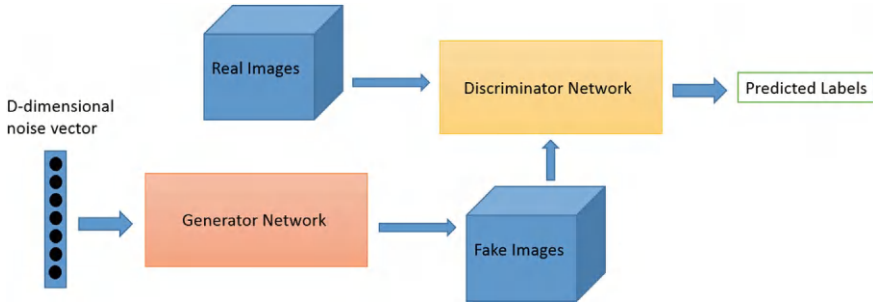


Fig. 1 Classical GAN architecture

- **Discriminator (D):** Its job is to classify the input it receives into real or fake data. It acts just like a binary classifier and learns to differentiate between real and generated samples.

These two engage in a two-player game where the generator tries to fool the discriminator (i.e. minimising the loss function) while the discriminator pushes for correctly identifying real and fake samples, or simply put, maximising the loss function. Mathematically, the problem is expressed as **min-max optimisation**:

$$\min_G \max_D E_{x \sim p_{data}(x)} [\log D(x)] + E_{z \sim p_z(z)} [\log(1 - D(G(z)))]$$

Here, $p_{data}(x)$ is real data distribution, and $p_z(z)$ represents noise distribution (Fig. 1).

3.2 Training Dynamics

Training a GAN is challenging due to the nature of loss functions, which, being adversarial, often result in unstable training where the generator produces a limited variety of outputs.

Stable training is only possible if both the generator and discriminator are equally strong otherwise, we may get poor results.

Some techniques for stabilising GAN training are:

- **Batch Normalisation:** Reduces the covariate shifts, improving stability.
- **Wasserstein GAN:** Addresses the problem of vanishing gradient using the Earth Mover's distance as a measure of divergence between real and fakes.
- **Spectral Normalisation:** Controls the Lipschitz constant of the discriminator, further stabilising training.

3.3 *GAN Variants*

Some major improvements in the structure of basic GAN architecture have laid the foundation for various variants.

- **Deep convolutional GAN:** Introduced the use of convolutional layers in GANs, enhancing the effects in image generation tasks.
- **Conditional GAN:** Allows the generation of data that follows desired characteristics based on some conditions.
- **Cycle GAN:** Designed for image-to-image translation without requiring paired data labels.
- **StyleGAN:** Incorporates a style-based architecture that helps in controlling image attributes, such as facial expressions.

3.4 *GAN Applications*

Some fields where GAN's success has brought major changes are:

- **Image generation:** Synthetic image creation has become a routine task using GAN, which helps in generating photorealistic images, synthetic faces and artistic styles.
- **Data augmentation:** New training examples can be generated using GAN in the works of machine learning for the use of underrepresented classes, thus preventing overfitting.
- **Video and audio generation:** Recent involvements of GAN have been there in the fields of realistic video and audio generation, hence expanding the horizons of its use.

3.5 *GAN-Challenges and Limitations*

Despite their huge success in modern generative models, GANs, too, suffer from limitations.

- **Mode collapse:** A situation in which the generator fails to cover the entire distribution of real data provided to it, producing limited or repetitive outputs.
- **Unstable training:** If the balance between the generator and discriminator is not taken care of, the non-convergence behaviour of loss function.
- **Evaluation metrics:** None of the available metrics perfectly quantify the quality and diversity of generated samples, even Inception score and Frechet Inception distance fall short.

4 Quantum GANs: The Convergence

The collaboration of quantum computing and Generative Adversarial Networks (Arjovsky et al., 2017; Huang & Wang, 2020) has brought out a new class of generative models known as **Quantum GANs** (Chen et al., 2018). These models harness the computational powers of quantum mechanics (Schuld & Petruccione, 2018) to improve the training dynamics and the efficiency (Zoufal et al., 2019) of classical GANs. QGANs (Lloyd & Weedbrook, 2018) promise to revolutionize (Cerezo et al., 2021) generative modelling, especially for tasks involving high-dimensional data.

4.1 QGANs—Conceptual Foundation

Quantum GANs (Kim et al., 2020; Wang et al., 2021) are designed based on the same framework as classical GANs, in which two networks—a quantum generator and a quantum discriminator (Cerezo et al., 2021; Schuld & Kloran, 2019) compete against each other in a two-player game (Neven & Aharonov, 2020). Either one or both networks can be quantum systems.

- **Quantum generator:** In QGANs, the generator uses quantum circuits to produce synthetic data. Due to the superposition of qubits, the generator can explore large solution spaces, which leads to faster convergence and improved exploration of complex data distributions.
- **Quantum discriminator:** The discriminator also used quantum measurements to classify real and fake data. They can utilise quantum states that encode both real and generated data, hence improving their ability to distinguish between them.

The key idea behind QGANs (Benedetti et al., 2019) is to capitalise on **quantum speedup** (Cai et al., 2019)—the potential for quantum computers to process information more efficiently than classical systems, thus making our framework more robust and reliable.

4.2 QGANs—Architecture

The architecture of QGANs is built by making some specific adaptations in classical GAN structures for quantum environments. Two primary architectures are proposed:

- **Hybrid Quantum–Classical GANs:** In this architecture, either one of the generators or discriminator is implemented as a quantum model, while the other remains classical. This approach acts as a starting point for many applications, allowing easier integration of quantum components with existing classical machine learning infrastructure.

- **Fully Quantum GANs:** Here, both the generator and the discriminator are quantum models. While they offer a theoretical advantage, fully quantum GANs are more challenging to implement due to current hardware limitations.

4.3 QGANs—Algorithms

Many algorithms, each offering different levels of quantum advantage, have been proposed:

- **Quantum generative adversarial networks:** This framework uses quantum circuits for the generator and discriminator, hence leveraging quantum measurements to generate and classify data.
- **Variational quantum generative adversarial networks:** This algorithm uses **variational quantum circuits**, which are parameterised circuits optimised using classical techniques. They aim to achieve a balance between the quantum speedup and variational circuits on noisy quantum devices.
- **Hybrid quantum GANs:** They utilise only one quantum network, keeping the other classical. They are easy to implement on near-term devices without requiring full quantum architectures (Fig. 2).

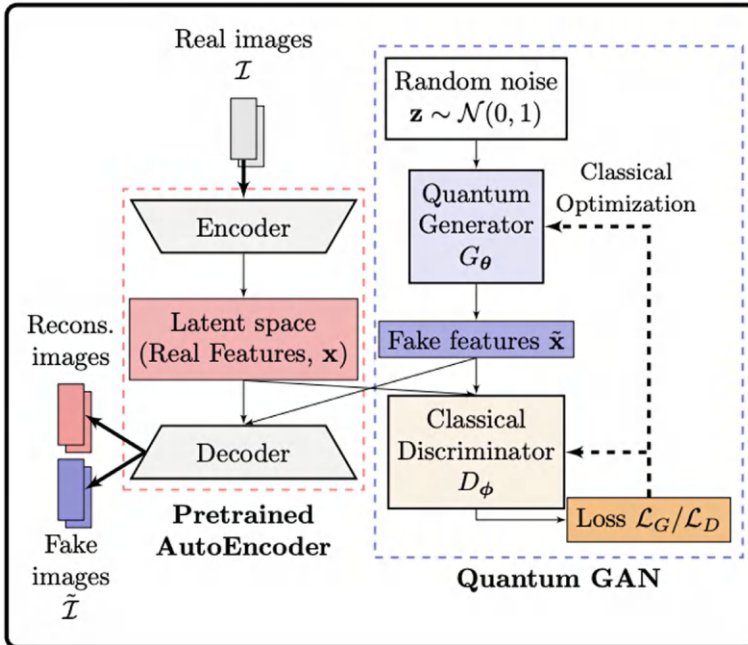


Fig. 2 Quantum GAN architecture

4.4 QGANs—Advantages

QGANs offer a great deal of advantages (Deng et al., 2018) over their classical counterparts (Farhi et al., 2014):

- **Quantum Parallelism:** Since qubits can exist in superposition, quantum generators can simultaneously explore many data points. This enhances the search through vast solution spaces, leading to faster and more stable training.
- **Optimisation:** Quantum computers leverage quantum algorithms (Shor, 1994) to find the optimal parameters for both generator and discriminator, hence significantly reducing training time for GANs.
- **Higher-order correlations:** QGANs enable the generation and discrimination of data with higher-order correlations, a task that is difficult for classical GANs. This can result in more realistic and diverse data generation.
- **High-dimensional data handling:** High-dimensional spaces, which are common in generative modelling, are efficiently handled using QGANs. The ability of quantum circuits to represent complex distributions (Kerendis & Prakash, 2020; Zoufal et al., 2019) with fewer resources than classical systems comes in handy.

5 Challenges and Limitations

The full realisation of QGANs (Cerezo et al., 2021) is still hindered by several technical and practical challenges, which stem from both the current state of quantum hardware and the fundamental complexities (Kerendis & Prakash, 2020) of GAN models.

5.1 Quantum Hardware Limitations

Quantum computing is still in its nascent stages, and current devices (Dunjko & Breigel, 2018) suffer from several drawbacks:

- **Limited Qubit count:** Most quantum devices today offer a very small number of qubits, which restricts the complexity of models that can be implemented, making it difficult to scale QGANs for real-world problems.
- **Quantum noise:** Quantum systems are noisy, and the qubits, too, are susceptible to error due to environmental interactions and imperfect gates. Noisy outputs produced due to this affect the performance of QGANs.
- **Short coherence time:** Coherence time is the period during which qubits maintain their quantum state. This is generally short, limiting the depth of quantum circuits that can be executed before the system's quantum information is lost.

5.2 Training Stability

Even in classical form, GANs are known to have unstable training, leading to non-convergence and mode collapse. QGANs inherit these challenges and also introduce additional complexities.

- **Quantum circuit optimisation:** Since there are no well-established quantum gradient descent techniques available, optimisation in QGANs becomes more challenging.
- **Gradient estimation and vanishing gradient:** Gradient estimation is not straightforward in QGANs, and issues like vanishing gradient can arise, particularly in deep quantum circuits.
- **Adversarial loss functions:** Adversarial loss functions in classical systems are based on probability distributions and divergence measures. Representing them in quantum systems is a whole start-from-scratch process, adding more complexity.

5.3 Data Representation and Encoding

The representation of classical data (Pudenz & Hollenberg, 2014) within quantum systems is still a challenge. This process is non-trivial and, hence, a complex one.

- **Quantum data encoding:** Classical data must be encoded into quantum states using techniques like amplitude encoding. Such methods have their trade-offs in terms of efficiency and dimensionality. Thus, finding an encoding scheme for high dimensional data is still an open research.
- **Data dimensionality:** Due to qubit limitations, quantum circuits are designed to operate in low dimensional spaces. Encoding of high-dimensional classical data into a quantum system often requires a dimensionality reduction, leading to the loss of important features.

5.4 Evaluation Metrics for QGANs

As discussed before, the performance measurement (Mescheder et al., 2018) of classical GANs is already a difficult task, and it does not get any easier for QGANs. Traditional metrics are not directly applicable to quantum-generated data.

- **Quantum classical translation:** When quantum states are produced by quantum generators, these states need to be measured and converted back to classical form for evaluation, which introduces biases in performance metrics.
- **Quantum-specific metrics:** To assess the quality of quantum-generated data, new evaluation metrics are needed. Such metrics are still under development, and their effectiveness is yet to be fully explored.

Future Area of Research: Fuzzy quantum computing, Generative Adversarial Networks (GANs), and fuzzy networks are at the forefront of technological innovation, each contributing uniquely to the field of artificial intelligence and data processing.

Fuzzy quantum computing integrates the principles of quantum mechanics with fuzzy logic to manage uncertainty and imprecision in quantum systems. This approach enhances quantum algorithms by incorporating degrees of truth rather than binary true/false values, making decision-making processes more flexible and robust.

The previous research in fuzzy set theory and its applications plays a significant role in addressing uncertainty and complexity in social networks and group decision-making processes. The methods developed in this area offer valuable insights into how fuzzy logic can be used to model and analyze systems where imprecision and ambiguity are prevalent. The related articles can be found

Quantum GANs (QGANs) extend classical GANs by utilizing quantum computing to potentially improve the efficiency and performance of generative models. QGANs leverage quantum bits (qubits) and quantum operations to explore a larger solution space more efficiently, leading to faster training times and the generation of more complex and realistic data².

Fuzzy networks combine fuzzy logic with neural networks to handle uncertainty in data. A notable development in this area is the **Hierarchical Fused Quantum Fuzzy Neural Network (HQFNN)**, which uses quantum neural networks to learn fuzzy membership functions. This model is particularly effective in dealing with ambiguous and noisy data, outperforming traditional methods in tasks like image classification².

6 Conclusions

We have discussed the groundbreaking step of converging quantum computing along with Generative Adversarial Networks and how we can use them for advanced generative modelling. The chapter explored the fundamental principles of both, highlighting how their convergence can do away with some prominent challenges faced by classical GANs, such as training instability and computational costs.

Quantum GANs offer promising benefits, such as the potential for exponential speed-up in training and handling complex data structures.

However, significant challenges faced include hardware limitations and architectural complexities. The noisy nature of quantum systems and the limited number of qubits add to the limitations, too.

Furthermore, designing evaluation metrics for QGANs remains an open problem.

Despite these challenges, the potential of QGANs is immense. As quantum technology continues to grow, we expect QGANs to play a transformative role in how we approach generative modelling in the future.

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Concepts in Fuzzy and Quantum Social Networks



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Abstract In recent years, the use of fuzzy graph theory in social network analysis has become increasingly prevalent. The study of social networks has numerous practical applications in real life. This chapter aims to explore various aspects of fuzzy social networks, including connectivity and the control of actors within a network. An small introduction to quantum social networks is presented. This emerging area of study has the ability to transform our understanding of social networks and their dynamics. Additionally, the potential applications of fuzzy social networks in detecting sarcasm and in the healthcare sector will be discussed.

1 Introduction

The study of social networks has its origins tracing back to the early 1930s. It has been informally explored through concepts such as socionomy and sociometry. Socionomy is a branch of science that focuses on the psychological properties of a population, while sociometry is a sub-field of socionomy that specifically examines interpersonal relationships within small groups. Moreno [10], coined the term “sociometry” to describe the mathematical techniques and analysis necessary for understanding the psychological properties of a population. In order to fully understand how a group functions, it is crucial to have a thorough understanding of its structure. Moreno also introduced the term “sociogram” to visually represent the interpersonal structure of groups. A sociogram is a diagram that uses points to represent individuals, also known as social units, and lines to connect related points and show relationships.

