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Physics—Problems, Solutions, and Computer Calculations

Volume 2 Waves, Sound, Electricity, Magnetism, and Optics

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We dedicate this book to our wives, husband, and all our children for their encouragement and understanding.

Preface

The book is an introductory physics study book. The book offers students of science and engineering the basic concepts and principles of introductory physics, presenting problems and their solutions by analytical and computer calculations. It is for introductory physics learning at early undergraduate university education.

The introductory physics topics are divided into two volumes:

Volume I Mechanics, properties of matter, and heat
Volume II Waves, sound, electricity, magnetism, and optics—which is the present volume.

Each chapter begins with the main points of the topic. These are summaries of concepts, principles, definitions, and formulae of the topic. Then, problems are posed and solved. Steps are detailed so that reasoning and understanding are built. Many figures are drawn to help in visualizing the physics problems and solutions. Calculations and solutions are also performed by computer using wxMaxima to instill computational skills. An appendix *Introduction to wxMaxima* is included to get students started with the software. Calculations by wxMaxima achieved the solutions themselves or for rechecking the values obtained analytically.

Our belief is that success in solving physics problems by analysis or computer calculation boosts confidence and motivation, a sense of victory, and a sense of "I can do this, let me try the other". In computer calculation, changing the values of a few physical quantities and redoing the calculation might change the physical scene into a new one or into a hard to comprehend situation. We are tempted to explore and experiment with different physical scenes and be creative, albeit by computer.

Chapter 1 solves problems on traveling waves, wave equations, and harmonic waves. Amplitude, angular frequency, propagation constant, speed, and direction of travel of the wave are determined from its equation. Animations of traveling waves are presented.

Chapter 2 solves problems on sound waves in air, their displacement, and pressure waves. Speed of sound in various media, intensity, and intensity level of sound are considered. Doppler's effects due to the relative motion of the sound source and observer are also discussed.

Problems on the superposition of waves and stationary waves are solved in Chap. 3. These include stationary waves in air column and string. Nodes and antinodes of the stationary waves are identified. Animations of these stationary waves are presented for insight into the physics.

Problems on electricity are solved in Chaps. 4–9. Chapter 4 discusses problems on electric charge, electrostatic force, and electric field. Vector additions and methods of calculus are used to calculate some of the electric fields.

Chapter 5 solves problems on Gauss's law and its application. Gauss's law states that electric flux through a closed surface is equal to the electric charge enclosed by the surface divided by the permittivity of free space. Using Gauss's law, electric fields of some symmetric charge distributions are calculated.

Chapter 6 solves problems on electric potential energy, electric potential difference, and electric potential. Every point in a region of an electric field is associated with an electric potential which is electric potential energy per unit charge at the point.

Chapter 7 discusses problems on capacitance, equivalent capacitance of capacitors in series and parallel, and energy in charged capacitors. Also discussed is the effect of inserting dielectric material between plates of capacitor.

Chapter 8 solves problems on electric current, current density, resistance, resistivity, and Ohm's law. Problems on increase in resistance due to rise in temperature, resistance temperature coefficient, and dissipation of electrical power by resistors are also solved.

Chapter 9 solves problems on direct current circuits by applying Kirchhoff's rules. The rules are (1) the sum of the currents into any junction is zero, and (2) the sum of potential differences across each element around a closed loop is zero. Problems to determine equivalent resistance of resistors in series and in parallel and to determine current and charge in direct current *RC* circuits are also tackled.

Problems on magnetism are solved in Chaps. 10–14. Problems on magnetic forces due to moving charged particles and current carrying conductors in magnetic fields are solved in Chap. 10. The torque due to the magnetic moment of current carrying loop in magnetic field is also discussed.

Chapter 11 solves problems on magnetic fields created by current carrying conductors and loops. The Biot–Savart law is applied to determine the magnetic fields. Magnetic fields in a current carrying solenoid and toroid are determined by applying Ampere's law.

Problems related to magnetic materials and how magnetic induction, magnetic field strength, and magnetization are affected when the materials are inserted in the core of current carrying solenoid and toroid are solved in Chap. 12.

Chapter 13 solves problems related to emf induced by changing the magnetic flux. Faraday's law states that the emf induced is equal to the negative time rate of change of the magnetic flux. Emf is induced in a moving conductor when the conductor cuts through the magnetic field lines. Emf is also induced in a rotating conducting loop when the loop cuts through the magnetic field lines.

Chapter 14 solves problems on electric inductance—a measure of resistance of a conducting coil to change in current or magnetic flux linkages per unit current of the coil. Problems on self- and mutual inductance, energy in inductor, and direct current *RL* circuit are solved.

Chapter 15 solves problems on series *RLC* alternating current circuits. Inductive and capacitive reactance, impedance, phase angle, power factor, root mean square current, and average power of the circuits are determined.

Chapter 16 solves problems on plane electromagnetic wave, associated Poynting vector, and radiation pressure. These include the determination of electric and magnetic field amplitudes and directions, intensity, energy density, and direction of propagation of the electromagnetic waves. An animation of a traveling plane electromagnetic wave is presented.

Problems on geometrical or ray optics are solved in Chaps. 17 and 18. Chapter 17 solves problems on light reflection, refraction, total internal reflection, dispersion, and polarization.

Chapter 18 solves problems on image formation by mirrors, spherical surfaces, and lenses using geometrical or ray optics. Calculations of image size, location, and magnification are performed. Spherical mirror, refraction at a spherical surface, lens maker, and thin lens equations are applied.

Problems on wave optics are solved in Chaps. 19 and 20. Chapter 19 solves problems on interference of light, a phenomenon due to the superposition of coherent lights. These include interference in Young's double-slit experiment, thin film, lens coating, air wedge, and Newton's rings experiment.

Problems on diffraction of light are solved in Chap. 20, the last chapter. Diffraction is the bending or spreading of light at an aperture or obstacle. Problems on diffraction by a single slit and diffraction by a grating and its resolving power are discussed.

We wish to acknowledge the advice from several of our colleagues and undergraduate students on the idea of the book. We are also grateful to the editorial staff of Springer Nature for their support.

Johor Bahru, Malaysia
2023

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Roslinda Zainal

About This Book

The book offers students of science and engineering the basic concepts and principles of introductory physics, presenting problems, and their solutions by analytical and computer calculations. It is for introductory physics learning in the first undergraduate year of university education. This volume covers topics of waves, sound, electricity, magnetism, and optics. Each chapter begins with the main points of the topic. These are summaries of concepts, principles, definitions, and formulae of the topic. Then, problems are posed and solved. Steps are detailed so that reasoning and understanding are built. There are 250 worked problems and 100 exercises in this volume. There are 280 figures drawn to help visualize the physics problem and solution. Calculation and solution are also performed by computer using wxMaxima to provide insight and instill computational skills. The knowledge and skills presented by the book are important foundations for further studies in science or engineering. Physics teachers would also find the book useful for their instruction.

Contents

About the Authors

Dr. Wan Muhamad Saridan Wan Hassan was a former associate professor and head of Department of Physics at Universiti Teknologi Malaysia, Skudai, Malaysia. He holds a bachelor's degree from Universiti Teknologi Malaysia, master of science in physics from University of California at Riverside, and Ph.D. from University of Aberdeen, Scotland. His fields of expertise are radiation physics and medical imaging. He retired from the Universiti Teknologi Malaysia in 2020.

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Mr. Muhammad Zaki Yaacob was a former physics senior lecturer at Universiti Teknologi Malaysia. He has B.Sc. (Hons) Universiti Kebangsaan Malaysia and M.Sc. (Physics) University of Sussex, England, degrees. His fields of expertise are solid state physics and electronics.

Roslinda Zainal is presently the Head of Physics Department, Universiti Teknologi Malaysia. She has B.Sc. (Industrial Physics), M.Sc. (Physics), and Ph.D. (Physics) degrees from Universiti Teknologi Malaysia. Her fields of expertise are electronics and lasers.

Chapter 1
Waves

Abstract This chapter solves problems on traveling waves, wave equations, and harmonic waves. Amplitude, angular frequency, propagation constant, speed, and direction of travel of the wave are determined from its equation. Animations of traveling waves are presented. Both solutions by analysis and computer calculation via wxMaxima are presented.

1.1 Basic Concepts and Formulae

(1) A transverse wave is a wave in which particles move or vibrate perpendicular to the direction of the wave propagation. Examples of transverse waves are waves of a stretched string and electromagnetic waves. The wave propagation along a string is perpendicular to the vibrations of a particle of the string. In electromagnetic wave, the electric and magnetic field vibrations are perpendicular to the direction of the wave propagation.

(2) A longitudinal wave is a wave in which the particles move or vibrate in a direction parallel to the wave. Sound wave is a longitudinal wave. In a sound wave, the air molecules or the air layer vibrations are in the same direction as the wave propagation direction.

(3) A one-dimensional wave propagating with speed v in the positive x direction is represented by

$$y(x, t) = f(x - vt). \tag{1.1}$$

It is a function of $x - vt$ where v is the speed of the wave, x is position, and t is time. $y(x, t)$ represents the wave; it is the particle displacement of a stretched string, the air pressure or air layer displacement in a sound wave, or the electric or the magnetic field in an electromagnetic wave. A profile or a snapshot of a wave is obtained if time t_0 is chosen and $y(x, t = t_0)$ is plotted.

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(4) Superposition principle states that the resultant wave is the addition of waves. When two waves superpose, interference could occur. The interference could be constructive or destructive.

(5) The speed of transverse wave in a string with tension F and mass per unit length μ is

$$v = \sqrt{\frac{F}{\mu}}. \tag{1.2}$$

(6) When a propagating pulse in a string hits a fixed end, the pulse is reflected and over turned. If the pulse hits a free end, it will be reflected but not overturned.

(7) Wave function of a one-dimensional harmonic wave moving to the right is

$$y(x, t) = A \sin \frac{2\pi}{\lambda}(x - vt) = A \sin(kx - \omega t), \tag{1.3}$$

where A is amplitude, λ is wavelength, k is propagation constant or wavenumber, and ω is angular frequency. If T is the period (the time for the wave to move a distance of one wavelength) and f is the frequency, then,

$$v = \frac{\lambda}{T} = \lambda f, \tag{1.4}$$

$$k = \frac{2\pi}{\lambda}, \tag{1.5}$$

$$\omega = \frac{2\pi}{T} = 2\pi f. \tag{1.6}$$

(8) A propagating wave in the positive x direction is a function of $x - vt$, $vt - x$, $kx - \omega t$, $\omega t - kx$, $t - x/v$ or $x/v - t$, while the one propagating in the negative x direction is a function of $x + vt$, $vt + x$, $kx + \omega t$, $\omega t + kx$, $t + x/v$ or $x/v + t$. Here, v, k, and ω are speed, propagation constant, and angular frequency of the wave, respectively.

(9) Power transferred by a harmonic wave in a stretched string is

$$P = \frac{1}{2}\mu\omega^2 A^2 v. \tag{1.7}$$

(10) The wave function $y(x, t)$ satisfies the linear wave equation,

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \quad \text{or} \quad \frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}, \tag{1.8}$$

where v is the speed of the wave.

(11) Inverse square law: A scientific law states that a quantity is inversely proportional to the square of the distance. For example, intensity of a wave I at a distance d from the source of the wave,

$$I \propto \frac{1}{d^2}. \tag{1.9}$$

Thus, at two distances, the law is written as

$$\frac{I_1}{I_2} = \frac{d_2^2}{d_1^2} \quad \text{or} \quad I_1 d_1^2 = I_2 d_2^2.$$

1.2 Problems and Solutions

Problem 1.1 Which of the following is not a propagating wave?

(a) $y(x, t) = 0.7 \sin(2x - 3t)$
(b) $y(x, t) = 5 \cos(3x) \sin(5t)$
(c) $y(x, t) = 3 \cos(2t + x)$
(d) $y(x, t) = 4e^{-3(x-2t)^2}$
(e) $y(x, t) = 12 \cos^2(t + 5x)$
(f) $y(x, t) = 2 \sin(3x + 10t)$.

Solution

Item (b) is not a propagating wave. A propagating wave is represented by

$$y(x, t) = f(x \pm vt),$$

where v is velocity of the wave, x is coordinate of position, and t is time. The plus sign is for a wave propagating in the negative x direction, while the negative sign is for the one in the positive x direction. In general, any function of $(Cx \pm Dt)$ where C and D are constants, is a propagating wave. Thus, all except (b) are propagating waves. A way to see that a wave is propagating is by plotting the wave profiles at increasing times. See Problems 1.8 and 1.9 as complete examples, in which we show that (f) is a sinusoidal wave propagating in the negative x direction, while (d) is a pulse propagating in the positive x direction.

Problem 1.2 Show that $y(x, t) = 3e^{-(x+7t)^2}$ satisfies the linear wave equation.

Solution

The linear wave equation is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2},$$

where v is the speed of the wave. For this problem,

$$\frac{\partial y}{\partial t} = -2(x+7t)(7)(3)e^{-(x+7t)^2} = -42\ (x+7t)e^{-(x+7t)^2},$$

$$\frac{\partial^2 y}{\partial t^2} = -42\left[7e^{-(x+7t)^2} - 2(x+7t)^2(7)e^{-(x+7t)^2}\right]$$

$$= -294\left[e^{-(x+7t)^2} - 2(x+7t)^2 e^{-(x+7t)^2}\right].$$

$$\frac{\partial y}{\partial x} = -2(x+7t)(3)e^{-(x+7t)^2} = -6\ (x+7t)e^{-(x+7t)^2},$$

$$\frac{\partial^2 y}{\partial x^2} = -6\left[e^{-(x+7t)^2} - 2\ (x+7t)^2 e^{-(x+7t)^2}\right].$$

This means that

$$\frac{\partial^2 y}{\partial t^2} \Big/ \frac{\partial^2 y}{\partial x^2} = \frac{-294}{-6} = 49,$$

and

$$\frac{\partial^2 y}{\partial t^2} = 49\frac{\partial^2 y}{\partial x^2}.$$

Therefore, $y(x,t) = 3e^{-(x+7t)^2}$ satisfies the linear wave equation and the wave velocity is $\sqrt{49} = 7$ units of velocity.

- wxMaxima codes:

```
(%i1) y: 3*exp(-(x+7*t)^2);
(y)   3*%e^(-(x+7*t)^2)
(%i3) diff(y,t); diff(y, t, 2);
(%o2) -42*(x+7*t)*%e^(-(x+7*t)^2)
(%o3) 588*(x+7*t)^2*%e^(-(x+7*t)^2)-294*%e^(-(x+7*t)^2)
(%i5) diff(y, x); diff(y, x, 2);
(%o4) -6*(x+7*t)*%e^(-(x+7*t)^2)
(%o5) 12*(x+7*t)^2*%e^(-(x+7*t)^2)-6*%e^(-(x+7*t)^2)
(%i7) diff(y,t,2)/diff(y,x,2)$ radcan(%);
(%o7) 49
(%i8) sqrt(%);
(%o8) 7
```

Comments on the codes:

(%i1) Assign the wave equation.

(%i3) Partial differentiation of the wave equation with respect to t and twice partial differentiation of the wave equation with respect to t.

(%i5) Partial differentiation of the wave equation with respect to x and twice partial differentiation of the wave equation with respect to x.

(%i7) Division of second partial derivative with respect to t by second partial derivative with respect to x, followed by simplification by radcan(%).

(%i8) Calculate the square root of 49.

Problem 1.3 A harmonic wave in one dimension is given by

$$y = 0.3\sin(4x + 8t),$$

where y and x are in meters and t in seconds. Determine

(a) wavelength
(b) frequency
(c) velocity
(d) direction of propagation of the wave.

Solution

(a) We compare general wave equations with the harmonic wave,

$$y = A\sin(kx + \omega t) = A\sin\left(\frac{2\pi}{\lambda}x + 2\pi f t\right),$$
$$y = 0.3\sin(4x + 8t).$$

From the comparison, the wavelength λ is calculated as follows:

$$k = \frac{2\pi}{\lambda} = 4.0\ \text{m}^{-1},$$
$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{4.0\ \text{m}^{-1}} = 1.6\ \text{m}.$$

(b) The frequency of the wave f is calculated as follows:

$$\omega = 2\pi f = 8.0\ \text{s}^{-1},$$
$$f = \frac{\omega}{2\pi} = \frac{8.0\ \text{s}^{-1}}{2\pi} = 1.3\ \text{Hz}.$$

(c) The velocity of the wave v is

$$v = \lambda f = \frac{\lambda}{2\pi}\cdot 2\pi f = \frac{\omega}{k} = \frac{8.0\ \text{s}^{-1}}{4.0\ \text{m}^{-1}} = 2.0\ \text{m s}^{-1}.$$

(d) The harmonic wave equation is in the form of $f(x + vt)$. Hence, the wave is propagating to the left, i.e. moving in the negative x direction.

- wxMaxima codes:

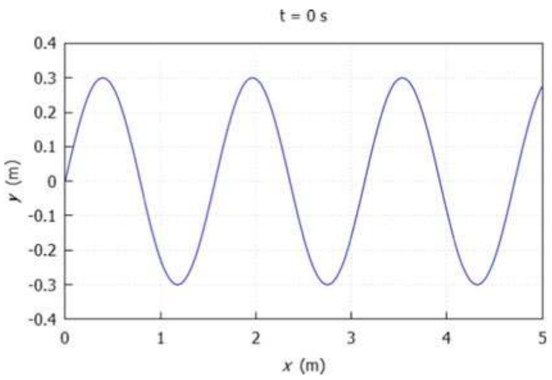
```
(%i1) fpprintprec:5;
(fpprintprec)     5
(%i2) k:4;
(k)   4
(%i4) lambda: 2*%pi/k$ float(%);
(%o4) 1.5708
(%i5) omega:8;
(omega) 8
(%i7) f: omega/(2*%pi)$ float(%);
(%o7) 1.2732
(%i8) v: lambda*f;
(v)   2
```

Comments on the codes:

(%i1) Set the floating point print precision to 5. With the fpprintprec: 5; command, numerical output is set to 5 digits.
(%i2) Assign propagation constant k the value 4.
(%i4) Calculate λ. Part (a).
(%i5) Assign ω the value 8.
(%i7) Calculate f. Part (b).
(%i8) Calculate v. Part (c).

- Animation of $y = 0.3\,\sin(4x + 8t)$ by wxMaxima:

```
(%i1) fpprintprec:2;
(fpprintprec)     2
(%i2) with_slider_draw(
         t, makelist(i,i,0,3,0.1),
         title=concat("t = ",t," s"),
         explicit(0.3*sin(4*x + 8*t), x,0,5),
         grid=true,
         yrange=[-0.4,0.4],
         xlabel="{/Helvetica-Italic x} (m)",
         ylabel="{/Helvetica-Italic y} (m)");
```



Comments on the codes:

To run the animation, copy the codes to the wxMaxima command window; press <shift> and <enter> keys simultaneously to run the codes; right click the graphic that appears and choose *Start Animation.*

Problem 1.4 A simple harmonic wave in one dimension is given by,

$$y = 1.5 \sin(4t - 8x),$$

where y and x are in centimeters, whereas t is in seconds. Determine

(a) wavelength
(b) frequency
(c) speed
(d) propagation direction of the wave.

Solution

(a) We compare general wave equations with the simple harmonic wave,

$$y = A \sin(\omega t - kx) = A \sin\left(2\pi f t - \frac{2\pi}{\lambda} x\right),$$
$$y = 1.5 \sin(4t - 8x).$$

It follows that the propagation constant is

$$k = \frac{2\pi}{\lambda} = 8.0 \text{ cm}^{-1}.$$

The wavelength is

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{8.0 \text{ cm}^{-1}} = 0.79 \text{ cm}.$$

(b) The angular frequency and frequency are

$$\omega = 2\pi f = 4.0 \text{ s}^{-1},$$
$$f = \frac{\omega}{2\pi} = \frac{4.0 \text{ s}^{-1}}{2\pi} = 0.64 \text{ s}^{-1}.$$

(c) The speed of the wave is

$$v = \frac{\omega}{k} = \frac{4.0 \text{ s}^{-1}}{8.0 \text{ cm}^{-1}} = 0.50 \text{ cm s}^{-1}.$$

(d) The simple harmonic wave equation is in the form of $f(\omega t - kx)$. The wave is moving in the positive x direction.

- wxMaxima codes:

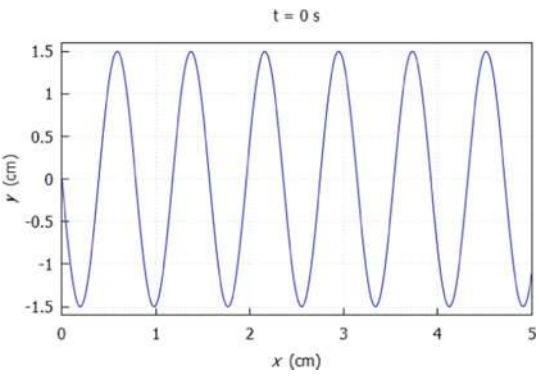
```
(%i1) fpprintprec:5;
(fpprintprec)     5
(%i2) k:8;
(k)    8
(%i4) lambda: 2*%pi/k$ float(%);
(%o4) 0.7854
(%i5) omega:4;
(omega) 4
(%i7) f: omega/(2*%pi)$ float(%);
(%o7) 0.63662
(%i9) v: omega/k$ float(%);
(%o9) 0.5
```

Comments on the codes:

(%i1) Set the floating point print precision to 5.
(%i2) Assign k.
(%i4) Calculate λ.
(%i5) Assign ω.
(%i7) Calculate f.
(%i9) Calculate v.

- Animation of $y = 1.5\sin(4t - 8x)$, by wxMaxima:

```
(%i1) fpprintprec:2;
(fpprintprec)     2
(%i2) with_slider_draw(
        t, makelist(i,i,0,3,0.1),
        title=concat("t = ",t," s"),
        explicit(1.5*sin(4*t - 8*x), x,0,5),
        grid=true,
        yrange=[-1.6,1.6],
        xlabel="{/Helvetica-Italic x}  (cm)",
        ylabel="{/Helvetica-Italic y}  (cm)");
```



Comments on the codes:

To run the animation, copy the codes to the wxMaxima command window; press <shift> and <enter> keys simultaneously to run the codes; right click the graphic that appears and choose *Start Animation.*

Problem 1.5 A harmonic wave is given by,

$$y = 0.3\sin(4x + 8t),$$

where y and x are in meters and t in seconds. The wave is propagating in a medium. What is

(a) the maximum speed of a particle in the medium?
(b) the maximum acceleration of a particle in the medium?

Solution

(a) The speed of the particle can be obtained by differentiating the given equation with respect to time. We get

$$\frac{dy}{dt} = 2.4\cos(4x + 8t).$$

This gives the speed of a particle at position x and time t. The maximum speed of a particle is 2.4 m s^{-1}.

(b) Acceleration of a particle is obtained by differentiating twice the given equation with respect to time. The acceleration of a particle at position x and time t is

$$\frac{d^2y}{dt^2} = -19.2\ \sin(4x + 8t).$$

The maximum acceleration of the particle is 19.2 m s^{-2}.

- wxMaxima codes:

```
(%i1) y(x,t):=0.3*sin(4*x + 8*t);
(%o1) y(x,t):=0.3*sin(4*x+8*t)
(%i2) diff(y(x,t),t,1);
(%o2) 2.4*cos(4*x+8*t)
(%i3) diff(y(x,t),t,2);
(%o3) -19.2*sin(4*x+8*t)
```

Comments on the codes:

(%i1) Define the wave function.
(%i2) Differentiate the wave function with respect to *t*.

(%i3) Twice differentiation of the wave function with respect to t.

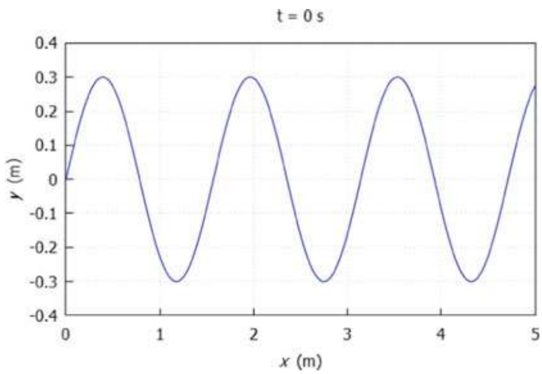
Consider any point of the wave, say point $x = 0$. Then, the displacement and the acceleration of the particle as time goes are

$$y = 0.3\ \sin(8t),$$
$$a = \frac{d^2y}{dt^2} = -19.2\ \sin(8t) = -64 \times 0.3\ \sin(8t) = -64y.$$

The acceleration is proportional to negative of the displacement. This means that the particle at the point oscillates according to a simple harmonic motion. Thus, the wave is called a harmonic wave.

- Animation of $y = 0.3\sin(4x + 8t)$, by wxMaxima:

```
(%i1) fpprintprec:2;
(fpprintprec)     2
(%i2) with_slider_draw(
         t, makelist(i,i,0,3,0.1),
         title=concat("t = ",t," s"),
         explicit(0.3*sin(4*x + 8*t), x,0,5),
         grid=true,
         yrange=[-0.4,0.4],
         xlabel="{/Helvetica-Italic x} (m)",
         ylabel="{/Helvetica-Italic y} (m)");
```



Comments on the codes:

To run the animation, copy the codes to the wxMaxima command window; press <shift> and <enter> keys simultaneously to run the codes; right click the graphic that appears and choose *Start Animation.*

Problem 1.6 The solar intensity on Earth is 1340 W m^{-2}. What is the solar intensity on Mars? Distance from Earth to the Sun is 93 million miles and the distance from Mars to the Sun is 142 million miles.

Solution

Using the inverse square law, $I \propto 1/r^2$ (Eq. 1.9), intensity is inversely proportional to distance square, we write

$$\frac{I_{Earth}}{I_{Mars}} = \frac{r_{Mars}^2}{r_{Earth}^2},$$

where I_{Earth} and I_{Mars} are the solar intensities on Earth and Mars, while r_{Earth} and r_{Mars} are the distances from the Sun to the Earth and Mars, respectively. The solar radiation intensity on Mars is calculated as follows:

$$\frac{1340 \text{ W m}^{-2}}{I_{Mars}} = \frac{(142 \text{ million miles})^2}{(93 \text{ million miles})^2}$$
$$I_{Mars} = 575 \text{ W m}^{-2}.$$

- wxMaxima codes:

```
(%i5) fpprintprec:5; ratprint:false; IEarth:1340; rEarth:93; rMars:142;
(fpprintprec)     5
(ratprint)  false
(IEarth)    1340
(rEarth)    93
(rMars)     142
(%i7) solve(IEarth/IMars=rMars^2/rEarth^2, IMars)$ float(%);
(%o7) [IMars=574.77]
```

Comments on the codes:

(%i5) Set the floating point precision to 5 and internal rational numbers print to false, and assign values of I_{Earth}, r_{Earth}, and r_{Mars}.

(%i7) Use the solve function to solve $I_{\text{Earth}}/I_{\text{Mars}} = r_{\text{Mars}}^2/r_{\text{Earth}}^2$ to find I_{Mars} and get the decimal value.

Problem 1.7 The intensity of light of a lamp at a distance of 10 m away is 2.0 W m^{-2}. What is the intensity at a distance of 20 m away?

Solution

Using the inverse square law, $I \propto 1/r^2$, we write

$$\frac{I_1}{I_2} = \frac{r_2^2}{r_1^2},$$

where I_1 is the intensity at r_1 away and I_2 is at r_2 away. We have

$$\frac{2.0 \text{ W m}^{-2}}{I_2} = \frac{(20 \text{ m})^2}{(10 \text{ m})^2}.$$

The light intensity at 20 m from the lamp is

$$I_2 = 0.50 \text{ Wm}^{-2}.$$

- wxMaxima codes:

```
(%i5) fpprintprec:5; ratprint:false; I1:2; r1:10; r2:20;
(fpprintprec)     5
(ratprint)  false
(I1)  2
(r1)  10
(r2)  20
(%i7) solve(I1/I2=r2^2/r1^2, I2)$  float(%);
(%o7) [I2=0.5]
```

Comments on the codes:

(%i5) Set the floating point precision to 5 and internal rational number print to false, and assign values of I_1, r_1, and r_2.
(%i7) Solve $I_1/I_2 = r_2^2/r_1^2$ for I_2 and get the decimal value.

Problem 1.8 Show graphically that the wave $y(x, t) = 2\sin(3x + 10t)$ is propagating in the negative x direction. Calculate the wave speed.

Solution

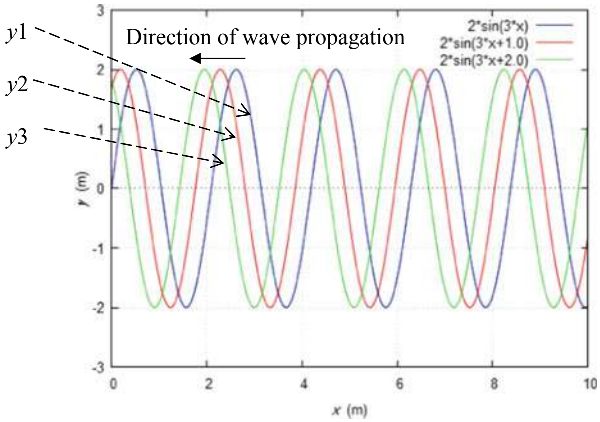
A way to determine the direction of wave propagation is by plotting wave profiles at increasing consecutive times. A wave profile is a snapshot of the wave at a specific time. From the profiles, we can determine whether the wave is propagating to the positive or negative x direction.

For this problem, we plot wave profiles at time $t = 0$, 0.1, and 0.2 s and labeled the profiles as $y1$, $y2$, and $y3$:

$$\begin{aligned} y(x,t) &= 2\sin(3x + 10t), \\ y1 &= 2\sin(3x), \\ y2 &= 2\sin(3x + 1), \\ y3 &= 2\sin(3x + 2). \end{aligned}$$

- Plot by wxMaxima:

```
(%i1) y(x,t):= 2*sin(3*x+10*t);
(%o1) y(x,t):=2*sin(3*x+10*t)
(%i4) y1: y(x,0); y2: y(x,0.1); y3: y(x,0.2);
(y1)  2*sin(3*x)
(y2)  2*sin(3*x+1.0)
(y3)  2*sin(3*x+2.0)
(%i5) wxplot2d([y1,y2,y3], [x,0,10], [y,-3,3], grid2d,
[xlabel,"{/Helvetica-Italic x} (m)"],[ylabel,"{/Helvetica-Italic y} (m)"]);
```



Comments on the codes:

(%i1) Define $y(x, t)$.
(%i4) Assign $y1$, $y2$, and $y3$.
(%i5) Plot $y1$, $y2$, and $y3$ for $0 \leq x \leq 10$.

It can be seen that profiles $y1$, $y2$, and $y3$ move in the negative x direction.

The speed of the wave is determined as follows. By comparing the wave and the general wave equation, wave propagation constant k and angular frequency ω can be determined:

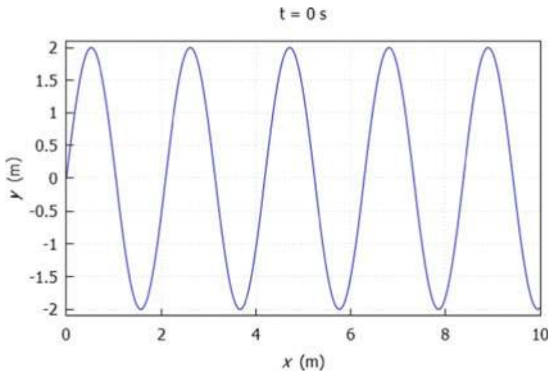
$$\begin{aligned} y(x, t) &= 2\sin(3x + 10t), \\ y(x, t) &= A\sin(kx + \omega t), \\ k &= 3.0 \text{ m}^{-1}, \\ \omega &= 10 \text{ s}^{-1}. \end{aligned}$$

The speed of the wave is

$$v = \frac{\omega}{k} = \frac{10 \text{ s}^{-1}}{3.0 \text{ m}^{-1}} = 3.3 \text{ m s}^{-1}.$$

- An animation of the wave $y(x, t) = 2\sin(3x + 10t)$ propagating to the left by wxMaxima:

```
(%i1) fpprintprec:2;
(fpprintprec)     2
(%i2) with_slider_draw(
          t, makelist(i,i,0,3,0.1),
          title=concat("t = ",t," s"),
          explicit(2*sin(3*x +10*t), x,0,10),
          grid=true,
          yrange=[-2.1,2.1],
          xlabel="{/Helvetica-Italic x}  (m)",
          ylabel="{/Helvetica-Italic y}  (m)");
```



Comments on the codes:

To run the animation, copy the codes to the wxMaxima command window; press <shift> and <enter> keys simultaneously to run the codes; right click the graphic that appears and choose *Start Animation.*

Problem 1.9

(a) Plot wave profiles of

$$y(x, t) = 4e^{-3(x-2t)^2},$$

at $t = 0$ and $t = 1.0$ s. In which direction is the wave moving?

(b) Show that $y(x, t)$ satisfies the wave equation.

(c) Calculate the wave speed.

Solution

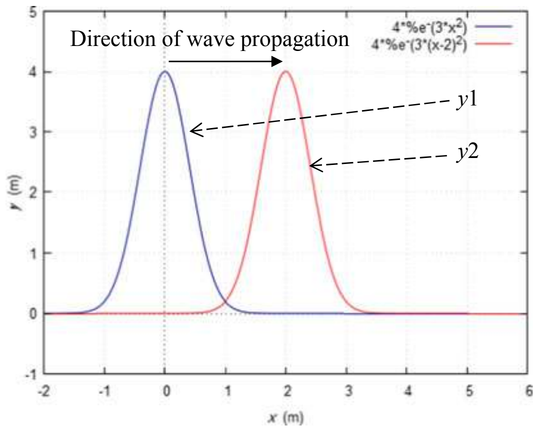
(a) Wave profiles are obtained by substituting values for time into the equation. We labeled the profiles as $y1$ and $y2$,

$$\text{at } t = 0 \text{ s}, \quad y1 = y(x, 0) = 4e^{-3x^2}$$

$$\text{at } t = 1.0 \text{ s}, \quad y2 = y(x, 1) = 4e^{-3(x-2)^2}$$

- Wave profile plots by wxMaxima:

```
(%i1) y(x,t) := 4*exp(-3*(x-2*t)^2);
(%o1) y(x,t):=4*exp((-3)*(x-2*t)^2)
(%i3) y1: y(x,0); y2: y(x,1);
(y1)  4*%e^(-3*x^2)
(y2)  4*%e^(-3*(x-2)^2)
(%i4) wxplot2d([y1,y2],[x,-2,6], [y,-1,5], grid2d,
[xlabel,"{/Helvetica-Italic x}(m)"], [ylabel,"{/Helvetica-Italic y} (m)"]);
```



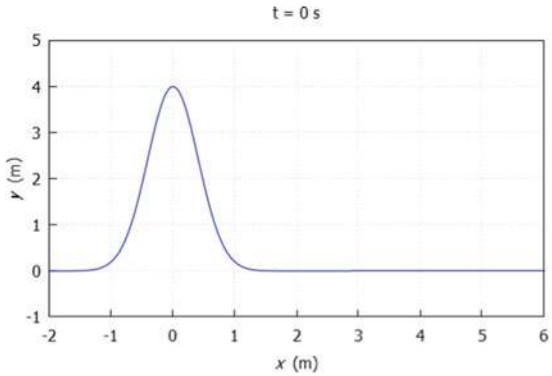
Comments on the codes:

(%i1) Define $y(x, t)$.
(%i3) Assign $y1$ and $y2$.
(%i4) Plot $y1$ and $y2$ for $-4 \leq x \leq 4$ m.

It can be seen that the wave is moving in the positive x direction. The wave is a propagating pulse in the positive x direction.

- An animation of the pulse $y(x, t) = 4e^{-3(x-2t)^2}$ moving to the right by wxMaxima:

```
(%i1) fpprintprec:2;
(fpprintprec)     2
(%i2) with_slider_draw(
         t, makelist(i,i,0,3,0.1),
         title=concat("t = ",t," s"),
         explicit(4*exp(-3*(x-2*t)^2), x,-2,6),
         grid=true,
         yrange=[-1,5],
         xlabel="{/Helvetica-Italic x} (m)",
         ylabel="{/Helvetica-Italic y} (m)");
```



Comments on the codes:

To run the animation, copy the codes to the wxMaxima command window; press <shift> and <enter> keys simultaneously to run the codes; right click the graphic that appears and choose *Start Animation.*

(b) The wave equation is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}. \tag{1.10}$$

It must be shown that the expression,

$$y(x, t) = 4e^{-3(x-2t)^2}, \tag{1.11}$$

satisfies the wave equation. That is, showing that Eq. (1.11) satisfies Eq. (1.10). Calculate the second partial derivative of y with respect to x,

$$\frac{\partial y}{\partial x} = 4e^{-3(x-2t)^2} \cdot [-6(x-2t)] = -24\,(x-2t)e^{-3(x-2t)^2},$$

$$\frac{\partial^2 y}{\partial x^2} = -24\ [e^{-3(x-2t)^2} - 6(x-2t)e^{-3(x-2t)^2}(x-2t)]$$
$$= -24\ [e^{-3(x-2t)^2} - 6(x-2t)^2 e^{-3(x-2t)^2}].$$

Calculate the second derivative of y with respect to t,

$$\frac{\partial y}{\partial t} = 4e^{-3(x-2t)^2}.[-6(x-2t)(-2)] = 48\ (x-2t)e^{-3(x-2t)^2},$$
$$\frac{\partial^2 y}{\partial t^2} = 48\ [-2e^{-3(x-2t)^2} + 12(x-2t)e^{-3(x-2t)^2}(x-2t)]$$
$$= -96\ [e^{-3(x-2t)^2} - 6(x-2t)^2 e^{-3(x-2t)^2}].$$

Using these results and Eq. (1.10), we have

$$\frac{\partial^2 y}{\partial t^2} / \frac{\partial^2 y}{\partial x^2} = v^2 = 4.$$

This means that (1.11) satisfies Eq. (1.10) with $v^2 = 4$.

- wxMaxima codes:

```
(%i1) y(x,t) := 4*exp(-3*(x-2*t)^2);
(%o1) y(x,t):=4*exp((-3)*(x-2*t)^2)
(%i3) diff(y(x,t),x,1)$ radcan(%);
(%o3) -(24*x-48*t)*%e^(-3*x^2+12*t*x-12*t^2)
(%i5) diff(y(x,t),x,2)$ radcan(%);
(%o5) (144*x^2-576*t*x+576*t^2-24)*%e^(-3*x^2+12*t*x-12*t^2)
(%i7) diff(y(x,t),t,1)$ radcan(%);
(%o7) (48*x-96*t)*%e^(-3*x^2+12*t*x-12*t^2)
(%i9) diff(y(x,t),t,2)$ radcan(%);
(%o9) (576*x^2-2304*t*x+2304*t^2-96)*%e^(-3*x^2+12*t*x-12*t^2)
(%i11) diff(y(x,t),t,2)/diff(y(x,t),x,2)$ radcan(%);
(%o11)4
```

Comments on the codes:

(%i1)	Define $y(x, t)$.
(%i3), (%i5)	Differentiate $y(x, t)$ once and twice with respect to x and simplify.
(%i7), (%i9)	Differentiate $y(x, t)$ once and twice with respect to t and simplify.
(%i11)	Calculate $\frac{\partial^2 y}{\partial t^2} / \frac{\partial^2 y}{\partial x^2}$ and simplify.

(c) The speed of the wave is

$$v^2 = 4,$$
$$v = 2.0\ \mathrm{m\ s^{-1}}.$$

Alternative method: Compare $(kx - \omega t)$ with $(x - 2t)$ in Eq. (1.11). We get the propagation constant and angular frequency of the wave. The speed of the wave is calculated as follows,

$$k = 1.0\ \text{m}^{-1}, \quad \omega = 2.0\ \text{s}^{-1},$$
$$v = \frac{\omega}{k} = \frac{2.0\ \text{s}^{-1}}{1.0\ \text{m}^{-1}} = 2.0\ \text{m s}^{-1}.$$

Problem 1.10 You will learn later in Modern Physics or Quantum Mechanics that the wave function,

$$\psi = Ae^{i(kx-\omega t)},$$

represents a particle moving in the positive x direction. Show that ψ satisfies the wave equation.

Solution

We differentiate ψ twice with respect to x and with respect to t and see if ψ satisfies the wave equation:

$$\frac{\partial \psi}{\partial t} = -i\omega Ae^{i(kx-\omega t)}, \quad \frac{\partial^2 \psi}{\partial t^2} = -\omega^2 Ae^{i(kx-\omega t)}.$$
$$\frac{\partial \psi}{\partial x} = ikAe^{i(kx-\omega t)}, \quad \frac{\partial^2 \psi}{\partial x^2} = -k^2 Ae^{i(kx-\omega t)}.$$

This means that

$$\frac{\partial^2 \psi}{\partial t^2} = \left(\frac{\omega}{k}\right)^2 \frac{\partial^2 \psi}{\partial x^2} = v^2 \frac{\partial^2 \psi}{\partial x^2}.$$

Indeed, ψ satisfies the wave equation.

- wxMaxima codes:

```
(%i1) psi(x,t):= A*exp(%i*(k*x-omega*t));
(%o1) psi(x,t):=A*exp(%i*(k*x-omega*t))
(%i2) expression1: diff(psi(x,t),t,2);
(expression1) -A*omega^2*%e^(%i*(k*x-omega*t))
(%i3) expression2: diff(psi(x,t),x,2);
(expression2) -A*k^2*%e^(%i*(k*x-omega*t))
(%i4) expression1/expression2;
(%o4) omega^2/k^2
```

Comments on the codes:

(%i1)	Define $\psi(x, t)$.
(%i2), (%i3)	Calculate $\partial^2\psi/\partial t^2$ and $\partial^2\psi/\partial x^2$.
(%i4)	Calculate $\partial^2\psi/\partial t^2/\partial^2\psi/\partial x^2$.

1.3 Summary

- A one-dimensional wave propagating with speed v in the positive x direction is represented by

$$y(x, t) = f(x - vt).$$

- Wave function of a one-dimensional harmonic wave moving to the right is

$$y(x, t) = A \sin \frac{2\pi}{\lambda}(x - vt) = A \sin(kx - \omega t),$$

where A is amplitude, λ is wavelength, k is propagation constant or wavenumber, and ω is angular frequency. If T is the period and f is the frequency, then,

$$v = \frac{\lambda}{T} = \lambda f,$$
$$k = \frac{2\pi}{\lambda},$$
$$\omega = \frac{2\pi}{T} = 2\pi f.$$

- The wave function $y(x, t)$ satisfies the linear wave equation,

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2},$$

where v is the speed of the wave.

1.4 Exercises

Exercise 1.1

(a) Show that wave function

$$y = 3\ \cos(2t + x)$$

satisfies the wave equation,

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}.$$

(b) What is the speed of the wave?
(c) In which direction is the wave traveling?
(Answer: (b) $v = 2.0$ m s^{-1}; (c) negative x direction)

Exercise 1.2 A metal string of length 14 m and mass 0.30 kg is fixed between two nails. The tension in the string is 40 N. What is the speed of a pulse on this string?
(Answer: 43 m s^{-1})

Exercise 1.3 A wave traveling in one dimension is given by

$$y(x, t) = 3e^{-2(x+0.5t)^2}.$$

(a) Sketch the wave profiles at $t = 0$ and $t = 1.0$ s.
(b) In which direction is the wave moving?
(c) Show that $y(x, t)$ satisfies the wave equation.

(Answer: (b) the negative x direction)

Exercise 1.4 A traveling wave in the positive x direction is given by

$$y = f(x - vt).$$

Using partial differentiation and the chain rule of calculus, show that y satisfies the one-dimensional wave equation,

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}.$$

Exercise 1.5 A transverse harmonic wave on a stretched string has a wavelength of 0.080 m, a frequency of 160 Hz, and an amplitude of 6.0×10^{-3} m. The mass per unit length of the string is 2.0×10^{-4} kg m^{-1}. Calculate

(a) speed of the wave
(b) power transferred by the wave
(Answer: (a) 13 m s^{-1}; (b) 4.7×10^{-2} W)

Chapter 2
Sound Wave

Abstract This chapter solves problems on sound waves in air, their displacement, and pressure waves. Speed of sound in various media, intensity and intensity level of sound are considered. Doppler's effects due to the relative motion of the sound source and observer are also discussed. Both solutions by analysis and computer calculation via wxMaxima are presented.

2.1 Basic Concepts and Formulae

(1) Sound wave is a longitudinal mechanical wave, moving through a compressible medium. The speed of the wave depends on the compressibility and density of the medium. The speed of sound wave v in a medium of compressibility B and density ρ is,

$$v = \sqrt{\frac{B}{\rho}}. \tag{2.1}$$

The speed of sound in an ideal gas is,

$$v = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{\gamma RT}{M}}, \tag{2.2}$$

where γ is the ratio of molar specific heats (molar specific heat at constant pressure C_p divided by molar specific heat at constant volume C_V, that is, $\gamma = C_p/C_V$), p is the pressure, ρ is the density, T is the absolute temperature, M is the molar mass of the gas, and $R = 8.31$ J mol^{-1} K^{-1} is the molar gas constant. The ratio of molar specific heats γ is called the adiabatic constant of a gas.

For monatomic gas $\gamma = 1.67$, and for diatomic gas $\gamma = 1.40$. For air, $\gamma = 1.40$ and $M = 28.8 \times 10^{-3}$ kg mol^{-1}.

W. M. S. Wan Hassan et al., *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-43165-4_2

Speed of a longitudinal wave in a fluid is

$$v = \sqrt{\frac{B}{\rho}}, \tag{2.3}$$

where B is the bulk modulus of the fluid and ρ is its density.

Speed of a longitudinal wave in a solid rod is

$$v = \sqrt{\frac{Y}{\rho}}, \tag{2.4}$$

where Y is the Young modulus of the solid rod and ρ is its density.

Speed of a transverse wave on a string is

$$v = \sqrt{\frac{T}{\mu}}, \tag{2.5}$$

where T is tension in the string and μ is mass per unit length of the string.

The Average power of a transverse sinusoidal wave on a string is,

$$P = \frac{1}{2}\sqrt{\mu T}\,\omega^2 A^2, \tag{2.6}$$

where ω is the angular frequency and A is the amplitude of the wave.

(2) For a harmonic sound wave, change of pressure from its equilibrium value is,

$$\Delta p = \Delta p_m \sin(kx - \omega t), \tag{2.7}$$

where Δp_m is the pressure amplitude, k is the propagation constant, and ω is angular frequency.

Displacement of the air layer is,

$$s = s_m \cos(kx - \omega t), \tag{2.8}$$

where s_m is the displacement amplitude. Thus, the pressure wave has a phase difference of 90° with displacement wave.

The pressure amplitude is given by

$$\Delta p_m = \rho v \omega\, s_m = k\rho v^2 s_m = \frac{2\pi}{\lambda}\rho v^2 s_m = 2\pi\rho v f s_m. \tag{2.9}$$

where ρ is the density of air, v is the speed of sound, k is the propagation constant, λ is the wavelength, and f is the frequency of sound.

The intensity of harmonic sound wave, i.e. the sound power per unit area is

$$I = \frac{1}{2}\rho(\omega s_m)^2 v = \frac{(\Delta p_m)^2}{2\rho v}. \tag{2.10}$$

(3) The intensity level β of a sound with intensity I is defined as

$$\beta = 10\ \log_{10}\left(\frac{I}{I_0}\right) = 10\ \log_{10}\left(\frac{I}{10^{-12}\ \mathrm{W\ m^{-2}}}\right). \tag{2.11}$$

Unit of β is decibel (dB). The sound intensity $I_0 = 10^{-12}$ W m^{-2} is the threshold of a human hearing a sound, that is, the lowest intensity human could hear.

(4) Doppler effect is the change in frequency of sound heard by an observer due to relative motion between the sound source and the observer.

If the observer moves with a speed of v_o and the sound source is at rest, the frequency heard f' is

$$f' = \left(1 \pm \frac{v_o}{v}\right)f = \left(\frac{v \pm v_o}{v}\right)f, \tag{2.12}$$

where f is the frequency of sound from the source and v is the speed of sound in air. The (+) sign applies if the observer is moving toward the source and the (−) applies if the observer is moving away from the source.

If the sound source is moving with speed v_s and the observer is at rest, the frequency heard f' is

$$f' = \left(\frac{1}{1 \mp \frac{v_s}{v}}\right)f = \left(\frac{v}{v \mp v_s}\right)f. \tag{2.13}$$

The (–) sign applies if the sound source is moving toward the observer, while the (+) sign applies if the sound source is moving away from the observer.

If both observer and sound source are moving, the frequency heard by the observer f' is

$$f' = \left(\frac{v \pm v_o}{v \mp v_s}\right)f, \tag{2.14}$$

where v_o is the speed of the observer, v_s is the speed of the sound source, v is the speed of sound in air, and f is the frequency of the sound source. For the numerator, use the (+) sign if the observer is moving toward or the (−) sign if the observer is moving away from the sound source. For the denominator, use the (−) sign if the sound source is moving toward or the (+) sign if the sound source is moving away from the observer.

2.2 Problems and Solutions

Problem 2.1 A wire of length 40 cm has a mass of 1.0 g. The wire is stretched between two nails such that the tension in the wire is 2500 N. What is the speed of a transverse wave of the wire?

Solution

The mass per unit length of the wire is

$$\mu = \frac{m}{l} = \frac{1.0 \times 10^{-3}\ \text{kg}}{40 \times 10^{-2}\ \text{m}} = 2.5 \times 10^{-3}\ \text{kg m}^{-1}.$$

The speed of waves on the wire is, Eq. (2.5),

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{2500\ \text{N}}{2.5 \times 10^{-3}\ \text{kg m}^{-1}}} = 1000\ \text{m s}^{-1}.$$

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; m:1e-3; l:40e-2; T:2500;
(fpprintprec)     5
(m)    0.001
(l)    0.4
(T)    2500
(%i5) mu: m/l;
(mu)   0.0025
(%i6) v: sqrt(T/mu);
(v)    1000.0
```

Comments on the codes:

(%i4) Set the floating point print precision to 5, and assign values of mass m, length l, and tension T.
(%i5) Calculate mass per unit length μ.
(%i6) Calculate the speed of the wave v.

Problem 2.2

(a) Given the bulk modulus of water is 2.1×10^9 N m^{-2}, calculate the speed of sound in water.
(b) The speed of sound in steel is 5.9×10^3 m s^{-1} and the density of steel is 7.9×10^3 kg m^{-3}. Calculate the bulk modulus of steel.

Solution

(a) The speed of sound in water is, Eq. (2.3),

$$v = \sqrt{\frac{B_{water}}{\rho_{water}}} = \sqrt{\frac{2.1 \times 10^9 \text{ N m}^{-2}}{1000 \text{ kg m}^{-3}}} = 1450 \text{ m s}^{-1}.$$

(b) The speed of sound in a material is, Eq. (2.3),

$$v = \sqrt{\frac{B}{\rho}}.$$

The bulk modulus of steel is calculated as follows:

$$\begin{aligned} v_{steel} &= \sqrt{\frac{B_{steel}}{\rho_{steel}}}, \\ B_{steel} &= v_{steel}^2 \rho_{steel} = (5.9 \times 10^3 \text{ m/s})^2 (7.9 \times 10^3 \text{ kg/m}^3) \\ &= 2.8 \times 10^{11} \text{ N m}^{-2}. \end{aligned}$$

◆ wxMaxima codes:

```
(%i3) fpprintprec:5; B_water:2.1e9; rho_water:1000;
(fpprintprec)      5
(B_water)    2.1*10^9
(rho_water) 1000
(%i4) v_water: sqrt(B_water/rho_water);
(v_water)    1449.1
(%i6) v_steel:5.9e3; rho_steel:7.9e3;
(v_steel)    5900.0
(rho_steel) 7900.0
(%i7) B_steel: v_steel^2*rho_steel;
(B_steel)    2.75*10^11
```

Comments on the codes:

(%i3) Set the floating point print precision to 5 and assign values of B_{water} and ρ_{water}.
(%i4) Calculate v_{water}.
(%i6) Assign v_{steel} and ρ_{steel}.
(%i7) Calculate B_{steel}.

Problem 2.3 The density of helium gas at standard temperature and pressure is 0.179 kg m^{-3}. Determine the speed of sound in the gas at that temperature and pressure. What is the speed of sound in helium at 20°C? Assume helium is an ideal gas. The Adiabatic constant of helium gas is $\gamma = 1.67$.

Solution

Speed of sound wave in an ideal gas is (Eq. 2.2)

$$v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{\gamma p}{\rho}},$$

where $B = \gamma p$ is the gas bulk modulus, ρ its density, p is pressure, and γ is the adiabatic constant of the gas. For monatomic gas, $\gamma = 1.67$; for diatomic gas, $\gamma = 1.4$.

Helium gas is monatomic, thus, the speed of sound wave in helium gas at standard temperature (0°C) and pressure (1.013×10^5 Pa) is, Eq. (2.2),

$$v_0 = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{1.67(1.013 \times 10^5 \text{ Pa})}{0.179 \text{ kg/m}^3}} = 972 \text{ m s}^{-1}, \tag{2.15}$$

where the standard pressure is $p = 1.013 \times 10^5$ Pa and $\gamma = 1.67$ for helium gas. For an ideal gas, $pV = \mu RT$, where p is the pressure of the gas, V is its volume, T is its temperature, μ is number of moles, and R is the universal gas constant. We have

$$\frac{\gamma p}{\rho} = \frac{\gamma}{\rho}\frac{\mu RT}{V} = \frac{\gamma \mu RT}{m} = \frac{\gamma RT}{M},$$

where m is mass of the gas and M is its molar mass. Therefore, the speed of sound in an ideal gas is, Eq. (2.2),

$$v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{\gamma RT}{M}}.$$

This means that the speed of sound in an ideal gas is proportional to the square root of the gas temperature, $v \propto \sqrt{T}$. We thus write

$$\frac{v_0}{v_{20}} = \frac{\sqrt{273 \text{ K}}}{\sqrt{(20 + 273) \text{ K}}}, \tag{2.16}$$

where v_0 and v_{20} are speeds of sound wave in helium gas at 0°C and 20°C, respectively. The speed of a sound wave in helium gas at 20°C is

$$v_{20} = v_0 \frac{\sqrt{(20 + 273)\text{K}}}{\sqrt{273 \text{ K}}} = (972 \text{ m/s}) \frac{\sqrt{(20 + 273)\text{K}}}{\sqrt{273 \text{ K}}} = 1007 \text{ m s}^{-1}.$$

Here, the speed of sound at 0°C in Eq. (2.15) is substituted in Eq. (2.16) to arrive at the answer.

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; ratprint:false; gamma:1.67; p:1.013e5; rho:0.179;
(fpprintprec)     5
(ratprint)  false
(gamma)     1.67
(p)   1.013*10^5
(rho) 0.179
(%i6) v0: sqrt(gamma*p/rho);
(v0)  972.16
(%i8) solve(v0/v20 = sqrt(273)/sqrt(20+273), v20)$ float(%);
(%o8) [v20=1007.1]
```

Comments on the codes:

(%i5) Set the floating point print precision to 5 and internal rational number print to false, and assign values of γ, p, and ρ.
(%i6) Calculate v_0.
(%i8) Solve $v_0/v_{20} = \sqrt{273}/\sqrt{20+273}$ for v_{20}.

Problem 2.4 Hydrogen gas consists of diatomic molecules with a relative molecular mass of 2. Calculate the speed of sound in hydrogen gas at 27°C. Given that the adiabatic constant of hydrogen gas is $\gamma = 1.40$ and the universal gas constant is $R = 8.31$ J mol^{-1} K^{-1}.

Solution

The speed of sound in hydrogen gas at 27°C is, Eq. (2.2),

$$v = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{\gamma RT}{M}} = \sqrt{\frac{1.4(8.31\ \text{J mol}^{-1}\ \text{K}^{-1})(27+273)\text{K}}{2.00\times 10^{-3}\ \text{kg/mol}}} = 1321\ \text{m s}^{-1}.$$

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; gamma:1.4; R:8.31; T:27+273; M:2e-3;
(fpprintprec)     5
(gamma)     1.4
(R)   8.31
(T)   300
(M)   0.002
(%i6) v: sqrt(gamma*R*T/M);
(v)   1321.0
```

Comments on the codes:

(%i5) Set the floating point print precision to 5 and assign values of γ, R, T, and M.
(%i6) Calculate v.

Problem 2.5 The speed of sound in air at 0°C is 331 m s^{-1}. What is the speed of sound in air at 37°C?

Solution

The relation between the speed of sound in a gas, v, and the gas temperature, T, is (Eq. 2.2)

$$v \propto \sqrt{T}.$$

We thus write

$$\frac{v_{37}}{v_0} = \sqrt{\frac{(37+273)\text{K}}{(0+273)\text{K}}}.$$

The speed of sound in air at 37°C is

$$v_{37} = v_0\sqrt{\frac{310\ \text{K}}{273\ \text{K}}} = (331\ \text{m/s})\sqrt{\frac{310\ \text{K}}{273\ \text{K}}} = 353\ \text{m s}^{-1}.$$

◆ wxMaxima codes:

```
(%i3) fpprintprec:5; ratprint:false; v0:331;
(fpprintprec)     5
(ratprint)  false
(v0)  331
(%i5) solve(v37/v0 = sqrt((37+273)/273), v37)$ float(%);
(%o5) [v37=352.72]
```

Comments on the codes:

(%i3) Set the floating point print precision to 5 and internal rational number print to false, and assign the value of v_0.
(%i5) Solve $\frac{v_{37}}{v_0} = \sqrt{\frac{37+273}{273}}$ for v_{37}.

Problem 2.6 The temperature of air is 10.0°C. The temperature of air then increases, and the velocity of sound increases by 1%. Calculate the increase in temperature.

Solution

The relationship between the speed of sound in air, v, and the temperature of air, T, is (Eq. 2.2)

$$v \propto \sqrt{T},$$

or

$$v = k\sqrt{T},$$

where k is a constant. We write

$$\frac{v_x - v_{10}}{v_{10}} = \frac{1}{100},$$

where v_x and v_{10} are the speeds of sound at increased temperature and at 10°C (283 K), respectively. Thus,

$$\frac{k(283+\theta)^{1/2} - k(283)^{1/2}}{k(283)^{1/2}} = \frac{1}{100},$$

where θ is the increase in temperature. This equation is solved for the increase in temperature,

$$\left(1+\frac{\theta}{283}\right)^{1/2} - 1 = \frac{1}{100}$$
$$\left(1+\frac{\theta}{283}\right)^{1/2} = 1.01$$
$$\theta = 5.7°\text{C}.$$

The air temperature is 10.0°C + 5.7°C = 15.7°C.

◆ wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)     5
(ratprint)  false
(%i4) solve(((283+theta)^0.5 - 283^0.5)/283^0.5 = 1/100, theta)$ float(%);
(%o4) [theta=5.6883]
```

Comments on the codes:

(%i2) Set the floating point print precision to 5 and internal rational number print to false.

(%i4) Solve $\frac{(283+\theta)^{0.5}-(283)^{0.5}}{(283)^{0.5}} = \frac{1}{100}$ for θ.

Problem 2.7 The maximum pressure variation of a sound wave is 30.0 Pa. The speed of sound in air is 330 m s^{-1} and the density of air is 1.22 kg m^{-3}. Calculate

(a) maximum displacement of the air layer if the frequency of the sound is 500 Hz.
(b) intensity of the sound.

Solution

(a) The relation between maximum pressure variation of sound wave, Δp_m (also called pressure amplitude) and maximum displacement of the air layer, s_m (also called displacement amplitude), is, Eq. (2.9),

$$\Delta p_m = k\rho v^2 s_m = \frac{2\pi}{\lambda}\rho v^2 s_m = 2\pi\rho v f s_m,$$

where k is the propagation constant, ρ is the air density, v is the speed of sound, λ is the wavelength, and f is the frequency of sound. The maximum displacement of the air layer (the displacement amplitude) is calculated as follows:

$$30 \text{ Pa} = 2\pi(1.22 \text{ kg m}^{-3})(330 \text{ m s}^{-1})(500 \text{ s}^{-1})s_m,$$
$$s_m = 2.4 \times 10^{-5} \text{ m}.$$

(b) The intensity of sound is, Eq. (2.10),

$$I = \frac{(\Delta p_m)^2}{2\rho v} = \frac{(30 \text{ Pa})^2}{2(1.22 \text{ kg m}^{-3})(330 \text{ m s}^{-1})} = 1.1 \text{ W m}^{-2}.$$

◆ wxMaxima codes:

```
(%i6) fpprintprec:5; ratprint:false; deltapm:30; v:330; rho:1.22; f:500;
(fpprintprec)     5
(ratprint)  false
(deltapm)   30
(v)   330
(rho) 1.22
(f)   500
(%i8) solve(deltapm=2*%pi*rho*v*f*sm, sm)$ float(%);
(%o8) [sm=2.3719*10^-5]
(%i9) I:  deltapm^2/(2*rho*v);
(I)   1.1177
```

Comments on the codes:

(%i6) Set the floating point print precision to 5 and internal rational number print to false, and assign values of Δp_m, v, ρ, and f.

(%i8) Solve $\Delta p_m = 2\pi\rho v f s_m$ for s_m.
(%i9) Calculate I.

Problem 2.8 The intensity of a sound wave is 1.00×10^{-2} W m^{-2} and its frequency is 600 Hz. What are the pressure amplitude and the displacement amplitude? The speed of sound is 330 m s^{-1}, and density of air is 1.22 kg m^{-3}.

Solution

The relationship between the intensity of sound, I, and pressure amplitude, Δp_m, is, Eq. (2.10),

$$I = \frac{(\Delta p_m)^2}{2\rho v},$$

where ρ is the density of air and v is the speed of sound in air. The pressure amplitude is calculated as follows:

$$1.00 \times 10^{-2} \text{ W m}^{-2} = \frac{(\Delta p_m)^2}{2(1.22 \text{ kg m}^{-3})(330 \text{ m s}^{-1})},$$
$$\Delta p_m = 2.84 \text{ N m}^{-2}.$$

The displacement amplitude is, Eq. (2.9),

$$s_m = \frac{\Delta p_m}{2\pi f \rho v} = \frac{2.84 \text{ Pa}}{2\pi(600 \text{ s}^{-1})(1.22 \text{ kg m}^{-3})(330 \text{ m s}^{-1})} = 1.87 \times 10^{-6} \text{ m},$$

where f is the frequency of sound.

◆ wxMaxima codes:

```
(%i6) fpprintprec:5; ratprint:false; I:1e-2; f:600; v:330; rho:1.22;
(fpprintprec)     5
(ratprint)  false
(I)   0.01
(f)   600
(v)   330
(rho) 1.22
(%i8) solve(I=deltapm^2/(2*rho*v), deltapm)$ float(%);
(%o8) [deltapm=-2.8376,deltapm=2.8376]
(%i9) deltapm: rhs(%[2]);
(deltapm) 2.8376
(%i11) sm: deltapm/(2*%pi*f*rho*v)$ float(%);
(%o11) 1.8696*10^-6
```

Comments on the codes:

(%i6) Set the floating point print precision to 5 and internal rational number print to false, and assign values of I, f, v, and ρ.

(%i8) Solve $I = \frac{(\Delta p_m)^2}{2\rho v}$ for Δp_m.
(%i9) Assign Δp_m.
(%i11) Calculate s_m.

Problem 2.9 An audio speaker emits sound at an intensity level of 70 dB. A lecturer talks at an intensity level of 40 dB. Compare the sound intensity of the speaker to that of the lecturer.

Solution

The definition of the intensity level β of a sound source with intensity I is, Eq. (2.11),

$$\beta = 10\ \log_{10}\left(\frac{I}{10^{-12}\ \text{W m}^{-2}}\right).$$

For the speaker and the lecturer we can write these two equations,

$$70 = 10\ \log_{10}\left(\frac{I_{speaker}}{10^{-12}\ \text{W m}^{-2}}\right),$$

$$40 = 10\ \log_{10}\left(\frac{I_{lecturer}}{10^{-12}\ \text{W m}^{-2}}\right).$$

The ratio of the sound intensity of the speaker to that of the lecturer is calculated as follows:

$$\begin{aligned} 70 - 40 &= 10\ \log_{10}\left(\frac{I_{speaker}}{10^{-12}\ \text{W m}^{-2}}\right) - 10\ \log_{10}\left(\frac{I_{lecturer}}{10^{-12}\ \text{W m}^{-2}}\right) \\ &= 10 \log_{10}\left(\frac{I_{speaker}}{I_{lecturer}}\right), \\ \frac{I_{speaker}}{I_{lecturer}} &= 1000. \end{aligned}$$

◆ wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)     5
(ratprint)  false
(%i3) log10(x) := log(x)/log(10);
(%o3) log10(x):=log(x)/log(10)
(%i5) solve(70=10*log10(Ispeaker/1e-12), Ispeaker)$ float(%);
(%o5) [Ispeaker=1.0*10^-5]
(%i7) solve(40=10*log10(Ilecturer/1e-12), Ilecturer)$ float(%);
(%o7) [Ilecturer=1.0*10^-8]
(%i8) ratio: %o5/%o7;
(ratio) [Ispeaker/Ilecturer=1000.0]
```

Comments on the codes:

(%i2) Set the floating point print precision to 5 and internal rational number print to false.

(%i3) In wxMaxima, the built-in function of base e logarithm of x is log(x). There is no built-in function for base 10 logarithm of x. We define base 10 logarithm of x by log10(x): = log(x)/log(10) which means $\log_{10}(x) = \log_e(x)/\log_e(10)$.

(%i5), (%i7) Solve $70 = 10 \log_{10}(I_{speaker}/10^{-12})$ and $40 = 10 \log_{10}(I_{lecturer}/10^{-12})$ for $I_{speaker}$ and $I_{lecturer}$, respectively.

(%i8) Calculate ratio $I_{speaker}/I_{lecturer}$.

Problem 2.10 A sound of intensity 1.20 W m^{-2} hurts the ears. What is the intensity level?

Solution

The intensity level β of the sound in dB is, Eq. (2.11),

$$\beta = 10\ \log_{10}\left(\frac{I}{10^{-12}\ \text{W m}^{-2}}\right) = 10\ \log_{10}\left(\frac{1.2\ \text{W m}^{-2}}{10^{-12}\ \text{W m}^{-2}}\right) = 121\ \text{dB}.$$

Sound with an intensity level of 121 dB hurts the ears.

◆ wxMaxima codes:

```
(%i3) fpprintprec:5; log10(x):=log(x)/log(10); I:1.2;
(fpprintprec)     5
(%o2) log10(x):=log(x)/log(10)
(I)   1.2
(%i5) beta: 10*log10(I/1e-12); float(%);
(beta) 278.13/log(10)
(%o5) 120.79
```

Comments on the codes:

(%3) Set the floating point print precision to 5, define $\log_{10}(x)$, and assign the value of I.

(%i5) Calculate β.

Problem 2.11 A source emits sound uniformly in all directions. At a distance of 4.0 m from the source, the sound intensity level is 90 dB.

(a) Calculate the sound intensity at the point.
(b) At which point is the intensity level 70 dB?

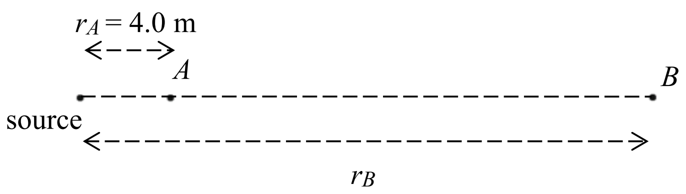


Fig. 2.1 Sound intensity and intensity level, Problem 2.11

Solution

(a) Fig. 2.1 shows the sound source and points in question.

Sound intensity level β (in dB) at a point with intensity I is defined as, Eq. (2.11),

$$\beta = 10\ \log_{10}\left(\frac{I}{I_0}\right) = 10\ \log_{10}\left(\frac{I}{10^{-12}\ \mathrm{W\ m^{-2}}}\right),$$

where $I_0 = 10^{-12}$ W m^{-2} is the reference intensity.

For point A, we write

$$90 = 10\ \log_{10}\left(\frac{I_A}{10^{-12}\ \mathrm{W\ m^{-2}}}\right),$$

where I_A is the sound intensity at point A. The sound intensity at point A is

$$\begin{aligned} I_A &= 10^9 \times 10^{-12}\ \mathrm{W\ m^{-2}} \\ &= 1.0 \times 10^{-3}\ \mathrm{W\ m^{-2}}. \end{aligned}$$

◆ wxMaxima codes:

```
(%i3) fpprintprec:5; ratprint:false; log10(x):=log(x)/log(10);
(fpprintprec)     5
(ratprint)  false
(%o3) log10(x):=log(x)/log(10)
(%i5) solve(90=10*log10(IA/1e-12), IA)$ float(%);
(%o5) [IA=0.001]
```

Comments on the codes:

(%i3) Set the floating point print precision to 5 and internal rational number print to false, and define $\log_{10}(x)$.

(%i5) Solve $90 = 10\ \log_{10}(I_A/10^{-12})$ for I_A.

(b) Sound intensity level at point A is

$$90 = 10\ \log_{10}\left(\frac{I_A}{I_0}\right) = 10\ \log_{10}\left(\frac{P}{4\pi r_A^2 I_0}\right),$$

where P is the power of the sound source and $4\pi r_A^2$ is the area of the surface of a sphere of radius r_A.

The sound intensity level at point B is

$$70 = 10\ \log_{10}\left(\frac{I_B}{I_0}\right) = 10\ \log_{10}\left(\frac{P}{4\pi r_B^2 I_0}\right),$$

where $4\pi r_B^2$ is the area of the surface of a sphere of radius r_B. From the two equations,

$$90 - 70 = 10\ \log_{10}\left(\frac{r_B^2}{r_A^2}\right) = 20\ \log_{10}\left(\frac{r_B}{r_A}\right) = 20\ \log_{10}\left(\frac{r_B}{4.0\text{ m}}\right).$$

Thus, the distance at which the sound intensity level is 70 dB is

$$r_B = 40\,\text{m}.$$

◆ wxMaxima codes:

```
(%i3) fpprintprec:5; ratprint:false; log10(x):=log(x)/log(10);
(fpprintprec)     5
(ratprint)  false
(%o3) log10(x):=log(x)/log(10)
(%i5) rA:4; I0:1e-12;
(rA)  4
(I0)  10.0*10^-13
(%i7) solve(90=10*log10(P/(4*%pi*rA^2*I0)), P)$ float(%);
(%o7) [P=0.20106]
(%i8) P: rhs(%[1]);
(P)   0.20106
(%i10) solve(70=10*log10(P/(4*%pi*rB^2*I0)), rB)$ float(%);
(%o10) [rB=-40.0,rB=40.0]
```

Comments on the codes:

(%i3) Set the floating point print precision to 5 and internal rational number print to false, and define $\log_{10}(x)$.

(%i5) Assign values of r_A and I_0.

(%i7) Solve $90 = 10\ \log_{10}\left(\frac{P}{4\pi r_A^2 I_0}\right)$ for P.

(%i8) Assign value of P.

(%i10) Solve $70 = 10\ \log_{10}\left(\frac{P}{4\pi r_B^2 I_0}\right)$ for r_B.

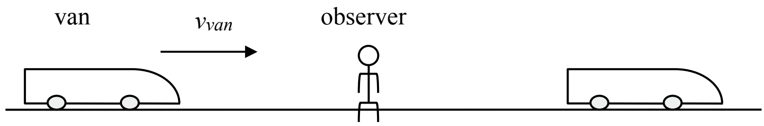


Fig. 2.2 A van approaching and leaving an observer, Problem 2.12

Problem 2.12 A van sounds its siren as it approaches and leaves a stationary observer. On approach, the observer hears a sound of frequency 219 Hz. On leaving, the observer hears a sound of frequency 184 Hz. The speed of sound in air is 340 m s^{-1}. Determine the speed of the van and the frequency of the siren.

Solution

Figure 2.2 shows the van approaching and leaving the observer. The speed of the van is v_{van}.

The frequency of sound heard by the observer as the van approaches him is, Eq. (2.13),

$$f_{approach} = \left(\frac{v}{v - v_{van}}\right) f,$$

where v_{van}, v, and f are the speed of the van, speed of sound, and frequency of the siren, respectively.

The frequency of sound heard by the observer as the van leaves him is, Eq. (2.13),

$$f_{leave} = \left(\frac{v}{v + v_{van}}\right) f.$$

Substituting known values in both equations gives

$$219 \text{ Hz} = \left(\frac{340 \text{ m s}^{-1}}{340 \text{ m s}^{-1} - v_{van}}\right) f,$$

$$184 \text{ Hz} = \left(\frac{340 \text{ m s}^{-1}}{340 \text{ m s}^{-1} + v_{van}}\right) f.$$

The speed of the van and frequency of the siren are obtained by solving both equations for v_{van} and f,

$$v_{van} = 29.5 \text{ m s}^{-1},$$

$$f = 200 \text{ Hz}.$$

◆ wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)     5
(ratprint)  false
(%i4) solve([219=340/(340-vvan)*f,184=340/(340+vvan)*f],[vvan,f])$
float(%);
(%o4)[[vvan=29.529,f=199.98]]
```

Comments on the codes:

(%i2) Set the floating point print precision to 5 and internal rational number print to false.

(%i4) Solve $219 = \left(\frac{340}{340-v_{van}}\right) f$ and $184 = \left(\frac{340}{340+v_{van}}\right) f$ for v_{van} and f.

Problem 2.13 A bird flies away from a boy toward a cliff at a speed of 15.0 m s^{-1}. The bird emits sound of frequency 800 Hz, as illustrated in Fig. 2.3.

(a) What is the sound frequency heard by the boy for sound coming directly from the bird?

(b) What is the sound frequency heard by the boy from the echo reflected by the cliff to the boy? The speed of sound in air is 340 m s^{-1}.

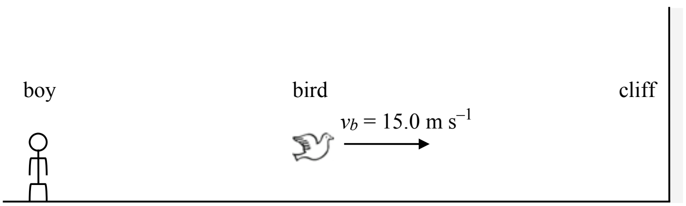


Fig. 2.3 A bird flying away from a boy toward a cliff, Problem 2.13

Solution

(a) Frequency of sound heard by the boy for the sound coming directly from the bird is, Eq. (2.13),

$$f_{boy1} = \left(\frac{v}{v + v_{bird}}\right) f = \left(\frac{340 \text{ m s}^{-1}}{340 \text{ m s}^{-1} + 15.0 \text{ m s}^{-1}}\right)(800 \text{ Hz}) = 766 \text{ Hz}.$$

Here, v is speed of sound, v_{bird} is speed of the bird, and f is frequency of sound emitted by the bird.

(b) Sound that hits the cliff is reflected without any frequency change, as such; the boy will hear the same frequency as any person stationed at the cliff. Therefore, for the sound reflected from the cliff, the boy will hear the sound of frequency, Eq. (2.13),

$$f_{boy2} = \left(\frac{v}{v - v_{bird}}\right) f = \left(\frac{340 \text{ m s}^{-1}}{340 \text{ m s}^{-1} - 15.0 \text{ m s}^{-1}}\right)(800 \text{ Hz}) = 837 \text{ Hz}.$$

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; ratprint:false; v:340; vbird:15; f:800;
(fpprintprec)     5
(ratprint)   false
(v)          340
(vbird)      15
(f)          800
(%i7) fboy1: v/(v+vbird)*f$ float(%);
(%o7) 766.2
(%i9) fboy2: v/(v-vbird)*f$ float(%);
(%o9) 836.92
```

Comments on the codes:

(%i5)	Set the floating point print precision to 5 and internal rational number print to false, and assign values of v, v_{bird}, and f.
(%i7), (%i9)	Calculate f_{boy1} and f_{boy2}.

Problem 2.14 A train approaches, goes through, and leaves a tunnel of a hill at a speed of 30.0 m s^{-1}. It sounds its siren with a frequency of 1000 Hz when it approaches and leaves the tunnel. Calculate

(a) the sound frequency heard by the train driver as the train approaches the tunnel, for the sound reflected by the hill.
(b) the sound frequency heard by the train driver as the train leaves the tunnel, for the sound reflected by the hill. The speed of sound in air is 330 m s^{-1}.

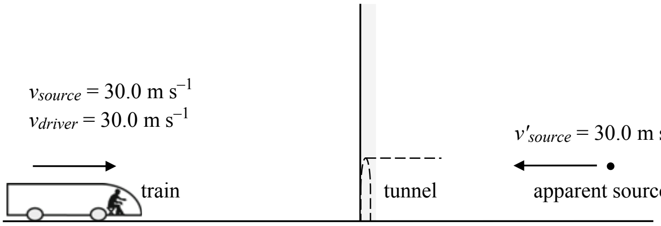


Fig. 2.4 A train approaching a tunnel in a hill, Problem 2.14

Solution

(a) Fig. 2.4 shows the train approaching the tunnel.

The speed of the observer (driver) v_{driver} and the sound source v_{source} is the speed of the train. The sound emitted by the siren goes toward the cliff, gets reflected, and goes back to the train. Thus, there is an apparent sound source moving at the speed of $v'_{source} = 30.0\ \text{m s}^{-1}$ toward the driver.

Therefore, the sound frequency heard by the driver is, Eq. (2.14),

$$f' = \left(\frac{v + v_{driver}}{v - v'_{source}}\right) f = \left(\frac{330\ \text{m s}^{-1} + 30.0\ \text{m s}^{-1}}{330\ \text{m s}^{-1} - 30.0\ \text{m s}^{-1}}\right) (1000\ \text{Hz}) = 1200\ \text{Hz}.$$

where v is the speed of sound in air and f is the frequency of the siren.

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; v:330; vdriver:30; vsourceprime:30; f:1000;
(fpprintprec)     5
(v)   330
(vdriver)   30
(vsourceprime)    30
(f)   1000
(%i6) fprime: (v+vdriver)/(v-vsourceprime)*f;
(fprime) 1200
```

Comments on the codes:

(%i5) Set the floating point print precision to 5 and assign values of v, v_{driver}, v'_{source}, and f.
(%i6) Calculate f'.

(b) Fig. 2.5 shows the train leaving the tunnel.

In this case, there is an apparent sound source moving at a speed of $v'_{source} = 30.0\ \text{m s}^{-1}$ away from the train. Thus, the sound frequency heard by the driver is, Eq. (2.14),

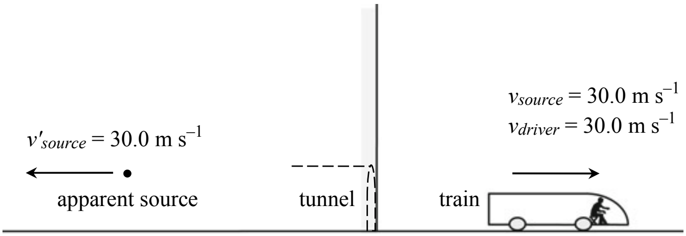


Fig. 2.5 A train leaving a tunnel in a hill, Problem 2.14

$$f' = \left(\frac{v - v_{driver}}{v + v'_{source}}\right) f = \left(\frac{330\ \mathrm{m\ s^{-1}} - 30.0\ \mathrm{m\ s^{-1}}}{330\ \mathrm{m\ s^{-1}} + 30.0\ \mathrm{m\ s^{-1}}}\right)(1000\ \mathrm{Hz}) = 833\ \mathrm{Hz}.$$

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; v:330; vdriver:30; vsourceprime:30; f:1000;
(fpprintprec)     5
(v)   330
(vdriver)   30
(vsourceprime)    30
(f)   1000
(%i7) fprime: (v-vdriver)/(v+vsourceprime)*f$ float(%);
(%o7) 833.33
```

Comments on the codes:

(%i5) Set the floating point print precision to 5 and assign values of v, v_{driver}, v'_{source}, and f.

(%i6) Calculate f'.

2.3 Summary

- Sound waves are longitudinal waves, and the disturbances are the displacements of air layer or variations in air pressure.
- The intensity level β of a sound with intensity I is defined as

$$\beta = 10\ \log_{10}\left(\frac{I}{10^{-12}\ \mathrm{W\ m^{-2}}}\right).$$

  Unit of β is decibel (dB). The sound intensity of 10^{-12} W m^{-2} is the threshold of a human hearing a sound, that is, the lowest intensity a human could hear.
- Doppler effect: When the observer and sound source are in relative motion in a medium where the speed of sound is v, the frequency heard by the observer f' is

$$f' = \left(\frac{v \pm v_o}{v \mp v_s}\right) f,$$

where v_o is the speed of the observer and v_s is the speed of the sound source. For the numerator, use the (+) sign if the observer is moving toward or the (−) sign if the observer is moving away from the sound source. For the denominator, use the (−) sign if the sound source is moving toward or the (+) sign if the sound source is moving away from the observer.

2.4 Exercises

Exercise 2.1 Calculate the speed of sound in oxygen gas at 27°C. Molecular weight of oxygen molecules is 32 kg kmol^{-1}. Molar gas constant is $R = 8.31$ J K^{-1} mol^{-1}.
(Answer: 330 m s^{-1})

Exercise 2.2 A car at a speed of 27 m s^{-1} is moving toward a stationary siren, and the driver hears a sound of frequency 400 Hz, as illustrated in Fig. 2.6. The speed of sound in air is 330 m s^{-1}. What is the frequency of sound emitted by the siren?
(Answer: 370 Hz)

Exercise 2.3 An ambulance with its siren on is speeding at 120 km h^{-1}, as shown in Fig. 2.7. A driver of a car moving at a speed of 90 km h^{-1} in front of the ambulance hears the siren sound of 600 Hz. The speed of sound in air is 330 m s^{-1}. What is the frequency of sound emitted by the siren?
(Answer: 580 Hz)

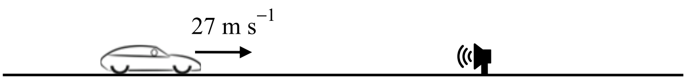


Fig. 2.6 A car moving toward a stationary siren, Exercise 2.2

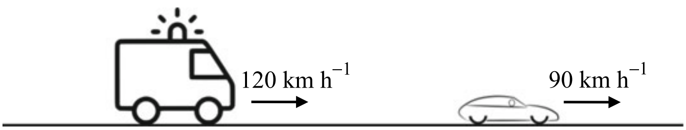


Fig. 2.7 An ambulance speeds toward a moving car, Exercise 2.3

Exercise 2.4

(a) Calculate the intensity of a 50 dB sound of an electric speaker.
(b) The speaker has an area of 120 cm^2. What is the acoustic power output of the speaker?

(Answer: (a) 1.0×10^{-7} W m^{-2}; (b) 1.2×10^{-9} W)

Exercise 2.5 What is the displacement amplitude of the air layer of a 50 dB, 800 Hzsound wave? The speed of sound in air is 330 m s^{-1} and the density of air is 1.29 kg m^{-3}.

(Answer: 4.3×10^{-9} m)

Chapter 3
Superposition and Stationary Wave

Abstract Problems on superposition of waves and stationary waves are solved in this chapter. These include stationary waves in air column and string. Nodes and antinodes of the stationary waves are identified. Animations of these stationary waves are presented for insight into the physics. Formation of sound beats is discussed. Both analytical and computer solutions are presented.

3.1 Basic Concepts and Formulae

(1) The superposition of two waves with the same amplitude and frequency will give a resultant wave that has the same frequency and the amplitude that depends on phase difference ϕ of the two waves. If the waves are

$$y_1 = A\sin(kx - \omega t), \tag{3.1}$$

$$y_2 = A\sin(kx - \omega t - \phi), \tag{3.2}$$

then the resultant wave is

$$y = y_1 + y_2 = 2A\,\cos\frac{\phi}{2}\,\sin(kx - \omega t - \frac{\phi}{2}). \tag{3.3}$$

The amplitude of the resultant wave is $2A\,\cos(\phi/2)$ that depends on phase difference ϕ. Constructive interference occurs if the two waves are in phase, that is, when $\phi = 0, 2\pi, 4\pi, \ldots$ Destructive interference occurs if the two waves differ in phase by 180° or π rad, that is, when $\phi = \pi, 3\pi, 5\pi, \ldots$

(2) A stationary wave is formed from the superposition of two harmonic waves having the same frequency, amplitude, and wavelength and moving in opposite directions. For example, a stationary wave formed from a superposition of

$$y_1 = A\sin(kx - \omega t), \tag{3.4}$$

W. M. S. Wan Hassan et al., *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-43165-4_3

$$y_2 = A\sin(kx + \omega t), \tag{3.5}$$

is,

$$y = y_1 + y_2 = 2A\ \sin kx\ \cos\omega t. \tag{3.6}$$

The amplitude of the stationary wave is $2A \sin kx$. This means that the amplitude varies with position x. Points at which the amplitudes are at maxima (called antinodes) are $x = \frac{n\pi}{2k} = n\frac{\lambda}{4}$ (n are odd numbers). Points at which the amplitudes are zero (called nodes) are $x = \frac{n\pi}{k} = n\frac{\lambda}{2}$ (n are integers).

(3) Stationary waves could be found in string, air column, metal rod, etc.

(a) Stretched string is of length L with the two ends fixed. Natural frequencies are

$$f_n = \frac{n}{2L}\sqrt{\frac{F}{\mu}}, \quad n = 1, 2, 3, \ldots \tag{3.7}$$

where F is tension in the string and μ is mass per unit length of the string. Natural frequencies form harmonic series, that is, $f_1, 2f_1, 3f_1, \ldots$

(b) Air column in a pipe with open ends. Natural frequencies are

$$f_n = n\frac{v}{2L}, \quad n = 1, 2, 3, \ldots \tag{3.8}$$

where v is speed of sound in air and L is length of the pipe.

(c) Air column in a pipe with one end closed. Natural frequencies are

$$f_n = n\frac{v}{4L}, \quad n = 1, 3, 5, \ldots \tag{3.9}$$

where v is speed of sound in air and L is length of the pipe.

(4) An oscillating system is resonant with a driving force when the frequency of the driving force is the same as natural frequency of the oscillating system. At resonance, the amplitude of the oscillating system is very big.

(5) Beats are formed from the superposition of two waves with a small frequency difference, moving in the same direction. For sound waves, they are periodic loud and silent sound as time passes. This means that beats are wave interference in time. If the waves are

$$y_1 = A\cos 2\pi f_1 t, \tag{3.10}$$

$$y_2 = A \cos 2\pi f_2 t, \tag{3.11}$$

then the superposition of the waves is

$$y = y_1 + y_2 = 2A \cos 2\pi \left(\frac{f_1 - f_2}{2} \right) t \cos 2\pi \left(\frac{f_1 + f_2}{2} \right) t. \tag{3.12}$$

Frequency of the beat is

$$f_1 - f_2, \tag{3.13}$$

because $|\cos 2\pi((f_1 - f_2)/2)| = 1$ at a rate of $f_1 - f_2$. When $|\cos 2\pi((f_1 - f_2)/2)| = 1$ the sound is loud and when $|\cos 2\pi((f_1 - f_2)/2)| = 0$ there is silence.

(6) Any periodic wave can be represented by a combination of sinusoidal waves forming a harmonic series. This is called Fourier synthesis.

3.2 Problems and Solutions

Problem 3.1 Two harmonic waves are given as

$$y_1 = 5 \sin[\pi(4x - 1200t)],$$

$$y_2 = 5 \sin[\pi(4x - 1200t - 0.25)],$$

where x, y_1, and y_2 are in meters and t in seconds. The two waves are superposed.

(a) Calculate the amplitude and frequency of the resultant wave y.
(b) At $x = 0$ m, plot y_1, y_2, and y against t for $0 \leq t \leq 0.005$ s.

Solution

(a) The resultant wave is obtained by adding the two waves,

$$\begin{aligned}
y &= y_1 + y_2 \\
&= 5 \sin[\pi(4x - 1200t)] + 5 \sin[\pi(4x - 1200t - 0.25)] \\
&= 5 \sin(4\pi x - 1200\pi t) + 5 \sin(4\pi x - 1200\pi t - 0.25\pi) \\
&= 5\ [\sin(4\pi x - 1200\pi t) + \sin(4\pi x - 1200\pi t - 0.25\pi)] \\
&= 5\ [2 \sin(4\pi x - 1200\pi t - 0.125\pi)\cos(0.125\pi)] \\
&= 10 \cos(0.125\pi) \sin(4\pi x - 1200\pi t - 0.125\pi) \\
&= 9.24 \sin(4\pi x - 1200\pi t - 0.125\pi).
\end{aligned}$$

Trigonometric identity $\sin\theta + \sin\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$ is used in the calculation, see Appendix D.

Comparing the resultant wave and a general wave, we have

$$9.24\sin(4\pi x - 1200\pi t - 0.125\pi) \equiv A\sin(kx - \omega t + \phi),$$
$$A = 9.24.$$

Therefore, the amplitude of the resultant wave is 9.24 m.

Comparing the resultant wave and a general wave, we have

$$9.24\sin(4\pi x - 1200\pi t - 0.125\pi) \equiv A\sin(kx - \omega t + \phi),$$

$$\omega = 2\pi f = 1200\pi.$$

The frequency of the resultant wave is

$$f = \frac{\omega}{2\pi} = \frac{1200\pi}{2\pi} = 600 \text{ Hz}.$$

◆ wxMaxima codes:

```
(%i1) fpprintprec:5;
(fpprintprec)     5
(%i2) amplitude:float(10*cos(0.125*%pi));
(amplitude) 9.2388
(%i3) omega:1200*%pi;
(omega) 1200*%pi
(%i4) solve(omega=2*%pi*f, f);
(%o4) [f=600]
```

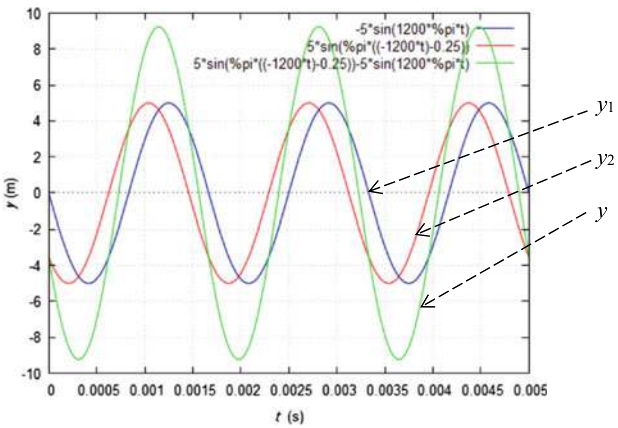
Comments on the codes:

(%i1) Set the floating point print precision to 5.
(%i2) Calculate the amplitude.
(%i3) Assign ω.
(%i4) Solve $\omega = 2\pi f$ for f.

(b) To plot curves of y_1, y_2, and y against t, we set $x = 0$ m, define y_1 and y_2, calculate y, and plot the three curves against time.

◆ wxMaxima codes:

```
(%i1) x:0;
(x)   0
(%i2) y1: 5*sin(%pi*(4*x-1200*t));
(y1)  -5*sin(1200*%pi*t)
(%i3) y2: 5*sin(%pi*(4*x-1200*t-0.25));
(y2)  5*sin(%pi*(-1200*t-0.25))
(%i4) y: y1+y2;
(y)   5*sin(%pi*(-1200*t-0.25))-5*sin(1200*%pi*t)
(%i5) wxplot2d([y1,y2,y], [t,0,0.005], grid2d,
[xlabel,"{/Helvetica-Italic t} (s)"],[ylabel,"{/Helvetica-Italic y} (m)"]);
```



Comments on the codes:

(%i1)	Assign $x = 0$.
(%i2), (%i3)	Assign y_1 and y_2.
(%i4)	Calculate y.
(%i5)	Plot y_1, y_2, and y for $0 \leq t \leq 0.005$ s.

Problem 3.2 A harmonic wave is described by,

$$y_1 = 8\sin(0.2\pi x - 160\pi t),$$

where y_1 and x are in meters and t in seconds. Find an expression for wave y_2 having the same frequency, amplitude, and wavelength as y_1 that will give a resultant wave with an amplitude of $8\sqrt{3}$ m when added with y_1.

Solution

Let the two waves be

$$y_1 = A\sin(kx - \omega t),$$

$$y_2 = A\sin(kx - \omega t - \phi).$$

The resultant wave is

$$y = y_1 + y_2 = \left(2A\cos\frac{\phi}{2}\right)\sin\left(kx - \omega t - \frac{\phi}{2}\right).$$

We have used the trigonometry identity $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ to arrive at the result; see Appendix D. The amplitude of the resultant wave is

$$2A\cos\frac{\phi}{2} = 16\cos\frac{\phi}{2} = 8\sqrt{3}.$$

This means that

$$\cos\frac{\phi}{2} = \frac{\sqrt{3}}{2},$$
$$\frac{\phi}{2} = \frac{\pi}{6},$$
$$\phi = \frac{\pi}{3}.$$

The expression for wave y_2 is

$$y_2 = 8\sin\left(0.2\pi x - 160\pi t - \frac{\pi}{3}\right),$$

because

$$\begin{aligned} y = y_1 + y_2 &= 8\sin(kx - \omega t) + 8\sin\left(kx - \omega t - \frac{\phi}{2}\right) \\ &= \left(16\cos\frac{\phi}{2}\right)\sin\left(kx - \omega t - \frac{\phi}{2}\right) \\ &= 8\sqrt{3}\sin\left(0.2\pi x - 160\pi t - \frac{\pi}{6}\right) \\ &= 13.9\sin(0.628x - 503t - 0.524). \end{aligned}$$

Problem 3.3 A stationary wave is formed from the superposition of two waves,

$$y_1 = 4\sin(3x - 2t),$$

$$y_2 = 4\sin(3x + 2t),$$

where x and y are in centimeters and t in seconds. Find,

(a) maximum displacement at $x = 2.3$ cm
(b) locations of nodes and antinodes.

Solution

(a) By adding or superposing both waves, we get the resultant wave,

$$\begin{aligned} y &= y_1 + y_2 \\ &= 4\ \sin(3x - 2t) + 4\ \sin(3x + 2t) \\ &= 8\ \sin 3x\ \cos 2t. \end{aligned}$$

Here, the use of trigonometry identity $\sin\theta + \sin\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$ is made in the addition; see Appendix D. The wave is a stationary wave with amplitude of 8.0 cm. Maximum displacement at $x = 2.3$ cm is calculated as follows:

$$\begin{aligned} y_{max} &= 8\ \sin 3x \Big|_{x=2.3} \\ &= 8\ \sin(6.9) \\ &= 4.6\ \text{cm}. \end{aligned}$$

Note that the maximum value of $\cos 2t$ is 1.

◆ wxMaxima codes:

```
(%i1) fpprintprec:5;
(fpprintprec)     5
(%i3) y1: 4*sin(3*x-2*t); y2: 4*sin(3*x+2*t);
(y1)  4*sin(3*x-2*t)
(y2)  4*sin(3*x+2*t)
(%i6) y: y1+y2$ trigexpand(%)$ trigsimp(%);
(%o6) 8*cos(2*t)*sin(3*x)
(%i7) x: 2.3;
(x)   2.3
(%i8) y: 8*sin(3*x);
(y)   4.6275
```

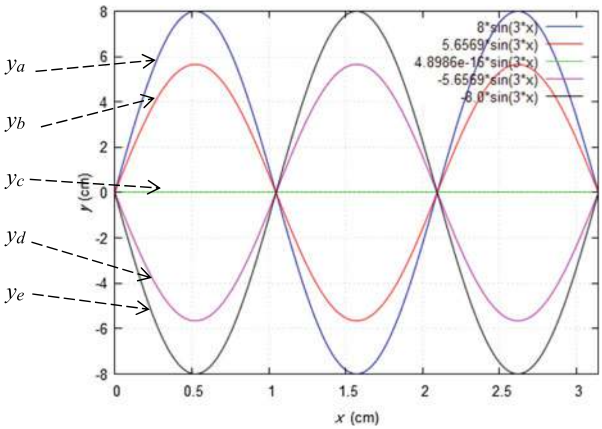
Comments on the codes:

(%i1)	Set the floating point print precision to 5.
(%i3)	Define y_1 and y_2.
(%i6)	Calculate y and simplify.
(%i7), (%i8)	Assign value of x and calculate y.

To plot the stationary wave $y = 8\ \sin 3x\ \cos 2t$, choose time $t_1 = 0$, $t_2 = \pi/8$, $t_3 = \pi/4$, $t_4 = 3\pi/8$, $t_5 = \pi/2$ s, and plot y against x for each t. We labelled the curves as y_a, y_b, y_c, y_d, and y_e.

◆ wxMaxima codes:

```
(%i6) fpprintprec:5; t1:0; t2:float(%pi/8); t3:float(2*%pi/8);
t4:float(3*%pi/8); t5:float(4*%pi/8);
(fpprintprec)     5
(t1)  0
(t2)  0.3927
(t3)  0.7854
(t4)  1.1781
(t5)  1.5708
(%i7) ya: 8*cos(2*t1)*sin(3*x);
(ya)  8*sin(3*x)
(%i8) yb: 8*cos(2*t2)*sin(3*x);
(yb)  5.6569*sin(3*x)
(%i9) yc: 8*cos(2*t3)*sin(3*x);
(yc)  4.8986*10^-16*sin(3*x)
(%i10) yd: 8*cos(2*t4)*sin(3*x);
(yd)  -5.6569*sin(3*x)
(%i11) ye: 8*cos(2*t5)*sin(3*x);
(ye)  -8.0*sin(3*x)
(%i12) wxplot2d([ya,yb,yc,yd,ye], [x,0,%pi], grid2d, [xlabel,"{/Helvetica-
Italic x}  (cm)"], [ylabel,"{/Helvetica-Italic y} (cm)"]);
```

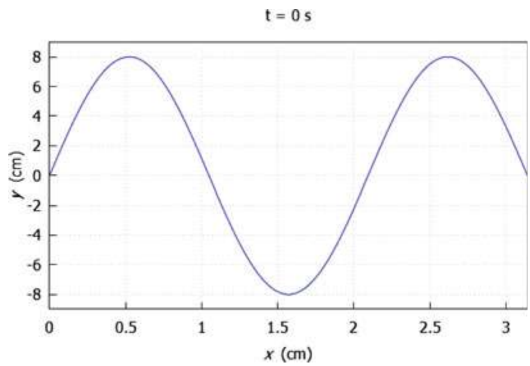


Comments on the codes:

(%i6)	Set the floating point print precision to 5 and assign values of t_1, t_2, t_3, t_4, and t_5.
(%i7), (%i8), (%i9), (%i10), (%i11)	Assign y_a, y_b, y_c, y_d, and y_e.
(%i12)	Plot y_a, y_b, y_c, y_d, and y_e for $0 \le x \le \pi$ cm.

♦ Animation of the stationary wave, $y = 8\ \sin 3x\ \cos 2t$, by wxMaxima:

```
(%i1) fpprintprec:2;
(fpprintprec)     2
(%i2) with_slider_draw(
              t, makelist(i,i,0,3,0.1),
              title=concat("t = ",t," s"),
              explicit(8*sin(3*x)*cos(2*t),x,0,%pi),
              grid=true,
              yrange=[-9,9],
              xlabel="{/Helvetica-Italic x} (cm)",
              ylabel="{/Helvetica-Italic y} (cm)");
```



Comment on the codes:

To run the animation, copy the codes to the wxMaxima command window; press <shift> and <enter> keys simultaneously to run the codes; right click the graphic that appears and choose *Start Animation.*

(b) Nodes are formed at points that satisfy

$$\sin 3x = 0,$$

or

$$3x = n\pi, \quad n = 0, 1, 2, 3, \ldots$$

This means that, to have sin $3x = 0$, 3x must be zero or a multiple of π. The nodes are at

$$x = n\left(\frac{\pi}{3}\right)\text{ cm} = 0, 1.05, 2.09, 3.14\,\text{cm}, \ldots \quad n = 0, 1, 2, 3, \ldots$$

Antinodes are formed at points that satisfy

$$\sin 3x = 1,$$

or,

$$3x = m\left(\frac{\pi}{2}\right), \quad m = 1, 3, 5, \ldots$$

This means that, to have $\sin 3x = 1$, 3x must be an odd multiple of $\pi/2$. The antinodes are at

$$x = m\left(\frac{\pi}{6}\right) \text{ cm} = 0.52, 1.57, 2.62 \text{ cm}, \ldots \quad m = 1, 3, 5, \ldots$$

Problem 3.4 An organ pipe 3.00 m long is closed at one end and opened at the other. Air column in the pipe vibrates and forms stationary waves. Obtain the first three harmonics. The speed of sound in air is 330 m s^{-1}.

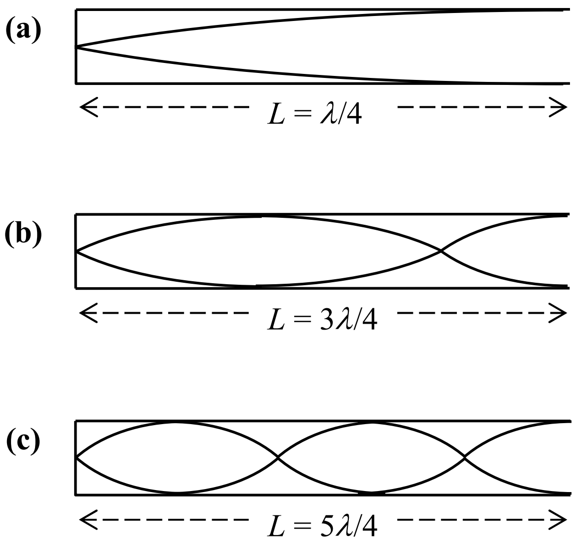
Solution

Figure 3.1a shows the stationary wave of fundamental frequency in the pipe.

The stationary wave is 1/4 of the complete wave occupying the pipe length L. We write

$$\frac{\lambda}{4} = L,$$

Fig. 3.1 **a** Fundamental frequency, **b** first overtone, and **c** second overtone of stationary sound wave of a pipe closed at one end, Problem 3.4



$$\lambda = 4L.$$

The fundamental frequency or frequency of the first harmonic is

$$f_0 = \frac{v}{\lambda} = \frac{v}{4L} = \frac{330 \text{ m/s}}{4(3.00 \text{ m})} = 27.5 \text{ Hz}.$$

Next, the first overtone or the second harmonic is formed by 3/4 of the complete wave occupying length L, as shown in Fig. 3.1b. We write

$$\frac{3\lambda}{4} = L,$$
$$\lambda = \frac{4L}{3}.$$

The frequency of the first overtone or the second harmonic is

$$f_1 = \frac{v}{\lambda} = \frac{3v}{4L} = \frac{3(330 \text{ m/s})}{4(3.00 \text{ m})} = 82.5 \text{ Hz} = 3f_0.$$

Lastly, the second overtone or the third harmonic is formed by 5/4 of the complete wave occupying pipe length L, as in Fig. 3.1c. We write

$$\frac{5\lambda}{4} = L,$$
$$\lambda = \frac{4L}{5}.$$

The frequency of the second overtone or the third harmonic is

$$f_2 = \frac{v}{\lambda} = \frac{5v}{4L} = \frac{5(330 \text{ m/s})}{4(3.00 \text{ m})} = 138 \text{ Hz} = 5f_0.$$

◆ wxMaxima codes:

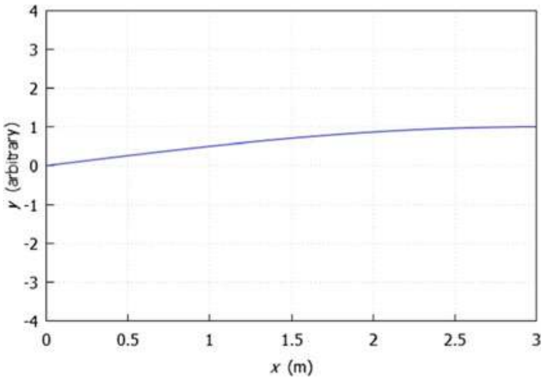
```
(%i3) fpprintprec:5; v:330; L:3;
(fpprintprec)     5
(v)   330
(L)   3
(%i4) f0: float(v/(4*L));
(f0)  27.5
(%i5) f1: float(3*v/(4*L));
(f1)  82.5
(%i6) f2: float(5*v/(4*L));
(f2)  137.5
```

Comments on the codes:

(%i3)	Set the floating point print precision to 5 and assign values of v and L.
(%i4), (%i5), (%i6)	Calculate f_0, f_1, and f_2.

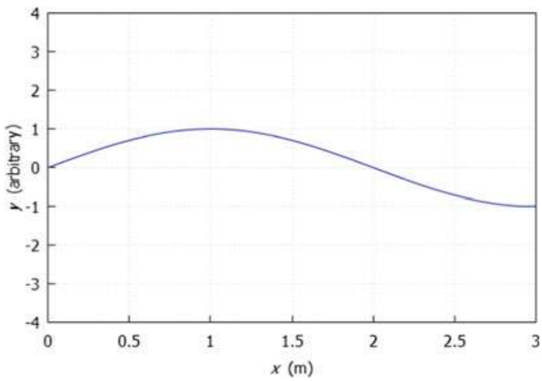
◆ Animation of the vibrating air column of fundamental frequency (first harmonic):

```
(%i1) with_slider_draw(
            t, makelist(i,i,0,2*%pi,0.5),
            explicit(sin(2*%pi*x/12)*cos(t),x,0,3),
            grid=true,
            yrange=[-4,4],
            xlabel="{/Helvetica-Italic x} (m)",
            ylabel="{/Helvetica-Italic y} (arbitrary)");
```



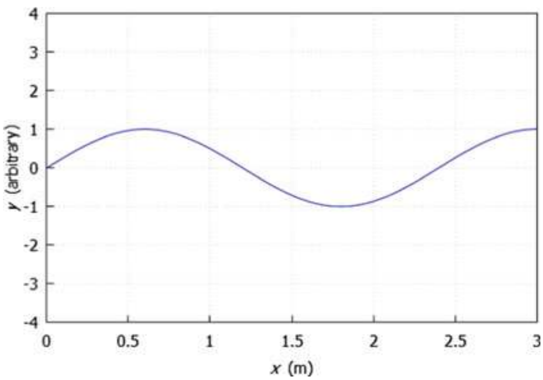
◆ Animation of the vibrating air column of first overtone (second harmonic):

```
(%i1) with_slider_draw(
           t, makelist(i,i,0,2*%pi,0.5),
           explicit(sin(2*%pi*x/4)*cos(t),x,0,3),
           grid=true,
           yrange=[-4,4],
           xlabel="{/Helvetica-Italic x} (m)",
           ylabel="{/Helvetica-Italic y} (arbitrary)");
```



◆ Animation of the vibrating air column of second overtone (third harmonic):

```
(%i1) with_slider_draw(
            t, makelist(i,i,0,2*%pi,0.5),
            explicit(sin(2*%pi*x/(12/5))*cos(t),x,0,3),
            grid=true,
            yrange=[-4,4],
            xlabel="{/Helvetica-Italic x} (m)",
            ylabel="{/Helvetica-Italic y} (arbitrary)");
```



Comment on the codes:

To run any of the animations, copy the codes to the wxMaxima command window; press <shift> and <enter> keys simultaneously to run the codes; right-click the graphic that appears and choose *Start Animation*. The y-axis represents the longitudinal displacement of the air layer, while the x-axis is the distance along the pipe.

Problem 3.5 Air column in an open end pipe 3.00 m long vibrates and forms stationary sound waves. Determine the first three harmonics of the sound. The speed of sound in air is 330 m s^{-1}.

Solution

Figure 3.2a shows the stationary wave of the fundamental frequency in the pipe. There are two antinodes and one node. This stationary wave is 1/2 of the complete wave occupying length L. We write

$$\frac{\lambda}{2} = L,$$
$$\lambda = 2L.$$

The frequency of the first harmonic (fundamental frequency) is

$$f_0 = \frac{v}{\lambda} = \frac{v}{2L} = \frac{330 \text{ m/s}}{2(3.00 \text{ m})} = 55.0 \text{ Hz}.$$

The second harmonic is shown in Fig. 3.2b. The full wave occupies the length L, so we write

$$\lambda = L.$$

The frequency of the second harmonic (first overtone) is

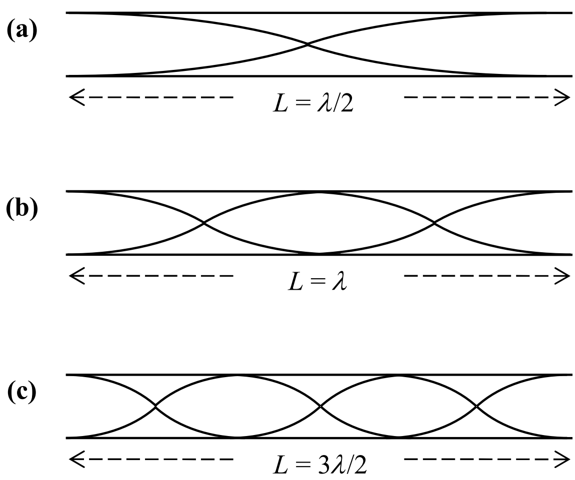


Fig. 3.2 **a** Fundamental frequency, **b** first overtone, and **c** second overtone of stationary sound wave of an opened end pipe, Problem 3.5

$$f_1 = \frac{v}{\lambda} = \frac{v}{L} = \frac{330 \text{ m/s}}{3.00 \text{ m}} = 110 \text{ Hz} = 2f_0.$$

The third harmonic is shown in Fig. 3.2c. Here, 3/2 of the full wave occupies the pipe length L. This means that

$$\frac{3\lambda}{2} = L,$$
$$\lambda = \frac{2L}{3}.$$

The frequency of the third harmonic (second overtone) is

$$f_2 = \frac{v}{\lambda} = \frac{3v}{2L} = \frac{3(330 \text{ m/s})}{2(3.00 \text{ m})} = 165 \text{ Hz} = 3f_0.$$

◆ wxMaxima codes:

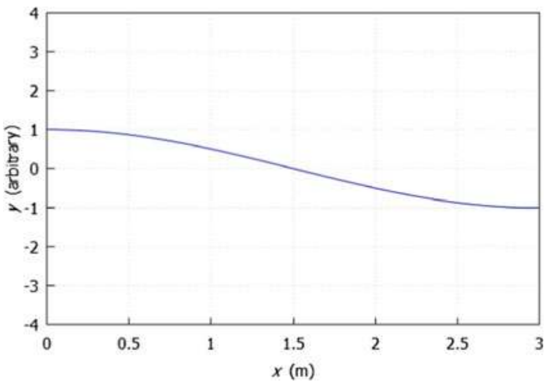
```
(%i3) fpprintprec:5; v:330; L:3;
(fpprintprec)     5
(v)   330
(L)   3
(%i4) f0: float(v/(2*L));
(f0)  55.0
(%i5) f1: float(v/L);
(f1)  110.0
(%i6) f2: float(3*v/(2*L));
(f2)  165.0
```

Comments on the codes:

(%i3)	Set the floating point print precision to 5 and assign values of v and L.
(%i4), (%i5), (%i6)	Calculate f_0, f_1, and f_2.

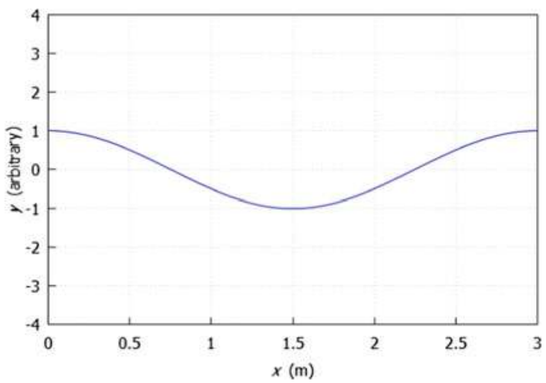
◆ Animation of the vibrating air column of first harmonic (fundamental frequency):

```
(%i1) with_slider_draw(
                    t, makelist(i,i,0,2*%pi,0.5),
                    explicit(cos(2*%pi*x/6)*cos(t),x,0,3),
                    grid=true,
                    yrange=[-4,4],
                    xlabel="{/Helvetica-Italic x} (m)",
                    ylabel="{/Helvetica-Italic y} (arbitrary)");
```



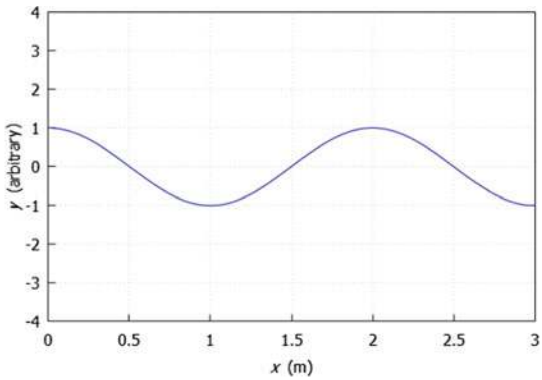
♦ Animation of the vibrating air column of second harmonic (first overtone):

```
(%i1) with_slider_draw(
                    t, makelist(i,i,0,2*%pi,0.5),
                    explicit(cos(2*%pi*x/3)*cos(t),x,0,3),
                    grid=true,
                    yrange=[-4,4],
                    xlabel="{/Helvetica-Italic x} (m)",
                    ylabel="{/Helvetica-Italic y} (arbitrary)");
```



♦ Animation of the vibrating air column of third harmonic (second overtone):

```
(%i1) with_slider_draw(
                    t, makelist(i,i,0,2*%pi,0.5),
                    explicit(cos(2*%pi*x/2)*cos(t),x,0,3),
                    grid=true,
                    yrange=[-4,4],
                    xlabel="{/Helvetica-Italic x} (m)",
                    ylabel="{/Helvetica-Italic y} (arbitrary)");
```



Comment on the codes:

To run any of the animations, copy the codes to the wxMaxima command window; press <shift> and <enter> keys simultaneously to run the codes; right-click the graphic that appears and choose *Start Animation*. The *y*-axis represents the longitudinal displacement of the air layer, while the *x*-axis is the distance along the pipe.

Problem 6.6 Two sound waves interfere at a point. The wavelengths of the first and second waves are 25.0 cm and 24.9 cm, respectively. What is the beat frequency? The speed of sound in air is 330 m s^{-1}.

Solution

The frequency of the first sound wave is

$$f_1 = \frac{v}{\lambda_1} = \frac{330 \text{ m/s}}{25.0 \times 10^{-2} \text{ m}} = 1320.0 \text{ Hz}.$$

The frequency of the second sound wave is

$$f_2 = \frac{v}{\lambda_2} = \frac{330 \text{ m/s}}{24.9 \times 10^{-2} \text{ m}} = 1325.3 \text{ Hz}.$$

The beat frequency is

$$|f_1 - f|_2 = 5.3\,\text{Hz}.$$

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; v:330; lambda1:25e-2; lambda2:24.9e-2;
(fpprintprec)     5
(v)   330
(lambda1)   0.25
(lambda2)   0.249
(%i5) f1: v/lambda1;
(f1)  1320.0
(%i6) f2: v/lambda2;
(f2)  1325.3
(%i7) beat: abs(f1-f2);
(beat) 5.3012
```

Comments on the codes:

(%i4)	Set the floating point print precision to 5 and assign values of v, λ_1, and λ_2.
(%i5), (%i6), (%i7)	Calculate f_1, f_2, and the beat frequency.

Problem 3.7 A wire 0.50 m long with a mass per unit length of 1.0×10^{-4} kg m^{-1} is tied between two nails. The tension in the wire is 4.0 N. The wire is plucked and sound can be heard. Calculate the fundamental frequency, and first and second overtones.

Solution

Figure 3.3 shows the wire vibrating with stationary waves at (a) fundamental frequency, (b) first overtone, and (c) second overtone.

The speed of transverse wave in the wire is (Eq. 2.5)

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{4.0\text{ N}}{1.0 \times 10^{-4}\text{ kg m}^{-1}}} = 200\text{ m s}^{-1}.$$

From Fig. 3.3a, the wavelength is $\lambda_0 = 2L = 2(0.50\text{ m}) = 1.0$ m. This means that the fundamental frequency is

$$f_0 = \frac{v}{\lambda_0} = \frac{200\text{ m s}^{-1}}{1.0\text{ m}} = 200\text{ Hz}.$$

From Fig. 3.3b, the wavelength is $\lambda_1 = L = 0.50$ m. The frequency of the first overtone is

$$f_1 = \frac{v}{\lambda_1} = \frac{200\text{ m s}^{-1}}{0.50\text{ m}} = 400\text{ Hz}.$$

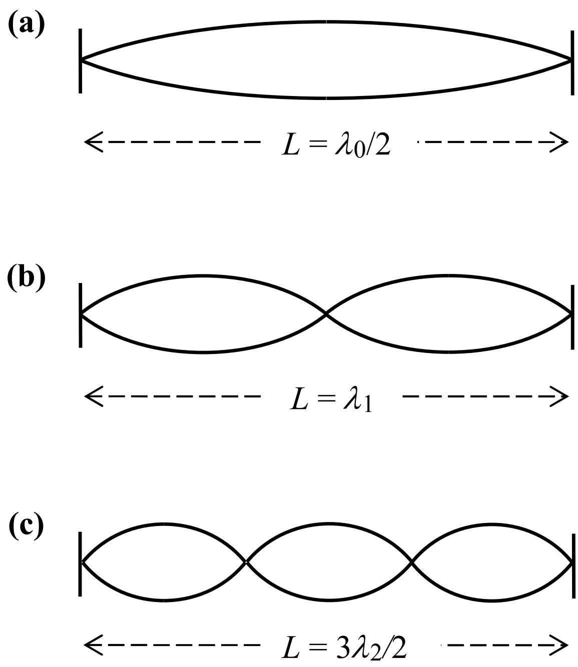


Fig. 3.3 **a** Fundamental frequency, **b** first over tone, and **c** second overtone of a vibrating wire, Problem 3.7

From Fig. 3.3c, the wavelength is $\lambda_2 = 2L/3 = 0.33$ m. The frequency of the second overtone is

$$f_2 = \frac{v}{\lambda_2} = \frac{200\ \text{m s}^{-1}}{0.33\ \text{m}} = 600\ \text{Hz}.$$

◆ wxMaxima codes:

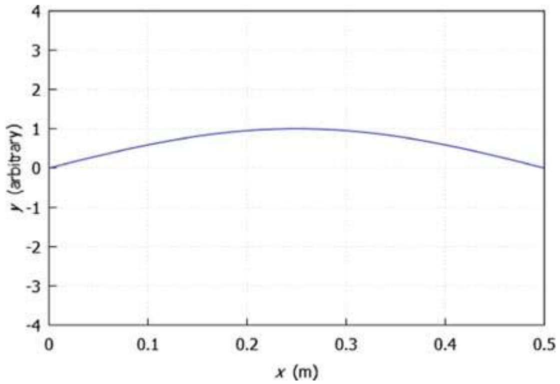
```
(%i4) fpprintprec:5; T:4; L:0.5; mu:1e-4;
(fpprintprec)     5
(T)   4
(L)   0.5
(mu)  1.0*10^-4
(%i5) v: sqrt(T/mu);
(v)   200.0
(%i6) lambda0: 2*L;
(lambda0) 1.0
(%i7) f0: v/lambda0;
(f0)  200.0
(%i8) lambda1: L;
(lambda1) 0.5
(%i9) f1: v/lambda1;
(f1)  400.0
(%i10) lambda2: 2*L/3;
(lambda2) 0.33333
(%i11) f2: v/lambda2;
(f2)  600.0
```

Comments on the codes:

(%i4)	Set the floating point print precision to 5 and assign values of T, L, and μ.
(%i5), (%i6), (%i7), (%i8), (%i9), (%i10), (%i11)	Calculate v, λ_0, f_0, λ_1, f_1, λ_2, and f_2.

