TEXTBOOK

Wolfgang Cassing

Theoretical Physics Compact II

Electrodynamics

Springer

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Dedicated to Prof. Dr. Achim Weiguny

Preface

This book provides a textbook on electrodynamics and is in particular suited for bachelor students in their second year of studies in theoretical physics. The mathematical requirements include a knowledge of differentiation and integration, elementary linear algebra and concepts of vector analysis. Mathematical proofs are kept as simple as possible, however, still kept stringent.

After introducing the concept of 'charge' of mass points and strategies to measure 'charge' the electric field $\mathbf{E}(\mathbf{r})$ is defined and discussed for a system of fixed point charges or a continuous charge distribution $\rho(\mathbf{r})$ (electrostatics). It is shown that the divergence of the electric field is proportional to the charge density $\rho(\mathbf{r})$ and the field itself emerges as the negative gradient of the scalar potential $\Phi(\mathbf{r})$. By combining these findings the Poisson equation is derived for static charge distributions. In the case of constant currents $\mathbf{j}(\mathbf{r})$ a magnetic field $\mathbf{B}(\mathbf{r})$ emerges that is discussed and evaluated for simple examples (magnetostatics). This field is found to be characterized by a vanishing divergence and thus can be written as the rotation of a vector field $\mathbf{A}(\mathbf{r})$. In case of time-dependent charge distributions $\rho(\mathbf{r};t)$ and currents $\mathbf{j}(\mathbf{r};t)$ the sources are coupled by a continuity equation, which implies the conservation of the total charge. In this case (electrodynamics) the electric and magnetic fields are coupled via Faraday's law of induction and the basic equations for the $\mathbf{E}$ and $\mathbf{B}$ fields emerge in the form of Maxwell's equations, which—together with the Lorentz force—completely describe electrodynamics on the classical level. However, these coupled differential equations are difficult to solve directly. By introduction of time-dependent scalar and vector potentials $\Phi(\mathbf{r};t)$ and $\mathbf{A}(\mathbf{r};t)$ general wave equations are derived by exploiting the fact, that these potentials have a gauge freedom, i.e. they provide the same $\mathbf{E}$ and $\mathbf{B}$ fields in case of gauge transformations. In this context the Coulomb and Lorentz conventions (gauges) are discussed. It is, furthermore, shown that energy, momentum and angular momentum have to be assigned to the electromagnetic field and that a radiation pressure appears for radiation fields. This paves the way for an interpretation of the fields as 'photons' or 'γ-quanta'.

The wave equations are first solved in vacuum, i.e. without external sources, and polarized plane waves are found as basic solutions, that are characterized by an angular frequency ω, a wave number $\mathbf{k}$ and a polarization vector orthogonal to the direction of propagation $\mathbf{k}$. A superposition of plane waves—in terms of a Fourier series—then provides the general solution in vacuum in terms of wave packets, which can be used for the transmission of information. The general solution of the inhomogeneous wave equations is obtained with the help of retarded Green's functions, that lead to the retarded potentials known as Liénard—Wichert potentials. A solution for a system of moving point charges is computed explicitly and it is shown that accelerated charges produce (or absorb) electromagnetic radiation. The latter is analyzed with respect to the frequency ω and angular distribution for electric and magnetic dipole radiation as well as for electric quadrupole radiation.

The electromagnetic field in matter is discussed in the second part of this book and macroscopic space-time averages are introduced for the macroscopic electric and magnetic fields $\vec{\mathscr{E}}$ and $\vec{\mathscr{B}}$. By a separation of 'localized' and 'free' charge carriers the Maxwell equations for the macroscopic fields are derived, that include a dielectric polarization $\vec{\mathscr{P}}$ and magnetization $\vec{\mathscr{M}}$, which add to the auxiliary fields $\vec{\mathscr{D}}$ and $\vec{\mathscr{H}}$ and are easier to control experimentally than the fields $\vec{\mathscr{E}}$ and $\vec{\mathscr{B}}$. The energy, momentum and angular momentum of the matter fields are evaluated and Kirchhoff's rules are derived from charge and energy conservation. The electric and magnetic properties of matter are analyzed in terms of material equations which are solved in linear response theory, giving either the electric conductivity, the electric polarization or magnetization. Furthermore, the properties of the electromagnetic field at interfaces are derived and discussed explicitly for linear and isotropic media. In this context the laws for reflection and refraction of light are derived as well as the propagation of electromagnetic waves in conductive materials.

In the last part of this book a covariant formulation of electrodynamics is presented and it is shown that the basic equations are invariant with respect to Lorentz transformations, which demonstrates that they have the same form in every inertial system and thus satisfy Einstein's principle of special relativity.

In the appendices simple introductions (as well as examples) are given for volume integrals in different coordinate systems, surface integrals as well as path integrals. Gauss's theorem and Stoke's theorem are presented and verified with the help of examples.

Acknowledgements This book results from the collaboration with many students and collaborators throughout about 35 years of common teaching and research. It follows the drafts of my teacher Prof. Dr. Achim Weiguny to whom this volume is dedicated. Special thanks go to my daughter Marie for preparing some of the figures and helpful comments on notations and presentations.

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1. Introduction to Electrodynamics

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In this chapter we introduce the 'charge' of mass points and provide a survey of the different parts of this book, that address particular questions in the context of electrodynamics in vacuum and in matter and finally lead to a covariant formulation of electrodynamics, which is consistent with Einstein's theory of special relativity.

1.1 Electric Charge

While in mechanics the property **mass** of mass points is of primary interest, the **charge** of mass points is the starting point of electrodynamics. It has a number of fundamental properties which are characterized by a variety of experimental measurements:

(1.) There are 2 types of **charges**: positive and negative. Charges of the same signs repel each other, charges of different signs attract each other.

(2.) The total charge of a system of mass points is the algebraic sum of the individual charges; the charge is a **scalar**.

(3.) The total charge of a closed system is constant and independent of the motion of the system.

(4.) Charge only appears as a multiple of an **elementary charge** e,

$$q = ne;\hat{n} = 0, \pm 1, \pm 2, \pm 3, \ldots,$$

where $-e$ is the charge of an electron.

The classic proof for this **quantization** of the charge is the **Millikan experiment**. Elementary particles **quarks** have fractional charges, i.e. $q = \pm 1/3e$ or $q = \pm 2/3e$, however, **quarks** are not observable as free particles in the energy range we are interested in here.

1.2 Electrostatics

The most simple problem in electrodynamics is the case of stationary charges, which we denote by **electrostatics**. Inserting a **test charge** q into the region of one (or more) spatially fixed point charges, then a force $\mathbf{F}$ acts on this test charge, which in general depends on its position **r**:

$$\mathbf{F} = \mathbf{F}(\mathbf{r}).$$

If one replaces q with another test charge q', one finds for the force $\mathbf{F}'$ acting on q':

$$\mathbf{F}'/q' = \mathbf{F}/q.$$

This finding suggests to introduce the concept of the **electric field**

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{q}\mathbf{F}(\mathbf{r}).}$$

This field—created by the stationary point charges—assigns a triple of real numbers to each space point **r**, which transforms as a vector.

> The task of electrostatics is to find the general connection between a charge distribution $\rho(\mathbf{r})$ and the electric field $\mathbf{E}(\mathbf{r})$ and to calculate the field $\mathbf{E}(\mathbf{r})$ from a given charge distribution $\rho(\mathbf{r})$ (e.g. a homogeneous spatial sphere).

1.3 Magnetostatics

Moving charges in the form of **stationary currents** are the origin of magnetostatic fields that we will introduce in analogy to electrostatic fields. We start from the following experimental observation: Inserting a test charge q into the environment of a conductor with a stationary electric current, the force acting on q at position $\mathbf{r}$ can be written as

$$\mathbf{F}(\mathbf{r}) = q(\mathbf{v} \times \mathbf{B}(\mathbf{r})).$$

where $\mathbf{v}$ is the velocity of the test charge and $\mathbf{B}(\mathbf{r})$ (independent of $\mathbf{v}$) the vector field generated by the given stationary electric current.

The task of magnetostatics is to find the general connection between a stationary electric current distribution $\mathbf{j}(\mathbf{r})$ and the magnetic field $\mathbf{B}(\mathbf{r})$ and to calculate the field $\mathbf{B}(\mathbf{r})$ for a given current distribution (e.g. for a stationary circulating current).

1.4 Concept of the Electromagnetic Field

One might get the impression that the electric and magnetic fields are independent quantities. The following simple considerations, however, show that this is not the case:

(1.) If a point charge Q is at rest in an inertial system Σ, the force acting on a test charge q for an observer in Σ is due to an electric field $\mathbf{E} \neq 0$, but there is no magnetic field $\mathbf{B}$. For other observers in an inertial system Σ' moving relative to Σ with velocity $\mathbf{v}$ the charge is moving. The observer in Σ' therefore measures a force due to both an electric field $\mathbf{E}' \neq 0$ and a magnetic field $\mathbf{B}' \neq 0$. The interaction between the charge Q and a test charge q will be seen as **electrical** interaction (mediated by the field $\mathbf{E}$) by an observer in Σ, whereas an observer in Σ' will detect both **electric** and **magnetic** interactions (mediated by the fields $\mathbf{E}'$ and $\mathbf{B}'$). This consideration shows that electric and magnetic fields must be considered as a unit, i.e. as the **electromagnetic** field.

Note: For the case discussed above, for a stationary current in the conductor there is no electric field because no **charge accumulation** occurs in the conductor, such that the positive and negative charge carriers (lattice building blocks—located in the conductor—and conduction electrons) compensate each other.

(2.) The mutual dependence of electric and magnetic fields inevitably occurs in case of arbitrary charge and current distributions $\rho(\mathbf{r})$ and $\mathbf{j}(\mathbf{r})$. The conservation of charge then results in a connection between ρ and $\mathbf{j}$, since the charge in a certain volume V can only decrease (increase), if a corresponding current flows out (in) through the surface of V. But then $\mathbf{E}$ and $\mathbf{B}$ can no longer be calculated independent of each other.

1.5 Maxwell's Equations

The general connections between the fields $\mathbf{E}$, $\mathbf{B}$ and the charges or currents (the **sources** of the electromagnetic field) are described by the **Maxwell equations**. The following task arises:

(1.) to formulate the Maxwell equations and to justify them experimentally,

(2.) to examine their invariance properties, which directly leads to the special theory of relativity. The investigation will show that the transition from an inertial system Σ to another inertial system Σ' must be described by a **Lorentz transformation**, i.e. the **same physics holds** for all observers in inertial systems.

(3.) The energy, momentum and angular momentum balance for a charged system of mass points will lead to assign **energy, momentum and angular momentum** to the electromagnetic field. From these terms such phenomena emerge as **radiation pressure**, which leads to the introduction of **photons**.

(4.) Solutions of Maxwell's equations. Examples are the propagation of electromagnetic waves or the radiation of an oscillating electric dipole in the vacuum.

1.6 The Electromagnetic Field in Matter

The Maxwell equations basically determine the fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$, if the charge distribution $\rho(\mathbf{r},t)$ and the current distribution $\mathbf{j}(\mathbf{r},t)$ are known. In practice the following problems arise:

(1.) For a system of *N* charged mass points one would have to solve Newton's equations of motion to get $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ **microscopically** in order to be able to calculate the electromagnetic fields. For matter of **macroscopic** dimensions (e.g. the dielectric medium between the plates of a capacitor or the iron core of a coil carrying a current) we are dealing with $10^{20} - 10^{25}$ mass points and charges!

(2.) The microscopically calculated functions $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ will in general fluctuate strongly for small spatial and temporal distances. The solution of Maxwell equations (multidimensional integrations) will then be practically impossible or not economical!

A way out of this problem is the following compromise: We discard the knowledge of the electromagnetic field in microscopic dimensions (volumes of 10^{-24} cm^3, times of 10^{-8} sec) and are satisfied with average values (10^{-6} cm^3, 10^{-3} sec). Instead of $\rho(\mathbf{r},t)$, $\mathbf{j}(\mathbf{r},t)$, $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ then averages appear in the form,

$$< \rho(\mathbf{r},t) >= \frac{1}{\Delta V \Delta t} \int d^3\xi d\tau \ \rho(\mathbf{r} + \overrightarrow{\xi}, t + \tau),$$

and correspondingly for $< \mathbf{j}(\mathbf{r},t) >$, $< \mathbf{E}(\mathbf{r},t) >$ and $< \mathbf{B}(\mathbf{r},t) >$. From Maxwell's equations for **microscopic** fields then equations of a similar structure arise for the **macroscopic** electromagnetic field. The distributions $< \rho >$ and $< \mathbf{j} >$ then are defined by the experimental setup (and resolution).

In this context it is useful to introduce as auxiliary variables the **dielectric displacement field** $\mathbf{D}$ and the **magnetic field strength**$\mathbf{H}$ in addition to the average values of the **fundamental** fields, the **electric field strength** $\mathbf{E}$ and the **magnetic induction** $\mathbf{B}$. This requires an additional determination of the relation between the different fields; these equations are obtained by assuming a linear connection of $\mathbf{E}$ and $\mathbf{D}$ or $\mathbf{B}$ and $\mathbf{H}$, characterized by the **dielectric constant** ϵ and the **permeability** μ. In the most simple case (Ohm's law) one establishes a linear relationship between the macroscopic current and the electric field strength, i.e. another material constant is introduced: the **electric conductivity** σ (as a proportionality constant). The actual calculation of these material constants (ϵ, μ, σ)—on the basis of the atomic structure of matter—belongs to the field of atomic and solid state physics and uses methods of statistical mechanics.

This results in the following **tasks**:

(1.) Transition from the microscopic to the macroscopic Maxwell equations.

(2.) Introduction of material constants and their calculation from the atomic structure of matter for simple models.

(3.) Behavior of the fields at interfaces between different media. As an example we will derive the laws of reflection and refraction in optics.

1.7 Covariant Formulation of Electrodynamics

The Maxwell equations are not **Gallilei** invariant but **Lorentz** invariant. In the last part of this book we will give a fully covariant formulation of electrodynamics and show the compatibility with Einstein's principle of special relativity.

Part I
Electrostatics

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2. Coulomb’s Law

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In this chapter we will introduce the electric field $\mathbf{E}(\mathbf{r})$ for a system of stationary point charges and calculate the total energy of the system. Furthermore, an asymptotic multipole expansion of the field will be analyzed for localized charge distributions and lead to characteristic properties of the systems, i.e. the total charge Q, the electric dipole moment $\mathbf{d}$ and the electric quadrupole tensor Q_{ij}.

2.1 Conservation of Charge and Charge Invariance

In the introduction we have briefly summarized the basic properties of the electric charge. For experimental tests, however, one first needs a rule for the measurement of **charge**. Such a prescription will be given in the next subchapter. Some observations for charge conservation are worth pointing out before.

Impressive evidence for charge conservation is pair creation and pair annihilation. For example, an electron (e^-) and a positron (e^+) annihilate to a high-energy **massive** photon which is uncharged; the opposite occurs when pairs are created (e.g. in π^+, π^- meson annihilation). In all these reactions always the same charge shows up (see Fig. 2.1).

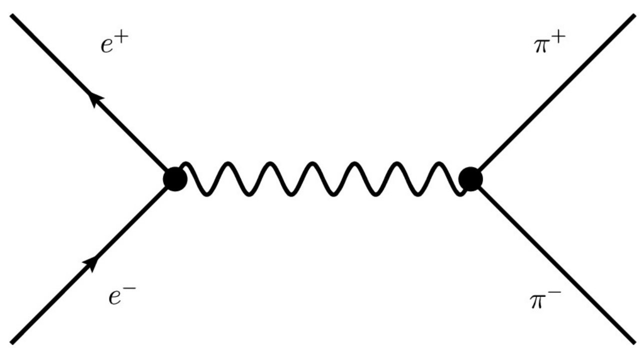

Fig. 2.1 Electron-positron annihilation into a $\pi^+\pi^-$ pair and vice versa. The total charge is zero throughout

The charge invariance is shown, for example, by the fact that atoms and molecules are charge neutral, although the state of motion of photons and electrons are very different. Particularly clear is the example of the helium atom (^4He) and the deuterium molecule (D_2). Both consist of 2 protons, 2 neutrons and 2 electrons and are therefore electrically neutral, although the state of motion of the protons in the nucleus of the helium atom and the D_2 molecule are very different: the ratio of the kinetic energies is about 10^6, the average distance of the protons in the D_2 molecule is of the order of 10^{-8} cm, in the *He* core of 10^{-13} cm.

2.2 Coulomb Force

As an experimentally proven **basis for electrostatics** we use **Coulomb's** law for the force between 2 point charges q_1 and q_2,

$$\mathbf{F}_{12} = \Gamma_e \ \frac{q_1 q_2}{r_{12}^3} \ \mathbf{r}_{12}, \tag{2.1}$$

where $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$ denotes the vector of their relative distance (see Fig. 2.2).

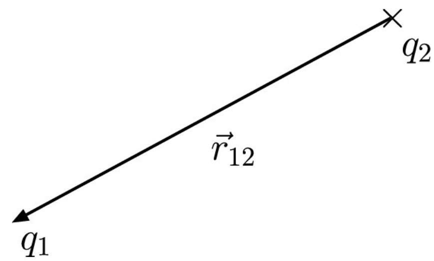

Fig. 2.2 Relative vector $\mathbf{r}_{12}$ between two point charges q_1 and q_2

Properties:

(1.) Attraction (repulsion) for opposite (equal) charges.

(2.) $\mathbf{F}_{12} = -\mathbf{F}_{21}$: actio = reactio ($\rightarrow$ momentum conservation).

(3.) Central force: a point charge (described by the scalar quantities m, q) has no preferred direction in space ($\rightarrow$ conservation of angular momentum).

Note: For (fast) moving charges (2.1) no longer holds. The electromagnetic field then has to be included in the momentum and angular momentum balance.

Equation (2.1) has to be supplemented by the **superposition principle**:

$$\mathbf{F}_1 = \mathbf{F}_{21} + \mathbf{F}_{31} \tag{2.2}$$

for the force exerted by 2 point charges q_2 and q_3 on q_1.

Prescription for the measurement of charge:

Comparing 2 charges q, q'—by measuring the forces exerted by a fixed charge Q—we find according to (2.1):

$$\frac{q}{q'} = \frac{F}{F'}. \tag{2.3}$$

Thus ratios of charges can be determined by force measurements: Choosing a **unit charge** (charge of the electron or positron) we can measure charges relative to this unit charge.

Unit systems:

In order to define the constant Γ_e there are basically 2 common options:

(i) **cgs-** (Gauß) system: Here we choose Γ_e as a dimensionless constant; the special choice

$$\Gamma_e = 1, \tag{2.4}$$

then determines (by (2.1)) the dimension of the charge to

$$[q] = [\text{force}]^{1/2}[\text{length}] = \text{dyn}^{1/2} \times \text{cm}. \tag{2.5}$$

The electrostatic unit then is the charge that exerts the force 1 dyn on an equal charge at a distance of 1 cm. This system is preferred in fundamental physics.

In applied electrodynamics (electrical engineering) one uses the (ii) **MKSA**—system in which, in addition to the mechanical units (meter, kilogram, second) still the charge unit **Coulomb** = ampère · second shows up. 1 ampère is the electrical current that deposits 1,118 mg of silver per second from a silver nitrate solution. Writing

$$\Gamma_e = \frac{1}{4\pi\epsilon_0}, \tag{2.6}$$

the constant ϵ_0 has the value

$$\epsilon_0 = 8.854 \cdot 10^{-12} \frac{\text{Coulomb}^2}{\text{Newton}\cdot\text{meter}^2}. \tag{2.7}$$

2.3 The Electric Field of a System of Point Charges

The force exerted by N point charges q_i–located at positions $\mathbf{r}_i$–on a test charge q at the position $\mathbf{r}$ according to (2.1) and (2.2) is:

$$\mathbf{F}(\mathbf{r}) = q\Gamma_e \sum_{i=1}^{N} \frac{q_i(\mathbf{r}-\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}_i|^3} = q\mathbf{E}(\mathbf{r}), \tag{2.8}$$

where we denote

$$\mathbf{E}(\mathbf{r}) = \sum_{i=1}^{N} \frac{q_i}{4\pi\epsilon_0} \frac{(\mathbf{r}-\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}_i|^3} \tag{2.9}$$

as (static) **electric field**, which is generated by the point charges q_i at the position $\mathbf{r}$. According to (2.8) $\mathbf{E}(\mathbf{r})$ is a vector field since q is a scalar. For a given charge q (2.8) shows how to measure an electric field. In this case the test charge has to be 'small' such that its influence on the field $\mathbf{E}$ can be neglected. Simple examples for the electrostatic field are shown in Fig. 2.3.

In analogy to the theory of gravity in mechanics one can obtain the vector function $\mathbf{E}(\mathbf{r})$ from the

electric potential

$$\Phi(\mathbf{r}) = \sum_{i=1}^{N} \frac{q_i}{4\pi\epsilon_0} \frac{1}{|\mathbf{r}-\mathbf{r}_i|}, \tag{2.10}$$

which is a scalar function, by differentiation:

$$\mathbf{E}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}). \tag{2.11}$$

The (potential) energy of the resting mass points with the charges q_i then is

$$U = \frac{1}{2}\sum_{i\neq j}^{N} \frac{q_i q_j}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}_i-\mathbf{r}_j|} = \frac{1}{2}\sum_{i=1}^{N} q_i\Phi(\mathbf{r}_i), \tag{2.12}$$

where $\Phi(\mathbf{r}_i)$ is the potential at position $\mathbf{r}_i$.

Note: In principle the **self-energy** for $i = j$ in the right expression has to be subtracted again from (2.12).

Examples:

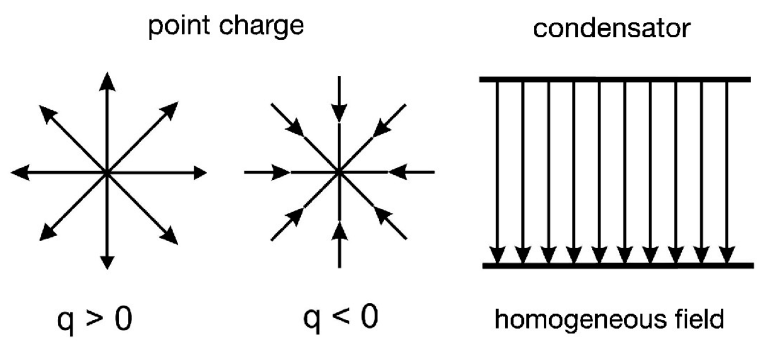

Fig. 2.3 Illustration for the electrostatic field in case of point charges and a condensator

2.4 Continuous Charge Distributions

In this case we replace the sum over charges by a volume integral over the charge distribution

$$\sum_i q_i \;\ldots . \rightarrow \int dV \rho(\mathbf{r}) \;\ldots \tag{2.13}$$

with the normalization

$$Q = \sum_i q_i = \int dV \rho(\mathbf{r}). \tag{2.14}$$

Equations (2.9), (2.10), (2.12) then turn to:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int dV'\rho(\mathbf{r}')\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}, \tag{2.15}$$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int dV'\rho(\mathbf{r}')\frac{1}{|\mathbf{r}-\mathbf{r}'|} \tag{2.16}$$

and

$$U = \frac{1}{2}\int dV \rho(\mathbf{r})\Phi(\mathbf{r}). \tag{2.17}$$

Example: homogeneously charged sphere

$$\rho(\mathbf{r}) = \rho_0 \quad \text{for} \quad |\mathbf{r}| \leq \mathrm{R}; \quad \rho(\mathbf{r}) = 0 \quad \text{else}. \tag{2.18}$$

The integration in (2.16) gives:

$$\Phi(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0|\mathbf{r}|} \quad \text{for} \quad \mathrm{r} \geq \mathrm{R}; \quad \Phi(\mathbf{r}) = \frac{\rho_0}{\epsilon_0}\left(\frac{\mathrm{R}^2}{2} - \frac{\mathrm{r}^2}{6}\right) \quad \text{for} \quad \mathrm{r} \leq \mathrm{R} \tag{2.19}$$

with

$$Q = \int dV \rho(\mathbf{r}) = \frac{4\pi}{3}\rho_0 R^3. \tag{2.20}$$

The electric field $\mathbf{E}$ then follows from (2.11):

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0}\frac{\mathbf{r}}{|\mathbf{r}|^3} \quad \text{for} \quad \mathrm{r} \geq \mathrm{R}; \quad \mathbf{E}(\mathbf{r}) = \frac{\rho_0}{3\epsilon_0}\mathbf{r} \quad \text{for} \quad \mathrm{r} \leq \mathrm{R}. \tag{2.21}$$

The energy U then reads (using (2.17) and (2.19)):

$$\begin{aligned} U &= \frac{\rho_0}{2}\int dV \;\; \Phi(\mathbf{r}) = \frac{4\pi\rho_0^2}{2\epsilon_0}\int_0^R r^2 dr \left(\frac{R^2}{2} - \frac{r^2}{6}\right) \\ &= 2\pi\frac{\rho_0^2}{\epsilon_0}\frac{2R^5}{15} = \frac{3}{5}\frac{Q^2}{4\pi\epsilon_0}\frac{1}{R}. \end{aligned} \tag{2.22}$$

Application: Determination of the classical electron radius.

According to (2.22) the **self-energy** of a point-like charge becomes infinite for $R \to 0$. According to the theory of relativity the energy of a stationary particle, e.g. an electron, with rest mass m_0 is linked to its self-energy by

$$E_0 = m_0c^2 \equiv U_e = \frac{3}{5}\frac{e^2}{4\pi\epsilon_0}\frac{1}{R_0}. \tag{2.23}$$

A strictly point-like (charged) particle then will have an infinitely large rest mass according to (2.22)! On the other hand, to regain the total (finite) rest mass of an electron by its electrostatic energy, one can introduce a finite radius R_0, the **classical electron radius**,

$$R_0 = \frac{3}{5}\frac{e^2}{4\pi\epsilon_0}\frac{1}{m_0c^2} \approx 10^{-13} \;\; \text{cm} = 1\,\text{fm}. \tag{2.24}$$

Thus for dimensions $< 10^{-13}$ cm we have to expect deviations from Coulomb's law.

2.5 Multipole Expansion

We consider a charge distribution (discrete or continuous) limited to a finite volume V and examine its potential $\Phi(\mathbf{r})$ at a point P far outside of the volume V (see Fig. 2.4).

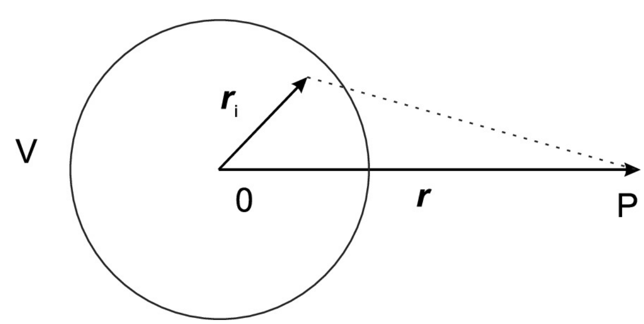

Fig. 2.4 The charge distribution $\rho(\mathbf{r})$, localized in the volume V, is analyzed in a distant point P

We can use the coordinate origin O as the center of charge, defined by

$$\mathbf{r}_q = \frac{\sum_i |q_i| \mathbf{r}_i}{\sum_i |q_i|}. \tag{2.25}$$

As long as $r_i \ll r$, we can expand (2.10) in a Taylor series,

$$\Phi(\mathbf{r}) = \Phi_0(\mathbf{r}) + \Phi_1(\mathbf{r}) + \Phi_2(\mathbf{r}) + \Phi_3(\mathbf{r}) + \ldots \tag{2.26}$$

Using

$$f(\mathbf{r} - \mathbf{a}) = \sum_{n=0}^{\infty} \frac{1}{n!} (-\mathbf{a} \cdot \nabla_r)^n \; f(\mathbf{r}) \tag{2.27}$$

for a scalar function $f(\mathbf{r})$ (infinitely differentiable) or in our case

$$\frac{1}{|\mathbf{r}-\mathbf{r}_i|} = \sum_{n=0}^{\infty} \frac{1}{n!} (-\mathbf{r}_i \cdot \nabla_r)^n \; \frac{1}{r} = \frac{1}{r} - \mathbf{r}_i \cdot \nabla_r \frac{1}{r} + \frac{1}{2} (\mathbf{r}_i \cdot \nabla_r)^2 \frac{1}{r} \cdots \tag{2.28}$$

$$= \frac{1}{r} - \left(x_i \frac{\partial}{\partial x} + y_i \frac{\partial}{\partial y} + z_i \frac{\partial}{\partial z} \right) \frac{1}{r} + \frac{1}{2} \left(x_i \frac{\partial}{\partial x} + y_i \frac{\partial}{\partial y} + z_i \frac{\partial}{\partial z} \right)^2 \frac{1}{r} \cdots$$

the first terms are:
(1.) **The monopole term**

$$\Phi_0(\mathbf{r}) = \sum_i \frac{q_i}{4\pi\epsilon_0 r} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r} \tag{2.29}$$

describes a point charge Q localized at the origin O. In 0th order approximation of the Taylor expansion every charge distribution looks like a point charge if viewed from a sufficiently large distance!
(2.) **Dipole term**
The linear term in the coordinates $\mathbf{r}_i$ of the point charges has the following form:

$$\Phi_1(\mathbf{r}) = \tag{2.30}$$

$$-\sum_i \frac{q_i}{4\pi\epsilon_0} x_i \frac{\partial}{\partial x} \left(\frac{1}{|\mathbf{r}-\mathbf{r}_i|} \right)_{r_i=0} - \sum_i \frac{q_i}{4\pi\epsilon_0} y_i \frac{\partial}{\partial y} \left(\frac{1}{|\mathbf{r}-\mathbf{r}_i|} \right)_{r_i=0} - \sum_i \frac{q_i}{4\pi\epsilon_0} z_i \frac{\partial}{\partial z} \left(\frac{1}{|\mathbf{r}-\mathbf{r}_i|} \right)_{r_i=0}$$

$$= -\sum_i \frac{q_i}{4\pi\epsilon_0} (x_i \frac{\partial}{\partial x} + y_i \frac{\partial}{\partial y} + z_i \frac{\partial}{\partial z}) \frac{1}{r}$$

$$= -\frac{\mathbf{d}}{4\pi\epsilon_0} \cdot \nabla(\frac{1}{r}) = \frac{\mathbf{d} \cdot \mathbf{r}}{4\pi\epsilon_0 r^3} = \frac{d \;\; \cos\theta}{4\pi\epsilon_0 r^2}$$

where the vector $\mathbf{d}$, the **dipole moment**, is given by

$$\mathbf{d} = \sum_i q_i \mathbf{r}_i. \tag{2.31}$$

The angle θ—used in the transformations in (2.30)—is the angle between $\mathbf{r}$ and $\mathbf{d}$ (see Fig. 2.5).

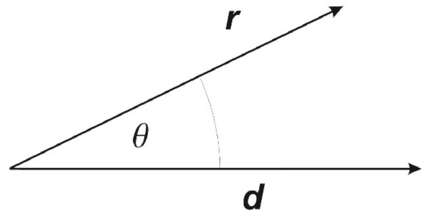

Fig. 2.5 Illustration of the angle θ in (2.30)

(Example for $\partial/\partial x(1/r)$):

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{\partial r}{\partial x}\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{\partial r}{\partial x} = -\frac{1}{r^2}\frac{x}{r}. \tag{2.32}$$

Dependence of the dipole moment on the coordinate origin: If we shift the origin O by the spatial vector $\mathbf{a}$, the dipole moment becomes

$$\mathbf{d}' = \sum_i q_i(\mathbf{r}_i - \mathbf{a}) = \sum_i q_i \mathbf{r}_i - \mathbf{a}\sum_i q_i = \mathbf{d} - \mathbf{a}Q. \tag{2.33}$$

If $Q \neq 0$, we can choose **a** such that **d**' = 0. On the other hand, if Q = 0, then $\mathbf{d} = \mathbf{d}'$ independent of the origin and the dipole moment describes a real **internal** property of the system under consideration (e.g. a charge neutral molecule).

Extraction of the dipole moment: We determine the centers of mass of the positive and negative charge carriers. If these coincide then according to (2.31) **d** = 0. Otherwise its connecting line gives the direction of **d**, its distance is a measure for the magnitude of the dipole moment **d**.

Example: molecules (see Fig. 2.6).

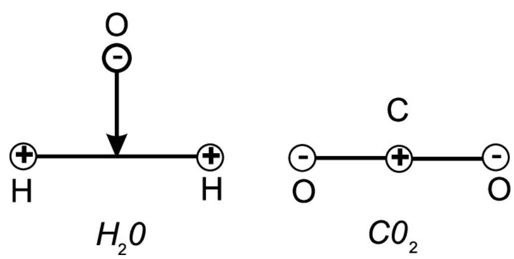

Fig. 2.6 Illustration for the positive and negative centers of the charge distribution for H_2O and CO_2

(3.) **Quadrupole term**:
We look at the quadratic term in the Taylor series (2.28) and get

$$\begin{aligned} 4\pi\epsilon_0\Phi_2(\mathbf{r}) &= \frac{1}{2}\Big\{Q_{xx}\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) + Q_{yy}\frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) + Q_{zz}\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) \\ &\quad + Q_{xy}\frac{\partial^2}{\partial x\partial y}\left(\frac{1}{r}\right) + Q_{yz}\frac{\partial^2}{\partial y\partial z}\left(\frac{1}{r}\right) + Q_{zx}\frac{\partial^2}{\partial z\partial x}\left(\frac{1}{r}\right), \\ &\quad + Q_{yx}\frac{\partial^2}{\partial y\partial x}\left(\frac{1}{r}\right) + Q_{zy}\frac{\partial^2}{\partial z\partial y}\left(\frac{1}{r}\right) + Q_{xz}\frac{\partial^2}{\partial x\partial z}\left(\frac{1}{r}\right)\Big\}, \end{aligned} \tag{2.34}$$

where

(2.35)

$$Q_{xx} = \sum_i q_i \ x_i^2; \quad Q_{xy} = \sum_i q_i \ x_i y_i; \quad etc.$$

are the components of the **quadrupole tensor**. This tensor is symmetric and real and therefore can always be diagonalized:

$$Q_{mn} = Q_m \delta_{mn}, \tag{2.36}$$

i.e.

$$4\pi\epsilon_0 \Phi_2(\mathbf{r}) = \tfrac{1}{2}\{Q_x \tfrac{\partial^2}{\partial x^2} + Q_y \tfrac{\partial^2}{\partial y^2} + Q_z \tfrac{\partial^2}{\partial z^2}\}(\tfrac{1}{r}) \tag{2.37}$$

in the principal axis system, in which the quadrupole tensor is diagonal. There is a wide analogy to the **inertia tensor** in mechanics.

Physical normalization of the diagonal elements: The relation

$$\Delta(\tfrac{1}{r}) = (\tfrac{\partial^2}{\partial x^2} + \tfrac{\partial^2}{\partial y^2} + \tfrac{\partial^2}{\partial z^2})\tfrac{1}{r} = (\tfrac{\partial}{\partial x}\tfrac{x}{r}\tfrac{\partial}{\partial r} + \tfrac{\partial}{\partial y}\tfrac{y}{r}\tfrac{\partial}{\partial r} + \tfrac{\partial}{\partial z}\tfrac{z}{r}\tfrac{\partial}{\partial r})\tfrac{1}{r} \tag{2.38}$$

$$= -\frac{3}{r^3} + 3\frac{x^2+y^2+z^2}{r^5} = 0 \quad \text{for} \quad \mathrm{r} \neq 0$$

allows to replace the components (2.35) in (2.36) in the principal axis system by:

$$Q_x = \sum_i q_i (x_i^2 - \tfrac{r_i^2}{3}) = \tfrac{1}{3}\sum_i q_i \{2x_i^2 - y_i^2 - z_i^2\}, \tag{2.39}$$

$$Q_y = \frac{1}{3}\sum_i q_i\{2y_i^2 - x_i^2 - z_i^2\}, Q_z = \frac{1}{3}\sum_i q_i\{2z_i^2 - x_i^2 - y_i^2\},$$

without changing $\Phi_2(\mathbf{r})$ because an additional term $-r^2\Delta(\frac{1}{r})$ in (2.37) has no effect on $\Phi_2(\mathbf{r})$. The Eqs. (2.39) show that the eigenvalues Q_m describe the deviations from spherical symmetry because for a spherical charge distribution we get:

$$\sum_i q_i \ x_i^2 = \sum_i q_i \ y_i^2 = \sum_i q_i \ z_i^2 = \tfrac{1}{3}\sum_i q_i \ r_i^2 \ \to Q_m = 0. \tag{2.40}$$

Special case: Axial symmetry e.g. around the z-axis. Then with

$$< x^2 >_q := \frac{1}{3}\sum_i q_i x_i^2 =< y^2 >_q, < z^2 >_q := \frac{1}{3}\sum_i q_i z_i^2$$

$$Q_x =< x^2 >_q - < z^2 >_q = Q_y = -\tfrac{1}{2}Q_z = -\tfrac{1}{2}\big(2 < z^2 >_q -2 < x^2 >_q\big), \tag{2.41}$$

i.e. the quadrupole term $\Phi_2(\mathbf{r})$ is given by a single number, the **quadrupole moment**. For this case the angular dependence $\Phi_2(\mathbf{r})$ is easy to specify: We form

$$\tfrac{\partial^2}{\partial x^2}(\tfrac{1}{r}) = -\tfrac{\partial}{\partial x}(\tfrac{x}{r^3}) = -\tfrac{1}{r^3} + 3\tfrac{x^2}{r^5} = \tfrac{3x^2-r^2}{r^5}, \tag{2.42}$$

$$\frac{\partial^2}{\partial y^2}(\frac{1}{r}) = \frac{3y^2 - r^2}{r^5}, \frac{\partial^2}{\partial z^2}(\frac{1}{r}) = \frac{3z^2 - r^2}{r^5},$$

and find with (2.41):

$$\Phi_2(\mathbf{r}) = \tfrac{1}{4\pi\epsilon_0}\tfrac{1}{2r^5}\left(Q_x(3x^2 - r^2) + Q_y(3y^2 - r^2) + Q_z(3z^2 - r^2)\right) \tag{2.43}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{Q_z}{2r^5}\left(-\frac{1}{2}(3x^2 - r^2) - \frac{1}{2}(3y^2 - r^2) + (3z^2 - r^2)\right)$$

$$= \frac{1}{4\pi\epsilon_0}\frac{Q_z}{4r^5}\left(-3x^2 - 3y^2 + 6z^2 - 3z^2 + 3z^2)\right)$$

$$= \frac{3Q_z(3z^2 - r^2)}{4\pi\epsilon_0 \cdot 4r^5} = \frac{Q_0}{4\pi\epsilon_0} \cdot \frac{(3\,\cos^2\,\theta - 1)}{2r^3},$$

where

$$Q_0 = \tfrac{3}{2}Q_z \tag{2.44}$$

is the **quadrupole moment** of the axially symmetric charge distribution.

Equation (2.43) shows the characteristic r-dependence for the quadrupole term; the angular dependence is clearly different from the dipole term.

Continuous charge distributions:

In analogy to Sect. 2.4 we get for a continuous (spatially localized) charge distribution ($dV = d^3r$):

$$\mathbf{d} = \textstyle\int dV \rho(\mathbf{r})\ \mathbf{r} \tag{2.45}$$

instead of Eqs. (2.31) and (2.39) is replaced by:

$$Q_x = \frac{1}{3}\int dV\rho(\mathbf{r})\ (2x^2 - y^2 - z^2),$$

$$Q_y = \frac{1}{3}\int dV\rho(\mathbf{r})\ (2y^2 - x^2 - z^2),$$

$$Q_z = \tfrac{1}{3}\textstyle\int dV\rho(\mathbf{r})\ (2z^2 - x^2 - y^2). \tag{2.46}$$

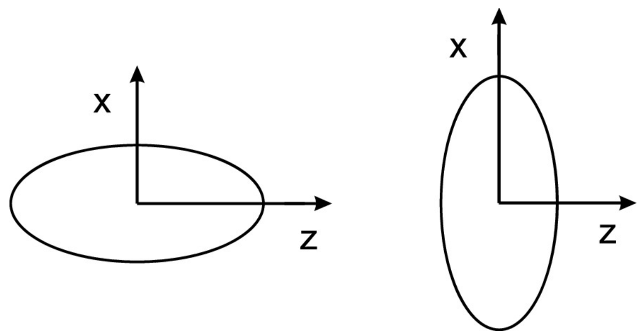

Fig. 2.7 Examples for axially symmetric charge distributions with positive (left) and negative (right) quadrupole moment

Example: A series of atomic nuclei is (axially symmetrical) deformed and is characterized electrostatically by a quadrupole moment Q_0. On the other hand an atomic nucleus (by Coulomb excitation) can by excited to a quadrupole-like deformed (rotating) state. Such deformed and rotating atomic nuclei decay to the ground state by the emission of electromagnetic radiation. The deviation from spherical symmetry can be both **positive**, $Q_0 > 0$ (left Fig. 2.7), as well as **negative**, $Q_0 < 0$ (right Fig. 2.7), which clearly shows a shape of a **cigar** or corresponds to a **disk**.

In summarizing this chapter we have introduced the electric field $\mathbf{E}(\mathbf{r})$ for a system of stationary point charges and calculated the total energy of the system. Furthermore, an asymptotic multipole expansion of the field has been analyzed for localized charge distributions and lead to characteristic properties of the systems, i.e. the total charge Q, the electric dipole moment $\mathbf{d}$ and the electric quadrupole tensor Q_{ij}.

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3. Basics of Electrostatics

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In this chapter we will address more formal aspects of electrostatics and introduce the flux of a vector field. The Gauss' law will allow to relate the flux of the electrostatic field through a closed surface to the total charge within the volume bordered by the surface considered. Some applications of the Gauss' law are discussed and differential equations for the electrostatic field $\mathbf{E}$ and its potential Φ are derived.

3.1 Flux of a Vector Field

In the following we want to look for equivalent formulations of Coulomb's law. To this aim we introduce the concept of the **flux of a vector field**.

Let a vector field $\mathbf{A}(\mathbf{r})$ be defined on a surface *F*, which is **finite** and **two-sided**, i.e. *F* has a finite area and a **top** and **bottom** defined by the surface normals. **Counterexample**: the **Möbius' band** has not a well defined top and bottom.

The flux of the vector field $\mathbf{A}$ through the surface *F* we define by the surface integral

$$\int_F \mathbf{A}(\mathbf{r}) \cdot d\mathbf{f} = \int_F A_n(\mathbf{r}) df, \tag{3.1}$$

where A_n is the component of $\mathbf{A}$ in direction of the surface normal.

To interpret (3.1) we consider a fluid flowing with the velocity $\mathbf{v}(\mathbf{r})$ and the density $\rho(\mathbf{r})$. Let's choose the vector field as

$$\mathbf{A}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{v}(\mathbf{r}), \tag{3.2}$$

then

(3.3)

$$\int_F \mathbf{A}(\mathbf{r}) \cdot d\mathbf{f} = \int_F \rho(\mathbf{r})\mathbf{v}(\mathbf{r}) \cdot d\mathbf{f}$$

is the amount of fluid flowing through F per unit of time. Equation (3.3) shows that only the area perpendicular to the flow contributes.

3.2 Gauss' Law

We now choose the electrostatic field $\mathbf{E}(\mathbf{r})$ (for $\mathbf{A}(\mathbf{r})$) and for F a **closed** surface with the properties mentioned above. Then the **electric flux**

$$\Psi = \oint_F \mathbf{E}(\mathbf{r}) \cdot d\mathbf{f} = \oint_F E_n(\mathbf{r})\ df \tag{3.4}$$

is linked to the total charge Q in the volume V by the

Gauss' law:

$$\Psi = \oint_F \mathbf{E}(\mathbf{r}) \cdot d\mathbf{f} = \frac{Q}{\epsilon_0}. \tag{3.5}$$

Proof

1st step: Let q be a single point charge at the center of a sphere with radius R. Then at every point on the surface of the sphere $\mathbf{E}(\mathbf{r})$ is parallel to the (outer) surface normal $\mathbf{n}$ (see Fig. 3.1) and for the magnitude of $\mathbf{E}$ we have:

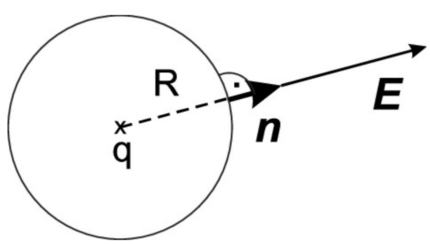

Fig. 3.1 For a single charge q in the center of the sphere of radius R the electric field is oriented in the direction of the surface vector $\mathbf{n}$

$$E = E_n = \frac{q}{4\pi\epsilon_0 R^2}. \tag{3.6}$$

Then

$$\Psi = \oint \frac{q}{4\pi\epsilon_0 R^2} R^2 d\Omega = \frac{q}{4\pi\epsilon_0} \oint d\Omega = \frac{q}{\epsilon_0}, \tag{3.7}$$

is independent of the radius R of the sphere.

2nd step: We replace the spherical surface with any closed area within the framework of the requirements stated in Sect. 3.1 (see Fig. 3.2). Detail (see Fig. 3.3):

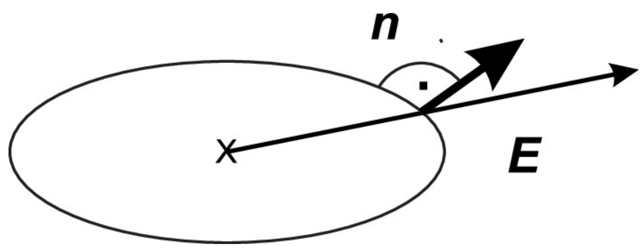

Fig. 3.2 Illustration of a closed area around the charge q in the center

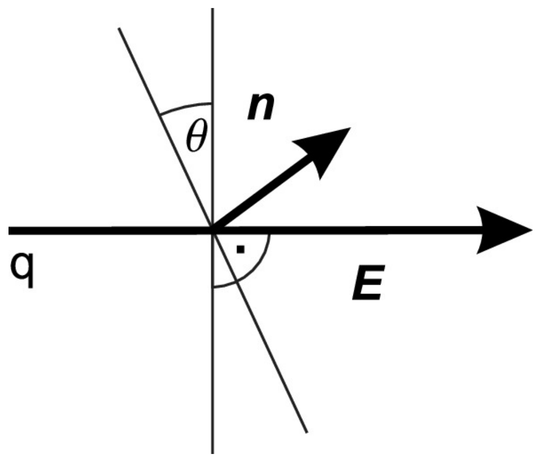

Fig. 3.3 Detail of the angles at the surface of the area of Fig. 3.2

Then

$$\Psi = \oint_F \frac{q\cos\theta}{4\pi\epsilon_0 R^2} df = \oint_F \frac{q}{4\pi\epsilon_0 R^2} df' = \frac{q}{4\pi\epsilon_0} \oint d\Omega = \frac{q}{\epsilon_0}, \tag{3.8}$$

since the area vector $\mathbf{df}'$ is parallel to $\mathbf{E}$ and the angle between $\mathbf{df}$ and $\mathbf{df'}$ is given by θ.

3rd step: The charge q is located outside of F.