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The term “social network” was first used in 1954 by Barnes [15] and refers to the set of “actors” and “ties” among those actors [5].

A classical social network is composed of social units known as nodes or actors, which are connected by arcs called links or edges. These arcs represent the relationships between two social units. While the social network structure can be represented using crisp graphs, there may be uncertainties in describing both the social units (referred to as vertices) and the links (referred to as edges). To address this issue, fuzzy sets are used to describe actors and links, resolving any inherent vagueness. As a result, social networks are often modeled using fuzzy graphs and are called as “Fuzzy Social Network.” Once a social network is formed, understanding its patterns and dynamics becomes crucial in gaining insights. This study of the structure of a social network is referred to as “Social Network Analysis.” Social network analysis allows for an understanding of various interactions among actors, such as how information flows among them. Some important measures in fuzzy social network analysis include centrality measures, clique analysis, ego network analysis, and cohesion measures. Centrality measures help identify the most influential actors, clique analysis is used to find subgroups of densely connected vertices, and cohesion analysis measures the strength of a network.

One limitation of classical social networks is their inability to explain the reasons behind the connections or lack thereof between certain actors (nodes) when depicted graphically. To address this issue, Cabello et al. [4], coined the concept of a quantum social network, where each actor i is assigned a quantum state $|\psi_i\rangle$ and it is characterised by yes-no test T_i . In quantum mechanics states are represented as vectors in Hilbert space. The working of quantum social network involves incorporating quantum mechanics into social networks. This is achieved by representing relationships as quantum states. Each individual can be modeled as a quantum entity, allowing for superpositions of social states. The work on fully fuzzy linear systems and the signed decomposition method [16] is crucial for modeling the complex interrelations within social networks, where interactions can be represented as linear systems with fuzzy parameters. This framework supports a more realistic analysis of decision-making processes, particularly when dealing with multiple interdependent and uncertain factors.

Fuzzy linear programming methods [18] are directly applicable to optimization problems in group decision-making, where the objectives and constraints are often imprecise. These techniques help navigate complex decision-making environments within social networks, where individual preferences and goals may conflict or remain undefined.

The exploration of bipolar fuzzy systems [19] further enhances the understanding of social interactions, where positive and negative relationships coexist. Bipolar fuzzy logic provides a robust tool for modeling such dual dynamics, which is essential for understanding group dynamics and decision-making in environments where both cooperation and conflict exist.

Pythagorean fuzzy systems [20] and trapezoidal approximations [21] provide additional methods for handling high levels of uncertainty in decision-making processes. These approaches offer flexible tools for analyzing complex decision

problems, ensuring that a wide range of uncertain factors are considered when making group decisions.

Mahapatra et al. [22] offers a new and improved definition, introducing the generalized neutrosophic planar graph (GNPG). Here, planarity is evaluated more precisely by considering the true, false, and indeterminacy values for the degree of planarity, which gives a clearer measure of the graph's overall planarity. Link prediction is a crucial task in social networks, with various methods available to predict connections. The RSM index [26] is one such method, useful in fuzzy environments, but it lacks consideration for falsity and indeterminacy value common in uncertain settings in modified RSM index [28]. A COVID-19 infection example is provided to demonstrate how directed fuzzy graph colouring can model transmission influence and control by Mathapatra et al. [23, 24]. A new concept of the chromatic number for neutrosophic graphs and discusses the differences between fuzzy colouring in fuzzy graphs and colouring in NG [25, 27, 28]. Identifying central nodes in social networks is essential for tasks like business growth, news dissemination, and advertising. Traditional centrality methods focus on a node's direct reach but often overlook uncertainty in relationships. A new centrality measure is introduced for neutrosophic graphs and linguistic graph [17, 29–31], which includes self-weight or individual merit within the network.

In the last two decades, the use of social networking platforms has significantly increased, resulting in a vast amount of data available for analysis. The insights obtained from the analysis of social networks can help in understanding in social behavioral patterns of various groups which can be beneficial in various sectors such as marketing, policy framing, and healthcare. As a result, social network analysis has become a central focus of research. This chapter aims to discuss various concepts related to fuzzy and quantum social networks and their analysis. In Sect. 2, we will review several definitions associated with fuzzy graphs. Section 3 will provide a brief overview of fuzzy social networks and explore different types of connectivity within them. In Sect. 4, we will explore the concepts related to quantum social network. In Sect. 5, we will explore the practical applications of fuzzy social networks in sarcasm detection and healthcare. Finally, this chapter will conclude with remarks in Sect. 6.

2 Preliminaries

This section introduces basic terminologies from [9].

Definition 1 (*Fuzzy Set*) Let X be a non-empty set and $\mu : X \rightarrow [0, 1]$ be a membership map. A fuzzy set $\mathcal{A}$ is the ordered pair (X, μ) given by

$$\mathcal{A} = \{(x, \mu(x)) : x \in X\}.$$

Definition 2 (*Fuzzy Subset*) A fuzzy subset of set X is a mapping $\mu_X : X \rightarrow [0, 1]$, where $\mu_X(x)$ denotes the ‘membership degree’ of x in X .

Definition 3 ([9], Definition 1.1) Let μ and ν be two ‘fuzzy subsets’ of a set X . Then

- (1) $\mu_X \subseteq \nu_X$ if $\mu_X(x) \leq \nu_X(x)$ for all $x \in X$,
- (2) $\mu_X \subset \nu_X$ if $\mu_X(x) \leq \nu_X(x)$ for all $x \in X$ and there is at least one $x \in X$ such that $\mu_X(x) < \nu_X(x)$,
- (3) $\mu_X = \nu_X$ if $\mu_X(x) = \nu_X(x)$ for all $x \in X$.

Notation 1 We use the symbols ‘ $\vee$ ’ and ‘ $\wedge$ ’ to represent the maximum and minimum of two membership degrees respectively. Therefore, for a fuzzy set (X, μ_X) , the maximum membership degree of $x_1, x_2 \in X$, is denoted as $\mu_X(x_1) \vee \mu_X(x_2) = \max\{\mu_X(x_1), \mu_X(x_2)\}$ while their minimum membership degree is denoted as $\mu_X(x_1) \wedge \mu_X(x_2) = \min\{\mu_X(x_1), \mu_X(x_2)\}$.

Definition 4 (Fuzzy Relation) Let X and Y be sets with ‘fuzzy subsets’ μ_X and μ_Y respectively. A fuzzy relation ρ from ‘fuzzy subset μ_X ’ into the ‘fuzzy subset μ_Y ’ is a ‘fuzzy subset’ ρ of $X \times Y$, hence it is a mapping

$$\rho : X \times Y \rightarrow [0, 1]$$

such that

$$\rho(x, y) \leq \mu_X(x) \wedge \mu_Y(y) \text{ for all } x \in X, y \in Y.$$

We say ρ is a fuzzy relation on μ_X , whenever $X = Y$ and $\mu_X = \mu_Y$, and denote it by ρ_X .

Definition 5 (Fuzzy Composition) [[9], Definition 1.4] Let $\rho_1 : X \times Y \rightarrow [0, 1]$ be a fuzzy relation from ‘fuzzy subset μ_X ’ into ‘fuzzy subset μ_Y ’ and $\rho_2 : Y \times Z \rightarrow [0, 1]$ be a fuzzy relation from ‘fuzzy subset μ_Y ’ into ‘fuzzy subset μ_Z .’ The composition of ρ_1 with ρ_2 , is a fuzzy relation from the ‘fuzzy subset μ_X ’ into ‘fuzzy subset μ_Z ,’ given by $\rho_1 \circ \rho_2 : X \times Z \rightarrow [0, 1]$ and defined as

$$\rho_1 \circ \rho_2(x, z) = \sup \{ \rho_1(x, y) \wedge \rho_2(y, z) : y \in Y \}.$$

Definition 6 (Fuzzy Equivalence Relation) Consider a ‘fuzzy subset μ_X ’ of set X and let ρ_X be a ‘fuzzy relation’ on X . Then ρ_X is called an fuzzy equivalence relation on X if

- (1) ρ_X is reflexive: $\rho_X(x, x) = \mu_X(x)$ for all $x \in X$
- (2) ρ_X is symmetric: $\rho_X(x, y) = \rho_X(y, x)$ for all $x, y \in X$
- (3) ρ_X is transitive: $\rho_X \circ \rho_X(x, y) \leq \rho_X(x, y)$ for all $x, y \in X$.

Kaufmann [8] introduced the notion of ‘fuzzy graph’ first and it was based on Zadeh’s concept of ‘fuzzy relations.’ Later, Rosenfeld [14] expanded on this idea by considering ‘fuzzy relations’ on ‘fuzzy sets’ and developing the theory of ‘fuzzy graphs.’

Definition 7 (Fuzzy Graph) A fuzzy graph G consists of non-empty set V together with pair of functions μ_V , which is a ‘fuzzy subset’ of V and ρ_V , which is a ‘fuzzy relation’ on μ_V .

Notation 2 A fuzzy graph (FG) G on V with functions μ_V and ρ_V is denoted as $G = (V, \mu_V, \rho_V)$. The ‘fuzzy subset’ μ_V is called as fuzzy vertex set of G and ‘fuzzy relation’ ρ_V is called as fuzzy edge set of G .

A FG $G = (V, \mu_V, \rho_V)$, is a weighted graph where a edge $(x, y) \in V \times V$ has the weight $\rho_V(x, y)$. It can be directed or undirected.

Example 1 Consider a fuzzy graph $G = (V, \mu_V, \rho_V)$ illustrated in Fig. 1, where $V = \{u, v, w\}$. Suppose the membership degree of each vertices in G is $\mu_V(u) = 0.4$, $\mu_V(v) = 0.5$, and $\mu_V(w) = 0.8$. Since $\rho_V(u, v) \leq \min \{\mu_V(u), \mu_V(v)\}$, thus $\rho_V(u, v) \leq 0.4$. Similarly, $\rho_V(u, w) \leq 0.4$, and $\rho_V(v, w) \leq 0.5$. Set $\rho_V(u, v) = 0.3$, $\rho_V(u, w) = 0.4$ and $\rho_V(v, w) = 0.1$. Thus, Fig. 1, depicts an ‘undirected fuzzy graph.’

Graphs are excellent way to visualise the relation between nodes, but it has computational limits. In order to enhance computational ability of graphs in computer algorithms, they are represented using matrices. Similar to matrices associated with crisp graph, Pal et al. [12], defined matrix associated with fuzzy graphs.

Definition 8 (Adjacency matrix) Let $G = (V, \mu_V, \rho_V)$ denote a fuzzy undirected graph where $V = \{v_1, \dots, v_n\}$. The ‘adjacency matrix’ associated with G is defined as $\mathcal{A}[G] = [\mathbf{r}_{\mathcal{A}(i)}][\mathbf{a}_{ij}]_{n \times n}$, where

$$\mathbf{r}_{\mathcal{A}(i)} = \mu_V(v_i) \text{ and } \mathbf{a}_{ij} = \rho_V(v_i, v_j)$$

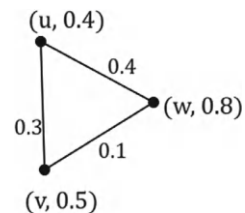
for all $1 \leq i, j \leq n$.

Example 2 The ‘adjacency matrix’ of G , as shown in Fig. 1 of Example 1, is

$$\text{represented as follows: } \mathcal{A}[G] = \begin{matrix} & \begin{matrix} u(0.4) & v(0.5) & w(0.8) \end{matrix} \\ \begin{matrix} u(0.4) \\ v(0.5) \\ w(0.8) \end{matrix} & \begin{pmatrix} 0.0 & 0.3 & 0.4 \\ 0.3 & 0.0 & 0.1 \\ 0.4 & 0.1 & 0.0 \end{pmatrix} \end{matrix}$$

Similarly one can extend this concept of ‘adjacency matrix’ to directed fuzzy graph. The ‘adjacency matrix’ that is associated with an undirected FG is symmetric whereas, ‘adjacency matrix’ associated with an directed FG need not be symmetric.

Fig. 1 An fuzzy undirected graph $G = (V, \mu_V, \rho_V)$



3 Fuzzy Social Network

Networks are represented by matrices and visualized by graphs. A ‘fuzzy social network (FSN)’ is a directed labelled ‘fuzzy graph’. In this section we explore various concepts associated with fuzzy social network.

Definition 9 (*Fuzzy Social Network (FSN)* [5], Sect. 3) A FSN is a ‘fuzzy relational structure’ $\mathcal{U} = (U, \{A_i\}_{i \in I})$, where,

- (1) U is an non-empty ‘finite set of nodes’ called as actors of network.
- (2) $\{A_i\}_{i \in I}$ is a family of ‘fuzzy relations’ between actors.

A FSN can be either directed or undirected. In a directed FSN, the ‘fuzzy relation’ between actors is directional, meaning that ties are oriented from one actor to another. This type of ‘fuzzy relation’ is referred to as a directed fuzzy relation. In the following part of this section, we will review the definitions of undirected and directed fuzzy social networks.

Definition 10 (*Undirected Fuzzy Social Network (UFSN)* [7], Definition 1) An UFSN is the ‘fuzzy relational structure’ $\tilde{G} = (V, \tilde{E})$, where $V = \{v_1, \dots, v_n\}$ is

the set of actors (or nodes) and $\tilde{E} = \begin{pmatrix} \tilde{e}_{11} & \cdots & \tilde{e}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{e}_{n1} & \cdots & \tilde{e}_{nn} \end{pmatrix}$ is an ‘undirected fuzzy relation’

on V .

Definition 11 (*Directed Fuzzy Social Network (DFS N)* [7], Definition 2) An DFSN on n actors is the ‘fuzzy relational structure’ $\tilde{G}_{dn} = (V, \tilde{E}_{dn})$, where $V =$

$\{v_1, \dots, v_n\}$ is the set of actors (or nodes) and $\tilde{E}_{dn} = \begin{pmatrix} \tilde{e}_{11} & \cdots & \tilde{e}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{e}_{n1} & \cdots & \tilde{e}_{nn} \end{pmatrix}$ is an ‘directed

fuzzy relation’ on V .

One type of matrices associated with fuzzy graphs is the incidence matrix, which is defined as follows.

Definition 12 Let $\mu(\tilde{e}_{ij})$ denote the weight or strength of edge e_{ij} connecting the

vertex v_i to the vertex v_j , then the matrix $\mu_{\tilde{E}_{dn}} = \begin{pmatrix} \mu(\tilde{e}_{11}) & \cdots & \mu(\tilde{e}_{1n}) \\ \vdots & \ddots & \vdots \\ \mu(\tilde{e}_{n1}) & \cdots & \mu(\tilde{e}_{nn}) \end{pmatrix}$ is called the incidence matrix or membership function of $\tilde{E}_{dn}$.