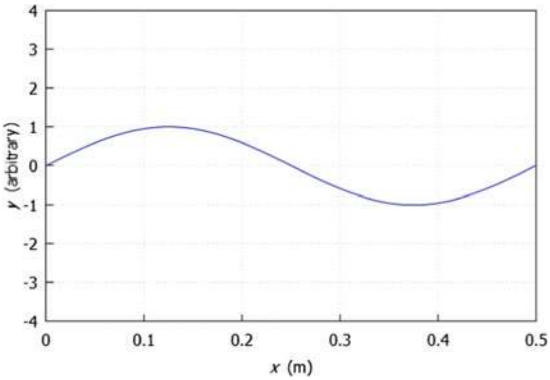
♦ Animation of the vibrating wire (fundamental frequency):

```
(%i1) with_slider_draw(
                t, makelist(i,i,0,2*%pi,0.5),
                explicit(sin(2*%pi*x/1.0)*cos(t),x,0,0.5),
                grid=true,
                yrange=[-4,4],
                xlabel="{/Helvetica-Italic x} (m)",
                ylabel="{/Helvetica-Italic y} (arbitrary)");
```



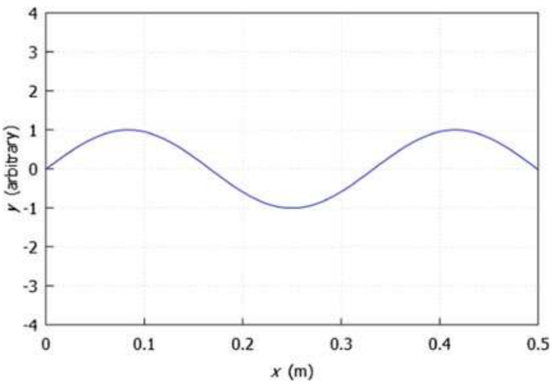
♦ Animation of the vibrating wire (first overtone):

```
(%i1) with_slider_draw(
                t, makelist(i,i,0,2*%pi,0.5),
                explicit(sin(2*%pi*x/0.5)*cos(t),x,0,0.5),
                grid=true,
                yrange=[-4,4],
                xlabel="{/Helvetica-Italic x} (m)",
                ylabel="{/Helvetica-Italic y} (arbitrary)");
```



◆ Animation of the vibrating wire (second overtone):

```
(%i1) with_slider_draw(
                t, makelist(i,i,0,2*%pi,0.5),
                explicit(sin(2*%pi*x/0.333)*cos(t),x,0,0.5),
                grid=true,
                yrange=[-4,4],
                xlabel="{/Helvetica-Italic x} (m)",
                ylabel="{/Helvetica-Italic y} (arbitrary)");
```



Comment on the codes:

To run any of the animations, copy the codes to the wxMaxima command window; press <shift> and <enter> keys simultaneously to run the codes; right-click the graphic that appears and choose *Start Animation*. The *y*-axis represents the transverse displacement of the string, while the *x*-axis is the distance along the string.

Problem 3.8 Two sound waves superpose at a point. Variations of the air pressures with time of the two waves at the point are

$$p_1 = 30\cos(2\pi f_1 t),$$
$$p_2 = 30\cos(2\pi f_2 t),$$

where $f_1 = 55.0$ Hz, $f_2 = 50.0$ Hz, and 30 Pa is the pressure amplitude of the sound waves. Show that the beat frequency is $f_1 - f_2 = 55.0$ Hz – 50.0 Hz = 5.0 Hz.

Solution

This is the superposition of two waves whose frequencies differ a bit at a point. We calculate the superposition of the two pressure variations,

$$\begin{aligned} p &= p_1 + p_2 \\ &= 30\cos(2\pi f_1 t) + 30\cos(2\pi f_2 t) \\ &= 60\cos(\pi(f_1 + f_2)t) \cdot \cos(\pi(f_1 - f_2)t). \end{aligned}$$

We have used trigonometric identity $\cos\theta + \cos\phi = 2\cos[(\theta + \phi)/2].\cos[(\theta - \phi)/2]$ to get the result, see Appendix D. We write

$$p = [60\cos(\pi(f_1 - f_2)t)] \cdot \cos(\pi(f_1 + f_2)t).$$

The expression in the square brackets is the amplitude. The amplitude is a maximum if $\cos(\pi(f_1 - f_2)t)$ is 1 or -1, that is, twice in a cycle of the cosine function. The frequency of the cosine function is $(f_1 - f_2)/2$. Thus, the frequency of maximum amplitude (the beat frequency) is twice of $(f_1 - f_2)/2$, that is, $(f_1 - f_2)$. The beat frequency is

$$f_{beat} = f_1 - f_2 = 55.0\text{ Hz} - 50.0\text{ Hz} = 5.0\text{ Hz}.$$

Alternative argument: The intensity of sound is proportional to the amplitude square. In our case, the intensity is proportional to $[60\cos(\pi(f_1 - f_2)t)]^2$. The intensity is a maximum if $\cos^2(\pi(f_1 - f_2)t) = 1$. This occurs $f_1 - f_2$ times a second. Therefore, the beat frequency is $f_1 - f_2$.

Problem 3.9 For Problem 3.8, show graphically that the beat frequency is $f_1 - f_2 = 5.0$ Hz.

Solution

We separately plot pressure variation against the time of the two waves,

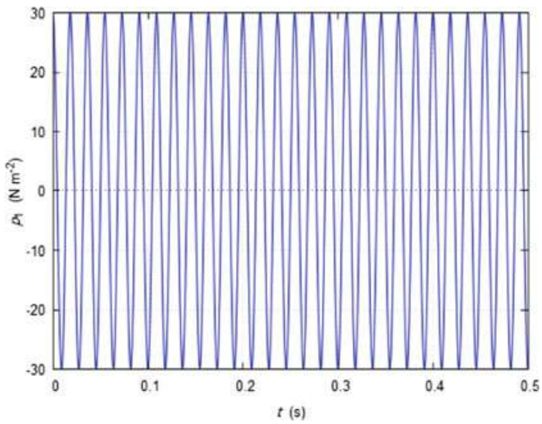
$$p_1 = 30\cos(2\pi f_1 t) = 30\cos(2\pi(55)t),$$
$$p_2 = 30\cos(2\pi f_2 t) = 30\cos(2\pi(50)t).$$

We then plot the sum of the two pressure variations against time,

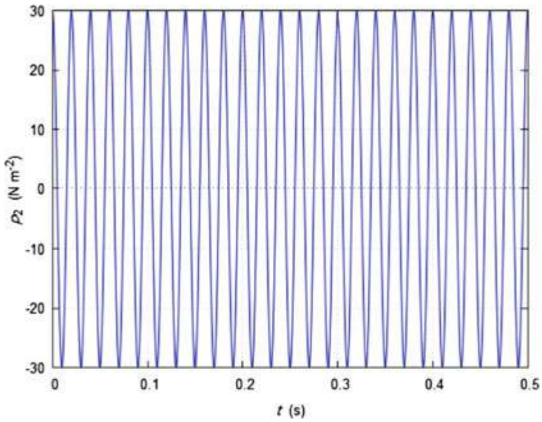
$$p = p_1 + p_2.$$

♦ wxMaxima codes:

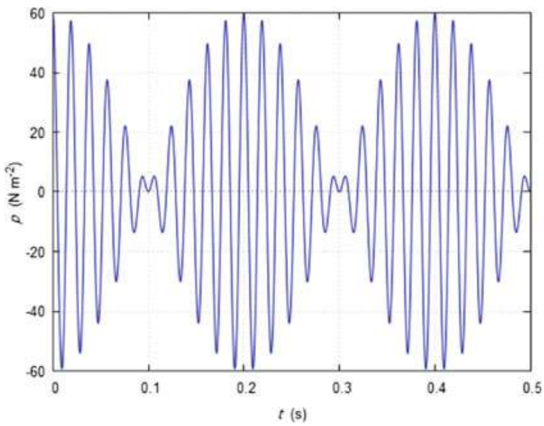
```
(%i2) f1:55; f2:50;
(f1)  55
(f2)  50
(%i5) p1: 30*cos(2*%pi*f1*t); p2: 30*cos(2*%pi*f2*t); p: p1+p2;
(p1)  30*cos(110*%pi*t)
(p2)  30*cos(100*%pi*t)
(p)   30*cos(110*%pi*t)+30*cos(100*%pi*t)
(%i6) wxplot2d(p1, [t,0,0.5], grid2d, [xlabel,"{/Helvetica-Italic t}
(s)"], [ylabel,"{/Helvetica-Italic p}_1  (N m^{-2})"]);
```




```
(%i7) wxplot2d(p2, [t,0,0.5], grid2d, [xlabel,"{/Helvetica-Italic t}
(s)"], [ylabel,"{/Helvetica-Italic p}_2  (N m^{-2})"]);
```




```
(%i8) wxplot2d(p, [t,0,0.5], grid2d, [xlabel,"{/Helvetica-Italic t}  (s)"],
[ylabel,"{/Helvetica-Italic p}  (N m^{-2})"]);
```



Comments on the codes:

(%i2)	Assign values of f_1 and f_2.
(%i5)	Define p_1, p_2, and p in terms of t.
(%i6), (%i7), (%i8)	Plot p_1, p_2, and p for $0 \leq t \leq 0.5$ s.

From the last graphic, time interval between beats is $T = 0.4$ s – 0.2 s $= 0.2$ s. The beat frequency is

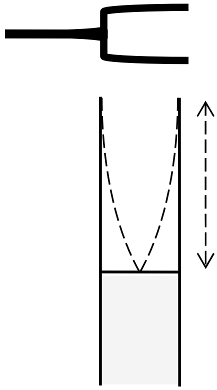


Fig. 3.4 A tuning fork and an air column at resonance, Problem 3.10

$$f = \frac{1}{T} = \frac{1}{0.2\ \text{s}} = 5.0\ \text{Hz}.$$

Problem 3.10 A tuning fork vibrating at a frequency of 256 Hz produces a loud sound when placed near a 0.31 m air column. Calculate the speed of sound in air.

Solution

Figure 3.4 shows the tuning fork, the air column, and the stationary wave when the loud sound is heard.

The length of the air column corresponds to 1/4 wavelength of the sound wave. The wavelength of the sound is

$$L = \frac{\lambda}{4},$$
$$\lambda = 4L = 4(0.31\ \text{m}) = 1.24\ \text{m}.$$

The speed of the sound is

$$v = \lambda f = (1.24\ \text{m})(256\ \text{s}^{-1}) = 317\ \text{m s}^{-1}.$$

◆ wxMaxima codes:

```
(%i3) fpprintprec:5; L:0.31; f:256;
(fpprintprec)     5
(L)   0.31
(f)   256
(%i4) lambda: 4*L;
(lambda) 1.24
(%i5) v: lambda*f;
(v)   317.44
```

Comments on the codes:

(%i3)	Set the floating point print precision to 5 and assign values of L and f.
(%i4), (%i5)	Calculate λ and v.

Problem 3.11 A tuning fork vibrating at a frequency of 320 Hz is placed near a measuring cylinder filled with water. Consecutive loud sounds are heard when the water levels are at 20 cm and 73 cm marks. Determine the speed of sound in air.

Solution

Figure 3.5 shows the stationary waves at both resonances. A loud sound is heard when there is a resonance. When the water level is at the 73 cm mark, ¼ of a complete wave resonates; when the water level is at the 20 cm mark, ¾ of a complete wave resonates.

From the figure, one-half of the wavelength of the stationary wave is

$$\frac{\lambda}{2} = 73\text{ cm} - 20\text{ cm} = 53\text{ cm}.$$

The wavelength of the sound wave is

$$\lambda = 2(53\text{ cm}) = 106\text{ cm}.$$

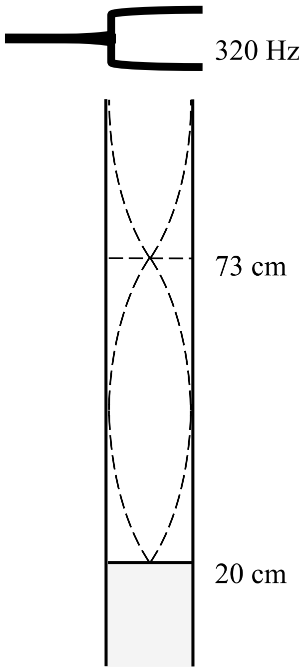


Fig. 3.5 A tuning fork and an air column at two resonances, Problem 3.11

The speed of sound in air is

$$v = \lambda f = (1.06\ \text{m})(320\ \text{s}^{-1}) = 340\ \text{m s}^{-1}.$$

◆ wxMaxima codes:

```
(%i2) fpprintprec:5; f:320;
(fpprintprec)     5
(f) 320
(%i3) lambda: 2*(0.73-0.2);
(lambda) 1.06
(%i4) v: lambda*f;
(v) 339.2
```

Comments on the codes:

(%i2)	Set the floating point print precision to 5 and assign the value of f.
(%i3), (%i4)	Calculate λ and v.

Problem 3.12 Two waves represented by

$$y_1(x, t) = 1.5\cos(2t - 3x + \pi/3),$$
$$y_2(x, t) = 1.5\cos(2t - 3x),$$

superpose and form a new resultant wave. Find the resultant wave.

Solution

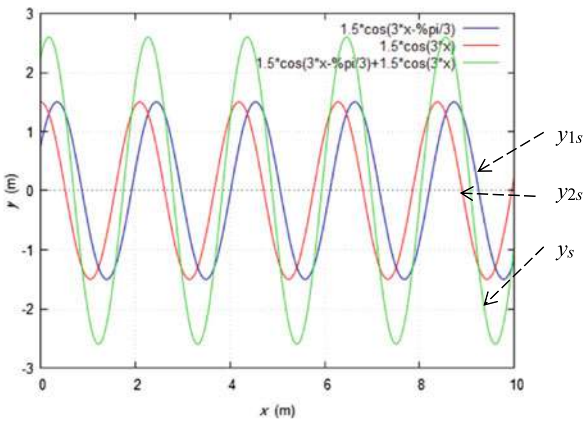
Superposition of waves is obtained by summing the two waves,

$$\begin{aligned} y &= y_1(x, t) + y_2(x, t) \\ &= 1.5\cos(2t - 3x + \pi/3) + 1.5\cos(2t - 3x) \\ &= 3\cos\left(2t - 3x + \frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right) \\ &= 2.6\ \cos\left(2t - 3x + \frac{\pi}{6}\right). \end{aligned}$$

The trigonometric identity $\cos\theta + \cos\phi = 2\cos[(\theta + \phi)/2].\cos[(\theta - \phi)/2]$ has been used to get the result; see Appendix D. The resultant wave is a traveling sinusoidal wave and is not a stationary wave. The amplitude is 2.6 m and the wavelength and frequency are the same as those of the summed waves.

◆ wxMaxima codes:

```
(%i1) fpprintprec:5;
(fpprintprec)     5
(%i2) amplitude: float(3*cos(%pi/6));
(amplitude) 2.5981
(%i4) y1(x,t):=1.5*cos(2*t-3*x+%pi/3); y2(x,t):=1.5*cos(2*t-3*x);
(%o3) y1(x,t):=1.5*cos(2*t-3*x+%pi/3)
(%o4) y2(x,t):=1.5*cos(2*t-3*x)
(%i6) y1s: y1(x,0);  y2s: y2(x,0);
(y1s) 1.5*cos(3*x-%pi/3)
(y2s) 1.5*cos(3*x)
(%i7) ys: y1s+y2s;
(ys)  1.5*cos(3*x-%pi/3)+1.5*cos(3*x)
(%i8) wxplot2d([y1s, y2s, ys],  [x,0,10], grid2d, [xlabel,"{/Helvetica-
Italic x}  (m)"], [ylabel,"{/Helvetica-Italic y}  (m)"]);
```



Comments on the codes:

(%i1)	Set the floating point print precision to 5.
(%i2)	Calculate the amplitude of the resultant wave.
(%i4)	Define $y_1(x, t)$ and $y_2(x, t)$.
(%i6), (%i7)	Assign $y_{1s} = y_1(x, 0)$, $y_{2s} = y_2(x, 0)$, and $y_s = y_{1s} + y_{2s}$.
(%i8)	Plot y_{1s}, y_{2s}, and y_s for $0 \le x \le 10$ m.

Problem 3.13 Two sources of waves are located at points A and B separated by 20 m, as shown in Fig. 3.6. The vibrations of the sources are

$$y_{A\ vibration} = 30\cos[2\pi(95)t],$$
$$y_{B\ vibration} = 20\cos[2\pi(90)t].$$

Waves of the two vibrations propagate along the positive x direction at a speed of 300 m s^{-1}. Determine the vibrations at point C, 10 m to the right of B.

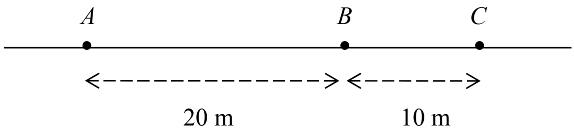


Fig. 3.6 Two wave sources at points A and B, Problem 3.13

Solution

Waves from A propagate to the right; the wave equation is,

$$\begin{aligned} y_A &= 30\cos\left[2\pi(95)\left(t-\frac{x}{v}\right)\right] \\ &= 30\cos\left[2\pi(95)\left(t-\frac{x}{300}\right)\right]. \end{aligned}$$

We get this from the general equation of wave moving to the right, by substituting $f = 95$ Hz and speed of the wave $v = 300\ \mathrm{m\ s^{-1}}$,

$$\begin{aligned} y_A &= A\cos\,(\omega t - kx) \\ &= A\cos\left[\omega\left(t-\frac{x}{v}\right)\right] \\ &= A\cos\left[2\pi f\left(t-\frac{x}{v}\right)\right] \\ &= 30\ \cos\left[2\pi(95)\left(t-\frac{x}{300}\right)\right]. \end{aligned}$$

Vibrations at point C due to waves from A are obtained by inserting the value of $x = 20\ \mathrm{m} + 10\ \mathrm{m} = 30\ \mathrm{m}$,

$$y_{A,C} = 30\ \cos\left[2\pi(95)\left(t-\frac{30}{300}\right)\right].$$

Waves from B travel to the right and the wave equation is

$$\begin{aligned} y_B &= 20\ \cos\left[2\pi(90)\left(t-\frac{x}{v}\right)\right] \\ &= 20\ \cos\left[2\pi(90)\left(t-\frac{x}{300}\right)\right]. \end{aligned}$$

Vibrations at point C due to waves from B are obtained by inserting the value of $x = 10$ m,

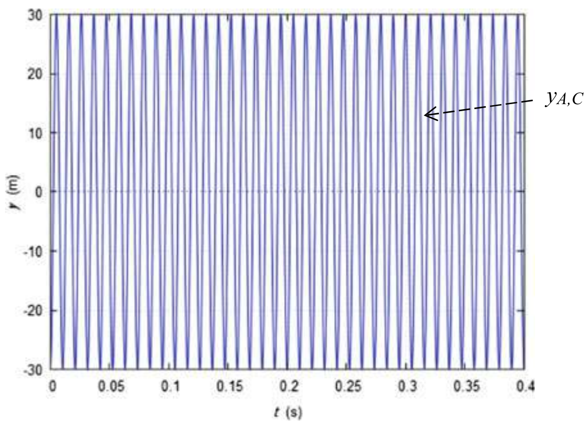
$$y_{B,C} = 20\ \cos\left[2\pi(90)\left(t-\frac{10}{300}\right)\right].$$

Thus, vibrations at point C are

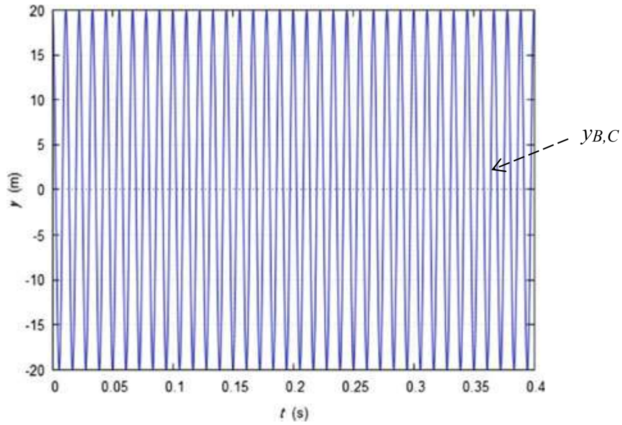
$$y_{C\ vibration} = y_{A,C} + y_{B,C}$$
$$= 30\ \cos\left[2\pi(95)\left(t - \frac{30}{300}\right)\right] + 20\ \cos\left[2\pi(90)\left(t - \frac{10}{300}\right)\right].$$

◆ Plots of $y_{A,C}$, $y_{B,C}$, and $y_{C\ vibration}$ against time by wxMaxima:

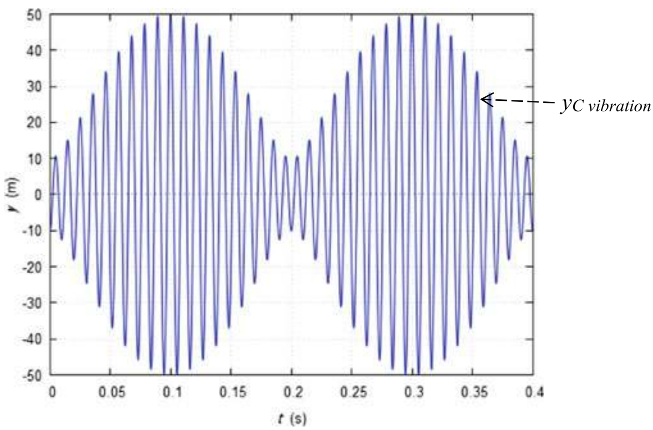
```
(%i1) yAC: 30*cos(2*%pi*95*(t-30/300));
(yAC) 30*cos(190*%pi*(t-1/10))
(%i2) yBC: 20*cos(2*%pi*90*(t-10/300));
(yBC) 20*cos(180*%pi*(t-1/30))
(%i3) yCvibration: yAC + yBC;
(yCvibration) 20*cos(180*%pi*(t-1/30))+30*cos(190*%pi*(t-1/10))
(%i4) wxplot2d(yAC, [t,0,0.4], grid2d, [xlabel,"{/Helvetica-Italic t}
(s)"], [ylabel,"{/Helvetica-Italic y}  (m)"]);
```




```
(%i5) wxplot2d(yBC, [t,0,0.4], grid2d, [xlabel,"{/Helvetica-Italic t}
(s)"], [ylabel,"{/Helvetica-Italic y}  (m)"]);
```




```
(%i6) wxplot2d(yCvibration, [t,0,0.4], grid2d, [xlabel,"{/Helvetica-Italic
t}  (s)"], [ylabel,"{/Helvetica-Italic y}  (m)"]);
```



Comments on the codes:

(%i1), (%i2)	Define $y_{A,C}$ and $y_{B,C}$.
(%i3)	Calculate $y_{C\ vibration}$.
(%i4), (%i5), (%i6)	Plot $y_{A,C}$, $y_{B,C}$, and $y_{C\ vibration}$ against t for $0 \leq t \leq 0.4$ s.

The vibrations at C have beats in them, as shown in the plot. These vibrations have a beat frequency of 95.0 Hz – 90.0 Hz = 5.0 Hz. From the plot, the period of the beat is $T = 0.3$ s – 0.1 s = 0.2 s, giving the beat frequency as $1/T = 1/0.2$ s = 5.0 Hz.

Problem 3.14 Two wave sources are located at points A and B a distance 20 m apart, as shown in Fig. 3.7. The vibrations at A and B are

$$y_{A\ vibration} = 0.06\ \sin\ (\pi t),$$
$$y_{B\ vibration} = 0.02\ \sin\ (\pi t).$$

Waves originating from vibrations at A move in the positive x direction with a speed of 3.0 m s^{-1}, while waves originating from vibrations at B move in the negative x direction with the same speed. Determine the vibrations at a point C, 8.0 m to the left of B.

Solution

Vibrations at A generate waves that propagate to the right. The equation of the traveling wave is

$$\begin{aligned} y_A &= 0.06\ \sin\left[\pi\left(t - \frac{x}{v}\right)\right] \\ &= 0.06\ \sin\left[\pi\left(t - \frac{x}{3}\right)\right]. \end{aligned}$$

We have used the fact that a wave traveling to the right is a function of $t - x/v$, so we replaced t with $t - x/v$ in $y_{A\ vibration}$ to get y_A. We then substitute the speed of wave $v = 3.0$ m s^{-1} in the equation. Vibrations at point C due to waves coming from A is obtained by substituting $x = 12$ m into the equation,

$$y_{A,C} = 0.06\ \sin\left[\pi\left(t - \frac{12}{3}\right)\right].$$

Waves from B move to the left to C. The wave is represented by,

$$\begin{aligned} y_B &= 0.02\ \sin\left[\pi\left(t + \frac{x}{v}\right)\right] \\ &= 0.02\ \sin\left[\pi\left(t + \frac{x}{3}\right)\right]. \end{aligned}$$

Again, we have used the fact that a wave traveling to the left is a function of $t + x/v$, so we replaced t with $t + x/v$ in $y_{B\ vibration}$ to get y_B. We also substituted the

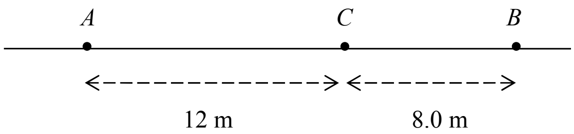


Fig. 3.7 Two wave sources at points A and B, Problem 3.14

speed of wave $v = 3.0$ m s^{-1} in the equation. Vibrations at C due to waves from B are obtained by substituting $x = -8.0$ m into the equation,

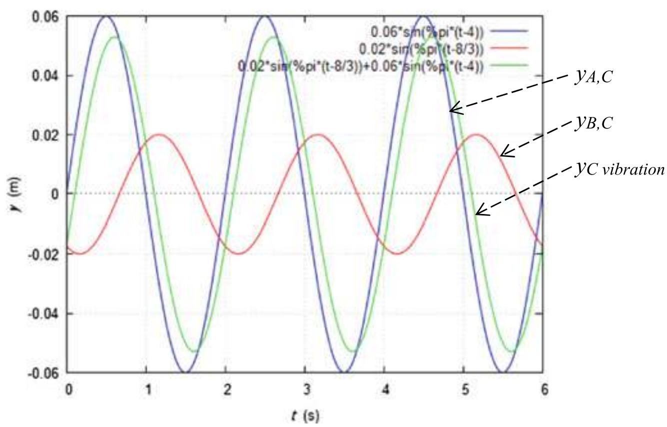
$$y_{B,C} = 0.02\ \sin\left[\pi\left(t - \frac{8}{3}\right)\right].$$

Therefore, vibrations at C due to vibrations of sources at A and B are,

$$\begin{aligned} y_{C\ vibration} &= y_{A,C} + y_{B,C} \\ &= 0.06\ \sin\left[\pi\left(t - \frac{12}{3}\right)\right] + 0.02\ \sin\left[\pi\left(t - \frac{8}{3}\right)\right] \\ &= 0.05 \sin\,(\pi t) - 0.017 \cos\,(\pi t). \end{aligned}$$

♦ Plots of $y_{A,C}$, $y_{B,C}$, and $y_{C\ vibration}$ against time by wxMaxima:

```
(%i1) yAC: 0.06*sin(%pi*(t-12/3));
(yAC) 0.06*sin(%pi*(t-4))
(%i2) yBC: 0.02*sin(%pi*(t-8/3));
(yBC) 0.02*sin(%pi*(t-8/3))
(%i3) yCvibration: yAC + yBC;
(yCvibration) 0.02*sin(%pi*(t-8/3))+0.06*sin(%pi*(t-4))
(%i4) wxplot2d([yAC, yBC, yCvibration], [t,0,6], grid2d,
[xlabel,"{/Helvetica-Italic t}  (s)"],
[ylabel,"{/Helvetica-Italic y}  (m)"]);
```



Comments on the codes:

(%i1), (%i2), (%i3)   Define $y_{A,C}$, $y_{B,C}$, and $y_{C\ vibration}$.

(%i4) Plot $y_{A,C}$, $y_{B,C}$, and $y_{C\,vibration}$ for $0 \le t \le 6$ s.

The plots show that vibrations at point C are sinusoidal vibrations (simple harmonic motion) with the same frequency as the frequency of sources at A and B.

3.3 Summary

- Superposition is the combination of two or more waves at the same location.
- The wave that results from the superposition of two sine waves that differ only by a phase shift is a wave with an amplitude that depends on the phase difference.
- A stationary wave is formed from the superposition of two sine waves having the same frequency, amplitude, and wavelength and moving in opposite directions. The wave varies in amplitude but does not propagate.

3.4 Exercises

Exercise 3.1 What is the wave obtained from the superposition of the following two traveling waves,

$$y_1 = A\cos(\omega t - kx + \phi),$$
$$y_2 = A\cos(\omega t - kx)?$$

(Answer: $2A\cos(\frac{\phi}{2})\cos(\omega t - kx + \frac{\phi}{2})$)

Exercise 3.2 A string is fixed between two nails. A transverse wave along the string to one of the nails is represented by

$$y_1 = A\cos(kx - \omega t),$$

while the one reflected from the nail is represented by

$$y_2 = A\cos(kx + \omega t).$$

What is the superposition of the two waves?
(Answer: $2A\cos(kx)\cos(\omega t)$)

Exercise 3.3 A 50.0 cm long wire with a mass per unit length of 1.00×10^{-4} kg m^{-1} vibrates under a tension of 4.00 N as shown in Fig. 3.8. Find the fundamental frequency of the vibrations.
(Answer: 200 Hz)

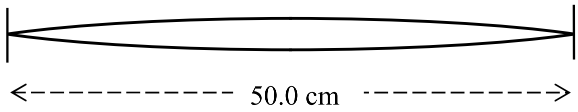


Fig. 3.8 A vibrating wire, Exercise 3.3

Exercise 3.4 The equation for a stationary wave on a string is,

$$y = 0.12 \ \sin(5x) \ \cos(200t).$$

where y and x are in meters and t in seconds. Find

(a) amplitude of vibration at the antinodes,
(b) distance between antinodes,
(c) wavelength,
(d) frequency,
(e) speed of the wave.

(Answer: (a) 0.12 m; (b) 0.63 m; (c) 1.3 m; (d) 32 Hz; (e) 40 m s^{-1})

Exercise 3.5 The second overtone produced by a vibrating string 2.0 m long is 900 Hz. Determine

(a) fundamental and first overtone frequencies,
(b) speed of the wave in the string.

(Answer: (a) 300 Hz, 600 Hz; (b) 1.2×10^3 m s^{-1})

Chapter 4
Electric Field

Abstract This chapter discusses problems on electric charge, electrostatic force, and electric field. Vector additions and methods of calculus are used to calculate some of the electric fields. Both analytical and computer calculations are presented.

4.1 Basic Concepts and Formulae

(1) Electric charge has the following properties:

   (a) Charge is conserved.
   (b) Charges of different signs (+ and −) attract each other. Charges of the same signs (+ and +, or − and −) repel each other.
   (c) Charge is quantized; it exists as multiple of electronic charges. An electronic charge is the charge of an electron. An electron has an electric charge of 1.6022×10^{-19} C. Thus, 1 C is the charge of 6.2415×10^{18} electrons.
   (d) The force between two charges varies with the inverse square of their separation distance.

(2) A conductor is a material in which electrical charges can move freely. Examples of conductors are copper, aluminum, and silver.
(3) An insulator is a material in which charges cannot move freely. Examples of insulators are glass, rubber, and wood.
(4) Coulomb's law states that the magnitude of electrostatic force F between two charges q_1 and q_2, separated by a distance r, is,

$$F = k\frac{q_1 q_2}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}, \tag{4.1}$$

where k is Coulomb's constant,

$$k = \frac{1}{4\pi\varepsilon_0} = 8.9876 \times 10^9 \text{ N m}^2 \text{ C}^{-2} \approx 9 \times 10^9 \text{ N m}^2 \text{ C}^{-2},$$

W. M. S. Wan Hassan et al., *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-43165-4_4

and ε_0 is the permittivity of free space,

$$\varepsilon_0 = 8.8542 \times 10^{-12}\ \mathrm{C^2\ N^{-1}\ m^{-2}}.$$

(5) The smallest unit of charge in the universe is the charge of an electron or a proton. The magnitude of the charge is,

$$e = 1.6022 \times 10^{-19}\ \mathrm{C}.$$

Charge on other entities is the multiple of this unit of charge. The charge is quantized.

(6) Electric field $\boldsymbol{E}$ at a point is the electric force $\boldsymbol{F}$ acting on test charge q at the point divided by the charge,

$$\boldsymbol{E} = \frac{\boldsymbol{F}}{q}. \tag{4.2}$$

(7) Electric field due to charge q at a point a distance r from the charge is

$$\boldsymbol{E} = k\frac{q}{r^2}\hat{\boldsymbol{r}}, \tag{4.3}$$

where $\hat{\boldsymbol{r}}$ is the unit vector in the direction of the charge to the point. For positive point charge, the electric field vector is directed in a radial way away from the point charge.

(8) Electric field at a point of observation due to many point charges is the vector sum of electric fields of each charge at the point,

$$\boldsymbol{E} = k\sum_i \frac{q_i}{r_i^2}\hat{\boldsymbol{r}}_i. \tag{4.4}$$

(9) Electric field of a continuous charge distribution at a point is,

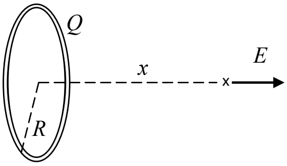
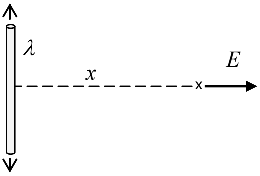
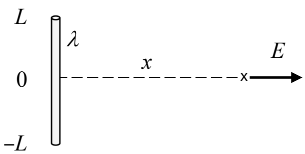
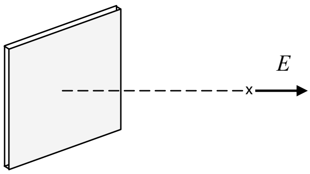
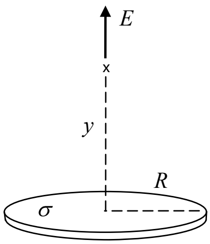
$$\boldsymbol{E} = k\int \frac{dq}{r^2}\hat{\boldsymbol{r}}, \tag{4.5}$$

where dq is infinitesimal charge of the charge distribution and r is distance from the infinitesimal charge to the point of observation. Table 4.1 shows the electric fields of a few charge distribution configurations.

(10) Electric field lines are used to indicate electric field in space. Electric field vector $\boldsymbol{E}$ is tangent to electric field line. The number of electric field lines per unit area through a surface perpendicular to the electric field lines is proportional to the magnitude of the electric field.

(11) A particle of mass m and charge q in electric field $\boldsymbol{E}$ will move with acceleration,

Table 4.1 Electric fields of a few charge configurations

	Configuration	Electric field
(a)		A ring with charge Q, $E = kQ\frac{x}{(x^2+R^2)^{3/2}}$
(b)		A long wire with charge per unit length λ, $E = \frac{2k\lambda}{x} = \frac{\lambda}{2\pi\varepsilon_0 x}$
(c)		A wire of length $2L$ with charge per unit length λ, $E = \frac{2k\lambda L}{x\sqrt{L^2+x^2}}$
(d)		A wide insulator plate with charge per unit area σ, $E = 2\pi k\sigma = \frac{\sigma}{2\varepsilon_0}$
(e)		An insulator circular disk with charge per unit area σ, $E = 2\pi k\sigma\left(1 - \frac{y}{\sqrt{y^2+R^2}}\right) = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{y}{\sqrt{y^2+R^2}}\right)$

$$\boldsymbol{a} = \frac{q\boldsymbol{E}}{m}.$$

In a uniform electric field, the acceleration a is constant and the motion of the particle is similar to the motion of a projectile in a uniform gravitational field.

4.2 Problems and Solutions

Problem 4.1

(a) How many electrons are there in a charge of -1.0 C?
(b) Calculate the repulsive force between two particles, each with a -1.0 C charge, separated by a distance of 1.0 m.

Solution

(a) An electron has a charge of -1.6×10^{-19} C. The number of electrons in -1.0 C charge is

$$\frac{-1.0\ \text{C}}{-1.6 \times 10^{-19}\ \text{C}} = 6.2 \times 10^{18}.$$

(b) Figure 4.1 shows the two particles and the repulsive force $\boldsymbol{F}$ on one of them.

The magnitude of repulsive force between the two particles is, Eq. (4.1),

$$F = k\frac{q_1 q_2}{r^2} = (9 \times 10^9\ \text{N m}^2\ \text{C}^{-2})\frac{(1.0\ \text{C})(1.0\ \text{C})}{(1.0\ \text{m})^2} = 9.0 \times 10^9\ \text{N}.$$

◆ wxMaxima codes:

```
(%i6) fpprintprec:5; e:-1.6e-19; k:9e9; q1:1; q2:1; r:1;
(fpprintprec)     5
(e)   -1.6*10^-19
(k)   9.0*10^9
(q1)  1
(q2)  1
(r)   1
(%i7) -1/e;
(%o7) 6.25*10^18
(%i8) F: k*q1*q2/r^2;
(F)   9.0*10^9
```

Comments on the codes:

(%i6) Set the floating point print precision to 5 and assign values of e, k, q_1, q_2, and r.

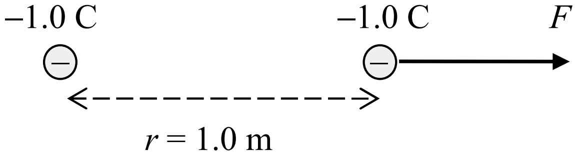


Fig. 4.1 Two particles with an equal electric charge on each, Problem 4.1

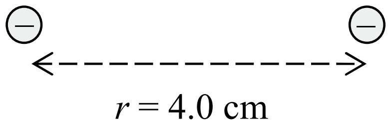


Fig. 4.2 Two spheres, each has equal number of electrons, Problem 4.2

(%i7) Calculate the number of electrons. Part (a).
(%i8) Calculate the Coulomb force F between two particles. Part (b).

Problem 4.2 Two small spheres are separated by a distance of 4.0 cm. Each sphere carries an equal number of electrons. How many electrons are there on each of them so that the repulsive force is 1.0×10^{-19} N?

Solution

Figure 4.2 shows the two spheres separated by a distance of 4.0 cm. Let us have n electrons on each of them. The charge on each sphere is $-ne$ where $-e$ is the charge of an electron.

Applying Eq. (4.1), the magnitude of the Coulomb force, that is, the repulsive force between the spheres is

$$F = k\frac{q_1 q_2}{r^2} = k\frac{(ne)(ne)}{r^2} = k\frac{e^2 n^2}{r^2}.$$

Substituting known values and solving for n give

$$1.0 \times 10^{-19}\ \text{N} = (9 \times 10^9\ \text{N m}^2\ \text{C}^{-2})\frac{(1.6 \times 10^{-19}\ \text{C})^2 n^2}{(4.0 \times 10^{-2}\ \text{m})^2},$$
$$n = 833.$$

◆ wxMaxima codes:

```
(%i6) fpprintprec:5; ratprint:false; F:1e-19; e:1.6e-19; k:9e9; r:4e-2;
(fpprintprec)     5
(ratprint)  false
(F)   1.0*10^-19
(e)   1.6*10^-19
(k)   9.0*10^9
(r)   0.04
(%i8) solve(F=k*e^2*n^2/r^2, n)$ float(%);
(%o8) [n=-833.33,n=833.33]
```

Comments on the codes:

(%i6) Set the floating point print precision to 5 and internal rational number print to false, and assign values of F, e, k, and r.
(%i8) Solve $F = ke^2n^2/r^2$ for n.

Problem 4.3

(a) Calculate the magnitude of electric force on a particle of charge $q_1 = 2.0 \times 10^{-6}$ C due to a second particle of charge $q_2 = 3.0 \times 10^{-6}$ C. Both charges are separated by a distance of 5.0 m.
(b) A third particle of charge $q_3 = -4.0 \times 10^{-6}$ C is placed between both particles, 3.0 m from the first and 2.0 m from the second. What is the electric force on the first particle?

Solution

(a) Figure 4.3a shows the first and second particles.

By Coulomb's law, electric force on the first particle due to the second particle is, Eq. (4.1),

$$F_{12} = k\frac{q_1 q_2}{r^2} = (9 \times 10^9 \text{ N m}^2 \text{ C}^{-2})\frac{(2.0 \times 10^{-6}\text{ C})(3.0 \times 10^{-6}\text{ C})}{(5.0\text{ m})^2}$$
$$= 2.2 \times 10^{-3}\text{ N to the left.}$$

The electric force is toward the left as the same charges repel. Due to the same charges, the second particle pushes the first toward the left.

(b) Figure 4.3b shows the three particles. From part (a), electric force on the first particle due to the second particle is

$$F_{12} = 2.2 \times 10^{-3}\text{N toward the left.}$$

Electric force on the first particle due to the third particle is,

$$F_{13} = k\frac{q_1 |q_3|}{r^2} = (9 \times 10^9 \text{ N m}^2 \text{ C}^{-2})\frac{(2.0 \times 10^{-6}\text{ C})(4.0 \times 10^{-6}\text{ C})}{(3.0\text{ m})^2}$$
$$= 8.0 \times 10^{-3}\text{ N to the right.}$$

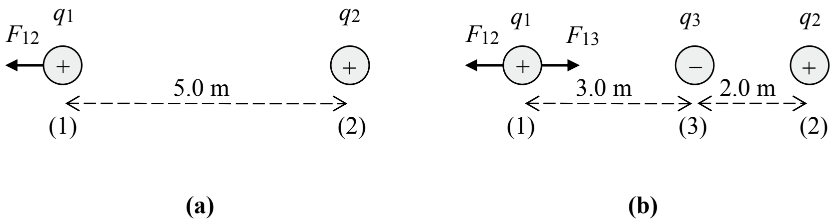


Fig. 4.3 Determining electric force of cases **a** and **b**, Problem 4.3

The absolute value of q_3 is used because we want to calculate the magnitude F_{13}. The electric force is toward the right as opposite charges attract each other. Due to their opposite charges, the third particle pulls the first toward the right.

The resultant electric force on the first particle is,

$$\begin{aligned} F_{13} - F_{12} &= 8.0 \times 10^{-3}\ \text{N} - 2.2 \times 10^{-3}\ \text{N} \\ &= 5.8 \times 10^{-3}\ \text{N to the right.} \end{aligned}$$

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; k:9e9; q1:2e-6; q2:3e-6; q3:-4e-6;
(fpprintprec)     5
(k)   9.0*10^9
(q1)  2.0*10^-6
(q2)  3.0*10^-6
(q3)  -4.0*10^-6
(%i6) F12: k*q1*q2/5^2;
(F12) 0.00216
(%i7) F13: k*q1*abs(q3)/3^2;
(F13) 0.008
(%i8) resultant_force: F13-F12;
(resultant_force) 0.00584
```

Comments on the codes:

(%i5)	Set the floating point print precision to 5 and assign values of k, q_1, q_2, and q_3.
(%i6), (%i7)	Calculate F_{12} and F_{13}.
(%i8)	Calculate resultant electric force $F_{13} - F_{12}$.

Problem 4.4 Two charges q_1 and q_2, each of 1.0×10^{-9} C, are separated by a distance of 8.0 cm. A third charge $q_3 = 5.0 \times 10^{-11}$ C is placed 5.0 cm from both of them. Calculate the electric force on the third charge.

Solution

Figure 4.4 shows the three charges and the forces on the third charge.

The magnitude of electric force on the third charge due to the first is

$$\begin{aligned} F_{31} &= k\frac{q_1 q_3}{r_{13}^2} = (9 \times 10^9\ \text{N m}^2\ \text{C}^{-2})\frac{(1.0 \times 10^{-9}\ \text{C})(5.0 \times 10^{-11}\ \text{C})}{(5.0 \times 10^{-2}\ \text{m})^2} \\ &= 1.8 \times 10^{-7}\ \text{N}. \end{aligned}$$

The force is expressed as a vector

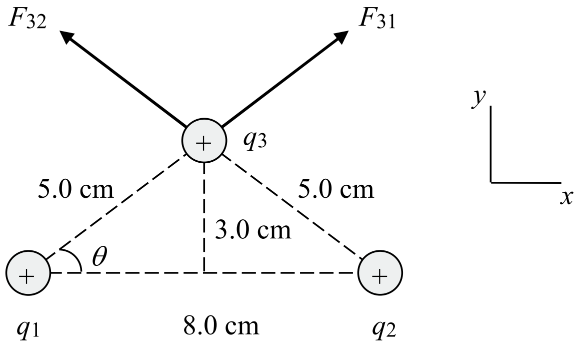


Fig. 4.4 Determining electric force on q_3, Problem 4.4

$$\boldsymbol{F}_{31} = F_{31}\ \cos\theta\ \mathbf{i} + F_{31}\sin\theta\ \mathbf{j}\ = \left[1.8\times10^{-7}\left(\frac{4}{5}\right)\mathbf{i} + 1.8\times10^{-7}\left(\frac{3}{5}\right)\mathbf{j}\right]\mathrm{N}$$
$$= (1.4\times10^{-7}\ \mathbf{i} + 1.1\times10^{-7}\ \mathbf{j})\ \mathrm{N}.$$

The magnitude of electric force on the third charge due to the second F_{32} is equal to F_{31},

$$F_{32} = 1.8\times10^{-7}\ \mathrm{N}.$$

In vector form, this force is

$$\boldsymbol{F}_{32} = -F_{32}\ \cos\theta\ \mathbf{i} + F_{32}\sin\theta\ \mathbf{j}$$
$$= (-1.4\times10^{-7}\ \mathbf{i} + 1.1\times10^{-7}\ \mathbf{j})\ \mathrm{N}.$$

Therefore, the resultant electrostatic force on the third charge is

$$\boldsymbol{F} = \boldsymbol{F}_{31} + \boldsymbol{F}_{32} = 2.2\times10^{-7}\ \mathbf{j}\ \mathrm{N}.$$

The force is in the positive y direction, that is, the $\mathbf{j}$ direction.

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; k:9e9; q1:1e-9; q2:1e-9; q3:5e-11;
(fpprintprec)     5
(k)   9.0*10^9
(q1)  1.0*10^-9
(q2)  1.0*10^-9
(q3)  5.0*10^-11
(%i6) F31: k*q1*q3/5e-2^2;
(F31) 1.8*10^-7
(%i7) F31vec: [F31*(4/5),F31*(3/5)];
(F31vec) [1.44*10^-7,1.08*10^-7]
(%i8) F32: F31;
(F32) 1.8*10^-7
(%i9) F32vec: [-F32*(4/5),F32*(3/5)];
(F32vec) [-1.44*10^-7,1.08*10^-7]
(%i10) Fvec: F31vec + F32vec;
(Fvec) [0.0,2.16*10^-7]
```

Comments on the codes:

(%i5)	Set the floating point print precision to 5 and assign values of k, q_1, q_2, and q_3.
(%i6), (%i7)	Calculate F_{31} and assign vector $\boldsymbol{F}_{31}$.
(%i8), (%i9)	Assign F_{32} and vector $\boldsymbol{F}_{32}$.
(%i10)	Calculate vector $\boldsymbol{F}$.

Problem 4.5 A particle of charge $q_1 = 5.0 \times 10^{-3}$ C is located at the origin. A second particle of charge $q_2 = -3.0 \times 10^{-3}$ C is placed at coordinate (3, 4) m. Calculate the electric force acting on the second particle.

Solution

Figure 4.5 shows the two particles and the electric force acting on the second particle.

In vector form, the electrostatic force acting on the second particle due to the first particle is,

$$\boldsymbol{F}_{21} = k\frac{q_1 q_2}{r_{12}^2}\hat{\boldsymbol{r}}_{12} = k\frac{q_1 q_2}{r_{12}^3}\boldsymbol{r}_{12},$$

where

$$\begin{aligned}
\boldsymbol{r}_{12} &= (3\,\mathbf{i} + 4\,\mathbf{j})\text{ m},\\
r_{12} &= \sqrt{3^2 + 4^2}\text{m} = 5\text{ m},\\
\hat{\boldsymbol{r}}_{12} &= \frac{\boldsymbol{r}_{12}}{r_{12}}.
\end{aligned}$$

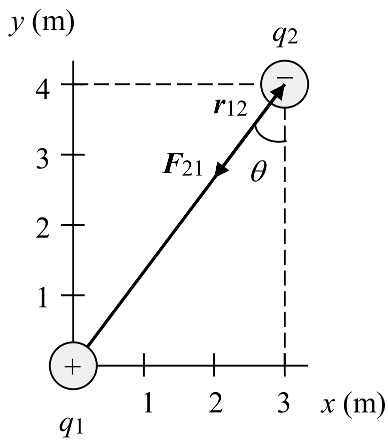


Fig. 4.5 Determining electric force on q_2, Problem 4.5

This gives the electrostatic force acting on the second particle due to the first particle as

$$\boldsymbol{F}_{21} = (9 \times 10^9 \text{ N m}^2 \text{ C}^{-2}) \frac{(5.0 \times 10^{-3} \text{ C})(-3.0 \times 10^{-3} \text{ C})}{(5.0 \text{ m})^3} (3\,\mathbf{i} + 4\,\mathbf{j}) \text{ m}$$
$$= (-3240\,\mathbf{i} - 4320\,\mathbf{j}) \text{ N}.$$

The magnitude of the electrostatic force is

$$F_{21} = \sqrt{(-3240)^2 + (-4320)^2} \text{ N} = 5.4 \times 10^3 \text{ N}.$$

The angle is

$$\theta = \tan^{-1}\left(\frac{3}{4}\right) = 0.64 \text{ rad} = 37^\circ.$$

Alternative solution: This problem can be solved without resorting to vectors, as well. The distance between the charges is 5.0 m and the magnitude of the electrostatic force is, Eq. (4.1),

$$F_{21} = k\frac{q_1 q_2}{r_{12}^2} = (9 \times 10^9 \text{ N m}^2 \text{ C}^{-2}) \frac{(5.0 \times 10^{-3} \text{ C})(3.0 \times 10^{-3} \text{ C})}{(5.0 \text{ m})^2}$$
$$= 5.4 \times 10^3 \text{ N}.$$

The force is attractive because the signs of charges are not the same. The direction of the force is from the second to the first particle, along the line connecting the two particles.

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; q1:5e-3; q2:-3e-3; k:9e9;
(fpprintprec)     5
(q1)  0.005
(q2)  -0.003
(k)   9.0*10^9
(%i6) r12vector:[3,4]; r12:5;
(r12vector) [3,4]
(r12) 5
(%i7) F21vector: k*q1*q2/r12^3*r12vector;
(F21vector) [-3240.0,-4320.0]
(%i8) F21: sqrt(F21vector[1]^2 + F21vector[2]^2);
(F21) 5400.0
(%i9) theta: float(atan(3/4));
(theta) 0.6435
(%i10) theta_deg: float(theta*180/%pi);
(theta_deg) 36.87
(%i11) F21: k*q1*abs(q2)/r12^2;
(F21) 5400.0
```

Comments on the codes:

(%i4)	Set the floating point print precision to 5 and assign values of q_1, q_2, and k.
(%i6)	Assign vector $\boldsymbol{r}_{12}$ and its magnitude r_{12}.
(%i7), (%i8)	Calculate force $\boldsymbol{F}_{21}$ and magnitude F_{21}.
(%i9), (%i10)	Calculate angle θ.
(%i11)	Calculate magnitude F_{21} directly.

Problem 4.6 Four charges are fixed on a plane as in Fig. 4.6a. What is the resultant electric force on the first charge? The charges are $q_1 = 5.0 \times 10^{-3}$ C, $q_2 = -6.0 \times 10^{-3}$ C, $q_3 = -3.0 \times 10^{-3}$ C, and $q_4 = 4.0 \times 10^{-3}$ C.

Solution

The resultant electric force on the first charge due to charges 2, 3, and 4 is

$$\begin{aligned}
\boldsymbol{F} &= \boldsymbol{F}_{12} + \boldsymbol{F}_{13} + \boldsymbol{F}_{14} \\
&= k\left[\frac{q_2 q_1}{r_{21}^3}\boldsymbol{r}_{21} + \frac{q_3 q_1}{r_{31}^3}\boldsymbol{r}_{31} + \frac{q_4 q_1}{r_{41}^3}\boldsymbol{r}_{41}\right] \\
&= (9 \times 10^9 \text{ N m}^2\text{ C}^{-2})\left[\frac{(-6.0 \times 10^{-3}\text{ C})(5.0 \times 10^{-3}\text{ C})}{(3.0\text{ m})^3}(-3\,\mathbf{i}\text{ m})\right. \\
&\quad + \frac{(-3.0 \times 10^{-3}\text{ C})(5.0 \times 10^{-3}\text{ C})}{(5.0\text{ m})^3}(-3\,\mathbf{i} - 4\,\mathbf{j})\text{ m} \\
&\quad \left. + \frac{(4.0 \times 10^{-3}\text{ C})(5.0 \times 10^{-3}\text{ C})}{(4.0\text{ m})^3}(-4\,\mathbf{j}\text{ m})\right] \\
&= (33240\,\mathbf{i} - 6930\,\mathbf{j})\text{ N}.
\end{aligned}$$

The magnitude of the force is

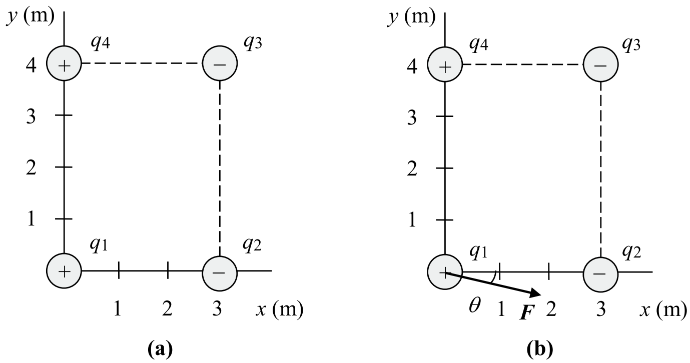


Fig. 4.6 **a** Configuration of four charges; **b** determining electric force on q_1, Problem 4.6

$$F = \sqrt{(33240)^2 + (-6930)^2}\ \text{N} = 33955\ \text{N},$$

and the direction is

$$\theta = \tan^{-1}\left(\frac{-6930}{33240}\right) = -0.21\ \text{rad} = -12^\circ.$$

The resultant electric force F and its direction θ are shown in Fig. 4.6b.

◆ wxMaxima codes:

```
(%i6) fpprintprec:5; k:9e9; q1:5e-3; q2:-6e-3; q3:-3e-3; q4:4e-3;
(fpprintprec)     5
(k)   9.0*10^9
(q1)  0.005
(q2)  -0.006
(q3)  -0.003
(q4)  0.004
(%i12) r21vec:[-3,0]; r21:3; r31vec:[-3,-4]; r31:5; r41vec:[0,-4]; r41:4;
(r21vec) [-3,0]
(r21) 3
(r31vec) [-3,-4]
(r31) 5
(r41vec) [0,-4]
(r41) 4
(%i13) F12vec: k*q2*q1/r21^3*r21vec;
(F12vec) [3.0*10^4,0]
(%i14) F13vec: k*q3*q1/r31^3*r31vec;
(F13vec) [3240.0,4320.0]
(%i15) F14vec: k*q4*q1/r41^3*r41vec;
(F14vec) [0,-1.125*10^4]
(%i16) Fvec: F12vec + F13vec + F14vec;
(Fvec) [3.324*10^4,-6930.0]
(%i17) F: sqrt(Fvec[1]^2 + Fvec[2]^2);
(F)   3.3955*10^4
(%i18) theta: atan(Fvec[2]/Fvec[1]);
(theta) -0.20554
(%i19) theta_deg: float(theta*180/%pi);
(theta_deg) -11.777
```

Comments on the codes:

(%i6)	Set the floating point print precision to 5 and assign values of k, q_1, q_2, q_3, and q_4.
(%i12)	Assign vector $\boldsymbol{r}_{21}$ and its length r_{21}, vector $\boldsymbol{r}_{31}$ and its length r_{31}, and vector $\boldsymbol{r}_{41}$ and its length r_{41}.
(%i13), (%i14), (%i15)	Calculate vectors $\boldsymbol{F}_{12}$, $\boldsymbol{F}_{13}$, and $\boldsymbol{F}_{14}$.
(%i16), (%i17)	Calculate vector $\boldsymbol{F}$ and its magnitude F.
(%i18), (%i19)	Calculate θ and convert the angle to degree.

Alternative solution: This problem can also be solved without using vectors. Figure 4.7 shows the four charges and three electric forces acting on the first charge.

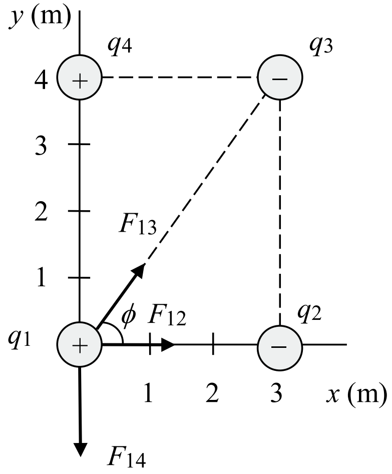


Fig. 4.7 Determining electric force on q_1, Problem 4.6

The magnitudes of all three electric forces on the first charge are

$$F_{12} = k\frac{q_1|q_2|}{r_{12}^2} = (9 \times 10^9 \text{ N m}^2 \text{ C}^{-2})\frac{(5.0 \times 10^{-3} \text{ C})(6.0 \times 10^{-3} \text{ C})}{(3.0 \text{ m})^2} = 30000 \text{ N},$$

$$F_{13} = k\frac{q_1|q_3|}{r_{13}^2} = (9 \times 10^9 \text{ N m}^2 \text{ C}^{-2})\frac{(5.0 \times 10^{-3} \text{ C})(3.0 \times 10^{-3} \text{ C})}{(5.0 \text{ m})^2} = 5400 \text{ N},$$

$$F_{14} = k\frac{q_1 q_4}{r_{14}^2} = (9 \times 10^9 \text{ N m}^2 \text{ C}^{-2})\frac{(5.0 \times 10^{-3} \text{ C})(4.0 \times 10^{-3} \text{ C})}{(4.0 \text{ m})^2} = 11250 \text{ N}.$$

The resultant electric force in the x direction is

$$F_x = F_{12} + F_{13}\cos\phi = [30000 + 5400\ (3/5)] \text{ N} = 33240 \text{ N}.$$

The resultant electric force in the y direction is

$$F_y = -F_{14} + F_{13}\sin\phi = [-11250 + 5400\ (4/5)] \text{ N} = -6930 \text{ N}.$$

Therefore, the magnitude of resultant electric force is

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{33240^2 + 6930^2} \text{ N} = 33955 \text{ N},$$

and the angle between the resultant electric force and the x-axis is

$$\theta = \tan^{-1}\left(\frac{-6930}{33240}\right) = -0.21 \text{ rad} = -12^\circ.$$

The resultant electric force F and its direction θ are shown in Fig. 4.6b.

◆ wxMaxima codes:

```
(%i6) fpprintprec:5; k:9e9; q1:5e-3; q2:-6e-3; q3:-3e-3; q4:4e-3;
(fpprintprec)     5
(k)   9.0*10^9
(q1)  0.005
(q2)  -0.006
(q3)  -0.003
(q4)  0.004
(%i7) F12: k*q1*abs(q2)/3^2;
(F12) 3.0*10^4
(%i8) F13: k*q1*abs(q3)/5^2;
(F13) 5400.0
(%i9) F14: k*q1*q4/4^2;
(F14) 1.125*10^4
(%i10) Fx: F12 + F13*3/5;
(Fx)  3.324*10^4
(%i11) Fy: -F14 + F13*4/5;
(Fy)  -6930.0
(%i12) F: sqrt(Fx^2 + Fy^2);
(F)   3.3955*10^4
(%i13) theta: atan(Fy/Fx);
(theta) -0.20554
(%i14) theta_deg: float(theta*180/%pi);
(theta_deg) -11.777
```

Comments on the codes:

(%i6)	Set the floating point print precision to 5, and assign values of k, q_1, q_2, q_3, and q_4.
(%i7), (%i8), (%i9)	Calculate F_{12}, F_{13}, and F_{14}.
(%i10), (%i11)	Calculate F_x and F_y.
(%i12)	Calculate magnitude F.
(%i13), (%i14)	Calculate θ and convert the angle to degree.

Problem 4.7 Two spheres having the same mass of 0.10 g and the same electric charge are suspended by a 50 cm thread as shown in Fig. 4.8. The angle between the thread and the vertical is 10° due to repulsion between the spheres. Calculate,

(a) charge on the sphere
(b) tension in the thread.

Solution

(a) Let the charge on the sphere be q. Figure 4.9 shows forces acting on one of the spheres. The forces are the weight of the sphere mg, tension in the thread T, and electrostatic repulsive force F.

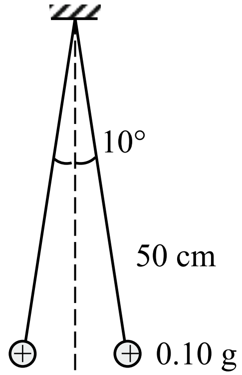


Fig. 4.8 Two separated charged spheres on threads, Problem 4.7

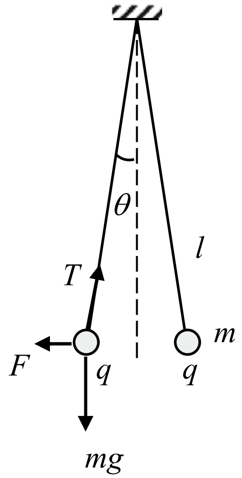


Fig. 4.9 Forces on one of the spheres, Problem 4.7

The vector sum of the three forces is zero, because the sphere is in equilibrium. Thus, the net force in the x and y directions are zero and we write

$$\sum F_x = T \sin\theta - F = 0, \tag{1}$$

$$\sum F_y = T \cos\theta - mg = 0. \tag{2}$$

The two equations give

$$F = mg \tan\theta. \tag{3}$$

ByCoulomb's law, the electrostatic force is,

$$F = \frac{kq^2}{r^2} = \frac{kq^2}{(2l\sin\theta)^2} \tag{4}$$

since the distance between the spheres is $2l \sin\theta$, where l is the length of the thread.

The charge q can be calculated from Eqs. (3) and (4),

$$\frac{kq^2}{(2l\sin\theta)^2} = mg\tan\theta,$$
$$q^2 = \frac{4l^2 mg\sin^2\theta\tan\theta}{k}$$
$$= \frac{4(0.50\text{ m})^2(0.10\times10^{-3}\text{ kg})(9.8\text{ m/s}^2)\sin^2 10^\circ\tan 10^\circ}{9\times10^9\text{ N m}^2\text{ C}^{-2}},$$
$$q = 2.4\times10^{-8}\text{ C}.$$

(b) Tension in the thread is calculated from Eq. (4.2) as follows:

$$T\cos\theta - mg = 0,$$
$$T = \frac{mg}{\cos\theta} = \frac{(0.10\times10^{-3}\text{ kg})(9.8\text{ m/s}^2)}{\cos 10^\circ}$$
$$= 1.0\times10^{-3}\text{ N}.$$

◆ wxMaxima codes:

```
(%i7) fpprintprec:5; ratprint:false; k:9e9; g:9.8; m:0.1e-3; l:0.5;
theta:float(10/180*%pi);
(fpprintprec)     5
(ratprint)  false
(k)   9.0*10^9
(g)   9.8
(m)   1.0*10^-4
(l)   0.5
(theta) 0.17453
(%i9) solve(k*q^2/(2*l*sin(theta))^2 = m*g*tan(theta), q)$ float(%);
(%o9) [q=-2.4061*10^-8,q=2.4061*10^-8]
(%i10) T: m*g/cos(theta);
(T) 9.9512*10^-4
```

Comments on the codes:

(%i7) Set the floating point print precision to 5 and internal rational number print to false, and assign values of k, g, m, l, and θ.

(%i9) Solve $kq^2/(2l\sin\theta)^2 = mg\tan\theta$ for q. Part (a).

(%i10) Calculate tension T. Part (b).

◆ Alternative calculation:

```
(%i7) fpprintprec:5; ratprint:false; k:9e9; g:9.8; m:0.1e-3; l:0.5;
theta:float(10/180*%pi);
(fpprintprec)     5
(ratprint)  false
(k)   9.0*10^9
(g)   9.8
(m)   1.0*10^-4
(l)   0.5
(theta) 0.17453
(%i9) solve([T*sin(theta)-F=0, T*cos(theta)-m*g=0,
F=k*q^2/(2*l*sin(theta))^2], [q,T,F])$ float(%);
(%o9) [[q=2.4061*10^-8,T=9.9512*10^-4,F=1.728*10^-4],[q=-2.4061*10^-8,T=
9.9512*10^-4,F=1.728*10^-4]]
```

Comments on the codes:

(%i7) Set the floating point print precision to 5 and internal rational number print to false, and assign values of k, g, m, l, and θ.
(%i9) Solve Eqs. (1), (2), and (4) for q, T, and F.
(%o9) The solutions.

Problem 4.8 A particle of charge q and a thin rod of length L that has charge Q are arranged as in Fig. 4.10. Calculate the electric force on the particle.

Solution

The charged rod and the particle are redrawn in Fig. 4.11. We want to calculate electric force on the particle due to the whole length of the charged rod. To do this, we consider an infinitesimal element of the rod, calculate the force due to the element, and do the integration for the whole rod.

For the infinitesimal length of the rod dx, the infinitesimal charge is

$$dQ = \frac{dx}{L} Q.$$

By Coulomb's law, electric force on the particle due to this infinitesimal charge is

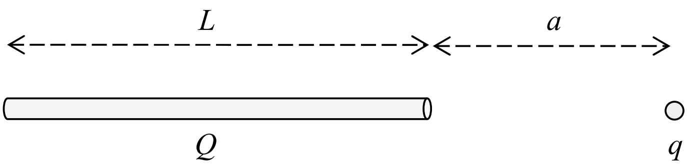


Fig. 4.10 A charged particle and a charged rod, Problem 4.8

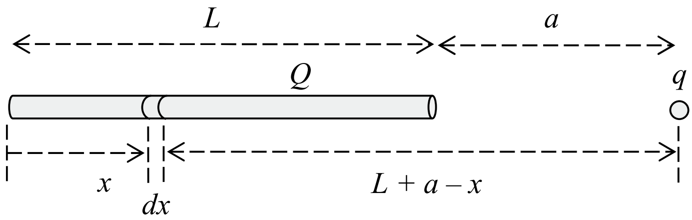


Fig. 4.11 Determining electrical force on a charged particle due to a charged rod, Problem 4.8

$$dF = \frac{kq\,dQ}{(L+a-x)^2} = \frac{kQq\,dx}{L(L+a-x)^2}.$$

This force is toward the right. The force is toward the right as charges of the same sign repel. Due to the same signs, the infinitesimal charge pushes the particle toward the right.

The force on the particle due to the whole length of the rod is

$$F = \int dF = \frac{kQq}{L}\int_0^L \frac{dx}{(L+a-x)^2}.$$

The integral can be calculated by substitution,

$$u = \frac{1}{L+a-x} = (L+a-x)^{-1},$$
$$du = (-1)(L+a-x)^{-2}(-1)dx = \frac{dx}{(L+a-x)^2}.$$

Therefore, the electric force on the particle is

$$\begin{aligned} F &= \frac{kQq}{L}\int_{1/(L+a)}^{1/a} du = \frac{kQq}{L}[u]_{1/(L+a)}^{1/a} \\ &= \frac{kQq}{L}\left[\frac{1}{a} - \frac{1}{L+a}\right] \\ &= \frac{kQq}{a(L+a)}. \end{aligned}$$

The direction of the force is toward the right.

♦ wxMaxima codes:

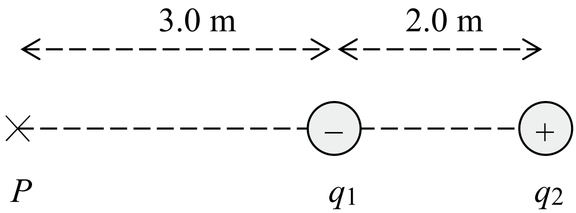


Fig. 4.12 Two charged particles, Problem 4.9

```
(%i1) F: integrate(k*Q*q/L, u, 1/(L+a), 1/a);
(F)   (Q*(1/a-1/(a+L))*k*q)/L
(%i2) ratsimp(%);
(%o2) (Q*k*q)/(a^2+L*a)
```

Comments on the codes:

(%i1) Calculate the integration $F = \int_{1/(L+a)}^{1/a} \frac{kQq}{L} du$.

(%i2) Simplify the result.

Problem 4.9 Two particles of charges $q_1 = -2.0 \times 10^{-6}$ C and $q_2 = 5.0 \times 10^{-6}$ C are arranged as in Fig. 4.12.

(a) Calculate the electric field at point *P*.
(b) A third particle of charge $q_3 = 1.0 \times 10^{-6}$ C is placed at *P*. What is the electric force acting on the particle?

Solution

(a) Figure 4.13 shows the two charges, point *P*, and electric fields at *P*.

The magnitude of the electric field due to q_1 at point *P* is, Eq. (4.3),

$$E_1 = k\frac{|q_1|}{r^2} = \left(9 \times 10^9 \frac{\text{N m}^2}{\text{C}^2}\right) \frac{(2.0 \times 10^{-6}\text{ C})}{(3.0\text{ m})^2} = 2000\text{ N C}^{-1}.$$

The direction is toward the right because q_1 is negative. This electric field is written as

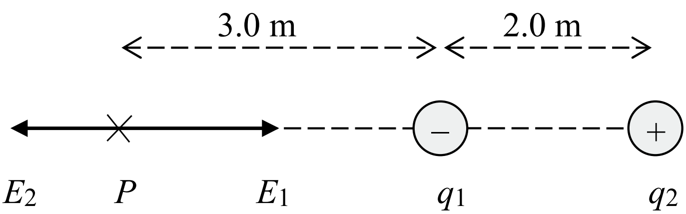


Fig. 4.13 Determining electric field at point *P*, Problem 4.9

$$\boldsymbol{E}_1 = 2000\,\mathbf{i}\,\mathrm{N\,C^{-1}}.$$

The magnitude of the electric field due to q_2 at point P is, Eq. (4.3),

$$E_2 = k\frac{q_2}{r^2} = \left(9 \times 10^9 \frac{\mathrm{N\,m^2}}{\mathrm{C^2}}\right)\frac{(5.0 \times 10^{-6}\ \mathrm{C})}{(5.0\ \mathrm{m})^2} = 1800\ \mathrm{N\ C^{-1}}.$$

The direction is toward the left because q_2 is positive. This electric field is written as,

$$\boldsymbol{E}_2 = -1800\,\mathbf{i}\,\mathrm{N\,C^{-1}}.$$

Thus, the electric field at P due to both charges is,

$$\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2 = (2000 - 1800)\ \mathbf{i}\ \mathrm{N\ C^{-1}} = 200\ \mathbf{i}\ \mathrm{N\ C^{-1}}.$$

The direction of the field is toward the right.

(b) Force on the third charged particle placed at point P is the charge multiplied by the electric field there,

$$\boldsymbol{F} = q_3\boldsymbol{E} = (1.0 \times 10^{-6}\ \mathrm{C})(200\ \mathbf{i}\ \mathrm{N\ C^{-1}}) = 2.0 \times 10^{-4}\ \mathbf{i}\ \mathrm{N}.$$

The force is toward the right.

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; k:9e9; q1:-2e-6; q2:5e-6;
(fpprintprec)     5
(k)   9.0*10^9
(q1)  -2.0*10^-6
(q2)  5.0*10^-6
(%i5) E1: k*abs(q1)/3^2;
(E1)  2000.0
(%i6) E2: -k*q2/5^2;
(E2)  -1800.0
(%i7) E: E1+E2;
(E)   200.0
(%i8) q3:1e-6;
(q3)  1.0*10^-6
(%i9) F: q3*E;
(F)   2.0*10^-4
```

Comments on the codes:

(%i4)	Set the floating point print precision to 5 and assign values of k, q_1 and q_2.
(%i5), (%i6), (%i7)	Calculate E_1, E_2, and E. Part (a).

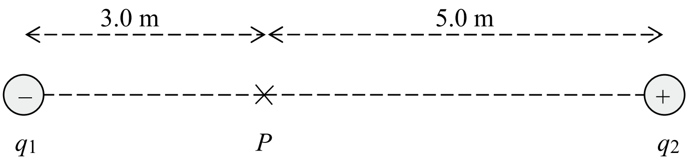


Fig. 4.14 Two charged particles' configuration, Problem 4.10

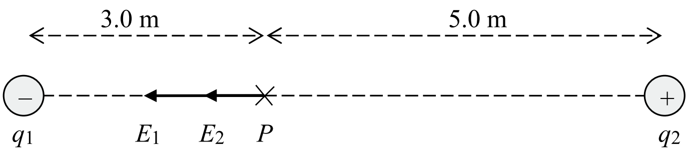


Fig. 4.15 Determining electric field at point P, Problem 4.10

(%i8) Assign q_3.
(%i9) Calculate F. Part (b).

Problem 4.10 Two particles of charges $q_1 = -2.0 \times 10^{-6}$ C and $q_2 = 5.0 \times 10^{-6}$ C are arranged as in Fig. 4.14. Calculate the electric field at point P.

Solution

Figure 4.15 shows the two charged particles, point P, and electric fields due to the charges at the point.

The magnitude of electric field due to charge q_1 at point P is, Eq. (4.3),

$$E_1 = k\frac{|q_1|}{r^2} = \left(9 \times 10^9 \frac{\text{N m}^2}{\text{C}^2}\right)\frac{(2.0 \times 10^{-6}\text{ C})}{(3.0\text{ m})^2} = 2000\text{ N C}^{-1}.$$

The direction of the field is toward the left because q_1 is negative. The electric field is written as

$$\boldsymbol{E}_1 = -2000\,\mathbf{i}\,\text{N C}^{-1}.$$

The magnitude of electric field due to charge q_2 at point P is, Eq. (4.3),

$$E_2 = k\frac{q_2}{r^2} = \left(9 \times 10^9 \frac{\text{N m}^2}{\text{C}^2}\right)\frac{(5.0 \times 10^{-6}\text{ C})}{(5.0\text{ m})^2} = 1800\text{ N C}^{-1}.$$

The direction of the field is toward the left because q_2 is positive. The electric field is written as

$$\boldsymbol{E}_2 = -1800\,\mathbf{i}\,\text{N C}^{-1}.$$

Therefore, the electric field due to both charges at point P is

$$\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2 = (-2000 - 1800)\ \mathbf{i}\ \mathrm{N\ C^{-1}} = -3800\ \mathbf{i}\ \mathrm{N\ C^{-1}}$$

The direction of the field is toward the left.

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; k:9e9; q1:-2e-6; q2:5e-6;
(fpprintprec)     5
(k)   9.0*10^9
(q1)  -2.0*10^-6
(q2)  5.0*10^-6
(%i5) E1: -k*abs(q1)/3^2;
(E1)  -2000.0
(%i6) E2: -k*q2/5^2;
(E2)  -1800.0
(%i7) E: E1+E2;
(E)   -3800.0
```

Comments on the codes:

(%i4)	Set the floating point print precision to 5 and assign values of k, q_1, and q_2.
(%i5), (%i6), (%i7)	Calculate E_1, E_2, and E.

Problem 4.11 Charges of 5.0 and −8.0 μC are placed on the x-axis at $x = 0$ and $x = 1.0$ m, respectively. Where should a third charge be placed so that the electrical force on it is zero?

Solution

Figure 4.16 shows the two charges on the x-axis. Any charged object will not be acted by any electrical force if it is placed in zero electric field region, because $F = qE$. Therefore, we need to find a point of zero electric field on the x-axis.

Let P be the point where the electric field is zero and l the distance from P to the first charge. We assume l to be a positive number. At P, the electric fields due to 5.0 and −8.0 μC charges must be of the same magnitude but opposite in sign. So,

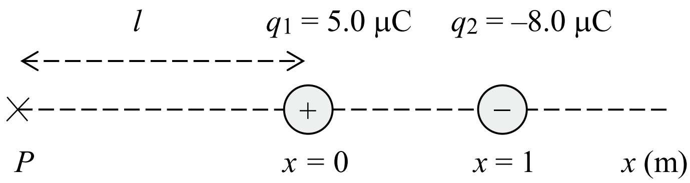


Fig. 4.16 Determining point of zero electric field, Problem 4.11

$$E_1 = |E_2|,$$
$$k\frac{q_1}{l^2} = k\frac{|q_2|}{(l+1)^2}.$$

This gives

$$\frac{5}{l^2} = \frac{8}{(l+1)^2},$$
$$l = 3.8 \text{ m or } -0.44 \text{ m}.$$

From Fig. 4.16, the solution $l = 3.8$ m corresponds to the position of the arbitrary third charge at $x = -3.8$ m. At this point, the electric field is zero. Any charged particle placed at the point will not be acted by any electrical force. The −0.44 m value is not accepted because we require l to be a positive number.

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; ratprint:false; k:9e9; q1:5e-6; q2:-8e-6;
(fpprintprec)     5
(ratprint)  false
(k)   9.0*10^9
(q1)  5.0*10^-6
(q2)  -8.0*10^-6
(%i7) solve(k*q1/l^2 = k*abs(q2)/(l+1)^2, l)$ float(%);
(%o7) [l=-0.44152,l=3.7749]
```

Comments on the codes:

(%i5) Set the floating point print precision to 5 and internal rational number print to false, and assign values of k, q_1, and q_2.
(%i7) Solve $kq_1/l^2 = k|q_2|/(l+1)^2$ for l.

Problem 4.12 Figure 4.17 shows three charges $q_1 = 4.0\ \mu$C, $q_2 = -6.0\ \mu$C, and $q_3 = 8.0\ \mu$C, on a plane. Calculate the electric field at the origin.

Solution

Electric field due to the distribution of point charges can be calculated by the formula, Eq. (4.4),

$$\boldsymbol{E} = k\sum_i \frac{q_i}{r_i^3}\boldsymbol{r}_i = k\sum_i \frac{q_i}{r_i^2}\hat{\boldsymbol{r}}_i,$$

where q_i is the i th charge, $\boldsymbol{r}_i$ is the displacement vector from the charge to the observation point, and $\hat{\boldsymbol{r}}_i = \frac{\boldsymbol{r}_i}{r_i}$ is the unit vector. The electric field at the origin due to the three charges is

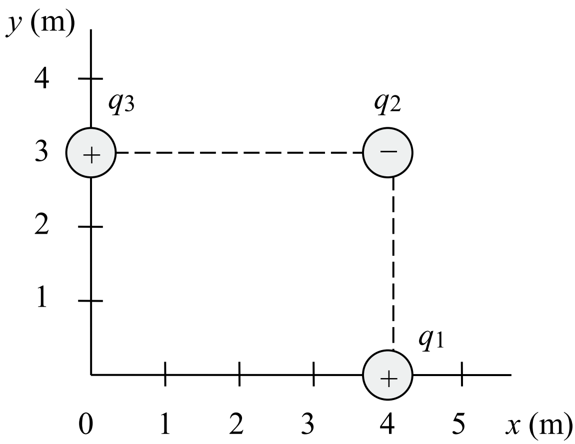


Fig. 4.17 Configuration of three charges, Problem 4.12

$$
\begin{aligned}
\boldsymbol{E} &= k\left[\frac{q_1}{r_1^3}\boldsymbol{r}_1 + \frac{q_2}{r_2^3}\boldsymbol{r}_2 + \frac{q_3}{r_3^3}\boldsymbol{r}_3\right] \\
&= \left(9 \times 10^9\ \frac{\mathrm{N\ m^2}}{\mathrm{C^2}}\right)\left[\ \frac{(4.0 \times 10^{-6}\ \mathrm{C})}{(4.0\ \mathrm{m})^3}(-4\ \mathbf{i}\ \mathrm{m})\right. \\
&\quad + \frac{(-6.0 \times 10^{-6}\ \mathrm{C})}{(5.0\ \mathrm{m})^3}(-4\ \mathbf{i}\ - 3\ \mathbf{j})\ \mathrm{m} \\
&\quad \left. + \frac{(8.0 \times 10^{-6}\ \mathrm{C})}{(3.0\ \mathrm{m})^3}(-3\ \mathbf{j}\ \mathrm{m})\ \right] \\
&= (-522\ \mathbf{i}\ - 6704\ \mathbf{j})\ \mathrm{N\ C^{-1}}.
\end{aligned}
$$

The magnitude of the electric field is

$$E = \sqrt{(-522)^2 + (-6704)^2}\ \mathrm{N\ C^{-1}} = 6724\ \mathrm{N\ C^{-1}},$$

and the angle of the electric field with the x-axis is

$$\theta = \tan^{-1}\left(\frac{-6704}{-522}\right) = 266°.$$

The electric field $\boldsymbol{E}$ and the angle θ are shown in Fig. 4.18.

◆ wxMaxima codes:

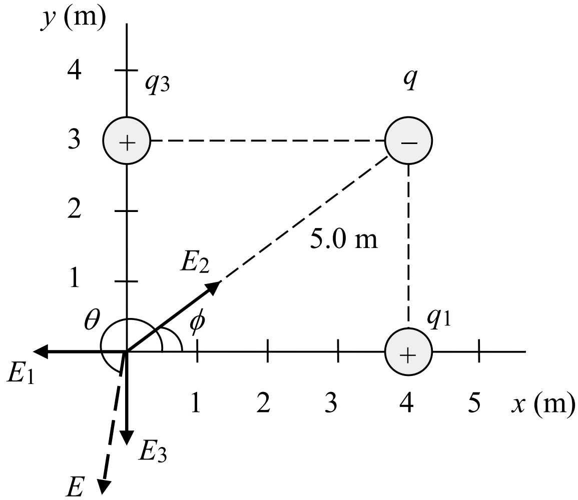


Fig. 4.18 Determining electric field at the origin due to three charges, Problem 4.12

```
(%i5) fpprintprec:5; k:9e9; q1:4e-6; q2:-6e-6; q3:8e-6;
(fpprintprec)     5
(k)    9.0*10^9
(q1)   4.0*10^-6
(q2)   -6.0*10^-6
(q3)   8.0*10^-6
(%i8) r1vec:[-4,0]; r2vec:[-4,-3]; r3vec:[0,-3];
(r1vec) [-4,0]
(r2vec) [-4,-3]
(r3vec) [0,-3]
(%i9) Evec: k*(q1/4^3*r1vec + q2/5^3*r2vec + q3/3^3*r3vec);
(Evec) [-522.0,-6704.0]
(%i10) E: sqrt(Evec[1]^2 + Evec[2]^2);
(E)    6724.3
(%i11) theta: atan(Evec[2]/Evec[1]);
(theta) 1.4931
(%i12) theta_deg: float(theta*180/%pi);
(theta_deg) 85.548
(%i13) 180+theta_deg;
(%o13) 265.55
```

Comments on the codes:

(%i5)	Set the floating point print precision to 5 and assign values of k, q_1, q_2, and q_3.
(%i8)	Assign vectors $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, and $\boldsymbol{r}_3$.
(%i9)	Calculate electric field vector $\boldsymbol{E}$.
(%i10)	Calculate the magnitude of the electric field E.
(%i11), (%i12), (%i13)	Calculate angle θ between $\boldsymbol{E}$ and the x-axis.

Alternative solution: Another way to tackle the problem is shown in Fig. 4.18. First, we calculate the magnitudes of electric fields due to the three charges at the origin. Then, we add the x and y components of the fields.

In the figure, E_1, E_2, and E_3 are electric fields at the origin due to charges q_1, q_2, and q_3, respectively. The magnitudes of the electric fields are

$$E_1 = k\frac{q_1}{r_1^2} = \left(9 \times 10^9 \ \frac{\text{N m}^2}{\text{C}^2}\right)\frac{(4.0 \times 10^{-6} \text{ C})}{(4.0 \text{ m})^2} = 2250 \text{ N C}^{-1},$$

$$E_2 = k\frac{|q_2|}{r_2^2} = \left(9 \times 10^9 \ \frac{\text{N m}^2}{\text{C}^2}\right)\frac{(6.0 \times 10^{-6} \text{ C})}{(5.0 \text{ m})^2} = 2160 \text{ N C}^{-1},$$

$$E_3 = k\frac{q_3}{r_3^2} = \left(9 \times 10^9 \ \frac{\text{N m}^2}{\text{C}^2}\right)\frac{(8.0 \times 10^{-6} \text{ C})}{(3.0 \text{ m})^2} = 8000 \text{ N C}^{-1}.$$

The resultant electric field in the x direction at the origin is

$$E_x = -E_1 + E_2 \ \cos\phi = [-2250 + 2160 \ (4/5)] \text{ N C}^{-1} = -522 \text{ N C}^{-1}.$$

The resultant electric field in the y direction at the origin is

$$E_y = E_2 \sin\phi - E_3 = [2160 \ (3/5) - 8000] \text{ N C}^{-1} = -6704 \text{ N C}^{-1}.$$

Thus, the electric field at the origin is

$$\boldsymbol{E} = (-522 \, \mathbf{i} \ - 6704 \, \mathbf{j}) \text{ N C}^{-1}.$$

The electric field $\boldsymbol{E}$ is shown in Fig. 4.18.

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; k:9e9; q1:4e-6; q2:-6e-6; q3:8e-6;
(fpprintprec)     5
(k)   9.0*10^9
(q1)  4.0*10^-6
(q2)  -6.0*10^-6
(q3)  8.0*10^-6
(%i8) E1:k*q1/4^2; E2:k*abs(q2)/5^2; E3:k*q3/3^2;
(E1)  2250.0
(E2)  2160.0
(E3)  8000.0
(%i10) Ex:-E1+E2*(4/5); Ey:E2*(3/5)-E3;
(Ex)  -522.0
(Ey)  -6704.0
```

Comments on the codes:

(%i5) Set the floating point print precision to 5 and assign values of k, q_1, q_2, and q_3.

(%i8) Calculate magnitudes of electric fields E_1, E_2, and E_3.

(%i10) Calculate E_x and E_y, the components of $\boldsymbol{E}$.

Problem 4.13 Figure 4.19 shows two charges, each of $+q$, separated by a distance of $2a$.

(a) Determine the electric field at point P a distance x away.
(b) What is the field when $x \gg a$?

Solution

(a) Figure 4.20 shows the two charges, point P, electric fields due to both charges at P, i.e. E_1 and E_2, and related distances and angle.

The magnitude of the electric field at P due to the first (top) charge is,

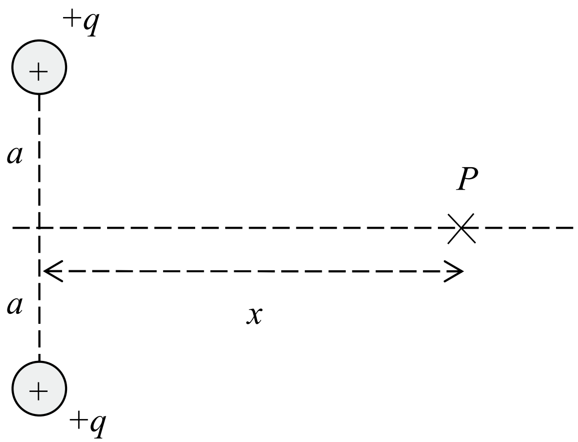


Fig. 4.19 Configuration of two charges, Problem 4.13

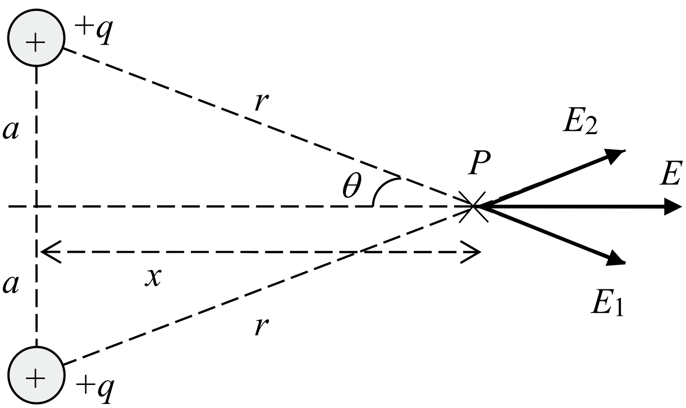


Fig. 4.20 Determining electric field at point P, Problem 4.13

$$E_1 = k\frac{q}{r^2} = k\frac{q}{a^2+x^2}.$$

This electric field expressed as a vector is

$$\begin{aligned}\boldsymbol{E}_1 &= E_1 \cos\theta\ \mathbf{i} - E_1 \sin\theta\ \mathbf{j} \\ &= k\frac{q}{a^2+x^2}\cdot\frac{x}{\sqrt{a^2+x^2}}\ \mathbf{i} - k\frac{q}{a^2+x^2}\cdot\frac{a}{\sqrt{a^2+x^2}}\ \mathbf{j}.\end{aligned}$$

The magnitude of the electric field at P due to the second (bottom) charge is

$$E_2 = k\frac{q}{r^2} = k\frac{q}{a^2+x^2}.$$

This electric field expressed as a vector is

$$\begin{aligned}\boldsymbol{E}_2 &= E_2 \cos\theta\ \mathbf{i} + E_2 \sin\theta\ \mathbf{j} \\ &= k\frac{q}{a^2+x^2}\cdot\frac{x}{\sqrt{a^2+x^2}}\ \mathbf{i} + k\frac{q}{a^2+x^2}\cdot\frac{a}{\sqrt{a^2+x^2}}\ \mathbf{j}.\end{aligned}$$

The electric field at point P is the vector sum of the two electric fields,

$$\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2 = \frac{2kqx}{(a^2+x^2)^{3/2}}\ \mathbf{i}.$$

The magnitude of the electric field is $\frac{2kqx}{(a^2+x^2)^{3/2}}$ in the positive x direction.

◆ wxMaxima codes:

```
(%i1) E1vec: [k*q*x/(a^2+x^2)/sqrt(a^2+x^2),
              -k*q*x/(a^2+x^2)/sqrt(a^2+x^2)];
(E1vec) [(k*q*x)/(x^2+a^2)^(3/2),-(k*q*x)/(x^2+a^2)^(3/2)]
(%i2) E2vec: [k*q*x/(a^2+x^2)/sqrt(a^2+x^2),
              k*q*x/(a^2+x^2)/sqrt(a^2+x^2)];
(E2vec) [(k*q*x)/(x^2+a^2)^(3/2),(k*q*x)/(x^2+a^2)^(3/2)]
(%i3) Evec: E1vec + E2vec;
(Evec) [(2*k*q*x)/(x^2+a^2)^(3/2),0]
```

Comments on the codes:

(%i1) Assign electric field vector $\boldsymbol{E}_1$.
(%i2) Assign electric field vector $\boldsymbol{E}_2$.
(%i3) Calculate electric field vector $\boldsymbol{E}$.

(b) When $x \gg a$, $\frac{x}{(a^2+x^2)^{3/2}} \approx \frac{1}{x^2}$, so the electric field is,

$$\boldsymbol{E} = \frac{2kq}{x^2}\ \mathbf{i}.$$

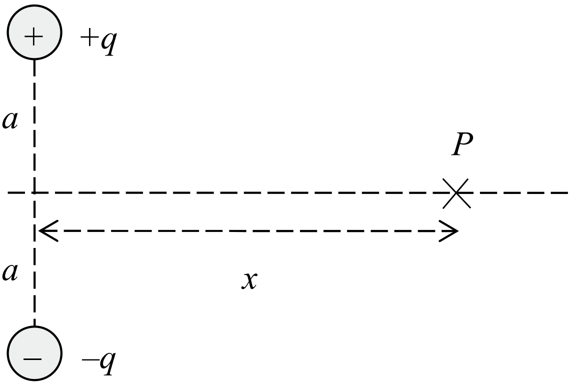


Fig. 4.21 An electric dipole, Problem 4.14

Problem 4.14 Figure 4.21 is an electric dipole, that is, two electric charges of the same magnitude but opposite in signs, separated by a distance of $2a$.

(a) Determine the electric field at point P a distance x away from the center of the electric dipole.
(b) What is the electric field if $x \gg a$?

Solution

(a) Figure 4.22 shows the electric dipole and electric fields at point P.

The magnitude of electric field at point P due to top charge is

$$E_1 = k\frac{q}{r^2} = k\frac{q}{a^2 + x^2}.$$

This electric field is written in vector form as

$$\boldsymbol{E}_1 = E_1 \cos\theta \ \mathbf{i} - E_1 \sin\theta \ \mathbf{j}.$$

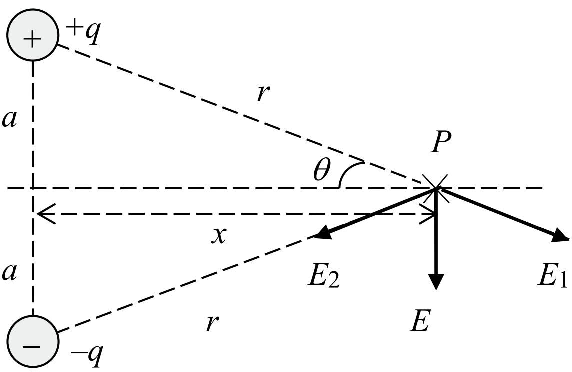


Fig. 4.22 Determining electric field at point P of an electric dipole, Problem 4.14

The magnitude of electric field at point P due to bottom charge is the same,

$$E_2 = k\frac{q}{r^2} = k\frac{q}{a^2 + x^2} = E_1.$$

The electric field is written in vector form as

$$\boldsymbol{E}_2 = -E_1 \cos\theta \, \mathbf{i} - E_1 \sin\theta \, \mathbf{j}.$$

The electric field at point P is the vector sum of $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$,

$$\begin{aligned}\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2 = -2E_1 \sin\theta \, \mathbf{j} = -2 \times \frac{kq}{a^2 + x^2} \cdot \frac{a}{(a^2 + x^2)^{1/2}} \, \mathbf{j} \\ = -\frac{2kqa}{(a^2 + x^2)^{3/2}} \, \mathbf{j}.\end{aligned}$$

This is the electric field due to an electric dipole at a distance x away. The direction of the electric field is to the negative y direction, that is, $-\mathbf{j}$ direction.

◆ wxMaxima codes:

```
(%i1) E1: k*q/(a^2+x^2);
(E1)   (k*q)/(x^2+a^2)
(%i2) E1vec: [E1*x/sqrt(a^2+x^2), -E1*a/sqrt(a^2+x^2)];
(E1vec) [(k*q*x)/(x^2+a^2)^(3/2),-(a*k*q)/(x^2+a^2)^(3/2)]
(%i3) E2: E1;
(E2)   (k*q)/(x^2+a^2)
(%i4) E2vec: [-E2*x/sqrt(a^2+x^2), -E2*a/sqrt(a^2+x^2)];
(E2vec) [-(k*q*x)/(x^2+a^2)^(3/2),-(a*k*q)/(x^2+a^2)^(3/2)]
(%i5) Evec: E1vec + E2vec;
(Evec) [0,-(2*a*k*q)/(x^2+a^2)^(3/2)]
```

Comments on the codes:

(%i1), (%i2) Assign E_1 and vector $\boldsymbol{E}_1$.
(%i3), (%i4) Assign E_2 and vector $\boldsymbol{E}_2$.
(%i5) Calculate vector $\boldsymbol{E}$.

(b) If $x \gg a$, then $\frac{a}{(a^2+x^2)^{3/2}} \approx \frac{a}{x^3}$. The electric field at a point far away from the electric dipole is,

$$\boldsymbol{E} = -\frac{2kqa}{x^3} \, \mathbf{j} = -\frac{qa}{2\pi\varepsilon_0 x^3} \, \mathbf{j}.$$

This means that at a point far away from the electric dipole, the magnitude of the electric field is inversely proportional to the cube of the distance.

Problem 4.15

(a) Figure 4.23 shows a wire of length $2L$ with linear charge density λ. Calculate the electric field at point P, a distance y away from the wire.
(b) What is the electric field if the wire is very long?

Solution

Figure 4.24 shows the wire and other quantities needed to solve the problem. Consider an element of the wire of length dx. Electric field at P due to this element is calculated. The electric field due to the whole wire is then calculated by the integration of all elements.

The electric charge of wire element dx is,

$$dq = \lambda \, dx,$$

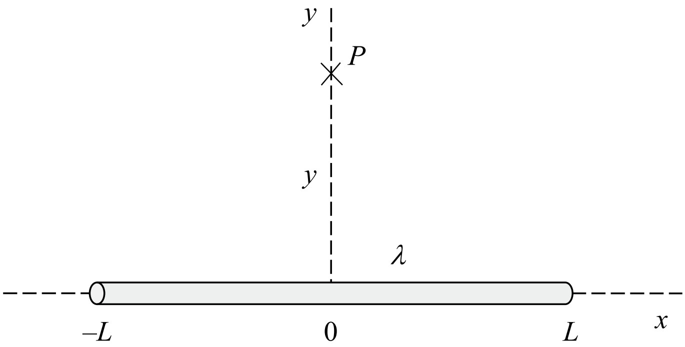


Fig. 4.23 Wire of length $2L$ and charge density λ, Problem 4.15

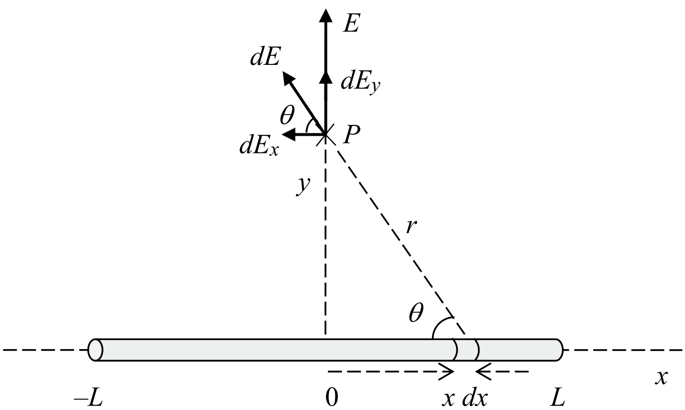


Fig. 4.24 Determining electric field at point P, Problem 4.15

where λ is the linear charge density. The charge produces the electric field dE at point P. By Coulomb's law, the magnitude of the field is

$$dE = \frac{k\,dq}{r^2} = \frac{k\lambda\,dx}{x^2+y^2}.$$

The electric field dE is resolved into the x component of dE_x and the y component of dE_y. Considering the whole wire, the x component of the field vanishes by symmetry.

The electric field at P is the integral of the y component of the field for the whole wire,

$$\begin{aligned} E &= \int dE_y = \int dE\sin\theta = \int_{-L}^{L} \frac{k\lambda\,dx}{x^2+y^2}\cdot\frac{y}{\sqrt{x^2+y^2}} \\ &= \frac{k\lambda}{y}\int_{-L}^{L}\frac{y^2\,dx}{(x^2+y^2)^{3/2}} = \frac{k\lambda}{y}\left[\frac{x}{(x^2+y^2)^{1/2}}\right]_{-L}^{L} \\ &= \frac{2k\lambda L}{y(L^2+y^2)^{1/2}}. \end{aligned}$$

The electric field is in the positive y direction. This is entry (c) of Table 4.1.

(b) If the wire is very long, $L \to \infty$, $\frac{L}{(L^2+y^2)^{1/2}} \to 1$, and the electric field is,

$$E = \frac{2k\lambda}{y} = \frac{\lambda}{2\pi\varepsilon_0 y}.$$

This is entry (b) of Table 4.1. This result can be obtained by applying Gauss's law as well, as shown in Problem 5.6.

◆ wxMaxima codes:

```
(%i2) assume(L>0); integrate(y^2/(x^2+y^2)^(3/2), x, -L, L);
(%o1) [L>0]
(%o2) (2*L*y^2*sqrt(y^2+L^2))/(y^4+L^2*y^2)
(%i3) ratsimp(%);
(%o3) (2*L)/sqrt(y^2+L^2)
(%i4) E: k*lambda/y*%;
(E)   (2*L*k*lambda)/(y*sqrt(y^2+L^2))
(%i5) limit( E, L, inf);
(%o5) (2*k*lambda)/y
```

Comments on the codes:

(%i2) Calculate definite integral $\int_{-L}^{L} \frac{y^2\,dx}{(x^2+y^2)^{3/2}}$.

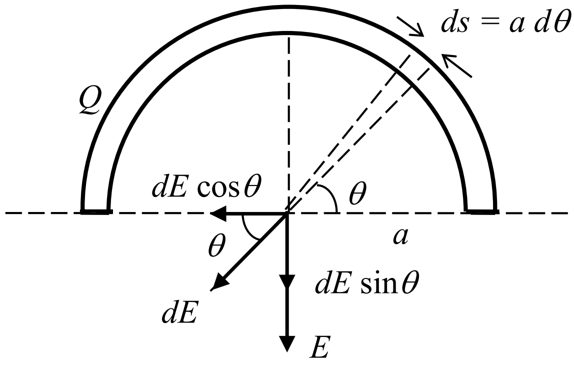


Fig. 4.25 Determining electric field at center of curvature of a semicircular charged wire, Problem 4.16

(%i3) Simplify the result.
(%i4) Calculate E.
(%i5) Calculate the limit of E as L goes to infinity.

Problem 4.16 Electric charge Q distributes uniformly on a semicircular wire. The radius of the semicircle is a. Determine the electric field at the center of curvature of the wire.

Solution

Figure 4.25 shows the semicircular wire and quantities to solve the problem. A wire element of length ds is considered, and the electric field at the center of curvature due to the element is calculated. The electric field is obtained by integration of the whole length of the wire.

Linear charge density (charge per unit length) of the wire is

$$\lambda = \frac{Q}{\pi a}.$$

The length element $ds = a\,d\theta$ is in the first quadrant. The charge of the length element is

$$dq = \lambda\, ds = \lambda a\, d\theta.$$

Applying Coulomb's law, the electric field dE due to the length element is

$$dE = \frac{k\,dq}{a^2} = \frac{k}{a^2}\,\lambda a\,d\theta = \frac{k\lambda}{a}d\theta.$$

The electric field dE is resolved into $dE \cos\theta$ and $dE \sin\theta$. By symmetry, the $dE \cos\theta$ vanishes when quadrants one and two are considered. Only the $dE \sin\theta$ component contributes to the field. By symmetry, the electric field at the center of curvature is

$$E = 2\int_0^{\pi/2} dE\sin\theta = 2\int_0^{\pi/2} \frac{k\lambda}{a}\sin\theta\, d\theta = \frac{2k\lambda}{a}[-\cos\theta]_0^{\pi/2} = \frac{2k\lambda}{a}$$
$$= \frac{2kQ}{a^2\pi}$$
$$= \frac{Q}{2\varepsilon_0\pi^2 a^2}.$$

The field is in the negative y direction.

◆ wxMaxima codes:

```
(%i1) E: integrate(2*k*lambda/a*sin(theta), theta, 0, %pi/2);
(E)    (2*k*lambda)/a
```

Comment on the codes:

(%i1) Calculate the integral $E = \int_0^{\pi/2} 2\frac{k\lambda}{a}\sin\theta\, d\theta$.

Problem 4.17 A wire is bent into an arc of a circle of radius a, Fig. 4.26. Charge on the wire is Q. Determine the electric field at the center of curvature of the wire.

Solution

Figure 4.27 shows the charged wire, the electric fields, and other quantities to solve the problem. A length element of the wire ds is considered and the electric field due to the element is calculated. The effective electric field is calculated by integrating the field due to the element for the whole wire.

Length of the wire is

$$\frac{\pi/2}{2\pi}\cdot 2\pi a = \frac{\pi a}{2}.$$

Linear charge density (charge per unit length) is

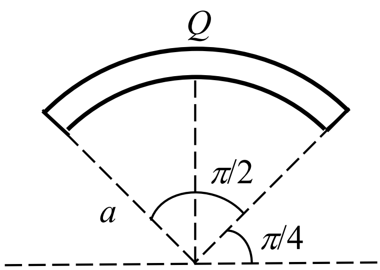


Fig. 4.26 A charged arc, Problem 4.17

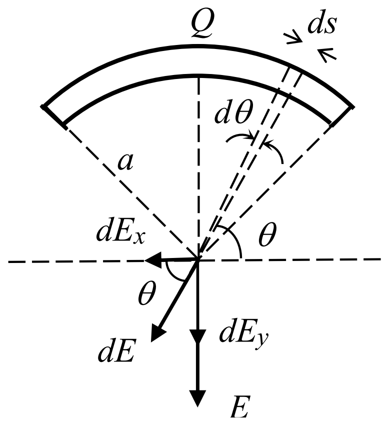


Fig. 4.27 Determining electric field at center of curvature of a charged arc, Problem 4.17

$$\lambda = \frac{Q}{\pi a/2} = \frac{2Q}{\pi a}.$$

For length element $ds = a\ d\theta$, the charge it carries is

$$dq = \lambda\ ds = \lambda a\ d\theta.$$

Applying Coulomb's law, the electric field dE due to length element ds at the center of curvature is

$$dE = \frac{k\ dq}{a^2} = \frac{k\lambda a\ d\theta}{a^2} = \frac{k\lambda}{a}d\theta.$$

The x component of the field is

$$dE_x = -dE\cos\theta = -\frac{k\lambda}{a}\cos\theta\ d\theta.$$

The x component of the electric field due to the whole wire is

$$E_x = \int dE_x = \int_{\pi/4}^{3\pi/4} -\frac{k\lambda}{a}\cos\theta\ d\theta = -\frac{k\lambda}{a}[\sin\theta]\begin{matrix}3\pi/4\\ \pi/2\end{matrix} = 0.$$

The y component of the field is

$$dE_y = -dE\sin\theta = -\frac{k\lambda}{a}\sin\theta\ d\theta.$$

The y component of the electric field due to the whole wire is

$$E_y = \int dE_y = \int_{\pi/4}^{3\pi/4} -\frac{k\lambda}{a}\sin\theta \, d\theta = \frac{k\lambda}{a}[\cos\theta]\begin{matrix}3\pi/4\\ \pi/2\end{matrix}$$

$$= \frac{k\lambda}{a}[\cos(3\pi/4) - \cos(\pi/2)]$$

$$= -\frac{\sqrt{2}k\lambda}{a} = -\frac{\sqrt{2}k}{a} \cdot \frac{2Q}{\pi a}$$

$$= -\frac{2\sqrt{2}kQ}{\pi a^2}.$$

The direction of the field is in the negative y direction or the $-\mathbf{j}$ direction. This means that the electric field at the center of curvature is

$$\boldsymbol{E} = \boldsymbol{E}_x + \boldsymbol{E}_y = -\frac{2\sqrt{2}kQ}{\pi a^2}\mathbf{j}.$$

◆ wxMaxima codes:

```
(%i1) Ex: integrate(-k*lambda/a*cos(theta), theta, %pi/4, 3*%pi/4);
(Ex)   0
(%i2) Ey: integrate(-k*lambda/a*sin(theta), theta, %pi/4, 3*%pi/4);
(Ey)   -(sqrt(2)*k*lambda)/a
```

Comments on the codes:

(%i1), (%i2) Calculate $E_x = \int_{\pi/4}^{3\pi/4} -\frac{k\lambda}{a}\cos\theta \, d\theta$ and $E_y = \int_{\pi/4}^{3\pi/4} -\frac{k\lambda}{a}\sin\theta \, d\theta$.

Problem 4.18 A ring of radius R carries a charge of Q. Determine the electric field along the axis of the ring.

Solution

Figure 4.28 shows the ring of charge Q and radius R. To solve the problem, a length element of the ring ds is considered. The electric field due to the element at P is calculated. Then, the effective electric field is calculated by integrating the field due to the element for the whole ring.

Linear charge density of the ring is

$$\lambda = \frac{Q}{2\pi R},$$

because the length (circumference) of the ring is $2\pi R$. The charge of the length element ds is

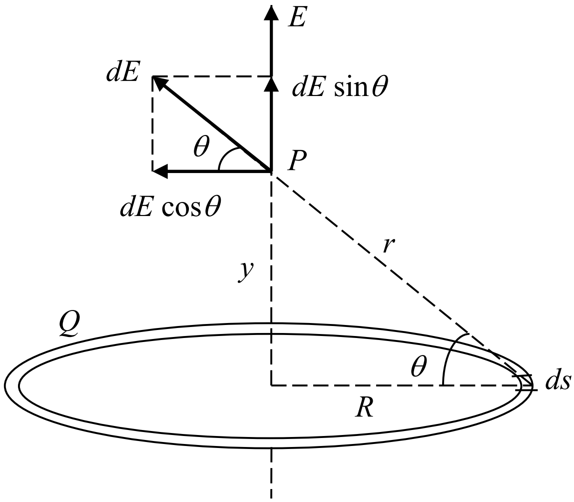


Fig. 4.28 Determining electric field along the axis of a charged ring, Problem 4.18

$$dq = \lambda \, ds.$$

Due to this charge, an electric field *dE* is present at point *P*,

$$dE = \frac{k \, dq}{r^2} = \frac{k\lambda \, ds}{r^2}.$$

The electric field *dE* is resolved into horizontal component $dE \cos\theta$ and vertical component $dE \sin\theta$. When all elements are summed, the horizontal component vanishes. This is by symmetry of the problem. The vertical component needs to be summed. The electric field at point *P* is

$$\begin{aligned} E &= \int dE \sin\theta = \int \frac{k\lambda \sin\theta}{r^2} ds = \int_0^{2\pi R} \frac{k\lambda y}{r^3} ds = \frac{k\lambda y}{r^3} \cdot 2\pi R \\ &= \frac{2\pi k\lambda R y}{(y^2 + R^2)^{3/2}} \\ &= \frac{kQy}{(y^2 + R^2)^{3/2}}. \end{aligned}$$

The direction of the field is upward. This is entry (a) of Table 4.1.

If point *P* is far away from the ring, $y \gg R$, and the electric field is $E = kQ/y^2$. This means that at a far distance, the electric field of a charged ring is just like the field of a point charge.

◆ wxMaxima codes:

```
(%i1) r: sqrt(y^2 + R^2);
(r) sqrt(y^2+R^2)
(%i2) E: integrate(k*lambda*y/r^3, s, 0, 2*%pi*R);
(E) (2*%pi*R*k*y*lambda)/(y^2+R^2)^(3/2)
(%i3) lambda: Q/(2*%pi*R);
(lambda) Q/(2*%pi*R)
(%i4) E: integrate(k*lambda*y/r^3, s, 0, 2*%pi*R);
(E) (Q*k*y)/(y^2+R^2)^(3/2)
```

Comments on the codes:

(%i1), (%i2) Assign r and calculate $E = \int_0^{2\pi R} \frac{k\lambda y}{r^3} ds$.

(%i3), (%i4) Assign λ and calculate $E = \int_0^{2\pi R} \frac{k\lambda y}{r^3} ds$.

Problem 4.19 A disk of radius R has a charge per unit area σ, as shown in Fig. 4.29. Determine the electric field at point P, a distance of y from the disk.

Solution

Figure 4.30 shows the disk and other quantities needed to solve the problem. The disk is divided into rings. A ring of radius r with thickness dr is considered. The electric field due to this ring is calculated and the electric field due to the disk is calculated by summing the fields due to the rings.

A ring of radius r and thickness dr has a charge of,

$$dq = \sigma \cdot 2\pi r\ dr.$$

The electric field at P due to the ring is,

$$dE = \frac{ky\ dq}{(y^2 + r^2)^{3/2}} = \frac{2\pi ky\sigma r\ dr}{(y^2 + r^2)^{3/2}}.$$

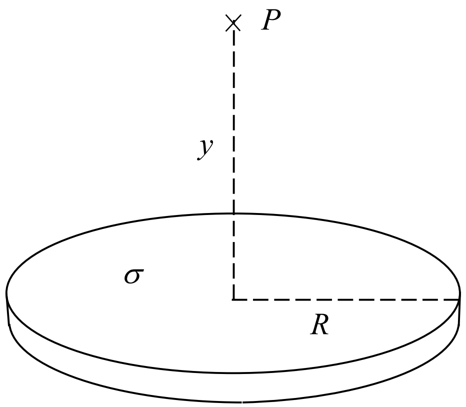


Fig. 4.29 A charged disk, Problem 4.19

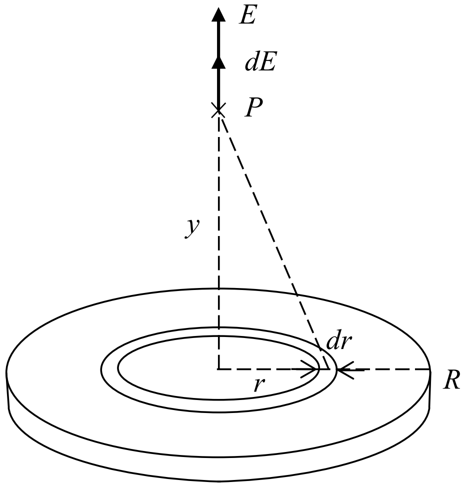


Fig. 4.30 Determining electric field at point P due to a charged disk, Problem 4.19

This is obtained by using the result of Problem 4.18 or Table 4.1a that gives the electric field along the axis of a charged ring. The electric field at P due to the disk is obtained by integration of dE, that is,

$$E = \int dE = 2\pi k y \sigma \int_0^R \frac{r\,dr}{(y^2 + r^2)^{3/2}} = \left[-2\pi k y \sigma \frac{1}{\sqrt{y^2 + r^2}} \right]_0^R$$

$$= 2\pi k \sigma \left(1 - \frac{y}{\sqrt{y^2 + R^2}} \right)$$

$$= \frac{\sigma}{2\varepsilon_0} \left(1 - \frac{y}{\sqrt{y^2 + R^2}} \right),$$

where $k = 1/(4\pi\varepsilon_0)$. The direction of the electric field is vertically upward. This result is the same as Table 4.1e.

If the disk is very wide, $R \gg y$, then the electric field becomes $E = \frac{\sigma}{2\varepsilon_0}$. This is the same as Table 4.1d.

◆ wxMaxima codes:

```
(%i3) assume(R>0); assume(y>0); E: 2*%pi*k*y*sigma*
integrate(r/(y^2+r^2)^(3/2), r, 0, R);
(%o1) [R>0]
(%o2) [y>0]
(E)   2*%pi*k*sigma*y*(1/y-1/sqrt(y^2+R^2))
```

Comment on the codes:

(%i3) Calculate the definite integral $E = 2\pi k y\sigma \int_0^R \frac{r\,dr}{(y^2+r^2)^{3/2}}$.

4.3 Summary

- The electrostatic force between two charges of q_1 and q_2, separated by a distance of r, is

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}.$$

- The electric field at a point is the force experienced by a unit positive test charge placed at the point.
- The magnitude of electric field at a distance of r from a point charge of q is

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}.$$

4.4 Exercises

Exercise 4.1 Two charges are separated by a certain distance. The magnitude of their charges is halved and their separation is doubled. What happens to the electric force between the charges?

(Answer: The electric force decreases by a factor of 16)

Exercise 4.2 Three charges, $q_1 = 5.0 \times 10^{-3}$ C, $q_2 = -3.0 \times 10^{-3}$ C, and $q_3 = 2.0 \times 10^{-3}$ C, are fixed at (0, 0), (3, 4) m, and (3, 0) m, respectively, as shown in Fig. 4.31. Calculate the electric force on charge q_3.

(Answer: $\boldsymbol{F} = 1.0 \times 10^4\,\mathbf{i} + 3.4 \times 10^3\,\mathbf{j}$ N, $F = 1.1 \times 10^4$ N, $\theta = 19°$)

Exercise 4.3 Two charges, $q_1 = 5.0 \times 10^{-3}$ C and $q_2 = -3.0 \times 10^{-3}$ C, are fixed at (0, 0) and (3, 4) m, respectively, as shown in Fig. 4.32. Calculate the electric field at point P.

(Answer: $\boldsymbol{E} = (5.0 \times 10^6\,\mathbf{i} + 1.7 \times 10^6\,\mathbf{j})$ N C^{-1}, $E = 5.3 \times 10^{-6}$ N C^{-1}, $\theta = 19°$)

Exercise 4.4 Figure 4.33 shows three equal point charges, q, fixed at corners of a square of side l. Find the electric field at the center of the square.

(Answer: $E = 2kq/l^2$, $\theta = 45°$)

Exercise 4.5 A proton is placed in a region of uniform electric field 400 N C^{-1}. What is the acceleration of the proton?

(Answer: 3.8×10^{10} m s^{-2})

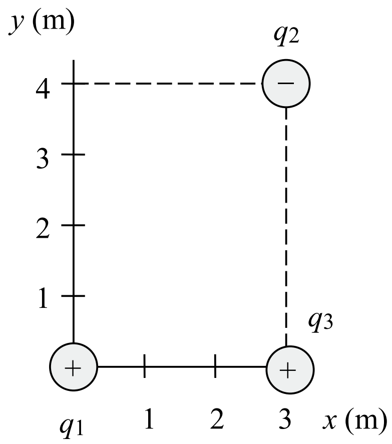


Fig. 4.31 Configuration of three charges, Exercise 4.2

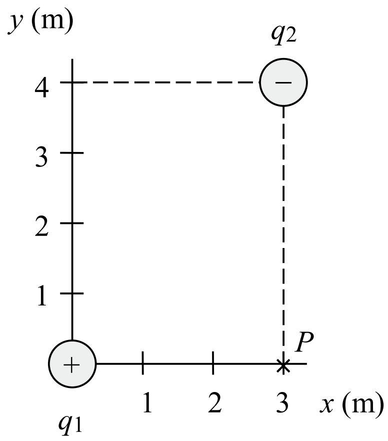


Fig. 4.32 Configuration of two charges, Exercise 4.3

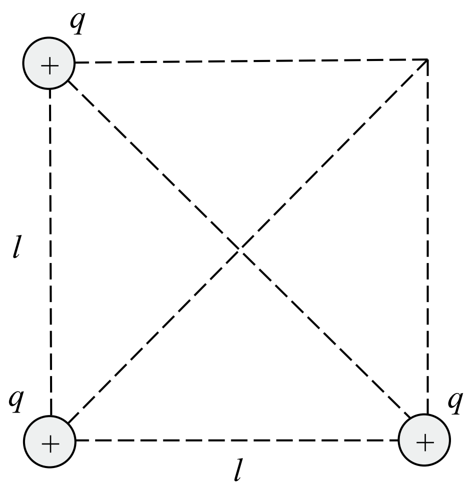


Fig. 4.33 Configuration of three charges, Exercise 4.4

Chapter 5
Gauss's Law

Abstract This chapter solves problems on Gauss's law and its application. Gauss's law states that electric flux through a closed surface is equal to the electric charge enclosed by the surface divided by permittivity of free space. Using Gauss's law, electric fields of some symmetric charge distributions are calculated. Solutions are by analysis and computer calculation.

5.1 Basic Concepts and Formulae

(1) Electric flux is the number of electric field lines through a surface that is perpendicular to the field lines. This is written as

$$\Phi = \int_{surface} \boldsymbol{E} \cdot \boldsymbol{dA}, \tag{5.1}$$

where $\boldsymbol{E}$ is electric field and $\boldsymbol{dA}$ is surface element vector. The surface element vector is normal to the surface element and its magnitude is the area of the surface element dA. For a surface of area A with its normal at an angle θ with a uniform electric field, the electric flux is,

$$\Phi = \boldsymbol{E} \cdot \boldsymbol{A} = EA\cos\theta. \tag{5.2}$$

(2) Gauss's law states that net electric flux Φ through a closed surface (Gauss's surface) is the net charge in the closed surface divided by ε_0, that is,

$$\Phi = \oint \boldsymbol{E} \cdot \boldsymbol{dA} = \frac{q}{\varepsilon_0}, \tag{5.3}$$

where the constant,

$$\varepsilon_0 = 8.8542 \times 10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}},$$

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https://doi.org/10.1007/978-3-031-43165-4_5

is the permittivity of free space and it is related to Coulomb's constant k by

$$k = \frac{1}{4\pi\varepsilon_0} = 8.9876 \times 10^9 \text{ N m}^2 \text{ C}^{-2} \approx 9 \times 10^9 \text{ N m}^2 \text{ C}^{-2}.$$

(3) Using Gauss's law, electric fields of symmetrical charge distributions can be calculated. Table 5.1 lists a few electric fields that can be derived by application of Gauss's law.

(4) A conductor is in electrostatic equilibrium. The followings apply:

 (a) Electric field is zero inside the conductor.
 (b) For an isolated conductor, excess charge resides on the surface of the conductor.
 (c) Electric field outside the surface of a conductor is perpendicular to the surface and the magnitude is σ/ε_0 where σ is charge per unit area.
 (d) The charge on a conductor accumulates at sharp points, that is, regions where a radius of curvature of the surface is smallest.

5.2 Problems and Solutions

Problem 5.1 Figure 5.1 shows a wedge-shaped closed surface in a uniform electric field of 50 N C^{-1}. Calculate the electric flux across each surface and the flux through the wedge.

Solution

Figure 5.2 shows the wedge-shaped closed surface, electric field, surface element vectors, and related angles.

Electric flux is, Eq. (5.1),

$$\Phi = \int \boldsymbol{E} \cdot \boldsymbol{dA}.$$

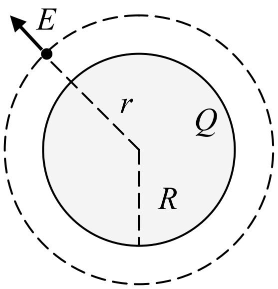
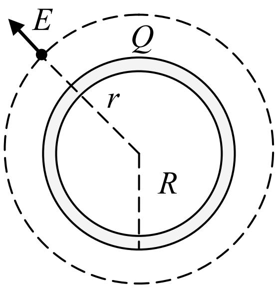
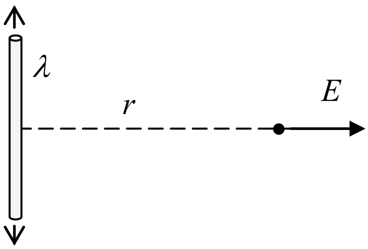
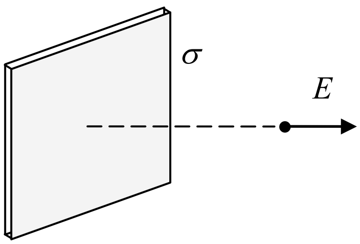
For surfaces *abe*, *befc*, and *dcf*, the electric field $\boldsymbol{E}$ and surface element vector $\boldsymbol{dA}$ are perpendicular to each other. Thus, $\boldsymbol{E} \cdot \boldsymbol{dA} = 0$ and the electric fluxes across these surfaces are zero,

$$\Phi_{abe} = \Phi_{befc} = \Phi_{dcf} = 0.$$

For rectangular surface *abcd*, the electric flux across it is,

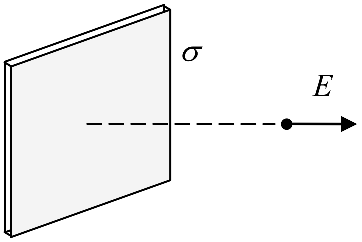
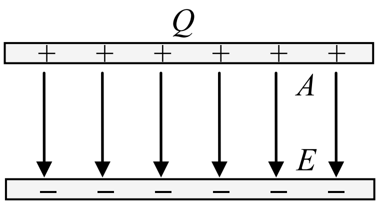
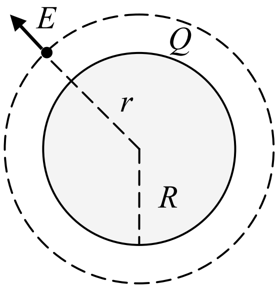
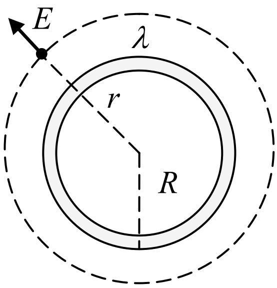
$$\Phi_{abcd} = \int \boldsymbol{E} \cdot \boldsymbol{dA} = \int 50\, dA \cos 180^\circ = -50 \int dA = -50\,(1 \times 2) \text{ N m}^2 \text{ C}^{-1}$$
$$= -100 \text{ N m}^2 \text{ C}^{-1}.$$

Table 5.1 Electric fields of a few charge configurations derivable by Gauss's law

	Configuration	Electric field
(a)		A spherical insulator of radius R with uniform charge distribution and total charge Q $E = \frac{kQ}{r^2} = \frac{Q}{4\pi\varepsilon_0 r^2}, \quad r > R$ $E = \frac{kQ}{R^3} r = \frac{Q}{4\pi\varepsilon_0 R^3} r, \quad r \leq R$ where r is distance of observation point to center of the sphere
(b)		A spherical shell of radius R with charge Q $E = \frac{kQ}{r^2} = \frac{Q}{4\pi\varepsilon_0 r^2}, \quad r \geq R$ $E = 0, \quad r < R$ where r is distance of observation point to center of the shell
(c)		A long rod with charge per unit length λ $E = \frac{2k\lambda}{r} = \frac{\lambda}{2\pi\varepsilon_0 r}$ where r is perpendicular distance from the rod to observation point
(d)		A charged insulator plate, with a charge per unit area of σ $E = \frac{\sigma}{2\varepsilon_0}$ outside the plate

(continued)

Table 5.1 (continued)

	Configuration	Electric field
(e)		A conducting plate with a charge per unit area of σ $E = \frac{\sigma}{\varepsilon_0}$ outside the plate $E = 0$ in the plate
(f)		A parallel plate capacitor with a charge of Q $E = \frac{Q}{\varepsilon_0 A}$ between the plates
(g)		A solid conducting sphere of a radius R and a charge of Q $E = \frac{kQ}{r^2} = \frac{Q}{4\pi\varepsilon_0 r^2}, \quad r > R$ $E = 0, \quad r \le R$ where r is the distance of observation point to the center of the sphere
(h)		A cylindrical shell of radius R with a charge per unit length of λ $E = \frac{2k\lambda}{r} = \frac{\lambda}{2\pi\varepsilon_0 r}, \quad r \ge R$ $E = 0, \quad r < R$ where r is distance of the observation point to the axis of the shell

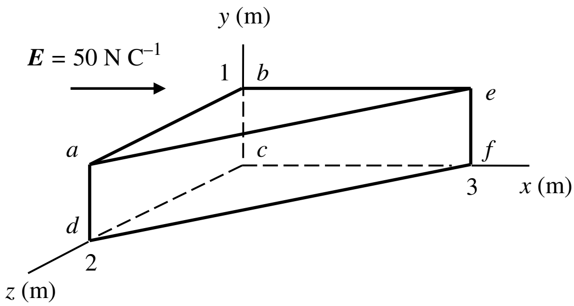


Fig. 5.1 A wedge-shaped closed surface in a uniform electric field, Problem 5.1

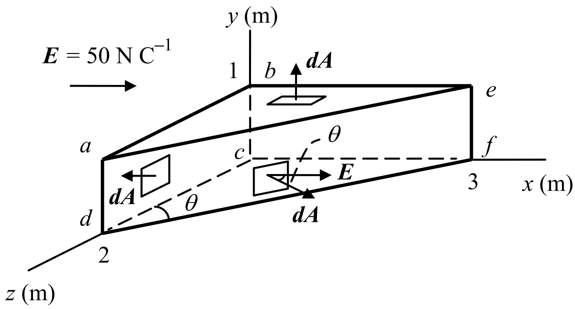


Fig. 5.2 Determining electric fluxes, Problem 5.1

For rectangular surface *aefd*, the electric flux across it is,

$$\Phi_{aefd} = \int \boldsymbol{E}\cdot \boldsymbol{dA} = \int 50\, dA\ \cos\theta = 50\ \cos\theta \int dA$$
$$= 50 \cdot \frac{2}{\sqrt{13}} \cdot (1 \times \sqrt{13})\ \mathrm{N\ m^2\ C^{-1}}$$
$$= 100\ \mathrm{N\ m^2\ C^{-1}}.$$

The electric flux across the wedge closed surface is zero because the sum of electric fluxes of the five surfaces is zero.

Problem 5.2 Figure 5.3 shows imaginary surfaces of a cylinder of length L and radius r in the region of uniform electric field E_0 in the positive x direction. Calculate the electric flux across,

(a) surface 1
(b) surface 2
(c) surface 3
(d) enclosed surface of the cylinder.

Solution

Figure 5.4 shows the cylinder surfaces, surface element vectors, and the electric field.

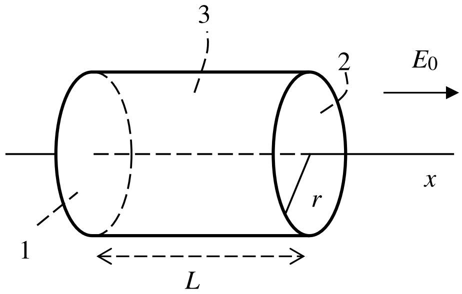


Fig. 5.3 Imaginary surfaces of a cylinder, Problem 5.2

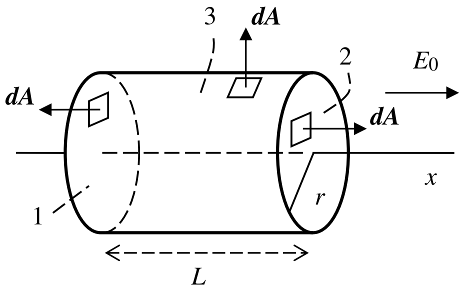


Fig. 5.4 Determining electric fluxes, Problem 5.2

Electric flux is, Eq. (5.1),

$$\Phi = \int \boldsymbol{E} \cdot \boldsymbol{dA},$$

where ***dA*** is surface element vector and ***E*** is electric field. The electric field is,

$$\boldsymbol{E} = E_0\mathbf{i}.$$

(a) For surface 1, $\boldsymbol{dA} = -dA\mathbf{i}$. The electric flux across surface 1 is,

$$\Phi = \int \boldsymbol{E} \cdot \boldsymbol{dA} = \int E_0\mathbf{i} \cdot (-dA\ \mathbf{i}) = -E_0 \int dA = -E_0\pi r^2.$$

(b) For surface 2, $\boldsymbol{dA} = dA\mathbf{i}$. The electric flux across surface 2 is,

$$\Phi = \int \boldsymbol{E} \cdot \boldsymbol{dA} = \int E_0\mathbf{i} \cdot dA\ \mathbf{i} = E_0 \int dA = E_0\pi r^2.$$

(c) For surface 3, ***dA*** is perpendicular to ***E***. The electric flux is zero.
(d) From the three results, electric flux through the enclosed surface of the cylinder is zero, that is,

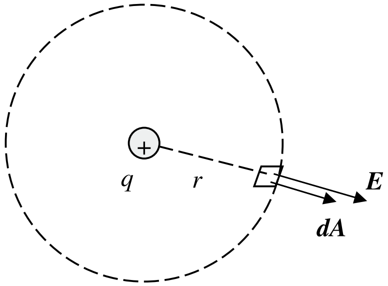


Fig. 5.5 A point charge and an imaginary Gauss's surface, Problem 5.3

$$-E_0\pi r^2 + E_0\pi r^2 + 0 = 0.$$

Problem 5.3 Apply Coulomb's law to determine the electric field around an isolated point charge q.

Solution

Figure 5.5 shows the point charge, imaginary Gauss's surface, and surface element vector to solve the problem.

The imaginary Gauss's surface is the surface of a sphere of radius r. Electric field $\boldsymbol{E}$ due to the positive point charge is directed out in a radial way from the charge and is normal to the sphere surface. This means that $\boldsymbol{E}$ is parallel to $\boldsymbol{dA}$. Equation (5.3),

$$\oint \boldsymbol{E} \cdot \boldsymbol{dA} = \frac{q}{\varepsilon_0},$$

becomes,

$$\oint E \cdot dA = \frac{q}{\varepsilon_0}.$$

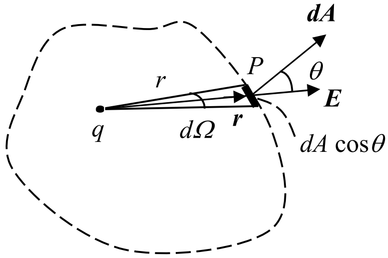
But by symmetry, E is constant at every point on the surface. Thus,

$$\oint E \cdot dA = E \oint dA = E \cdot 4\pi r^2 = \frac{q}{\varepsilon_0},$$

where $\oint dA = 4\pi r^2$ is area of the spherical surface. Therefore, the electric field is,

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} = k\frac{q}{r^2}.$$

Fig. 5.6 A point charge and an imaginary closed Gauss's surface, Problem 5.4



Problem 5.4 Derive Gauss's law using electric field of a point charge q.

Solution

Figure 5.6 shows a point charge q inside an arbitrary imaginary closed surface (Gauss's surface). The surface element vector is $\boldsymbol{dA}$, position vector of point P on the surface is $\boldsymbol{r}$, solid angle subtended by dA is $d\Omega$, angle between $\boldsymbol{dA}$ and $\boldsymbol{r}$ is θ, and unit vector is $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$.

The electric field at point P due to the charge is, Eq. (1.3),

$$\boldsymbol{E} = k\frac{q}{r^2}\hat{\boldsymbol{r}} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}}.$$

The electric flux across surface dA is,

$$d\Phi = \boldsymbol{E}\cdot\boldsymbol{dA} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}}\cdot\boldsymbol{dA} = \frac{q}{4\pi\varepsilon_0}\frac{dA\cos\theta}{r^2} = \frac{q}{4\pi\varepsilon_0}d\Omega,$$

where $d\Omega$ is a solid angle element subtended by $\boldsymbol{dA}$. The electric flux for the whole closed surface is,

$$\Phi = \oint d\Phi = \oint \boldsymbol{E}\cdot\boldsymbol{dA} = \oint \frac{q}{4\pi\varepsilon_0}d\Omega = \frac{q}{4\pi\varepsilon_0}\oint d\Omega = \frac{q}{4\pi\varepsilon_0}\cdot 4\pi = \frac{q}{\varepsilon_0}.$$

Therefore, Gauss's law has been derived, that is,

$$\oint \boldsymbol{E}\cdot\boldsymbol{dA} = \frac{q}{\varepsilon_0}.$$

Problem 5.5 A solid non-conductor sphere of radius R has uniform charge density of ρ.

(a) Using Gauss's law, determine the electric field outside and inside the sphere.
(b) What is the electric field at the surface of the sphere?
(c) Sketch the variation of electric field versus radial distance.

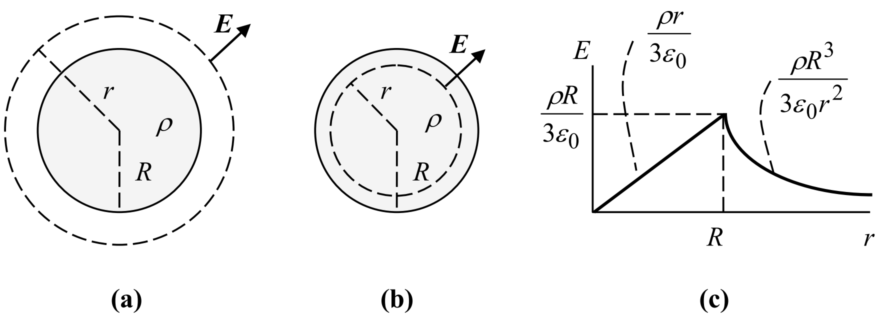


Fig. 5.7 Gauss's surface **a** out of and **b** in the charged sphere, and **c** curve of E against r, Problem 5.5

Solution

(a) Figure 5.7a shows the charged sphere of radius R with charge density ρ enclosed by imaginary spherical Gauss's surface of radius r.

Electric field outside of the charged sphere is calculated as follows. Take a Gauss's surface in the form of a surface of a sphere of radius r. Apply Gauss's law, Eq. (5.3),

$$\oint \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{q}{\varepsilon_0},$$
$$E \cdot 4\pi r^2 = \frac{\frac{4}{3}\pi R^3 \rho}{\varepsilon_0}.$$

The electric field for the region outside the charged sphere is

$$E(r) = \frac{\rho R^3}{3\varepsilon_0 r^2}, \qquad r > R. \tag{1}$$

Electric field in the charged sphere is calculated as follows. Take a Gauss's surface in the form of a surface of a sphere of radius r as shown in Fig. 5.7b. Apply Gauss's law,

$$\oint \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{q}{\varepsilon_0},$$
$$E \cdot 4\pi r^2 = \frac{\frac{4}{3}\pi r^3 \rho}{\varepsilon_0}.$$

The electric field in the charged sphere is,

$$E(r) = \frac{\rho}{3\varepsilon_0} r, \qquad r \le R. \tag{2}$$

(b) Electric field at the surface of the charged sphere can be calculated using Eq. (1). At the surface, $r = R$, and the electric field is,

$$E(R) = \frac{\rho R^3}{3\varepsilon_0 R^2} = \frac{\rho R}{3\varepsilon_0}.$$

The result is obtained by using Eq. (2) as well,

$$E(R) = \frac{\rho}{3\varepsilon_0} R.$$

(c) Using results of parts (a) and (b), the curve of electric field versus radial distance can be sketched and this is shown in Fig. 5.7c.

Let the total charge on the non-conductor sphere be Q. We have,

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3}.$$

The electric field for the region outside the charged sphere becomes

$$E(r) = \frac{\rho R^3}{3\varepsilon_0 r^2} = \frac{Q}{4\pi\varepsilon_0 r^2}, \qquad r > R.$$

The electric field in the charged sphere becomes

$$E(r) = \frac{\rho}{3\varepsilon_0} r = \frac{Q}{4\pi\varepsilon_0 R^3} r, \qquad r \leq R.$$

These results are entries (a) of Table 5.1.

Problem 5.6 A wire a has charge per unit length of λ. Determine the electric field around the wire by Gauss's law.

Solution

Figure 5.8 shows the wire with a charge per unit length of λ. An imaginary closed Gauss's surface is in the form of curved cylindrical surface of length L and radius r (surface 1) and two circular surfaces of radius r (surfaces 2 and 3). In the figure, surface element vectors $\boldsymbol{dA}$ and electric fields $\boldsymbol{E}$ are indicated as well.

By symmetry, the electric field is radial and perpendicular to the wire. By Gauss's law, Eq. (5.3),

$$\oint \boldsymbol{E} \cdot \boldsymbol{dA} = \frac{q}{\varepsilon_0},$$

$$\int_1 \boldsymbol{E} \cdot \boldsymbol{dA} + \int_2 \boldsymbol{E} \cdot \boldsymbol{dA} + \int_3 \boldsymbol{E} \cdot \boldsymbol{dA} = \frac{\lambda L}{\varepsilon_0},$$

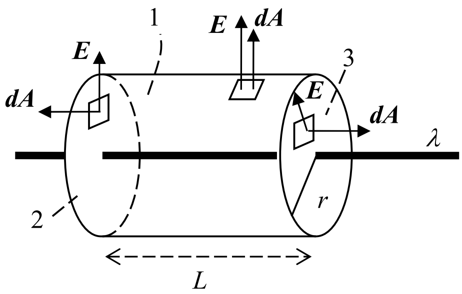


Fig. 5.8 A long charged wire and an imaginary closed Gauss's surface, Problem 5.6

$$E \cdot 2\pi r L + 0 + 0 = \frac{\lambda L}{\varepsilon_0}.$$

The integral of surface 1 is $E \cdot 2\pi r L$ because the surface element vector is parallel to the electric field, while those of surfaces 2 and 3 are zero because surface element vectors are perpendicular to the electric fields. Therefore, the electric field around a charged wire is

$$E = \frac{\lambda L}{\varepsilon_0 \cdot 2\pi r L} = \frac{\lambda}{2\pi \varepsilon_0 r} = \frac{2k\lambda}{r},$$

where r is the distance of the observation point from the long wire. This is the same as entry (b) of Table 4.1 or Problem 4.15(b).

Problem 5.7 A solid spherical insulator of radius a having a uniform charge density is shown in Fig. 5.9. The total charge on the sphere is Q. Concentric to the sphere is a spherical shell conductor with inner and outer radii of b and c ($b < c$). Determine

(a) the electric field for regions $r < a$ and $a < r < b$

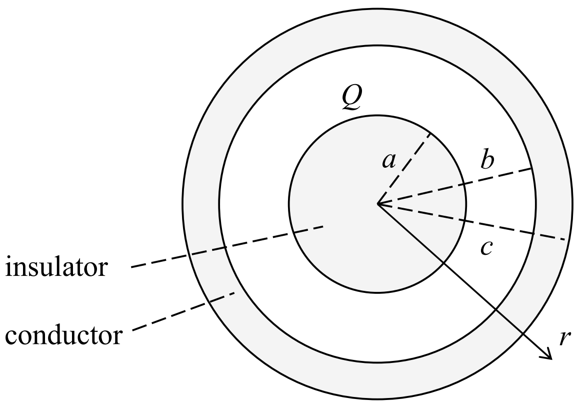


Fig. 5.9 A non-conducting charged sphere and a conducting spherical shell, Problem 5.7

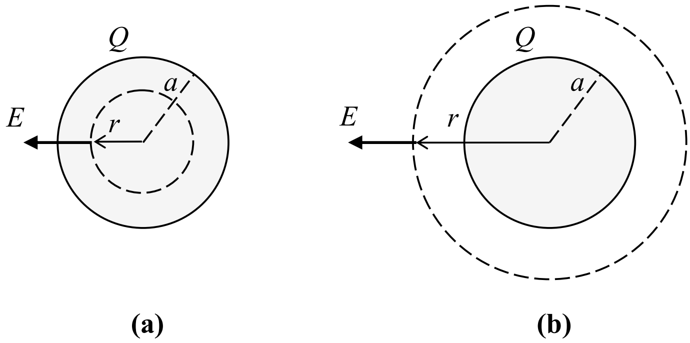


Fig. 5.10 Gauss's surface **a** in and **b** out of the non-conducting charged sphere, Problem 5.7

(b) surface charge densities of the inner and outer surfaces of the shell.

Solution

(a) Figure 5.10a shows the charged solid spherical insulator. Consider region $r < a$ in the sphere.

Charge density of the solid sphere is,

$$\rho = \frac{Q}{\frac{4}{3}\pi a^3}.$$

Consider an imaginary closed Gauss's surface in the form of a surface of a sphere of radius r. Charge in the closed Gauss's surface is

$$q = \rho \cdot \frac{4}{3}\pi r^3 = \frac{Q}{\frac{4}{3}\pi a^3} \cdot \frac{4}{3}\pi r^3 = \frac{r^3}{a^3}Q.$$

By Gauss's law, Eq. (5.3),

$$\oint \boldsymbol{E} \cdot \boldsymbol{dA} = \frac{q}{\varepsilon_0},$$

$$E \cdot 4\pi r^2 = \frac{r^3 Q}{\varepsilon_0 a^3}.$$

The electric field is

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{a^3}r = \frac{kQ}{a^3}r, \qquad (r < a).$$

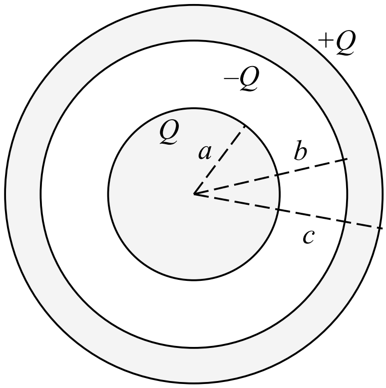


Fig. 5.11 Charges on the inner and outer surfaces of the spherical shell, Problem 5.7

Figure 5.10b considers the region outside the solid sphere but inside the spherical shell. Take an imaginary closed Gauss's surface as the surface of a sphere of radius r. By Gauss's law, Eq. (5.3),

$$\oint \boldsymbol{E} \cdot \boldsymbol{dA} = \frac{q}{\varepsilon_0}$$
$$E \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0}.$$

This gives the electric field as

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} = \frac{kQ}{r^2}, \qquad (a < r < b).$$

(b) Figure 5.11 shows that charge Q on solid insulator sphere induces charge $-Q$ and $+Q$ on the spherical conductor shell. The inner and outer surfaces of the conductor shell have charges of $-Q$ and $+Q$, respectively.

Therefore, the inner surface charge density σ_b and the outer surface charge density σ_c are

$$\sigma_b = \frac{-Q}{4\pi b^2}, \quad \sigma_c = \frac{+Q}{4\pi c^2}.$$

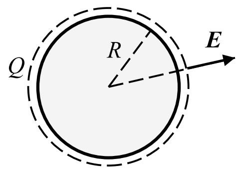
Problem 5.8 A metal sphere of 0.50 cm radius has an 8.0 nC charge on it. Calculate the electric field at the surface of the sphere.

Solution

Figure 5.12 shows the metal sphere of radius R with charge Q. An imaginary closed Gauss's surface in the form of a surface of a sphere very near the surface of the metal sphere and the electric field are shown as well.

Applying Gauss's law to the closed Gauss's surface gives, Eq. (5.3),

Fig. 5.12 A charged metal sphere and an imaginary Gauss's surface, Problem 5.8



$$\oint \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{q}{\varepsilon_0},$$
$$E \cdot 4\pi R^2 = \frac{Q}{\varepsilon_0},$$
$$E = \frac{Q}{4\pi R^2 \varepsilon_0} = \frac{kQ}{R^2}.$$

Inserting known numerical values, the electric field at the surface of the metal sphere is

$$E = \frac{kQ}{R^2} = \frac{(9 \times 10^9 \text{ N m}^2 \text{ C}^{-2})(8.0 \times 10^{-9} \text{ C})}{(0.50 \times 10^{-2} \text{ m})^2}$$
$$= 2.9 \times 10^6 \text{ N C}^{-1}.$$

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; k:9e9; Q:8e-9; R:0.5e-2;
(fpprintprec)     5
(k)   9.0*10^9
(Q)   8.0*10^-9
(R)   0.005
(%i5) E: k*Q/R^2;
(E)   2.88*10^6
```

Comments on the codes:

(%i4) Set the floating point print precision to 5 and assign values of k, Q, and R.
(%i5) Calculate $E = kQ/R^2$.

Problem 5.9 Dielectric strength of air is 3.0×10^6 N C^{-1}. Calculate the maximum charge on a metal sphere of radius 0.50 cm. The dielectric strength of a material is the maximum electric field that the material can withstand without undergoing electrical breakdown and becoming electrically conductive.

Solution

Dielectric strength of air equals 3.0×10^6 N C^{-1} means if the electric field in air exceeds the value, sparks will be produced. Using result of Problem 5.8, the maximum amount of charge on a sphere before sparks are produced is,

$$E = \frac{kQ}{R^2},$$
$$Q = \frac{ER^2}{k} = \frac{(3.0 \times 10^6 \text{ N C}^{-1})(0.50 \times 10^{-2} \text{ m})^2}{(9 \times 10^9 \text{ N m}^2 \text{ C}^{-2})}$$
$$= 8.3 \times 10^{-9} \text{ C}.$$

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; k:9e9; E:3e6; R:0.5e-2;
(fpprintprec)     5
(k)   9.0*10^9
(E)   3.0*10^6
(R)   0.005
(%i5) Q: E*R^2/k;
(Q)   8.3333*10^-9
```

Comments on the codes:

(%i4) Set the floating point print precision to 5 and assign values of k, E, and R.
(%i5) Calculate $Q = ER^2/k$.

Problem 5.10 The electric field between plates of a parallel plate capacitor is 300 kV m^{-1}. The area of the plate is 600 cm^2. What is the charge on the plate?

Solution

Figure 5.13 shows the parallel plate capacitor, electric field $\boldsymbol{E}$, and charge Q. An imaginary closed Gauss's surface is the surface of a box around the upper plate of the capacitor as shown. The area of the bottom surface of the box is A.

By Gauss's law, the electric field can be calculated as follows

$$\oint \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{q}{\varepsilon_0},$$
$$E \cdot A = \frac{Q}{\varepsilon_0},$$
$$E = \frac{Q}{\varepsilon_0 A}.$$

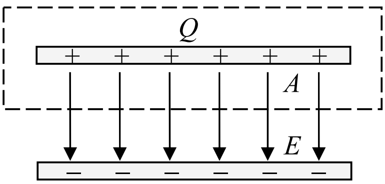


Fig. 5.13 A parallel plate capacitor and an imaginary Gauss's surface, Problem 5.10

The charge on the plate is

$$\begin{aligned} Q &= \varepsilon_0 A E \\ &= (8.8542 \times 10^{-12}\ \mathrm{C^2\ N^{-1}\ m^{-2}})(600 \times 10^{-4}\ \mathrm{m^2})(300 \times 10^3\ \mathrm{V\ m^{-1}}) \\ &= 1.6 \times 10^{-7}\ \mathrm{C}. \end{aligned}$$

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; epsilon0:8.8542e-12; A:600e-4; E:300e3;
(fpprintprec)     5
(epsilon0)  8.8542*10^-12
(A)   0.06
(E)   3.0*10^5
(%i5) Q: epsilon0*A*E;
(Q)   1.5938*10^-7
```

Comments on the codes:

(%i4) Set the floating point print precision to 5 and assign values of ε_0, A, and E.
(%i5) Calculate $Q = \varepsilon_0 A E$.

5.3 Summary

- Gauss's law says that if q is the total charge enclosed in a closed surface, then the total outward electric flux through the closed surface is q/ε_0, that is,

$$\oint_{surface} \boldsymbol{E} \cdot \boldsymbol{dA} = \frac{q}{\varepsilon_0}.$$

- The electric fields of some symmetrical charge distributions can be calculated by applying Gauss's law.

5.4 Exercises

Exercise 5.1 Figure 5.14 shows a 5.0 μC charge placed at the center of an imaginary cube. What is the electric flux through the cube surfaces?
(Answer: $\Phi = 5.6 \times 10^5\ \mathrm{N\ m^2\ C^{-1}}$)

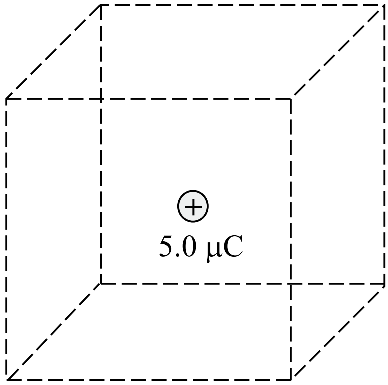


Fig. 5.14 A charge and an imaginary cube surfaces, Exercise 5.1

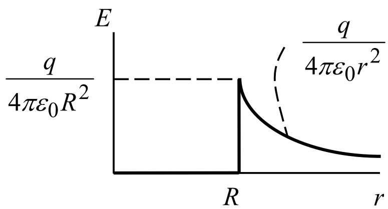


Fig. 5.15 Curve of E against r, Exercise 5.4

Exercise 5.2 Charge q is distributed uniformly throughout a non-conducting sphere of radius R. Using Gauss's law, calculate the electric field at a point $R/2$ from the center of the sphere.

(Answer: $E = \frac{q}{8\pi\varepsilon_0 R^2}$)

Exercise 5.3 Charge q is distributed uniformly throughout a spherical insulating shell with outer radius R. What is the electric flux through the outer surface of the shell and the electric field at the outer surface?

(Answer: $\Phi = \frac{q}{\varepsilon_0},\ E = \frac{q}{4\pi\varepsilon_0 R^2}$)

Exercise 5.4 A solid conducting sphere of radius R has a charge of q. Show that the electric field E as a function of distance r from the center of the sphere is as in Fig. 5.15.

Exercise 5.5 The electric field 2.0 cm from a uniformly charged long wire is $30\ \mathrm{N\,C^{-1}}$. What is the electric field 6.0 cm from the wire?

(Answer: $E = 10\ \mathrm{N\,C^{-1}}$)

Chapter 6
Electric Potential

Abstract This chapter solves the problem of electric potential energy, electric potential difference, and electric potential. Every point in a region of electric field is associated with an electric potential which is electric potential energy per unit charge at the point. Potential difference is the difference in electric potential of two points in the region of electric field. Solutions by analysis and computer calculation are presented.

6.1 Basic Concepts and Formulae

(1) When a positive test charge q_0 is moved from point A to point B in an electric field $\boldsymbol{E}$, the change in electric potential energy is,

$$\Delta U = U_B - U_A = -q_0 \int_A^B \boldsymbol{E} \cdot \boldsymbol{ds}, \tag{6.1}$$

where U_A and U_B are potential energies at points A and B, respectively, and $\boldsymbol{ds}$ is elementary displacement.

(2) Potential difference ΔV between points A and B in the electric field $\boldsymbol{E}$ is the change in potential energy divided by the test charge q_0,

$$\Delta V = V_B - V_A = \frac{\Delta U}{q_0} = \frac{U_B}{q_0} - \frac{U_A}{q_0} = -\int_A^B \boldsymbol{E} \cdot \boldsymbol{ds}, \tag{6.2}$$

where $V_A = U_A/q_0$ and $V_B = U_B/q_0$ are potentials at points A and B, respectively. The unit of electric potential is volt (V) or joule/coulomb (J C^{-1}).

For uniform electric field $\boldsymbol{E}$, potential difference between points A and B is

W. M. S. Wan Hassan et al., *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-43165-4_6

$$\Delta V = -E \cdot d, \quad (6.3)$$

where d is displacement along $\boldsymbol{E}$. Thus, for two parallel plates at a potential difference of ΔV separated by a distance of d, the magnitude of the uniform magnetic field between the plates is $E = \Delta V/d$.

(3) Equipotential surface is a surface with the same electric potential. Equipotential surface is perpendicular to electric field line.

(4) Electric potential due to charge q at a distance r from the charge is

$$V = \frac{kq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}, \quad (6.4)$$

where,

$$k = \frac{1}{4\pi\varepsilon_0} = 8.9876 \times 10^9 \text{ N m}^2 \text{ C}^{-2} \approx 9 \times 10^9 \text{ N m}^2 \text{ C}^{-2},$$

is Coulomb's constant, and

$$\varepsilon_0 = 8.8542 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2},$$

is the permittivity of free space or the permittivity constant.

Electric potential due to a number of charges is the sum of electric potential due to each charge,

$$V = k\sum_i \frac{q_i}{r_i}. \quad (6.5)$$

(5) Electric potential energy U of charge q_1 and q_2 separated by a distance of r_{12} is

$$U = k\frac{q_1 q_2}{r_{12}}. \quad (6.6)$$

U is the work done to bring the charges from infinite separation to separation of r_{12}. Electric potential energy for the distribution of point charges is the sum of the potential energy of every charge pair.

(6) Electric potential due to continuous charge distribution is,

$$V = k\int \frac{dq}{r}, \quad (6.7)$$

where dq is the charge element of the continuous charge distribution and r is the distance of the element from the observation point.

(7) If electric potential as a function of coordinates is known, the electric field component can be calculated from the derivative of the potential with respect to the coordinate,

$$E_x = -\frac{dV}{dx}, \qquad E_y = -\frac{dV}{dy}, \qquad E_z = -\frac{dV}{dz}. \tag{6.8}$$

(8) Every point on the surface of a charged conductor in electrostatic equilibrium has the same potential. The potential is constant at any point in the conductor and it is the potential at the surface of the conductor.

(9) The work done to transport an electron through a potential difference of 1 V is 1 electronvolt (eV). This means that

$$1\,\text{eV} = e\,\Delta V = (1.602 \times 10^{-19}\ \text{C})(1\ \text{V}) = 1.602 \times 10^{-19}\ \text{J}.$$

(10) Electric potentials due to four charge distributions are given in Table 6.1.

6.2 Problems and Solutions

Problem 6.1 Show that the electric potential at a distance r from a charge q is $V = kq/r$.

Solution

Figure 6.1 shows charge q, point P at a distance of r' away from the charge, and the electric field $E = kq/r'^2$ due to the charge.

Electric potential at point r is defined by,

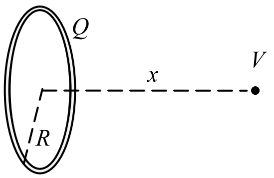
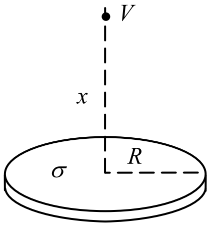
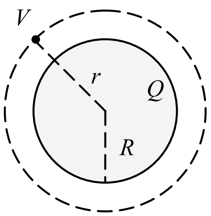
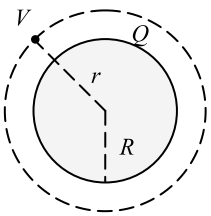
$$V - V_\infty = -\int_\infty^r \boldsymbol{E} \cdot d\boldsymbol{s}.$$

This corresponds to the work done to bring a +1 C charge from ∞ to point r. In the equation, $V_\infty = 0$, $ds = -dr'$, therefore,

$$V = -\int_\infty^r \frac{kq}{r'^2} dr' = \left[\frac{kq}{r'}\right]_\infty^r = \frac{kq}{r}.$$

We have shown that the electric potential at distance of r from a charge q is $V = kq/r$.

Table. 6.1 Electric potentials of a few charge configurations

	Configuration	Electric potential
(a)		A ring of radius R uniformly charged, total charge Q, $V = k\frac{Q}{\sqrt{x^2+R^2}}$, where x is the distance along the ring axis from the ring center
(b)		A uniformly charged disk of radius R, charge density per unit area σ, $V = 2\pi k\sigma\left(\sqrt{x^2+R^2}-x\right)$, where x is the distance along the disk axis from the disk center
(c)		A solid insulator sphere of radius R, uniformly charged, total charge Q, charge density ρ, $V = k\frac{Q}{r} = \frac{\rho R^3}{3\varepsilon_0 r}$, $r \geq R$, $V = \frac{kQ}{2R}\left(3-\frac{r^2}{R^2}\right) = \frac{\rho}{6\varepsilon_0}(3R^2-r^2)$, $r < R$
(d)		A solid conducting sphere of radius R, with charge Q, $V = k\frac{Q}{r}$, $r \geq R$, $V = \frac{kQ}{R}$, $r < R$

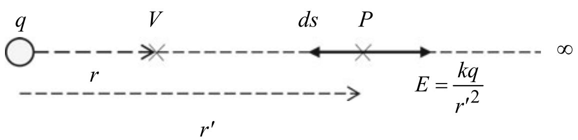


Fig. 6.1 Electric potential V at r away from a charge, Problem 6.1

♦ wxMaxima codes:

```
(%i1) assume(r>0);
(%o1) [r>0]
(%i2) V: integrate(-k*q/rprime^2, rprime, inf, r);
(V)   (k*q)/r
```

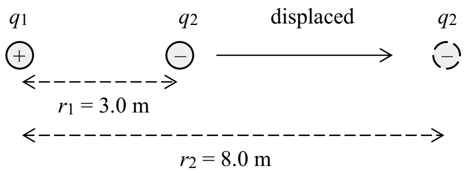


Fig. 6.2 System of two charges, before and after one of them is displaced, Problem 6.2

Comment on the codes:

(%i1) Assume r to be positive.
(%i2) Calculate the definite integral $V = -\int_{\infty}^{r} \frac{kq}{r'^2} dr'$.

Problem 6.2 Two charges $q_1 = 4.0 \times 10^{-4}$ C and $q_2 = -8.0 \times 10^{-4}$ C are separated by a distance of 3.0 m. What is the electric potential energy of the two charges? One of the charges is displaced so that the separation becomes 8.0 m. What is the change in electric potential energy?

Solution

Figure 6.2 shows the charges in the initial and final instances.

The electric potential energy at the initial instance is, Eq. (6.6),

$$\begin{aligned} U_{initial} &= k\frac{q_1 q_2}{r_1} = (9 \times 10^9 \text{ N m}^2 \text{ C}^{-2})\frac{(4.0 \times 10^{-4} \text{ C})(-8.0 \times 10^{-4} \text{ C})}{3.0 \text{ m}} \\ &= -960 \text{ J}. \end{aligned}$$

The electric potential energy at the final instance is

$$\begin{aligned} U_{final} &= k\frac{q_1 q_2}{r_2} = (9 \times 10^9 \text{ N m}^2 \text{ C}^{-2})\frac{(4.0 \times 10^{-4} \text{ C})(-8.0 \times 10^{-4} \text{ C})}{8.0 \text{ m}} \\ &= -360 \text{ J}. \end{aligned}$$

The change in electric potential energy is

$$\begin{aligned} U_{final} - U_{initial} &= -360 \text{ J} - (-960 \text{ J}) \\ &= 600 \text{ J}. \end{aligned}$$

This means that an increase in the separation of two oppositely signed charges amounts to an increase in the electric potential energy.

◆ wxMaxima codes:

```
(%i6) fpprintprec:5; k:9e9; q1:4e-4; q2:-8e-4; r1:3; r2:8;
(fpprintprec)     5
(k)   9.0*10^9
(q1)  4.0*10^-4
(q2)  -8.0*10^-4
(r1)  3
(r2)  8
(%i8) Uinitial:k*q1*q2/r1; Ufinal:k*q1*q2/r2;
(Uinitial) -960.0
(Ufinal) -360.0
(%i9) Ufinal-Uinitial;
(%o9) 600.0
```

Comments on the codes:

(%i6) Set the floating point print precision to 5 and assign values of k, q_1, q_2, r_1, and r_2.
(%i8) Calculate $U_{initial} = kq_1q_2/r_1$ and $U_{final} = kq_1q_2/r_2$.
(%i9) Calculate the change in electric potential energy.

Problem 6.3 A charge of 5.0 μC is located at (0, 0) and a charge of –8.0 μC is located at (1, 0) m. A third charge of 2.0 μC is moved from point *A* (0.5, 0) m to point *B* (0, 1) m. What is the work done?

Solution

Figure 6.3 shows the two charges $q_1 = 5.0$ μC and $q_2 = –8.0$ μC, points *A* and *B*, and the third charge $q_3 = 2.0$ μC moved from *A* to *B*.

We calculate the electric potentials at points *A* and *B* due to charge 1 and 2. We then calculate the work done to bring charge 3 from *A* to *B*.

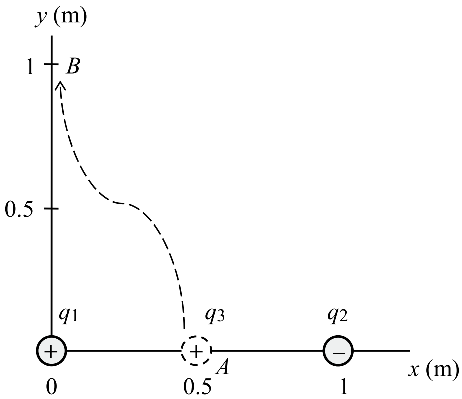


Fig. 6.3 Configuration of three charges, q_3 is moved from point *A* to point *B*, Problem 6.3

The electric potential at point A due to charges 1 and 2 is, Eq. (6.5),

$$\begin{aligned}V_A &= \frac{kq_1}{r_{1A}} + \frac{kq_2}{r_{2A}} \\ &= \left(9 \times 10^9 \frac{\text{N m}^2}{\text{C}^2}\right)\left(\frac{5.0 \times 10^{-6}\ \text{C}}{0.50\ \text{m}}\right) + \left(9 \times 10^9 \frac{\text{N m}^2}{\text{C}^2}\right)\left(\frac{-8.0 \times 10^{-6}\ \text{C}}{0.50\ \text{m}}\right) \\ &= -54000\ \text{V}.\end{aligned}$$

The electric potential at point B due to charges 1 and 2 is,

$$\begin{aligned}V_B &= \frac{kq_1}{r_{1B}} + \frac{kq_2}{r_{2B}} \\ &= \left(9 \times 10^9 \frac{\text{N m}^2}{\text{C}^2}\right)\left(\frac{5.0 \times 10^{-6}\ \text{C}}{1.0\ \text{m}}\right) + \left(9 \times 10^9 \frac{\text{N m}^2}{\text{C}^2}\right)\left(\frac{-8.0 \times 10^{-6}\ \text{C}}{\sqrt{2}\ \text{m}}\right) \\ &= -5912\ \text{V}.\end{aligned}$$

The work to bring charge 3 from point A to B is,

$$\begin{aligned}W_{AB} &= q_3(V_B - V_A) = (2.0 \times 10^{-6}\ \text{C})[-5912 - (-54000)]\ \text{V} \\ &= 9.6 \times 10^{-2}\ \text{J}.\end{aligned}$$

The work done is the change in electric potential energy, $q_3 V_B - q_3 V_A$.

◆ wxMaxima codes:

```
(%i9) fpprintprec:5; k:9e9; q1:5e-6; q2:-8e-6; q3:2e-6; r1A:0.5; r2A:0.5;
r1B:1; r2B:float(sqrt(2));
(fpprintprec)     5
(k)   9.0*10^9
(q1)  5.0*10^-6
(q2)  -8.0*10^-6
(q3)  2.0*10^-6
(r1A) 0.5
(r2A) 0.5
(r1B) 1
(r2B) 1.4142
(%i11) VA: k*q1/r1A + k*q2/r2A; VB: k*q1/r1B + k*q2/r2B;
(VA)  -5.4*10^4
(VB)  -5911.7
(%i12) WAB: q3*(VB-VA);
(WAB) 0.096177
```

Comments on the codes:

(%i9) Set the floating point print precision to 5 and assign values of k, q_1, q_2, q_3, r_{1A}, r_{2A}, r_{1B}, and r_{2B}.

(%i11) Calculate electric potentials $V_A = kq_1/r_{1A} + kq_2/r_{2A}$ and $V_B = kq_1/r_{1B} + kq_2/r_{2B}$.

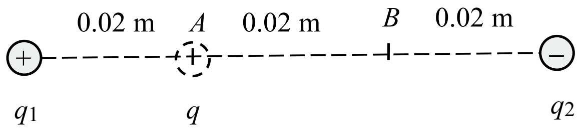


Fig. 6.4 Configuration of three charges. Charge q moves from A to B, q_1 and q_2 are fixed, Problem 6.4

(%i12) Calculate work $W_{AB} = q_3(V_B - V_A)$.

Problem 6.4 Figure 6.4 shows two fixed particles of charges $q_1 = 8.0 \times 10^{-9}$ C and $q_2 = -8.0 \times 10^{-9}$ C, separated by a distance of 0.06 m. A third particle of mass $m = 0.002$ kg with a charge of $q = 3.0 \times 10^{-9}$ C is released from point A and moves to point B. Determine

(a) electric potentials at A and B,
(b) velocity of the third particle at B, and
(c) work done by the electric field to move the third particle from A to B.

Solution

(a) The electric potential at point A due to charges 1 and 2 is, Eq. (6.5),

$$V_A = k\left(\frac{q_1}{r_{1A}} + \frac{q_2}{r_{2A}}\right) = \left(9 \times 10^9 \frac{\text{N m}^2}{\text{C}^2}\right)\left(\frac{8.0 \times 10^{-9}\ \text{C}}{0.02\ \text{m}} + \frac{-8.0 \times 10^{-9}\ \text{C}}{0.04\ \text{m}}\right)$$
$$= 1800\ \text{V}.$$

The electric potential at point B due to charges 1 and 2 is,

$$V_B = k\left(\frac{q_1}{r_{1B}} + \frac{q_2}{r_{2B}}\right) = \left(9 \times 10^9 \frac{\text{N m}^2}{\text{C}^2}\right)\left(\frac{8.0 \times 10^{-9}\ \text{C}}{0.04\ \text{m}} + \frac{-8.0 \times 10^{-9}\ \text{C}}{0.02\ \text{m}}\right)$$
$$= -1800\ \text{V}.$$

The electric potential of point A is higher than that of B.

(b) When the third particle is at point A, it has electric potential energy. This energy is converted to kinetic energy plus electric potential energy when it reaches point B. We write,

$$K_A + U_A = K_B + U_B,$$
$$0 + qV_A = \frac{1}{2}mv^2 + qV_B,$$

where K is kinetic energy, U is electric potential energy, v is velocity of the particle at point B, and q and m are charge and mass of the particle, respectively. The velocity of the third particle at point B is,

$$v = \sqrt{\frac{2q}{m}(V_A - V_B)}$$

$$= \sqrt{\frac{2(3.0 \times 10^{-9}\ \text{C})}{0.002\ \text{kg}}[1800\ \text{V} - (-1800\ \text{V})]}$$
$$= 0.10\ \text{m s}^{-1}.$$

(c) Work done by the electric field to move the third particle is,

$$W_{AB} = q(V_A - V_B) = (3.0 \times 10^{-9}\ \text{C})[1800 - (-1800)]\ \text{V} = 1.1 \times 10^{-5}\ \text{J}.$$

◆ wxMaxima codes:

```
(%i11) fpprintprec:5; ratprint:false; k:9e9; q1:8e-9; q2:-8e-9; q:3e-9;
r1A:0.02; r2A:0.04; r1B:0.04; r2B:0.02; m:0.002;
(fpprintprec)     5
(ratprint)  false
(k)    9.0*10^9
(q1)   8.0*10^-9
(q2)   -8.0*10^-9
(q)    3.0*10^-9
(r1A) 0.02
(r2A) 0.04
(r1B) 0.04
(r2B) 0.02
(m)    0.002
(%i13) VA: k*(q1/r1A + q2/r2A); VB: k*(q1/r1B + q2/r2B);
(VA)  1800.0
(VB)  -1800.0
(%i15) solve(q*VA = 0.5*m*v^2 + q*VB, v)$ float(%);
(%o15) [v=-0.10392,v=0.10392]
(%i16) WAB: q*(VA-VB);
(WAB) 1.08*10^-5
```

Comments on the codes:

(%i11) Set the floating point print precision to 5 and internal rational number print to false, and assign values of k, q_1, q_2, q, r_{1A}, r_{2A}, r_{1B}, r_{2B}, and m.
(%i13) Calculate electric potentials $V_A = k(q_1/r_{1A} + q_2/r_{2A})$ and $V_B = k(q_1/r_{1B} + q_2/r_{2B})$.
(%i15) Solve $qV_A = 0.5 \times mv^2 + qV_B$ for v.
(%i16) Calculate work $W_{AB} = q(V_A - V_B)$.

Problem 6.5

(a) For a two-charge system shown in Fig. 6.5, determine the electric potential energy and electric potential of the system. Determine also the electric field at point A.
(b) If $q_2 = –5.0 \times 10^{-6}$ C, determine all quantities in (a).

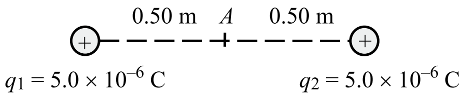


Fig. 6.5 Configuration of two charges, Problem 6.5

Solution

(a) The electric potential energy of the two-charge system is, Eq. (6.6),

$$\begin{aligned} U &= k\frac{q_1 q_2}{r_{12}} = \left(9 \times 10^9 \frac{\text{N m}^2}{\text{C}^2}\right)\frac{(5.0 \times 10^{-6}\ \text{C})(5.0 \times 10^{-6}\ \text{C})}{1.0\ \text{m}} \\ &= 0.22\ \text{J}. \end{aligned}$$

The electric potential at point A due to charges q_1 and q_2 is, Eq. (6.5),

$$\begin{aligned} V_A &= k\left(\frac{q_1}{r_{1A}} + \frac{q_2}{r_{2A}}\right) = \left(9 \times 10^9 \frac{\text{N m}^2}{\text{C}^2}\right)\left(\frac{5.0 \times 10^{-6}\ \text{C}}{0.50\ \text{m}} + \frac{5.0 \times 10^{-6}\ \text{C}}{0.50\ \text{m}}\right) \\ &= 1.8 \times 10^5\ \text{V}. \end{aligned}$$

The electric field at point A is, Eq. (1.4),

$$\boldsymbol{E}_A = \frac{kq_1}{r_{1A}^2}\mathbf{i} - \frac{kq_2}{r_{2A}^2}\mathbf{i} = 0.$$

(b) If $q_2 = -5.0 \times 10^{-6}$ C, the electric potential energy of the two-charge system is, Eq. (6.6),

$$\begin{aligned} U &= k\frac{q_1 q_2}{r_{12}} = \left(9 \times 10^9 \frac{\text{N m}^2}{\text{C}^2}\right)\frac{(5.0 \times 10^{-6}\ \text{C})(-5.0 \times 10^{-6}\ \text{C})}{1.0\ \text{m}} \\ &= -0.22\ \text{J}. \end{aligned}$$

The electric potential at point A due to charges q_1 and q_2 is, Eq. (6.5),

$$\begin{aligned} V_A &= k\left(\frac{q_1}{r_{1A}} + \frac{q_2}{r_{2A}}\right) = \left(9 \times 10^9 \frac{\text{N m}^2}{\text{C}^2}\right)\left(\frac{5.0 \times 10^{-6}\ \text{C}}{0.50\ \text{m}} + \frac{-5.0 \times 10^{-6}\ \text{C}}{0.50\ \text{m}}\right) \\ &= 0\ \text{V}. \end{aligned}$$

The electric field at point A is, Eq. (1.4),

$$\begin{aligned}\boldsymbol{E}_A &= k\frac{q_1}{r_{1A}^2}\,\mathbf{i} + k\frac{q_2}{r_{2A}^2}\,\mathbf{i}\\ &= \left(9\times 10^9\frac{\text{N m}^2}{\text{C}^2}\right)\frac{5.0\times 10^{-6}\ \text{C}}{(0.50\ \text{m})^2}\,\mathbf{i} - \left(9\times 10^9\frac{\text{N m}^2}{\text{C}^2}\right)\frac{(-5,0\times 10^{-6}\ \text{C})}{(0.50\ \text{m})^2}\,\mathbf{i}\\ &= 3.6\times 10^5\ \text{N C}^{-1}\,\mathbf{i}.\end{aligned}$$

◆ wxMaxima codes:

```
(%i7) fpprintprec:5; k:9e9; q1:5e-6; q2:5e-6; r12:1; r1A:0.5; r2A:0.5;
(fpprintprec)     5
(k)   9.0*10^9
(q1)  5.0*10^-6
(q2)  5.0*10^-6
(r12) 1
(r1A) 0.5
(r2A) 0.5
(%i8) U: k*q1*q2/r12;
(U)   0.225
(%i9) VA: k*(q1/r1A + q2/r2A);
(VA)  1.8*10^5
(%i10) EA: k*q1/r1A^2 - k*q2/r2A^2;
(EA)  0.0
(%i11) q2:-5e-6;
(q2)  -5.0*10^-6
(%i12) U: k*q1*q2/r12;
(U)   -0.225
(%i13) VA: k*(q1/r1A + q2/r2A);
(VA)  0.0
(%i14) EA: k*q1/r1A^2 - k*q2/r2A^2;
(EA)  3.6*10^5
```

Comments on the codes:

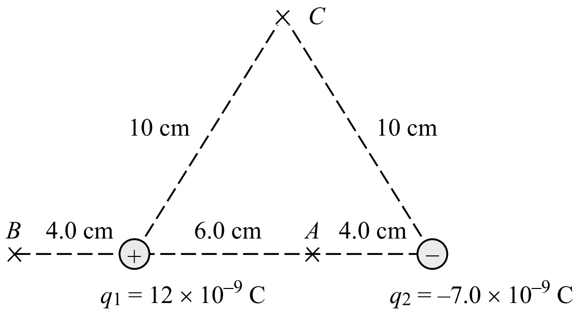
(%i7)	Set the floating point print precision to 5 and assign values of k, q_1, q_2, r_{12}, r_{1A}, and r_{2A}.
(%i8)	Calculate electric potential energy $U = kq_1q_2/r_{12}$.
(%i9)	Calculate electric potential $V_A = k(q_1/r_{1A} + q_2/r_{2A})$.
(%10)	Calculate electric field E_A.
(%i11)	Reassign q_2.
(%i12), (%i13), (%i14)	Recalculate U, V_A, and E_A.

Problem 6.6

(a) For a two-charge system consisting of q_1 and q_2 shown in Fig. 6.6, what is the potential difference between points B and A, between points B and C?

(b) If a charge of 4.0×10^{-9} C is placed at point A, what is the electric potential energy of the charge?

Fig. 6.6 A two-charge system, Problem 6.6



Solution

(a) The electric potentials at points A, B, and C due to charges 1 and 2 are, Eq. (6.5),

$$V_A = k\sum_i \frac{q_i}{r_i} = \left(9\times 10^9 \frac{\text{N m}^2}{\text{C}^2}\right)\left(\frac{12\times 10^{-9}\text{ C}}{6.0\times 10^{-2}\text{ m}} + \frac{-7.0\times 10^{-9}\text{ C}}{4.0\times 10^{-2}\text{ m}}\right)$$
$$= 225 \text{ V},$$

$$V_B = \left(9\times 10^9 \frac{\text{N m}^2}{\text{C}^2}\right)\left(\frac{12\times 10^{-9}\text{ C}}{4.0\times 10^{-2}\text{ m}} + \frac{-7.0\times 10^{-9}\text{ C}}{14\times 10^{-2}\text{ m}}\right)$$
$$= 2250 \text{ V},$$

$$V_C = \left(9\times 10^9 \frac{\text{N m}^2}{\text{C}^2}\right)\left(\frac{12\times 10^{-9}\text{ C}}{10\times 10^{-2}\text{ m}} + \frac{-7.0\times 10^{-9}\text{ C}}{10\times 10^{-2}\text{ m}}\right)$$
$$= 450 \text{ V}.$$

The potential difference between points B and A is

$$V_{BA} = V_B - V_A = 2250 \text{ V} - 225 \text{ V} = 2025 \text{ V}.$$

The potential difference between points B and C is

$$V_{BC} = V_B - V_C = 2250 \text{ V} - 450 \text{ V} = 1800 \text{ V}.$$

(b) The electric potential energy of the charge $q = 4.0\times 10^{-9}$ C at point A is

$$U_A = qV_A = (4.0\times 10^{-9}\text{ C})(225\text{ V}) = 9.0\times 10^{-7}\text{ J}.$$

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; k:9e9; q1:12e-9; q2:-7e-9; q:4e-9;
(fpprintprec)     5
(k)   9.0*10^9
(q1)  1.2*10^-8
(q2)  -7.0*10^-9
(q)   4.0*10^-9
(%i6) VA: k*(q1/6e-2 + q2/4e-2);
(VA)  225.0
(%i7) VB: k*(q1/4e-2 + q2/14e-2);
(VB)  2250.0
(%i8) VC: k*(q1/10e-2 + q2/10e-2);
(VC)  450.0
(%i9) VBA: VB-VA;
(VBA) 2025.0
(%i10) VBC: VB-VC;
(VBC) 1800.0
(%i11) UA: q*VA;
(UA)  9.0*10^-7
```

Comments on the codes:

(%i5)	Set the floating point print precision to 5 and assign values of k, q_1, q_2, and q.
(%i6), (%i7), (%i8)	Calculate potentials V_A, V_B, and V_C.
(%i9), (%i10)	Calculate potential differences V_{BA} and V_{BC}.
(%11)	Calculate potential energy U_A.

Problem 6.7 Three charges, q_1, q_2, and q_3 are arranged on the circumference of a circle of radius 3.0 m as shown in Fig. 6.7. Calculate

(a) electric potential at points A and B and
(b) electric potential energy of the three charges.

Solution

(a) The electric potential of a system of discrete charges is $V = k\sum_i \frac{q_i}{r_i}$, Eq. (6.5).

The potential at point A is,

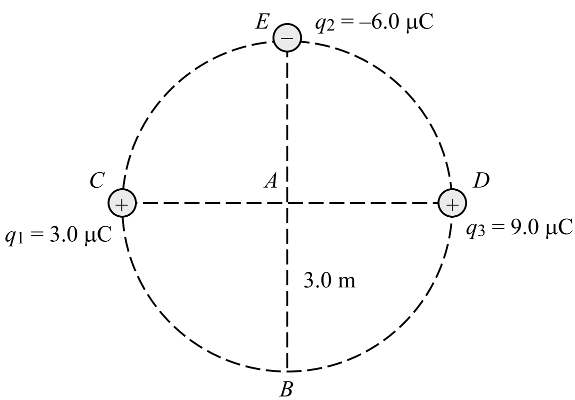


Fig. 6.7 A three-charge system, Problem 6.7

$$
\begin{aligned}
V_A &= k\left(\frac{q_1}{AC} + \frac{q_2}{AE} + \frac{q_3}{AD}\right) \\
&= \left(9 \times 10^9 \frac{\text{N m}^2}{\text{C}^2}\right)\left(\frac{3.0 \times 10^{-6}\ \text{C}}{3.0\ \text{m}} - \frac{6.0 \times 10^{-6}\ \text{C}}{3.0\ \text{m}} + \frac{9.0 \times 10^{-6}\ \text{C}}{3.0\ \text{m}}\right) \\
&= 18000\ \text{V}.
\end{aligned}
$$

The potential at point B is,

$$
\begin{aligned}
V_B &= k\left(\frac{q_1}{BC} + \frac{q_2}{BE} + \frac{q_3}{BD}\right) \\
&= \left(9 \times 10^9 \frac{\text{N m}^2}{\text{C}^2}\right)\left(\frac{3.0 \times 10^{-6}\ \text{C}}{\sqrt{18}\ \text{m}} - \frac{6.0 \times 10^{-6}\ \text{C}}{6.0\ \text{m}} + \frac{9.0 \times 10^{-6}\ \text{C}}{\sqrt{18}\ \text{m}}\right) \\
&= 16456\ \text{V}.
\end{aligned}
$$

(b) The electric potential energy of the three charges is, Eq. (6.6),

$$
\begin{aligned}
U &= k\left(\frac{q_1 q_2}{CE} + \frac{q_2 q_3}{DE} + \frac{q_1 q_3}{CD}\right) \\
&= \left(9 \times 10^9 \frac{\text{N m}^2}{\text{C}^2}\right)\left(\frac{3.0(-6.0)}{\sqrt{18}} + \frac{(-6.0)(9.0)}{\sqrt{18}} + \frac{3.0(9.0)}{6.0}\right) \times 10^{-12} \frac{\text{C}^2}{\text{m}} \\
&= -0.11\ \text{J}.
\end{aligned}
$$

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; k:9e9; q1:3e-6; q2:-6e-6; q3:9e-6;
(fpprintprec)     5
(k)   9.0*10^9
(q1)  3.0*10^-6
(q2)  -6.0*10^-6
(q3)  9.0*10^-6
(%i8) AC: 3; AE:3; AD: 3;
(AC)  3
(AE)  3
(AD)  3
(%i9) VA: k*(q1/AC + q2/AE + q3/AD);
(VA)  1.8*10^4
(%i12) BC: float(sqrt(18)); BE: 6; BD: float(sqrt(18));
(BC)  4.2426
(BE)  6
(BD)  4.2426
(%i13) VB: k*(q1/BC + q2/BE + q3/BD);
(VB)  1.6456*10^4
(%i16) CE: float(sqrt(18));  DE: float(sqrt(18)); CD: 6;
(CE)  4.2426
(DE)  4.2426
(CD)  6
(%i17) U: k*(q1*q2/CE + q2*q3/DE + q1*q3/CD);
(U)   -0.11224
```

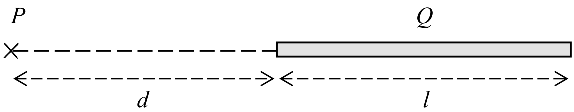


Fig. 6.8 A charged wire and point P, Problem 6.8

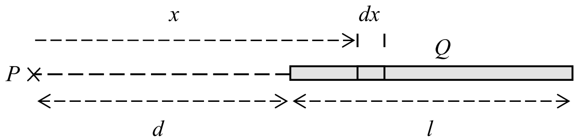


Fig. 6.9 Determining electric potential at point P due to a charged wire, Problem 6.8

Comments on the codes:

(%i5) Set the floating point print precision to 5 and assign values of k, q_1, q_2, and q_3.
(%i8) Assign distances AC, AE, and AD.
(%i9) Calculate potential V_A.
(%i12) Assign distances BC, BE, and BD.
(%i13) Calculate potential V_B.
(%i16) Assign distances CE, DE, and CD.
(%i17) Calculate potential energy U.

Problem 6.8 A wire of length l has a charge of Q distributed uniformly along its length as shown in Fig. 6.8.

(a) Determine the electric potential at point P.
(b) What is the electric potential at P if $d >> l$?

Solution

(a) Figure 6.9 shows the wire, element of the wire dx, and x the position of the element with respect to P.

The linear charge density is,

$$\lambda = \frac{Q}{l}.$$

Consider the wire element of length dx at coordinate x away from point P. The amount of charge of the element is

$$dq = \lambda\, dx.$$

The potential at point P due to the element is (Eq. 6.4)

$$dV = \frac{k\, dq}{x} = \frac{k\lambda\, dx}{x}.$$

Therefore, the electric potential at point P due to the whole wire is

$$V_P = \int dV = k\lambda \int_d^{d+l} \frac{dx}{x} = [k\lambda \ln x]_d^{d+l} = k\lambda \ln\left(\frac{d+l}{d}\right)$$
$$= \frac{kQ}{l} \ln\left(\frac{d+l}{d}\right).$$

◆ wxMaxima codes:

```
(%i3) assume (l>0); assume(d>0); lambda: Q/l;
(%o1) [l>0]
(%o2) [d>0]
(lambda) Q/l
(%i4) VP: k*lambda*integrate(1/x, x, d, d+l);
(VP)   (Q*k*(log(l+d)-log(d)))/l
```

Comments on the codes:

(%i3) Assume l and d positive and assign λ.

(%i4) Calculate $V_P = k\lambda \int_d^{d+l} \frac{dx}{x}$.

(b) From part (a),

$$V_P = \frac{kQ}{l} \ln\left(\frac{d+l}{d}\right) = \frac{kQ}{l} \ln\left(1 + \frac{l}{d}\right).$$

If $d >> l$, $\ln\left(1 + \frac{l}{d}\right) = \frac{l}{d} - \frac{1}{2}(\frac{l}{d})^2 + \ldots \approx \frac{l}{d}$. See Appendix D for the series expansion. Thus, the electric potential at P when d is much greater than l is

$$V_P = \frac{kQ}{l} \cdot \frac{l}{d} = \frac{kQ}{d}.$$

Problem 6.9 A long wire has a linear charge density of λ. Determine the potential difference at radial distances of r_A and r_B from the wire.

Solution

Figure 6.10 shows a long wire with a linear charge density of λ, and radial distances of r_A, r_B, and r of the problem.

From Table 4.1b of Chap. 4, the electric field at a radial distance of r from a long wire is,

$$E = \frac{2k\lambda}{r}.$$

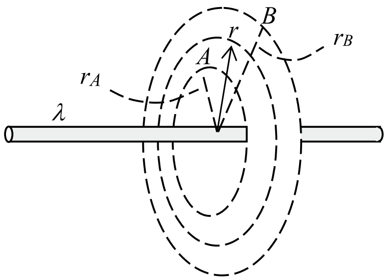


Fig. 6.10 Determining potential difference between two points, Problem 6.9

The direction of the electric field is radial and perpendicular to the wire. The potential difference between points B and A is,

$$V_{BA} = V_B - V_A = -\int_{r_A}^{r_B} E \cdot ds = -\int_{r_A}^{r_B} \frac{2k\lambda}{r} dr = [-2k\lambda \ln r]_{r_A}^{r_B}$$
$$= -2k\lambda \ln\left(\frac{r_B}{r_A}\right).$$

Thus, the potential difference between points A and B is,

$$V_{AB} = V_A - V_B$$
$$= 2k\lambda \ln\left(\frac{r_B}{r_A}\right).$$

The potential at point A is higher than that at point B because r_B is greater than r_A, that is, point A is nearer than point B to the charged wire.

Problem 6.10 Linear charge density of the ring shown in Fig. 6.11 is λ. The radius of the ring is R. Determine the electric potential at point P, a distance of x away from the center of the ring.

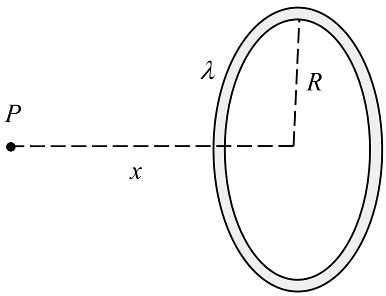


Fig. 6.11 Electric potential at point P due to a charged ring, Problem 6.10

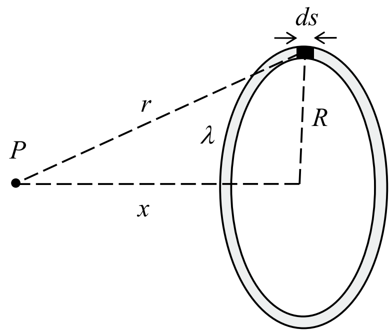


Fig. 6.12 Determining electric potential at point P due to a charged ring, Problem 6.10

Solution

Figure 6.12 shows the ring and the ring element of length ds needed to solve the problem.

The electrical charge of element ds is

$$dq = \lambda\, ds.$$

The electric potential at P due to the element is (Eq. 6.4)

$$dV = \frac{k\,dq}{r} = \frac{k\lambda\,ds}{\sqrt{R^2 + x^2}}.$$

Therefore, the electric potential at point P due to the whole ring is

$$V = \int dV = \frac{k\lambda}{\sqrt{R^2 + x^2}} \int_0^{2\pi R} ds = \frac{2\pi k\lambda R}{\sqrt{R^2 + x^2}}$$
$$= \frac{\lambda R}{2\varepsilon_0\sqrt{R^2 + x^2}}.$$

If the charge on the ring is Q, the electric potential is

$$V = \frac{2\pi k\lambda R}{\sqrt{R^2 + x^2}} = \frac{2\pi k R}{\sqrt{R^2 + x^2}}\frac{Q}{2\pi R}$$
$$= \frac{kQ}{\sqrt{R^2 + x^2}}.$$

This is entry (a) of Table 6.1.

Problem 6.11 A solid non-conducting sphere of radius R has a uniform charge density ρ. Determine,

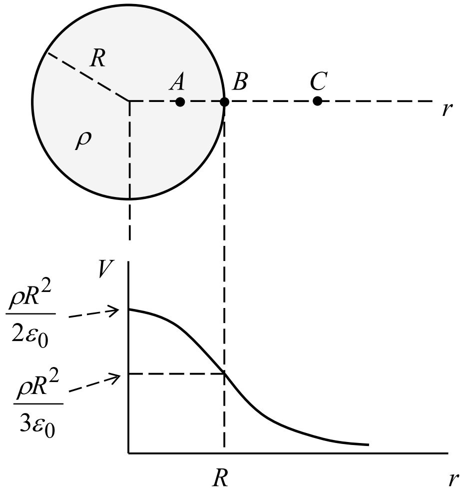


Fig. 6.13 Determining electric potential in and out of a charged non-conducting sphere, Problem 6.11

(a) electric potential outside the sphere
(b) electric potential in the sphere.

Solution

(a) Figure 6.13 shows the insulator sphere, point A in the sphere, point B on the surface of the sphere, and point C outside the sphere.

The total charge of the sphere is

$$Q = \frac{4}{3}\pi R^3 \rho.$$

The electric field out of the sphere is (Eq. 1.3)

$$E_{out} = k\frac{Q}{r^2}, \quad r > R.$$

The potential difference between point C and infinity is calculated as follows:

$$V_C - V_\infty = -\int_\infty^{r_C} E_{out} \cdot dr = -\int_\infty^{r_C} \frac{kQ}{r^2} dr = \left[\frac{kQ}{r}\right]_\infty^{r_C} = \frac{kQ}{r_C},$$

where r_C is the distance from the center of the sphere to point C (Fig. 6.13). Taking $V_\infty = 0$, the potential beyond the sphere is

$$V_C = \frac{kQ}{r} = \frac{\rho R^3}{3\varepsilon_0 r}, \quad r > R,$$

where $k = 1/(4\pi\varepsilon_0)$ and $Q = 4\pi R^3\rho/3$. On the surface of the sphere, $r = R$, so the electric potential at point B is

$$V_B = \frac{\rho R^2}{3\varepsilon_0} = \frac{kQ}{R},$$

where $\rho = Q/(4\pi R^3/3)$ and $\varepsilon_0 = 1/(4\pi k)$.

◆ wxMaxima codes:

```
(%i2) assume(rC>0); VC:-integrate(k*Q/r^2, r, inf, rC);
(%o1) [rC>0]
(VC)  (Q*k)/rC
```

Comments on the codes:

(%i2) Assume $r_C > 0$ and calculate $V_C = -\int_{\infty}^{r_C} \frac{kQ}{r^2} dr$.
(VC) The result.

(b) From Problem 5.5, Chap. 5, the electric field within the sphere is

$$E_{in} = \frac{\rho r}{3\varepsilon_0}, \quad r < R.$$

The potential difference between point A and a point at infinity is calculated as follows:

$$V_A - V_\infty = (V_A - V_B) + (V_B - V_\infty) = -\int_R^{r_A} E_{in} \cdot dr + V_B$$
$$= -\int_R^{r_A} \frac{\rho r}{3\varepsilon_0} dr + \frac{\rho R^2}{3\varepsilon_0}$$
$$= \frac{\rho}{6\varepsilon_0}(R^2 - r_A^2) + \frac{\rho R^2}{3\varepsilon_0},$$

where r_A is the distance from the center of the sphere to point A (Fig. 6.13). Setting $V_\infty = 0$, the electric potential within the sphere is

$$V_A = \frac{\rho}{6\varepsilon_0}(3R^2 - r^2)$$
$$= \frac{kQ}{2R}\left(3 - \frac{r^2}{R^2}\right), \quad r < R,$$

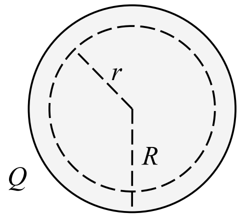


Fig. 6.14 A non-conducting charged sphere, Problem 6.12

where $\rho = Q/(4\pi R^3/3)$ and $\varepsilon_0 = 1/(4\pi k)$. At the center of the sphere, $r = 0$, the electric potential there is

$$V_0 = \frac{\rho R^2}{2\varepsilon_0} = \frac{3kQ}{2R},$$

where $\rho = Q/(4\pi R^3/3)$ and $\varepsilon_0 = 1/(4\pi k)$. In Fig. 6.13, the curve of electric potential V against r for the sphere is shown as well. These results are the same as entry (c) of Table 6.1.

Problem 6.12 Show that the energy needed to construct a uniformly charged non-conducting solid sphere of radius R and charge Q is

$$U = \frac{3}{5}\frac{kQ^2}{R}.$$

Solution

Figure 6.14 shows the solid non-conducting sphere of radius R and charge Q. Also shown is an imaginary sphere of radius r in the solid sphere.

The charge of the imaginary sphere is

$$q = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3} \cdot Q = \frac{r^3}{R^3}Q.$$

Differentiation of q with respect to r gives

$$dq = \frac{3Qr^2}{R^3}dr.$$

The electric potential at the surface of the imaginary sphere is (Eq. 6.4)

$$V = \frac{kq}{r} = \frac{k}{r}\frac{r^3 Q}{R^3} = \frac{kQr^2}{R^3}.$$

The energy to construct a sphere of charge Q and radius R is calculated as follows:

$$U = \int V\, dq = \int_0^R \frac{kQr^2}{R^3} \cdot \frac{3Qr^2}{R^3} dr = \frac{3kQ^2}{R^6} \int_0^R r^4 dr = \left[\frac{3kQ^2}{R^6}\frac{r^5}{5}\right]_0^R$$
$$= \frac{3}{5}\frac{kQ^2}{R}.$$

◆ wxMaxima code:

```
(%i1) U: 3*k*Q^2/R^6*integrate(r^4, r, 0, R);
(%o1) (3*k*Q^2)/(5*R)
```

Comment on the code:

(%i1) Calculate definite integration $U = \frac{3kQ^2}{R^6} \int_0^R r^4 dr$.

Problem 6.13 A small sphere of mass 1.0×10^{-4} kg and charge $+2.4 \times 10^{-9}$ C is suspended by a thread between two vertical parallel plates separated by a distance of 10 cm, as shown in Fig. 6.15. What is the angle between the thread and the vertical if the potential difference between the plates is 10 kV?

Solution

Figure 6.16 shows the forces acting on the sphere, the parallel plates, and the quantities needed to solve the problem. Here, d is the separation and ΔV is the potential difference between the plates.

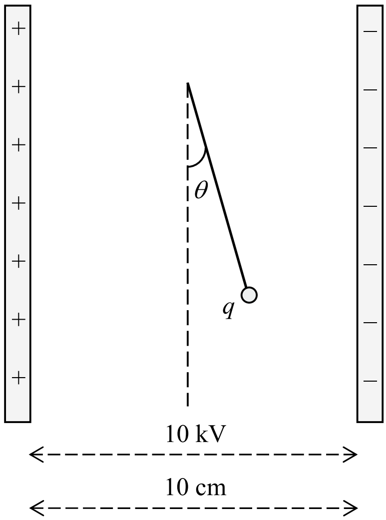


Fig. 6.15 A charged sphere suspended by a thread between two parallel plates, Problem 6.13

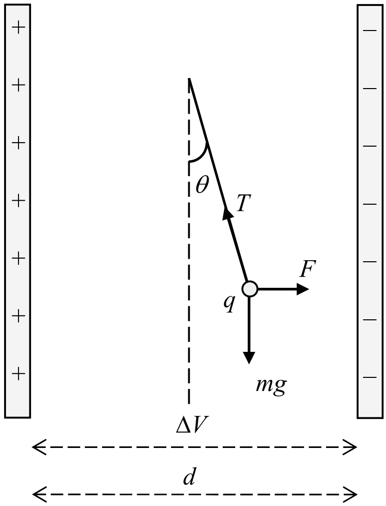


Fig. 6.16 Forces on the charged sphere, Problem 6.13

The forces are weight of the sphere mg, tension in the thread T, and electrostatic force F due to the charged sphere in an electric field. The electrostatic force is (Eqs. 4.2 and 6.3),

$$F = qE = q\frac{\Delta V}{d}, \tag{1}$$

where q is the charge of the sphere, ΔV is the potential difference between plates, and d is the separation distance. The sphere is in equilibrium, so,

$$\sum F_x = F - T\sin\theta = 0, \tag{2}$$

$$\sum F_y = T\cos\theta - mg = 0. \tag{3}$$

From these equations,

$$\tan\theta = \frac{F}{mg} = \frac{q\Delta V}{dmg} = \frac{(2.4\times 10^{-9}\ \text{C})(10\times 10^{3}\ \text{V})}{(0.10\ \text{m})(1.0\times 10^{-4}\ \text{kg})(9.8\ \text{m/s}^2)} = 0.24.$$

Therefore, the angle between the thread and the vertical is,

$$\theta = 14^\circ.$$

◆ wxMaxima codes:

```
(%i6) fpprintprec:5; m:1e-4; q:2.4e-9; g:9.8; d:10e-2; dV:10e3;
(fpprintprec)     5
(m)   1.0*10^-4
(q)   2.4*10^-9
(g)   9.8
(d)   0.1
(dV)  1.0*10^4
(%i7) tantheta: q*dV/(d*m*g);
(tantheta) 0.2449
(%i8) theta: atan(tantheta);
(theta) 0.24017
(%i9) theta_deg: float(theta*180/%pi);
(theta_deg) 13.761
```

Comments on the codes:

(%i6)	Set the floating point print precision to 5 and assign values of m, q, g, d, and ΔV.
(%i7), (%i8), (%i9)	Calculate angle θ.

Further question: Calculate the tension in the string T.

Solution: Solving Eqs. (1), (2) and (3) for θ, T, and F, one obtains the angle, the tension in the string, and the force as

$$\theta = 0.24 \text{ rad} = 14^\circ,$$
$$T = 1.0 \times 10^{-3} \text{ N},$$
$$F = 2.4 \times 10^{-4} \text{ N}.$$

◆ wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)     5
(ratprint)  false
(%i5) eq1: F=q*dV/d; eq2: F-T*sin(theta)=0; eq3: T*cos(theta)-m*g=0;
(eq1) F=(dV*q)/d
(eq2) F-T*sin(theta)=0
(eq3) T*cos(theta)-g*m=0
(%i6) solve([eq1,eq2,eq3], [sin(theta), F, T]);
(%o6) [[sin(theta)=(dV*q*cos(theta))/(d*g*m),F=(dV*q)/d,
        T=(g*m)/cos(theta)]]
(%i7) subst([m=1e-4, q=2.4e-9, g=9.8, d=10e-2, dV=10e3], %);
(%o7) [[sin(theta)=0.2449*cos(theta),F=2.4*10^-4,T=(9.8*10^-4)/cos(theta)]]
(%i10) tantheta: 0.2449; theta:atan(tantheta);
theta_deg:float(theta*180/%pi);
(tantheta)  0.2449
(theta)     0.24017
(theta_deg) 13.761
(%i11)      T: 9.8*10^-4/cos(theta);
(T)   0.001009
```

Comments on the codes:

(%i2) Set the floating point print precision to 5 and internal rational number print to false.
(%i5) Assign Eqs. (1), (2) and (3) as eq1, eq2, and eq3.
(%i6) Solve Eqs. (1), (2) and (3) for $\sin\theta$, F, and T in symbols.
(%i7) Substitute values of m, q, g, d, and ΔV into the solution.
(%i10) Calculate θ in rad and degree.
(%i11) Calculate tension T.

6.3 Summary

- Electric potential is electric potential energy per unit charge. Electric potential due to charge of q at a distance of r from the charge is,

$$V = \frac{kq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}.$$

- Electric potential energy U of charge q_1 and q_2 separated by a distance of r_{12} is,

$$U = k\frac{q_1 q_2}{r_{12}}.$$

- Potential difference between points A and B is the work done against electric forces in carrying a unit positive test charge from A to B, that is, $V_B - V_A = \Delta V$. The work W done against electric forces to carry a charge q from point A to B is $W = q(V_B - V_A) = q\,\Delta V$.

6.4 Exercises

Exercise 6.1 Two charges, $q_1 = 5.0 \times 10^{-3}$ C and $q_2 = -3.0 \times 10^{-3}$ C are fixed at (0, 0) and (3, 4) m, respectively, as in Fig. 6.17. Calculate the electric potential energy of the two charges and the electric potential at point P.

(Answer: $U = -2.7 \times 10^4$ J, $V_P = 8.2 \times 10^6$ V)

Exercise 6.2 Two charges, $q_1 = 5.0 \times 10^{-3}$ C and $q_2 = -3.0 \times 10^{-3}$ C, are placed at (0, 0) and (3, 4) m, respectively. Charge q_2 is then moved from (3, 4) m to (3, 0) m. Calculate the change in electric potential energy.

(Answer: $\Delta U = -1.8 \times 10^4$ J)

Exercise 6.3 Figure 6.18 shows three charges q_1, q_2, and q_3 at the vertices of an equilateral triangle with sides of length l. Calculate work needed to move charge q_3 from point A to point B.

(Answer: $W = (q_1 + q_2)q_3 k/l$)

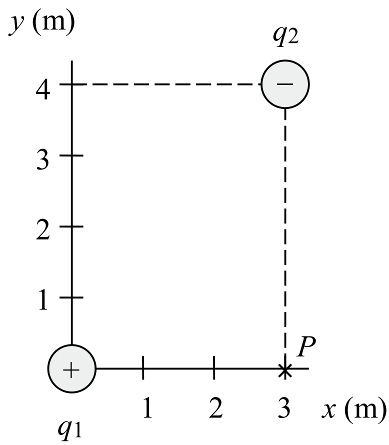


Fig. 6.17 Configuration of two charges, Exercise 6.1

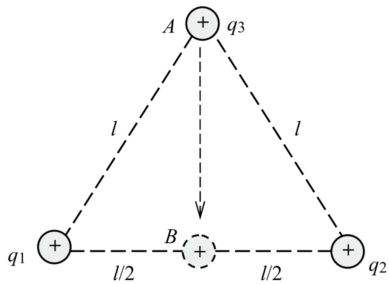


Fig. 6.18 Configuration of three charges, Exercise 6.3

Exercise 6.4 Figure 6.19 shows a region of uniform electric field $E = 2000\ \text{N C}^{-1}$. What is the potential difference between points A and B, A and C, and B and C?

(Answer: $V_{AB} = 0$, $V_{AC} = 800$ V, $V_{BC} = 800$ V)

Exercise 6.5 Two conducting plates are separated by a distance of 30 cm in a vacuum and are at a potential difference of 1.0 kV. An oxygen ion, with charge +2, starts

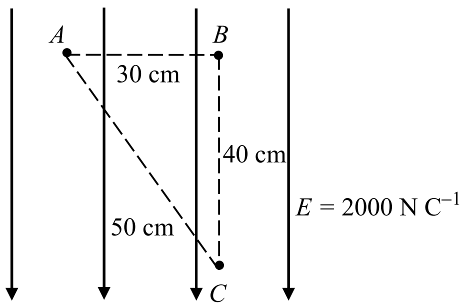


Fig. 6.19 Points A, B, and C in a uniform electric field, Exercise 6.4

from rest on the surface of the positive plate and accelerates to the negative plate. What is the final kinetic energy of the oxygen ion? If the separation distance of the plates is reduced to 10 cm and the potential difference is kept the same, will the final kinetic energy of the oxygen ion change?

(Answer: 2.0 keV or 3.2×10–16 J, no)

Chapter 7
Capacitance and Dielectric

Abstract This chapter solves problems on capacitance, equivalent capacitance of capacitors in series and parallel, and energy in charged capacitors. Also discussed is the effect of inserting dielectric material between the plates of a capacitor. Both analytical solutions and computer calculations by wxMaxima of the problems are presented.

7.1 Basic Concepts and Formulae

(1) A capacitor consists of two conductors with the same charges but opposite in signs, separated by a small gap. The two conductors have potential difference of V. Capacitance C is the magnitude of charge Q of either conductor divided by the magnitude of the potential difference V,

$$C = \frac{Q}{V}. \tag{7.1}$$

SI unit for capacitance is coulomb per volt (C V^{-1}) or farad (F):

$$1\,\mathrm{F} = 1\,\mathrm{C\,V^{-1}}. \tag{7.2}$$

(2) For conductors that are separated by vacuum or air, the capacitance is as follows:

(a) Parallel plate capacitor: Area of plate A and separation between plates d:

$$C = \frac{\varepsilon_0 A}{d}. \tag{7.3}$$

(b) Cylindrical capacitor: Length l, inner radius a, and outer radius b:

$$C = \frac{l}{2k\ln\left(\frac{b}{a}\right)} = \frac{2\pi\varepsilon_0 l}{\ln\left(\frac{b}{a}\right)}. \tag{7.4}$$

W. M. S. Wan Hassan et al., *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-43165-4_7

(c) Spherical capacitor: Inner radius a and outer radius b:

$$C = \frac{ab}{k(b-a)} = \frac{4\pi\varepsilon_0 ab}{b-a}. \tag{7.5}$$

(d) Isolated charged sphere of radius R:

$$C = 4\pi\varepsilon_0 R. \tag{7.6}$$

where

$$k = \frac{1}{4\pi\varepsilon_0} = 8.9876 \times 10^9\ \mathrm{N\,m^2\,C^{-2}} \approx 9 \times 10^9\ \mathrm{N\,m^2\,C^{-2}},$$

is Coulomb's constant, and

$$\varepsilon_0 = 8.8542 \times 10^{-12}\ \mathrm{C^2\,N^{-1}\,m^{-2}},$$

is the is the permittivity of free space.

(3) For capacitors connected in parallel, the potential difference across each capacitor is the same. Equivalent capacitance C_p is

$$C_p = C_1 + C_2 + C_3 + \dots \tag{7.7}$$

For For capacitors connected in series, the charge on each capacitor is the same. Equivalent capacitance C_s can be calculated by the following formula:

$$\frac{1}{C_s} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots \tag{7.8}$$

(4) Work is done in charging a capacitor because charge is moved from a conductor at low potential to another conductor at high potential. The work done to charge a capacitor C to charge Q is the electric potential energy U in the capacitor,

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV = \frac{1}{2}CV^2. \tag{7.9}$$

(5) When a dielectric material is inserted between the plates of a capacitor, the capacitance increases by a factor of K,

$$C = KC_0, \tag{7.10}$$

where C_0 is the capacitance without the dielectric material, and K is dielectric constant of the dielectric material.

7.2 Problems and Solutions

Problem 7.1

(a) A capacitor is charged until its charge is 40 μC and the potential difference across it is 100 V. What is the capacitance? What is the electric potential energy stored?
(b) At other times the charge in the capacitor is 80 μC. What is the capacitance?

Solution

(a) The capacitance is, Eq. (7.1),

$$C = \frac{Q}{V} = \frac{40 \times 10^{-6}\ \text{C}}{100\ \text{V}} = 4.0 \times 10^{-7}\ \text{F} = 0.40\ \mu\text{F}.$$

The electric potential energy in the capacitor is, Eq. (7.9),

$$U = \frac{1}{2}QV = \frac{1}{2}(40 \times 10^{-6}\ \text{C})(100\ \text{V}) = 2.0 \times 10^{-3}\ \text{J}.$$

This energy is the same in value as the work done to charge the capacitor.

◆ wxMaxima codes:

```
(%i3) fpprintprec:5; Q:40e-6; V:100;
(fpprintprec)     5
(Q)    4.0*10^-5
(V)    100
(%i4) C: Q/V;
(C)    4.0*10^-7
(%i5) U: 1/2*Q*V;
(U)    0.002
```

Comments on the codes:

(%i3) Set the floating point print precision to 5, and assign charge $Q = 40 \times 10^{-6}$ C and potential difference $V = 100$ V.

(%i4) Calculate capacitance $C = Q/V$.

(%i5) Calculate electric energy $U = ½\, QV$.

(b) Capacitance of a capacitor is a fixed quantity. So the capacitance is 0.40 μF. If the charge increases, the potential difference across the capacitor increases as well,

$$C = \frac{Q_1}{V_1} = \frac{Q_2}{V_2} = \text{fixed value}.$$

If the charge is 80 μC, the potential difference across the capacitor is 200 V, so that the capacitance is fixed at 0.40 μF.

Problem 7.2 The space between plates of a parallel plate capacitor is filled with an insulator with dielectric constant of 100. The area of the plate is 0.50 cm^2.

(a) The capacitance is 40 pF. What is the thickness of the insulator?
(b) Dielectric strength of the insulator is 6.0×10^6 V m^{-1}. What are the maximum charge, energy, and energy density of the capacitor?

Solution

(a) The capacitance of a parallel plate capacitor filled with material of dielectric constant K is, Eq. (7.3) and (7.10),

$$C = \frac{K\varepsilon_0 A}{d},$$

where A is the area of one of the plates and d is the distance between plates. Thus, inserting given numerical values gives

$$(40 \times 10^{-12}\ \mathrm{F}) = \frac{(100)(8.85 \times 10^{-12}\ \mathrm{C^2\ N^{-1}\ m^{-2}})(0.50 \times 10^{-4}\ \mathrm{m^2})}{d}.$$

The thickness of the insulator is

$$d = 1.1 \times 10^{-3}\ \mathrm{m}.$$

(b) Dielectric strength of an insulator is the maximum electric field the insulator material is able to sustain before its insulating properties begin to fail. The maximum charge of the capacitor is, (Eqs. 7.1 and 6.3),

$$\begin{aligned} q_{max} &= CV_{max} = CE_{max}d \\ &= (40 \times 10^{-12}\ \mathrm{F})(6.0 \times 10^6\ \mathrm{V\ m^{-1}})(1.1 \times 10^{-3}\ \mathrm{m}) \\ &= 2.7 \times 10^{-7}\ \mathrm{C}. \end{aligned}$$

Here, $V_{max} = E_{max}d$, that is, the maximum potential difference equals the maximum electric field times the distance, as in Eq. (6.3), Chap. 6.

The maximum energy stored in the capacitor is, Eq. (7.9),

$$U_{max} = \frac{1}{2}\frac{q_{max}^2}{C} = \frac{1}{2}\frac{(2.7 \times 10^{-7}\ \mathrm{C})^2}{40 \times 10^{-12}\ \mathrm{F}} = 8.8 \times 10^{-4}\ \mathrm{J}.$$

The maximum energy density of the capacitor is

$$u = \frac{\text{energy}}{\text{volume}} = \frac{U_{max}}{Ad} = 1.6 \times 10^4\ \mathrm{J\ m^{-3}}.$$

◆ wxMaxima codes:

```
(%i6) fpprintprec:5; ratprint:false; K:100; epsilon0:8.85e-12; A:0.5e-4;
C:40e-12;
(fpprintprec)     5
(ratprint)  false
(K)   100
(epsilon0)  8.85*10^-12
(A)   5.0*10^-5
(C)   4.0*10^-11
(%i8) solve(C = K*epsilon0*A/d, d)$ float(%);
(%o8) [d=0.0011062]
(%i9) d: rhs(%[1]);
(d)   0.0011062
(%i10) qmax: C*6e6*d;
(qmax) 2.655*10^-7
(%i11) Umax: 1/2*qmax^2/C;
(Umax) 8.8113*10^-4
(%i12) u: Umax/(A*d);
(u) 1.593*10^4
```

Comments on the codes:

(%i6) Set the floating point print precision to 5 and internal rational number print to false, and assign values of K, ε_0, A, and C.

(%i7) Solve $C = K\varepsilon_0 A/d$ for d.

(%i9) Assign the value of the solution to d.

(%i10) Calculate the maximum charge of the capacitor q_{max}.

(%i11) Calculate the maximum energy of the capacitor U_{max}.

(%i12) Calculate energy density of the capacitor u.

Problem 7.3 A capacitor consists of two coaxial thin cylindrical shells of radii a and b, ($a < b$). The length of both cylinders is l, and $l >> a$, $l >> b$.

(a) Determine the capacitance.
(b) The space between the shells is filled with a material with dielectric constant K. What is the new capacitance?

Solution

(a) Fig. 7.1 shows the cross section of the capacitor. The inner and outer radii of the thin cylindrical shells are a and b, respectively. The charge is Q and charge per unit length is $\lambda = Q/l$. An imaginary cylinder of radius r is also shown. The electric field at the surface of this cylinder is $E = 2k\lambda/r$; see Table 5.1(h), Chap. 5.

The potential difference between outer and inner cylinders is, Eq. (6.2),

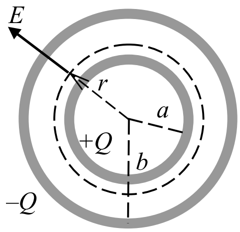


Fig. 7.1 A two coaxial cylindrical shell capacitor, Problem 7.3

$$
\begin{aligned}
V_b - V_a &= -\int_a^b E \cdot ds = -\int_a^b \frac{2k\lambda}{r} dr = [-2k\lambda \ln r]_a^b \\
&= 2k\lambda \ln\left(\frac{a}{b}\right) \\
&= 2k\frac{Q}{l} \ln\left(\frac{a}{b}\right).
\end{aligned}
$$

The value of $V_b - V_a$ is negative. The capacitance is, Eq. (7.1),

$$C = -\frac{Q}{V_b - V_a} = \frac{l}{2k\ln\left(\frac{b}{a}\right)} = \frac{2\pi\varepsilon_0 l}{\ln\left(\frac{b}{a}\right)}.$$

This is as in Eq. (7.4).

♦ wxMaxima codes:

```
(%i2) k: 1/(4*%pi*epsilon0); lambda: Q/l;
(k)   1/(4*%pi*epsilon0)
(lambda)    Q/l
(%i6) assume(a>0); assume(b>0); assume((b-a)>0); potential_difference: -
integrate(2*k*lambda/r,r,a,b);
(%o3) [a>0]
(%o4) [b>0]
(%o5) [b>a]
(potential_difference)  -(Q*(log(b)-log(a)))/(2*%pi*epsilon0*l)
(%i7) C: -Q/potential_difference;
(C)   (2*%pi*epsilon0*l)/(log(b)-log(a))
```

Comments on the codes:

(%i2) Assign $k = 1/(4\pi\varepsilon_0)$ and $\lambda = Q/l$.

(%i6) Calculate the potential difference $-\int_a^b \frac{2k\lambda}{r} dr$.

(%i7) Calculate the capacitance C.

(b) When the space between the cylindrical shells is filled with the material of dielectric constant K, the capacitance increases to, Eq. (7.10),

$$C_D = KC = \frac{Kl}{2k\ln\left(\frac{b}{a}\right)} = \frac{2\pi\,\varepsilon_0 Kl}{\ln\left(\frac{b}{a}\right)}.$$

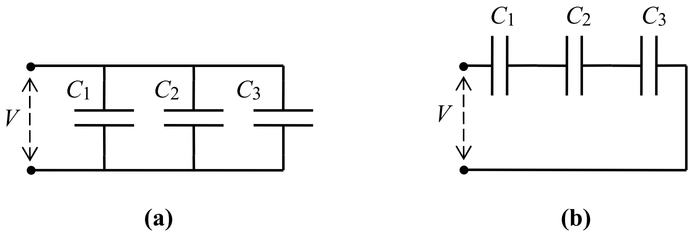


Fig. 7.2 Capacitors in parallel (a) and in series (b), Problem 7.4

Problem 7.4 Determine the equivalent capacitance for systems in Fig. 7.2.

Solution

(a) Fig. 7.2(a) is three capacitors connected in parallel. Potential difference across each capacitor is the same, that is, V. The charges in the capacitors are

$$q_1 = C_1 V \quad \text{in capacitor}\, C_1,$$
$$q_2 = C_2 V \quad \text{in capacitor}\, C_2,$$
$$q_3 = C_3 V \quad \text{in capacitor}\, C_3.$$

The total charge in the system is

$$q = q_1 + q_2 + q_3 = V(C_1 + C_2 + C_3).$$

The equivalent capacitance for the capacitors connected in parallel is

$$C_{equivalent} = \frac{\text{charge}}{\text{potential difference}} = \frac{V(C_1 + C_2 + C_3)}{V}$$
$$= C_1 + C_2 + C_3.$$

Figure 7.3 shows this equivalence.

(b) Fig. 7.2(b) is three capacitors connected in series. The sum of potential difference across each capacitor is the potential difference across all capacitors in series V,

$$V = V_1 + V_2 + V_3.$$

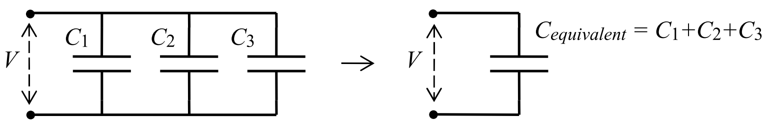


Fig. 7.3 Equivalent capacitance of capacitors in parallel, Problem 7.4

The charge in each capacitor is the same, let's say q. So we write

$$V = \frac{q}{C_1} + \frac{q}{C_2} + \frac{q}{C_3}.$$

The equivalent capacitance for the capacitors connected in series is

$$C_{equivalent} = \frac{\text{charge}}{\text{potential difference}} = \frac{q}{\frac{q}{C_1} + \frac{q}{C_2} + \frac{q}{C_3}}$$
$$= \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}}.$$

We can also write

$$\frac{1}{C_{equivalent}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}.$$

Figure 7.4 shows this equivalence.

Problem 7.5

(a) A parallel plate capacitor has a plate area of A and separation distance of plates of d. What is the capacitance?
(b) Two pieces of dielectric materials with dielectric constants K_1 and K_2, each of area A and thickness $d/2$, are inserted in the capacitor. What is the new capacitance?

Solution

(a) Fig. 7.5(a) shows the parallel plate capacitor.

The electric field in the region between the plates is, see Table 5.1(f), Chap. 5,

$$E = \frac{V}{d} = \frac{\sigma}{\varepsilon_0} = \frac{Q}{A\varepsilon_0},$$

where Q is charge in the capacitor and σ is surface charge density. The capacitance is

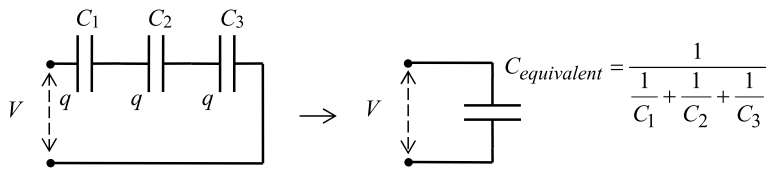


Fig. 7.4 Equivalent capacitance of capacitors in series, Problem 7.4

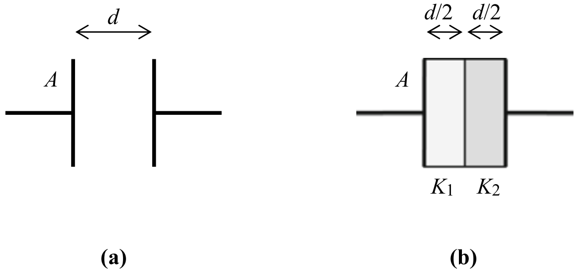


Fig. 7.5 Parallel plate capacitor (a), and parallel plate capacitor with two dielectric materials (b), Problem 7.5

$$C = \frac{Q}{V} = \frac{\varepsilon_0 A}{d}.$$

(b) When the two dielectric materials are inserted, Fig. 7.5(b), two capacitors connected in series are created. For the first dielectric material, the capacitance is

$$C_1 = \frac{K_1 \varepsilon_0 A}{d/2}.$$

For the second dielectric material, the capacitance is

$$C_2 = \frac{K_2 \varepsilon_0 A}{d/2}.$$

Effective capacitance C is calculated as follows, Eq. (7.8),

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2},$$
$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{\frac{K_1 \varepsilon_0 A}{d/2} \cdot \frac{K_2 \varepsilon_0 A}{d/2}}{\frac{K_1 \varepsilon_0 A}{d/2} + \frac{K_2 \varepsilon_0 A}{d/2}} = \frac{2\varepsilon_0 A}{d}\left(\frac{K_1 K_2}{K_1 + K_2}\right).$$

◆ wxMaxima codes:

```
(%i1) C1: K1*epsilon0*A/(d/2);
(C1)  (2*A*K1*epsilon0)/d
(%i2) C2: K2*epsilon0*A/(d/2);
(C2)  (2*A*K2*epsilon0)/d
(%i3) solve(1/C = 1/C1 + 1/C2, C);
(%o3) [C=(2*A*K1*K2*epsilon0)/((K2+K1)*d)]
```

Comments on the codes:

(%i1), (%i2) Assign C_1 and C_2.

(%i3) Solve $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$ for C.

Problem 7.6 Figure 7.6 shows two capacitors $C_1 = 6.0$ pF and $C_2 = 2.0$ pF in parallel at potential difference of 120 V. Determine

(a) charge in each capacitor
(b) equivalent capacitance
(c) energy stored in both capacitors.

Solution

(a) The potential difference across each capacitor is the same, that is, $V = 120$ V. The charges in the capacitors are

$$q_1 = C_1 V = (6.0 \times 10^{-12}\ \text{F})(120\ \text{V}) = 7.2 \times 10^{-10}\ \text{C},$$

$$q_2 = C_2 V = (2.0 \times 10^{-12}\ \text{F})(120\ \text{V}) = 2.4 \times 10^{-10}\ \text{C}.$$

(b) The total charge in the system is

$$q = q_1 + q_2 = 9.6 \times 10^{-10}\ \text{C}.$$

Therefore, the equivalent capacitance is

$$C_{equivalent} = \frac{q}{V} = \frac{9.6 \times 10^{-10}\ \text{C}}{120\ \text{V}} = 8.0 \times 10^{-12}\ \text{F}.$$

Alternative solution: For capacitors connected in parallel, the equivalent capacitance is, Eq. (7.7),

$$C_{equivalent} = C_1 + C_2 + ...$$

So we get

$$C_{equivalent} = C_1 + C_2 = (6.0 + 2.0) \times 10^{-12}\ \text{F} = 8.0 \times 10^{-12}\ \text{F}.$$

(c) Electrical energy stored in a capacitor is $\frac{1}{2}qV = \frac{1}{2}CV^2 = \frac{1}{2}\frac{q^2}{C}$, Eq. (7.9). For the first capacitor, energy stored is

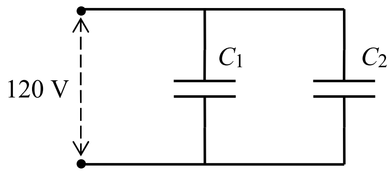


Fig. 7.6 Two capacitors in parallel at 120 V, Problem 7.6

$$U_1 = \frac{1}{2}q_1 V = \frac{1}{2}(7.2 \times 10^{-10} \text{ C})(120 \text{ V}) = 4.32 \times 10^{-8} \text{ J}.$$

For the second capacitor, energy stored is

$$U_2 = \frac{1}{2}q_2 V = \frac{1}{2}(2.4 \times 10^{-10} \text{ C})(120 \text{ V}) = 1.44 \times 10^{-8} \text{ J}.$$

The total energy stored is

$$U = U_1 + U_2 = 4.32 \times 10^{-8} \text{ J} + 1.44 \times 10^{-8} \text{ J} = 5.8 \times 10^{-8} \text{ J}.$$

Alternative solution: Equivalent capacitance has been calculated in part (b). Thus, the energy stored is, Eq. (7.9),

$$U = \frac{1}{2}C_{equivalent}V^2 = \frac{1}{2}(8.0 \times 10^{-12} \text{ F})(120 \text{ V})^2 = 5.8 \times 10^{-8} \text{ J}.$$

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; C1:6e-12; C2:2e-12; V:120;
(fpprintprec)     5
(C1)   6.0*10^-12
(C2)   2.0*10^-12
(V)    120
(%i8) q1: C1*V; q2: C2*V; q: q1+q2; Cequivalent: q/V;
(q1)   7.2*10^-10
(q2)   2.4*10^-10
(q)    9.6*10^-10
(Cequivalent) 8.0*10^-12
(%i11) U1: 0.5*q1*V; U2: 0.5*q2*V; U: U1+U2;
(U1)   4.32*10^-8
(U2)   1.44*10^-8
(U)    5.76*10^-8
(%i12) U: 0.5*Cequivalent*V^2;
(U)    5.76*10^-8
```

Comments on the codes:

(%i4) Set the floating point precision to 5 and assign values of C_1, C_2, and V.

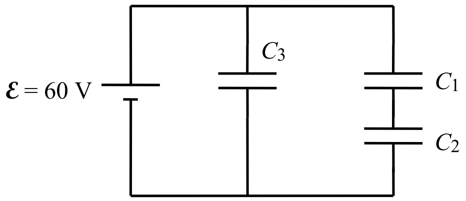
(%i8) Calculate q_1, q_2, q, and $C_{equivalent}$.

(%i11) Calculate U_1, U_2, and U.

(%i12) Another calculation of U.

Problem 7.7 Three capacitors $C_1 = 6.0\ \mu\text{F}$, $C_2 = 12\ \mu\text{F}$, and $C_3 = 16\ \mu\text{F}$ are connected to a battery with emf $\varepsilon = 60$ V as in Fig. 7.7. Determine

Fig. 7.7 Three capacitors connected to a battery, Problem 7.7



(a) equivalent capacitance
(b) charge in equivalent capacitance
(c) charge in each capacitor
(d) voltage across each capacitor
(e) energy to charge each capacitor
(f) total energy in the three capacitors.

Solution

(a) From Fig. 7.7, capacitors C_1 and C_2 are in series. Let the equivalent capacitance of both capacitors be $C_{equivalent1}$. This gives, Eq. (7.8),

$$\frac{1}{C_{equivalent1}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{6\ \mu\text{F}} + \frac{1}{12\ \mu\text{F}} = \frac{1}{4\ \mu\text{F}}$$
$$C_{equivalent1} = 4.0\ \mu\text{F}.$$

Next, capacitors C_3 and $C_{equivalent1}$ are in parallel. Let their equivalent capacitance be $C_{equivalent}$. We have, Eq. (7.7),

$$C_{equivalent} = C_3 + C_{equivalent1} = 16\ \mu\text{F} + 4\ \mu\text{F} = 20\ \mu\text{F}.$$

(b) The charge in the equivalent capacitor is

$$Q_{equivalent} = C_{equivalent}\,\mathcal{E} = (20 \times 10^{-6}\ \text{F})(60\ \text{V}) = 1.2 \times 10^{-3}\ \text{C}.$$

(c) The voltage across C_3 is $\varepsilon = 60$ V. So, the charge in C_3 is

$$Q_3 = C_3\mathcal{E} = (16 \times 10^{-6}\ \text{F})(60\ \text{V}) = 9.6 \times 10^{-4}\ \text{C}.$$

The charges in C_1 and C_2 are the same because C_1 and C_2 are in series, and the charge equals the charge in equivalent capacitance of both capacitors. Because the voltage across $C_{equivalent1}$ is $\varepsilon = 60$ V, the charge is

$$Q_{equivalent1} = Q_1 = Q_2 = C_{equivalent1}\,\mathcal{E} = (4.0 \times 10^{-6}\ \text{F})(60\ \text{V})$$
$$= 2.4 \times 10^{-4}\ \text{C}$$

(d) The voltage across capacitor C_1 is

$$V_1 = \frac{Q_1}{C_1} = \frac{2.4 \times 10^{-4}\ \text{C}}{6.0 \times 10^{-6}\ \text{F}} = 40\ \text{V}.$$

The voltage across capacitor C_2 is

$$V_2 = \frac{Q_2}{C_2} = \frac{2.4 \times 10^{-4}\ \text{C}}{12 \times 10^{-6}\ \text{F}} = 20\ \text{V}.$$

The voltage across capacitor C_3 is

$$V_3 = \mathcal{E} = 60\ \text{V}.$$

(e) The electric energy needed to charge capacitor C_1 is, Eq. (7.9),

$$U_1 = \frac{1}{2} Q_1 V_1 = \frac{1}{2}(2.4 \times 10^{-4}\ \text{C})(40\ \text{V}) = 4.8 \times 10^{-3}\ \text{J}.$$

The electric energy needed to charge capacitor C_2 is

$$U_2 = \frac{1}{2} Q_2 V_2 = \frac{1}{2}(2.4 \times 10^{-4}\ \text{C})(20\ \text{V}) = 2.4 \times 10^{-3}\ \text{J}.$$

The electric energy needed to charge capacitor C_3 is

$$U_3 = \frac{1}{2} Q_3 V_3 = \frac{1}{2}(9.6 \times 10^{-4}\ \text{C})(60\ \text{V}) = 2.9 \times 10^{-2}\ \text{J}.$$

(f) The electric energy stored in the three capacitors is the sum of energies in (e),

$$U = U_1 + U_2 + U_3 = 3.6 \times 10^{-2}\ \text{J}.$$

Alternative calculation: The energy obtained in part (f) is the energy stored in the equivalent capacitor $C_{equivalent}$ that has a charge of $Q_{equivalent}$ stored in it. Thus, using part (b) we have, Eq. (7.9),

$$U = \frac{1}{2} Q_{equivalent}\, \mathcal{E} = \frac{1}{2}(1.2 \times 10^{-3}\ \text{C})(60\ \text{V}) = 3.6 \times 10^{-2}\ \text{J}.$$

◆ wxMaxima codes:

```
(%i6) fpprintprec:5; ratprint:false; Epsilon:60; C1:6e-6; C2:12e-6; C3:16e-
6;
(fpprintprec)     5
(ratprint)  false
(Epsilon)   60
(C1)  6.0*10^-6
(C2)  1.2*10^-5
(C3)  1.6*10^-5
(%i8) solve(1/Cequivalent1 = 1/C1 + 1/C2, Cequivalent1)$ float(%);
(%o8) [Cequivalent1=4.0*10^-6]
(%i9) Cequivalent1: rhs(%[1]);
(Cequivalent1) 4.0*10^-6
(%i10) Cequivalent: C3 + Cequivalent1;
(Cequivalent) 2.0*10^-5
(%i11) Qequivalent: Cequivalent*Epsilon;
(Qequivalent) 0.0012
(%i12) Q3: C3*Epsilon;
(Q3) 9.6*10^-4
(%i15) Qequivalent1:Cequivalent1*Epsilon; Q1:Qequivalent1; Q2:Qequivalent1;
(Qequivalent1) 2.4*10^-4
(Q1) 2.4*10^-4
(Q2) 2.4*10^-4
(%i18) V1:Q1/C1; V2:Q2/C2; V3:Epsilon;
(V1)  40.0
(V2)  20.0
(V3)  60
(%i22) U1:1/2*Q1*V1; U2:1/2*Q2*V2; U3:1/2*Q3*V3; U:U1+U2+U3;
(U1)  0.0048
(U2)  0.0024
(U3)  0.0288
(U)   0.036
(%i23) U:1/2*Qequivalent*Epsilon;
(U)   0.036
```

Comments on the codes:

(%i6) Set the floating point print precision to 5 and internal rational number print to false, and assign values of ε, C_1, C_2, and C_3.

(%i8) Solve $\frac{1}{C_{equivalent1}} = \frac{1}{C_1} + \frac{1}{C_2}$ for $C_{equivalent1}$.

(%i9) Assign value of $C_{equivalent1}$.

(%i10), (%i11), (%i12) Calculate $C_{equivalent}$, $Q_{equivalent}$, and Q_3.

(%i15) Assign $Q_{equivalent1}$, Q_1, and Q_2.

(%i18) Calculate V_1, V_2, and V_3.

(%i22) Calculate U_1, U_2, U_3, and U.

(%i23) Another calculation of U.

Problem 7.8

(a) A parallel plate capacitor consists of two metal plates each of area 0.06 m^2. The separation distance between plates is 1.0 cm and the capacitor is connected to a voltage source of 100 V dc. Determine the capacitance, charge on the plate, electric field between plates, and energy in the capacitor.

(b) The DC voltage source is disconnected from the capacitor. A material with dielectric constant 4.2 is inserted between the plates of the capacitor. Determine the new voltage, capacitance, charge, electric field, and energy stored.

Solution

(a) Capacitance of the parallel plate capacitor is, Eq. (7.3),

$$C_0 = \frac{\varepsilon_0 A}{d} = \left(8.85 \times 10^{-12} \frac{\text{C}^2}{\text{N m}^2}\right)\left(\frac{0.06 \text{ m}^2}{0.01 \text{ m}}\right) = 5.3 \times 10^{-11} \text{ F}.$$

The charge in the capacitor is, Eq. (7.1),

$$Q_0 = C_0 V_0 = (5.3 \times 10^{-11} \text{ F})(100 \text{ V}) = 5.3 \times 10^{-9} \text{ C}.$$

The electric field between plates is, Eq. (3.3),

$$E_0 = \frac{V_0}{d} = \frac{100 \text{ V}}{0.01 \text{ m}} = 1.0 \times 10^4 \text{ V m}^{-1}.$$

The electrical energy stored in the capacitor is, Eq. (7.9),

$$U_0 = \frac{1}{2} Q_0 V_0 = \frac{1}{2}(5.3 \times 10^{-9} \text{ C})(100 \text{ V}) = 2.7 \times 10^{-7} \text{ J}.$$

(b) When the voltage source is disconnected, the charge on a plate is still the same as the charge calculated in (a), that is,

$$Q = Q_0 = 5.3 \times 10^{-9} \text{ C}.$$

But the capacitance increases to, Eq. (7.10),

$$C = K C_0 = (4.2)(5.3 \times 10^{-11} \text{ F}) = 2.2 \times 10^{-10} \text{ F}.$$

The voltage across the capacitor decreases to

$$V = \frac{Q}{C} = \frac{5.3 \times 10^{-9} \text{ C}}{2.2 \times 10^{-10} \text{ F}} = 24 \text{ V}.$$

The electric field decreases to

$$E = \frac{V}{d} = \frac{24 \text{ V}}{0.01 \text{ m}} = 2.4 \times 10^3 \text{ V m}^{-1}.$$

The electric energy stored in the capacitor decreases to

$$U = \frac{1}{2}QV = \frac{1}{2}(5.3 \times 10^{-9} \text{ C})(24 \text{ V}) = 6.3 \times 10^{-8} \text{ J}.$$

◆ wxMaxima codes:

```
(%i6) fpprintprec:5; epsilon0:8.85e-12; A:0.06; d:0.01; V0:100; K:4.2;
(fpprintprec)     5
(epsilon0)  8.85*10^-12
(A)   0.06
(d)   0.01
(V0)  100
(K)   4.2
(%i10) C0:epsilon0*A/d; Q0:C0*V0; E0:V0/d; U0:0.5*Q0*V0;
(C0)  5.31*10^-11
(Q0)  5.31*10^-9
(E0)  1.0*10^4
(U0)  2.655*10^-7
(%i15) Q:Q0; C:K*C0; V:Q/C; E:V/d; U:0.5*Q*V;
(Q)   5.31*10^-9
(C)   2.2302*10^-10
(V)   23.81
(E)   2381.0
(U)   6.3214*10^-8
```

Comments on the codes:

(%i6) Set the floating point print precision to 5 and assign values of ε_0, A, d, V_0, and K.

(%i10) Calculate C_0, Q_0, E_0, and U_0.

(%i15) Assign $Q = Q_0$ and calculate C, V, E, and U.

Problem 7.9 A capacitor $C_1 = 6.0$ μF is fully charged and the potential difference across it is $V_0 = 80$ V. The capacitor is then connected to an uncharged capacitor $C_2 = 12$ μF. Determine the charge, voltage, and energy of the capacitors in the initial and final situations.

Solution

Figure 7.8 shows the initial and final situations.