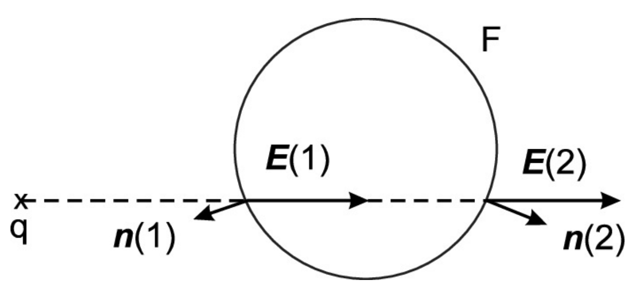

Fig. 3.4 Orientation of the $\mathbf{E}$ field on the surface of a sphere which does not include the charge q

Taking into account the respective normal direction (according to the outside normal) we find that e.g. the contributions from the area around points 1 and 2 (see Fig. 3.4) cancel each other since the field strength $\mathbf{E}(\mathbf{r})$ (from q) is always radially directed and drops like $1/R^2$ while *df* increases with R^2. Thus we get for the electrical flux

$$\Psi = 0. \tag{3.9}$$

4th step: For *N* point charges q_i within the volume *V* with surface $F = \partial V$ we find according to the superposition principle:

$$\boxed{\Psi = \frac{1}{\epsilon_0} \sum_i q_i = \frac{Q}{\epsilon_0}.} \tag{3.10}$$

3.3 Applications of Gauss' Law

For symmetrical charge distributions Eq. (3.5) offers the possibility to calculate the field strength $\mathbf{E}(\mathbf{r})$ with a rather low effort. We consider 2 **examples**:

1. **Field of a homogeneously charged sphere**.
Let

$$\rho(\mathbf{r}) = \rho(r) \quad \text{for} \quad \mathrm{r} \leq \mathrm{R}, \quad \rho(\mathbf{r}) = 0 \quad \text{else.} \tag{3.11}$$

Due to the spherical symmetry $\mathbf{E}(\mathbf{r})$ is directed radially, such that

$$\Psi = \oint_{F(r)} \mathbf{E}(\mathbf{r}) \cdot d\mathbf{f} = 4\pi r^2 E(r) = \frac{Q_r}{\epsilon_0} = \frac{4\pi}{\epsilon_0} \int_0^r r'^2 \rho(r') \; dr', \tag{3.12}$$

where $E(r) = |\mathbf{E}(|\mathbf{r}|)|$ and Q_r is the charge contained in a concentric sphere with radius *r*. For points with $r \geq R$, $Q_r = Q$ is the total charge and it follows from (3.12):

$$E(r) = \frac{Q}{4\pi\epsilon_0 r^2} \quad \text{for} \quad \mathrm{r} \geq \mathrm{R}. \tag{3.13}$$

For $r \leq R$ the result depends on the special form of $\rho(r)$. As an example we choose

$$\rho(r) = \rho_0 = \text{const}, \tag{3.14}$$

then:

(3.15)

$$Q_r = \frac{4\pi}{3} r^3 \rho_0,$$

and $E(r)$ as in (2.21):

$$E(r) = \frac{Q_r}{4\pi r^2 \epsilon_0} = \frac{\rho_0}{3\epsilon_0} r. \tag{3.16}$$

2. **Homogeneously charged, infinitely extended plane**.

For symmetry reasons $\mathbf{E}$ is perpendicular to the charged plane, the magnitude $E = |\mathbf{E}|$ is the same for the points 1 and 2 at a distance r from the plane (see Fig. 3.5). The Gauss' law then gives:

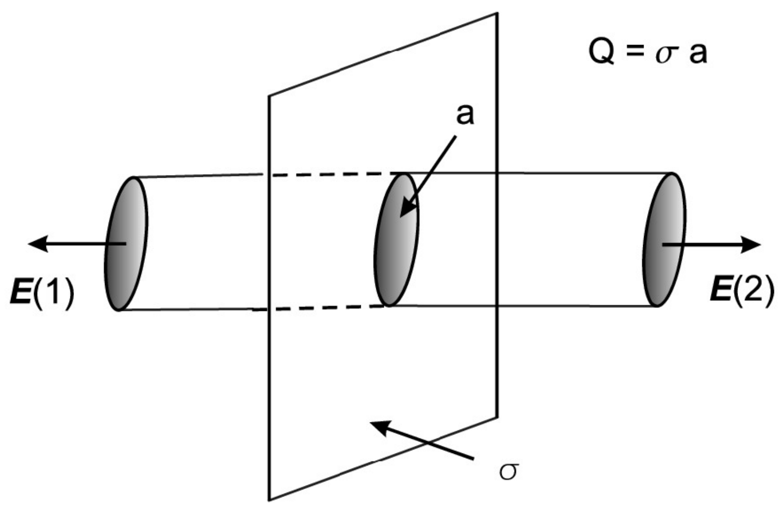

Fig. 3.5 Illustration of the electric field for a charged, infinitely extended plane

$$\Psi = \oint_F \mathbf{E} \cdot d\mathbf{f} = aE(1) + aE(2) = \frac{Q}{\epsilon_0} = \frac{\sigma a}{\epsilon_0}, \tag{3.17}$$

if a is the cylinder base area and σ the surface charge density. We don't get any contribution from the cylinder mantle because $\mathbf{E}$ has no component in the direction of the normal on the cylinder mantle. Result:

$$E = \frac{\sigma}{2\epsilon_0} \tag{3.18}$$

independent of r.

3.4 Differential Equations for $\mathbf{E}$ and Φ

We want to represent the Gauss' law (3.5) in differential form. To this aim we transform the surface integral into a volume integral over the enclosed surface $F = \partial V$ of the volume V (**Gauss' formula**):

$$\oint_F \mathbf{E}(\mathbf{r}) \cdot d\mathbf{f} = \int_V \nabla \cdot \mathbf{E}(\mathbf{r}) \; dV = \frac{Q}{\epsilon_0}. \tag{3.19}$$

With

$$Q = \int_V \rho(\mathbf{r}) \; dV \tag{3.20}$$

then follows:

$$\int_V (\nabla \cdot \mathbf{E}(\mathbf{r}) - \frac{\rho(\mathbf{r})}{\epsilon_0}) \; dV = 0. \tag{3.21}$$

Since Eq. (3.21) must hold for any volume V the integrand has to disappear:

$$\boxed{\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0}.} \tag{3.22}$$

Equation (3.22) does not change if we add to $\mathbf{E}(\mathbf{r})$ any divergence-free vector function $\mathbf{E}'(\mathbf{r})$; Eq. (3.22) is therefore not sufficient to determine the electrical field. Another differential relation for $\mathbf{E}(\mathbf{r})$ we get from (cf. (2.11))

$$\mathbf{E}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}) \tag{3.23}$$

with the vector indentity

$$\nabla \times (\nabla f) = 0, \tag{3.24}$$

i.e.:

$$\boxed{\nabla \times \mathbf{E}(\mathbf{r}) = 0.} \tag{3.25}$$

However, to calculate the electromagnetic field from a given charge distribution $\rho(\mathbf{r})$ from (3.22) and (3.25) is quite complex.

In practice it is more convenient to go a step ahead to the potential $\Phi(\mathbf{r})$ and calculate the field strength $\mathbf{E}(\mathbf{r})$ by differentiation according to(3.23). Inserting

(3.23) in (3.22) we get the **Poisson equation**

$$\nabla \cdot (\nabla \Phi)(\mathbf{r}) = \Delta \Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0} \tag{3.26}$$

with the abbreviation

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{3.27}$$

Having found a solution to (3.26) we can always add any solution of the homogeneous equation (**Laplace equation**)

$$\Delta \Phi(\mathbf{r}) = 0 \tag{3.28}$$

and get a new solution of (3.26). This ambiguity can be avoided by specifying boundary conditions. For a further discussion see Chap. 4.

3.5 Energy of the Electrostatic Field

In order to transfer a point charge q_1 from infinity to a charge q_2 by the distance r_{12}, one needs (or gains) the energy

$$U = \frac{q_1 q_2}{4\pi\epsilon_0 r_{12}}. \tag{3.29}$$

In order to get a specific charge distribution of N point charges q_i (characterized by the mutual distances of the charges q_i) one needs (or gains) the energy

$$U = \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{4\pi\epsilon_0 r_{ij}}, \tag{3.30}$$

where the factor 1/2 ensures that double counting is avoided. The restriction $i \neq j$ excludes **self-energies** of the point charges.

We can interpret U as the **potential energy** of a system of charged mass points. On the other hand the energy U can also be considered as the energy **stored** in the electric field in form of **field energy**.

To analyse the connection between the two perspectives quantitatively we reformulate (3.30) (see Chap. 2) by:

$$U = \frac{1}{2}\sum_i q_i \Phi(\mathbf{r}_i) = \frac{1}{2}\int_V \rho(\mathbf{r})\Phi(\mathbf{r})\ dV, \tag{3.31}$$

where $\Phi(\mathbf{r}_i)$ is the potential at position $\mathbf{r}_i$ of the point charge *i*, which the other point charges have created. Now we can rewrite Eqs. (3.31) with (3.26) as:

$$U = -\frac{\epsilon_0}{2}\int_V \Phi(\mathbf{r})\Delta\Phi(\mathbf{r})\ dV. \tag{3.32}$$

Equation (3.32) completely describes the energy *U* in terms of the potential Φ , i.e. by the electrostatic field without reference to the charges. Instead of the potential Φ we can express the potential *U* by the field strength $\mathbf{E}$ using the identity

$$\nabla\cdot(f\nabla g) = (\nabla f)\cdot(\nabla g) + f\Delta g \tag{3.33}$$

for $f = g = \Phi$, i.e. $\Phi\Delta\Phi = \nabla\cdot(\Phi\nabla\Phi) - (\nabla\Phi)^2$ leading to:

$$U = \frac{\epsilon_0}{2}\int_V (\nabla\Phi(\mathbf{r}))^2 dV - \frac{\epsilon_0}{2}\int_V \nabla\cdot(\Phi(\mathbf{r})\nabla\Phi(\mathbf{r}))\ dV, \tag{3.34}$$

and using the Gauss' formula,

$$\int_V \nabla\cdot(\Phi(\mathbf{r})\nabla\Phi(\mathbf{r}))\ dV = \oint_F \Phi(\mathbf{r})\nabla\Phi(\mathbf{r})\cdot d\mathbf{f} \tag{3.35}$$

with $F = \partial V$ denoting the surface of *V*. Now if all charges are enclosed in a finite volume the surface integral (3.35) decreases with increasing volume *V*, since $\Phi(\mathbf{r})\nabla\Phi(\mathbf{r})$ drops as $\sim R^{-3}$ with increasing distance *R* from the charge center, while the surface only increases with R^2. In the limit $V \to \infty$ we then obtain

$$U = \frac{\epsilon_0}{2}\int_V (\nabla\Phi(\mathbf{r}))^2\ dV = \frac{\epsilon_0}{2}\int_V \mathbf{E}^2(\mathbf{r})\ dV \tag{3.36}$$

as the energy stored in the field. The quantity $\epsilon_0 E^2(\mathbf{r})/2$ then gives the **energy density**.

3.6 Multipoles in the External Electric Field

If a spatially localized charge distribution ρ is placed in an **external** electrostatic field given by its potential Φ_a, then (according to Sect. 3.5) we get for its energy

$$U = \int_V \rho(\mathbf{r})\Phi_a(\mathbf{r})dV, \tag{3.37}$$

if we assume that the external field is not changed (noticeably) by $\rho(\mathbf{r})$ and the charges—generating the external field Φ_a—are outside the area of V. This explains the absence of the factor 1/2 in (3.37) compared to (3.31). Furthermore, let Φ_a be slowly changing in the volume V such that we can expand Φ_a in a Taylor series with respect to the center of the charge distribution ρ:

$$\Phi_a(\mathbf{r}) = \Phi_a(0) + \sum_{i=1}^{3} x_i \frac{\partial \Phi_a}{\partial x_i}(0) + \frac{1}{2}\sum_{i,j=1}^{3} x_i x_j \frac{\partial^2 \Phi_a}{\partial x_i \partial x_j}(0) + \ldots \tag{3.38}$$

Since in the region of the volume V we have for the **external field**

$$\nabla \cdot \mathbf{E}_a = 0 \tag{3.39}$$

in line with our assumption, we can rewrite Eq. (3.38) as follows (see Sect. 2.5):

$$\Phi_a(\mathbf{r}) = \Phi_a(0) - \sum_{i=1}^{3} x_i E_{ia}(0) - \frac{1}{2}\sum_{i,j=1}^{3} (x_i x_j - \frac{r^2}{3}\delta_{ij}) \frac{\partial E_{ia}}{\partial x_j}(0) + \ldots \tag{3.40}$$

with $E_{ia}(0) = -\partial/\partial x_i \Phi_a(0)$. The combination of (3.37) and (3.40) gives:

$$U = \int_V \rho(\mathbf{r})\Phi_a(\mathbf{r})\ dV \tag{3.41}$$

$$= \int_V \rho(\mathbf{r}) \left(\Phi_a(0) - \sum_{i=1}^{3} x_i E_{ia}(0) - \frac{1}{2}\sum_{i,j=1}^{3} (x_i x_j - \frac{r^2}{3}\delta_{ij}) \frac{\partial E_{ia}}{\partial x_j}(0) + \ldots \right)\ dV$$

$$= Q\Phi_a(0) - \sum_{i=1}^{3} d_i E_{ia}(0) - \frac{1}{2}\sum_{i,j=1}^{3} Q_{ij} \frac{\partial E_{ia}}{\partial x_j}(0) + \ldots$$

Equation (3.41) shows how the multipole moments of a charge distribution $\rho(\mathbf{r})$ interact with an external field $\mathbf{E}_a$: the total charge Q with the potential Φ_a, the dipole moment $\mathbf{d}$ with the field strength $\mathbf{E}_a$, the quadrupole tensor Q_{ij} with the field gradient $\partial E_{ia}/\partial x_j$ etc.

Examples: Atomic dipoles in external electric fields, interaction of the nuclear quadrupole moment with the electron shell or with time-dependent electrical fields (e.g. in nuclear reactions with center of mass energies below the Coulomb barrier).

In summarizing this chapter we have introduce the flux of a vector field and derived the Gauss' law which relates the flux of the electrostatic field through a closed surface to the total charge within the volume bordered by the surface considered. Some applications of the Gauss' law have been presented and differential equations for the electrostatic field $\mathbf{E}$ and its potential Φ have been derived. Furthermore, the energy density of the electrostatic field has been computed and the interaction of a static charge configuration with an external field been discussed.

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4. Boundary Value Problems in Electrostatics

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The solution of the Poisson equation is in general subject to boundary conditions that have to be fulfilled simultaneously. Apart from a discussion of the uniqueness of the solution in this chapter we will present three practical methods for the calculation of $\Phi(\mathbf{r})$ in case of specific boundary conditions, i.e. the mirror method, the inversion method and the separation of variables.

4.1 Uniqueness

In the following we want to show that the Poisson equation or the Laplace equation has a unique solution for $\Phi(\mathbf{r})$, if one of the following boundary conditions hold:

(i) **Dirichlet condition**

$$\Phi(\mathbf{r}) \text{ is given on a closed area } \; F \tag{4.1}$$

or

(ii) **von Neumann condition**

$$\nabla\Phi(\mathbf{r}) \text{ is given on a closed area } \; F \; . \tag{4.2}$$

Proof We assume that there are 2 solutions $\Phi_1(\mathbf{r})$ and $\Phi_2(\mathbf{r})$ of

$$\Delta\Phi(\mathbf{r}) = -\tfrac{\rho}{\epsilon_0} \tag{4.3}$$

with the same boundary conditions given by (4.1) or (4.2). Then we obtain for the difference $U(\mathbf{r}) = \Phi_1(\mathbf{r}) - \Phi_2(\mathbf{r})$:

$$\Delta U(\mathbf{r}) = 0 \tag{4.4}$$

in the volume V enclosed by F. Furthermore, due to the boundary conditions we either have

$$U(\mathbf{r}) = 0 \,\text{on}\ \ \mathrm{F} \tag{4.5}$$

or

$$\nabla U(\mathbf{r}) = 0 \,\text{on}\ \ \mathrm{F}. \tag{4.6}$$

With the identity

$$\nabla \cdot (U(\mathbf{r}) \nabla U(\mathbf{r})) = (\nabla U)^2(\mathbf{r}) + U(\mathbf{r}) \Delta U(\mathbf{r}) \tag{4.7}$$

and (4.4) we obtain:

$$\int_V (\nabla U)^2(\mathbf{r})\ dV = \int_V \nabla \cdot (U(\mathbf{r}) \nabla U(\mathbf{r}))\ dV = \oint_F U \nabla U(\mathbf{r}) \cdot d\mathbf{f} = 0 \tag{4.8}$$

using the Gauss' formula, if one of the two conditions (4.5) or (4.6) is fulfilled. Thus:

$$\int_V (\nabla U)^2(\mathbf{r})\ dV = 0, \tag{4.9}$$

i.e. in V:

$$\nabla U(\mathbf{r}) = 0, \tag{4.10}$$

since $(\nabla U)^2(\mathbf{r}) \geq 0$. This leads to

$$U(\mathbf{r}) = \text{const} \tag{4.11}$$

and $\Phi_1(\mathbf{r})$ and $\Phi_2(\mathbf{r})$ differ by at most a (physically insignificant) constant.

Special case: $V \to \infty$.

If V is the entire 3-dim. space, then the solution of the Poisson equation is unique, if $\rho(\mathbf{r})$ is of finite range and $\Phi(\mathbf{r})$ asymptotically drops so fast that

$$r^2\Phi(\mathbf{r})\frac{\partial\Phi}{\partial n}(\mathbf{r}) \to 0 \text{ for } \mathrm{r} \to \infty, \tag{4.12}$$

where $\partial\Phi/\partial n$ denotes the normal derivative of $\Phi(\mathbf{r})$. The proof above follows directly when considering that the surface grows for a fixed volume as r^2.

4.2 Mirror Method

This method consists in introducing so-called **mirror charges** of suitable size outside the area under investigation in such a way that the required boundary conditions are met. This procedure is allowed because one can solve the (inhomogeneous) Poisson equation by adding a solution of the (homogeneous) Laplace equation (cf. Sect. 3.4). The mirror method provides the solution of the Laplace equation which, together with the selected special solution of the Poisson equation, fulfills the required boundary conditions.

As a simple **example** let's consider a point charge q at a distance a from a conducting plane, which has the potential $\Phi = 0$ on the plane (Fig. 4.1 left). The mirror charge q' then is introduced mirror-symmetrical to q with respect to the plane (see Fig. 4.1 right).

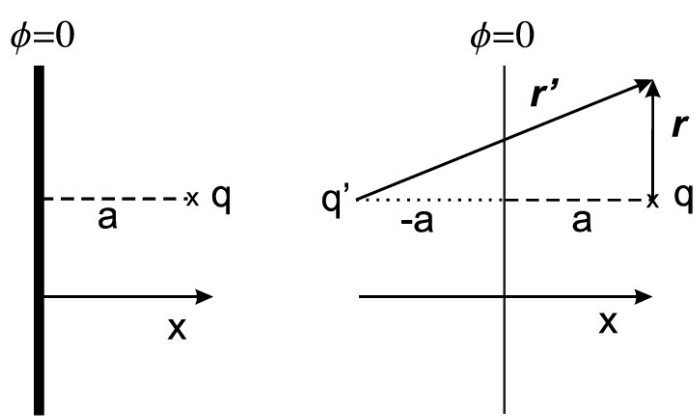

Fig. 4.1 Point charge q at a distance a from a conducting plane, which has the potential $\Phi = 0$ (left). The mirror charge q' then is introduced mirror-symmetrical to q with respect to the plane (right)

Then the potential at point P is:

$$4\pi\epsilon_0\Phi(P) = \frac{q}{r} + \frac{q'}{r'} \tag{4.13}$$

and we get Φ = 0 for all points of the conducting plane, x = 0, choosing:

$$q' = -q. \tag{4.14}$$

In the region x > 0 (which is of interest), $q/(4\pi\epsilon_0 r)$ is a special solution of the Poisson equation, $q'/(4\pi\epsilon_0 r')$ a solution of the Laplace equation, which ensures that for $x = 0$ the required boundary condition is fulfilled.

For the *x*-component of the electric field $\mathbf{E}$ one gets from (4.13) and (4.14):

$$E_x(P) = -\frac{\partial\Phi}{\partial x} = \frac{q}{4\pi\epsilon_0}\left(\frac{x-a}{r^3} - \frac{x+a}{r^3}\right), \tag{4.15}$$

thus for the plane ($x = 0$),

$$E_x(x = 0) = -\frac{2qa}{4\pi\epsilon_0 r^3}. \tag{4.16}$$

The components in the $x = 0$ plane (in *y*, *z*-direction) disappear because the electric field is perpendicular to the plane (otherwise there would be a current in the surface layer). Equation (4.16) implies that according to the Gauss' law (cf. Sect. 3.2) in the plane $x = 0$ a charge with the (spatially dependent) charge density

$$\sigma = \epsilon_0 E_x(x = 0) = -\frac{qa}{2\pi r^3} \tag{4.17}$$

appears, which is **induced** by the presence of the point charge *q* at distance *a*.

4.3 Inversion Method

Let $\Phi(r, \theta, \phi)$ be the potential at the position $\mathbf{r} = (r, \theta, \phi)$ generated by point charges q_i:

$$\Phi(r, \theta, \phi) = \sum_i \frac{q_i}{4\pi\epsilon_0\sqrt{r^2 + r_i^2 - 2rr_i\cos\gamma_i}}; \tag{4.18}$$

here (r_i, θ_i, ϕ_i) denote the positions of the point charges q_i and γ_i the angle between $\mathbf{r}$ and $\mathbf{r}_i$. Then

$$\bar{\Phi}(r, \theta, \phi) = \frac{a}{r}\Phi(\frac{a^2}{r}, \theta, \phi) \tag{4.19}$$

is the potential, that the point charges

$$\bar{q} = \frac{aq_i}{r_i} \tag{4.20}$$

at $(a^2/r_i, \theta_i, \phi_i)$ generate in the position (r, θ, ϕ).

Proof We combine Eqs. (4.19) and (4.18) to

$$\bar{\Phi}(r, \theta, \phi) = \frac{a}{r}\sum_i \frac{q_i}{4\pi\epsilon_0\sqrt{a^4/r^2 + r_i^2 - 2a^2 r_i \cos\gamma_i/r}} \tag{4.21}$$

$$= \sum_i \frac{aq_i/r_i}{4\pi\epsilon_0\sqrt{r^2 + a^4/r_i^2 - 2a^2 r \cos\gamma_i/r_i}}.$$

As an **example** we consider a point charge outside a conducting sphere, which has the potential $\Phi = 0$ on its surface. We replace the sphere with a point charge $\bar{q}$, where its size and position is chosen such that the resulting potential of q and $\bar{q}$ on the surface disappears. The potential at position (r, θ, ϕ), which is generated from the point charge q at $(r_q, 0, 0)$, is denoted by $\Phi(r, \theta, \phi)$. Placing the charge $\bar{q} = -Rq/r_q$ at the position $(R^2/r_q, 0, 0)$ (see Fig. 4.2).

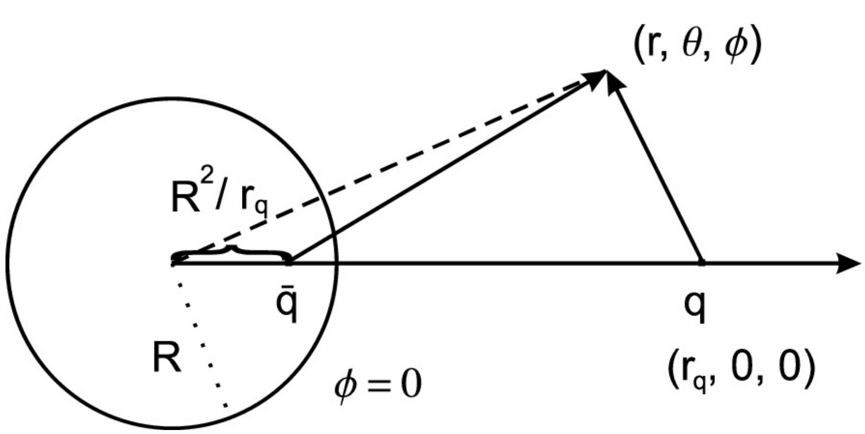

Fig. 4.2 Geometry of a conducting sphere of radius R with a vanishing potential Φ on its surface and the position of the charge $\bar{q}$

then the potential, generated by $\bar{q}$ at position (r, θ, ϕ), is (according to (4.19)):

$$\bar{\Phi}(r, \theta, \phi) = -\frac{R}{r}\Phi(\frac{R^2}{r}, \theta, \phi). \tag{4.22}$$

On the spherical surface, $r = R$, it is:

$$\bar{\Phi}(R, \theta, \phi) = -\Phi(R, \theta, \phi), \tag{4.23}$$

such that

$$\bar{\Phi}(R, \theta, \phi) + \Phi(R, \theta, \phi) = 0. \tag{4.24}$$

The solution of the Poisson equation outside the conducting sphere then is:

$$\bar{\Phi}(r, \theta, \phi) + \Phi(r, \theta, \phi) \tag{4.25}$$

with

$$\Phi(r, \theta, \phi) = \frac{q}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_q|}. \tag{4.26}$$

4.4 Separation of Variables

In the following example we are looking for solutions of the Laplace equation,

$$\Delta\Phi(\mathbf{r}) = 0, \tag{4.27}$$

and for simplicity assume that $\Phi(\mathbf{r})$ does not depend on z,

$$\Phi(\mathbf{r}) = \Phi(x, y). \tag{4.28}$$

Then (4.27) simplifies in cartesian coordinates to:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi(x,y)=0. \tag{4.29}$$

Since (4.29) does not contain a **mixed term** $\partial^2/\partial x\partial y$, it is obvious to use the following **separation Ansatz**:

$$\Phi(x,y)=f(x)g(y); \tag{4.30}$$

then (4.29) reads as:

$$g(y)\frac{\partial^2}{\partial x^2}f(x)+f(x)\frac{\partial^2}{\partial y^2}g(y)=0. \tag{4.31}$$

With the exception of zero's of *f* and *g* Eq. (4.31) is equivalent to:

$$\frac{1}{f(x)}\frac{\partial^2 f}{\partial x^2}+\frac{1}{g(y)}\frac{\partial^2 g}{\partial y^2}=0. \tag{4.32}$$

The 1st term in (4.32) depends only on *x*, the 2nd only on *y*; since *x* and *y* are independent variables it follows from (4.32):

$$\frac{1}{f(x)}\frac{\partial^2 f}{\partial x^2}=\mathrm{const}=-\frac{1}{\mathrm{g(y)}}\frac{\partial^2 \mathrm{g}}{\partial \mathrm{y}^2}. \tag{4.33}$$

If we choose the constant in (4.33) to be real and positive (= k^2), we get the following differential equations:

$$\frac{\partial^2 f}{\partial x^2}-k^2 f(x)=0;\frac{\partial^2 g}{\partial y^2}+k^2 g(y)=0 \tag{4.34}$$

with the solutions:

$$f(x)=a\ \ \exp(\mathrm{kx})+\mathrm{b}\ \ \exp(-\mathrm{kx});\mathrm{g(y)}=\mathrm{c}\ \ \sin(\mathrm{ky})+\mathrm{d}\ \ \cos(\mathrm{ky}). \tag{4.35}$$

The integration constants *a*, *b*, *c*, *d* and the separation constant *k* have to be determined by boundary conditions. As an **example** let's consider a rectangular cylinder, which is infinitely extended in the *z* direction (with edge lengths x_0 and y_0) and the boundary conditions at $y=0$ and $y=y_0$:

$$\Phi(x,0)=\Phi(x,y_0)=0. \tag{4.36}$$

Then

$$d = 0; \sin(\mathrm{ky_0}) = 0 \to \mathrm{k} = \frac{\mathrm{n}\pi}{\mathrm{y_0}} = \mathrm{k_n}. \tag{4.37}$$

Furthermore, the boundary conditions at $x = 0$ and $x = x_0$ are chosen as:

$$\Phi(0, y) = 0; \Phi(x_0, y) = V(y), \tag{4.38}$$

where *V*(*y*) is any given function. From (4.38) it follows that

$$a = -b \to f = a\{\exp(\mathrm{k_n x}) - \exp(-\mathrm{k_n x})\}. \tag{4.39}$$

In order to fulfill the 4th condition, we expand Φ in a Fourier series:

$$\Phi(x, y) = \textstyle\sum_{n=1}^{\infty} A_n \sin(\mathrm{k_n y}) \sinh(\mathrm{k_n x}); \mathrm{A_n} = 2\mathrm{a_n c_n}, \tag{4.40}$$

and determine the coefficients A_n by requiring

$$\Phi(x_0, y) = V(y) = \textstyle\sum_{n=1}^{\infty} A_n \sin(\mathrm{k_n y}) \sinh(\mathrm{k_n x_0}). \tag{4.41}$$

According to the inverse theorem for the Fourier series in the sin-functions we get:

$$A_n = \frac{2}{y_0 \sinh(\mathrm{k_n x_0})} \textstyle\int_0^{y_0} V(y) \ \sin(\mathrm{k_n y})\, \mathrm{dy}. \tag{4.42}$$

In case of boundary conditions of spherical symmetry one solves the Laplace equation using a separation Ansatz in spherical coordinates; the same procedure is followed in case of axial symmetry.

Overview of Electrostatics

(1.) Basis: **Coulomb law**

$$\mathbf{F}(\mathbf{r}) = q\mathbf{E}(\mathbf{r}) \text{ with } \mathbf{E}(\mathbf{r}) = \sum_{\mathrm{i}} \frac{\mathrm{q_i}(\mathbf{r} - \mathbf{r}_\mathrm{i})}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}_\mathrm{i}|^3}$$

(2.) **Field equations**:

(a) integral equations:

$$\oint_S \mathbf{E}(\mathbf{r}) \cdot d\mathbf{s} = 0; \oint_F \mathbf{E}(\mathbf{r}) \cdot d\mathbf{f} = \frac{Q}{\epsilon_0}$$

(b) differential equations:

$$\nabla \times \mathbf{E}(\mathbf{r}) = 0; \nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0}$$

(3.) **Electrostatic potential**:

$$\mathbf{E}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}) \to \Delta\Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}. \,\mathrm{Poisson\,equation}$$

(4.) **Field energy**:

$$\frac{1}{2}\sum_{i \neq j} \frac{q_i q_j}{4\pi\epsilon_0 r_{ij}} \to \frac{1}{2}\int_V \rho(\mathbf{r})\Phi(\mathbf{r})\ dV \to \frac{\epsilon_0}{2}\int_V \mathbf{E}(\mathbf{r})^2\ dV$$

The potential energy of the point charges $\to$ electrostatic field energy.

In summarizing this chapter we have discussed the uniqueness of the solution for the potential Φ in case of boundary conditions and presented three practical methods for the calculation of $\Phi(\mathbf{r})$, i.e. the mirror method, the inversion method and the separation of variables.

Part II
Magnetostatics

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5. Ampère’s Law

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In this chapter we will present the basic equations of magnetostatics for the case of stationary electric currents and introduce the magnetostatic field $\mathbf{B}(\mathbf{r})$ by Ampère’s law as well as the magnetic dipole moment emerging from circulating currents.

5.1 Electric Current and Conservation of Charge

Electric currents are caused by moving charge carriers. Charge carriers can be, for example: ions in a particle accelerator, an electrolyte or a gas, electrons in a metal, etc. The origins for the motion of charges are primarily electric fields but it might also involve material transport of charged objects. We define the **electric current** as the amount of charge that flows through the conductor area per unit of time.

We will initially consider the most simple case of a charge carrier with the same charge q and constant velocity $\mathbf{v}$. Let $\mathbf{a}$ be the vector perpendicular to the area of the conductive medium, where the magnitude of a indicates the size of the area and n is the density of the charge carriers. During the time Δt then the charge carriers in the volume $\Delta V = (\mathbf{a} \cdot \mathbf{v})\Delta t$ pass the conductor cross section, i.e. $n(\mathbf{a} \cdot \mathbf{v})\Delta t$. Thus the charge current is

$$I(a) = \frac{nq(\mathbf{a}\cdot\mathbf{v})\Delta t}{\Delta t} = nq(\mathbf{a} \cdot \mathbf{v}). \tag{5.1}$$

In the more general case with n_i charge carriers q_i with velocity $\mathbf{v}_i$ per unit volume this becomes:

$$I(a) = \mathbf{a} \cdot (\sum_i n_i q_i \mathbf{v}_i). \tag{5.2}$$

Equations (5.1) and (5.2) suggest to introduce the **current density j** as

$$\mathbf{j} := \sum_i n_i q_i \mathbf{v}_i, \tag{5.3}$$

which is related (for $q_i = q$) to the average velocity

$$< \mathbf{v} >= \frac{1}{n} \sum_i n_i \mathbf{v}_i \tag{5.4}$$

and the charge density ρ by:

$$\mathbf{j} = nq < \mathbf{v} >= \rho < \mathbf{v} >. \tag{5.5}$$

Equation (5.5) shows that high absolute velocities of the charge carriers do not imply a high current since only the average value of the velocities of the charge carriers are essential. For example, if the velocities of the charge carriers are uniformly distributed in all directions, then $< \mathbf{v} >= 0$ and therefore also $\mathbf{j}$ = 0. In the general case ρ and $< \mathbf{v} >$ is space- and time-dependent, thus

$$\mathbf{j} = \mathbf{j}(\mathbf{r}, t). \tag{5.6}$$

The law of conservation of charge we can formulate in terms of the charge and current density as follows: We consider an arbitrary finite volume V with surface $F = \partial V$. The amount of charge contained inside is $Q = Q(t)$. If V does not depend on time the change in the amount of charge contained in V per unit of time is:

$$\frac{dQ}{dt} = \int_V \frac{\partial \rho(\mathbf{r}, t)}{\partial t} \; dV. \tag{5.7}$$

Since charge cannot be created or destroyed, the decrease (increase) of the charge contained in V is equal to the amount of charge flowing out (in) through F (in the period of time considered). The latter is given by the surface integral of the current density, which—according to Gauss' formula—can be transformed to a volume integral:

$$\oint_F \mathbf{j}(\mathbf{r}, t) \cdot d\mathbf{f} = \int_V \nabla \cdot \mathbf{j}(\mathbf{r}, t) \; dV. \tag{5.8}$$

Then the charge balance reads:

$$\boxed{-\frac{dQ}{dt} = -\int_V \frac{\partial \rho(\mathbf{r}, t)}{\partial t} \; dV = \int_V \; \nabla \cdot \mathbf{j}(\mathbf{r}, t) \; dV \qquad (5.9)}$$

or, since V can be chosen arbitrarily, we get the **continuity equation**:

$$\boxed{\nabla \cdot \mathbf{j}(\mathbf{r}, t) + \frac{\partial \rho(\mathbf{r}, t)}{\partial t} = 0. \qquad (5.10)}$$

While (5.9) describes the conservation of charge in **integral** form, equation (5.10) describes the charge conservation in **differential** form.

Special cases:

(i) **Electrostatics**: stationary charges

$$\mathbf{j} = 0 \rightarrow \frac{\partial \rho}{\partial t} = 0 \rightarrow \rho = \rho(\mathbf{r}) \tag{5.11}$$

(ii) **Magnetostatics**: stationary currents

$$\mathbf{j} = \mathbf{j}(\mathbf{r}) \text{ and } \nabla \cdot \mathbf{j} = 0 \rightarrow \frac{\partial \rho}{\partial t} = 0. \tag{5.12}$$

For a stationary current, $\nabla \cdot \mathbf{j}$ is constant in time, and this constant must be zero everywhere because charge is not created or destroyed.

5.2 Ampère's Law

Let's consider a stationary current distribution $\mathbf{j} = \mathbf{j}(\mathbf{r})$. To eliminate electrostatic effects we assume that the density of the moving charge carriers, which build up the current, is compensated by resting charge carriers of opposite sign (e.g. moving conduction electrons and resting lattice ions in a metallic conductor). On a moving test charge q—in the vicinity of the current flowing through the conductor—then acts a force, which is found experimentally to be:

$$\mathbf{F}(\mathbf{r}) = q(\mathbf{v} \times \mathbf{B}(\mathbf{r})) \tag{5.13}$$

with

$$\mathbf{B}(\mathbf{r}) = \Gamma_m \int_V \frac{\mathbf{j}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, dV' \tag{5.14}$$

as the **magnetic induction**. The Eqs. (5.13) and (5.14)—as the basis of magnetostatics—is experimentally verified on the same level as

$$\mathbf{F}(\mathbf{r}) = q\mathbf{E}(\mathbf{r}) \tag{5.15}$$

with

$$\mathbf{E}(\mathbf{r}) = \Gamma_e \int_V \frac{\rho(\mathbf{r}')\ (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\ dV' \tag{5.16}$$

in electrostatics! While (5.15) serves as a rule for measuring the electrostatic field $\mathbf{E}(\mathbf{r})$, (5.13) provides a rule for the measurement of the magnetic induction $\mathbf{B}(\mathbf{r})$.

Unit systems:

If one has defined Γ_e, i.e. one has defined the unit charge, then in (5.13) and (5.14) all quantities are fixed w.r.t. their units. Thus Γ_m can no longer be chosen independently but in the

(i) **MKSA** system is

$$\Gamma_m = \frac{\mu_0}{4\pi}\Gamma_e = \frac{1}{4\pi\epsilon_0} \tag{5.17}$$

with

$$\mu_0 = 4\pi \cdot 10^{-7} \frac{\mathrm{m\,kg}}{\mathrm{Coul.}^2} \tag{5.18}$$

as the **magnetic permeability**.

(ii) **cgs** system:

$$\Gamma_m = \frac{1}{c^2}\Gamma_e = 1 \tag{5.19}$$

with the velocity of light *c*.

Note: Equation (5.14) contains—as in (5.16)—the superposition principle: the fields of two current distributions $\mathbf{j}_1(\mathbf{r})$ and $\mathbf{j}_2(\mathbf{r})$ superimpose linearly, since $\mathbf{j}(\mathbf{r}) = \mathbf{j}_1(\mathbf{r}) + \mathbf{j}_2(\mathbf{r})$ is the resulting current distribution. Furthermore, the ratio Γ_m/Γ_e must be a constant independent of the unit system. With (5.17) and (5.19) and $\Gamma_e = 1/(4\pi\epsilon_0)$ or Γ_e = 1 in the cgs system (see Sect. 2.2) we obtain the relationship

$$\frac{\Gamma_m}{\Gamma_e} = \epsilon_0\mu_0 = \frac{1}{c^2}. \tag{5.20}$$

This fundamental relation already points to a connection with Einstein's theory of special relativity.

In the following the vector field $\mathbf{B}(\mathbf{r})$ will be calculated for a couple of simple current distributions.

5.3 Biot-Savart

For a **thin** conductor we can immediately integrate over the area f of the conductor and (instead of (5.14)) obtain

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_L \frac{d\mathbf{l}' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{5.21}$$

with dl' in direction of the conductor and

$$I = \int_f \mathbf{j} \cdot d\mathbf{f}' \tag{5.22}$$

for the current strength (cf. Fig. 5.1).

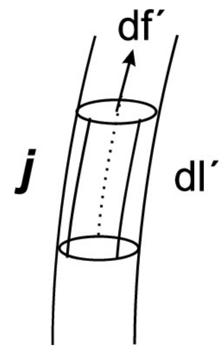

Fig. 5.1 Sketch for a conductor with a finite (thin) area df'

If the conductor is straight, it follows from (5.21) or also (5.14), that the field lines of $\mathbf{B}(\mathbf{r})$ run concentrically around the conductor. So we only need to calculate the magnitude $B(R)$, since all contributions to the integral (5.21) have the same direction for a straight conductor. From Fig. 5.2 then follows (with $d = |\mathbf{r}-\mathbf{r}'|$ and $|d\mathbf{l}' \times (\mathbf{r}-\mathbf{r}')| = d \sin(\theta) dz$):

$$B(R) = \frac{\mu_0 I}{4\pi} \int_L \frac{\sin\theta}{d^2} \; dz \; . \tag{5.23}$$

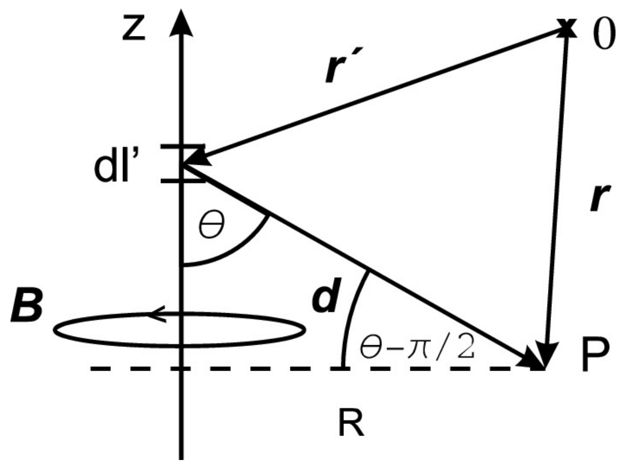

Fig. 5.2 Choice of the integration variable to calculate equation (5.25)

We carry out the remaining integration for an infinitely long conductor: With

$$R = d\ \sin\theta; \mathrm{z} = \mathrm{d}\ \cos\ (\theta) = \mathrm{R}\ \ \mathrm{cotg}\,\theta \rightarrow \mathrm{dz} = \frac{-\mathrm{R}}{\sin^2\theta}\mathrm{d}\theta \tag{5.24}$$

we get for the field strength at point P with distance R:

$$B(R) = \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty}\frac{\sin^3\theta}{R^2}\ dz = \frac{\mu_0 I}{4\pi}\int_{\pi}^{0}\frac{\sin^3\theta}{R^2}\ \frac{dz}{d\theta}d\theta$$

$$= \frac{\mu_0 I}{4\pi}\int_0^{\pi}\frac{\mathrm{R}\sin\theta}{R^2}\ d\theta = \frac{\mu_0 I}{4\pi R}\int_{-1}^{1}d(\cos\theta) = \frac{\mu_0 \mathrm{I}}{2\pi \mathrm{R}}. \tag{5.25}$$

This is the formula of **Biot** and **Savart** for a thin, straight, infinitely long conductor.

5.4 Force and Torque on a Current in the Magnetic Field

Based on the force experienced by a charge q_i moving with velocity $\mathbf{v}_i$ in the magnetic field $\mathbf{B}$,

$$\mathbf{F_i} = q_i(\mathbf{v}_i \times \mathbf{B}(\mathbf{r}_i)), \tag{5.26}$$

the force on a current with the current density $\mathbf{j}$ is obtained as:

(5.27)

$$\mathbf{F} = \sum_i q_i(\mathbf{v}_i \times \mathbf{B}(\mathbf{r}_i)) = \int_V \mathbf{j}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})\ dV,$$

where the volume V has to be chosen such that it completely includes the current.

Example: For a thin wire, where the $\mathbf{B}$ field does not change (significantly) over its area, we can (as in Sect. 5.3) carry out 2 of the 3 integrations in (5.27):

$$\mathbf{F} = I \int_L d\mathbf{l} \times \mathbf{B}. \tag{5.28}$$

The remaining line integral along the conductor L is easy for a straight conductor, if $\mathbf{B}$ does not change along L:

$$\mathbf{F} = (\mathbf{I} \times \mathbf{B})L, \tag{5.29}$$

where L is the length of the conductor. The force is therefore perpendicular to the current direction and to the $\mathbf{B}$ field; it has a maximum, if $\mathbf{I}$ is perpendicular to $\mathbf{B}$, and disappears when $\mathbf{I}$ runs parallel to $\mathbf{B}$.

On the charge q_i with velocity $\mathbf{v}_i$ in the field $\mathbf{B}$ acts the torque:

$$\mathbf{N}_i = \mathbf{r}_i \times \mathbf{F}_i = \mathbf{r}_i \times (q_i\mathbf{v}_i \times \mathbf{B}(\mathbf{r}_i)); \tag{5.30}$$

correspondingly for the current density $\mathbf{j}(\mathbf{r})$:

$$\boxed{\mathbf{N} = \sum_i \mathbf{r}_i \times (q_i\mathbf{v}_i \times \mathbf{B}(\mathbf{r}_i)) = \int_V \mathbf{r} \times (\mathbf{j}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}))\ dV.} \tag{5.31}$$

Simple examples are (rectangular or circular) current loops in a homogeneous $\mathbf{B}$ field.

For the practical evaluation of (5.31) it is expedient to employ the identity ('bac-cab rule')

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \tag{5.32}$$

to transform (5.31) to:

$$\boxed{\mathbf{N} = \int_V \{(\mathbf{r} \cdot \mathbf{B})\mathbf{j} - (\mathbf{r} \cdot \mathbf{j})\mathbf{B}\} dV.} \tag{5.33}$$

For a stationary, spatially limited current, the 2nd term in (5.33) disappears. To show this we use the relationship $(n, m = 1, 2, 3)$

(5.34)

$$\int_V x_n j_m \; dV = \int_V x_n \nabla \cdot (x_m \mathbf{j}) \; dV = \int_V \nabla \cdot (x_n x_m \mathbf{j}) \; dV - \int_V x_m j_n \; dV$$

$$= \oint_F x_n x_m \mathbf{j} \cdot d\mathbf{f} - \int_V x_m j_n \; dV = -\int_V x_m j_n \; dV$$

taking advantage of $\nabla \cdot \mathbf{j} = 0$, the product rule, the Gauss' formula and the disappearance of $\mathbf{j}$ on the surface F. For $n = m$ it follows from (5.34)

$$\int_V (\mathbf{r} \cdot \mathbf{j}) \; dV = 0, \tag{5.35}$$

such that in (5.33) the 2nd term (approximately) disappears for a homogeneous (weakly changing) field. Correspondingly, it follows from (5.34) for $m \neq n$:

$$\int_V (\mathbf{r} \cdot \mathbf{B}) \mathbf{j} \; dV = -\int_V (\mathbf{j} \cdot \mathbf{B}) \mathbf{r} \; dV, \tag{5.36}$$

such that (with the 'bac-cab' rule):

$$\int_V (\mathbf{B} \cdot \mathbf{r}) \; \mathbf{j} \; dV = \tfrac{1}{2} \int_V \{(\mathbf{B} \cdot \mathbf{r}) \; \mathbf{j} - (\mathbf{B} \cdot \mathbf{j}) \; \mathbf{r}\} \; dV = -\tfrac{1}{2} \mathbf{B} \times \int_V (\mathbf{r} \times \mathbf{j}) \; dV \tag{5.37}$$

with Eq. (5.32). Result:

$$\boxed{\mathbf{N} = \left(\tfrac{1}{2} \int_V (\mathbf{r} \times \mathbf{j}(\mathbf{r})) \; dV \right) \times \mathbf{B} = \mathbf{m} \times \mathbf{B}} \tag{5.38}$$

with the **magnetic dipole moment**

$$\boxed{\mathbf{m} = \tfrac{1}{2} \int_V (\mathbf{r} \times \mathbf{j}(\mathbf{r})) \; dV.} \tag{5.39}$$

For a plane current (e.g. circulating current in the (x,y) plane) $\mathbf{m}$ is perpendicular to the current plane (in the direction $\mathbf{e}_z$) (see Fig. 5.3).

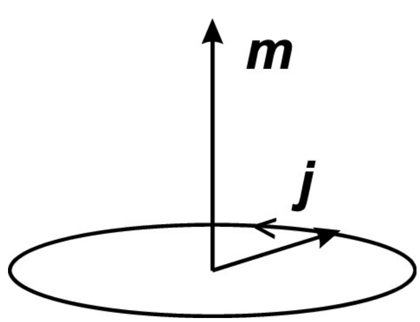

Fig. 5.3 The magnetic dipole moment $\mathbf{m}$ of a circulating current

If the current-carrying conductor is thin, we get (after integration over the conductor area):

$$\mathbf{m} = \frac{I}{2} \oint_L (\mathbf{r} \times d\mathbf{l}) = \frac{I}{2} \int_0^{2\pi} d\phi \; r^2 \; \mathbf{e}_z = \pi r^2 I \; \mathbf{e}_z, \tag{5.40}$$

and for the magnitude of $\mathbf{m}$:

$$m = IF, \tag{5.41}$$

where I is the current and $F = \pi r^2$ the area formed by the closed current (cf. the area law for the motion of a mass point in a central field!). For a particle of mass M and charge q with angular momentum $\mathbf{L}$ of a closed (periodic) orbit we can replace $\mathbf{r} \times q\mathbf{v}$ by $(\mathbf{r} \times M\mathbf{v})q/M = \mathbf{L}q/M$ and get

$$\mathbf{m} = \frac{q}{2M}\mathbf{L}.$$

Applications: Measurement of currents.

5.5 Forces Between Currents

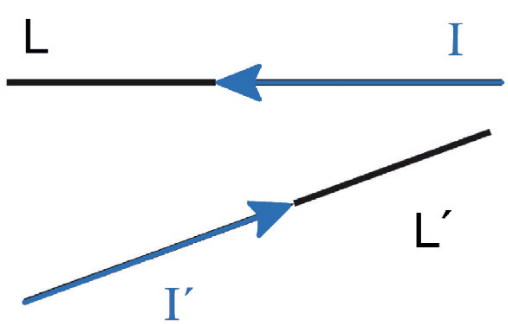

Fig. 5.4 Example for the interaction between two directed currents

With (5.21) and (5.28) the force of a current I' on a current I in thin conductors (see Fig. 5.4) reads as:

$$\mathbf{F} = I \int_L d\mathbf{l} \times \mathbf{B} = \frac{\mu_0 I I'}{4\pi} \int_L \int_{L'} \frac{d\mathbf{l} \times (d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}'))}{|\mathbf{r} - \mathbf{r}'|^3}. \tag{5.42}$$

Equation (5.42) can be symmetrized using the 'bac-cab' rule (5.32):

$$\boxed{\mathbf{F} = \frac{\mu_0 I I'}{4\pi} \int_L \int_{L'} \frac{(d\mathbf{l} \cdot d\mathbf{l}')(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3},} \tag{5.43}$$

since

$$\int_L \frac{d\mathbf{l} \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} = -\int_L \nabla\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) \cdot d\mathbf{l} = 0 \tag{5.44}$$

for closed or infinitely long conductor circuits.