3.1 Connectivity in Fuzzy Social Network

In this subsection, the criteria for determining when a FSN is considered connected will be discussed

Definition 13 (*Fuzzy Directed Walk* [7], Definition 3) A fuzzy directed walk in a DFSN is a sequence of ‘alternating actors’ and ‘fuzzy relation’ so that each ‘fuzzy relation’ has its origin at the previous actor and its terminus at the subsequent actors such that all the ‘fuzzy relations’ are pointing in the same direction.

Example 3 Consider directed fuzzy social network $\widetilde{G}_{d_4} = (V, \widetilde{E}_{d_4})$ given in Fig. 2, where $V = \{a, b, c, d\}$. Some fuzzy directed walk in $\widetilde{G}_{d_4}$ are: $a - b - c - d$, $b - c - d - a - b - d$, $a - b - d$. In a fuzzy directed walk, same actors and same fuzzy relations can be used multiple times.

Definition 14 (*Fuzzy Directed path* [7], Definition 5) A fuzzy directed path in a DFSN is a ‘fuzzy directed walk’ in which no actor and no ‘fuzzy relation’ is included more than once.

Example 4 In Fig. 2 of Example 3, fuzzy directed walk $a - b - c - d$ is a fuzzy directed path but fuzzy directed walk $b - c - d - a - b - d$ is not a fuzzy directed path as it includes actors b and d more than once.

Definition 15 (*Fuzzy Semipath*) A fuzzy semipath that joins actors v and w in a DFSN is a sequence of ‘distinct actors’, such that all successive pairs of actors being connected either by a ‘fuzzy relation’ from the first to the second or by a ‘fuzzy relation’ from the second to the first.

Example 5 In Fig. 2 of Example 3, actors b and d are joined by semipaths $b - d$, $b - c - d$, $b - a - d$.

Remark 3 In a fuzzy walk, all actors point in the same direction. However, in a ‘fuzzy semipath,’ the direction of a ‘fuzzy relation’ is not relevant. It is important to note that every ‘fuzzy path’ is a ‘semipath,’ but the converse may not always be true.

Hu and Zhang [7] proposed four type of connectivity in DFSN. We discuss each of them.

Definition 16 (*Fuzzy Weakly Connected*) Let $\widetilde{G}_{dn} = (V, \widetilde{E}_{dn})$ be a DFSN and $v, w \in V$ be two actors in $\widetilde{G}_{dn}$. Then v and w are said to be fuzzy weakly connected if there is a ‘fuzzy semipath’ between v to w .

Example 6 Consider DFSN $\widetilde{G}_{d3} = (V, \widetilde{E}_{d3})$ given in Fig. 3, where $V = \{u, v, w\}$. Actors u and w are fuzzy weakly connected as there is a path from u to v given by $u - v$ and there is also a path from w to v given by $w - v$, hence there is a semipath between u and w .

Fig. 2 Walks in fuzzy directed social network

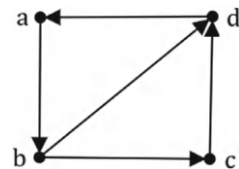
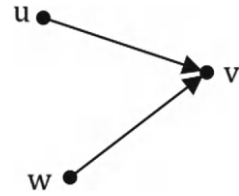


Fig. 3 Weakly connected actors in fuzzy directed social network



Definition 17 (*Fuzzy Unilaterally Connected*) Let $\widetilde{G}_{dn} = (V, \widetilde{E}_{dn})$ be a DFSN and $u, w \in V$ be two actors in $\widetilde{G}_{dn}$. Then u and w are said to be fuzzy unilaterally connected if there exists a ‘fuzzy directed path’ starting from u and ending at w **or** there is a ‘fuzzy directed path’ starting from w and ending at u .

Example 7 Consider DFSN $\widetilde{G}_{d_3} = (V, \widetilde{E}_{d_3})$ given in Fig. 4, where $V = \{u, v, w\}$. Since there is a ‘fuzzy directed path’ from u to w , therefore u and w are fuzzy unilaterally connected. Actors v and w are also fuzzy unilaterally connected in $\widetilde{G}_{d_3}$.

Definition 18 (*Fuzzy Strongly Connected*) Let $\widetilde{G}_{dn} = (V, \widetilde{E}_{dn})$ be a DFSN and $u, w \in V$ be two actors in $\widetilde{G}_{dn}$. Then u and w are said to be fuzzy strongly connected if there exists a ‘fuzzy directed path’ starting from u and ending at w **and** there is a ‘fuzzy directed path’ starting from w and ending at u .

Fig. 4 A fuzzy directed social network $\widetilde{G}_{d_3} = (V, \widetilde{E}_{d_3})$ in which actors u and w are fuzzy unilaterally connected



In Definition 18, the path starting from v and ending at w may contain ‘actors’ and ‘fuzzy relations’ that are different from ‘actors’ and ‘fuzzy relations’ present in path starting from w and ending at v .

Example 8 Consider DFSN $\widetilde{G}_{d_3} = (V, \widetilde{E}_{d_3})$ given in Fig. 5, where $V = \{u, v, w\}$. Since there exists a ‘fuzzy directed path’ from u to w given by $u - v - w$ and there exists a ‘fuzzy directed path’ from w to u , given by $w - u$, therefore u and w are ‘fuzzy strongly connected.’

Definition 19 (*Fuzzy Recursively Connected*) Let $\widetilde{G}_{dn} = (V, \widetilde{E}_{dn})$ be a DFSN and $u, w \in V$ be two actors in $\widetilde{G}_{dn}$. Then u and w are said to be fuzzy recursively connected if they are ‘strongly connected’ and the ‘fuzzy directed path’ from u to w uses same actors and fuzzy relations present in ‘fuzzy directed path’ from w to u but in reverse order.

Example 9 Consider DFSN $\widetilde{G}_{d_3} = (V, \widetilde{E}_{d_3})$ given in Fig. 6, where $V = \{u, v, w\}$. There is a ‘fuzzy directed path’ from u to w such that actor u is connected to actor v and then actor v is connected to actor w . There is also a ‘fuzzy directed path’ from w to u involving same set of actors and ‘fuzzy relations’ in exactly reverse order. Therefore u and w are fuzzy recursively connected.

A DFSN denoted as $\widetilde{G}_{dn} = (V, \widetilde{E}_{dn})$ is considered to be “weakly connected,” “unilaterally connected,” “strongly connected,” “recursively connected” if all pairs of actors in $\widetilde{G}_{dn}$ are also “weakly connected,” “unilaterally connected,” “strongly connected,” “recursively connected,” respectively.

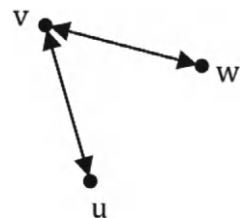
Structural Hole

Structural holes are a crucial aspect of network analysis, as they play a significant role in determining an actor’s influence within a social network. In order to understand

Fig. 5 A fuzzy directed social network $\widetilde{G}_{d_3}$ in which actors u and w are strongly connected



Fig. 6 A fuzzy directed social network $\widetilde{G}_{d_3}$ in which actors u and w are recursively connected



the connectedness of a network, it is important to consider the number of structural holes occupied by actors in a directed fuzzy network. There are two types of structural holes that exist within a DFSN given in following part of this subsection.

Definition 20 (*Unidirectional Fuzzy Structural Hole* [7], Definition 8) Let $\widetilde{G_{dn}} = (V, \widetilde{E_{dn}})$ be a DFSN and $u, v, w \in V$ be actors in $\widetilde{G_{dn}}$ such that u and w are unilaterally connected. Then v is said to occupy a ‘unidirectional fuzzy structural hole’ from u to w if v lies on the path from u to w ,

Example 10 In Fig. 4 of Example 7, actor v occupies the ‘unidirectional fuzzy structural hole.’

Definition 21 (*Bidirectional Fuzzy Structural Hole* [7], Definition 9) Let $\widetilde{G_{dn}} = (V, \widetilde{E_{dn}})$ be a DFSN and $u, v, w \in V$ be actors in $\widetilde{G_{dn}}$ such that u and w are ‘recursively connected.’ Then v is said to occupy a bidirectional fuzzy structural hole from u to w if v lies on the path between u to w ,

Example 11 In Fig. 6 of Example 9, actor v occupies the ‘bidirectional fuzzy structural hole.’

Analyzing the number of “unidirectional and bidirectional structural holes” occupied by actors in a network can help determine the level of control held by these actors. In their study, Hu and Zhang [7] developed an algorithm for identifying the number of unidirectional and bidirectional structural holes occupied by an actor in a network. They applied this algorithm to a case study of the G. Y. research team, which consisted of 8 researchers. Their study focused on unilateral and recursive paths involving at least three actors. This case study is further elaborated in Example 12.

Example 12 Consider social network of research team consisting of 8 actors which is a DFSN $\widetilde{G_{d8}} = (V, \widetilde{E_{d8}})$ as shown in Fig. 7 ([7], Fig. 5).

In Fig. 7, only actor v_4 and v_5 are connected by a recursive path with at least three actors. Since v_6 lies on the path between v_4 and v_5 , thus actor v_6 occupies a ‘bidirectional structural hole.’

For finding number of ‘unidirectional structural hole’ occupied by actor v_6 , consider all unilateral paths with at least three actors on which v_6 lies.

Unilateral paths originating from v_1 are: $v_1 - v_4 - v_6$, $v_1 - v_4 - v_6 - v_5$, and $v_1 - v_4 - v_6 - v_5 - v_2$. Since first path is contained in second and second is contained in third, therefore only third path is considered towards calculation of number of unidirectional structural hole of v_6 . Thus there is one unilateral path originating from v_1 on which v_6 lies.

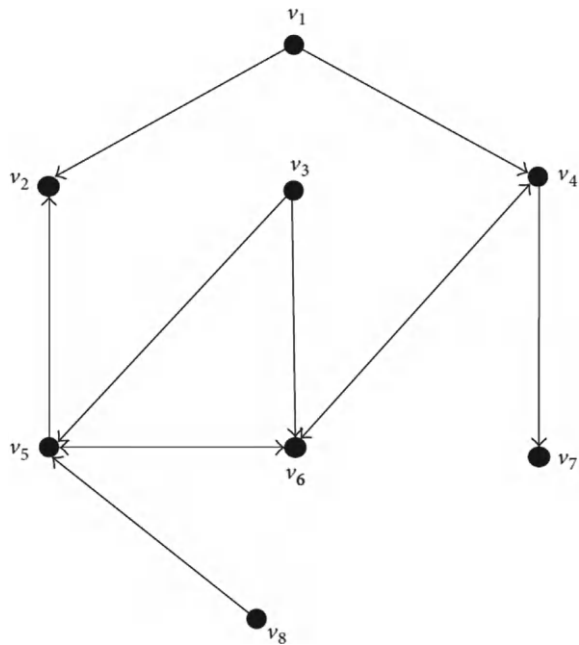
There does not exist unilateral path with at least three actors originating from v_2 .

There are three distinct paths with at least three actors originating from v_3 viz., $v_3 - v_6 - v_4 - v_7$, $v_3 - v_5 - v_6 - v_4 - v_7$ and $v_3 - v_6 - v_5 - v_2$ on which v_6 lies.

There does not exist unilateral path with at least three actors originating from v_4 , v_5 , v_6 and v_7 .

There is only one unilateral path originating from v_8 on which v_6 lies and the path is given by $v_8 - v_5 - v_6 - v_4 - v_7$.

Fig. 7 Social network of research team ([7], Fig. 5)



Therefore, v_6 lies on five unilateral paths with at least three actors hence the number of unidirectional structural hole occupied by v_6 is 5.

Similarly the number of unidirectional and bidirectional structural hole is calculated for each researcher and it is shown in Table 1.

It is evident form Table 1, only v_6 occupies bidirectional structural hole and hence can receive and send maximum information. Therefore v_6 can have maximum control in the network. Actors v_4 , v_5 and v_6 occupy unidirectional structural hole and hence influence network. They act as a hub and intermediaries in the network.

Table 1 Number of fuzzy structural hole in a social network of research team

Actors	Number of unidirectional fuzzy structural hole	Number of bidirectional fuzzy structural hole
v_1	0	0
v_2	0	0
v_3	0	0
v_4	7	0
v_5	7	0
v_6	5	1
v_7	0	0
v_8	0	0

4 Quantum Theory in Social Network

The Quantum Social Network utilizes principles of ‘quantum mechanics,’ such as ‘superposition,’ ‘entanglement,’ ‘interference,’ ‘multiverse,’ ‘tunneling,’ and ‘uncertainty principle,’ to describe the dynamics of a social network [1]. A quantum mathematical model is created by defining the initial configuration of space and modeling nodes (or agents) within it. A function, similar to the wave function in quantum mechanics, is used to define the properties of these nodes. The functioning of the quantum social network is based on the assumption that nodes are entangled in the same way as particles in quantum mechanics. Here entanglement refers to the correlation between particles, where the state of one immediately affects the state of another, regardless of distance. We will now explore more about quantum social networks in the following part of this section.

4.1 States in Quantum Social Network

In this subsection, we will explore concepts related to quantum social networks.

Definition 22 (*Qubit*) A qubit or quantum bit is the basic unit of information in quantum computing.

A qubit has two states $|0\rangle$ and $|1\rangle$. There is the possibility of qubits becoming entangled with one another, which means that they have a relationship that prevents them from acting independently. A multi-qubit system is a collection of multiple qubits which is treated as single system. An n -qubit can be represented as a vector $(x_1, \dots, x_n)^T$ where T is the transpose of the vector. We use tensor product to represent an n -qubit. If $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ are qubits then their tensor product is given as

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}.$$

One can extend this tensor product for multi-qubit systems in an inductive manner.

In a ‘quantum system’ with n qubits, there are total 2^n basis states. For example in a system with 2 qubits $|0\rangle$ and $|1\rangle$ represented by vectors $(1 \ 0)^T$ and $(0 \ 1)^T$ respectively there are $2^2 = 4$ basis states in total which are given as $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$, and they are binary equivalent of 0, 1, 2, 3 respectively, where,

$$|00\rangle = |0\rangle \otimes |0\rangle, |01\rangle = |0\rangle \otimes |1\rangle, |10\rangle = |1\rangle \otimes |0\rangle, |11\rangle = |1\rangle \otimes |1\rangle.$$

Thus, we have

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = a |00\rangle + b |01\rangle + c |10\rangle + d |11\rangle .$$

We recall that a Hilbert space $\mathcal{H}$ is a metric space that is complete w.r.t. distance function induced from inner product.