In the initial situation, the charge in capacitor C_1 is

$$Q_0 = C_1 V_0 = (6.0 \times 10^{-6} \text{ F})(80 \text{ V}) = 4.8 \times 10^{-4} \text{ C}.$$

The voltage across capacitor C_1 is

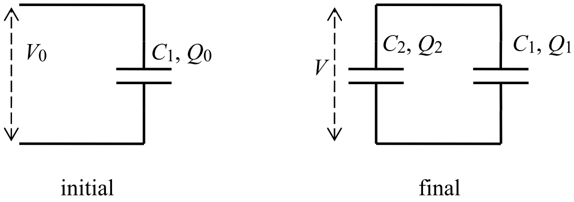


Fig. 7.8 Capacitor C_1 in initial, and capacitors C_1 and C_2 in final situations, Problem 7.9

$$V_0 = 80 \text{ V}.$$

The electrical energy in capacitor C_1 is, Eq. (7.9),

$$U_0 = \frac{1}{2} Q_0 V_0 = \frac{1}{2}(4.8 \times 10^{-4} \text{ C})(80 \text{ V}) = 1.9 \times 10^{-2} \text{ J}.$$

In the final situation, the charge Q_0 is distributed to capacitors C_1 and C_2. This means that

$$Q_0 = Q_1 + Q_2,$$

and

$$C_1 V_0 = C_1 V + C_2 V.$$

The voltage is

$$V = \frac{C_1}{C_1 + C_2} V_0 = \frac{6.0 \ \mu\text{F}}{6.0 \ \mu\text{F} + 6.0 \ \mu\text{F}} 80 \text{ V} = 27 \text{ V}.$$

The voltage across C_1 and C_2 is 27 V. The charges in C_1 and C_2 are

$$Q_1 = C_1 V = (6.0 \times 10^{-6} \text{ F})(27 \text{ V}) = 1.6 \times 10^{-4} \text{ C},$$

$$Q_2 = C_2 V = (12 \times 10^{-6} \text{ F})(27 \text{ V}) = 3.2 \times 10^{-4} \text{ C}.$$

The energy stored in capacitors C_1 and C_2 are

$$U_1 = \frac{1}{2} Q_1 V = \frac{1}{2}(1.6 \times 10^{-4} \text{ C})(27 \text{ V}) = 2.1 \times 10^{-3} \text{ J},$$

$$U_2 = \frac{1}{2} Q_2 V = \frac{1}{2}(3.2 \times 10^{-4} \text{ C})(27 \text{ V}) = 4.3 \times 10^{-3} \text{ J}.$$

The total energy stored in the two capacitors is

$$U = U_1 + U_2 = 6.4 \times 10^{-3}\ \text{J}.$$

The total energy can also be calculated as follows

$$\begin{aligned}
U &= U_1 + U_2 = \frac{1}{2}V(Q_1 + Q_2) = \frac{1}{2}V \cdot V(C_1 + C_2) \\
&= \frac{1}{2}\left(\frac{C_1}{C_1 + C_2}V_0\right)^2 (C_1 + C_2) = \frac{1}{2}\frac{C_1^2 V_0^2}{(C_1 + C_2)} \\
&= \frac{1}{2}\frac{(6.0 \times 10^{-6}\ \text{F})^2(80\ \text{V})^2}{(6.0 \times 10^{-6}\ \text{F} + 12 \times 10^{-6}\ \text{F})} \\
&= 6.4 \times 10^{-3}\ \text{J}.
\end{aligned}$$

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; C1:6e-6; V0:80; C2:12e-6;
(fpprintprec)     5
(C1)   6.0*10^-6
(V0)   80
(C2)   1.2*10^-5
(%i6) Q0:C1*V0; U0:0.5*Q0*V0;
(Q0)   4.8*10^-4
(U0)   0.0192
(%i9) V:C1*V0/(C1+C2); Q1:C1*V; Q2:C2*V;
(V)    26.667
(Q1)   1.6*10^-4
(Q2)   3.2*10^-4
(%i12) U1:0.5*Q1*V; U2:0.5*Q2*V; U:U1+U2;
(U1)   0.0021333
(U2)   0.0042667
(U)    0.0064
(%i13) U:1/2*C1^2*V0^2/(C1+C2);
(U)    0.0064
```

Comments on the codes:

(%i4) Set the floating point print precision to 5 and assign values of C_1, V_0, and C_2.

(%i6) Calculate Q_0 and W_0.

(%i9) Calculate V, Q_1, and Q_2.

(%i12) Calculate U_1, U_2, and U.

(%i13) Another calculation of U.

Problem 7.10 A parallel plate capacitor has a charge of q and a plate area of A.

(a) Show that force on one plate due to the charge on the other plate is

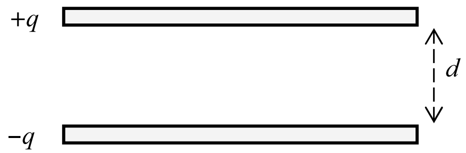


Fig. 7.9 A parallel plate capacitor, Problem 7.10

$$F = \frac{q^2}{2\varepsilon_0 A}.$$

(b) Calculate F, for a 2.0 pF capacitor with a plate area of 3.0 cm^2 and a potential difference of 100 V.

Solution

(a) Fig. 7.9 shows the parallel plate capacitor. Charge in the capacitor is q and separation between plates is d.

The electric energy in the parallel plate capacitor is, Eq. (7.9),

$$U = \frac{1}{2}\frac{q^2}{C} = \frac{1}{2}\frac{q^2 d}{\varepsilon_0 A},$$

where we have used the fact that for parallel plate capacitor the capacitance is $C = \varepsilon_0 A/d$, Eq. (7.3). The energy is the work to separate the plates from distance 0 to d. The work is

$$U = \int_0^d F\, dx,$$

where F is the attractive electric force between the plates and dx is the elementary displacement. The distance d is small and F does not vary much and is almost constant. Thus, the work is

$$U = F\int_0^d dx = Fd.$$

We equate the work done and the electric energy,

$$Fd = \frac{1}{2}\frac{q^2 d}{\varepsilon_0 A},$$

and calculate the attractive electric force between the plates to be

$$F = \frac{1}{2}\frac{q^2}{\varepsilon_0 A}.$$

(b) From part (a) and the given numerical values, the force F is

$$F = \frac{1}{2}\frac{q^2}{\varepsilon_0 A} = \frac{1}{2}\frac{C^2V^2}{\varepsilon_0 A} = \frac{1}{2}\frac{(2.0 \times 10^{-12}\ \mathrm{F})^2(100\ \mathrm{V})^2}{\left(8.85 \times 10^{-12}\frac{\mathrm{C}^2}{\mathrm{N\,m}^2}\right)(3.0 \times 10^{-4}\ \mathrm{m}^2)}$$
$$= 7.5 \times 10^{-6}\ \mathrm{N}.$$

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; C:2e-12; V:100; epsilon0:8.85e-12; A:3e-4;
(fpprintprec)     5
(C)   2.0*10^-12
(V)   100
(epsilon0)  8.85*10^-12
(A)   3.0*10^-4
(%i6) F: 0.5*C^2*V^2/(epsilon0*A);
(F)   7.533*10^-6
```

Comments on the codes:

(%i5) Set the floating point print precision to 5 and assign values of C, V, ε_0, and A.

(%i6) Calculate the force between the plates F.

Problem 7.11 Show that the energy of a conducting sphere of radius R and a charge of Q in vacuum is

$$U = \frac{1}{2}\frac{kQ^2}{R}.$$

Solution

Figure 7.10 shows the conducting sphere of radius R and a charge of Q.

The electric potential at the surface of the sphere is, Table 6.1(d),

$$V = \frac{kQ}{R}.$$

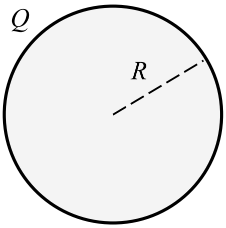


Fig. 7.10 A conducting charged sphere, Problem 7.11

The capacitance of the sphere is, Eq. (7.1),

$$C = \frac{Q}{V} = \frac{Q}{kQ/R} = \frac{R}{k}.$$

Thus, the electric energy of the sphere is, Eq. (7.9),

$$U = \frac{1}{2}CV^2 = \frac{1}{2}\frac{R}{k}\left(\frac{kQ}{R}\right)^2$$
$$= \frac{1}{2}\frac{kQ^2}{R}.$$

7.3 Summary

- A capacitor stores electrical charge and electrical energy. The capacitance of a capacitor is the amount of charge it stores per unit potential difference between the plates, that is, $C = Q/V$. The SI unit of capacitance is farad (F), 1 F = 1 C/V.
- The electric potential energy U in the capacitor is

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV = \frac{1}{2}CV^2.$$

- When a material with dielectric constant of K is inserted between plates of a capacitor, the capacitance increases by a factor of K.

7.4 Exercises

Exercise 7.1 What is the equivalent capacitance of the capacitors in Fig. 7.11? (Answer: $C_{equivalent} = \frac{(C_1+C_2)C_3}{C_1+C_2+C_3}$)

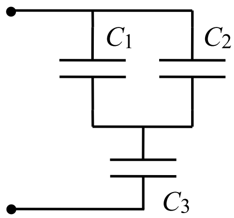


Fig. 7.11 Three capacitors, Exercise 7.1

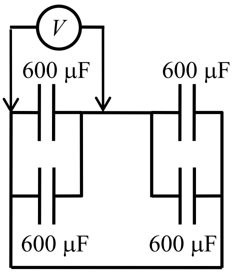


Fig. 7.12 Four charged capacitors, Exercise 7.3

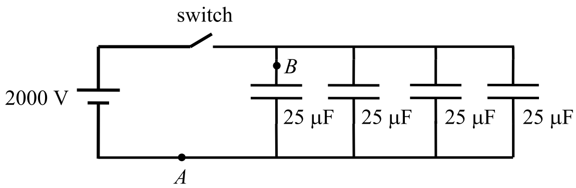


Fig. 7.13 Four capacitors in parallel connected to a 2000 V source, Exercise 7.4

Exercise 7.2 A parallel plate capacitor has a plate area of 0.20 m^2 and a plate separation of 0.10 mm. The charge on the capacitor is 4.1×10^{-6} C. What are the electric field between the plates and the potential difference across the plates?

(Answer: $E = 2.3 \times 10^6$ V m^{-1}, $V = 230$ V)

Exercise 7.3 Figure 7.12 shows four charged 600 μF capacitors. The reading of the voltmeter is 1200 V. Find the charge and the energy stored in each capacitor.

(Answer: 0.72 C, 430 J)

Exercise 7.4 Four 25 μF capacitors are connected to a 2000 V dc source and a switch as in Fig. 7.13. How many coulombs of charge pass through points A and B after the switch is closed?

(Answer: 0.20 C, 0.050 C)

Exercise 7.5 A mica sheet 0.10 mm thick and of dielectric constant 6.0 filled the space of a parallel plate capacitor. The plate area is 0.20 m^2. Calculate the capacitance.

(Answer: 1.1×10^{-7} F)

Chapter 8
Current and Resistance

Abstract This chapter solves problems on electric current, current density, resistance, resistivity, and Ohm's law. Problems with an increase in resistance due to a rise in temperature, resistance temperature coefficient, and dissipation of electrical power by a resistor are also solved. Solutions are by analysis and computer calculation.

8.1 Basic Concepts and Formulae

(1) Electric current I in a conductor is defined as

$$I = \frac{dQ}{dt}, \tag{8.1}$$

where dQ is the charge that passes across the cross-section of the conductor in time dt. SI unit for electric current is ampere (A).

$$1\,\text{A} = 1\,\text{C s}^{-1}. \tag{8.2}$$

(2) Electric current in a conductor is the movement of charge carriers

$$I = nqv_d A, \tag{8.3}$$

where n is the density of charge carriers (number of charge carriers per unit volume), q is the charge of the carrier, v_d is the drift velocity of the carrier, and A is the cross-sectional area of the conductor.

(3) Current density J in a conductor is current per unit area,

$$J = \frac{I}{A} = nqv_d. \tag{8.4}$$

(4) Current density in a conductor is proportional to the electric field

W. M. S. Wan Hassan et al., *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-43165-4_8

$$J = \sigma E, \tag{8.5}$$

where σ is a constant called conductivity of the material. Reciprocal of conductivity σ is resistivity ρ

$$\rho = \frac{1}{\sigma} = \frac{E}{J}. \tag{8.6}$$

A material obeys Ohm's law if its conductivity does not depend on the electric field.

(5) Resistance R of a conductor is the potential difference V across the conductor divided by the current flow I in it

$$R = \frac{V}{I}. \tag{8.7}$$

If the resistance does not depend on the applied voltage, Ohm's law is obeyed. Ohm's law is written as

$$V = IR. \tag{8.8}$$

(6) A conductor with a cross-sectional area of A and a length of l has a resistance of

$$R = \rho \frac{l}{A} = \frac{l}{\sigma A}, \tag{8.9}$$

where ρ is the resistivity and σ is the conductivity of the conductor. SI unit for resistance is ohm (Ω) or volt per ampere ($V\ A^{-1}$).

$$1\ \Omega = 1\ \mathrm{V\ A}^{-1} \tag{8.10}$$

(7) Resistivity, ρ, of a conductor changes with temperature, T, approximately according to the equation,

$$\rho = \rho_0[1 + \alpha(T - T_0)], \tag{8.11}$$

where α is the resistance temperature coefficient and ρ_0 is resistivity at temperature T_0. Therefore, the resistance, R, of a conductor changes with temperature, T, in the same way

$$R = R_0[1 + \alpha(T - T_0)], \tag{8.12}$$

where α is the resistance temperature coefficient and R_0 is the resistance at temperature T_0.

(8) If the potential difference across a resistor is V and the current through the resistor is I, then the power or rate of energy given to the resistor is

$$P = IV = I^2R = \frac{V^2}{R}, \tag{8.13}$$

where R is the resistance. The power is equal to the rate of electric energy dissipated by the resistor as heat energy.

8.2 Problems and Solutions

Problem 8.1 An electric current of 5.0 A flows in a copper wire of a cross-sectional area of 3.0×10^{-6} m^2. The number of free electrons in copper is 8.5×10^{28} electrons per m^3. Calculate the drift velocity of electron in copper wire.

Solution

The relationship between electric current I, number of charge carriers per unit volume n, drift velocity of charge carrier v_d, charge of the carrier q, and cross-sectional area of the conductor A, is given by, Eq. (8.3)

$$I = nqv_dA.$$

The drift velocity v_d of an electron in copper is

$$v_d = \frac{I}{nqA} = \frac{5.0 \text{ A}}{(8.5 \times 10^{28} \text{ m}^{-3})(1.6 \times 10^{-19} \text{ C})(3.0 \times 10^{-6} \text{ m}^2)}$$
$$= 1.2 \times 10^{-4} \text{ m s}^{-1}.$$

The conventional current flow is opposite in direction to the flow of electrons.

◆ wxMaxima codes:

```
(%i5) fpprintprec:5;  A:3e-6; I:5; n:8.5e28; q:1.6e-19;
(fpprintprec)     5
(A)   3.0*10^-6
(I)   5
(n)   8.5*10^28
(q)   1.6*10^-19
(%i6) vd: I/(n*q*A);
(vd)  1.2255*10^-4
```

Comments on the codes:

(%i5) Set floating point print precision to 5 and assign values of A, I, n, and q.
(%i6) Calculate the drift velocity v_d.

Problem 8.2

(a) The resistivity of copper at 25°C is 1.7×10^{-8} Ω m. What is the electric field in a copper wire so that the electric current density is 3.0×10^7 A m^{-2}?
(b) At 25°C, what is the resistance of a copper wire of length 2.0 m and a radius of 1.0 mm?
(c) What is the wire resistance at 65°C?

The Resistance temperature coefficient of copper is 3.9×10^{-3} K^{-1}.

Solution

(a) Resistivity ρ is defined as electric field E in the conductor divided by current density J, Eq. (8.6),

$$\rho = \frac{E}{J}.$$

The electric field E in the copper wire is

$$E = \rho J = (1.7 \times 10^{-8}\ \Omega\ \text{m})(3.0 \times 10^7\ \text{A m}^{-2}) = 0.51\ \text{V m}^{-1}.$$

(b) Resistance of the copper wire at 25°C is, Eq. (8.9),

$$R_{25} = \frac{\rho l}{A} = \frac{(1.7 \times 10^{-8}\ \Omega\ \text{m})(2.0\ \text{m})}{\pi (1.0 \times 10^{-3}\ \text{m})^2} = 1.1 \times 10^{-2}\ \Omega,$$

where l is the length and A is the cross sectional area of the copper wire.
(c) Resistance of the copper wire at 65°C is, Eq. (8.12),

$$\begin{aligned} R_{65} &= R_{25}[1 + \alpha(65°\text{C} - 25°\text{C})] \\ &= 1.1 \times 10^{-2}\ \Omega\ [1 + 3.9 \times 10^{-3}\ \text{K}^{-1}(65 - 25)\ \text{K}] \\ &= 1.3 \times 10^{-2}\ \Omega. \end{aligned}$$

◆ wxMaxima codes:

```
(%i6) fpprintprec:5; rho:1.7e-8; J:3e7; l:2; r:1e-3; alpha:3.9e-3;
(fpprintprec)     5
(rho) 1.7*10^-8
(J)   3.0*10^7
(l)   2
(r)   0.001
(alpha) 0.0039
(%i7) E: rho*J;
(E)   0.51
(%i8) R25: rho*l/float(%pi*r^2);
(R25) 0.010823
(%i9) R65: R25*(1+alpha*(65-25));
(R65) 0.012511
```

Comments on the codes:

(%i6)	Set floating point print precision to 5 and assign values of ρ, J, l, r, and α.
(%i7), (%i8), (%i9)	Calculate E, R_{25}, and R_{65}.

Problem 8.3 Figure 8.1 shows a dry cell with internal resistance r and a resistor R in a circuit. There is a voltmeter V and an ammeter A to measure the potential difference and current, and a switch S to open or close the circuit. When S is opened the voltmeter reading is 1.6 V, and when S is closed the reading is 1.4 V. Determine,

(a) emf of the cell
(b) internal resistance of the cell r
(c) reading of the ammeter when S is closed.

Solution

(a) The emf of a cell is the open circuit voltage, that is, the voltage of the cell which is not connected to any load in a circuit. So, the emf of the cell is

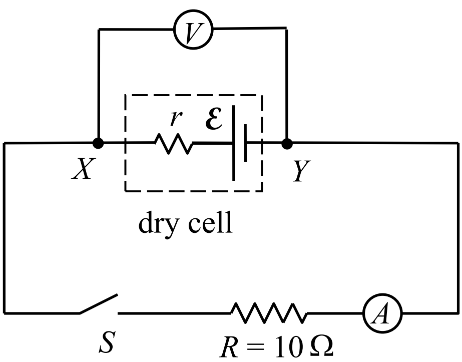


Fig. 8.1 A circuit consisting of a dry cell with internal resistance, a resistor, and a switch. A voltmeter and an ammeter are also connected, Problem 8.3

$$\mathcal{E} = 1.6\,\text{V}.$$

(b) When the switch is closed, the potential difference across the cell is

$$V_{XY} = \mathcal{E} - Ir,$$

where r is the internal resistance of the cell and I is the current. Current I is

$$I = \frac{\Sigma\mathcal{E}}{\Sigma R} = \frac{\mathcal{E}}{r+R}.$$

The potential difference across the cell is

$$V_{XY} = \mathcal{E} - \left(\frac{\mathcal{E}}{r+R}\right)r,$$

and the internal resistance of the cell is

$$r = R\left(\frac{\mathcal{E}}{V_{XY}} - 1\right) = 10\ \Omega\left(\frac{1.6\ \text{V}}{1.4\ \text{V}} - 1\right) = 1.4\ \Omega.$$

(c) Current flow when the switch is closed is

$$I = \frac{\mathcal{E} - V_{XY}}{r} = \frac{1.6\ \text{V} - 1.4\ \text{V}}{1.4\ \Omega} = 0.14\ \text{A}.$$

The current is measured by the ammeter, thus, the ammeter reading is 0.14 A.

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; ratprint:false; emf:1.6; VXY:1.4; R:10;
(fpprintprec)     5
(ratprint)  false
(emf) 1.6
(VXY) 1.4
(R)   10
(%i7) solve([VXY=emf-I*r, I=emf/(r+R)], [r,I])$ float(%);
(%o7) [[r=1.4286,I=0.14]]
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of $\mathcal{E}$, V_{XY}, and R.

(%i7) Solve $V_{XY} = \mathcal{E} - Ir$ and $I = \frac{\mathcal{E}}{r+R}$ for r and I.

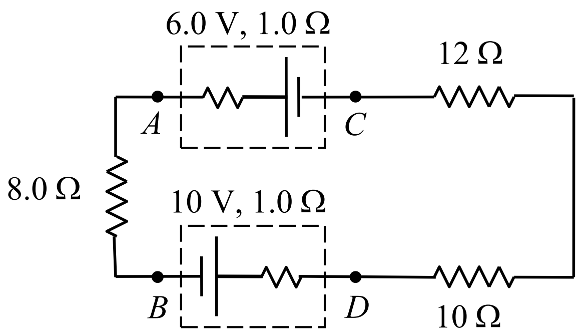


Fig. 8.2 Determining electric current and potential difference, Problem 8.4

Problem 8.4 For the circuit in Fig. 8.2, determine,

(a) current in the 8.0 Ω resistor
(b) potential difference V_{AB} and V_{AC}.

Solution

(a) To get the current, calculate the total emf and resistance, and divide the two. Counter clockwise from point C, the total emf is

$$\Sigma\mathcal{E} = 6.0\,\text{V} + 10\,\text{V} = 16\,\text{V}.$$

Total resistance is,

$$\Sigma R = (1.0 + 8.0 + 1.0 + 10 + 12)\ \Omega = 32\ \Omega.$$

The current in the circuit is

$$I = \frac{\Sigma\mathcal{E}}{\Sigma R} = \frac{16\ \text{V}}{32\ \Omega} = 0.50\ \text{A}.$$

The direction of the current is counter clockwise. The current flow in the circuit is the current in the 8.0 Ω resistor, that is, 0.50 A.

(b) Counter clockwise from point A to B, we have

$$V_A - (0.50\ \text{A})(8.0\ \Omega) = V_B,$$

where V_A is the electric potential at point A, $-(0.50\ \text{A})(8.0\ \Omega)$ is the potential drop by the 8.0 Ω resistor, and V_B is the potential at point B. The potential difference between points A and B is

$$V_{AB} = V_A - V_B = (0.50\ \text{A})(8.0\ \Omega) = 4.0\ \text{V}.$$

Counter clockwise from point A to C, we have

$$V_A - (0.50\ \text{A})(8.0\ \Omega) - (0.50\ \text{A})(1.0\ \Omega) + 10\ \text{V} \\ -(0.50\ \text{A})(10\ \Omega) - (0.50\ \text{A})(12\ \Omega) = V_C,$$

where V_A is the electric potential at point A, $-(0.50$ A)(8.0 Ω) is the potential drop by the 8.0 Ω resistor, $-(0.50$ A)(1.0 Ω) is the potential drop by the 1.0 Ω resistor of the bottom cell, 10 V is the emf of the bottom cell, $-(0.50$ A)(10 Ω) is the potential drop by the 10 Ω resistor, $-(0.50$ A)(12 Ω) is the potential drop by the 12 Ω resistor, and V_C is the potential at point C. Thus, the potential difference between points A and C is

$$\begin{aligned} V_{AC} &= V_A - V_C \\ &= (0.50\ \text{A})(8.0\ \Omega) + (0.50\ \text{A})(1.0\ \Omega) - 10\ \text{V} \\ &\quad + (0.50\ \text{A})(10\ \Omega) + (0.50\ \text{A})(12\ \Omega) \\ &= 5.5\ \text{V}. \end{aligned}$$

◆ wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)     5
(ratprint)  false
(%i6) emfsum: 6+10; Rsum: 1+8+1+10+12; I: emfsum/Rsum; float(%);
(emfsum)    16
(Rsum)      32
(I)   1/2
(%o6) 0.5
(%i8) solve([VA-I*8=VB, VAB=VA-VB], [VAB,VA])$ float(%);
(%o8) [[VAB=4.0,VA=VB+4.0]]
(%i10) solve([VA-I*8-I*1+10-I*10-I*12=VC, VAC=VA-VC], [VAC,VA])$ float(%);
(%o10) [[VAC=5.5,VA=0.5*(2.0*VC+11.0)]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i6) Calculate $\Sigma\,\mathcal{E}, \Sigma\,R$, and I.
(%i8) Solve $V_A - I(8) = V_B$ and $V_{AB} = V_A - V_B$ for V_{AB} and V_A.
(%i10) Solve $V_A - I(8) - I(1) + 10 - I(10) - I(12) = V_C$ and $V_{AC} = V_A - V_C$ for V_{AC} and V_A.

Alternative calculation: To calculate the potential difference V_{AC}, it is easier to move from point C to A counter clockwise. We get

$$V_C + 6.0\ \text{V} - (0.50\ \text{A})(1.0\ \Omega) = V_A,$$

where V_C is the potential of point C, 6.0 V is the emf of the top cell, $-(0.50$ A)(1.0 Ω) is the potential drop by the 1.0 Ω internal resistance, and V_A is the potential of point A. This means that

$$V_{AC} = V_A - V_C = 5.5 \text{ V}.$$

◆ wxMaxima codes:

```
(%i2) fpprintprec: 5; ratprint:false;
(fpprintprec)     5
(ratprint)  false
(%i6) emfsum: 6+10; Rsum: 1+8+1+10+12; I: emfsum/Rsum; float(%);
(emfsum)    16
(Rsum)      32
(I)   1/2
(%o6) 0.5
(%i8) solve([VC+6-I*1=VA, VAC=VA-VC], [VAC, VA])$ float(%);
(%o8) [[VAC=5.5,VA=0.5*(2.0*VC+11.0)]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.

(%i6) Calculate $\Sigma\,\mathcal{E}, \Sigma\,R$, and I.

(%i8) Solve $V_C + 6 - I(1) = V_A$ and $V_{AC} = V_A - V_C$ for V_{AC} and V_A.

Problem 8.5 The filament of a light bulb has a resistance of 10 Ω at 20°C. The resistance temperature coefficient of the filament is 4.5×10^{-3} K^{-1}. Calculate:

(a) resistance of the filament at 520°C.

(b) reduction of current through the filament at 520°C compared to at 20°C by assuming that the potential difference across the filament is constant at 110 V.

(c) dissipated power by the filament at 520°C.

Solution

(a) The resistance of the filament at 520°C is, Eq. (8.12),

$$\begin{aligned} R_{520} &= R_{20}[1 + \alpha(520°\text{C} - 20°\text{C})] \\ &= 10\ \Omega\,[1 + 4.5 \times 10^{-3}\ \text{K}^{-1}(520 - 20)\ \text{K}] \\ &= 32\ \Omega. \end{aligned}$$

(b) The electric current at 20°C is, Eq. (8.8),

$$I_{20} = \frac{V}{R_{20}} = \frac{110\ \text{V}}{10\ \Omega} = 11\ \text{A}.$$

The electric current at 520°C is,

$$I_{520} = \frac{V}{R_{520}} = \frac{110\ \text{V}}{32.5\ \Omega} = 3.4\ \text{A}.$$

Reduction in current is 11 A – 3.4 A = 7.6 A.

(c) The dissipated power by the filament at 520°C is, Eq. (8.13),

$$P = VI_{520} = \frac{V^2}{R_{520}} = \frac{(110\ \text{V})^2}{32\ \Omega} = 370\ \text{W}.$$

◆ wxMaxima codes:

```
(%i5) fpprintprec: 5; R20:10; alpha:4.5e-3; V:110;
(fpprintprec)     5
(ratprint)  false
(R20) 10
(alpha) 0.0045
(V)   110
(%i6) R520: R20*(1+alpha*(520-20));
(R520)32.5
(%i7) I20: V/R20;
(I20) 11
(%i8) I520: V/R520;
(I520)3.3846
(%i9) current_reduction: I20-I520;
(current_reduction) 7.6154
(%i10)P: V^2/R520;
(P) 372.31
```

Comments on the codes:

(%i5)	Set floating point print precision to 5, assign values of R_{20}, α, and V.
(%i6), (%i7), (%i8)	Calculate R_{520}, I_{20}, and I_{520}.
(%i9)	Calculate current reduction.
(%i10)	Calculate dissipated power.

Problem 8.6 The resistance of a nichrome wire is $R = 72\ \Omega$. What are the rates of electric energy dissipated in these two situations?

(a) Potential difference of 120 V is set across the wire.

(b) The wire is cut into one half and a potential difference of 120 V is set across each one half of the wire.

Solution

(a) Figure 8.3a shows the nichrome wire and the potential difference set across its length

The rate of electrical energy dissipated by the wire is, Eq. (8.13),

$$P_a = \frac{V^2}{R} = \frac{(120\ \text{V})^2}{72\ \Omega} = 200\ \text{W}.$$

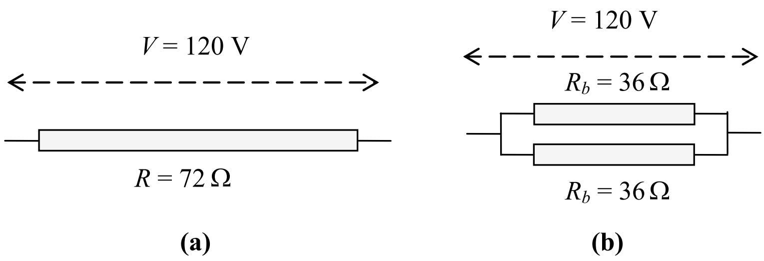


Fig. 8.3 Potential difference across **a** one piece and **b** two pieces of nichrome wire, Problem 8.6

(b) Figure 8.3b shows the two halves of the wire and the potential difference set on them. The wire was cut into two equal parts, the resistance of each part is $R_b = 36\ \Omega$. The rate of electric energy dissipated by one part is

$$P = \frac{V^2}{R_b} = \frac{(120\ \text{V})^2}{36\ \Omega} = 400\ \text{W}.$$

Therefore, the rate of energy dissipated by both parts is,

$$P_b = 2P = 800\ \text{W}.$$

◆ wxMaxima codes:

```
(%i2) V: 120; R: 72;
(V)   120
(R)   72
(%i3) Pa: V^2/R;
(Pa)  200
(%i4) Rb: 36;
(Rb)  36
(%i5) P: V^2/Rb;
(P)   400
(%i6) Pb: 2*P;
(Pb)  800
```

Comments on the codes:

(%i2)	Assign values of V and R.
(%i3,), (%i4), (%i5), (%i6)	Calculate P_a, R_b, P, and P_b.

Problem 8.7 Resistance of a mercury column at 15°C is 10 Ω. What is the resistance of the column at 30 and 0°C? Resistance temperature coefficient of mercury is $0.0072°\text{C}^{-1}$ at 0°C.

Solution

Using Eq. (8.12), $R = R_0[1 + \alpha(T - T_0)]$,resistances of mercury column at 15 and 30°C are written as,

$$R_{15} = R_0\ (1 + 15\,\alpha),$$

$$R_{30} = R_0\ (1 + 30\,\alpha),$$

where R_0 is the resistance of the column at 0°C and α is the resistance temperature coefficient. The two equations give

$$\frac{R_{30}}{R_{15}} = \frac{R_0(1 + 30\,\alpha)}{R_0(1 + 15\,\alpha)},$$

$$\begin{aligned} R_{30} &= R_{15}\left(\frac{1 + 30\,\alpha}{1 + 15\,\alpha}\right) = 10\ \Omega\left(\frac{1 + 30^\circ\mathrm{C}\ \ (0.0072\ ^\circ\mathrm{C}^{-1})}{1 + 15^\circ\mathrm{C}\ \ (0.0072\ ^\circ\mathrm{C}^{-1})}\right) \\ &= 11\ \Omega. \end{aligned}$$

The resistance of mercury column at 30°C is 11 Ω. The resistance of mercury column at 0°C is calculated as follows:

$$\begin{aligned} R_{15} &= R_0\ (1 + 15\,\alpha) \\ 10\ \Omega &= R_0[1 + 15^\circ\mathrm{C}\ (0.0072\ ^\circ\mathrm{C}^{-1})] \\ R_0 &= 9.0\ \Omega. \end{aligned}$$

◆ wxMaxima codes:

```
(%i4) fpprintprec: 5; ratprint: false; R15: 10; alpha: 0.0072;
(fpprintprec)     5
(ratprint)  false
(R15) 10
(alpha) 0.0072
(%i5) R30: R15*(1+30*alpha)/(1+15*alpha);
(R30) 10.975
(%i7) solve(R15=R0*(1+15*alpha), R0)$ float(%);
(%o7) [R0=9.0253]
```

Comments on the codes:

(%i4) Set floating point print precision to 5 and internal rational number print to false, assign values of R_{15} and α.
(%i5) Calculate R_{30}.
(%i7) Solve $R_{15} = R_0\ (1 + 15\,\alpha)$ for R_0.

Problem 8.8 A light bulb operates at 240 V, 100 W with the filament temperature at 2000°C. The resistance temperature coefficient of the filament is $5.00 \times 10^{-3}\ \mathrm{K}^{-1}$ at 15.0°C. What is the current in the light bulb when it is switched on at 15.0°C?

Solution

Using $P = VI = V^2/R$ and $V = IR$, Eqs. (8.13) and (8.8), respectively, the filament resistance R_{2000} and the current in the filament I_{2000} at 2000°C are,

$$R_{2000} = \frac{V^2}{P} = \frac{(240\ \mathrm{V})^2}{100\ \mathrm{W}} = 576\ \Omega,$$

$$I_{2000} = \frac{V}{R_{2000}} = \frac{240\ \mathrm{V}}{576\ \Omega} = 0.417\ \mathrm{A}.$$

The filament resistance R_{15} at 15.0°C is calculated as follows,

$$\begin{aligned} R_{2000} &= R_{15}\ [1 + \alpha(\theta_2 - \theta_1)] \\ 576\ \Omega &= R_{15}[1 + 5.00 \times 10^{-3}\ \mathrm{K}^{-1}\ (2000 - 15.0)\mathrm{K}] \\ R_{15} &= 52.7\ \Omega. \end{aligned}$$

Therefore, the current when the light bulb is switched on at 15.0°C is

$$I_{15} = \frac{V}{R_{15}} = \frac{240\ \mathrm{V}}{52.7\ \Omega} = 4.55\ \mathrm{A}.$$

◆ wxMaxima codes:

```
(%i6) fpprintprec: 5; V: 240; P: 100; theta2: 2000; theta1: 15; alpha: 5e-
3;
(fpprintprec)     5
(V)   240
(P)   100
(theta2)    2000
(theta1)    15
(alpha)     0.005
(%i7) R2000: float(V^2/P);
(R2000) 576.0
(%i8) I2000: float(V/R2000);
(I2000) 0.41667
(%i9) R15: R2000/(1 + alpha*(theta2-theta1));
(R15) 52.723
(%i10) I15: V/R15;
(I15) 4.5521
```

Comments on the codes:

(%i6) Set floating point print precision to 5, assign values of V, P, θ_2, θ_1, and α.

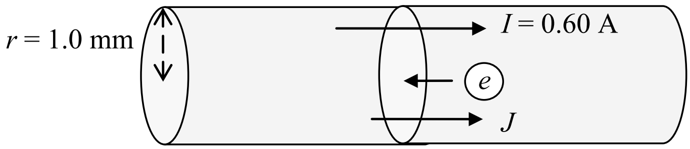


Fig. 8.4 Electron flow and electric current, Problem 8.9

(%i7), (%i8), (%i9), (%i10) Calculate R_{2000}, I_{2000}, R_{15}, and I_{15}.

Problem 8.9 A current of 0.60 A flows in a wire of radius 1.0 mm. Calculate the number of electrons crossing a cross-section of the wire per second. What is the current density?

Solution

Figure 8.4 shows electron flow and electric current in the wire. The Direction of electric current is opposite to that of electron flow.

Current of $I = 0.60$ A means a flow of 0.60 coulombs of charge per second. The number of electrons flowing per second is obtained by dividing the value by the magnitude of electron charge

$$n = \frac{0.60\ \text{C/s}}{1.6 \times 10^{-19}\ \text{C}} = 3.7 \times 10^{18}\ \text{electrons per second.}$$

The current density J is current per cross-sectional area, Eq. (8.4),

$$J = \frac{I}{A} = \frac{I}{\pi r^2} = \frac{0.60\ \text{A}}{\pi(1.0 \times 10^{-3}\ \text{m})^2} = 1.9 \times 10^5\ \text{A m}^{-2}.$$

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; I:0.6; r:1e-3; e:1.6e-19;
(fpprintprec)     5
(I)   0.6
(r)   0.001
(e)   1.6*10^-19
(%i5) n: I/e;
(n)   3.75*10^18
(%i6) J: I/float(%pi*r^2);
(J)   1.9099*10^5
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of I, r, and e.
(%i5), (%i6) Calculate n and J.

Problem 8.10 Electric current of 5.0 A flows in a copper wire of a cross-sectional area of 3.0 mm^2. Calculate the drift velocity of the electrons in the wire. Atomic mass of copper is 63.6 kg $kmol^{-1}$ and density of copper is 8920 kg m^{-3}.

Solution

The current density J is, Eq. (8.4),

$$J = \frac{I}{A} = \frac{5.0 \text{ A}}{3.0 \times 10^{-6} \text{ m}^2} = 1.7 \times 10^6 \text{ A m}^{-2}.$$

Current density J and drift velocity v_d of electron are related as follows, Eq. (8.4),

$$J = nev_d,$$

where n is the number of electrons per unit volume of copper, e is the electron charge, and v_d is the drift velocity of electrons. The number of copper atoms per unit volume is,

$$n = \frac{N_A \rho}{M} = \frac{(6.02 \times 10^{26} \text{ atom/kmol})(8920 \text{ kg/m}^3)}{63.6 \text{ kg/kmol}}$$
$$= 8.4 \times 10^{28} \text{ atoms m}^{-3},$$

where N_A is the Avogadro number, ρ is the density of copper, and M is the molar mass of copper. Assume that there is one free electron for each copper atom that creates the current. Then, the value is the number of charge carriers per unit volume, that is, $n = 8.4 \times 10^{28}$ free electrons per m^3. Therefore, the drift velocity of electrons in copper is

$$v_d = \frac{J}{ne} = \frac{1.7 \times 10^6 \text{ A m}^{-2}}{(8.4 \times 10^{28} \text{ electrons m}^{-3})(1.6 \times 10^{-19} \text{ C})}$$
$$= 1.2 \times 10^{-4} \text{ m s}^{-1}.$$

◆ wxMaxima codes:

```
(%i7) fpprintprec:5; I:5; A:3e-6; M:63.6; rho:8920; NA: 6.02e26; e:1.6e-19;
(fpprintprec)     5
(I)    5
(A)    3.0*10^-6
(M)    63.6
(rho)  8920
(NA)   6.02*10^26
(e)    1.6*10^-19
(%i8)  J: I/A;
(J)    1.6667*10^6
(%i9)  n: NA*rho/M;
(n)    8.4431*10^28
(%i10) vd: J/(n*e);
(vd)   1.2337*10^-4
```

Comments on the codes:

(%i7) Set floating point print precision to 5, assign values of I, A, M, ρ, N_A, and e.
(%i8), (%i9), (%i10) Calculate J, n, and v_d.

Problem 8.11 Calculate the resistance of 100 m of silver wire with a cross-sectional area of 0.30 mm^2. Resistivity of silver is 1.6×10^{-8} Ω m.

Solution

The resistance of the silver wire is, Eq. (8.9),

$$R = \frac{\rho l}{A} = \frac{(1.6 \times 10^{-8}\ \Omega\ \text{m})(100\ \text{m})}{0.30 \times 10^{-6}\ \text{m}^2} = 5.3\ \Omega.$$

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; l:100; A:0.3e-6; rho:1.6e-8;
(fpprintprec)     5
(l)   100
(A)   3.0*10^-7
(rho) 1.6*10^-8
(%i5) R: rho*l/A;
(R)   5.3333
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of l, A, and ρ.
(%i5) Calculate R.

8.3 Summary

- Electrical current is the rate at which charge flows

$$I = \frac{dq}{dt}.$$

- The unit for current is ampere (A)

$$1\ \text{A} = 1\ \text{C s}^{-1}.$$

- Resistance R of a cylinder of length l and cross-sectional area A and resistivity ρ is

$$R = \frac{\rho l}{A}.$$

- Temperature affects resistivity ρ and resistance R of a material

$$\rho = \rho_0(1 + \alpha(T - T_0)),$$

$$R = R_0(1 + \alpha(T - T_0)),$$

where ρ_0, R_0, and T_0 are original resistivity, resistance, and temperature, respectively, and α is the temperature coefficient of resistivity.
- Ohm's law gives the relationship among current I, voltage V, and resistance R in a simple circuit as

$$V = IR.$$

- Electric power is the rate at which the electric energy is consumed by a load or supplied to a load. Power dissipated by a resistor is

$$P = IV = I^2R = V^2/R.$$

8.4 Exercises

Exercise 8.1 A current of 6.0 A is maintained in a wire for 50 s. At this time, how much charge and how many electrons flow through the wire?
(Answer: 300 C, 1.9×10^{21} electrons)

Exercise 8.2 A current of 2.5 A flows in a metal rod of diameter 0.20 cm and length 1.5 m when the potential difference between the rod ends is 40 V, Fig. 8.5. Calculate

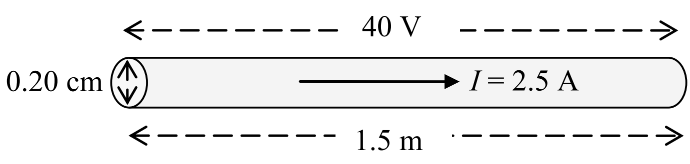


Fig. 8.5 Electric current in a metal rod, Exercise 8.2

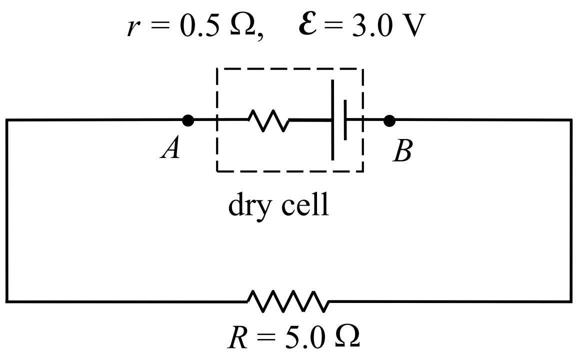


Fig. 8.6 A circuit consisting of a dry cell with internal resistance and a resistor, Exercise 8.4

(a) current density
(b) electric field in the rod
(c) resistivity of the metal

(Answer: (a) $J = 8.0 \times 10^5$ A m^{-2} (b) $E = 27$ V m^{-1} (c) $\rho = 3.4 \times 10^{-5}$ Ω m)

Exercise 8.3 A copper wire is 10 m long and 0.25 mm in diameter. Resistivity of copper is 1.7×10^{-8} Ω m. Calculate the resistance of the wire.
(Answer: 3.5 Ω)

Exercise 8.4 A dry cell with an emf of 3.0 V and an internal resistance of 0.5 Ω is connected to a 5.0 Ω resistor, Fig. 8.6. What is the potential difference between points A and B?
(Answer: 2.7 V)

Exercise 8.5 The resistance of a metal wire at 20°C is 1.64 Ω and at 150°C is 2.41 Ω. Find the resistance of the wire at 0°C and the temperature coefficient of resistance.
(Answer: $R_0 = 1.52$ Ω, $\alpha = 3.89 \times 10^{-3}$°C^{-1})

Chapter 9
Direct Current Circuit

Abstract This chapter solves problems on direct current circuits by applying Kirchhoff's rules. The rules are (1) the sum of the currents into any junction is zero and (2) the sum of potential differences across each element around a closed loop is zero. Problems to determine the equivalent resistance of resistors in series and in parallel and to determine current and charge in direct current *RC* circuits are also tackled. Solutions are by analytical means and computer calculation.

9.1 Basic Concepts and Formulae

(1) Electromotive force (emf) of a battery is the voltage across its terminals when the current is zero. The emf is the open circuit voltage of a battery.

(2) Equivalent resistance R_{series} of two or more resistors connected in series is

$$R_{series} = R_1 + R_2 + R_3 + \dots \tag{1}$$

Equivalent resistance $R_{parallel}$ of two or more resistors connected in parallel is given by

$$\frac{1}{R_{parallel}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \dots \tag{2}$$

(3) Electric circuits can be analyzed by Kirchhoff's rules that say:

(a) the sum of currents into a junction is the sum of currents out of the junction.
(b) the sum of potential differences across every element of a closed loop is zero.

(4) If a resistor is tracked in the direction of current, the change in potential across the resistor is $-IR$, that is, there is a voltage drop. If a resistor is tracked in the opposite direction to the current, the change in potential is IR, that is, there

W. M. S. Wan Hassan et al., *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-43165-4_9

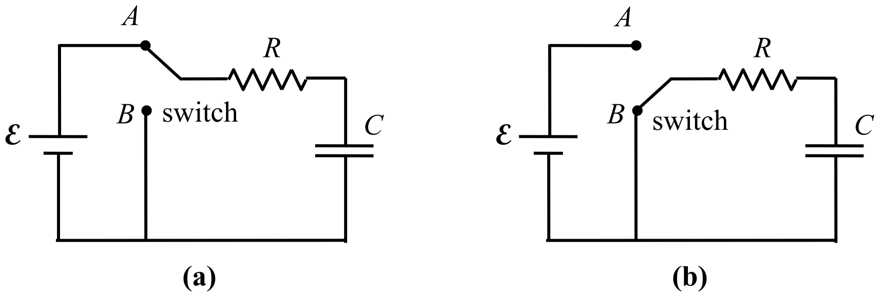


Fig. 9.1 A direct current *RC* circuit, **a** charging, **b** discharging

is a voltage rise. If an emf source is tracked in the emf direction (negative to positive), the change in potential is $\mathcal{E}$. If an emf source is tracked in the opposite direction to the emf direction (positive to negative), the change in potential is $-\mathcal{E}$.

(5) A capacitor C is connected to a resistor R and a battery with emf $\mathcal{E}$, as shown in Figure 9.1a. This is called an RC circuit. The current I in the circuit and the charge Q in the capacitor vary with time as

$$I(t) = \frac{\mathcal{E}}{R} \cdot e^{-\frac{t}{RC}} = I_{max} \cdot e^{-t/\tau}, \tag{3}$$

$$Q(t) = C\,\mathcal{E} \cdot (1 - e^{-\frac{t}{RC}}) = Q_{max} \cdot (1 - e^{-t/\tau}), \tag{4}$$

where $I_{max} = \mathcal{E}/R$ is the maximum current, $Q_{max} = C\mathcal{E}$ is the maximum charge of the capacitor, and $\tau = RC$ is the time constant of the circuit.

(6) When the capacitor is discharged, Figure 9.1b, the charge in the capacitor Q and current in the circuit I change with time as

$$Q(t) = Q_0 \cdot e^{-\frac{t}{RC}} = Q_0 \cdot e^{-t/\tau}, \tag{5}$$

$$I(t) = I_0 \cdot e^{-\frac{t}{RC}} = \frac{Q_0}{RC} \cdot e^{-t/\tau}, \tag{6}$$

where $I_0 = Q_0/(RC)$ is the initial current in the circuit and Q_0 the is initial charge in the capacitor.

9.2 Problems and Solutions

Problem 9.1 For the circuit in Fig. 9.2a, determine the current in each resistor.

Solution

There are two cells that drive the currents. We assumed two counter clockwise loops A and B and currents I_1, I_2, and I_3 at junction C, Fig. 9.2b. Applying Kirchhoff's rule, at junction C

$$I_2 = I_1 + I_3, \tag{1}$$

that is, current going in I_2 equals currents going out $I_1 + I_3$.

For loop A, we have

$$\mathcal{E}_1 - R_1 I_1 + R_3 I_3 = 0,$$

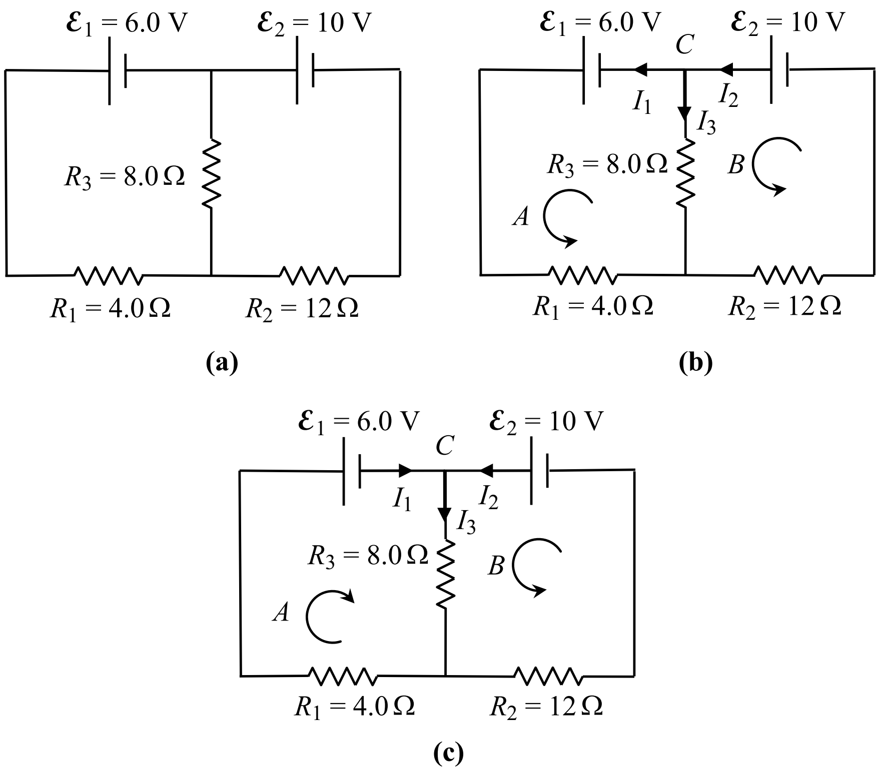


Fig. 9.2 Determining currents in a circuit, **a** the circuit, **b** setting currents and loops for Kirchhoff's rule analysis, **c** alternative Kirchhoff's rule analysis, Problem 9.1

$$6.0\text{ V} - (4.0\ \Omega)I_1 + (8.0\ \Omega)I_3 = 0. \tag{2}$$

Here, starting at the 6.0 V cell, we have a 6.0 V potential rise by the cell, $-(4.0\ \Omega)I_1$ potential drop by the 4.0 Ω resistor, and $+(8.0\ \Omega)I_3$ potential rise by the 8.0 Ω resistor.

For loop B, we have

$$\mathcal{E}_2 - R_3 I_3 - R_2 I_2 = 0,$$

$$10\text{ V} - (8.0\ \Omega)I_3 - (12\ \Omega)I_2 = 0. \tag{3}$$

That is, starting from the 10 V cell, we have a 10 V potential rise by the cell, $-(8.0\ \Omega)I_3$ potential drop by the 8.0 Ω resistor, and another $-(12\ \Omega)I_2$ potential drop by the 12 Ω resistor.

Solving Eqs. (1), (2), and (3), gives currents in resistors R_1, R_2, and R_3 as

$$I_1 = 1\frac{3}{22}\text{A} = 1.1\text{ A},\ I_2 = \frac{21}{22}\text{A} = 0.95\text{ A},\ \text{and}\ I_3 = -\frac{2}{11}\text{A} = -0.18\text{ A}.$$

The solutions say that the directions of I_1 and I_2 are the same as the ones assumed in Fig. 9.2(b), while the direction of I_3 is the opposite, and hence the negative sign in the current.

The current is 1.1 A from right to left in the 4.0 Ω resistor, 0.95 A from left to right in the 12 Ω resistor, and 0.18 A from bottom to top in the 8.0 Ω resistor.

◆ wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)     5
(ratprint)  false
(%i4) solve([I2=I1+I3, 6-4*I1+8*I3=0, 10-8*I3-12*I2=0], [I1,I2,I3])$
float(%);
(%o4) [[I1=1.1364,I2=0.95455,I3=-0.18182]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Solve Eqs. (1), (2), and (3) for I_1, I_2, and I_3.

What if you assume different current directions and loops? Will the currents be the same? Fig. 9.2(c) shows an example. Applying Kirchhoff's rules at junction C,

$$I_3 = I_1 + I_2. \tag{4}$$

For loop A,

$$-R_3I_3 - R_1I_1 - \mathcal{E}_1 = 0,$$

$$-(8.0\ \Omega)I_3 - (4.0\ \Omega)I_1 - 6.0\ \text{V} = 0, \tag{5}$$

For loop B,

$$\mathcal{E}_2 - R_3I_3 - R_2I_2 = 0,$$

$$10\ \text{V} - (8.0\ \Omega)I_3 - (12\ \Omega)I_2 = 0. \tag{6}$$

Solving Eqs. (4), (5), and (6) gives currents in resistors R_1, R_2, and R_3 as

$$I_1 = -1\frac{3}{22}\text{A} = -1.1\ \text{A}, \quad I_2 = \frac{21}{22}\text{A} = 0.95\ \text{A},\ \text{and}\ I_3 = -\frac{2}{11}\text{A} = -0.18\ \text{A}.$$

The solutions say that the direction of I_2 is the same as the one assumed in Fig. 9.2c, while those of I_1 and I_3 are the opposite. These results are in physics terms the same as those of Fig. 9.2b.

◆ wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)     5
(ratprint)  false
(%i4) solve([I3=I1+I2, -8*I3-4*I1-6=0, 10-8*I3-12*I2=0], [I1,I2,I3])$
float(%);
(%o4) [[I1=-1.1364,I2=0.95455,I3=-0.18182]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.

(%i4) Solve Eqs. (4), (5), and (6) for I_1, I_2, and I_3.

Problem 9.2 For Fig. 9.3, determine,

(a) the equivalent resistance between points X and Y.
(b) the potential difference between points X and A if the current through 8.0 Ω resistor is 0.50 A.

Solution

(a) We calculate equivalent resistances in stages, and at each stage, substitute the equivalent resistance into the circuit until the final equivalent resistance R_e is obtained, Fig. 9.4.

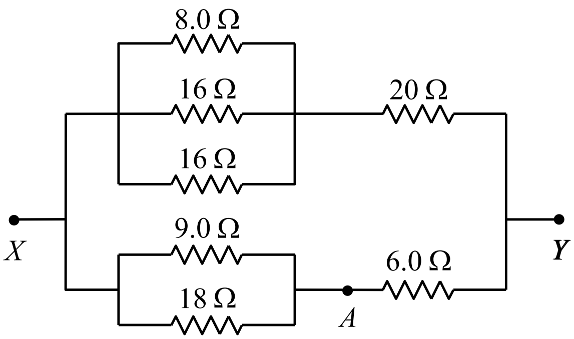


Fig. 9.3 A network of resistors, Problem 9.2

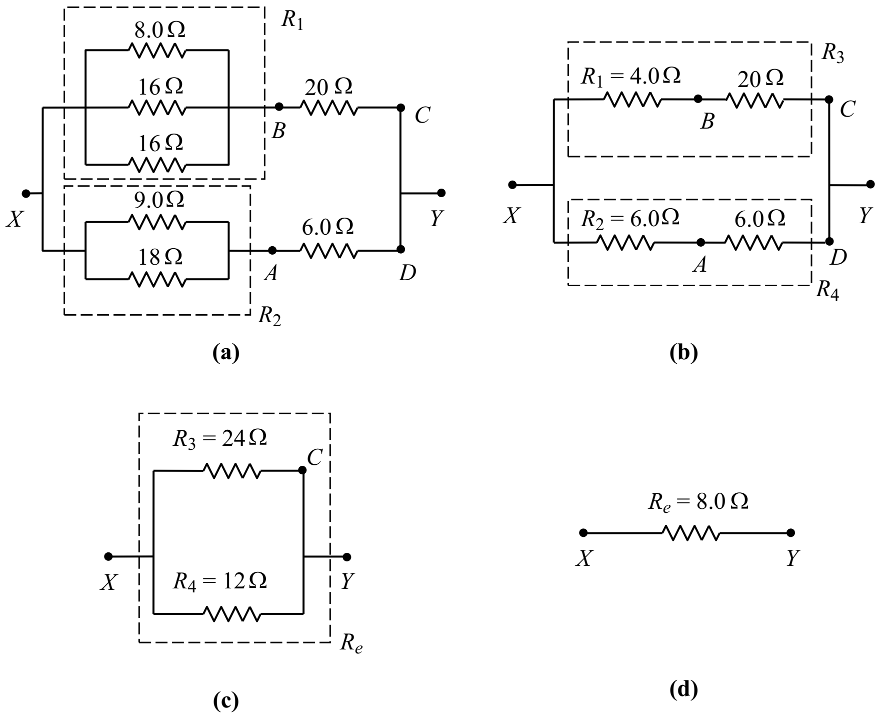


Fig. 9.4 Determining equivalent resistance, Problem 9.2

In Fig. 9.4a, the 8.0, 16, and 16 Ω resistors are in parallel. Let their equivalent resistance be R_1 with value

$$R_1 = \frac{1}{\frac{1}{8.0} + \frac{1}{16} + \frac{1}{16}} \Omega = 4.0\ \Omega.$$

The 9.0 and 18 Ω resistors are in parallel. Let their equivalent resistance be R_2

$$R_2 = \frac{1}{\frac{1}{9.0} + \frac{1}{18}}\,\Omega = 6.0\ \Omega.$$

In Fig. 9.4b, R_1 and the 20 Ω resistors are in series. Let their equivalent resistance be R_3

$$R_3 = 4.0\ \Omega + 20\ \Omega = 24\ \Omega.$$

Similarly, R_2 and the 6.0 Ω resistors are in series. Their equivalent resistance is R_4

$$R_4 = 6.0\ \Omega + 6.0\ \Omega = 12\ \Omega.$$

In Fig. 9.4c and d, the equivalent resistance between points X and Y is

$$R_e = \frac{1}{\frac{1}{R_3} + \frac{1}{R_4}} = \frac{1}{\frac{1}{24} + \frac{1}{12}}\,\Omega = 8.0\ \Omega.$$

(b) In Fig. 9.4a, with the info that current through 8.0 Ω resistor is 0.50 A, using $V = IR$, then

$$V_{XB} = (8.0\ \Omega)(0.50\ \text{A}) = 4.0\ \text{V}.$$

Therefore, the current through the three resistors in parallel is

$$I_{XB} = 0.50\ \text{A} + \frac{4.0\ \text{V}}{16\ \Omega} + \frac{4.0\ \text{V}}{16\ \Omega} = 1.0\ \text{A},$$

where we have used $I = V/R$ to calculate the second and third currents. Because $I_{BC} = I_{XB} = 1.0$ A, applying $V = IR$, we write

$$V_{BC} = (20\ \Omega)I_{BC} = (20\ \Omega)(1.0\ \text{A}) = 20\ \text{V}.$$

Thus, the voltage between points X and Y is

$$V_{XY} = V_{XB} + V_{BC} = 4.0\ \text{V} + 20\ \text{V} = 24\ \text{V}.$$

Because $V_{XY} = V_{XD} = 24$ V, applying $I = V/R$, we write

$$I_{XD} = \frac{V_{XD}}{R_{XD}} = \frac{24\ \text{V}}{12\ \Omega} = 2.0\ \text{A},$$

and $I_{XD} = I_{AD} = I_{XA} = 2.0$ A as well. The potential difference between points X and A is

$$V_{XA} = I_{XA}R_{XA} = I_{XA}R_2 = (2.0\ \text{A})(6.0\ \Omega) = 12\ \text{V}.$$

◆ wxMaxima codes:

```
(%i1) R1: 1/(1/8 + 1/16 + 1/16);
(R1)  4
(%i2) R2: 1/(1/9 + 1/18);
(R2)  6
(%i3) R3: 4 + 20;
(R3)  24
(%i4) R4: 6 + 6;
(R4)  12
(%i5) Re: 1/(1/24 + 1/12);
(Re)  8
(%i6) VXB: 8*0.5;
(VXB) 4.0
(%i7) VBC: 20*1;
(VBC) 20
(%i8) VXY: VXB + 20;
(VXY) 24.0
(%i9) IXD: 24/12;
(IXD) 2
(%i10)VXA: 2*6;
(VXA) 12
```

Comments on the codes:

(%i1), (%i2), (%i3), (%i4), (%i5) Calculate R_1, R_2, R_3, R_4, and R_e.
(%i6), (%i7), (%i8), (%9), (%i10) Calculate V_{XB}, V_{BC}, V_{XY}, I_{XD}, and V_{XA}.

Problem 9.3 Figure 9.5a and b show two configurations of resistors. Each resistor is 3.0 Ω and has a maximum output power of 48 W. What are the maximum power and voltage of terminals of each configuration?

Solution

(a) The maximum current that flows in the 3.0 Ω resistor on the right of Fig. 9.5(a) is, Eq. (8.13),

$$I_{max} = \sqrt{\frac{P_{max}}{R}} = \sqrt{\frac{48\ \text{W}}{3.0\ \Omega}} = 4.0\ \text{A},$$

where application of $P = I^2R$ is made. The two 3.0 Ω resistors on the left are in parallel, the current through each of them is 2.0 A, and the power of each is

$$P = I^2R = (2.0\ \text{A})^2(3.0\ \Omega) = 12\ \text{W}.$$

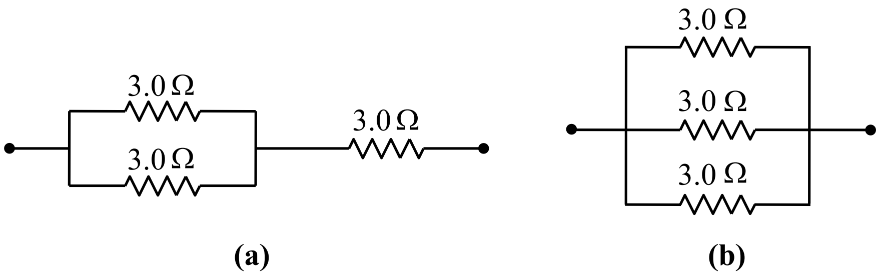


Fig. 9.5 Power and voltage of configuration of resistors (a) and (b), Problem 9.3

Therefore, the maximum electrical power of the three resistors of Fig. 9.5a is

$$P_{total} = 48\,\mathrm{W} + 12\,\mathrm{W} + 12\,\mathrm{W} = 72\,\mathrm{W}.$$

The voltage between the terminals is

$$V = (3.0\,\Omega)(2.0\,\mathrm{A}) + (3.0\,\Omega)(4.0\,\mathrm{A}) = 18\mathrm{V}.$$

◆ wxMaxima codes:

```
(%i2) Pmax:48; R:3;
(Pmax) 48
(R)    3
(%i3) Imax: sqrt(Pmax/R);
(Imax) 4
(%i4) I: Imax/2;
(I)    2
(%i5) P: I^2*R;
(P)    12
(%i6) Ptotal: 48 + P + P;
(Pmax) 72
(%i7) V: 3*2 + 3*4;
(V)    18
```

Comments on the codes:

(%i2)	Assign values of P_{max} and R.
(%i3), (%i4), (%i5), (%i6), (%i7)	Calculate I_{max}, I, P, P_{total}, and V.

(b) For the configuration of Fig. 9.5b, the maximum current that flows in each of the 3.0 Ω resistors in parallel is, Eq. (8.13),

$$I_{max} = \sqrt{\frac{P_{max}}{R}} = \sqrt{\frac{48\ \mathrm{W}}{3.0\ \Omega}} = 4.0\ \mathrm{A}$$

The current through the configuration is 4.0 A + 4.0 A + 4.0 A = 12 A. The maximum electric power of the configuration is

$$P_{total} = 48\,\text{W} + 48\,\text{W} + 48\,\text{W} = 144\,\text{W}.$$

The voltage between the terminals is, Eq. (5.13)

$$V = \frac{P}{I} = \frac{48\ \text{W}}{4.0\ \text{A}} = 12\ \text{V}.$$

◆ wxMaxima codes:

```
(%i2) Pmax:48; R:3;
(Pmax) 48
(R)    3
(%i3) Imax: sqrt(Pmax/R);
(Imax) 4
(%i4) Ptotal: 48 + 48 + 48;
(Ptotal)    144
(%i5) V: 48/4;
(V)    12
```

Comments on the codes:

(%i2)	Assign values of P_{max} and R.
(%i3), (%i4), (%i5)	Calculate I_{max}, P_{total}, and V.

Problem 9.4 Three identical resistors are connected in series and a potential difference of V is applied, the dissipated power is 10 W. What is the dissipated power if the three resistors are connected in parallel with the same potential difference applied?

Solution

Figure 9.6a shows the resistors in series, while Fig. 9.6b shows the resistors in parallel. Each resistor has resistance R.

For resistors connected in series, Fig. 9.6(a), the equivalent resistance is, Eq. (1),

$$R_{series} = R + R + R = 3R.$$

The dissipated power is, Eq. (8.13),

$$P_{series} = \frac{V^2}{R_{equivalent}} = \frac{V^2}{R_{series}} = \frac{V^2}{3R} = 10\ \text{W}.$$

This means that each resistor dissipates 10/3 W = 3.3 W of electrical power. For resistors connected in parallel, Fig. 9.6b, the equivalent resistance is, Eq. (2),

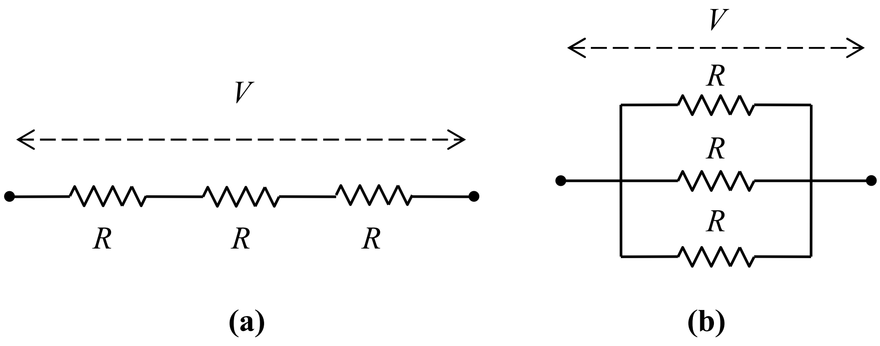


Fig. 9.6 Three resistors in series (**a**), and in parallel (**b**), Problem 9.4

$$R_{parallel} = \frac{1}{\frac{1}{R} + \frac{1}{R} + \frac{1}{R}} = \frac{R}{3}.$$

The dissipated power is, Eq. (8.13),

$$P_{parallel} = \frac{V^2}{R_{equivalent}} = \frac{V^2}{R_{parallel}} = \frac{V^2}{R/3} = \frac{3V^2}{R}.$$

The ratio of dissipated powers for resistors in parallel to that in series is

$$\frac{P_{parallel}}{P_{series}} = \frac{(3V^2/R)}{(V^2/3R)} = 9.$$

Thus, the power for resistors in parallel is

$$P_{parallel} = 9P_{series} = 9\,(10\text{ W}) = 90\text{ W}.$$

This means that each resistor dissipates 90/3 W = 30 W of electrical power.

◆ wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)     5
(ratprint)  false
(%i4) solve([Pseries = 10, V^2/(3*R)=10, Pparallel=3*V^2/R],
[Pparallel,Pseries,V])$ float(%);
(%o4)[[Pparallel=90.0,Pseries=10.0,V=5.4772*sqrt(R)],
     [Pparallel=90.0,Pseries=10.0,V=-5.4772*sqrt(R)]]
```

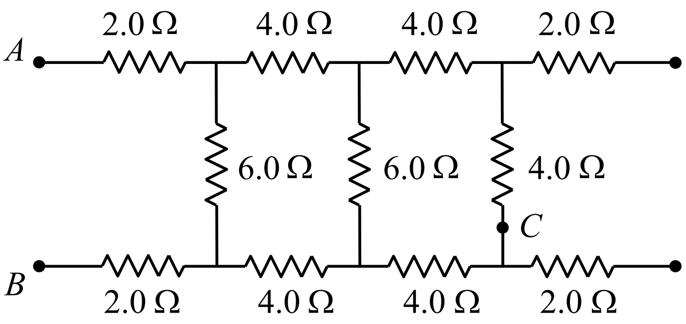


Fig. 9.7 A network of eleven resistors, Problem 9.5

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.

(%i4) Solve $P_{series} = 10, \frac{V^2}{3R} = 10$, and $P_{parallel} = \frac{3V^2}{R}$ for $P_{parallel}$, P_{series}, and V.

Another question: If the resistors are replaced by identical light bulbs, how does the brightness of the bulbs in series compare with the bulbs in parallel?

Answer: The bulbs in series are less bright than the ones in parallel because the power of bulbs in series is less than the ones in parallel.

Problem 9.5 Eleven resistors are arranged as in Fig. 9.7, determine:

(a) the resistance between points A and B.

(b) potential difference between points A and B that causes current of 1.0 A in point C.

Solution

(a) Figure 9.8 shows a way to simplify the circuit in stages. Resistors in series and resistors in parallel are replaced by their equivalent resistances, this is repeated until a single equivalent resistance is obtained.

In Figs. 9.7 and 9.8a

$$R_1 = 4.0\ \Omega + 4.0\ \Omega + 4.0\ \Omega = 12\ \Omega.$$

The two 2.0 Ω resistors on the right of Fig. 9.7 are not included, because if A and B are the terminals of emf, no current will flow in the two resistors. In Figs. 9.8a and b, the 6.0 Ω resistor and R_1 are in parallel, thus

$$R_2 = \frac{1}{\frac{1}{R_1} + \frac{1}{6.0\ \Omega}} = \frac{1}{\frac{1}{12\ \Omega} + \frac{1}{6.0\ \Omega}} = 4.0\ \Omega.$$

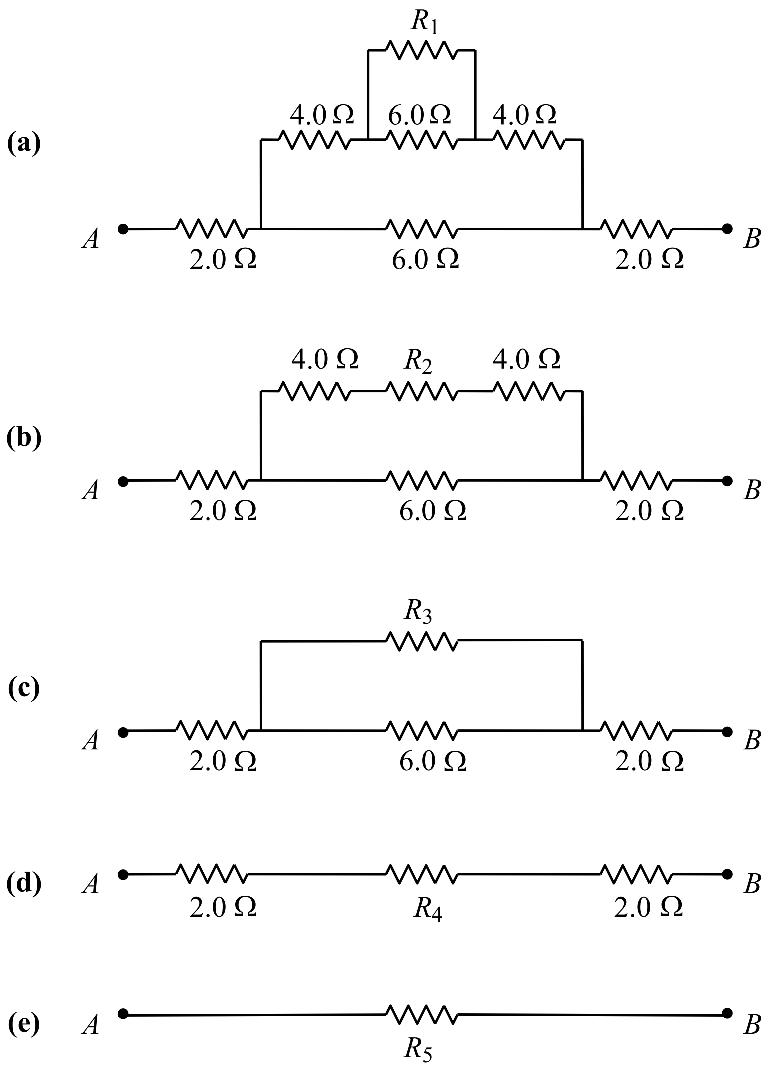


Fig. 9.8 Simplifying the resistor network in stages, Problem 9.5

In Figs. 9.8b and c, the two 4.0 Ω resistors and R_2 are in series, thus

$$R_3 = 4.0\ \Omega + R_2 + 4.0\ \Omega = 12\ \Omega.$$

In Figs. 9.8c and d, the 6.0 Ω resistocr and R_3 are in parallel, thus

$$R_4 = \frac{1}{\frac{1}{R_3} + \frac{1}{6.0\ \Omega}} = \frac{1}{\frac{1}{12\ \Omega} + \frac{1}{6.0\ \Omega}} = 4.0\ \Omega.$$

Lastly, in Figs. 9.8d and 9.8e, the equivalent resistance between points A and B is

$$R_5 = 2.0\ \Omega + R_4 + 2.0\ \Omega = 8.0\ \Omega.$$

◆ wxMaxima codes:

```
(%i1) R1: 4+4+4;
(R1)   12
(%i2) R2: 1/(1/R1 + 1/6);
(R2)   4
(%i3) R3: 4+R2+4;
(R3)   12
(%i4) R4: 1/(1/R3 + 1/6);
(R4)   4
(%i5) R5: 2+R4+2;
(R5)   8
```

Comments on the codes:

(%i1), (%i2), (%i3), (%i4) (%i5) Calculate R_1, R_2, R_3, R_4, and R_5.

(b) In Figs. 9.7 and 9.8a,

$$\text{current in } C = \text{current in } R_1 = 1.0\ \text{A}.$$

In Fig. 9.8a and b

$$\begin{aligned}\text{potential difference across } R_2 &= \text{potential difference across } R_1\\ &= (1.0\ \text{A})R_1 = (1.0\ \text{A})(12\ \Omega)\\ &= 12\ \text{V}.\end{aligned}$$

In Fig. 9.8b and c

$$\text{current in } R_3 = \text{current in } R_2 = \frac{12\ \text{V}}{R_2} = \frac{12\ \text{V}}{4.0\ \Omega} = 3.0\ \text{A}.$$

$$\text{potential difference across } R_3 = (3.0\ \text{A})(12\ \Omega) = 36\ \text{V}.$$

In Fig. 9.8c and d

$$\text{potential difference across } R_4 = \text{potential difference across } R_3 = 36\ \text{V}.$$

$$\text{current in } R_4 = \frac{36\ \text{V}}{R_4} = \frac{36\ \text{V}}{4.0\ \Omega} = 9.0\ \text{A}.$$

In Fig. 9.8d and e

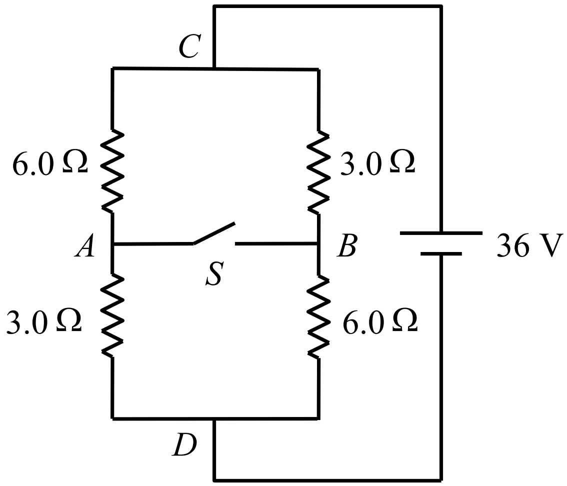


Fig. 9.9 Circuit of Problem 9.6

$$\text{current in } R_5 = \text{current in } R_4 = 9.0 \text{ A}.$$

Thus, the potential difference between points A and B is

$$(9.0 \text{ A})R_5 = (9.0 \text{ A})(8.0\ \Omega) = 72 \text{ V}.$$

Problem 9.6 For Fig. 9.9, determine:

(a) the potential difference between points A and B, V_{AB} when switch S is opened.
(b) the current in switch S when the switch is closed.

Solution

(a) When switch S is opened, using $I = V/R$, the current in point A or B is

$$\frac{36 \text{ V}}{(3.0 + 6.0)\ \Omega} = 4.0 \text{ A}.$$

The voltage drop across CA, using $V = IR$, is

$$(4.0 \text{ A})(6.0\ \Omega) = 24 \text{ V}.$$

The potential at point A is

$$V_A = 36 \text{ V} - 24 \text{ V} = 12 \text{ V}.$$

The voltage drop across CB is

$$(4.0 \text{ A})(3.0\ \Omega) = 12 \text{ V}.$$

The potential at point B is

$$V_B = 36\,\text{V} - 24\,\text{V} = 24\,\text{V}.$$

Therefore, the potential difference between points A and B is

$$V_{AB} = V_A - V_B = 12\,\text{V} - 24\,\text{V} = -12\,\text{V}.$$

(b) Figure 9.10a shows currents and resistors for the circuit in Fig. 9.9 when switch S is closed. Directions of currents and circuit loops are assigned for application of Kirchhoff's rules.

For top and bottom loops, the Kirchhoff's rules give

$$-(3.0\ \Omega)I_1 + (6.0\ \Omega)(I - I_1) = 0, \tag{1}$$

$$-(6.0\ \Omega)I_2 + (3.0\ \Omega)(I - I_2) = 0. \tag{2}$$

Top loop, Eq. (1): In a clockwise direction, starting from point C, there is a potential drop of $-(3.0\ \Omega)I_1$ across the 3.0 Ω resistor and potential rise of $+(6.0\ \Omega)(I - I_1)$ across the 6.0 Ω resistor.

Bottom loop, Eq. (2): In a clockwise direction, starting from point B, there is a potential drop of $-(6.0\ \Omega)I_2$ across the 6.0 Ω resistor and potential rise of $+(3.0\ \Omega)(I - I_2)$ across the 3.0 Ω resistor.

Solving Eqs. (1) and (2) gives I_1 and I_2 in terms of I

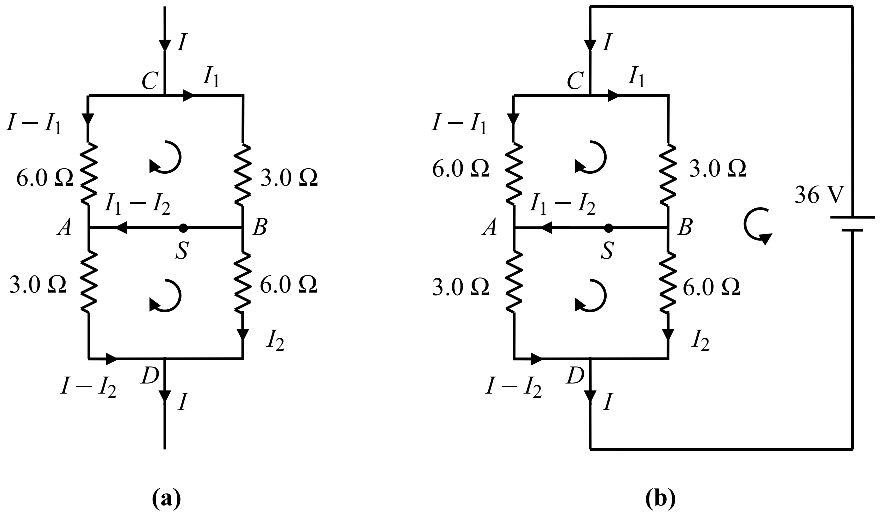


Fig. 9.10 **a** Analysis by Kirchhoff's rule, **b** alternative analysis by Kirchhoff's rule, Problem 9.6

$$I_1 = \frac{2}{3}I, \quad I_2 = \frac{1}{3}I.$$

The equivalent resistance between points C and D is

$$R = \frac{1}{\frac{1}{3.0} + \frac{1}{6.0}}\ \Omega + \frac{1}{\frac{1}{6.0} + \frac{1}{3.0}}\ \Omega = 4.0\ \Omega.$$

The current in C or D is

$$I = \frac{V_{CD}}{R} = \frac{36\ \text{V}}{4.0\ \Omega} = 9.0\ \text{A}.$$

Thus, the current through switch S is

$$I_1 - I_2 = \frac{1}{3}I = \frac{1}{3}(9.0\ \text{A}) = 3.0\ \text{A}.$$

The direction of the current is from point B to A. The other currents are

$$I_1 = \frac{2}{3}I = \frac{2}{3}(9.0\ \text{A}) = 6.0\ \text{A}.$$

$$I_2 = \frac{1}{3}I = \frac{1}{3}(9.0\ \text{A}) = 3.0\ \text{A}.$$

◆ wxMaxima codes:

```
(%i1) solve([-3*I1+6*(I-I1)=0, -6*I2+3*(I-I2)=0 ], [ I1, I2]);
(%o1) [[I1=(2*I)/3,I2=I/3]]
(%i2) R: 1/(1/3 + 1/6) + 1/(1/6 + 1/3);
(R)   4
(%i3) I: 36/R;
(I)   9
(%i4) 1/3*I;
(%o4) 3
```

Comments on the codes:

(%i1)	Solve Eqs. (1) and (2) for I_1 and I_2.
(%i2), (%i3)	Calculate equivalent resistance R and current I.
(%i4)	Calculate the current in switch S.

Alternative calculation: Fig. 9.10b shows the circuit with the 36 Vvoltage source and a third loop. Using Kirchhoff's rules in the three loops, we have

$$-(3.0\ \Omega)I_1 + (6.0\ \Omega)(I - I_1) = 0, \tag{1}$$

$$-(6.0\ \Omega)I_2 + (3.0\ \Omega)(I - I_2) = 0, \tag{2}$$

$$36\ \text{V} - (3.0\ \Omega)I_1 - (6.0\ \Omega)I_2 = 0. \tag{3}$$

Top loop, Eq. (1): In a clockwise direction, starting from point C, there is a potential drop of $-(3.0\ \Omega)I_1$ across the 3.0 Ω resistor and potential rise of $+ (6.0\ \Omega)(I - I_1)$ across the 6.0 Ω resistor.

Bottom loop, Eq. (2): In a clockwise direction, starting from point B, there is a potential drop of $-(6.0\ \Omega)I_2$ across the 6.0 Ω resistor and potential rise of $+ (3.0\ \Omega)(I - I_2)$ across the 3.0 Ω resistor.

Right loop, Eq. (3): In counter clockwise direction, starting from the voltage source, there is a potential rise of + 36 V across the voltage source, a potential drop of $- (3.0\ \Omega)I_1$ across the 3.0 Ω resistor, and a potential drop of $-(6.0\ \Omega)I_2$ across the 6.0 Ω resistor.

Solving Eqs. (1), (2), and (3) gives

$$I_1 = 6.0\ \text{A}, \quad I_2 = 3.0\ \text{A}, \quad I = 9.0\ \text{A}.$$

Therefore, the current in switch S is

$$I_1 - I_2 = 3.0\ \text{A}.$$

◆ wxMaxima codes:

```
(%i1) solve([-3*I1+6*(I-I1)=0, -6*I2+3*(I-I2)=0,36-3*I1-6*I2=0 ],
[I, I1, I2]);
(%o1) [[I=9,I1=6,I2=3]]
(%i3) I1: 6 ; I2: 3;
(I1)  6
(I2)  3
(%i4) I1-I2;
(%o4) 3
```

Comments on the codes:

(%i1) Solve Eqs. (1), (2), and (3) for I, I_1, and I_2.
(%i4) Calculate the current in switch S.

Problem 9.7 Figure 9.11 is an RC series circuit with $\varepsilon = 12$ V, $R = 1.4$ MΩ, and $C = 1.8$ μF.

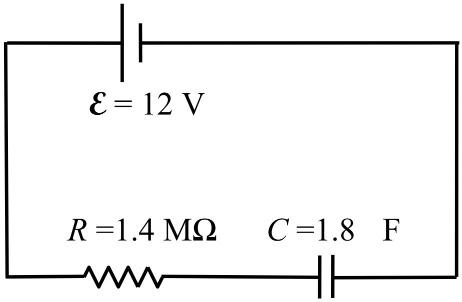


Fig. 9.11 An *RC* circuit, Problem 9.7

(a) Calculate the time constant τ.
(b) What is the maximum charge Q_{max} in the capacitor?
(c) Calculate the time to charge the capacitor to 16 μC.
(d) Plot curves of charge in the capacitor and current in the circuit against time for 0 to 10 s.

Solution

(a) The time constant is, Eq. (4),

$$\tau = RC = (1.4 \times 10^6\ \Omega)(1.8 \times 10^{-6}\text{F}) = 2.5\ \text{s}.$$

(b) The maximum charge in the capacitor is, Eq. (4),

$$Q_{max} = C\,\mathcal{E} = (1.8 \times 10^{-6}\ \text{F})(12\ \text{V}) = 2.2 \times 10^{-5}\ \text{C} = 22\ \mu\text{C}.$$

(c) The time to charge the capacitor to 16 μC is calculated as follows, Eq. (4),

$$\begin{aligned} Q &= Q_{max} \cdot (1 - e^{-t/\tau}), \\ e^{-t/\tau} &= 1 - \frac{Q}{Q_{max}}, \\ t &= -\tau \ln\left(1 - \frac{Q}{Q_{max}}\right) = -(2.5\ \text{s}) \ln\left(1 - \frac{16\ \mu\text{C}}{22\ \mu\text{C}}\right) \\ &= 3.4\ \text{s}. \end{aligned}$$

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; ratprint:false; emf:12; R:1.4e6; C:1.8e-6;
(fpprintprec)     5
(ratprint)  false
(emf) 12
(R)   1.4*10^6
(C)   1.8*10^-6
(%i6) tau: R*C;
(tau) 2.52
(%i7) Qmax: C*emf;
(Qmax) 2.16*10^-5
(%i8) Q: 16e-6;
(Q)   1.6*10^-5
(%i9) t: -tau*log(1- Q/Qmax);
(t)   3.4018
```

Comments on the codes:

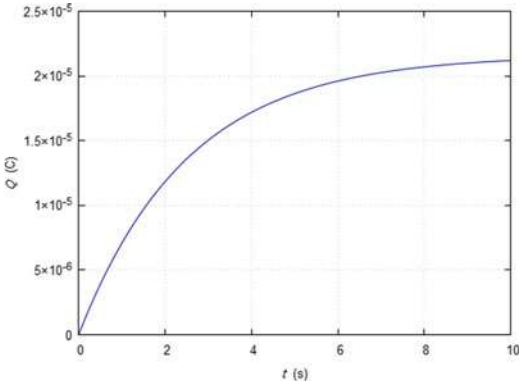
(%i5)	Set floating point print precision to 5, internal rational number print to false, and assign ε, *R*, and *C*.
(%i6), (%i7)	Calculate τ and Q_{max}.
(%i8), (%i9)	Assign *Q* and calculate *t*.

(d) The charge *Q* in the capacitor varies with time *t* as, Eq. (4),

$$Q = Q_{max} \cdot (1 - e^{-t/\tau}).$$

◆ Curve of charge in the capacitor against time by wxMaxima is as follows:

```
(%i4) fpprintprec:5;  emf:12; R:1.4e6; C:1.8e-6;
(fpprintprec)     5
(emf) 12
(R)   1.4*10^6
(C)   1.8*10^-6
(%i5) tau: R*C;
(tau) 2.52
(%i6) Qmax: C*emf;
(Qmax) 2.16*10^-5
(%i7) Q: Qmax*(1-exp(-t/tau));
(Q)   2.16*10^-5*(1-%e^(-0.39683*t))
(%i8) wxplot2d(Q, [t,0,10], grid2d, [xlabel,"{/Helvetica-Italic t}  (s)"],
[ylabel,"{/Helvetica-Italic Q}  (C)"]);
```



Comments on the codes:

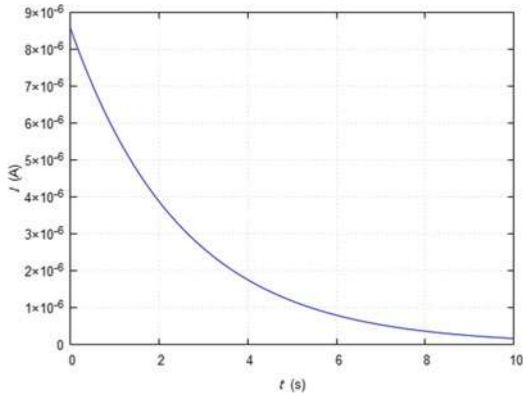
(%i4)	Set floating point print precision to 5, and assign values of ε, R, and C.
(%i5), (%i6)	Calculate τ and Q_{max}.
(%i7)	Define Q.
(%i8)	Plot Q against t for $0 \le t \le 10$ s.

The current I in the RC circuit varies with time t as, Eq. (3),

$$I = \frac{\mathcal{E}}{R} \cdot e^{-t/\tau}.$$

◆ Curve of current in the circuit against time by wxMaxima is as follows:

```
(%i4) fpprintprec:5;  emf:12; R:1.4e6; C:1.8e-6;
(fpprintprec)     5
(emf) 12
(R)   1.4*10^6
(C)   1.8*10^-6
(%i5) tau: R*C;
(tau) 2.52
(%i6) I: emf/R*exp(-t/tau);
(I)   8.5714*10^-6*%e^(-0.39683*t)
(%i7) wxplot2d(I, [t,0,10], grid2d, [xlabel,"{/Helvetica-Italic t}  (s)"],
[ylabel,"{/Helvetica-Italic I}  (A)"]);
```



Comments on the codes:

(%i4) Set floating point print precision to 5, and assign values of ε, *R*, and *C*.
(%i5) Calculate τ.
(%i6) Define *I*.
(%i7) Plot *I* against *t* for $0 \leq t \leq 10$ s.

Alternative solution: For a series RC circuit of Fig. 9.11, the circuit equation is

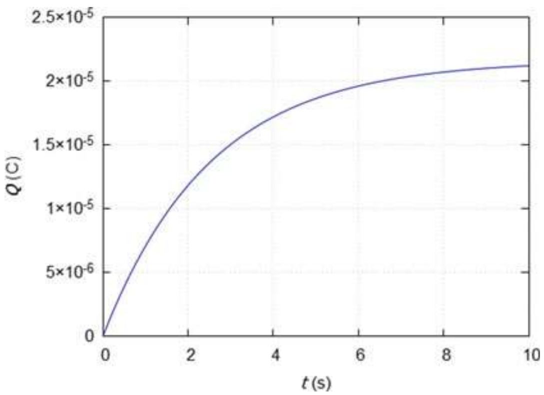
$$\mathcal{E} = RI + \frac{Q}{C} = R\frac{dQ}{dt} + \frac{Q}{C},$$

where *I* is the current in the circuit, *Q* is the charge in the capacitor, and ε **E** is the emf of the battery. Thus, *RI* is the potential drop across the resistor and *Q*/*C* is the potential drop across the capacitor. Electric current is the time rate of charge flow, $I = dQ/dt$. The initial condition is, at $t = 0$ s, $Q = 0$ C.

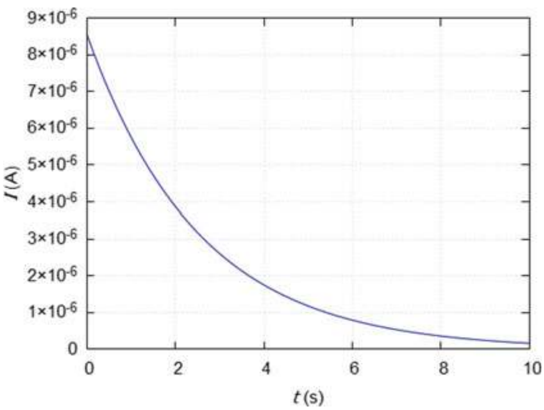
The equation is a first-order differential equation, where charge *Q* is the dependent variable and *t* is the independent variable. This can be solved using predefined functions *ode*2 and *ic*1 of wxMaxima. See *Solving first order ordinary differential equation* in Appendix A.

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; ratprint:false; emf:12; R:1.4e6; C:1.8e-6;
(fpprintprec)     5
(ratprint)  false
(emf) 12
(R)   1.4*10^6
(C)   1.8*10^-6
(%i6) sol: ode2(emf=R*'diff(Q,t) + Q/C, Q, t);
(sol) Q=%e^(-(25*t)/63)*((27*%e^((25*t)/63))/1250000+%c)
(%i7) ic1(sol, t=0, Q=0);
(%o7) Q=(%e^(-(25*t)/63)*(27*%e^((25*t)/63)-27))/1250000
(%i8) Q: rhs(%);
(Q)   (%e^(-(25*t)/63)*(27*%e^((25*t)/63)-27))/1250000
(%i9) wxplot2d(Q, [t,0,10], grid2d, [xlabel,"{/Helvetica-Italic t} (s)"],
[ylabel,"{/Helvetica-Italic Q} (C)"] );
```




```
(%i10) I: diff(Q,t);
(I)   3/350000-(%e^(-(25*t)/63)*(27*%e^((25*t)/63)-27))/3150000
(%i11) wxplot2d(I, [t,0,10], grid2d, [xlabel,"{/Helvetica-Italic t} (s)"],
[ylabel,"{/Helvetica-Italic I} (A)"] );
```



Comments on the codes:

(%i5)	Set floating point print precision to 5, internal rational number print to false, and assign values of ε, *R*, and *C*.
(%i6)	Solve ODE $\mathcal{E} = R\ dQ/dt + Q/C$, and get a general solution.
(%i7)	Set the initial condition and get a particular solution.
(%i8), (%i9)	Assign the solution to Q and plot Q against t for $0 \le t \le 10$ s.
(%i10), (%i11)	Calculate I and plot I against t for $0 \le t \le 10$ s.

Problem 9.8 Figure 9.12 is an *RC* circuit with a capacitance of $C = 1.02$ μF and a battery of emf $\varepsilon = 20.0$ V. The capacitor is fully charged to a charge of $Q_0 = C\varepsilon$. At time $t = 0$ s, the switch is moved from point *A* to *B*. The Current *I* decreases to half of its initial value in 40 μs.

(a) What is the charge in the capacitor at $t = 0$?
(b) Calculate resistance *R*
(c) What is the charge in the capacitor at $t = 60$ μs?

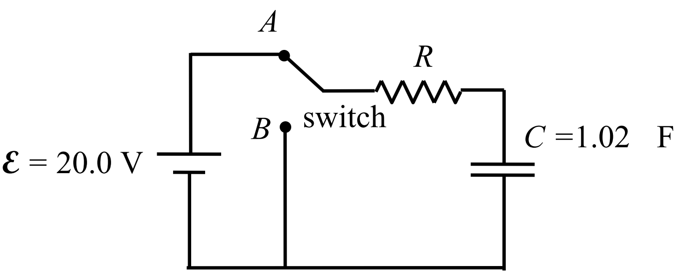


Fig. 9.12 An *RC* circuit, Problem 9.8

Solution

(a) At $t = 0$, the charge is, Eq. (4),

$$Q_0 = C\,\mathcal{E} = (1.02 \times 10^{-6}\text{ F})(20\text{ V}) = 2.04 \times 10^{-5}\text{ C} = 20.4\ \mu\text{C}.$$

(b) From the question, at time $t = 40\ \mu\text{s}$, $I = 0.5I_0$, thus, Eq. (6),

$$\begin{aligned} I &= I_0 \cdot e^{-\frac{t}{RC}}, \\ 0.5I_0 &= I_0 \cdot e^{-\frac{40\times10^{-6}\text{ s}}{R(1.02\times10^{-6}\text{ F})}}, \\ \ln 2 &= \frac{40 \times 10^{-6}\text{ s}}{R(1.02 \times 10^{-6}\text{ F})}, \\ R &= \frac{40 \times 10^{-6}\text{ s}}{(1.02 \times 10^{-6}\text{ F})\ln 2} = 57\ \Omega. \end{aligned}$$

(c) At $t = 60\ \mu\text{s}$, the charge in the capacitor is, Eq. (5),

$$Q = Q_0 \cdot e^{-\frac{t}{RC}} = 20.4\ \mu\text{C} \cdot e^{-\frac{60\times10^{-6}\text{ s}}{(57\ \Omega)(1.02\times10^{-6}\text{ F})}} = 7.2\ \mu\text{C}.$$

◆ wxMaxima codes:

```
(%i4) fpprintprec:5;  emf:20; C:1.02e-6; t:40e-6;
(fpprintprec)     5
(emf) 20
(C)   1.02*10^-6
(t)   4.0*10^-5
(%i5) Q0: C*emf;
(Q0)  2.04*10^-5
(%i7) R: t/(C*log(2)); float(%);
(R)   39.216/log(2)
(%o7) 56.576
(%i8) t: 60e-6;
(t)   6.0*10^-5
(%i10) Q: Q0*exp(-t/(R*C)); float(%);
(Q)   2.04*10^-5*%e^(-1.5*log(2))
(%o10) 7.2125*10^-6
```

Comments on the codes:

(%i4)	Set floating point print precision to 5, and assign values of ε, C, and t.
(%i5), (%i7)	Calculate Q_0 and R.
(%i8), (%i10)	Assign t and calculate Q.

Problem 9.9 Figure 9.13 shows a circuit consisting of a resistance $R = 1.70\ \text{M}\Omega$ and a capacitance $C = 2.30\ \mu\text{F}$, and a switch. The capacitor has a charge of $50\ \mu\text{C}$.

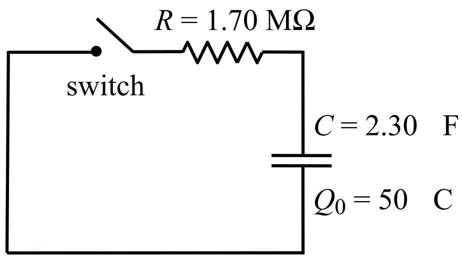


Fig. 9.13 Discharging an *RC* circuit, Problem 9.9

The switch is closed at $t = 0$ s, so that the circuit is completed. Plot curves of charge in the capacitor versus time and current in the circuit versus time for the discharging process.

Solution

For the discharging process of an *RC* circuit, charge and current vary with time as, Eq. (5) and (6),

$$Q = Q_0 \cdot e^{-\frac{t}{RC}} = Q_0 \cdot e^{-\frac{t}{\tau}}, \tag{1}$$

$$I = I_0 \cdot e^{-\frac{t}{RC}} = \frac{Q_0}{RC} \cdot e^{-\frac{t}{\tau}}. \tag{2}$$

The time constant of the *RC* circuit is

$$\tau = RC = (1.70 \times 10^6\ \Omega)(2.30 \times 10^{-6}\ \text{F}) = 3.91\ \text{s}.$$

The initial current is

$$I_0 = \frac{Q_0}{RC} = \frac{50 \times 10^{-6}\ \text{C}}{(1.70 \times 10^6\ \Omega)(2.30 \times 10^{-6}\ \text{F})} = 1.28 \times 10^{-5}\ \text{A}.$$

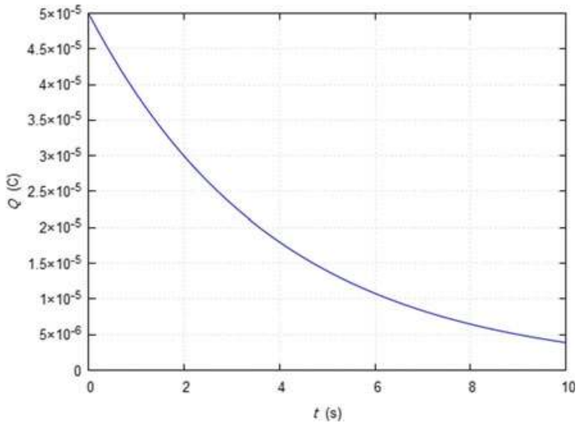
The curves to be plotted are

$$Q = (50 \times 10^{-6}\ \text{C}) \cdot e^{-\frac{t}{3.91\ \text{s}}}, \tag{3}$$

$$I = (1.28 \times 10^{-5}\ \text{A}) \cdot e^{-\frac{t}{3.91\ \text{s}}}. \tag{4}$$

♦ Plot of curve (3) i.e. curve of Q against time t for $0 \le t \le 10$ s by wxMaxima is as follows:

```
(%i4) fpprintprec:5; R:1.7e6; C:2.3e-6; Q0:50e-6;
(fpprintprec)     5
(R)   1.7*10^6
(C)   2.3*10^-6
(Q0)  5.0*10^-5
(%i5) tau: R*C;
(tau) 3.91
(%i6) Q: Q0*exp(-t/tau);
(Q)   5.0*10^-5*%e^(-0.25575*t)
(%i7) wxplot2d(Q, [t,0,10], grid2d, [xlabel,"{/Helvetica-Italic t}  (s)"],
[ylabel,"{/Helvetica-Italic Q}  (C)"]);
```

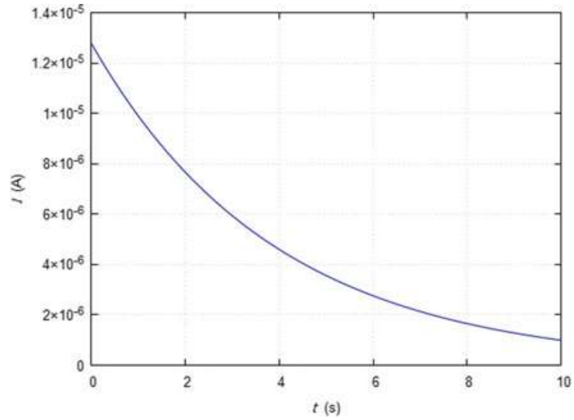


Comments on the codes:

(%i4) Set floating point print precision to 5, and assign values of R, C, and Q_0.
(%i5) Calculate τ.
(%i6) Assign Q.
(%i7) Plot Q against t for $0 \le t \le 10$ s.

♦ Plot of curve (4) i.e. curve of I against time t for $0 \le t \le 10$ s by wxMaxima is as follows:


```
(%i4) fpprintprec:5; R:1.7e6; C:2.3e-6; Q0:50e-6;
(fpprintprec)     5
(R)   1.7*10^6
(C)   2.3*10^-6
(Q0)  5.0*10^-5
(%i5) tau: R*C;
(tau) 3.91
(%i6) Q: Q0*exp(-t/tau);
(Q)   5.0*10^-5*%e^(-0.25575*t)
(%i7) wxplot2d(Q, [t,0,10], grid2d, [xlabel,"{/Helvetica-Italic t}  (s)"],
[ylabel,"{/Helvetica-Italic Q}  (C)"]);
```



Comments on the codes:

(%i4)	Set floating point print precision to 5, and assign values of *R*, *C*, and Q_0.
(%i5), (%i6)	Calculate τ and I_0.
(%i7)	Assign *I*.
(%i8)	Plot *I* against *t* for $0 \le t \le 10$ s.

Alternative solution: The circuit equation of the problem is,

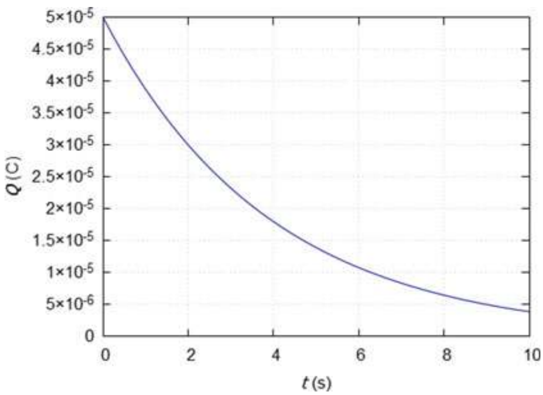
$$RI + \frac{Q}{C} = 0,$$
$$R\frac{dQ}{dt} + \frac{Q}{C} = 0.$$

This is a first-order ordinary differential equation with *Q* and *t* as its dependent and independent variables, respectively. The initial condition is, at $t = 0$ s, $Q = Q_0 = 50 \times 10^{-6}$ C.

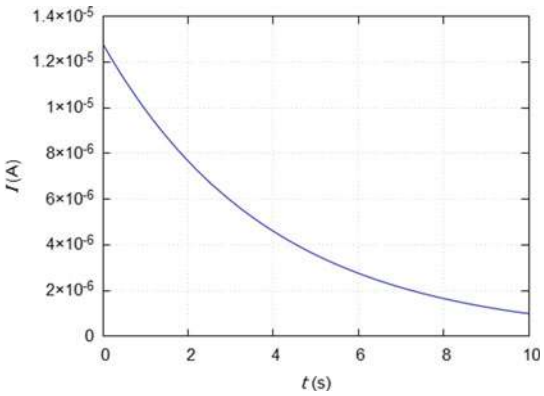
The first-order ordinary differential equation can be solved via predefined functions *ode*2 and *ic*1 of wxMaxima. See *Solving first order ordinary differential equation* in Appendix A.

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; ratprint:false; R:1.7e6; C:2.3e-6; Q0:50e-6;
(fpprintprec)     5
(ratprint)  false
(R)   1.7*10^6
(C)   2.3*10^-6
(Q0)  5.0*10^-5
(%i7) sol: ode2(R*'diff(Q,t) + Q/C = 0, Q, t)$ expand(%);
(%o7) Q=%c*%e^(-(100*t)/391)
(%i9) ic1(sol, t=0, Q=Q0)$ expand(%);
(%o9) Q=%e^(-(100*t)/391)/20000
(%i10) Q: rhs(%);
(Q)   %e^(-(100*t)/391)/20000
(%i11) wxplot2d(Q, [t,0,10], grid2d, [xlabel,"{/Helvetica-Italic t} (s)"],
[ylabel,"{/Helvetica-Italic Q} (C)"] );
```




```
(%i12) I: abs(diff(Q,t));
(I)   %e^(-(100*t)/391)/78200
(%i13) wxplot2d(I, [t,0,10], grid2d, [xlabel,"{/Helvetica-Italic t} (s)"],
[ylabel,"{/Helvetica-Italic I} (A)"] );
```



Comments on the codes:

(%i5)	Set floating point print precision to 5, internal rational number print to false, and assign values of R, C, and Q_0.
(%i7)	Solve ODE $R\, dQ/dt + Q/C = 0$ and get a general solution.
(%i9)	Set the initial condition and get a particular solution.
(%i10), (%i11)	Assign the solution to Q and plot Q against t for $0 \leq t \leq 10$ s.
(%i12), (%i13)	Calculate I and plot I against t for $0 \leq t \leq 10$ s.

Problem 9.10 Show that the resistance of an infinite network shown in Fig. 9.14 is $(1 + \sqrt{3})R$.

Solution

Figure 9.15 shows the infinite resistor network. Additional observation points C and D are also marked.

Resistance between A and B is the sum of R (top left), parallel resistors consisting of R and effective resistance between C and D (R_{CD}), and R (bottom left). We write

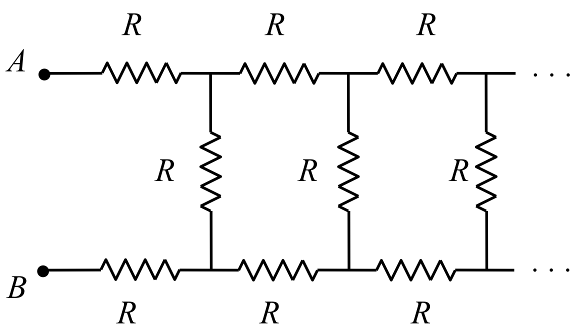


Fig. 9.14 An infinite network of resistors, Problem 9.10

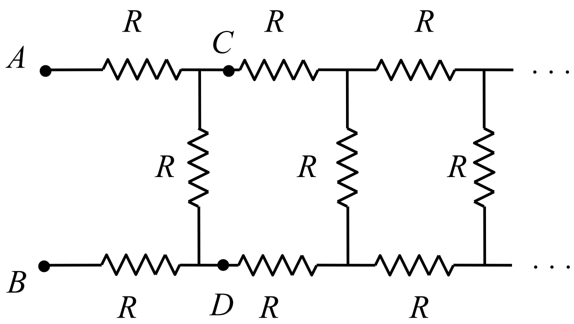


Fig. 9.15 Determining equivalent resistance, Problem 9.10

$$
\begin{aligned}
R_{AB} &= R + \frac{1}{\frac{1}{R} + \frac{1}{R_{CD}}} + R \\
&= 2R + \frac{R R_{CD}}{R + R_{CD}} \\
&= \frac{2R^2 + 3R R_{CD}}{R + R_{CD}}, \\
R R_{AB} + R_{AB} R_{CD} &= 2R^2 + 3R R_{CD}.
\end{aligned}
$$

The network is infinite, so $R_{AB} \approx R_{CD}$, and let us call them R_∞. The resistance can be calculated by the quadratic formula

$$
\begin{aligned}
&R R_\infty + R_\infty^2 = 2R^2 + 3R R_\infty, \\
&R_\infty^2 - 2R R_\infty - 2R^2 = 0, \\
&R_\infty = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{2R + \sqrt{4R^2 + 8R^2}}{2}, \\
&R_\infty = (1 + \sqrt{3})R = 2.73R.
\end{aligned}
$$

We have shown that for an infinite network of resistors of Fig. 9.15 the resistance $R_{\mathrm{AB}} = R_{\mathrm{CD}} = R_\infty = (1 + \sqrt{3})R$.

♦ wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)     5
(ratprint)  false
(%i4) solve(R*Rinf + Rinf^2 = 2*R^2 + 3*R*Rinf, Rinf); float(%);
(%o3) [Rinf=(1-sqrt(3))*R,Rinf=(sqrt(3)+1)*R]
(%o4) [Rinf=-0.73205*R,Rinf=2.7321*R]
```

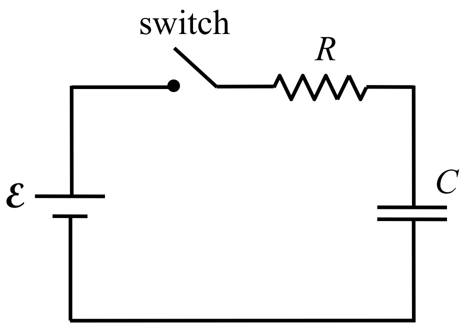


Fig. 9.16 Charging an *RC* circuit, Problem 9.11

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.

(%i4) Solve $RR_\infty + R_\infty^2 = 2R^2 + 3RR_\infty$ for R_∞.

Problem 9.11 Figure 9.16 shows an *RC* circuit. The circuit consists of a resistance *R*, a capacitance *C*, and a cell with emf ε. At time $t = 0$ s, the switch is closed. Show that the charge *q* in the capacitor and the current *i* in the circuit are given by

$$q = C\,\mathcal{E} \cdot (1 - e^{-\frac{t}{RC}}),$$

$$i = \frac{\mathcal{E}}{R} \cdot e^{-\frac{t}{RC}}.$$

Solution

When the switch is closed, the equation of the circuit is

$$\mathcal{E} = Ri + \frac{q}{C},$$

where *i* is the current in the circuit, *q* is the charge in the capacitor, and ε **E** is the emf of the battery. Here, *Ri* is the potential drop across the resistor and q/C is the potential drop across the capacitor. As electric current is the time rate of charge flow, $i = dq/dt$, the equation is written as

$$\mathcal{E} = R\frac{dq}{dt} + \frac{q}{C},$$

$$\frac{dq}{dt} + \frac{q}{RC} - \frac{\mathcal{E}}{R} = 0. \tag{1}$$

Equation (1) is a first-order ordinary differential equation, with time t as the independent variable and q as the dependent variable.

Let us guess a solution of the form

$$q = C\mathcal{E} - Ke^{-t/(RC)}. \tag{2}$$

where K is a constant and $C\varepsilon$ is the charge in a fully charged capacitor. Equation (2) says that q increases with t, and as t is very big q becomes $C\varepsilon$. The time derivative of q is

$$\frac{dq}{dt} = \frac{K}{RC}e^{-t/(RC)}. \tag{3}$$

If we substitute (2) and (3) into (1), we get

$$\frac{dq}{dt} + \frac{q}{RC} - \frac{\mathcal{E}}{R} = \frac{K}{RC}e^{-t/(RC)} + \frac{C\,\mathcal{E} - Ke^{-t/(RC)}}{RC} - \frac{\mathcal{E}}{R} = 0.$$

This shows that (2) is a solution of (1).

The initial condition says that at time $t = 0$ s, the charge $q = 0$ C. Substituting these values into Eq. (2) gives,

$$\begin{aligned} 0 &= C\,\mathcal{E} - K, \\ K &= C\,\mathcal{E}. \end{aligned}$$

Thus, Eq. (2) becomes

$$\begin{aligned} q &= C\,\mathcal{E} - Ke^{-t/(RC)} \\ &= C\,\mathcal{E} - C\,\mathcal{E}e^{-t/(RC)} \\ &= C\,\mathcal{E} \cdot (1 - e^{-t/(RC)}). \end{aligned}$$

Electric current is obtained by differentiating the charge with respect to time

$$i = \frac{dq}{dt} = \frac{\mathcal{E}}{R} \cdot e^{-t/(RC)}.$$

We have shown that that the charge q in the capacitor and the current i in the circuit are given by

$$q = C\,\mathcal{E} \cdot (1 - e^{-\frac{t}{RC}}),$$

$$i = \frac{\mathcal{E}}{R} \cdot e^{-\frac{t}{RC}}.$$

This is the same as note (5) at the beginning of this chapter.

These results can also be obtained using predefined functions *ode*2 and *ic*1 of wxMaxima. See *Solving first order ordinary differential equation* in Appendix A. The first order ordinary differential equation to be solved is $\frac{dq}{dt} + \frac{q}{RC} - \frac{\mathcal{E}}{R} = 0$ and the initial condition is $t = 0$ s, $q = 0$ C. Charge q is the dependent variable and time t is the independent variable.

◆ wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)     5
(ratprint)  false
(%i3) sol: ode2('diff(q,t) + q/(RC)-emf/R = 0, q, t);
(sol) q=%e^(-t/RC)*((RC*emf*%e^(t/RC))/R+%c)
(%i5) ic1(sol, t=0, q=0)$ expand(%);
(%o5) q=(RC*emf)/R-(RC*emf*%e^(-t/RC))/R
(%i6) q: rhs(%);
(q)   (RC*emf)/R-(RC*emf*%e^(-t/RC))/R
(%i8) i: diff(q,t)$ expand(%);
(%o8) (emf*%e^(-t/RC))/R
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i3) Get a general solution of ODE $\frac{dq}{dt} + \frac{q}{RC} - \frac{\mathcal{E}}{R} = 0$.
(%i5) Set the initial condition and get a particular solution.
(%i6) Assign the solution to q.
(%i8) Calculate i.

The codes show that the charge is, (%o5),

$$q = \frac{RC\,\mathcal{E}}{R} - \frac{RC\,\mathcal{E}e^{-t/(RC)}}{R} = C\,\mathcal{E} \cdot (1 - e^{-t/(RC)}),$$

and the current is, (%o8),

$$i = \frac{\mathcal{E}\,e^{-t/(RC)}}{R} = \frac{\mathcal{E}}{R} \cdot e^{-t/(RC)}.$$

Problem 9.12 Figure 9.17 shows a circuit consisting of a resistance R and a capacitance C, and a switch. The capacitor has an initial charge of Q_0. The switch is closed at $t = 0$ s so that the circuit is completed. Show that the charge Q in the capacitor and current I in the circuit change with time t as

$$Q(t) = Q_0 \cdot e^{-\frac{t}{RC}},$$

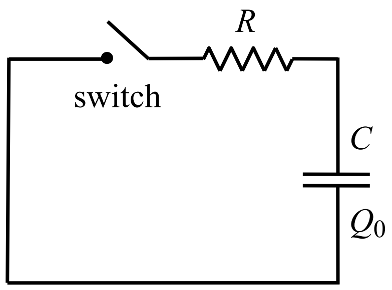


Fig. 9.17 Discharging an *RC* circuit, Problem 9.12

$$I(t) = \frac{Q_0}{RC} \cdot e^{-\frac{t}{RC}}.$$

Solution

The circuit equation of the problem is,

$$RI + \frac{Q}{C} = 0,$$

$$R\frac{dQ}{dt} + \frac{Q}{C} = 0. \qquad (1)$$

This is a first-order ordinary differential equation with Q and t as its dependent and independent variables, respectively. The initial condition is, at $t = 0$ s, $Q = Q_0$.

Let us guess a solution of the form

$$Q = Ke^{-t/(RC)}, \qquad (2)$$

where K is a constant. Equation (2) says that Q decreases with t, and as t is very big Q becomes 0 C. The time derivative of Q is

$$\frac{dQ}{dt} = -\frac{K}{RC}e^{-t/(RC)}. \qquad (3)$$

Substituting (2) and (3) into (1) gives

$$R\frac{dQ}{dt} + \frac{Q}{C} = R\left(-\frac{K}{RC}e^{-t/(RC)}\right) + \frac{Ke^{-t/(RC)}}{C} = 0.$$

This shows that (2) is a general solution of the ordinary differential equation (1). Substituting initial condition at $t = 0$ s, $Q = Q_0$ into (2) gives

$$Q = Ke^{-t/(RC)},$$
$$Q_0 = K,$$

and the particular solution becomes

$$Q = Q_0 \cdot e^{-t/(RC)}.$$

This is the variation of the charge in the capacitor with time. The current in the circuit is

$$I = \frac{dQ}{dt} = -\frac{Q_0}{RC} \cdot e^{-t/(RC)},$$

and its magnitude is

$$I = \frac{Q_0}{RC} \cdot e^{-t/(RC)}.$$

These results are in note (6) at the beginning of the chapter.

The first-order ordinary differential equation can be solved via predefined functions *ode*2 and *ic*1 of wxMaxima. See *Solving first order ordinary differential equation* in Appendix A.

◆ wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)     5
(ratprint)  false
(%i3) sol: ode2(R*'diff(Q,t) + Q/C = 0, Q, t);
(sol) Q=%c*%e^(-t/(C*R))
(%i4) ic1(sol, t=0, Q=Q0);
(%o4) Q=Q0*%e^(-t/(C*R))
(%i5) Q: rhs(%);
(Q)   Q0*%e^(-t/(C*R))
(%i6) I: abs(diff(Q,t));
(I)   (abs(Q0)*%e^(-t/(C*R)))/(abs(C)*abs(R))
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i3) Get a general solution of ODE $R\ dQ/dt + Q/C = 0$.
(%i4) Set the initial condition and get a particular solution.
(%i5) Assign the solution to Q.
(%i6) Calculate I.

9.3 Summary

- Potential difference across a conductor of resistance R carrying current I is $V = IR$.
- Resistors R_1, R_2, R_3, … connected in series have an equivalent resistance R_{eqv} of

$$R_{eqv} = R_1 + R_2 + R_3 + \ldots$$

- Resistors R_1, R_2, R_3, … connected in parallel have an equivalent resistance R_{eqv} that can be obtained from

$$\frac{1}{R_{eqv}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots$$

- Kirchoff's rules can be used to solve direct current circuit. The rules are (1) the sum of the currents into any junction is zero and (2) the sum of potential differences across each element around a closed loop is zero.
- In an RC circuit, the current I in the circuit and the charge Q in the capacitor vary with time as:

$$I(t) = \frac{\mathcal{E}}{R} \cdot e^{-\frac{t}{RC}} = I_{max} \cdot e^{-t/\tau},$$

$$Q(t) = C\,\mathcal{E} \cdot (1 - e^{-\frac{t}{RC}}) = Q_{max} \cdot (1 - e^{-t/\tau}),$$

where $I_{max} = \varepsilon/R$ is the maximum current, $Q_{max} = C\varepsilon$ is the maximum charge of the capacitor, and $\tau = RC$ is the time constant of the circuit.

9.4 Exercises

Exercise 9.1 Calculate electric currents through points A, B, C, and D of Fig. 9.18.
(Answer: $I_A = 2.5$ A, $I_B = 1.7$ A, $I_C = 5.2$ A, $I_D = 1.0$ A)

Exercise 9.2 What is the equivalent resistance between points A and B of resistors in Fig. 9.19.
(Answer: 0.85 Ω)

Exercise 9.3 Calculate the current in the 3.0, 6.0, and 12 Ω resistors in the circuit shown in Fig. 9.20.
(Answer: 1.5 A, 1.0 A, 0.5 A)

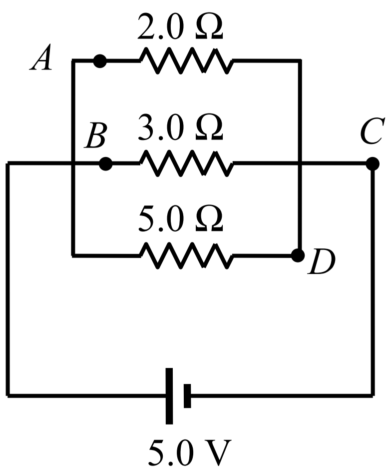


Fig. 9.18 Circuit of Exercise 9.1

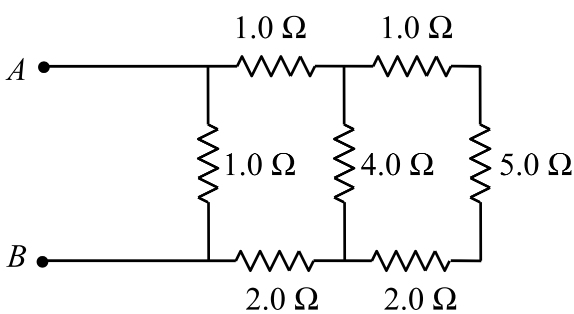


Fig. 9.19 Network of resistors, Exercise 9.2

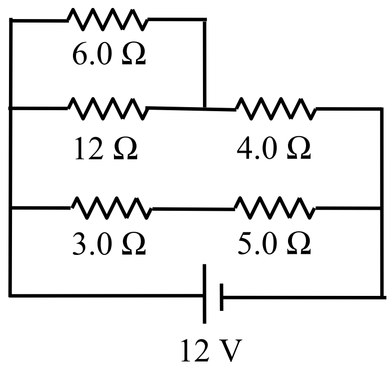


Fig. 9.20 Circuit of Exercise 9.3

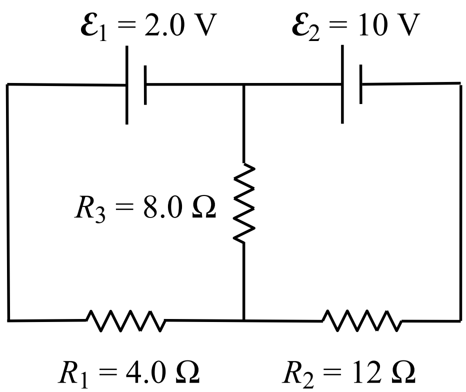


Fig. 9.21 Circuit of Exercise 9.4

Exercise 9.4 Determine currents in each resistor of the circuit in Fig. 9.21. Use Kirchhoff's rule.