Equation (5.43) changes the sign when the two currents are exchanged, i.e. of I and I' as well as of $\mathbf{r}$ and $\mathbf{r}'$. This reflects the actio-reactio principle, which holds for electrostatic as well as for magnetostatic interactions. However, it will be broken in case of arbitrary time-dependent current and charge distributions (see Chap. 7).

In summarizing this chapter we have introduced the magnetic induction $\mathbf{B}(\mathbf{r})$ for stationary currents and calculated the torque exerted by $\mathbf{B}(\mathbf{r})$ on a current $\mathbf{j}(\mathbf{r})$ as well as the magnetic dipole moment $\mathbf{m}$ emerging from a circulating current.

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6. Basic Equations of Magnetostatics

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In this chapter we will focus on the mathematical aspects of magnetostatics, introduce the vector potential $\mathbf{A}(\mathbf{r})$ and analyse the multipole expansion of the vector potential.

6.1 Divergence of the Magnetic Induction

Equation (5.14) can be rewritten as follows:

$$\boxed{\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{j}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\ dV' = \frac{\mu_0}{4\pi}\nabla\times\left(\int_V \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\ dV'\right). \tag{6.1}}$$

The proof is given by differentiations corresponding to the operation $\nabla\times$ in the integral: With

$$\frac{\partial}{\partial x}\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\frac{(x-x')}{|\mathbf{r}-\mathbf{r}'|^3},\ \frac{\partial}{\partial y}\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\frac{(y-y')}{|\mathbf{r}-\mathbf{r}'|^3},\ \frac{\partial}{\partial z}\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\frac{(z-z')}{|\mathbf{r}-\mathbf{r}'|^3}, \tag{6.2}$$

we find

$$\nabla\times\mathbf{j}(\mathbf{r}')\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\times\mathbf{j}(\mathbf{r}') = \mathbf{j}(\mathbf{r}')\times\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{6.3}$$

According to (6.1) $\mathbf{B}(\mathbf{r})$ can now be written in the form

$$\boxed{\mathbf{B}(\mathbf{r}) = \nabla\times\mathbf{A}(\mathbf{r}) \tag{6.4}}$$

with the vector field defined by

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\ dV'. \tag{6.5}}$$

Then

$$\boxed{\nabla\cdot\mathbf{B}(\mathbf{r}) = \nabla\cdot(\nabla\times\mathbf{A}(\mathbf{r})) = 0. \tag{6.6}}$$

Equation (6.6) formally corresponds to

(6.7)

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0},$$

and shows that there are no **magnetic charges**. Let's formulate the corresponding integral statement to (6.6):

$$\int_V \nabla \cdot \mathbf{B}(\mathbf{r})\ dV = \oint_F \mathbf{B}(\mathbf{r}) \cdot d\mathbf{f} = 0. \tag{6.8}$$

We find that the flux of the magnetic induction through a closed surface F disappears. The comparison with:

$$\oint_F \mathbf{E}(\mathbf{r}) \cdot d\mathbf{f} = \frac{Q}{\epsilon_0}, \tag{6.9}$$

explains this statement.

6.2 Rotation of $\mathbf{B}$

In electrostatics we found

$$\nabla \times \mathbf{E}(\mathbf{r}) = 0 \tag{6.10}$$

or equivalently

$$\oint_S \mathbf{E}(\mathbf{r}) \cdot d\mathbf{s} = 0 \tag{6.11}$$

according to the formula of Stokes. Accordingly, we want to examine the line integral

$$\oint_S \mathbf{B}(\mathbf{r}) \cdot d\mathbf{s} \tag{6.12}$$

over a closed path $S = \partial F$, that encloses an area F, and use Stokes' formula to calculate $\nabla \times \mathbf{B}(\mathbf{r})$.

We first consider an infinitely long, thin, straight conductor. For that we have found

$$\mathbf{B}(r) = \frac{I\mu_0}{2\pi r}\mathbf{e}_\phi, \tag{6.13}$$

where r is the distance from the conductor, I is the current and $\mathbf{e}_\phi$ indicates the direction: the field lines run concentric around the conductor. We first consider a closed path S in the plane perpendicular to the conductor, which includes the conductor (see Fig. 6.1).

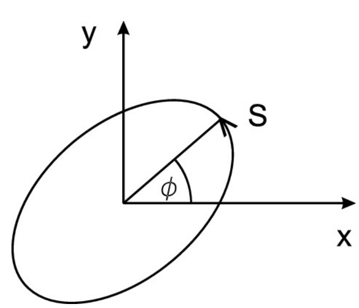

Fig. 6.1 Integration in the plane (x, y) perpendicular to the conductor

Then (with $d\mathbf{s} = \mathbf{e}_\phi r d\phi$):

$$\oint_S \mathbf{B}(r) \cdot d\mathbf{s} = \frac{I\mu_0}{2\pi} \oint_S \frac{\mathbf{e}_\phi \cdot d\mathbf{s}}{r} = \frac{I\mu_0}{2\pi} \oint d\phi = I\mu_0. \tag{6.14}$$

If S does not include the current we obtain:

$$\oint_S \mathbf{B}(r) \cdot d\mathbf{s} = 0 \ . \tag{6.15}$$

This is immediately clear for the following path (see Fig. 6.2).

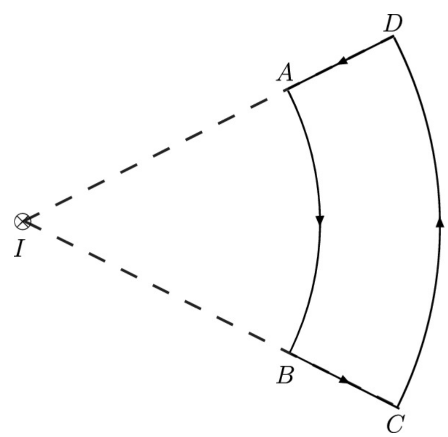

Fig. 6.2 Example of a path with a vanishing path integral of $\mathbf{B}(r)$—generated by the current I

The distances AD, BC do not contribute to the integral since they are perpendicular to $\mathbf{B}$. Using (6.14) we find that the contributions of AB, DC compensate each other due to the opposite directions of circulation and the $1/r$ dependence of $B(r)$.

These results can be generalized to several currents of the type discussed above due to the superposition principle and closed space curves S can be composed of plane segments. Without discussing the details of the general proof—which is the task of mathematics—we find the general result:

$$\oint_S \mathbf{B}(\mathbf{r}) \cdot d\mathbf{s} = \mu_0 I \ , \tag{6.16}$$

where I is the current strength of the current enclosed by S.

Remark: If S circulates the current n times, then I has to be replaced by nI.

The integral statement (6.16)—analogous to Eq. (6.11)—we can transform to a differential relationship using **Stokes' law**. The latter allows to transform the line integral above into a surface integral ($S = \partial F$):

$$\oint_S \mathbf{B}(\mathbf{r}) \cdot d\mathbf{s} = \int_F (\nabla \times \mathbf{B}(\mathbf{r})) \cdot d\mathbf{f} \ , \tag{6.17}$$

where F is an arbitrary smooth orientable surface with the closed path S as a borderline. F and $S = \partial F$ are in the definition domain of the continuously differentiable vector field $\mathbf{B}(\mathbf{r})$. With (6.17) this gives for (6.16):

$$\oint_S \mathbf{B}(\mathbf{r}) \cdot d\mathbf{s} = \int_F (\nabla \times \mathbf{B}(\mathbf{r})) \cdot d\mathbf{f} = \mu_0 I = \mu_0 \int_F \mathbf{j}(\mathbf{r}) \cdot d\mathbf{f}, \tag{6.18}$$

or, since F can be chosen arbitrarily:

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{j}(\mathbf{r}). \tag{6.19}$$

In contrast to the electrostatic field $\mathbf{E}$ with $\nabla \times \mathbf{E}(\mathbf{r}) = 0$ the $\mathbf{B}$ field **is thus not vortex-free**!

6.3 Vector Potential

Instead of computing $\mathbf{B}(\mathbf{r})$ for a given current distribution $\mathbf{j}(\mathbf{r})$ from (6.6) and (6.19) we want to calculate $\mathbf{B}(\mathbf{r})$ from an auxiliary vector field corresponding to the electrostatic potential $\Phi(\mathbf{r})$ in electrostatics. To this aim we introduce the **vector potential** $\mathbf{A}(\mathbf{r})$, from which the magnetic induction can be obtained by differentiation. In Sect. 6.1 we have already briefly introduced:

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}), \tag{6.20}$$

and now want to find a differential equation for the vector potential $\mathbf{A}(\mathbf{r})$ from which $\mathbf{A}(\mathbf{r})$ for a given current distribution $\mathbf{j}(\mathbf{r})$ can be calculated. We form:

$$\nabla \times (\nabla \times \mathbf{A}(\mathbf{r})) = \mu_0 \mathbf{j}(\mathbf{r}) = \nabla(\nabla \cdot \mathbf{A}(\mathbf{r})) - \Delta \mathbf{A}(\mathbf{r}). \tag{6.21}$$

The 1st term on the right side in (6.21) can be eliminated and the desired differential equation be simplified by exploiting the fact, that $\mathbf{A}(\mathbf{r})$ is not uniquely defined by (6.20). The field $\mathbf{B}(\mathbf{r})$ doesn't change when applying the **gauge transformation**

$$\mathbf{A}(\mathbf{r}) \to \mathbf{A}'(\mathbf{r}) = \mathbf{A}(\mathbf{r}) + \nabla \chi(\mathbf{r}), \tag{6.22}$$

where $\chi(\mathbf{r})$ is an arbitrary (at least twice partially differentiable) scalar function, since:

$$\nabla \times \mathbf{A}'(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) + \nabla \times (\nabla \chi(\mathbf{r})) = \nabla \times \mathbf{A}(\mathbf{r}) + 0. \tag{6.23}$$

In case of

$$\nabla \cdot \mathbf{A}(\mathbf{r}) \neq 0, \tag{6.24}$$

we can choose $\chi(\mathbf{r})$ such that

$$\nabla \cdot \mathbf{A}'(\mathbf{r}) = \nabla \cdot \mathbf{A}(\mathbf{r}) + \nabla \cdot (\nabla \chi(\mathbf{r})) = 0. \tag{6.25}$$

We find the scalar function $\chi(\mathbf{r})$ of interest by solving a differential equation of the type (3.26):

$$\nabla \cdot (\nabla\chi(\mathbf{r})) = \Delta\chi(\mathbf{r}) = -\nabla \cdot \mathbf{A}(\mathbf{r}), \tag{6.26}$$

where $-\nabla \cdot \mathbf{A}(\mathbf{r})$ has to be considered as a given inhomogeneity. It thus can always be achieved (without changing the physics, i.e. the $\mathbf{B}$ field) that:

$$\boxed{\Delta\mathbf{A}(\mathbf{r}) = -\mu_0\mathbf{j}(\mathbf{r}).} \tag{6.27}$$

The vector Eq. (6.27) consists out of 3 components, where each equation is again the well-known Poisson equation (3.26).

6.4 Multipole Expansion

In analogy to the case of electrostatics one is often interested in the $\mathbf{B}$ field at a large distance from the (spatially localized) current distribution **j**. It is then useful to expand the vector potential $\mathbf{A}(\mathbf{r})$ into a Taylor series (as for $\Phi(\mathbf{r})$):

$$\mathbf{A}(\mathbf{r}) = \mathbf{A}_0(\mathbf{r}) + \mathbf{A}_1(\mathbf{r}) + \ldots. \tag{6.28}$$

$$= \frac{\mu_0}{4\pi}\int_V dV'\ \mathbf{j}(\mathbf{r}')\Big(\frac{1}{r} - (x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + z'\frac{\partial}{\partial z})\frac{1}{r} + \cdots\Big)$$

with the

monopole term:

$$\mathbf{A}_0(\mathbf{r}) = \tfrac{\mu_0}{4\pi r}\int_V \mathbf{j}(\mathbf{r}')\ dV' \tag{6.29}$$

as the 1st term in the expansion of Eq. (6.5). Now for each component $i = 1, 2, 3$

$$\int_V j_i(\mathbf{r}')\ dV' = \int_V \nabla' \cdot (x_i'\ \mathbf{j}(\mathbf{r}'))\ dV' = \oint_F x_i'\ \mathbf{j}(\mathbf{r}') \cdot d\mathbf{f}' = 0 \tag{6.30}$$

because

$$\nabla' \cdot (x_i'\mathbf{j}(\mathbf{r}') = j_i(\mathbf{r}') + x_I'\ \nabla \cdot \mathbf{j}(\mathbf{r}'),$$

$\nabla \cdot \mathbf{j} = 0$, the formula of Gauss and the fact that $\mathbf{j} \neq 0$ only within V and disappears on the surface $F = \partial V$. We then get:

$$\boxed{\mathbf{A}_0 = 0\ ,} \tag{6.31}$$

since there are no magnetic monopoles opposite to electric charges in electrodynamics (2.29).

Dipole component:

$$\mathbf{A}_1(\mathbf{r}) = \tfrac{\mu_0}{4\pi}\int_V dV'\ \mathbf{j}(\mathbf{r}')\ (-\mathbf{r}' \cdot \nabla_r)\tfrac{1}{r} = \tfrac{\mu_0}{4\pi r^3}\int_V (\mathbf{r} \cdot \mathbf{r}')\ \mathbf{j}(\mathbf{r}')\ dV'. \tag{6.32}$$

We transform the integral (6.32) according to (5.37):

$$\int_V (\mathbf{r}\cdot\mathbf{r}')\mathbf{j}(\mathbf{r}')\ dV' = \frac{1}{2}\int_V \{(\mathbf{r}\cdot\mathbf{r}')\mathbf{j}(\mathbf{r}') - (\mathbf{r}\cdot\mathbf{j}(\mathbf{r}'))\ \mathbf{r}'\}\ dV' = \frac{1}{2}\int_V \{\mathbf{r}\times(\mathbf{j}(\mathbf{r}')\times\mathbf{r}')\}\ dV' \tag{6.33}$$

$$= -\frac{1}{2}\mathbf{r}\times\int_V (\mathbf{r}'\times\mathbf{j}(\mathbf{r}'))\ dV' = \left(\frac{1}{2}\int_V (\mathbf{r}'\times\mathbf{j}(\mathbf{r}'))\ dV'\right)\times\mathbf{r} = \mathbf{m}\times\mathbf{r}\ .$$

Result:

$$\mathbf{A}_1(\mathbf{r}) = \mathbf{m}\times\left(\frac{\mu_0}{4\pi}\frac{\mathbf{r}}{r^3}\right) \tag{6.34}$$

with the magnetic dipole moment $\mathbf{m}$ of (5.39). Compare the result with equation (2.30)!
Analysis of the dipole moment $\mathbf{m}$:
For N point charges q_i the magnetic dipole moment $\mathbf{m}$ is given by:

$$\mathbf{m} = \frac{1}{2}\ \sum_{i=1}^{N} q_i(\mathbf{r}_i\times\mathbf{v}_i). \tag{6.35}$$

Furthermore, $\mathbf{m}$ can be connected to the angular momentum $\mathbf{L}$ of the N charged mass points if $M_i = M$ and $q_i = q$, i.e.:

$$\mathbf{m} = \frac{q}{2M}\mathbf{L} = \frac{q}{2M}\sum_{i=1}^{N}\ M(\mathbf{r}_i\times\mathbf{v}_i)\ . \tag{6.36}$$

The orbital angular momentum of a system of (identical) charged particles is thus linked to a magnetic moment in the direction of $\mathbf{L}$. This statement also holds in the atomic range, e.g. for the electrons of an atom. However, not every magnetic moment corresponds to an orbital angular momentum according to (6.36). Elementary particles (such as electrons) have an **internal** magnetic dipole moment, which is not related to the orbital angular momentum but is linked to the **spin** of these particles by:

$$\mathbf{m}_s = g\frac{q}{2M}\mathbf{s}, \tag{6.37}$$

where $\mathbf{s}$ is the spin vector and g the **gyromagnetic ratio**. Esperimentally one finds $g \approx 2.0024$ for electrons, which can also be calculated within the scope of quantum electrodynamics (QED).

6.5 Energy of a Dipole in the External Magnetic Field

For the force $\mathbf{F}$ on a magnetic dipole $\mathbf{m}$ in a (spatially weakly changing) field $\mathbf{B}$ one finds:

$$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B}). \tag{6.38}$$

For the proof we go back to (5.27) and proceed as in the calculation of $\mathbf{N}$ in equation (5.38). From (6.38) we get the potential energy of the dipole in the $\mathbf{B}$ field as ($\mathbf{F} = -\nabla U$):

(6.39)

$$U = -(\mathbf{m} \cdot \mathbf{B}),$$

in analogy to $-(\mathbf{d} \cdot \mathbf{E})$ as the energy of an electric dipole in the electrostatic field (3.41). Thus the dipole will preferentially be oriented in the direction of the field since this gives the lowest possible energy.

Overview of magnetostatics

(1.) Basis: **Ampère's law**

$$\mathbf{F}(\mathbf{r}) = q(\mathbf{v} \times \mathbf{B}(\mathbf{r})) \text{ with } \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathrm{V}} \frac{\mathbf{j}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, \mathrm{dV}'$$

for stationary currents with $\nabla \cdot \mathbf{j}(\mathbf{r}) = -\partial \rho / \partial t = 0$.

(2.) **Field equations**: From

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) \text{ with } \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathrm{V}} \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, \mathrm{dV}'$$

we obtain the

(a) **differential** relations:

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0; \nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{j}(\mathbf{r})$$

or in

(b) **integral** form:

$$\oint_F \mathbf{B}(\mathbf{r}) \cdot d\mathbf{f} = 0; \oint_S \mathbf{B}(\mathbf{r}) \cdot d\mathbf{s} = \mu_0 I$$

(3.) **Vector potential**:

$$\nabla \times (\nabla \times \mathbf{A}(\mathbf{r})) = \mu_0 \mathbf{j}(\mathbf{r}) \rightarrow \Delta \mathbf{A}(\mathbf{r}) = -\mu_0 \mathbf{j}(\mathbf{r})$$

for $\nabla \cdot \mathbf{A}(\mathbf{r}) = 0$ (i.e. in **Coulomb gauge**).

In summarizing this chapter we have introduced the vector potential $\mathbf{A}(\mathbf{r})$ and analysed the multipole expansion of the vector potential. Furthermore, we have calculated the energy of a dipole in the external magnetic field.

Part III
Basics of Electrodynamics

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7. Maxwell's Equations

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In this chapter we will extend the previous cases of static charges or stationary currents and derive the field equations for $\mathbf{E}(\mathbf{r};t)$ and $\mathbf{B}(\mathbf{r};t)$ for arbitrary space-time dependent sources $\rho(\mathbf{r};t)$ and $\mathbf{j}(\mathbf{r};t)$ on the basis of Faraday's law of induction.

7.1 Concept of the Electromagnetic Field

As the definition of the fields $\mathbf{E}(\mathbf{r};t)$ and $\mathbf{B}(\mathbf{r};t)$ we use–in extension of the Eqs. (2.8) and (5.13)–the relation (**Lorentz force**)

$$\mathbf{F}(\mathbf{r},t) = q(\mathbf{E}(\mathbf{r},t) + (\mathbf{v} \times \mathbf{B}(\mathbf{r},t))). \tag{7.1}$$

Since $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ are linked by the continuity equation

$$\frac{\partial \rho(\mathbf{r},t)}{\partial t} + \nabla \cdot \mathbf{j}(\mathbf{r},t) = 0, \tag{7.2}$$

it is clear that the electric and the magnetic field can no longer be treated separately: The **Maxwell equations** are a system of coupled differential equations for the fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$.

7.2 Faraday's Law of Induction

We start with the following experimental observation: If the magnetic flux (Sect. 6.1) through a closed conductor circuit changes in time, then an electric field is **induced** along the conductor circuit, which generates an **induction current** in the conductor. Quantitatively:

$$-k\frac{d}{dt}\left(\int_F \mathbf{B}(\mathbf{r},t) \cdot d\mathbf{f}\right) = \oint_S \mathbf{E}'(\mathbf{r},t) \cdot d\mathbf{s}, \tag{7.3}$$

where:
(i) F is any smooth area with the conductor circuit S as boundary;
(ii) $\mathbf{E}'(\mathbf{r},t)$ is the induced electric field strength relative to a coordinate system Σ' moving along with the conductor S;
(iii) k is a constant that depends on the unit system, i.e.:

$$k = 1 \quad \text{in the} \quad \text{MKSA system}\,; \quad \mathrm{k} = \tfrac{1}{\mathrm{c}} \quad \text{in the} \quad \text{cgs system} \tag{7.4}$$

Equation (7.1) refers to the MKSA system and has to be replaced in the cgs system by

$$\mathbf{F}(\mathbf{r},t) = q(\mathbf{E}(\mathbf{r},t) + \tfrac{1}{c}(\mathbf{v}\times\mathbf{B}(\mathbf{r},t))) = q(\mathbf{E}(\mathbf{r},t) + (\overrightarrow{\beta}\times\mathbf{B}(\mathbf{r},t))) \tag{7.5}$$

with $\overrightarrow{\beta} = \mathbf{v}/c$. All following formulae refer to the MKSA system.
(iv) The sign in (7.3) reflects the **Lenz' rule**.
From (i) it follows that also for time-dependent fields

$$\nabla\cdot\mathbf{B}(\mathbf{r},t) = 0 \tag{7.6}$$

as in magnetostatics. If F_1 and F_2 are arbitrary surfaces with boundary S, it follows from (i):

$$\int_{F_1}\mathbf{B}(\mathbf{r},t)\cdot d\mathbf{f}_1 = \int_{F_2}\mathbf{B}(\mathbf{r},t)\cdot d\mathbf{f}_2. \tag{7.7}$$

Taking into account the orientation of the surfaces, Gauss' theorem then gives for the volume defined by the different surfaces F_1 and F_2:

$$0 = \oint_{F_1}\mathbf{B}(\mathbf{r},t)\cdot d\mathbf{f}_1 - \oint_{F_2}\mathbf{B}(\mathbf{r},t)\cdot d\mathbf{f}_2 = \int_V \nabla\cdot\mathbf{B}(\mathbf{r},t)\;dV, \text{q. e. d.} \tag{7.8}$$

The universal validity of $\nabla\cdot\mathbf{B}(\mathbf{r},t) = 0$ was already expected due to the interpretation in Sect. 6.1.

Discussion of the Law of Induction
Case 1: Time-varying field $\mathbf{B}(\mathbf{r},t)$ with a **stationary** conductor circuit S.
Then $\mathbf{E}'(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t)$ is the induced field strength in the laboratory system Σ and it follows according to the formula from Stokes:

$$\oint_S \mathbf{E}(\mathbf{r},t)\cdot d\mathbf{s} = \int_F(\nabla\times\mathbf{E}(\mathbf{r},t))\cdot d\mathbf{f} = -\int_F \frac{\partial\mathbf{B}(\mathbf{r},t)}{\partial t}\cdot d\mathbf{f}, \tag{7.9}$$

or, since F is arbitrary ($S = \partial F$),

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}. \tag{7.10}$$

Equation (7.10) shows the expected connection of the fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$.

Note: Equation (7.10) holds regardless of whether the conductor circuit actually exists or not; the conductor circuit only serves to detect the induced field!

Application: Betatron.

Charged particles are accelerated in the induced electric field $\mathbf{E}(\mathbf{r}, t)$ from a time-dependent magnetic field $\mathbf{B}(\mathbf{r}, t)$.

Case 2: **Moving** conductor circuit S in a constant, time-independent $\mathbf{B}$ field.

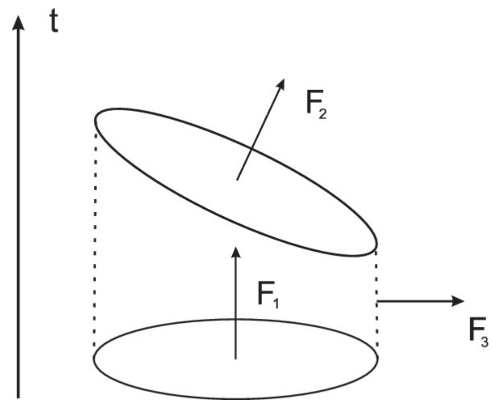

Fig. 7.1 Illustration of a moving conductor circuit S with the surfaces F_1 and F_2 at times t_1 and t_2, which define an enclosed volume V

Explanation: F_1 and F_2 are arbitrary surfaces with boundaries S_1 and S_2, F_3 is the lateral surface, which connects S_1 and S_2. The arrows give the orientations of the surfaces (see Fig. 7.1).

According to the formula of Gauss we get:

$$-\int_{F_1} \mathbf{B}(\mathbf{r}, t) \cdot d\mathbf{f}_1 + \int_{F_2} \mathbf{B}(\mathbf{r}, t) \cdot d\mathbf{f}_2 + \int_{F_3} \mathbf{B}(\mathbf{r}, t) \cdot d\mathbf{f}_3 = \int_V (\nabla \cdot \mathbf{B}(\mathbf{r}, t))\ dV = 0 \tag{7.11}$$

considering (7.6).

For the temporal change of the flux it follows (with $d\mathbf{f}_3/dt = \mathbf{v} \times d\mathbf{s}$):

$$\frac{d}{dt} \int_F \mathbf{B}(\mathbf{r}, t) \cdot d\mathbf{f} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \{\int_{F_2} \mathbf{B}(\mathbf{r}, t) \cdot d\mathbf{f}_2 - \int_{F_1} \mathbf{B}(\mathbf{r}, t) \cdot d\mathbf{f}_1\} \tag{7.12}$$

$$= -\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{F_3} \mathbf{B}(\mathbf{r}, t) \cdot d\mathbf{f}_3 = \oint_S \mathbf{B}(\mathbf{r}, t) \cdot (\mathbf{v} \times d\mathbf{s}),$$

and (7.3) takes the form:

$$\begin{aligned} \oint_S \mathbf{E}'(\mathbf{r}, t) \cdot d\mathbf{s} &= -\oint_S \mathbf{B}(\mathbf{r}, t) \cdot (\mathbf{v} \times d\mathbf{s}) \\ &= -\oint_S d\mathbf{s} \cdot (\mathbf{B}(\mathbf{r}, t) \times \mathbf{v}) = \oint_S d\mathbf{s} \cdot (\mathbf{v} \times \mathbf{B}(\mathbf{r}, t)). \end{aligned} \tag{7.13}$$

Equation (7.13) allows to compute the potential $\oint \mathbf{E}' \cdot d\mathbf{s}$ (**electromotive force**) that is induced by a constant magnetic field in a moving conductor loop.

Application: Alternating current (AC) generator.
By combining case 1 and case 2 we get:

$$\oint_S \mathbf{E}' \cdot d\mathbf{s} = -\int_F \frac{\partial B}{\partial t} \cdot d\mathbf{f} + \oint_S d\mathbf{s} \cdot (\mathbf{v} \times \mathbf{B}) = \oint_S \mathbf{E} \cdot d\mathbf{s} + \oint_S (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{s}. \tag{7.14}$$

Since the conductor loop S can be chosen arbitrarily we obtain:

$$\boxed{\mathbf{E}' = \mathbf{E} + (\mathbf{v} \times \mathbf{B}).} \tag{7.15}$$

This connection between the (induced) electrical field strength $\mathbf{E}'$ in the moving system Σ' and the (induced) field strength $\mathbf{E}$ and the magnetic induction $\mathbf{B}$ in the laboratory system Σ can be explained for $v \ll c$ within the framework of Galilei's principle of relativity:

The force on a charge carrier q of the conductor circuit S in the laboratory system Σ is:

$$\mathbf{F}(\mathbf{r}, t) = q(\mathbf{E}(\mathbf{r}, t) + (\mathbf{v} \times \mathbf{B}(\mathbf{r}, t))), \tag{7.16}$$

whereas in the moving system Σ' it is:

$$\mathbf{F}'(\mathbf{r}, t) = q\mathbf{E}'(\mathbf{r}, t), \tag{7.17}$$

since q rests in Σ', i.e. $\mathbf{v}'_q = 0$. For $\mathbf{v}$ = const. the connection between Σ and Σ' is given by a **Galilei transformation**, which keeps the forces invariant, i.e.

$$\mathbf{F}(\mathbf{r}, t) = \mathbf{F}'(\mathbf{r}, t), \tag{7.18}$$

and from which (7.15) follows directly.

7.3 Extension of Ampère's law

Ampère's law of magnetostatics

$$\nabla \times \mathbf{B}(\mathbf{r}, t) = \mu_0 \mathbf{j}(\mathbf{r}, t) \tag{7.19}$$

only holds for stationary currents. From

$$\nabla \cdot (\nabla \times \mathbf{B}(\mathbf{r}, t)) = \mu_0 \nabla \cdot \mathbf{j}(\mathbf{r}, t) \tag{7.20}$$

follows, with the identity (for an arbitrary vector field $\mathbf{a}(\mathbf{r}, t)$)

$$\nabla \cdot (\nabla \times \mathbf{a}) = 0, \tag{7.21}$$

directly $\nabla \cdot \mathbf{j}(\mathbf{r}, t) = 0$, i.e. stationary currents. In general, however, the continuity equation applies

$$\nabla \cdot \mathbf{j}(\mathbf{r}, t) = -\frac{\partial \rho(\mathbf{r},t)}{\partial t}, \tag{7.22}$$

such that (7.19) has to be modified for non-stationary currents.

This extension is straight forward when keeping the Gauss' law of electrostatics (Sect. 3.4):

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = \frac{\rho(\mathbf{r},t)}{\epsilon_0}, \tag{7.23}$$

which is supported by the charge invariance. Now combining (7.22) and (7.23) we get:

$$\boxed{\nabla \cdot \mathbf{j}(\mathbf{r}, t) + \frac{\partial \rho(\mathbf{r},t)}{\partial t} = \nabla \cdot (\mathbf{j}(\mathbf{r}, t) + \epsilon_0 \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t}) = 0.} \tag{7.24}$$

We therefore replace

$$\mathbf{j}(\mathbf{r}, t) \to \mathbf{j}(\mathbf{r}, t) + \epsilon_0 \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t}, \tag{7.25}$$

in order to obtain again a current with vanishing divergence as in magnetostatics. In accordance with the conservation of charge we extend (7.19) as follows:

$$\boxed{\nabla \times \mathbf{B}(\mathbf{r}, t) = \mu_0 \mathbf{j}(\mathbf{r}, t) + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t}.} \tag{7.26}$$

Ampère's law (7.26) finds its experimental confirmation in the existence of electromagnetic waves (see Chap. 10).

7.4 Overview of Maxwell's Equations

Homogeneous equations:

$$\boxed{\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0,} \tag{7.27}$$

which corresponds to the absence of magnetic monopoles.

$$\boxed{\nabla \times \mathbf{E}(\mathbf{r}, t) + \frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t} = 0,} \tag{7.28}$$

which corresponds to the law of induction.

Inhomogeneous equations:

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = \frac{\rho(\mathbf{r},t)}{\epsilon_0}, \tag{7.29}$$

which corresponds to the Gauss' law;

$$\nabla \times \mathbf{B}(\mathbf{r}, t) - \mu_0 \epsilon_0 \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t} = \mu_0 \mathbf{j}(\mathbf{r}, t), \tag{7.30}$$

which corresponds to the Ampère-Maxwell law.

In (7.29) and (7.30) the conservation of charge (7.22) is already implicitly included. (7.28) and (7.30) show that a time-dependent magnetic field $\mathbf{B}(\mathbf{r}, t)$ induces an electric field $\mathbf{E}(\mathbf{r}, t)$ and vice versa. The Eqs. (7.27)–(7.30) together with the **Lorentz force**

$$\mathbf{F}(\mathbf{r}, t) = q(\mathbf{E}(\mathbf{r}, t) + (\mathbf{v} \times \mathbf{B}(\mathbf{r}, t))). \tag{7.31}$$

completely describe the electromagnetic interaction of charged particles in the context of classical physics.

In summarizing this chapter we have extended the previous cases of static charges or stationary currents and derived the field equations for $\mathbf{E}(\mathbf{r};t)$ and $\mathbf{B}(\mathbf{r};t)$ for arbitrary space-time dependent sources $\rho(\mathbf{r};t)$ and $\mathbf{j}(\mathbf{r};t)$ on the basis of Faraday's law of induction.

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8. The Electromagnetic Potentials

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Instead of solving the coupled differential Eqs. (7.27)–(7.30) for $\mathbf{E}(\mathbf{r};t)$ and $\mathbf{B}(\mathbf{r};t)$ directly, it is more convenient—in analogy to the procedure in electrostatics and magnetostatics—to employ **electromagnetic potentials** which, however, are not unique. It will be shown that specific gauge transformation are allowed that do not change the physical fields $\mathbf{E}$ and $\mathbf{B}$.

8.1 Scalar Potential and Vector Potential

Since in general we have

$$\nabla \cdot \mathbf{B}(\mathbf{r};t) = 0, \tag{8.1}$$

we can get a vector potential $\mathbf{A} = \mathbf{A}(\mathbf{r}, t)$ via the relation

$$\boxed{\mathbf{B}(\mathbf{r};t) = \nabla \times \mathbf{A}(\mathbf{r};t).} \tag{8.2}$$

Then (7.28) can be written as

$$\nabla \times \left(\mathbf{E}(\mathbf{r};t) + \tfrac{\partial \mathbf{A}(\mathbf{r};t)}{\partial t}\right) = 0, \tag{8.3}$$

and the vector function $(\mathbf{E}(\mathbf{r};t) + \partial \mathbf{A}(\mathbf{r};t)/\partial t)$ can be written as a gradient of a scalar function $\Phi = \Phi(\mathbf{r}, t)$:

$$\mathbf{E}(\mathbf{r};t) + \tfrac{\partial \mathbf{A}(\mathbf{r};t)}{\partial t} = -\nabla \Phi(\mathbf{r};t), \tag{8.4}$$

or

$$\boxed{\mathbf{E}(\mathbf{r};t) = -\frac{\partial \mathbf{A}(\mathbf{r};t)}{\partial t} - \nabla\Phi(\mathbf{r};t). \qquad (8.5)}$$

Thus $\mathbf{E}(\mathbf{r};t)$ and $\mathbf{B}(\mathbf{r};t)$ can be expressed by the vector potential $\mathbf{A}(\mathbf{r};t)$ and the scalar potential $\Phi(\mathbf{r};t)$, and now we have to set up differential equations that allow to calculate $\mathbf{A}(\mathbf{r};t)$ and $\Phi(\mathbf{r};t)$, if the sources $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},\mathbf{t})$ are given.

To this aim we use the inhomogeneous Eqs. (7.29) and (7.30). From (7.29) it follows with $\mathbf{E}(\mathbf{r};t)$ from (8.5):

$$\Delta\Phi(\mathbf{r};t) + \nabla\cdot\frac{\partial \mathbf{A}(\mathbf{r};t)}{\partial t} = -\frac{\rho(\mathbf{r};t)}{\epsilon_0} \qquad (8.6)$$

and from (7.30) with (8.2):

$$\nabla\times(\nabla\times\mathbf{A}(\mathbf{r};t)) + \mu_0\epsilon_0\left(\nabla\frac{\partial\Phi(\mathbf{r};t)}{\partial t} + \frac{\partial^2\mathbf{A}(\mathbf{r};t)}{\partial t^2}\right) = \mu_0\mathbf{j}(\mathbf{r};t). \qquad (8.7)$$

With the identity

$$\nabla\times(\nabla\times\mathbf{a}) = -\Delta\mathbf{a} + \nabla(\nabla\cdot\mathbf{a}) \qquad (8.8)$$

Equation (8.7) turns to:

$$\Delta\mathbf{A}(\mathbf{r};t) - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}(\mathbf{r};t)}{\partial t^2} - \nabla\left(\nabla\cdot\mathbf{A}(\mathbf{r};t) + \mu_0\epsilon_0\frac{\partial\Phi(\mathbf{r};t)}{\partial t}\right) = -\mu_0\mathbf{j}(\mathbf{r};t). \quad (8.9)$$

Thus the 8 Maxwell equations for $\mathbf{E}(\mathbf{r};t)$ and $\mathbf{B}(\mathbf{r};t)$ are converted to 4 equations for the potentials $\mathbf{A}(\mathbf{r};t)$ and $\Phi(\mathbf{r};t)$, which, however, are linked to each other.

In order to decouple these equations we make use of the fact that the Maxwell equations are invariant with respect to **gauge transformations**:

$$\mathbf{A}(\mathbf{r};t) \to \mathbf{A}(\mathbf{r};t) + \nabla\chi(\mathbf{r},t), \qquad (8.10)$$

$$\Phi(\mathbf{r};t) \to \Phi(\mathbf{r};t) - \frac{\partial\chi(\mathbf{r},t)}{\partial t}, \qquad (8.11)$$

where $\chi(\mathbf{r}, t)$ is an arbitrary function that is twice continuously differentiable.

8.2 Lorentz Convention

Equation (8.9) suggests to choose $\chi(\mathbf{r}, t)$ such that

$$\nabla \cdot \mathbf{A}(\mathbf{r};t) + \mu_0\epsilon_0 \frac{\partial\Phi(\mathbf{r};t)}{\partial t} = 0, \tag{8.12}$$

which corresponds to the **Lorentz convention**. We then obtain decoupled equations from (8.9) and (8.6):

$$\Delta\mathbf{A}(\mathbf{r};t) - \mu_0\epsilon_0 \frac{\partial^2\mathbf{A}(\mathbf{r};t)}{\partial t^2} = -\mu_0\mathbf{j}(\mathbf{r};t). \tag{8.13}$$

$$\Delta\Phi(\mathbf{r};t) - \mu_0\epsilon_0 \frac{\partial^2\Phi(\mathbf{r};t)}{\partial t^2} = -\frac{\rho(\mathbf{r};t)}{\epsilon_0}, \tag{8.14}$$

which have the same mathematical structure. They simplify to the time-independent fields in Eqs. (3.26) and (6.27) of electrostatics or magnetostatics. The Lorentz convention (8.12) is used also in the relativistic formulation of electrodynamics employing $\mu_0\epsilon_0 = c^{-2}$.

Construction of $\chi(\mathbf{r}, t)$: If

$$\nabla \cdot \mathbf{A}(\mathbf{r};t) + \mu_0\epsilon_0 \frac{\partial\Phi(\mathbf{r};t)}{\partial t} \neq 0 \tag{8.15}$$

we preform a gauge transformation and require:

$$\nabla \cdot \mathbf{A}(\mathbf{r};t) + \Delta\chi(\mathbf{r};t) + \mu_0\epsilon_0 \frac{\partial\Phi(\mathbf{r};t)}{\partial t} - \mu_0\epsilon_0 \frac{\partial^2\chi(\mathbf{r};t)}{\partial t^2} = 0. \tag{8.16}$$

Equation (8.16) is an inhomogeneous, partial differential equation of 2nd order of the form

$$\Delta\chi(\mathbf{r}, t) - \mu_0\epsilon_0 \frac{\partial^2\chi(\mathbf{r},t)}{\partial t^2} = f(\mathbf{r}, t). \tag{8.17}$$

For a given inhomogeneity

$$f(\mathbf{r},t) = -\nabla \cdot \mathbf{A}(\mathbf{r};t) - \mu_0\epsilon_0 \frac{\partial\Phi(\mathbf{r};t)}{\partial t} \tag{8.18}$$

the solution is not unique, since for every solution of (8.17) another arbitrary solution of the homogeneous equation

$$\Delta\chi(\mathbf{r};t) - \mu_0\epsilon_0 \frac{\partial^2\chi(\mathbf{r};t)}{\partial t^2} = 0 \tag{8.19}$$

can be added. This situation is referred to as **gauge freedom**.

8.3 Coulomb Gauge

In atomic and nuclear physics the gauge $\chi(\mathbf{r},t)$ usually is chosen such that

$$\boxed{\nabla \cdot \mathbf{A}(\mathbf{r};t) = 0.} \tag{8.20}$$

Then (8.6) transforms to

$$\Delta\Phi(\mathbf{r};t) = -\frac{\rho(\mathbf{r};t)}{\epsilon_0}, \tag{8.21}$$

with the known (particular) solution:

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\ dV'. \tag{8.22}$$

Equation (8.9) then reads

$$\begin{aligned}\Delta\mathbf{A}(\mathbf{r};t) - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}(\mathbf{r};t)}{\partial t^2} &= -\mu_0\mathbf{j}(\mathbf{r},t) + \epsilon_0\mu_0\nabla\frac{\partial\Phi(\mathbf{r},t)}{\partial t} \\ &= -\mu_0\mathbf{j}(\mathbf{r},t) - \frac{\epsilon_0\mu_0}{4\pi\epsilon_0}\int_V \frac{\partial\rho(\mathbf{r}',t)/\partial t\ (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\ dV' \\ &= -\mu_0\mathbf{j}(\mathbf{r},t) + \frac{\mu_0}{4\pi}\int_V \frac{(\nabla\cdot\mathbf{j}(\mathbf{r}',t))(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\ dV'.\end{aligned} \tag{8.23}$$

Application: In source-free areas where

(8.24)

$$\rho(\mathbf{r};t) = 0; \quad \mathbf{j}(\mathbf{r};t) = 0,$$

Equations (8.22) and (8.23) reduce to:

$$\Phi = 0; \quad \Delta\mathbf{A}(\mathbf{r};t) - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{A}(\mathbf{r};t)}{\partial t^2} = 0. \tag{8.25}$$

The solutions of (8.25) are **electromagnetic waves**, e.g. in form of transverse plane waves (see Chap. 10).

Construction of $\chi(\mathbf{r}, t)$: If the solution $\mathbf{A}(\mathbf{r}, t)$ of (8.9) does not fulfill the gauge condition (8.20), we perform the transformation (8.10), (8.11) and require

$$\nabla \cdot \mathbf{A}(\mathbf{r};t) + \Delta\chi(\mathbf{r};t) = 0, \tag{8.26}$$

or

$$\Delta\chi(\mathbf{r}, t) = -\nabla \cdot \mathbf{A}(\mathbf{r}, t). \tag{8.27}$$

This is a special case of (8.17) with $-\nabla \cdot \mathbf{A}(\mathbf{r}, t)$ as inhomogeneity.

Note: To any solution of (8.27) one can still add any solution of the homogeneous equation,

$$\Delta\chi(\mathbf{r}, t) = 0, \tag{8.28}$$

(**gauge freedom**).

8.4 Law of Induction, Self-induction

The magnetic flux—as the decisive quantity of the law of induction—can be determined with the vector potential as follows (arguments of the integrands suppressed):

$$\int_F \mathbf{B} \cdot d\mathbf{f} = \int_F (\nabla \times \mathbf{A}) \cdot d\mathbf{f} = \oint_S \mathbf{A} \cdot d\mathbf{s}, \tag{8.29}$$

by applying Stokes' law. The right side of (8.29) shows explicitly that the flux only depends on the path (conductor loop) S, but not on the special shape of the surface F with boundary $S = \partial F$.

For the case of self-induction one has to calculate the vector potential $\mathbf{A}(\mathbf{r}, t)$ for a given current density $\mathbf{j}(\mathbf{r}, t)$ from (8.13) or (8.23) and then to calculate the integral (8.29). For a given current the result only depends on the conductor geometry. Since

$$\oint_S (\nabla\chi) \cdot d\mathbf{s} = \int_F (\nabla \times \nabla\chi) \cdot d\mathbf{f} = 0, \tag{8.30}$$

it is independent of the choice of the gauge, i.e. with respect to the transformation $\mathbf{A} \to \mathbf{A}(\mathbf{r};t) + \nabla\chi(\mathbf{r};t)$.

In summary, we have rewritten the coupled Maxwell equations for the fields $\mathbf{E}$ and $\mathbf{B}$ in terms of inhomogeneous wave equations for the scalar potential Φ and the vector potential $\mathbf{A}$, which can be decoupled by a specific choice of the gauge χ due to the **gauge freedom**, which leaves the physical fields $\mathbf{E}$ and $\mathbf{B}$ invariant.

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9. Energy, Momentum and Angular Momentum

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In this chapter we will calculate the energy, momentum and angular momentum of the electromagnetic field, which will provide the basis for the description of electromagnetic phenomena in the atomic domain by **particles** (denoted by **photons**).

9.1 Energy

In Sect. 3.5 we have attributed an energy to the electrostatic field characterized by the energy density

$$\omega_{el}(\mathbf{r};t) = \frac{\epsilon_0}{2}\mathbf{E}^2(\mathbf{r};t). \tag{9.1}$$

In analogy we can assign an energy to the magnetostatic field. We want to skip this step and go straight on to the energy balance for an arbitrary electromagnetic field.

We first consider a point charge q, which is moving with the velocity $\mathbf{v}$ in an electromagnetic field $\{\mathbf{E}, \mathbf{B}\}$. The work done by the field on the charge is given by:

$$\frac{dW}{dt} = \mathbf{F}\cdot\mathbf{v} = q(\mathbf{E} + (\mathbf{v}\times\mathbf{B}))\cdot\mathbf{v} = q\mathbf{E}\cdot\mathbf{v}, \tag{9.2}$$

since the magnetic field does not contribute to the work. Correspondingly, the following holds for a current density $\mathbf{j}(\mathbf{r}, t)$ (arguments of the integrands suppressed):

$$\frac{dW_M}{dt} = \int_V (\mathbf{E}\cdot\mathbf{j})\ dV. \tag{9.3}$$

The work done by the field on the moving point charges is at the expense of the electromagnetic field; its explicit form for the energy we will derive below.

We first eliminate the current density $\mathbf{j}$ in (9.3) with the help from Eq. (7.30):

$$\int_V (\mathbf{E}\cdot\mathbf{j})\ dV = \int_V \left(\frac{1}{\mu_0}\mathbf{E}\cdot(\nabla\times\mathbf{B}) - \epsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}\right)\ dV. \tag{9.4}$$

This expression, which only includes the fields $\mathbf{E}$ and $\mathbf{B}$, can be symmetrized with respect to $\mathbf{E}$ and $\mathbf{B}$ with the relations

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}) \tag{9.5}$$

and

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{9.6}$$

Result:

$$\boxed{\frac{dW_M}{dt} = \int_V (\mathbf{E} \cdot \mathbf{j})\ dV = -\int_V \left(\frac{1}{2\mu_0} \frac{\partial \mathbf{B}^2}{\partial t} + \frac{\epsilon_0}{2} \frac{\partial \mathbf{E}^2}{\partial t} + \frac{1}{\mu_0} \nabla \cdot (\mathbf{E} \times \mathbf{B}) \right)\ dV.} \tag{9.7}$$

Interpretation:
Case 1: $V \to \infty$.
From (9.3) and (9.7) the **field energy** becomes:

$$W_F = \int_V (\frac{1}{2\mu_0} \mathbf{B}^2 + \frac{\epsilon_0}{2} \mathbf{E}^2)\ dV, \tag{9.8}$$

if the fields decay asymptotically fast enough such that the $\nabla\cdot$—term in (9.7) disappears. With the help of the Gauss' formula,

$$\int_V \nabla \cdot (\mathbf{E} \times \mathbf{B})\ dV = \oint_F (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{f}, \tag{9.9}$$

with F denoting the surface of the (finite) volume V, one finds that the fields $\mathbf{E}$ and $\mathbf{B}$ must decay faster than $1/R$ because df increases with R^2 (see Sect. 3.5). The requirement above is met for static fields, but not for radiation fields (see Chap. 12). In (9.8) we now can introduce the **energy density of the electromagnetic field**,

$$\boxed{\omega_F = \frac{1}{2\mu_0} \mathbf{B}^2 + \frac{\epsilon_0}{2} \mathbf{E}^2} \tag{9.10}$$

which results from an electric component (cf. (9.1))

$$\boxed{\omega_{el} = \frac{\epsilon_0}{2} \mathbf{E}^2} \tag{9.11}$$

and an magnetic component

$$\boxed{\omega_{mag} = \frac{1}{2\mu_0} \mathbf{B}^2.} \tag{9.12}$$

Case 2: V finite.

We keep the interpretation of (9.10) and write, since the volume V can be chosen arbitrarily, (9.7) as a (differential) energy balance:

$$\boxed{\mathbf{E} \cdot \mathbf{j} + \frac{\partial \omega_F}{\partial t} + \nabla \cdot \mathbf{S} = 0. \qquad (9.13)}$$

with

$$\boxed{\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}). \qquad (9.14)}$$

Interpretation of (9.13): The field energy in a volume V can change because energy —in the form of electromagnetic radiation (Chap. 12)—flows in (out), as described by the term $\nabla \cdot \mathbf{S}$, and/or that work is being done on point charges described by $\mathbf{E} \cdot \mathbf{j}$. In analogy to the charge conservation (Sect. 5.1) we denote by $\mathbf{S}$ the **energy current density** (Poynting vector). The energy balance shows that the energy of the closed system (point charges plus electromagnetic field) is a conserved quantity.