Definition 23 In quantum mechanical system, states are represented using unit vectors in Hilbert Space. If $\{|e_1\rangle, |e_2\rangle, \dots, |e_n\rangle\}$ is the orthonormal basis of the Hilbert space $\mathcal{H}$ then the state of a agent (or node) ψ is given by

$$|\psi\rangle = \sum_i c_i |\psi_j\rangle \text{ where } c_i \in \mathbb{C} \text{ and } \sum_i |c_i|^2 = 1$$

We¹ use tensor product to find the state of the system. For example in a Hilbert space of two dimension, if $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ and $|\phi\rangle = \gamma |0\rangle + \delta |1\rangle$ are state of two agents, then state of the system is:

$$\begin{aligned} |\psi\phi\rangle &= |\psi\rangle \otimes |\phi\rangle \\ &= (\alpha |0\rangle + \beta |1\rangle) \otimes (\gamma |0\rangle + \delta |1\rangle) \\ &= \alpha\gamma |00\rangle + \alpha\delta |01\rangle + \beta\gamma |10\rangle + \beta\delta |11\rangle . \end{aligned}$$

In a system consisting of n agents, such that agent i has quantum state $|\psi_i\rangle$ for $1 \leq i \leq n$, the state of the system is given as:

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \dots \otimes |\psi_n\rangle .$$

4.2 Quantum Social Network Models

Some popular applications that utilize quantum social networks include the Quantum Walk Model, Q.NET Project, and Quantum-PSO based unsupervised clustering. In this section, we will provide an outline of each of these models.

4.2.1 Quantum Walk Model

When quantum mechanics principles are applied to classical random walks, the outcome is a quantum walk. This type of walk involves superpositions of paths and inferences. Quantum walks can be either discrete-time or continuous-time. ‘Discrete-time quantum walks’ evolve step by step, while ‘continuous-time quantum walks’ evolve

¹ $\mathbb{C}$ denotes the set of complex numbers.

continuously over time [13]. A quantum walk model is utilized to simulate the formation of consensus in group decision making and the spread of ideas in social networks.

4.2.2 Q.NET Project

The aim of Q.NET project is to develop a platform that is open source and it can model nodes and their interactions. This is achieved by identifying the ‘quantum physics models’ that are suitable to model the evolution and behaviour of organisational communities. Q.NET framework built’s an actor-agent model based on specification of its location along different directions using vector representation of agent’s state [3] and it has six phases. The framework uses ‘digital representation of the social network’ to visualise the results emerging during analysis and simulation phase.

4.2.3 Quantum-PSO Based Unsupervised Clustering

‘Quantum particle swarm optimization (QPSO)’ is used to obtain clusters in the social network data set based on similarity maximization. QPSO is obtained by applying principles of ‘quantum mechanics’ to classical PSO. The goal of QPSO is to form clusters in a social network using quantum theory such that nodes of same cluster have maximum similarity in them with respect to profile attribute [11]. Cluster detection has applications in areas such as prediction and prevention of disease, commodity recommendation, etc. The working of QPSO model involves obtaining the ‘mean best position’ of the population at the $t - th$ iteration for the d dimension which is given by

$$mbest_d^t = \left(\frac{1}{M} \sum_{i=1}^M P_{i1}, \dots, \frac{1}{M} \sum_{i=1}^M P_{id} \right)$$

where, M is the number of particles in the system, P_{id} is the personal best position of $i - th$ particle of $d - th$ dimension and it is given as

$$P_{id} = \varphi P_{id} + (1 - \varphi) P_{gd} \quad (1)$$

In Eq. 1, P_{gd} denotes the ‘global best position’ of the $i - th$ particle for $d - th$ dimension and $\varphi \in (0, 1)$. In quantum space, positions of particles after the particle get through stochastic simulation of Monte Carlo measurement. For further information on the computational aspects, please see reference [11].

5 Applications of Fuzzy Social Network

This section describes two real-life applications that involve fuzzy social networks.

5.1 Sarcasm Detection

Social media data is frequently utilized to analyze customer sentiment through their written content. It can serve as a valuable tool for making social marketing decisions. However, accurately interpreting customer sentiment from social media text can be challenging when sarcasm is present. Sarcasm involves using words that convey the opposite of their intended meaning and is often employed to mock or amuse a situation. In order to address this issue, Amina et al. [2], proposed a fuzzy approach for detecting sarcasm in social networks. They developed a mathematical model known as the ‘SSC model’ which calculates a sarcasm score. The model focuses on four features of a user’s text message and assigns a sarcasm score to each feature using an appropriate machine learning technique. For more technical details on the score calculations, please see the reference [2]. The feature-based sarcasm scores are as follows:

- (1) v_1 = Sarcasm score based on text feature of a message.
- (2) v_2 = Sarcasm score based on historical feature of a message.
- (3) v_3 = Sarcasm score based on identification feature of a message.
- (4) v_4 = Sarcasm score based on response feature of a message.

The degree of sarcasm of text t is given as:

$$SarcasmDegree(t) = \sum_{i=1}^4 w_i v_i,$$

where the weights w_1, w_2, w_3, w_4 represent degree of importance of v_1, v_2, v_3, v_4 respectively. A message is classified as non-sarcastic or undecidable or sarcastic. There are five threshold values t_1, t_2, t_3, t_4, t_5 . Let $x = SarcasmDegree(t)$, denote the sarcasm degree where $x \in [0, 1]$. The membership function for the three classes are:

$$F_{nonsarcastic}(x) = \begin{cases} 1 & \text{if } x \leq t_1 \\ \frac{t_3-x}{t_3-t_1} & \text{if } t_1 < x < t_3 \\ 0 & \text{if } x \geq t_3 \end{cases}$$

$$F_{undecidable}(x) = \begin{cases} 0 & \text{if } x \leq t_1 \text{ or } x \geq t_5 \\ \frac{x-t_1}{t_2-t_1} & \text{if } t_1 < x < t_2 \\ 1 & \text{if } t_2 \leq x \leq t_4 \\ \frac{t_5-x}{t_5-t_4} & \text{if } t_4 < x < t_5 \end{cases}$$

$$F_{sarcastic}(x) = \begin{cases} 0 & \text{if } x \leq t_3 \\ \frac{x-t_3}{t_5-t_3} & \text{if } t_3 < x < t_5 \\ 1 & \text{if } x \geq t_5 \end{cases}$$

where $F_{nonsarcastic}(x)$ is the membership value of non-sarcastic class, $F_{undecidable}(x)$ is the membership value of undecidable class and $F_{sarcastic}(x)$ is the membership value of sarcastic class. The membership value of x is determined using all three fuzzy membership functions and the final decision about text classification is based on the maximum membership value of x . Therefore a social text message is sarcasm if $F_{Sarcasm}(x)$ is maximum of $\{F_{nonsarcasm}(x), F_{undecidable}(x), F_{sarcastic}(x)\}$.

5.2 Healthcare Fuzzy Social Network

Link prediction is a crucial task in dynamic social networks. It involves predicting the edges that will be added to the network in the next time interval, based on a snapshot of the network at a given time, denoted as t [6]. Therefore, in dynamic social networks, the development of methods for predicting links between two actors is a significant area of study.

The task in healthcare social network is to find link between two patients (actors) on the basis of similarity such as medical advice, geographical location, age, etc. Dhouioui et al. [6], proposed a model that generates link between actors in dynamic social network based on friendship score between them. The link between actors A and B is based on score of the function $R_{friend}(A, B)$, which is computed using weights w_1, w_2 such that $w_1 + w_2 = 1$. We revisit the algorithm for determining $R_{friend}(A, B)$. Let

$$R_{similarity}(A, B) = p_1 SimLoc(A, B) + p_2 CloseGeo(A, B) + p_3 Age(A, B) + p_4 GenInt(A, B). \quad (2)$$

with weights p_1, p_2, p_3, p_4 such that $\sum_{i=1}^4 p_i = 1$. In Eq. 2, $SimLoc$ represents level of the similarity of users location, $CloseGeo$, represents how two users are close geographically, Age represents how closely actors A and B are belonging to the same age interval, and $GenInt$ refers if they have the same general interest. The four weights p_1, p_2, p_3, p_4 are all taken equal and they have common value 0.25. Hence from Eq. 2,

$$R_{similarity}(A, B) = \begin{cases} 1 & \text{if } SimLoc(A, B) = CloseGeo(A, B) = 1 \\ & \text{and } Age(A, B) = GenInt(A, B) = 1 \\ 0 & \text{Otherwise} \end{cases}$$

The next parameter $R_{friendshipchance}$ indicates whether or not a friendship is possible between two actors. Dhouioui et al. [6], developed a truth table for determining the minimal and maximal chances of friendship. We will refer to Table I and Table II from [6], using 'T' the true and 'F' for false. To keep it short we will use the abbreviations:

M.F.M.C.: “Mutual Friends with Maximal Chance”

M.I.M.C.: “Mutual Interactions with Maximal Chance”

M.E.M.C.: “Mutual Events with Maximal Chance”

M.F.Mi.C.: “Mutual Friends with Minimal Chance”

M.I.Mi.C.: “Mutual Interactions with Minimal Chance”

M.E.Mi.C.: “Mutual Events with Minimal Chance”

The truth table for minimal friendship chance is given in Table 2 and truth table for maximal friendship chance is given in Table 3.

$$R_{friendshipchance}(A, B) = \begin{cases} 1 & \text{if } chance = 1 \\ 0 & \text{Otherwise} \end{cases} \quad (3)$$

There must at least be two minimal chances or one maximal chance in order that a relation of friendship is possible. Finally the friendship score between actors A and B is computed as:

$$R_{friend}(A, B) = w_1 R_{friendshipchance}(A, B) + w_2 R_{similarity}(A, B), \quad (4)$$

where, $R_{friendshipchance}(A, B)$ is computed using Eq. 3, and $R_{similarity}(A, B)$ is computed using Eq. 2.

Table 2 Truth table of minimal chance

M.F.Mi.C.	M.I.Mi.C.	M.E.Mi.C.	Friendship chance
F	F	F	F
T	F	F	F
F	T	F	F
T	T	F	T
F	F	T	F
T	F	T	T
F	T	T	T
T	T	T	T

Table 3 Truth table of maximal chance

M.F.Mi.C.	M.I.Mi.C.	M.E.Mi.C.	Friendship chance
F	F	F	F
T	F	F	T
F	T	F	T
T	T	F	T
F	F	T	T
T	F	T	T
F	T	T	T
T	T	T	T

6 Concluding Remarks

The field of fuzzy social networks and quantum social networks is currently a popular area of research. Both types of social networks have numerous applications, but utilizing quantum social networks offers some benefits over fuzzy social networks. Quantum social networks provide improved security and privacy, as well as the ability to handle multiple states simultaneously, resulting in faster data processing and transfer. Additionally, they can enhance the decision-making capabilities of a network. Combining principles of fuzzy logic and quantum mechanics can produce more effective models. With the passage of time, innovative applications of these principles will only continue to grow.

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Quantum Coopetition Graphs



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Abstract The prey-predator model of competition graphs and collaboration model of graphs have been discussed as the backbone of the study. In classical competition graphs, competition arises when predators share a common preys (nodes). This model is extended to the quantum graphs, where vertices with dynamic weights and entangled edges. The concept of quantum coopetition graphs with the dynamics of competition and collaboration has been introduced here. The properties of the relevant concepts have been discussed.

Keywords Quantum graphs · Coopetition · Competition graphs · Collaboration · Dynamic weights

1 Introduction

The study of competition graphs within biological and social systems are significant for researchers. Competition graphs represent the competitive relationships between nodes through prey-predator dynamics. On the other hand, coopetition graphs extend this concept by incorporating both competitive and collaborative interactions.

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A competition graph is derived from a directed graph $D = (V, A)$, where V is the set of vertices and A is the set of directed edges. In a competition graph, an edge exists between two vertices if they have a common out-neighbor in the directed graph. Mathematically, for a directed graph D , the competition graph $C(D)$ is an undirected graph $G = (V, E)$ where:

$$E = \{(u, v) \mid \exists w \in V \text{ such that } (u, w) \in A \text{ and } (v, w) \in A\}$$

This means that vertices u and v are adjacent in the competition graph if there is a vertex w such that both u and v have directed edges to w .

Coopetition graphs is a generalised version which represents both competitive and collaborative interactions. Semi-directed graphs have some directed and some undirected edges. Competitions are represented by directed edges, and collaborations are represented by undirected edges. Mathematically, for a semi-directed graph $D = (V, A, E)$, where A is the set of directed edges and E is the set of undirected edges, the coopetition graph $C(D)$ with vertex set V is defined as follows:

$$E_c = \{(u, v) \mid (u, v) \in E \text{ and } \exists w \in V \text{ such that } (u, w) \in A \text{ and } (v, w) \in A\}$$

This means that vertices u and v are adjacent in the coopetition graph if they have a common out-neighbor in the directed part of the semi-directed graph.

Additionally, an undirected edge (collaboration) in a semi-directed graph results in a bunch node in coopetition graphs containing the end vertices:

If $(u, v) \in E$, then $\{u, v\}$ forms a bunch node in $C(D)$.

If there is a clique of undirected edges (collaborative groups), it results in a bunch node containing all the vertices of the clique in coopetition graphs:

If $\{u_1, u_2, \dots, u_k\}$ is a clique in E , then $\{u_1, u_2, \dots, u_k\}$ forms a bunch node in $C(D)$.

In a quantum graph, vertices can represent quantum states or particles (in reality, nodes of any dynamical systems), and edges can represent entanglement or other quantum correlations.

A quantum graph [27] with dynamic vertices and entangled edges is defined as follows:

1. **Vertices with Dynamic Weights (dweight):** Each vertex v_i has a dynamic weight $\text{dweight}_i(t)$ that changes over time:

$$0 \leq \text{dweight}_i(t) \leq 1$$

2. **Edges and Entanglement:** Edges e_{ij} between vertices v_i and v_j can be entangled, represented by a correlation function $g_{ij}(t)$:

$$g_{ij}(t) = h(\text{dweight}_i(t), \text{dweight}_j(t))$$

3. **Graph Structure:** The graph G is represented as $G = (V, E)$, with an adjacency matrix $A(t)$:

$$A_{ij}(t) = \begin{cases} g_{ij}(t) & \text{if } e_{ij} \text{ exists and is entangled} \\ 0 & \text{otherwise} \end{cases}$$

The primary objectives of this chapter are to: Discuss the preliminaries of coopetition graphs; Define the concept of quantum coopetition graphs; Investigate the properties of quantum coopetition graphs.

2 Literature Review

Competition graphs are directly related to represent ecological (prey-predator model) and social systems. Authors provided discussed the properties of competition graphs [4], role of the graph in modeling complex biological interactions [5], in competitive interactions [6].