(Answer: Current in R_1 is 0.68 A to the right, current in R_2 is 0.77 A to the right, current in R_3 is 0.09 A from top to bottom)

Exercise 9.5 Figure 9.22 shows an *RC* circuit during charging and discharging. The emf is $\varepsilon = 6.0$ V, resistance is $R = 5.0 \times 10^5$ Ω, and the capacitance is $C = 8.0 \times 10^{-6}$ F.

(a) Calculate the time constant of the circuit.
(b) What are the maximum current in the circuit and the maximum charge of the capacitor?
(c) Get equations of current against time and charge against time for charging and discharging.

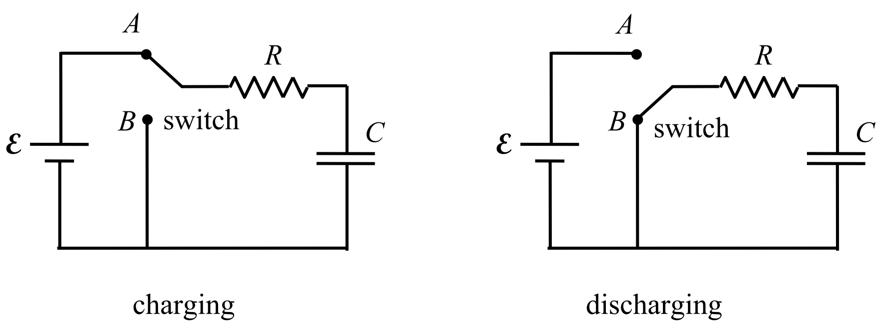


Fig. 9.22 Charging and discharging of an *RC* circuit, Exercise 9.5

(Answer: (a) time constant is 4.0 s, (b) $I_{max} = 1.2 \times 10^{-5}$ A, $Q_{max} = 4.8 \times 10^{-5}$ C, (c) charging: $I = 1.2 \times 10^{-5} \exp(-0.25t)$, $Q = 4.8 \times 10^{-5} [1 - \exp(-0.25t)]$, discharging: $I = 1.2 \times 10^{-5} \exp(-0.25t)$, $Q = 4.8 \times 10^{-5} \exp(-0.25t)$)

Chapter 10
Magnetic Field

Abstract Problems with magnetic forces due to moving charged particles and current carrying conductors in magnetic fields are solved in this chapter. The torque due to the magnetic moment of the current carrying loop in the magnetic field is also discussed. Both analytical solutions and computer calculations by wxMaxima are presented.

10.1 Basic Concepts and Formulae

(1) Magnetic force that acts on a charge q moving with velocity $\boldsymbol{v}$ in a magnetic field $\boldsymbol{B}$ is

$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}. \tag{10.1}$$

The magnitude of the force is,

$$F = qvB \sin\theta, \tag{10.2}$$

where θ is the small angle between $\boldsymbol{v}$ and $\boldsymbol{B}$. SI unit for B is weber per meter square (Wb m^{-2}) or tesla (T),

$$1\,\text{T} = 1\,\text{Wb}\,\text{m}^{-2} = 1\,\text{N}\,\text{A}^{-1}\text{m}^{-1}. \tag{10.3}$$

(2) Magnetic force that acts on a straight conductor of length l carrying a current I in a uniform magnetic field $\boldsymbol{B}$ is

$$\boldsymbol{F} = I\boldsymbol{l} \times \boldsymbol{B}, \tag{10.4}$$

where vector $\boldsymbol{l}$ is in the same direction as the current.

The magnitude of the force is

$$F = IlB \sin\theta, \tag{10.5}$$

W. M. S. Wan Hassan et al., *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-43165-4_10

where θ is the small angle between $\boldsymbol{l}$ and $\boldsymbol{B}$.

(3) For any wire carrying current I in a uniform external magnetic field $\boldsymbol{B}$, the magnetic field $\boldsymbol{dF}$ on a small segment $\boldsymbol{ds}$ of the wire is

$$\boldsymbol{dF} = I\ \boldsymbol{ds} \times \boldsymbol{B}. \tag{10.6}$$

(4) Magnetic force acting on any current carrying closed loop in a uniform external magnetic field is zero.

(5) Magnetic moment $\boldsymbol{\mu}$ of a loop carrying current I is

$$\boldsymbol{\mu} = I\boldsymbol{A}, \quad \mu = IA, \tag{10.7}$$

where $\boldsymbol{A}$ is the area vector normal to the plane of the loop and A is the area of the loop. The area vector is defined as

$$\boldsymbol{A} = A\, n, \tag{10.8}$$

where **n** is the unit vector normal to the plane of the loop.

(6) Torque $\boldsymbol{\tau}$ on a loop in a uniform magnetic field $\boldsymbol{B}$ is

$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \boldsymbol{B} = I\boldsymbol{A} \times \boldsymbol{B}. \tag{10.9}$$

(7) When a charged particle moves in a magnetic field, the work done by the magnetic force on the particle is zero because displacement is always perpendicular to the magnetic force. Magnetic force changes the direction of velocity, but the speed remains the same. If velocity v of the particle is perpendicular to magnetic field B, the particle will move in a circular path whose plane is perpendicular to the magnetic field. The radius of the circular path is

$$r = \frac{mv}{qB}, \tag{10.10}$$

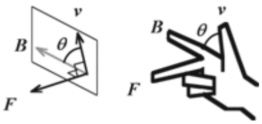
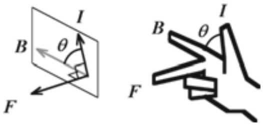
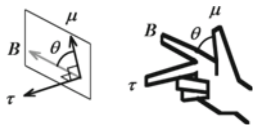
where m and q are the mass and charge of the particle, respectively. The angular frequency (cyclotron frequency) of the rotating particle is

$$\omega = \frac{qB}{m}. \tag{10.11}$$

(8) A particle of charge q moving at a velocity of $\boldsymbol{v}$ in the region of magnetic field $\boldsymbol{B}$ and electric field $\boldsymbol{E}$ is acted by Lorentz force which is given by,

$$\boldsymbol{F} = q\boldsymbol{E} + q\boldsymbol{v} \times \boldsymbol{B}. \tag{10.12}$$

Table 10.1 Directions of vector cross products

	Directions of vectors	Note
(a)		$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$ $\boldsymbol{v}$ – velocity of particle of charge q $\boldsymbol{B}$ – magnetic field $\boldsymbol{F}$ – magnetic force acting on the charged particle
(b)		$\boldsymbol{F} = I\boldsymbol{l} \times \boldsymbol{B}$ $\boldsymbol{l}$ – direction of $\boldsymbol{l}$ is the direction of current I $\boldsymbol{B}$ – magnetic field $\boldsymbol{F}$ – magnetic force acting on the conductor
(c)		$\boldsymbol{\tau} = \boldsymbol{\mu} \times \boldsymbol{B} = I\boldsymbol{A} \times \boldsymbol{B}$ $\boldsymbol{\mu}$ – magnetic moment $\boldsymbol{B}$ – magnetic field $\boldsymbol{\tau}$ – torque acting on the magnetic moment

(9) Table 10.1 shows how to determine the direction of the magnetic force on a charged particle moving in a magnetic field, the direction of the magnetic force on a current carrying conductor in a magnetic field, and the direction of the torque on a magnetic moment (a current carrying loop) in a magnetic field. All by the right-hand rule.

10.2 Problems and Solutions

Problem 10.1 A positron moves with a velocity of $v = 3.0 \times 10^5$ m s^{-1} in a uniform magnetic field of $B = 2.0 \times 10^3$ Gauss as shown in Fig. 10.1. Calculate the magnetic force on the positron.

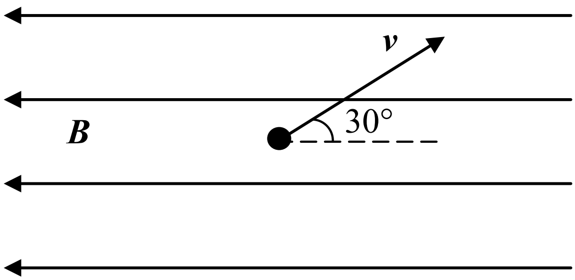


Fig. 10.1 A positron moving in a uniform magnetic field, Problem 10.1

Solution

A positron has the charge of an electron but of a positive sign, that is, $+1.6 \times 10^{-19}$ C. There is a magnetic force acting on a charged particle moving in a magnetic field. The magnitude of the magnetic force is, Eq. (10.2),

$$
\begin{aligned}
F &= qvB \sin\theta \\
&= (1.6 \times 10^{-19}\ \text{C})(3.0 \times 10^5\ \text{m/s}) \\
&\quad \times (2.0 \times 10^3\ \text{gauss} \times \frac{1\ \text{T}}{10^4\ \text{gauss}}) \sin(180^\circ - 30^\circ) \\
&= 4.8 \times 10^{-15}\ \text{N}.
\end{aligned}
$$

Conversion of unit 1 T = 10^4 gauss is used, see Appendix C. The direction of the force is out of the plane of the paper. This is determined by the right-hand rule, Table 10.1(a).

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; q:1.6e-19; v:3e5; B:2e3/1e4;
theta:(180-30)/180*float(%pi);
(fpprintprec)     5
(q)    1.6*10^-19
(v)    3.0*10^5
(B)    0.2
(theta) 2.618
(%i6) F: q*v*B*sin(theta);
(F)    4.8*10^-15
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of charge q, its speed v, magnetic field B, and angle θ in radian.
(%i6) Calculate the magnitude of magnetic force F.

Alternative calculation: Express velocity of the positron and magnetic field as vectors, and do the vector cross product, Eq. (10.1),

$$\boldsymbol{v} = 3.0 \times 10^5 \cos 30^\circ\ i\ +\ 3.0 \times 10^5 \sin 30^\circ\ j,$$

$$\boldsymbol{B} = -0.20\, i,$$

$$
\begin{aligned}
\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B} &= 1.6 \times 10^{-19} \begin{vmatrix} i & j & k \\ 3.0 \times 10^5 \cos\ 30^\circ & 3.0 \times 10^5 \sin 30^\circ & 0 \\ -0.20 & 0 & 0 \end{vmatrix} \\
&= 4.8 \times 10^{-15}\ k\ \text{N}.
\end{aligned}
$$

The magnetic force on the proton is 4.8×10^{-15} N in the positive z direction (out of the plane of the paper).

◆ wxMaxima codes:

```
(%i2) fpprintprec:5; load("vect");
(fpprintprec)     5
(%o2) "C:/maxima-5.43.0/share/maxima/5.43.0/share/vector/vect.mac"
(%i4) q:1.6e-19; angle:30/180*%pi;
(q) 1.6*10^-19
(angle) %pi/6
(%i5) v: [3e5*cos(angle), 3e5*sin(angle), 0];
(v)   [1.5*10^5*sqrt(3),1.5*10^5,0]
(%i6) B: [-0.2, 0, 0];
(B)   [-0.2,0,0]
(%i8) F: q*v~B; express(%);
(F)   -1.6*10^-19*[-0.2,0,0]~[1.5*10^5*sqrt(3),1.5*10^5,0]
(%o8) [0,0,4.8*10^-15]
```

Comments on the codes:

(%i2)	Set floating point print precision to 5 and load the "vect" vector package.
(%i4)	Assign values of charge of positron q and angle in radian.
(%i5) (%i6)	Assign velocity $\boldsymbol{v}$ and magnetic field $\boldsymbol{B}$.
(%i8)	Calculate the vector cross product $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$.

Problem 10.2 Calculate the force on a 8.0×10^{-18} C charged particle moving with a velocity of $3.0 \times 10^5\,\mathbf{i}$ m s^{-1} in a uniform magnetic field of $3.0\,\mathbf{j}$ T.

Solution

The magnetic force on the charged particle is, Eq. (10.1),

$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B} = 8.0 \times 10^{-18} \begin{vmatrix} i & j & k \\ 3.0 \times 10^5 & 0 & 0 \\ 0 & 3.0 & 0 \end{vmatrix}$$
$$= 7.2 \times 10^{-12}\, k \text{ N}.$$

The magnetic force on the charged particle is 7.2×10^{-12} N in the positive z direction (out of the plane of the paper).

◆ wxMaxima codes:

```
(%i2) fpprintprec:5; load("vect");
(fpprintprec)     5
(%o2) "C:/maxima-5.43.0/share/maxima/5.43.0/share/vector/vect.mac"
(%i5) q:8e-18; v: [3e5, 0, 0]; B: [0, 3, 0];
(q)    8.0*10^-18
(v)    [3.0*10^5,0,0]
(B)    [0,3,0]
(%i7) F: q*v~B; express(%);
(F)    -8.0*10^-18*[0,3,0]~[3.0*10^5,0,0]
(%o7) [0,0,7.2*10^-12]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and load the "vect" vector package.
(%i5) Assign values of q, $\boldsymbol{v}$, and $\boldsymbol{B}$.
(%i7) Calculate $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$.

Problem 10.3 An electron is moving with a velocity of $\boldsymbol{v} = 4.0 \times 10^5\,\mathbf{i}$ m s^{-1} in a uniform magnetic field of $\boldsymbol{B} = 3.0\,\mathbf{k}$ Wb m^{-2}. Calculate

(a) acceleration of the electron
(b) radius of the circular path traced by the electron.

Solution

(a) Force on the electron is, Eq. (10.1),

$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B} = -1.6 \times 10^{-19} \begin{vmatrix} i & j & k \\ 4.0 \times 10^5 & 0 & 0 \\ 0 & 0 & 3.0 \end{vmatrix}$$

$$= 1.9 \times 10^{-13}\, j \text{ N}.$$

Acceleration of the electron is

$$\boldsymbol{a} = \frac{\boldsymbol{F}}{m_e} = \frac{1.9 \times 10^{-13}}{9.1 \times 10^{-31}}\, j \text{ m s}^{-2} = 2.1 \times 10^{17}\, j \text{ m s}^{-2}.$$

where m_e is the mass of the electron. This is the centripetal acceleration of the electron.

(b) The magnitude of centripetal acceleration is $a = v^2/r$, where v is the speed of the electron and r is the radius of the circular path. The radius of the circular path of the electron is

$$r = \frac{v^2}{a} = \frac{(4.0 \times 10^5 \text{ m/s})^2}{2.1 \times 10^{17} \text{ m/s}^2} = 7.6 \times 10^{-7} \text{ m}.$$

◆ wxMaxima codes:

```
(%i3) fpprintprec:5; me:9.1e-31; load("vect");
(fpprintprec)     5
(me)  9.1*10^-31
(%o3) "C:/maxima-5.43.0/share/maxima/5.43.0/share/vector/vect.mac"
(%i6) q:-1.6e-19; v:[4e5, 0, 0]; B:[0, 0, 3];
(q)   -1.6*10^-19
(v)   [4.0*10^5,0,0]
(B)   [0,0,3]
(%i8) F: q*v~B; express(%);
(F)   1.6*10^-19*[0,0,3]~[4.0*10^5,0,0]
(%o8) [0,1.92*10^-13,0]
(%i10) a: F/me; express(%);
(a)   1.7582*10^11*[0,0,3]~[4.0*10^5,0,0]
(%o10) [0,2.1099*10^17,0]
(%i11) a_mag: 2.1099*10^17;
(a_mag) 2.1099*10^17
(%i12) r: 4e5^2/a_mag;
(r)   7.5833*10^-7
```

Comments on the codes:

(%i3)	Set floating point print precision to 5, assign m_e, and load "vect" vector package.
(%i6)	Assign values of q, $\boldsymbol{v}$, and $\boldsymbol{B}$.
(%i8), (%i10)	Calculate $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$ and $\boldsymbol{a} = \boldsymbol{F}/m_e$.
(%i11)	Assign magnitude of acceleration a.
(%i12)	Calculate r.

Further question: What difference will it be, if you have a proton instead of an electron?

Answer: We redo the calculations. Force on the proton is

$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B} = 1.6 \times 10^{-19} \begin{vmatrix} i & j & k \\ 4.0 \times 10^5 & 0 & 0 \\ 0 & 0 & 3.0 \end{vmatrix}$$

$$= -1.9 \times 10^{-13}\, j \text{ N}.$$

The magnetic force on the proton is the same in magnitude but opposite in direction to that of the electron.

The centripetal acceleration of the proton is

$$\boldsymbol{a} = \frac{\boldsymbol{F}}{m_p} = \frac{-1.9 \times 10^{-13}}{1.67 \times 10^{-27}}\, j \text{ m s}^{-2} = -1.1 \times 10^{14}\, j \text{ m s}^{-2}.$$

where m_p is the mass of the proton. The magnitude of proton centripetal acceleration is smaller than that of the electron.

The magnitude of centripetal acceleration is $a = v^2/r$, where v is the speed of the proton and r is the radius of the circular path. The radius of the circular path of the proton is

$$r = \frac{v^2}{a} = \frac{(4 \times 10^5 \text{ m/s})^2}{1.1 \times 10^{14} \text{ m/s}^2} = 1.4 \times 10^{-3} \text{ m.}$$

The radius of the circular path of the proton is bigger than that of the electron.

◆ wxMaxima codes:

```
(%i3) fpprintprec:5; mp:1.67e-27; load("vect");
(fpprintprec)     5
(mp)  1.67*10^-27
(%o3) "C:/maxima-5.43.0/share/maxima/5.43.0/share/vector/vect.mac"
(%i6) q:1.6e-19; v:[4e5, 0, 0]; B:[0, 0, 3];
(q)   1.6*10^-19
(v)   [4.0*10^5,0,0]
(B)   [0,0,3]
(%i8) F: q*v~B; express(%);
(F)   -1.6*10^-19*[0,0,3]~[4.0*10^5,0,0]
(%o8) [0,-1.92*10^-13,0]
(%i10) a: F/mp; express(%);
(a)   -9.5808*10^7*[0,0,3]~[4.0*10^5,0,0]
(%o10) [0,-1.1497*10^14,0]
(%i11) a_mag: 1.1497*10^14;
(a_mag) 1.1497*10^14
(%i12) r: 4e5^2/a_mag;
(r)   0.0013917
```

Comments on the codes:

(%i3)	Set floating point print precision to 5, assign m_p, and load "vect" vector package.
(%i6)	Assign values of q, $\boldsymbol{v}$, and $\boldsymbol{B}$.
(%i8), (%i10)	Calculate $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$ and $\boldsymbol{a} = \boldsymbol{F}/m_p$.
(%i11)	Assign magnitude of acceleration a.
(%i12)	Calculate r.

Problem 10.4 He^{2+} ion is moving at a velocity of 1.0×10^5 m s^{-1} perpendicular to a magnetic field of 1.0 T. Calculate the magnitude of the magnetic force on the ion.

Solution

The magnitude of the magnetic force on the helium ion is

$$\begin{aligned} F &= qvB \sin\theta \\ &= (2 \times 1.6 \times 10^{-19} \text{ C})(1.0 \times 10^5 \text{ m/s})(1.0 \text{ T}) \sin 90^\circ \\ &= 3.2 \times 10^{-14} \text{ N.} \end{aligned}$$

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; q:2*1.6e-19;  v:1e5;  B:1;  theta:float(%pi/2);
(fpprintprec)     5
(q)   3.2*10^-19
(v)   1.0*10^5
(B)   1
(theta) 1.5708
(%i6) F: q*v*B*sin(theta);
(F)   3.2*10^-14
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of q, v, B, and θ.
(%i6) Calculate the magnitude of magnetic force, F.

Problem 10.5 A particle of mass 1.0 g and charge of 2.5×10^{-8} C moves with a horizontal velocity of 6.0×10^4 m s^{-1} in a region that has both gravitational and magnetic fields. What is the magnitude and direction of the magnetic field so that the particle stays moving in a horizontal path?

Solution

Figure 10.2(a) shows the particle moving horizontally with a velocity of $\boldsymbol{v}$ to the right. The particle will stay in the horizontal path if the weight of the particle, mg, is balanced by the magnetic force, F_m. The gravitational field and the weight of the particle are in the downward direction. To get a magnetic force in the upward direction, the magnetic field $\boldsymbol{B}$ must be into the plane of the paper as indicated by crosses in Fig. 10.2(a). This can be deduced by the right-hand rule, Fig. 10.2(b).

If there is no magnetic field, the particle will move to the right and downward in a parabolic path due to gravitational force, mg. To balance this force, magnetic force, F_m, is needed,

$$F_m = qvB.$$

The weight of the particle must be equal in magnitude to the magnetic force,

$$mg = qvB,$$

and the magnetic field is,

$$B = \frac{mg}{qv} = \frac{(1.0 \times 10^{-3}\ \text{kg})(9.8\ \text{m/s}^2)}{(2.5 \times 10^{-8}\ \text{C})(6.0 \times 10^4\ \text{m/s})} = 6.5\ \text{T}.$$

The direction of the magnetic field is into the plane of the paper and this can be deduced by the right-hand rule, Fig. 10.2(b).

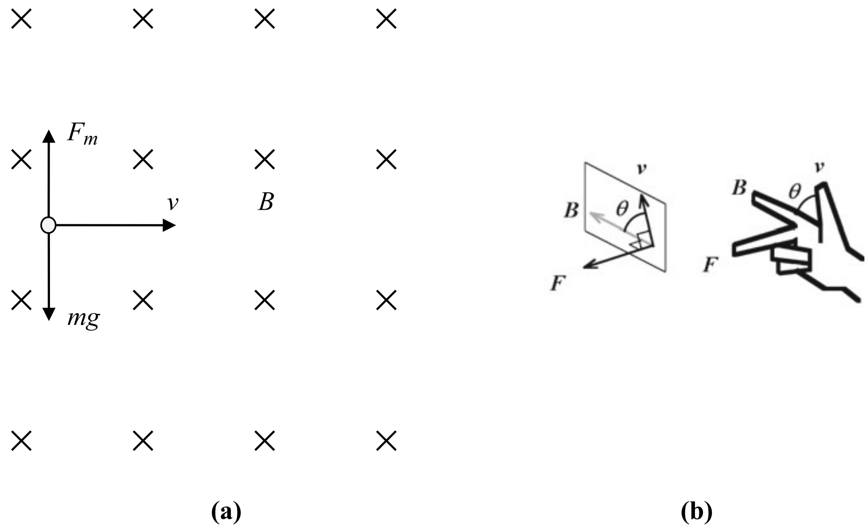


Fig. 10.2 (a) A charged particle moving in a region of magnetic and gravitational fields, (b) directions of $\boldsymbol{v}$, $\boldsymbol{B}$, and magnetic force, Problem 10.5

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; m:1e-3;  g:9.8;  q:2.5e-8;  v:6e4;
(fpprintprec)     5
(m)    0.001
(g)    9.8
(q)    2.5*10^-8
(v)    6.0*10^4
(%i6) B: m*g/(q*v);
(B)    6.5333
```

Comments on the codes:

(%i5) Set floating point print precision to 5 and assign values of m, g, q, and v.
(%i6) Calculate the magnitude of the magnetic field, B.

Problem 10.6 An electron is moving with a velocity of $v = 1.0 \times 10^7$ m s^{-1} at point P, Fig. 10.3. Calculate

(a) the magnitude and direction of the magnetic field that causes the electron to follow a semicircular path.
(b) the time taken for the electron to travel from point P to Q in a semicircular path.

Solution

(a) When the electron moves in a circular path, the magnetic force on the electron is the centripetal force. This means that

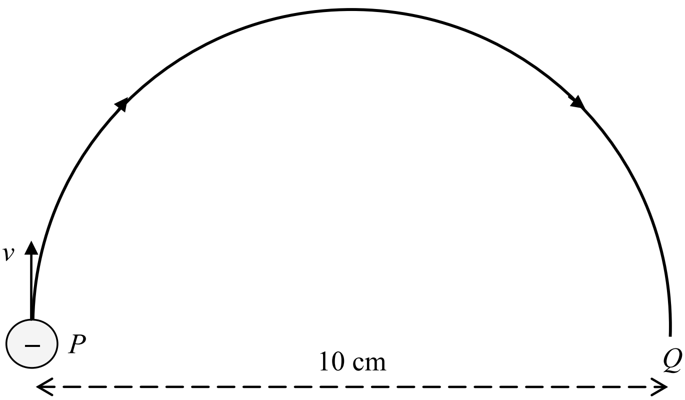


Fig. 10.3 An electron semicircular path in a region of uniform magnetic field, Problem 10.6

$$qvB = m\frac{v^2}{R},$$

where q, v, and m are the magnitude of charge, speed, and mass of the electron, respectively, B is the magnetic field, and R is the radius of the semicircle. The magnetic field is

$$B = \frac{mv}{qR} = \frac{(9.1 \times 10^{-31}\ \text{kg})(1.0 \times 10^{7}\ \text{m/s})}{(1.6 \times 10^{-19}\ \text{C})(5.0 \times 10^{-2}\ \text{m})} = 1.1 \times 10^{-3}\ \text{T}.$$

The direction of the magnetic field is in the plane of the paper. This can be deduced by the right-hand rule.

(b) The time of travel is

$$\text{time} = \frac{\text{distance}}{\text{speed}} = \frac{\pi R}{\text{speed}} = \frac{\pi(5.0 \times 10^{-2}\ \text{m})}{1.0 \times 10^{7}\ \text{m/s}} = 1.6 \times 10^{-8}\ \text{s}.$$

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; m:9.1e-31; v:1e7; q:1.6e-19; R:5e-2;
(fpprintprec)     5
(m)   9.1*10^-31
(v)   1.0*10^7
(q)   1.6*10^-19
(R)   0.05
(%i6) B: m*v/(q*R);
(B)   0.0011375
(%i8) time: %pi*R/v; float(%);
(time) 5.0*10^-9*%pi
(%o8) 1.5708*10^-8
```

Comments on the codes:

(%i5) Set floating point print precision to 5, and assign values of m, v, q, and R.

(%i6) Calculate the magnitude of magnetic field B.

(%i8) Calculate the time of travel.

Problem 10.7 An electron is accelerated from rest by a potential difference of 3750 V. The electron enters a region with magnetic field $B = 4.0 \times 10^{-3}$ T that is perpendicular to the velocity of the electron. Calculate the radius of the circular path of the electron.

Solution

Figure 10.4 shows the electron being accelerated by a potential difference of 3750 V, enters a region of uniform magnetic field, and moves in a circular path. The direction of magnetic field B is into the plane of the paper. The radius of the circular path is R.

The velocity of the electron when it enters the region of uniform magnetic field is

$$v = \sqrt{\frac{2eV}{m}},$$

where m and e are the mass and magnitude of charge of the electron, respectively, and V is the potential difference. This is obtained by equating the potential energy of the electron with its kinetic energy

$$eV = \frac{1}{2}mv^2.$$

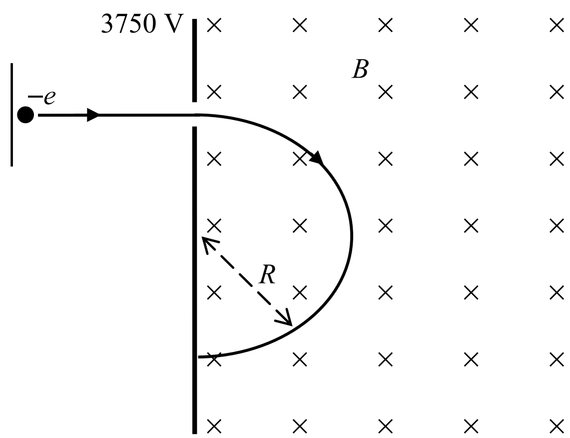


Fig. 10.4 An electron accelerated by a potential difference and its path in a region of uniform magnetic field, Problem 10.7

The magnetic force that acts on the electron in the region of the magnetic field is the centripetal force

$$F = evB = m\frac{v^2}{R}.$$

Therefore, the radius of the circular path of the electron is

$$\begin{aligned} R &= \frac{mv}{eB} = \frac{m}{eB}\sqrt{\frac{2eV}{m}} = \sqrt{\frac{2mV}{B^2 e}} \\ &= \sqrt{\frac{2(9.1 \times 10^{-31}\ \text{kg})\ (3750\ \text{V})}{(4.0 \times 10^{-3}\ \text{T})^2 (1.6 \times 10^{-19}\ \text{C})}} \\ &= 5.2 \times 10^{-2}\ \text{m}. \end{aligned}$$

◆ wxMaxima codes:

```
(%i6) fpprintprec:5; ratprint:false; m:9.1e-31; V:3750; B:4e-3; e:1.6e-19;
(fpprintprec)     5
(ratprint)  false
(m)   9.1*10^-31
(V)   3750
(B)   0.004
(e)   1.6*10^-19
(%i7) v: sqrt(2*e*V/m);
(v)   3.6314*10^7
(%i9) solve(e*v*B=m*v^2/R, R)$ float(%);
(%o9) [R=0.051633]
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, and assign values of m, V, B, and e.

(%i7) Calculate speed, v.

(%i9) Solve $evB = mv^2/R$ for R.

Problem 10.8 Derive an expression for the cyclotron frequency of a particle of mass m and a charge of q, moving with a speed of v in a plane perpendicular to a uniform magnetic field of B.

Solution

Figure 10.5 shows the particle of charge $+q$ and mass m moving with a speed of v in a magnetic field of B in a cyclotron. The magnetic field is in the plane of the paper. The radius of the circular path of the particle is R.

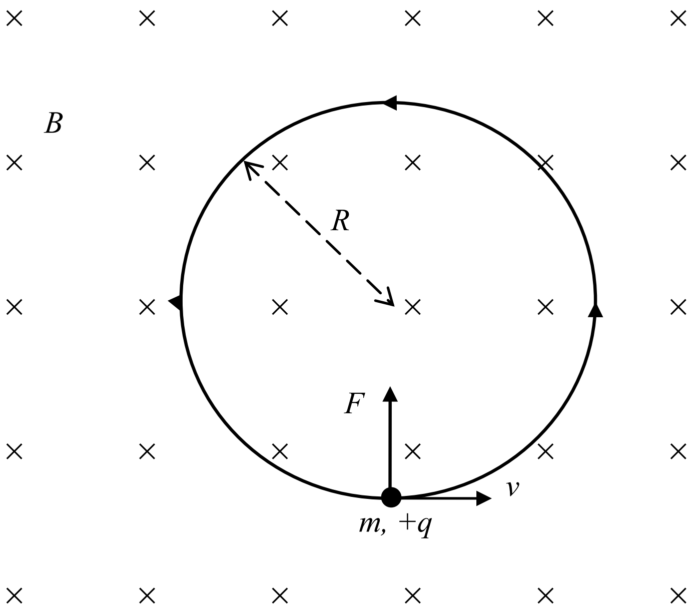


Fig. 10.5 Circular path of a charged particle in a cyclotron, Problem 10.8

Magnetic force on the particle is

$$F = qvB.$$

The direction of the force is toward the center of the circle. This is the centripetal force acting on the particle. The source of the force is the charged particle motion in a magnetic field. Centripetal acceleration of the particle is

$$a = \frac{F}{m} = \frac{qvB}{m}.$$

Radius of the circular path is calculated as follows

$$a = \frac{v^2}{R},$$
$$R = \frac{v^2}{a} = \frac{v^2}{qvB/m} = \frac{mv}{qB}.$$

Therefore, the cyclotron frequency is

$$f = \frac{v}{2\pi R} = \frac{vqB}{2\pi mv} = \frac{qB}{2\pi m}.$$

◆ wxMaxima codes:

```
(%i1) R: m*v/(q*B);
(R)    (m*v)/(B*q)
(%i2) f: v/(2*%pi*R);
(f)    (B*q)/(2*%pi*m)
```

Comments on the codes:

(%i1) Assign R.

(%i2) Calculate cyclotron frequency, f.

Problem 10.9 A particle of charge q and mass m is accelerated from rest by a potential difference of V. The particle then enters a region of uniform magnetic field of B that is perpendicular to the direction of particle motion. The particle enters the magnetic field region along the x-axis at $x = 0$. Show that the y coordinate of the particle position after time t is

$$y = Bx^2\left(\frac{q}{8mV}\right)^{1/2}.$$

Solution

Figure 10.6 shows the particle of charge $+q$ and mass m accelerated by a potential difference of V. It enters into the region of magnetic field B with speed v and gets deflected. The magnetic field is into the plane of the paper, R is the radius of the circular path of the particle, and C is the center of the circle. The magnetic force acting on the particle is toward point C as can be deduced by the right-hand rule.

The speed of the particle, v, on entering the region of the magnetic field is calculated as follows

$$qV = \frac{1}{2}mv^2,$$
$$v = \sqrt{\frac{2qV}{m}}.$$

That is, the electric potential energy of the particle, qV, is converted to kinetic energy, $mv^2/2$. When the particle is in the region of the magnetic field, the magnetic force is the centripetal force. Thus, we can calculate the radius of the circular segment

$$qvB = \frac{mv^2}{R},$$
$$R = \frac{mv}{qB} = \frac{m}{qB}\sqrt{\frac{2qV}{m}} = \sqrt{\frac{2mV}{qB^2}}.$$

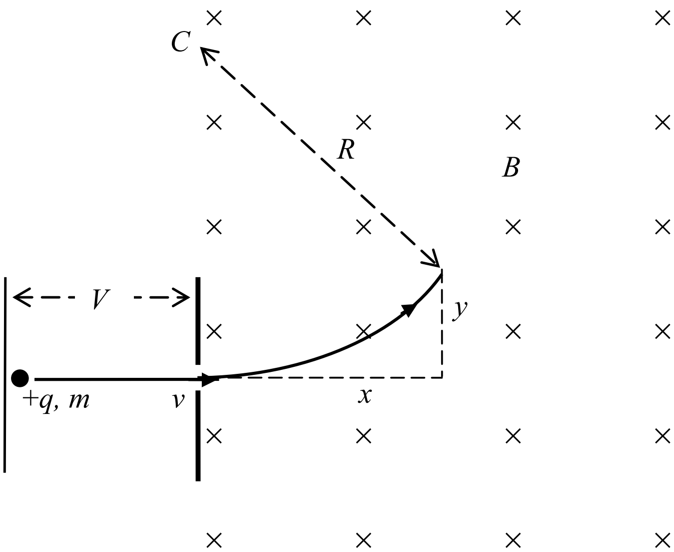


Fig. 10.6 Path of a charged particle accelerated by a potential difference and in a region of uniform magnetic field, Problem 10.9

From Figure 10.6 and Pythagoras theorem, we write

$$\begin{aligned} R^2 &= (R-y)^2 + x^2 \\ &= R^2 - 2Ry + y^2 + x^2, \\ 2Ry &= y^2 + x^2 \\ &\approx x^2. \end{aligned}$$

The last expression is obtained because $x^2 >> y^2$. Therefore, the y coordinate is

$$\begin{aligned} y = \frac{x^2}{2R} &= \frac{x^2}{2\sqrt{\frac{2mV}{qB^2}}} \\ &= Bx^2\left(\frac{q}{8mV}\right)^{1/2}. \end{aligned}$$

◆ wxMaxima codes:

```
(%i1) solve([R=sqrt(2*m*V/(q*B^2)), 2*R*y=x^2], [y,R]);
(%o1) [[y=(abs(B)*x^2)/(2^(3/2)*sqrt((V*m)/q)),
        R=(sqrt(2)*sqrt((V*m)/q))/abs(B)]]
```

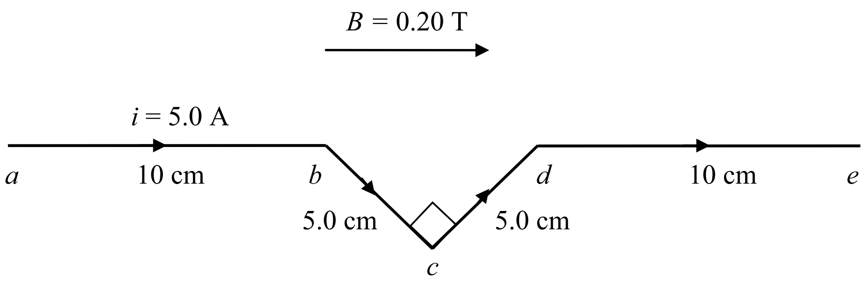


Fig. 10.7 A current carrying wire in a region of uniform magnetic field, Problem 10.10

Comments on the codes:

(%i1) Solve $R = \sqrt{\frac{2mV}{qB^2}}$ and $2Ry = x^2$ for y and R.

(%o1) The solutions.

Problem 10.10 Figure 10.7 shows a wire carrying a current of $i = 5.0$ A in a magnetic field of $B = 0.20$ T in the x direction. Calculate the magnetic force on each segment of the wire.

Solution

Magnetic force on a current-carrying conductor in the region with magnetic field is, Eq. (10.4) and (10.5)

$$\boldsymbol{F} = i\boldsymbol{l} \times \boldsymbol{B}, \quad F = ilB \sin\theta.$$

For wire segment *ab*, the force acting on it is

$$F_{ab} = ilB \sin 0 = 0,$$

because the current and the magnetic field are parallel to each other.

The force acting on wire segment *de* is zero as well,

$$F_{de} = 0.$$

For wire segment *bc*, the force acting on it is

$$F_{bc} = ilB \sin\theta = (5.0 \text{ A})(5.0 \times 10^{-2} \text{ m})(0.20 \text{ T}) \sin 45^\circ = 3.5 \times 10^{-2} \text{ N}.$$

The direction of the force is out of the plane of the paper. This can be deduced by the right-hand rule.

Lastly, for wire segment *cd*, the force acting on it is

$$F_{cd} = ilB\sin\theta = (5.0\ \text{A})(5.0\times10^{-2}\ \text{m})(0.20\ \text{T})\sin 45° = 3.5\times10^{-2}\ \text{N}.$$

The direction of the force is into the plane of the paper, determined by the right-hand rule.

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; i:5; l:5e-2; B:0.2; theta:float(45/180*%pi);
(fpprintprec)     5
(i)    5
(l)    0.05
(B)    0.2
(theta) 0.7854
(%i6) Fbc: i*l*B*sin(theta);
(Fbc) 0.035355
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of i, l, B, and θ.

(%i6) Calculate F_{bc}.

Problem 10.11 Figure 10.8 shows a current of $i = 5.0$ A in wire *abcde*. The wire is in a uniform magnetic field of $B = 0.15$ T pointing to the right. Calculate the force on each segment of the wire.

Solution

The magnetic force on a wire segment $\boldsymbol{l}$ carrying a current of i in a magnetic field of $\boldsymbol{B}$ is, Eq. (10.4) and (10.5),

$$\boldsymbol{F} = i\boldsymbol{l}\times\boldsymbol{B}, \quad F = ilB\sin\theta.$$

For wire segment *ab*, the magnetic force acting on it is

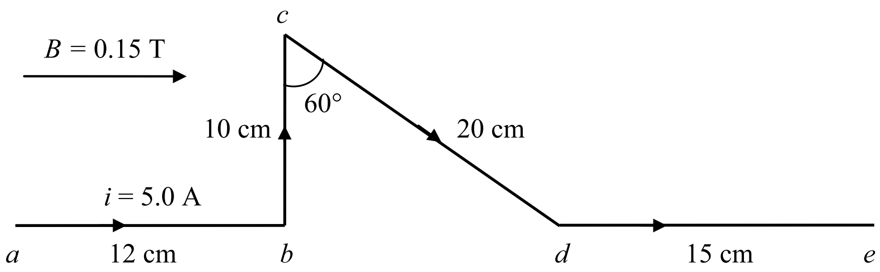


Fig. 10.8 A current carrying wire in a region of uniform magnetic field, Problem 10.11

$$F_{ab} = ilB\sin\theta = (5.0\text{ A})(0.12\text{ m})(0.15\text{ T})\sin 0^\circ = 0,$$

because $\theta = 0$. The same goes to wire segment *de*

$$F_{de} = 0.$$

For wire segment *bc*, the magnetic force acting on it is

$$F_{bc} = ilB\sin\theta = (5.0\text{ A})(0.10\text{ m})(0.15\text{ T})\sin 90^\circ = 7.5\times 10^{-2}\text{ N}.$$

pointing into the plane of the paper as determined by the right-hand rule.

Lastly, for wire segment *cd*, the magnetic force acting on it is

$$F_{cd} = ilB\sin\theta = (5.0\text{ A})(0.20\text{ m})(0.15\text{ T})\sin 30^\circ = 7.5\times 10^{-2}\text{ N}.$$

pointing out of the plane of the paper as determined by the right-hand rule.

◆ wxMaxima codes:

```
(%i1) fpprintprec:5;
(fpprintprec)     5
(%i2) Fbc: 5*0.1*0.15*sin(90/180*%pi);
(Fbc) 0.075
(%i3) Fcd: 5*0.2*0.15*sin(30/180*%pi);
(Fcd) 0.075
```

Comments on the codes:

(%i1) Set floating point print precision to 5.

(%i2), (%i3) Calculate F_{bc} and F_{cd}.

Problem 10.12 An imaginary cube of side 1.0 m is in a uniform magnetic field of $B = 2.0$ T pointing in the positive of x direction, as shown in Fig. 10.9. A current of $i = 3.0$ A flows in the wire loop *abcdefa* as shown. Calculate:

(a) the magnitude and direction of magnetic force acting on each segment of the wire.
(b) the resultant magnetic force on the wire loop.

Solution

(a) The magnetic force acting on a wire segment $\boldsymbol{l}$ carrying current i in magnetic field $\boldsymbol{B}$ is, Eq. (10.4) and (10.5),

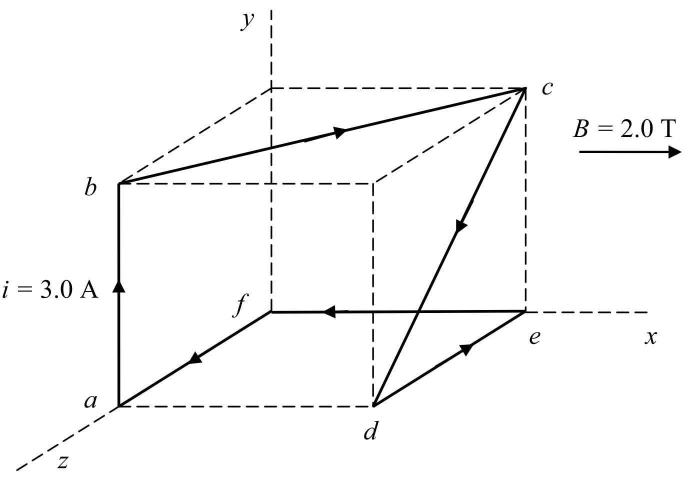


Fig. 10.9 A current carrying loop in a uniform magnetic field, Problem 10.12

$$\boldsymbol{F} = i\boldsymbol{l} \times \boldsymbol{B}, \quad F = ilB\sin\theta.$$

For wire segment *ab*,

$$\boldsymbol{F}_{ab} = i\boldsymbol{l} \times \boldsymbol{B} = (3.0\text{ A})(1.0\ j\text{ m}) \times (2.0\ i\text{ T}) = -6.0\ k\text{ N}.$$

For wire segment *bc*,

$$\boldsymbol{F}_{bc} = i\boldsymbol{l} \times \boldsymbol{B} = (3.0\text{ A})(1.0\ i\text{ m} - 1.0\ k\text{ m}) \times (2.0\ i\text{ T}) = -6.0\ j\text{ N}.$$

For wire segment *cd*,

$$\boldsymbol{F}_{cd} = i\boldsymbol{l} \times \boldsymbol{B} = (3.0\text{ A})(-1.0\ j\text{ m} + 1.0\ k\text{ m}) \times (2.0\ i\text{ T}) = (6.0\ k + 6.0\ j)\text{ N}.$$

For wire segment *de*,

$$\boldsymbol{F}_{de} = i\boldsymbol{l} \times \boldsymbol{B} = (3.0\text{ A})(-1.0\ k\text{ m}) \times (2.0\ i\text{ T}) = -6.0\ j\text{ N}.$$

For wire segment *ef*,

$$\boldsymbol{F}_{ef} = i\boldsymbol{l} \times \boldsymbol{B} = (3.0\text{ A})(-1.0\ i\text{ m}) \times (2.0\ i\text{ T}) = 0.$$

Lastly, for wire segment *fa*,

$$\boldsymbol{F}_{fa} = i\boldsymbol{l} \times \boldsymbol{B} = (3.0\text{ A})(1.0\ k\text{ m}) \times (2.0\ i\text{ T}) = 6.0\ j\text{ N}.$$

(b) The resultant magnetic force on the loop is

$$\begin{aligned}\boldsymbol{F}_{ab} + \boldsymbol{F}_{bc} + \boldsymbol{F}_{cd} + \boldsymbol{F}_{de} + \boldsymbol{F}_{ef} + \boldsymbol{F}_{fa} &= (-6.0 + 6.0)k \text{ N} \\ &\quad + (-6.0 + 6.0 - 6.0 + 6.0)\,j \text{ N} \\ &= 0.\end{aligned}$$

This shows that the resultant force on a current carrying a closed loop in a uniform magnetic field is zero.

◆ wxMaxima codes:

```
(%i1) load("vect");
(%o1) "C:\maxima-5.43.0\share\maxima\5.43.0\share\vector\vect.mac"
(%i3) Fab: 3*[0,1,0]~[2,0,0]; express(%);
(Fab) 3*[0,1,0]~[2,0,0]
(%o3) [0,0,-6]
(%i5) Fbc: 3*[1,0,-1]~[2,0,0]; express(%);
(Fbc) 3*[1,0,-1]~[2,0,0]
(%o5) [0,-6,0]
(%i7) Fcd: 3*[0,-1,1]~[2,0,0]; express(%);
(Fcd) 3*[0,-1,1]~[2,0,0]
(%o7) [0,6,6]
(%i9) Fde: 3*[0,0,-1]~[2,0,0]; express(%);
(Fde) 3*[0,0,-1]~[2,0,0]
(%o9) [0,-6,0]
(%i11) Fef: 3*[-1,0,0]~[2,0,0]; express(%);
(Fef) 3*[-1,0,0]~[2,0,0]
(%o11) [0,0,0]
(%i13) Ffa: 3*[0,0,1]~[2,0,0]; express(%);
(Ffa) 3*[0,0,1]~[2,0,0]
(%o13) [0,6,0]
(%i15) Fab+Fbc+Fcd+Fde+Fef+Ffa; express(%);
(%o14) 3*[1,0,-1]~[2,0,0]+3*[0,1,0]~[2,0,0]+3*[0,0,1]~[2,0,0]
+3*[0,0,-1]~[2,0,0]+3*[0,-1,1]~[2,0,0]+3*[-1,0,0]~[2,0,0]
(%o15) [0,0,0]
```

Comments on the codes:

(%i1)	Load "vect" package.
(%i3), (%i5), (%i7), (%i9), (%i11), (%i13)	Calculate $\boldsymbol{F}_{ab}$, $\boldsymbol{F}_{bc}$, $\boldsymbol{F}_{cd}$, $\boldsymbol{F}_{de}$, $\boldsymbol{F}_{ef}$, and $\boldsymbol{F}_{fa}$.
(%i15)	Calculate vector sum $\boldsymbol{F}_{ab} + \boldsymbol{F}_{bc} + \boldsymbol{F}_{cd} + \boldsymbol{F}_{de} + \boldsymbol{F}_{ef} + \boldsymbol{F}_{fa}$.

Problem 10.13 Figure 10.10 shows a metal conductor of mass m and length L that is free to slide on wires connected to a battery. Current I flows in the wire. The system is in a region of uniform magnetic field B pointing vertically downward. There is a very small friction f between the metal conductor and the wires. What is the acceleration of the metal conductor?

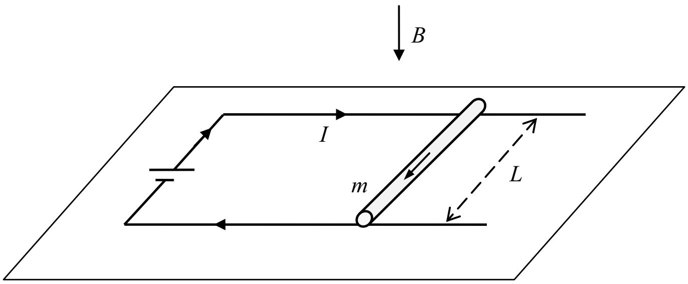


Fig. 10.10 A current carrying metal conductor in a region of uniform magnetic field, Problem 10.13

Solution

Force acting on the metal conductor is, Equation (10.5),

$$F = IlB \sin\theta = ILB \sin 90^\circ = ILB,$$

pointing to the right, obtained by applying the right-hand rule. The resultant force on the metal conductor is

$$F - f = ILB - f,$$

pointing to the right, where f is friction pointing to the left. The acceleration of the metal conductor is

$$a = \frac{\text{force}}{\text{mass}} = \frac{ILB - f}{m},$$

to the right. The metal conductor will move to the right.

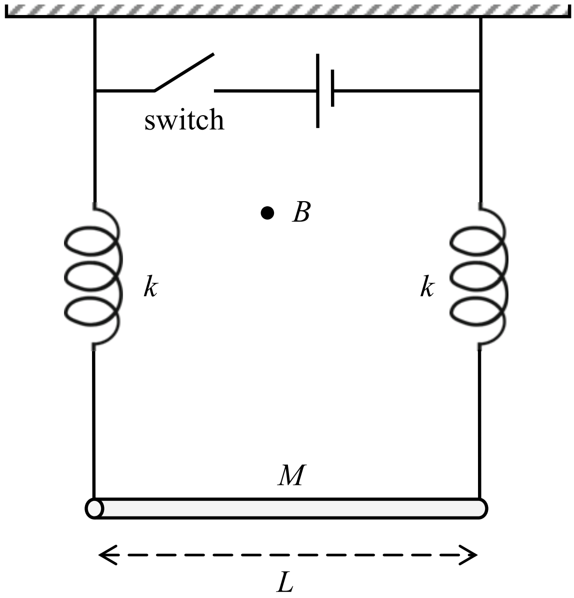
Problem 10.14 Figure 10.11 shows a conducting rod of mass M and length L, suspended by two identical springs, connected to a battery and a switch. The force constant of the spring is k. The system is in a uniform magnetic field B pointing out of the plane of the paper.

(a) What will happen when the switch is closed?
(b) Calculate the tension in each spring when the switch is closed.

Solution

(a) When the switch is closed, a counter clockwise current I flows in the system. As the rod is in a magnetic field region, a magnetic force in a downward direction acts on the rod. This can be deduced by the right-hand rule. The magnitude of the magnetic force is, Eq. (10.5),

Fig. 10.11 A conducting rod suspended by springs and connected to a battery and a switch in a uniform magnetic field, Problem 10.14



$$F = IlB\sin\theta = ILB\sin 90^\circ = ILB.$$

The magnetic force is balanced by the elastic forces of the two springs. The elastic force in each spring is one half of the magnetic force i.e. *ILB*/2. The extension of each spring is then

$$\text{extension} = \frac{\text{force}}{k} = \frac{ILB/2}{k} = \frac{ILB}{2k}.$$

So, when the switch is closed, a magnetic force *IBL* acts on the rod in the downward direction and each supporting spring stretches by *ILB*/(2*k*).

(b) When the switch is opened, that is, when there is no current, tension in each spring is

$$\frac{Mg}{2}.$$

When the switch is closed, the tension in each spring increases by

$$\text{force constant} \times \text{extension} = k \times \frac{ILB}{2k} = \frac{ILB}{2}.$$

Therefore, the tension in each spring when the switch is closed is the sum of the two

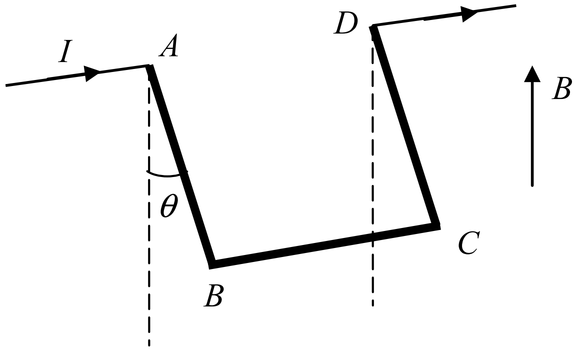


Fig. 10.12 A current carrying frame in a uniform magnetic field, Problem 10.15

$$\frac{Mg}{2} + \frac{ILB}{2}.$$

Problem 10.15 Figure 10.12 shows a conducting frame *ABCD* pivoted at *AD*. Each segment of the frame is of the same length and its linear density is $\lambda = 0.10$ kg m^{-1}. The frame is in a region of uniform magnetic field $B = 1.0 \times 10^{-2}$ T pointing vertically upward. What is the slant angle θ of the frame from the vertical when current $I = 10$ A flows in the frame?

Solution

Magnetic forces acting on segments *AB* and *CD* do not deflect the frame. This is because the forces are the same in magnitudes but opposite in directions, and they cancel each other. Right-hand rule shows that the magnetic force on segment *AB* points to the left, while the one on segment *CD* points to the right. Magnetic force acting on segment *BC* does deflect the frame. Let $AB = BC = CD = L$. Force acting on segment *BC* is, Eq. (10.5),

$$F_{magnet} = ILB.$$

This force points to the right as determined by the right-hand rule, Fig. 10.13. The figure shows a side view of the frame and the forces acting on it.

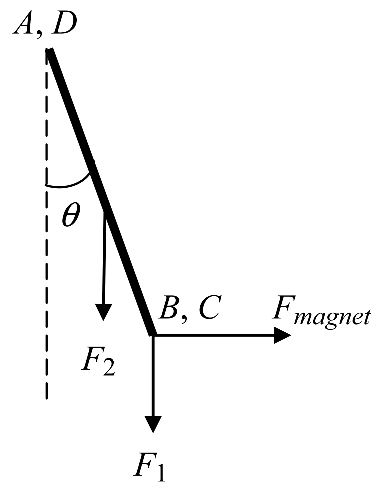
Another forces that acts on the segments is their weight. The weight of segment *BC* is

$$F_1 = \lambda Lg.$$

The weight of segments *AB* and *CD* is

$$F_2 = 2\lambda Lg.$$

Fig. 10.13 Side view of the frame and the forces acting on it, Problem 10.15



In equilibrium, the torque about AD is zero. We write

$$F_{magnet}\ L\cos\theta - F_2\left(\frac{L}{2}\right)\sin\theta - F_1 L\sin\theta = 0,$$

that is

$$(ILB)L\cos\theta - (2\lambda Lg)\left(\frac{L}{2}\right)\sin\theta - (\lambda Lg)L\sin\theta = 0.$$

This gives

$$\tan\theta = \frac{IB}{2\lambda g} = \frac{(10\ \text{A})(1.0\times10^{-2}\ \text{T})}{2(0.10\ \text{kg/m})(9.8\ \text{m/s}^2)} = 0.051.$$

Thus, the slant angle is,

$$\theta = \tan^{-1}(0.051) = 2.9^\circ.$$

◆ wxMaxima codes:

```
(%i6) fpprintprec:5; ratprint:false; I:10; B:10e-3; lambda:0.1; g:9.8;
(fpprintprec)     5
(ratprint)   false
(I)    10
(B)    0.01
(lambda) 0.1
(g)    9.8
(%i8) solve(I*L*B*L*cos(theta)-2*lambda*L*g*L/2*sin(theta)-
lambda*L*g*L*sin(theta)=0, sin(theta))$ float(%);
(%o8) [sin(theta)=0.05102*cos(theta)]
(%i9) theta_rad: atan(0.05102);
(theta_rad) 0.050976
(%i10) theta_degree: float(theta_rad/%pi*180);
(theta_degree) 2.9207
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, and assign values of I, B, λ, and g.

(%i8) Solve $(ILB)L\cos\theta - (2\lambda Lg)\left(\frac{L}{2}\right)\sin\theta - (\lambda Lg)L\sin\theta = 0$ for $\sin\theta$.

(%i9), (%i10) Calculate θ and convert the angle to degree.

Problem 10.16 The plane of 5.0 × 8.0 cm rectangular wire loop is parallel to a magnetic field of 0.15 T. If the loop carries a current of 10 A, what is the torque?

Solution

Figure 10.14 shows the current carrying loop in the magnetic field. Positive x direction (**i**) is to the right, positive y (**j**) is upward and positive z (**k**) is out of the plane of the paper.

Torque is calculated as follows, Equation (10.9),

$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \boldsymbol{B} = I\boldsymbol{A} \times \boldsymbol{B},$$

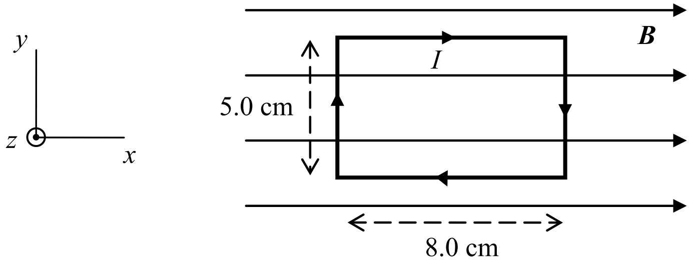


Fig. 10.14 A current carrying loop in a region of uniform magnetic field, Problem 10.16

$$\tau = IAB \ \sin\theta = (10\text{ A})(0.050\text{ m} \times 0.080\text{ m})(0.15\text{ T}) \ \sin 90^\circ$$
$$= 6.0 \times 10^{-3}\text{ N m}.$$

The direction of the torque is the negative *y* direction. Here, $\boldsymbol{\mu}$ is into the plane of the paper, $\boldsymbol{B}$ to the right, thus, torque to the negative *y* direction. See the right-hand rule, Table 10.1(c).

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; I:10; A:0.05*0.08; B:0.15; theta:float(90/180*%pi);
(fpprintprec)     5
(I)   10
(A)   0.004
(B)   0.15
(theta) 1.5708
(%i6) tau: I*A*B*sin(theta);
(tau) 0.006
```

Comments on the codes:

(%i5) Assign values of I, A, B, and θ.

(%i6) Calculate $\tau = IAB \ \sin\theta$.

Alternative calculation: Express the magnetic moment and magnetic field in terms of unit vectors and do the vector cross product.

$$\boldsymbol{\mu} = I\boldsymbol{A} = -(10\text{ A})(0.050\text{ m} \times 0.080\text{ m})k$$
$$= -0.040\ k\ \text{A m}^2,$$
$$\boldsymbol{B} = 0.15\ i\ \text{T},$$
$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \boldsymbol{B} = -0.040\ k\ \text{A m}^2 \times 0.15\ i\ \text{T}$$
$$= -6.0 \times 10^{-3}\ j\ \text{N m}.$$

where we have used the fact that $\mathbf{k} \times \mathbf{i} = \mathbf{j}$. The direction of the torque is to the negative *y* direction.

◆ wxMaxima codes:

```
(%i1) IA: 10*0.05*0.08;
(IA)  0.04
(%i2) mu: [0,0,-IA];
(mu)  [0,0,-0.04]
(%i3) B: [0.15,0,0];
(B)   [0.15,0,0]
(%i4) load("vect");
(%o4) "C:\maxima-5.43.0\share\maxima\5.43.0\share\vector\vect.mac"
(%i6) tau: mu~B; express(%);
(tau) [0,0,-0.04]~[0.15,0,0]
(%o6) [0,-0.006,0]
```

Comments on the codes:

(%i1) Assign *IA*.

(%i2), (%i3) Assign vectors $\boldsymbol{\mu}$ and $\boldsymbol{B}$.

(%i4) Load "vect" package.

(%i6) Calculate $\boldsymbol{\tau} = \boldsymbol{\mu} \times \boldsymbol{B}$.

Problem 10.17 A wire is shaped into the letter "M" and it carries a current of $I = 15$ A. The wire is placed in a region of uniform magnetic field of $B = 2.5$ T, as in Fig. 10.15(a). Calculate the magnitude and direction of the net magnetic force that acts on the wire.

Solution

For a wire of arbitrary shape carrying a current of I in a magnetic field of $\boldsymbol{B}$, the magnetic force acting on the wire is

$$\boldsymbol{F} = I\boldsymbol{l}_0 \times \boldsymbol{B},$$

where $\boldsymbol{l}_0$ is the vector from one end of the wire to the other.

For this problem, $\boldsymbol{l}_0$ is shown in Fig. 10.15(b). Thus, the force acting on the wire is

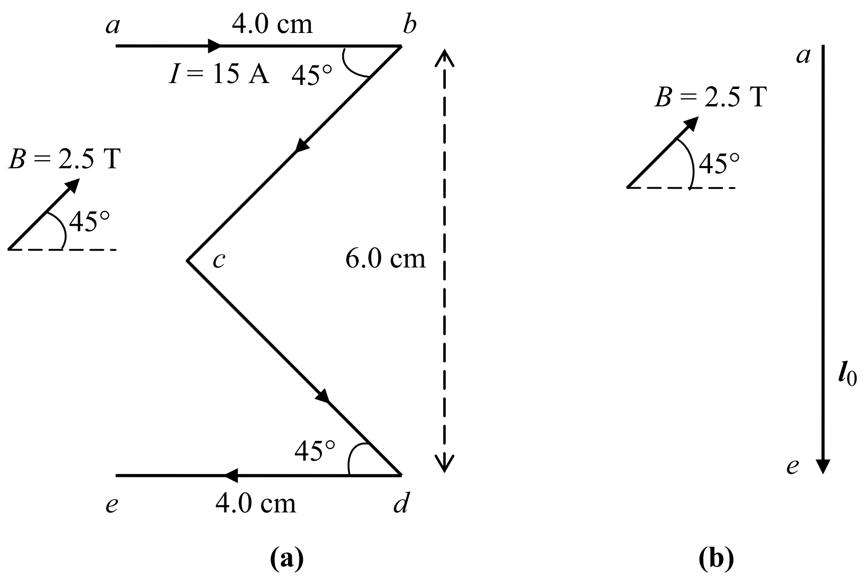


Fig. 10.15 **a** An "M" shaped current carrying wire in a region of uniform magnetic field, **b** a current carrying vector from *a* to *e* in a region of uniform magnetic field, Problem 10.17

$$F = Il_0B \sin 135^\circ = (15\ \text{A})(0.060\ \text{m})(2.5\ \text{T}) \sin 135^\circ = 1.6\ \text{N}.$$

pointing out of the plane of the paper.

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; I:15; l0:0.06; B:2.5;
(fpprintprec)     5
(I)   15
(l0)  0.06
(B)   2.5
(%i5) F: I*l0*B*sin(135*float(%pi)/180);
(F)   1.591
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of I, l_0, and B.

(%i5) Calculate F.

Alternative calculation: The force can also be obtained by summing the magnetic forces of all segments of the wire.

For wire segment *ab*, the magnetic force is

$$F_{ab} = IlB \sin\theta = (15\ \text{A})(0.040\ \text{m})(2.5\ \text{T}) \sin 45^\circ = 1.1\ \text{N}.$$

pointing out of the plane of the paper.

For wire segment *bc*, the magnetic force is

$$F_{bc} = 0.$$

For wire segment *cd*, the magnetic force is

$$F_{cd} = IlB \sin\theta = (15\ \text{A})(\sqrt{18} \times 10^{-2}\ \text{m})(2.5\ \text{T}) \sin 90^\circ = 1.6\ \text{N}.$$

pointing out of the plane of the paper.

For wire segment *de*, the magnetic force is

$$F_{de} = IlB \sin\theta = (15\ \text{A})(0.040\ \text{m})(2.5\ \text{T}) \sin 135^\circ = 1.1\ \text{N}.$$

pointing into the plane of the paper.

The sum or resultant of these magnetic forces is 1.6 N pointing out of the plane of the paper.

◆ wxMaxima codes:

```
(%i7) fpprintprec:5; I:15; B:2.5; ab:4e-2; bc:float(sqrt(18))*1e-2;
cd:float(sqrt(18))*1e-2; de:4e-2;
(fpprintprec)     5
(I)   15
(B)   2.5
(ab)  0.04
(bc)  0.042426
(cd)  0.042426
(de)  0.04
(%i8) Fab: I*ab*B*sin(45/180*float(%pi));
(Fab) 1.0607
(%i9) Fbc: I*bc*B*sin(%pi);
(Fbc) 0
(%i10) Fcd: I*cd*B*sin(90/180*float(%pi));
(Fcd) 1.591
(%i11) Fde: I*de*B*sin(225/180*float(%pi));
(Fde) -1.0607
(%i12) Fab + Fbc + Fcd + Fde;
(%o12) 1.591
```

Comments on the codes:

(%i7)	Set floating point print precision to 5, assign values of I, B, lengths ab, bc, cd, and de.
(%i8), (%i9), (%i10), (%i11)	Calculate F_{ab}, F_{bc}, F_{cd}, and F_{de}.
(%i12)	Calculate $F_{ab} + F_{bc} + F_{cd} + F_{de}$.

10.3 Summary

– Magnetic force on a charge q moving with a velocity $\boldsymbol{v}$ in a magnetic field $\boldsymbol{B}$ is

$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}.$$

– Magnetic force on a conductor of length $\boldsymbol{l}$ carrying a current I in a magnetic field $\boldsymbol{B}$ is

$$\boldsymbol{F} = I\boldsymbol{l} \times \boldsymbol{B}.$$

– Magnetic moment $\boldsymbol{\mu}$ of a loop carrying current I is

$$\boldsymbol{\mu} = I\boldsymbol{A}, \quad \mu = IA,$$

where $\boldsymbol{A}$ is the area vector normal to the plane of the loop and A is the area of the loop. The area vector is defined as

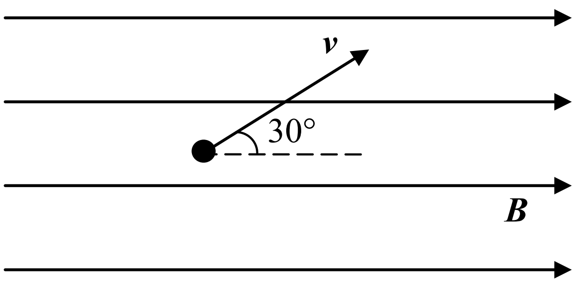


Fig. 10.16 A proton moving in a region of uniform magnetic field, Exercise 10.1

$$\boldsymbol{A} = A\,n,$$

where **n** is the unit vector normal to the plane of the loop.

– Torque $\boldsymbol{\tau}$ on a loop in a uniform magnetic field $\boldsymbol{B}$ is

$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \boldsymbol{B} = I\boldsymbol{A} \times \boldsymbol{B}.$$

10.4 Exercises

Exercise 10.1 A proton moves with velocity $v = 5.0 \times 10^6$ m s^{-1} in magnetic field $B = 2.0$ T as shown in Fig. 10.16. Calculate the magnetic force on the proton.

(Answer: 8.0×10^{-13} N into the plane of the paper)

Exercise 10.2 A proton moves in a circular orbit of radius 6.0 cm in a uniform magnetic field of 0.50 T, as shown in Fig. 10.17. What are the speed, angular frequency, and period of revolution of the proton?

(Answer: 2.9×10^6 m s^{-1}, 4.8×10^7 rad s^{-1}, 1.3×10^{-7} s)

Exercise 10.3 A wire of length $l = 2.0$ m carries a current of $I = 5.0$ A in a uniform magnetic field of $B = 0.030$ T, Fig. 10.18. What is the magnetic force acting on the wire?

(Answer: 0.26 N out of the plane of the paper)

Exercise 10.4 A 2.0 keV alpha particle enters a region of uniform magnetic field of 0.15 T. The direction of the magnetic field is perpendicular to the alpha direction of motion. An alpha particle has a charge of $+\,2e$ and a mass of 6.68×10^{-27} kg. Calculate the radius of the alpha particle path in the magnetic field.

(Answer: 43 mm)

Exercise 10.5 A coil of 40 turns of area 800 mm^2 has a current flow of 0.5 A. The coil is a region of uniform magnetic field of 0.30 T with the coil plane parallel to the direction of the field. What is the torque on the coil?

(Answer: 4.8×10^{-3} N m)

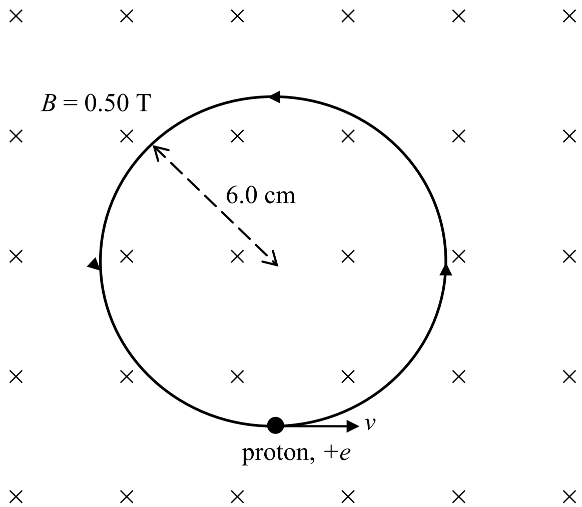


Fig. 10.17 Circular path of a proton in a uniform magnetic field, Exercise 10.2

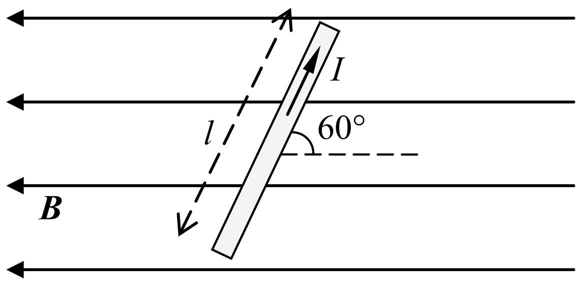


Fig. 10.18 A current carrying wire in a region of uniform magnetic field, Exercise 10.3

Chapter 11
Sources of Magnetic Field

Abstract This chapter solves problems on magnetic fields created by current-carrying conductors and loops. The Biot–Savart law is applied to determine the magnetic fields. Magnetic fields in a current-carrying solenoid and toroid are determined by applying Ampere's law. Solutions are obtained by analysis and computer calculation.

11.1 Basic Concepts and Formulae

(1) Biot–Savart law states that magnetic field $\boldsymbol{dB}$ at point P due to infinitesimal element $\boldsymbol{ds}$ of a conductor carrying current I shown in Fig. 11.1 is

$$\boldsymbol{dB} = k_m \frac{I\ \boldsymbol{ds} \times \boldsymbol{r}}{r^3} = k_m \frac{I\ \boldsymbol{ds} \times \hat{\boldsymbol{r}}}{r^2} = \frac{\mu_0}{4\pi} \frac{I\ \boldsymbol{ds} \times \hat{\boldsymbol{r}}}{r^2}, \tag{11.1}$$

where $\boldsymbol{r}$ is a vector from element $\boldsymbol{ds}$ to point P, $r = |\boldsymbol{r}|$, $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$.

Magnetic constant $k_m = \mu_0/(4\pi) = 10^{-7}$ Wb A^{-1} m^{-1}.

Permeability of free space $\mu_0 = 4\pi k_m = 4\pi \times 10^{-7}$ Wb A^{-1} m^{-1}.

Magnetic field $\boldsymbol{B}$ due to the whole length of the conductor is the integration of $\boldsymbol{dB}$,

$$\boldsymbol{B} = \int \boldsymbol{dB} = k_m I \int \frac{\boldsymbol{ds} \times \hat{\boldsymbol{r}}}{r^2} = \frac{\mu_0 I}{4\pi} \int \frac{\boldsymbol{ds} \times \hat{\boldsymbol{r}}}{r^2}. \tag{11.2}$$

(2) Table 11.1 gives magnetic fields of common current-carrying conductor configurations obtained by application of the Biot–Savart law.

(3) Force per unit length between two parallel long wires, separated by a distance of a, and carrying currents of I_1 and I_2,

$$\frac{F}{l} = \frac{2k_m I_1 I_2}{a} = \frac{\mu_0 I_1 I_2}{2\pi a}. \tag{11.3}$$

W. M. S. Wan Hassan et al., *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-43165-4_11

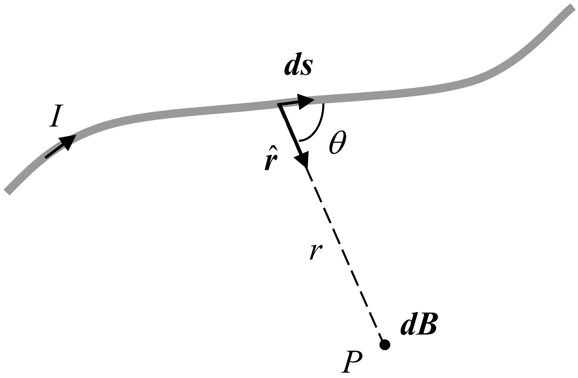
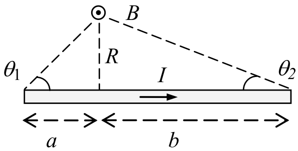
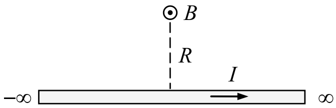
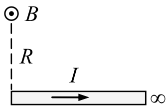
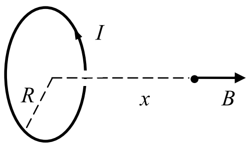
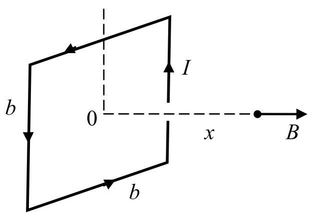


Fig. 11.1 The magnetic field dB at point P due to element ds carrying a current I is given by the Biot–Savart law

Table 11.1 Magnetic fields of a few configurations of current-carrying conductors

	Configuration	Magnetic field
(a)		$B = \frac{\mu_0 I}{4\pi R}(\cos\theta_1 - \cos\theta_2)$ $= \frac{\mu_0 I}{4\pi R}\left(\frac{a}{\sqrt{a^2+R^2}} + \frac{b}{\sqrt{b^2+R^2}}\right)$
(b)		$B = \frac{\mu_0 I}{2\pi R} = \frac{2k_m I}{R}$
(c)		$B = \frac{\mu_0 I}{4\pi R} = \frac{k_m I}{R}$
(d)		$B = \frac{\mu_0 I}{2}\frac{R^2}{(x^2+R^2)^{3/2}}$ At the center of the ring, $B = \frac{\mu_0 I}{2R}$ In terms of magnetic dipole moment, $\mu = IA = I\pi R^2$, $B = \frac{\mu_0}{2\pi}\frac{\mu}{(x^2+R^2)^{3/2}}$ At a point far away from the magnetic dipole moment, $x >> R$, $B = \frac{\mu_0}{2\pi}\frac{\mu}{x^3}$
(e)		$B = \frac{\mu_0 I b^2}{2\pi[x^2+(b/2)^2]\sqrt{x^2+2(b/2)^2}}$ At the center of the square loop, $B = 2\sqrt{2}\frac{\mu_0 I}{\pi b}$

The force is attractive if the currents are in the same direction, but repulsive if the currents are in opposite directions.

(4) Ampere's law states that the line integral ***B*.*ds*** along a closed path is $\mu_0 I$, that is

$$\oint \boldsymbol{B} \cdot \boldsymbol{ds} = \mu_0 I, \tag{11.4}$$

where I is the current through a surface bounded by the closed path.

(5) Application of Ampere's law gives the magnetic fields inside a solenoid as

$$B_{solenoid} = \mu_0 \frac{N}{l} I = \mu_0 n I, \tag{11.5}$$

where N is the number of turns of the wire, l is the length of the solenoid, and n is the number of turns per unit length.

Application of Ampere's law gives the magnetic field inside a toroid as

$$B_{toroid} = \frac{\mu_0 N I}{2\pi r}, \tag{11.6}$$

where N is the number of turns of the wire and r is the radius of the toroid.

(6) Magnetic flux Φ_m through a surface is defined by the surface integration

$$\Phi_m = \int \boldsymbol{B} \cdot \boldsymbol{dA}. \tag{11.7}$$

(7) Gauss's magnetic law states that the net magnetic flux through any closed surface is zero.

(8) Direction of the magnetic field of a current-carrying loop or wire can be determined by the right-hand rule, as shown in Fig. 11.2a or b.

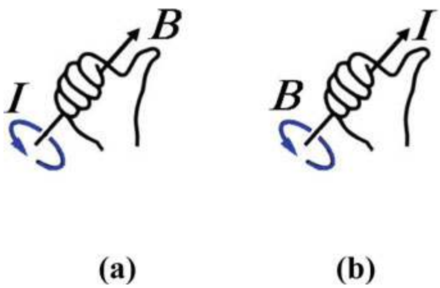


Fig. 11.2 The right-hand rule determines the direction of the magnetic field of a current-carrying loop or wire. **a** For a loop or coil carrying current in the direction of the four fingers, the magnetic field is in the direction of the thumb. **b** For wire or conductor carrying current in the direction of the thumb, the magnetic field around the wire is in the direction of the four fingers

11.2 Problems and Solutions

Problem 11.1 Figure 11.3a shows a long wire carrying a current of 10 A. What is the magnetic field at point P, 2.0 m from the wire?

Solution

The magnetic field at point P due to a current-carrying wire is, Table 11.1(b),

$$B = \frac{2k_m I}{r} = \frac{2(10^{-7}\ \text{T m A}^{-1})(10\ \text{A})}{2.0\ \text{m}} = 1.0 \times 10^{-6}\ \text{T}.$$

The magnetic field points out of the plane of the paper as determined by the right-hand rule, Fig. 11.3b.

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; km:1e-7; I:10; r:2;
(fpprintprec)     5
(km)   1.0*10^-7
(I)    10
(r)    2
(%i5) B: 2*km*I/r;
(B)    1.0*10^-6
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of k_m, I, and r.
(%i5) Calculate B.

Problem 11.2 A long wire carries a current of $I = 10$ A along the negative y-axis as shown in Fig. 11.4. The wire is in a region of uniform magnetic field $B_0 = 1.0 \times 10^{-6}$ T in the positive x direction. Determine the resultant magnetic field at the point:

(a) P (0, 0, 2.0 m).
(b) Q (2.0 m, 0, 0).
(c) R (0, 0, −1.0 m).

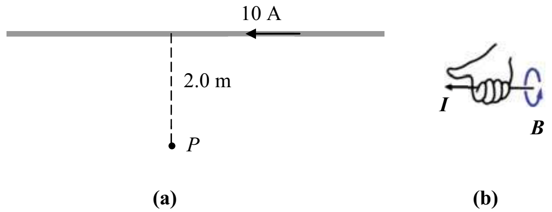


Fig. 11.3 **a** A current-carrying wire, **b** directions of current and magnetic field, Problem 11.1

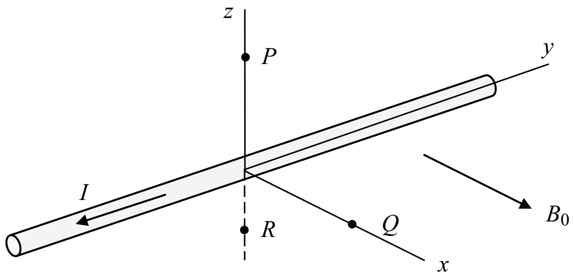


Fig. 11.4 A current-carrying wire in a region of uniform magnetic field, Problem 11.2

Solution

(a) Figure 11.5 shows the current-carrying wire and the related magnetic fields.

The magnetic field at a distance of r from a long wire carrying a current of I is, Table 11.1(b),

$$B = 2\left(\frac{\mu_0}{4\pi}\right)\frac{I}{r} = 2k_m\frac{I}{r}.$$

At point P (0, 0, 2.0 m), the magnetic field due to the current is

$$B_1 = 2k_m\frac{I}{r} = 2\ (10^{-7}\ \text{Wb A}^{-1}\ \text{m}^{-1})\frac{(10\ \text{A})}{(2.0\ \text{m})} = 1.0 \times 10^{-6}\ \text{T},$$

pointing to the negative x direction. The magnetic field of the region is

$$B_0 = 1.0 \times 10^{-6}\ \text{T},$$

pointing to the positive x direction. Thus, the magnetic field at point P is

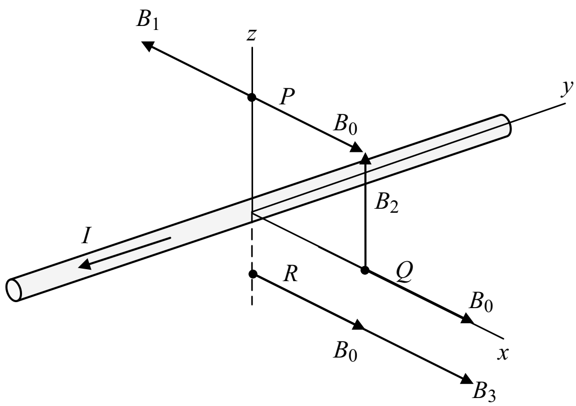


Fig. 11.5 Magnetic fields at points P, Q, and R, Problem 11.2

$$B_P = B_0 - B_1 = 1.0 \times 10^{-6}\ \mathrm{T} - 1.0 \times 10^{-6}\ \mathrm{T} = 0.$$

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; km:1e-7; I:10; r:2;
(fpprintprec)     5
(km)  1.0*10^-7
(I)   10
(r)   2
(%i7) B1: 2*km*I/r; B0: 1e-6; BP: B0-B1;
(B1)  10.0*10^-7
(B0)  10.0*10^-7
(BP)  0.0
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of k_m, I, and r.
(%i7) Calculate B_1, assign B_0, and calculate B_P.

(b) At point Q (2.0 m, 0, 0), the magnetic field due to the current is

$$B_2 = 2\ (10^{-7}\ \mathrm{Wb\ A^{-1}\ m^{-1}})\frac{(10\ \mathrm{A})}{(2.0\ \mathrm{m})} = 1.0 \times 10^{-6}\ \mathrm{T},$$

pointing to the positive z direction. The region's magnetic field is

$$B_0 = 1.0 \times 10^{-6}\ \mathrm{T},$$

pointing to the positive x direction. Thus, the magnetic field at point Q is

$$\boldsymbol{B}_Q = B_0\mathbf{i} + B_2\mathbf{k} = (1.0 \times 10^{-6}\ \mathbf{i} + 1.0 \times 10^{-6}\ \mathbf{k})\ \mathrm{T}.$$

The magnitude of the magnetic field is

$$B_Q = \sqrt{(1.0 \times 10^{-6})^2 + (1.0 \times 10^{-6})^2}\ \mathrm{T} = 1.4 \times 10^{-6}\ \mathrm{T}.$$

This means that the angle between $\boldsymbol{B}_Q$ and the x-axis is 45°.

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; km:1e-7; I:10; r:2;
(fpprintprec)     5
(km)  1.0*10^-7
(I)   10
(r)   2
(%i7) B2: 2*km*I/r; B0: 1e-6; BQ: sqrt(B0^2+B2^2);
(B2)  10.0*10^-7
(B0)  10.0*10^-7
(BQ)  1.4142*10^-6
```

Comments on the codes:

(%i4) Set floating point print precision to 5, and assign values of k_m, I, and r.
(%i7) Calculate B_2, assign B_0, and calculate B_Q.

(c) At point R (0, 0, −1.0 m), the magnetic field due to the current is

$$B_3 = 2\ (10^{-7}\ \text{Wb A}^{-1}\ \text{m}^{-1})\frac{(10\ \text{A})}{(1.0\ \text{m})} = 2.0 \times 10^{-6}\ \text{T},$$

pointing to the positive x direction. The magnetic field of the region is

$$B_0 = 1.0 \times 10^{-6}\ \text{T},$$

pointing to the positive x direction. Thus, the magnetic field at point R is

$$B_R = B_0 + B_3 = 1.0 \times 10^{-6}\ \text{T} + 2.0 \times 10^{-6}\ \text{T} = 3.0 \times 10^{-6}\ \text{T},$$

pointing to the positive x direction. This means that

$$\boldsymbol{B}_R = 3.0 \times 10^{-6}\ \mathbf{i}\ \text{T}.$$

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; km:1e-7; I:10; r:1;
(fpprintprec)     5
(km)  1.0*10^-7
(I)   10
(r)   1
(%i7) B3: 2*km*I/r; B0: 1e-6; BR: B0+B3;
(B3)  2.0*10^-6
(B0)  10.0*10^-7
(BR)  3.0*10^-6
```

Comments on the codes:

(%i4) Set floating point print precision to 5, and assign values of k_m, I, and r.
(%i7) Calculate B_3, assign B_0, and calculate B_R.

Problem 11.3 Two parallel long wires separated by a distance of 20 cm have currents of 5.0 and 7.0 A flowing in the same direction, which are shown in Fig. 11.6. Determine:

(a) the magnetic field at point P.
(b) the point where the magnetic field is zero.

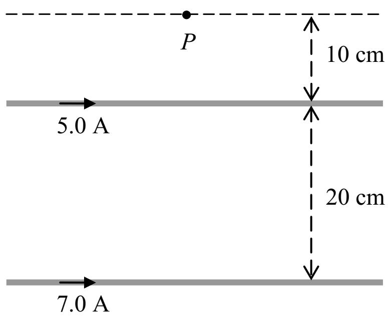


Fig. 11.6 Two current-carrying parallel wires, Problem 11.3

Solution

(a) Magnetic field at point P due to wire with 5.0 A current is, Table 11.1(b),

$$B_1 = \frac{\mu_0 I}{2\pi r} = (4\pi \times 10^{-7}\ \text{Wb A}^{-1}\ \text{m}^{-1})\frac{(5.0\ \text{A})}{2\pi(0.10\ \text{m})} = 1.0 \times 10^{-5}\ \text{T},$$

pointing out of the page. The magnetic field at point P due to wire with 7.0 A current is,

$$B_2 = \frac{\mu_0 I}{2\pi r} = (4\pi \times 10^{-7}\ \text{Wb A}^{-1}\ \text{m}^{-1})\frac{(7.0\ \text{A})}{2\pi(0.30\ \text{m})} = 4.7 \times 10^{-6}\ \text{T},$$

pointing out of the page. Thus, the magnetic field at point P due to currents in both wires is

$$B_P = B_1 + B_2 = 1.0 \times 10^{-5}\ \text{T} + 4.7 \times 10^{-6}\ \text{T} = 1.5 \times 10^{-5}\ \text{T},$$

pointing out of the page.

◆ wxMaxima codes:

```
(%i2) fpprintprec:5; mu0:4*%pi*1e-7;
(fpprintprec)     5
(mu0) 4.0*10^-7*%pi
(%i3) B1: mu0*5/(2*%pi*0.1);
(B1)  1.0*10^-5
(%i4) B2: mu0*7/(2*%pi*0.3);
(B2)  4.6667*10^-6
(%i5) BP: B1+B2;
(BP)  1.4667*10^-5
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and assign the value of μ_0.

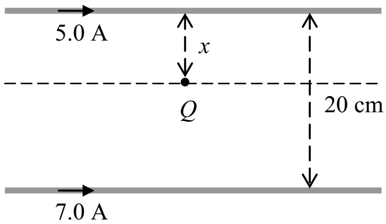


Fig. 11.7 Determining point of zero magnetic field, Problem 11.3

(%i3), (%i4), (%5) Calculate B_1, B_2, and B_P.

(b) The magnetic field is zero somewhere at point Q, a distance of x from the top wire, Fig. 11.7. At this point, the magnetic field due to the top wire is into the page, while the field due to the bottom wire is out of the page. Adding the two fields gives zero magnetic field.

The magnetic field at point Q due to wire with 5.0 A current is

$$B_1 = \frac{\mu_0 I}{2\pi r} = (4\pi \times 10^{-7}\ \text{Wb A}^{-1}\ \text{m}^{-1})\frac{(5.0\ \text{A})}{2\pi x},$$

pointing into the page. The magnetic field at point Q due to wire with 7.0 A current is

$$B_2 = \frac{\mu_0 I}{2\pi r} = (4\pi \times 10^{-7}\ \text{Wb A}^{-1}\ \text{m}^{-1})\frac{(7.0\ \text{A})}{2\pi(0.20\ \text{m} - x)},$$

pointing out of the page. At point Q, $B_1 = B_2$ because the field is zero, therefore

$$B_1 = B_2$$
$$\frac{5.0\ \text{A}}{2\pi x} = \frac{7.0\ \text{A}}{2\pi(0.20\ \text{m} - x)}$$
$$x = 0.083\ \text{m}.$$

The magnetic field is zero at point Q where $x = 0.083$ m.

♦ wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)     5
(ratprint)  false
(%i4) solve(5/x = 7/(0.2-x), x)$ float(%);
(%o4) [x=0.083333]
```

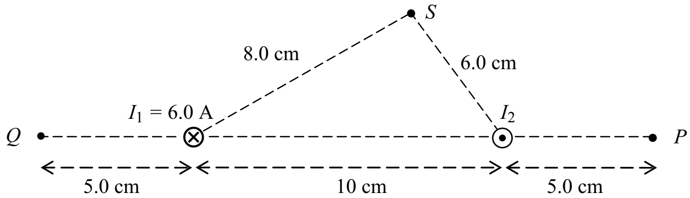


Fig. 11.8 Two current carrying wires, Problem 11.4

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Solve $5/x = 7/(0.2 - x)$ for x.

Problem 11.4 Electric currents flowing in two long parallel wires separated by a distance of 10 cm are illustrated in Fig. 11.8. The wire on the left carries a current of $I_1 = 6.0$ A into the plane of the paper. The magnetic field at point P is zero. Determine:

(a) the direction and magnitude of the current in the wire on the right, I_2
(b) the magnetic field at point Q
(c) the magnetic field at point S.

Solution

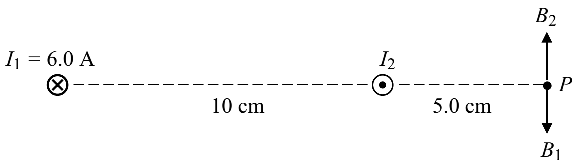
(a) Figure 11.9 shows the two current-carrying wires, point P, and the relevant magnetic fields.

The magnetic field due to current I_1 at point P is

$$B_1 = \frac{2k_m I_1}{r_1},$$

where r_1 is the distance from the left wire to point P. The field points downward. To get a zero magnetic field at point P, the magnetic field due to current I_2 must point upward and this means that current I_2 must flow out of the plane of the paper. The magnitude of the magnetic field due to I_2 must be the same as B_1. Thus

Fig. 11.9 Determining I_2, Problem 11.4



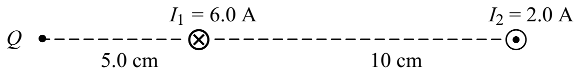


Fig. 11.10 Determining magnetic field at point Q, Problem 11.4

$$B_2 = \frac{2k_m I_2}{r_2},$$

where r_2 is the distance from the right wire to point P, and

$$B_1 = B_2,$$
$$\frac{2k_m I_1}{r_1} = \frac{2k_m I_2}{r_2},$$
$$I_2 = \frac{r_2 I_1}{r_1} = \frac{(5.0 \text{ cm})(6.0 \text{ A})}{(15 \text{ cm})} = 2.0 \text{ A}.$$

The direction of I_2 is out of the plane of the paper.

(b) Figure 11.10 shows the two current-carrying wires and point Q.

The magnetic field at Q due to current I_1 points upward while the one due to I_2 points downward. Thus, the magnetic field at point Q due to currents I_1 and I_2 is

$$B_Q = \frac{2k_m I_1}{r_1} - \frac{2k_m I_2}{r_2} = 2k_m\left(\frac{I_1}{r_1} - \frac{I_2}{r_2}\right)$$
$$= 2(10^{-7} \text{ Wb A}^{-1} \text{ m}^{-1})\left(\frac{6.0 \text{ A}}{0.050 \text{ m}} - \frac{2.0 \text{ A}}{0.15 \text{ m}}\right)$$
$$= 2.1 \times 10^{-5} \text{ T}.$$

pointing upward. Here, r_1 is the distance from the left wire to point Q and r_2 is the distance of the right wire to point Q.

♦ wxMaxima codes:

```
(%i6) fpprintprec:5; km:1e-7; I1:6; I2:2; r1:0.05; r2:0.15;
(fpprintprec)     5
(km)  1.0*10^-7
(I1)  6
(I2)  2
(r1)  0.05
(r2)  0.15
(%i7) BQ: 2*km*(I1/r1-I2/r2);
(BQ)  2.1333*10^-5
```

Comments on the codes:

(%i6) Set floating point print precision to 5, assign values of k_m, I_1, I_2, r_1, and r_2.
(%i7) Calculate B_Q.

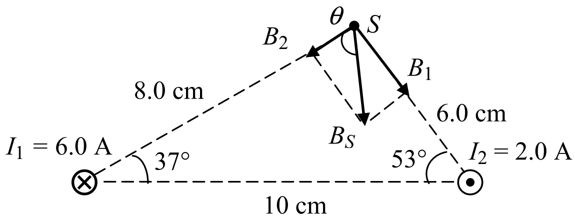


Fig. 11.11 Determining magnetic field at point *S*, Problem 11.4

(c) Figure 11.11 shows the two current-carrying wires, point *S*, and the relevant magnetic fields.

At point *S*, the magnetic field due to current I_1 is

$$B_1 = \frac{2k_m I}{r_1} = 2(10^{-7}\ \text{Wb A}^{-1}\ \text{m}^{-1})\left(\frac{6.0\ \text{A}}{0.080\ \text{m}}\right) = 1.5 \times 10^{-5}\ \text{T}.$$

and the field due to current I_2 is

$$B_2 = \frac{2k_m I}{r_2} = 2(10^{-7}\ \text{Wb A}^{-1}\ \text{m}^{-1})\left(\frac{2.0\ \text{A}}{0.060\ \text{m}}\right) = 6.7 \times 10^{-6}\ \text{T}.$$

Thus, the magnitude of the magnetic field at point *S* is

$$B_S = \sqrt{B_1^2 + B_2^2} = \sqrt{(1.5 \times 10^{-5}\ \text{T})^2 + (6.7 \times 10^{-6}\ \text{T})^2} = 1.6 \times 10^{-5}\ \text{T}.$$

Angle θ is calculated as follows:

$$\tan\theta = \frac{B_1}{B_2} = \frac{1.5 \times 10^{-5}\ \text{T}}{6.7 \times 10^{-6}\ \text{T}} = 2.25, \quad \theta = \tan^{-1}(2.25) = 1.2\ \text{rad} = 66°.$$

The direction of $\boldsymbol{B}_S$ is not vertically downward, but at $66° - 53° = 13°$ from the vertical.

◆ wxMaxima codes:

```
(%i2) fpprintprec:5; km:1e-7;
(fpprintprec)     5
(km)  1.0*10^-7
(%i3) B1: 2*km*6/0.08;
(B1)  1.5*10^-5
(%i4) B2: 2*km*2/0.06;
(B2)  6.6667*10^-6
(%i5) BS: sqrt(B1^2+B2^2);
(BS)  1.6415*10^-5
(%i6) theta_rad: atan(B1/B2);
(theta_rad) 1.1526
(%i7) theta_deg: float(theta_rad*180/%pi);
(theta_deg) 66.038
```

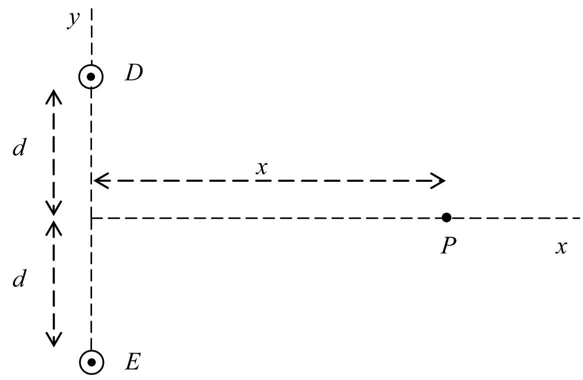


Fig. 11.12 Two long current-carrying wires, Problem 11.5

Comments on the codes:

(%i2)	Set floating point print precision to 5 and assign k_m.
(%i3), (%i4), (%i5)	Calculate B_1, B_2, and B_S.
(%i6), (%i7)	Calculate θ and convert the angle to degree.

Problem 11.5 Figure 11.12 shows a cross-section of two long wires D and E separated by a distance of $2d$. Each wire carries current I out of the plane of the paper.

(a) Find the magnetic field at point P. At what point along the x-axis the magnetic field is zero?
(b) If the direction of current in wire E is into the plane of the paper, what is the magnetic field at point P? Where along the x-axis the magnetic field is a maximum? What is the magnetic field?

Solution

(a) Figure 11.13 shows the two wires, point P, and magnetic fields at P.

The magnetic fields due to currents in wires D and E at point P are, Table 11.1(b),

$$B_D = \frac{\mu_0 I}{2\pi\sqrt{d^2+x^2}}, \quad B_E = \frac{\mu_0 I}{2\pi\sqrt{d^2+x^2}},$$

respectively. Both B_D and B_E are resolved into x and y components and added. The x components vanished. The y components give the resultant magnetic field at point P,

$$B_P = 2B_D \cos\,\theta = 2\frac{\mu_0 I}{2\pi\sqrt{d^2+x^2}}\frac{x}{\sqrt{d^2+x^2}} = \frac{\mu_0 I x}{\pi(d^2+x^2)},$$

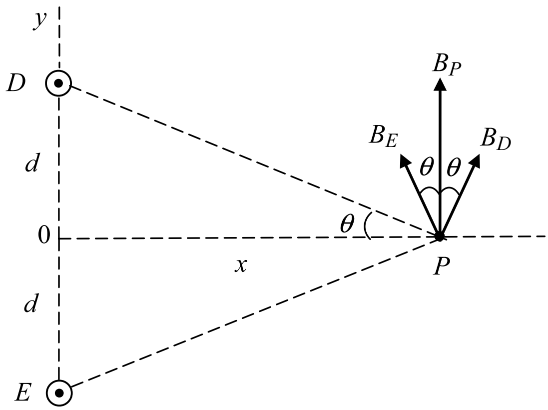


Fig. 11.13 Magnetic fields at point P due to current carrying wires, part (a) Problem 11.5

pointing to the positive y direction (upward direction).

The magnetic field is zero at $x = 0$. The magnetic fields due to currents in wires D and E at $x = 0$ are

$$B_D = \frac{\mu_0 I}{2\pi d}, \quad B_E = \frac{\mu_0 I}{2\pi d}.$$

B_D points to the positive x direction while B_E to the negative. Thus, the magnetic field is zero at $x = 0$.

(b) Figure 11.14 shows the two wires, point P, and the relevant magnetic fields.

The magnetic fields due to currents in wires D and E at point P are

$$B_D = \frac{\mu_0 I}{2\pi\sqrt{d^2 + x^2}}, \quad B_E = \frac{\mu_0 I}{2\pi\sqrt{d^2 + x^2}},$$

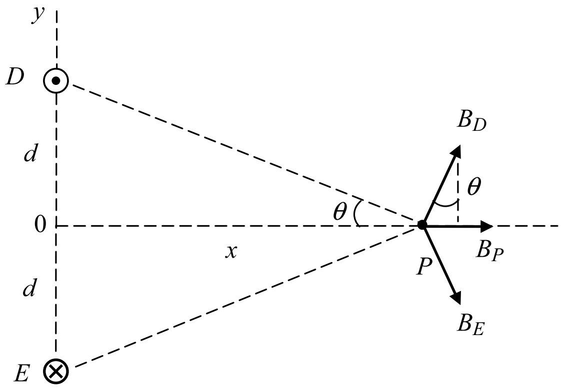


Fig. 11.14 Magnetic fields at point P due to current carrying wires, part (b) Problem 11.5

respectively. Both B_D and B_E are resolved into x and y components and added. The y components vanished. The x components give the resultant magnetic field at point P. The resultant magnetic field at point P is

$$B_P = 2B_D \sin\theta = 2\frac{\mu_0 I}{2\pi\sqrt{d^2+x^2}}\frac{d}{\sqrt{d^2+x^2}} = \frac{\mu_0 I d}{\pi(d^2+x^2)},$$

pointing to the positive x direction (to the right).

At $x = 0$, the magnetic fields due to currents in wires D and E are

$$B_D = \frac{\mu_0 I}{2\pi d}, \quad B_E = \frac{\mu_0 I}{2\pi d}.$$

Both fields point to the positive x direction. Thus, the magnetic field is a maximum at $x = 0$, with the value $B_{max} = \frac{\mu_0 I}{\pi d}$ in the positive x direction.

Problem 11.6 Currents of 10 A flowing in wires of various configurations are illustrated in Fig. 11.15. Calculate the magnetic field at point P in each configuration.

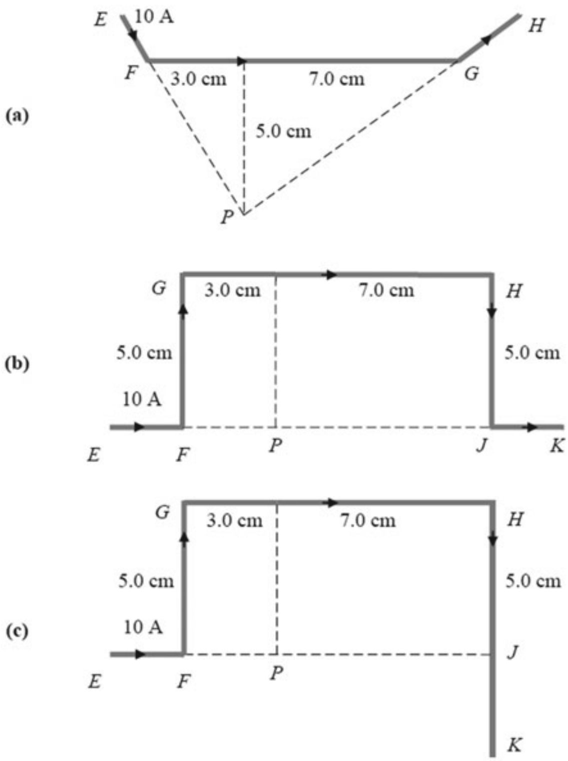


Fig. 11.15 Three configurations of current carrying wires, Problem 11.6

Solution

(a) In Fig. 11.15a, wire segments *EF* and *GH* do not give any magnetic field at point *P* because the currents are toward or away from point *P*. This is because $\boldsymbol{ds} \times \boldsymbol{r}$ is zero for both segments and according to Biot–Savart law will give no magnetic field, Eq. (11.1). The magnetic field at point *P* is from wire segment *FG*, that is, Table 11.1(a),

$$
\begin{aligned}
B_P &= \frac{\mu_0 I}{4\pi R}\left(\frac{a}{\sqrt{a^2+R^2}} + \frac{b}{\sqrt{b^2+R^2}}\right) \\
&= \frac{\mu_0 I}{4\pi(0.050\text{ m})} \\
&\quad \times \left[\frac{0.030\text{ m}}{\sqrt{(0.030\text{ m})^2+(0.050\text{ m})^2}} + \frac{0.070\text{ m}}{\sqrt{(0.070\text{ m})^2+(0.050\text{ m})^2}}\right] \\
&= 2.7\times 10^{-5}\text{ T.}
\end{aligned}
$$

In the calculation, $\mu_0 = 4\pi \times 10^{-7}$ Wb A^{-1} m^{-1} and $I = 10$ A.

◆ wxMaxima codes:

```
(%i3) fpprintprec:5; mu0:float(4*%pi*1e-7); I:10;
(fpprintprec)     5
(mu0) 1.2566*10^-6
(I)   10
(%i4) BP: mu0*I/(4*float(%pi)*0.05)*(0.03/sqrt(0.03^2+0.05^2)
          + 0.07/sqrt(0.07^2+0.05^2));
(BP)  2.6565*10^-5
```

Comments on the codes:

(%i3) Set floating point print precision to 5, and assign values of μ_0 and I.
(%i4) Calculate magnetic field B_P.

(b) In Fig. 11.15b, wire segments *EF* and *JK* do not contribute to the magnetic field at point *P* because the currents are toward or away from point *P*. The magnetic field at point *P* is

$$
\begin{aligned}
B_P &= B_{FG} + B_{GH} + B_{HJ} \\
&= \frac{\mu_0 I}{4\pi(0.030\text{ m})}\frac{0.050\text{ m}}{\sqrt{(0.050\text{ m})^2+(0.030\text{ m})^2}} \\
&\quad + \frac{\mu_0 I}{4\pi(0.050\text{ m})}\left[\frac{0.030\text{ m}}{\sqrt{(0.030\text{ m})^2+(0.050\text{ m})^2}}\right. \\
&\quad \left. + \frac{0.070\text{ m}}{\sqrt{(0.070\text{ m})^2+(0.050\text{ m})^2}}\right]
\end{aligned}
$$

$$+\frac{\mu_0 I}{4\pi(0.070\text{ m})}\frac{0.050\text{ m}}{\sqrt{(0.050\text{ m})^2+(0.070\text{ m})^2}}$$
$$=6.3\times10^{-5}\text{ T}.$$

◆ wxMaxima codes:

```
(%i3) fpprintprec:5; mu0:float(4*%pi*1e-7); I:10;
(fpprintprec)     5
(mu0) 1.2566*10^-6
(I)   10
(%i4) BP: mu0*I/(4*float(%pi)*0.03)*0.05/sqrt(0.05^2+0.03^2)
     +mu0*I/(4*float(%pi)*0.05)*(0.03/sqrt(0.03^2+0.05^2)
     +0.07/sqrt(0.07^2+0.05^2))
     +mu0*I/(4*float(%pi)*0.07)*0.05/sqrt(0.05^2+0.07^2);
(BP)  6.3451*10^-5
```

Comments on the code:

(%i3) Set floating point print precision to 5, and assign values of μ_0 and I.
(%i4) Calculate magnetic field B_P.

(c) In Fig. 11.15c, wire segment EF does not contribute to the magnetic field at point P because the current is toward point P. The magnetic field at point P is

$$\begin{aligned}B_P &= B_{FG}+B_{GH}+B_{HK}\\ &=\frac{\mu_0 I}{4\pi(0.030\text{ m})}\frac{0.050\text{ m}}{\sqrt{(0.050\text{ m})^2+(0.030\text{ m})^2}}\\ &\quad+\frac{\mu_0 I}{4\pi(0.050\text{ m})}\\ &\quad\times\left[\frac{0.030\text{ m}}{\sqrt{(0.030\text{ m})^2+(0.050\text{ m})^2}}+\frac{0.070\text{ m}}{\sqrt{(0.070\text{ m})^2+(0.050\text{ m})^2}}\right]\\ &\quad+\frac{\mu_0 I}{4\pi(0.070\text{ m})}\\ &\quad\times\left[\frac{0.050\text{ m}}{\sqrt{(0.050\text{ m})^2+(0.070\text{ m})^2}}+1\right]\\ &=7.8\times10^{-5}\text{ T}.\end{aligned}$$

◆ wxMaxima codes:

```
(%i3) fpprintprec:5; mu0:float(4*%pi*1e-7); I:10;
(fpprintprec)     5
(mu0) 1.2566*10^-6
(I)   10
(%i4) BP: mu0*I/(4*float(%pi)*0.03)*0.05/sqrt(0.05^2+0.03^2)
      +mu0*I/(4*float(%pi)*0.05)*(0.03/sqrt(0.03^2+0.05^2)
      +0.07/sqrt(0.07^2+0.05^2))
      +mu0*I/(4*float(%pi)*0.07)*(0.05/sqrt(0.05^2+0.07^2)+1);
(BP)  7.7737*10^-5
```

Comments on the codes:

(%i3) Set floating point print precision to 5, and assign values of μ_0 and I.
(%i4) Calculate magnetic field B_P.

Alternative calculation for part (c): The magnetic field can also be calculated as follows:

$$\begin{aligned} B_P &= B_{FG} + B_{GH} + B_{HJ} + B_{JK} \\ &= \text{magnetic field of part (b)} + B_{JK} \\ &= 6.3 \times 10^{-5}\ \text{T} + \frac{\mu_0 I}{4\pi(0.070\ \text{m})} \\ &= 7.8 \times 10^{-5}\ \text{T}. \end{aligned}$$

The magnetic field B_{JK} is obtained using Table 11.1(c), that is, the magnetic field due to the current in wire segment *JK* is one half of the field due to the current in a long wire.

◆ wxMaxima codes:

```
(%i3) fpprintprec:5; mu0:float(4*%pi*1e-7); I:10;
(fpprintprec)     5
(mu0) 1.2566*10^-6
(I)   10
(%i4) BP: 6.3451*10^-5 + mu0*I/(4*float(%pi)*0.07);
(BP)  7.7737*10^-5
```

Comments on the codes:

(%i3) Set floating point print precision to 5, and assign values of μ_0 and I.
(%i4) Calculate magnetic field B_P.

Problem 11.7 A current I flows in a square shaped wire loop of side b. What is the magnetic field at the center of the loop? What is the magnetic field if $I = 10$ A and $b = 4.0$ cm?

Solution

Figure 11.16 shows the current carrying square wire loop, point P, and relevant magnetic field B_P. We want to calculate the magnetic field at point P. Using the right-hand rule, the magnetic field is in the plane of the paper. The field is contributed by four wire segments *EF*, *FG*, *GH*, and *HE*.

The magnetic field at point P due to current I in wire segment *EF* is, Table 11.1(a),

$$B_{EF} = \frac{\mu_0 I}{4\pi(b/2)}\left[\frac{b/2}{\sqrt{(b/2)^2 + (b/2)^2}} + \frac{b/2}{\sqrt{(b/2)^2 + (b/2)^2}}\right] = \frac{\mu_0 I}{\sqrt{2}\pi b},$$

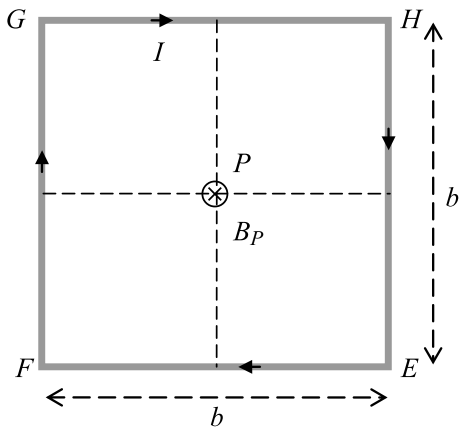


Fig. 11.16 A square wire loop carrying electric current *I*, Problem 11.7

pointing into the plane of the paper. Magnetic fields due to currents in wire segments *FG*, *GH*, and *HE* are the same as this. Therefore, the magnetic field at point *P* is

$$B_P = 4B_{EF} = 2\sqrt{2}\frac{\mu_0 I}{\pi b},$$

pointing into the plane of the paper.

If $I = 10$ A and $b = 4.0$ cm, the magnetic field is

$$B_P = 2\sqrt{2}\frac{\mu_0 I}{\pi b} = 2\sqrt{2}\frac{(4\pi \times 10^{-7}\text{Wb A}^{-1}\text{ m}^{-1})(10\text{ A})}{\pi(0.040\text{ m})} = 2.8 \times 10^{-4}\text{ T},$$

pointing into the plane of the paper.

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; mu0:float(4*%pi*1e-7); I:10; b:0.04;
(fpprintprec)     5
(mu0) 1.2566*10^-6
(I)   10
(b)   0.04
(%i6) BP:2*sqrt(2)*mu0*I/(%pi*b); float(%);
(BP)  (1.5708*10^-4*2^(5/2))/%pi
(%o6) 2.8284*10^-4
```

Comments on the codes:

(%i4) Set floating point print precision to 5, and assign values of μ_0, *I*, and *b*.
(%i6) Calculate magnetic field B_P.

Problem 11.8 Figure 11.17 shows a square wire loop of side *b* carrying current *I*. What is the magnetic field at point *P* a distance *x* away from the loop?

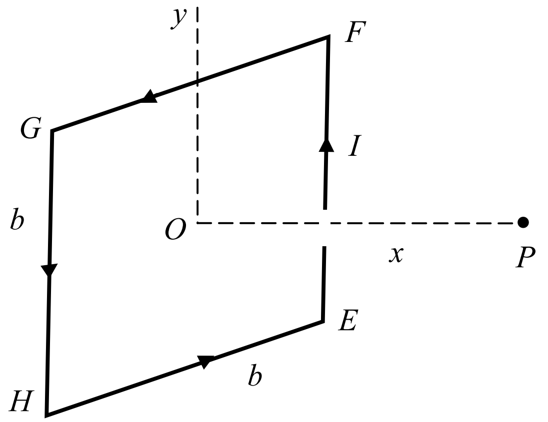


Fig. 11.17 A square wire loop carrying electric current I, Problem 11.8

Solution

Figure 11.18 shows a cross-section of the loop across segments FG and HE. Current directions in wire segments FG and HE are out of and into the plane of the paper, respectively.

The magnitude of the magnetic field at point P due to current flow in wire segment FG is, Table 11.1(a),

$$B_{FG} = \frac{\mu_0 I}{4\pi\sqrt{x^2 + (b/2)^2}}\left[\frac{2(b/2)}{\sqrt{(b/2)^2 + x^2 + (b/2)^2}}\right]$$
$$= \frac{\mu_0 I}{4\pi\sqrt{x^2 + (b/2)^2}}\frac{b}{\sqrt{x^2 + 2(b/2)^2}}.$$

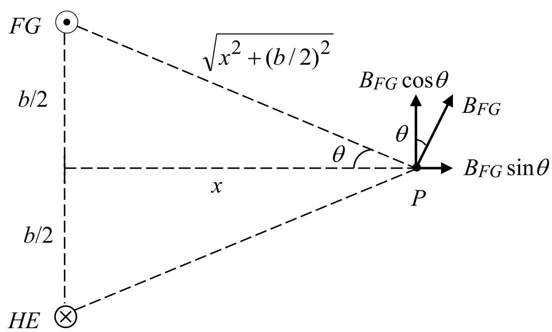


Fig. 11.18 Magnetic fields at P due to currents in wire segments FG and HE, Problem 11.8

The x component of B_{FG} is

$$\begin{aligned}B_{FG,x} &= B_{FG}\sin\theta \\ &= \frac{\mu_0 I}{4\pi\sqrt{x^2+(b/2)^2}}\frac{b}{\sqrt{x^2+2(b/2)^2}}\frac{b/2}{\sqrt{x^2+(b/2)^2}} \\ &= \frac{\mu_0 I b^2}{8\pi[x^2+(b/2)^2]\sqrt{x^2+2(b/2)^2}}.\end{aligned}$$

The y component of B_{FG} is

$$\begin{aligned}B_{FG,y} &= B_{FG}\cos\theta \\ &= \frac{\mu_0 I}{4\pi\sqrt{x^2+(b/2)^2}}\frac{b}{\sqrt{x^2+2(b/2)^2}}\frac{x}{\sqrt{x^2+(b/2)^2}} \\ &= \frac{\mu_0 I b x}{4\pi[x^2+(b/2)^2]\sqrt{x^2+2(b/2)^2}}.\end{aligned}$$

The magnitude of the magnetic field at point P due to current flow in wire segment HE is

$$B_{HE} = B_{FG}.$$

The x component of B_{HE} is

$$B_{HE,x} = B_{FG,x}.$$

The y component of B_{HE} is the same in magnitude but opposite in direction to the y component of B_{FG}. The y components cancel each other and do not contribute to the magnetic field at point P. The same argument is for wire segments EF and GH. Thus, the magnetic field at point P due to the current flow in the square loop is

$$B_P = 4B_{FG,x} = \frac{\mu_0 I b^2}{2\pi[x^2+(b/2)^2]\sqrt{x^2+2(b/2)^2}},$$

pointing to the positive x direction.

As a check, the magnetic field at the center of the loop is obtained by substituting $x = 0$ in the equation, that is

$$B_{P,x=0} = \frac{\mu_0 I b^2}{2\pi[(b/2)^2]\sqrt{2(b/2)^2}} = 2\sqrt{2}\frac{\mu_0 I}{\pi b}.$$

This is the same as the one discussed in Problem 11.7.

◆ wxMaxima codes:

```
(%i2) BFGx:mu0*I*b^2/(8*%pi)/(x^2+(b/2)^2)/sqrt(x^2+2*(b/2)^2); BP:4*BFGx;
(BFGx) (I*b^2*mu0)/(8*%pi*(x^2+b^2/4)*sqrt(x^2+b^2/2))
(BP)   (I*b^2*mu0)/(2*%pi*(x^2+b^2/4)*sqrt(x^2+b^2/2))
(%i3) x: 0;
(x)    0
(%i5) BFGx:mu0*I*b^2/(8*%pi)/(x^2+(b/2)^2)/sqrt(x^2+2*(b/2)^2); BP:4*BFGx;
(BFGx) (I*mu0)/(sqrt(2)*%pi*abs(b))
(BP)   (2^(3/2)*I*mu0)/(%pi*abs(b))
```

Comments on the codes:

(%i2) Assign $B_{FG,x}$ and calculate B_P.
(%i3) Assign $x = 0$.
(%i5) Calculate $B_{FG,x}$ and B_P.

Problem 11.9 A wire is shaped into a regular polygon of n sides. The edges of the polygon touch an imaginary circle of radius R. A current I flows in the wire.

(a) Calculate the magnetic field at the center of the polygon.
(b) What is the magnetic field at the center of the polygon when n is very large?
(c) What is the magnetic field at the center of a square wire loop with side b?

Solution

(a) Figure 11.19 shows one of the sides of the polygon, other sides are not shown. The length of the side is $2b$, the perpendicular distance of the side to the center is a, and the angle subtending the side is θ.

The angle is

$$\theta = \frac{2\pi}{n}.$$

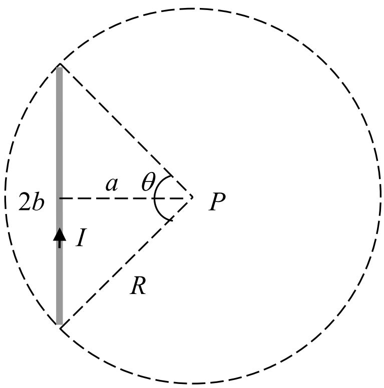


Fig. 11.19 One of the sides of the polygon carrying current I, Problem 11.9

because the polygon has n sides. The perpendicular distance of the side to the center of the polygon is

$$a = R\cos\left(\frac{\theta}{2}\right) = R\cos\left(\frac{\pi}{n}\right).$$

The length of the side is

$$2b = 2R\sin\left(\frac{\theta}{2}\right) = 2R\sin\left(\frac{\pi}{n}\right).$$

The magnetic field at point P due to current on this side is, using Table 11.1(a),

$$\begin{aligned} B_{side} &= \frac{\mu_0 I}{4\pi a}\left(\frac{b}{(a^2+b^2)^{1/2}} + \frac{b}{(a^2+b^2)^{1/2}}\right) \\ &= \frac{\mu_0 I}{2\pi}\frac{R\sin\left(\frac{\pi}{b}\right)}{R\cos\left(\frac{\pi}{n}\right)(R^2\sin^2\left(\frac{\pi}{b}\right) + R^2\cos^2\left(\frac{\pi}{n}\right))^{1/2}} \\ &= \frac{\mu_0 I}{2\pi R}\tan\left(\frac{\pi}{n}\right). \end{aligned}$$

There are n sides, thus, the magnetic field at the center of the polygon due to the current is

$$B_P = \frac{\mu_0 n I}{2\pi R}\tan\left(\frac{\pi}{n}\right),$$

pointing into the plane of the paper.

(b) When n is very large, that is $n \to \infty$, $n\tan(\pi/n) \to \pi$. This is because as $n \to \infty$, $\tan(\pi/n) \to \pi/n$. Thus, the magnetic field at the center of the polygon, when n is very large, is

$$B_P = \frac{\mu_0 I}{2R}.$$

When n is very large, the polygon becomes a ring. This result is the same as the magnetic field at the center of a current-carrying ring, Table 11.1(d). The limits can be calculated by the L'Hospital's rule that you learn in calculus

$$\begin{aligned} \lim_{n\to\infty} n\tan\left(\frac{\pi}{n}\right) &= \lim_{n\to\infty}\frac{\tan\left(\frac{\pi}{n}\right)}{1/n} = \lim_{n\to\infty}\frac{-\frac{\pi}{n^2}\sec^2\left(\frac{\pi}{n}\right)}{-1/n^2} \\ &= \lim_{n\to\infty}\frac{\pi}{\cos^2(\pi/n)} \\ &= \pi. \end{aligned}$$

◆ wxMaxima codes:

```
(%i1) limit(n*tan(%pi/n), n, inf);
(%o1) %pi
(%i2) BP: mu0*n*I/(2*%pi*R)*tan(%pi/n);
(BP)  (I*mu0*tan(%pi/n)*n)/(2*%pi*R)
(%i3) limit(BP, n, inf);
(%o3) (I*mu0)/(2*R)
```

Comments on the codes:

(%i1) Calculate $\lim_{n\to\infty} n \tan(\pi/n)$.
(%i2) Assign B_P.
(%i3) Calculate $\lim_{n\to\infty} B_P$.

(c) For a square loop of side b, we have

$$R = \frac{\sqrt{2}}{2}b, \quad n = 4,$$

and the magnetic field at the center of a square wire loop is

$$B_P = \frac{\mu_0 n I}{2\pi R}\tan\left(\frac{\pi}{n}\right) = \frac{\mu_0(4)I(2)}{2\pi\sqrt{2}b}\tan\left(\frac{\pi}{4}\right) = \frac{\mu_0(4)I(2)}{2\pi\sqrt{2}b}(1)$$
$$= 2\sqrt{2}\frac{\mu_0 I}{\pi b}.$$

This is the same result as in Problem 11.7.

Problem 11.10

(a) A circular loop of radius R carrying a current of I is shown in Fig. 11.20a. What is the magnetic field B at the center of the loop?
(b) A circular coil of radius 0.20 m has a current of 5.0 A. What is the magnetic field at the center of the coil?
(c) Calculate the magnetic field at the center of the coil, if the coil contains 50 windings.

Solution

(a) Figure 11.20b shows the current carrying loop, a length element $\boldsymbol{ds}$ of the loop, and unit vector $\hat{\boldsymbol{r}}$ from $\boldsymbol{ds}$ to the center of the loop. Using the Biot–Savart law, the magnetic field at the center of the loop due to the length element is

$$\boldsymbol{dB} = \frac{\mu_0}{4\pi}\frac{I\ \boldsymbol{ds}\times\hat{\boldsymbol{r}}}{r^2},$$

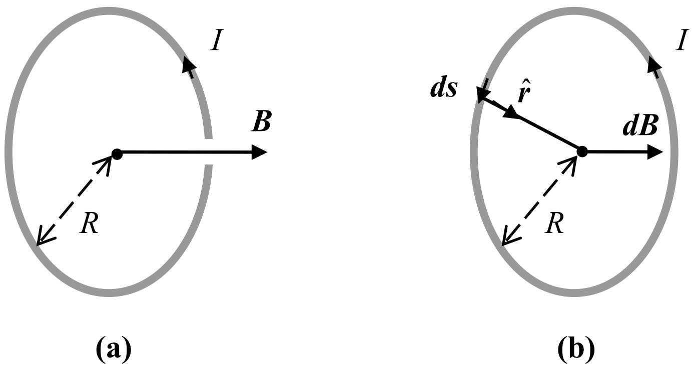


Fig. 11.20 **a** Current carrying circular loop, **b** determining magnetic field at the center of the circular loop, Problem 11.10

$$dB = \frac{\mu_0}{4\pi}\frac{I \sin 90^\circ \; ds}{R^2} = \frac{\mu_0}{4\pi}\frac{I \; ds}{R^2}.$$

Here, $\left|\boldsymbol{ds} \times \hat{\boldsymbol{r}}\right| = \sin 90^\circ ds = ds$ because $\boldsymbol{ds}$ is perpendicular to $\hat{\boldsymbol{r}}$ and $r = R$. The magnetic field at the center of the loop due to the whole loop is obtained by integrating the dB

$$B = \int dB = \frac{\mu_0}{4\pi}\frac{I}{R^2}\int ds = \frac{\mu_0}{4\pi}\frac{I}{R^2}(2\pi R)$$
$$= \frac{\mu_0 I}{2R}.$$

This is the same as Table 11.1(d).

(b) The magnetic field at the center of the coil is

$$B = \frac{\mu_0 I}{2R} = \frac{2\pi k_m I}{R} = \frac{2\pi(10^{-7}\text{ T m A}^{-1})(5.0\text{ A})}{0.20\text{ m}} = 1.6 \times 10^{-5}\text{ T}.$$

♦ wxMaxima codes:

```
(%i4) fpprintprec:5; km:1e-7; I:5; R:0.2;
(fpprintprec)     5
(km)  1.0*10^-7
(I)   5
(R)   0.2
(%i5) B: 2*float(%pi)*km*I/R;
(B)   1.5708*10^-5
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of k_m, I, and R.