9.2 Momentum

In addition to energy we can also assign a momentum to the electromagnetic field. We start again with the momentum balance for a point charge q with velocity $\mathbf{v}$. According to Newton the change in the momentum of the point charge is:

$$\mathbf{F} = \frac{d\mathbf{p}_M}{dt} = q(\mathbf{E} + (\mathbf{v} \times \mathbf{B})). \qquad (9.15)$$

Corresponding, for N point charges, characterized by a current density $\mathbf{j}$ and charge density ρ, we obtain:

$$\boxed{\frac{d\mathbf{P}_M}{dt} = \int_V (\rho\mathbf{E} + (\mathbf{j} \times \mathbf{B}))\ dV. \qquad (9.16)}$$

In analogy to Sect. 9.1 we try to eliminate ρ and $\mathbf{j}$ such that the right side in (9.16) only contains the fields $\mathbf{E}$ and $\mathbf{B}$.

We use

$$\rho = \epsilon_0 \nabla \cdot \mathbf{E} \qquad (9.17)$$

and

$$\mathbf{j} = \frac{1}{\mu_0}\nabla \times \mathbf{B} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \qquad (9.18)$$

The result,

$$\frac{d\mathbf{P}_M}{dt} = \int_V \Big(\epsilon_0 \mathbf{E}(\nabla\cdot\mathbf{E}) + \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} - \epsilon_0(\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B})\Big)dV, \tag{9.19}$$

we can symmetrize with respect to $\mathbf{E}$ and $\mathbf{B}$ by adding in (9.19) the (disappearing) term,

$$\frac{1}{\mu_0}\mathbf{B}(\nabla\cdot\mathbf{B}), \tag{9.20}$$

and using the product rule in

$$-\epsilon_0\big(\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B}\big) = -\epsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}) + \epsilon_0\big(\mathbf{E}\times\frac{\partial\mathbf{B}}{\partial t}\big). \tag{9.21}$$

Inserting the law of induction

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{9.22}$$

we obtain the result:

$$\boxed{\begin{aligned}\frac{d\mathbf{P}_M}{dt} = \int_V \Big\{\Big[\epsilon_0\mathbf{E}(\nabla\cdot\mathbf{E}) \quad &+ \frac{1}{\mu_0}\mathbf{B}(\nabla\cdot\mathbf{B}) + \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} \\ &+ \epsilon_0(\nabla\times\mathbf{E})\times\mathbf{E}\big] - \epsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}\Big)\Big\}\ dV.\end{aligned}} \tag{9.23}$$

For the interpretation of (9.23) we sum up the [....] terms as follows:

$$(\mathbf{E}(\nabla\cdot\mathbf{E}) + \mathbf{E}\times(\nabla\times\mathbf{E}))_i = E_i\sum_{m=1}^{3}\frac{\partial E_m}{\partial x_m} - \sum_{m=1}^{3}\frac{\partial E_m}{\partial x_i}E_m + \sum_{m=1}^{3}\frac{\partial E_i}{\partial x_m}E_m \tag{9.24}$$

$$= \sum_{m=1}^{3}\frac{\partial}{\partial x_m}(E_iE_m) - \frac{1}{2}\frac{\partial}{\partial x_i}(\sum_m E_m^2) = \sum_{m=1}^{3}\frac{\partial}{\partial x_m}(E_iE_m - \frac{1}{2}E^2\delta_{im}).$$

We proceed for the $\mathbf{B}$ terms accordingly. The result is ($i = 1, 2, 3$):

$$\boxed{\frac{d}{dt}(\mathbf{P}_M + \mathbf{P}_F)_i = \int_V\sum_{m=1}^{3}\frac{\partial}{\partial x_m}T_{im}\ dV} \tag{9.25}$$

with the tensor

$$\boxed{\begin{aligned}T_{im} &= \epsilon_0(E_iE_m - \tfrac{1}{2}E^2\delta_{im}) + \frac{1}{\mu_0}(B_iB_m - \tfrac{1}{2}B^2\delta_{im}) \text{ and} \\ \mathbf{P}_F &= \epsilon_0\int_V(\mathbf{E}\times\mathbf{B})\ dV.\end{aligned}} \tag{9.26}$$

Case 1: $V \to \infty$.

As in Sect. 9.1 the right side in (9.25) disappears, if the fields $\mathbf{E}$ and $\mathbf{B}$ drop faster than $1/R$. Then the momentum balance is:

$$\mathbf{P}_M + \mathbf{P}_F = \text{const.} \tag{9.27}$$

Equation (9.27) suggests to interpret $\mathbf{P}_F$ as the momentum of the electromagnetic field. For the complete system (point charges plus field) then the total momentum, which is composed additively of particle and field momentum, is a conserved quantity.

Case 2: V finite.

We use the Gauss' theorem to rewrite the right side in (9.25):

$$\tfrac{d}{dt}(\mathbf{P}_M + \mathbf{P}_F)_i = \oint_F \sum_{m=1}^{3} T_{im} n_m \; df \tag{9.28}$$

where n_m are the components of the normal vector of the surface F of V. Since the left side of (9.28) is a force, we can attribute $T_{im} n_m$ to the **pressure** of the field (**radiation pressure**). The electromagnetic field can transfer not only energy but also momentum to an absorber!

Remark: The fact that the **momentum density**

$$\overrightarrow{\pi}_F = \epsilon_0 (\mathbf{E} \times \mathbf{B}) \tag{9.29}$$

and the energy current density $\mathbf{S}$ only differ by a constant factor,

$$\overrightarrow{\pi}_F = \epsilon_0 \mu_0 \mathbf{S} = \tfrac{1}{c^2} \mathbf{S}, \tag{9.30}$$

is not a coincidence but arises inevitably within the framework of the relativistic formulation (Chap. 19).

9.3 Angular Momentum

The change in the angular momentum of a point charge q in the electromagnetic field is given by:

$$\tfrac{d\mathbf{l}_M}{dt} = \mathbf{r} \times \tfrac{d\mathbf{p}_M}{dt} = q\mathbf{r} \times (\mathbf{E} + (\mathbf{v} \times \mathbf{B})). \tag{9.31}$$

Correspondingly, for N point charges, which are represented by ρ and $\mathbf{j}$ in a volume V, we get:

(9.32)

$$\frac{d\mathbf{L}_M}{dt} = \int_V \mathbf{r} \times (\rho\mathbf{E} + (\mathbf{j} \times \mathbf{B}))\ dV.$$

If we eliminate ρ and $\mathbf{j}$ again and symmetrize the result with respect to $\mathbf{E}$ and $\mathbf{B}$, we obtain (in analogy to Sect. 9.2):

$$\begin{aligned}\frac{d\mathbf{L}_M}{dt} = \int_V \mathbf{r} \quad &\times\{\epsilon_0\mathbf{E}(\nabla\cdot\mathbf{E}) + \frac{1}{\mu_0}\mathbf{B}(\nabla\cdot\mathbf{B}) + \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} \\ &+\epsilon_0(\nabla\times\mathbf{E})\times\mathbf{E} - \epsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B})\}\ dV.\end{aligned} \tag{9.33}$$

If the fields drop asymptotically fast enough, i.e. stronger than $1/R$ for $V \to \infty$, we obtain

$$\frac{d}{dt}(\mathbf{L}_M + \mathbf{L}_F) = 0, \tag{9.34}$$

with

$$\mathbf{L}_F = \epsilon_0 \int_V \mathbf{r} \times (\mathbf{E}\times\mathbf{B})\ dV = \int_V (\mathbf{r} \times \vec{\pi}_F)\ dV \tag{9.35}$$

as **angular momentum** of the field.

The sum of the mechanical angular momentum $\mathbf{L}_M$ and that of the field $\mathbf{L}_F$ is a conserved quantity:

$$\mathbf{L}_M + \mathbf{L}_F = \text{const}. \tag{9.36}$$

9.4 Summary

In the absence of other forces the conservation laws for energy, momentum and angular momentum hold for a closed system (point charges plus field). Since the energy, momentum and angular momentum of the point charges change in time, we have to assign energy, momentum and angular momentum to the electromagnetic field itself in order to guarantee the conservation laws for the entire system. The basic quantities

energy density

$$\omega_F(\mathbf{r};t) = \frac{\epsilon_0}{2}\mathbf{E}^2(\mathbf{r};t) + \frac{1}{2\mu_0}\mathbf{B}^2(\mathbf{r};t), \tag{9.37}$$

momentum density

$$\vec{\pi}_F(\mathbf{r};t) = \epsilon_0(\mathbf{E}(\mathbf{r};t)\times\mathbf{B}(\mathbf{r};t)) = \frac{1}{c^2}\mathbf{S}(\mathbf{r};t) \tag{9.38}$$

and

angular momentum density

$$\overrightarrow{\lambda}_F(\mathbf{r};t) = \epsilon_0 \mathbf{r} \times (\mathbf{E}(\mathbf{r};t) \times \mathbf{B}(\mathbf{r};t)) = \mathbf{r} \times \overrightarrow{\pi}_F(\mathbf{r};t) \tag{9.39}$$

can be found from the respective balances using Maxwell's equations.

The fact, that one can assign **mechanical** quantities such as energy, momentum and angular momentum to the Maxwell field, provides the basis for the description of electromagnetic phenomena in the atomic domain by **particles**, which are denoted by **photons** (after quantization).

Part IV
Electromagnetic Radiation

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10. The Electromagnetic Field in Vacuum

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In this chapter we will discuss the general solutions to the wave equations for the electromagnetic fields in vacuum, i.e. in a source-free region, which are the basis for the transmission of information.

10.1 Homogeneous Wave Equations

In the vacuum ($\rho = 0$; $\mathbf{j} = \vec{0}$) the Maxwell equations are

$$\nabla \cdot \mathbf{E} = 0; \quad \nabla \cdot \mathbf{B} = 0; \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}; \quad \nabla \times \mathbf{B} = \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{10.1}$$

To obtain a **decoupling** of $\mathbf{E}$ and $\mathbf{B}$ we form

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \Delta \mathbf{B} = -\epsilon_0 \mu_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}. \tag{10.2}$$

The result is a **homogeneous wave equation**

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B} = 0; \quad \frac{1}{c^2} = \epsilon_0 \mu_0. \tag{10.3}$$

We proceed in the same way for the $\mathbf{E}$ field. We then get

$$\boxed{\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B} = 0; \quad \nabla \cdot \mathbf{B} = 0} \tag{10.4}$$

and

$$\boxed{\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E} = 0; \quad \nabla \cdot \mathbf{E} = 0.} \tag{10.5}$$

For the associated potentials one finds according to Chap. 9:

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = 0; \quad \nabla \cdot \mathbf{A} = 0 \tag{10.6}$$

$$\Phi = 0 \tag{10.7}$$

in Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$).

We thus have differential equations of the type

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) f(\mathbf{r}, t) = 0, \tag{10.8}$$

where f is representative for any component of $\mathbf{E}$, $\mathbf{B}$ or $\mathbf{A}$. The solutions for $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{A}$, however, are still subject to the additional condition that the divergence disappears (**transversality condition**).

10.2 Plane Waves

An important type of solutions of (10.8) are **plane waves**,

$$f = f(\mathbf{q} \cdot \mathbf{r} \mp ct) \tag{10.9}$$

for any (at least twice differentiable) function f and vectors $\mathbf{q}$ with $\mathbf{q}^2 = 1$.

Proof With the abbreviation

$$\xi = \mathbf{q} \cdot \mathbf{r} \mp ct \tag{10.10}$$

we form:

$$\nabla f = \mathbf{q}\frac{df}{d\xi}; \quad \Delta f = \mathbf{q}^2\frac{d^2 f}{d\xi^2}; \quad \mp\frac{1}{c}\frac{\partial f}{\partial t} = \frac{df}{d\xi}; \quad \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = \frac{d^2 f}{d\xi^2}, \quad \text{q. e. d.} \tag{10.11}$$

Thus

$$\mathbf{B} = \mathbf{B}_0 f(\mathbf{r}, t) \tag{10.12}$$

is a solution of (10.3); similar solutions hold for $\mathbf{E}$ and $\mathbf{A}$.

Properties of the solutions:

(i) Plane waves.

Functions of the type (10.9) describe plane waves whose wavefronts are planes: The points $\mathbf{r}$, in which $f(\mathbf{r}, t)$ has the same value at a fixed time t lie on a plane (Hesse's normal form)

$$\mathbf{q} \cdot \mathbf{r} = \quad \text{const}, \tag{10.13}$$

which is perpendicular to $\mathbf{q}$. Depending on the choice of the sign in (10.9) we get waves that run in the $\pm\mathbf{q}$ direction.

(ii) Transversality of electromagnetic waves.

From $\nabla \cdot \mathbf{B} = 0$ it follows with (10.12)

$$(\mathbf{B}_0 \cdot \mathbf{q}) \frac{df}{d\xi} = 0, \tag{10.14}$$

thus

$$\boxed{\mathbf{B} \cdot \mathbf{q} = 0;} \tag{10.15}$$

correspondingly for $\mathbf{E}$ and $\mathbf{A}$ because of $\nabla \cdot \mathbf{E} = 0$ and the Coulomb gauge requires $\nabla \cdot \mathbf{A} = 0$.

(iii) Orthogonality of $\mathbf{E}$ and $\mathbf{B}$.

From

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{10.16}$$

we get for the plane wave solutions

$$\mathbf{E} = \mathbf{E}_0 \; f(\mathbf{q} \cdot \mathbf{r} - ct); \quad \mathbf{B} = \mathbf{B}_0 \; g(\mathbf{q} \cdot \mathbf{r} - ct) \tag{10.17}$$

the relationship

$$(\mathbf{q} \times \mathbf{E}_0) \frac{df}{d\xi} = c\mathbf{B}_0 \frac{dg}{d\xi}, \tag{10.18}$$

therefore $\mathbf{E} \perp \mathbf{B}$ with (10.15). $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{q}$ form an orthogonal tripod.

Choosing as solutions of the Maxwell equations $f(\xi) = g(\xi) =$sin (ξ) (or cos (ξ)) we get (at fixed time t) the following result for direction and amplitude of the $\mathbf{E}$ and $\mathbf{B}$ fields in space: $\mathbf{E}$ and $\mathbf{B}$ oscillate in space and time orthogonal to the direction of propagation $\sim \mathbf{q}$) and also orthogonal to each other.

Comments:

(1.) In addition to plane waves there are also spherical waves that are solutions of (10.8); they have the form ($r = |\mathbf{r}|$):

$$\frac{f(r-ct)}{r}, \tag{10.19}$$

where f is any (at least twice differentiable) function. The proof is analogous to (10.11) in spherical coordinates.

(2.) The existence of electromagnetic waves (e.g. light waves, radio waves, microwaves, γ radiation etc.) proves the validity of the relation $\nabla \times \mathbf{B} = \epsilon_0 \mu_0 \partial \mathbf{E} / \partial t$ in vacuum, which is crucial in the derivation of the wave equations. This provides an experimental confirmation for the **Maxwell-Ampère law** (7.26).

10.3 Monochromatic Plane Waves

A special form of the plane wave is (e.g. for the electrical field strength)

$$\mathbf{E} = \mathbf{E}_0 \; \exp \; (i(\mathbf{k} \cdot \mathbf{r} \mp \omega t)). \tag{10.20}$$

In (10.20)

$$\mathbf{k} = k\mathbf{q}, \tag{10.21}$$

and ω and $\mathbf{k}$ are connected by the **dispersion relation**

$$\omega^2 = k^2 c^2, \tag{10.22}$$

as can be seen immediately when inserting (10.20) into the wave Eq. (10.5). A plane wave of the type (10.20) is called **monochromatic**. Corresponding solutions can be found for $\mathbf{A}$ and $\mathbf{B}$, i.e.

$$\mathbf{B} = \mathbf{B}_0 \exp\left(i(\mathbf{k}\cdot\mathbf{r} \mp \omega t)\right); \quad \mathbf{A} = \mathbf{A}_0 \exp\left(i(\mathbf{k}\cdot\mathbf{r} \mp \omega t)\right).$$

Remark: $\mathbf{E}$, $\mathbf{A}$ and $\mathbf{B}$ are real vector fields by definition. The complex notation in Eq. (10.20) is to be understood in such a way that the physical vector field is described by the real part of (10.20). The complex notation is often more convenient than the real one (e.g. when differentiating); this does not create problems as long as only linear operations are carried out.

When calculating physical quantities such as the energy current density (see below) products of vector fields appear. Time averages of such products can be expressed in complex notation as follows: For two vector fields (of the same frequency)

$$\mathbf{a}(\mathbf{r},t) = \mathbf{a}_0(\mathbf{r}) \exp\left(-i\omega t\right); \quad \mathbf{b}(\mathbf{r},t) = \mathbf{b}_0(\mathbf{r}) \exp\left(-i\omega t\right) \tag{10.23}$$

the temporal average of the product is given by (with $\tau = 2\pi/\omega$):

$$\tfrac{1}{\tau}\int_0^\tau dt\ \Re\mathbf{a}(t)\cdot\Re\mathbf{b}(t) =: (\Re\mathbf{a})\cdot(\Re\mathbf{b}) = \tfrac{1}{2}\Re(\mathbf{a}\cdot\mathbf{b}^*), \tag{10.24}$$

since in

$$(\Re\mathbf{a})\cdot(\Re\mathbf{b}) = \tfrac{1}{4}\left(\mathbf{a}_0 \exp\left(-i\omega t\right) + \mathbf{a}_0^* \exp\left(i\omega t\right)\right)\cdot\left(\mathbf{b}_0 \exp\left(-i\omega t\right) + \mathbf{b}_0^* \exp\left(i\omega t\right)\right) \tag{10.25}$$

$$= \frac{1}{4}\left(\mathbf{a}_0\cdot\mathbf{b}_0 \exp\left(-2i\omega t\right) + \mathbf{a}_0^*\cdot\mathbf{b}_0^* \exp\left(2i\omega t\right) + \mathbf{a}_0\cdot\mathbf{b}_0^* + \mathbf{a}_0^*\cdot\mathbf{b}_0\right)$$

mixed terms with the time factors $\exp\left(\pm 2i\omega t\right)$ disappear after time averaging, i.e.

$$\tfrac{1}{\tau}\int_0^\tau dt\ \ \exp\left(\pm 2i\omega t\right) = \tfrac{1}{\pm 2i\omega\tau}\exp\left(\pm 2i\omega t\right)\Big|_0^\tau = \tfrac{1}{\pm 4\pi i}\left(\exp\left(\pm 4i\pi\right) - 1\right) = 0 \tag{10.26}$$

and only

$$(\Re\mathbf{a})\cdot(\Re\mathbf{b}) = \tfrac{1}{4}(\mathbf{a}\cdot\mathbf{b}^* + \mathbf{a}^*\cdot\mathbf{b}) = \tfrac{1}{2}(\Re\mathbf{a}\cdot\Re\mathbf{b} + \Im\mathbf{a}\cdot\Im\mathbf{b}) = \tfrac{1}{2}\Re(\mathbf{a}\cdot\mathbf{b}^*) \tag{10.27}$$

remains.

Terminology: The quantity *k* of the **wave vector k** is called **wave number** and is linked to the **wavelength** λ by

$$\lambda = \frac{2\pi}{k}. \tag{10.28}$$

With (10.22) we get

$$\tau = \frac{2\pi}{\omega} \tag{10.29}$$

for the connection of the **angular frequency** ω with the **oscillation period** τ. Instead of ω the **frequency** $\nu = \omega/(2\pi)$ can also be used. Based on (10.20) one can see that τ describes the temporal periodicity of the wave at fixed position $\mathbf{r}$,

$$\exp\,(i\omega(t+\tau)) = \exp\,(i\omega t + 2\pi i) = \exp\,(i\omega t); \tag{10.30}$$

in analogy λ gives the spatial periodicity:

$$\exp\,(ik(z+\lambda)) = \exp\,(ikz + 2\pi i) = \exp\,(ikz) \tag{10.31}$$

for a wave in *z* direction at fixed time *t*.

The quantity

$$\phi(\mathbf{r}, t) = \mathbf{k} \cdot \mathbf{r} - \omega t \tag{10.32}$$

is called the **phase** of the wave. The **phase velocity** v_{ph} is the velocity at which a wave point moves for a given fixed phase. To determine v_{ph} we consider again a plane wave in *z* direction and form the total differential of $\phi(z,t)$:

$$d\phi(z,t) = \frac{\partial\phi}{\partial z} dz + \frac{\partial\phi}{\partial t} dt = kdz - \omega dt. \tag{10.33}$$

For a constant phase ϕ we get:

$$v_{ph} = \frac{dz}{dt} = \frac{\omega}{k} = c\ ; \tag{10.34}$$

the phase velocity is equal to the velocity of light *c*.

Remark: Strictly speaking, a plane wave is extended infinitely perpendicular to the direction of propagation; any practically feasible wave, however, is limited in space. Nevertheless, the plane wave is a reasonable approximation if the extension of the real wave (perpendicular to the direction of propagation) is large compared to any **obstacles** (e.g. a thin gap in a plate or a grid), by which the wave can be disturbed.

For monochromatic plane waves the relations

$$\mathbf{B} = \nabla \times \mathbf{A}; \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}, \tag{10.35}$$

in complex representation turn to

$$\mathbf{B} = i(\mathbf{k} \times \mathbf{A}); \quad \mathbf{E} = i\omega\mathbf{A}. \tag{10.36}$$

Energy and momentum of the wave can be calculated easily using (10.36) and (10.24). For the time average of the energy density

$$\omega_F = \frac{1}{\tau}\int_0^\tau \omega_F \; dt \tag{10.37}$$

(in real representation),

$$\omega_F = \frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2 \tag{10.38}$$

we get (with $\mathbf{A} \cdot \mathbf{k} = 0$):

$$\omega_F = \frac{\epsilon_0}{4}Re(\omega^2\mathbf{A}\cdot\mathbf{A}^* + c^2k^2\mathbf{A}\cdot\mathbf{A}^*) = \frac{\epsilon_0}{2}\omega^2|\mathbf{A}_0|^2 = \frac{\epsilon_0}{2}|\mathbf{E}_0|^2. \tag{10.39}$$

In analogy we obtain for the energy current density (9.14)

$$\mathbf{S} = \frac{\omega}{2\mu_0}|\mathbf{A}_0|^2 \; \mathbf{k} = \frac{\epsilon_0 c}{2}|\mathbf{E}_0|^2 \; \mathbf{q} \tag{10.40}$$

and directly by (9.30) for the momentum density

$$\pi_F = \frac{\epsilon_0}{2c}|\mathbf{E}_0|^2 \; \mathbf{q} = \frac{1}{c^2}\mathbf{S}. \tag{10.41}$$

By comparing (10.39) with (10.41) we find that the energy is transported with the velocity *c*. In contrast to ω_F, $\mathbf{S}$ and π_F, the time average of the angular momentum density (9.39) depends on the position and is not suited for the characterization of a plane wave. However, the angular momentum of the field has significance for spherical waves, where it plays an analogous role as the momentum for plane waves.

10.4 Polarization

Due to the transversality and the orthogonality of $\mathbf{E}$ and $\mathbf{B}$ we can describe a monochromatic plane wave in the form (10.20):

$$\mathbf{E} = \mathbf{e}_1 E_0 \exp\left(i(\mathbf{k}\cdot\mathbf{r} - \omega t)\right); \quad \mathbf{B} = \mathbf{e}_2 B_0 \exp\left(i(\mathbf{k}\cdot\mathbf{r} - \omega t)\right) \tag{10.42}$$

with

(10.43)

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}; \quad \mathbf{e}_i \cdot \mathbf{k} = 0.$$

Such a wave is called **linearly polarized**. An **equivalent**, linearly independent plane wave with equal wave vector $\mathbf{k}$ is obtained by moving $\mathbf{E}$ in the $\mathbf{e}_2$ direction and $\mathbf{B}$ in the $\mathbf{e}_1$ direction. The general polarization state of a monochromatic plane wave then results from the superposition principle, e.g. for the electric field:

$$\mathbf{E} = (\mathbf{e}_1 E_1 + \mathbf{e}_2 E_2) \exp\left(i(\mathbf{k} \cdot \mathbf{r} - \omega t)\right) \tag{10.44}$$

with E_l (l = 1,2) as arbitrary complex numbers $E_l = |E_l| \exp\left(i\phi_l\right)$. Equation (10.44) describes all possible polarization states:

(1.) **Linear polarization** occurs if

$$\phi_1 = \phi_2. \tag{10.45}$$

The direction and magnitude of $\mathbf{E}$ then are given by (see Fig. 10.1)

$$\theta = \arctan\left(\frac{\mathrm{E}_2}{\mathrm{E}_1}\right); \quad \mathrm{E} = \sqrt{\mathrm{E}_1^2 + \mathrm{E}_2^2} \tag{10.46}$$

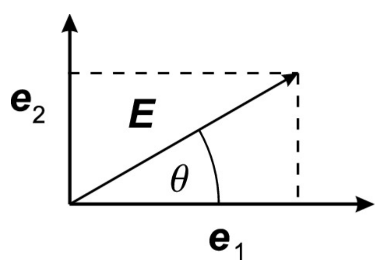

Fig. 10.1 Example for a linearly polarized wave

(2.) **Circular polarization**
Exists if:

$$E_1 = E_2; \quad \phi_1 - \phi_2 = \pm\frac{\pi}{2}; \tag{10.47}$$

then (with $\exp\left(\pm i\pi/2\right) = \pm i$)

$$\mathbf{E} = E_0(\mathbf{e}_1 \pm i\mathbf{e}_2) \exp\left(i(\mathbf{k} \cdot \mathbf{r} - \omega t)\right), \tag{10.48}$$

or in real representation

$$E_x = E_0 \cos\left(kz - \omega t\right); \quad E_y = \mp E_0 \sin\left(kz - \omega t\right), \tag{10.49}$$

if $\mathbf{k}$ points in the *z* direction. The direction of rotation is fixed in (10.48) by the choice of the sign; we get **left or right handed circular polarization**, i.e. $\mathbf{E}-$ and $\mathbf{B}-$ field rotate around the *z* axis in space and time (see Fig. 10.2).

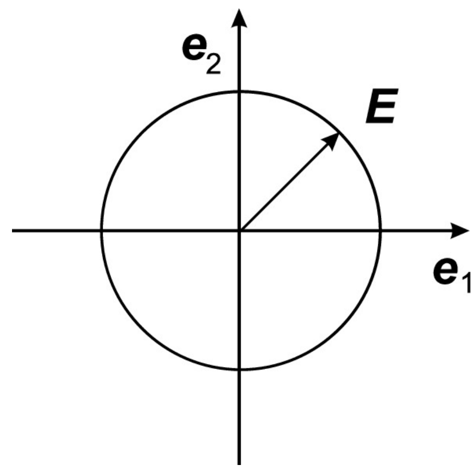

Fig. 10.2 Example for a circularly polarized wave

(3.) **Elliptic polarization** occurs for

$$E_1 \neq E_2; \quad \phi_1 - \phi_2 \neq 0. \tag{10.50}$$

$\mathbf{E}$ then describes an elliptical orbit for fixed z, its position relative to $\mathbf{e}_1$ by $\phi_1 - \phi_2$ and their principal axis ratio is determined by E_1/E_2.

In summary, the monochromatic plane waves (10.20) provide a convenient basis for the construction of wave packets in the vacuum by suitable superposition.

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11. Wave Packets in Vacuum

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An important application for electromagnetic radiation is the transmission of information. Monochromatic plane waves are not suited for this task because they contain practically no information other than their period (ω). One can, however, **modulate** monochromatic plane waves and by this transfer information. Such a superposition of plane waves is most conveniently described in terms of **Fourier series** or **Fourier integrals** (in the continuum). In this chapter we will introduce such Fourier series, analyze their properties and find the general solution to the homogeneous wave equation.

11.1 Transmission of Information by Electromagnetic Waves

In the most simple case one forms a superposition of 2 monochromatic waves,

$$f(t) = f_0 \cos(\omega_1 t) + f_0 \cos(\omega_2 t) \tag{11.1}$$

describing the wave at fixed position. Equation (11.1) can be represented as an **amplitude-modulated oscillation**:

$$f(t) = 2f_0 \cos(\omega_m t) \cos(\omega_0 t) = 2f_0(\cos((\omega_1 - \omega_2)/2) \cos((\omega_1 + \omega_2)/2)) \tag{11.2}$$

$$= f_0(\cos(\omega_0 + \omega_m) + \cos(\omega_0 - \omega_m))$$

with

$$\omega_m = \frac{\omega_1 - \omega_2}{2}; \quad \omega_0 = \frac{\omega_1 + \omega_2}{2}; \quad \omega_1 = \omega_0 + \omega_m; \quad \omega_2 = \omega_0 - \omega_m. \tag{11.3}$$

If we choose $\omega_1 \approx \omega_2$ then (11.2) is an almost harmonic oscillation of frequency ω_0 (**carrier frequency**), whose amplitude changes with the **modulation frequency** ω_m. We get the image of a **levitation** (see Fig. 11.1).

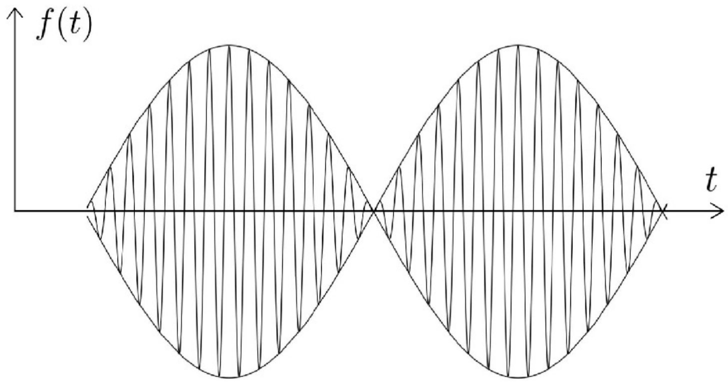

Fig. 11.1 Superposition of two frequencies with $\omega_1 \approx \omega_2$

More complicated oscillations and therefore more possibilities for the transmission of information arise for a **superposition** of several different vibration frequencies.

11.2 Fourier Series and Fourier Integrals

Starting from a **fundamental frequency** $\omega = 2\pi/T$ we form

$$f(t) = \sum_{n=-\infty}^{\infty} f_n \exp\left(-i\omega_n t\right); \quad \omega_n = n\omega. \tag{11.4}$$

The **Fourier series** (11.4) converges uniformly (and thus also pointwise), if $f(t)$ is periodic with period T and smooth piecewise. The (weaker) requirement of convergence is satisfied for periodic functions $f(t)$ that are finite and continuous in the interval $[0, T]$.

The **Fourier coefficients** f_n for a given function $f(t)$, which satisfies the requirements above, can be calculated as follows:

$$f_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\exp\left(i\omega_n t\right)\, dt. \tag{11.5}$$

Proof With

$$\frac{1}{T}\int_{-T/2}^{T/2} \exp\left(i\omega(m-n)t\right)\, dt = \delta_{mn} \tag{11.6}$$

we get:

$$\frac{1}{T}\int_{-T/2}^{T/2} f(t)\exp\left(i\omega_m t\right)\, dt = \sum_n f_n\delta_{mn} = \ f_m. \tag{11.7}$$

As an example of Fourier series we consider the periodic ‘triangle function’

(11.8)

$$f(x) = 1 + \frac{2\epsilon}{\pi} x$$

in the interval $[-\pi, \pi]$ with $\epsilon = 1$ for $x < 0$ and $\epsilon = -1$ for $x \geq 0$. The corresponding Fourier series (11.4) (in real representation) is:

$$f(x) = \lim_{N\to\infty} \frac{8}{\pi^2} \sum_{n=0}^{N} \frac{1}{(2n+1)^2} \cos\left((2n+1)x\right). \tag{11.9}$$

The approximation of the function *f*(*x*) by (11.9) is shown in Fig. 11.2 for the case $N = 1$ as well as for $N = 3$, i.e. only the first four vibration modes are taken into account. However, we can see from Fig. 11.2 that already $N = 3$ gives a useful approximation.

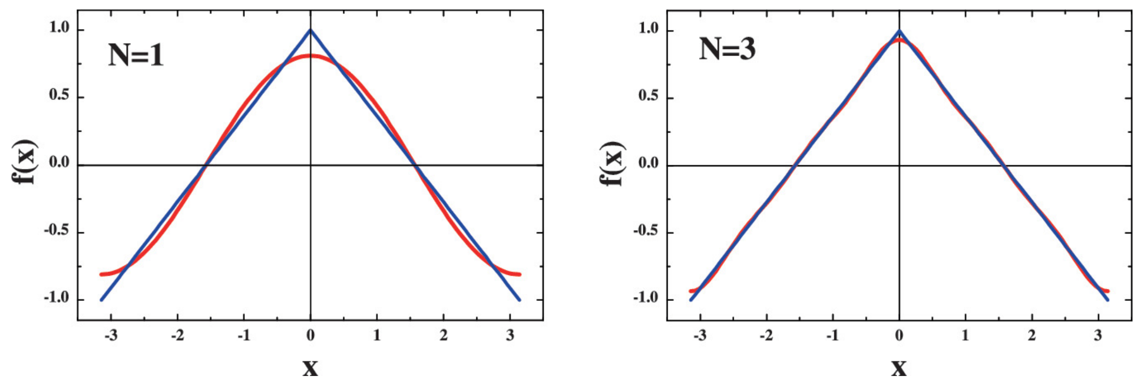

Fig. 11.2 Illustration of the Fourier series (11.9) for $N = 1$ and $N = 3$

Non-periodic functions can be represented (for very weak assumptions, see below) by **Fourier integrals**, which result from (11.4) in the limit $T \to \infty$.

With the distance $\Delta\omega = 2\pi/T$ of neighboring frequencies ω_n we define

$$\tilde{f}(\omega_i) = \lim_{T\to\infty} \left(\frac{T}{2\pi} f_i\right), \tag{11.10}$$

and obtain

$$f(t) = \sum_{n=-\infty}^{\infty} \tilde{f}(\omega_n) \exp\left(-i\omega_n t\right) \Delta\omega \tag{11.11}$$

as the **Riemann sum** of the **Fourier integral**

$$\boxed{f(t) = \int_{-\infty}^{\infty} \tilde{f}(\omega) \exp\left(-i\omega t\right) d\omega.} \tag{11.12}$$

For the inverse of (11.12) we obtain:

$$\boxed{\tilde{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \exp\left(i\omega t\right) dt.} \tag{11.13}$$

The function $\tilde{f}(\omega)$ is called the **Fourier transform** to $f(t)$. It exists and (11.12) **converges in the root mean square** for all square integrable functions $f(t)$ with

$$\int_{-\infty}^{\infty} |f(t)|^2 \ dt < \infty; \tag{11.14}$$

$\tilde{f}(\omega)$ then is also square integrable.

Example: square wave pulse

$$f(t) = 1 \quad \text{for} \quad -\tfrac{\tau}{2} \leq \mathrm{t} \leq \tfrac{\tau}{2}; \quad \mathrm{f(t)} = 0 \quad \text{else.} \tag{11.15}$$

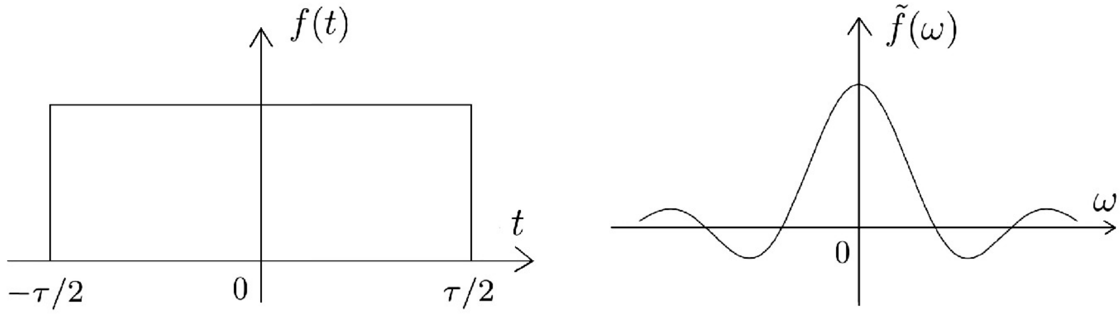

Fig. 11.3 The square wave pulse $f(t)$ (left) and its Fourier transform $\tilde{f}(\omega)$ (right)

Then we get

$$\tilde{f}(\omega) = \tfrac{1}{2\pi} \int_{-\tau/2}^{\tau/2} \exp\ (i\omega t)\ dt = \tfrac{1}{\pi\omega} \tfrac{\exp(i\omega t)}{2i} \Big|_{-\tau/2}^{\tau/2} = \tfrac{\sin(\omega\tau/2)}{\pi\omega}. \tag{11.16}$$

The width $\Delta\omega$ of $\tilde{f}(\omega)$ can be estimated from Fig. 11.3 (from the distance of the first zeros) by:

$$\boxed{\Delta\omega \approx \tfrac{2\pi}{\tau} \quad \text{or} \quad \Delta\omega\Delta \mathrm{t} \approx 2\pi \ \text{ or } \ \Delta\nu\Delta\mathrm{t} \approx 1\ .} \tag{11.17}$$

A narrower (wider) signal $f(t)$ leads to wider (narrower) frequency spectrum $\tilde{f}(\omega)$. This **uncertainty relation** does not only hold for the example (11.15) but is a characteristic property of the Fourier transformation (see quantum mechanics).

Note: The Fourier transform is often used in the symmetric form

$$f(t) = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega)\ \exp\ (-i\omega t)\ d\omega \tag{11.18}$$

with

$$\tilde{f}(\omega) = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\ \exp\ (i\omega t)\ dt. \tag{11.19}$$

11.3 Spectral Decomposition of Plane Waves

The Fourier series of a periodic function $f(\mathbf{q}\cdot\mathbf{r} - ct) = f(\xi)$, which represents a plane wave, is:

$$f(\xi) = \sum_{n=-\infty}^{\infty} f_n \exp\left(i\omega_n \xi/c\right) \tag{11.20}$$

with

$$\xi = \mathbf{q}\cdot\mathbf{r} - ct, \quad \omega_n = n\omega \tag{11.21}$$

and the Fourier coefficients f_n are given by:

$$f_n = \frac{\omega}{2\pi c} \int_{-\pi c/\omega}^{\pi c/\omega} f(\xi) \exp\left(-i\omega_n \xi/c\right)\, d\xi. \tag{11.22}$$

The Fourier integral is used for aperiodic plane waves:

$$f(\xi) = \int_{-\infty}^{\infty} \tilde{f}(\omega) \exp\left(i\omega\xi/c\right)\, d\omega \tag{11.23}$$

with the inverse

$$\tilde{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) \exp\left(-i\omega\xi/c\right)\, d\xi/c. \tag{11.24}$$

The spatial or temporal extent of the wave then is determined by:

$$\Delta\xi\Delta\omega \approx 2\pi c, \tag{11.25}$$

i.e.

$$\Delta t\Delta\omega \approx 2\pi \tag{11.26}$$

for a fixed position **r** and a fixed time *t*:

$$\Delta z\Delta k \approx 2\pi, \tag{11.27}$$

if the wave travels in *z* direction.

Note: It is important for the transmission of information that the plane wave packets of the form (11.23) keep their shape and do not **disintegrate** (see quantum mechanics):

$$f(\mathbf{r}, 0) = f(\mathbf{r} + \mathbf{q}ct, t), \tag{11.28}$$

since *f* only depends on the argument ξ (11.21). This property no longer holds for the propagation of electromagnetic waves in matter (see Part V)!

11.4 δ-Distribution

The Fourier transform (11.12), (11.13) leads to the following mathematical problem: Inserting (11.13) in (11.12), we must get (after exchanging the order of integration)

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t')\ \exp\left(-i\omega(t-t')\right)\ d\omega dt' = \int_{-\infty}^{\infty} f(t')\delta(t-t')\ dt' \quad (11.29)$$

with

$$\boxed{\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-i\omega(t-t')\right)\ d\omega \quad (11.30)}$$

for any square integrable functions $f(t)$. The quantity $\delta(t-t')$ obviously is not an ordinary function, but a **distribution**, which strictly speaking can **only be defined in connection with the integration** in (11.29).

The δ-distribution, defined in (11.29), can be represented by any sequence of continuous functions δ_n, for which holds:

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f(t')\ \delta_n(t-t')\ dt' = f(t). \quad (11.31)$$

Examples:

(1) Rectangle

$$\delta_n(t) = n \quad \text{for} \quad |\mathrm{t}| < \frac{1}{2\mathrm{n}}; \quad \delta_\mathrm{n}(\mathrm{t}) = 0 \quad \text{else.} \quad (11.32)$$

(2) Gauss' function

$$\delta_n(t) = n\ \exp\left(-\pi t^2 n^2\right). \quad (11.33)$$

(3) The representation

$$\delta_n(t) = \frac{1}{\pi}\frac{\sin(nt)}{t} = \frac{1}{2\pi}\int_{-n}^{n} \exp\left(i\omega t\right)\ d\omega = \frac{1}{2it\pi}\left(\exp\left(int\right) - \exp\left(-int\right)\right) \quad (11.34)$$

leads to the notation (11.30).

Warning: Equations (11.31)–(11.34) have to be understood in such a way that the t'-integration is carried out **before** the limit $n\to\infty$ is taken!

Calculation rules:

(1) $\delta(t) = \delta(-t)$

(2) $\delta(at) = \frac{1}{|a|}\delta(t)$

(3) $\delta(t^2 - a^2) = \frac{1}{2|a|}(\delta(t+a) + \delta(t-a)); \quad a \neq 0\,.$

(4) $\delta(f(t)) = \sum_k \frac{1}{|f'(t_k)|}\delta(t - t_k),$

where the t_k are all (simple) zeros of $f(t)$, i.e. $f(t_k) = 0$.

11.5 General Solution of the Homogeneous Wave Equation

The plane waves examined in Sect. 11.3 are indeed limited in time and space in their direction of propagation, but not in the plane perpendicular to the propagation. They are not suited for the transmission of information, since their infinite surface extent would require an infinitely large energy. Signals of finite energy can only be obtained for fields that are limited in space and time (**wave packets**), which can be built by superposition of monochromatic plane waves. Starting from the two basic solutions $\exp\,(i(\mathbf{k}\cdot\mathbf{r} \mp \omega t)$ with a fixed $\mathbf{k}$ we expand the vector potential (in the extension of the Fourier transform to 3 dimensions) as

$$\begin{aligned}\mathbf{A}(\mathbf{r},t) \quad &= \frac{1}{2(2\pi)^{3/2}}\int d^3k\;[\mathbf{A}_+(\mathbf{k})\,\exp\,(i(\mathbf{k}\cdot\mathbf{r}-\omega t)) \\ &+\mathbf{A}_-(\mathbf{k})\,\exp\,(i(\mathbf{k}\cdot\mathbf{r}+\omega t))].\end{aligned} \tag{11.35}$$

Due to (10.22) the expansion (11.35) covers all possible ω values. To obtain a real function $\mathbf{A}(\mathbf{r},t)$, we replace in the 2nd term of (11.35) $\mathbf{k}$ by $-\mathbf{k}$:

$$\begin{aligned}\mathbf{A}(\mathbf{r},t) \quad &= \frac{1}{2(2\pi)^{3/2}}\int d^3k\;[\mathbf{A}_+(\mathbf{k})\,\exp\,(i(\mathbf{k}\cdot\mathbf{r}-\omega t)) \\ &+\mathbf{A}_-(-\mathbf{k})\,\exp\,(-i(\mathbf{k}\cdot\mathbf{r}-\omega t))].\end{aligned} \tag{11.36}$$

$\mathbf{A}(\mathbf{r},t)$ becomes real if $(2\Re z = z + z^*)$

$$\mathbf{A}_+(\mathbf{k}) = \mathbf{A}_-^*(-\mathbf{k}) = \mathbf{A}(\mathbf{k}), \tag{11.37}$$

thus

$$\begin{aligned}\mathbf{A}(\mathbf{r},t) \quad &= \frac{1}{2(2\pi)^{3/2}}\int d^3k\;[\mathbf{A}(\mathbf{k})\,\exp\,(i(\mathbf{k}\cdot\mathbf{r}-\omega t)) \\ &+\mathbf{A}^*(\mathbf{k})\,\exp\,(-i(\mathbf{k}\cdot\mathbf{r}-\omega t))].\end{aligned} \tag{11.38}$$

With (11.38) we have found the **general solution** of the homogeneous wave Eq. (10.6). The Coulomb gauge requires additionally

$$\boxed{\mathbf{k} \cdot \mathbf{A}(\mathbf{k}) = 0. \qquad (11.39)}$$

It is important for the formulation of quantum mechanics, where the electromagnetic field is described by **photons**, that the energy, momentum and angular momentum of the field emerge additively from the contributions of the monochromatic plane waves.

We demonstrate this for the case of the energy and rewrite (11.38) again as,

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{2(2\pi)^{3/2}} \int d^3k \ [\mathbf{A}(\mathbf{k}, t) + \mathbf{A}^*(-\mathbf{k}, t)] \ \ \exp\,(i\mathbf{k} \cdot \mathbf{r}) \qquad (11.40)$$

with the abbreviations

$$\mathbf{A}(\mathbf{k}, t) = \mathbf{A}(\mathbf{k}) \exp\,(-i\omega t); \quad \mathbf{A}^*(-\mathbf{k}, t) = \mathbf{A}^*(-\mathbf{k}) \exp\,(i\omega t). \qquad (11.41)$$

Then according to (9.10) the energy of the electromagnetic field is given by

$$\begin{aligned} W \ &= \int \omega_F(\mathbf{r}) \ d^3r = \int [\tfrac{\epsilon_0}{2} (\tfrac{\partial \mathbf{A}}{\partial t})^2 + \tfrac{1}{2\mu_0} (\nabla \times \mathbf{A})^2] \ d^3r \\ &= \tfrac{1}{4(2\pi)^3} \int d^3r \int d^3k \int d^3k' [\tfrac{\epsilon_0}{2} (-i\omega \mathbf{A}(\mathbf{k}, t) \\ &+ i\omega \mathbf{A}^*(-\mathbf{k}, t))(-i\omega' \mathbf{A}(\mathbf{k}', t)) + i\omega' \mathbf{A}^*(-\mathbf{k}', t)) \\ &+ \tfrac{1}{2\mu_0} (i\mathbf{k} \times \mathbf{A}(\mathbf{k}, t) + i\mathbf{k} \times \mathbf{A}^*(-\mathbf{k}, t))(i\mathbf{k}' \times \mathbf{A}(\mathbf{k}', t)) \\ &+ i\mathbf{k}' \times \mathbf{A}^*(-\mathbf{k}', t))] \exp\,(i(\mathbf{k} + \mathbf{k}') \cdot \mathbf{r}), \end{aligned} \qquad (11.42)$$

where [....] still depends on $\mathbf{k}$ and $\mathbf{k}'$. After performing the integration $\int d^3r$ we get – due to

$$\frac{1}{(2\pi)^3} \int d^3r \ \ \exp\,(i(\mathbf{k} + \mathbf{k}') \cdot \mathbf{r}) = \delta^3(\mathbf{k} + \mathbf{k}') \qquad (11.43)$$

and $\delta^3(\mathbf{k}) = \delta(k_x)\delta(k_y)\delta(k_z)$ – only contributions for $\mathbf{k} = -\mathbf{k}'$ and thus $\omega = \omega'$:

$$W = \frac{1}{4} \int d^3k \ [\tfrac{\epsilon_0}{2} (-i\omega \mathbf{A}(\mathbf{k}, t) + i\omega \mathbf{A}^*(-\mathbf{k}, t))(-i\omega \mathbf{A}(-\mathbf{k}, t)) + i\omega \mathbf{A}^*(\mathbf{k}, t)) \qquad (11.44)$$

$$+ \frac{1}{2\mu_0} (i\mathbf{k} \times \mathbf{A}(\mathbf{k}, t) + i\mathbf{k} \times \mathbf{A}^*(-\mathbf{k}, t))(-i\mathbf{k} \times \mathbf{A}(-\mathbf{k}, t)) - i\mathbf{k} \times \mathbf{A}^*(\mathbf{k}, t))]$$

$$= \frac{\epsilon_0}{8} \int d^3k \ [(i\omega)^2 (\mathbf{A}(\mathbf{k}, t) - \mathbf{A}^*(-\mathbf{k}, t))(\mathbf{A}(-\mathbf{k}, t)) - \mathbf{A}^*(\mathbf{k}, t))$$

$$-(ikc)^2(\mathbf{A}(\mathbf{k},t)+\mathbf{A}^*(-\mathbf{k},t))(\mathbf{A}(-\mathbf{k},t))+\mathbf{A}^*(\mathbf{k},t))]$$
$$=\frac{\epsilon_0}{4}\int d^3k\ \omega^2[\mathbf{A}(\mathbf{k})\cdot\mathbf{A}^*(\mathbf{k})+\mathbf{A}^*(-\mathbf{k})\cdot\mathbf{A}(-\mathbf{k})]$$
$$=\frac{\epsilon_0}{2}\int d^3k\omega^2[\mathbf{A}(\mathbf{k})\cdot\mathbf{A}^*(\mathbf{k})].$$

Here we have used

$$\frac{\partial}{\partial t}\mathbf{A}(\mathbf{k},t)=-i\omega\mathbf{A}(\mathbf{k},t) \tag{11.45}$$

and

$$\nabla\times\mathbf{A}(\mathbf{k},t)\ \exp\ (i\mathbf{k}\cdot\mathbf{r})=i(\mathbf{k}\times\mathbf{A}(\mathbf{k},t))\ \exp\ (i\mathbf{k}\cdot\mathbf{r}) \tag{11.46}$$

as well as $\mathbf{k}\cdot\mathbf{A}(\mathbf{k})=0$ (11.39) and $\epsilon_0\mu_0=c^{-2}$.