Graph theory with uncertainty, named fuzzy graphs are also useful in optimization techniques. Fuzzy sets (FS) have become a popular approach for incorporating fuzzy information as part of an expert's evaluative measures. For example, the development of fuzzy fractional differential equations using fuzzy Laplace transforms is fit for modeling systems where the influence of individuals or groups evolves fractionally over time under uncertainty [10]. In [29, 38] studied fuzzy variable precision rough sets, in [30] introduced hesitant bipolar-valued fuzzy sets and bipolar-valued hesitant fuzzy sets, and studied multi-granulation bipolar-valued fuzzy probabilistic rough sets and their corresponding three-way decisions.

The introduction of predictor-corrector methods (in solving fuzzy differential equations) has direct applications in social groups [11, 13–17]. These methods allow for a more accurate simulation of how consensus forms over time, particularly in environments where group dynamics are shaped by uncertainty and imprecision.

Authors [9, 28] have studied the use of fuzzy logic in the optimization of industrial processes. In 2016, Samanta et al. introduced on the completeness and regularity of generalized fuzzy graphs offered significant insights into the structural properties of these graphs [19]. In 2021, they introduced a measure of influences in social networks, which has practical implications for understanding and analyzing social dynamics [20], vague graphs and strengths [21], and a new approach to social networks based on fuzzy graphs [8] have further enriched the literature. Moreover, their work on competition cluster hypergraphs [23], the concept of fuzzy planar graphs [24], and the coloring of COVID-19 affected regions based on fuzzy directed graphs [25]

demonstrate their ongoing commitment to advancing fuzzy graph theory and its applications. Their research on generalized fuzzy trees [26].

In 2011, they introduced a full fuzzy method for solving differential equations based on Taylor expansion for handling uncertainties in differential equations [12]. Other notable works include their research on fuzzy integer linear programming problems [17], the trapezoidal approximation of fuzzy numbers [18].

Several studies relevance of coopetition in various fields in medical diagnosis [39], theoretical framework of coopetition [40], in network structures [41], properties of coopetition [42], graph neural networks to financial fraud detection [43]. in signal processing [44], in material science [45], in natural language processing [46], secure communication aspects [47]. Coopetition represent a model competitive and cooperative interactions among entities [35–37] and in social networks to business ecosystems [1–3, 34].

Competition graphs have been a subject of extensive research due to their applications in various domains in fuzzy environments [22, 29–31]. Kalampakas et al. (2014) contributed on syntactic recognition of fuzzy graphs, which are crucial for modeling competitive interactions [32]. Their earlier work on the syntactic recognizability of graphs with fuzzy attributes also provides a basis for understanding competition in fuzzy graph theory [33].

Collaboration and competition are found in real worlds more often. Recent advancements in the field have focused on fuzzy competition graphs [7, 8], m -step fuzzy competition graphs [8]. Samanta, Dubey, and Das (2022) demonstrated the relevance of competition and cooperation models as coopetition graphs [9]. But in literature we found the coopetition graphs in static model. In this study, coopetition system is formed in dynamical systems as quantum coopetition graphs.

3 Quantum Competition Graphs

Quantum Directed Graphs extend the concept of quantum graphs by incorporating directionality in the edges. For a quantum directed graph D , the quantum competition graph $QC(D)$ is an undirected graph $G = (V, E)$ where:

$$E = \{(u, v) \mid \exists w \in V \text{ such that } (u, w) \in A \text{ and } (v, w) \in A \text{ and } q(u, v, w) \neq 0\}$$

Here, $q(u, v, w)$ represents a quantum correlation function that measures the influence of quantum entanglement between vertices u and v through a common neighbor w . This function could depend on the dynamic weights of the vertices and the nature of their quantum entanglement.

3.1 Example 1: Quantum Sensors Network

We consider a network of quantum sensors, let $q(u, v, w) = \frac{1}{1+e^{-\alpha(w_u+w_v-w_w)}}$, where w_u , w_v , and w_w are the dynamic weights of vertices u , v , and w respectively, and α is a parameter that controls the sensitivity to weight differences. This function models the probability that the competitive interaction between u and v is influenced by their entanglement with w (Table 1).

3.2 Example 2: Quantum Communication Network

In a quantum communication network, vertices represent communication nodes, and edges represent the potential for information transfer, influenced by quantum entanglement. The quantum correlation function $q(u, v, w)$ could be defined as:

$$q(u, v, w) = \text{Tr}(\rho_{uw} \cdot \rho_{vw}),$$

where ρ_{uw} and ρ_{vw} are the density matrices representing the quantum states between u and w , and v and w respectively. This function measures the overlap of quantum states, indicating the strength of entanglement (Table 2).

3.3 Example 3: Quantum Biological Systems

In a quantum biological system, vertices could represent different biological entities, and edges represent competitive interactions influenced by quantum coherence. The correlation function $q(u, v, w)$ might be:

$$q(u, v, w) = \cos(\theta_{uw} - \theta_{vw}),$$

Table 1 Quantum sensors network

Vertices	Common neighbor	Quantum correlation function
u, v	w	$q(u, v, w) = \frac{1}{1+e^{-\alpha(w_u+w_v-w_w)}}$

Table 2 Quantum communication network

Vertices	Common neighbor	Quantum correlation function
u, v	w	$q(u, v, w) = \text{Tr}(\rho_{uw} \cdot \rho_{vw})$

Table 3 Quantum biological systems

Vertices	Common neighbor	Quantum correlation function
u, v	w	$q(u, v, w) = \cos(\theta_{uw} - \theta_{vw})$

where θ_{uw} and θ_{vw} are the phases of the quantum states between u and w , and v and w respectively. This function captures the phase coherence between the entities, influencing their competitive interactions (Table 3).

4 Quantum Coopetition Graphs

A coopetition graph is derived from a semi-directed graph $D = (V, A, E)$, where V is the set of vertices, A is the set of directed edges representing competition, and E is the set of undirected edges representing collaboration.

For a semi-directed graph D , the quantum coopetition graph $QC(D)$ is an undirected graph $G = (V, E_c)$ where:

$$E_c = \{(u, v) \mid (u, v) \in E \text{ or } (\exists w \in V \text{ such that } (u, w) \in A \text{ and } (v, w) \in A \text{ and } q(u, v, w) \neq 0)\}$$

Here, $q(u, v, w)$ represents a quantum correlation function that measures the influence of quantum entanglement between vertices u and v through a common neighbor w . This function could depend on the dynamic weights of the vertices and the nature of their quantum entanglement.

Additionally, an undirected edge (collaboration) in a semi-directed graph results in a bunch node in quantum coopetition graphs containing the end vertices:

If $(u, v) \in E$, then $\{u, v\}$ forms a bunch node in $QC(D)$.

If there is a clique of undirected edges (collaborative groups), it results in a bunch node containing all the vertices of the clique in quantum coopetition graphs:

If $\{u_1, u_2, \dots, u_k\}$ is a clique in E , then $\{u_1, u_2, \dots, u_k\}$ forms a bunch node in $QC(D)$.

4.1 Examples

4.1.1 Example 1: Quantum Sensors Network

Consider a network of quantum sensors, where each sensor (vertex) has a dynamic weight representing its measurement accuracy over time. The edges represent competitive and collaborative interactions between sensors, influenced by their quantum

entanglement. The adjacency matrix $A(t)$ captures the time-dependent competitive structure, while $E(t)$ captures the collaborative structure.

For instance, let $q(u, v, w) = \frac{1}{1+e^{-\alpha(w_u+w_v-w_w)}}$, where w_u , w_v , and w_w are the dynamic weights of vertices u , v , and w respectively, and α is a parameter that controls the sensitivity to weight differences. This function models the probability that the competitive interaction between u and v is influenced by their entanglement with w .

4.1.2 Example 2: Quantum Communication Network

In a quantum communication network, vertices represent communication nodes, and edges represent the potential for information transfer, influenced by quantum entanglement. The quantum correlation function $q(u, v, w)$ could be defined as:

$$q(u, v, w) = \text{Tr}(\rho_{uw} \cdot \rho_{vw}),$$

where ρ_{uw} and ρ_{vw} are the density matrices representing the quantum states between u and w , and v and w respectively. This function measures the overlap of quantum states, indicating the strength of entanglement.

4.1.3 Example 3: Quantum Biological Systems

In a quantum biological system, vertices could represent different biological entities, and edges represent competitive and collaborative interactions influenced by quantum coherence. The correlation function $q(u, v, w)$ might be:

$$q(u, v, w) = \cos(\theta_{uw} - \theta_{vw}),$$

where θ_{uw} and θ_{vw} are the phases of the quantum states between u and w , and v and w respectively. This function captures the phase coherence between the entities, influencing their competitive and collaborative interactions.

These examples illustrate how quantum coopetition graphs can model complex systems where interactions are influenced by quantum properties, providing a richer framework for analyzing such systems.

4.2 *t*-Coopetition Number

The ***t*-coopetition number** of a semi-directed graph $D = (V, A, E)$ is defined as the maximum number of pairwise simultaneous competition and collaboration relationships that satisfy a threshold t for quantum correlation function. *t*-coopetition

number will be counted for each undirected edge $(u, v) \in E$, there exists a common out-neighborhood in the semi directed graph.

Formally, the t -coopetition number $\kappa_t(D)$ is given by:

$$\kappa_t(D) = \max \{|E_c| \mid E_c \subseteq E \text{ and } \forall (u, v) \in E_c, \exists w \in V \text{ such that } (u, w) \in A \text{ and } (v, w) \in A \text{ and } q(u, v, w) > t\}$$

4.2.1 Remark 1: Upper Bound on t -Coopetition Number

Remark For any semi-directed graph $D = (V, A, E)$, the t -coopetition number $\kappa_t(D)$ is bounded above by the number of undirected edges E .

Proof By definition, the t -coopetition number $\kappa_t(D)$ is the maximum size of a subset $E_c \subseteq E$ such that for each $(u, v) \in E_c$, there exists a vertex $w \in V$ with $(u, w) \in A$ and $(v, w) \in A$ and $q(u, v, w) > t$. Since E_c is a subset of E , it follows that:

$$\kappa_t(D) \leq |E|$$

Thus, the t -coopetition number is bounded above by the total number of undirected edges in the graph.

4.2.2 Remark 2: t -Coopetition Number in Bipartite Graphs

Remark For a bipartite semi-directed graph $B = (V_1 \cup V_2, A, E)$, the t -coopetition number $\kappa_t(B)$ is zero if there are no common out-neighbors for any pair of vertices from different partitions.

Proof In a bipartite graph, vertices are divided into two disjoint sets V_1 and V_2 such that every edge connects a vertex in V_1 to a vertex in V_2 . If there are no common out-neighbors for any pair of vertices (u, v) where $u \in V_1$ and $v \in V_2$, then for any undirected edge $(u, v) \in E$, there does not exist a vertex $w \in V$ such that $(u, w) \in A$ and $(v, w) \in A$. Therefore, no edge can be included in E_c , and:

$$\kappa_t(B) = 0$$

4.3 Theorem 1: Existence of t -Coopetition Number

Theorem For any quantum coopetition graph $D = (V, A, E)$, the t -coopetition number $\kappa_t(D)$ exists and is finite.

Proof By definition, the t -coopetition number $\kappa_t(D)$ is the maximum size of a subset $E_c \subseteq E$ such that for each $(u, v) \in E_c$, there exists a vertex $w \in V$ with $(u, w) \in A$

and $(v, w) \in A$ and $q(u, v, w) > t$. Since E is finite, any subset E_c is also finite. Therefore, $\kappa_t(D)$ is finite.

4.4 Theorem 2: t -Coopetition Number in Complete Quantum Coopetition Graphs

Theorem For a complete quantum coopetition graph K_n with n vertices, the t -coopetition number $\kappa_t(K_n)$ is $\binom{n}{2}$ if all pairwise quantum correlation values exceed t .

Proof In a complete quantum coopetition graph K_n , every pair of vertices (u, v) is connected by an undirected edge, and there exists a directed edge from each vertex to every other vertex. Therefore, for any pair (u, v) , we can always find a common out-neighbor w such that $(u, w) \in A$ and $(v, w) \in A$. If the quantum correlation value $q(u, v, w)$ exceeds t for all such pairs, then every undirected edge (u, v) can be included in E_c . Hence:

$$\kappa_t(K_n) = \binom{n}{2}$$

where $\binom{n}{2}$ is the number of ways to choose 2 vertices out of n , representing the total number of undirected edges in K_n .

4.5 Theorem 3: Lower Bound on t -Coopetition Number

Theorem For any quantum coopetition graph $D = (V, A, E)$, the t -coopetition number $\kappa_t(D)$ is at least the number of edges in any subgraph where all quantum correlation values exceed t .

Proof We consider a subgraph $D' = (V', A', E')$ of D where for every edge $(u, v) \in E'$, there exists a vertex $w \in V'$ such that $(u, w) \in A'$ and $(v, w) \in A'$ and $q(u, v, w) > t$. By definition, all edges in E' satisfy the condition for the t -coopetition number. Therefore, $E' \subseteq E_c$ and:

$$\kappa_t(D) \geq |E'|$$

5 Conclusions

We began by defining the structure of quantum coopetition graphs, derived from semi-directed graphs that include both directed and undirected edges. The directed edges represent competitive interactions, while the undirected edges represent collaborative

interactions. We introduced the quantum correlation function ($q(u, v, w)$), which measures the influence of quantum entanglement between vertices through a common neighbor.

We defined t-coopetition number and its properties. The properties of quantum coopetition graphs in representing real-world systems have some limitations. The theories of quantum coopetition graphs will be helpful in network theory representing collaborations and competitions, in dynamic networks such as brain networks, in biological networks. Additionally, it will be useful in quantum computing where qubits may be of coopetition in nature.

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Quantum Computing and Fuzzy Logic



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Abstract This chapter discusses quantum computing and fuzzy logic. The quantum bits (qubits) inherently contains uncertainty. This motivates the study to define quantum fuzzy logic. Besides few imporatan properties of quantum fuzzy logic has been discussed.

Keywords Quantum computing · Fuzzy logic · Quantum fuzzy logic · Quantum algorithms

1 Introduction

Quantum computing follows the principles of quantum mechanics such as superposition and entanglement. These principles allow quantum computers to process information in ways that classical computers cannot. For instance, a quantum bit or qubit can exist in a superposition of states, as follows:

$$|\phi\rangle = \gamma|0\rangle + \delta|1\rangle$$

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where γ and δ are complex numbers satisfying $|\gamma|^2 + |\delta|^2 = 1$. This property allows quantum computers to execute parallel computations, potentially solving certain problems much faster than classical computers.