(%i5) Calculate magnetic field B.

(c) If the number of windings is $N = 50$, the magnetic field at the center of the coil is

$$B = \frac{\mu_0 N I}{2R} = \frac{2\pi k_m N I}{R} = \frac{2\pi(10^{-7}\ \mathrm{T\ m\ A^{-1}})(50)(5.0\ \mathrm{A})}{0.20\ \mathrm{m}} = 7.9 \times 10^{-4}\ \mathrm{T}.$$

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; km:1e-7; I:5; R:0.2; N:50;
(fpprintprec)     5
(km)   1.0*10^-7
(I)    5
(R)    0.2
(N)    50
(%i6) B: 2*float(%pi)*km*N*I/R;
(B)    7.854*10^-4
```

Comments on the codes:

(%i5) Set floating point print precision to 5, and assign values of k_m, I, R, and N.
(%i6) Calculate magnetic field B.

Problem 11.11

(a) A circular loop of radius R carrying a current of I is shown in Fig. 11.21a. What is the magnetic field at point P a distance x on the central axis of the loop?
(b) A circular loop of radius 0.20 m carrying a current of 3.0 A is shown in Fig. 11.21b. What is the magnetic field at a point 0.50 m on the central axis of the loop?

Solution

(a) Figure 11.21c shows the cross-section of the current loop, length element $\boldsymbol{ds}$, and the magnetic field $\boldsymbol{dB}$. The direction of the current is out of the page in the length element and into the page on the opposite side of the loop. By the Biot–Savart law, the magnetic field due to length element $\boldsymbol{ds}$ is

$$\boldsymbol{dB} = \frac{\mu_0}{4\pi}\frac{I\,\boldsymbol{ds} \times \hat{\boldsymbol{r}}}{r^2},$$
$$dB = \frac{\mu_0}{4\pi}\frac{I \sin 90^\circ\, ds}{r^2} = \frac{\mu_0}{4\pi}\frac{I\,ds}{R^2 + x^2}.$$

$\boldsymbol{dB}$ is resolved into $dB\cos\theta$ and $dB\sin\theta$. By symmetry, the $dB\sin\theta$ component will sum up to zero when the whole loop is considered. Thus, the magnetic field at point P a distance x on the central axis of the loop is

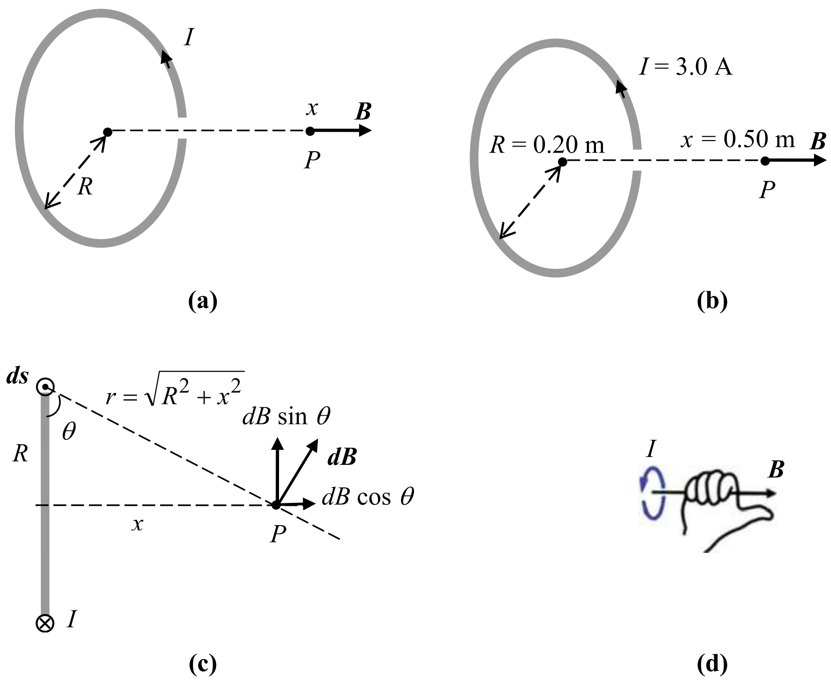


Fig. 11.21 **a** Magnetic field at point P, **b** magnetic field at $x = 0.50$ m, **c** determining magnetic field at point P, **d** direction of the magnetic field, Problem 11.11

$$
\begin{aligned}
B &= \int dB = \int dB \cos\theta \\
&= \int \frac{\mu_0}{4\pi} \frac{I\ ds}{R^2 + x^2} \frac{R}{\sqrt{R^2 + x^2}} = \frac{\mu_0}{4\pi} \frac{I\ (2\pi R)}{R^2 + x^2} \frac{R}{\sqrt{R^2 + x^2}} \\
&= \frac{\mu_0 I R^2}{2(R^2 + x^2)^{3/2}}.
\end{aligned}
$$

The direction of the magnetic field is to the positive x direction (to the right). This result is the same as Table 11.1(d).

(b) Using the result of part (a) and Fig. 11.21b the magnetic field at point P is

$$
\begin{aligned}
B &= \frac{\mu_0 I R^2}{2(R^2 + x^2)^{3/2}} = \frac{(4\pi \times 10^{-7}\ \text{T m A}^{-1})(3.0\ \text{A})(0.20\ \text{m})^2}{2[(0.20\ \text{m})^2 + (0.50\ \text{m})^2]^{3/2}} \\
&= 4.8 \times 10^{-7}\ \text{T}.
\end{aligned}
$$

The direction of the magnetic field is determined by the right-hand rule, Fig. 11.21d. The curled fingers are the direction of the current and the thumb is the direction of the magnetic field. Thus, the magnetic field points to the right.

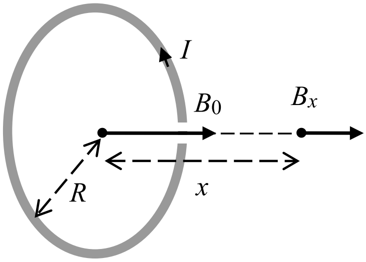


Fig. 11.22 A current carrying circular coil, Problem 11.12

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; mu0:float(4*%pi*1e-7); I:3; R:0.2; x:0.5;
(fpprintprec)     5
(mu0) 1.2566*10^-6
(I)   3
(R)   0.2
(x)   0.5
(%i6) B: mu0*I*R^2/(2*(R^2+x^2)^(3/2));
(B)   4.828*10^-7
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of μ_0, I, R, and x.
(%i6) Calculate magnetic field B.

Problem 11.12 A circular coil of radius R has a current of I flowing in it. Where along its axis the magnetic field is one half of the magnetic field at the center?

Solution

Figure 11.22 shows the coil carrying a current of I, a magnetic field at the center of the coil, B_0, and a magnetic field at a distance x from the coil, B_x.

The magnetic field along the central axis of the coil is, Table 11.1(d),

$$B_x = \frac{\mu_0 I R^2}{2(R^2 + x^2)^{3/2}}.$$

Letting $x = 0$, the magnetic field at the center of the coil is

$$B_0 = \frac{\mu_0 I}{2R}.$$

We require that

$$\frac{B_x}{B_0} = \frac{\mu_0 I R^2}{2(R^2 + x^2)^{3/2}} \cdot \frac{2R}{\mu_0 I} = \frac{R^3}{(R^2 + x^2)^{3/2}} = \frac{1}{2}.$$

Solving the equation for x gives

$$2R^3 = (R^2 + x^2)^{3/2},$$
$$x = \sqrt{2^{2/3} - 1}\; R = 0.77R.$$

This means that, at a distance of $0.77R$ from the coil, the magnetic field is one half of the one at the center.

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; assume(R>0); solve(2*R^3=(R^2+x^2)^(3/2), x)$ float(%);
(fpprintprec)     5
(%o2) [R>0]
(%o4) [x=-0.62996*(0.76642-2.2599*%i)*R,x=0.62996*(0.76642-2.2599*%i)*R,
       x=-0.62996*(2.2599*%i+0.76642)*R,x=0.62996*(2.2599*%i+0.76642)*R,
       x=-0.76642*R,x=0.76642*R]
```

Comments on the codes:

(%i4) Set floating point print precision to 5, and solve $2R^3 = (R^2 + x^2)^{3/2}$ for x.
(%o4) The solutions.

Problem 11.13 Two rings of radii 0.10 and 0.20 m, separated by a distance of 1.0 m have currents of 3.0 and 5.0 A, respectively, in opposite directions, as illustrated in Fig. 11.23. Determine the magnetic fields at points P, Q, and R along the axis of the rings.

Solution

The magnetic field along the central axis of a current-carrying ring is, Table 11.1(d),

$$B = \frac{\mu_0 I R^2}{2(R^2 + x^2)^{3/2}}.$$

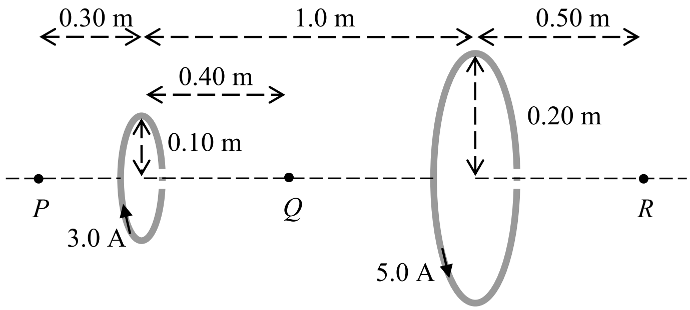


Fig. 11.23 Two current-carrying rings, Problem 11.13

The magnetic fields are added by vector addition. We will use these two facts to calculate magnetic fields due to two current-carrying rings. Let the left be the negative x direction and the right the positive one.

The magnetic field at point P due to currents in the small and big rings is

$$\begin{aligned} B_P &= -\frac{(4\pi \times 10^{-7})(3.0\text{ A})(0.10\text{ m})^2}{2[(0.10\text{ m})^2 + (0.30\text{ m})^2]^{3/2}} + \frac{(4\pi \times 10^{-7})(5.0\text{ A})(0.20\text{ m})^2}{2[(0.20\text{ m})^2 + (1.3\text{ m})^2]^{3/2}} \\ &= -5.4 \times 10^{-7}\text{ T}, \end{aligned}$$

pointing to the left.

The magnetic field at point Q due to currents in the small and big rings is

$$\begin{aligned} B_Q &= -\frac{(4\pi \times 10^{-7})(3.0\text{ A})(0.10\text{ m})^2}{2[(0.10\text{ m})^2 + (0.40\text{ m})^2]^{3/2}} + \frac{(4\pi \times 10^{-7})(5.0\text{ A})(0.20\text{ m})^2}{2[(0.20\text{ m})^2 + (0.60\text{ m})^2]^{3/2}} \\ &= 2.3 \times 10^{-7}\text{ T}, \end{aligned}$$

pointing to the right.

The magnetic field at point R due to currents in the small and big rings is

$$\begin{aligned} B_R &= -\frac{(4\pi \times 10^{-7})(3.0\text{ A})(0.10\text{ m})^2}{2[(0.10\text{ m})^2 + (1.5\text{ m})^2]^{3/2}} + \frac{(4\pi \times 10^{-7})(5.0\text{ A})(0.20\text{ m})^2}{2[(0.20\text{ m})^2 + (0.50\text{ m})^2]^{3/2}} \\ &= 8.0 \times 10^{-7}\text{ T}, \end{aligned}$$

pointing to the right.

◆ wxMaxima codes:

```
(%i2) fpprintprec:5; mu0:float(4*%pi*1e-7);
(fpprintprec)     5
(mu0) 1.2566*10^-6
(%i3) BP: -mu0*3*0.1^2/(2*(0.1^2+0.3^2)^(3/2))
        +mu0*5*0.2^2/(2*(0.2^2+1.3^2)^(3/2));
(BP)  -5.4085*10^-7
(%i4) BQ:-mu0*3*0.1^2/(2*(0.1^2+0.4^2)^(3/2))
       +mu0*5*0.2^2/(2*(0.2^2+0.6^2)^(3/2));
(BQ)  2.2781*10^-7
(%i5) BR: -mu0*3*0.1^2/(2*(0.1^2+1.5^2)^(3/2))
        +mu0*5*0.2^2/(2*(0.2^2+0.5^2)^(3/2));
(BR)  7.9911*10^-7
```

Comments on the codes:

(%i2)	Set floating point print precision to 5, assign value of μ_0.
(%i3), (%i4), (%i5)	Calculate B_P, B_Q, and B_R.

Additional question: What are the magnetic fields if the direction of the 5.0 A current is reversed?

Answer: We redo the calculations. The magnetic field at point P due to currents in the small and big rings is

$$\begin{aligned} B_P &= -\frac{(4\pi \times 10^{-7})(3.0\ \text{A})(0.10\ \text{m})^2}{2[(0.10\ \text{m})^2 + (0.30\ \text{m})^2]^{3/2}} - \frac{(4\pi \times 10^{-7})(5.0\ \text{A})(0.20\ \text{m})^2}{2[(0.20\ \text{m})^2 + (1.3\ \text{m})^2]^{3/2}} \\ &= -6.5 \times 10^{-7}\ \text{T}, \end{aligned}$$

pointing to the left.

The magnetic field at point Q due to currents in the small and big rings is

$$\begin{aligned} B_Q &= -\frac{(4\pi \times 10^{-7})(3.0\ \text{A})(0.10\ \text{m})^2}{2[(0.10\ \text{m})^2 + (0.40\ \text{m})^2]^{3/2}} - \frac{(4\pi \times 10^{-7})(5.0\ \text{A})(0.2)^2}{2[(0.20\ \text{m})^2 + (0.60\ \text{m})^2]^{3/2}} \\ &= -7.7 \times 10^{-7}\ \text{T}, \end{aligned}$$

pointing to the left.

The magnetic field at point R due to currents in the small and big rings is,

$$\begin{aligned} B_R &= -\frac{(4\pi \times 10^{-7})(3.0\ \text{A})(0.10\ \text{m})^2}{2[(0.10\ \text{m})^2 + (1.5\ \text{m})^2]^{3/2}} - \frac{(4\pi \times 10^{-7})(5.0\ \text{A})(0.2)^2}{2[(0.20\ \text{m})^2 + (0.50\ \text{m})^2]^{3/2}} \\ &= -8.1 \times 10^{-7}\ \text{T}, \end{aligned}$$

pointing to the left.

◆ wxMaxima codes:

```
(%i2) fpprintprec:5; mu0:float(4*%pi*1e-7);
(fpprintprec)     5
(mu0) 1.2566*10^-6
(%i3) BP: -mu0*3*0.1^2/(2*(0.1^2+0.3^2)^(3/2))
          -mu0*5*0.2^2/(2*(0.2^2+1.3^2)^(3/2));
(BP)  -6.513*10^-7
(%i4) BQ: -mu0*3*0.1^2/(2*(0.1^2+0.4^2)^(3/2))
          -mu0*5*0.2^2/(2*(0.2^2+0.6^2)^(3/2));
(BQ)  -7.6565*10^-7
(%i5) BR: -mu0*3*0.1^2/(2*(0.1^2+1.5^2)^(3/2))
          -mu0*5*0.2^2/(2*(0.2^2+0.5^2)^(3/2));
(BR)  -8.1021*10^-7
```

Comments on the codes:

(%i2)	Set floating point print precision to 5, assign value of μ_0.
(%i3), (%i4), (%i5)	Calculate B_P, B_Q, and B_R.

Problem 11.14 A solenoid of length 20 cm has 500 turns of wire. The current of 5.0 A flows in the solenoid. What is the magnetic field in the solenoid?

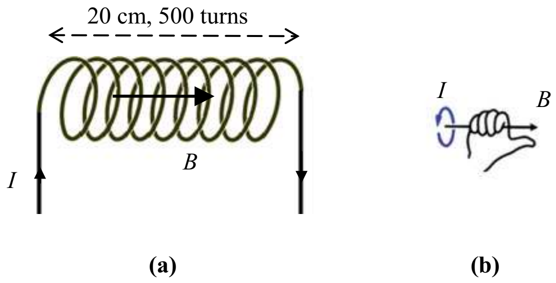


Fig. 11.24 **a** Magnetic field of a solenoid, **b** direction of the magnetic field, Problem 11.14

Solution

Figure 11.24a shows the solenoid with current I.

The magnetic field in the solenoid is, Eq. (11.5),

$$B = \mu_0 n I = (4\pi \times 10^{-7}\ \mathrm{T\ m\ A^{-1}})\left(\frac{500}{0.20\ \mathrm{m}}\right)(5.0\ \mathrm{A}) = 1.6 \times 10^{-2}\ \mathrm{T},$$

pointing to the right. The direction is determined by the right-hand rule, as illustrated by Fig. 11.24b.

◆ wxMaxima codes:

```
(%i6) fpprintprec:5; mu0:float(4*%pi*1e-7); N:500; l:0.2; n: N/l; I:5;
(fpprintprec)     5
(mu0) 1.2566*10^-6
(N)   500
(l)   0.2
(n)   2500.0
(I)   5
(%i7) B: mu0*n*I;
(B)   0.015708
```

Comments on the codes:

(%i6) Set floating point print precision to 5, and assign the value of μ_0, N, l, n, and I.

(%i7) Calculate the magnetic field of a solenoid B.

Problem 11.15 Current of 3.0 A flows in a solenoid of length 60 cm, radius 2.0 cm, and 1000 turns of winding. On the axis of the solenoid, there is a long wire carrying a current of 50 A. Determine the magnetic field at a point 1.0 cm from the wire.

Solution

Figure 11.25 shows the solenoid, the wire, and the relevant magnetic fields. The magnetic field due to the current carrying solenoid is B_s, the magnetic field due to the current carrying long wire is B_w, and the resultant magnetic field is B.

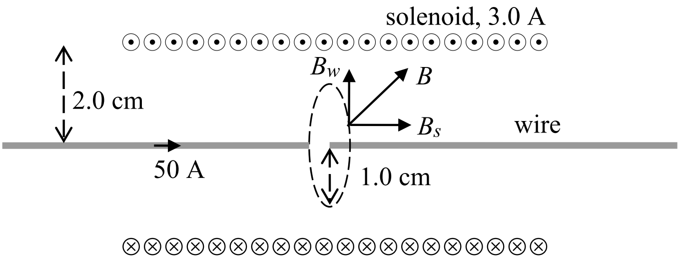


Fig. 11.25 A current-carrying wire in a solenoid, Problem 11.15

The magnetic field in the solenoid due to the current flow in it is

$$B_s = \mu_0 n I_s = (4\pi \times 10^{-7}\ \text{T m A}^{-1})\left(\frac{1000}{0.60\ \text{m}}\right)(3.0\ \text{A}) = 6.3 \times 10^{-3}\ \text{T}.$$

The magnetic field due to the current carrying long wire at a distance of 1.0 cm from the wire is

$$B_w = \frac{\mu_0 I_w}{2\pi r} = \frac{(4\pi \times 10^{-7}\ \text{T m A}^{-1})(50\ \text{A})}{2\pi(0.010\ \text{m})} = 1.0 \times 10^{-3}\ \text{T}.$$

Both B_s and B_w are perpendicular to each other. The resultant magnetic field is

$$B = \sqrt{B_s^2 + B_w^2} = \sqrt{(6.3 \times 10^{-3}\ \text{T})^2 + (1.0 \times 10^{-3}\ \text{T})^2} = 6.4 \times 10^{-3}\ \text{T}.$$

◆ wxMaxima codes:

```
(%i6) fpprintprec:5; mu0:float(4*%pi*1e-7); n:1000/0.6; Is:3; Iw:50; r:0.01;
(fpprintprec)     5
(mu0) 1.2566*10^-6
(n)   1666.7
(Is)  3
(Iw)  50
(r)   0.01
(%i7) Bs: mu0*n*Is;
(Bs)  0.0062832
(%i8) Bw: mu0*Iw/(2*float(%pi)*r);
(Bw)  0.001
(%i9) B: sqrt(Bs^2+Bw^2);
(B)   0.0063623
```

Comments on the codes:

(%i6) Set floating point print precision to 5, assign values of μ_0, n, I_s, I_w, and r.

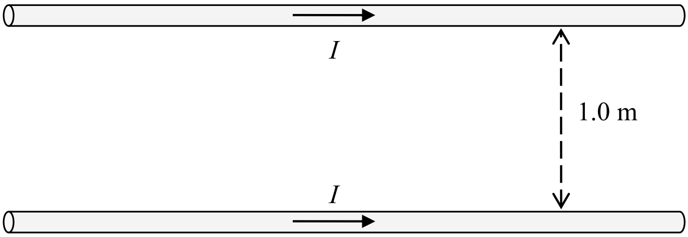


Fig. 11.26 Two current carrying wires, Problem 11.16

(%i7), (%i8) Calculate the magnetic field of the solenoid and the wire B_s and B_w.
(%i9) Calculate the resultant magnetic field B.

Problem 11.16 Figure 11.26 shows two long wires separated by a distance of 1.0 m with current I flowing in each. Each wire is attracted toward the other by 2.0×10^{-7} N per meter. What is the current?

Solution

Force per unit length between two parallel current-carrying wires is, Eq. (11.3),

$$\frac{F}{l} = \frac{2k_m I_1 I_2}{a}.$$

where I_1 and I_2 are the currents in the wires and a is the separation distance between the wires. For this problem, the currents are the same, so

$$\frac{F}{l} = \frac{2k_m I^2}{a}.$$

We can calculate the current,

$$2.0 \times 10^{-7}\ \text{N/m} = \frac{2(10^{-7}\ \text{T m A}^{-1})I^2}{1.0\ \text{m}}$$
$$I = 1.0\ \text{A}$$

In fact, these values had been used to define a current of 1.0 A. That is, 1.0 A is current in two parallel long wires separated by a distance of 1.0 m that gives rise to an attractive force of 2.0×10^{-7} N per meter between them.

◆ wxMaxima codes:

```
(%i1) ratprint:false;
(ratprint)  false
(%i2) solve(2e-7 = 2*1e-7*I^2,  I);
(%o2) [I=-1,I=1]
```

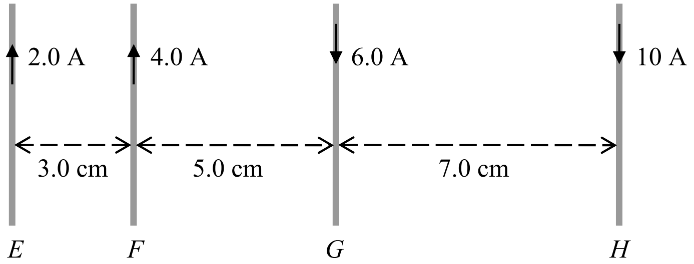


Fig. 11.27 Four parallel current carrying wires, Problem 11.17

Comments on the codes:

(%i1) Set internal rational number print to false.
(%i2) Solve $2.0 \times 10^{-7} = 2 \times 10^{-7} \times I^2$ for I.

Problem 11.17 Four long parallel wires E, F, G, and H carrying different currents are shown in Fig. 11.27. Calculate the force on the 10 cm of wire E.

Solution

The magnetic field along wire E, due to currents in wires F, G, and H is, Table 11.1(b),

$$B_E = \frac{\mu_0}{2\pi}\left(\frac{4.0\ \text{A}}{0.030\ \text{m}} - \frac{6.0\ \text{A}}{0.080\ \text{m}} - \frac{10\ \text{A}}{0.15\ \text{m}}\right) = -1.7 \times 10^{-6}\ \text{T}.$$

into the plane of the paper. The magnitude of the magnetic force on the 10 cm of wire E is

$$F = ILB_E = (2.0\ \text{A})(0.10\ \text{m})(1.7 \times 10^{-6}\ \text{T}) = 3.3 \times 10^{-7}\ \text{N}.$$

pointing to the left.

◆ wxMaxima codes:

```
(%i2) fpprintprec:5; mu0:float(4*%pi*1e-7);
(fpprintprec)      5
(mu0) 1.2566*10^-6
(%i3) BE: mu0/float(2*%pi)*(4/0.03-6/0.08-10/0.15);
(BE)  -1.6667*10^-6
(%i4) F: 2*0.1*abs(BE);
(F)   3.3333*10^-7
```

Comments on the codes:

(%i2) Set floating point print precision to 5, and assign the value of μ_0.
(%i3) Calculate magnetic field, B_E.

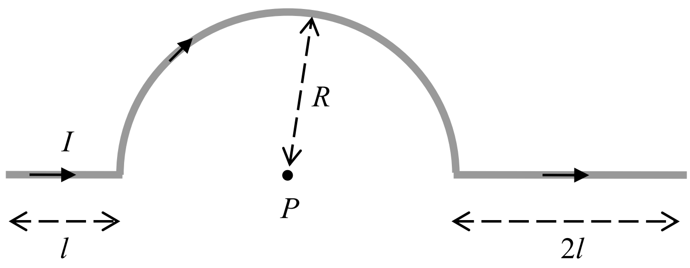


Fig. 11.28 A current carrying wire, Problem 11.18

(%i4) Calculate F.

Problem 11.18

(a) A wire shown in Fig. 11.28 carries a current of I. The lengths of straight wire segments are l and $2l$ while the radius of the semicircular segment is R. Determine the magnetic field at point P.

(b) What is the magnitude of the magnetic field at point P if $I = 5.0$ A, $l = 10$ cm, and $R = 15$ cm?

Solution

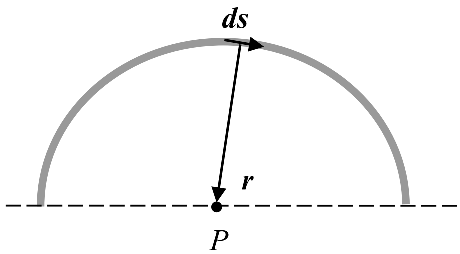
(a) Currents in the straight wire segments of lengths l and $2l$ do not contribute to the magnetic field at point P because the directions of the currents are toward and away from P. Such directions will give $\boldsymbol{ds} \times \boldsymbol{r} = 0$ for the segments and zero magnetic field at point P according to Biot–Savart law. The magnetic field due to the current in the semicircular segment at point P is calculated by Biot–Savart law, as illustrated in Fig. 11.29.

The current in wire element $\boldsymbol{ds}$ produces elementary magnetic field $\boldsymbol{dB}$ at point P, Eq. (11.1),

$$\boldsymbol{dB} = \frac{\mu_0 I}{4\pi} \frac{\boldsymbol{ds} \times \boldsymbol{r}}{r^3} = \frac{\mu_0 I}{4\pi} \frac{\boldsymbol{ds} \times \hat{\boldsymbol{r}}}{r^2}.$$

For this wire segment,

Fig. 11.29 Determining magnetic field due to current in the semicircular segment, Problem 11.18



$$dB = \frac{\mu_0 I}{4\pi} \frac{ds}{R^2},$$

because $\boldsymbol{ds}$ and $\boldsymbol{r}$ are perpendicular to each other and $r = R$. The magnetic field at point P due to the semicircular segment is

$$B = \int dB = \frac{\mu_0 I}{4\pi R^2} \int ds = \frac{\mu_0 I}{4\pi R^2}(\pi R) = \frac{\mu_0 I}{4R}.$$

The direction of the field is into the plane of the paper as indicated by the right-hand rule.

(b) Substituting the given numerical values, the magnitude of the magnetic field at point P is

$$B = \frac{\mu_0 I}{4R} = \frac{(4\pi \times 10^{-7}\ \text{T m/A})(5.0\ \text{A})}{4(0.15\ \text{m})} = 1.0 \times 10^{-5}\ \text{T}.$$

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; mu0:float(4*%pi*1e-7); I:5; R:0.15;
(fpprintprec)     5
(mu0) 1.2566*10^-6
(I)    5
(R)    0.15
(%i5) B: mu0*I/(4*R);
(B)    1.0472*10^-5
```

Comments on the codes:

(%i4) Set floating point print precision to 5, and assign values of μ_0, I, and R.
(%i5) Calculate B.

Problem 11.19 Show that the magnetic field inside a solenoid is $B = \mu_0 n I$, where n is the number of turns of the winding per unit length and I is the current in the solenoid. Use the Ampere's law.

Solution

Figure 11.30 shows part of a solenoid that has n turns of winding per meter with a current of I in it. We consider an imaginary rectangular Amperian closed loop *abcda* of length l and width w.

Using the Ampere's law (Eq. 11.4), we have

$$\oint_{abcda} \boldsymbol{B} \cdot \boldsymbol{ds} = \mu_0 I_{enclosed},$$

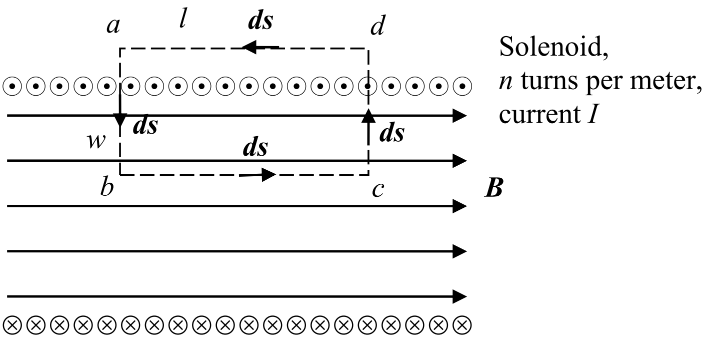


Fig. 11.30 A solenoid and an Amperian closed loop *abcda*, Problem 11.19

$$\int_a^b \boldsymbol{B} \cdot \boldsymbol{ds} + \int_b^c \boldsymbol{B} \cdot \boldsymbol{ds} + \int_c^d \boldsymbol{B} \cdot \boldsymbol{ds} + \int_d^a \boldsymbol{B} \cdot \boldsymbol{ds} = \mu_0 nlI,$$

$$0 + Bl + 0 + 0 = \mu_0 nlI.$$

Here, the current enclosed in the Amperian loop is *nlI* because there are *nl* wires each with current *I*. The line integrals along *ab* and *cd* are zero because ***B*** is zero or ***B*** is perpendicular to ***ds***, and the line integral along *da* is zero because ***B*** is zero outside the solenoid. The line integral along *bc* is *Bl* because ***B*** and ***ds*** are in the same direction and parallel to each other. Therefore, the magnetic field of a solenoid is

$$B = \mu_0 nI.$$

This is Eq. (11.5) given in point (5) at the beginning of this chapter.

Problem 11.20

(a) Use Biot–Savart law to find the magnetic field at point *P* of Fig. 11.31a. The wire has a current of *I*, its length is $a + b$, and its perpendicular distance from point *P* is *R*.
(b) What is the magnetic field at point *Q* of Fig. 11.31b? The wire has a current of *I*, its length is $2a$, and its perpendicular distance from point *Q* is *R*.
(c) What is the magnetic field at point *Q* of Fig. 11.31b if *a* is very large?
(d) What is the magnetic field at point *S* of Fig. 11.31c?

Solution

(a) We redraw Fig. 11.31a as Fig. 11.31d to do calculation according to the Bio–Savart law. Figure 11.31d shows the current *I* in a wire of length $a + b$, length element ***ds***, and vector ***r*** from ***ds*** to point *P*. By Biot–Savart law, the magnetic field ***dB*** due to length element ***ds*** is,

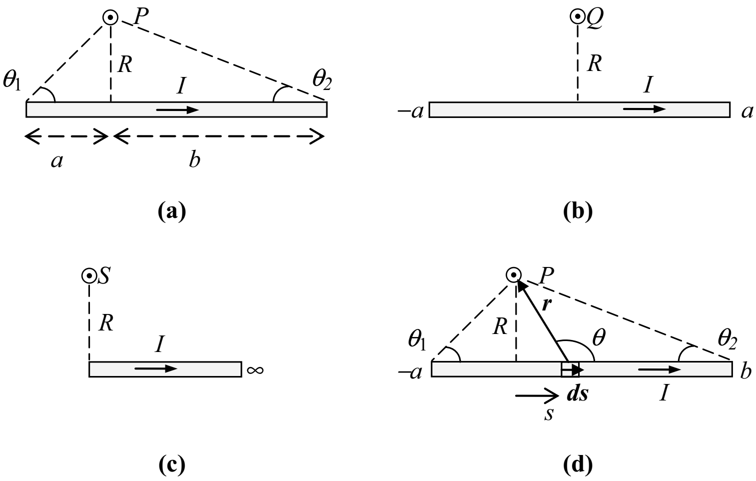


Fig. 11.31 Magnetic fields of four current-carrying wires using the Biot–Savart law, Problem 11.20

$$
\boldsymbol{dB} = \frac{\mu_0 I}{4\pi} \frac{\boldsymbol{ds} \times \boldsymbol{r}}{r^3} = \frac{\mu_0 I}{4\pi} \frac{\boldsymbol{ds} \times \hat{\boldsymbol{r}}}{r^2},
$$
$$
dB = \frac{\mu_0 I}{4\pi} \frac{\sin\theta \, ds}{r^2} = \frac{\mu_0 I}{4\pi} \frac{R \, ds}{r^3} = \frac{\mu_0 I}{4\pi} \frac{R \, ds}{(R^2 + s^2)^{3/2}}.
$$

The magnetic field at point P in Fig. 11.31a or Fig. 11.31d is

$$
B_P = \int dB = \frac{\mu_0 I R}{4\pi} \int_{-a}^{b} \frac{ds}{(R^2 + s^2)^{3/2}} = \frac{\mu_0 I}{4\pi R} \left[\frac{s}{(R^2 + s^2)^{1/2}} \right]_{-a}^{b}
$$
$$
= \frac{\mu_0 I}{4\pi R} \left(\frac{a}{(R^2 + a^2)^{1/2}} + \frac{b}{(R^2 + b^2)^{1/2}} \right)
$$
$$
= \frac{\mu_0 I}{4\pi R} (\cos\theta_1 - \cos\theta_2).
$$

This is the same as Table 11.1(a).

(b) Using the result of part (a), the magnetic field at point Q in Fig. 11.31b is

$$
B_Q = \frac{\mu_0 I}{4\pi R} \left(\frac{a}{(R^2 + a^2)^{1/2}} + \frac{a}{(R^2 + a^2)^{1/2}} \right)
$$
$$
= \frac{\mu_0 I}{2\pi R} \frac{a}{(R^2 + a^2)^{1/2}}.
$$

(c) Using the result of part (b), the magnetic field at point Q becomes

$$B_Q = \frac{\mu_0 I}{2\pi R},$$

as $a \to \infty$. This is because $a/(R^2 + a^2)^{1/2} \to 1$ as $a \to \infty$. This is the magnetic field around a long straight wire, Table 11.1(b).

(d) By symmetry and using the result of part (c), the magnetic field at point S in Fig. 11.31c is one half of that at point Q,

$$B_S = \frac{1}{2} B_Q = \frac{\mu_0 I}{4\pi R}.$$

This is the magnetic field of semi-infinite straight wire, Table 11.1(c).

11.3 Summary

- Biot–Savart's law states that the magnetic field $\boldsymbol{dB}$ due to a segment $\boldsymbol{ds}$ of a conductor carrying a current of I is given by

$$\boldsymbol{dB} = k_m \frac{I\, \boldsymbol{ds} \times \boldsymbol{r}}{r^3} = k_m \frac{I\, \boldsymbol{ds} \times \hat{\boldsymbol{r}}}{r^2} = \frac{\mu_0}{4\pi} \frac{I\, \boldsymbol{ds} \times \hat{\boldsymbol{r}}}{r^2}.$$

- Force per unit length between two parallel long wires, separated by a distance of a, and carrying currents of I_1 and I_2 is

$$\frac{F}{l} = \frac{2k_m I_1 I_2}{a} = \frac{\mu_0 I_1 I_2}{2\pi a}.$$

- Ampere's law states that the line integral of $\boldsymbol{B}$ around a closed path is $\mu_0 I$, that is

$$\oint \boldsymbol{B} \cdot \boldsymbol{ds} = \mu_0 I,$$

where I is the current through a surface bounded by the closed path.

- The magnetic fields inside a solenoid is

$$B_{solenoid} = \mu_0 \frac{N}{l} I = \mu_0 n I,$$

where N is the number of turns of the wire, l is the length of the solenoid, and n is the number of turns per unit length.

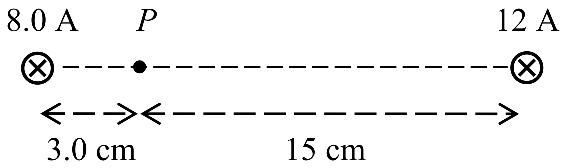


Fig. 11.32 Two parallel current carrying wires, Exercise 11.1

11.4 Exercises

Exercise 11.1 Figure 11.32 shows two long wires 18 cm apart carrying currents of 8.0 and 12 A into the plane of the paper.

(a) Calculate the magnetic field at point P
(b) At what point on the line joining the wires is the magnetic field zero?

(Answer: (a) $B_P = 3.7 \times 10^{-5}$ T in the negative y direction,
(b) 7.2 cm from the wire with 8.0 A current)

Exercise 11.2 Figure 11.33 shows a coil of radius $R = 20$ cm carrying a current of $I = 0.25$ A in counter clockwise direction. How many turns must there be in the coil so that the magnetic field B at the center of the coil is 4.0×10^{-5} T?

(Answer: 51 turns)

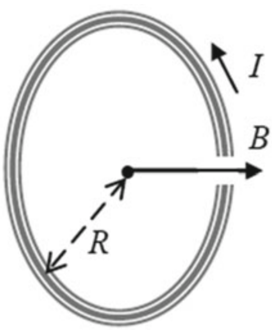
Exercise 11.3 Figure 11.34 shows a coil of radius 2.0 cm concentric with a coil of radius 7.0 cm. Each coil has 100 turns and the electric current in the larger coil is 5.0 A in counterclockwise direction. What is the current in the smaller coil so that the magnetic field B at the center of the coils is 2.0×10^{-3} T?

(Answer: 0.79 A, clockwise)

Exercise 11.4 The wire shown in Fig. 11.35 carries an electric current of 15 A. Calculate the magnetic field at point P.

(Answer: 3.5×10^{-4} T into the plane of the paper)

Fig. 11.33 A current-carrying coil, Exercise 11.2



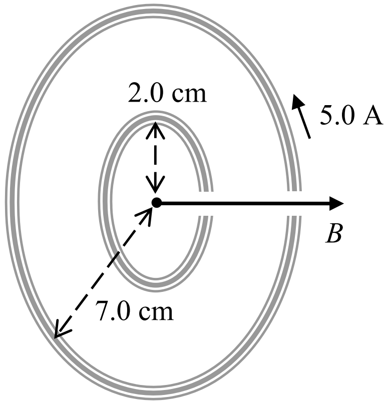


Fig. 11.34 Two concentric current carrying coils, Exercise 11.3

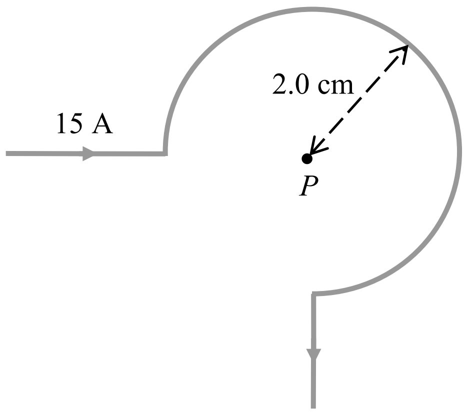
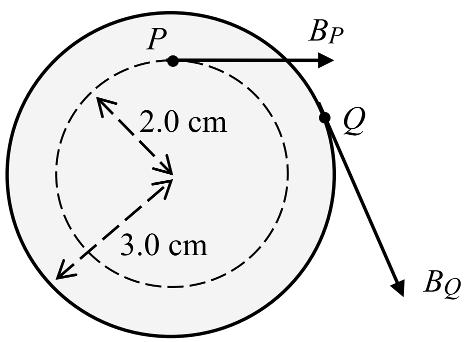


Fig. 11.35 Current carrying wire of Exercise 11.4

Fig. 11.36 Cross section of a current carrying conductor, Exercise 11.5



Exercise 11.5 Figure 11.36 shows the cross-section of a long conductor of radius 3.0 cm carrying a current of 5.0×10^2 A into the plane of the paper. Use Ampere's law to calculate the magnetic field B_P at point P and magnetic field B_Q at the surface.

(Answer: $B_P = 2.2 \times 10^{-3}$ T, $B_Q = 3.3 \times 10^{-3}$ T)

Chapter 12
Magnetic Properties of Matter

Abstract Problems related to magnetic materials and how magnetic induction, magnetic field strength, and magnetization are affected when the materials are inserted in the core of the current carrying solenoid and toroid are solved in this chapter. Both analytical solutions and computer calculations are presented.

12.1 Basic Concepts and Formulae

(1) The fundamental source of all magnetic fields is the magnetic dipole moment in atoms of materials. There are two types of magnetic dipole moments: spin magnetic dipole moment and orbital magnetic dipole moment.

(2) There are three types of magnetism in materials.

(a) Diamagnetism: Diamagnetic material shows its magnetic properties only when placed in an external magnetic field B_{ext}. In an external magnetic field, the material produces magnetic dipoles in the opposite direction to that of the external magnetic field. As a result, the material is pushed from the region of higher magnetic field. Examples of diamagnetic materials are gold, bismuth, mercury, water, glass, and helium.

(b) Paramagnetism: In a paramagnetic material, atoms have permanent magnetic dipole moments randomly oriented so that the net effect is no magnetic field. External magnetic field B_{ext} can align some of the atomic magnetic dipole moments to produce net magnetic dipole moments in the B_{ext} direction. The paramagnetic material is attracted to a region of higher magnetic field.

The alignment of atomic magnetic dipole moments increases with an increase in B_{ext} and decreases with an increase in temperature T. The extent a volume V of material has magnetic properties is given by magnetization M

$$M = \frac{\text{magnetic dipole moment } m}{V}. \tag{12.1}$$

W. M. S. Wan Hassan et al., *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-43165-4_12

If all N atomic magnetic dipoles of a sample are aligned with B_{ext}, the sample is saturated and the maximum magnetization is

$$M_{max} = \frac{Nm_a}{V}, \quad (12.2)$$

where m_a is the atomic magnetic dipole moment.

For small B_{ext}/T, where T is the absolute temperature of the material and B_{ext} is the external magnetic field, the magnetization is

$$M = C\frac{B_{ext}}{T}. \quad (12.3)$$

This is called Curie's law and the constant C is the Curie constant.

Examples of paramagnetic materials are aluminum, magnesium, oxygen, transition elements, and rare earth elements.

(c) Ferromagnetism: In a ferromagnetic material, most of the magnetic dipole moments of the atoms are self aligned in small regions called domains. Magnetic dipole moments are mainly from the spin magnetic dipole moments. Each domain behaves as a permanent magnet and the domains are randomly oriented if no external magnetic field is applied. The domains are partially aligned when the external magnetic field is applied so that the internal magnetic field becomes stronger. Ferromagnetic materials are used in magnetic devices. Examples are iron, nickel, and cobalt.

(3) The magnetic field of a material with relative permeability K_m is

$$B = K_m B_0, \quad (12.4)$$

where B_0 is the magnetic field without the material. Permeability μ of the material is

$$\mu = K_m \mu_0, \quad (12.5)$$

where $\mu_0 = 4\pi \times 10^{-7}$ H m^{-1} is permeability of free space.

(4) In a material medium, the relationship between magnetic induction $\boldsymbol{B}$, magnetic field strength $\boldsymbol{H}$, and magnetization $\boldsymbol{M}$ is

$$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}). \quad (12.6)$$

$\boldsymbol{B}$ is called magnetic induction or magnetic flux density.
$\boldsymbol{H}$ is called magnetic field strength, magnetic intensity, or magnetizing field.
$\boldsymbol{M}$ is called magnetization, magnetic polarization, or magnetic dipole moment per unit volume.
μ_0 is the permeability of free space.

(5) In an isotropic medium, $\boldsymbol{B}$, $\boldsymbol{H}$, and $\boldsymbol{M}$ are in the same direction. We have

$$B = \mu_0(H + M) = \mu_0(H + \chi_m H) = \mu_0(1 + \chi_m)H = \mu H, \quad (12.7)$$

$$\mu = \mu_0(1 + \chi_m), \quad (12.8)$$

$$K_m = \frac{\mu}{\mu_0} = 1 + \chi_m. \quad (12.9)$$

μ is the permeability of the medium,
μ_0 is the permeability of free space,
K_m is the relative permeability of the medium,
χ_m is the magnetic susceptibility of the medium.

(6) Ampere's law for magnetic field intensity is

$$\oint \boldsymbol{H} \cdot d\boldsymbol{s} = I. \quad (12.10)$$

The line integral $\boldsymbol{H} \cdot d\boldsymbol{s}$. along a closed path is the current I, where I is the current through a surface bounded by the closed path.

(7) For an air core solenoid

$$B_0 = \mu_0 n I = \mu_0 \frac{N}{l} I, \quad (12.11)$$

$$H = \frac{B_0}{\mu_0} = nI = \frac{N}{l} I, \quad (12.12)$$

where B_0 is the magnetic induction, H is the magnetic field strength, μ_0 is permeability of free space, N and l are the number of wire turns and length of the solenoid, respectively, and n is the number of wire turns per unit length.

12.2 Problems and Solutions

Problem 12.1 A permanent magnet made of ferromagnetic material has magnetization $M = 8.0 \times 10^5$ A m^{-1}. The magnet is in cube form with sides of 2.0 cm.

(a) Calculate the magnetic dipole moment
(b) Estimate the magnetic field of the permanent magnet at a point 10 cm away.

Solution

(a) Magnetic dipole moment m is magnetization M multiplied by volume V, Eq. (12.1),

$$m = MV = (8.0 \times 10^5 \text{ A/m})(2.0 \times 10^{-2} \text{ m})^3 = 6.4 \text{ A m}^2.$$

(b) The magnetic field of a magnetic dipole moment μ at a point far away from the dipole is, entry (d) of Table 11.1,

$$B = \frac{\mu_0 \mu}{2\pi x^3},$$

where x is the distance of the dipole to the observation point. Substituting magnetic dipole moment m from part (a), we get an estimate of the magnetic field of the permanent magnet

$$\begin{aligned} B &= \frac{\mu_0 m}{2\pi x^3} \\ &= \frac{(4\pi \times 10^{-7} \text{ T m/A})(6.4 \text{ A m}^2)}{2\pi (0.10 \text{ m})^3} \\ &= 1.3 \times 10^{-3} \text{ T}. \end{aligned}$$

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; M:8e5; V:(2e-2)^3; x:0.1; mu0:float(4*%pi*1e-7);
(fpprintprec)     5
(M)    8.0*10^5
(V)    8.0*10^-6
(x)    0.1
(mu0)  1.2566*10^-6
(%i6)  m: M*V;
(m)    6.4
(%i7)  B: mu0*m/(2*float(%pi)*0.1^3);
(B)    0.00128
```

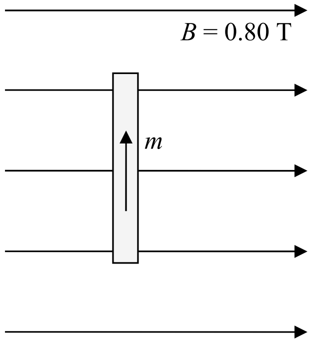


Fig. 12.1 An iron bar in a region of uniform magnetic field, Problem 12.2

Comments on the codes:

(%i5)	Set floating point print precision to 5, and assign values of M, V, x, and μ_0.
(%i6), (%i7)	Calculate m and B.

Problem 12.2 An iron atom has a magnetic dipole moment m_a of 1.83×10^{-23} A m^2.

(a) Determine the magnetic dipole moment m of a $9.0 \times 1.2 \times 1.0$ cm iron bar if the bar is 100% saturated magnetically.
(b) Calculate the torque τ on the iron bar if it is placed in a region of magnetic field $B = 0.80$ T as illustrated in Fig. 12.1. The density of iron is 7.8 g cm^{-3} and the molar mass of iron is 55.845 g mol^{-1}.

Solution

(a) If the iron bar is fully magnetized, all dipoles are aligned. Total dipole moment m is the number of atoms N multiplied by the dipole moment of an atom m_a

$$
\begin{aligned}
m &= N m_a = \frac{N_A \rho V}{M_m} m_a \\
&= \frac{(6.02 \times 10^{23}\ \text{atom/mol})(7.8\ \text{g/cm}^3)(9.0\ \text{cm})(1.2\ \text{cm})(1.0\ \text{cm})}{55.845\ \text{g/mol}} \\
&\quad \times (1.83 \times 10^{-23}\ \text{A m}^2\text{/atom}) \\
&= 17\ \text{A m}^2.
\end{aligned}
$$

Here, the number of iron atoms $N = N_A \rho V / M_m$, where ρ, V, and M_m are the density, volume, and molar mass of iron, respectively, and $N_A = 6.02 \times 10^{23}$ atom/mol is the Avogadro number.

(b) Torque τ on the iron bar if it is placed in the magnetic field of 0.80 T is, Eq. (11.9),

$$\tau = mB \sin\theta = (17\ \text{A m}^2)(0.80\ \text{T})(\sin 90^\circ) = 13\ \text{N m}.$$

◆ wxMaxima codes:

```
(%i7) fpprintprec:5; NA:6.02e23; rho:7.8; V:9*1.2*1; Mm:55.845;
m_a:1.83e-23; B:0.8;
(fpprintprec)     5
(NA)  6.02*10^23
(rho) 7.8
(V)   10.8
(Mm)  55.845
(m_a) 1.83*10^-23
(B)   0.8
(%i8) m: NA*rho*V*m_a/Mm;
(m)   16.618
(%i9) tau: m*B*sin(90*%pi/180);
(tau) 13.294
```

Comment on the codes:

(%i7) Set floating point print precision to 5, and assign values of N_A, ρ, V, M_m, m_a, and B.

(%i8), (%i9) Calculate m and τ.

Problem 12.3 A thin toroid has 285 turns per meter of wire wound around an iron core. The current of 3.0 A flows in it. If the relative permeability of iron is $K_m = \mu/\mu_0 = 2200$, what is the magnetic field in the toroid?

Solution

The magnetic field of a thin and long toroid is the same as that of a solenoid. The magnetic field of the iron core toroid is, Eqs. (11.5), (12.4), and (12.5),

$$B = \mu n I = K_m \mu_0 n I = (2200)(4\pi \times 10^{-7} \frac{\text{T m}}{\text{A}})(285\ \text{m}^{-1})(3.0\ \text{A}) = 2.4\ \text{T}.$$

where μ is the permeability of iron and n is the number of turns per unit length of wire.

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; Km:2200; mu0:float(4*%pi*1e-7); n:285; I:3;
(fpprintprec)     5
(Km)  2200
(mu0) 1.2566*10^-6
(n)   285
(I)   3
(%i6) B: Km*mu0*n*I;
(B)   2.3637
```

Comment on the codes:

(%i5) Set floating point print precision to 5, and assign values of K_m, μ_0, n, and I.
(%i6) Calculate B.

Problem 12.4 An iron core solenoid of length 38 cm and diameter 1.8 cm has 640 turns of wire. The magnetic field in the solenoid is 2.2 T when the current is 48 A. What is the permeability of iron at the field strength?

Solution

The magnetic field in an iron core solenoid is

$$B = \mu n I = \mu \frac{N}{l} I,$$

where μ is the permeability of iron, N is the number of turns of wire, l is the length of the solenoid, and I is current in the solenoid. The permeability of iron is

$$\mu = \frac{Bl}{NI} = \frac{(2.2\ \text{T})(0.38\ \text{m})}{640\ (48\ \text{A})} = 2.7 \times 10^{-5}\ \text{T m A}^{-1}.$$

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; l:0.38; N:640; B:2.2; I:48;
(fpprintprec)     5
(l)   0.38
(N)   640
(B)   2.2
(I)   48
(%i6) mu: (B*l)/(N*I);
(mu)  2.7214*10^-5
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of l, N, B, and I.
(%i6) Calculate μ.

Problem 12.5

(a) A 1.0 m long solenoid has 10^4 turns of copper wire. A current of 10 A flows in the solenoid. The cross section of the solenoid is 10 cm^2. Calculate:

   (i) magnetic field strength of the solenoid
   (ii) torque on the solenoid when it is placed perpendicular to an external magnetic field of $B_{ext} = 1.0 \times 10^{-2}$ T.

(b) The core of the solenoid is then filled with a magnetic material and the flux density in the material is $B = 1.5$ T. Calculate:

(i) magnetization in the material.
(ii) torque on the solenoid and the magnetic material when they are placed in the external magnetic field of $B_{ext} = 1.0 \times 10^{-2}$ T. Axis of the solenoid and external magnetic field are perpendicular to each other.

Solution

(a)

(i) Magnetic field strength of the air core solenoid is, Eq. (12.12),

$$H = nI = \frac{N}{l} I = \left(\frac{10^4 \text{ turns}}{1.0 \text{ m}} \right) (10 \text{ A}) = 1.0 \times 10^5 \text{ A m}^{-1}.$$

(ii) Magnetic moment of a loop is IA Eq. (10.7), where I is the current in the loop and A is the area of the loop. Thus, the magnetic moment of a turn of the solenoid is IA, and the magnetic moment of the solenoid is

$$m_0 = NIA = (10^4 \text{ turns})(10 \text{ A})(10 \times 10^{-4} \text{ m}^2) = 100 \text{ A m}^2.$$

Torque on the solenoid in an external magnetic field is Eq. (10.9)

$$\begin{aligned} \tau &= m_0 B_{\text{ext}} \sin 90^\circ \\ &= (100 \text{ A m}^2)(1.0 \times 10^{-2} \text{T}) \\ &= 1.0 \text{ N m}. \end{aligned}$$

(b)

(i) For a solenoid filled with a magnetic material Eq. (12.7), we have

$$B = \mu_0 (H + M),$$
$$M = \frac{B}{\mu_0} - H.$$

where M is the magnetization of the material, B is the magnetic induction, and H is the magnetic field strength. The magnetization of the material is

$$\begin{aligned} M &= \frac{B}{\mu_0} - H = \frac{1.5 \times 10^{-2} \text{ T}}{4\pi \times 10^{-7} \text{ T m/A}} - 1.0 \times 10^5 \text{ A m}^{-1} \\ &= 1.1 \times 10^6 \text{ A m}^{-1}. \end{aligned}$$

(ii) Magnetic moment of magnetic material m_m is Eq. (12.1)

$$\begin{aligned} m_m &= MV = MlA = (1.1 \times 10^6 \text{ A m}^{-1})(1.0 \text{ m})(10 \times 10^{-4} \text{ m}^2) \\ &= 1.1 \times 10^3 \text{ A m}^2. \end{aligned}$$

Magnetic moment m of the solenoid with the magnetic material core is

$$m = m_0 + m_m = 100 \text{ A m}^2 + 1.1 \times 10^3 \text{ A m}^2 = 1.2 \times 10^3 \text{ A m}^2.$$

Torque on the magnetic material cored solenoid is Eq. (10.9)

$$\begin{aligned}\tau_T &= m B_{\text{ext}} \sin 90^\circ \\ &= (1.2 \times 10^3 \text{ A m}^2)(1.0 \times 10^{-2}\text{T}) \\ &= 12 \text{ N m}.\end{aligned} \tag{12.1}$$

◆ wxMaxima codes:

```
(%i8) fpprintprec:5; l:1; N:1e4; I:10; A:10e-4; Bext:1e-2; B:1.5;
mu0:float(4*%pi*1e-7);
(fpprintprec)     5
(l)    1
(N)    1.0*10^4
(I)    10
(A)    0.001
(Bext) 0.01
(B)    1.5
(mu0) 1.2566*10^-6
(%i9) H: (N/l)*I;
(H)    1.0*10^5
(%i10) m0: N*I*A;
(m0)  100.0
(%i11) torque: m0*Bext;
(torque) 1.0
(%i12) M: B/mu0 - H;
(M) 1.0937*10^6
(%i13) mm: M*l*A;
(mm) 1093.7
(%i14) m: m0 + mm;
(m) 1193.7
(%i15) torqueT: m*Bext;
(torqueT) 11.937
```

Comments on the codes:

(%i8)	Set floating point print precision to 5, and assign values of l, N, I, A, B_{ext}, B, and μ_0.
(%i9), (%i10), (%i11), (%i12), (%i13), (%i14), (%i15)	Calculate H, m_0, τ, M, m_m, m, and τ_T.

Problem 12.6 A toroid is made of an iron core of length 60 cm, a cross section of 4.0 cm^2, and an air gap of 1.0 cm.

(a) If the toroid has 500 turns of wire and the current is 20 A, what is the magnetic flux density B_g in the gap? The relative permeability of iron is 3000.

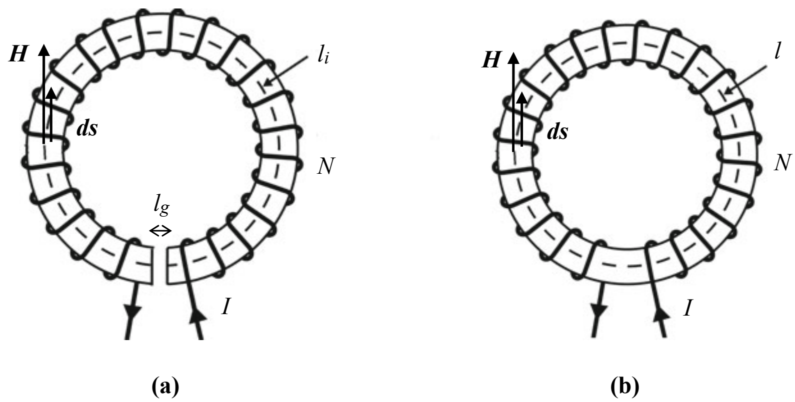


Fig. 12.2 **a** An iron core toroid with an air gap, **b** an iron core toroid, Problem 12.6

(b) If there is no air gap, what is the magnetic flux density B_i in the iron core?

Solution

(a) Fig. 12.2a shows the iron core toroid and the air gap. Let the magnetic flux density and the magnetic field intensity of iron be B_i and H_i, respectively, while the magnetic flux density and the magnetic field intensity of air gap be B_g and H_g, respectively. The lengths of iron and air gap are l_i and l_g, respectively.

Ampere's law for magnetic field intensity is, Eq. (12.10),

$$\oint \boldsymbol{H} \cdot \boldsymbol{ds} = I.$$

The line integral ***H.ds*** along a closed path is the current *I*, where *I* is the current enclosed by the closed path.

For this problem, the imaginary closed path in the dashed circle of the toroid, Fig. 12.2a. The magnetic field intensity ***H*** is parallel to line element ***ds*** and ***H.ds*** is $H \times$ length. The magnetic field intensity is the magnetic flux density divided by the permeability of the material, $H = B/\mu$. The current enclosed in the closed path is *NI*, where *N* is the number of turns of the wire. We have

$$\oint \boldsymbol{H} \cdot \boldsymbol{ds} = I_{enclosed},$$

$$H_i l_i + H_g l_g = NI,$$

$$\frac{B_i l_i}{\mu} + \frac{B_g l_g}{\mu_0} = NI,$$

$$\frac{B_i l_i}{K_m \mu_0} + \frac{B_g l_g}{\mu_0} = NI.$$

Assuming $B_i \approx B_g$, we have

$$\frac{B_g l_i}{K_m \mu_0} + \frac{B_g l_g}{\mu_0} = NI.$$

Thus, the magnetic flux density B_g of the air gap is

$$B_g = \frac{NI}{\left(\frac{l_i}{K_m\mu_0} + \frac{l_g}{\mu_0}\right)} = \frac{500(20\ \text{A})}{\left(\frac{0.60\ \text{m}}{3000\times 4\pi\times 10^{-7}\ \text{H m}^{-1}} + \frac{0.01\ \text{m}}{4\pi\times 10^{-7}\ \text{H m}^{-1}}\right)}$$
$$= 1.2\ \text{T}.$$

(b) If there is no air gap, the magnetic flux density B_i in the iron core toroid is, Fig. 12.2b,

$$\oint \boldsymbol{H}\cdot d\boldsymbol{s} = I_{enclosed},$$
$$H_i l = NI,$$
$$\frac{B_i l}{K_m \mu_0} = NI,$$
$$B_i = \frac{NIK_m\mu_0}{l} = \frac{500(20\ \text{A})(3000)(4\pi\times 10^{-7}\ \text{H m}^{-1})}{(0.60+0.01)\ \text{m}}$$
$$= 62\ \text{T}.$$

◆ wxMaxima codes:

```
(%i7) fpprintprec:5; N:500; I:20; li:0.6; lg:0.01; Km:3000;
mu0:float(4*%pi*1e-7);
(fpprintprec)     5
(N)   500
(I)   20
(li)  0.6
(lg)  0.01
(Km)  3000
(mu0) 1.2566*10^-6
(%i8) Bg: N*I/(li/(Km*mu0) + lg/mu0);
(Bg)  1.232
(%i9) Bi: N*I*Km*mu0/(li+lg);
(Bi)  61.802
```

Comments on the codes:

(%i7) Set floating point print precision to 5, and assign values of N, I, l_i, l_g, K_m, and μ_0.

(%i8), (%i9) Calculate B_g and B_i.

Problem 12.7 An electric power cable carries a current of 95 A to the west, 8.5 m above the ground.

(a) What is the magnitude and direction of the magnetic field due to the cable at the surface of the earth? Compare the field with the earth's magnetic field $B_{earth} = 0.50 \times 10^{-4}$ T. The earth's magnetic field points to the north.
(b) At what height above the ground the magnetic field is zero?

Solution

(a) Assume the cable is straight and long, from east to west, carrying a current from east to west, at a height of 8.5 m above the ground. The magnetic field due to the current in the cable is (Table 11.1b)

$$B_{cable} = \frac{\mu_0 I}{2\pi r} = \frac{\mu_0 I}{2\pi h} = \frac{(4\pi \times 10^{-7}\ \text{T m/A})(95\ \text{A})}{2\pi (8.5\ \text{m})} = 2.2 \times 10^{-6}\ \text{T}.$$

pointing to the south as determined by the right-hand rule. Assume the earth's magnetic field points to the north with magnitude $B_{earth} = 0.50 \times 10^{-4}$ T. At the earth's surface, the ratio of the magnetic field due to current in the cable to the magnetic field of the earth is

$$\frac{B_{cable}}{B_{earth}} = \frac{2.2 \times 10^{-6}\ \text{T}}{0.50 \times 10^{-4}\ \text{T}} = 0.045.$$

Magnetic field due to the current in the cable is 4% of the earth's magnetic field.
(b) To get zero magnetic field, $B_{cable} = B_{earth}$, and the two magnetic fields are in opposite directions. Thus

$$B_{earth} = B_{cable} = \frac{\mu_0 I}{2\pi r},$$
$$r = \frac{\mu_0 I}{2\pi B_{earth}} = \frac{(4\pi \times 10^{-7}\ \text{T m/A})(95\ \text{A})}{2\pi (0.50 \times 10^{-4}\ \text{T})} = 0.38\ \text{m}.$$

The magnetic field is zero at a distance of 0.38 m below the cable or at a height of 8.5 – 0.38 = 8.1 m from the ground.

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; I:95; h:8.5; mu0:float(4*%pi*1e-7); Bearth:0.5e-4;
(fpprintprec)     5
(I)   95
(h)   8.5
(mu0) 1.2566*10^-6
(Bearth) 5.0*10^-5
(%i6) Bcable: mu0*I/(2*float(%pi)*h);
(Bcable) 2.2353*10^-6
(%i7) Bcable/Bearth;
(%o7) 0.044706
(%i8) r: mu0*I/(2*float(%pi)*Bearth);
(r)   0.38
(%i9) height: h-r;
(height) 8.12
```

Comments on the codes:

(%i5)	Set floating point print precision to 5, and assign values of I, h, μ_0, and B_{earth}.
(%i6), (%i7)	Calculate B_{cable} and B_{cable}/B_{earth}.
(%i8), (%i9)	Calculate r and $h - r$.

Problem 12.8 Internal and external radii of an air core solenoid in the form of a toroid are 15 and 18 cm, respectively, as shown in Fig. 12.3. The toroid has 250 turns of wire and it carries a current of 8.5 A. What are the magnetic fields at points (a) 12 cm, (b) 16 cm, and (c) 20 cm from the center of the toroid.

Solution

(a) At $r = 0.12$ m, the magnetic field is zero.
(b) At $r = 0.16$ m, the magnetic field is

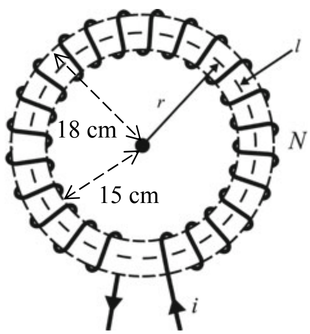


Fig. 12.3 Air core solenoid in the form of a toroid, Problem 12.8

$$B = \frac{\mu_0 NI}{l} = \frac{\mu_0 NI}{2\pi r} = \frac{\mu_0 NI}{2\pi r} = \frac{(4\pi \times 10^{-7}\ \text{H/m})(250)(8.5\ \text{A})}{2\pi(0.16\ \text{m})}$$
$$= 2.7 \times 10^{-3}\ \text{T.}$$

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; mu0:float(4*%pi*1e-7); N:250; I:8.5; r:0.16;
(fpprintprec)     5
(mu0) 1.2566*10^-6
(N)   250
(I)   8.5
(r)   0.16
(%i6) B: mu0*N*I/(2*float(%pi)*r);
(B)   0.0026562
```

Comments on the codes:

(%i5) Set floating point print precision to 5, and assign values of μ_0, N, I, and r.
(%i6) Calculate B.

(c) At $r = 0.20$ m, the magnetic field is zero.

Problem 12.9 A current of 2.4 A flows in a magnetic metal core solenoid in the form of a toroid, as shown in Fig. 12.4. The number of wire turns is 500 and the radius of the toroid is 25 cm. The magnetic field of the solenoid is 1.9 T. Calculate:

(a) relative permeability
(b) susceptibility of the metal.

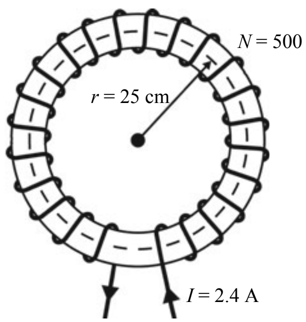


Fig. 12.4 A magnetic material core toroid, Problem 12.9

Solution

(a) The magnetic field of a magnetic metal cored solenoid or toroid is, (Eqs. 12.4 and 8.5),

$$B = K_m B_0 = \frac{K_m \mu_0 N I}{l} = \frac{K_m \mu_0 N I}{2\pi r}.$$

Here, K_m is the relative permeability of the magnetic metal, B_0 is the magnetic field of air core solenoid or toroid, N and l are the number of turns of wire and length of the toroid, respectively, and μ_0 is the permeability of free space. The relative permeability of the magnetic metal is calculated as follows:

$$B = \frac{K_m \mu_0 N I}{2\pi r}$$
$$1.9\ \text{T} = \frac{K_m (4\pi \times 10^{-7}\ \text{H/m})(500)(2.4\ \text{A})}{2\pi (0.25\ \text{m})}$$
$$K_m = 1979.$$

(b) Magnetic susceptibility χ_m of the metal is, Eq. (12.9),

$$\chi_m = K_m - 1 = 1979 - 1 = 1978.$$

◆ wxMaxima codes:

```
(%i7) fpprintprec:5; ratprint:false; B:1.9; mu0:float(4*%pi*1e-7); N:500;
I:2.4; r:0.25;
(fpprintprec)     5
(ratprint)  false
(B)   1.9
(mu0) 1.2566*10^-6
(N)   500
(I)   2.4
(r)   0.25
(%i9) solve(B=Km*mu0*N*I/(2*%pi*r), Km)$ float(%);
(%o9) [Km=1979.2]
(%i10) Km: 1979.2;
(Km)  1979.2
(%i11) Xm: Km-1;
(Xm)  1978.2
```

Comments on the codes:

(%i7)	Set floating point print precision to 5, internal rational number print to false, and assign values of B, μ_0, N, I, and r.
(%i9)	Solve $B = \frac{K_m \mu_0 N I}{2\pi r}$ for K_m.
(%i10), (%i11)	Assign K_m and calculate χ_m.

Problem 12.10 A solenoid with a silicon iron core has 60 turns of wire per cm. A current of 0.15 A flows in the solenoid. The relative permeability of silicon iron is 5200. Calculate:

(a) magnetic field of the solenoid without silicon iron core
(b) magnetic field of the solenoid with silicon iron core
(c) magnetization of silicon iron.

Solution

(a) The magnetic field of the solenoid without silicon iron core is, Eq. (8.5),

$$B_0 = \mu_0 n I = (4\pi \times 10^{-7}\ \text{H/m})\left(\frac{60}{0.010\ \text{m}}\right)(0.15\ \text{A})$$
$$= 1.1 \times 10^{-3}\ \text{T}.$$

(b) The magnetic field of the solenoid with silicon iron core is, Eq. (12.4),

$$B = K_m B_0 = K_m \mu_0 n I = (5200)(4\pi \times 10^{-7}\ \text{H/m})\left(\frac{60}{0.010\ \text{m}}\right)(0.15\ \text{A})$$
$$= 5.9\ \text{T}.$$

(c) The magnetization M is calculated as follows (Eq. 12.7),

$$\begin{aligned} B &= \mu_0(H + M) \\ &= \mu_0 H + \mu_0 M \\ &= B_0 + \mu_0 M, \\ M &= \frac{B - B_0}{\mu_0} = \frac{5.9\ \text{T} - 1.1 \times 10^{-3}\ \text{T}}{4\pi \times 10^{-7}\ \text{H/m}} \\ &= 4.7 \times 10^6\ \text{A m}^{-1}. \end{aligned}$$

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; n:60/0.01; I:0.15; Km:5200; mu0:float(4*%pi*1e-7);
(fpprintprec)     5
(n)   6000.0
(I)   0.15
(Km)  5200
(mu0) 1.2566*10^-6
(%i6) B0: mu0*n*I;
(B0)  0.001131
(%i7) B: Km*B0;
(B)   5.8811
(%i8) M: (B-B0)/mu0;
(M)   4.6791*10^6
```

Comments on the codes:

(%i5)	Set floating point print precision to 5, and assign values of n, I, K_m, and μ_0.
(%i6), (%i7), (%i8)	Calculate B_0, B, and M.

12.3 Summary

- Materials are classified as paramagnetic, diamagnetic, or ferromagnetic.
- Magnetic field of a material with relative permeability K_m is

$$B = K_m B_0,$$

where B_0 is the magnetic field without the material. Permeability of the material is

$$\mu = K_m \mu_0,$$

where $\mu_0 = 4\pi \times 10^{-7}$ H m^{-1} is permeability of free space.
- In a material medium, the relationship between magnetic induction $\boldsymbol{B}$, magnetization $\boldsymbol{M}$, and magnetic field strength $\boldsymbol{H}$ is

$$\boldsymbol{B} = \mu_0(\boldsymbol{M} + \boldsymbol{H}).$$

12.4 Exercises

Exercise 12.1 Calculate magnetizing field H and magnetic flux density B at the center of a 20 turns per cm solenoid carrying a current of 0.15 A.
(Answer: $H = 3.0 \times 10^2$ A m^{-1}, $B = 3.8 \times 10^{-4}$ T)

Exercise 12.2 An iron core of magnetic permeability 6.0×10^{-3} H m^{-1} is inserted in a 20 turns per cm solenoid carrying a current of 0.15 A of Exercise 12.1. Calculate magnetizing field H, magnetic flux density B, and magnetization M in the iron core.
(Answer: $H = 3.0 \times 10^2$ A m^{-1}, $B = 1.8$ T, $M = 1.4 \times 10^6$ A m^{-1})

Exercise 12.3 A 0.6 m long solenoid has 1800 turns of copper wire. An iron rod with a relative permeability of 500 is inserted into the solenoid and a current of 1.0 A flows in the wire. What are magnetizing field H, magnetic flux density B, magnetic dipole moment per unit volume M, and average magnetic dipole moment per atom m_a? The number density of iron is 8.48×10^{28} atoms per m^3.
(Answer: $H = 3.0 \times 10^3$ A m^{-1}, $B = 1.9$ T, $M = 1.5 \times 10^6$ A m^{-1}, $m_a = 1.8 \times 10^{-23}$ A m^2)

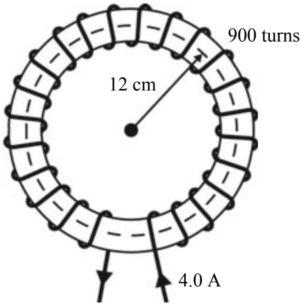


Fig. 12.5 An iron core toroid, Exercise 12.4

Exercise 12.4 An iron ring of radius 12 cm is wound with 900 turns of copper wire, as shown in Fig. 12.5. The relative permeability of the iron core is 250 and a current of 4.0 A flows in the wire. What are the magnetizing field H and the magnetic flux density B in the iron core?

(Answer: $H = 4.8 \times 10^3$ A m^{-1}, $B = 1.5$ T)

Exercise 12.5 A piece of iron of length of 1.0 cm is sawed out from the iron ring of Exercise 12.4, such that there is an air gap of length 1.0 cm in the ring. What is the magnetic flux density in the air gap?

(Answer: $B = 0.35$ T)

Chapter 13
Faraday's Law

Abstract This chapter solves problems related to emf induced by changing magnetic flux. Faraday's law states that the emf induced is equal to the negative time rate of change of the magnetic flux. Emf is induced in a moving conductor when the conductor cuts through the magnetic field lines. Emf is also induced in a rotating conducting loop when the loop cuts through the magnetic field lines. Solutions are obtained by analysis and computer calculation of wxMaxima.

13.1 Basic Concepts and Formulae

(1) Faraday's law of induction states that the induced electromotive force (emf) $\mathcal{E}$ in a loop is proportional to the rate of change of the magnetic flux of the loop. This is written as

$$\mathcal{E} = -\frac{d\Phi_m}{dt}, \tag{13.1}$$

where Φ_m is the magnetic flux that can be calculated by

$$\Phi_m = \int \boldsymbol{B} \cdot \boldsymbol{dA}. \tag{13.2}$$

Here, $\boldsymbol{B}$ is the magnetic field and $\boldsymbol{dA}$ is the surface element vector. The surface element vector is normal to the surface element and its magnitude is the area of the surface element dA. The magnetic flux for a given area is equal to the area times the component of the magnetic field perpendicular to the area.

If the loop is a coil of N turns, the induced emf is

$$\mathcal{E} = -N\frac{d\Phi_m}{dt}. \tag{13.3}$$

W. M. S. Wan Hassan et al., *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-43165-4_13

(2) When a conductor of length l moves with velocity v in a uniform magnetic field B, an emf is induced in the rod

$$\mathcal{E} = -Blv, \tag{13.4}$$

where B, v, and the rod are perpendicular to each other.

(3) Lenz's law states that the directions of induced current and emf in a conductor are opposite to the change that produced them.

13.2 Problems and Solutions

Problem 13.1 Figure 13.1 shows a conducting rod moving to the right at a speed of $v = 4.0\ \text{m s}^{-1}$ in a region of uniform magnetic field $B = 0.50$ T pointing into the plane of the paper. The length of the rod is $l = 1.5$ m.

(a) Determine the equivalent non-electrostatic electric field E_{ne} in the rod.
(b) Calculate the electrostatic electric field E_e in the rod.
(c) What is the motional emf in the rod?
(d) Determine the potential difference between the rod's ends. Which end has a higher electric potential?

Solution

(a) As the rod is moved to the right at a velocity of $\boldsymbol{v}$, a charge q in the rod is acted by a magnetic force $q\boldsymbol{v} \times \boldsymbol{B}$ in the upward direction. This is a non-electrostatic force. Thus, the charge is in a non-electrostatic electric field of

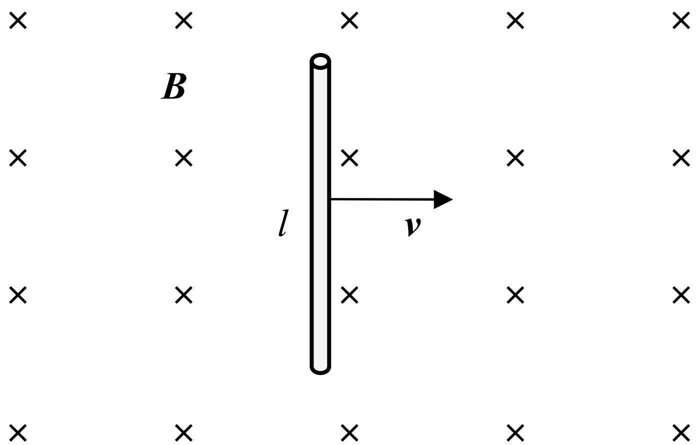


Fig. 13.1 A conducting rod moving in a region of uniform magnetic field, Problem 13.1

$$\boldsymbol{E}_{ne} = \frac{\text{force}}{\text{charge}} = \frac{q\boldsymbol{v} \times \boldsymbol{B}}{q} = \boldsymbol{v} \times \boldsymbol{B} = 4.0\ \mathbf{i}\ \text{m s}^{-1} \times 0.50\ \mathbf{j}\ \text{T} = 2.0\ \mathbf{k}\ \text{V m}^{-1}.$$

in the upward direction.

- wxMaxima codes:

```
(%i2) fpprintprec:5; load("vect");
(fpprintprec)     5
(%o2) "C:/maxima-5.43.0/share/maxima/5.43.0/share/vector/vect.mac"
(%i4) v:[4,0,0]; B:[0,0.5,0];
(v)   [4,0,0]
(B)   [0,0.5,0]
(%i6) Ene: v~B; express(%);
(Ene) -[0,0.5,0]~[4,0,0]
(%o6) [0,0,2.0]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and load "vect" package.
(%i4) Assign vectors $\boldsymbol{v}$ and $\boldsymbol{B}$.
(%i6) Calculate non-electrostatic electric field, $\boldsymbol{E}_{ne}$.

(b) The electrostatic electric field in the rod is,

$$\boldsymbol{E}_e = -2.0\ \mathbf{k}\ \text{V m}^{-1},$$

in the downward direction. As the rod moves, the $\boldsymbol{E}_{ne}$ field causes the positive charges to be accumulated at the top end of the rod, while the electrons at the bottom end. The accumulation creates the electrostatic electric field $\boldsymbol{E}_e$, until the resultant force on each charge is zero. Eventually, $\boldsymbol{E}_e = -\boldsymbol{E}_{ne}$.

(c) The motional emf is,

$$\mathcal{E} = \int \boldsymbol{E}_{ne} \cdot \boldsymbol{ds} = vBl = (4.0\ \text{m s}^{-1})(0.50\ \text{T})(1.5\ \text{m}) = 3.0\ \text{V}.$$

(d) The potential difference between the ends of the rod is 3.0 V. The top end is of higher electric potential than the bottom because the positive charges accumulate there.

Problem 13.2 Fig. 13.2 shows a conducting rod *ab* moving at speed $v = 4.0$ m s^{-1} to the right while touching conductor *cdef* in a region of uniform magnetic field $B = 0.50$ T into the plane of the paper. The length of rod *ab* is $l = 0.50$ m.

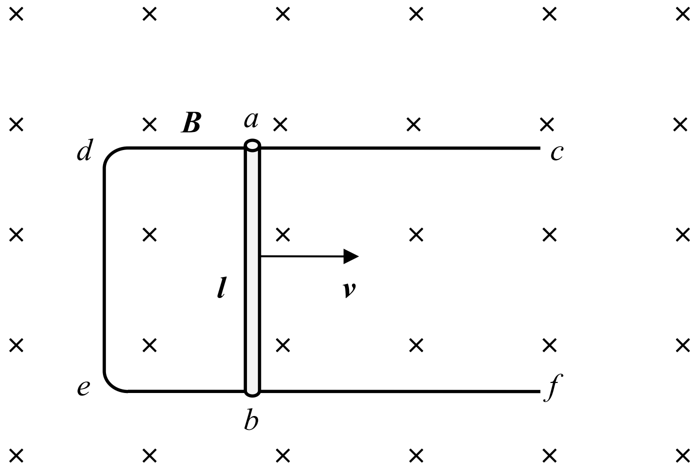


Fig. 13.2 A conducting rod moving in a region of uniform magnetic field. The rod touches a conductor while in motion, Problem 13.2

(a) Determine the magnitude and direction of induced emf in the rod.
(b) If the circuit resistance is $R = 0.20\ \Omega$ and friction is negligible, calculate the force needed to sustain the motion of the rod.
(c) Determine the rate of mechanical work done and compare it with the rate of electrical energy dissipation.

Solution

(a) The magnitude of induced emf in the rod is

$$\mathcal{E} = lE_{ne} = lvB = (0.50\ \text{m})(4.0\ \text{m s}^{-1})(0.50\ \text{T}) = 1.0\ \text{V},$$

and the direction is from b to a. Here, $E_{ne} = vB$ is the non-electrostatic electric field in the rod due to its motion in a magnetic field

(b) When there is induced emf, the counter clockwise current in the circuit is,

$$I = \frac{\mathcal{E}}{R} = \frac{1.0\ \text{V}}{0.20\ \Omega} = 5.0\ \text{A}.$$

Current I flows in the loop *bade*. Due to current I flowing in the rod, there exists magnetic force of

$$F = IlB = (5.0\ \text{A})(0.50\ \text{m})(0.50\ \text{T}) = 1.25\ \text{N},$$

to the left acting on the rod. To sustain the motion of the rod, the force of 1.25 N to the right must be applied to the rod.

(c) The rate of mechanical work done is

$$Fv = (1.25\ \text{N})(4.0\ \text{m s}^{-1}) = 5.0\ \text{W}.$$

The rate of electrical energy dissipation is

$$I^2R = (5.0\ \text{A})^2(0.20\ \Omega) = 5.0\ \text{W}.$$

Both rates are equal in value.

- wxMaxima codes:

```
(%i5) fpprintprec:5; v:4; B:0.5; l:0.5; R:0.2;
(fpprintprec)     5
(v)    4
(B)    0.5
(l)    0.5
(R)    0.2
(%i6) emf: l*v*B;
(emf) 1.0
(%i7) I: emf/R;
(I)    5.0
(%i8) F: I*l*B;
(F)    1.25
(%i9) F*v;
(%o9) 5.0
(%i10) I^2*R;
(%o10) 5.0
```

Comments on the codes:

(%i5)	Set floating point print precision to 5, assign values of v, B, l, and R.
(%i6), (%i7), (%i8), (%i9), (%i10)	Calculate emf, I, F, $F \times v$, and $I^2 \times R$.

Problem 13.3 A coil of 100 turns and a cross-sectional area of 20 cm^2 is rotated in and earth magnetic field of 6.0×10^{-5} T in 0.020 s. Initially, the plane of the coil is perpendicular to the earth's magnetic field and finally, the plane is parallel to the field. What is the average induced emf in the coil?

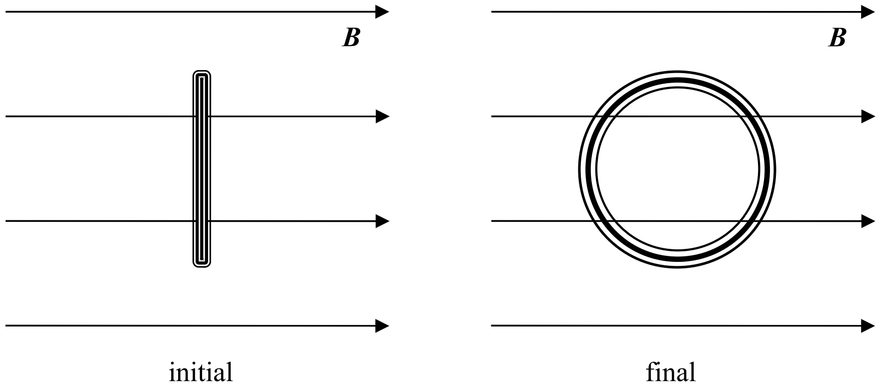


Fig. 13.3 A coil in initial and final situations, Problem 13.3

Solution

Figure 13.3 shows the coil in the initial and final situations. The earth magnetic field is indicated as *B*.

Induced emf is calculated by the rate of change of the magnetic flux through the coil. Initially, the magnetic flux through the coil is

$$\Phi_{init} = BAN = (6.0 \times 10^{-5}\ \text{T})(20 \times 10^{-4}\ \text{m}^2)(100) = 1.2 \times 10^{-5}\ \text{Wb},$$

because the plane of the coil is perpendicular to the magnetic field. Finally, the flux through the coil is zero because the plane of the coil is parallel to the magnetic field

$$\Phi_{final} = 0.$$

Using Faraday's law, the average induced emf in the coil is, Eq. (13.1),

$$\mathcal{E} = -\frac{\Delta\Phi_m}{\Delta t} = -\frac{(\Phi_{final} - \Phi_{init})}{\Delta t} = -\frac{(0 - 1.2 \times 10^{-5}\ \text{Wb})}{0.020\ \text{s}}$$
$$= 6.0 \times 10^{-4}\ \text{V}.$$

- wxMaxima codes:

```
(%i5) fpprintprec:5; N:100; A:20e-4; B:6e-5; delta_t:0.02;
(fpprintprec)     5
(N)    100
(A)    0.002
(B)    6.0*10^-5
(delta_t) 0.02
(%i6) phi_init: B*A*N;
(phi_init) 1.2*10^-5
(%i7) phi_final: 0;
(phi_final) 0
(%i8) emf: -(phi_final - phi_init)/delta_t;
(emf) 6.0*10^-4
```

Comments on the codes:

(%i5) Set floating point print precision to 5, and assign values of N, A, B, and Δt.
(%i6) Calculate Φ_{init}.
(%i7) Assign Φ_{final}.
(%i8) Calculate emf.

Problem 13.4 A coil of radius 0.10 m consists of 50 turns of wire. The resistance of the coil is 3.0 Ω. A Magnetic field perpendicular to the plane of the coil is created such that its magnitude varies from zero to 0.50 Wb m^{-2} in 0.20 s.

(a) Calculate the average induced emf in the coil.
(b) What is the induced current in the coil?

Solution

(a) Figure 13.4 shows the coil and the magnetic field.
Induced emf is given by Faraday's law as, Eq. (13.3),

$$\mathcal{E} = -N\frac{d\Phi_m}{dt}.$$

For this problem,

$$\begin{aligned}\mathcal{E} &= -N\frac{d\Phi_m}{dt} = -NA\frac{dB}{dt} = -NA\left(\frac{B_{final} - B_{init}}{\Delta t}\right)\\ &= \frac{-50\pi(0.10\text{ m})^2(0.50\text{ Wb/m}^2 - 0)}{0.20\text{ s}}\\ &= -3.9\text{ V}.\end{aligned}$$

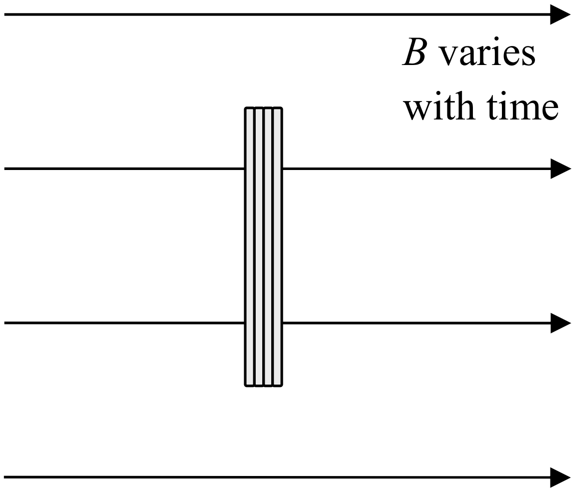


Fig. 13.4 A coil in a varying magnetic field, Problem 13.4

(b) The induced current in the coil is

$$I = \frac{\mathcal{E}}{R} = \frac{-3.9 \text{ V}}{3.0\ \Omega} = -1.3 \text{ A}.$$

- wxMaxima codes:

```
(%i8) fpprintprec:5; r:0.1; A:float(%pi*r^2); N:50; R:3; Binit:0;
Bfinal:0.5; delta_t:0.2;
(fpprintprec)     5
(r)    0.1
(A)    0.031416
(N)    50
(R)    3
(Binit)      0
(Bfinal)     0.5
(delta_t)    0.2
(%i9) emf: -N*A*(Bfinal-Binit)/delta_t;
(emf) -3.927
(%i10) I: emf/R;
(I) -1.309
```

Comments on the codes:

(%i8) Set floating point print precision to 5, and assign values of r, A, N, R, B_{init}, B_{final}, and Δt.

(%i9), (%i10) Calculate emf and I.

Problem 13.5 A coil of cross-sectional area A is placed in a region of magnetic field that is perpendicular to the plane of the coil. The magnetic field varies with time according to,

$$B = B_0 e^{-t/\tau},$$

where B_0 and τ are constants and t is time. Determine the induced emf in the coil as a function of time.

Solution

The magnetic flux is

$$\Phi_m = AB = AB_0 e^{-t/\tau}.$$

Using Faraday's law, the induced emf in the coil is, Eq. (13.1),

$$\mathcal{E} = -\frac{d\Phi_m}{dt} = -\frac{d(AB_0 e^{-t/\tau})}{dt} = \frac{AB_0}{\tau} e^{-t/\tau}.$$

- wxMaxima codes:

```
(%i1) phi_m: A*B0*exp(-t/tau);
(phi_m) A*B0*%e^(-t/tau)
(%i2) emf: -diff(phi_m,t,1);
(emf) (A*B0*%e^(-t/tau))/tau
```

Comments on the codes:

(%i1) Define Φ_m.
(%i2) Calculate emf.

Problem 13.6 A metal rod of length $l = 0.30$ m is pivoted at one of its ends and rotated at angular speed $\omega = 3.0$ rad s^{-1}, as illustrated in Fig. 13.5a. The rod is in the region of uniform magnetic field $B = 1.0 \times 10^{-3}$ T out of the plane of the paper. Calculate the potential difference between the ends of the rod. Which end has higher electric potential?

Solution

Using Faraday's law, induced emf in the metal rod is, Eq. (13.1),

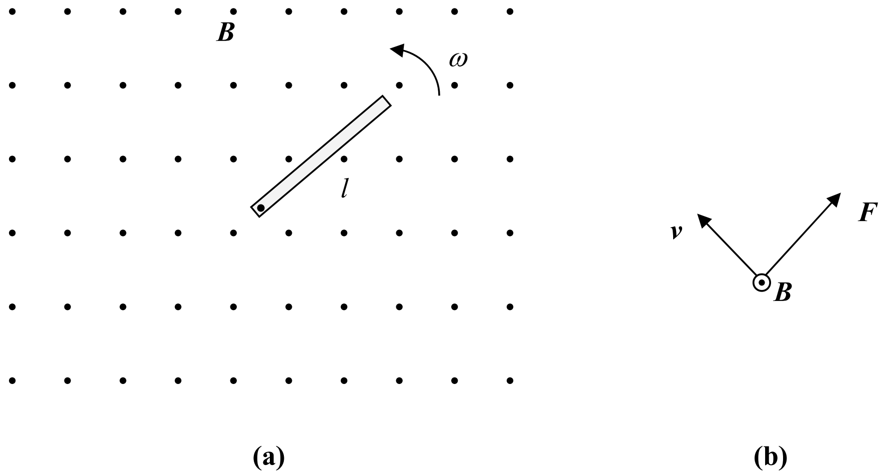


Fig. 13.5 **a** A metal rod rotating in a region of uniform magnetic field, **b** the force on a moving positive charge, Problem 13.6

$$\mathcal{E} = -\frac{d\Phi_m}{dt} = -\frac{d(AB)}{dt} = -B\frac{dA}{dt},$$

where dA/dt is the rate of area swept by the rotating rod. Let the time interval be dt and the rod rotates by $d\theta$. The area swept by the rod in time interval dt is,

$$dA = \frac{1}{2}(l)(l\ d\theta) = \frac{1}{2}l^2 d\theta,$$

because the area swept is a sector or a triangle with a base length of l and height $l\ d\theta$. This means that

$$\frac{dA}{dt} = \frac{1}{2}l^2\frac{d\theta}{dt} = \frac{1}{2}l^2\omega.$$

Another way to get dA/dt is as follows. The number of revolutions of the rod in a second is $\omega/(2\pi)$ and the area swept in one revolution is πl^2. Therefore, the area swept in a second is

$$\frac{\omega}{2\pi}(\pi l^2) = \frac{1}{2}l^2\omega.$$

The induced emf in the metal rod is

$$\mathcal{E} = -\frac{d\Phi_m}{dt} = -B\frac{dA}{dt} = -\frac{1}{2}Bl^2\omega.$$

The potential difference between the rod ends is

$$|\mathcal{E}| = \frac{1}{2}Bl^2\omega = \frac{1}{2}(1.0 \times 10^{-3}\ \text{T})(0.30\ \text{m})^2(3.0\ \text{rad/s})$$
$$= 1.3 \times 10^{-4}\ \text{V}.$$

The rotating end has a higher electric potential than the pivoted end. The rotation causes the positive charges to be accumulated at the rotating end and electrons to the pivoted end. You can verify this from $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$. For a positive charge, v is north-westerly and B is out of the plane of the paper, hence the force on the positive charge is north-easterly, Fig. 13.5b.

- wxMaxima codes:

```
(%i4) fpprintprec:5; l:0.3; omega:3; B:1e-3;
(fpprintprec)     5
(l)   0.3
(omega) 3
(B)   0.001
(%i5) emf: B*l^2*omega/2;
(emf) 1.35*10^-4
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of l, ω, and B.
(%i5) Calculate emf.

Problem 13.7 A long rectangular conducting loop is pulled from rest by a constant force $\boldsymbol{F}$ from a region of uniform magnetic field $\boldsymbol{B}$, as shown in Fig. 13.6. The width of the loop is l, mass m, and resistance R.

(a) Calculate the terminal velocity of the loop.
(b) Determine an equation of the velocity of the loop as a function of time.
(c) Let $F = 0.001$ N, $B = 1.0$ T, $l = 0.20$ m, $m = 0.03$ kg, $R = 0.50\ \Omega$. Calculate the terminal velocity of the loop and draw velocity against the time curve.

Solution

(a) When the loop is pulled to the right, current of magnitude

$$I = \frac{\mathcal{E}}{R} = \frac{d\Phi_m}{dt}/R = \frac{d(xlB)}{dt}/R = \frac{dx}{dt}lB/R = \frac{vlB}{R}, \tag{1}$$

is induced on the left side of the loop. The velocity of the loop is $v = dx/dt$ and the induced emf on the left side of the loop is $\mathcal{E} = vlB$. The direction of the current is bottom-up on the left side of the loop or clockwise in the loop. You can check this by $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$. For a positive charge on the left side of the loop,

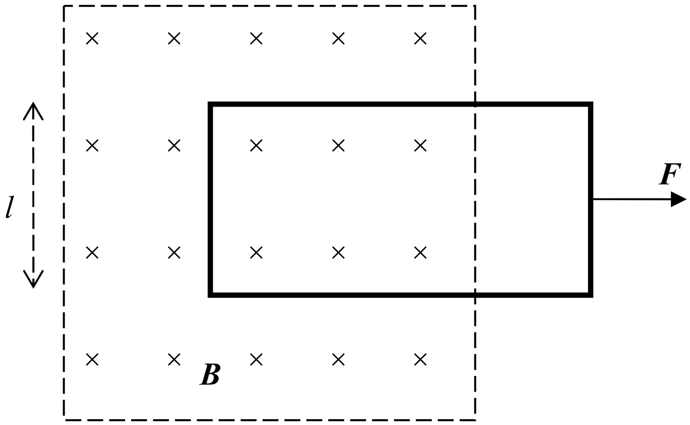


Fig. 13.6 A rectangular conducting loop is pulled out of a magnetic field region, Problem 13.7

v is to the right and ***B*** is into the plane of the paper. Therefore, the force on the positive charge is upward. Due to this electrical current, the left side of the loop is acted by magnetic force of magnitude

$$F_m = IlB = \frac{vlB}{R} lB = \frac{vl^2B^2}{R}, \tag{2}$$

to the left. Here, the current in Eq. (1) is inserted in Eq. (2) to get the magnetic force.

Terminal velocity is attained when the magnitudes of F and F_m are the same. The terminal velocity of the loop v_T is calculated as follows

$$F = F_m = \frac{v_T l^2 B^2}{R},$$
$$v_T = \frac{FR}{l^2B^2}. \tag{3}$$

(b) Net force acting on the loop is

$$F - F_m,$$

to the right. There is no electrical or magnetic force acting on the top and bottom sides of the loop. Using the Newton's second law, we write

$$F - F_m = ma,$$
$$F - \frac{l^2B^2}{R} v = m\frac{dv}{dt}.$$

To get velocity v as a function of time t, we do the integration as follows

$$\frac{dv}{dt} = \frac{F}{m} - \frac{l^2B^2}{mR}v,$$

$$\int_0^v \frac{dv}{\frac{F}{m} - \frac{l^2B^2}{mR}v} = \int_0^t dt,$$

$$-\frac{mR}{l^2B^2}\left[\ln\left(\frac{F}{m} - \frac{l^2B^2}{mR}v\right)\right]_0^v = t,$$

$$-\frac{mR}{l^2B^2}\left[\ln\left(\frac{F}{m} - \frac{l^2B^2}{mR}v\right) - \ln\left(\frac{F}{m}\right)\right] = t,$$

$$-\frac{mR}{l^2B^2}\ln\left(1 - \frac{l^2B^2}{FR}v\right) = t,$$

$$1 - \frac{l^2B^2}{FR}v = \exp\left(-\frac{l^2B^2}{mR}t\right),$$

$$v = \frac{FR}{l^2B^2}\left[1 - \exp\left(-\frac{l^2B^2}{mR}t\right)\right] \tag{4}$$

$$= v_T\left[1 - \exp\left(-\frac{l^2B^2}{mR}t\right)\right]. \tag{5}$$

Thus, the velocity of the loop increases with time and attains terminal velocity $v_T = \frac{FR}{l^2B^2}$.

(c) With $F = 0.001$ N, $B = 1.0$ T, $l = 0.20$ m, $m = 0.03$ kg, $R = 0.50\ \Omega$, the velocity against time curve is

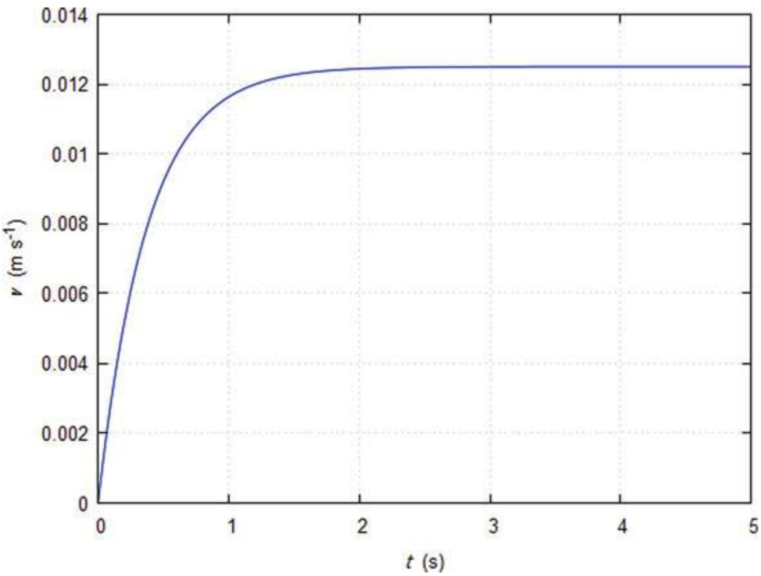
$$\begin{aligned} v &= \frac{FR}{l^2B^2}\left[1 - \exp\left(-\frac{l^2B^2}{mR}t\right)\right] \\ &= \frac{(0.001\ \text{N})(0.50\ \Omega)}{(0.20\ \text{m})^2(1.0\ \text{T})^2}\left[1 - \exp\left(-\frac{(0.20\ \text{m})^2(1.0\ \text{T})^2}{(0.03\ \text{kg})(0.50\ \Omega)}t\right)\right] \\ &= 0.0125\ \text{m/s}\cdot[1 - \exp(-2.6667t)]. \end{aligned}$$

where the terminal velocity is

$$v_T = \frac{FR}{l^2B^2} = \frac{(0.001\ \text{N})(0.50\ \Omega)}{(0.20\ \text{m})^2(1.0\ \text{T})^2} = 0.0125\ \text{m s}^{-1}.$$

- Curve of v against t is drawn by wxMaxima:

```
(%i6) fpprintprec:5; F:0.001; B:1; l:0.2; m:0.03; R:0.5;
(fpprintprec)     5
(F)   0.001
(B)   1
(l)   0.2
(m)   0.03
(R)   0.5
(%i7) v: F*R/(l^2*B^2)*(1 - exp(-l^2*B^2*t/(m*R)));
(v)   0.0125*(1-%e^(-2.6667*t))
(%i8) wxplot2d(v, [t, 0, 5], grid2d, [xlabel,"{/Helvetica-Italic t}  (s)"],
[ylabel,"{/Helvetica-Italic v}  (m s^{-1})"]);
```



Comments on the codes:

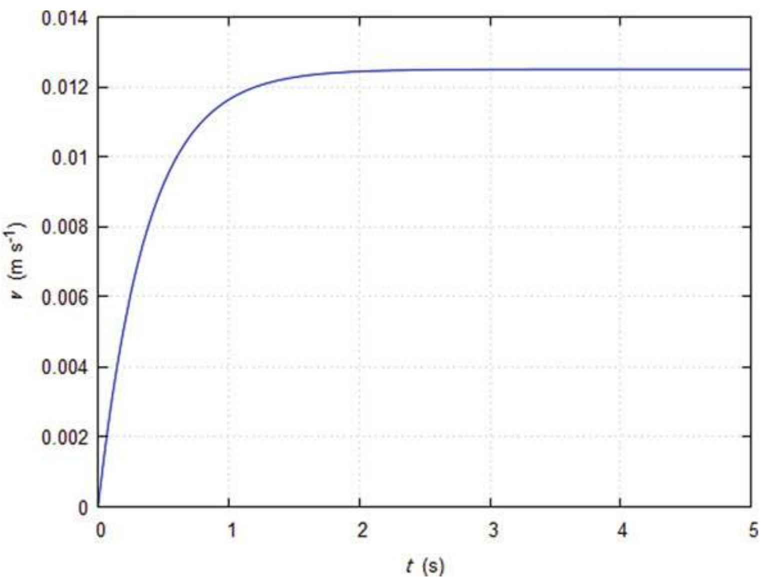
(%i6) Set floating point print precision to 5, and assign values of F, B, l, m, and R.
(%i7) Define v as in Eq. (4).
(%i8) Plot v against t for $0 \leq t \leq 5$ s.

Alternative solution: Parts (b) and (c) can be solved by predefined functions *ode*2 and *ic*1 of wxMaxima. See *Solving first order ordinary differential equation* in Appendix A. The first-order ordinary differential equation to be solved is

$\frac{dv}{dt} = \frac{F}{m} - \frac{l^2 B^2}{mR} v$, where v is the dependent variable and t independent variable, while the initial condition is $t = 0$ s, $v = 0$ m/s.

- wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)     5
(ratprint)  false
(%i3) sol: ode2('diff(v,t)=F/m - l^2*B^2*v/(m*R), v, t);
(sol) v=%e^(-
(B^2*l^2*t)/(R*m))*((F*R*%e^((B^2*l^2*t)/(R*m)))/(B^2*l^2)+%c)
(%i5) ic1(sol, t=0, v=0); expand(%);
(%o4) v=(%e^(-(B^2*l^2*t)/(R*m))*(F*R*%e^((B^2*l^2*t)/(R*m))-
F*R))/(B^2*l^2)
(%o5) v=(F*R)/(B^2*l^2)-(F*R*%e^(-(B^2*l^2*t)/(R*m)))/(B^2*l^2)
(%i6) rhs(%);
(%o6) (F*R)/(B^2*l^2)-(F*R*%e^(-(B^2*l^2*t)/(R*m)))/(B^2*l^2)
(%i7) v: subst([F=0.001, B=1, l=0.2, m=0.03, R=0.5], %);
(v)   0.0125-0.0125*%e^(-2.6667*t)
(%i8) wxplot2d(v, [t, 0, 5], grid2d, [xlabel,"{/Helvetica-Italic t}  (s)"],
[ylabel,"{/Helvetica-Italic v}  (m s^{-1})"]);
```



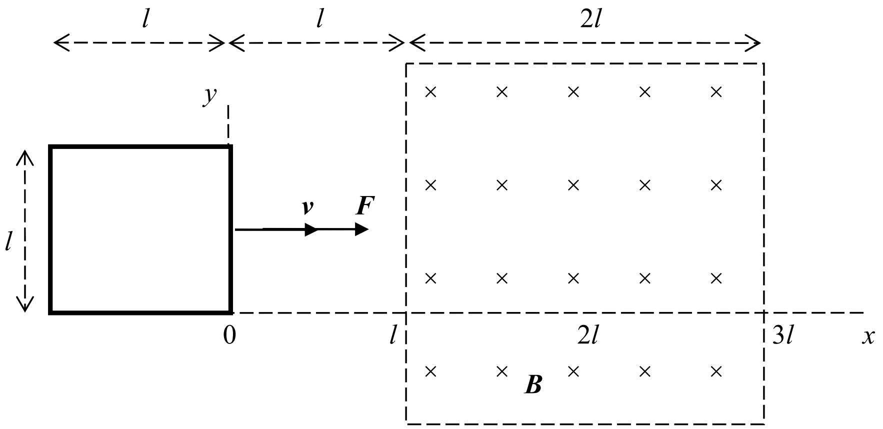


Fig. 13.7 A square conducting loop is pulled into a region of uniform magnetic field, Problem 13.8

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false,
(%i3) Solve the ordinary differential equation $\frac{dv}{dt} = \frac{F}{m} - \frac{l^2 B^2}{mR} v$ for a general solution, the dependent variable is v and the independent variable is t.
(%i5) Set the initial condition and get a particular solution.
(%o5) The solution is $v = \frac{FR}{B^2 l^2} - \frac{FR \exp\left(-\frac{B^2 l^2 t}{Rm}\right)}{B^2 l^2} = \frac{FR}{l^2 B^2}\left[1 - \exp\left(-\frac{l^2 B^2}{mR} t\right)\right]$.
(%i7) Substitute values of F, B, l, m, and R into the solution.
(%i9) Plot v against t for $0 \le t \le 5$ s.

Problem 13.8 Figure 13.7 shows a square conducting loop of side l, resistance R, and a uniform magnetic field region $\boldsymbol{B}$ of width $2l$. The direction of the magnetic field is in the plane of the paper. The loop is pulled with constant velocity $\boldsymbol{v}$ by an external force $\boldsymbol{F}$ to the right as shown. Sketch

(a) a curve of external force F against x for $0 \le x \le 5l$.
(b) a curve of current i in the loop as a function of x for $0 \le x \le 5l$.

Solution

(a) When the loop is outside the region of the uniform magnetic field, the external force is zero. When the right side of the loop enters the magnetic field region, the external force is, Eqs. (10.5) and (13.4),

$$F = IlB = \frac{\mathcal{E}}{R} lB = \frac{vlB}{R} lB = \frac{vl^2 B^2}{R}.$$

When the whole loop is in the magnetic field region, the current in the loop is zero and the external force is zero as well. When the right side of the loop exits

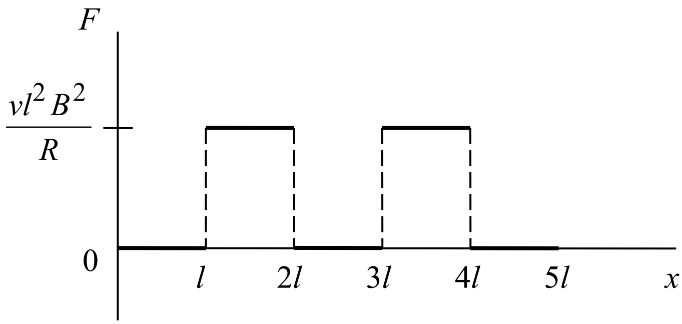


Fig. 13.8 Curve of F against x, Problem 13.8

the region of magnetic field and the left side is still in the region, the external force is,

$$F = \frac{vl^2B^2}{R}.$$

When the whole loop is outside the region of magnetic field, the external force is zero. The curve of F against x is shown in Fig. 13.8.

(b) When the loop is outside the region of the magnetic field, the current is zero. When the right side of the loop enters the magnetic field region, the current in the counter clockwise directions is

$$i = \frac{\mathcal{E}}{R} = \frac{vlB}{R}, \text{ counter-clockwise, positive.}$$

When the whole loop is in the magnetic field region, the current is zero. When the right side of the loop exits the region of the magnetic field while the left side is still in the region, the current flowing in the loop in clockwise direction is

$$i = \frac{\mathcal{E}}{R} = \frac{vlB}{R}, \text{ clockwise, negative.}$$

When the whole loop is outside the region of the magnetic field, the current is zero. The curve of current i against x is shown in Fig. 13.9.

Problem 13.9 A rectangular metal coil of size 10×15 cm has 20 turns. The coil is rotated about the x-axis in a uniform magnetic field of 0.05 T. The maximum induced emf in the coil is 20 mV. What is the angular speed of rotation?

Solution

Figure 13.10 shows the coil rotating in uniform magnetic field $\boldsymbol{B}$ at angular speed ω about the x-axis.

The magnetic flux through the coil at this instance is

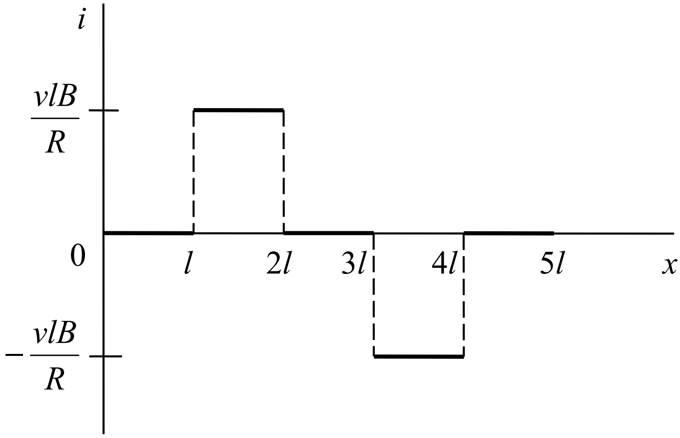


Fig. 13.9 Curve of i against x, Problem 13.8

Fig. 13.10 A rectangular coil rotating in a region of uniform magnetic field, Problem 13.9

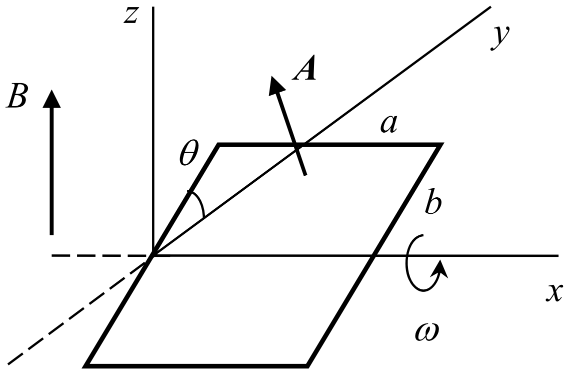
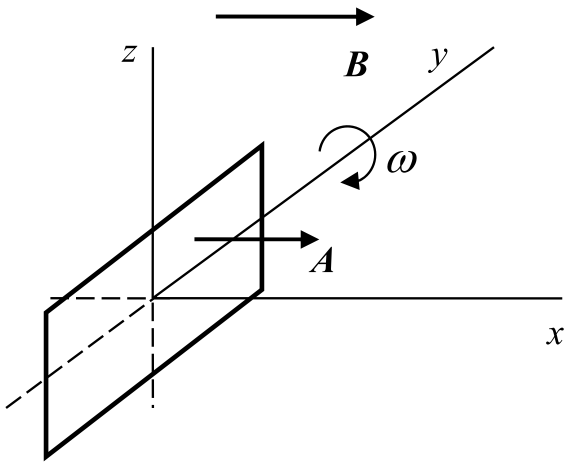


Fig. 13.11 A rectangular wire loop rotating in a region of uniform magnetic field, Problem 13.10



$$\Phi_m = N\boldsymbol{B} \cdot \boldsymbol{A} = NBab\cos\theta = NBab\cos\omega t,$$

where a, b, and N are the width, length, and number of turns of the coil, respectively. The rate of change of magnetic flux is

$$\frac{d\Phi_m}{dt} = -\omega abNB\sin\omega t.$$

Using Faraday's law, the induced emf in the coil is, Eq. (13.1),

$$\mathcal{E} = -\frac{d\Phi_m}{dt} = \omega abNB\sin\omega t.$$

The maximum emf is

$$\mathcal{E}_{max} = \omega abNB.$$

The angular speed of rotation of the coil is

$$\omega = \frac{\mathcal{E}_{max}}{abNB} = \frac{20 \times 10^{-3}\ \text{V}}{(0.10\ \text{m} \times 0.15\ \text{m})(20)(0.050\ \text{T})} = 1.3\ \text{rad s}^{-1},$$

or,

$$f = \frac{\omega}{2\pi} = 0.21\ \text{revolution per second.}$$

- wxMaxima codes:

```
(%i6) fpprintprec:5; emf:20e-3; a:0.1; b:0.15; N:20; B:0.05;
(fpprintprec)     5
(emf) 0.02
(a)   0.1
(b)   0.15
(N)   20
(B)   0.05
(%i7) omega: emf/(a*b*N*B);
(omega) 1.3333
(%i8) f: omega/(2*float(%pi));
(f)   0.21221
```

Comments on the codes:

(%i6)	Set floating point print precision to 5, and assign values of $\mathcal{E}$, a, b, N, and B.
(%i7) and (%i8)	Calculate ω and f.

Problem 13.10

(a) A rectangular wire loop of area A and resistance R rotates at constant angular speed ω about the y-axis, as illustrated in Fig. 13.11. The loop is in a uniform magnetic field B in the x direction. Determine expressions for

(i) magnetic flux Φ_m through the loop as a function of time. At time $t = 0$, the position of the loop is as shown in the figure.
(ii) rate of change of the magnetic flux $d\Phi_m/dt$.
(iii) induced emf in the loop.
(iv) torque τ such that the loop rotates at a constant angular speed.
(v) induced emf in the loop if the angular speed is twice a much.

(b) If $A = 400\ \text{cm}^2$, $R = 2.0\ \Omega$, $\omega = 10\ \text{rad s}^{-1}$ and $B = 0.50\ \text{T}$, determine maximum:

(i) flux through the loop.
(ii) induced emf
(iii) torque.

Show that in one revolution of the loop, the work done by the torque is equal to the electrical energy dissipated by the loop.

(c) What is the maximum induced emf if the angular speed of the loop is still $10\ \text{rad s}^{-1}$ but the loop is rotated about

(i) z axis?
(ii) x axis?

(d) Plot all quantities in part (a) using wxMaxima.

Solution

(a) (i) Magnetic flux is defined as

$$\Phi_m = \boldsymbol{B} \cdot \boldsymbol{A} = BA\cos\theta,$$

where θ is the angle between vectors $\boldsymbol{A}$ and $\boldsymbol{B}$. The magnetic flux is

$$\Phi_m = BA\cos\omega t,$$

because $\theta = \omega t$.

(ii) The rate of change of the magnetic flux is obtained by differentiating the flux with respect to time

$$\frac{d\Phi_m}{dt} = \frac{d}{dt} BA\cos\omega t = -\omega BA\sin\omega t.$$

(iii) The induced emf in the loop is obtained by Faraday's law

$$\mathcal{E} = -\frac{d\Phi_m}{dt} = \omega BA\sin\omega t.$$

(iv) The electric current flowing in the loop is

$$I = \frac{\mathcal{E}}{R} = \frac{\omega BA}{R}\sin\omega t.$$

The torque needed such that the loop rotates with a constant angular speed is

$$\tau = |I\boldsymbol{A} \times \boldsymbol{B}| = IAB\sin\omega t = (AB\sin\omega t)\frac{\omega AB}{R}\sin\omega t$$
$$= \frac{\omega A^2 B^2}{R}\sin^2\omega t.$$

(v) If the angular speed of the loop is $\omega' = 2\omega$, the induced emf becomes,

$$\mathcal{E}' = \omega' BA\sin\omega' t = 2\omega BA\sin 2\omega t.$$

- wxMaxima codes:

```
(%i1) phim: B*A*cos(omega*t);
(phim) A*B*cos(omega*t)
(%i2) dphim_over_dt: diff(phim, t);
(dphim_over_dt)   -A*B*omega*sin(omega*t)
(%i3) emf: -dphim_over_dt;
(emf) A*B*omega*sin(omega*t)
(%i4) I: emf/R;
(I)   (A*B*omega*sin(omega*t))/R
(%i5) tau: I*A*B*sin(omega*t);
(tau) (A^2*B^2*omega*sin(omega*t)^2)/R
(%i6) omegaprime: 2*omega;
(omegaprime) 2*omega
(%i7) emfprime: omegaprime*B*A*sin(omegaprime*t);
(emfprime)  2*A*B*omega*sin(2*omega*t)
```

Comments on the codes:

(%i1)–(%i3) Assign Φ_m, $\frac{d\Phi_m}{dt}$, $\mathcal{E}$.
(%i4)–(%i7) Assign I, τ, ω', ε'

(b) (i) The maximum magnetic flux across the loop is

$$\Phi_{m,max} = BA = (0.50\ \text{T})(400 \times 10^{-4}\ \text{m}^2) = 2.0 \times 10^{-2}\ \text{Wb}.$$

(ii) The maximum induced emf is

$$\mathcal{E}_{max} = \omega BA = (10\ \text{s}^{-1})(0.50\ \text{T})(400 \times 10^{-4}\ \text{m}^2) = 2.0 \times 10^{-1}\ \text{V}.$$

(iii) The maximum torque is

$$\tau_{max} = \frac{\omega B^2 A^2}{R} = \frac{(10\ \text{s}^{-1})(0.50\ \text{T})^2(400 \times 10^{-4}\ \text{m}^2)^2}{2.0\ \Omega}$$
$$= 2.0 \times 10^{-3}\ \text{N m}.$$

Work done by the torque in a revolution is

$$W = \int_0^{2\pi} \tau\ d\theta = \int_0^{2\pi} \frac{\omega A^2 B^2}{R}\ \sin^2\theta\ d\theta = \frac{\pi\omega A^2 B^2}{R}$$
$$= \frac{\pi(10\ \text{s}^{-1})(400 \times 10^{-4}\ \text{m}^2)^2(0.50\ \text{T})^2}{2.0\ \Omega} = 6.3 \times 10^{-3}\ \text{J}.$$

The electrical energy dissipated by the loop in a revolution is

$$W_e = \int_0^{2\pi/\omega} RI^2\ dt = \int_0^{2\pi/\omega} \frac{\omega^2 A^2 B^2}{R}\ \sin^2\omega t\ dt = \frac{\pi\omega A^2 B^2}{R}$$
$$= \frac{\pi(10\ \text{s}^{-1})(400 \times 10^{-4}\ \text{m}^2)^2(0.50\ \text{T})^2}{2.0\ \Omega} = 6.3 \times 10^{-3}\ \text{J}.$$

The numerical values of work and energy are the same.

- wxMaxima codes:

```
(%i6) fpprintprec:5; ratprint:false; A:400e-4; R:2; omega:10; B:0.5;
(fpprintprec)     5
(ratprint)  false
(A)   0.04
(R)   2
(omega)     10
(B)   0.5
(%i7) phim: B*A*cos(omega*t);
(phim) 0.02*cos(10*t)
(%i8) dphimdt: diff(phim, t);
(dphimdt) -0.2*sin(10*t)
(%i9) emf: -dphimdt;
(emf) 0.2*sin(10*t)
(%i10) I: emf/R;
(I) 0.1*sin(10*t)
(%i11) torque: I*A*B*sin(omega*t);
(torque) 0.002*sin(10*t)^2
(%i12) phimmax: B*A;
(phimmax) 0.02
(%i13) emfmax: omega*B*A;
(emfmax) 0.2
(%i14) torquemax: omega*B^2*A^2/R;
(torquemax) 0.002
(%i15) W: integrate(omega*A^2*B^2/R*(sin(theta))^2, theta, 0,
float(2*%pi));
(W) 0.0062832
(%i16) We: integrate(omega^2*A^2*B^2/R*(sin(omega*t))^2, t, 0,
float(2*%pi)/omega);
(We) 0.0062832
```

Comments on the codes:

(%i6)	Set floating point print precision to 5, internal rational number print to false, and assign values of A, R, ω, and B.
(%i7)–(%i11)	Assign Φ_m, $d\Phi_m/dt$, $\mathcal{E}$, I, and τ .
(%i12)–(%i14)	Calculate $\Phi_{m,max}$, $\mathcal{E}_{max}$, and τ_{max}.
(%i15) and (%i16)	Calculate $W = \int_0^{2\pi} \frac{\omega A^2 B^2}{R} \sin^2\theta\, d\theta$ and $W_e = \int_0^{2\pi/\omega} \frac{\omega^2 A^2 B^2}{R} \sin^2 \omega t\ dt$.

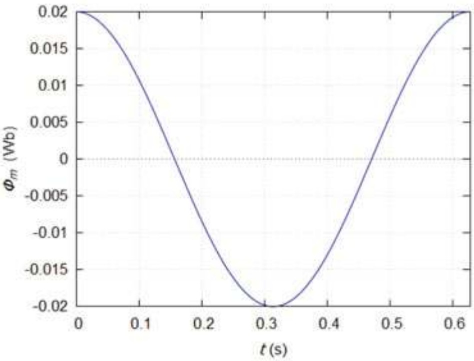
(c) (i) If the loop is rotated about the z-axis, $\mathcal{E}_{max}$ is 2.0×10^{-1} V still because the rate of magnetic flux change is still the same.

(ii) If the loop is rotated about the x axis, there is no flux change, thus, the emf is zero.

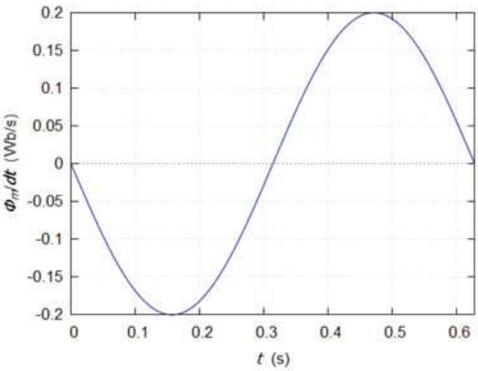
(d) We set the values of A, R, ω, and B as in part (b), i.e. $A = 400$ cm^2, $R = 2.0\ \Omega$, $\omega = 10$ rad s^{-1}, $B = 0.50$ T, and plot the curves by wxMaxima.

- wxMaxima codes:

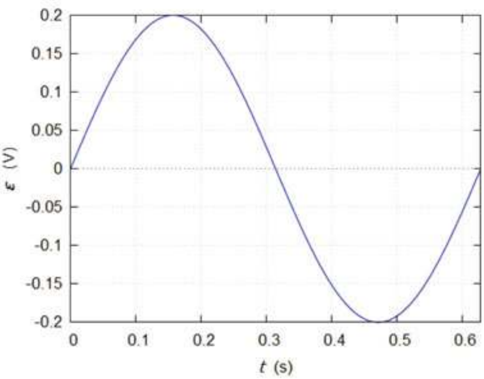
```
(%i5) fpprintprec:5; A:400e-4; R:2; omega:10; B:0.5;
(fpprintprec)     5
(A)   0.04
(R)   2
(omega) 10
(B)   0.5
(%i6) phim: B*A*cos(omega*t);
(phim) 0.02*cos(10*t)
(%i7) dphim_over_dt: diff(phim, t);
(dphim_over_dt) -0.2*sin(10*t)
(%i8) emf: -dphim_over_dt;
(emf) 0.2*sin(10*t)
(%i9) I: emf/R;
(I)   0.1*sin(10*t)
(%i10) tau: I*A*B*sin(omega*t);
(tau) 0.002*sin(10*t)^2
(%i11) omegaprime: 2*omega;
(omegaprime) 20
(%i12) emfprime: omegaprime*B*A*sin(omegaprime*t);
(emfprime) 0.4*sin(20*t)
(%i13) wxplot2d(phim, [t,0,2*%pi/omega], grid2d, [xlabel,"{/Helvetica-
Italic t} (s)"], [ylabel,"{/Symbol-Italic F}_{/Helvetica-Italic m}
(Wb)"]);
```



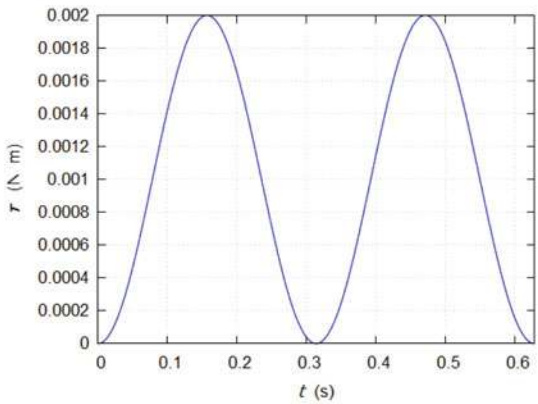

```
(%i14) wxplot2d(dphim_over_dt, [t,0,2*%pi/omega], grid2d,
[xlabel,"{/Helvetica-Italic t}  (s)"], [ylabel,"{/Symbol-Italic
F}_{/Helvetica-Italic m}/{/Helvetica-Italic dt}  (Wb/s)"]);
```



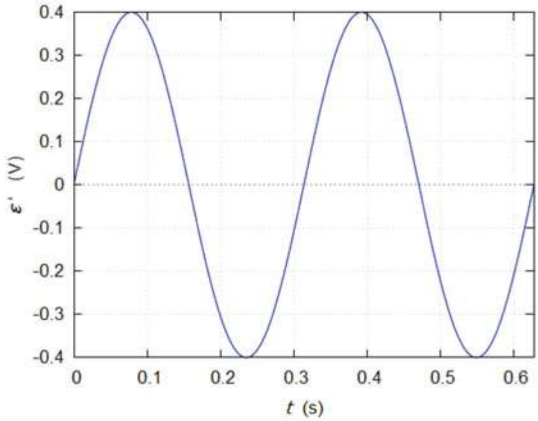

```
(%i15) wxplot2d(emf, [t,0,2*%pi/omega], grid2d, [xlabel,"{/Helvetica-Italic
t}  (s)"], [ylabel,"{/Symbol-Italic e}   (V)"]);
```




```
(%i16) wxplot2d(tau, [t,0,2*%pi/omega], grid2d, [xlabel,"{/Helvetica-Italic
t}  (s)"], [ylabel,"{/Symbol-Italic t}   (N  m)"]);
```




```
(%i17) wxplot2d(emfprime, [t,0,2*%pi/omega], grid2d, [xlabel,"{/Helvetica-
Italic t}  (s)"], [ylabel,"{/Symbol-Italic e} '   (V)"]);
```



Comments on the codes:

(%i5)	Set floating point print precision to 5, and assign values of A, R, ω, and B.
(%i6)–(%i10)	Assign Φ_m, $d\Phi_m/dt$, $\mathcal{E}$, I, and τ.
(%i11)	Assign ω'.
(%i12)	Calculate $\mathcal{E}'$.
(%i13)	Plot Φ_m against t for $0 \le t \le 2\pi/\omega$.
(%i14)	Plot $d\Phi_m/dt$ against t for $0 \le t \le 2\pi/\omega$.
(%i15)	Plot $\mathcal{E}$ against t for $0 \le t \le 2\pi/\omega$.
(%i16)	Plot τ against t for $0 \le t \le 2\pi/\omega$.
(%i17)	Plot $\mathcal{E}'$ against t for $0 \le t \le 2\pi/\omega$.

Problem 13.11 A deformable conducting loop of 20 cm radius is in a uniform magnetic field of 2.0 T and connected to a resistor of 1.2 Ω, as shown in Fig. 13.12.

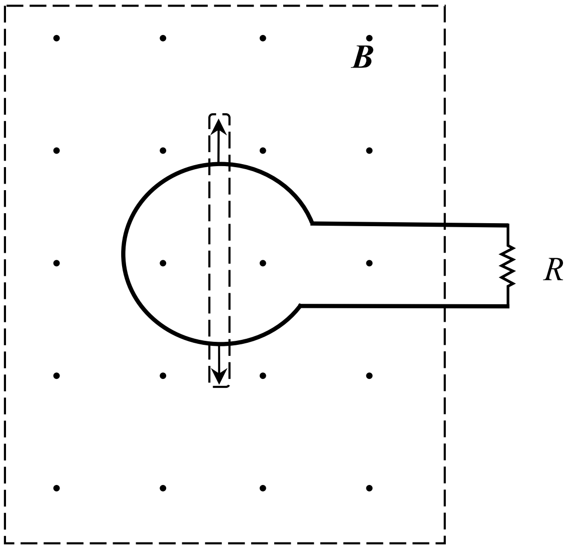


Fig. 13.12 A deformable conducting loop in a uniform magnetic field, Problem 13.11

The direction of the magnetic field is out of the page. The loop is pulled at two points as shown so that its area becomes zero in 0.20 s.

(a) Calculate the average induced emf.
(b) What is the electrical current through the resistor? Determine the direction of the current.

Solution

(a) When the loop is pulled and deformed, the magnetic flux through the loop changes, and the emf is induced. Initial magnetic flux is

$$\Phi_{init} = BA = B\pi r^2 = (2.0\ \text{T})\pi(0.20\ \text{m}^2) = 2.5 \times 10^{-1}\ \text{Wb}.$$

The final magnetic flux is zero because the area is zero

$$\Phi_{final} = 0.$$

By Faraday's law, average induced emf is, Eq. (13.1),

$$\mathcal{E} = -\frac{\Delta\Phi_m}{\Delta t} = -\frac{(\Phi_{final} - \Phi_{init})}{\Delta t} = -\frac{(0 - 2.5 \times 10^{-1}\ \text{Wb})}{0.20\ \text{s}} = 1.3\ \text{V}.$$

(b) The induced current through the resistor R is

$$I = \frac{\mathcal{E}}{R} = \frac{1.3\ \text{V}}{1.2\ \Omega} = 1.0\ \text{A}.$$

The direction of the current is determined as follows. When the loop is pulled, the magnetic flux through it decreases because the area decreases. By Lenz's law, induced current is such that it increases the flux. To increase the flux, the direction of current is counter clockwise. Thus, the current flows through the resistor from bottom to top.

- wxMaxima codes:

```
(%i5) fpprintprec:5; r:0.2; B:2; R:1.2; delta_t:0.2;
(fpprintprec)     5
(r)   0.2
(B)   2
(R)   1.2
(delta_t)   0.2
(%i7) phiinit: B*float(%pi)*r^2; phifinal:0;
(phiinit)   0.25133
(phifinal)  0
(%i8) emf: -(phifinal-phiinit)/delta_t;
(emf) 1.2566
(%i9) I: emf/R;
(I)   1.0472
```

Comments on the codes:

(%i5)	Set floating point print precision to 5, and assign values of r, B, R, and Δt.
(%i7)	Assign values of Φ_{init} and Φ_{final}.
(%i8) and (%i9)	Calculate emf and I.

Problem 13.12 Figure 13.13 shows the cross-section of a long and straight solenoid of radius R. The magnetic field of the solenoid is increasing at a rate of dB/dt.

(a) What is the rate of change of magnetic flux through the circle of radius r_1 in the solenoid? What is the induced electric field at r_1? Determine the direction of the induced electric field.
(b) What is the induced electric field at r_2?
(c) Sketch a curve of the induced electric field against distance from the axis of the solenoid r for $0 \le r \le 2R$.
(d) Determine induced emf in circular paths of radii $R/2$, R, and $2R$.

Solution

(a) For the circle of radius r_1, the rate of change of magnetic flux is

$$\frac{d\Phi_m}{dt} = A\frac{dB}{dt} = \pi r_1^2 \frac{dB}{dt}.$$

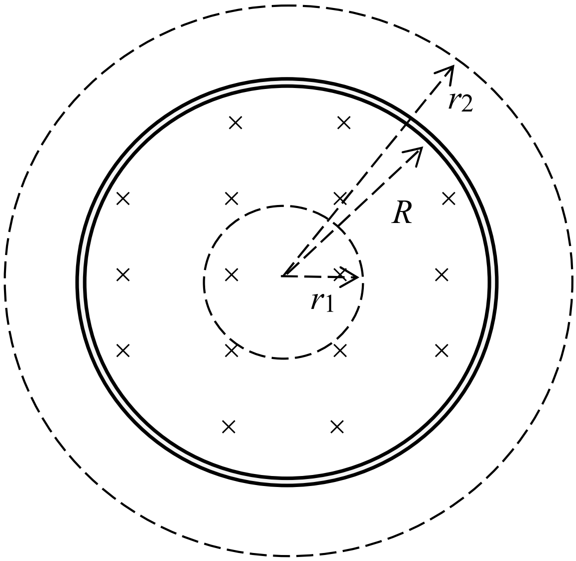


Fig. 13.13 Cross section of a solenoid, Problem 13.12

Induced electric field E_{ne} (induced non-electrostatic electric field) at r_1 is obtained by line integral of the electric field along the circular path of radius r_1, as illustrated in Fig. 13.14, and

$$\oint E_{ne} \cdot ds = E_{ne} \cdot 2\pi r_1.$$

The integration is equal in magnitude to the induced emf

$$\mathcal{E} = \frac{d\Phi_m}{dt} = \pi r_1^2 \frac{dB}{dt}.$$

This means that

$$E_{ne} \cdot 2\pi r_1 = \pi r_1^2 \frac{dB}{dt},$$

The induced electric field is

$$E_{ne} = \frac{\pi r_1^2 \frac{dB}{dt}}{2\pi r_1} = \frac{r_1}{2} \frac{dB}{dt}.$$

The direction of the electric field is shown in Fig. 13.14.

(b) For circle of radius r_2, the induced electric field at distance r_2 is calculated the same way. The rate of change of magnetic flux in a circular path of radius r_2 is

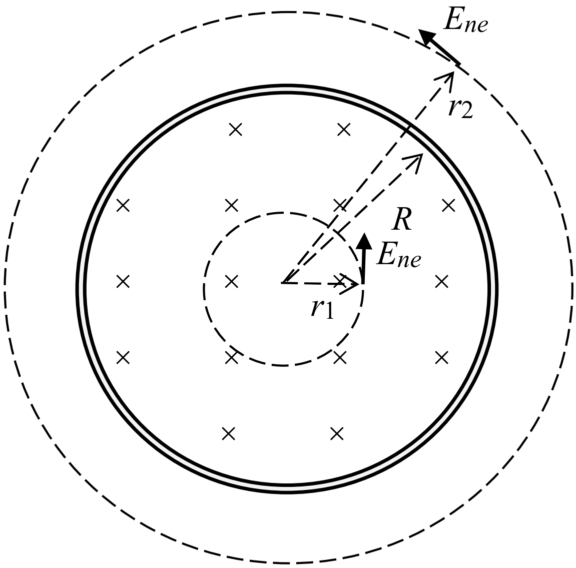


Fig. 13.14 Determining induced electric fields in and out of the solenoid, Problem 13.12

$$\frac{d\Phi_m}{dt} = A\frac{dB}{dt} = \pi R^2 \frac{dB}{dt}.$$

The line integral of the electric field along the circular path of radius r_2 is

$$\oint E_{ne} \cdot ds = E_{ne} \cdot 2\pi r_2.$$

This means that

$$\mathcal{E} = \oint E_{ne} \cdot ds = \frac{d\Phi_m}{dt},$$

giving

$$E_{ne} \cdot 2\pi r_2 = \pi R^2 \frac{dB}{dt},$$

and

$$E_{ne} = \frac{R^2}{2r_2}\frac{dB}{dt}.$$

(c) From the results of (a) and (b), the curve of induced electric field E_{ne} against distance r from the solenoid axis is shown in Fig. 13.15.

(d) Induced emf is calculated using

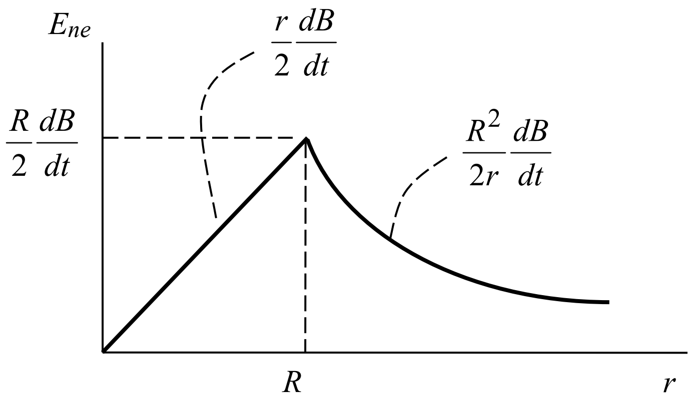


Fig. 13.15 Curve of the induced electric field against distance from the center of the solenoid, Problem 13.12

$$\mathcal{E} = \oint E_{ne} \cdot ds = E_n \cdot 2\pi r,$$

where E_{ne} and r follow the chosen path. For $r = R/2$,

$$\mathcal{E} = \left(\frac{r}{2}\frac{dB}{dt}\right)(2\pi r) = \frac{R/2}{2}\frac{dB}{dt} \cdot 2\pi R/2 = \frac{\pi R^2}{4}\frac{dB}{dt}.$$

For $r = R$,

$$\mathcal{E} = \left(\frac{r}{2}\frac{dB}{dt}\right)(2\pi r) = \frac{R}{2}\frac{dB}{dt} \cdot 2\pi R = \pi R^2\frac{dB}{dt},$$

or can also be calculated as follows

$$\mathcal{E} = \left(\frac{R^2}{2r}\frac{dB}{dt}\right)(2\pi r) = \frac{R^2}{2R}\frac{dB}{dt} \cdot 2\pi R = \pi R^2\frac{dB}{dt}.$$

For $r = 2R$,

$$\mathcal{E} = \left(\frac{R^2}{2r}\frac{dB}{dt}\right)(2\pi r) = \frac{R^2}{2(2R)}\frac{dB}{dt} \cdot 2\pi(2R) = \pi R^2\frac{dB}{dt}.$$

Problem 13.13 A long solenoid with a cross-section of 6.0 cm^2 and 10 turns of wire per cm, carries a current of 0.25 A. Ten turns of insulated wire is wound around the solenoid. What is the induced emf in the insulated wire if the current of the solenoid drops to zero in 0.05 s?

Solution

The initial magnetic field of the solenoid is

$$B_{init} = \mu_0 n I = (4\pi \times 10^{-7}\ \text{Wb A}^{-1}\ \text{m}^{-1})\left(\frac{10}{0.01\ \text{m}}\right)(0.25\ \text{A})$$
$$= 3.1 \times 10^{-4}\ \text{T}.$$

After 0.05 s, the current and the magnetic field were zero. The induced emf in the insulated wire is

$$\mathcal{E} = -\frac{\Delta \Phi_m}{\Delta t} = -NA\frac{\Delta B}{\Delta t} = -NA\frac{(B_{final} - B_{init})}{\Delta t}$$
$$= -10(6.0 \times 10^{-4}\ \text{m}^2)\frac{(0 - 3.1 \times 10^{-4}\ \text{T})}{0.05\ \text{s}} = 3.8 \times 10^{-5}\ \text{V}.$$

- wxMaxima codes:

```
(%i7) fpprintprec:5; mu0:float(4*%pi*1e-7); n:10/0.01; I:0.25;
delta_t:0.05; N:10; A:6e-4;
(fpprintprec)     5
(mu0) 1.2566*10^-6
(n)   1000.0
(I)   0.25
(delta_t) 0.05
(N)   10
(A)   6.0*10^-4
(%i8) Binit: mu0*n*I;
(Binit) 3.1416*10^-4
(%i9) emf: -N*A*(0-Binit)/delta_t;
(emf) 3.7699*10^-5
```

Comments on the codes:

(%i7)	Set floating point print precision to 5, and assign values of μ_0, n, I, Δt, N, and A.
(%i8) and (%i9)	Calculate B_{init} and emf.

13.3 Summary

- Faraday's law states that the induced electromotive force (emf) $\mathcal{E}$ in a loop is proportional to the rate of change of the magnetic flux of the loop

$$\mathcal{E} = -\frac{d\Phi_m}{dt}.$$

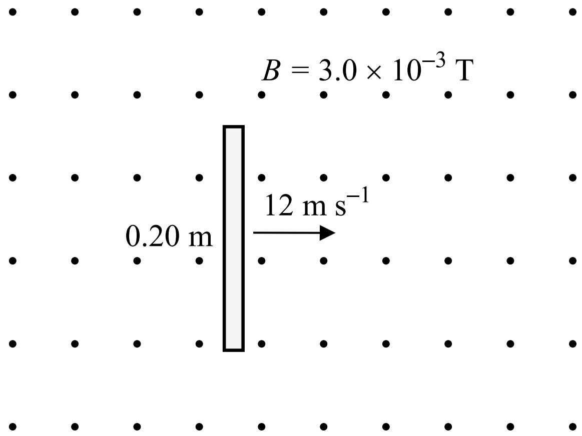


Fig. 13.16 A conductor moving in a region of uniform magnetic field, Exercise 13.3

- Motional emf $\mathcal{E}$ is induced in a conductor when the conductor of length l moves with velocity v in a uniform magnetic field B

$$\mathcal{E} = -Blv.$$

13.4 Exercises

Exercise 13.1 The magnetic field of a region is $\boldsymbol{B} = 0.0040\,\mathbf{i} - 0.0055\,\mathbf{j} + 0.0075\,\mathbf{k}$ T. A loop of area 0.024 m^2 lies flat on the xy plane. What is the magnetic flux that passes through the loop?

(Answer: 1.8×10^{-4} Wb)

Exercise 13.2 A coil with a radius of 1.0 cm has 50 loops of wire on it. It is placed in a magnetic field $B = 0.30$ T such that the magnetic flux through the coil is maximum. The coil is then rotated so that the flux is zero in 0.020 s. Calculate the average emf induced between the terminals of the coil.

(Answer: 0.24 V)

Exercise 13.3 A conductor of length 0.20 m is moving at a velocity of 12 m s^{-1} perpendicular to a magnetic field of 3.0×10^{-3} T, as shown in Fig. 13.16. The magnetic field is out of the plane of the paper. Determine the induced emf and its direction.

(Answer: 7.2×10^{-3} V, from top to bottom of the conductor)

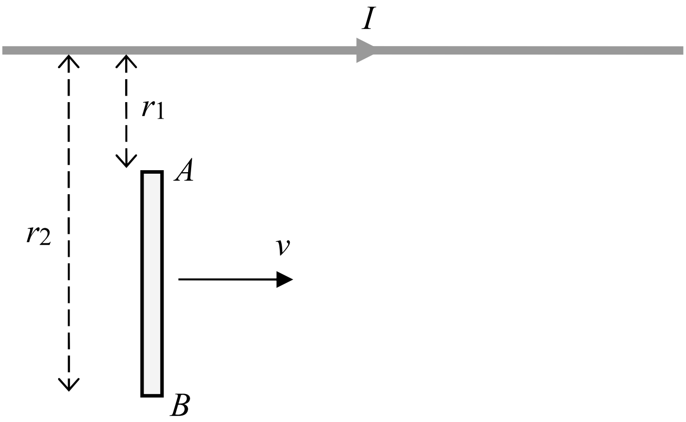


Fig. 13.17 A conductor moving parallel to a current-carrying wire, Exercise 13.4

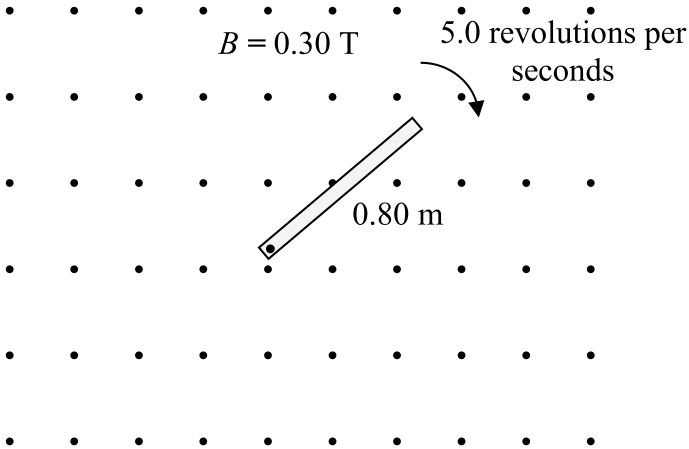


Fig. 13.18 A conductor rotating in a region of uniform magnetic field, Exercise 13.5

Exercise 13.4 Conductor AB moves with speed v near a wire carrying current I, as illustrated in Fig. 13.17. Motional emf is induced in the conductor. Show that the potential difference between points A and B is

$$V_A - V_B = \frac{\mu_0 I v}{2\pi} \ln\left(\frac{r_2}{r_1}\right).$$

Exercise 13.5 A conductor of length 0.80 m is pivoted at one of its ends and rotated at 5.0 revolutions per second, as shown in Fig. 13.18. The conductor is in the region of the uniform magnetic field of 0.30 T out of the plane of the paper. Calculate the potential difference between the ends of the rod. Which end has higher electric potential?

(Answer: 3.0 V, the pivoted end)

Chapter 14
Inductance

Abstract This chapter solves problems on electric inductance—a measure of resistance of a conducting coil to change in current or magnetic flux linkages per unit current of the coil. Problems on self and mutual inductance, energy in inductor, and direct current *RL* circuit are solved. Solutions by analysis and computer calculation are presented.

14.1 Basic Concepts and Formulae

(1) When electric current in a coil changes with time, emf is induced in the coil and is given by

$$\mathcal{E} = -L\frac{dI}{dt}, \tag{14.1}$$

where L is inductance of the coil. Inductance is a measure of resistance of a device to change in current. SI unit for inductance is henry (H).

$$1\ \mathrm{H} = 1\ \mathrm{V\ s\ A^{-1}}. \tag{14.2}$$

(2) Inductance of a coil is

$$L = \frac{N\Phi_m}{I}, \tag{14.3}$$

where Φ_m is the magnetic flux through the coil, N is the number of turns, and I is the current in the coil. This means that inductance is magnetic flux linkages per unit current.

For an air core solenoid, the self-inductance is

$$L = \frac{\mu_0 N^2 A}{l}, \tag{14.4}$$

W. M. S. Wan Hassan et al., *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-43165-4_14

where N is the number of turns, A is the cross-sectional area, l is the length of the solenoid, and $\mu_0 = 4\pi \times 10^{-7}$ H m^{-1} is the free space permeability.

For a solenoid with core of material with permeability μ, the self-inductance is

$$L = \frac{\mu N^2 A}{l} = \frac{K_m \mu_0 N^2 A}{l}, \tag{14.5}$$

where $K_m = \mu/\mu_0$ is the relative permeability of the core material.

(3) Direct current *RL* circuit: For resistance R and inductance L connected in series to a battery of emf $\mathcal{E}$, the current of the circuit increases with time as

$$I(t) = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau}), \tag{14.6}$$

where $\tau = L/R$ is the time constant of the *RL* circuit. If the battery is taken out and the circuit is completed, the current will decrease as

$$I(t) = \frac{\mathcal{E}}{R} e^{-t/\tau}. \tag{14.7}$$

(4) The energy stored in the magnetic field of an inductor carrying current I is

$$U_m = \frac{1}{2} L I^2. \tag{14.8}$$

(5) The energy density (that is, the energy per unit volume) at a point with magnetic field B is

$$u_m = \frac{B^2}{2\mu_0}. \tag{14.9}$$

(6) When two coils are near to each other, changing current in the first coil will induce emf in the second coil. The emf induced in the second coil is

$$\mathcal{E}_2 = -M \frac{dI_1}{dt}, \tag{14.10}$$

where dI_1/dt is the rate of change of current in the first coil and M is the mutual inductance.

(7) When a charged capacitor is connected to an inductor and the circuit is completed, the charge of the capacitor and the current in the circuit oscillate with time as follows:

$$Q = Q_m \cos(\omega t + \phi), \tag{14.11}$$

$$I = \frac{dQ}{dt} = -\omega\, Q_m \sin(\omega t + \phi), \tag{14.12}$$

where Q_m is the maximum charge of the capacitor, ϕ is phase constant, and

$$\omega = \frac{1}{\sqrt{LC}}, \tag{14.13}$$

is the frequency of oscillation. These equations are obtained by solving,

$$L\frac{dI}{dt} + \frac{Q}{C} = 0, \tag{14.14}$$

or,

$$L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0. \tag{14.15}$$

The energy in an LC circuit is mutually exchanged from the capacitor to the inductor and vice versa, but the total energy is constant and it is

$$\begin{aligned} U = U_C + U_L &= \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}LI^2 \\ &= \frac{1}{2}\frac{Q_m^2}{C}\cos^2(\omega t + \phi) + \frac{1}{2}LI_m^2 \sin^2(\omega t + \phi) \\ &= \frac{1}{2}\frac{Q_m^2}{C} \\ &= \frac{1}{2}LI_m^2, \end{aligned} \tag{14.16}$$

where U_C and U_L are energies in the capacitor and the inductor, respectively, and I_m and Q_m are the maximum current in the circuit and maximum charge in the capacitor, respectively.

When the charge in the capacitor is Q_m, the current is zero momentarily, and the total energy is $\frac{1}{2}\frac{Q_m^2}{C}$. All the energy is in the capacitor. When the charge in the capacitor is zero, the current is a maximum I_m, and the total energy is $\frac{1}{2}LI_m^2$, All the energy is in the inductor.

(8) For an RLC circuit, charge in the capacitor and current in the circuit decrease with time similar to a damped harmonic motion.

14.2 Problems and Solutions

Problem 14.1 A solenoid has cross-sectional area of A, number of turns N, and length l.

(a) Calculate the self-inductance of the solenoid
(b) Determine the self-inductance if the core of the solenoid is filled with a material of permeability μ.

Solution

(a) The relation between induced emf $\mathcal{E}$ and the rate of change of current dI/dt is, Eq. (14.1),

$$\mathcal{E} = -L\frac{dI}{dt},$$

where L is inductance. By Faraday's law, the induced emf is the rate of magnetic flux change,

$$\mathcal{E} = -N\frac{d\Phi_m}{dt}.$$

So we write

$$L = N\left|\frac{d\Phi_m}{dI}\right|.$$

That is, inductance is change of magnetic flux per unit current.

For a solenoid, let the current changes from zero to I and magnetic flux from zero to Φ_m. Then, the self-inductance is

$$L = N\frac{\Phi_m}{I} = \frac{NBA}{I} = \frac{N(\mu_0 nI)A}{I} = \frac{N\mu_0 IA}{I}\left(\frac{N}{l}\right) = \frac{\mu_0 N^2 A}{l}.$$

(b) If the core of the solenoid is a material with permeability μ, the self-inductance is

$$L = \frac{\mu N^2 A}{l}.$$

Problem 14.2 A 30 cm long solenoid is built by winding 2000 turns of insulated wire to an iron rod of cross-sectional area 1.5 cm^2.

(a) The relative permeability of iron is 600. Calculate the self-inductance of the solenoid.

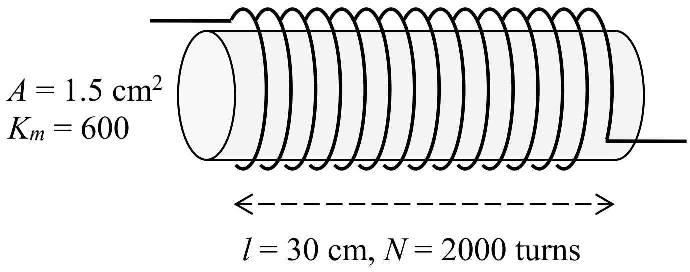


Fig. 14.1 An iron core solenoid, Problem 14.2

(b) If an electric current through the solenoid decreases from 0.60 A to 0.10 A in 0.03 s, what is the emf induced in the solenoid?

Solution

(a) Figure 14.1 shows the iron core solenoid.

Self-inductance of a solenoid is, Eq. (14.4),

$$L = \frac{\mu N^2 A}{l} = \frac{K_m \mu_0 N^2 A}{l},$$

where N, A, and l are number of turns, cross-sectional area, and length of the solenoid, while μ and K_m are permeability and relative permeability of iron, respectively. The self-inductance of the iron core solenoid is

$$L = \frac{600(4\pi \times 10^{-7}\ \text{H/m})(2000)^2(1.5 \times 10^{-4}\ \text{m}^2)}{0.30\ \text{m}} = 1.5\ \text{H}.$$

◆ wxMaxima codes:

```
(%i6) fpprintprec:5; l:0.3; N:2000; A:1.5e-4; Km:600; mu0:float(%pi*4e-7);
(fpprintprec)     5
(l)   0.3
(N)   2000
(A)   1.5*10^-4
(Km)  600
(mu0) 1.2566*10^-6
(%i7) L: Km*mu0*N^2*A/l;
(L)   1.508
```

Comment on the codes:

(%i6) Set floating point print precision to 5, assign values of l, N, A, K_m, and μ_0.
(%i7) Calculate self-inductance L.

(b) Average emf induced in the solenoid is, Eq. (14.1),

$$\mathcal{E} = -L\frac{dI}{dt} = -(1.5\text{ H})\left(\frac{0.10\text{ A} - 0.60\text{ A}}{0.03\text{ s}}\right) = 25\text{ V}.$$

◆ wxMaxima code:

```
(%i2) fpprintprec:5; emf:-1.5*(0.1-0.6)/0.03;
(fpprintprec)     5
(emf) 25.0
```

Comment on the code:

(%i2) Set floating point print precision to 5 and calculate emf.

Problem 14.3 A coil with resistance of 15 Ω and inductance of 0.60 H is connected to a 120 V DC voltage source and a switch. Determine the rate of increase of the current in the coil,

(a) immediately after the switch is closed.
(b) when the current is 90% of its maximum.

Solution

(a) Figure 14.2 shows the coil, the DC voltage source, and the circuit.
The loop equation of the circuit is

$$\mathcal{E} - L\frac{di}{dt} - iR = 0,$$

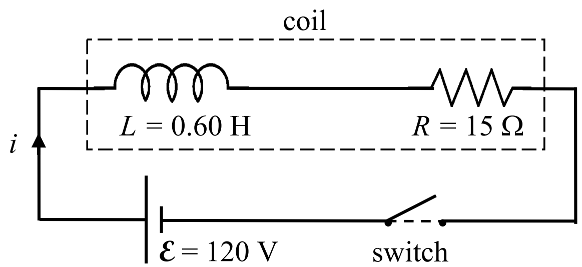


Fig. 14.2 A coil (with inductance and resistance) connected to a voltage source and a switch, Problem 14.3

where $\mathcal{E}$ is the emf of the voltage source, $L\ di/dt$ and iR are potential drops across L and R, respectively. Both L and R are physical properties of the coil. Thus, we write

$$\mathcal{E} = L\frac{di}{dt} + iR.$$

Immediately after the switch is closed, i is zero. The equation becomes

$$\mathcal{E} = L\frac{di}{dt}.$$

Therefore, the rate of change of electric current in the coil at the instance is

$$\frac{di}{dt} = \frac{\mathcal{E}}{L} = \frac{120\ \text{V}}{0.60\ \text{H}} = 200\ \text{A s}^{-1}.$$

(b) Maximum current flows in the circuit some time after the switch is closed. At the instance, current does not change, that is, $di/dt = 0$. The equation gives the maximum current as

$$i_{max} = \frac{\mathcal{E}}{R} = \frac{120\ \text{V}}{15\ \Omega} = 8.0\ \text{A}.$$

The rate of increase of current when the current is 90% of its maximum is calculated as follows:

$$\begin{aligned}\mathcal{E} &= L\frac{di}{dt} + iR\\ 120\ \text{V} &= 0.6\ \text{H} \times \frac{di}{dt} + (0.9 \times 8.0\ \text{A})(15\ \Omega)\\ \frac{di}{dt} &= 20\ \text{A s}^{-1}.\end{aligned}$$

◆ wxMaxima codes:

```
(%i2) ratprint:false; solve(120=0.6*di_dt + 0.9*8*15, di_dt);
(ratprint)  false
(%o2) [di_dt=20]
```

Comments on the codes:

(%i2) Set internal rational number print to false and solve $120 = 0.6 \times di/dt + (0.9 \times 8 \times 15)$ for di/dt.

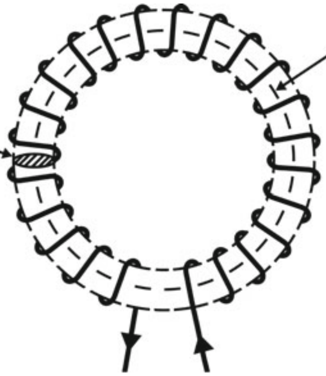


Fig. 14.3 An air core toroid, Problem 14.4

Problem 14.4 An air core toroid has 100 turns of wire, cross-sectional area of $10\ \text{cm}^2$, and average length of 0.50 m.

(a) Calculate the self-inductance of the toroid.
(b) If current in the toroid increases from zero to 1.0 A in 0.10 s, what is the self-induced emf?

Solution

(a) Figure 14.3 shows the air core toroid.
The self-inductance of the toroid is, Eq. (14.4),

$$L = \frac{\mu_0 N^2 A}{l} = \frac{(4\pi \times 10^{-7}\ \text{H m}^{-1})(100)^2(10 \times 10^{-4}\ \text{m}^2)}{0.50\ \text{m}}$$
$$= 2.5 \times 10^{-5}\ \text{H}.$$

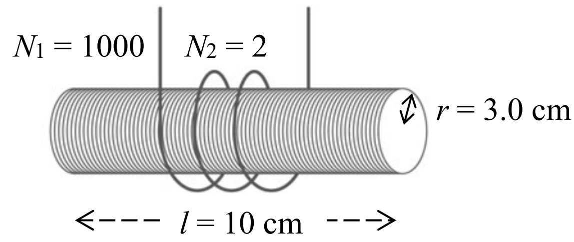
(b) The self-induced emf is, Eq. (14.1),

$$\mathcal{E} = -L\frac{dI}{dt} = -(2.5 \times 10^{-5}\ \text{H})\left(\frac{1.0\ \text{A}}{0.10\ \text{s}}\right) = -2.5 \times 10^{-4}\ \text{V}.$$

◆ wxMaxima codes:

```
(%i7) fpprintprec:5; mu0:float(4e-7*%pi); N:100; A:10e-4; l:0.5; dI:1;
dt:0.1;
(fpprintprec)     5
(mu0) 1.2566*10^-6
(N)   100
(A)   0.001
(l)   0.5
(dI)  1
(dt)  0.1
(%i8) L: mu0*N^2*A/l;
(L)   2.5133*10^-5
(%i9) emf: -L*dI/dt;
(emf) -2.5133*10^-4
```

Fig. 14.4 A solenoid and two turns of coil, Problem 14.5



Comments on the codes:

(%i7) Set floating point print precision to 5, assign values of μ_0, N, A, l, dI, and dt.

(%i8), (%i9) Calculate L and emf.

Problem 14.5 A solenoid of length 10 cm and a radius of 3.0 cm has 1000 turns of wire. Two turns of coil are wound around the solenoid, as shown in Fig. 14.4. Calculate

(a) self-inductance of the solenoid.
(b) mutual inductance of the solenoid and the coil.

Solution

(a) Consider a solenoid of length l with N_1 turns of wire and cross-sectional area $A = \pi r^2$, Fig. 14.4. The magnetic field of the solenoid is

$$B = \frac{\mu_0 N_1 i}{l},$$

when current i flows in the solenoid. The magnetic flux of the solenoid is

$$\Phi_m = BA = \frac{\mu_0 N_1 A i}{l}.$$

The magnetic flux linkage of the solenoid with itself is

$$N_1 \Phi_m = \frac{\mu_0 N_1^2 A i}{l}.$$

Thus, the self-inductance L of the solenoid is

$$L = \frac{\text{magnetic flux linkage}}{i} = \frac{\mu_0 N_1^2 A}{l},$$

as in Eq. (14.4). Substituting the numerical values, the self-inductance L of the solenoid is

$$\begin{aligned} L &= \frac{\mu_0 N_1^2 A}{l} = \frac{(4\pi \times 10^{-7}\ \text{H m}^{-1})(1000)^2 \pi (0.030\ \text{m})^2}{0.10\ \text{m}} \\ &= 3.6 \times 10^{-2}\ \text{H}. \end{aligned}$$

(b) Now consider a coil of N_2 turns wound around the solenoid in part (a), as shown in Fig. 14.4. When current i flows in the solenoid, the magnetic flux linkage of the coil is

$$N_2 \Phi_m = \frac{\mu_0 N_1 N_2 A i}{l}.$$

Thus, the mutual inductance M of the solenoid and the coil is

$$M = \frac{\text{magnetic flux linkage}}{i} = \frac{\mu_0 N_1 N_2 A}{l}.$$

Substituting the numerical values, the mutual inductance M of the solenoid and the coil is

$$\begin{aligned} M &= \frac{\mu_0 N_1 N_2 A}{l} = \frac{(4\pi \times 10^{-7}\ \text{H m}^{-1})(1000)(2)\pi (0.030\ \text{m})^2}{0.10\ \text{m}} \\ &= 7.1 \times 10^{-5}\ \text{H}. \end{aligned}$$

◆ wxMaxima codes:

```
(%i6) fpprintprec:5; mu0:float(4e-7*%pi); l:0.1; A:float(%pi*0.03^2);
N1:1000; N2:2;
(fpprintprec)     5
(mu0) 1.2566*10^-6
(l)   0.1
(A)   0.0028274
(N1)  1000
(N2)  2
(%i7) L: mu0*N1^2*A/l;
(L)   0.035531
(%i8) M: mu0*N1*N2*A/l;
(M)   7.1061*10^-5
```

Comments on the codes:

(%i6) Set floating point print precision to 5, assign values of μ_0, l, A, N_1, and N_2.

(%i7), (%i8) Calculate L and M.

Problem 14.6 The current in the solenoid in Problem 14.5 increases from zero to 4.0 A in 0.10 s. Calculate

(a) self-induced emf in the solenoid
(b) induced emf in the two-turn coil.

Solution

(a) The self-induced emf in the solenoid is

$$\mathcal{E}_1 = -L\frac{dI}{dt} = -(3.6\times 10^{-2}\ \text{H})\left(\frac{4.0\ \text{A}}{0.10\ \text{s}}\right) = -1.4\ \text{V}.$$

(b) The induced emf in the two-turn coil is

$$\mathcal{E}_2 = -M\frac{dI}{dt} = -(7.1\times 10^{-5}\ \text{H})\left(\frac{4.0\ \text{A}}{0.10\ \text{s}}\right) = -2.8\times 10^{-3}\ \text{V}.$$

◆ wxMaxima codes:

```
(%i6) fpprintprec:5; mu0:float(4e-7*%pi); l:0.1; A:float(%pi*0.03^2);
N1:1000; N2:2;
(fpprintprec)     5
(mu0) 1.2566*10^-6
(l)   0.1
(A)   0.0028274
(N1)  1000
(N2)  2
(%i7) L: mu0*N1^2*A/l;
(L)   0.035531
(%i8) M: mu0*N1*N2*A/l;
(M)   7.1061*10^-5
(%i10) dI:4; dt:0.1;
(dI)  4
(dt)  0.1
(%i11) emf_1: -L*dI/dt;
(emf_1) -1.4212
(%i12) emf_2: -M*dI/dt;
(emf_2) -0.0028424
```

Comments on the codes:

(%i6)	Set floating point print precision to 5, assign values of μ_0, l, A, N_1, N_2.
(%i7), (%i8)	Calculate L and M.
(%i10)	Assign dI and dt.
(%i11), (%i12)	Calculate $\mathcal{E}_1$ and $\mathcal{E}_2$.

Problem 14.7 Inductance and resistance of a coil are 0.20 H and 3.0 Ω, respectively. The coil is connected to a dc source of 90 V.

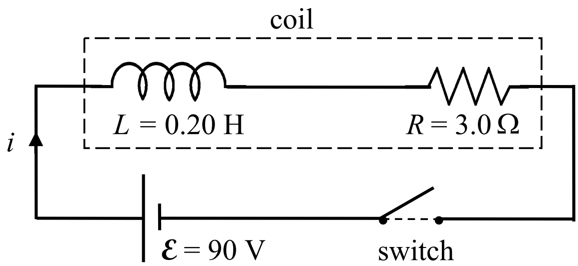


Fig. 14.5 A coil (with inductance and resistance) connected to a voltage source and a switch, Problem 14.7

(a) Calculate the rate of current increase in the coil, immediately after the switch is closed.
(b) What is the current in the coil when the rate of current increase is 100 A s^{-1}?

Solution

The coil, voltage source, and the circuit are shown in Fig. 14.5.

(a) The loop equation of the circuit is

$$\mathcal{E} - L\frac{di}{dt} - iR = 0,$$

where $\mathcal{E}$ is the emf, $L\,di/dt$ is the potential drop across L, and iR is the potential drop across R. This gives

$$\mathcal{E} = L\frac{di}{dt} + iR.$$

Immediately after the switch is closed, that is immediately as the circuit is completed, the current i is zero. The equation becomes

$$\mathcal{E} = L\frac{di}{dt}.$$

Thus, the rate of current increase in the coil at the moment is

$$\frac{di}{dt} = \frac{\mathcal{E}}{L} = \frac{90\text{ V}}{0.20\text{ H}} = 450\text{ A s}^{-1}.$$

(b) The circuit equation is

$$\mathcal{E} = L\frac{di}{dt} + iR.$$

If the rate of change of electric current is known, current in the circuit can be calculated. For the problem, the current through the coil is calculated as follows:

$$\mathcal{E} = L\frac{di}{dt} + iR$$
$$90\text{ V} = (0.20\text{ H})(100\text{ A s}^{-1}) + i(3.0\ \Omega)$$
$$i = 23\text{ A.}$$

♦ wxMaxima codes:

```
(%i5) fpprintprec:5; ratprint:false; emf:90; L:0.2; R:3;
(fpprintprec)     5
(ratprint)  false
(emf) 90
(L)   0.2
(R)   3
(%i6) di_over_dt: emf/L;
(di_over_dt) 450.0
(%i7) di_over_dt: 100;
(di_over_dt) 100
(%i9) solve(emf = L*di_over_dt + i*R, i)$ float(%);
(%o9) [i=23.333]
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of $\mathcal{E}$, L, and R.
(%i6) Calculate *di*/*dt*.
(%i7) Assign *di*/*dt*.
(%i9) Solve $\mathcal{E} = L\frac{di}{dt} + iR$ for i.

Problem 14.8 A coil with inductance of 0.60 H carries a current of 5.0 A. Calculate the energy in the coil.

Solution

The coil is assumed to be a pure inductor and has negligible resistance. Energy in the magnetic field of the inductor is, Eq. (14.8),

$$U_m = \frac{1}{2}LI^2 = \frac{1}{2}(0.60\text{ H})(5.0\text{ A})^2 = 7.5\text{ J.}$$

♦ wxMaxima codes:

```
(%i3) fpprintprec:5; L:0.6; I:5;
(fpprintprec)     5
(L)   0.6
(I)   5
(%i4) Um: 0.5*L*I^2;
(Um)  7.5
```

Comments on the codes:

(%i3) Set floating point print precision to 5, assign values of L and I.
(%i4) Calculate U_m.

Problem 14.9

(a) A coil has a self-inductance of 0.009 H. Calculate the back emf induced in the coil when the current in the coil is increasing at a rate of 110 A s^{-1}.
(b) What is the energy in the coil when the current is 6.0 A?

Solution

(a) Induced back emf is

$$\mathcal{E} = -L\frac{di}{dt} = -(0.009\ \text{H})(110\ \text{A s}^{-1}) = -0.99\ \text{V}.$$

(b) Energy in the coil is

$$U_m = \frac{1}{2}LI^2 = \frac{1}{2}(0.009\ \text{H})(6.0\ \text{A})^2 = 0.16\ \text{J}.$$

◆ wxMaxima codes:

```
(%i3) fpprintprec:5; L:0.009; di_over_dt:110;
(fpprintprec)     5
(L)   0.009
(di_over_dt) 110
(%i4) emf: -L*di_over_dt;
(emf) -0.99
(%i5) I:6;
(I)   6
(%i6) Um: 0.5*L*I^2;
(Um)  0.162
```

Comments on the codes:

(%i3) Set floating point print precision to 5, assign values of L and di/dt.
(%i4) Calculate back emf.
(%i5), (%i6) Assign I and calculate U_m.

Problem 14.10

(a) Show that for an RL circuit of Fig. 14.6, the current and the rate of change of current as the switch is closed are

$$i = \frac{\mathcal{E}}{R}(1 - e^{-Rt/L}),$$

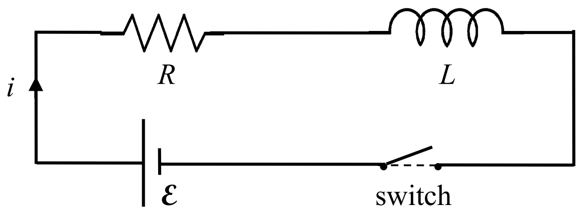


Fig. 14.6 An *RL* circuit, Problem 14.10

$$\frac{di}{dt} = \frac{\mathcal{E}}{L} e^{-Rt/L}.$$

In the figure, $\mathcal{E}$ is the emf, R the resistance, L the inductance, and i the electric current.

(b) For $\mathcal{E} = 100$ V, $R = 5.0\ \Omega$, and $L = 0.20$ H, plot curves of current against time and rate of change of current against time.

Solution

(a) At any time as the switch is closed, the potential difference across the resistor is iR and across the inductor is $L\, di/dt$. Thus, for the circuit,

$$\mathcal{E} = iR + L\frac{di}{dt}.$$

The rate of change of current is

$$\frac{di}{dt} = \frac{\mathcal{E}}{L} - \frac{R}{L}i.$$

The equation is written as

$$\frac{di}{(\mathcal{E}/R) - i} = \frac{R}{L}dt.$$

At $t = 0$, $i = 0$, so integration gives

$$\int_0^i \frac{di}{(\mathcal{E}/R) - i} = \int_0^t \frac{R}{L}dt,$$

$$[-\ln\,(\mathcal{E}/R - i)]_0^i = \left[\frac{R}{L}t\right]_0^t,$$

$$-\ln(\mathcal{E}/R - i) + \ln(\mathcal{E}/R) = \frac{R}{L}t,$$

$$\ln(\mathcal{E}/R - i) - \ln(\mathcal{E}/R) = -\frac{R}{L}t,$$

$$\ln(1 - Ri/\mathcal{E}) = -\frac{R}{L}t,$$

$$1 - \frac{R}{\mathcal{E}}i = e^{-Rt/L},$$

$$i = \frac{\mathcal{E}}{R}(1 - e^{-Rt/L}).$$

This is the equation for current i against time t. Initially the current is zero, and it increases to a steady value of $\mathcal{E}/R$.

Differentiating the equation with respect to time gives the rate of change of current di/dt,

$$\begin{aligned}\frac{di}{dt} &= \frac{\mathcal{E}}{R}(-e^{-Rt/L})\ (-R/L)\\ &= \frac{\mathcal{E}}{L}e^{-Rt/L}.\end{aligned}$$

(b) For $\mathcal{E} = 100$ V, $R = 5.0\ \Omega$, and $L = 0.20$ H, the curve of current against time is

$$\begin{aligned}i &= \frac{\mathcal{E}}{R}(1 - e^{-Rt/L})\\ &= \frac{100}{5}(1 - e^{-5t/0.2})\\ &= 20\ (1 - e^{-25t})\ \text{A}.\end{aligned}$$

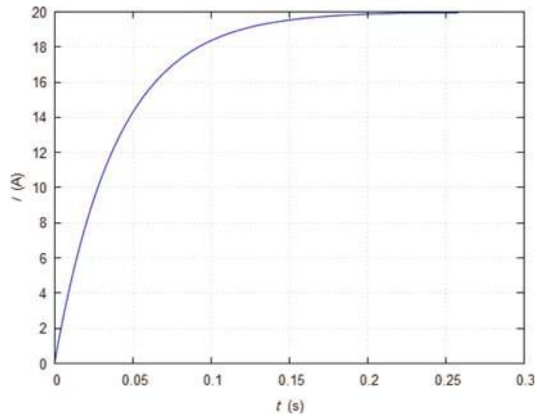
The curve of rate of change of current against time is

$$\begin{aligned}\frac{di}{dt} &= \frac{\mathcal{E}}{L}e^{-Rt/L}\\ &= \frac{100}{0.2}e^{-5t/0.2}\\ &= 500\ e^{-25t}\ \text{A s}^{-1}.\end{aligned}$$

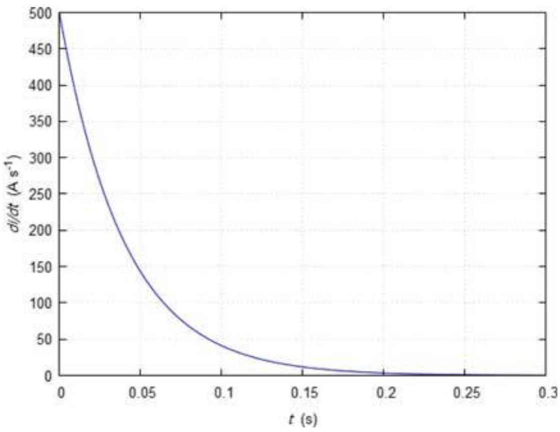
Curves of current against time and rate of current change against time are plotted by wxMaxima.

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; emf:100; R:5; L:0.2;
(fpprintprec)     5
(emf) 100
(R)   5
(L)   0.2
(%i5) i: emf/R*(1-exp(-R*t/L));
(i)   20*(1-%e^(-25.0*t))
(%i6) di_over_dt: emf/L*exp(-R*t/L);
(di_over_dt) 500.0*%e^(-25.0*t)
(%i7) wxplot2d(i, [t,0,0.3], grid2d, [xlabel,"{/Helvetica-Italic t}  (s)"],
[ylabel,"{/Helvetica-Italic i}  (A)"]);
```




```
(%i8) wxplot2d(di_over_dt, [t,0,0.3], grid2d, [xlabel,"{/Helvetica-Italic
t}  (s)"], [ylabel,"{/Helvetica-Italic di/dt}  (A s^{-1})"]);
```



Comments on the codes:

(%i4)	Set floating point print precision to 5, assign values of $\mathcal{E}$, *R*, and *L*.
(%i5), (%i6)	Define *i* and *di*/*dt*.
(%i7)	Plot curve of *i* against *t* for $0 \leq t \leq 0.3$ s.
(%i8)	Plot curve of *di*/*dt* against *t* for $0 \leq t \leq 0.3$ s.

Alternative solution: For an *RL* circuit of Fig. 14.6, when the switch is closed, the circuit equation is

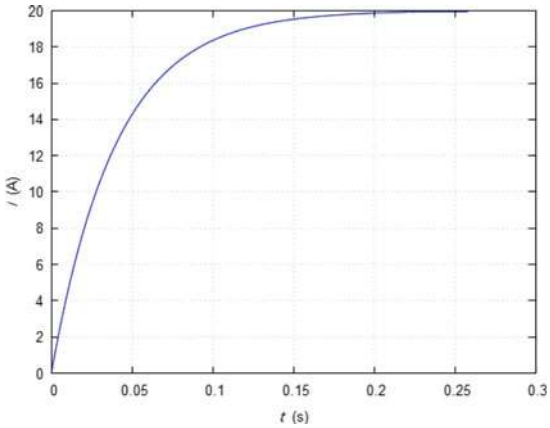
$$\mathcal{E} = iR + L\frac{di}{dt}. \tag{14.17}$$

Here, *iR* is the potential difference across the resistor, *L di*/*dt* is the potential difference across the inductor, and $\mathcal{E}$ is the emf of the cell. The initial condition is, at $t = 0$ s, $i = 0$ A. Equation (14.17) is a first-order ordinary differential equation (ODE), with *i* as dependent variable and *t* as independent variable.

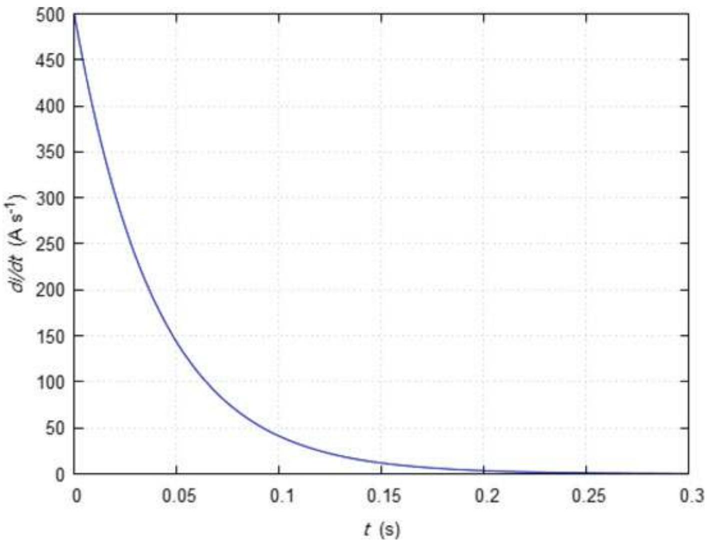
To solve the ODE, predefined functions *ode*2 and *ic*1 of wxMaxima can be used. See *Solving first-order ordinary differential equation* in Appendix A.

◆ wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)     5
(ratprint)  false
(%i4) sol: ode2(emf=i*R + L*'diff(i,t), i,t)$ expand(%);
(%o4) i=%c*%e^(-(R*t)/L)+emf/R
(%i6) ic1(sol, t=0, i=0)$ expand(%);
(%o6) i=emf/R-(emf*%e^(-(R*t)/L))/R
(%i7) i: rhs(%);
(i)   emf/R-(emf*%e^(-(R*t)/L))/R
(%i8) di_over_dt: diff(i,t);
(di_over_dt) (emf*%e^(-(R*t)/L))/L
(%i11) emf: 100; R:5; L:0.2;
(emf) 100
(R)   5
(L)   0.2
(%i12) wxplot2d(i, [t,0,0.3], grid2d,[xlabel,"{/Helvetica-Italic t} (s)"],
[ylabel,"{/Helvetica-Italic i}(A)"]);
```




```
(%i13) wxplot2d(di_over_dt, [t,0,0.3], grid2d, [xlabel,"{/Helvetica-Italic
t}  (s)"], [ylabel,"{/Helvetica-Italic di/dt}  (A s^{-1})"]);
```



Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Solve $\mathcal{E} = iR + L\frac{di}{dt}$ and get a general solution.
(%i6) Set the initial condition and get a particular solution.
(%i7) Assign the solution to i.
(%i8) Calculate di/dt.
(%i11) Assign values of $\mathcal{E}$, R, and L.
(%i12) Plot i against t for $0 \le t \le 0.3$ s.
(%i13) Plot di/dt against t for $0 \le t \le 0.3$ s.

Problem 14.11 After the *RL* circuit of Problem 14.10 attains steady current, the voltage source is removed and the circuit is completed. Obtain an expression for the current decay against time and draw the curve.

Solution

When the voltage source is removed and the circuit is completed, $\mathcal{E} = 0$, and we have

$$\mathcal{E} = iR + L\frac{di}{dt},$$
$$0 = iR + L\frac{di}{dt}.$$

The equation is written as

$$\frac{di}{i} = -\frac{R}{L}dt.$$

At $t = 0$, $i = \mathcal{E}/R$, and integration gives

$$\int_{\mathcal{E}/R}^{i} \frac{di}{i} = \int_{0}^{t} -\frac{R}{L}dt,$$

$$[\ln\, i]\,_{\mathcal{E}/R}^{i} = \left[-\frac{R}{L}t\right]_0^t,$$

$$\ln\left(\frac{i}{\mathcal{E}/R}\right) = -Rt/L,$$

$$i = \frac{\mathcal{E}}{R}e^{-Rt/L}.$$

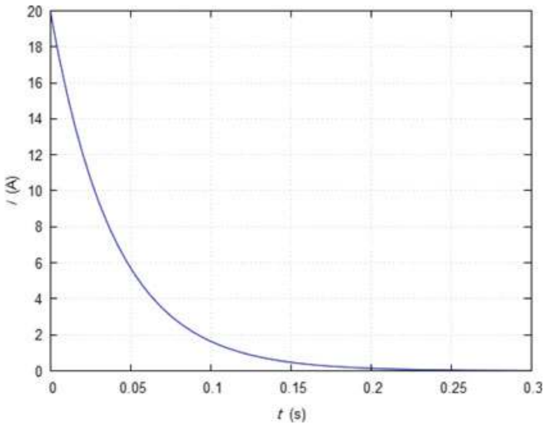
The expression is an exponential decay of current with time, with an initial current of $\mathcal{E}/R$. Using numerical values of Problem 14.10, that is, $\mathcal{E} = 100$ V, $R = 5.0\ \Omega$, and $L = 0.20$ H, one gets

$$\begin{aligned} i &= \frac{\mathcal{E}}{R}e^{-Rt/L} \\ &= \frac{100}{5}e^{-5t/0.2} \\ &= 20\ e^{-25t}\ \text{A}. \end{aligned}$$

A curve of electric current against time is obtained by wxMaxima.

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; emf:100; R:5; L:0.2;
(fpprintprec)     5
(emf) 100
(R)   5
(L)   0.2
(%i5) i: emf/R*exp(-R*t/L);
(i)   20*%e^(-25.0*t)
(%i6) wxplot2d(i, [t,0,0.3], grid2d, [xlabel,"{/Helvetica-Italic t}  (s)"],
[ylabel,"{/Helvetica-Italic i}  (A)"]);
```



Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of $\mathcal{E}$, R, and L.
(%i5), (%i6) Assign i and plot i against t for $0 \le t \le 0.3$ s.

Alternative solution: The circuit equation is

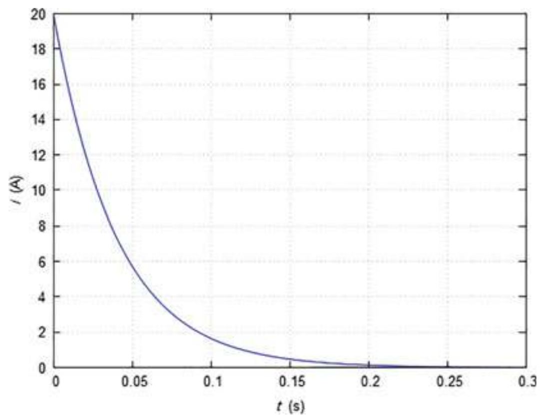
$$0 = iR + L\frac{di}{dt},$$

and the initial condition is $t = 0$ s, $i = \mathcal{E}/R$. Here, iR is the potential difference across the resistor and $L\,di/dt$ is the potential difference across the inductor. The equation is a first-order ordinary differential equation (ODE), with i as dependent variable and t as independent variable.

To solve the ODE, predefined functions *ode*2 and *ic*1 of wxMaxima can be used. See *Solving first-order ordinary differential equation* in Appendix A.

◆ wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)     5
(ratprint)  false
(%i4) sol: ode2(0=i*R + L*'diff(i,t), i,t)$ expand(%);
(%o4) i=%c*%e^(-(R*t)/L)
(%i6) ic1(sol, t=0, i=emf/R)$ expand(%);
(%o6) i=(emf*%e^(-(R*t)/L))/R
(%i7) i: rhs(%);
(i)   (emf*%e^(-(R*t)/L))/R
(%i10) emf: 100; R:5; L:0.2;
(emf) 100
(R)   5
(L)   0.2
(%i11) wxplot2d(i, [t,0,0.3], grid2d, [xlabel,"{/Helvetica-Italic t}
(s)"], [ylabel,"{/Helvetica-Italic i}  (A)"]);
```



Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Solve $0 = iR + L\ di/dt$ and get a general solution.
(%i6) Set the initial condition and get a particular solution.
(%i7) Assign the solution to *i*.
(%i10) Assign values of $\mathcal{E}$, *R*, and *L*.
(%i11) Plot *i* against *t* for $0 \leq t \leq\ 0.3$ s.

14.3 Summary

- Electric current changes in a coil induce an emf $\mathcal{E}$ in the coil itself,

$$\mathcal{E} = -L\frac{dI}{dt},$$

where *L* is the self-inductance of the inductor (coil) and *dI*/*dt* is the rate of change of current through it. By Faraday's law, the induced emf is also the time rate of magnetic flux change,

$$\mathcal{E} = -N\frac{d\Phi_m}{dt}.$$

Thus,

$$L = N\left|\frac{d\Phi_m}{dI}\right|.$$

This means that inductance is change of magnetic flux per unit current.

- For a solenoid, the self-inductance is

$$L = N\frac{\Phi_m}{I} = \frac{NBA}{I} = \frac{N(\mu_0 nI)A}{I} = \frac{N\mu_0 IA}{I}\left(\frac{N}{l}\right) = \frac{\mu_0 N^2 A}{l}.$$

- A change in current dI_1/dt in circuit 1 induces an emf $\mathcal{E}_2$ in circuit 2,

$$\mathcal{E}_2 = -M\frac{dI_1}{dt}.$$

where M is the mutual inductance between the two circuits.
- The energy U stored in an inductor is

$$U = \frac{1}{2}LI^2.$$

14.4 Exercises

Exercise 14.1 A solenoid of length 10 cm and cross-sectional area 1.0 cm^2 has 1000 turns of wire per meter. Calculate the inductance of the solenoid.
(Answer: $L = 1.3 \times 10^{-5}$ H)

Exercise 14.2 Figure 14.7 shows an *RL* circuit with resistor $R = 5.0\ \Omega$, inductor $L = 3.0 \times 10^{-2}$ H, and battery of emf $\mathcal{E} = 60$ V. The switch is closed at $t = 0$ s, find

(a) the time constant of the circuit
(b) the current in the circuit at $t = 3.0 \times 10^{-3}$ s.
(c) the energy stored in the inductor at $t = 3.0 \times 10^{-3}$ s.

(Answer: (a) 6.0×10^{-3} s; (b) 4.7 A; (c) 0.33 J)

Exercise 14.3 An emf of 10 V is induced in a coil when the current in it changes at the rate of 32 A s^{-1}. What is the inductance of the coil?
(Answer: $L = 0.31$ H)

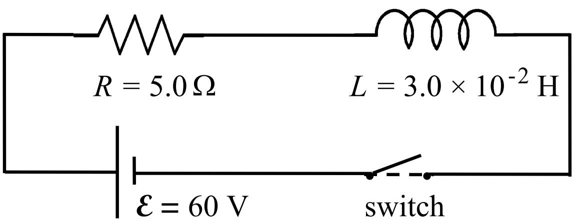


Fig. 14.7 An *RL* circuit, Exercise 14.2

Exercise 14.4 A current of 3.0 A creates a magnetic flux of 1.4×10^{-4} Wb in a coil of 500 turns. What is the inductance of the coil?

(Answer: $L = 0.023$ H)

Exercise 14.5 The average emf induced in a circuit is 250 V when the current in the circuit changes from 24 A to zero in 3.0×10^{-3} s. Calculate the self-inductance of the circuit and the energy stored in the magnetic field.

(Answer: $L = 0.031$ H, $U = 9.0$ J)

Chapter 15
Alternating Current Circuit

Abstract This chapter solves problems on series *RLC* alternating current circuits. Inductive and capacitive reactance, impedance, phase angle, power factor, root mean square current, and average power of the circuits are determined. Solutions by analysis and computer calculation are presented.

15.1 Basic Concepts and Formulae

(1) In an alternating current (AC) circuit having a voltage generator and a resistor, electric current is in phase with the voltage. The voltage and the current attain peak values at the same time. The root mean square current I_{rms} and the root mean square voltage V_{rms} of sinusoidal current and voltage are

$$I_{\text{rms}} = \frac{I_{\max}}{\sqrt{2}} = 0.707 I_{\max}, \tag{15.1}$$

$$V_{\text{rms}} = \frac{V_{\max}}{\sqrt{2}} = 0.707 V_{\max}, \tag{15.2}$$

where $I_{\max}$ and $V_{\max}$ are peak (maximum) current and peak (maximum) voltage, respectively. $I_{\max}$ and $V_{\max}$ are also called current amplitude and voltage amplitude, respectively.

(2) In an AC circuit having a voltage generator and an inductor, the current lags the voltage by 90°. The voltage attains a maximum value a quarter of a period earlier than the current.

(3) In an AC circuit having a voltage generator and a capacitor, the current leads the voltage by 90°. The current attains a maximum value a quarter of a period earlier than the voltage.

(4) In an AC circuit having an inductor and a capacitor, inductive reactance X_L and capacitive reactance X_C are defined as,

$$X_L = \omega L = 2\pi f L, \tag{15.3}$$

W. M. S. Wan Hassan et al., *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-43165-4_15

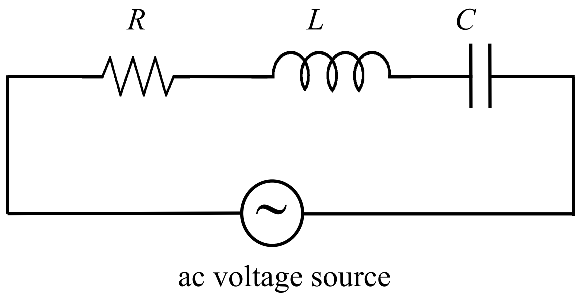


Fig. 15.1 Alternating current series *RLC* circuit

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}, \tag{15.4}$$

where f and ω are frequency and angular frequency of the AC voltage source, respectively.

(5) In an AC series *RLC* circuit, as shown in Fig. 15.1, a circuit that has a resistor, an inductor, and a capacitor in series connected to an AC voltage source, the impedance Z is,

$$Z = \sqrt{R^2 + (X_L - X_C)^2}. \tag{15.5}$$

The voltage and the current in the circuit differ in phase by phase angle ϕ, where,

$$\tan\phi = \frac{X_L - X_C}{R}. \tag{15.6}$$

This means that ϕ is the phase angle between the voltage and the current.

Average power of the circuit is,

$$\begin{aligned} P_{\text{average}} &= I_{\text{rms}} V_{\text{rms}} \cos\phi = Z\ I_{\text{rms}}^2 \cos\phi = I_{\text{rms}}^2 R \\ &= \frac{1}{2} I_{\max} V_{\max} \cos\ \phi. \end{aligned} \tag{15.7}$$

This is the power output of the resistor in the series *RLC* circuit. There is no loss of energy by the pure inductor and capacitor.

The quantity $\cos\phi$ is called the power factor,

$$\text{Power factor} = \cos\phi = \frac{R}{Z} = \frac{R}{\sqrt{R^2 + (X_L - X_C)^2}}. \tag{15.8}$$

The root mean square current is

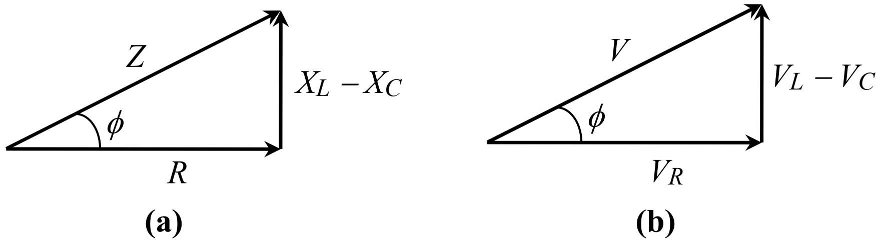


Fig. 15.2 **a** In an AC series *RLC* circuit, impedance *Z*, resistance *R*, inductive reactance X_L, capacitive reactance X_C, and phase angle ϕ can be represented by vectors, so do **b** voltages across the circuit *V*, across the resistor V_R, across the inductor V_L, and across the capacitor V_C

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{Z} = \frac{V_{\text{rms}}}{\sqrt{R^2 + (X_L - X_C)^2}}. \tag{15.9}$$

(6) In an AC series *RLC* circuit, impedance *Z*, resistance *R*, inductive reactance X_L, capacitive reactance X_C, and phase angle ϕ can be represented by vectors as shown in Fig. 15.2a. Similarly, voltages across the circuit *V*, across the resistor V_R, across the inductor V_L, and across the capacitor V_C can be represented by vectors as shown in Fig. 15.2b.

(7) An AC series *RLC* circuit is in resonance when inductive reactance is equal to capacitive reactance. The current is $I_{\text{rms}} = V_{\text{rms}}/R$, $X_L = X_C$, and the resonant frequency of the circuit is,

$$\omega_0 = \frac{1}{\sqrt{LC}}. \tag{15.10}$$

The current of an AC series *RLC* circuit is a maximum when angular frequency of the AC generator is equal to ω_0, that is, when the angular frequency of the generator is the same as the resonant frequency.

15.2 Problems and Solutions

Problem 15.1 When an AC voltmeter is connected across the terminals of an AC source of frequency 50 Hz, the reading is 160 V. Write an equation for the voltage of the AC source.

Solution

General equation for an AC voltage source is,

$$V = V_0 \sin(\omega t + \phi),$$

where V_0 is voltage amplitude, ω is angular frequency, and ϕ is phase angle. The angular frequency is,

$$\omega = 2\pi f = 2\pi(50\ \text{s}^{-1}) = 314\ \text{rad s}^{-1}.$$

The AC voltmeter measures effective voltage or the root mean square voltage V_{rms} of the voltage source. The relation between V_{rms} with the voltage amplitude V_0 is,

$$V_0 = \sqrt{2}\ V_{\text{rms}}.$$

This means that,

$$V_0 = \sqrt{2} \times 160\ \text{V} = 226\ \text{V}.$$

The equation for the AC voltage source is

$$V = 226 \sin(314t + \phi).$$

◆ wxMaxima codes:

```
(%i1) fpprintprec:5;
(fpprintprec)     5
(%i2) omega: float(2*%pi*50);
(omega) 314.16
(%i3) V0: float(sqrt(2)*160);
(V0)  226.27
```

Comments on the codes:

(%i1)	Set floating point print precision to 5.
(%i2), (%i3)	Calculate ω and V_0.

Problem 15.2 A 30 Ω resistor is connected in series with an AC ammeter A and a voltage source $V = 60 \sin(100\pi t)$. What is the reading of the ammeter?

Solution

The relevant circuit is shown in Fig. 15.3.

The AC voltage source is

$$V = V_0 \sin(\omega t) = 60 \sin(100\pi t).$$

The voltage amplitude or the peak voltage is $V_0 = 60$ V. The rms voltage across the resistor is,

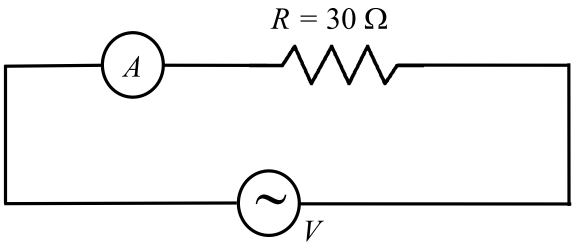


Fig. 15.3 A resistor connected to an ammeter and a voltage source, Problem 15.2

$$V_{\text{rms}} = \frac{V_0}{\sqrt{2}} = \frac{60\ \text{V}}{\sqrt{2}} = 42\ \text{V}.$$

Using Ohm's law, the rms current is,

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{R} = \frac{42\ \text{V}}{30\ \Omega} = 1.4\ \text{A}.$$

This is the reading of the AC ammeter.

◆ wxMaxima codes:

```
(%i3) fpprintprec:5; V0:60; R:30;
(fpprintprec)     5
(V0)   60
(R)    30
(%i4) Vrms: float(V0/sqrt(2));
(Vrms) 42.426
(%i5) Irms: float(Vrms/R);
(Irms) 1.4142
```

Comments on the codes:

(%i3) Set floating point print precision to 5, assign values of V_0 and R.
(%i4), (%i5) Calculate V_{rms} and I_{rms}.

Problem 15.3 A 50 Ω resistor is connected to a 15 V variable frequency voltage generator. Calculate the current in the resistor when the frequency of the voltage generator is

(a) 100 Hz.
(b) 100 kHz.

Solution

Figure 15.4 shows the resistor, the voltage generator, and the complete circuit.

(a) For a pure resistor, Ohm's law is obeyed, $V = IR$. The voltage across the resistor and the current in it do not depend on frequency of the source. When the

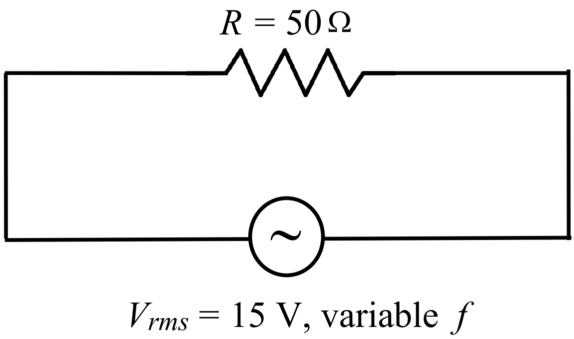


Fig. 15.4 A resistor connecter to an AC variable frequency voltage generator, Problem 15.3

frequency of the voltage generator is $f = 100$ Hz, the I_{rms} of the resistor or the I_{rms} in the circuit is

$$I_{\mathrm{rms}} = \frac{V_{\mathrm{rms}}}{R} = \frac{15\ \mathrm{V}}{50\ \Omega} = 0.30\ \mathrm{A}.$$

(b) When the frequency of the voltage generator is $f = 100$ kHz, the I_{rms} is the same,

$$I_{\mathrm{rms}} = \frac{V_{\mathrm{rms}}}{R} = \frac{15\ \mathrm{V}}{50\ \Omega} = 0.30\ \mathrm{A}.$$

◆ wxMaxima codes:

```
(%i3) fpprintprec:5; Vrms:15; R:50;
(fpprintprec)     5
(Vrms) 15
(R)   50
(%i4) Irms: float(Vrms/R);
(Irms) 0.3
```

Comments on the codes:

(%i3) Set floating point print precision to 5, assign values of V_{rms} and R.
(%i4) Calculate I_{rms}.

Problem 15.4 A 2.0 mH inductor is connected to a 15 V variable frequency voltage generator. Calculate the current in the circuit when the frequency of the voltage generator is

(a) 100 Hz.
(b) 100 kHz.

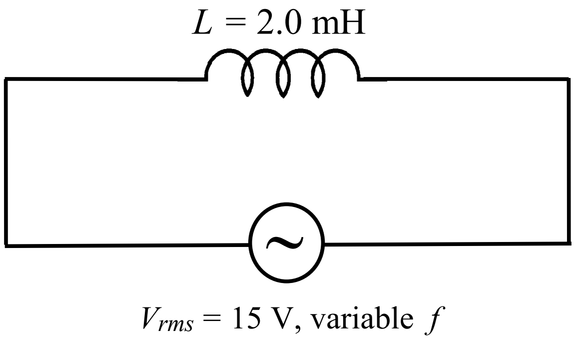


Fig. 15.5 An inductor connector to an AC variable frequency voltage source, Problem 15.4

Solution

Figure 15.5 shows the inductor, the variable frequency voltage source, and the complete circuit.

(a) The inductive reactance of the inductor at $f = 100$ Hz is

$$X_L = \omega L = 2\pi f L = 2\pi(100\ \text{s}^{-1})(2.0 \times 10^{-3}\ \text{H}) = 1.3\ \Omega.$$

Therefore, the current in the circuit is

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{X_L} = \frac{15\ \text{V}}{1.3\ \Omega} = 12\ \text{A}.$$

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; f:100; L:2e-3; Vrms:15;
(fpprintprec)     5
(f)   100
(L)   0.002
(Vrms) 15
(%i5) XL: float(2*%pi*f*L);
(XL)  1.2566
(%i6) Irms: float(Vrms/XL);
(Irms) 11.937
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of f, L, and V_{rms}.
(%i5), (%i6) Calculate X_L and I_{rms}.

(b) Inductive reactance of the inductor at $f = 100$ kHz is,

$$X_L = \omega L = 2\pi f L = 2\pi(100 \times 10^3\ \text{s}^{-1})(2.0 \times 10^{-3}\ \text{H}) = 1.3 \times 10^3\ \Omega.$$

The current in the circuit is,

$$I_{\rm rms} = \frac{V_{\rm rms}}{X_L} = \frac{15\ \rm V}{1.3 \times 10^3\ \Omega} = 1.2 \times 10^{-2}\ \rm A.$$

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; f:100e3; L:2e-3; Vrms:15;
(fpprintprec)     5
(f)   1.0*10^5
(L)   0.002
(Vrms) 15
(%i5) XL: float(2*%pi*f*L);
(XL)  1256.6
(%i6) Irms: float(Vrms/XL);
(Irms) 0.011937
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of f, L, and $V_{\rm rms}$.
(%i5), (%i6) Calculate X_L and $I_{\rm rms}$.

Problem 15.5 A 0.30 μF capacitor is connected to a 15 V variable frequency voltage generator. Calculate the current in the circuit when the frequency of the voltage generator is

(a) 100 Hz.
(b) 100 kHz.

Solution

Figure 15.6 shows the capacitor, the voltage generator, and the complete circuit.

(a) The capacitive reactance of the capacitor at $f = 100$ Hz is

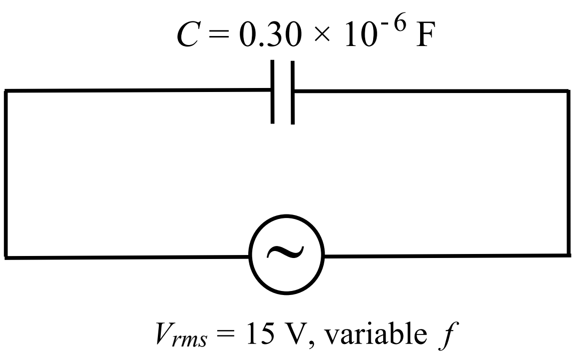


Fig. 15.6 A capacitor connected to an AC variable frequency voltage generator, Problem 15.5

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C} = \frac{1}{2\pi(100\ \text{s}^{-1})(0.30 \times 10^{-6}\ \text{F})} = 5.3 \times 10^3\ \Omega.$$

Thus, the current in the circuit is

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{X_C} = \frac{15\ \text{V}}{5.3 \times 10^3\ \Omega} = 2.8 \times 10^{-3}\ \text{A}.$$

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; f:100; C:0.3e-6; Vrms:15;
(fpprintprec)     5
(f)   100
(C)   3.0*10^-7
(Vrms) 15
(%i5) XC: 1/float(2*%pi*f*C);
(XC)  5305.2
(%i6) Irms: float(Vrms/XC);
(Irms) 0.0028274
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of f, C, and V_{rms}.
(%i5), (%i6) Calculate X_C and I_{rms}.

(b) The capacitive reactance of the capacitor at $f = 100$ kHz is

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C} = \frac{1}{2\pi(100 \times 10^3\ \text{s}^{-1})(0.30 \times 10^{-6}\ \text{F})} = 5.3\ \Omega.$$

The current in the circuit is,

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{X_C} = \frac{15\ \text{V}}{5.3 \times 10^6\ \Omega} = 2.8\ \text{A}.$$

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; f:100e3; C:0.3e-6; Vrms:15;
(fpprintprec)     5
(f)   1.0*10^5
(C)   3.0*10^-7
(Vrms) 15
(%i5) XC: 1/float(2*%pi*f*C);
(XC)  5.3052
(%i6) Irms: float(Vrms/XC);
(Irms) 2.8274
```

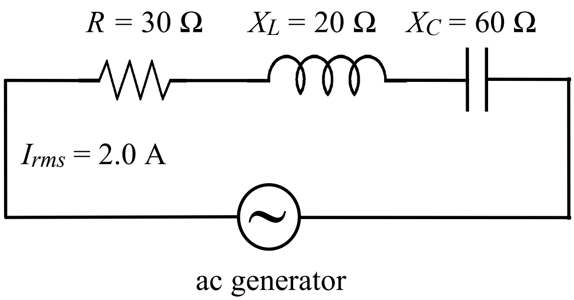


Fig. 15.7 Alternating current series *RLC* circuit, Problem 15.6

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of f, C, and V_{rms}.
(%i5), (%i6) Calculate X_C and I_{rms}.

Problem 15.6 Resistance, inductive reactance, capacitive reactance, and effective current of an ac series *RLC* circuit are 30, 20, 60 Ω, and 2.0 A, respectively. For the circuit calculate

(a) impedance.
(b) power factor.
(c) power dissipated by the resistor.
(d) maximum voltages across the resistor, inductor, and capacitor.

Solution

Figure 15.7 is the circuit meant by the question.

(a) The impedance of the circuit is

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(30\ \Omega)^2 + (20\ \Omega - 60\ \Omega)^2} = 50\ \Omega.$$

(b) The power factor is obtained by calculation of phase angle ϕ followed by calculation of cosine of the angle,

$$\tan\phi = \frac{X_L - X_C}{R} = \frac{20\ \Omega - 60\ \Omega}{30\ \Omega} = -4/3, \quad \phi = -53^{\circ},$$

Power factor $= \cos\phi = \cos(-53°) = 0.6$.
The power factor can also be calculated as follows:
Power factor $= \cos\phi = \frac{R}{Z} = \frac{30\ \Omega}{50\ \Omega} = 0.6$.

(c) The power dissipated by the resistor is

$$P = I_{\text{rms}}^2 R = (2.0\ \text{A})^2(30\ \Omega) = 120\ \text{W}.$$

It can also be calculated as follows:

$$P = Z\, I_{\text{rms}}^2 \cos\phi = (50\ \Omega)(2.0\ \text{A})^2(0.6) = 120\ \text{W}.$$

(d) The maximum voltage across the resistor is calculated as follows:

$$V_{R,\text{rms}} = I_{\text{rms}} R = (2.0\ \text{A})(30\ \Omega) = 60\ \text{V},$$
$$V_{R,\text{max}} = \sqrt{2} V_{R,\text{rms}} = \sqrt{2}(60\ \text{V}) = 85\ \text{V}.$$

The maximum voltage across the inductor is calculated as follows:

$$V_{L,\text{rms}} = I_{\text{rms}} X_L = (2.0\ \text{A})(20\ \Omega) = 40\ \text{V},$$
$$V_{L,\text{max}} = \sqrt{2} V_{l,\text{rms}} = \sqrt{2}(40\ \text{V}) = 57\ \text{V}.$$

The maximum voltage across the capacitor is calculated as follows:

$$V_{C,\text{rms}} = I_{\text{rms}} X_C = (2.0\ \text{A})(60\ \Omega) = 120\ \text{V},$$
$$V_{C,\text{max}} = \sqrt{2} V_{C,\text{rms}} = \sqrt{2}(120\ \text{V}) = 170\ \text{V}.$$

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; R:30; XL:20; XC:60; Irms:2;
(fpprintprec)     5
(R)   30
(XL)  20
(XC)  60
(Irms) 2
(%i6) Z: sqrt(R^2 + (XL-XC)^2);
(Z)   50
(%i8) phi: float(atan((XL-XC)/R)); phi_deg: float(phi*180/%pi);
(phi) -0.9273
(phi_deg)   -53.13
(%i9) power_factor: float(cos(phi));
(power_factor)    0.6
(%i10) power_factor1: float(R/Z);
(power_factor1)   0.6
(%i11) P: float(Irms^2*R);
(P)   120.0
(%i12) P1: float(Z*Irms^2*cos(phi));
(P1)  120.0
(%i13) VRrms: float(Irms*R);
(VRrms) 60.0
(%i14) VRmax: float(sqrt(2)*VRrms);
(VRmax) 84.853
(%i15) VLrms: float(Irms*XL);
(VLrms) 40.0
(%i16) VLmax: float(sqrt(2)*VLrms);
(VLmax) 56.569
(%i17) VCrms: float(Irms*XC);
(VCrms) 120.0
(%i18) VCmax: float(sqrt(2)*VCrms);
(VCmax) 169.71
```

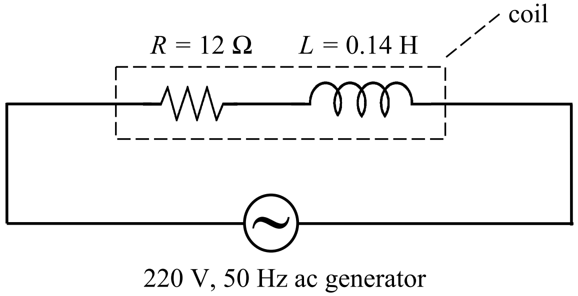


Fig. 15.8 Alternating current series *RL* circuit, Problem 15.7

Comments on the codes:

(%i5)	Set floating point print precision to 5, assign values of R, X_L, X_C, and I_{rms}.
(%i6), (%i8), (%i9), (%i10)	Calculate Z, ϕ, and power factors.
(%i11), (%i12)	Calculate power dissipated.
(%i13), (%i14)	Calculate $V_{R,\text{rms}}$ and $V_{R,\text{max}}$.
(%i15), (%i16)	Calculate $V_{L,\text{rms}}$ and $V_{L,\text{max}}$.
(%i17), (%i18)	Calculate $V_{C,\text{rms}}$ and $V_{C,\text{max}}$.

Problem 15.7 A coil with 0.14 H inductance and 12 Ω resistance is connected to a 220 V, 50 Hz AC source. Calculate

(a) current in the coil.
(b) phase angle between voltage and current.
(c) power factor.
(d) loss of electrical power of the coil.

Solution

Figure 15.8 shows the coil connected to the AC source. The coil has both resistance and inductance.

(a) The inductive reactance of the coil is

$$X_L = \omega L = 2\pi f L = 2\pi(50\ \text{s}^{-1})(0.14\ \Omega) = 44\ \Omega.$$

The impedance of the circuit is

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(12\ \Omega)^2 + (44\ \Omega - 0)^2} = 46\ \Omega.$$

Thus, the current in the coil is

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{Z} = \frac{220\ \text{V}}{46\ \Omega} = 4.8\ \text{A}.$$

(b) The phase angle between voltage and current is

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) = \tan^{-1}\left(\frac{44\ \Omega - 0}{12\ \Omega}\right) = 1.3\ \text{rad} = 75^\circ.$$

(c) The power factor is

$$\text{power factor} = \cos\phi = \cos 75^\circ = 0.26.$$

(d) The loss of electrical power is

$$P = I_{rms}^2 R = (4.8\ \text{A})^2(12\ \Omega) = 280\ \text{W}.$$

◆ wxMaxima codes:

```
(%i6) fpprintprec:5; L:0.14; R:12; Vrms:220; f:50; XC:0;
(fpprintprec)     5
(L)   0.14
(R)   12
(Vrms) 220
(f)   50
(XC)  0
(%i7) XL: float(2*%pi*f*L);
(XL)  43.982
(%i8) Z: float(sqrt(R^2 + (XL-XC)^2));
(Z)   45.59
(%i9) Irms: float(Vrms/Z);
(Irms) 4.8256
(%i10) phi: float(atan((XL-XC)/R));
(phi) 1.3044
(%i11) phi_deg: float(phi*180/%pi);
(phi_deg) 74.739
(%i12) power_factor: float(cos(phi));
(power_factor) 0.26322
(%i13) P: float(Irms^2*R);
(P) 279.44
```

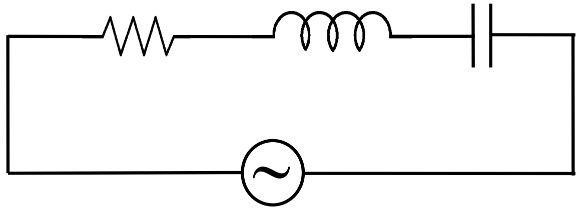
Comments on the codes:

(%i6)	Set floating point print precision to 5, assign values of L, R, V_{rms}, f, and X_C.
(%i7), (%i8), (%i9)	Calculate X_L, Z, and I_{rms}.
(%i10), (%i11)	Calculate ϕ and convert the angle to degree.
(%i12), (%i13)	Calculate power factor and loss of electrical power P.

Problem 15.8 An *RLC* series circuit consists of a 100 Ω resistor, a 0.10 H inductor, a 20 μF capacitor, and a 220 V, 50 Hz AC source. Calculate

(a) current.
(b) power loss.

Fig. 15.9 Alternating current series *RLC* circuit, Problem 15.8



(c) phase angle.
(d) voltages across the resistor, inductor, and capacitor.

Solution

Figure 15.9 shows the *RLC* circuit with the AC generator.

(a) The inductive and capacitive reactances are

$$X_L = \omega L = 2\pi f L = 2\pi(50\ \text{s}^{-1})(0.10\ \text{H}) = 31\ \Omega,$$

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C} = \frac{1}{2\pi(50\ \text{s}^{-1})(20 \times 10^{-6}\ \text{F})} = 159\ \Omega.$$

The impedance of the circuit is

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(100\ \Omega)^2 + (31\ \Omega - 159\ \Omega)^2} = 162\ \Omega.$$

Thus, the current in the circuit is

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{Z} = \frac{220\ \text{V}}{162\ \Omega} = 1.36\ \text{A}.$$

(b) The power loss is the one lost by the resistor,

$$P = I_{\text{rms}}^2 R = (1.36\ \text{A})^2(100\ \Omega) = 184\ \text{W}.$$

(c) The phase angle is,

$$\tan\phi = \frac{X_L - X_C}{R} = \frac{31\ \Omega - 159\ \Omega}{100\ \Omega} = -1.28.$$
$$\phi = -52^\circ.$$

(d) The voltages across the resistor, inductor, and capacitor are,

$$V_R = I_{\text{rms}} R = (1.36 \text{ A})(100 \ \Omega) = 136 \text{ V}.$$

$$V_L = I_{\text{rms}} X_L = (1.36 \text{ A})(31 \ \Omega) = 43 \text{ V}.$$

$$V_C = I_{\text{rms}} X_C = (1.36 \text{ A})(159 \ \Omega) = 216 \text{ V}.$$

◆ wxMaxima codes:

```
(%i6) fpprintprec:5; R:100; L:0.1; C:20e-6; Vrms:220; f:50;
(fpprintprec)     5
(R)   100
(L)   0.1
(C)   2.0*10^-5
(Vrms) 220
(f)   50
(%i7) XL: float(2*%pi*f*L);
(XL)  31.416
(%i8) XC: float(1/(2*%pi*f*C));
(XC)  159.15
(%i9) Z: float(sqrt(R^2 + (XL-XC)^2));
(Z)   162.23
(%i10) Irms: float(Vrms/Z);
(Irms) 1.3561
(%i11) P: float(Irms^2*R);
(P)   183.91
(%i12) phi: float(atan((XL-XC)/R));
(phi) -0.9066
(%i13) phi_deg: float(phi*180/%pi);
(phi_deg) -51.945
(%i14) VR: float(Irms*R);
(VR)  135.61
(%i15) VL: float(Irms*XL);
(VL)  42.604
(%i16) VC: float(Irms*XC);
(VC)  215.84
```

Comments on the codes:

(%i6)	Set floating point print precision to 5, assign values of R, L, C, V_{rms}, and f.
(%i7), (%i8), (%i9)	Calculate X_L, X_C, and Z.
(%i10), (%i11)	Calculate I_{rms} and P.
(%12), (%i13)	Calculate ϕ and convert the angle to degree.
(%i14), (%i15), (%i16)	Calculate V_R, V_L, and V_C.

Problem 15.9 Calculate the resonant frequency of a circuit consisting of a 40 mH inductor and a 600 pF capacitor. The resistance of the circuit is negligible.

Solution

Resonance of an AC circuit that consists of a resistor, an inductor, and a capacitor is attained when the impedance is a minimum. The impedance is

$$Z = \sqrt{R^2 + (X_L - X_C)^2}.$$

The impedance is a minimum when inductive reactance is equal to capacitive reactance, that is,

$$X_L = X_C$$
$$2\pi f_0 L = \frac{1}{2\pi f_0 C}$$
$$f_0 = \frac{1}{2\pi\sqrt{LC}}.$$

Thus, the resonant frequency of the circuit is,

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{(40 \times 10^{-3}\ \text{H})(600 \times 10^{-12}\ \text{F})}} = 3.2 \times 10^4\ \text{s}^{-1}.$$

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; ratprint:false; L:40e-3; C:600e-12;
(fpprintprec)     5
(ratprint)  false
(L)   0.04
(C)   6.0*10^-10
(%i6) XL: 2*%pi*f0*L; XC: 1/(2*%pi*f0*C);
(XL)  0.08*%pi*f0
(XC)   (8.3333*10^8)/(%pi*f0)
(%i8) solve(XL=XC, f0)$ float(%);
(%o8) [f0=-3.2487*10^4,f0=3.2487*10^4]
(%i9) f0: float(1/(2*%pi*sqrt(L*C)));
(f0)  3.2487*10^4
```

Comments on the codes:

(%i4) Set floating point print precision to 5, internal rational number print to false, assign values of L and C.

(%i6) Calculate X_L and X_C.

(%i8) Solve $X_L = X_C$ for f_0.

(%i9) Direct calculation of resonant frequency, f_0.

Problem 15.10 The impedance of an *RLC* series circuit at resonant frequency of 60 Hz is 8.0 Ω. The impedance of the circuit at frequency 80 Hz is 10 Ω. Calculate the inductance and capacitance of the circuit.

Solution

When the circuit is in resonance at $f_0 = 60$ Hz, the inductive reactance is equal to the capacitive reactance. We have

$$X_L = X_C \Rightarrow 2\pi f_0 L = \frac{1}{2\pi f_0 C},$$

$$f_0^2 = \frac{1}{4\pi^2 LC},$$

$$(60\ \mathrm{s}^{-1})^2 = \frac{1}{4\pi^2 LC}, \tag{1}$$

and the impedance is the resistance,

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = R = 8.0\ \Omega.$$

At frequency 80 Hz,

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \Rightarrow Z^2 = R^2 + (X_L - X_C)^2,$$

$$Z^2 = R^2 + (2\pi f L - \frac{1}{2\pi f C})^2,$$

$$(10\ \Omega)^2 = (8.0\ \Omega)^2 + [2\pi(80\ \mathrm{s}^{-1})L - \frac{1}{2\pi(80\ \mathrm{s}^{-1})C}]^2. \tag{2}$$

Solving Eqs. (1) and (2) for L and C, gives inductance and capacitance as,

$$L = 0.027\ \mathrm{H},$$

$$C = 2.6 \times 10^{-4}\ \mathrm{F}.$$

♦ wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)      5
(ratprint)  false
(%i4) solve([60^2=1/(4*%pi^2*L*C), 10^2=8^2+(2*%pi*80*L-1/(2*%pi*80*C))^2],
[L,C])$ float(%);
(%o4) [[L=-0.027284,C=-2.5789*10^-4],[L=0.027284,C=2.5789*10^-4]]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Solve Eqs. (1) and (2) for L and C.

15.3 Summary

- The voltage amplitude in an AC circuit is

$$V_{\max} = I_{\max} Z,$$

where Z is the impedance of the circuit.
- For a series RLC circuit, the impedance is

$$Z = \sqrt{R^2 + (X_L - X_C)^2},$$

where $X_L = \omega L$ is inductive reactance and $X_C = 1/(\omega C)$ is capacitive reactance. The phase angle ϕ between the voltage and current is given by

$$\tan\phi = \frac{X_L - X_C}{R}.$$

The average power of the circuit is

$$P_{\text{average}} = I_{\text{rms}} V_{\text{rms}} \cos\phi = Z\ I_{\text{rms}}^2 \cos\phi = I_{\text{rms}}^2 R$$
$$= \frac{1}{2} I_{\max} V_{\max} \cos\phi.$$

The quantity $\cos\phi$ is called the power factor,

$$\text{power factor} = \cos\phi = \frac{R}{Z} = \frac{R}{\sqrt{R^2 + (X_L - X_C)^2}}.$$

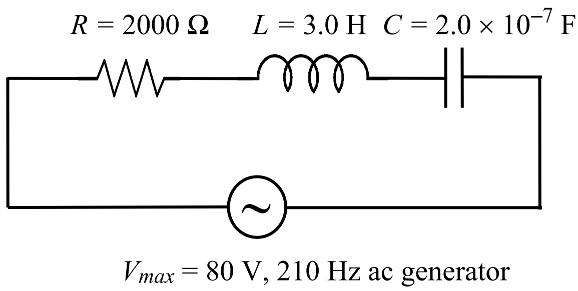
The root mean square current is

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{Z} = \frac{V_{\text{rms}}}{\sqrt{R^2 + (X_L - X_C)^2}}.$$

- The natural angular frequency ω_0 of oscillation of an LC circuit is

$$\omega_0 = \frac{1}{\sqrt{LC}}.$$

Fig. 15.10 Alternating current series *RLC* circuit, Exercise 15.3



15.4 Exercises

Exercise 15.1 An alternating current of effective value 5.0 A passes through a 25 Ω resistor. Find

(a) the maximum potential difference across the resistor.
(b) the power dissipated by the resistor.

(Answer: (a) $V_{\max} = 180$ V; (b) $P = 620$ W)

Exercise 15.2 A series circuit consisting of a 100 Ω resistor, a 0.10 H inductor, and a 20 μF capacitor is connected across a 220 V rms, 50 Hz power source. Calculate current in the circuit and the average power loss by the circuit.
(Answer: 1.4 A, 180 W)

Exercise 15.3 Figure 15.10 shows an *RLC* AC circuit with a 2000 Ω resistor, a 3.0 H inductor, and a 2.0×10^{-7} F capacitor. The voltage source is of amplitude 80 V and the frequency is 210 Hz. Determine

(a) phase angle between the voltage and the current.
(b) voltage amplitudes across the resistor, inductor, and capacitor.
(c) average power dissipated by the circuit.

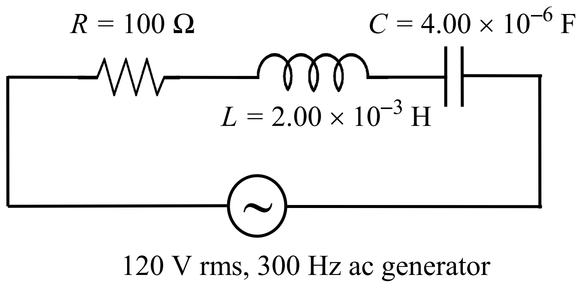
(Answer: (a) 4.8°; (b) $V_R = 80$ V, $V_L = 158$ V, $V_C = 151$ V; (c) 1.6 W)

Exercise 15.4 A series *RLC* circuit has a 100 Ω resistor, a 2.00×10^{-3} H inductor, and a 4.00×10^{-6} F capacitor connected to a 120 V rms AC source at 300 Hz, as shown in Fig. 15.11. Calculate

(a) impedance of the circuit..
(b) power factor of the circuit.
(c) root mean square current.
(d) average power dissipated by the circuit.

(Answer: (a) 163 Ω; (b) 0.613; (c) 0.736 A; (d) 54.1 W)

Fig. 15.11 Alternating current series *RLC* circuit, Exercise 15.4



Exercise 15.5 What is the resonant frequency of the *RLC* circuit of Fig. 15.11 in Exercise 4? If the AC generator operates at the resonant frequency with the same 120 V rms voltage what are the root mean square current and the average power dissipated by the circuit?

(Answer: 1780 Hz, 1.20 A, 144 W)

Chapter 16
Electromagnetic Wave

Abstract This chapter solves problems on plane electromagnetic wave, associated Poynting vector, and radiation pressure. These include determination of electric and magnetic field amplitudes and directions, intensity, energy density, and direction of propagation of the electromagnetic waves. Both solutions by analysis and computer calculation are presented. An animation of traveling plane electromagnetic wave is presented.

16.1 Basic Concepts and Formulae

(1) Laws of electromagnetism can be summarized as four equations called Maxwell's equations. Table 16.1 lists the four Maxwell's equations in integral and differential forms and their meanings.

$\varepsilon_0 = 8.85 \times 10^{-12}$ F m^{-1} is permittivity of free space.

$\mu_0 = 4\pi \times 10^{-7}$ H m^{-1} is permeability of free space.

Lorentz force: A particle of charge q moving with a velocity of $\boldsymbol{v}$ in an electric field $\boldsymbol{E}$ and a magnetic field $\boldsymbol{B}$ experiences a force of

$$\boldsymbol{F} = q\boldsymbol{E} + q\boldsymbol{v} \times \boldsymbol{B}.$$

(2) Electromagnetic waves have the following properties:

(a) Electric field E and magnetic field B satisfy the following wave equations,

$$\frac{\partial^2 E}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}, \tag{16.1}$$

$$\frac{\partial^2 B}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 B}{\partial t^2}. \tag{16.2}$$

W. M. S. Wan Hassan et al., *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-43165-4_16

Table 16.1 Four Maxwell's equations and their meanings

Integral equation form	Differential equation form	Meaning
$\oint E \cdot dA = \frac{q}{\varepsilon_0}$	$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$	Gauss' law for electricity. Electric flux from a volume is proportional to the charge in the volume
$\oint B \cdot dA = 0$	$\nabla \cdot B = 0$	Gauss' law for magnetism. Magnetic flux through a closed surface is zero. There is no magnetic monopole
$\oint B \cdot ds = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$	$\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$	Ampere's circuit law. Magnetic field induced in a closed loop is proportional to the electric current and displacement current enclosed in the loop
$\oint E \cdot ds = -\frac{d\Phi_m}{dt}$	$\nabla \times E = -\frac{\partial B}{\partial t}$	Maxwell–Faraday equation. Emf induced in a closed loop is proportional to the rate of change of magnetic flux enclosed in the loop

(b) Electromagnetic waves move in vacuum with the speed of light *c*,

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \text{ m s}^{-1}. \tag{16.3}$$

(c) Electric and magnetic fields of an electromagnetic wave are perpendicular to each other and the fields are perpendicular to the direction of wave propagation. Electromagnetic waves are transverse waves. Instantaneous magnitudes of the electric and magnetic fields satisfy

$$\frac{E}{B} = c. \tag{16.4}$$

(d) Electromagnetic waves carry energy. The rate of energy across unit area is given by Poynting vector $\boldsymbol{S}$,

$$\boldsymbol{S} = \frac{1}{\mu_0} \boldsymbol{E} \times \boldsymbol{B}. \tag{16.5}$$

Direction of $\boldsymbol{S}$ can be determined if directions of $\boldsymbol{E}$ and $\boldsymbol{B}$ are known.

If an electromagnetic wave with average Poynting vector value of $S_{average}$ is incident on an area *A*, the the power received by the area is,

$$\text{power} = S_{average} \times A.$$

Average Poynting vector value $S_{average}$ is the intensity *I* of the electromagnetic wave.

(e) The energy density of electromagnetic waves is

$$u = u_E + u_B = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\frac{B^2}{\mu_0}$$
$$= \varepsilon_0 E^2 = \frac{B^2}{\mu_0}$$
$$= \sqrt{\frac{\varepsilon_0}{\mu_0}} EB. \tag{16.6}$$

(f) Electromagnetic waves carry momentum and exert pressure on incident surface. For electromagnetic waves with Poynting vector of magnitude S incident normally to a surface and fully absorbed by the surface, the radiation pressure p is

$$p = \frac{S}{c}. \tag{16.7}$$

(3) Electric and magnetic fields of a plane sinusoidal electromagnetic wave propagating in the positive x direction (**i** direction) are written as

$$\boldsymbol{E} = E_{max} \cos(kx - \omega t)\, \mathbf{j}, \tag{16.8}$$

$$\boldsymbol{B} = B_{max} \cos(kx - \omega t)\, \mathbf{k}, \tag{16.9}$$

where ω and k are angular frequency and propagation constant, respectively.

Frequency f, period T, wavelength λ, and speed c of the wave are related as,

$$\frac{\omega}{k} = \lambda f = \frac{\lambda}{T} = c. \tag{16.10}$$

(4) Intensity of a plane sinusoidal electromagnetic wave is average value of the Poynting vector,

$$I = S_{average} = \frac{E_{max} B_{max}}{2\mu_0}$$
$$= \frac{E_{max}^2}{2\mu_0 c} = \frac{c B_{max}^2}{2\mu_0}$$
$$= \frac{E_{rms}^2}{\mu_0 c} = \frac{c B_{rms}^2}{\mu_0}. \tag{16.11}$$

(5) The main origin of electromagnetic waves is acceleration or oscillation of electric charges. For example, radio waves emitted by an antenna are due to continuous oscillations of electrons (negatively charged particles) in the antenna.

(6) Electromagnetic spectrum consists of waves with wide ranges of frequencies or wavelengths. These include radio waves, microwaves, infrared, visible light, ultraviolet, X-rays, and gamma rays. Frequency f and wavelength λ of the waves are related by

$$c = f\lambda. \tag{16.12}$$

16.2 Problems and Solutions

Problem 16.1 The frequency of a sinusoidal plane electromagnetic wave is 80 MHz. The wave travels in the positive x direction. At a point on the x-axis, at an instance, the maximum value of electric field is 750 N C^{-1} in the y direction.

(a) Calculate the wavelength and period of the wave.
(b) Determine the magnitude and direction of the magnetic field.
(c) Get expressions for the electric and magnetic fields of the wave.

Solution

(a) The wavelength of the electromagnetic wave is, Eq. (16.10),

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8 \text{ m/s}}{80 \times 10^6 \text{ s}^{-1}} = 3.7 \text{ m}.$$

The period of the wave is

$$T = \frac{1}{f} = \frac{1}{80 \times 10^6 \text{ s}^{-1}} = 1.2 \times 10^{-8} \text{ s}.$$

(b) The magnitude of the magnetic field is, Eq. (16.4),

$$B_{max} = \frac{E_{max}}{c} = \frac{750 \text{ N/C}}{3 \times 10^8 \text{ m/s}} = 2.5 \times 10^{-6} \text{ T}.$$

The magnetic field B is in the positive z direction.

(c) The propagation constant k is

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{3.7\ \text{m}} = 1.7\ \text{m}^{-1}.$$

The angular frequency ω is

$$\omega = 2\pi f = 2\pi(80 \times 10^6 s^{-1}) = 5.0 \times 10^8\ \text{s}^{-1}.$$

The expression for electric field is

$$\begin{aligned} E &= E_{max}\cos(kx - \omega t) \\ &= 750\ \cos(1.7x - 5.0 \times 10^8 t). \end{aligned}$$

The expression for magnetic field is

$$\begin{aligned} B &= B_{max}\cos(kx - \omega t) \\ &= 2.5 \times 10^{-6}\cos(1.7x - 5.0 \times 10^8 t). \end{aligned}$$

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; c:3e8; f:80e6; Emax:750;
(fpprintprec)     5
(c)   3.0*10^8
(f)   8.0*10^7
(Emax) 750
(%i5) lambda: c/f;
(lambda) 3.75
(%i6) T: 1/f;
(T)   1.25*10^-8
(%i7) Bmax: Emax/c;
(Bmax) 2.5*10^-6
(%i8) k: float(2*%pi/lambda);
(k)   1.6755
(%i9) omega: float(2*%pi*f);
(omega) 5.0265*10^8
```

Comments on the codes:

(%i4)	Set floating point print precision to 5, assign values of c, f, and E_{max}.
(%i5), (%i6), (%i7), (%i8), (%i9)	Calculate λ, T, B_{max}, k, and ω.

Problem 16.2 A radio station emits radio waves of frequency 104.1 MHz. Calculate the wavelength and the number of wave peaks per second passing through a point 5.0 km away from the station.

Solution

The wavelength of the radio wave is, Eq. (16.10),

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8 \text{m/s}}{104.1 \times 10^6 \text{s}^{-1}} = 2.9 \text{ m}.$$

Number of wave peaks per second passing through a point is the frequency of the wave that is 104.1×10^6 Hz.

◆ wxMaxima codes:

```
(%i3) fpprintprec:5; c:3e8; f:104.1e6;
(fpprintprec)     5
(c)   3.0*10^8
(f)   1.041*10^8
(%i4) lambda: c/f;
(lambda) 2.8818
```

Comments on the codes:

(%i3) Set floating point print precision to 5, assign values of c and f.
(%i4) Calculate λ.

Problem 16.3 Average power output of an electromagnetic radiation point source is $P_{average} = 900$ W. At a point 3.5 m from the source, calculate

(a) the maximum electric and magnetic fields
(b) the energy density.

Solution

(a) The intensity of an electromagnetic radiation at a distance r from the point source of power $P_{average}$ is

$$I = \frac{P_{average}}{4\pi r^2}.$$

The electromagnetic point wave radiates equally in all directions, and on the surface of a sphere of radius r the power per unit area is $P_{average}/(4\pi r^2)$.