In summary, Eq. (11.44) describes the field energy as a sum (integral) of the individual contributions (10.39) of the monochromatic waves involved. Together with the corresponding equations for momentum and angular momentum this provides the basis for the description of the electromagnetic field by independent **particles** (**photons**) (see quantum electrodynamics). The energy *W* itself is independent of time consistent with the conservation of energy.

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12. Solutions of the Inhomogeneous Wave Equations

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Whereas in the previous chapter we have derived the general solution of the homogeneous wave equation we now aim at solving the inhomogeneous wave equations for arbitrary sources $\rho(\mathbf{r};t)$ and $\mathbf{j}(\mathbf{r};t)$. To this aim we will use the method of Green's functions that will lead to retarded potentials. The latter will be employed to calculate the electromagnetic radiation from moving point charges.

In the presence of charges we have to solve the inhomogeneous equations (cf. (8.13), (8.14))

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = -\mu_0 \mathbf{j}, \tag{12.1}$$

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi = -\frac{\rho}{\epsilon_0} \tag{12.2}$$

with the secondary condition (Lorentz convention)

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0. \tag{12.3}$$

The problem is thus the **solution of an inhomogeneous wave equation**

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi(\mathbf{r},t) = -\omega(\mathbf{r},t), \tag{12.4}$$

where Ψ stands for Φ and the components A_i, while ω stands for ρ/ϵ_0 and the components $\mu_0 j_i$. The general solution of (12.4) arises from a general solution of the homogeneous wave Eq. (10.8) (discussed in Chap. 11) and a special solution of the inhomogeneous wave equation.

To construct a special solution of (12.4) we use the **method of Green's functions**: With the definition of the Green's function:

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r},\mathbf{r}';t,t') = -\delta^3(\mathbf{r}-\mathbf{r}')\ \delta(t-t') =: -\delta^4(x-x') \tag{12.5}$$

(and the four-vector $x = (t, \mathbf{x})$) we can write as a (formal) special solution:

$$\Psi(\mathbf{r},t) = \int G(\mathbf{r},\mathbf{r}';t,t')\ \omega(\mathbf{r}',t')\ d^3r' dt', \tag{12.6}$$

as is directly confirmed by substituting (12.6) in (12.4). This is done by exchanging the order of the integration with respect to $\mathbf{r}'$, t' and the differentiation with respect to $\mathbf{r}$, t, which has to be done with caution.

12.1 Construction of $G(\mathbf{r}, \mathbf{r}'; t, t')$

Let's first note 2 fundamental properties of G:

$$G = G(\mathbf{r} - \mathbf{r}'; t - t') \tag{12.7}$$

due to the invariance of (12.5) with respect to space and time translations;

$$G(\mathbf{r} - \mathbf{r}'; t - t') = 0 \quad \text{for} \ \ \mathrm{t} < \mathrm{t}' \tag{12.8}$$

due to the **principle of causality**.

As a preliminary exercise we consider the (known) case of static fields, e.g. the electrostatic field. The Poisson equation

$$\Delta\Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0} \tag{12.9}$$

has the (special) solution

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \ d^3r', \tag{12.10}$$

i.e. the Coulomb potential of a charge distribution $\rho(\mathbf{r})$. We can write Eq. (12.10) as

$$\Phi(\mathbf{r}) = \frac{1}{\epsilon_0} \int G(\mathbf{r}, \mathbf{r}') \ \rho(\mathbf{r}') \ d^3r' \tag{12.11}$$

with the Green's function

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}, \tag{12.12}$$

which satisfies the differential equation

$$\Delta G(\mathbf{r}, \mathbf{r}') = -\delta^3(\mathbf{r} - \mathbf{r}'). \tag{12.13}$$

Proof

(i) $R \neq 0$, where

$$\mathbf{R} = \mathbf{r} - \mathbf{r}'. \tag{12.14}$$

Then:

$$\Delta\left(\frac{1}{R}\right)=\nabla\cdot\left(\nabla\frac{1}{R}\right)=-\nabla\cdot\left(\frac{\mathbf{R}}{R^3}\right)=-\frac{1}{R^3}\nabla\cdot\mathbf{R}+\frac{3}{R^4}\left(\mathbf{R}\cdot\frac{\mathbf{R}}{R}\right)=-\frac{3}{R^3}+\frac{3}{R^3}=0,\ \text{q. e. d.} \tag{12.15}$$

(ii) Due to (12.15) in a volume integral of type

$$\int f(\mathbf{R})\Delta\left(\frac{1}{R}\right)\ d^3R \tag{12.16}$$

the integration range can be restricted to a small sphere of radius a with center at $R=0$,

$$\int f(\mathbf{R})\Delta\left(\frac{1}{R}\right)\ d^3R = \lim_{a\to 0}\int_{\text{sphere(a)}} f(\mathbf{R})\Delta\left(\frac{1}{R}\right)\ d^3R. \tag{12.17}$$

If $f(\mathbf{R})$ is continuous around 0, one can extract $f(\mathbf{R})\approx f(\mathbf{R}=0)$ from the integral

$$\int f(\mathbf{R})\Delta\left(\frac{1}{R}\right)\ d^3R = \lim_{a\to 0} f(\mathbf{R}=0)\int_{\text{sphere(a)}}\Delta\left(\frac{1}{R}\right)\ d^3R\ , \tag{12.18}$$

and obtaines by the Gauss's theorem:

$$\int_{\text{sphere(a)}}\Delta\left(\frac{1}{R}\right)\ d^3R=\int_{\text{sphere(a)}}\nabla\cdot\left(\nabla\left(\frac{1}{R}\right)\right)\ d^3R \tag{12.19}$$

$$=\int_{F(a)}\left(\nabla\left(\frac{1}{R}\right)\right)\cdot d\mathbf{f}=-\int_{F(a)}\frac{1}{R^2}R^2 d\Omega=-4\pi.$$

Thus:

$$\int f(\mathbf{R})\ \Delta\left(\frac{1}{R}\right)\ d^3R=-4\pi f(0). \tag{12.20}$$

Since (12.5) is the wave equation for a time and spatially point-like source, $G(\mathbf{r}-\mathbf{r}';t-t')$ must represent a spherical wave, which reaches the position $\mathbf{r}$ at time $t=t'+|\mathbf{r}-\mathbf{r}'|/c$, if the perturbation is at the position $\mathbf{r}'$ at time t'. We start with the Ansatz:

$$G(R,\tau)=\frac{g(\tau-R/c)}{R}=\frac{g(t-t'-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|} \tag{12.21}$$

with $\tau=t-t'$ and $R=|\mathbf{r}-\mathbf{r}'|$. To determine the function g we insert (12.21) in (12.5) and form:

$$\begin{aligned}\Delta G &= g\Delta\left(\frac{1}{R}\right)+\frac{1}{R}\Delta g+2\nabla\left(\frac{1}{R}\right)\cdot\nabla g\\ &=-4\pi g\ \delta(R)+\frac{1}{R}\frac{\partial^2}{\partial R^2}g+\frac{2}{R^2}\frac{\partial}{\partial R}g-\frac{2}{R^2}\frac{\partial}{\partial R}g;\end{aligned} \tag{12.22}$$

here we have used:

$$\Delta=\frac{\partial^2}{\partial R^2}+\frac{2}{R}\frac{\partial}{\partial R}+\frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{R^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{12.23}$$

$$=\frac{\partial^2}{\partial R^2}+\frac{2}{R}\frac{\partial}{\partial R}+\quad\text{angular term}$$

and

$$\frac{\partial g}{\partial x}=\frac{\partial g}{\partial R}\cdot\frac{x}{R},\ \frac{\partial g}{\partial y}=\frac{\partial g}{\partial R}\cdot\frac{y}{R},\ \frac{\partial g}{\partial z}=\frac{\partial g}{\partial R}\cdot\frac{z}{R}\ . \tag{12.24}$$

This gives

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G = -4\pi g(\tau - R/c)\ \delta^3(\mathbf{R}), \tag{12.25}$$

since

$$\frac{1}{R}\left(\frac{\partial^2}{\partial R^2} - \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2}\right)g(\tau - R/c) = 0 \tag{12.26}$$

for arbitrary (differentiable) functions $g(\tau - R/c)$. The comparison with (12.5) gives:

$$4\pi g(\tau - R/c) = \delta(\tau - R/c) \tag{12.27}$$

thus:

$$G(\mathbf{r}-\mathbf{r}';t-t') = \frac{\delta(t-t'-|\mathbf{r}-\mathbf{r}'|/c)}{4\pi|\mathbf{r}-\mathbf{r}'|} \quad \text{for } \ \mathrm{t} > \mathrm{t}' \tag{12.28}$$

$$G(\mathbf{r}-\mathbf{r}';t-t') = 0 \quad \text{for } \ \mathrm{t} < \mathrm{t}'.$$

Remark:

When deriving (12.27) the differentiability of $g(\tau - R/c)$ is assumed. This requirement is actually fulfilled because the δ-distribution can be differentiated (any number of times) in the sense:

$$\int f(x)\ \delta^{(n)}(x)\ dx = (-)^n \int f^{(n)}\ \delta(x)\ dx; \tag{12.29}$$

assuming that f is differentiable (any number of times).

Interpretation of G:

The inhomogeneity in (12.5) represents a point-like source, which at time t' at position $\mathbf{r}'$ is switched on for an (infinitesimal) short time. The perturbation caused by this source propagates as a spherical wave with velocity c. It follows that:

(i) The spherical wave $G(\mathbf{r}-\mathbf{r}';t-t')$ must disappear for $t < t'$ according to the principle of causality.

(ii) It must arrive at position $\mathbf{r}$ at time $t = t' + |\mathbf{r}-\mathbf{r}'|/c$ since electromagnetic waves move with the (finite) velocity of light c in the vacuum.

(iii) Since the energy of the wave is distributed on a spherical surface, G should disappear asymptotically like R^{-1}.

The **retarded** Green's function (12.28) satisfies these requirements exactly. Equation (12.6) shows, how to get the potentials $\mathbf{A}(\mathbf{r},t)$, $\Phi(\mathbf{r},t)$ for given source distributions $\rho(\mathbf{r},t)$, $\mathbf{j}(\mathbf{r},\mathbf{t})$ from the contributions of point-like sources.

12.2 Retarded Potentials

With (12.6) and (12.28) the (asymptotically vanishing) solutions of (12.1) and (12.2) for localized charge and current distributions are

$$\boxed{\Phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}',t') \ \delta(t-t'-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|} \ d^3r'dt'} \tag{12.30}$$

and

$$\boxed{\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}',t') \ \delta(t-t'-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|} \ d^3r'dt'.} \tag{12.31}$$

The solutions (12.30) and (12.31) are linked together by (12.3) or the conservation of charge (7.2). We will examine the integrations in (12.30) and (12.31) for 2 practically important special cases with particular attention to the argument of the δ-distribution, that incorporates the **retardation**.

When ignoring the retardation in (12.30) and (12.31),

$$\delta(t - t' - \frac{|\mathbf{r}-\mathbf{r}'|}{c}) \rightarrow \delta(t - t'), \tag{12.32}$$

we obtain the **quasi-stationary fields**:

$$\boxed{\Phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|} \ d^3r',} \tag{12.33}$$

$$\boxed{\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|} \ d^3r',} \tag{12.34}$$

which appear in the theory of electrical networks and machines. The approach (12.32) is justified if $\rho(\mathbf{r}',t)$ and $\mathbf{j}(\mathbf{r}',t)$ are (practically) not changing during the time, that an electromagnetic wave needs to travel the distance $|\mathbf{r}-\mathbf{r}'|$.

Example 1 Time-periodic source distributions of the form

$$\rho(\mathbf{r},t) = \rho(\mathbf{r}) \exp\,(-i\omega t); \quad \mathbf{j}(\mathbf{r},t) = \mathbf{j}(\mathbf{r}) \exp\,(-i\omega t). \tag{12.35}$$

Then we get from (12.30), (12.31):

$$\boxed{\Phi(\mathbf{r},t) = \Phi(\mathbf{r}) \exp\,(-i\omega t); \quad \mathbf{A}(\mathbf{r},t) = \mathbf{A}(\mathbf{r}) \exp\,(-i\omega t)} \tag{12.36}$$

with $k = \omega/c$ and

$$\int dt' \quad \exp\,(-i\omega t') \ \delta(t-t' - |\mathbf{r}-\mathbf{r}'|/c) = \exp\,(-i\omega t) \ \exp\,(i\omega|\mathbf{r} - \mathbf{r}'|/c)$$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')\exp(ik|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|} \ d^3r', \tag{12.37}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}')\exp(ik|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|} \ d^3r'. \tag{12.38}$$

The associated differential equations result from Eqs. (12.1), (12.2) and (12.35):

$$(\Delta + k^2)\Psi(\mathbf{r}) = -\gamma(\mathbf{r}), \tag{12.39}$$

where Ψ stands for Φ and A_i and γ for ρ and j_i. We can write the solutions (12.37) and (12.38) with the Green's function belonging to (12.39),

$$G(\mathbf{r}, \mathbf{r}'; k) = \frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{4\pi|\mathbf{r}-\mathbf{r}'|}, \tag{12.40}$$

as

$$\Psi(\mathbf{r}) = \int \gamma(\mathbf{r}') \ G(\mathbf{r}, \mathbf{r}'; k) \ d^3r'. \tag{12.41}$$

The discussion of the integrals (12.37), (12.38) for $\Phi(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$ will be taken up in Chap. 13.

Example 2: Fields of moving point charges:

For a point charge q moving on the path $\mathbf{r}(t)$ we can write:

$$\rho(\mathbf{r}, t) = q \ \delta^3(\mathbf{r} - \mathbf{r}(t)); \quad \mathbf{j}(\mathbf{r}, t) = q \ \mathbf{v}(t) \ \delta^3(\mathbf{r} - \mathbf{r}(t)). \tag{12.42}$$

Then in (12.30) the d^3r'—integration can be carried out:

$$\begin{aligned}\Phi(\mathbf{r}, t) &= \frac{q}{4\pi\epsilon_0} \int \frac{\delta^3(\mathbf{r}'-\mathbf{r}(t'))\delta(t-t'-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|} \ d^3r'dt' \\ &= \frac{q}{4\pi\epsilon_0} \int \frac{\delta(t-t'-|\mathbf{r}-\mathbf{r}(t')|/c)}{|\mathbf{r}-\mathbf{r}(t')|} dt',\end{aligned} \tag{12.43}$$

and for $\mathbf{A}(\mathbf{r}, t)$ we get in analogy

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 q}{4\pi} \int \frac{\mathbf{v}(t') \ \delta(t-t'-|\mathbf{r}-\mathbf{r}(t')|/c)}{|\mathbf{r}-\mathbf{r}(t')|} \ dt'. \tag{12.44}$$

To perform the t'—integration we use

$$\int_{-\infty}^{\infty} g(x) \ \delta(f(x)) \ dx = \sum_i \frac{g(x_i)}{|f'(x_i)|}, \tag{12.45}$$

where x_i are (simple) zeros of $f(x)$, i.e. $f(x_i) = 0$ and $f'(x_i) \neq 0$. Then we obtain:

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0} \sum_i \frac{1}{R(t_i')\kappa(t_i')} \tag{12.46}$$

with

$$\mathbf{R}(t_i') = \mathbf{r} - \mathbf{r}(t_i'); \quad \kappa(t_i') = 1 - \frac{\mathbf{R}(t_i')\cdot\mathbf{v}(t_i')}{cR(t_i')} = \left| \left(\frac{df}{dt'} \right)_{t'=t_i'} \right| . \tag{12.47}$$

In (12.47) the t_i' are solutions of the equation $f(t') = t' - t + R(t')/c = 0$. In analogy we get:

$$\mathbf{A}(\mathbf{r},t) = \frac{q\mu_0}{4\pi} \sum_i \frac{\mathbf{v}(t_i')}{R(t_i')\kappa(t_i')} . \tag{12.48}$$

The potentials (12.46) and (12.48) (**Liénard–Wichert potentials**) are written in shorthand form as:

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{R\kappa} \right)_{ret}; \quad \mathbf{A}(\mathbf{r},t) = \frac{q\mu_0}{4\pi} \left(\frac{\mathbf{v}}{R\kappa} \right)_{ret} . \tag{12.49}$$

The limit $v \to 0$ gives

$$\mathbf{A} \to 0; \quad \Phi(\mathbf{r},t) \to \frac{q}{4\pi\epsilon_0 R} , \tag{12.50}$$

i.e. the Coulomb potential known from electrostatics (as well as a vanishing vector potential).

12.3 Electromagnetic Radiation from Moving Point Charges

If the energy flow through an infinitely distant surface does not disappear,

$$\lim_{R\to\infty} \int \mathbf{S} \cdot d\mathbf{f} \neq 0, \tag{12.51}$$

we encounter radiation of electromagnetic waves caused by localized charge and current distributions. This implies that the fields $\mathbf{E}, \mathbf{B}$ then do not decrease stronger than R^{-1}, since the surface increases like R^2. Such fields are called **radiation fields** in contrast to the **static** fields that decrease with R^{-2}.

We now want to show that accelerated point charges **radiate**; to do this we have to calculate the fields associated to

$$\mathbf{B} = \nabla \times \mathbf{A}; \quad \mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} , \tag{12.52}$$

where we want to use the form (12.43), (12.44) for Φ and $\mathbf{A}$. With the abbreviations

$$\nabla f(R) = \mathbf{n} \frac{\partial f}{\partial R}; \quad \mathbf{n} = \frac{\mathbf{R}}{R} \tag{12.53}$$

we obtain:

$$-\nabla\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0} \int dt' \left(\frac{\mathbf{n}(t')}{R(t')^2} \delta\left(t' - t + \frac{R(t')}{c}\right) - \frac{\mathbf{n}(t')}{cR(t')} \delta'\left(t' - t + \frac{R(t')}{c}\right) \right) \tag{12.54}$$

and

$$\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t) = -\frac{\mu_0 q}{4\pi}\int dt' \left(\frac{\mathbf{v}(t')}{R(t')}\ \delta'\left(t'-t+\frac{R(t')}{c}\right)\right), \tag{12.55}$$

such that

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= \frac{q}{4\pi\epsilon_0}\int dt' \left(\frac{\mathbf{n}(t')}{R(t')^2}\delta\left(t'-t+\frac{R(t')}{c}\right)\right. \\ &\left. + \frac{\mathbf{v}(t')/c-\mathbf{n}(t')}{cR(t')}\delta'\left(t'-t+\frac{R(t')}{c}\right)\right).\end{aligned} \tag{12.56}$$

Here $\delta'(t'-t+R(t')/c)$ is the derivative (12.29) defined with respect to the argument $\xi = t'-t+R(t')/c$. In analogy:

$$\begin{aligned}&\mathbf{B}(\mathbf{r},t) = \\ &-\frac{\mu_0 q}{4\pi}\int dt'\ (\mathbf{n}(t')\times\mathbf{v}(t'))\left(\frac{1}{R(t')^2}\delta\left(t'-t+\frac{R(t')}{c}\right) - \frac{1}{cR(t')}\delta'\left(t'-t+\frac{R(t')}{c}\right)\right).\end{aligned} \tag{12.57}$$

To carry out the t' integration we use:

$$\delta'(\xi) = \frac{1}{\kappa(t')}\ \frac{d}{dt'}\delta\left(t'-t+\frac{R(t')}{c}\right) \tag{12.58}$$

with $\kappa(t')$ (12.47). With (12.29), (12.45) and (12.49) we obtain:

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= \frac{q}{4\pi\epsilon_0}\left(\frac{\mathbf{n}(t)}{\kappa(t)R(t)^2} + \frac{1}{\kappa(t)c}\ \frac{d}{dt'}\left(\frac{-\mathbf{v}(t')/c+\mathbf{n}(t')}{\kappa(t')R(t')}\right)\right)_{ret}. \\ \mathbf{B}(\mathbf{r},t) &= \frac{\mu_0 q}{4\pi}\left(\frac{\mathbf{v}\times\mathbf{n}(t)}{\kappa(t)R(t)^2} + \frac{1}{\kappa(t)c}\ \frac{d}{dt'}\left(\frac{\mathbf{v}(t')\times\mathbf{n}(t')}{\kappa(t')R(t')}\right)\right)_{ret}.\end{aligned} \tag{12.59}$$

To differentiate with respect to t' we compute

$$-\frac{d\mathbf{n}}{dt'} = \frac{\mathbf{R}}{R^2}(\mathbf{n}\cdot\mathbf{v}) - \frac{\mathbf{v}}{R} = \frac{1}{R}[(\mathbf{n}\cdot\mathbf{v})\mathbf{n}-\mathbf{v}] \tag{12.60}$$

and

$$\frac{d}{dt'}(\kappa R) = \frac{v^2}{c} - \mathbf{n}\cdot\mathbf{v} - \frac{R}{c}(\mathbf{n}\cdot\mathbf{a}) \tag{12.61}$$

with the acceleration $\mathbf{a} = d/dt'\mathbf{v}$. We insert (12.60), (12.61) into (12.59) and reorder in powers of R^{-1} to get:

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{c^2\kappa^3 R}\left[(\mathbf{n}\cdot\mathbf{a})\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right)-\kappa\mathbf{a}\right]\right)_{ret} + \quad O(R^{-2}); \tag{12.62}$$

The latter terms, which decrease like R^{-2}, are not interesting with respect to the condition (12.51). Correspondingly for $\mathbf{B}$ we get:

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0 q}{4\pi}\left(\frac{1}{c^2\kappa^3 R}[(\mathbf{n}\cdot\mathbf{a})(\mathbf{v}\times\mathbf{n}) - \kappa c\ (\mathbf{n}\times\mathbf{a})]\right)_{ret} + \quad O(R^{-2}). \tag{12.63}$$

To calculate the **energy flux density** we use the identity

$$\mathbf{n}\times\left(\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right)\times\mathbf{a}\right) = (\mathbf{n}\cdot\mathbf{a})\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right) - \kappa\mathbf{a} \tag{12.64}$$

as well as the relation

$$\mathbf{B} = \frac{1}{c}(\mathbf{n}\times\mathbf{E}), \tag{12.65}$$

which follows directly from (12.62), (12.63) for the asymptotic region. We then find for the **Poynting vector**

$$\mathbf{S} = \frac{\mathbf{E}\times\mathbf{B}}{\mu_0} = \frac{\mathbf{E}\times(\mathbf{n}\times\mathbf{E})}{\mu_0 c} = \frac{1}{\mu_0 c}\left(\mathbf{n}E^2 - \mathbf{E}(\mathbf{n}\cdot\mathbf{E})\right) = \frac{\mathbf{n}}{\mu_0 c}E^2 \tag{12.66}$$

with (12.62) and (12.64):

$$\mathbf{S} = \frac{q^2\mathbf{n}}{16\pi^2\epsilon_0 c^3\kappa^6 R^2}\left(\mathbf{n}\times\left[\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right)\times\mathbf{a}\right]\right)^2. \tag{12.67}$$

Since $|\mathbf{S}| \sim R^{-2}$ the condition (12.51) is fulfilled and we get the result that accelerated point charges, $\mathbf{a} \neq 0$, radiate. Point charges moving on straight lines and uniformly ($\mathbf{a}$ = 0) do not radiate as a simple consequence: The rest system of the point charge then is an inertial system, in which the electric field is the Coulomb field and the magnetic field, by definition, disappears such that $\mathbf{S} = 0$.

Examples:

(1.) Bremsstrahlung occurs when a charged particle (e.g. electron) is decelerated in an external field (e.g. when colliding with some target). This results in the continuous **Röntgen spectrum**.

(2.) Synchrotron radiation

The motion of charged particles on a circular paths is also an accelerated motion. The resulting radiation is a major problem in cyclic particle accelerators (synchrotrons); some of the energy supplied is **lost** by radiation. On the other hand, for highly relativistic electron beams the synchrotron radiation is focused (with suitable deflection magnets) at small forward angles such that a suitable high-energy photon beam is created!

(3.) Radiation damping:

In the classical atomic model the bound electrons move in circular or elliptical orbits around the atomic nucleus. Then they radiate continuously—as accelerated charges—electromagnetic waves. The resulting energy loss leads to unstable orbits and ultimately to the collapse of the atoms in the classical model. This contradiction to experimental observation can only be solved in quantum theory or quantum electrodynamics (QED).

In summarizing this chapter we have derived the solution of the inhomogeneous wave equation for arbitrary sources $\rho(\mathbf{r};t)$ and $\mathbf{j}(\mathbf{r};t)$ employing the method of Green's functions. This lead to retarded (Liénard–Wichert) potentials, which have been used to calculate the electromagnetic radiation from accelerated point charges and for time-periodic source distributions.

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13. Multipole Radiation

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In this chapter we will classify the leading order multipoles for magnetic and electric radiation emerging from accelerated electric and magnetic dipole moments as well as electric quadrupole moments.

13.1 Long Wave-Length Approximation

For a source distribution of the form

$$\rho(\mathbf{r},t)=\rho(\mathbf{r})\exp\left(-i\omega t\right);\mathbf{j}(\mathbf{r},t)=\mathbf{j}(\mathbf{r})\exp\left(-i\omega t\right) \tag{13.1}$$

we have found in Sect. 12.3:

$$\Phi(\mathbf{r},t)=\Phi(\mathbf{r})\exp\left(-i\omega t\right);\mathbf{A}(\mathbf{r},t)=\mathbf{A}(\mathbf{r})\exp\left(-i\omega t\right) \tag{13.2}$$

and $(k=\omega/c)$

$$\boxed{\begin{aligned}\Phi(\mathbf{r})&=\tfrac{1}{4\pi\epsilon_0}\int\tfrac{\rho(\mathbf{r}')\exp(ik|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\ d^3r', & (13.3)\\ \mathbf{A}(\mathbf{r})&=\frac{\mu_0}{4\pi}\int\frac{\mathbf{j}(\mathbf{r}')\exp\left(ik|\mathbf{r}-\mathbf{r}'|\right)}{|\mathbf{r}-\mathbf{r}'|}\ d^3r'.\end{aligned}}$$

When discussing (13.3) we may restrict to $\mathbf{A}(\mathbf{r})$, since $\mathbf{A}(\mathbf{r})$ and $\Phi(\mathbf{r})$ are directly connected by the Lorentz convention: From

$$\nabla\cdot\mathbf{A}+\tfrac{1}{c^2}\tfrac{\partial\Phi}{\partial t}=0 \tag{13.4}$$

follows with (13.2)

$$\Phi(\mathbf{r}) = \frac{c^2}{i\omega}\ \nabla \cdot \mathbf{A}(\mathbf{r}), \tag{13.5}$$

and is therefore also known when $\mathbf{A}(\mathbf{r})$ is determined. For the further evaluation of (13.3) we employ the **long wave-length approximation**

$$\boxed{d \ll \lambda = \frac{2\pi}{k}\,, \tag{13.6}}$$

where d indicates the radius of a sphere, which determines the charge and current distribution inside.

Examples:

For the optical radiation of atoms we have $d \approx 10^{-8}$ cm, $\lambda \approx 10^{-5}$ cm; in analogy we find for the γ radiation of atomic nuclei: $d \approx 10^{-13}$ cm, $\lambda \approx 10^{-11}$ cm.

When discussing (13.3) the lengths d, λ and r are essential. We investigate the following cases:

Case 1: $d < r \ll \lambda$ (**near zone**)

Then

$$k|\mathbf{r}-\mathbf{r}'| \ll 1 \tag{13.7}$$

and we get directly:

$$\boxed{\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\ d^3r', \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\ d^3r'. \tag{13.8}}$$

The spatial component of the potentials shows the same structure according to (13.8) as in electrostatics and magnetostatics. Given the time dependency (13.2) one deals with **quasi-static fields**, for which $\mathbf{E}$, $\mathbf{B}$ decay as R^{-2}, such that the radiation condition (12.51) is not fulfilled.

Case 2: $d \ll \lambda \ll r$ (**far zone**)

Since

$$kr \gg 1 \tag{13.9}$$

we can employ the Taylor expansion

$$|\mathbf{r}-\mathbf{r}'| = \sum_n \frac{(-)^n}{n!}(\mathbf{r}' \cdot \nabla)^n r \approx r - \frac{\mathbf{r}\cdot\mathbf{r}'}{r} \cdots \tag{13.10}$$

in (13.3):

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \mathbf{A}_0(\mathbf{r}) + \mathbf{A}_1(\mathbf{r}) + \cdots \\ &= \tfrac{\mu_0}{4\pi} \tfrac{\exp(ikr)}{r} \int d^3r' \; \mathbf{j}(\mathbf{r}')\{1 + (\tfrac{1}{r} - ik)(\mathbf{e}\cdot\mathbf{r}') + \cdots\}\end{aligned} \tag{13.11}$$

$$\approx \frac{\mu_0}{4\pi} \frac{\exp\,(ikr)}{r} \int d^3r' \; \mathbf{j}(\mathbf{r}')\{1 - i\frac{\omega}{c}(\mathbf{e}\cdot\mathbf{r}')\}$$

with $k = \omega/c$ and the direction vector

$$\mathbf{e} = \tfrac{\mathbf{r}}{r}. \tag{13.12}$$

13.2 Electric Dipole—Radiation

In the first term in (13.11)

$$\mathbf{A}_0(\mathbf{r}) = \tfrac{\mu_0}{4\pi} \tfrac{\exp(ikr)}{r} \int d^3r' \; \mathbf{j}(\mathbf{r}') \tag{13.13}$$

we can rewrite

$$\textstyle\int_V j_i \; d^3r' = \int_V \nabla' \cdot (x_i'\mathbf{j}) \; d^3r' - \int_V x_i'(\nabla' \cdot \mathbf{j}) \; d^3r' \tag{13.14}$$

$$= \int_F x_i'(\mathbf{j}\cdot d\mathbf{f}') - \int_V x_i'(\nabla' \cdot \mathbf{j}) \; d^3r' = -i\omega \int_V x_i' \rho(\mathbf{r}') \; d^3r',$$

due to charge conservation

$$\nabla \cdot \mathbf{j} - i\omega\rho = 0. \tag{13.15}$$

With (2.31) we then get:

$$\mathbf{A}_0(\mathbf{r}) = -i\omega \tfrac{\mu_0}{4\pi} \; \tfrac{\exp(ikr)}{r} \; \mathbf{d}, \tag{13.16}$$

where the electric dipole moment $\mathbf{d}$ is involved.
For the fields we get (following (13.9) and considering only terms $\sim r^{-1}$):

$$\mathbf{B}_0(\mathbf{r}) = \nabla \times \mathbf{A}_0 = \frac{\mu_0}{4\pi c}\ \omega^2 \frac{\exp(ikr)}{r}\ (\mathbf{e} \times \mathbf{d}). \tag{13.17}$$

With

$$\Phi_0(\mathbf{r}) = \frac{c^2}{i\omega} \nabla \cdot \mathbf{A}(\mathbf{r}) = -i\frac{\mu_0 c}{4\pi}\ \omega \frac{\exp(ikr)}{r}\ (\mathbf{e} \cdot \mathbf{d}) \tag{13.18}$$

we obtain from (8.5) the $\mathbf{E}$ field:

$$\mathbf{E}_0(\mathbf{r}) = -\nabla\Phi_0(\mathbf{r}) - \frac{\partial A_0}{\partial t} = \frac{\mu_0}{4\pi}\omega^2 \frac{\exp(ikr)}{r}(\mathbf{d} - \mathbf{e}(\mathbf{e} \cdot \mathbf{d})) = c\ (\mathbf{B}_0 \times \mathbf{e}), \tag{13.19}$$

using $\mathbf{e} \times \mathbf{d} \times \mathbf{e} = \mathbf{d} - \mathbf{e}(\mathbf{e} \cdot \mathbf{d})$. We now can calculate the **energy current density**

$$\mathbf{S} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} = \frac{c}{\mu_0}(\mathbf{B}_0 \times \mathbf{e}) \times \mathbf{B}_0 = \frac{c}{\mu_0}(\mathbf{e}B_0^2 - \mathbf{B}_0(\mathbf{B}_0 \cdot \mathbf{e}). \tag{13.20}$$

We use the real parts of (13.17) and (13.19) and find with (13.2) and $(\mathbf{e} \times \mathbf{d} \times \mathbf{e}) \times (\mathbf{e} \times \mathbf{d})$ = $\mathbf{e}d^2\ \sin^2\ \theta$:

$$\mathbf{S}_0 = \frac{\mu_0}{16\pi^2 c}\omega^4 d^2\ \sin^2\ \theta\ \ \frac{\cos^2(kr - \omega t)}{r^2}\mathbf{e}, \tag{13.21}$$

where θ is the angle enclosed by $\mathbf{e}$ and $\mathbf{d}$. For the time average we get:

$$\mathbf{S}_0 = \frac{\mu_0}{16\pi^2 c}\omega^4 d^2\ \ \frac{\sin^2\theta}{2r^2}\mathbf{e}. \tag{13.22}$$

The dipole does not radiate in the direction of $\mathbf{d}$ (θ = 0), but dominantly perpendicular to $\mathbf{d}$ ($\theta = 90°$). The $\sin^2\ \theta$—dependence is characteristic for dipole radiation.

Comments:

(1.) Typical for radiation fields is that $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{S}$ form an orthogonal tripod (cf. Sect. 10.3).

(2.) An oscillating dipole (with frequency ω) can be realized only by accelerated point charges. Equation (13.22) is therefore conformal with the general statement (12.67).

(3.) The radiation of the lowest multipolarity is dipole radiation (l=1), not monopole radiation (l=0)! Quantum theory shows how this multipolarity of the radiation and the angular momentum of the photons are related. Since photons have an intrinsic angular momentum (**spin** $1\hbar$), there is no **angular momentum-free** radiation, i.e. monopole radiation. The spin of photons is directly linked to the fact that radiation fields are described by vector fields.

13.3 Magnetic Dipole and Electric Quadrupole Radiation

The 2nd term of the expansion (13.11) is

$$\mathbf{A}_1(\mathbf{r}) = -i\omega \frac{\mu_0}{4\pi c} \frac{\exp(ikr)}{r} \int \mathbf{j}(\mathbf{r}')(\mathbf{e}\cdot\mathbf{r}') \; d^3r'; \tag{13.23}$$

the remaining integral is determined by the magnetic dipole moment and the electric quadrupole tensor. To identify this, we use the identity:

$$(\mathbf{e}\cdot\mathbf{r}')\mathbf{j} = \tfrac{1}{2}(\mathbf{r}'\times\mathbf{j})\times\mathbf{e} + \tfrac{1}{2}((\mathbf{e}\cdot\mathbf{r}')\mathbf{j} + (\mathbf{e}\cdot\mathbf{j})\mathbf{r}'), \tag{13.24}$$

which transforms the integrand in (13.23) into an antisymmetric and a symmetric part with respect to $\mathbf{r}'$. With the definition (5.39) of the magnetic dipole moment the antisymmetric part becomes:

$$\mathbf{A}_1^{(m)}(\mathbf{r}) = -i\omega \frac{\mu_0}{4\pi c} \frac{\exp(ikr)}{r} (\mathbf{m}\times\mathbf{e}). \tag{13.25}$$

The magnetic dipole component of the vector potential formally transfers to the electric dipole component (13.16) when replacing

$$\tfrac{1}{c}(\mathbf{m}\times\mathbf{e}) \to \mathbf{d}. \tag{13.26}$$

With (13.26), (13.17) and (13.19) we find for the field strengths:

$$\mathbf{B}_1^{(m)}(\mathbf{r}) = \frac{\mu_0}{4\pi c^2}\omega^2 \frac{\exp(ikr)}{r} (\mathbf{e}\times(\mathbf{m}\times\mathbf{e})) \tag{13.27}$$

and

$$\mathbf{E}_1^{(m)}(\mathbf{r}) = c\ (\mathbf{B}_1^{(m)} \times \mathbf{e}). \tag{13.28}$$

In analogy to (13.22) one determines the energy radiated over time:

$$\mathbf{S}_1^{(m)} = \frac{\mu_0}{16\pi^2 c^3}\omega^4 m^2 \frac{\sin^2\theta}{2r^2}\mathbf{e}, \tag{13.29}$$

where θ now is the angle between $\mathbf{m}$ and $\mathbf{e}$. The comparison of (13.29) and (13.22) shows that electric and magnetic dipole radiation do not differ in their frequency and angle dependencies. The only difference is in the **polarization**: for an electric dipole the vector of the electric field is in the plane spanned by $\mathbf{e}$ and $\mathbf{d}$, for a magnetic dipole, however, perpendicular to the plane spanned by $\mathbf{e}$ and $\mathbf{m}$.

We now consider the 2nd term in (13.24), which is given by

$$\mathbf{A}_1^{(e)}(\mathbf{r}) = -i\omega\frac{\mu_0}{4\pi c}\frac{\exp(ikr)}{2r}\int\{\mathbf{j}(\mathbf{e}\cdot\mathbf{r}') + \mathbf{r}'(\mathbf{e}\cdot\mathbf{j})\}\ d^3r'. \tag{13.30}$$

The integral in (13.30) can be reduced to the one introduced in Sect. 2.5 for the electric quadrupole tensor. In analogy to (13.14) we rewrite:

$$\int_V j_i x'_m\ d^3r' = \int_V\ x'_m \nabla'\cdot(x'_i\mathbf{j})\ d^3r' - \int_V\ x'_m x'_i(\nabla'\cdot\mathbf{j})\ d^3r' \tag{13.31}$$

$$= -\int_V\ x'_i j_m\ d^3r' - i\omega\int_V\ x'_m x'_i \rho(\mathbf{r}')\ d^3r',$$

where a partial integration (1st term) and the charge conservation (2nd term) are used. Thus:

$$\int_V\{j_i x'_m + x'_i j_m\}\ d^3r' = -i\omega\int_V\ x'_m x'_i \rho\ d^3r', \tag{13.32}$$

and we can write (13.30) as:

$$\mathbf{A}_1^{(e)}(\mathbf{r}) = -\frac{\mu_0}{4\pi c}\ \omega^2\ \frac{\exp(ikr)}{2r}\ \int(\mathbf{e}\cdot\mathbf{r}')\mathbf{r}'\rho(\mathbf{r}')\ d^3r' \tag{13.33}$$

For the fields we obtain with (13.9)

$$\boxed{\mathbf{B}_1^{(e)}(\mathbf{r}) = \nabla \times \mathbf{A}_1^{(e)}(\mathbf{r}) = ik\ \left(\mathbf{e} \times \mathbf{A}_1^{(e)}(\mathbf{r})\right) \qquad (13.34)}$$

and

$$\boxed{\mathbf{E}_1^{(e)}(\mathbf{r}) = i\frac{c^2}{\omega}\nabla \times \mathbf{B}_1^{(e)}(\mathbf{r}) = c\ (\mathbf{B}_1^{(e)}(\mathbf{r}) \times \mathbf{e}), \qquad (13.35)}$$

since in a charge-free space we have:

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}. \tag{13.36}$$

Note: With the abbreviations

$$f(r) = -\frac{\mu_0}{4\pi c}\ \omega^2 \frac{\exp(ikr)}{r}, \mathbf{v}(\mathbf{r}) = \int d^3r'(\mathbf{e}\cdot\mathbf{r}')\mathbf{r}'\rho(\mathbf{r}') \tag{13.37}$$

the magnetic field reads:

$$\begin{aligned}\mathbf{B}_1^{(e)} &= (\nabla f) \times \mathbf{v} + f\ \nabla \times \mathbf{v} = ikf(\mathbf{r})(\mathbf{e} \times \mathbf{v}) + O(r^{-2}) \\ &= ik(\mathbf{e} \times \mathbf{A}_1^{(e)}) + O(r^{-2}),\end{aligned} \tag{13.38}$$

since all derivatives are of order $O(r^{-1})$. In a similar way one proceeds in (13.35) with the calculation of $\mathbf{E}_1^{(e)}$.

With the help of the quadrupole tensor, given by its components

$$Q_{mn} = \int_V\ \rho(\mathbf{r}')\left(x'_m x'_n - \frac{1}{3}r'^2\delta_{mn}\right)\ d^3r', \tag{13.39}$$

we obtain for $\mathbf{B}_1^{(e)}$:

$$\mathbf{B}_1^{(e)}(\mathbf{r}) = -i\frac{\mu_0}{4\pi c^2}\ \omega^3\ \frac{\exp(ikr)}{2r}\ (\mathbf{e} \times \mathbf{Q}), \tag{13.40}$$

with the vector $\mathbf{Q}$ given by its components ($m = 1, 2, 3$)

$$Q_m = \sum_{n=1}^{3} Q_{mn}e_n. \tag{13.41}$$

Note that the 2nd term in (13.39) does not contribute to (13.40).

As above we now can calculate the **energy current density**

$$\mathbf{S}_1^{(e)} = \frac{1}{\mu_0}(\Re\mathbf{E}_1^{(e)} \times \Re\mathbf{B}_1^{(e)}) = \frac{c}{\mu_0}(\Re\mathbf{B}_1^{(e)} \times \mathbf{e}) \times \Re\mathbf{B}_1^{(e)}. \tag{13.42}$$

With

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \tag{13.43}$$

we get

$$\mathbf{S}_1^{(e)} = \frac{c}{\mu_0}\left(\Re\mathbf{B}_1^{(e)}\right)^2 \mathbf{e} = \frac{\mu_0}{16\pi^2 c^3}\ \omega^6\ \frac{\cos^2(kr-\omega t)}{4r^2}(\mathbf{e} \times \mathbf{Q})^2 \mathbf{e}. \tag{13.44}$$

By time averaging this leads to

$$\mathbf{S}_1^{(e)} = \frac{\mu_0}{16\pi^2 c^3}\omega^6\ \frac{1}{8r^2}(\mathbf{e} \times \mathbf{Q})^2 \mathbf{e}. \tag{13.45}$$

The difference to dipole radiation is obvious for the dependence on frequency ω. For the discussion of the angular dependence we consider the case of **axial symmetry** (cf. Sect. 2.5)

$$Q_{mn} = 0 \ \text{ for } \ \mathrm{m} \neq \mathrm{n}; \mathrm{Q}_{11} = \mathrm{Q}_{22} = -\frac{\mathrm{Q}_{33}}{2} = -\frac{\mathrm{Q}_0}{3}. \tag{13.46}$$

From

$$(\mathbf{e} \times \mathbf{Q})^2 = Q^2 - (\mathbf{e} \cdot \mathbf{Q})^2 \tag{13.47}$$

then follows

$$Q^2 = \frac{Q_0^2}{9}(e_1^2 + e_2^2) + \frac{4}{9}Q_0^2 e_3^2 = \frac{Q_0^2}{9}(\sin^2\theta + 4\cos^2\theta) \tag{13.48}$$

as well as

$$\mathbf{e} \cdot \mathbf{Q} = -\frac{Q_0}{3}\sin^2\theta + \frac{2}{3}Q_0\cos^2\theta; \tag{13.49}$$

thus

(13.50)

$$(\mathbf{e} \times \mathbf{Q})^2 = Q_0^2 \sin^2 \theta \cos^2 \theta.$$

Result:

$$\boxed{\mathbf{S}_1^{(e)} = \frac{\mu_0}{16\pi^2 c^3}\ \omega^6\ \frac{Q_0^2}{8r^2} \sin^2 \theta \cos^2 \theta\ \mathbf{e}.} \tag{13.51}$$

The electric quadrupole radiation differs from the electric and magnetic dipole radiation both in its frequency dependence as well as in its angular distribution.

Applications in atomic and nuclear physics

Atoms and nuclei can emit or absorb electromagnet radiation. The multipole expansion is a suitable tool for the description of the electromagnetic fields. In atomic physics the dipole radiation dominates: The comparison of (13.22) and (13.51) shows that the electric dipole radiation is stronger—by about a factor of order $(kd_0)^{-2}$—than the electric quadrupole radiation. The electric dipole radiation also dominates the magnetic dipole radiation, which in line with (13.22) and (13.29) is smaller by the factor $(v/c)^2$. The relations are more complex in nuclear physics. A thorough discussion of the multipole radiation here is only possible within the framework of **quantum theory**.

In summarizing this chapter we have classified the leading order multipoles for magnetic and electric radiation emerging from accelerated electric and magnetic dipole moments as well as electric quadrupole moments with respect to their frequency and angular dependence.

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14. Systematics of the Multipole Expansion

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In this chapter we will generalize the multipole expansion discussed in the previous chapter and introduce a particular set of **orthogonal functions** on the spherical surface denoted by **spherical harmonics**. These functions are of general use also in other areas of physics.

14.1 Multipole Expansion of Static Fields

For a localized charge distribution $\rho(\mathbf{r})$ we have the potential

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\ d^3r', \tag{14.1}$$

which at a sufficiently large distance from the charges ($r \gg r'$) can be expanded in a Taylor series:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int\ \rho(\mathbf{r}')\ \left(\sum_n \frac{(-)^n}{n!} (\mathbf{r}' \cdot \nabla)^n \frac{1}{r}\right)\ d^3r' \tag{14.2}$$

$$= \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(\mathbf{r}') \left(\frac{1}{r} + \frac{(\mathbf{r}\cdot\mathbf{r}')}{r^3} + \frac{3(\mathbf{r}\cdot\mathbf{r}')^2 - r^2 r'^2}{2r^5} + \cdots\right)$$

We now rewrite the expansion (14.2) in **spherical coordinates**

$$x = r\ \sin\,\theta\ \cos\,\phi;y = r\ \sin\,\theta\ \sin\,\phi;z = r\ \cos\,\theta \tag{14.3}$$

$$x' = r'\ \sin\,\theta'\ \cos\,\phi';y' = r'\ \sin\,\theta'\ \sin\,\phi';z' = r'\ \cos\,\theta'.$$

In these coordinates (14.2) can be represented as follows:

(14.4)

$$\Phi(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{q_{lm}}{\epsilon_0(2l+1)} Y_{lm}(\theta,\phi)\ \frac{1}{r^{l+1}}$$

with the expansion coefficients

$$q_{lm} = \int d^3r'\ \rho(\mathbf{r}')r'^l\ Y^*_{lm}(\theta',\phi'). \tag{14.5}$$

In (14.4) we have used that (for $r > r'$) the function $|\mathbf{r}-\mathbf{r}'|^{-1}$ can be expanded by spherical harmonics Y_{lm}:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r} \sum_{l=0}^{\infty} \sum_{m=-l}^{l}\ \frac{4\pi}{2l+1} \left(\frac{r'}{r}\right)^l\ Y^*_{lm}(\theta',\phi')\ Y_{lm}(\theta,\phi), \tag{14.6}$$

which is equivalent to the cartesian expansion in (14.2), but explicitly only depends on the spherical coordinates $r, \theta, \phi, r', \theta', \phi'$. Another advantage of the expansion (14.6) is that the spherical harmonics Y_{lm} for different (l, m)—in a particular sense—are **orthogonal** to each other (see below). The scalar field $\Phi(\mathbf{r})$ then can also be written as:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \left(\rho(\mathbf{r}')\ \frac{1}{r} \sum_{l=0}^{\infty} \sum_{m=-l}^{l}\ \frac{4\pi}{2l+1} \left(\frac{r'}{r}\right)^l\ Y^*_{lm}(\theta',\phi')\ Y_{lm}(\theta,\phi)\right)\ d^3r' \tag{14.7}$$

$$= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{q_{lm}}{\epsilon_0(2l+1)} Y_{lm}(\theta,\phi)\ \frac{1}{r^{l+1}}$$

with the expansion coefficients q_{lm} from (14.5) (q.e.d.).