Fuzzy logic, introduced by Lotfi Zadeh in 1965, provides a tool for reasoning with uncertainty. classical binary logic operates on precise true or false values, while fuzzy logic deals with degrees of truth. A fuzzy set A in a universe of discourse X is characterized by a membership function $\mu_A : X \rightarrow [0, 1]$, where $\mu_A(x)$ represents the degree of membership of element x in set A . This allows for more flexible and human-like reasoning in complex systems.

The primary objectives of this chapter are to:

- review the current state of quantum computing and fuzzy logic.
- propose a new model for Quantum Fuzzy Logic.
- discuss the potential applications and properties of Quantum Fuzzy Logic.

2 Literature Review

Research on quantum computing is significantly advanced in recent years. Quantum computation and quantum information have been studied in [1]. Recent research has focused on improving quantum algorithms and hardware [2, 44, 45].

Quantum computers use quantum bits (qubits) that can exist in multiple states simultaneously due to superposition and entanglement [3]. García et al. (2023) conducted a systematic literature review identifying various quantum implementations of classical machine learning algorithms [4]. Quantum computing uses the principles of quantum mechanics in processing large datasets and solving optimization problems [41, 42, 47]. Quantum algorithms are designed to solve specific problems more efficiently than classical algorithms. Shor's algorithm for factoring large numbers and Grover's algorithm for database search are notable examples. These algorithms demonstrate the potential for exponential speedup in solving certain computational problems [5]. Additionally, recent advancements in quantum algorithms have expanded their applicability to areas such as cryptography, and optimization [3]. Sood and Chauhan (2023) emphasize the need for advancements in quantum hardware and error correction techniques to achieve practical quantum advantage [5]. Research is ongoing to improve new quantum hardware architectures [3]. A review by Gill et al. (2020) discusses the potential impact of quantum computing on these fields and highlights ongoing research efforts to develop quantum algorithms tailored to specific applications [3].

Authors [9, 10] developed a fuzzy logic inference model to evaluate the effect of extrinsic factors on the transmission of infectious diseases. This model uses fuzzy logic to handle the uncertainty and variability in environmental and social factors, providing a more accurate assessment of disease transmission risks.

Fuzzy sets (FS) have become a popular approach for incorporating fuzzy information as part of an expert's evaluative measures. Fuzzy logic is essential in fields

such as medical diagnosis and sentiment analysis [39, 46]. Recent studies on quantum fuzzy neural networks for multimodal sentiment and sarcasm detection [46] and the application of quantum graph neural networks in financial fraud detection [43] are useful for researchers. For example, the development of fuzzy fractional differential equations using fuzzy Laplace transforms is fit for modeling systems where the influence of individuals or groups evolves fractionally over time under uncertainty [23]. In [29, 31] studied fuzzy variable precision rough sets, in [30, 32] introduced hesitant bipolar-valued fuzzy sets and bipolar-valued hesitant fuzzy sets, and studied multi-granulation bipolar-valued fuzzy probabilistic rough sets and their corresponding three-way decisions.

The introduction of predictor-corrector methods (in solving fuzzy differential equations) has direct applications in social groups [24–28, 33, 34]. Pythagorean fuzzy systems [35, 40] provide methods in decision-making processes.

Fuzzy graph theory [6–8, 11, 14] such as fuzzy competition graphs [16], fuzzy planar graphs [15] and applications of edge coloring in fuzzy graphs [12, 13] is used to solve some real world problems.

Samanta et al. proposed generalized fuzzy graph theory connecting relationships of edges and end vertices [17–21]. Generalised fuzzy graphs are further extended to quantum graphs with dynamic weights of vertices and edges changing over times [22]. Some other related articles are [36–38].

2.1 Recent Developments in Quantum Computing

Milestone	Details
Quantum supremacy [48]	Achieved by Google in 2019. Google’s quantum processor, Sycamore, performed a specific task in 200 seconds that would take the world’s fastest supercomputer approximately 10,000 years to complete
Quantum Algorithms [49]	Developed for optimization, machine learning, and cryptography. Demonstrates the potential of quantum computers to solve problems intractable for classical computers
Shor’s algorithm [50]	• Purpose: Integer factorization • Efficiency: Polynomial time complexity $O((\log N)^3)$ • Classical Comparison: Exponentially faster than classical algorithms with complexity $O(e^{(\log N)^{1/3}(\log \log N)^{2/3}})$ • Formula: $N = p \times q$, where p and q are prime factors
Grover’s algorithm [51]	• Purpose: Unstructured search problems • Efficiency: Quadratic speedup with time complexity $O(\sqrt{N})$ • Classical Comparison: Faster than classical $O(N)$ time • Grover Operator: $G = H^{\otimes n} (2 0\rangle\langle 0 - I) H^{\otimes n} O$ • Components: Hadamard gate H, identity matrix I, and oracle function O
Impact	• Potential: Solving complex problems more efficiently than classical computers • Future: Continued advancements in algorithms and hardware to expand capabilities and applications

2.2 Recent Developments in Fuzzy Logic

Advancement	Details
Fuzzy Inference Systems (FIS)	• Framework for reasoning with fuzzy logic • Components: Fuzzification, rule evaluation, and defuzzification • Example: Triangular membership function $\mu_A(x) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a < x \leq b \\ \frac{c-x}{c-b} & \text{if } b < x \leq c \\ 0 & \text{if } x > c \end{cases}$ where a , b , and c are parameters
Fuzzy Clustering Algorithms	• Example: Fuzzy C-Means (FCM) algorithm • Purpose: Partition dataset into clusters with fuzzy boundaries • Objective Function: $J_m = \sum_{i=1}^N \sum_{j=1}^C u_{ij}^m \ x_i - c_j\ ^2$ where N is the number of data points, C is the number of clusters, u_{ij} is the membership degree of data point x_i in cluster j, c_j is the centroid of cluster j, and m is the fuzziness parameter
Neuro-Fuzzy Systems	• Integration of fuzzy logic with neural networks • Combines learning capabilities of neural networks with reasoning capabilities of fuzzy logic • Representation: Multi-layer feedforward network • Learning Algorithm: Adjusts parameters of membership functions and fuzzy rules to minimize error
Impact	• Versatility and effectiveness in handling uncertainty and imprecision • Continued research expected to enhance capabilities and applications

3 New Idea on Quantum Fuzzy Logic

The proposed Quantum Fuzzy Logic framework aims to combine the principles of quantum computing and fuzzy logic to create a new computational tool. This framework includes the superposition and entanglement properties of quantum mechanics to handle uncertainty more effectively.

3.1 Quantum Fuzzy States

In classical fuzzy logic, a fuzzy state is represented by a membership function $\mu(x)$ that assigns a degree of membership to each element x in a set. In Quantum Fuzzy Logic, we extend this concept by representing fuzzy states as quantum superpositions. A quantum fuzzy state $|\psi\rangle$ can be expressed as:

$$|\psi\rangle = \sum_i \gamma_i |x_i\rangle$$

where γ_i are complex amplitudes such that $\sum_i |\gamma_i|^2 = 1$, and $|x_i\rangle$ are the basis states representing different fuzzy sets.

Example: Consider a fuzzy set of temperatures (cold, warm, hot). In Quantum Fuzzy Logic, a state could be a superposition of these temperatures:

$$|\psi\rangle = \frac{1}{\sqrt{3}}|cold\rangle + \frac{1}{\sqrt{3}}|warm\rangle + \frac{1}{\sqrt{3}}|hot\rangle$$

3.2 Quantum Fuzzy Inference

The inference process in Quantum Fuzzy Logic involves the application of quantum gates to manipulate quantum fuzzy states. For example, a quantum fuzzy inference rule can be represented as a unitary transformation V that operates on the quantum fuzzy state:

$$|\psi'\rangle = V|\psi\rangle$$

where V is designed to encode the fuzzy logic rules.

3.3 Quantum Entanglement of Fuzzy Variables

Quantum entanglement can be used to correlate fuzzy variables, leading to more accurate modeling of complex systems. Consider two fuzzy variables A and B with corresponding quantum states $|\phi_A\rangle$ and $|\phi_B\rangle$. An entangled state $|\Phi\rangle$ can be formed as:

$$|\Phi\rangle = \sum_{k,l} \delta_{kl} |x_k\rangle_A \otimes |y_l\rangle_B$$

where δ_{kl} are complex coefficients that define the entanglement between the states $|x_k\rangle_A$ and $|y_l\rangle_B$.

Example: If A and B represent two interdependent variables like humidity and temperature, their entangled state could capture the correlation between these variables more effectively than classical methods.

3.4 Measurement and Defuzzification

Measurement in Quantum Fuzzy Logic involves collapsing the quantum fuzzy state to a classical fuzzy state. The probability of measuring a particular fuzzy state $|x_i\rangle$ is given by $|\gamma_i|^2$. Defuzzification can then be performed using classical fuzzy logic techniques on the measured state.

4 Quantum Fuzzy Logic in Temperature Control

4.1 Problem Description

We consider a temperature control system that needs to maintain a comfortable room temperature. The system uses fuzzy logic to handle the uncertainty in temperature readings and control actions. We will extend this to Quantum Fuzzy Logic to include quantum superposition and entanglement for more efficient processing.

4.2 Classical Fuzzy Logic

In classical fuzzy logic, we define fuzzy sets for temperature states: *Cold*, *Comfortable*, and *Hot*. Each state is represented by a membership function.

Membership Functions

The membership functions for the fuzzy sets can be defined as follows:

$$\mu_{\text{Cold}}(T) = \begin{cases} 1 & \text{if } T \leq 15 \\ \frac{20-T}{5} & \text{if } 15 < T \leq 20 \\ 0 & \text{if } T > 20 \end{cases}$$

$$\mu_{\text{Comfortable}}(T) = \begin{cases} 0 & \text{if } T \leq 15 \text{ or } T > 25 \\ \frac{T-15}{5} & \text{if } 15 < T \leq 20 \\ \frac{25-T}{5} & \text{if } 20 < T \leq 25 \end{cases}$$

$$\mu_{\text{Hot}}(T) = \begin{cases} 0 & \text{if } T \leq 20 \\ \frac{T-20}{5} & \text{if } 20 < T \leq 25 \\ 1 & \text{if } T > 25 \end{cases}$$

4.3 Quantum Fuzzy Logic

In Quantum Fuzzy Logic, we represent the fuzzy states as quantum superpositions. Let's define a quantum fuzzy state for the temperature control system.

Quantum Fuzzy State

A quantum fuzzy state $|\psi\rangle$ can be expressed as:

$$|\psi\rangle = \alpha_{\text{Cold}}|\text{Cold}\rangle + \alpha_{\text{Comfortable}}|\text{Comfortable}\rangle + \alpha_{\text{Hot}}|\text{Hot}\rangle$$

where $\alpha_{\text{Cold}}, \alpha_{\text{Comfortable}}, \alpha_{\text{Hot}}$ are complex amplitudes such that $|\alpha_{\text{Cold}}|^2 + |\alpha_{\text{Comfortable}}|^2 + |\alpha_{\text{Hot}}|^2 = 1$.

Example: Suppose the initial state is equally likely to be any of the three temperature states:

$$|\psi\rangle = \frac{1}{\sqrt{3}}|\text{Cold}\rangle + \frac{1}{\sqrt{3}}|\text{Comfortable}\rangle + \frac{1}{\sqrt{3}}|\text{Hot}\rangle$$

4.4 Quantum Fuzzy Inference

The inference process involves applying quantum gates to the quantum fuzzy state. Let's define a unitary transformation U that represents a fuzzy rule.

Unitary Transformation

Suppose we have a fuzzy rule that increases the probability of the *Comfortable* state. The unitary transformation U can be designed as:

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Applying U to $|\psi\rangle$:

$$|\psi'\rangle = U|\psi\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

4.5 Measurement and Defuzzification

Measurement collapses the quantum fuzzy state to a classical fuzzy state. The probability of measuring a particular state is given by the square of the amplitude.

Example: The probability of measuring the *Cold* state is $\left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3}$.

Defuzzification can then be performed using classical fuzzy logic techniques on the measured state.

Theorem 1 Superposition Principle in Quantum Fuzzy Logic

Statement: In Quantum Fuzzy Logic, any linear combination of quantum fuzzy states is also a valid quantum fuzzy state.

Proof Let $|\psi_1\rangle$ and $|\psi_2\rangle$ be two quantum fuzzy states:

$$|\psi_1\rangle = \sum_i \alpha_i |x_i\rangle, \quad |\psi_2\rangle = \sum_j \beta_j |y_j\rangle$$

We consider a linear combination of these states:

$$|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle$$

where c_1 and c_2 are complex coefficients. Then,

$$|\psi\rangle = c_1 \sum_i \alpha_i |x_i\rangle + c_2 \sum_j \beta_j |y_j\rangle = \sum_i (c_1 \alpha_i) |x_i\rangle + \sum_j (c_2 \beta_j) |y_j\rangle$$

Since $\sum_i |c_1 \alpha_i|^2 + \sum_j |c_2 \beta_j|^2 = 1$, $|\psi\rangle$ is a valid quantum fuzzy state. ■

Theorem 2 Entanglement Correlation in Quantum Fuzzy Logic

Statement: In Quantum Fuzzy Logic, the measurement of one entangled fuzzy variable affects the state of the other entangled fuzzy variable.

Proof We consider an entangled state $|\Psi\rangle$ of two fuzzy variables A and B :

$$|\Psi\rangle = \sum_{i,j} \beta_{ij} |x_i\rangle_A \otimes |y_j\rangle_B$$

If we measure A and find it in state $|x_k\rangle_A$, the state of B collapses to:

$$|\psi_B\rangle = \frac{1}{\sqrt{p_k}} \sum_j \beta_{kj} |y_j\rangle_B$$

where $p_k = \sum_j |\beta_{kj}|^2$ is the probability of measuring A in state $|x_k\rangle_A$. This shows that the measurement of A affects the state of B . ■

Theorem 3 Linearity of Quantum Fuzzy Operators

Statement: Quantum fuzzy operators are linear operators on the space of quantum fuzzy states.

Proof Let $\hat{O}$ be a quantum fuzzy operator and $|\psi_1\rangle$ and $|\psi_2\rangle$ be quantum fuzzy states. Consider the linear combination of these states:

$$|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle$$

Applying the operator $\hat{O}$ to $|\psi\rangle$:

$$\hat{O}|\psi\rangle = \hat{O}(c_1|\psi_1\rangle + c_2|\psi_2\rangle) = c_1\hat{O}|\psi_1\rangle + c_2\hat{O}|\psi_2\rangle$$

Since $\hat{O}$ acts linearly on the quantum fuzzy states, it preserves the linearity property. ■

Theorem 4 Unitarity of Quantum Fuzzy Transformations

Statement: Quantum fuzzy transformations are unitary, preserving the norm of quantum fuzzy states.