Explanations:

(1.) Since the vectors $\mathbf{r}$ and $\mathbf{r}'$ in (14.2) appear in a completely symmetric way (via the scalar product $(\mathbf{r}\cdot\mathbf{r}')$), $\Phi(\mathbf{r})$ must depend in the same manner on θ, ϕ as on θ', ϕ'.

(2.) The index $l \geq 0$ classifies the asymptotic behavior of each individual term in the Taylor series.

(3.) The index *m* numbers the components of the multipole moments for fixed *l*. For every *l* there are $2(l+1)$ values of *m*: $-l, -l+1, \ldots, l-1, l$. In the 2nd term of the expansion the 3 components of the dipole moment $\mathbf{d}$ relate to $m = -1, 0, +1$.

The explicit comparison of (14.2) and (14.4) shows that the functions Y_{lm} for low *l*, *m* have the following form:

(14.8)

$$l=0:Y_{00}=\frac{1}{\sqrt{4\pi}}$$

$$l=1:Y_{10}=\sqrt{\frac{3}{4\pi}}\cos\theta;Y_{11}=-\sqrt{\frac{3}{8\pi}}\sin\theta\ \ \exp(i\phi)$$

$$l=2:Y_{22}=\sqrt{\frac{15}{32\pi}}\sin^2\theta\ \ \exp(2i\phi);Y_{21}=-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\ \ \exp(i\phi);$$

$$Y_{20}=\sqrt{\frac{5}{4\pi}}(\frac{3}{2}\cos^2\theta-\frac{1}{2}),$$

if we determine the phase by

$$Y_{l-m}(\theta,\phi)=(-)^m Y_{lm}^*(\theta,\phi). \tag{14.9}$$

The combination occurring in (14.4)

$$Y_{lm}(\theta,\phi)\ Y_{lm}^*(\theta',\phi')+Y_{l-m}(\theta,\phi)\ Y_{l-m}^*(\theta',\phi')$$

thus is real and symmetric with respect to (θ,ϕ) and (θ',ϕ').

The lowest expansion coefficients q_{lm} are with (14.8), (14.9):

$$l=0:q_{00}=\sqrt{\frac{1}{4\pi}}Q$$

with

$$Q=\int\rho(\mathbf{r}')\ d^3r' \tag{14.10}$$

as the total charge.

$$l=1:q_{11}=-\sqrt{\frac{3}{8\pi}}\int d^3r'\ \rho(\mathbf{r}')(x'-iy')=-\sqrt{\frac{3}{8\pi}}(d_x-id_y);$$

$$q_{10}=\sqrt{\frac{3}{4\pi}}\int d^3r'\ \rho(\mathbf{r}')z'=\sqrt{\frac{3}{4\pi}}d_z,$$

where d_x, d_y, d_z are the components of the dipole moment $\mathbf{d}$.

$$l = 2 : q_{22} = \sqrt{\frac{15}{32\pi}} \int d^3r' \;\; \rho(\mathbf{r}')(x' - iy')^2 = \sqrt{\frac{15}{32\pi}}(Q_{11} - Q_{22} - 2iQ_{12});$$

$$q_{21} = -\sqrt{\frac{15}{8\pi}} \int d^3r' \;\; \rho(\mathbf{r}')z'(x' - iy') = -\sqrt{\frac{15}{8\pi}}(Q_{13} - iQ_{23});$$

$$q_{20} = \frac{3}{2}\sqrt{\frac{5}{4\pi}} \int d^3r' \;\; \rho(\mathbf{r}')(z'^2 - \frac{r'^2}{3}) = \frac{3}{2}\sqrt{\frac{5}{4\pi}}Q_{33}.$$

Furthermore, due to (14.9) we obtain:

$$q_{lm} = (-)^m q^*_{l-m}. \tag{14.11}$$

14.2 General Properties of Spherical Harmonics

To determine the spherical harmonics Y_{lm} we use the fact that outside the area of the charges the Laplace equation for $\Phi(\mathbf{r})$ (14.4) holds,

$$\Delta\Phi = 0. \tag{14.12}$$

Since the functions r^{-l-1} for different *l* values and the functions exp $(im\phi)$ for different *m* values are linearly independent, we must have according to (14.12):

$$\Delta(r^{-l-1}Y_{lm}(\theta, \phi)) = 0. \tag{14.13}$$

The Δ operator in spherical coordinates has the explicit form:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin^2\theta}\Big(\sin\theta\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{\partial^2}{\partial\phi^2}\Big). \tag{14.14}$$

If we carry out the *r*-differentiations in (14.13) we are left with

$$\Big(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + l(l+1)\Big)Y_{lm} = 0, \tag{14.15}$$

which is the determining differential equation for the Y_{lm}. Within the separation Ansatz

$$Y_{lm}(\theta,\phi) = \exp\ (im\phi)\ \ F_{lm}(\theta) \tag{14.16}$$

(14.15) reduces to

$$\boxed{\Big(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\,\theta\frac{\partial}{\partial\theta}) - \frac{m^2}{\sin^2\theta} + l(l+1)\Big)F_{lm}(\theta) = 0. \tag{14.17}}$$

Remarks:

(1.) Apart from $r^{-l-1}Y_{lm}(\theta,\phi)$ the function $r^l Y_{lm}(\theta,\phi)$ is also a solution of (14.12), i.e.

$$\Delta(r^l Y_{lm}(\theta,\phi)) = 0. \tag{14.18}$$

However, from these two linearly independent solutions in our context only $r^{-l-1}Y_{lm}$ is useful due to the boundary condition

$$\Phi(r,\theta,\phi) \to 0 \,\text{for}\ \ \text{r} \to \infty. \tag{14.19}$$

(2.) The index *m* in (14.16) must be an integer since Φ is a unique periodic function:

$$\Phi(r,\theta,\phi) = \Phi(r,\theta,\phi+2\pi). \tag{14.20}$$

In order to solve Eq. (14.17) we further introduce:

$$\xi = \cos\,\theta;\text{i. e.}\ \frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta} = -\frac{\mathrm{d}}{\mathrm{d}\xi}, \tag{14.21}$$

such that (14.17) turns to:

$$\boxed{\Big(\frac{d}{d\xi}((1-\xi^2)\frac{d}{d\xi}) - \frac{m^2}{1-\xi^2} + l(l+1)\Big)F_{lm}(\xi) = 0. \tag{14.22}}$$

For $l = m$ one immediately finds the solution (except for a normalization factor):

$$\boxed{F_{ll} = (1-\xi^2)^{l/2} = (1 - \cos^2\,\theta)^{l/2} = (\sin\,\theta)^l. \tag{14.23}}$$

The solutions to $m \neq l$ are then obtained by recursion (except for a normalization factor):

$$\boxed{F_{lm-1} = \big(-\frac{d}{d\theta} - m\ \ \text{cotg}\,\theta\big)F_{lm}. \tag{14.24}}$$

The proof (by substituting (14.24) in (14.17)) is just as elementary as lengthy (see quantum mechanics). It is worth to note about the result (14.24) that the F_{lm} are polynomials in cos θ, sin θ of order l, since the differentiation with respect to θ and the multiplication by cotg θ does not change the order of F_{ll}. For all m values with $|m| > l$ the function F_{lm} vanishes as a result of the recursion process, which justifies the finite summation over m from $-l$ to l in (14.6).

Note:

Equation (14.22) has—in addition to the solution discussed here—as a differential equation of 2nd order still a 2nd basic solution. This has singularities for $\theta = 0, \pi$ and is not suitable for our problem.

An important property of spherical harmonics is their **orthogonality**. To define this let's consider a sequence of functions $f_1(x), f_2(x), \cdots, f_n(x), \cdots$, which are continuous in the interval $[a, b]$. We then define the **dot product** of two functions f_n, f_m by:

$$(f_m, f_n) = \int_a^b f_m^*(x) f_n(x) \; dx. \tag{14.25}$$

The **norm** of f_n is introduced by:

$$(f_n, f_n) = \int_a^b |f_n(x)|^2 \; dx \geq 0. \tag{14.26}$$

Two functions are called **orthogonal** if

$$(f_m, f_n) = 0. \tag{14.27}$$

Note: The terminology above is in analogy to vectors in vector spaces of finite dimension.

For the spherical harmonics $Y_{lm}(\theta, \phi)$ now the following relation holds:

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\,\theta d\theta \;\; Y_{lm}^* Y_{l'm'} = \int_0^{2\pi} d\phi \int_{-1}^{1} d\,\cos\,\theta \;\; Y_{lm}^* Y_{l'm'} = \delta_{ll'}\delta_{mm'}, \tag{14.28}$$

where the integral for $l = l'$ and $m = m'$ specifies the normalization of Y_{lm}. Due to

$$\int_0^{2\pi} d\phi \;\; \exp\,(i(m - m')\phi) = 2\pi \,\text{for}\;\; \mathrm{m} = \mathrm{m}' \tag{14.29}$$

and $= 0$ for $m \neq m'$ the orthogonality with respect to m is immediately clear. The (normalized) functions (with respect to the ϕ-dependence) then are

$$\frac{1}{\sqrt{2\pi}} \exp\left(im\phi\right). \tag{14.30}$$

We discard here the explicit calculation of the normalization factors for the functions $F_{lm}(\theta)$ and refer the reader to the specific literature (or quantum mechanics).

We show the orthogonality with respect to l as follows: We take advantage of the fact that the solution of the real differential equation (14.22) can always be chosen to be real and consider the difference from

$$\int_{-1}^{+1} d\xi\ F_{l'm}\Big(\tfrac{d}{d\xi}(1-\xi^2)\tfrac{d}{d\xi} - \tfrac{m^2}{1-\xi^2} + l(l+1)\Big)F_{lm} = 0 \tag{14.31}$$

and

$$\int_{-1}^{+1} d\xi\ F_{lm}\Big(\tfrac{d}{d\xi}(1-\xi^2)\tfrac{d}{d\xi} - \tfrac{m^2}{1-\xi^2} + l'(l'+1)\Big)F_{l'm} = 0. \tag{14.32}$$

We get

$$[l(l+1) - l'(l'+1)] \int_{-1}^{+1} d\xi\ F_{l'm}F_{lm} = 0; \tag{14.33}$$

where the 1st term in (14.31) (or (14.32)) was transformed by 2-fold partial integration and that there are no contributions from the integrated terms due to the factor $(1-\xi^2)$. For $l \neq l'$ we get from (14.33)

$$\int_{-1}^{+1} d\xi\ F_{l'm}F_{lm} = 0, \text{q. e. d.} \tag{14.34}$$

14.3 Multipole Expansion of the Radiation Field

The multipole solutions from Chap. 13 (in the source-free space) satisfy the differential equation

$$\Delta\mathbf{A} + k^2\mathbf{A} = 0. \tag{14.35}$$

Apart from plane waves also spherical waves are solutions, which we will discuss and construct below.

With the Ansatz

$$\mathbf{A}(\mathbf{r}) = \mathbf{a}_{lm}\ f_l(r)\ Y_{lm}(\theta,\phi) \tag{14.36}$$

Equation (14.35) turns to

$$\Big(\tfrac{d^2}{dr^2} + \tfrac{2}{r}\tfrac{d}{dr} + k^2 - \tfrac{l(l+1)}{r^2}\Big)f_l(r) = 0, \tag{14.37}$$

using (14.14) and (14.15). Equation (14.37) becomes simplified when using—instead of f_l—the function

$$g_l = r f_l \tag{14.38}$$

which leads to:

$$\left(\frac{d^2}{dr^2} + k^2 - \frac{l(l+1)}{r^2}\right) g_l(r) = 0. \tag{14.39}$$

Case 1: l=0

Then the solutions $g_0(r)$ are immediately apparent: $\sin(kr)$ and $\cos(kr)$ or $\exp(\pm ikr)$.

Case 2: $l \neq 0$

In the variable

$$\rho = kr \tag{14.40}$$

Equation (14.39) reads

$$\left(\frac{d^2}{d\rho^2} + 1 - \frac{l(l+1)}{\rho^2}\right) g_l(\rho) = 0. \tag{14.41}$$

To solve (14.41) we define the operators

$$\boxed{d_l^+ = \frac{d}{d\rho} - \frac{l}{\rho}; d_l^- = \frac{d}{d\rho} + \frac{l}{\rho}; \tag{14.42}}$$

then (14.41) can be written as

$$(d_l^+ d_l^- + 1) g_l = 0, \tag{14.43}$$

using

$$\frac{d}{d\rho}\left(\frac{l}{\rho} g_l\right) = -\frac{l}{\rho^2} g_l + \frac{l}{\rho}\frac{dg_l}{d\rho}. \tag{14.44}$$

In a similar way we get:

$$(d_{l+1}^- d_{l+1}^+ + 1) g_l = 0. \tag{14.45}$$

By applying the **operation** d_{l+1}^+ on (14.45) we obtain,

$$(d_{l+1}^+ d_{l+1}^- d_{l+1}^+ + d_{l+1}^+) g_l = (d_{l+1}^+ d_{l+1}^- + 1)(d_{l+1}^+ g_l) = 0 \tag{14.46}$$

and comparing with (14.43), we get (except for a constant factor)

$$g_{l+1} = d^{+}_{l+1} g_l. \tag{14.47}$$

Equation (14.47) allows to calculate $g_l(\rho)$ by recursion from $g_0(\rho)$. Depending on the choice of the basic solution g_0 one constructs the following functions:

solution overview:

g_0	$(-)^l g_l(\rho)/\rho$	symbol
$\sin\rho$	spherical Bessel functions	$j_l(\rho)$
$-\cos\rho$	spherical Neumann functions	$n_l(\rho)$
$\exp(\pm i\rho)$	spherical Hankel functions	$h^{\pm}_l(\rho)$

For an easy calculation of the lowest order Bessel functions j_l and Neumann functions n_l we write (14.42) as follows,

$$d^{+}_l = \rho^l \tfrac{d}{d\rho} \rho^{-l}, \tag{14.48}$$

such that with (14.47)

$$g_l = \rho^l \tfrac{d}{d\rho} \rho^{-l} \cdots \rho \tfrac{d}{d\rho} \rho^{-1} g_0 \tag{14.49}$$

or

$$\tfrac{g_l}{\rho} = \rho^l \left(\tfrac{1}{\rho} \tfrac{d}{d\rho} \right)^l \left(\tfrac{g_0}{\rho} \right). \tag{14.50}$$

One finds by simple differentiation:

$$j_0 = \frac{\sin\rho}{\rho}; n_0 = -\frac{\cos\rho}{\rho};$$

$$j_1 = \frac{\sin\rho}{\rho^2} - \frac{\cos\rho}{\rho}; n_1 = -\frac{\cos\rho}{\rho^2} - \frac{\sin\rho}{\rho};$$

$$j_2 = \left(\frac{3}{\rho^3} - \frac{1}{\rho} \right) \sin\rho - \frac{3\cos\rho}{\rho^2}; n_2 = \left(-\frac{3}{\rho^3} + \frac{1}{\rho} \right) \cos\rho - \frac{3\sin\rho}{\rho^2}.$$

The procedure to construct the Hankel functions $h^{\pm}_l$ with the basic solution $g_0(\rho) = \exp(\pm i\rho)$ is analogous.

The general solution of (14.35) can now can be written as

$$\mathbf{A}(\mathbf{r}) = \sum_{l,m}\left(\mathbf{a}_{lm}h_l^+(\rho) + \mathbf{b}_{lm}h_l^-(\rho)\right)\ Y_{lm}(\theta,\phi), \tag{14.51}$$

where the angular dependence is determined by the $Y_{lm}(\theta,\phi)$ and the radial dependence (for fixed *l*) by the Hankel functions $h_l^{\pm}(kr)$. For radiation problems $\mathbf{b}_{lm}$ = 0, since h_l^- describes an **incoming** spherical wave. The remaining coefficients $\mathbf{a}_{lm}$ of the **outgoing** spherical wave $h_l^+(kr)$ are determined from the multipole moments by comparing (14.51) and (13.11) for $\rho = kr \gg 1$.

14.4 Expansion of a Plane Wave in Spherical Harmonics

The functions $h_l^{\pm}(\rho)Y_{lm}(\theta,\phi)$ form—like the plane waves $\exp\,(\pm i\mathbf{k}\cdot\mathbf{r})$—a complete basis; the general solution of the wave equation can be derived from one basis or the other by superposition. The choice of a basis depends on the specific problem (e.g. the boundary conditions).

We want to show the connection between the two basic systems by expanding a plane wave in spherical harmonics. For simplicity, we choose $\mathbf{k} = (0,0,k)$, then in

$$\exp\,(i\mathbf{k}\cdot\mathbf{r}) = \exp\,(ikz) = \exp\,(ikr\,\cos\,\theta) \tag{14.52}$$

the angle ϕ no longer shows up and the expansion has the (ϕ-independent) form:

$$\exp\,(ikz) = \sum_l a_l\ j_l(kr)\ Y_{l0}(\theta). \tag{14.53}$$

The Neumann functions do not appear because the n_l become singular for $r \to 0$ (e.g. $n_0(\rho) \to -1/\rho$ for $\rho \to 0$). The coefficients are:

$$a_l = i^l(2l+1)\sqrt{\tfrac{4\pi}{2l+1}}. \tag{14.54}$$

To construct the Hankel functions $h_l^{\pm}$ with the basic solution $g_0(\rho) = \exp\,(\pm i\rho)$ one exploits the orthogonality of the functions for the proof of (14.54): In general we can determine the coefficients of an expansion

$$g(x) = \sum_m c_m f_m(x) \tag{14.55}$$

within a (complete) orthonormal system of functions $f_m(x)$ with

$$(f_n, f_m) = \delta_{nm} \tag{14.56}$$

by

$$c_n = (f_n, g) = \int_a^b f_n^*(x) g(x)\ dx. \tag{14.57}$$

For the example above (14.53) we get:

$$\boxed{a_l j_l(kr) = \int_0^\pi \sin\theta d\theta\ Y_{l0}(\theta) \exp\left(ikr \cos\theta\right); \qquad (14.58)}$$

using an expansion for small values of *r* on the right and left sides of (14.58) the θ integration can be carried out and the result just becomes (14.54).

14.5 Benefits of the Expansion in Spherical Harmonics

Knowing the angular dependence of the potentials Φ (14.4) or $\mathbf{A}$ (14.51) for fixed *r*, one immediately can decide which multipole moments are contained in the source of the field. For example with

$$(Y_{lm}, \Phi) \sim q_{lm}, \tag{14.59}$$

due to the orthogonality of the Y_{lm}, we can **to pick out** a specific term from the expansion (14.4).

In summarizing this chapter we have generalized the multipole expansion discussed in the previous chapter and introduced a particular set of **orthogonal functions** on the spherical surface denoted by **spherical harmonics**. As applications we have presented the general multipole expansion of the radiation field and the expansion of a plane wave in spherical harmonics.

Part V
The Electromagnetic Field in Matter

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15. Macroscopic Fields

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In principle, the Maxwell equations of Part III allow to calculate the electromagnetic field for any arrangement of matter as soon as the charge density $\rho(\mathbf{r},t)$ and the current density $\mathbf{j}(\mathbf{r},t)$ are known exactly. In such a **microscopic** theory the matter—in the area of space-time under consideration—is decomposed into point charges (electrons and atomic nuclei) and their state of motion then defines the charge density $\rho(\mathbf{r},t)$ and current density $\mathbf{j}(\mathbf{r},t)$. For distributions of matter of **macroscopic** dimensions (e.g. a capacitor with a dielectric or current-carrying coil with an iron core) such a **microscopic** calculation is neither feasible in practice nor desirable, since experimentally only spatial and temporal averages of the fields can be controlled. We will therefore examine in the following space-time averages of the fields and derive the Maxwell equations for the macroscopic fields.

15.1 Macroscopic Averages

are integrals of the form

$$\boxed{< f(\mathbf{r},t) >= \frac{1}{\Delta V \Delta T} \int d^3\xi d\tau \ f(\mathbf{r}+\vec{\xi},t+\tau) \qquad (15.1)}$$

where ΔV indicates the volume, ΔT the time interval and f stands for the charge or current density or the components of the field strengths ($f=\rho,\mathbf{j},\mathbf{E},\mathbf{B}..$). In the following we want to establish the connections between the average values (15.1) for charge and current density on the one hand and the macroscopic fields on the other hand. The starting point are the **microscopic** Maxwell equations

$$\nabla \cdot \mathbf{B} = 0; \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{15.2}$$

and

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}; \nabla \times \mathbf{B} - \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j}. \tag{15.3}$$

If we assume that in (15.1) differentiations with respect to $\mathbf{r}$ and t may be carried out in the integral,

$$\frac{\partial}{\partial t} < f >=< \frac{\partial f}{\partial t} > ; \frac{\partial}{\partial x} < f >=< \frac{\partial f}{\partial x} > ; \frac{\partial}{\partial y} < f >=< \frac{\partial f}{\partial y} > ; \text{etc.}\,, \tag{15.4}$$

we get the following equations for the average values from (15.2) and (15.3):

$$\nabla \cdot < \mathbf{B} >= 0; \nabla \times < \mathbf{E} > + \frac{\partial < \mathbf{B} >}{\partial t} = 0 \tag{15.5}$$

and

$$\nabla \cdot < \mathbf{E} >= \frac{< \rho >}{\epsilon_0}; \nabla \times < \mathbf{B} > - \epsilon_0 \mu_0 \frac{\partial < \mathbf{E} >}{\partial t} = \mu_0 < \mathbf{j} >. \tag{15.6}$$

The homogeneous equations (15.2) remain the same in the transition from the **microscopic** fields $\mathbf{E}, \mathbf{B}$ to the **macroscopic** fields

$$\vec{\mathscr{E}} =< \mathbf{E} > ; \vec{\mathscr{B}} =< \mathbf{B} >. \tag{15.7}$$

In the inhomogeneous equations (15.6) we now have to suitably divide $< \rho >$ and $< \mathbf{j} >$ into free and bound charge carriers.

15.2 Free and Bound Charge Carriers

We first look at (15.6) and the connection between $\vec{\mathscr{E}}$ and its sources. To this aim we distinguish in the averaged charge density the density of the **bound** charges ρ_b and the average density of the **free** charge carriers ρ_f, i.e.

$$\boxed{< \rho >= \rho_b + \rho_f. \quad (15.8)}$$

Bound charge carriers are, for example, the lattice building blocks of an ion crystal (like NaCl with the lattice building blocks Na^+ and Cl^-) or the electrons of atoms and molecules. **Bound** does not imply that the charge carriers cannot move, but that there are strong forces that keep the charge carriers at their **equilibrium positions**; small periodic oscillations around these positions are possible (thermal fluctuations).

Freely moving charge carriers are e.g. conduction electrons in metals, ions in gases or electrolytes. They are characterized by the fact that they **form a macroscopic current under the influence of an external field**.

The density of free charges ρ_f is a macroscopic quantity that—in contrast to ρ_b—can be directly controlled in experiments. The charges on the plates of a capacitor e.g. can be specified from the **outside**. They create an electric field, which in a dielectric between the plates can generate or align electrical dipoles. The effect for an observer are **polarization charges** on the surfaces of the dielectric, which depend on special conditions (type of the dielectric, temperature of the environment, strength of the $\vec{\mathscr{E}}$ field) (Fig. 15.1).

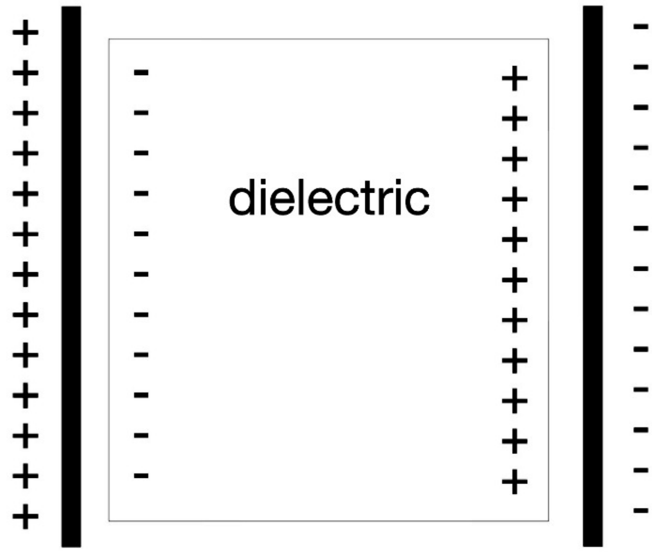

Fig. 15.1 Polarization of a dielectric in a capacitor

It is thus reasonable to combine the field from the (bound) polarization charges with the electric field generated by the free charges ρ_f on the plates. We choose the auxiliary field $\overrightarrow{\mathscr{P}}$ in such a way that:

$$\nabla \cdot \overrightarrow{\mathscr{P}} = -\rho_b \tag{15.9}$$

and

$$\overrightarrow{\mathscr{P}} = 0 \text{ where } \ \rho_{\mathrm{b}} = 0. \tag{15.10}$$

Then we obtain:

$$\nabla \cdot (\epsilon_0 \overrightarrow{\mathscr{E}} + \overrightarrow{\mathscr{P}}) = \rho_f \tag{15.11}$$

or after introduction of the **dielectric displacement** field

$$\overrightarrow{\mathscr{D}} := \epsilon_0 \overrightarrow{\mathscr{E}} + \overrightarrow{\mathscr{P}} \tag{15.12}$$

$$\nabla \cdot \overrightarrow{\mathscr{D}} = \rho_f. \tag{15.13}$$

We will show below that the **auxiliary field** $\overrightarrow{\mathscr{P}}$ is the density of the (macroscopic) dipole moment of the dielectric under consideration (**dielectric polarization**).

In analogy to (15.8) we write for the macroscopic current:

$$< \mathbf{j} >= \mathbf{j}_f + \mathbf{j}_b, \tag{15.14}$$

where $\mathbf{j}_f$ is the contribution of the free charge carriers (averaged according (15.1)). The contribution arising from bound charge carriers, i.e. (averaged) current density $\mathbf{j}_b$, is divided up again,

$$\mathbf{j}_b = \mathbf{j}_P + \mathbf{j}_M = \frac{\partial \overrightarrow{\mathscr{P}}}{\partial t} + \mathbf{j}_M. \tag{15.15}$$

Here $\mathbf{j}_P$ is the temporal change of the polarization $\overrightarrow{\mathscr{P}}$, i.e. due to the motion of the polarization charges,

$$\mathbf{j}_P = \frac{\partial \overrightarrow{\mathscr{P}}}{\partial t}. \tag{15.16}$$

The discussion of the remaining contribution $\mathbf{j}_M$, which results from molecular circular currents, i.e. **magnetic dipoles**, we postpone for later.

With (15.12), (15.14) and (15.16) the second inhomogeneous equation is written as follows:

$$\nabla \times \overrightarrow{\mathscr{B}} - \mu_0 \frac{\partial \overrightarrow{\mathscr{D}}}{\partial t} = \mu_0 \mathbf{j}_f + \mu_0 \mathbf{j}_M. \tag{15.17}$$

For the further transformation of (15.17) we employ the continuity equation for the free charge carriers:

(15.18)

$$\nabla \cdot \mathbf{j}_f + \frac{\partial \rho_f}{\partial t} = 0.$$

Then we get from (15.13) and (15.18)

$$\boxed{\nabla \cdot (\frac{\partial \overrightarrow{\mathscr{D}}}{\partial t} + \mathbf{j}_f) = 0, \qquad (15.19)}$$

such that the vector $\partial \overrightarrow{\mathscr{D}} / \partial t + \mathbf{j}_f$ can be represented as a rotation of a vector, which we denote by $\overrightarrow{\mathscr{H}}$, i.e.

$$\boxed{\nabla \times \overrightarrow{\mathscr{H}} = \frac{\partial \overrightarrow{\mathscr{D}}}{\partial t} + \mathbf{j}_f. \qquad (15.20)}$$

The connection between $\overrightarrow{\mathscr{B}}$ and $\overrightarrow{\mathscr{H}}$ according to (15.17) is:

$$\boxed{\nabla \times (\overrightarrow{\mathscr{B}} - \mu_0 \overrightarrow{\mathscr{H}}) = \mu_0 \mathbf{j}_M. \qquad (15.21)}$$

In analogy to the vector $\overrightarrow{\mathscr{P}}$ we introduce the magnetization $\overrightarrow{\mathscr{M}}$:

$$\boxed{\mu_0 \overrightarrow{\mathscr{M}} = \overrightarrow{\mathscr{B}} - \mu_0 \overrightarrow{\mathscr{H}}, \qquad (15.22)}$$

such that according to (15.9):

$$\nabla \times \overrightarrow{\mathscr{M}} = \mathbf{j}_M ; \overrightarrow{\mathscr{M}} = 0 \ \text{ where } \ \mathbf{j}_{\mathrm{M}} = 0. \qquad (15.23)$$

We will show later that $\overrightarrow{\mathscr{M}}$ is the density of the (macroscopic) magnetic dipole moment (**magnetization**).

Comments:

(1.) Only the fields $\overrightarrow{\mathscr{E}}$ and $\overrightarrow{\mathscr{B}}$ have a microscopic analogue, i.e. $\mathbf{E}$ and $\mathbf{B}$ (cf. (15.7)). $\overrightarrow{\mathscr{D}}$ and $\overrightarrow{\mathscr{H}}$ are only **auxiliary fields** that we have introduced to solve complicated electrical and magnetic properties of matter 'on average'.

(2.) A macroscopic polarization (or magnetization) can show up when existing electric (or magnetic) dipoles in the field are **aligned** or if dipoles are **induced** by an external field. Without an external field **permanent** dipoles are distributed statistically and—after averaging over a macroscopic volume—no polarization (or magnetization) results.

Summary of the macroscopic field equations:

Homogeneous equations:

$$\nabla \cdot \overrightarrow{\mathscr{B}} = 0; \nabla \times \overrightarrow{\mathscr{E}} + \frac{\partial \overrightarrow{\mathscr{B}}}{\partial t} = 0 \tag{15.24}$$

Inhomogeneous equations:

$$\nabla \cdot \overrightarrow{\mathscr{D}} = \rho_f; \nabla \times \overrightarrow{\mathscr{H}} - \frac{\partial \overrightarrow{\mathscr{D}}}{\partial t} = \mathbf{j}_f \tag{15.25}$$

Connections:

$$\overrightarrow{\mathscr{D}} = \epsilon_0 \overrightarrow{\mathscr{E}} + \overrightarrow{\mathscr{P}}; \overrightarrow{\mathscr{H}} = \frac{1}{\mu_0} \overrightarrow{\mathscr{B}} - \overrightarrow{\mathscr{M}}. \tag{15.26}$$

Equations (15.24), (15.25) have the same formal structure as (15.2), (15.3); we can therefore use the same methods for the solution.

Equations (15.2), (15.3) are, however, not yet sufficient to determine—for given $\rho_f, \mathbf{j}_f$—the 4 fields $\overrightarrow{\mathscr{E}}, \overrightarrow{\mathscr{D}}, \overrightarrow{\mathscr{B}}, \overrightarrow{\mathscr{H}}$ uniquely. To this aim we have to convert the formal connections (15.26) with the help of special models (for the matter under consideration) to explicit **material equations**. Simple examples will be discussed in the next chapters.

15.3 Polarization and Magnetization

For the interpretation of the polarization $\overrightarrow{\mathscr{P}}$ and magnetization $\overrightarrow{\mathscr{M}}$ we introduce via

$$\overrightarrow{\mathscr{B}} = \nabla \times \overrightarrow{\mathscr{A}} ; \overrightarrow{\mathscr{E}} = -\nabla\tilde{\Phi} - \frac{\partial\overrightarrow{\mathscr{A}}}{\partial t} \tag{15.27}$$

the macroscopic scalar potential $\tilde{\Phi}$ and vector potential $\overrightarrow{\mathscr{A}}$. For these potentials we obtain the inhomogeneous wave equations (in Lorentz gauge):

$$-\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\tilde{\Phi} = \frac{1}{\epsilon_0}\left(\rho_f - \nabla \cdot \overrightarrow{\mathscr{P}}\right), \tag{15.28}$$

$$-\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\overrightarrow{\mathscr{A}} = \mu_0\left(\mathbf{j}_f + \nabla \times \overrightarrow{\mathscr{M}} + \frac{\partial\overrightarrow{\mathscr{P}}}{\partial t}\right). \tag{15.29}$$

Special solutions are the retarded potentials (see Sect. 12.3)

$$\tilde{\Phi}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\left(\int d^3r' \, \frac{\rho_f(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|} - \int d^3r' \, \frac{\nabla'\cdot\overrightarrow{\mathscr{P}}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\right) \tag{15.30}$$

with the retarded time $t' = t + |\mathbf{r} - \mathbf{r}'|/c$, and

$$\begin{aligned}\overrightarrow{\mathscr{A}}(\mathbf{r},t) \quad &= \frac{\mu_0}{4\pi}\left(\int d^3r' \, \frac{\mathbf{j}_f(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|} + \int d^3r' \, \frac{\partial\overrightarrow{\mathscr{P}}(\mathbf{r}',t')/\partial t'}{|\mathbf{r}-\mathbf{r}'|}\right.\\ &\left. + \int d^3r' \, \frac{\nabla'\times\overrightarrow{\mathscr{M}}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\right).\end{aligned} \tag{15.31}$$

The term with $\overrightarrow{\mathscr{P}}$ (15.30) we reformulate by partial integration:

$$\int d^3r' \, \frac{\nabla'\cdot\overrightarrow{\mathscr{P}}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|} = \int d^3r' \, \frac{(\mathbf{r}-\mathbf{r}')\cdot\overrightarrow{\mathscr{P}}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|^3}, \tag{15.32}$$

where—for simplicity—we have neglected retardation ($t = t'$). For matter of finite extension no surface term arises from the partial integration.

> The comparison with Sect. 13.2 or Sect. 2.5 shows that $\overrightarrow{\mathscr{P}}$ has the interpretation of the density of the macroscopic electric dipole moment, as already mentioned above.

Accordingly, neglecting the retardation ($t = t'$) in the last term in (15.31):

$$\int d^3r' \, \frac{\nabla' \times \overrightarrow{\mathscr{M}}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|} = \int d^3r' \, \frac{(\mathbf{r}-\mathbf{r}') \times \overrightarrow{\mathscr{M}}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{15.33}$$

> The comparison with Sect. 6.4 or Sect. 13.3 shows that $\overrightarrow{\mathscr{M}}(\mathbf{r}, t)$ is the density of the macroscopic magnetic dipole moment.

It arises because either **permanent** magnetic dipoles are aligned in the field or **induced** by the field as in the case of electric dipoles.

The charge conservation for the bound charge carriers,

$$\nabla \cdot \mathbf{j}_b + \frac{\partial \rho_b}{\partial t} = 0, \tag{15.34}$$

follows from (15.9), (15.14) as well as (15.16) and (15.23).

In summary, we have extended the microscopic Maxwell equations to the macroscopic fields and introduced two auxiliary fields, i.e. the density of the (macroscopic) electric dipole moment $\overrightarrow{\mathscr{P}}$ (**dielectric polarization**) and the density of the (macroscopic) magnetic dipole moment $\overrightarrow{\mathscr{M}}$ (**magnetization**), that have to be specified separately by material equations.

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16. Energy, Momentum and Angular Momentum of $(\overrightarrow{\mathscr{E}}, \overrightarrow{\mathscr{B}})$

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In Chap. 9 we have introduced the energy, momentum and angular momentum of the microscopic field and applied this concept in Part IV to the radiation field in vacuum. In the following we want to discuss how the considerations in Chap. 9 can be transferred to the macroscopic field.

16.1 Energy

The starting point for the energy balance in Chap. 9 was that of the work done per unit of time of a (microscopic) field $(\mathbf{E}, \mathbf{B})$ on a system of charged mass points:

$$\frac{dW_M}{dt} = \int (\mathbf{j} \cdot \mathbf{E}) \; dV. \tag{16.1}$$

The basis of (16.1) is the Lorentz force, e.g. for a point charge q:

$$\mathbf{F} = q(\mathbf{E} + (\mathbf{v} \times \mathbf{B})), \tag{16.2}$$

whose magnetic component does not contribute to (16.1). From (16.2) we obtain with (15.1) the (average) force, which a macroscopic field $(\overrightarrow{\mathscr{E}}, \overrightarrow{\mathscr{B}})$ exerts on a point charge q moving with velocity $\mathbf{v}$:

$$\boxed{\overrightarrow{\mathscr{F}} = q\left(\overrightarrow{\mathscr{E}} + (\mathbf{v} \times \overrightarrow{\mathscr{B}})\right). \tag{16.3}}$$

The work per unit of time done by the macroscopic field on the **free** charges of the density ρ_f then is in analogy to (16.1):

$$\boxed{\frac{d\mathscr{W}_M}{dt} = \int (\mathbf{j}_f \cdot \overrightarrow{\mathscr{E}}) \; dV. \tag{16.4}}$$

The right hand side of (16.4) we can rewrite using (15.25) ($\mathbf{j}_f = \nabla \times \overrightarrow{\mathscr{H}} - \frac{\partial \overrightarrow{\mathscr{D}}}{\partial t}$) as:

$$\frac{d\mathscr{W}_M}{dt} = \int \left(\overrightarrow{\mathscr{E}} \cdot (\nabla \times \overrightarrow{\mathscr{H}}) - \overrightarrow{\mathscr{E}} \cdot \frac{\partial \overrightarrow{\mathscr{D}}}{\partial t} \right) \; dV. \tag{16.5}$$

As in Chap. 9 we can symmetrize (16.5) using the identity

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}) \tag{16.6}$$

and (15.24),

$$\nabla \times \overrightarrow{\mathscr{E}} = -\frac{\partial \overrightarrow{\mathscr{B}}}{\partial t}. \tag{16.7}$$

We get:

$$\frac{d\mathscr{W}_M}{dt} = -\int dV \left(\nabla \cdot (\overrightarrow{\mathscr{E}} \times \overrightarrow{\mathscr{H}}) + \overrightarrow{\mathscr{E}} \cdot \frac{\partial \overrightarrow{\mathscr{D}}}{\partial t} + \overrightarrow{\mathscr{H}} \cdot \frac{\partial \overrightarrow{\mathscr{B}}}{\partial t} \right). \tag{16.8}$$

The comparison with (9.7) shows that

$$\overrightarrow{\mathscr{S}} = \overrightarrow{\mathscr{E}} \times \overrightarrow{\mathscr{H}} \tag{16.9}$$

is the **energy current density of the macroscopic field** (Poynting vector). To interpret the remaining terms we consider the

approximation of **linear, isotropic media**:

$$\overrightarrow{\mathscr{D}} = \epsilon \overrightarrow{\mathscr{E}}; \qquad \overrightarrow{\mathscr{B}} = \mu \overrightarrow{\mathscr{H}}. \tag{16.10}$$

Then we get

$$\overrightarrow{\mathscr{E}} \cdot \frac{\partial \overrightarrow{\mathscr{D}}}{\partial t} + \overrightarrow{\mathscr{H}} \cdot \frac{\partial \overrightarrow{\mathscr{B}}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} (\overrightarrow{\mathscr{E}} \cdot \overrightarrow{\mathscr{D}} + \overrightarrow{\mathscr{H}} \cdot \overrightarrow{\mathscr{B}}) \tag{16.11}$$

and in analogy to (9.10) can interpret

$$\frac{1}{2} (\overrightarrow{\mathscr{E}} \cdot \overrightarrow{\mathscr{D}} + \overrightarrow{\mathscr{H}} \cdot \overrightarrow{\mathscr{B}}) \tag{16.12}$$

as the **energy density of the macroscopic field** (in the case of linear, isotropic media).

16.2 Momentum, Angular Momentum

According to (16.3)

$$\frac{d\mathbf{p}}{dt} = q \left(\overrightarrow{\mathscr{E}} + (\mathbf{v} \times \overrightarrow{\mathscr{B}}) \right) \tag{16.13}$$

is the change of the momentum of a test charge q with velocity $\mathbf{v}$ in the field $(\overrightarrow{\mathscr{E}}, \overrightarrow{\mathscr{B}})$. For the change in momentum of a system of free charges, described by $\rho_f, \mathbf{j}_f$, in the field $(\overrightarrow{\mathscr{E}}, \overrightarrow{\mathscr{B}})$ follows:

$$\frac{d\mathbf{P}_M}{dt} = \int dV \, \left(\rho_f \overrightarrow{\mathscr{E}} + (\mathbf{j}_f \times \overrightarrow{\mathscr{B}}) \right). \tag{16.14}$$

In analogy to chapter 9.2 we rewrite (16.14) with

$$\nabla \cdot \overrightarrow{\mathscr{D}} = \rho_f; \qquad \nabla \times \overrightarrow{\mathscr{H}} - \frac{\partial \overrightarrow{\mathscr{D}}}{\partial t} = \mathbf{j}_f \tag{16.15}$$

to get

$$\frac{d\mathbf{P}_M}{dt} = \int dV \, \left(\overrightarrow{\mathscr{E}}(\nabla \cdot \overrightarrow{\mathscr{D}}) + (\nabla \times \overrightarrow{\mathscr{H}}) \times \overrightarrow{\mathscr{B}} - \frac{\partial \overrightarrow{\mathscr{D}}}{\partial t} \times \overrightarrow{\mathscr{B}} \right). \tag{16.16}$$

We symmetrize (16.16) using

$$\nabla \cdot \overrightarrow{\mathscr{B}} = 0; \qquad \nabla \times \overrightarrow{\mathscr{E}} = -\frac{\partial \overrightarrow{\mathscr{B}}}{\partial t}, \tag{16.17}$$

$$\frac{d\mathbf{P}_M}{dt} = \int dV \, \left(\overrightarrow{\mathscr{E}}(\nabla \cdot \overrightarrow{\mathscr{D}}) + \overrightarrow{\mathscr{H}}(\nabla \cdot \overrightarrow{\mathscr{B}}) + (\nabla \times \overrightarrow{\mathscr{H}}) \times \overrightarrow{\mathscr{B}} + (\nabla \times \overrightarrow{\mathscr{E}}) \times \overrightarrow{\mathscr{D}} - \frac{\partial}{\partial t}(\overrightarrow{\mathscr{D}} \times \overrightarrow{\mathscr{B}}) \right). \tag{16.18}$$

As in Chap. 9 we then can interpret

$$\overrightarrow{\mathscr{D}} \times \overrightarrow{\mathscr{B}} \tag{16.19}$$

as the **momentum density of the macroscopic electromagnetic field** (cf. (9.38)).

The transfer from (9.39) to the case of the macroscopic field then is trivial, i.e.

$$\mathbf{r} \times \overrightarrow{\mathscr{D}} \times \overrightarrow{\mathscr{B}} \tag{16.20}$$

can be interpreted as the **angular momentum density of the macroscopic electromagnetic field**.

16.3 Kirchhoff's Rules

The theory of electrical circuits is based on the following rules:

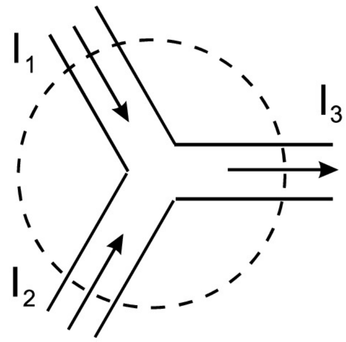

Fig. 16.1 Illustration of the knot rule

(1.) Kirchhoff's law (**knot rule**)
For a current splitting of stationary and quasi-stationary currents holds (Fig. 16.1):

$$\sum_{i=1}^{N} I_i = 0. \tag{16.21}$$

Proof
For stationary and quasi-stationary currents the continuity equation gives

$$\nabla \cdot \mathbf{j}_f = 0, \tag{16.22}$$

with the Gauss' integral theorem:

$$\int_F \mathbf{j}_f \cdot d\mathbf{f} = \sum_{i=1}^{N} I_i = 0. \tag{16.23}$$

Remark: The quasi-stationary case is defined such that in (15.19) $\partial\overrightarrow{\mathscr{D}}/\partial t$ can be neglected, which directly gives (16.22). In the stationary case $\partial\rho_f/\partial t = 0$ and (16.22) follows from (15.18). The basis of the first Kirchhoff's rule is the **conservation of charge**.
(2.) Kirchhoff's (**circuit rule**)
The sum of the voltages along a closed path in a circuit disappears,

$$\sum_j U_j = 0. \tag{16.24}$$

For U_j in (16.24) we can have (see Fig. 16.2)
(i) ohmic voltage (**resistance** R)

$$U_R = IR, \tag{16.25}$$

(ii) capacitor voltage (**capacity** C)

$$U_C = \frac{1}{C}\int I dt, \tag{16.26}$$

(iii) induced voltage (**inductance** L)

$$U_L = L\frac{dI}{dt} \tag{16.27}$$

as well as an external (battery) **voltage** U_B.

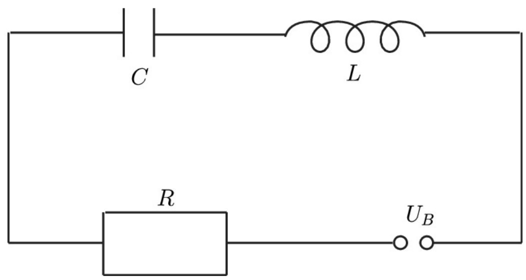

Fig. 16.2 Illustration of a resonant circuit with capacitor C, inductance L, ohmic resistance R and battery voltage U_B

Proof From

$$\nabla \times \overrightarrow{\mathscr{E}} = -\frac{\partial \overrightarrow{\mathscr{B}}}{\partial t} \tag{16.28}$$

we get with Stoke's integral theorem

$$\int_F (\nabla \times \overrightarrow{\mathscr{E}}) \cdot d\mathbf{f} = \oint_S \overrightarrow{\mathscr{E}} \cdot d\mathbf{s} = -\frac{\partial}{\partial t} \int_F \overrightarrow{\mathscr{B}} \cdot d\mathbf{f}. \tag{16.29}$$

The basis of Kirchhoff's 2nd rule is the law of induction or the **energy theorem**. E.g. carrying a charge q on a closed path through the circuit, then (16.24)—after multiplication by q—gives the energy balance.

Resonant circuit: For a circuit with a capacitor C, an inductance L and an ohmic resistance R we get according to (16.25),

$$\frac{1}{C} \int^t I dt' + IR + L\frac{dI}{dt} = 0. \tag{16.30}$$

After a further time differentiation (and division by L) this gives

$$\left(\frac{d^2}{dt^2} + \frac{R}{L}\frac{d}{dt} + \frac{1}{LC} \right) I(t) = 0, \tag{16.31}$$

which is the differential equation for a damped harmonic oscillator - with known solutions incorporating $\omega^2 = 1/(LC)$ and $\gamma = R/L$.

Note: The inductance L creates—except for superconducting materials and correspondingly low temperatures—a finite resistance R such that ordinary resonant circuits have an oscillation of finite lifetime $\tau = 1/\gamma$.

In summary, we have computed the energy, momentum and angular momentum of the macroscopic field and discussed Kirchhoff's rules in the context with charge and energy conservation.

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17. Electric and Magnetic Properties of Matter

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The macroscopic Maxwell equations

$$\nabla \cdot \overrightarrow{\mathscr{B}} = 0; \nabla \times \overrightarrow{\mathscr{E}} + \frac{\partial \overrightarrow{\mathscr{B}}}{\partial t} = 0 \tag{17.1}$$

and

$$\nabla \cdot \overrightarrow{\mathscr{D}} = \rho_f; \nabla \times \overrightarrow{\mathscr{H}} - \frac{\partial \overrightarrow{\mathscr{D}}}{\partial t} = \mathbf{j}_f \tag{17.2}$$

are not sufficient (as already mentioned in Sect. 16.2) to determine the fields $\overrightarrow{\mathscr{E}}, \overrightarrow{\mathscr{B}}, \overrightarrow{\mathscr{D}}$ and $\overrightarrow{\mathscr{H}}$ as long as there are no explicit **material equations**, which connect $\overrightarrow{\mathscr{E}}, \overrightarrow{\mathscr{B}}, \overrightarrow{\mathscr{D}}$ and $\overrightarrow{\mathscr{H}}$ to each other. The macroscopic sources ρ_f and $\mathbf{j}_f$ are often also not given as a function of $\mathbf{r}$ and t, but have to be calculated functionally from the fields. In this chapter we will present some simple cases for such material equations using linear response theory.

17.1 Material Equations

For the discussion of material equations we rewrite (17.2) such that the fields $\overrightarrow{\mathscr{E}}$ and $\overrightarrow{\mathscr{B}}$ are represented in their dependence on the macroscopic sources $\rho_f, \mathbf{j}_f, \overrightarrow{\mathscr{P}}$ and $\overrightarrow{\mathscr{M}}$:

$$\boxed{\begin{aligned} \nabla \cdot \overrightarrow{\mathscr{E}} &= \frac{1}{\epsilon_0}(\rho_f - \nabla \cdot \overrightarrow{\mathscr{P}}); \\ \nabla \times \overrightarrow{\mathscr{B}} - \epsilon_0\mu_0 \frac{\partial \overrightarrow{\mathscr{E}}}{\partial t} &= \mu_0(\mathbf{j}_f + \frac{\partial \overrightarrow{\mathscr{P}}}{\partial t} + \nabla \times \overrightarrow{\mathscr{M}}), \end{aligned} \quad (17.3)}$$

where $\rho_f, \mathbf{j}_f, \overrightarrow{\mathscr{P}}$ and $\overrightarrow{\mathscr{M}}$ are functionals of the fields $\overrightarrow{\mathscr{E}}$ and $\overrightarrow{\mathscr{B}}$; they can also dependent on **external** parameters such as the temperature T (see thermodynamics), i.e.:

$$\overrightarrow{\mathscr{P}} = \overrightarrow{\mathscr{P}}[\overrightarrow{\mathscr{E}}, \overrightarrow{\mathscr{B}}, T], \overrightarrow{\mathscr{M}} = \overrightarrow{\mathscr{M}}[\overrightarrow{\mathscr{E}}, \overrightarrow{\mathscr{B}}, T], \mathbf{j}_f = \mathbf{j}_f[\overrightarrow{\mathscr{E}}, \overrightarrow{\mathscr{B}}, T]. \tag{17.4}$$

Then ρ_f is also determined by $\mathbf{j}_f$ via the continuity equation:

$$\boxed{\nabla \cdot \mathbf{j}_f + \frac{\partial \rho_f}{\partial t} = 0. \qquad (17.5)}$$

In the following we will discuss simple models of material equations of the type (17.4). We will be guided by concepts of **linear response theory**, which are based mathematically on the **convolution theorem**.

We recall that in case of inhomogeneous differential equations of 2nd order, first of all a Green's function $G_0(x-x')$, i.e. the inverse of the differential operator D_0 of the homogeneous equation, is defined by:

$$D_0(x)G_0(x-x') = \delta^n(x-x'), \qquad (17.6)$$

and a special solution to the inhomogeneous equation (with inhomogeneity $f(x)$) then is given by

$$\Phi(x) = \int d^n x' \; G_0(x-x') \; f(x'), \qquad (17.7)$$

where n denotes the dimension of the problem. We already have exploited this procedure in Sect. 12.2 in connection with retarded potentials.

In the following we limit ourselves (for the sake of simplicity) to a single dimension, i.e. explicitly to the time variable t:

$$\Phi(t) = \int dt' \; G_0(t-t') \; f(t'), \qquad (17.8)$$

where it should be noted that $G_0(t-t') = 0$ for $t' > t$. Equation (17.8) is a folding integral of the type

$$a(t) = \int_{-\infty}^{\infty} b(t-t') \; c(t') \; dt'. \qquad (17.9)$$

After a Fourier transformation of the quantities $a(t)$, $b(t)$ and $c(t)$ we obtain the simple algebraic relation for the Fourier transforms $a(\omega)$, $b(\omega)$ and $c(\omega)$:

$$a(\omega) = \int_{-\infty}^{\infty} dt \;\; \exp\,(i\omega t) \;\; a(t) = \int dt \;\; \exp\,(i\omega t) \int dt' \;\; b(t-t')c(t') \qquad (17.10)$$

$$= \int dt \;\; \exp\,(i\omega t) \int dt' \; \int \frac{d\omega'}{2\pi} \;\; \exp\,(-i\omega'(t-t')) \; \int \frac{d\omega''}{2\pi} \;\; \exp\,(-i\omega'' t') \;\; b(\omega')c(\omega'')$$

$$= \int dt \;\; \exp\,(i\omega t) \; \int \frac{d\omega'}{2\pi} \;\; \exp\,(-i\omega' t) \; \int d\omega'' \;\; \delta(\omega''-\omega') \;\; b(\omega')c(\omega'')$$

$$= \int dt \;\; \exp\,(i\omega t) \; \int \frac{d\omega'}{2\pi} \;\; \exp\,(-i\omega' t) \;\; b(\omega')c(\omega')$$

$$= \int d\omega' \ \delta(\omega - \omega') \ b(\omega')c(\omega') = b(\omega)c(\omega).$$

The connection between (17.9) and (17.10) is denoted by **convolution theorem**. This folding theorem is used in **linear response theory** because for 'small perturbations' $f(t)$ in the system—described by the differential operator D_0—the solution is given explicitly in Fourier representation in the form (17.10). The Fourier transform $b(\omega)$ is also referred to as **response function**.

Example: To illustrate the role of the response function $b(\omega)$ we consider the simple example of the damped oscillator. The equation of motion for a damped harmonic oscillator under the influence of an external force $f(t)$ is:

$$M(\tfrac{d^2}{dt^2} q(t) + \omega_0^2 q(t)) + \gamma \ \tfrac{d}{dt} \ q(t) = f(t). \tag{17.11}$$

By Fourier transformation of $q(t)$ and $f(t)$ this leads to

$$\{M(\omega_0^2 - \omega^2) - i\gamma\omega\} q(\omega) = f(\omega), \tag{17.12}$$

or

$$q(\omega) = \tfrac{1}{M(\omega_0^2 - \omega^2) - i \ \gamma\omega} f(\omega) = b(\omega) f(\omega). \tag{17.13}$$

The response function $b(\omega)$ is thus given by

$$\boxed{b(\omega) = \tfrac{1}{M(\omega_0^2 - \omega^2) - i \ \gamma\omega} = -\tfrac{1}{M} \left(\tfrac{1}{\omega^2 - \omega_0^2 + i \ \gamma / M\omega} \right).} \tag{17.14}$$

As seen immediately the poles of $b(\omega)$ are in the lower complex half plane for $\gamma > 0$. Analogous examples will show up frequently (see below).

17.2 Ohm's Law; Electric Conductivity

In metals the conductivity is due to the existence of **free** electrons in the conduction band. The equation of motion for such a conduction electron *i* is:

$$M \tfrac{d\mathbf{v}_i}{dt} + \xi \mathbf{v}_i = e\mathbf{E}_i, \tag{17.15}$$

where $\mathbf{E}_i$ is the electric field acting on the electron *i* and the friction term ($\sim \mathbf{v}_i$) accounts for the net effect, that the conduction electrons lose energy by collisions with the ions from the grid.

From (17.15) we get with (5.4), (5.5) the equation for the current density (after division by *M*):

(17.16)

$$\frac{d\mathbf{j}_f}{dt} + \frac{\xi}{M}\mathbf{j}_f = n\frac{e^2}{M}\overrightarrow{\mathscr{E}},$$

where we have identified the average over $\mathbf{E}_i$ with the macroscopic field $\overrightarrow{\mathscr{E}}$. In Eq. (17.16) n denotes the density of conduction electrons, i.e. $\mathbf{j}_f = e < \mathbf{v} > n$. For static $\overrightarrow{\mathscr{E}}$ fields (17.16) has the stationary solution ($d\mathbf{j}_f/dt = 0$):

$$\mathbf{j}_f = \frac{ne^2}{\xi}\overrightarrow{\mathscr{E}} = \sigma_0\overrightarrow{\mathscr{E}} \tag{17.17}$$

with the

DC conductivity

$$\sigma_0 = \frac{ne^2}{\xi}. \tag{17.18}$$

For a time-periodic field

$$\overrightarrow{\mathscr{E}}(t) = \overrightarrow{\mathscr{E}_0}\,\exp\,(-i\omega t) \tag{17.19}$$

we expect (after Fourier transformation) a solution of (17.16) as:

$$\mathbf{j}_f(t) = \mathbf{j}_0\,\exp\,(-i\omega t). \tag{17.20}$$

Equation (17.16) then gives the relation

$$\mathbf{j}_f(\omega) = \sigma(\omega)\overrightarrow{\mathscr{E}}(\omega) \tag{17.21}$$

with the **frequency-dependent conductivity** (response function)

$$\sigma(\omega) = \frac{\sigma_0}{1-i\omega\tau} = \frac{\sigma_0(1+i\omega\tau)}{1+\omega^2\tau^2} = \frac{\sigma_0}{1+\omega^2\tau^2} + i\frac{\sigma_0\omega\tau}{1+\omega^2\tau^2}; \tag{17.22}$$

where the damping constant τ is determined by

$$\tau = \frac{M}{\xi}. \tag{17.23}$$

For low frequencies ($\omega\tau \ll 1$) the conductivity $\sigma(\omega)$ becomes real, $\sigma(\omega) \approx \sigma_0$, while vice versa for high frequencies ($\omega\tau \gg 1$) $\sigma(\omega)$ becomes purely imaginary such that $\mathbf{j}_f$ and

$\vec{\mathscr{E}}$ are out of phase by $\pi/2$. By measuring $\sigma(\omega)$ one can determine the density of the conduction electrons n as well as the decay time τ.

17.3 Dielectrics

A polarization $\vec{\mathscr{P}}$ arises under the influence of an electric field $\vec{\mathscr{E}}$ in a non-conductive, polarizable medium. We differentiate between 2 types:

1. Polarization by orientation

Already existing (**permanent**) electric dipoles are aligned in the $\vec{\mathscr{E}}$ field. The organizing influence of the field counteracts the thermal motion of the electric dipoles and the resulting macroscopic polarization is temperature dependent. For $\vec{\mathscr{E}} = 0$ the directions of the elementary dipoles are statistically distributed and $\vec{\mathscr{P}} = 0$. For finite $\vec{\mathscr{E}}$ fields the temperature-dependent polarization will be calculated explicitly in Sect. 17.5.

2. Induced polarization

In atoms or molecules the $\vec{\mathscr{E}}$ field shifts electrons and nuclei relative to each other and thus generates (**induces**) electric dipoles in the field direction (**induced polarization**). In this way, a temperature-independent polarization is created.

At low field strengths and/or high temperatures the linear relationship

$$\vec{\mathscr{P}} = \chi_e \epsilon_0 \vec{\mathscr{E}} \tag{17.24}$$

(linear response) is a good approximation of (17.4); χ_e is the **electric susceptibility** (response function) and in general temperature dependent. With (17.24) the following holds:

$$\vec{\mathscr{D}} = \epsilon \vec{\mathscr{E}} \tag{17.25}$$

with the **dielectric constant**

$$\epsilon = \epsilon_0(1 + \chi_e). \tag{17.26}$$

Remark:

Equations (17.24) and (17.25) assume an isotropic material. For anisotropic media χ_e and ϵ have to replaced by tensors $[\chi_e]_{ij}$ and ϵ_{ij}.

For rapidly oscillating fields ϵ (or the response function χ_e) turns out to be frequency-dependent,

$$\epsilon = \epsilon(\omega). \tag{17.27}$$

Example: For H_2O (at $T = 20\,^\circ C$) $\epsilon/\epsilon_0 \approx 40$ if one chooses ω as the frequency of the yellow *Na* line.

The frequency dependence of $\epsilon(\omega)$ can be explained using the following (very simplified) model for the structure of atoms and molecules: We assume that the electrons in an atom or molecule perform harmonic oscillations (see above). Then the equation of motion for the n-th electron of an atom (or molecule) under the influence of a periodic $\overrightarrow{\mathscr{E}}$ field is:

$$\frac{d^2}{dt^2}\mathbf{r}_n(t) + \gamma_n \frac{d}{dt}\mathbf{r}_n(t) + \omega_n^2 \mathbf{r}_n(t) = \frac{e}{M}\overrightarrow{\mathscr{E}_0}\exp\left(-i\omega t\right). \tag{17.28}$$

With the Ansatz

$$\mathbf{r}_n(t) = \mathbf{r}_n^0 \exp\left(-i\omega t\right) \tag{17.29}$$

(or Fourier transform) we find as a solution to (17.28)

$$\mathbf{r}_n(t) = \frac{e}{M(\omega_n^2 - \omega^2 - i\omega\gamma_n)}\ \overrightarrow{\mathscr{E}_0}\exp\left(-i\omega t\right) \tag{17.30}$$

and from this for the dipole moment

$$\mathbf{d} = \sum_{n=1}^{Z} e\mathbf{r}_n = \frac{e^2}{M}\overrightarrow{\mathscr{E}_0}\exp\left(-i\omega t\right)\sum_{n=1}^{Z}\frac{1}{(\omega_n^2 - \omega^2 - i\omega\gamma_n)}, \tag{17.31}$$

if Z electrons are in the atom (molecule). Then we obtain for the electric susceptibility (with $\overrightarrow{\mathscr{P}} = \chi_e \epsilon_0 \overrightarrow{\mathscr{E}}$, $\overrightarrow{\mathscr{P}} = N\mathbf{d}$):

$$\chi_e(\omega) = \frac{Ne^2}{\epsilon_0 M}\sum_{n=1}^{Z}\frac{1}{(\omega_n^2 - \omega^2 - i\omega\gamma_n)}, \tag{17.32}$$

where N is the number of atoms (molecules) per unit volume.

The damping term in (17.28) accounts for the fact that the atomic oscillators lose energy by collisions between the atoms or molecules. This implies that χ_e or ϵ becomes complex. As an example we will consider the absorption of electromagnetic waves in matter in the next chapter, where the imaginary part of ϵ is made explicit.

17.4 Para- and Diamagnetism

A magnetization $\overrightarrow{\mathscr{M}}$ can (in analogy to the case of polarization $\overrightarrow{\mathscr{P}}$) arise in the following way:

1. Magnetization by orientation (paramagnetism)

Permanent elementary magnetic dipoles are aligned against the thermal motion in an external magnetic field and lead to a macroscopic magnetization $\overrightarrow{\mathscr{M}}$. Without a magnetic field, the elementary dipole moments $\mathbf{m}$ are statistically distributed with respect to their direction and in the macroscopic average $\vec{\mathscr{M}} = 0$.

2. Induced magnetization (diamagnetism)
In the magnetic field the orbits of the electrons are changed, especially their angular momentum. Such a change in angular momentum is, according to (6.36), associated with a change in the magnetic dipole moment of the atom. Thus atoms, that don't have a permanent magnetic dipole moment, get—in the external magnetic field—an **induced** dipole moment. Its direction is given by the direction of the external field (see below).

The magnetization $\vec{\mathscr{M}}$ generally depends on the external field and the temperature. Since the fundamental field is the $\vec{\mathscr{B}}$ field, we should consider according to (17.4)

$$\vec{\mathscr{M}} = \vec{\mathscr{M}}[\vec{\mathscr{B}}, T]. \tag{17.33}$$

However, it is common to replace (17.33) by

$$\vec{\mathscr{M}} = \vec{\mathscr{M}}[\vec{\mathscr{H}}, T] \tag{17.34}$$

since $\vec{\mathscr{H}}$ is practically easier to control—via the current density $\mathbf{j}_f$—than $\vec{\mathscr{B}}$.

Note:
In the electrical case one considers—in accordance with the microscopic theory—

$$\vec{\mathscr{P}} = \vec{\mathscr{P}}[\vec{\mathscr{E}}, T], \tag{17.35}$$

since potential differences can be controlled more conveniently than the free charge density ρ_f and the associated $\vec{\mathscr{D}}$ field.

For sufficiently weak fields and/or high temperatures one expects

$$\vec{\mathscr{M}} = \chi_m \vec{\mathscr{H}} \tag{17.36}$$

to be a good approximation of (17.34); here χ_m is the **magnetic susceptibility**. With (17.36) and (15.22) we get:

$$\vec{\mathscr{B}} = \mu \vec{\mathscr{H}} = \mu_0 \vec{\mathscr{H}} + \mu_0 \vec{\mathscr{M}}, \tag{17.37}$$

where the **permeability** μ is related to χ_m by

$$\boxed{\mu = \mu_0 (1 + \chi_m).} \tag{17.38}$$

Finally, we want to study the diamagnetism more quantitatively. In a simple model we consider an electron, that is elastically bound to the atomic nucleus, under the influence of an external magnetic field:

$$\frac{d^2}{dt^2}\mathbf{r}_n + \omega_n^2 \mathbf{r}_n = \frac{e}{M}\mathbf{v}_n \times \mathbf{B} = 2\overrightarrow{\omega}_L \times \frac{d}{dt}\mathbf{r}_n \tag{17.39}$$

with the abbreviation

$$\overrightarrow{\omega}_L = -\frac{e}{2M}\mathbf{B}. \tag{17.40}$$

For the solution of the equation of motion (17.39) we go over to a system Σ' that is rotating with the angular velocity $\overrightarrow{\omega}_L$ relative to the laboratory system Σ. Due to

$$\mathbf{r} = \mathbf{r}'; \frac{d}{dt}\mathbf{r} = \frac{d}{dt}\mathbf{r}' + (\overrightarrow{\omega}_L \times \mathbf{r}'); \frac{d^2}{dt^2}\mathbf{r} = \frac{d^2}{dt^2}\mathbf{r}' + 2(\overrightarrow{\omega}_L \times \frac{d}{dt}\mathbf{r}') + \overrightarrow{\omega}_L \times (\overrightarrow{\omega}_L \times \mathbf{r}') \tag{17.41}$$

we get from (17.39):

$$\frac{d^2}{dt^2}\mathbf{r}' + \omega_n^2 \mathbf{r}'_n = \overrightarrow{\omega}_L \times (\overrightarrow{\omega}_L \times \mathbf{r}'_n). \tag{17.42}$$

Since in general $\omega_n \gg \omega_L$, we get approximately

$$\frac{d^2}{dt^2}\mathbf{r}' + \omega_n^2 \mathbf{r}'_n \approx 0; \tag{17.43}$$

the electron (almost) does not **see** the magnetic field in the rotating coordinate system Σ'; it oscillates with the undisturbed frequency ω_n. From the perspective of the laboratory system Σ, a rotation about the direction of $\mathbf{B}$ with the **Lamor frequency**ω_L has to added.

According to this decomposition of the motion we divide the magnetic moment of an atom with Z electrons in 2 parts:

$$\boxed{\begin{aligned} \mathbf{m} &= \frac{e}{2M}\sum_{i=1}^{Z}(\mathbf{r}'_i \times \mathbf{p}'_i) + \frac{e}{2}\sum_{i=1}^{Z}\mathbf{r}'_i \times (\overrightarrow{\omega}_L \times \mathbf{r}'_i) \\ &= \frac{e}{2M}\sum_{i=1}^{Z}\mathbf{l}'_i - \frac{e^2}{4M}\sum_{i=1}^{Z}\Big(\mathbf{B}r'^2_i - \mathbf{r}'_i(\mathbf{B}\cdot\mathbf{r}'_i)\Big). \end{aligned} \quad (17.44)}$$

The 1st term in (17.44) provides the permanent magnetic dipole moment; it is different from zero if the angular momentum (here: orbital angular momentum) of the undisturbed atom $L' = \sum_i \mathbf{l}'_i \neq 0$. The 2. term describes the induced moment $\mathbf{m}_{ind}$.

In order to discuss the induced moment $\mathbf{m}_{ind}$ we consider a spherically symmetric charge distribution. Then the mixed terms, e.g. $\sum_i x'_i y'_i$, in (17.44) give no contribution; the contribution of the remaining quadratic terms can be expressed by the average radius ρ of the atom. One gets:

$$\boxed{\mathbf{m}_{ind} = -\frac{e^2}{4M}\mathbf{B}\sum_{i=1}^{Z} r'^2_i = -\frac{e^2}{6M}\rho^2\mathbf{B}. \quad (17.45)}$$

The diamagnetism ($\chi_m < 0$) obviously counteracts the paramagnetism ($\chi_m \geq 0$). Atoms with L' = 0 are diamagnetic; if a substance turns out as paramagnetic, then the 1st term in (17.44) dominates the 2nd term.

Note:

Atomic nuclei can also have angular momentum. Due to (6.36) the associated moment of the nucleus is considerably smaller than that generated by the electrons, since the nucleon mass is $\approx$ 2000 times larger than the electron mass.

17.5 Temperature Dependence of the Polarization

We assume a homogeneous and isotropic medium; then we have for the polarization of **permanent dipoles** by orientation in the $\overrightarrow{\mathscr{E}}$ field:

$$\overrightarrow{\mathscr{P}} = N < \mathbf{p} >_T, \tag{17.46}$$

where N is the number of atoms per unit volume and $< \mathbf{p} >_T$ is determined by averaging the dipoles of magnitude p with respect to their direction to the $\overrightarrow{\mathscr{E}}$ field at a given temperature T,

$$\boxed{< \mathbf{p} >_T = \frac{\int d(\cos\theta)\cos\theta\exp(\xi\cos\theta)}{\int d(\cos\theta)\exp(\xi\cos\theta)} \; p} \tag{17.47}$$

with

$$\boxed{\xi = \frac{p\mathscr{E}}{kT}} \tag{17.48}$$

and θ as the angle between the dipole $\mathbf{p}$ and the field $\overrightarrow{\mathscr{E}}$. The weight factor

$$\boxed{\exp\,(\xi\,\cos\,\theta) = \exp\,(\frac{\mathbf{p}\cdot\overrightarrow{\mathscr{E}}}{k_BT}) = \exp\,(-\frac{H_{int}}{k_BT})} \tag{17.49}$$

is taken from thermodynamics; it is proportional to the probability that an elementary dipole forms the angle θ with the field $\overrightarrow{\mathscr{E}}$ at given temperature T. The denominator in (17.47) ensures the correct normalization (see thermodynamics); k_B is the Boltzmann constant.

The evaluation of the integrals in (17.47) results in:

$$\mathscr{P} = Np\frac{\int_{-1}^{1} dz\ \ z\exp(\xi z)}{\int_{-1}^{1} dz\exp(\xi z)} = Np\frac{\exp(\xi)+\exp(-\xi)-1/\xi(\exp(\xi)-\exp(-\xi))}{(\exp(\xi)-\exp(-\xi))} \tag{17.50}$$

$$= Np\ \{\coth\xi - \frac{1}{\xi}\}.$$

Discussion:

Case 1: strong fields, low temperatures, i.e.

$$\xi \gg 1, \tag{17.51}$$

such that—as expected—the **saturation value**

$$\mathscr{P} = Np\frac{\exp(\xi)+\exp(-\xi)-1/\xi(\exp(\xi)-\exp(-\xi))}{(\exp(\xi)-\exp(-\xi))} \approx Np \tag{17.52}$$

is reached.

Case 2: weak fields, high temperatures, i.e.

$$\xi \ll 1, \tag{17.53}$$

such that with

$$\coth\xi = \frac{\cosh\xi}{\sinh\xi} \approx \frac{1+1/2\xi^2}{\xi(1+1/6\xi^2)} \approx \frac{1+1/2\xi^2-1/6\xi^2}{\xi} = \frac{1}{\xi} + \frac{\xi}{3} \tag{17.54}$$

we get:

$$\boxed{\mathscr{P} = \frac{Np\xi}{3} = \frac{Np^2}{3k_BT}\mathscr{E} = \chi_e\epsilon_0\mathscr{E},} \tag{17.55}$$

i.e. the **linear response region** discussed in Eq. (17.24) with the temperature-dependent electric susceptibility

$$\boxed{\chi_e = \frac{Np^2}{3k_BT\epsilon_0}.} \tag{17.56}$$

Remark:

The procedure above can be directly applied to the treatment of the paramagnetism by replacing the energy of the dipole $\mathbf{p}$ in the $\overrightarrow{\mathscr{E}}$ field: $H_{int} = -\mathbf{p}\cdot\overrightarrow{\mathscr{E}}$, by $H_{int} = -\mathbf{m}\cdot\overrightarrow{\mathscr{B}}$, where $\mathbf{m}$ is the (permanent) magnetic dipole moment.

In summary, we have discussed simple models for material equations in linear response theory and derived explicit formulae for the conductivity and the temperature dependence of electric and magnetic (permanent) dipoles in an external electric or magnetic field.

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18. The Electromagnetic Field at Interfaces

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In this chapter we will analyze the properties of the macroscopic fields at interfaces between the vacuum and dielectrics or conducting materials for linear, isotropic media and in particular study the reflection and refraction of light. Furthermore, we will investigate the propagation of electromagnetic waves in conductive materials.

18.1 General Continuity Conditions

The macroscopic Maxwell equations result in a number of consequences for the behavior of the fields at the interface between two media with different electrical and magnetic properties. For the sake of simplicity it is assumed below that the interface is planar.

We get

1. Normal components of $\overrightarrow{\mathscr{B}}$ and $\overrightarrow{\mathscr{D}}$ from

$$\nabla \cdot \overrightarrow{\mathscr{B}} = 0;\nabla \cdot \overrightarrow{\mathscr{D}} = \rho_f. \tag{18.1}$$

To this aim we apply the Gauss' integral theorem to the volume (see Fig. 18.1):

The top surfaces (F_1, F_2) of a **box** with volume V and surface F is assumed to be symmetrically to the interface; size and shape of the top surfaces can be arbitrary. Reducing the height h of the box to infinitesimal size, we get (18.1):

$$\int_V \nabla \cdot \overrightarrow{\mathscr{B}}\ dV = \int_F \overrightarrow{\mathscr{B}} \cdot d\mathbf{f} = \int_{F_1} (\overrightarrow{\mathscr{B}}_n^{(1)} - \overrightarrow{\mathscr{B}}_n^{(2)})\ df = 0, \tag{18.2}$$

since for the surface normals $\mathbf{n}_1 = \mathbf{n} = -\mathbf{n}_2$. Since F_1 can be chosen arbitrarily we obtain:

$$\overrightarrow{\mathscr{B}}_n^{(1)} = \overrightarrow{\mathscr{B}}_n^{(2)}. \tag{18.3}$$

The normal component of $\overrightarrow{\mathscr{B}}$ is continuous at the interface.

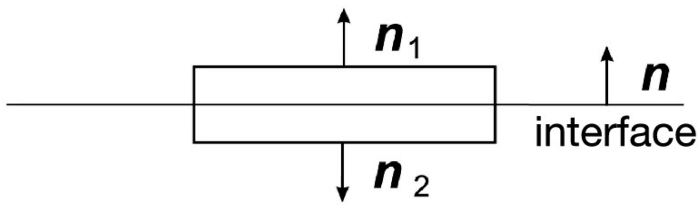

Fig. 18.1 Finite volume *V* including the interface of height *h*

In analogy we get

$$\int_V \nabla \cdot \overrightarrow{\mathscr{D}}\ dV = \int_F \overrightarrow{\mathscr{D}} \cdot d\mathbf{f} = \int_{F_1} (\overrightarrow{\mathscr{D}}_n^{(1)} - \overrightarrow{\mathscr{D}}_n^{(2)})\ df = Q_f, \tag{18.4}$$

for the normal component of $\overrightarrow{\mathscr{D}}$:

$$\overrightarrow{\mathscr{D}}_n^{(1)} - \overrightarrow{\mathscr{D}}_n^{(2)} = \gamma_f, \tag{18.5}$$

if γ_f is the surface charge density of the free charge carriers in the interface. For dielectrics with $\gamma_f = 0$ the normal component of $\overrightarrow{\mathscr{D}}$ is continuous; on the other hand, $\overrightarrow{\mathscr{D}}_n$ jumps at the interface of a conductor with a non-conductor by γ_f.

2. Tangential components of $\overrightarrow{\mathscr{E}}$ and $\overrightarrow{\mathscr{H}}$. We use:

$$\nabla \times \overrightarrow{\mathscr{E}} = -\frac{\partial \overrightarrow{\mathscr{B}}}{\partial t}; \nabla \times \overrightarrow{\mathscr{H}} = -\frac{\partial \overrightarrow{\mathscr{D}}}{\partial t} + \mathbf{j}_f, \tag{18.6}$$

and apply Stokes' integral theorem to the area in Fig. 18.2.

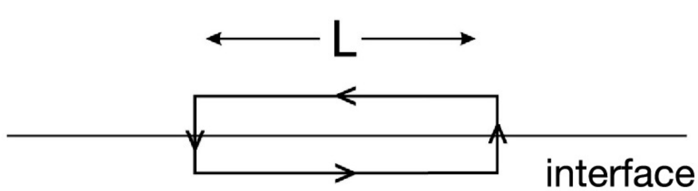

Fig. 18.2 Rectangular loop including the interface with length L and infinitesimal height h

A rectangular loop S has sides of length L parallel to the interface and the length h perpendicular to it. Then the integration over the flat surface F bordered by S in the limit $h \to 0$ leads to:

$$\int_F (\nabla \times \overrightarrow{\mathscr{E}}) \cdot d\mathbf{f} = \oint_S \overrightarrow{\mathscr{E}} \cdot d\mathbf{s} = \int_0^l ds \; (\overrightarrow{\mathscr{E}_t}^{(1)} - \overrightarrow{\mathscr{E}_t}^{(2)}) = 0, \tag{18.7}$$

since for $h \to 0$ the contents of the area F disappears, such that:

$$\int_F (\frac{\partial \overrightarrow{\mathscr{B}}}{\partial t}) \cdot d\mathbf{f} \to 0 \,\text{for}\;\; \text{h} \to 0. \tag{18.8}$$

Since L can be chosen arbitrarily in (18.7), the continuity of the tangential component of $\overrightarrow{\mathscr{E}}$ follows directly:

$$\boxed{\overrightarrow{\mathscr{E}_t}^{(1)} = \overrightarrow{\mathscr{E}_t}^{(2)}.} \tag{18.9}$$

In analogy the second equation of (18.6) results in

$$\int_0^l ds \; (\overrightarrow{\mathscr{H}_t}^{(1)} - \overrightarrow{\mathscr{H}_t}^{(2)}) = I_f, \tag{18.10}$$

if I_f is the current strength of the (free) current flowing in the interface perpendicular to the tangential component $\overrightarrow{\mathscr{H}_t}$ of $\overrightarrow{\mathscr{H}}$.

With the representation

$$I_f = \int_0^l i_f \; dl \tag{18.11}$$

follows from (18.10) accordingly

$$\boxed{\overrightarrow{\mathscr{H}_t}^{(1)} - \overrightarrow{\mathscr{H}_t}^{(2)} = i_f;} \tag{18.12}$$

where i_f is the surface current density in the interface perpendicular to $\overrightarrow{\mathscr{H}_t}$. We now examine the above results for linear, isotropic media.

18.2 Linear, Isotropic Media

If

$$\boxed{\overrightarrow{\mathscr{B}} = \mu \overrightarrow{\mathscr{H}}; \overrightarrow{\mathscr{D}} = \epsilon \overrightarrow{\mathscr{E}}} \tag{18.13}$$

holds, we obtain from (18.3), (18.5), (18.9) and (18.12):

$$\boxed{\mu_1 \overrightarrow{\mathscr{H}}_n^{(1)} = \mu_2 \overrightarrow{\mathscr{H}}_n^{(2)}; \frac{\overrightarrow{\mathscr{D}}_t^{(1)}}{\epsilon_1} = \frac{\overrightarrow{\mathscr{D}}_t^{(2)}}{\epsilon_2}} \tag{18.14}$$

and

$$\boxed{\epsilon_1 \overrightarrow{\mathscr{E}}_n^{(1)} - \epsilon_2 \overrightarrow{\mathscr{E}}_n^{(2)} = \gamma_f; \frac{\overrightarrow{\mathscr{B}}_t^{(1)}}{\mu_1} - \frac{\overrightarrow{\mathscr{B}}_t^{(2)}}{\mu_2} = \boldsymbol{i}_f.} \tag{18.15}$$

If Ohm's law applies,

$$\mathbf{j}_f = \sigma \overrightarrow{\mathscr{E}}, \tag{18.16}$$

then follows from (18.9) ($\overrightarrow{\mathscr{E}}_t^{(1)} = \overrightarrow{\mathscr{E}}_t^{(2)}$) for the tangential component of $\mathbf{j}_f$:

$$\boxed{\frac{\mathbf{j}_{ft}^{(1)}}{\sigma_1} = \frac{\mathbf{j}_{ft}^{(2)}}{\sigma_2}.} \tag{18.17}$$

For the normal component the continuity equation reads as:

$$\nabla \cdot \mathbf{j}_f + \frac{\partial \rho_f}{\partial t} = 0 \tag{18.18}$$

and by applying the Gauss' theorem (as for 1.) we get

$$j_{fn}^{(1)} - j_{fn}^{(2)} = -\frac{\partial \gamma_f}{\partial t}. \tag{18.19}$$

Especially for stationary currents with

$$\nabla \cdot \mathbf{j}_f = 0 \tag{18.20}$$

we get the continuity of the normal components

$$\boxed{\mathbf{j}_{fn}^{(1)} = \mathbf{j}_{fn}^{(2)}.} \tag{18.21}$$

Example: Conductor (1)—non-conductor (2)
Since no current can flow in the non-conductor, we have from (18.21)

$$\mathbf{j}_{fn}^{(1)} = \mathbf{j}_{fn}^{(2)} = 0, \tag{18.22}$$

and therefore with (18.16)

$$\overrightarrow{\mathscr{E}}_n^{(1)} = 0, \tag{18.23}$$

since $\sigma_1 \neq 0$. On the other hand, for $\overrightarrow{\mathscr{E}}_n^{(2)}$ we get from (18.15):

$$\epsilon_2 \mathscr{E}_n^{(2)} = -\gamma_f. \tag{18.24}$$

In case of electrostatics, due to $\mathbf{j}_f = 0$, we obtain

$$\overrightarrow{\mathscr{E}}_t^{(1)} = 0; \tag{18.25}$$

which demands (18.9)

$$\overrightarrow{\mathscr{E}}_t^{(2)} = 0. \tag{18.26}$$

Thus the $\overrightarrow{\mathscr{E}}$ field is perpendicular to the conductor surface; it is zero within the conductor.

18.3 Reflection and Refraction of Light

In the absence of free charges ($\rho_f = 0$) the Maxwell equations are:

$$\nabla \cdot \overrightarrow{\mathscr{B}} = 0; \nabla \cdot \overrightarrow{\mathscr{D}} = 0 \tag{18.27}$$

and

$$\nabla \times \overrightarrow{\mathscr{E}} = -\frac{\partial \overrightarrow{\mathscr{B}}}{\partial t}; \nabla \times \overrightarrow{\mathscr{H}} = \frac{\partial \overrightarrow{\mathscr{D}}}{\partial t}. \tag{18.28}$$

They are simplified by assuming linear, isotropic media

$$\overrightarrow{\mathscr{B}} = \mu \overrightarrow{\mathscr{H}}; \overrightarrow{\mathscr{D}} = \epsilon \overrightarrow{\mathscr{E}}, \tag{18.29}$$

to

$$\nabla \cdot \overrightarrow{\mathscr{H}} = 0; \nabla \cdot \overrightarrow{\mathscr{E}} = 0 \tag{18.30}$$

and

$$\nabla \times \overrightarrow{\mathscr{E}} = -\mu \frac{\partial \overrightarrow{\mathscr{H}}}{\partial t}; \nabla \times \overrightarrow{\mathscr{H}} = \epsilon \frac{\partial \overrightarrow{\mathscr{E}}}{\partial t}. \tag{18.31}$$

As in Chap. 10 the Eq. (18.31) can be decoupled in view of (18.30). We get the wave equations

$$\boxed{\Delta \overrightarrow{\mathscr{E}} - \frac{1}{c'^2} \frac{\partial^2}{\partial t^2} \overrightarrow{\mathscr{E}} = 0; \Delta \overrightarrow{\mathscr{H}} - \frac{1}{c'^2} \frac{\partial^2}{\partial t^2} \overrightarrow{\mathscr{H}} = 0,} \tag{18.32}$$

where c' is the phase velocity in the medium (see Sect. 10.3):

$$\boxed{\frac{1}{c'^2} = \epsilon \mu.} \tag{18.33}$$

Since in the following we study the behavior of the electromagnetic field for flat interfaces, we consider special solutions of (18.32) in the form of plane waves, e.g.:

$$\boxed{\overrightarrow{\mathscr{E}} = \overrightarrow{\mathscr{E}_0} \exp\{i(\mathbf{k} \cdot \mathbf{r} - \omega t)\},} \tag{18.34}$$

where the relationship between ω and $\mathbf{k}$ is

$$\tag{18.35}$$

$$\omega = kc'.$$

As in Chap. 10 one finds that $\overrightarrow{\mathscr{E}}, \overrightarrow{\mathscr{H}}$ and $\mathbf{k}$ are perpendicular to each other.

Equation (18.35) differs from (10.22) since there c is a constant while c' depends on the frequency ω, i.e. $\epsilon = \epsilon(\omega)$. The components of different frequency ω in a wave packet run with different velocities $c' = c'(\omega)$; the wave packet does not maintain its shape in time (**disintegration** of wave packets; see Sect. 11.3).

Note:

Depending on the form of $\epsilon(\omega)$, we can get $c' > c$. This is no contradiction to the theory of special relativity since the **phase velocity** v_{ph} is not identical to the **group velocity**

$$v_g = \left(\frac{d\omega}{dk}\right)_{k=k_0} \tag{18.36}$$

of a wave packet whose amplitude is concentrated in the vicinity of the wave number k_0; the energy transport in such a wave packet is determined by v_g and not by v_{ph}.

We now examine the behavior of a light wave, described by (18.34), at a flat interface (Fig. 18.3).

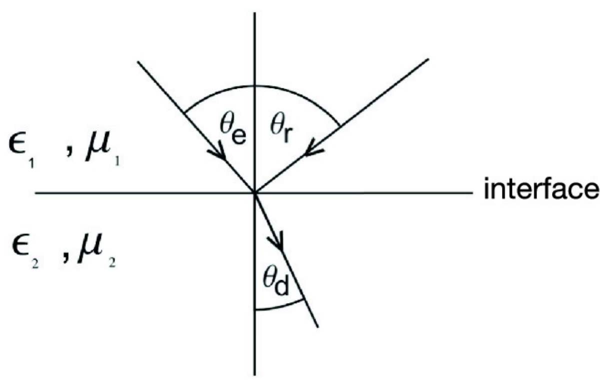

Fig. 18.3 Geometry for light reflection and transmission at a flat interface

From the continuity of the tangential component of $\overrightarrow{\mathscr{E}}$ (18.9) ($\overrightarrow{\mathscr{E}}_t^{(1)} = \overrightarrow{\mathscr{E}}_t^{(1)}$) we get

$$\overrightarrow{\mathcal{T}} \cdot (\overrightarrow{\mathscr{E}}_e + \overrightarrow{\mathscr{E}}_r) = \overrightarrow{\mathcal{T}} \cdot \overrightarrow{\mathscr{E}}_d \tag{18.37}$$

for all times t and all vectors $\mathbf{r}$ of the interface; let $\overrightarrow{\tau}$ be a unit vector parallel to the interface, then $\overrightarrow{\mathscr{E}_e}$, $\overrightarrow{\mathscr{E}_r}$ and $\overrightarrow{\mathscr{E}_d}$ denote the electric field strength of the incident, reflected and transmitted light wave. Placing the coordinate origin in the interface, we obtain from (18.37) and (18.34) for $\mathbf{r}$ = 0 the conservation of the frequency,

$$\boxed{\omega_e = \omega_r = \omega_d. \qquad (18.38)}$$

On the other hand, for $t = 0$ we get the phase equality

$$\boxed{\mathbf{k}_e \cdot \mathbf{r} = \mathbf{k}_r \cdot \mathbf{r} = \mathbf{k}_d \cdot \mathbf{r}, \qquad (18.39)}$$

i.e. the **coplanarity** of $\mathbf{k}_e$, $\mathbf{k}_r$ and $\mathbf{k}_d$; thus all $\mathbf{k}$-vectors must be in a plane. To prove this statement we choose a specific $\mathbf{r} = \mathbf{r}_0$ such that $\mathbf{k}_e \cdot \mathbf{r}_0$ = 0; then according to (18.39) all 3 vectors $\mathbf{k}_e$, $\mathbf{k}_r$ and $\mathbf{k}_d$ must be perpendicular to $\mathbf{r}_0$, which implies that $\mathbf{k}_e$, $\mathbf{k}_r$ and $\mathbf{k}_d$ lie in a plane. Then from (18.39) we obtain,

$$\boxed{k_e \sin\theta_e = k_r \sin\theta_r, \qquad (18.40)}$$

and with the conservation of the frequency (18.38) ($\omega_e = \omega_r$), $k_e = k_r$, the

reflection law:

$$\boxed{\theta_e = \theta_r. \qquad (18.41)}$$

Also from (18.38) ($\omega_e = \omega_d$) we get:

$$\frac{k_e}{k_d} = \frac{\sqrt{\epsilon_1\mu_1}}{\sqrt{\epsilon_2\mu_2}} = \frac{n_1}{n_2}, \qquad (18.42)$$

such that (with (18.39)) we obtain the

refraction law:

$$\frac{\sin\theta_e}{\sin\theta_d} = \frac{k_d}{k_e} = \frac{n_2}{n_1}. \qquad (18.43)$$

When evaluating the conditions contained in (18.37) for the amplitudes, one gets the **Fresnel formulas**, **Brewster's law** (generation of linearly polarized light) and the **total reflection** (fiber optics).

Note:

By (17.32) $\epsilon(\omega)$ in general is complex and thus also k. An electromagnetic wave in the medium consequently is weakened (absorption).

18.4 Propagation of Electromagnetic Waves in Conductive Materials

We consider an ohmic conductor with a plane interface and surface charge σ. In this case the Maxwell equations read:

$$\nabla \cdot \overrightarrow{\mathscr{H}} = 0; \nabla \cdot \overrightarrow{\mathscr{E}} = 0; \nabla \times \overrightarrow{\mathscr{H}} - \epsilon \frac{\partial \overrightarrow{\mathscr{E}}}{\partial t} - \sigma \overrightarrow{\mathscr{E}} = 0; \nabla \times \overrightarrow{\mathscr{E}} + \mu \frac{\partial \overrightarrow{\mathscr{H}}}{\partial t} = 0. \tag{18.44}$$

As long as no charge accumulation occurs, the surface charge density $\rho_f = 0$ (see section 5.2), although

$$\mathbf{j}_f = \sigma \overrightarrow{\mathscr{E}} \neq 0. \tag{18.45}$$

As a solution to (18.44) we take:

$$\overrightarrow{\mathscr{E}} = \overrightarrow{\mathscr{E}}_0 \exp\{i(\mathbf{k} \cdot \mathbf{r} - \omega t)\}, \tag{18.46}$$

also for $\overrightarrow{\mathscr{H}}$ and find with (18.44):

$$\overrightarrow{\mathscr{H}} = \frac{1}{\mu\omega}(\mathbf{k} \times \overrightarrow{\mathscr{E}}); i(\mathbf{k} \times \overrightarrow{\mathscr{H}}) + i\epsilon\omega\overrightarrow{\mathscr{E}} - \sigma\overrightarrow{\mathscr{E}} = 0. \tag{18.47}$$

Eliminating in the last expression $\overrightarrow{\mathscr{E}}$ or $\overrightarrow{\mathscr{H}}$ we obtain:

$$k^2 = \omega^2 \mu\epsilon \ (1 + i\frac{\sigma}{\omega\epsilon}), \tag{18.48}$$

using the transversality of $\overrightarrow{\mathscr{E}}$ and $\overrightarrow{\mathscr{H}}$ ($\overrightarrow{\mathscr{E}} \cdot \overrightarrow{\mathscr{H}} = 0$), which follows from (18.44). If we write

$$k = \alpha + i\beta; \alpha, \beta \text{ real,} \tag{18.49}$$

we get with

$$k^2 = \alpha^2 - \beta^2 + 2i\alpha\beta, \tag{18.50}$$

thus:

$$\alpha^2 - \beta^2 = \mu\epsilon\omega^2; 2\alpha\beta = \mu\omega\sigma. \tag{18.51}$$

Eliminating in the 1st equation $\alpha = \mu\omega\sigma/(2\beta)$ and using the 2. equation, this gives:

$$\beta^4 - \tfrac{1}{4}(\mu\omega\sigma)^2 + \beta^2\mu\epsilon\omega^2 = 0. \tag{18.52}$$

Since β is real, the only possible solution is:

$$\beta^2 = \tfrac{\mu\epsilon\omega^2}{2}\left(\sqrt{1 + \left(\tfrac{\sigma}{\epsilon\omega}\right)^2} - 1\right); \tag{18.53}$$

thus

$$\alpha^2 = \beta^2 + \mu\epsilon\omega^2 = \tfrac{\mu\epsilon\omega^2}{2}\left(\sqrt{1 + \left(\tfrac{\sigma}{\epsilon\omega}\right)^2} + 1\right). \tag{18.54}$$

For $\sigma \to 0$ follows:

$$\boxed{\beta \to 0; \alpha^2 \to \mu\epsilon\omega^2} \tag{18.55}$$

in accordance with (18.35). Since $\mu\omega\sigma \geq 0$, α and β must have the same sign according to (18.51). For $\beta \neq 0$ (i.e. $\sigma \neq 0$) a light wave—incident on a metal surface—is damped exponentially in the metal; for a plane wave travelling in the positive *x* direction then holds

$$\exp\{i(kx - \omega t)\} = \exp\{i(\alpha x - \omega t)\} \ \exp\{-\beta x\}, \tag{18.56}$$

since with $\alpha > 0$ also $\beta > 0$.

Limiting cases:

(1.) At high conductivity ($\sigma \to \infty$) the light wave is practically totally reflected since the **penetration depth** $d \sim \beta^{-1}$ vanishes.

(2.) For high frequencies ($\omega \to \infty$) the conductivity σ is frequency dependent according to (17.22): σ will be purely imaginary for $\omega \to \infty$, i.e. k^2 in (18.48) is real; the material becomes **transparent**. However, this fact implies that it is difficult to 'focus' hard X-rays.

As a result of the attenuation β, alternating currents can flow only in a surface layer of the conductor due to (18.45); the thickness of the layer is determined by β^{-1} (**skin effect**).

In summarizing this chapter we have analyzed the properties of the macroscopic fields at interfaces between the vacuum and dielectrics or conducting materials for linear, isotropic media and in particular studied the reflection and refraction of light at interfaces. Furthermore, we have investigated the propagation of electromagnetic waves in conductive materials and derived the skin effect.

Part VI
Relativistic Formulation of Electrodynamics

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19. Covariance of Electrodynamics

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In the following we want to show that the basic equations of electrodynamics have the same form in all inertial systems (**covariance of electrodynamics**) and thus obey the principle of special relativity. In preparation we examine the mathematical structure of the Lorentz transformations, define the four-current density, the four-potential and show the Lorentz invariance of the wave equations. In addition the transformation of the fields $\mathbf{E}$ and $\mathbf{B}$ are derived with help of the electromagnetic field-strength tensor.

19.1 Lorentz Group

First of all it should be shown that the Lorentz transformations are orthogonal complex transformations in a 4-dimensional pseudo-Euclidean vector space (**Minkowski space**). To this aim we introduce the following coordinates:

$$x_0 = ict, \qquad x_1 = x, \qquad x_2 = y, \qquad x_3 = z. \tag{19.1}$$

In these coordinates the length (squared) of a space-time vector in different reference systems Σ and Σ' can be written as:

$$\textstyle\sum_{\mu=0}^{3} x_\mu^2 = \sum_{\mu=0}^{3} x_\mu'^2. \tag{19.2}$$

A general Lorentz transformation

$$\boxed{x'_\mu = \textstyle\sum_\nu a_{\mu\nu} x_\nu; \qquad \mu, \nu = 0, 1, 2, 3} \tag{19.3}$$

must keep the **length** of the vector (x_0, x_1, x_2, x_3) invariant:

$$\boxed{\textstyle\sum_{\mu=0}^{3} x_\mu^2 = \mathbf{r}^2 - c^2 t^2 = \text{const.}} \tag{19.4}$$

In analogy to the 3-dimensional Euclidean space one can fulfill this condition in terms of an orthogonality relation for the transformation coefficients $a_{\mu\nu}$:

$$\textstyle\sum_{\nu=0}^{3} a_{\mu\nu}^T a_{\nu\lambda} = \delta_{\mu\lambda}, \tag{19.5}$$

where a^T is the transposed matrix to a. Equation (19.5) follows from:

$\sum_\mu x'^2_\mu = \sum_\mu \sum_{\nu\nu'} a_{\mu\nu} a_{\mu\nu'} x_\nu x_{\nu'} = \sum_{\nu\nu'} \{\sum_\mu a^T_{\nu\mu} a_{\mu\nu'}\} x_\nu x_{\nu'} = \sum_{\nu\nu'} \delta_{\nu\nu'} x_\nu x_{\nu'} = \sum_\nu x^2_\nu.$ (19.6)

For a Lorentz transformation in the x_1 direction with velocity $\beta = v/c$ the transformation matrix $a_{\mu\nu}$ has the special form

$$a_{\mu\nu} = \begin{pmatrix} \gamma & -i\gamma\beta & 0 & 0 \\ i\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{19.7}$$

with $\gamma^2 = 1/(1-\beta^2)$. The restriction in (19.7) to a boost in x_1-direction can be solved by replacing (19.7) with an orthogonal transformation in 3-dim space in the form of a rotation. The basis for this is the group property of Lorentz transformations:

(1.) If we carry out 2 Lorentz transformations one after the other,

$$x'_\mu = \sum_\nu a_{\mu\nu} x_\nu; \qquad x''_\rho = \sum_\nu a'_{\rho\nu} x'_\nu; \qquad (\Sigma \to \Sigma' \to \Sigma''), \tag{19.8}$$

the result

$$x''_\rho = \sum_{\nu,\mu} a'_{\rho\nu} a_{\nu\mu} x_\mu = \sum_\mu a''_{\rho\mu} x_\mu; \qquad (\Sigma \to \Sigma'') \tag{19.9}$$

is again a Lorentz transformation, since for the matrices a'', a' and a we have:

$$(a'')^T a'' = (a'a)^T (a'a) = a^T (a'^T a') a = a^T a = 1_4, \tag{19.10}$$

and as required

$$a^T a = 1_4; \qquad (a')^T a' = 1_4 \tag{19.11}$$

with 1_4 as the 4×4 identity matrix.