Proof Let U be a quantum fuzzy transformation and $|\psi\rangle$ be a quantum fuzzy state. The norm of $|\psi\rangle$ is given by:

$$\langle\psi|\psi\rangle = 1$$

Applying the transformation U :

$$|\psi'\rangle = U|\psi\rangle$$

The norm of the transformed state $|\psi'\rangle$ is:

$$\langle\psi'|\psi'\rangle = \langle\psi|U^\dagger U|\psi\rangle$$

Since U is unitary, $U^\dagger U = I$, where I is the identity operator. Therefore,

$$\langle\psi'|\psi'\rangle = \langle\psi|I|\psi\rangle = \langle\psi|\psi\rangle = 1$$

Thus, the norm is preserved. ■

Theorem 5 Quantum Fuzzy Entanglement Preservation

Statement: Quantum fuzzy entanglement is preserved under unitary transformations.

Proof Consider an entangled state $|\Psi\rangle$ of two fuzzy variables A and B :

$$|\Psi\rangle = \sum_{i,j} \beta_{ij} |x_i\rangle_A \otimes |y_j\rangle_B$$

Let U_A and U_B be unitary transformations acting on A and B , respectively. The transformed state is:

$$|\Psi'\rangle = (U_A \otimes U_B)|\Psi\rangle = \sum_{i,j} \beta_{ij} (U_A |x_i\rangle_A) \otimes (U_B |y_j\rangle_B)$$

Since U_A and U_B are unitary, they preserve the entanglement structure of the state. Therefore, $|\Psi'\rangle$ remains entangled. ■

Theorem 6 Quantum Fuzzy Measurement Collapse

Statement: Measurement of a quantum fuzzy state collapses it to a classical fuzzy state with probabilities given by the squared amplitudes.

Proof Let $|\psi\rangle$ be a quantum fuzzy state:

$$|\psi\rangle = \sum_i \alpha_i |x_i\rangle$$

The probability of measuring the state $|x_i\rangle$ is $|\alpha_i|^2$. Upon measurement, the state collapses to $|x_i\rangle$ with probability $|\alpha_i|^2$. The classical fuzzy state is then represented by the membership function $\mu(x_i) = |\alpha_i|^2$. ■

5 Conclusions

This chapter has discussed a new framework for Quantum Fuzzy Logic. One of the primary limitations is the current state of quantum hardware. Quantum computers are still in their infancy, with issues such as qubit coherence, error rates, and scalability posing significant hurdles. Another challenge lies in the theoretical foundation of Quantum Fuzzy Logic. The integration of quantum mechanics with fuzzy logic principles is not simple and demands a deep understanding of both fields. Researchers need to develop simple models that can accurately represent fuzzy logic operations within a quantum context, ensuring that the inherent uncertainties and probabilistic nature of quantum systems are appropriately handled.

In conclusion, while the combination of quantum computing and fuzzy logic presents significant challenges, advancing computational capabilities. By addressing the current limitations and focusing on future research directions, the field of Quantum Fuzzy Logic can be used for solving complex problems across various domains. The journey towards practical Quantum Fuzzy Logic is just beginning, and with continued effort and innovation, it holds the promise of transforming the complex computation.

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Quantum Graphs and Optimization



Sovan Samanta 

Abstract Quantum graph, characterized by dynamic vertices and entangled edges, is a generalisation of graph theory. These graphs exhibit time-dependent behavior, with each vertex assigning a dynamic weight that changes over time. Entanglement, a fundamental quantum phenomenon, is the correlation between adjacent vertices. By combining these unique features of quantum graphs, we discussed and modified optimization techniques in quantum environments.

Keywords Graph theory · Quantum graphs · Optimization

1 Introduction

Graph theory has been using in solving optimization problems since long time. From the ancient Seven Bridges of Königsberg problem to modern network flow algorithms, graphs provide a visual and conceptual framework for understanding complex systems. In this chapter, we discuss the optimization techniques using graphs and then using quantum graphs. There are many problems which have been solved using graph theory such as the Traveling Salesman Problem (TSP) (Given a list of cities and their pairwise distances, find the shortest route that visits each city exactly once and returns to the starting point. The TSP has applications in logistics, circuit design, and even DNA sequencing [1, 2]); Maximum Flow and Minimum Cut (Consider a network of pipes carrying water or a communication network transmitting data. How can we optimize the flow through these networks? Graphs provide efficient solutions through algorithms like Ford-Fulkerson and Edmonds-Karp); Facility Location and

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Graph Centers (Suppose we want to build warehouses to serve a set of cities efficiently. Where should we place these warehouses to minimize transportation costs? Graph centers-vertices with minimal average distance to other vertices-play a crucial role in solving this problem.).

Quantum computing [47], despite its limitations, promises exponential speedup for certain problems. We can imagine nodes representing variables and edges capturing interactions. Quantum graphs extend to encode optimization problems into graph structures. The problems solved by classical graph neural networks, inspire to enhance quantum graph optimization. Algorithms, introducing quantum entanglement and dynamic weights, will be useful for many complex dynamical problems.

In this chapter, we aim to:

1. understand the earlier contexts of graph-based optimization techniques.
2. investigate the algorithms of optimization such as shortest path problems, Minimum Spanning Tree Problems in quantum graphs.
3. introduce quantum graphs as an efficient tool to optimization.

2 Literature Review

Graph-based optimization techniques play a major role in solving a wide range of real-world problems. From transportation networks to social networks, graphs provide a natural representation for capturing relationships and dependencies. The historical evolution of optimization methods based on graph theory reveals significant milestones. Graph algorithms, such as Dijkstra's shortest path algorithm and Ford-Fulkerson's maximum flow algorithm, are the earlier techniques. These classical approaches continue to find applications in diverse domains.

Authors [38, 44] introduced techniques for addressing optimization problems in graph theory in solving complex graph-based problems, teaching-learning-based optimization (GT-TLBO) [39], for optimizing transportation networks [40], highlighted the potential of graph theory in enhancing the efficiency and reliability of renewable energy systems [41]. Few several researches have been found focusing on optimization and algebraic graph theory [42], fuzzy clustering in handling uncertainty and imprecision in graph-based data [43], the Schrödinger approach to enhance prediction accuracy [45]. Farhi, Gamarnik, and Gutmann (2020) discussed the necessity of considering the entire graph in quantum optimization [46].

Several optimization algorithms are present in the form of graph structures such as GCNs extend convolutional neural networks to graph-structured data. They aggregate information from neighboring nodes, allowing them to capture local and global dependencies. Applications include node classification, link prediction, and recommendation systems [3]. GraphSAGE learns node embeddings by sampling and aggregating features from a node's local neighborhood. It enables scalable graph-based learning and has been effective in semi-supervised node classification tasks [4]. Few other computer based techniques useful for large networks can be found in [48–50]

Despite their success, graph-based optimization techniques face challenges. Many real-world problems involve non-convex objective functions, making optimization challenging. Scaling to large graphs remains an open problem. Ensuring convergence in iterative graph algorithms is crucial.

Graph theory with uncertainty, named fuzzy graphs are more powerful tool in optimization techniques, capturing uncertainty. Fuzzy set theory is used to represent vague information. Fuzzy sets (FS) have become a popular approach for incorporating fuzzy information as part of an expert's evaluative measures [34]. For example, the development of fuzzy fractional differential equations using fuzzy Laplace transforms is fit for modeling systems where the influence of individuals or groups evolves fractionally over time under uncertainty [20]. In [5] studied fuzzy variable precision rough sets, in [6] introduced hesitant bipolar-valued fuzzy sets and bipolar-valued hesitant fuzzy sets, and studied multi-granulation bipolar-valued fuzzy probabilistic rough sets and their corresponding three-way decisions. The decision-theoretic rough sets and their applications are studied in [7].

The introduction of predictor-corrector methods (in solving fuzzy differential equations) has direct applications in social groups [21–25]. These methods allow for a more accurate simulation of how consensus forms over time, particularly in environments where group dynamics are shaped by uncertainty and imprecision. The bipolar fuzzy systems [11, 27] describes the concepts of positivity and negativity properties. Pythagorean fuzzy systems [28] and trapezoidal approximations [26] provide methods in decision-making processes. Some other related articles on fuzzy techniques in optimizations are [29–33, 35–37].

Fuzzy graph theory [9, 10] such as fuzzy competition graphs [13], fuzzy planar graphs [12, 15] and applications of edge coloring in fuzzy graphs [8] is used to solve some real world problems. Samanta et al. proposed generalized fuzzy graph theory connecting relationships of edges and end vertices [14, 16–18]. Generalised fuzzy graphs are further extended to quantum graphs with dynamic weights of vertices and edges changing over times [19].

3 Preliminaries

The shortest path problem aims to find the shortest path between two vertices in a graph. Given a graph $G = (V, E)$ and a weight function $w : E \rightarrow \mathbb{R}$, the goal is to find the path from a source vertex s to a target vertex t that minimizes the total weight of the edges in the path.

$$\min \sum_{(u,v) \in P} w(u, v)$$

subject to:

P is a path from s to t

The maximum flow problem aims to find the maximum flow from a source vertex to a sink vertex in a directed network (flow). Given a flow network $G = (V, E)$ with a capacity function $c : E \rightarrow \mathbb{R}^+$, the goal is to maximize the flow from s to t .

$$\max \sum_{v \in V} f(s, v)$$

subject to:

$$\sum_{v \in V} f(u, v) = \sum_{v \in V} f(v, u) \quad \forall u \in V \setminus \{s, t\}$$

$$0 \leq f(u, v) \leq c(u, v) \quad \forall (u, v) \in E$$

The minimum spanning tree (MST) problem aims to find a subset of edges that connects all vertices in the graph with the minimum total edge weight. Given a graph $G = (V, E)$ and a weight function $w : E \rightarrow \mathbb{R}$, the goal is to find a spanning tree $T \subseteq E$ that minimizes the total weight.

$$\min \sum_{(u,v) \in T} w(u, v)$$

subject to:

$$T \text{ is a spanning tree of } G$$

A Quantum Graph with dynamic vertices and entangled edges is defined as follows:

1. Vertices with Dynamic Weights (dweight): Each vertex v_i has a dynamic weight $\text{dweight}_i(t)$ that changes over time. The weight is a real number between 0 and 1:

$$0 \leq \text{dweight}_i(t) \leq 1$$

The weight follows a specific pattern, for example a sinusoidal function as follows:

$$\text{dweight}_i(t) = \frac{1}{2} (1 + \sin(\omega_i t + \phi_i))$$

where ω_i is the frequency and ϕ_i is the phase shift.

2. Edges and Entanglement: Edges e_{ij} between vertices v_i and v_j can be either entangled or not. If entangled, the edge is represented by a correlation function $g_{ij}(t)$ that describes the relationship between the weights of the connected vertices:

$$g_{ij}(t) = h(\text{dweight}_i(t), \text{dweight}_j(t))$$

where h is a function defining the nature of the entanglement or correlation.

3. Graph Structure: The graph G is represented as $G = (V, E)$, where V is the set of vertices and E is the set of edges. The adjacency matrix $A(t)$ of the graph is defined as:

$$A_{ij}(t) = \begin{cases} g_{ij}(t) & \text{if } e_{ij} \text{ exists and is entangled} \\ 0 & \text{otherwise} \end{cases}$$

4 Dijkstra's Algorithm for Quantum Graphs

The effect of changes over time in the quantum graph is important, we need to consider the cumulative effect of the dynamic weights and correlations over a time interval from 0 to t . This can be achieved by integrating the correlation function $g_{ij}(t)$ over the interval. The integration will capture the continuous changes in the dynamic weights and provide a more accurate representation of the edge weights over time (Table 1).

Consider a quantum graph G with 5 vertices $V = \{A, B, C, D, E\}$ and edges with dynamic weights and entanglement correlations. The goal is to find the shortest path from vertex A to vertex E . Let us assume the dweights of the vertices as follows:

Table 1 Quantum Dijkstra's algorithm with time integration

Quantum Dijkstra's Algorithm	
Input	Quantum graph G with vertices V and edges E
Output	Shortest path distances from s to all vertices
Steps	
1.	Initialize quantum graph G with vertices V and edges E
2.	Assign initial dynamic weights $\text{dweight}_i(0)$ to each vertex
3.	Initialize $d(s) = 0$ and $d(v) = \infty$ for all $v \neq s$
4.	Initialize priority queue Q with all vertices
5.	While Q is not empty do
6.	Extract vertex u with minimum $d(u)$ from Q
7.	For each neighbor v of u do
8.	Update $\text{dweight}_i(t+1) = \text{dweight}_i(t) + \alpha \nabla f(x_i)$
9.	Update $g_{ij}(t+1) = h(\text{dweight}_i(t+1), \text{dweight}_j(t+1))$
10.	Calculate the integrated correlation weight: $G_{ij}(t) = \int_0^t g_{ij}(\tau) d\tau$
11.	If $d(u) + G_{ij}(t) < d(v)$ then
12.	$d(v) = d(u) + G_{ij}(t)$
13.	Update priority queue Q
	End If
	End For
	End While

$$\text{dweight}_A(t) = \frac{1}{2} (1 + \sin(t))$$

$$\text{dweight}_B(t) = \frac{1}{2} (1 + \cos(t))$$

$$\text{dweight}_C(t) = \frac{1}{2} (1 + \sin(2t))$$

$$\text{dweight}_D(t) = \frac{1}{2} (1 + \cos(2t))$$

$$\text{dweight}_E(t) = \frac{1}{2} (1 + \sin(3t))$$

The correlation function $g_{ij}(t)$ is defined as:

$$g_{ij}(t) = |\text{dweight}_i(t) - \text{dweight}_j(t)|$$

The integrated correlation weight $G_{ij}(t)$ is:

$$G_{ij}(t) = \int_0^t |\text{dweight}_i(\tau) - \text{dweight}_j(\tau)| d\tau$$

The initial adjacency matrix $A(t)$ is shown below:

	A	B	C	D	E
A	0	$G_{AB}(t)$	$G_{AC}(t)$	∞	∞
B	$G_{AB}(t)$	0	∞	$G_{BD}(t)$	∞
C	$G_{AC}(t)$	∞	0	$G_{CD}(t)$	$G_{CE}(t)$
D	∞	$G_{BD}(t)$	$G_{CD}(t)$	0	$G_{DE}(t)$
E	∞	∞	$G_{CE}(t)$	$G_{DE}(t)$	0

Here is a visual representation of the quantum graph with curved and thick lines, and vertices as small circles (Fig. 1):

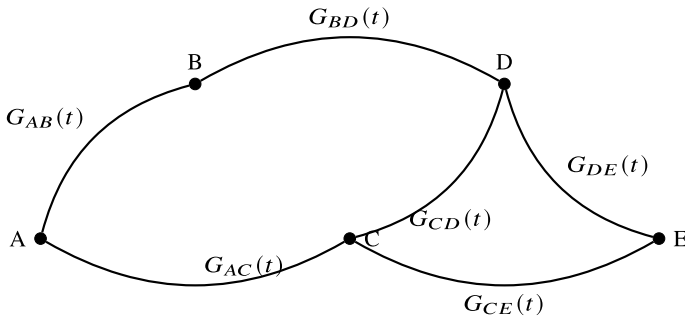


Fig. 1 Quantum graph with dynamic vertices and entangled edges

5 Maximum Flow Problem for Quantum Graphs

Given a quantum flow network $G = (V, E)$ with a source vertex s and a sink vertex t , and a capacity function $c : E \rightarrow \mathbb{R}^+$, the goal is to maximize the flow from s to t .

$$\max \sum_{v \in V} f(s, v)$$

subject to:

$$\sum_{v \in V} f(u, v) = \sum_{v \in V} f(v, u) \quad \forall u \in V \setminus \{s, t\}$$

$$0 \leq f(u, v) \leq c(u, v) \quad \forall (u, v) \in E$$

5.1 Ford-Fulkerson Method for Quantum Graphs

Quantum graphs are useful where optimizing the flow is crucial. Ford-Fulkerson Method for Quantum Graphs is given as follows (Table 2).