The connection between the elements of the group is therefore the (4×4) matrix multiplication.

(2.) The neutral element is the 1_4 matrix for Lorentz transformations with velocity $v = 0$.

(3.) For every transformation a there is an inverse transformation, since from (19.5) we have:

$$\det(a^T a) = (\det(a))^2 = 1, \tag{19.12}$$

thus:

$$\det(a) \neq 0. \tag{19.13}$$

(4.) Since the matrix multiplication is associative, it also applies to Lorentz transformations.

The orthogonal transformations in 3-dim. space (rotations and reflections) form a subgroup of the Lorentz group, represented by

(19.14)

$$d_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & d_{ik} \end{pmatrix}$$

with $i, k = 1, 2, 3$ and

$$\sum_{m=1}^{3} d_{im}^{T} d_{mj} = \delta_{ij}. \tag{19.15}$$

The general Lorentz transformation (19.3) with the condition (19.5) is obtained by combining (19.7) with (19.14), (19.15) and adding the **time reversal**

$$x'_i = x_i; \qquad x'_0 = -x_0; \qquad i = 1, 2, 3 \tag{19.16}$$

as well as reflections in space

$$x'_i = -x_i; \qquad x'_0 = x_0; \qquad i = 1, 2, 3. \tag{19.17}$$

The Lorentz transformations therefore include: rotations in 3-dim. space, space reflections and time reversal as well as the transition between inertial systems that move with constant velocity relative to each other.

Addition: For translations in space or time the condition (19.2) does not change because it only affects spatial and temporal distances. The group of the **homogeneous** Lorentz transformations (discussed above) we can therefore extend by **translations in space and time**. We then get the 10-parameter **Poincaré group**, which has 3 parameters for spatial rotations, 3 parameters for Lorentz boosts with the velocity **v** and 4 parameters for space-time translations. **Today it is considered as the basis invariance group for all physics**.

19.2 Lorentz Group (Four-Tensors)

Analogous to the case of the group of rotations, we now define tensors (of different rank) with respect to the Lorentz group:

(1.) **Lorentz scalar**

We call a quantity Ψ a **Lorentz scalar**, if Ψ does not change for Lorentz transformations,

$$\Psi \to \Psi' = \Psi. \tag{19.18}$$

An example for this is the electric charge (see Sect. 2.1).

(2.) **Lorentz vector**

We define a **Lorentz or four-vector** by the property that for Lorentz transformations its components A_μ transform as the components x_μ, i.e.

$$A_\mu \to A'_\mu = \sum_\nu a_{\mu\nu} A_\nu. \tag{19.19}$$

Examples:

(i) The partial derivatives of a Lorentz scalar Ψ with respect to the x_μ form the components of a four-vector:

$$\frac{\partial \Psi'}{\partial x'_\mu} = \sum_\nu \frac{\partial \Psi}{\partial x_\nu} \frac{\partial x_\nu}{\partial x'_\mu} = \sum_\nu a_{\mu\nu} \frac{\partial \Psi}{\partial x_\nu}, \tag{19.20}$$

using the inverse expression to (19.3),

$$x_\nu = \sum_\rho a_{\rho\nu} x'_\rho. \tag{19.21}$$

(ii) The **4-divergence** of a four-vector is a four-point scalar

$$\sum_\nu \frac{\partial A'_\nu}{\partial x'_\nu} = \sum_\nu \sum_{\mu,\mu'} a_{\nu\mu} a_{\nu\mu'} \frac{\partial A_\mu}{\partial x_{\mu'}} = \sum_\mu \frac{\partial A_\mu}{\partial x_\mu} \tag{19.22}$$

considering (19.5).

(iii) Choosing the components of the four-vector according to (19.19) as

$$A_\mu = \frac{\partial \Psi}{\partial x_\mu}, \tag{19.23}$$

then follows from (19.22):

$$\sum_\nu \frac{\partial^2}{\partial x_\nu^2} \Psi = \sum_\nu \frac{\partial^2}{\partial x_\nu'^2} \Psi'. \tag{19.24}$$

The operator

$$\left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) = \sum_\mu \frac{\partial^2}{\partial x_\mu^2}$$

is invariant for Lorentz transformations.

Thus for a four-vector with the components A_μ the wave equation

$$\sum_\nu \frac{\partial^2}{\partial x_\nu^2} A_\mu = \left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) A_\mu \tag{19.25}$$

transforms as like the μth component of a four-vector.

(iv) The dot product of two four-vectors is a four-point scalar:

$$\sum_\mu A'_\mu B'_\mu = \sum_\mu \sum_{\nu,\rho} a_{\mu\rho} a_{\mu\nu} A_\rho B_\nu = \sum_\nu A_\nu B_\nu. \tag{19.26}$$

(3.) **Lorentz tensors of 2nd rank**

Except for scalars (=tensors of 0th rank) and the vectors (=tensors of 1st rank) we will still encounter tensors of 2nd rank like the electromagnetic field-strength tensor (see below). They are defined as 4×4 matrices; their components $F_{\mu\nu}$ have the transformation property

$$F'_{\mu\nu} = \sum_{\lambda,\sigma} a_{\mu\lambda} a_{\nu\sigma} F_{\lambda\sigma}. \tag{19.27}$$

19.3 Four-Current Density

To prove the covariance of electrodynamics we investigate the transformation properties of the **sources j** and ρ of the electromagnetic field. The charge conservation serves as a starting point:

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0. \tag{19.28}$$

With the notation

$$j_0 = ic\rho; \qquad j_1 = j_x; \qquad j_2 = j_y; \qquad j_3 = j_z \tag{19.29}$$

we can write the continuity Eq. (19.28) in four-notation as

$$\boxed{\sum_\mu \frac{\partial}{\partial x_\mu} j_\mu = 0.} \tag{19.30}$$

Since charge invariance (19.30) must hold in every inertial frame (19.30) is invariant for Lorentz transformations. Then according to (19.22) the j_μ must be the components of a four-vector (**four-current density**).

Let's convince ourselves directly for a simple case. We consider a charge distribution at rest in the system Σ':

$$\boxed{j'_0 = ic\rho_0, \qquad j'_1 = j'_2 = j'_3 = 0.} \tag{19.31}$$

As components of a four-vector j'_μ transforms for the Lorentz transformation with velocity $\beta = v/c$ in x_1-direction as

$$x_0 = \gamma(i\beta x'_1 + x'_0); \qquad x_1 = \gamma(x'_1 - i\beta x'_0); \qquad x_2 = x'_2; \qquad x_3 = x'_3, \tag{19.32}$$

the same as

$$j_0 = ic\gamma\rho_0; \qquad j_1 = \gamma\rho_0 v; \qquad j_2 = 0; \qquad j_3 = 0. \tag{19.33}$$

A comparison with (19.29) gives:

$$\rho = \gamma\rho_0. \tag{19.34}$$

We know that a volume element dV_0 resting in Σ' for an observer in Σ has the size

$$dV = \frac{dV_0}{\gamma} \tag{19.35}$$

due to the length contraction. The charge invariance,

$$\int_V \rho \; dV = \int \gamma\rho_0 \frac{dV_0}{\gamma} = \int \rho_0 dV_0 \tag{19.36}$$

shows that $ic\rho$ can be viewed as the 0th component of a four-vector. Furthermore, with (19.34) we get

$$j_1 = \rho v \tag{19.37}$$

in accordance with the definition of the (ordinary) current density; the components of $\vec{j}$ are the 1, 2, 3 components of a four-vector.

19.4 Four-Potential

To determine the transformation properties of the vector potential $\mathbf{A}$ and the scalar potential Φ we employ the Lorentz gauge

$$\boxed{\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0.} \tag{19.38}$$

Then for $\mathbf{A}$ and Φ the following inhomogeneous wave equations hold:

$$\boxed{\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = -\mu_0 \mathbf{j}; \qquad \left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi = -\frac{\rho}{\epsilon_0}.} \tag{19.39}$$

Introducing in analogy to (19.29):

$$(A_\mu) = \left(\frac{i}{c}\Phi, \mathbf{A}\right), \tag{19.40}$$

the inhomogeneous wave equations can be summarized as:

$$\boxed{\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)A_\mu = -\mu_0 j_\mu} \tag{19.41}$$

using

$$\epsilon_0 \mu_0 = c^{-2}. \tag{19.42}$$

The right side of (19.41) shows the components of a four-vector and the differential operator $\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)$ according to (19.24) is a four-scalar, then the A_μ turn out to be the components of a four-vector.

The Lorentz convention (19.38) is now written as:

$$\boxed{\sum_\mu \frac{\partial}{\partial x_\mu} A_\mu = 0} \tag{19.43}$$

and is Lorentz-invariant according to (19.22).

Result:

The equations (19.30) and (19.43) are Lorentz-invariant, i.e. they do not change when moving from an inertial system to another. If in Σ holds

$$\sum_\mu \frac{\partial}{\partial x_\mu} j_\mu = 0; \qquad \sum_\mu \frac{\partial}{\partial x_\mu} A_\mu = 0, \tag{19.44}$$

then also in Σ':

$$\sum_\mu \frac{\partial}{\partial x'_\mu} j'_\mu = 0; \qquad \sum_\mu \frac{\partial}{\partial x'_\mu} A'_\mu = 0. \tag{19.45}$$

The 4 equations (19.41) are covariant because from (19.41) in Σ follows for Σ'

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) A'_\mu = -\mu_0 j'_\mu, \tag{19.46}$$

since:

$$\sum_\nu a_{\nu\mu}\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) A_\mu = \left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\sum_\nu a_{\nu\mu} A_\mu = \left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) A'_\mu \tag{19.47}$$

$$= -\mu_0 \sum_\nu a_{\nu\mu} j_\mu = -\mu_0 j'_\mu.$$

19.5 Plane Waves

A plane wave in vacuum is described (in an inertial system Σ) by

$$A_\mu(x_\rho) = A_\mu^{(0)} \exp\left(i(\mathbf{k}\cdot\mathbf{r} - \omega t)\right) = A_\mu^{(0)} \exp\left(i \sum_\lambda k_\lambda x_\lambda\right) \tag{19.48}$$

with the abbreviations:

$$k_0 = i\frac{\omega}{c}; \qquad k_1 = k_x; \qquad k_2 = k_y; \qquad k_3 = k_z. \tag{19.49}$$

Due to the covariance of the wave equation

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) A_\mu = 0 \tag{19.50}$$

it follows from (19.48) that in another system Σ' one again obtains a plane wave in line with (19.19):

$$\begin{aligned} A'_\mu(x'_\rho) &= \sum_\nu a_{\mu\nu} A_\nu(x_\lambda) = \sum_\nu a_{\mu\nu} A_\nu^{(0)} \exp\left(i \sum_\lambda k_\lambda x_\lambda\right) \\ &= A_\mu^{'(0)} \exp\left(i \sum_\lambda k'_\lambda x'_\lambda\right). \end{aligned} \tag{19.51}$$

The phase of the wave must be Lorentz-invariant:

$$\sum_\lambda k_\lambda x_\lambda = \sum_\lambda k'_\lambda x'_\lambda \tag{19.52}$$

as in case of a point-like source, where the wavefronts are spherical surfaces—moving with velocity *c*—in each inertial system.

Since (19.52) has the form of an (invariant) scalar product, the components k_μ are the components of a four-vector. For a boost in *x*-direction they transform as:

$$k'_x = \gamma(k_x - \tfrac{v}{c^2}\omega); \qquad k'_y = k_y; \qquad k'_z = k_z; \tag{19.53}$$

$$\omega' = \gamma(\omega - v k_x). \tag{19.54}$$

Using the dispersion relation (for massless photons),

$$\tfrac{\omega}{k} = c = \tfrac{\omega'}{k'}, \tag{19.55}$$

and denoting by ϕ and ϕ' the angles between $\mathbf{k}$ and $\mathbf{k}'$ with the direction of $\mathbf{v}$ (the *x*-direction in the present case) we get:

$$\omega' = \gamma\omega(1 - \beta \cos \phi) \tag{19.56}$$

and

$$\cos \phi' = \tfrac{k}{k'}\gamma(\cos \phi - \beta) = \tfrac{\cos\phi - \beta}{1 - \beta\cos\phi}. \tag{19.57}$$

Equation (19.56) describes the **Doppler-effect**, which apart from the **longitudinal** effect,

$$\omega' = \omega\tfrac{1 \mp \beta}{\sqrt{1-\beta^2}} \approx \omega(1 \mp \beta) \tag{19.58}$$

for $\beta \ll 1$ and $\phi = 0, \pi$, also includes a **transversal** effect,

$$\omega' = \tfrac{\omega}{\sqrt{1-\beta^2}}, \tag{19.59}$$

for $\phi = \pm\pi/2$, which is a typical relativistic phenomenon. This effect was proven in 1938 when studying the radiation of moving hydrogen atoms. As an example for the longitudinal effect we mention the **red shift** of light from distant galaxies, which shows that these galaxies are moving away from us.

19.6 Transformation of the Fields $\mathbf{E}$ and $\mathbf{B}$

Knowing $\mathbf{A}$ and Φ we can calculate the fields $\mathbf{E}$ and $\mathbf{B}$ by

$$\mathbf{B} = \nabla \times \mathbf{A}; \qquad \mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}. \tag{19.60}$$

We now want to rewrite (19.60) in the coordinates x_μ and the components of the four-potential A_μ. We get e.g.:

$$\frac{i}{c}E_1 = \frac{\partial A_1}{\partial x_0} - \frac{\partial A_0}{\partial x_1}; \qquad B_1 = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}. \tag{19.61}$$

Equation (19.61) suggests to introduce the following antisymmetric 4×4 matrix:

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} = -F_{\nu\mu}. \tag{19.62}$$

It has exactly 6 independent elements, for which according to 19.60), (19.61) one finds:

$$F_{\mu\nu} = \begin{pmatrix} 0 & \frac{i}{c}E_1 & \frac{i}{c}E_2 & \frac{i}{c}E_3 \\ -\frac{i}{c}E_1 & 0 & B_3 & -B_2 \\ -\frac{i}{c}E_2 & -B_3 & 0 & B_1 \\ -\frac{i}{c}E_3 & B_2 & -B_1 & 0 \end{pmatrix}. \tag{19.63}$$

The matrix (19.62) is a 2nd rank Lorentz tensor since:

$$F'_{\mu\nu} = \sum_{\lambda\rho} a_{\mu\lambda}a_{\nu\rho}\left\{\frac{\partial A_\rho}{\partial x_\lambda} - \frac{\partial A_\lambda}{\partial x_\rho}\right\} = \sum_{\lambda\rho} a_{\mu\lambda}a_{\nu\rho}F_{\lambda\rho}. \tag{19.64}$$

Thus we also know the transformation properties of the fields $\mathbf{E}$ and $\mathbf{B}$. For the special transformation (19.7) we find from (19.64) and (19.63)

$$E'_x = E_x; \qquad B'_x = B_x; \qquad E'_y = \gamma(E_y - vB_z); \qquad B'_y = \gamma(B_y + \frac{v}{c^2}E_z); \tag{19.65}$$

$$E'_z = \gamma(E_z + vB_y); \qquad B'_z = \gamma(B_z - \frac{v}{c^2}E_y).$$

In general, the parallel components (in the direction of $\mathbf{v}$) remain without change:

$$\mathbf{E}'_\parallel = \mathbf{E}_\parallel; \qquad \mathbf{B}'_\parallel = \mathbf{B}_\parallel, \tag{19.66}$$

while the transversal components change according to:

$$\mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp + (\mathbf{v}\times\mathbf{B})); \qquad \mathbf{B}'_\perp = \gamma(\mathbf{B}_\perp - \tfrac{1}{c^2}(\mathbf{v}\times\mathbf{E})). \tag{19.67}$$

The inverse transformation

$$\mathbf{E}_\perp = \gamma(\mathbf{E}'_\perp - (\mathbf{v}\times\mathbf{B}')); \qquad \mathbf{B}_\perp = \gamma(\mathbf{B}'_\perp + \tfrac{1}{c^2}(\mathbf{v}\times\mathbf{E}')) \tag{19.68}$$

is obtained in analogy to the case of coordinate transformations. The equations (19.67), (19.68) show the inevitable connection of the fields $\mathbf{E}$ and $\mathbf{B}$ in the **electromagnetic field**.

19.7 The Coulomb Field

The field of a point charge q resting in Σ' is:

$$\mathbf{E}'(\mathbf{r}') = \tfrac{q}{4\pi\epsilon_0}\tfrac{\mathbf{r}'}{r'^3}; \qquad \mathbf{B}'(\mathbf{r}') = 0. \tag{19.69}$$

According to (19.66), (19.68) in a system Σ' moving with the velocity $\mathbf{v} = (v, 0, 0)$ relative to Σ the field components become:

$$E_x = E'_x = \tfrac{q}{4\pi\epsilon_0}\tfrac{\gamma(x-vt)}{(\gamma^2(x-vt)^2+y^2+z^2)^{3/2}}; \tag{19.70}$$

$$E_y = \gamma E'_y = \frac{q}{4\pi\epsilon_0}\frac{\gamma y}{(\gamma^2(x-vt)^2+y^2+z^2)^{3/2}};$$

$$E_z = \gamma E'_z = \frac{q}{4\pi\epsilon_0}\frac{\gamma z}{(\gamma^2(x-vt)^2+y^2+z^2)^{3/2}}.$$

Here x', y', z' is written explicitly (after Lorentz transformation) as a function of *x*, *y*, *z*. The field appears in Σ as well as in Σ' as a central field; However, in Σ it is no longer isotropic because the factor γ^2 in (19.70) the *x* direction is specified compared to the *y* and *z* directions. According to (19.68) an observer in Σ sees a magnetic field:

$$\mathbf{B} = \tfrac{1}{c^2}(\mathbf{v}\times\mathbf{E}), \tag{19.71}$$

since the charge q is moving for him, i.e. represents a current. To illustrate (19.70) and (19.71) we consider the limiting case $\gamma \gg 1$:

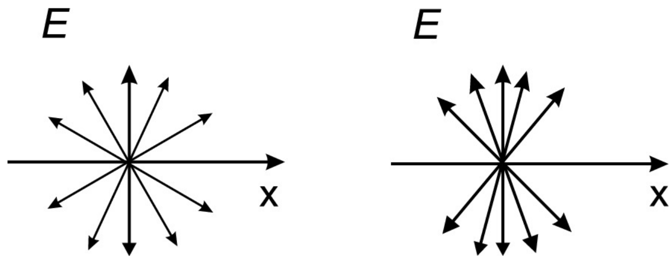

Fig. 19.1 Field lines of a static (left) and boosted (right) electric field

(i) Close to the *x* axis ($y, z \approx 0; x - vt \neq 0$) E_x becomes

$$E_x \approx \frac{1}{\gamma^2} \frac{q}{4\pi\epsilon_0} \frac{1}{(x-vt)^2}, \tag{19.72}$$

which, compared to the static field, reduces the field strength by a factor of γ^{-2}.

(ii) In the plane parallel to the $y - z$ plane to q we get:

$$E_y = \frac{\gamma y}{(y^2+z^2)^{3/2}}; \qquad E_z = \frac{\gamma z}{(y^2+z^2)^{3/2}}, \tag{19.73}$$

which, compared to the static field, results in an amplification by a factor of γ. The—radially directed—field lines therefore are **diluted** in the direction of the motion compared to the static field, but **condensed** perpendicular to it (Fig. 19.1).

For $\gamma \to \infty$ (ultra-relativistic case) $\mathbf{E} \perp \mathbf{B}$, such that with (19.71) the field lines of the $\mathbf{B}$-field run concentrically around *q* in the $y - z$ plane, i.e. perpendicular to the direction of motion.

Summary: The basic equations of electrodynamics are covariant with respect to Lorentz transformations and have the same form in every inertial system. They thus satisfy Einstein's principle of special relativity. In addition the transformation of the fields $\mathbf{E}$ and $\mathbf{B}$ have been derived with help of the electromagnetic field-strength tensor.

Appendix

Appendix

In this appendix we provide a brief introduction to volume, surface and path integrals which are of particular importance for mechanics and electrodynamics. Furthermore, the **Gauss' theorem** and **Stoke's theorem** are introduced and discussed in connection with a variety of examples, that should help the reader to solve physical problems.

A.1 Volume Integrals

In physics **volume integrals** are of particular interest; they are triple integrals e.g. over a spatial region V, i.e. $dV = d^3x$. A simple example in cartesian coordinates (x, y, z) for the integration of a scalar function $\rho(x,y,z)$ over a finite cuboid is:

$$\int\limits_V dV \rho(x,y,z) := \int\limits_{x_a}^{x_b} dx \int\limits_{y_a}^{y_b} dy \int\limits_{z_a}^{z_b} dz\, \rho(x,y,z) := \int\limits_{x_a}^{x_b} dx \left(\int\limits_{y_a}^{y_b} dy \left(\int\limits_{z_a}^{z_b} dz\, \rho(x,y,z) \right) \right) . \tag{A.1}$$

Here V is the volume of a cuboid extending in the x direction from x_a to x_b, in the y direction from y_a to y_b and in the z direction from z_a to z_b. The function $\rho(x,y,z)$ e.g. is a mass density or charge density defining the mass (or charge) at position $\mathbf{r} = (x,y,z)$. The mass or charge in an infinitesimal volume dV around the point $\mathbf{r}$ then is $\rho(x,y,z)\, dV = \rho(x,y,z)\, dx\, dy\, dz$.

Example 1: Volume V_Q of a cuboid of dimensions l, b, h.

We choose a cartesian coordinate system with its origin O in a corner of the cuboid and the adjacent edges with the positive coordinate semi-axes:

$$V_Q = \int\limits_0^l dx \int\limits_0^b dy \int\limits_0^h dz\, 1 = lbh\, . \tag{A.2}$$

Example 2: Volume of a cylinder with radius R and length L.

We choose the cartesian coordinates such that the z axis coincides with the cylinder axis and the bottom surface of the cylinder in the x-y plane is at $z = 0$, while the top surface is at $z = L$. The interior of the cylinder then is: $0 < z < L$, $-R < x < R$, $-\sqrt{R^2 - x^2} < y < \sqrt{R^2 - x^2}$. The volume in cartesian coordinates is calculated as:

$$V_C = \int_C dV = \int\limits_0^L dz \int\limits_{-R}^{R} dx \int\limits_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy\, 1 = L \int\limits_{-R}^{R} dx\, 2\sqrt{R^2 - x^2}\, . \tag{A.3}$$

This integral can in principle be solved by the substitution $x = R \cos\theta$ (recommended for practice). It's easier to compute the integral for a better choice of coordinates, i.e. here cylindrical coordinates:

$$V_C = \int_C dV = \int_0^R dr\ r\ \int_0^{2\pi} d\varphi \int_0^L dz = L \int_0^R dr\ r\ \int_0^{2\pi} d\varphi = 2\pi L \int_0^R dr\ r\ = \pi L R^2, \tag{A.4}$$

where the transformation of the integral to cylindrical coordinates includes a transformation determinant (r). Any integral of a scalar function $f(x, y, z)$ over a cylinder volume (radius R and length L) then can be written as

$$\int_C dx\, dy\, dz\ f(x,y,z) = \int_0^R r\, dr\ \int_0^{2\pi} d\varphi\ \int_0^L dz\ f(r\cos\varphi, r\sin\varphi, z)\, , \tag{A.5}$$

if one takes the cylinder base at $z = 0$.

Example 3: Integral of $f(x,y,z) = x^2 + y^2 - 2z^2$ over a cylinder volume:

$$\tag{A.6}$$

$$I = \int_Z dxdydz\ (x^2+y^2-2z^2) = \int_0^R r\,dr\ \int_0^{2\pi} d\varphi\ \int_0^L dz\ (r^2-2z^2)$$

$$= \int_0^R dr\ \int_0^{2\pi} d\varphi\ \int_0^L dz\ (r^3-2z^2r) = 2\pi \int_0^R dr\ \int_0^L dz\ (r^3-2z^2r) =$$

$$= 2\pi \int_0^R dr\ \ (r^3L - 2r\frac{L^3}{3}) = 2\pi(\frac{R^4}{4}L - R^2\frac{L^3}{3}) = \pi L(\frac{R^4}{2} - \frac{2}{3}R^2L^2)\ .$$

Example 4: Volume of a cone in cylindrical coordinates. Let the distance of the tip from the base be *h* and the radius of the base be *R*. We choose the coordinate system in such a way that the *z* axis is on the cone axis and the tip of the cone is at the origin. $z = h$ (floor), $r/z = R/h$ (A mantle); interior: $0 < z < h, r < zR/h$:

$$V_{cone} = \int_0^h dz \int_0^{2\pi} d\varphi \int_0^{zR/h} dr\, r = \int_0^h dz\, 2\pi\, \frac{z^2R^2}{2h^2} = \pi\, \frac{h^3}{3}\, \frac{R^2}{h^2} = \frac{1}{3}\, h\, \pi R^2\,. \tag{A.7}$$

Many problems in physics have spherical symmetry such that a proper choice are spherical coordinates. In this case the transformation determinant from cartesian to spherical coordinates is $r^2\ \sin\vartheta$. Each integral of a scalar function *f*(*x*, *y*, *z*) over a spherical volume *K*(*R*) with radius *R* then also can be written as

$$\begin{aligned}&\int_{K(R)} dx\,dy\,dz\ \ f(x,y,z)\\ &= \int_0^R r^2\,dr\ \int_0^{2\pi} d\varphi\ \int_0^{\pi}\ \sin\vartheta\ \ d\vartheta\ \ f(r\,\sin\vartheta\,\cos\varphi, r\,\sin\vartheta\,\sin\varphi, r\,\cos\vartheta)\ \ .\end{aligned} \tag{A.8}$$

Example 5: Sphere with radius $R \to \infty$ (and center at coordinate origin) with the mass density ρ =exp $(-r/r_0)$ g/cm^3 with $r_0 = 1$ cm. The mass density of the spherically symmetric system only depends on $r = |\mathbf{r}|$. We set $R = \infty$, $\rho_0 = 1$ g/cm^3, $a = r/r_0$, $(r_0 da = dr)$; the total mass *M* then is calculated as

$$M = \int_V dV\,\rho(r) = \int_0^{\pi} d\vartheta\ \ \sin\vartheta\ \int_0^{2\pi} d\varphi\ \int_0^{\infty} dr\,r^2\rho_0\ \exp\ (-r/r_0) \tag{A.9}$$

$$= 2\cdot 2\pi\, r_0^3\,\rho_0 \int_0^{\infty}\ a^2\ \exp\ (-a) = 4\pi r_0^3\,\rho_0\,2 = 8\pi\ \ [g] \approx 25\ \ [g]\ \ ,$$

with the help of the additional integral (partial integration):

$$\begin{aligned}\int_0^{\infty} da\,a^2\ \exp\ (-a)\ \ &= -a^2\ \exp\ (-a)\big|_0^{\infty} - \int_0^{\infty} da\,2a\,(-1)\ \exp\ (-a)\\ &= -2\int_0^{\infty} da\,(-1)\ \exp\ (-a) = 2.\end{aligned} \tag{A.10}$$

A.2 Surface Integrals of Scalar Functions

In a number of physical problems one has to integrate scalar functions *f*(*x*, *y*, *z*) over the surface of a geometric volume. The integration over the surface of a cuboid ∂Q contains 6 contributions, i.e.

$$\int_{F=\partial Q} dF\ \ f(x,y,z) = \int_{y_a}^{y_b} dy \int_{z_a}^{z_b} dz\ \ f(x=x_a,y,z) + \int_{y_a}^{y_b} dy \int_{z_a}^{z_b} dz\ \ f(x=x_b,y,z) \tag{A.11}$$

$$+ \int_{y_a}^{y_b} dy \int_{x_a}^{x_b} dx\ \ f(x,y,z=z_a) + \int_{y_a}^{y_b} dy \int_{x_a}^{x_b} dx\ \ f(x,y,z=z_b)$$

$$+\int_{z_a}^{z_b} dz \int_{x_a}^{x_b} dx \;\; f(x, y = y_a, z) + \int_{z_a}^{z_b} dz \int_{x_a}^{x_b} dx \;\; f(x, y = y_b, z) \;\; .$$

In the case of cylindrical coordinates we get for the integration over the cylinder surface 3 contributions from the bottom and top surfaces (at $z = z_a$ and $z = z_b$) and the lateral surface ($r = R$):

$$\int_{F=\partial C} dF \;\; f(r, \varphi, z) = \int_0^R \;\; rdr \int_0^{2\pi} f(r, \varphi, z = z_a) + \int_0^R \;\; rdr \int_0^{2\pi} d\varphi \;\; f(r, \varphi, z = z_b) + \int_{z_a}^{z_b} dz \int_0^{2\pi} d\varphi \;\; R \;\; f($$

In case of a sphere $K(R)$ of radius R there is only an integration over the spherical surface contribution (for $r = R$):

$$\int_{F=\partial K} dF \;\; f(r, \vartheta, \varphi) = \int_0^\pi \sin\vartheta \;\; d\vartheta \int_0^{2\pi} d\varphi \;\; R^2 \;\; f(r = R, \vartheta, \varphi) \;\; . \tag{A.13}$$

If $f(r, \vartheta, \varphi)$ does not depend on ϑ or just from cos ϑ, we substitute and get

$$\int_{F=\partial K} dF \;\; f(r, \cos\vartheta, \varphi) = -\int_{\cos(0)}^{\cos(\pi)} \;\; d\cos\vartheta \int_0^{2\pi} d\varphi \;\; R^2 \;\; f(r = R, \cos\vartheta, \varphi) \tag{A.14}$$

$$= \int_{-1}^{1} \;\; d\cos\vartheta \int_0^{2\pi} d\varphi \;\; R^2 \;\; f(r = R, \cos\vartheta, \varphi) \;\; .$$

In the special case that *f* only depends on *r* we obtain the simple result:

$$\int_{F=\partial K} dF \;\; f(r) = \int_{-1}^{1} \;\; d\cos\vartheta \int_0^{2\pi} d\varphi \;\; R^2 \;\; f(r = R) = 4\pi R^2 f(R) \;\; , \tag{A.15}$$

i.e. the function *f*(*r*) at the point $r = R$ is multiplied by the spherical surface $4\pi R^2$.

A.3 Surface Integrals of Vector Fields

Areas in a plane (e.g. the (*x*, *y*) plane) have apart from their surface area also an orientation in 3-dim. space, i.e. in $\pm z$-direction (in this case). For infinitesimal surfaces $d\mathbf{f}$ with area $|d\mathbf{f}|$ and area unit vector $\mathbf{f}/|\mathbf{f}|$—perpendicular to the surface—one defines the flux of a vector field $\mathbf{A}(\mathbf{r})$ by the infinitesimal area $d\mathbf{f}$ as a dot product, i.e.

$$d\Phi = \mathbf{A}(\mathbf{r}) \cdot d\mathbf{f} \;\; . \tag{A.16}$$

The generalization to finite surfaces *F* then is the sum over all infinitesimal surfaces, i.e. the surface integral of the vector field $\mathbf{A}(\mathbf{r})$

$$\Phi = \int_F d\Phi = \int_F \mathbf{A}(\mathbf{r}) \cdot d\mathbf{f} \;\; . \tag{A.17}$$

Example: We consider a circular area with radius *R* and center at $z = 0$ in the *x*, *y* plane. Let the circular area vector be in the $+z$ direction. The flux of the vector field $\mathbf{A}(\mathbf{r}) = (A_x(\mathbf{r}), A_y(\mathbf{r}), A_z(\mathbf{r}))$ through this circular area only provides the surface integral of the component $A_z(x, y, z = 0)$ since $\mathbf{A} \cdot \mathbf{e}_z = A_z$: (in cylindrical coordinates)

$$\Phi = \int_F A_z(x, y, z = 0) \;\; dxdy = \int_0^{2\pi} d\varphi \int_0^R dr \;\; r \;\; A_z(r\cos\varphi, r\sin\varphi, z = 0) \;\; . \tag{A.18}$$

In case of general areas the determination of the local surface vector is not unique. However, for closed surfaces this ambiguity can be avoided by defining the direction of the area vector to be ‘outwards’.

Gauss’ integral theorem:
It is often very helpful to establish a connection between a volume integral over the divergence of a vector field $\mathbf{A}(\mathbf{r})$ and a surface integral over the border of the volume $F = \partial V$ with the vector field $\mathbf{A}(\mathbf{r})$. Especially in electrodynamics this connection allows for a simple calculation of central quantities such as the

electric field $\mathbf{E}(\mathbf{r})$. Gauss' integral theorem provides this connection, i.e. the scalar flux of the field $\mathbf{A}(\mathbf{r})$ through the closed area $F = \partial V$ is equal to the integral of the divergence of $\mathbf{A}$ over the enclosed volume V:

$$\int_{\partial V} \mathbf{A} \cdot d\mathbf{f} = \int_V (\nabla \cdot \mathbf{A}(\mathbf{r}))\ dV. \tag{A.19}$$

The area element $d\mathbf{f}$ is a directed quantity and is oriented vertically (outwards). Depending on the symmetry of the given problem, appropriate coordinates have to be chosen.

Examples:
(1.) Let V be a cube with side length $2a$ and center at the coordinate origin and $\mathbf{A} = (0, 0, z)$. In this case it is convenient to use **cartesian coordinates**. With

$$\nabla \cdot \mathbf{A} = 0 + 0 + 1 = 1 \tag{A.20}$$

the volume integral becomes

$$\int_{-a}^{a} dx \int_{-a}^{a} dy \int_{-a}^{a} dz\ 1\ = (2a)(2a)(2a) = 8a^3. \tag{A.21}$$

The surface integral can be written as

$$\int_{F=\partial V} (A_x dydz + A_y dzdx + A_z dxdy). \tag{A.22}$$

Only the z component of the field contributes to the integral since $A_x = A_y = 0$. The The integration—to be carried out—therefore consists of two terms corresponding to the sides of the cube for $z = a$ and $z = -a$. The surface integral therefore is

$$\int_{-a}^{a} dy \int_{-a}^{a} dx A_z(a) - \int_{-a}^{a} dy \int_{-a}^{a} dx A_z(-a) = \int_{-a}^{a} dy \int_{-a}^{a} dx\ a - \int_{-a}^{a} dy \int_{-a}^{a} dx\ (-a) \tag{A.23}$$

$$= a(2a)(2a) + a(2a)(2a) = 8a^3,$$

where the minus sign in front of the second integral comes from the orientation of the side $z = -a$ in the negative z-direction.

(2.) Let V be a cylinder with radius R, height h and center at the coordinate origin and $\mathbf{A} = (x, y, z)$. In this case it is convenient to use **cylindrical coordinates**. With

$$\nabla \cdot \mathbf{A} = 1 + 1 + 1 = 3 \tag{A.24}$$

the volume integral is

$$3 \int_0^R\ rdr \int_0^{2\pi} d\varphi \int_{-h/2}^{h/2} dz = 3\left(\tfrac{R^2}{2}\right)(2\pi)\left(\tfrac{h}{2} - \tfrac{-h}{2}\right) = 3\pi R^2 h. \tag{A.25}$$

To calculate the surface integral, we must add the contributions of the top, the bottom and the mantle of the cylinder. For the bottom we get:

$$\int_0^{2\pi} d\varphi \int_0^R rdr \begin{pmatrix} x \\ y \\ h/2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \int_0^{2\pi} d\varphi \int_0^R rdr \tfrac{h}{2} = \tfrac{\pi R^2 h}{2}. \tag{A.26}$$

For the top

$$\int_0^{2\pi} d\varphi \int_0^R rdr \begin{pmatrix} x \\ y \\ -h/2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \int_0^{2\pi} d\varphi \int_0^R rdr \tfrac{h}{2} = \tfrac{\pi R^2 h}{2}. \tag{A.27}$$

The mantle gives

$$\int_0^{2\pi} d\varphi \int_{-h/2}^{h/2} dz R \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \\ z \end{pmatrix} \cdot \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$$
$$= \int_0^{2\pi} d\varphi \int_{-h/2}^{h/2} dz R^2(\cos^2\varphi + \sin^2\varphi) = 2\pi R^2 h. \tag{A.28}$$

The sum of all contributions also provides the value $3\pi R^2 h$ for this surface integral.

(3.) Let V be a sphere with radius R and center at the coordinate origin and $\mathbf{A} = (x, y, z)$. In this case, it is best to use **spherical coordinates**. With

$$\nabla \cdot \mathbf{A} = 1 + 1 + 1 = 3 \tag{A.29}$$

the volume integral is

$$3\int_0^R r^2 dr \int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\theta) = 3\left(\frac{R^3}{3}\right)(2\pi)2 = 4\pi R^3. \tag{A.30}$$

The surface integral can be written as

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\theta)\ \ R^2\ \ \mathbf{A}(R, \vartheta, \varphi) \cdot \mathbf{e}_r \tag{A.31}$$

$$= \int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\theta)\ \ R^2 \begin{pmatrix} R\cos\varphi\sin\theta \\ R\sin\varphi\sin\theta \\ R\cos\theta \end{pmatrix} \cdot \begin{pmatrix} \cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{pmatrix}$$

$$= \int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\theta)\ \ R^3((\cos^2\varphi + \sin^2\varphi)\sin^2\theta + \cos^2\theta)$$

$$= \int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\theta)\ \ R^3 = 4\pi R^3,$$

which is identical to (A.30).

A.4 Line Integrals and Stokes' Integral Theorem

An **oriented path** in space from a point A to a point B is characterized by vectors $\mathbf{r}(s)$, which provide a parameterization of the path such that $\mathbf{r}(a) = A$ and $\mathbf{r}(b) = B$. The **parameter** s runs through all values between a and b. For a closed path we have $\mathbf{r}(a) = A = B = \mathbf{r}(b)$.

Examples:

(1.) Every linear path from $A = (x_A, y_A, z_A)$ to $B = (x_B, y_B, z_B)$ is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_A \\ y_A \\ z_A \end{pmatrix} + s\begin{pmatrix} x_B - x_A \\ y_B - y_A \\ z_B - z_A \end{pmatrix} \qquad \text{with} \qquad s \in [0, 1]. \tag{A.32}$$

For $A = (0, 1, 0)$ and $B = (2, 2, 0)$ a corresponding parameterization is given by

$$\mathbf{r}(s) = \begin{pmatrix} x(s) \\ y(s) \\ z(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \qquad \text{with} \qquad s \in [0, 1]. \tag{A.33}$$

Closed pathes such as triangles, squares, polygons *etc*. then are constructed from parts of the form (A.32).

(2.) A circle in the $x-y$ plane with radius R and center in the coordinate origin has the standard parameterization,

$$\mathbf{r}(s)=\begin{pmatrix}x(s)\\y(s)\\z(s)\end{pmatrix}=\begin{pmatrix}R\cos s\\R\sin s\\0\end{pmatrix}\qquad\text{with}\qquad s\in[0,2\pi].\tag{A.34}$$

For parts of the circular arc the parameter s is limited to the corresponding interval $[\theta_A,\theta_B]$.

(3.) A helix in the z direction with radius R is described by

$$\mathbf{r}(s)=\begin{pmatrix}x(s)\\y(s)\\z(s)\end{pmatrix}=\begin{pmatrix}R\sin s\\R\cos s\\hs/(2\pi)\end{pmatrix}\qquad\text{with}\qquad s\in[0,2\pi],\tag{A.35}$$

where the screw reaches the height h after one revolution. In case of an infinite helical line $s\in[0,\infty]$ has to be set.

(4.) With the definition of an oriented path we can define the path integral (or **line integral**) of a vector field $\mathbf{E}$ along the path S as follows:

$$I_S=\int_S d\mathbf{r}\cdot\mathbf{E}(\mathbf{r})=\int_a^b\,ds\,\left(\tfrac{d\mathbf{r}}{ds}\cdot\mathbf{E}(\mathbf{r}(s))\right).\tag{A.36}$$

In this way, a **line integral** of a vector field turns out to be a simple one-dimensional integral, since the dot product of vectors is a scalar function (depending on the parameter s).

Physical example: The work done by a force $\mathbf{F}(\mathbf{r})$ on a body along a path S is

$$W=\int_S d\mathbf{r}\cdot\mathbf{F}(\mathbf{r})=\int_{s_a}^{s_b}ds\left(\tfrac{d\mathbf{r}}{ds}\cdot\mathbf{F}(\mathbf{r}(s))\right)=\int_{t_a}^{t_b}dt\left(\tfrac{d\mathbf{r}}{dt}\cdot\mathbf{F}(\mathbf{r}(t))\right).\tag{A.37}$$

In this case the parameter $s=t$ has the meaning of time and $\frac{d\mathbf{r}}{dt}$ that of a velocity.

Examples:

(1.) Let $\mathbf{E}=(y^2,y,z/4)$. We want to calculate the line integral of the vector field $\mathbf{E}$ along a line segment of $A=(0,0,0)$ to $B=(0,1,2)$. In this case a parameterization of the path is given by

$$\mathbf{r}(s)=\begin{pmatrix}x(s)\\y(s)\\z(s)\end{pmatrix}=s\begin{pmatrix}0\\1\\2\end{pmatrix}\qquad\text{with}\qquad s\in[0,1]\tag{A.38}$$

and for the derivative with respect to s follows

$$\tfrac{d\mathbf{r}}{ds}=\begin{pmatrix}0\\1\\2\end{pmatrix}.\tag{A.39}$$

For the line integral we then get from (A.36)

$$I_S=\int_0^1\,ds\begin{pmatrix}0\\1\\2\end{pmatrix}\cdot\begin{pmatrix}s^2\\s\\s/2\end{pmatrix}=\int_0^1\,ds\,\,(0+s+s)='2\int_0^1\,ds\,\,s=1.\tag{A.40}$$

(2.) Let $\mathbf{E}=(y,-x,z^3)$. We want to calculate the line integral of the vector field $\mathbf{E}$ along the positive semicircle from the point $A=(R,0,0)$ to $B=(-R,0,0)$ with radius R and center in the coordinate origin. In this case we use the parameterization of the path (A.34)

$$\mathbf{r}(s) = \begin{pmatrix} x(s) \\ y(s) \\ z(s) \end{pmatrix} = \begin{pmatrix} R\cos s \\ R\sin s \\ 0 \end{pmatrix} \qquad \text{with} \qquad s \in [0,\pi]\ ; \tag{A.41}$$

its derivative is

$$\frac{d\mathbf{r}}{ds} = \begin{pmatrix} -R\sin s \\ R\cos s \\ 0 \end{pmatrix} \qquad \text{with} \qquad s \in [0,\pi]. \tag{A.42}$$

The line integral follows from (A.36)

$$I_S = \int_0^\pi ds \begin{pmatrix} -R\sin s \\ R\cos s \\ 0 \end{pmatrix} \cdot \begin{pmatrix} R\sin s \\ -R\cos s \\ 0 \end{pmatrix} = -\int_0^\pi ds\ R^2(\sin^2 s + \cos^2 s) = -\pi R^2. \tag{A.43}$$

The **length of a path** is defined by

$$\int_S |d\mathbf{r}| = \int_a^b ds \left|\frac{d\mathbf{r}}{ds}\right| = \int_a^b ds \left(\frac{d\mathbf{r}}{ds}\cdot\frac{d\mathbf{r}}{ds}\right)^{1/2}. \tag{A.44}$$

As an example, let's consider the arc length of a circle with radius *R*:

$$L = \int_0^{2\pi} ds \left|\begin{pmatrix} -R\sin s \\ R\cos s \\ 0 \end{pmatrix}\right| = \int_0^{2\pi} ds \sqrt{R^2(\sin^2 s + \cos^2 s)} = 2\pi R. \tag{A.45}$$

Stokes' **theorem** provides an answer to the question with respect to the **path (in)dependence** of line integrals with fixed start and end points. This question is particularly important for the concept of work *W* (A.37), because for a given force $\mathbf{F}(\mathbf{r})$ one could perform different work to move a body from *A* to *B* in different ways.

We now consider two different paths S_1 and S_2, which connect points *A* and *B*. For an arbitrary vector field $\mathbf{E}$ usually we have

$$\int_{S_1} d\mathbf{r}\cdot\mathbf{E}(\mathbf{r}) \neq \int_{S_2} d\mathbf{r}\cdot\mathbf{E}(\mathbf{r}). \tag{A.46}$$

The necessary (and sufficient) condition for the path independence of the line integral of a vector field gives **Stokes theorem**:

$$\int_{S=\partial F} d\mathbf{r}\cdot\mathbf{E}(\mathbf{r}) = \int_F (\nabla\times\mathbf{E})\cdot d\mathbf{f}. \tag{A.47}$$

Here the line integral of $\mathbf{E}(\mathbf{r})$ along a **closed** path *S* is considered, which has an oriented area *F*. We write in short: $S = \partial F$. The direction of rotation of the curve is chosen such that the direction of rotation of the edge of the surface element with the surface normal forms a right-handed screw.

We claim now that the path independence of the line integral from $\mathbf{E}(\mathbf{r})$ exists if and only if

$$\nabla\times\mathbf{E} = 0. \tag{A.48}$$

For the **proof** let's consider two different paths S_1 and S_2 from *A* to *B*. Let's first take the path S_1 from *A* to *B* and then the path S_2 back from *B* to *A* (in the opposite direction); in this way we get a closed path with an enclosed area $F \neq 0$, i.e. according to Stokes' theorem:

$$\int_{S_1} d\mathbf{r}\cdot\mathbf{E}(\mathbf{r}) - \int_{S_2} d\mathbf{r}\cdot\mathbf{E}(\mathbf{r}) = \int_F (\nabla\times\mathbf{E})\cdot d\mathbf{f}. \tag{A.49}$$

Thus if Eq. (A.48) is satisfied, the right side of the Eq. (A.49) is zero, i.e. the difference of the path integrals must vanish.

Example: Let $\mathbf{E} = (z^5 + x, 2x, 0)$. We want to verify the theorem of Stokes for the path over the positive semicircle from the point $A = (R, 0, 0)$ to $B = (-R, 0, 0)$ with radius *R* and center in the coordinate origin and from a straight line from *B* to *A*.

For the line integral we first calculate the contribution from the semicircle. The corresponding parameterization is given by (A.34),

$$\begin{aligned} I_1 = & \int_0^\pi \, ds \begin{pmatrix} -R\sin s \\ R\cos s \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 + R\cos s \\ 2R\cos s \\ 0 \end{pmatrix} = \int_0^\pi \, ds \; R^2(-\cos s \sin s + 2\cos^2 s) \\ = & \; R^2\left[-\tfrac{\sin^2 s}{2} + s + \sin s \cos s\right]_0^\pi = \pi R^2. \end{aligned}$$

The contribution of the straight line can be calculated as

$$I_2 = \int_0^1 \, ds \begin{pmatrix} 2R \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 + 2Rs - R \\ 4Rs - 2R \\ 0 \end{pmatrix} = \int_0^1 \, ds \; 2R^2(2s-1) = 2R^2\left[s^2 - s\right]_0^1 = 0. \tag{A.50}$$

Now the calculation over the surface integral: Since the enclosed surface is oriented in the *z*-direction, we only need the *z*-component of the rotation of the field:

$$(\nabla \times \mathbf{E})_z = \left(\tfrac{\partial}{\partial x} E_y - \tfrac{\partial}{\partial y} E_x\right) = 2 + 0 = 2. \tag{A.51}$$

The surface integral thus is

$$I_F = 2\int_0^R \, r dr \int_0^\pi d\varphi = 2\left(\tfrac{R^2}{2}\right)\pi = \pi R^2 \tag{A.52}$$

in agreement with Stokes' theorem.

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