Table 2 Quantum ford-fulkerson method with time integration

Ford-Fulkerson Method for quantum Graphs	
Input	Quantum flow network G with vertices V and edges E
Output	Maximum flow from s to t
Steps	
1.	Initialize quantum flow network G with vertices V and edges E
2.	Assign initial dynamic weights $\text{dweight}_i(0)$ to each vertex
3.	Initialize flow $f(u, v) = 0$ for all $(u, v) \in E$
4.	While there exists an augmenting path P from s to t do
5.	Update $\text{dweight}_i(t + 1) = \text{dweight}_i(t) + \alpha \nabla f(x_i)$
6.	Update $g_{ij}(t + 1) = h(\text{dweight}_i(t + 1), \text{dweight}_j(t + 1))$
7.	Calculate the integrated correlation weight: $G_{ij}(t) = \int_0^t g_{ij}(\tau) d\tau$
8.	Find the residual capacity $c_f(P) = \min\{G_{ij}(t) - f(u, v) \mid (u, v) \in P\}$
9.	For each edge (u, v) in P do
10.	$f(u, v) = f(u, v) + c_f(P)$
11.	$f(v, u) = f(v, u) - c_f(P)$
	End For
	End While

5.2 Example

We consider a quantum flow network G with 10 vertices $V = \{S, A, B, C, D, E, F, G, H, T\}$ and edges with dynamic capacities and entanglement correlations. The goal is to find the maximum flow from source S to sink T . We assume the same set of dweight functions as follows:

$$\text{dweight}_S(t) = \sin(t) + 1$$

$$\text{dweight}_A(t) = \cos(t) + 1$$

$$\text{dweight}_B(t) = \frac{1}{2} \sin(t) + 1$$

$$\text{dweight}_C(t) = \frac{1}{2} \cos(t) + 1$$

$$\text{dweight}_D(t) = \sin(2t) + 1$$

$$\text{dweight}_E(t) = \cos(2t) + 1$$

$$\text{dweight}_F(t) = \frac{1}{3} \sin(t) + 1$$

$$\text{dweight}_G(t) = \frac{1}{3} \cos(t) + 1$$

$$\text{dweight}_H(t) = \sin(3t) + 1$$

$$\text{dweight}_T(t) = \cos(3t) + 1$$

The correlation function $g_{ij}(t)$ is defined as:

$$g_{ij}(t) = |\text{dweight}_i(t) - \text{dweight}_j(t)|$$

The integrated correlation weight $G_{ij}(t)$ is:

$$G_{ij}(t) = \int_0^t |\text{dweight}_i(\tau) - \text{dweight}_j(\tau)| d\tau$$

The initial adjacency matrix $A(t)$ is shown below (Fig. 2):

	S	A	B	C	D	E	F	G	H	T
S	0	$G_{SA}(t)$	$G_{SB}(t)$	∞	∞	∞	∞	∞	∞	∞
A	$G_{SA}(t)$	0	$G_{AB}(t)$	$G_{AC}(t)$	∞	∞	∞	∞	∞	∞
B	$G_{SB}(t)$	$G_{AB}(t)$	0	$G_{BC}(t)$	$G_{BD}(t)$	∞	∞	∞	∞	∞
C	∞	$G_{AC}(t)$	$G_{BC}(t)$	0	$G_{CD}(t)$	$G_{CE}(t)$	∞	∞	∞	∞
D	∞	∞	$G_{BD}(t)$	$G_{CD}(t)$	0	$G_{DE}(t)$	$G_{DF}(t)$	∞	∞	∞
E	∞	∞	∞	$G_{CE}(t)$	$G_{DE}(t)$	0	$G_{EF}(t)$	$G_{EG}(t)$	∞	∞
F	∞	∞	∞	∞	$G_{DF}(t)$	$G_{EF}(t)$	0	$G_{FG}(t)$	$G_{FH}(t)$	∞
G	∞	∞	∞	∞	∞	$G_{EG}(t)$	$G_{FG}(t)$	0	$G_{GH}(t)$	$G_{GT}(t)$
H	∞	∞	∞	∞	∞	∞	$G_{FH}(t)$	$G_{GH}(t)$	0	$G_{HT}(t)$
T	∞	∞	∞	∞	∞	∞	∞	$G_{GT}(t)$	$G_{HT}(t)$	0

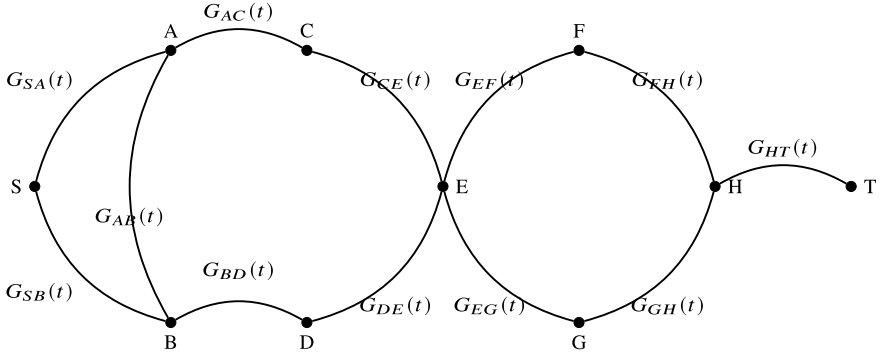


Fig. 2 Quantum flow network with dynamic vertices and entangled edges

6 Minimum Spanning Tree for Quantum Graphs

Given a quantum graph $G = (V, E)$ with a weight function $w : E \rightarrow \mathbb{R}$, the goal is to find a spanning tree $T \subseteq E$ that minimizes the total weight.

$$\min \sum_{(u,v) \in T} w(u, v)$$

subject to:

T is a spanning tree of G

6.1 Kruskal's Algorithm for Quantum Graphs

The Minimum Spanning Tree for Quantum Graphs is a significant area of research in quantum information theory. QKruskal's Algorithm for Quantum Graphs is described below (Table 3).

6.1.1 Example

We consider a quantum graph G with 10 vertices $V = \{A, B, C, D, E, F, G, H, I, J\}$ and edges with dynamic weights and entanglement correlations. The goal is to find the minimum spanning tree.

The dweights of the vertices are as follows:

$$\text{dweight}_A(t) = \sin(t) + 1$$

$$\text{dweight}_B(t) = \cos(t) + 1$$

Table 3 Quantum Kruskal's algorithm with time integration

Quantum Kruskal's Algorithm with Time Integration	
Input	Quantum graph G with vertices V and edges E
Output	Minimum spanning tree T
Steps	
1.	Initialize quantum graph G with vertices V and edges E
2.	Assign initial dynamic weights $\text{dweight}_i(0)$ to each vertex
3.	Sort edges E in non-decreasing order of $G_{ij}(0)$
4.	Initialize $T = \emptyset$
5.	For each edge (u, v) in sorted order do
6.	Update $\text{dweight}_i(t+1) = \text{dweight}_i(t) + \alpha \nabla f(x_i)$
7.	Update $g_{ij}(t+1) = h(\text{dweight}_i(t+1), \text{dweight}_j(t+1))$
8.	Calculate the integrated correlation weight: $G_{ij}(t) = \int_0^t g_{ij}(\tau) d\tau$
9.	If adding (u, v) to T does not form a cycle then
10.	Add (u, v) to T
	End If
	End For

$$\text{dweight}_C(t) = \frac{1}{2} \sin(t) + 1$$

$$\text{dweight}_D(t) = \frac{1}{2} \cos(t) + 1$$

$$\text{dweight}_E(t) = \sin(2t) + 1$$

$$\text{dweight}_F(t) = \cos(2t) + 1$$

$$\text{dweight}_G(t) = \frac{1}{3} \sin(t) + 1$$

$$\text{dweight}_H(t) = \frac{1}{3} \cos(t) + 1$$

$$\text{dweight}_I(t) = \sin(3t) + 1$$

$$\text{dweight}_J(t) = \cos(3t) + 1$$

The correlation function $g_{ij}(t)$ is defined as:

$$g_{ij}(t) = |\text{dweight}_i(t) - \text{dweight}_j(t)|$$

The integrated correlation weight $G_{ij}(t)$ is:

$$G_{ij}(t) = \int_0^t |\text{dweight}_i(\tau) - \text{dweight}_j(\tau)| d\tau$$

The initial adjacency matrix $A(t)$ is shown below:

	A	B	C	D	E	F	G	H	I	J
A	0	$G_{AB}(t)$	$G_{AC}(t)$	∞	∞	∞	∞	∞	∞	∞
B	$G_{AB}(t)$	0	$G_{BC}(t)$	$G_{BD}(t)$	∞	∞	∞	∞	∞	∞
C	$G_{AC}(t)$	$G_{BC}(t)$	0	$G_{CD}(t)$	$G_{CE}(t)$	∞	∞	∞	∞	∞
D	∞	$G_{BD}(t)$	$G_{CD}(t)$	0	$G_{DE}(t)$	$G_{DF}(t)$	∞	∞	∞	∞
E	∞	∞	$G_{CE}(t)$	$G_{DE}(t)$	0	$G_{EF}(t)$	$G_{EG}(t)$	∞	∞	∞
F	∞	∞	∞	$G_{DF}(t)$	$G_{EF}(t)$	0	$G_{FG}(t)$	$G_{FH}(t)$	∞	∞
G	∞	∞	∞	∞	$G_{EG}(t)$	$G_{FG}(t)$	0	$G_{GH}(t)$	$G_{GI}(t)$	∞
H	∞	∞	∞	∞	∞	$G_{FH}(t)$	$G_{GH}(t)$	0	$G_{HI}(t)$	$G_{HJ}(t)$
I	∞	∞	∞	∞	∞	∞	$G_{GI}(t)$	$G_{HI}(t)$	0	$G_{IJ}(t)$
J	∞	∞	∞	∞	∞	∞	∞	$G_{HJ}(t)$	$G_{IJ}(t)$	0

7 Optimization Technique Using Quantum Graphs

Consider an optimization problem where we aim to minimize a cost function $f(x)$ subject to certain constraints.

7.1 Problem Formulation

Let $G = (V, E)$ be a quantum graph with vertices V and edges E . Each vertex v_i has a dynamic weight $\text{dweight}_i(t)$, and each edge e_{ij} has an entanglement correlation function $g_{ij}(t)$.

The optimization problem can be formulated as:

$$\min_x f(x)$$

subject to:

$$g_{ij}(t) \geq \epsilon, \quad \forall e_{ij} \in E$$

where ϵ is a threshold for entanglement.

7.2 Quantum Graph-Based Algorithm

1. Initialization: Initialize the quantum graph G with vertices V and edges E . Assign initial dynamic weights $\text{dweight}_i(0)$ to each vertex.
2. Dynamic Weight Update: Update the dynamic weights $\text{dweight}_i(t)$ at each time step t based on the optimization process:

$$\text{dweight}_i(t+1) = \text{dweight}_i(t) + \alpha \nabla f(x_i)$$

where α is a learning rate and $\nabla f(x_i)$ is the gradient of the cost function at vertex v_i .

3. Entanglement Adjustment: Adjust the entanglement correlation functions $g_{ij}(t)$ based on the updated dynamic weights:

$$g_{ij}(t+1) = h(\text{dweight}_i(t+1), \text{dweight}_j(t+1))$$

4. Constraint Check: Ensure that the entanglement constraints are satisfied:

$$g_{ij}(t+1) \geq \epsilon, \quad \forall e_{ij} \in E$$

5. Iteration: Repeat steps 2-4 until convergence, i.e., until the change in the cost function $f(x)$ is below a predefined threshold.

7.3 Example Application

We consider a network of quantum sensors where each sensor's accuracy is represented by a dynamic weight. The goal is to optimize the sensor network's overall accuracy while maintaining a certain level of entanglement between sensors (Table 4).

Table 4 Quantum graph-based optimization algorithm

Quantum Graph-Based Optimization Algorithm	
Input	Quantum graph G with vertices V and edges E
Output	Optimized dynamic weights and entanglement correlations
Steps	
1.	Initialize quantum graph G with vertices V and edges E
2.	Assign initial dynamic weights $\text{dweight}_i(0)$ to each vertex
3.	While not converged do
4.	For each vertex v_i do
5.	Update $\text{dweight}_i(t+1) = \text{dweight}_i(t) + \alpha \nabla f(x_i)$
6.	End For
7.	For each edge e_{ij} do
8.	Update $g_{ij}(t+1) = h(\text{dweight}_i(t+1), \text{dweight}_j(t+1))$
9.	If $g_{ij}(t+1) < \epsilon$ then
10.	Adjust weights or entanglement to satisfy constraints
	End If
	End For
	End While

7.4 Comparison of Optimization Techniques: Quantum Graphs Versus Weighted Graphs

In comparing optimization techniques, quantum graphs leverage dynamic weights ($\text{dweight}_i(t)$) that follow a pattern and change over time, representing the probability amplitude of quantum states, and entanglement, which is captured by a time-dependent adjacency matrix $A(t)$. This dynamic nature allows for potentially higher performance due to adaptive adjustments. In contrast, weighted graphs use static weights (weight_i) for vertices and edges (w_{ij}), with a static adjacency matrix A , relying on fixed values throughout the optimization process. This static nature limits their adaptability and performance compared to the dynamic adjustments possible in quantum graphs.

8 Conclusion

The concept of quantum graphs opens up the possibilities for future research and practical applications. By combining quantum properties into graph-based optimization, we can achieve better solution quality. This chapter lays the foundation for further investigation into quantum graph-based optimization.

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