The intensity in terms of electric field amplitude of the electromagnetic wave is (Eq. 16.11),

$$I = \frac{E_{max}^2}{2\mu_0 c}.$$

The maximum electric field (the electric field amplitude) is calculated as follows:

$$\frac{P_{average}}{4\pi r^2} = \frac{E_{max}^2}{2\mu_0 c},$$
$$E_{max} = \left(\frac{\mu_0 c P_{average}}{2\pi r^2}\right)^{1/2}$$
$$= \left(\frac{(4\pi \times 10^{-7}\ \text{N/A}^2)(3 \times 10^8\ \text{m/s})(900\ \text{W})}{2\pi (3.5\ \text{m})^2}\right)^{1/2}$$
$$= 66\ \text{V m}^{-1}.$$

The maximum magnetic field (the magnetic field amplitude) is, Eq. (16.4),

$$B_{max} = \frac{E_{max}}{c} = \frac{66\ \text{V m}^{-1}}{3 \times 10^8\ \text{m/s}} = 2.2 \times 10^{-7}\ \text{T}.$$

(b) The energy density of the electromagnetic wave 3.5 m from the source is, Eq. (16.6),

$$u = \frac{B_{max}^2}{\mu_0} = \frac{(2.2 \times 10^{-7}\ \text{T})^2}{4\pi \times 10^{-7}\ \text{N/A}^2} = 3.9 \times 10^{-8}\ \text{J m}^{-3}.$$

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; c:3e8; mu0:float(4*%pi*1e-7); Paverage:900; r:3.5;
(fpprintprec)     5
(c)   3.0*10^8
(mu0) 1.2566*10^-6
(Paverage) 900
(r)   3.5
(%i6) Emax: float(sqrt(mu0*c*Paverage/(2*%pi*r^2)));
(Emax) 66.394
(%i7) Bmax: Emax/c;
(Bmax) 2.2131*10^-7
(%i8) u: Bmax^2/mu0;
(u) 3.8977*10^-8
```

Comments on the codes:

(%i5)	Set print point precision to 5, assign values of c, μ_0, $P_{average}$, and r.
(%i6), (%i7), (%i8)	Calculate E_{max}, B_{max}, and u.

Problem 16.4 The wavelength range of visible light is 390 nm (violet) to 780 nm (red). Determine the frequency range of visible light.

Solution

Frequencies of violet and red lights are calculated as follows, Eq. (16.12),

$$c = \lambda f,$$
$$f = \frac{c}{\lambda},$$
$$f_{violet} = \frac{c}{\lambda_{violet}} = \frac{3 \times 10^8 \text{ m/s}}{390 \times 10^{-9} \text{ m}} = 7.7 \times 10^{14} \text{ Hz},$$
$$f_{red} = \frac{c}{\lambda_{red}} = \frac{3 \times 10^8 \text{ m/s}}{780 \times 10^{-9} \text{ m}} = 3.8 \times 10^{14} \text{ Hz}.$$

Thus, the frequency range of visible light is 3.8×10^{14} Hz to 7.7×10^{14} Hz.

◆ wxMaxima codes:

```
(%i2) fpprintprec:5; c:3e8;
(fpprintprec)     5
(c)   3.0*10^8
(%i3) fviolet: c/390e-9;
(fviolet) 7.6923*10^14
(%i4) fred: c/780e-9;
(fred) 3.8462*10^14
```

Comments on the codes:

(%i2)	Set floating point print precision to 5 and assign value of c.
(%i3), (%i4)	Calculate f_{violet} and f_{red}.

Problem 16.5 A car moves at a speed of v toward an observer. An electromagnetic wave of frequency f is incident on the car, reflected from it, and is received by the observer. The frequency of the wave received is

$$f_{received} = \left(1 + \frac{2v}{c}\right) f.$$

Use the information to solve the following problem.

A 1000 MHzelectromagnetic wave is sent by a stationary observer to a car which moves toward him. The frequency received by the observer increased by 150 Hz. Calculate the speed of the car.

Solution

The speed of the car is calculated as follows:

$$f_{received} = \left(1 + \frac{2v}{c}\right) f,$$
$$(1000 \times 10^6 + 150) \text{ Hz} = \left(1 + \frac{2v}{3 \times 10^8 \text{ m/s}}\right) \left(1000 \times 10^6 \text{ Hz}\right),$$

$$\begin{aligned} v &= 22.5 \text{ m s}^{-1} \\ &= 22.5/1000 \times 3600 \text{ km h}^{-1} \\ &= 81 \text{ km h}^{-1}. \end{aligned}$$

♦ wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint: false;
(fpprintprec)     5
(ratprint)  false
(%i4) solve(1000e6+150=(1+2*v/3e8)*1000e6, v)$ float(%);
(%o4) [v=22.5]
(%i5) km_per_h: 22.5/1000*3600;
(km_per_h)  81.0
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Solve $(1000 \times 10^6 + 150) = (1 + \frac{2v}{3\times10^8})(1000 \times 10^6)$ for v.
(%i5) Convert speed to km/h.

Problem 16.6 Sunlight with intensity 1000 W m^{-2} falls on a 10 × 20 m roof. Calculate,

(a) power received by the roof.
(b) radiation pressure on the roof by assuming the light is completely absorbed by the roof.
(c) energy received by the roof in one hour.

Solution

(a) The intensity, I, or the average value of Poynting vector, $S_{average}$, of the sunlight is 1000 W m^{-2}. The power received by the roof is

$$\text{power} = S_{average} \times A = (1000 \text{ W/m}^2)(10 \times 20) \text{ m}^2 = 2.0 \times 10^5 \text{ W}.$$

(b) The radiation pressure on the roof is, Eq. (16.7),

$$p = \frac{S_{average}}{c} = \frac{1000 \text{ W/m}^2}{3 \times 10^8 \text{ m/s}} = 3.3 \times 10^{-6} \text{ N m}^{-2}.$$

(c) The energy received by the roof in one hour,

$$\text{energy} = \text{power} \times \text{time} = (2.0 \times 10^5 \text{ W})(3600 \text{ s}) = 7.2 \times 10^8 \text{ J}.$$

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; S:1000; A:10*20; c:3e8;
(fpprintprec)     5
(S)   1000
(A)   200
(c)   3.0*10^8
(%i5) Power: S*A;
(Power) 200000
(%i6) P: S/c;
(P) 3.3333*10^-6
(%i7) Energy: Power*3600;
(Energy) 720000000
```

Comments on the codes:

(%i4)	Set floating point print precision to 5, assign values of *S*, *A*, and *c*.
(%i5), (%i6), (%i7)	Calculate power, radiation pressure *P*, and energy.

Problem 16.7 The average solar energy falling on a surface in unit time and area is 1000 W m^{-2}.

(a) Calculate the energy that falls on a 5.0 m × 5.0 m surface in one hour.
(b) What is the momentum transferred to the surface in one hour?
(c) If all the energy is converted to electrical energy, how many bulbs of 100 W can be lighted?

Solution

(a) The energy U falling on the surface in one hour is

$$\begin{aligned} U &= \text{intensity} \times \text{area} \times \text{time} \\ &= (1000\ \text{W/m}^2)(5.0\ \text{m} \times 5.0\ \text{m})(3600\ \text{s}) = 9.0 \times 10^7\ \text{J}. \end{aligned}$$

(b) The momentum transferred to the surface in one hour is

$$\text{momentum} = \frac{U}{c} = \frac{9.0 \times 10^7\ \text{J}}{3 \times 10^8\ \text{m/s}} = 0.30\ \text{kg m s}^{-1}.$$

(c) The number of bulbs that can be lighted is

$$\frac{\text{power}}{\text{power of a bulb}} = \frac{\text{intensity} \times \text{area}}{\text{power of a bulb}} = \frac{(1000\ \text{W/m}^2)(5.0\ \text{m} \times 5.0\ \text{m})}{100\ \text{W}} = 250.$$

◆ wxMaxima codes:

```
(%i1) U: 1000*5*5*3600;
(U)   90000000
(%i2) momentum: U/3e8;
(momentum)  0.3
(%i3) 1000*5*5/100;
(%o3) 250
```

Comment on the codes:

(%i1), (%i2), (%i3) Calculate energy U, momentum, and number of bulbs.

Problem 16.8 A 6.0×10^7 Hz plane sinusoidal electromagnetic wave propagates in free space in the positive x direction. The magnetic field of the wave is in the z-axis and its amplitude is 5.0×10^{-7} Wb m^{-2}.

(a) Calculate the wavelength.
(b) Determine the electric field of the wave.
(c) Write expressions for the electric and magnetic fields of the wave.
(d) Calculate the Poynting vector and its average value.

Solution

(a) The wavelength of the electromagnetic wave is, Eq. (16.10),

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8 \text{ m/s}}{6.0 \times 10^7 \text{ s}^{-1}} = 5.0 \text{ m}.$$

(b) The amplitude of the electric field is, Eq. (16.4),

$$E_0 = cB_0 = 3 \times 10^8 \text{ m/s} \times 5.0 \times 10^{-7} \text{ Wb/m}^2 = 150 \text{ V m}^{-1}.$$

The electric field is in the positive y direction.

(c) General forms of the electric and magnetic fields of a plane sinusoidal electromagnetic wave are

$$E = E_0 \cos(kx - \omega t),$$
$$B = B_0 \cos(kx - \omega t),$$

where E_0 and B_0 are amplitudes of the electric and magnetic fields, while k and ω are propagation constant and angular frequency of the wave, respectively. The propagation constant and angular frequency are

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{5.0 \text{ m}} = 1.3 \text{ m}^{-1},$$
$$\omega = 2\pi f = 2\pi \times 6.0 \times 10^7 \text{ s}^{-1} = 3.8 \times 10^8 \text{ s}^{-1}.$$

The electric and magnetic fields of the electromagnetic wave are

$$\boldsymbol{E} = E_0 \cos(kx - \omega t)\, \mathbf{j} = 150 \cos(1.3x - 3.8 \times 10^8 t)\, \mathbf{j} \text{ V m}^{-1},$$
$$\boldsymbol{B} = B_0 \cos(kx - \omega t)\, \mathbf{k} = 5.0 \times 10^{-7} \cos(1.3x - 3.8 \times 10^8 t)\, \mathbf{k} \text{ Wb m}^{-2}.$$

(d) The Poynting vector is, Eq. (16.5),

$$\boldsymbol{S} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0}$$
$$= \frac{150 \cos(1.3x - 3.8 \times 10^8 t)\, \mathbf{j} \times 5.0 \times 10^{-7} \cos(1.3x - 3.8 \times 10^8 t)\, \mathbf{k}}{4\pi \times 10^{-7}}$$
$$= 60 \cos^2(1.3x - 3.8 \times 10^8 t)\, \mathbf{i}\ W \text{ m}^{-2}.$$

The average value of Poynting vector is, Eq. (16.11),

$$S_{average} = \frac{E_0 B_0}{2\mu_0} = \frac{150 \times 5.0 \times 10^{-7}}{4\pi \times 10^{-7}} = 30 \text{ W m}^{-2}.$$

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; c:3e8; f:6e7; B0:5e-7; mu0:float(4*%pi*1e-7);
(fpprintprec)     5
(c)    3.0*10^8
(f)    6.0*10^7
(B0)   5.0*10^-7
(mu0) 1.2566*10^-6
(%i6) lambda: c/f;
(lambda)    5.0
(%i7) E0: c*B0;
(E0)   150.0
(%i8) k: float(2*%pi/lambda);
(k)    1.2566
(%i9) omega: float(2*%pi*f);
(omega) 3.7699*10^8
(%i10) float(E0*B0/mu0);
(%o10) 59.683
(%i11) Saverage: E0*B0/(2*mu0);
(Saverage)  29.842
```

Comments on the codes:

(%i5)	Set floating point print precision to 5, assign values of c, f, B_0, and μ_0.
(%i6), (%i7), (%i8), (%i9), (%i11)	Calculate λ, E_0, k, ω, and $S_{average}$.

Problem 16.9 The electric fields of a plane electromagnetic wave propagating in free space are

$$E_x = E_y = 0,$$
$$E_z = 100\ \sin[8\pi \times 10^{14}(t - \frac{x}{3 \times 10^8})].$$

(a) Calculate the flux density of the wave.
(b) Determine the direction of propagation of the wave.
(c) Write the wave electric field in vector form.
(d) Write the wave magnetic field in vector form.
(e) Calculate Poynting vector of the wave.

Solution

(a) Flux density I of the wave is the average of Poynting vector $S_{average}$, Eqs. (16.11) and (16.3),

$$\begin{aligned} I = S_{average} &= \frac{1}{2} c\varepsilon_0 E_0^2 \\ &= \frac{1}{2}(3 \times 10^8\ \text{m/s})(8.85 \times 10^{-12}\ \text{F/m}) \times (100\ \text{V/m})^2 \\ &= 13\ \text{W m}^{-2}. \end{aligned}$$

(b) From the expression of E_z it is deduced that the wave is propagating in the positive x direction, that is, the **i** direction. This is the direction of Poynting vector $\boldsymbol{S}$.

(c) The electric field of the wave is

$$\boldsymbol{E} = 100\ \sin[8\pi \times 10^{14}(t - \frac{x}{3 \times 10^8})]\ \mathbf{k}\ \text{V m}^{-1}.$$

(d) The amplitude B_0 of the magnetic field is, Eq. (16.4),

$$B_0 = \frac{E_0}{c} = \frac{100\ \text{V/m}}{3 \times 10^8\ \text{m/s}} = 3.3 \times 10^{-7}\ \text{Wb m}^{-2}.$$

From the formula $\boldsymbol{S} = (1/\mu_0)\ \boldsymbol{E} \times \boldsymbol{B}$ and the right hand rule for cross product of two vectors, it is deduced that the direction of $\boldsymbol{B}$ is the negative y direction or –**j** direction. We write

$$B_x = B_z = 0,$$
$$B_y = -3.3 \times 10^{-7}\ \sin\left[8\pi \times 10^{14}\left(t - \frac{x}{3 \times 10^8}\right)\right]\ \text{Wb m}^{-2},$$
$$\boldsymbol{B} = -3.3 \times 10^{-7}\ \sin\left[8\pi \times 10^{14}\left(t - \frac{x}{3 \times 10^8}\right)\right]\ \mathbf{j}\ \text{Wb m}^{-2}.$$

(e) The Poynting vector is, Eq. (16.5),

$$\begin{aligned}\boldsymbol{S} &= \frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0}\\ &= \frac{1}{4\pi\times 10^{-7}}\times 100\ \sin\left[8\pi\times 10^{14}\left(t-\frac{x}{3\times 10^8}\right)\right]\mathbf{k}\\ &\quad\times(-3.3\times 10^{-7})\ \sin\left[8\pi\times 10^{14}\left(t-\frac{x}{3\times 10^8}\right)\right]\mathbf{j}\\ &= 27\ \sin^2\left[8\pi\times 10^{14}\left(t-\frac{x}{3\times 10^8}\right)\right]\mathbf{i}\ \mathrm{W\ m^{-2}}.\end{aligned}$$

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; c:3e8; epsilon0:8.85e-12; mu0:float(4*%pi*1e-7);
E0:100;
(fpprintprec)     5
(c)   3.0*10^8
(epsilon0)  8.85*10^-12
(mu0) 1.2566*10^-6
(E0)  100
(%i6) I: 0.5*c*epsilon0*E0^2;
(I)   13.275
(%i7) B0: E0/c;
(B0)  3.3333*10^-7
(%i8) E0*B0/mu0;
(%o8) 26.526
(%i9) load("vect");
(%o9) "C:/maxima-5.43.0/share/maxima/5.43.0/share/vector/vect.mac"
(%i10) Evec: [0,0,E0*sin(8*%pi*1e14*(t-x/3e8))];
(Evec) [0,0,100*sin(8.0*10^14*%pi*(t-3.3333*10^-9*x))]
(%i11) Bvec: [0, -B0*sin(8*%pi*1e14*(t-x/3e8)), 0];
(Bvec) [0,-3.3333*10^-7*sin(8.0*10^14*%pi*(t-3.3333*10^-9*x)),0]
(%i13) Svec: Evec~Bvec/mu0; express(%);
(Svec) 7.9577*10^5*[0,0,100*sin(8.0*10^14*%pi*(t-3.3333*10^-9*x))]
~[0,-3.3333*10^-7*sin(8.0*10^14*%pi*(t-3.3333*10^-9*x)),0]
(%o13) [26.526*sin(8.0*10^14*%pi*(t-3.3333*10^-9*x))^2,0,0]
```

Comments on the codes:

(%i5)	Set floating point print precision to 5, assign values of c, ε_0, μ_0, and E_0.
(%i6), (%i7), (%i8)	Calculate I, B_0, and E_0B_0/μ_0.
(%i9)	Load "vect" package.
(%i10, (%i11)	Assign vectors $\boldsymbol{E}$ and $\boldsymbol{B}$.
(%i13)	Calculate Poynting vector, $\boldsymbol{S}$.

Problem 16.10

(a) Determine the equation for electric field of a 104.1 MHz radio wave of propagating in the positive x direction. Root mean square value of the electric field is 3.0 mV m^{-1}.
(b) What is the equation of magnetic field of the radio wave?

Solution

(a) Assume the equations for the electric fields are,

$$E_x = E_z = 0,$$
$$E_y = E_0 \cos[\omega(t - \frac{x}{c})].$$

The electric field amplitude E_0 and angular frequency ω are,

$$E_0 = \sqrt{2}E_{rms} = \sqrt{2}(3.0 \text{ mV/m}) = 4.2 \text{ mV/m} = 4.2 \times 10^{-3} \text{ V m}^{-1},$$
$$\omega = 2\pi f = 2\pi(104.1 \times 10^6) = 6.5 \times 10^8 \text{ s}^{-1}.$$

The equations for the electric field of the radio wave are

$$E_y = 4.2 \times 10^{-3} \cos\left[6.5 \times 10^8\left(t - \frac{x}{3 \times 10^8}\right)\right] \text{ V m}^{-1},$$
$$\boldsymbol{E} = 4.2 \times 10^{-3} \cos\left[6.5 \times 10^8\left(t - \frac{x}{3 \times 10^8}\right)\right] \mathbf{j} \text{ V m}^{-1}.$$

(b) The amplitude of the magnetic field of the radio wave is, Eq. (16.4),

$$B_0 = \frac{E_0}{c} = \frac{4.2 \times 10^{-3} \text{ V/m}}{3 \times 10^8 \text{ m/s}} = 1.4 \times 10^{-8} \text{ Wb m}^{-2}.$$

The equations of the magnetic field of the radio wave are,

$$B_x = B_y = 0,$$
$$B_z = B_0 \cos\left[\omega\left(t - \frac{x}{c}\right)\right]$$
$$= 1.4 \times 10^{-8} \cos\left[6.5 \times 10^8\left(t - \frac{x}{3 \times 10^8}\right)\right] \text{ Wb m}^{-2},$$
$$\boldsymbol{B} = 1.4 \times 10^{-8} \cos\left[6.5 \times 10^8\left(t - \frac{x}{3 \times 10^8}\right)\right] \mathbf{k} \text{ Wb m}^{-2}.$$

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; f:104.1e6; c:3e8; Erms:3;
(fpprintprec)     5
(f)   1.041*10^8
(c)   3.0*10^8
(Erms) 3
(%i5) E0: float(sqrt(2)*Erms);
(E0)  4.2426
(%i6) omega: float(2*%pi*f);
(omega) 6.5408*10^8
(%i7) B0: E0/c;
(B0)  1.4142*10^-8
```

Comments on the codes:

(%i4)	Set floating point print precision to 5, assign values of f, c, and E_{rms}.
(%i5), (%i6), (%i7)	Calculate E_0, ω, and B_0.

Problem 16.11 A plane sinusoidal electromagnetic wave propagates in the positive x direction. The electric and magnetic fields of the wave are

$$E = E_{max}\cos(kx - \omega t),$$
$$B = B_{max}\cos(kx - \omega t),$$

where ω and k are angular frequency and propagation constant, respectively. Frequency f, wavelength λ, and speed c of the wave are related as

$$\frac{\omega}{k} = \lambda f = c.$$

Sketch the wave at time $t = 0$.

Solution

Figure 16.1 shows the sketch of the wave. This is a *snap shot* of the wave at time $t = 0$.

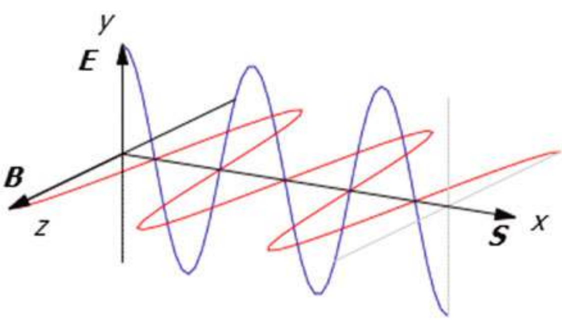


Fig. 16.1 Problem 16.11

The wave moves in the positive x direction with speed c. The electric field $\boldsymbol{E}$ is in the y-axis and the magnetic field $\boldsymbol{B}$ is in the z-axis. Directions of the fields satisfy

$$\boldsymbol{S} = \frac{1}{\mu_0} \boldsymbol{E} \times \boldsymbol{B}$$

and the right hand rule of cross product of two vectors.

The electric field is

$$\boldsymbol{E} = E_{max} \cos(kx - \omega t)\ \mathbf{j}$$

The magnetic field is,

$$\boldsymbol{B} = B_{max} \cos(kx - \omega t)\ \mathbf{k}$$

The Poynting vector is

$$\boldsymbol{S} = \frac{1}{\mu_0} \boldsymbol{E} \times \boldsymbol{B} = \frac{E_{max} B_{max}}{\mu_0} \cos^2(kx - \omega t)\ \mathbf{i}.$$

♦ The following wxMaxima codes give an animation of a travelling electromagnetic wave:

```
(%i1) with_slider_draw3d(
      d, makelist(i,i,0,5,0.5),
      axis_3d=false,
      zrange=[-1,1.3],
      yrange=[-1.3,1],
      color=blue,nticks=50,parametric(x,0,cos(x-d),x,0,5*%pi),
      color=red,nticks=50,parametric( y, -cos(y-d),0,y,0,5*%pi),
      color=black,parametric(0,y,0,y,-1,1),
      color=black,head_length=0.5,head_angle=15,vector([0,0,0], [0,-1,0]),
      color=grey,parametric(5*%pi,y,0,y,-1,1),
      color=black,parametric(0,0,z,z,-1,1),
      color=black,head_length=0.5,head_angle=15,vector([0,0,0], [0,0,1]),
      color=grey,parametric(5*%pi,0,z,z,-1,1),
      color=black,head_length=0.5,head_angle=15,vector([0,0,0], [19,0,0]),
      color=black,
      label(["{/Helvetica-Italic y}", -0.9 ,0,1.2]),
      label(["{/Helvetica-Italic-Bold E}", -1.8 ,0,0.8]),
      label(["{/Helvetica-Italic z}", 1.5 ,-1,-0.1]),
      label(["{/Helvetica-Italic-Bold B}",-1.5,-0.7,0.1]),
      label(["{/Helvetica-Italic x}", 20, 0, 0]),
      label(["{/Helvetica-Italic-Bold S}", 18,0,-0.2])  );
```

Comments on the codes:

To run the animation, copy the codes to the wxMaxima command window; press <shift> and <enter> keys simultaneously to run the codes; right click the graphic that appears and choose *Start Animation.*

16.3 Summary

- The four Maxwell's equations and the Lorentz force law encompass the major laws of electricity and magnetism.
- The origin of electromagnetic waves is acceleration, deceleration, or oscillation of electric charges.
- For plane electromagnetic waves, the directions of the electric and magnetic fields of the wave, and the direction of the wave propagation, are all mutually perpendicular. The electromagnetic wave is a transverse wave of oscillating electric and magnetic fields.
- The speed of the electromagnetic wave c is related to the electric field E and magnetic field B as

$$c = \frac{E}{B} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}.$$

- The wavelength λ, frequency f, and speed c of an electromagnetic wave is related as

$$c = \lambda f.$$

- The rate that electromagnetic energy passes through a unit area is given by Poynting's vector $\boldsymbol{S}$

$$\boldsymbol{S} = \frac{1}{\mu_0} \boldsymbol{E} \times \boldsymbol{B}.$$

- Intensity of a plane sinusoidal electromagnetic wave is the average value of the Poynting vector

$$\begin{aligned} I = S_{average} &= \frac{E_{max} B_{max}}{2\mu_0} \\ &= \frac{E_{max}^2}{2\mu_0 c} = \frac{c B_{max}^2}{2\mu_0} \\ &= \frac{E_{rms}^2}{\mu_0 c} = \frac{c B_{rms}^2}{\mu_0}. \end{aligned}$$

- For electromagnetic waves with Poynting vector S incident normally to a surface and fully absorbed by the surface, the radiation pressure p is

$$p = \frac{S}{c}.$$

16.4 Exercises

Exercise 16.1 A 60.0 W light bulb radiates light uniformly in all direction. Calculate the average intensity, rms value of the electric field, and rms value of magnetic field at a distance 0.400 m from the bulb.

(Answer: $I = 29.8$ W m^{-2}, $E_{rms} = 106$ V m^{-1}, $B_{rms} = 3.54 \times 10^{-7}$ T)

Exercise 16.2 The amplitude of electric field of a plane electromagnetic wave is 100 V m^{-1}. What is the intensity of the wave?

(Answer: 13 W m^{-2})

Exercise 16.3 Write the equations for the electric and magnetic fields of a plane radio wave from a 88.9 MHz radio station. The wave is traveling in the positive x direction and the rms electric field is 3.0×10^{-3} V m^{-1}.

(Answer: $E = 4.2 \times 10^{-3}\ \sin(1.9x - 5.6 \times 10^{8}t)\,\mathbf{j}$ V m^{-1},

$B = 1.4 \times 10^{-11}\ \sin(1.9x - 5.6 \times 10^{8}t)\,\mathbf{k}$ T)

Exercise 16.4 The electric field component $\boldsymbol{E}$ of a plane electromagnetic wave travelling in the positive z direction is given by

$$\boldsymbol{E} = 100\ \sin(9.4 \times 10^{6}z - 2.8 \times 10^{15}t)\,\mathbf{i}\ \text{V m}^{-1}.$$

(a) Determine the speed, frequency, wavelength, period, electric field amplitude.
(b) Write an expression for the magnetic field component $\boldsymbol{B}$ of the electromagnetic wave.

(Answer: (a) 3.0×10^{8} m s^{-1}, 4.5×10^{14} Hz, 6.7×10^{-7} m,

2.2×10^{-15} s, 100 V m^{-1};

(b) $\boldsymbol{B} = 3.3 \times 10^{-7}\ \sin(9.4 \times 10^{6}z - 2.8 \times 10^{15}t)\,\mathbf{j}$ T)

Exercise 16.5 Sunlight with energy flux of 1000 W m^{-2} incidents normally on a mirror of area 0.30 m^{2}.

(a) What is the energy delivered in one minute?
(b) Calculate the radiation pressure on the mirror.

(Answer: (a) 1.8×10^{4} J (b) 6.7×10^{-6} N m^{-2})

Chapter 17
Light Phenomena

Abstract This chapter solves problems on geometrical optics. These include problems on light reflection, refraction, total internal reflection, dispersion, and polarization. Problems are solved analytically and by computer calculation.

17.1 Basic Concepts and Formulae

(1) Lights are electromagnetic waves. Speed of light, c, in vacuum is,

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \text{ m s}^{-1},$$

where $\varepsilon_0 = 8.85 \times 10^{-12}$ F m^{-1} is permittivity of free space and $\mu_0 = 4\pi \times 10^{-7}$ H m^{-1} is permeability of free space. Electric and magnetic fields of a light wave are perpendicular to each other and the fields are perpendicular to direction of light propagation. Light waves are transverse waves.

The frequency of light is in the range of 4.0×10^{14} Hz (red) to 7.5×10^{14} Hz (violet). This corresponds to wavelength in the range of 7.5×10^{-7} m (red) to 4.0×10^{-7} m (violet).

(2) In geometrical optics, light travels in a medium in a straight line called ray. The ray model of light describes the path of light as straight lines. Geometrical optics deals with the ray aspect of light.

(3) Law of reflection states that angle of incident θ_i is equal to angle of reflection θ_r. Incident ray, reflected ray, and normal to the reflecting surface lie in the same plane, as shown in Figure 17.1.

(4) Law of refraction or Snell's law states that

$$n_1 \sin\theta_1 = n_2 \sin\theta_2, \tag{17.1}$$

W. M. S. Wan Hassan et al., *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-43165-4_17

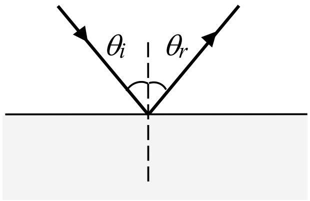


Fig. 17.1 Law of reflection, angle of incident is equal to angle of reflection

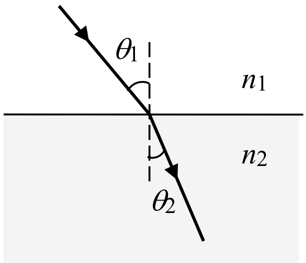


Fig. 17.2 Snell's law or law of refraction, $n_1 \sin\theta_1 = n_2 \sin\theta_2$

where θ_1 and θ_2 are angles of incidence and refraction, while n_1 and n_2 are indices of refraction of first and second media, respectively. Incident ray, refracted ray, and normal to the refracting surface lie in the same plane, as shown in Figure 17.2.

(5) Index of refraction (refractive index) of a medium is

$$n = \frac{c}{v}, \tag{17.2}$$

where c is speed of light in vacuum and v is speed of light in the medium. Also, index of refraction is

$$n = \frac{\lambda_0}{\lambda_n}, \tag{17.3}$$

where λ_0 is wavelength of light in vacuum and λ_n is wavelength of light in the medium.

(6) Huygen's principle states that every point on the wave front is a point wave source producing a wavelet. At a later time, the wave front is a surface tangent to the wavelets.

(7) Total internal reflection can occur when light travels from a medium with higher index of refraction to another with a lower one. The minimum incident angle, θ_c, for the total internal reflection is given by

$$\sin\theta_c = \frac{n_2}{n_1}, \qquad (n_1 > n_2) \tag{17.4}$$

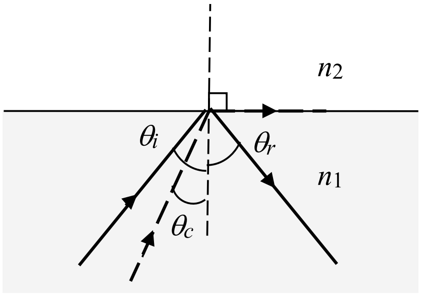


Fig. 17.3 Total internal reflection occurs when angle of incident θ_i is larger than critical angle θ_c. Angle of incident θ_i is equal to angle of reflection θ_r

where n_1 and n_2 are indices of refraction of light in medium 1 and 2, respectively, as shown in Figure 17.3. The incident and reflected lights are both in medium 1. The angle of incident is larger than θ_c for the total internal reflection to occur.

(8) Dispersion is spreading of white light into spectrum of wavelengths. Rainbows are produced by a refraction, reflection, and dispersion of sunlight into colors by water droplets in the air.

(9) Polarization is the attribute that wave oscillations have a definite direction relative to the direction of propagation of the wave. The direction of polarization is defined as the direction parallel to the electric field of the electromagnetic wave.

(10) Un-polarized light can be polarized by four processes: (a) selective absorption, (b) reflection, (c) double refraction, and (d) dispersion.

(11) When a polarized light of intensity I_0 is incident to a polarizer film, the intensity of light, I, that passes the film is

$$I = I_0 \cos^2 \theta, \tag{17.5}$$

where θ is the angle between the polarizer transmission axis and the electric field vector of incident light.

(12) A light reflected from a dielectric material, for example glass, is partially polarized. However, the reflected light is completely polarized if the incident angle is such that the angle between the reflected and the refracted lights is 90°. The incident angle is called the polarizing angle, θ_P, and,

$$n = \tan \theta_P, \tag{17.6}$$

where n is the index of refraction of the medium. The equation represents the Brewster's law. In other words, Brewster's law states that reflected light is completely polarized at the angle of reflection, θ_P, known as Brewster's angle, as illustrated in Figure 17.4.

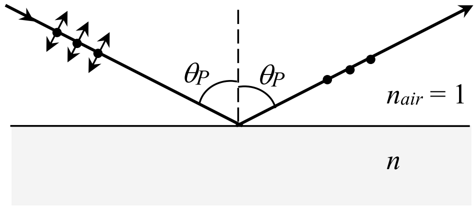


Fig. 17.4 Brewster's law, $n = \tan\theta_P$

17.2 Problems and Solutions

Problem 17.1 A yellow light beam of wavelength 5890 Å travels in air, water, glass, and air, as shown in Fig. 17.5.

(a) Calculate angles θ_2, θ_3, and θ_4 if the incident angle is 40° and refractive indices of air, water, and glass are 1.00, 1.33, and 1.52, respectively.
(b) What are wavelength and speed of the yellow light in water and glass?

Solution

(a) UsingSnell's law (Eq. 17.1) at the air–water interface,

$$n_{air}\sin 40^\circ = n_{water}\sin\theta_2, \tag{17.7}$$

giving,

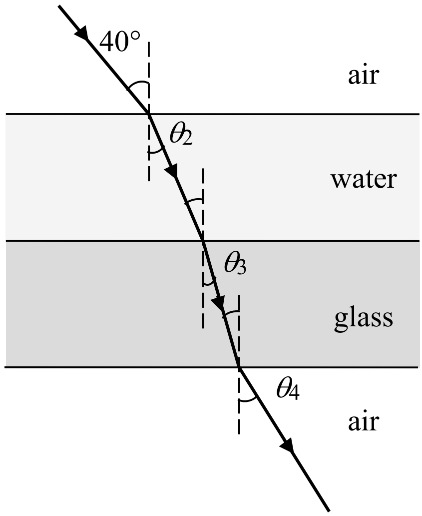


Fig. 17.5 A beam of light undergoing multiple refractions, Problem 17.1

$$1.00 \times \sin 40° = 1.33 \times \sin \theta_2,$$
$$\theta_2 = 28.9°.$$

At the water–glass interface,

$$n_{water} \sin \theta_2 = n_{glass} \sin \theta_3,$$
$$1.33 \times \sin 28.9° = 1.52 \times \sin \theta_3$$
$$\theta_3 = 25.0°. \tag{17.8}$$

At the glass–air interface,

$$n_{glass} \sin \theta_3 = n_{air} \sin \theta_4,$$
$$1.52 \times \sin 25° = 1.00 \times \sin \theta_4$$
$$\theta_4 = 40.0°. \tag{17.9}$$

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; ratprint:false; n_air:1; n_water:1.33; n_glass:1.52;
(fpprintprec)     5
(ratprint)  false
(n_air)     1
(n_water)   1.33
(n_glass)   1.52
(%i7) solve(n_air*sin(40*%pi/180)=n_water*sin(theta2), theta2)$ float(%);
solve: using arc-trig functions to get a solution.
Some solutions will be lost.
(%o7) [theta2=0.50442]
(%i8) theta2: rhs(%[1]);
(theta2) 0.50442
(%i9) theta2_deg: float(theta2*180/%pi);
(theta2_deg) 28.901
(%i11) solve(n_water*sin(theta2)=n_glass*sin(theta3), theta3)$ float(%);
solve: using arc-trig functions to get a solution.
Some solutions will be lost.
(%o11) [theta3=0.43663]
(%i12) theta3: rhs(%[1]);
(theta3) 0.43663
(%i13) theta3_deg: float(theta3*180/%pi);
(theta3_deg) 25.017
(%i15) solve(n_glass*sin(theta3)=n_air*sin(theta4), theta4)$ float(%);
solve: using arc-trig functions to get a solution.
Some solutions will be lost.
(%o15) [theta4=0.69813]
(%i16) theta4: rhs(%[1]);
(theta4) 0.69813
(%i17) theta4_deg: float(theta4*180/%pi);
(theta4_deg) 40.0
```

Comments on the codes:

(%i5)	Set floating point print precision to 5, internal rational number print to false, assign values of n_{air}, n_{water}, and n_{glass}.
(%i7), (%i8) (%i9)	Solve Eq. (17.7) for θ_2, assign value of θ_2, convert θ_2 to degree.
(%i11), (%i12) (%i13)	Solve Eq. (17.8) for θ_3, assign value of θ_3, convert θ_3 to degree.
(%i15), (%i16) (%i17)	Solve Eq. (17.9) for θ_4, assign value of θ_4, convert θ_4 to degree.

(b) The wavelength and speed of yellow light in water are (Eqs. 17.3 and 17.2),

$$\lambda_{water} = \frac{\lambda}{n_{water}} = \frac{5890\ \text{Å}}{1.33} = 4429\ \text{Å},$$
$$v_{water} = \frac{c}{n_{water}} = \frac{3 \times 10^8\ \text{m/s}}{1.33} = 2.26 \times 10^8\ \text{m s}^{-1}.$$

The wavelength and speed of yellow light in glass are,

$$\lambda_{glass} = \frac{\lambda}{n_{glass}} = \frac{5890\ \text{Å}}{1.52} = 3875\ \text{Å},$$
$$v_{glass} = \frac{c}{n_{glass}} = \frac{3 \times 10^8\ \text{m/s}}{1.52} = 1.97 \times 10^8\ \text{m s}^{-1}.$$

♦ wxMaxima codes:

```
(%i5) fpprintprec:5; lambda:5890; n_water:1.33; n_glass:1.52; c:3e8;
(fpprintprec)     5
(lambda)    5890
(n_water)   1.33
(n_glass)   1.52
(c)   3.0*10^8
(%i7) lambda_water: lambda/n_water; v_water: c/n_water;
(lambda_water) 4428.6
(v_water) 2.2556*10^8
(%i9) lambda_glass: lambda/n_glass; v_glass: c/n_glass;
(lambda_glass) 3875.0
(v_glass) 1.9737*10^8
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of λ, n_{water}, n_{glass}, and c.
(%i7) Calculate λ_{water} and v_{water}.
(%i9) Calculate λ_{glass} and v_{glass}.

Problem 17.2 A fish swims at a depth of 1.0 m in water. What is the apparent depth as seen from above? Refractive index of water is 1.33.

Solution

Figure 17.6 shows a refracted ray from the fish to the observer.

Using the law of refraction or Snell's law (Eq. 17.1), we write,

$$n_{water} \sin\theta_1 = n_{air} \sin\theta_2.$$

The angles θ_1 and θ_2 are small so that $\sin\theta_1 \approx \theta_1 \approx x/d$ and $\sin\theta_2 \approx \theta_2 \approx x/l$. We write,

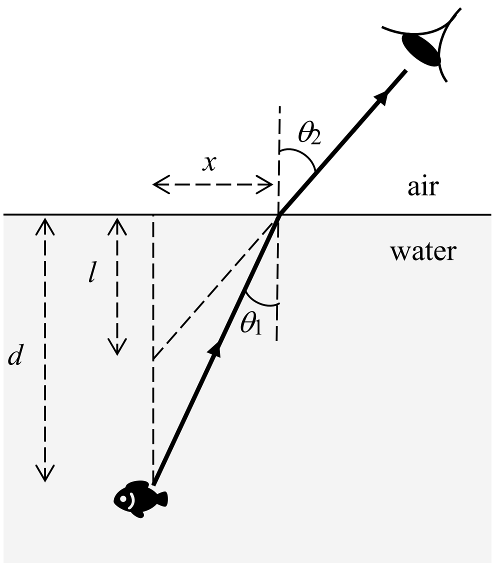
$$n_{water}\frac{x}{d} = n_{air}\frac{x}{l}.$$

The apparent depth is

$$l = \frac{n_{air}}{n_{water}}d = \frac{1.00}{1.33} \times 1.0\ \text{m} = 0.75\ \text{m}.$$

This also means that

Fig. 17.6 Refraction of light ray, Problem 17.2



$$\frac{n_{water}}{n_{air}} = \frac{\text{depth}}{\text{apparent depth}}.$$

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; ratprint:false; n_water:1.33; n_air:1; d:1;
(fpprintprec)     5
(ratprint)  false
(n_water)   1.33
(n_air)     1
(d)   1
(%i7) solve(n_water/d = n_air/l, l)$ float(%);
(%o7) [l=0.75188]
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of n_{water}, n_{air}, and d.
(%i7) Solve $n_{water}/d = n_{air}/l$ for l.

Problem 17.3 Calculate the critical angle of diamond. Indices of refraction of diamond and air are 2.42 and 1.00, respectively.

Solution

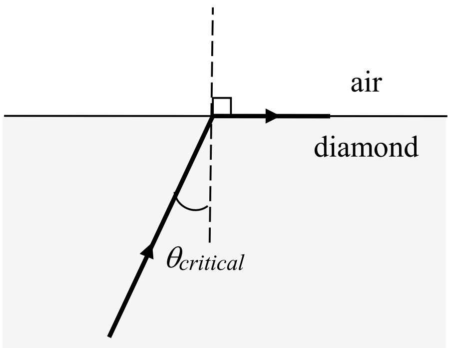
Figure 17.7 shows the critical angle of diamond.
UsingSnell's law (Eq. 17.1),

$$n_{diamond} \sin \theta_{diamond} = n_{air} \sin \theta_{air}.$$

At critical angle,

$$\begin{aligned} n_{diamond} \sin \theta_{critical} &= n_{air} \sin 90°, \\ 2.42 \times \sin \theta_{critical} &= 1.00 \times \sin 90°, \\ \theta_{critical} &= 24.4°. \end{aligned}$$

Fig. 17.7 Critical angle, Problem 17.3



Total internal reflection occurs if angle of incident is greater than 24.4°.

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; ratprint:false; n_diamond:2.42; n_air:1;
(fpprintprec)     5
(ratprint)  false
(n_diamond) 2.42
(n_air)     1
(%i6) solve(n_diamond*sin(theta_critical)=n_air*sin(90/180*%pi),
theta_critical)$ float(%);
solve: using arc-trig functions to get a solution.
Some solutions will be lost.
(%o6) [theta_critical=0.42599]
(%i7) theta_critical: rhs(%[1]);
(theta_critical) 0.42599
(%i8) theta_critical_deg: float(theta_critical*180/%pi);
(theta_critical_deg) 24.407
```

Comments on the codes:

(%i4)	Set floating point print precision to 5, internal rational number print to false, assign values of $n_{diamond}$ and n_{air}.
(%i6)	Solve $n_{diamond} \times \sin\theta_{critical} = n_{air} \times \sin 90°$ for $\theta_{critical}$.
(%i7), (%i8)	Assign value of $\theta_{critical}$ and convert the angle to degree.

Problem 17.4 Figure 17.8 shows an observer seeing the bottom edge of a cylindrical tumbler. The diameter of the tumbler is 5.0 cm. When water with refraction index of 1.33 completely fills the tumbler, the observer can see the center of bottom of the tumbler *P*. Calculate the height of the tumbler *h*.

Solution

Figure 17.9 shows a light ray from the center of the bottom of the tumbler *P* being refracted in water and air to the observer. A ray from the bottom edge of the tumbler straight to the observer when there is no water in the tumbler is shown as well.

Using Snell's law (Eq. 17.1),

$$\begin{aligned} n_{water} \sin\theta_{water} &= n_{air} \sin\theta_{air}, \\ 1.33 \times \sin\theta_{water} &= 1.00 \times \sin\theta_{air}, \\ 1.33 \times \frac{2.5\text{ cm}}{\sqrt{h^2 + (2.5\text{ cm})^2}} &= \frac{5.0\text{ cm}}{\sqrt{h^2 + (5.0\text{ cm})^2}}. \end{aligned}$$

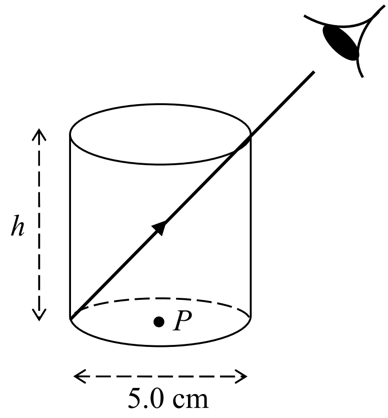


Fig. 17.8 Seeing the bottom edge of a cylindrical tumbler, Problem 17.4

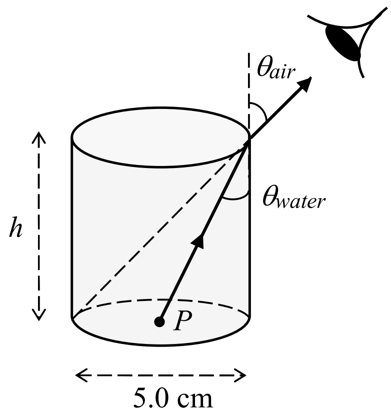


Fig. 17.9 Seeing the center of bottom of the tumbler *P*, Problem 17.4

Squaring the last equation and solving for h give the height of the glass,

$$1.33^2 \times \frac{(2.5\ \text{cm})^2}{h^2 + (2.5\ \text{cm})^2} = \frac{(5.0\ \text{cm})^2}{h^2 + (5.0\ \text{cm})^2},$$
$$h = 2.9\ \text{cm}.$$

♦ wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)     5
(ratprint)  false
(%i4) solve(1.33^2*2.5^2/(h^2 + 2.5^2) = 5^2/(h^2 + 5^2), h)$ float(%);
(%o4) [h=-2.9353,h=2.9353]
```

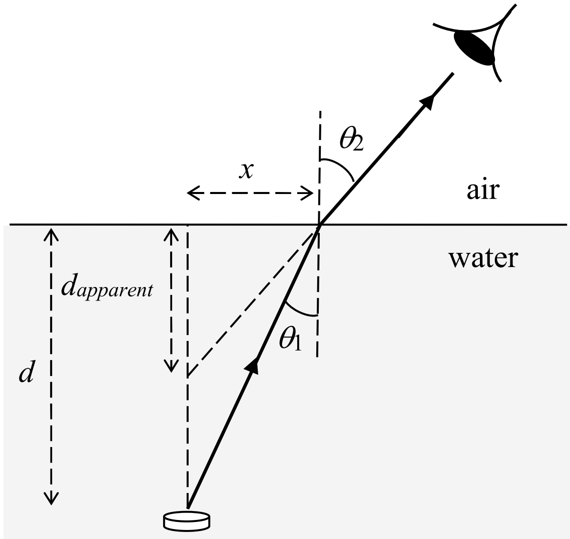


Fig. 17.10 Seeing a coin in water, Problem 17.5

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i4) Solve $1.33^2 \times 2.5^2/(h^2 + 2.5^2) = 5^2/(h^2 + 5^2)$ for h.

Problem 17.5 A coin is at the bottom of a water pool of 2.0 m deep. Calculate the apparent depth of the coin as seen from above. Index of refraction of water is 1.33.

Solution

Figure 17.10 shows the coin at the bottom of the pool and the refracted ray from the coin to the observer.

Using Snell's law (Eq. 17.1), we have,

$$n_{water} \sin\theta_1 = n_{air} \sin\theta_2.$$

Angles θ_1 and θ_2 are small so that $\sin\theta_1 \approx \theta_1 \approx x/d$ and $\sin\theta_2 \approx \theta_2 \approx x/d_{apparent}$. Here, d and $d_{apparent}$ are the depth and apparent depth of the coin, respectively. The equation becomes

$$n_{water}\frac{x}{d} = n_{air}\frac{x}{d_{apparent}}.$$

Therefore, the apparent depth of the coin is

$$d_{apparent} = \frac{n_{air}}{n_{water}}d = \frac{1}{1.33}(2.0\text{ m}) = 1.5\text{ m}.$$

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; n_air:1; n_water:1.33; d:2;
(fpprintprec)      5
(n_air)      1
(n_water)    1.33
(d)    2
(%i5) d_apparent: n_air/n_water*d;
(d_apparent) 1.5038
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of n_{air}, n_{water}, and d.
(%i5) Calculate $d_{apparent}$.

Problem 17.6 The wavelength of a red laser light in air is 632.8 nm.

(a) Calculate the frequency of the laser light.
(b) Determine the wavelength of the laser light in glass of refractive index 1.50.
(c) What is the speed of the laser light in the glass?

Solution

(a) The frequency of the laser light is

$$f_0 = \frac{c}{\lambda_0} = \frac{3 \times 10^8 \text{ m/s}}{632.8 \times 10^{-9} \text{ m}} = 4.74 \times 10^{14} \text{ s}^{-1}.$$

(b) The wavelength of the laser light in glass is calculated as follows (Eq. 17.3),

$$n_{glass} = \frac{\lambda_0}{\lambda_{glass}},$$

$$\lambda_{glass} = \frac{\lambda_0}{n_{glass}} = \frac{632.8 \times 10^{-9} \text{ m}}{1.50} = 422 \times 10^{-9} \text{ m}.$$

(c) The speed of the laser light in glass is calculated as follows (Eq. 17.2),

$$n_{glass} = \frac{c}{v_{glass}},$$

$$v_{glass} = \frac{c}{n_{glass}} = \frac{3 \times 10^8 \text{ m/s}}{1.50} = 2.00 \times 10^8 \text{ m s}^{-1}.$$

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; c:3e8; lambda0:632.8e-9; n_glass:1.5;
(fpprintprec)     5
(c)   3.0*10^8
(lambda0)   6.328*10^-7
(n_glass)   1.5
(%i5) f0: c/lambda0;
(f0)  4.7408*10^14
(%i6) lambda_glass: lambda0/n_glass;
(lambda_glass) 4.2187*10^-7
(%i7) v_glass: c/n_glass;
(v_glass) 2.0*10^8
```

Comments on the codes:

(%i4)	Set floating point print precision to 5, assign values of c, λ_0, and n_{glass}.
(%i5), (%i6), (%i7)	Calculate f_0, λ_{glass}, and v_{glass}.

Problem 17.7 A light beam of wavelength 550 nm is incident at 40° to glass and is refracted by 25°. Calculate the index of refraction of the glass and the wavelength of light in it.

Solution

Figure 17.11 shows the light beam traveling the air and the glass.

Using Snell's law (Eq. 17.1), the index of refraction of the glass is calculated as follows:

$$n_{air} \sin\theta_{air} = n_{glass} \sin\theta_{glass},$$
$$n_{glass} = \frac{n_{air} \sin\theta_{air}}{\sin\theta_{glass}} = \frac{1.00 \times \sin 40°}{\sin 25°} = 1.52.$$

The wavelength of light in the glass is (Eq. 17.3),

$$\lambda_{glass} = \frac{\lambda_{air}}{n_{glass}} = \frac{550 \text{ nm}}{1.52} = 362 \text{ nm}.$$

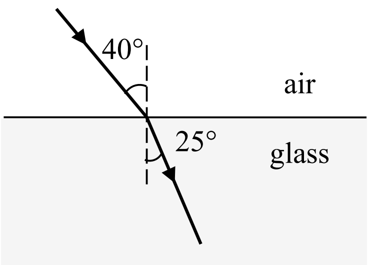


Fig. 17.11 Refraction of light, Problem 17.7

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; theta_air:float(40*%pi/180);
theta_glass:float(25*%pi/180); n_air:1; lambda_air:550;
(fpprintprec)     5
(theta_air) 0.69813
(theta_glass) 0.43633
(n_air) 1
(lambda_air) 550
(%i6) n_glass: n_air*sin(theta_air)/sin(theta_glass);
(n_glass) 1.521
(%i7) lambda_glass: lambda_air/n_glass;
(lambda_glass) 361.61
```

Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of θ_{air}, θ_{glass}, n_{air}, and λ_{air}.
(%i6), (%7) Calculate n_{glass} and λ_{glass}.

Problem 17.8 A beam of light of wavelength 590 nm is incident at 30° to water. The index of refraction of water is 1.33. Calculate

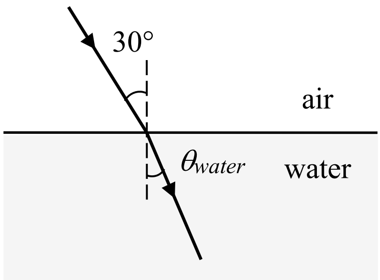
(a) the angle of refraction in water,
(b) the speed and wavelength of the light in water.

Solution

(a) Fig. 17.12 shows the ray of light traveling from air to water.
Using Snell's law (Eq. 17.1), the angle of refraction in water is calculated as follows,

$$\begin{aligned} n_{air} \sin\theta_{air} &= n_{water} \sin\theta_{water}, \\ \sin\theta_{water} &= \frac{n_{air}\sin\theta_{air}}{n_{water}} = \frac{1.00 \times \sin 30°}{1.33} = 1.52, \\ \theta_{water} &= \sin^{-1} 1.52 = 0.39 \text{ rad} = 22°. \end{aligned}$$

Fig. 17.12 Refraction of light, Problem 17.8



(b) The speed of light in water is (Eq. 17.2),

$$v_{water} = \frac{c}{n_{water}} = \frac{3 \times 10^8 \text{ m/s}}{1.33} = 2.26 \times 10^8 \text{ m s}^{-1}.$$

The wavelength of the light in water is (Eq. 17.3),

$$\lambda_{water} = \frac{\lambda_{air}}{n_{water}} = \frac{590 \text{ nm}}{1.33} = 444 \text{ nm}.$$

◆ wxMaxima codes:

```
(%i6) fpprintprec:5; theta_air:float(30*%pi/180); n_air:1; n_water:1.33;
lambda_air: 590; c: 3e8;
(fpprintprec)     5
(theta_air) 0.5236
(n_air) 1
(n_water) 1.33
(lambda_air) 590
(c)   3.0*10^8
(%i7) theta_water: asin(n_air*sin(theta_air)/n_water);
(theta_water) 0.38541
(%i8) theta_water_deg: float(theta_water*180/%pi);
(theta_water_deg) 22.082
(%i9) v_water: c/n_water;
(v_water) 2.2556*10^8
(%i10) lambda_water: lambda_air/n_water;
(lambda_water) 443.61
```

Comments on the codes:

(%i6)	Set floating point print precision to 5, assign values of θ_{air}, n_{air}, n_{water}, λ_{air}, and c.
(%i7), (%i8)	Calculate θ_{water} and convert the angle to degree.
(%i9), (%i10)	Calculate v_{water} and λ_{water}.

Problem 17.9 Figure 17.13 shows a light beam is refracted by a glass prism of 60°. Angle of incident is 30° and index of refraction of glass is 1.52. Calculate,

(a) the angle the light beam exits the prism θ_4,
(b) the angle of deviation of the light beam θ_D.

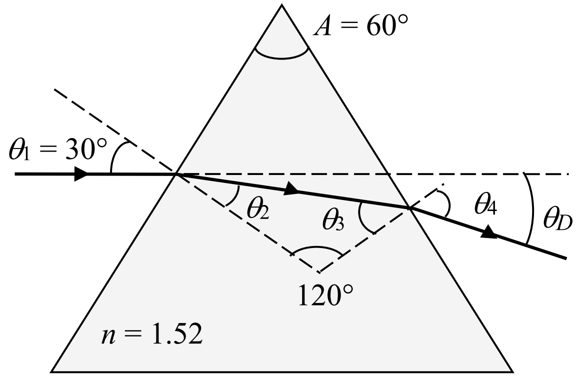


Fig. 17.13 Refraction of light by a glass prism, Problem 17.9

Solution

(a) Using Snell's law (Eq. 17.1) at the left surface of the prism,

$$
\begin{aligned}
n_{air}\sin\theta_1 &= n\sin\theta_2,\\
(1.00)\sin 30^\circ &= (1.52)\sin\theta_2,\\
\theta_2 &= 19^\circ.
\end{aligned}
$$

where θ_1 is angle of incident, θ_2 is angle of refraction, and n is refraction index of the glass. Also, from Fig. 17.13, we have,

$$
\begin{aligned}
\theta_2 + \theta_3 + 120^\circ &= 180^\circ,\\
\theta_3 &= 180^\circ - 120^\circ - \theta_2\\
&= 180^\circ - 120^\circ - 19^\circ\\
&= 41^\circ.
\end{aligned}
$$

Using Snell's law at the right surface of the prism,

$$
\begin{aligned}
n\sin\theta_3 &= n_{air}\sin\theta_4,\\
(1.52)\sin 41^\circ &= (1.00)\sin\theta_4,\\
\theta_4 &= 83^\circ.
\end{aligned}
$$

where θ_3 in angle of incident and θ_4 is angle of refraction.

(b) Deviation at the left surface of the prism is $\theta_1 - \theta_2$ and deviation at the right one is $\theta_4 - \theta_3$. Thus, the deviation of the beam is

$$
\begin{aligned}
\theta_D &= \theta_1 - \theta_2 + \theta_4 - \theta_3\\
&= \theta_1 + \theta_4 - (\theta_2 + \theta_3)\\
&= \theta_1 + \theta_4 - A
\end{aligned}
$$

$$= 30° + 83° - 60°$$
$$= 53°.$$

because $A = \theta_2 + \theta_3$.

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; n:1.52; theta1:float(30*%pi/180); A:60;
(fpprintprec)     5
(n) 1.52
(theta1) 0.5236
(A) 60
(%i6) theta2: asin(sin(theta1)/n); theta2_deg: float(theta2*180/%pi);
(theta2) 0.33519
(theta2_deg) 19.205
(%i8) theta3_deg: 180-120-theta2_deg; theta3: float(theta3_deg/180*%pi);
(theta3_deg) 40.795
(theta3) 0.71201
(%i10) theta4: asin(n*sin(theta3)); theta4_deg: float(theta4*180/%pi);
(theta4) 1.4533
(theta4_deg) 83.266
(%i11) thetaD_deg: 30 + theta4_deg - A;
(thetaD_deg) 53.266
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of n, θ_1 in radian, and A in degree.
(%i6) Calculate θ_2 and convert the angle to degree.
(%i8) Calculate θ_3 in degree and convert the angle to radian.
(%i10) Calculate θ_4 and convert the angle to degree.
(%i11) Calculate θ_D in degree.

Problem 17.10 Figure 17.14 shows a light beam being refracted symmetrically by a prism of angle A and refractive index n. The incident beam and the beam coming out of the prism are symmetric. The deviation of the beam is a minimum $\theta_{D,min}$. Show that the refractive index of the prism is

$$n = \sin\left(\frac{A + \theta_{D,min}}{2}\right)/\sin\left(\frac{A}{2}\right).$$

Solution

Figure 17.15 shows the beam, the prism, and the related angles when the deviation of the beam is at minimum.

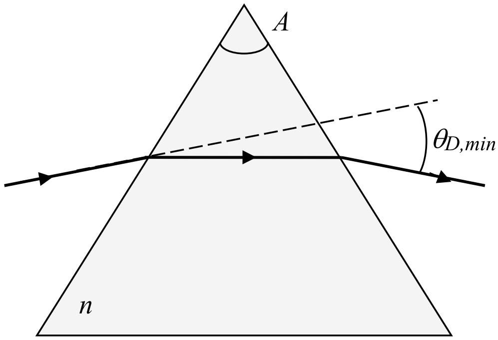


Fig. 17.14 Refraction of light by a glass prism, Problem 17.10

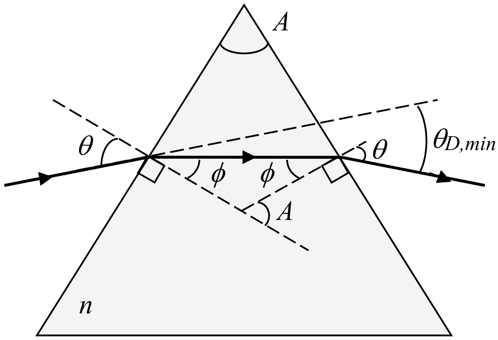


Fig. 17.15 Minimum deviation of the light beam, Problem 17.10

At the left side of the prism, angle of incident is θ and angle of refraction is ϕ. At the right side of the prism, angle of incident is ϕ and angle of refraction is θ. From trigonometry, we have

$$2\phi = A,$$
$$\phi = \frac{A}{2}.$$

Beam deviation at the left side of the prism is $\theta - \phi$ while beam deviation at the right side of the prism is $\theta - \phi$, giving total beam deviation as

$$\theta_{D,min} = (\theta - \phi) + (\theta - \phi) = 2\theta - 2\phi.$$

We calculate the angle θ,

$$\theta_{D,min} = 2\theta - 2\phi = 2\theta - A,$$
$$\theta = \frac{A + \theta_{D,min}}{2}.$$

Applying Snell's law (Eq. 17.1) at the right side of the prism, we obtain the refractive index of the prism,

$$\begin{aligned} n \ \sin\phi &= n_{air}\sin\theta, \\ n \ \sin\phi &= 1.00\times\sin\theta, \\ n \ \sin\left(\frac{A}{2}\right) &= \sin\left(\frac{A+\theta_{D,min}}{2}\right), \\ n &= \sin\left(\frac{A+\theta_{D,min}}{2}\right)/\left(\frac{A}{2}\right). \end{aligned}$$

◆ wxMaxima codes:

```
(%i1) theta: (A+theta_Dmin)/2;
(theta) (theta_Dmin+A)/2
(%i2) phi: A/2;
(phi) A/2
(%i3) n: sin(theta)/sin(phi);
(n)   sin((theta_Dmin+A)/2)/sin(A/2)
```

Comments on the codes:

(%i1), (%i2) Define θ and ϕ.
(%i3) Calculate n.

Problem 17.11 The minimum beam deviation of a 60° prism is 37°. What is the refractive index of the prism?

Solution

Using result of Problem 17.10, the refractive index of the prism is

$$\begin{aligned} n &= \sin\left(\frac{A+\theta_{D,min}}{2}\right)/\sin\left(\frac{A}{2}\right) \\ &= \sin\left(\frac{60^\circ+37^\circ}{2}\right)/\sin\left(\frac{60^\circ}{2}\right) \\ &= 1.5. \end{aligned}$$

◆ wxMaxima codes:

```
(%i3) fpprintprec:5; A:float(60*%pi/180); thetaDmin:float(37*%pi/180);
(fpprintprec)     5
(A)   1.0472
(thetaDmin) 0.64577
(%i4) n: sin((A+thetaDmin)/2)/sin(A/2);
(n)   1.4979
```

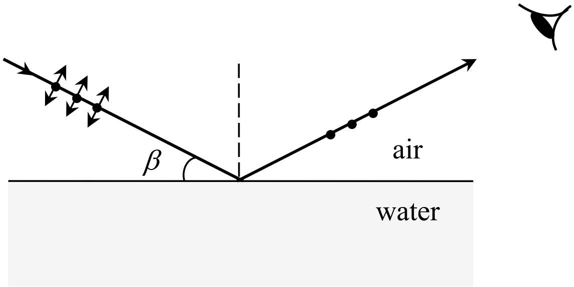


Fig. 17.16 A plane polarized light reflected from water, Problem 17.12

Comments on the codes:

(%i3) Set floating point print precision to 5, assign values of A and $\theta_{D,min}$.
(%i4) Calculate n.

Problem 17.12 Figure 17.16 shows a completely plane polarized light reflected from the surface of water. The index of reflection of water is 1.33. Calculate angle β.

Solution

Figure 17.17 shows the situation when a completely plane polarized light is obtained.

Brewster's law (Eq. 17.6) is satisfied. The polarizing angle θ_P can be calculated as follows:

$$\tan\theta_P = \frac{n_{water}}{n_{air}} = \frac{1.33}{1.00},$$
$$\theta_P = 53°.$$

The angle β is

$$\beta = 90° - \theta_P = 90° - 53° = 37°.$$

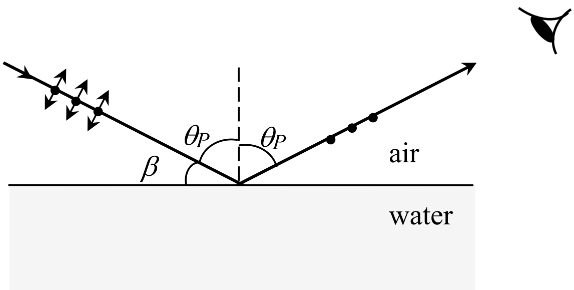


Fig. 17.17 The polarizing angle θ_P, Problem 17.12

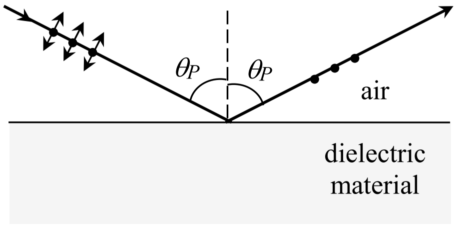


Fig. 17.18 Brewster's law, Problem 17.13

◆ wxMaxima codes:

```
(%i3) fpprintprec:5; n_water:1.33; n_air:1;
(fpprintprec)     5
(n_water) 1.33
(n_air) 1
(%i5) thetaP: atan(n_water/n_air); thetaP_deg: float(thetaP*180/%pi);
(thetaP) 0.92609
(thetaP_deg) 53.061
(%i6) beta: 90-thetaP_deg;
(beta) 36.939
```

Comments on the codes:

(%i3) Set floating point print precision to 5, assign values of n_{water} and n_{air}.
(%i5) Calculate θ_P and convert the angle to degree.
(%i6) Calculate β.

Problem 17.13 The polarizing angle for reflected rays of a dielectric material is 58°. What is the index of refraction of the material?

Solution

Using Brewster's law (Eq. 17.6), the index of refraction of the material is, Fig. 17.18,

$$n = \tan\theta_P = \tan\ 58^\circ = 1.60.$$

◆ wxMaxima codes:

```
(%i2) fpprintprec:5; thetaP:58;
(fpprintprec)     5
(thetaP) 58
(%i4) n: tan(thetaP*%pi/180); float(%);
(n)   tan((29*%pi)/90)
(%o4) 1.6003
```

Comment on the codes:

(%i2) Set floating point print precision to 5 and assign value of θ_P.
(%i4) Calculate *n*.

17.3 Summary

- Light is an electromagnetic wave propagating in vacuum with a speed of 3×10^8 m s^{-1}.
- When a light ray strikes a smooth surface, the angle of reflection equals the angle of incident.
- The law of refraction or Snell's law relates the indices of refraction for two media with the angles of incident and refraction of a light ray in them,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2.$$

- Total internal reflection occurs at the boundary between two media if the incident angle in the first medium is greater than the critical angle, θ_c.

$$\theta_c = \sin^{-1}\frac{n_2}{n_1}, \qquad (n_1 > n_2).$$

- Polarization is the attribute that wave oscillations have a definite direction relative to the direction of propagation of the wave.
- Brewster's law states that reflected light is completely polarized at the angle of reflection, θ_P,

$$\theta_P = \tan^{-1} n,$$

  where the light is incident from air and reflected from a medium with index of refraction, *n*.

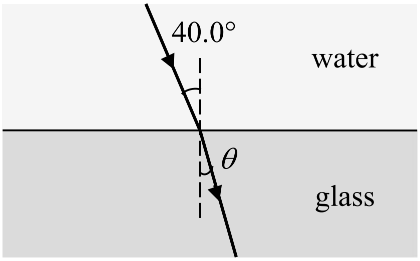


Fig. 17.19 Refraction of light, Exercise 17.1

17.4 Exercises

Exercise 17.1 A beam of light in water enters a glass slab at an angle of incident of 40.0°, Fig. 17.19. Index of refraction of water is 1.33 and that of glass is 1.50. What is the angle of refraction, θ?

(Answer: $\theta = 34.7°$)

Exercise 17.2 Index of refraction of benzene is 1.50. What is the speed of light in benzene?

(Answer: 2.00×10^8 m s^{-1})

Exercise 17.3 A person looks into a swimming pool at the 1.52 m deep level. How deep does it look to the person? Index of refraction of water is 1.33.

(Answer: 1.14 m)

Exercise 17.4 A vertically polarized light of intensity 100 W m^{-2} passes through a polarizer with its transmission axis at 35.0° to the vertical. What is the transmitted intensity of the light?

(Answer: 67.1 W m^{-2})

Exercise 17.5 Calculate Brewster's angle for light reflected from the top of a water surface. Index of refraction of water is 1.33.

(Answer: $\theta_P = 53.1°$)

Chapter 18
Mirror and Lens

Abstract This chapter solves problems on image formation by mirrors, spherical surfaces, and lenses using geometrical or ray optics. Calculations of image size, location, and magnification are performed. Spherical mirror, refraction at a spherical surface, lens maker, and thin lens equations are applied. Solutions are by analysis and computer calculation of wxMaxima.

18.1 Basic Concepts and Formulae

(1) Magnification M of a mirror or lens is defined as ratio of image height h' to object height h or ratio of image distance s' to object distance s,

$$M = \frac{h'}{h} = -\frac{s'}{s}. \tag{18.1}$$

Magnification of less than 1 is a minification while magnification of 1 means object and image are of the same size.

Figure 18.1 shows examples of magnification by (a) concave mirror, (b) convex mirror, (c) convex lens, and (d) concave lens. Here, f is focal length, F is focus point, and C is center of the curvature.

(2) For a spherical mirror of radius, R, the object distance, s, and image distance, s', obey the mirror equation,

$$\frac{1}{s} + \frac{1}{s'} = \frac{2}{R} = \frac{1}{f}, \tag{18.2}$$

where $f = R/2$ is focal length of the mirror. Sign convention for spherical mirror is as follows:

(a) s is + if the object is in front of the mirror (real object).
(b) s is − if the object is behind the mirror (virtual object).
(c) s' is + if the image is in front of the mirror (real image).

W. M. S. Wan Hassan et al., *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-43165-4_18

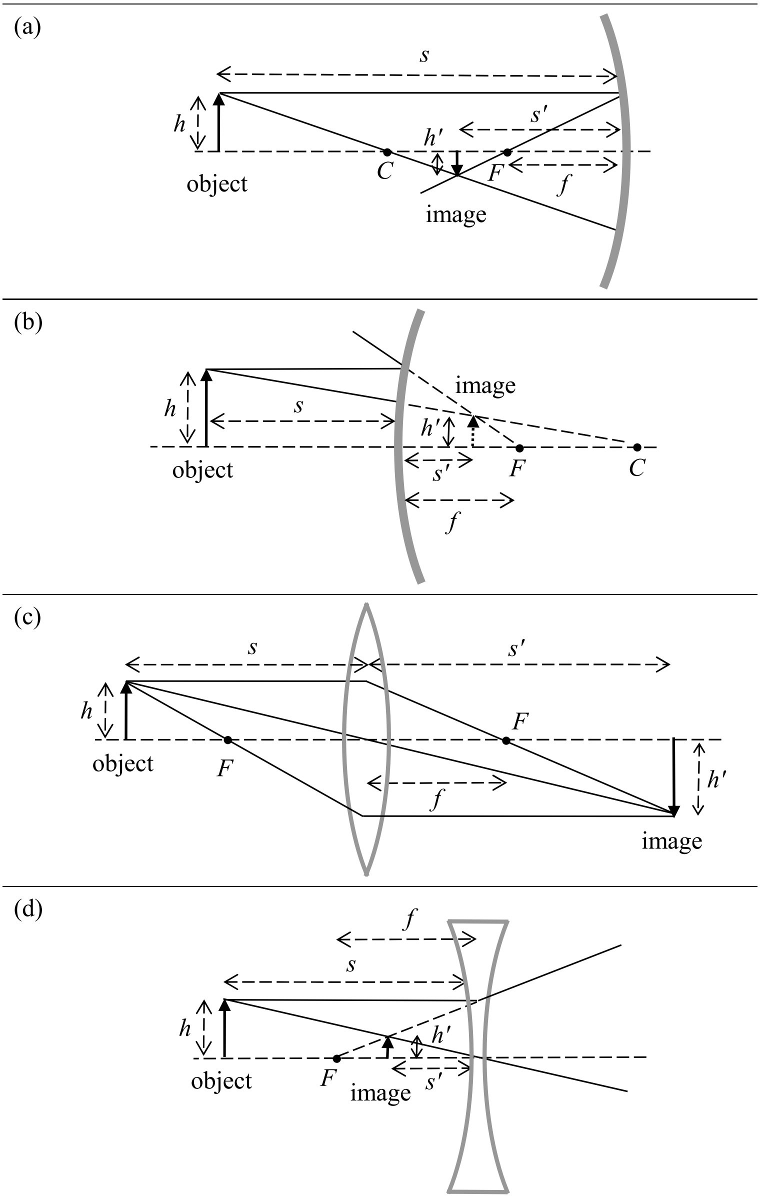


Fig. 18.1 Magnifications by concave and convex mirrors, convex and concave lenses

(d) s' is − if the image is behind the mirror (virtual image).
(e) f and R are + if the center of curvature is in front of the mirror (concave mirror).
(f) f and R are − if the center of curvature is behind the mirror (convex mirror).
(g) M is + means upright image.
(h) M is − means inverted image.

Figure 18.2 shows ray diagrams for image formation in (a) concave and (b) convex spherical mirrors. Here, f is focal length, F is focus point, C is center of curvature, and R is radius of curvature.

(3) For refraction at a spherical surface,

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R}, \tag{18.3}$$

where n_1 and n_2 are refractive indices of medium 1 and 2, and R is the radius of the spherical surface. Sign convention for spherical surface refraction,

(a) s is + if the object is in front of the surface (real object).

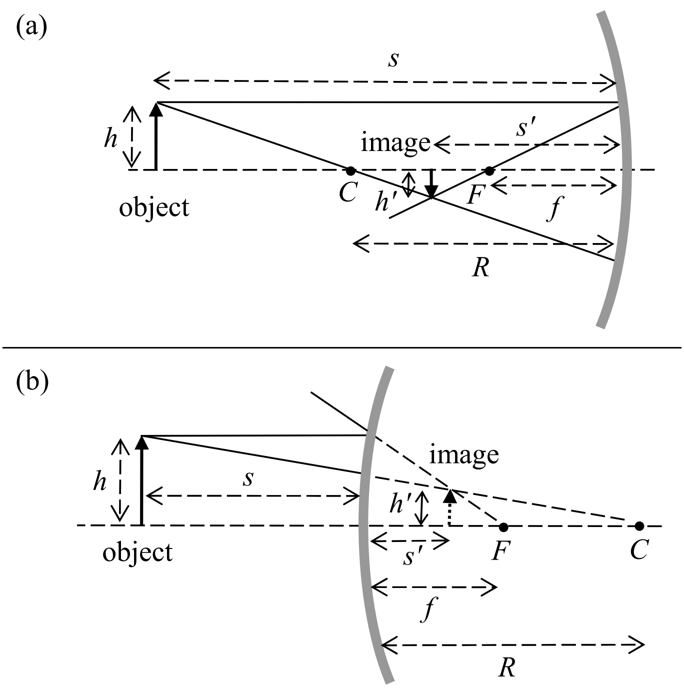


Fig. 18.2 Image formations by **a** concave and **b** convex mirrors

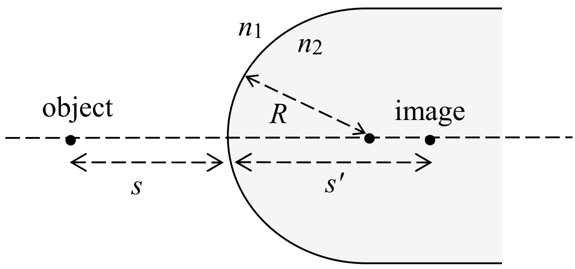


Fig. 18.3 Refraction at a spherical surface

(b) s is − if the object is behind the surface (virtual object).
(c) s' is + if the object is behind the surface (real image).
(d) s' is − if the object is in front of the surface (virtual image).
(e) R is + if the center of curvature is behind the surface.
(f) R is − if the center of curvature is in front of the surface.

Figure 18.3 shows the object and image locations in a refraction at a spherical surface of radius of curvature, R.

(4) Lens maker equation for thin lens,

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right). \tag{18.4}$$

The equation is for thin lens in air. Here, f is the focal length, n is index of refraction of the lens material, R_1 and R_2 are the radii of curvature of the first and second surfaces of the lens, respectively. The object is on the left of the lens. Radius of curvature is positive if the object faces convex surface and negative if it faces concave surface.

If the lens is in a medium with index of refraction n_{medium}, i.e. not in air, n is replaced with n/n_{medium}.

Figure 18.4 shows the quantities that affect the focal length according to the lens maker equation.

Thin lens equation: the object distance, s, image distance, s', and focal length of the lens, f, satisfy

$$\frac{1}{f} = \frac{1}{s} + \frac{1}{s'}. \tag{18.5}$$

Sign convention for thin lens,

(a) s is + if the object is in front of the lens.
(b) s is − if the object is behind the lens
(c) s' is + if the image is behind the lens.
(d) s' is − if the image is in front of the lens.
(e) R_1 and R_2 are + if the center of curvature is behind the lens

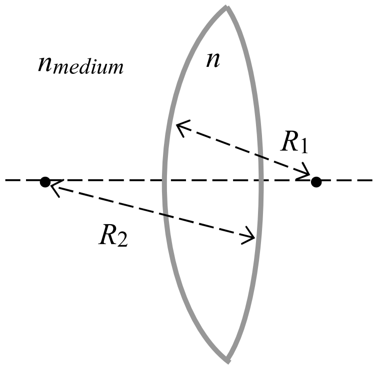


Fig. 18.4 Parameters of lens maker equation

(f) R_1 and R_2 are − if the center of curvature is in front of the lens.

Figure 18.5 shows ray diagrams for image formation in converging and diverging lenses.

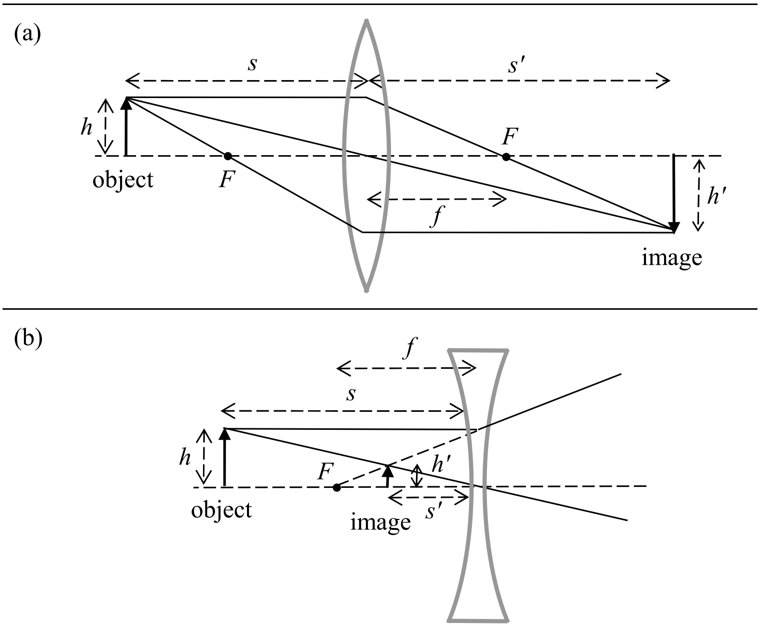


Fig. 18.5 Image formations by **a** convex and **b** concave lenses

18.2 Problems and Solutions

Problem 18.1 An object is placed 4.0 m in front of a concave mirror with radius of curvature 40 cm. Determine location of the image and the magnification.

Solution

The relation between object distance, *s*, image distance, *s*', and radius of curvature, *R*, of a concave mirror is (Eq. 18.2),

$$\frac{2}{R} = \frac{1}{s} + \frac{1}{s'}.$$

For this problem, the image distance, *s'*, is calculated as follows:

$$\frac{2}{0.40\ \text{m}} = \frac{1}{4.0\ \text{m}} + \frac{1}{s'},$$
$$s' = 0.21\ \text{m}.$$

The image is real, inverted, 0.21 m in front of the concave mirror.
The magnification is,

$$M = -\frac{s'}{s} = -\frac{0.21\ \text{m}}{4.0\ \text{m}} = -0.05.$$

Negative magnification means the image is inverted. The absolute value of magnification is less than 1.00 means the image is minified, that is, the image is smaller than the object.

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; ratprint:false; R:0.4; s:4;
(fpprintprec)     5
(ratprint)  false
(R)   0.4
(s)   4
(%i6) solve(2/R = 1/s + 1/s_prime, s_prime)$ float(%);
(%o6) [s_prime=0.21053]
(%i7) s_prime: rhs(%[1]);
(s_prime) 0.21053
(%i8) M: -s_prime/s;
(M)   -0.052632
```

Comments on the codes:

(%i4) Set floating point print precision to 5, internal rational number print to false, assign values of *R* and *s*.

(%i6) Solve $\frac{2}{R} = \frac{1}{s} + \frac{1}{s'}$ for *s'*.

(%i7), (%i8) Assign *s'* and calculate *M*.

Problem 18.2 The focal length of a concave mirror is 10 cm. Determine image distance and magnification if the object distance is (a) 25 cm, (b) 20 cm, (c) 10 cm, and (d) 5.0 cm.

Solution

(a) Using the concave mirror equation (Eq. 18.2), the image distance is calculated as follows:

$$\frac{1}{f} = \frac{1}{s} + \frac{1}{s'},$$
$$\frac{1}{10\text{ cm}} = \frac{1}{25\text{ cm}} + \frac{1}{s'},$$
$$s' = 17\text{ cm}.$$

The magnification is

$$M = -\frac{s'}{s} = -\frac{17\text{ cm}}{25\text{ cm}} = -0.67.$$

The image is smaller than the object as the magnitude of *M* is less than 1.0, inverted as *M* is negative, real, and in front of the concave mirror.

◆ wxMaxima codes:

```
(%i4) fpprintprec: 5; ratprint: false; f: 10; s: 25;
(fpprintprec)     5
(ratprint)  false
(f)   10
(s)   25
(%i6) solve(1/f = 1/s + 1/s_prime, s_prime)$ float(%);
(%o6) [s_prime=16.667]
(%i7) s_prime: rhs(%[1]);
(s_prime) 16.667
(%i8) M: -s_prime/s;
(M)   -0.66668
```

Comments on the codes:

(%i4) Set floating point print precision to 5, internal rational number print to false, assign values of *f* and *s*.
(%i6) Solve $\frac{1}{f} = \frac{1}{s} + \frac{1}{s'}$ for *s'*.
(%i7), (%i8) Assign *s'* and calculate *M*.

(b) Repeat the calculation for object distance, $s = 20$ cm, and we obtained the image distance, *s'*, as

$$\frac{1}{f} = \frac{1}{s} + \frac{1}{s'},$$
$$\frac{1}{10\text{ cm}} = \frac{1}{20\text{ cm}} + \frac{1}{s'},$$
$$s' = 20\text{ cm}.$$

The magnification is,

$$M = -\frac{s'}{s} = -\frac{20\text{ cm}}{20\text{ cm}} = -1.0.$$

The image is the same size as the object as the magnitude of M is 1.0, inverted as M is negative, real, in the front of the concave mirror.

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; ratprint:false; f:10; s:20;
(fpprintprec)     5
(ratprint)  false
(f)   10
(s)   20
(%i6) solve(1/f = 1/s + 1/s_prime, s_prime)$ float(%);
(%o6) [s_prime=20.0]
(%i7) s_prime: rhs(%[1]);
(s_prime) 20.0
(%i8) M: -s_prime/s;
(M)   -1.0
```

Comments on the codes:

(%i4) Set floating point print precision to 5, internal rational number print to false, assign values of f and s.
(%i6) Solve $\frac{1}{f} = \frac{1}{s} + \frac{1}{s'}$ for s'.
(%i7), (%i8) Assign s' and calculate M.

(c) Repeat the calculation for object distance, $s = 10$ cm, and we obtained,

$$\frac{1}{f} = \frac{1}{s} + \frac{1}{s'},$$
$$\frac{1}{10\text{ cm}} = \frac{1}{10\text{ cm}} + \frac{1}{s'},$$
$$s' = \infty.$$

Rays from an object at the focal point reflect off the concave mirror and neither converge nor diverge. After the reflection, the rays travel parallel to each other to infinity and do not result in formation of an image.

(d) Repeat the calculation for object distance, s = 5.0 cm, and we obtained the image distance, s', as

$$\frac{1}{f} = \frac{1}{s} + \frac{1}{s'},$$
$$\frac{1}{10\ \text{cm}} = \frac{1}{5\ \text{cm}} + \frac{1}{s'},$$
$$s' = -10\ \text{cm}.$$

The magnification is,

$$M = -\frac{s'}{s} = -\frac{-10\ \text{cm}}{5.0\ \text{cm}} = 2.0.$$

The image is bigger than the object as the magnitude of M is greater than 1.0, upright as M is positive, virtual, behind the concave mirror.

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; ratprint:false; f:10; s:5;
(fpprintprec)     5
(ratprint)  false
(f)   10
(s)   5
(%i6) solve(1/f = 1/s + 1/s_prime, s_prime)$ float(%);
(%o6) [s_prime=-10.0]
(%i7) s_prime: rhs(%[1]);
(s_prime) -10.0
(%i8) M: -s_prime/s;
(M)   2.0
```

Comments on the codes:

(%i4)	Set floating point print precision to 5, internal rational number print to false, assign values of f and s.
(%i6)	Solve $\frac{1}{f} = \frac{1}{s} + \frac{1}{s'}$ for s'.
(%i7), (%i8)	Assign s' and calculate M.

For reflection of light by a concave mirror, our results show that if the object is placed beyond focal point, the image is real, minified, and inverted; if the object is at the focal point, no image is formed (the image is at infinity); and if the object is placed within focal point, the image is virtual, magnified, and upright.

Problem 18.3 An object of 3.0 cm height is placed (a) 20 cm, (b) 8.0 cm, and (c) 6.0 cm in front of a convex mirror with a focal length of 8.0 cm. Determine location and size of the image.

Solution

(a) The focal length of the convex mirror is $f = -8.0$ cm. The object distance is $s = 20$ cm. The location of the image is calculated using the spherical mirror equation (Eq. 18.2),

$$\frac{1}{f} = \frac{1}{s} + \frac{1}{s'},$$
$$\frac{1}{-8.0 \text{ cm}} = \frac{1}{20 \text{ cm}} + \frac{1}{s'},$$
$$s' = -5.7 \text{ cm}.$$

The magnification is

$$M = -\frac{s'}{s} = -\frac{-5.7 \text{ cm}}{20 \text{ cm}} = 0.29.$$

The image size is

$$h' = Mh = 0.29(3.0 \text{ cm}) = 0.86 \text{ cm}.$$

The image is virtual behind the convex mirror (*s'* is negative), minified (*M* is less than 1.0), and upright (*M* is positive).

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; ratprint:false; f:-8; s:20; h:3;
(fpprintprec)     5
(ratprint)  false
(f)   -8
(s)   20
(h)   3
(%i7) solve(1/f = 1/s + 1/s_prime, s_prime)$ float(%);
(%o7) [s_prime=-5.7143]
(%i8) s_prime: rhs(%[1]);
(s_prime) -5.7143
(%i9) M: -s_prime/s;
(M)   0.28571
(%i10) h_prime: M*h;
(h_prime) 0.85715
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of f, s, and h.

(%i7) Solve $\frac{1}{f} = \frac{1}{s} + \frac{1}{s'}$ for *s'*.
(%i8), (%i9), (%i10) Assign *s'*, calculate *M* and *h'*.

(b) Repeat the calculation for object distance, $s = 8.0$ cm,

$$\frac{1}{f} = \frac{1}{s} + \frac{1}{s'},$$
$$\frac{1}{-8.0\text{ cm}} = \frac{1}{8.0\text{ cm}} + \frac{1}{s'},$$
$$s' = -4.0\text{ cm}.$$

The magnification is

$$M = -\frac{s'}{s} = -\frac{-4.0\text{ cm}}{8.0\text{ cm}} = 0.50.$$

The image size is

$$h' = Mh = 0.50\ (3.0\text{ cm}) = 1.5\text{ cm}.$$

The image is virtual, behind the mirror (*s'* is negative), minified (*M* is less than 1.0), and upright (*M* is positive).

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; ratprint:false; f:-8; s:8; h:3;
(fpprintprec)     5
(ratprint)  false
(f)   -8
(s)   8
(h)   3
(%i7) solve(1/f = 1/s + 1/s_prime, s_prime)$ float(%);
(%o7) [s_prime=-4.0]
(%i8) s_prime: rhs(%[1]);
(s_prime) -4.0
(%i9) M: -s_prime/s;
(M)   0.5
(%i10) h_prime: M*h;
(h_prime) 1.5
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of *f*, *s*, and *h*.
(%i7) Solve $\frac{1}{f} = \frac{1}{s} + \frac{1}{s'}$ for *s'*.
(%i8), (%i9), (%i10) Assign *s'*, calculate *M* and *h'*.

(c) Repeat the calculation for object distance, $s = 6.0$ cm,

$$\frac{1}{f} = \frac{1}{s} + \frac{1}{s'},$$
$$\frac{1}{-8.0\text{ cm}} = \frac{1}{6.0\text{ cm}} + \frac{1}{s'},$$
$$s' = -3.4\text{ cm}.$$

The magnification is

$$M = -\frac{s'}{s} = -\frac{-3.4\text{ cm}}{6.0\text{ cm}} = 0.57.$$

The image size is

$$h' = Mh = 0.57\ (3.0\text{ cm}) = 1.7\text{ cm}.$$

The image is virtual, behind the mirror (*s'* is negative), minified (*M* is less than 1.0), and upright (*M* is positive).

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; ratprint:false; f:-8; s:6; h:3;
(fpprintprec)     5
(ratprint)  false
(f)   -8
(s)   6
(h)   3
(%i7) solve(1/f = 1/s + 1/s_prime, s_prime)$ float(%);
(%o7) [s_prime=-3.4286]
(%i8) s_prime: rhs(%[1]);
(s_prime) -3.4286
(%i9) M: -s_prime/s;
(M)   0.57143
(%i10) h_prime: M*h;
(h_prime) 1.7143
```

Comments on the codes:

(%i5)	Set floating point print precision to 5, internal rational number print to false, assign values of f, s, and h.
(%i7)	Solve $\frac{1}{f} = \frac{1}{s} + \frac{1}{s'}$ for *s'*.
(%i8), (%i9), (%i10)	Assign *s'*, calculate *M* and *h'*.

For reflection of light by a convex mirror, the image is always upright, virtual, and minified.

Problem 18.4 Figure 18.6 shows an end of a glass rod formed into a convex surface of radius of curvature 6.0 cm. Index of refraction of glass is 1.5. An object is placed along the rod axis at (a) 20 cm, (b) 10 cm, and (c) 3.0 cm from the rod. Determine the location of the image.

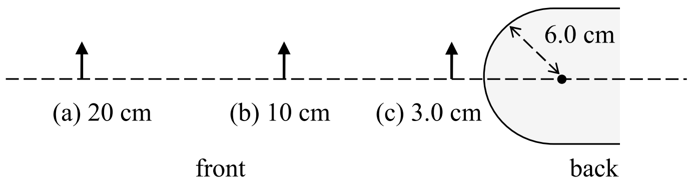


Fig. 18.6 Refraction at a spherical surface, Problem 18.4

Solution

(a) Refraction equation for spherical surface is (Eq. 18.3),

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R}.$$

For this problem,

$$\frac{n_{air}}{s} + \frac{n_{glass}}{s'} = \frac{n_{glass} - n_{air}}{R},$$
$$\frac{1.0}{20\text{ cm}} + \frac{1.5}{s'} = \frac{1.5 - 1.0}{6.0\text{ cm}},$$
$$s' = 45\text{ cm}.$$

The image is real, at the back of the convex surface.

♦ wxMaxima codes:

```
(%i6) fpprintprec:5; ratprint:false; n_air:1; n_glass:1.5; R:6; s:20;
(fpprintprec)     5
(ratprint)  false
(n_air)     1
(n_glass)   1.5
(R)   6
(s)   20
(%i8) solve(n_air/s+n_glass/s_prime=(n_glass-n_air)/R, s_prime)$ float(%);
(%o8) [s_prime=45.0]
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of n_{air}, n_{glass}, R, and s.

(%i8) Solve $\frac{n_{air}}{s} + \frac{n_{glass}}{s'} = \frac{n_{glass}-n_{air}}{R}$ for s'.

(b) Repeat the calculation for object distance, $s = 10$ cm,

$$\frac{n_{air}}{s} + \frac{n_{glass}}{s'} = \frac{n_{glass} - n_{air}}{R},$$
$$\frac{1.0}{10\text{ cm}} + \frac{1.5}{s'} = \frac{1.5 - 1.0}{6.0\text{ cm}},$$
$$s' = -90\text{ cm}.$$

The image is virtual, in front of the convex surface.

◆ wxMaxima codes:

```
(%i6) fpprintprec:5; ratprint:false; n_air:1; n_glass:1.5; R:6; s:10;
(fpprintprec)     5
(ratprint)  false
(n_air)     1
(n_glass)   1.5
(R)   6
(s)   10
(%i8) solve(n_air/s+n_glass/s_prime=(n_glass-n_air)/R, s_prime)$ float(%);
(%o8) [s_prime=-90.0]
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of n_{air}, n_{glass}, R, and s.

(%i8) Solve $\frac{n_{air}}{s} + \frac{n_{glass}}{s'} = \frac{n_{glass} - n_{air}}{R}$ for s'.

(c) Repeat the calculation for object distance, $s = 3.0$ cm,

$$\frac{n_{air}}{s} + \frac{n_{glass}}{s'} = \frac{n_{glass} - n_{air}}{R},$$
$$\frac{1.0}{3.0\text{ cm}} + \frac{1.5}{s'} = \frac{1.5 - 1.0}{6.0\text{ cm}},$$
$$s' = -6.0\text{ cm}.$$

The image is virtual, in front of the convex surface.

◆ wxMaxima codes:

```
(%i6) fpprintprec:5; ratprint:false; n_air:1; n_glass:1.5; R:6; s:3;
(fpprintprec)     5
(ratprint)  false
(n_air)     1
(n_glass)   1.5
(R)   6
(s)   3
(%i8) solve(n_air/s+n_glass/s_prime=(n_glass-n_air)/R, s_prime)$ float(%);
(%o8) [s_prime=-6.0]
```

Comments on the codes:

(%i6) Set floating point print precision to 5, internal rational number print to false, assign values of n_{air}, n_{glass}, R, and s.

(%i8) Solve $\frac{n_{air}}{s} + \frac{n_{glass}}{s'} = \frac{n_{glass} - n_{air}}{R}$ for s'.

Problem 18.5 Figure 18.7 shows a 2.0 cm diameter coin embedded in a glass ball of 30 cm radius. The coin is 20 cm from the surface of the ball and the refractive index of the glass is 1.5. Determine the location and size of the image.

Solution

Figure 18.8 shows rays of light from the coin in the glass, refracted at the spherical surface as the rays go into the air to the observer.

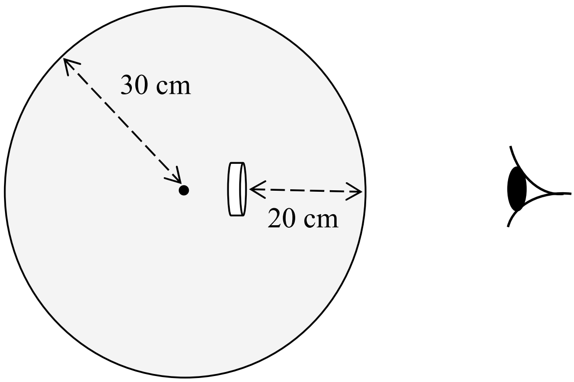


Fig. 18.7 A coin embedded in a glass ball, Problem 18.5

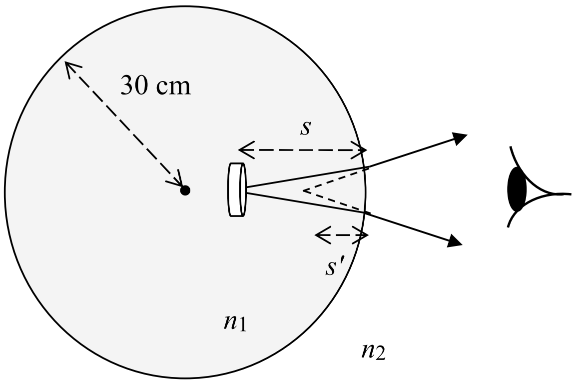


Fig. 18.8 Refraction at a spherical surface, Problem 18.5

Using the refraction equation of spherical surface (Eq. 18.3), we have

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R},$$
$$\frac{1.5}{20 \text{ cm}} + \frac{1.0}{s'} = \frac{1.0 - 1.5}{-30 \text{ cm}},$$
$$s' = -17 \text{ cm}.$$

The image is virtual, in the ball. The size of the image is

$$h' = Mh = -\frac{s'}{s}h = -\frac{-17 \text{ cm}}{20 \text{ cm}}(2.0 \text{ cm}) = 1.7 \text{ cm}.$$

◆ wxMaxima codes:

```
(%i7) fpprintprec:5; ratprint:false; n1:1.5; n2:1; s:20; R:-30; h:2;
(fpprintprec)     5
(ratprint)  false
(n1)  1.5
(n2)  1
(s)   20
(R)   -30
(h)   2
(%i9) solve(n1/s + n2/s_prime = (n2 - n1)/R, s_prime)$ float(%);
(%o9) [s_prime=-17.143]
(%i10) s_prime: rhs(%[1]);
(s_prime) -17.143
(%i11) h_prime: -s_prime/s*h;
(h_prime) 1.7143
```

Comments on the codes:

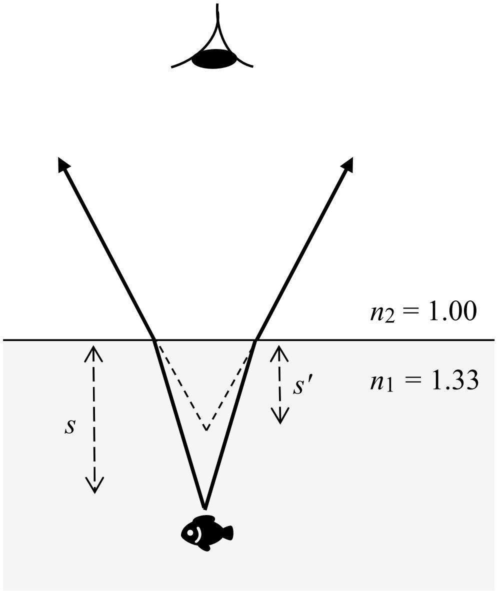
(%i7)	Set floating point print precision to 5, internal rational number print to false, assign values of n_1, n_2, s, R, and h.
(%i9)	Solve $\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R}$ for *s'*.
(%i10), (%i11)	Assign *s'* and calculate *h'*.

Problem 18.6 Figure 18.9 shows a fish at depth, *s*, in water. Refractive index of water is 1.33. Determine the apparent depth of the fish.

Solution

Using the refraction equation of spherical surface (Eq. 18.3), with radius of curvature, $R = \infty$, for flat surface, we get,

Fig. 18.9 Refraction of light, Problem 18.6



$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R},$$
$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{\infty},$$
$$\frac{n_1}{s} + \frac{n_2}{s'} = 0,$$
$$s' = -\frac{n_2}{n_1}s = -\frac{1.00}{1.33}s = -0.75s.$$

This means that the apparent depth is 0.75 of the real depth.

♦ wxMaxima codes:

```
(%i4) fpprintprec:5; ratprint:false; n1:1.33; n2:1;
(fpprintprec)     5
(ratprint)  false
(n1)  1.33
(n2)  1
(%i6) solve(n1/s + n2/s_prime = 0, s_prime)$ float(%);
(%o6) [s_prime=-0.75188*s]
```

Comments on the codes:

(%i4) Set floating point print precision to 5, internal rational number print to false, assign values of n_1 and n_2.

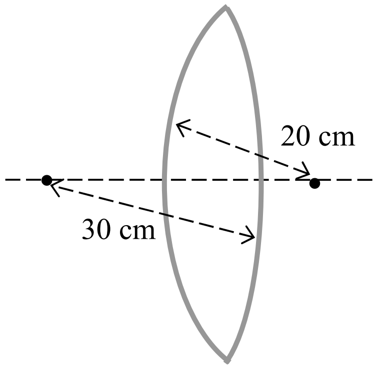


Fig. 18.10 Convex lens of Problem 18.7

(%i6) Solve $\frac{n_1}{s} + \frac{n_2}{s'} = 0$ for *s'*.

Problem 18.7 Figure 18.10 shows a convex lens made of glass of refractive index 1.5, with radii of curvature 20 cm and 30 cm. Calculate focal length of the lens.

Solution

We apply the lens maker equation for this problem. Lens maker equation is (Eq. 18.4),

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

Here, f is the focal length, n is index of refraction of the lens material, R_1 and R_2 are the radii of curvature of the first and second surfaces of the lens, respectively. The object is assumed on the left of the lens. Radius of curvature is positive if the object faces convex surface and negative if it faces concave surface. We have, $R_1 = +20$ cm as the object faces convex first surface, $R_2 = -30$ cm as the object faces concave second surface, and $n = 1.5$. The focal length of the lens is calculated as follows,

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right),$$
$$\frac{1}{f} = (1.5-1)\left(\frac{1}{+20\text{ cm}} - \frac{1}{-30\text{ cm}}\right),$$
$$f = +24\text{ cm}.$$

The lens is a converging lens because the focal length is a positive number.

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; ratprint:false; n:1.5; R1:20; R2:-30;
(fpprintprec)     5
(ratprint)  false
(n)   1.5
(R1)  20
(R2)  -30
(%i7) solve(1/f = (n-1)*(1/R1 - 1/R2), f)$ float(%);
(%o7) [f=24.0]
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of n, R_1, and R_2.

(%i7) Solve $\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$ for f.

Problem 18.8 The focal length of a biconvex lens made of glass with refractive index 1.52 is 25 cm.

(a) Calculate the radius of curvature of the lens.
(b) What is the focal length of the lens if the lens is in water? Refractive index of water is 1.33.

Solution

(a) Lens maker equation is (Eq. 18.4),

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

Here, f is the focal length, n is index of refraction of the lens material, R_1 and R_2 are the radii of curvature of the first and second surfaces of the lens. The object is on the left of the lens. Radius of curvature is positive if the object faces convex surface and negative if it faces concave surface. Setting r as a positive number, we have, $R_1 = +r$ as positive because the object faces convex first surface and $R_2 = -r$ as negative because the object faces concave second surface. Thus, the radius of curvature is calculated as

$$\frac{1}{25\text{ cm}} = (1.52-1)\left(\frac{1}{+r} - \frac{1}{-r}\right) = 0.52\left(\frac{2}{r}\right),$$
$$r = 26\text{ cm}.$$

The radii of curvature of the lens are $R_1 = 26$ cm and $R_2 = -26$ cm.

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; ratprint:false; n:1.52; f:25;
(fpprintprec)     5
(ratprint)  false
(n)   1.52
(f)   25
(%i6) solve(1/f = (n-1)*(1/r - 1/(-r)), r)$ float(%);
(%o6) [r=26.0]
```

Comments on the codes:

(%i4) Set floating point print precision to 5, internal rational number print to false, assign values of n and f.

(%i6) Solve $\frac{1}{f} = (n-1)\left(\frac{1}{r} - \frac{1}{-r}\right)$ for r.

(b) If the lens is in medium with index of refraction n_{medium}, i.e. not in air, replace n with n/n_{medium}. In water, the focal length is calculated as

$$\frac{1}{f} = (\frac{n}{n_{water}} - 1)\left(\frac{2}{R_1}\right) = (\frac{1.52}{1.33} - 1)\left(\frac{2}{26\ \text{cm}}\right),$$
$$f = 91\ \text{cm}.$$

The lens still converges the light rays in water ($f = 91$ cm), although weaker than in air ($f = 25$ cm).

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; ratprint:false; n:1.52; n_water:1.33; R1:26;
(fpprintprec)     5
(ratprint)  false
(n) 1.52
(n_water) 1.33
(R1)  26
(%i7) solve(1/f = (n/n_water-1)*(2/R1), f)$ float(%);
(%o7) [f=91.0]
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of n, n_{water}, and R_1.

(%i7) Solve $\frac{1}{f} = (\frac{n}{n_{water}} - 1)\left(\frac{2}{R_1}\right)$ for f.

Problem 18.9 A glass lens has convex and concave surfaces. Radii of curvature of convex and concave surfaces are 30 and 25 cm, respectively. Index of refraction of glass is 1.52. Calculate the focal length of the lens.

Solution

Figure 18.11 shows the lens.

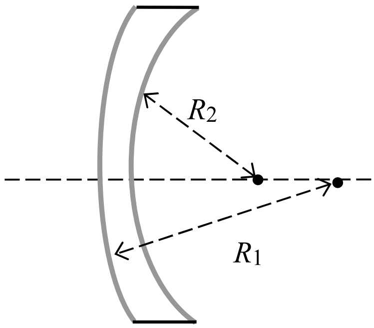


Fig. 18.11 Glass lens of Problem 18.9

Using the lens maker equation (Eq. 18.4) with $R_1 = 30$ cm, $R_2 = 25$ cm, and $n = 1.52$,

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right),$$
$$\frac{1}{f} = (1.52-1)\left(\frac{1}{30\text{ cm}} - \frac{1}{25\text{ cm}}\right),$$
$$f = -288\text{ cm}.$$

It is a diverging lens, because f is negative.

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; ratprint:false; n:1.52; R1:30; R2:25;
(fpprintprec)     5
(ratprint)   false
(n)   1.52
(R1)   30
(R2)   25
(%i7) solve(1/f = (n-1)*(1/R1-1/R2), f)$ float(%);
(%o7) [f=-288.46]
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of n, R_1, and R_2.
(%i7) Solve $\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$ for f.

Problem 18.10

(a) The distance between a converging lens and a screen is 20 cm to get a focused image of a distant object on the screen. What is the focal length of the lens?
(b) An object is located 100 cm from the converging lens. Determine the location of the image.

Solution

(a) Parallel rays from distant object reach the lens. The parallel rays are converged to the focal point of the lens. Thus, the focal length of the lens is 20 cm.

(b) Using the thin lens equation (Eq. 18.5), the image distance, *s'*, is calculated as follows:

$$\frac{1}{f} = \frac{1}{s} + \frac{1}{s'},$$
$$\frac{1}{20\ \text{cm}} = \frac{1}{100\ \text{cm}} + \frac{1}{s'},$$
$$s' = 25\ \text{cm}.$$

The image is 25 cm at the back of the lens, minified, inverted, and real.

◆ wxMaxima codes:

```
(%i4) fpprintprec:5; ratprint:false; f:20; s:100;
(fpprintprec)     5
(ratprint)   false
(f)    20
(s)    100
(%i6) solve(1/f = 1/s + 1/s_prime, s_prime)$ float(%);
(%o6) [s_prime=25.0]
```

Comments on the codes:

(%i4) Set floating point print precision to 5, internal rational number print to false, assign values of f and s.

(%i6) Solve $\frac{1}{f} = \frac{1}{s} + \frac{1}{s'}$ for *s'*.

Problem 18.11 Two lenses with focal lengths of 10 cm and 20 cm separate by 18 cm, as illustrated in Fig. 18.12. An object is located at 15 cm from the lens system. Determine the location of the image and the magnification.

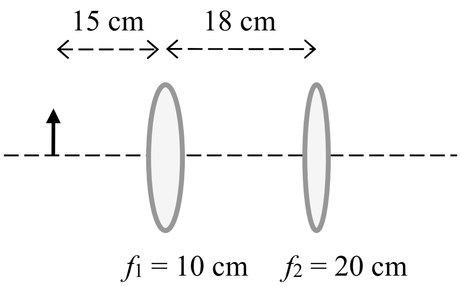


Fig. 18.12 Two lenses of Problem 18.11

Solution

Using thin lens equation (Eq. 18.5), we calculate location of the image as the light rays go through the first lens (the left lens),

$$\frac{1}{f_1} = \frac{1}{s_1} + \frac{1}{s_1'},$$
$$\frac{1}{10\text{ cm}} = \frac{1}{15\text{ cm}} + \frac{1}{s_1'},$$
$$s_1' = 30\text{ cm}.$$

The image is 30 cm on the right of the first lens, that is, 30 cm – 18 cm = 12 cm on the right of the second lens. This image is the virtual object of the second lens. We write, for the second lens, the object distance as $s_2 = -12$ cm. Now, we calculate the location of the final image through the second lens, again, using the thin lens equation,

$$\frac{1}{f_2} = \frac{1}{s_2} + \frac{1}{s_2'},$$
$$\frac{1}{20\text{ cm}} = \frac{1}{-12\text{ cm}} + \frac{1}{s_2'},$$
$$s_2' = 7.5\text{ cm}.$$

Thus, the final image is 7.5 cm on the right of the second lens. Magnifications of first, second, and both lenses are

$$M_1 = \frac{-s_1'}{s_1} = \frac{-30\text{ cm}}{15\text{ cm}} = -2.0,$$
$$M_2 = \frac{-s_2'}{s_2} = \frac{-7.5\text{ cm}}{-12\text{ cm}} = 0.63,$$
$$M_1 M_2 = (-2.00)(0.62) = -1.3.$$

The final image is real, inverted, and magnified, behind the second lens.

◆ wxMaxima codes:

```
(%i5) fpprintprec:5; ratprint:false; f1:10; f2:20; s1:15;
(fpprintprec)     5
(ratprint)  false
(f1)  10
(f2)  20
(s1)  15
(%i7) solve(1/f1 = 1/s1 + 1/s1_prime, s1_prime)$ float(%);
(%o7) [s1_prime=30.0]
(%i8) s1_prime: rhs(%[1]);
(s1_prime)  30.0
(%i9) s2: -(s1_prime-18);
(s2)  -12
(%i11) solve(1/f2 = 1/s2 + 1/s2_prime, s2_prime)$ float(%);
(%o11) [s2_prime=7.5]
(%i12) s2_prime: rhs(%[1]);
(s2_prime)  7.5
(%i13) M1: -s1_prime/s1;
(M1)  -2.0
(%i14) M2: -s2_prime/s2;
(M2)  0.625
(%i15) M1*M2;
(%o15) -1.25
```

Comments on the codes:

(%i5)	Set floating point print precision to 5, internal rational number print to false, assign values of f_1, f_2, and s_1.
(%i7)	Solve $\frac{1}{f_1} = \frac{1}{s_1} + \frac{1}{s_1'}$ for s_1’.
(%i8), (%i9)	Assign s_1’ and s_2.
(%i11)	Solve $\frac{1}{f_2} = \frac{1}{s_2} + \frac{1}{s_2'}$ for s_2’.
(%i12)	Assign s_2’.
(%i13), (%i14), (%i15)	Calculate M_1, M_2, and M_1M_2.

Problem 18.12 State the sign convention for the lens maker formula.

Solution

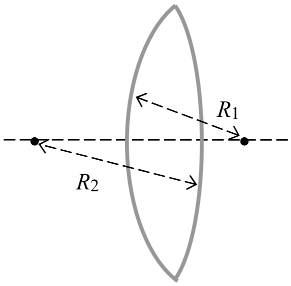
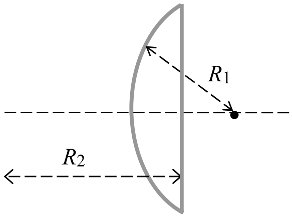
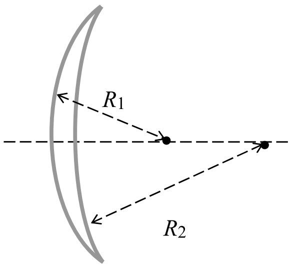
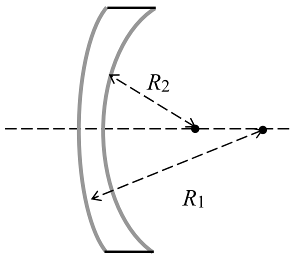
The lens maker formula for this lens is (Eq. 18.4),

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right),$$

where f is the focal length, n is index of refraction of the lens material, R_1 and R_2 are radii of curvature of the first and second surfaces of the lens, respectively. The object is on the left of the lens. Radius of curvature is positive if the object faces convex surface and negative if it faces concave surface. If the lens is in medium with index of refraction n_{medium}, i.e. not in air, n is replaced with n/n_{medium}. Table 18.1 gives examples of application of the formula. Index of refraction of lens material is $n = 1.52$.

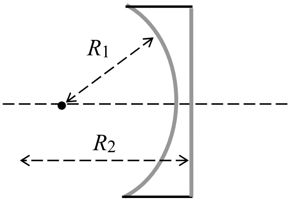
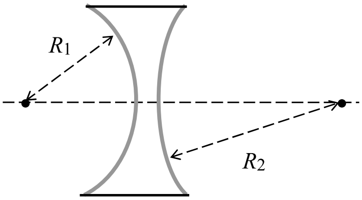
◆ wxMaxima codes:

Table 18.1 Focal lengths of lenses calculated by the lens maker formula

	Lens dimensions	Type	R_1	R_2	f
(a)		Biconvex	+20 cm	−30 cm	+23 cm
(b)		Planoconvex	+20 cm	∞	+38 cm
(c)			+20 cm	+30 cm	+115 cm
(d)			+30 cm	+20 cm	−115 cm

(continued)

Table 18.1 (continued)

	Lens dimensions	Type	R_1	R_2	f
(e)		Planoconcave	−20 cm	∞	−38 cm
(f)		Biconcave	−20 cm	+30 cm	−23 cm

```
(%i3) fpprintprec:5; ratprint:false; n:1.52;
(fpprintprec)     5
(ratprint)  false
(n)   1.52
(%i5) R1:20; R2:-30;
(R1)  20
(R2)  -30
(%i7) solve(1/f = (n-1)*(1/R1 - 1/R2), f)$ float(%);
(%o7) [f=23.077]
(%i9) R1:20; R2:inf;
(R1)  20
(R2)  inf
(%i11) solve(1/f = (n-1)*(1/R1 - 0), f)$ float(%);
(%o11) [f=38.462]
(%i13) R1:20; R2:30;
(R1)  20
(R2)  30
(%i15) solve(1/f = (n-1)*(1/R1 - 1/R2), f)$ float(%);
(%o15) [f=115.38]
(%i17) R1:30; R2:20;
(R1)  30
(R2)  20
(%i19) solve(1/f = (n-1)*(1/R1 - 1/R2), f)$ float(%);
(%o19) [f=-115.38]
(%i21) R1:-20; R2:inf;
(R1)  -20
(R2)  inf
(%i23) solve(1/f = (n-1)*(1/R1 - 0), f)$ float(%);
(%o23) [f=-38.462]
(%i25) R1:-20; R2:30;
(R1)  -20
(R2)  30
(%i27) solve(1/f = (n-1)*(1/R1 - 1/R2), f)$ float(%);
(%o27) [f=-23.077]
```

Comments on the codes:

(%i3)	Set floating point print precision to 5, internal rational number print to false, assign value of *n*.
(%i5), (%i7)	Calculate *f*, case (a) Table 18.1.
(%i9), (%i11)	Calculate *f*, case (b) Table 18.1.
(%i13), (%i15)	Calculate *f*, case (c) Table 18.1.
(%i17), (%i19)	Calculate *f*, case (d) Table 18.1.
(%i21), (%i23)	Calculate *f*, case (e) Table 18.1.
(%i25), (%i27)	Calculate *f*, case (f) Table 18.1.

18.3 Summary

- The spherical mirror equation: For a spherical mirror of radius, *R*, the object distance, *s,* and image distance, *s'*, obey

$$\frac{1}{s} + \frac{1}{s'} = \frac{2}{R} = \frac{1}{f},$$

where $f = R/2$ is focal length of the mirror.

- Refraction at a spherical surface,

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R},$$

where n_1 and n_2 are refractive indices of medium 1 and 2, respectively, and *R* is the radius of the spherical surface.
- Lens maker equation for thin lens,

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

- Thin lens equation: The object distance, *s*, image distance, *s*', and focal length of the lens, *f*, satisfy

$$\frac{1}{f} = \frac{1}{s} + \frac{1}{s'}.$$

18.4 Exercises

Exercise 18.1 An object is placed 3.5 m in front of a concave mirror with radius of curvature 30 cm. Determine location of the image and the magnification.

(Answer: $s' = 16$ cm, $M = -0.045$)

Exercise 18.2 An object is placed 60 cmfrom a convex mirror and a magnification of 0.25 is obtained. Determine the location of the image and the focal length of the mirror.

(Answer: $s' = -15$ cm, $f = -20$ cm)

Exercise 18.3 Figure 18.13 shows a lens made of glass of refractive index 1.5, with radii of curvature 30 and 55 cm.

(a) Determine focal length of the lens.
(b) An object is placed 80 cm in front of the lens. Calculate the location of the image and the magnification.

(Answer: (a) $f = 39$ cm; (b) $s' = 75$ cm, $M = -0.94$)

Exercise 18.4 An object is located in a medium whose index of refraction is 1.5, 20 cm from the surface whose radius is 30 cm, as shown in Fig. 18.14. Determine the location of the image and the magnification as seen by the observer.

(Answer: $s' = -17$ cm in the medium, $M = 0.86$)

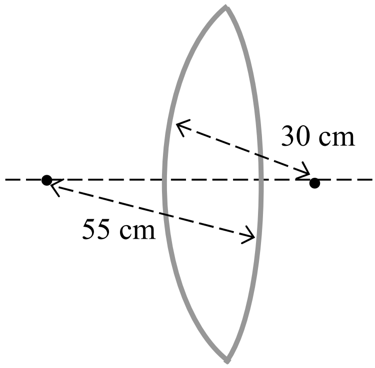


Fig. 18.13 Glass lens of Exercise 18.3

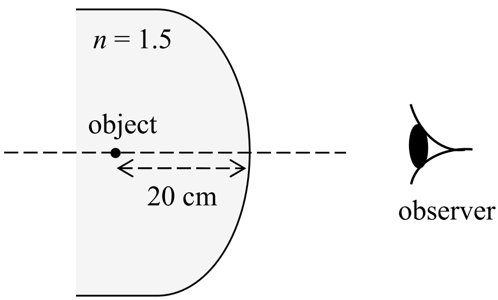


Fig. 18.14 Refraction at a spherical surface, Exercise 18.4

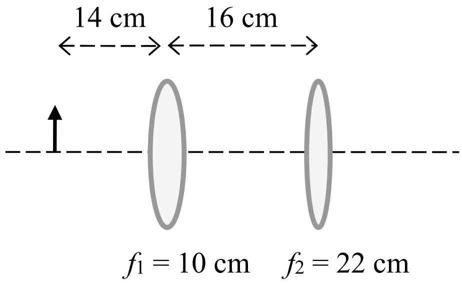


Fig. 18.15 Two lenses of Exercise 18.5

Exercise 18.5 Two lenses with focal lengths of 10 and 22 cm separate by a distance of 16 cm, as shown in Fig. 18.15. An object is placed at 14 cm from the lens system. Determine the location of the image and the magnification.

(Answer: A real image 10 cm to the right of the right lens, $M = -1.3$)

Chapter 19
Interference of Light

Abstract Problems on interference of light are solved in this chapter. Light interference is a phenomenon due to superposition of coherent lights. These include interference in Young's double slit experiment, thin film, lens coating, air wedge, and Newton's rings experiment. Both solutions by analysis and computer calculation via wxMaxima are presented.

19.1 Basic Concepts and Formulae

(1) Interference of light is an effect of superposition of light waves at a point. Persistent interference pattern exists if,

   (a) wave sources are coherence (that is, the phase difference of sources is constant),
   (b) the sources are monochromatic (that is, the same wavelength), and.
   (c) linear superposition principle is obeyed.

(2) Young's double-slit experiment: Two slits separated by a small distance d illuminated by a monochromatic light, as illustrated in Fig. 19.1. Interference pattern of bright and dark bands are formed on a screen. To get a constructive interference on the screen, path difference of light from the two slits must be zero or integer multiple of wavelength, λ. The path difference of light from the two slits is d sin θ. This means that a condition for bright band to be formed on the screen (a constructive interference) is,

$$d \sin\theta = m\lambda, \quad m = 0, \pm1, \pm2, \ldots \tag{19.1}$$

where m is order number. The central bright fringe with $\theta = 0$, $m = 0$, is called the zeroth order maximum. The first maxima on both sides of the zeroth order maximum is called the first order maxima with $m = \pm1$.

W. M. S. Wan Hassan et al., *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-43165-4_19

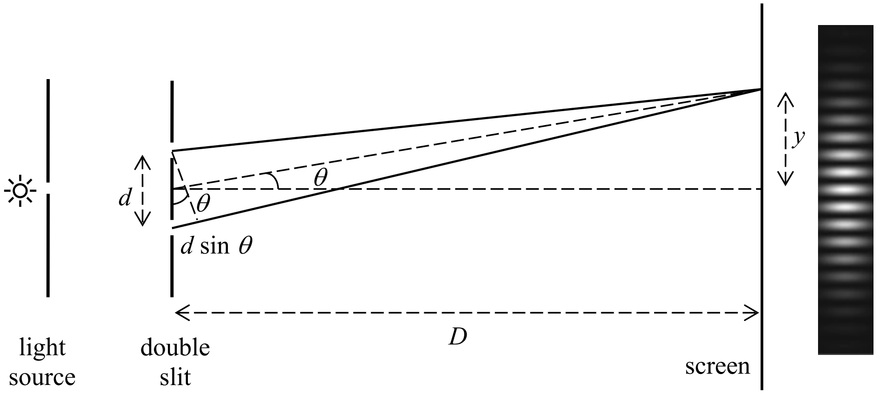


Fig. 19.1 Young's double-slit interference experiment. Light of wavelength λ is incident on a double slit separated by a distance d. Interference pattern is observed on a screen a distance D away. θ is the angle between the fringe and the central bright fringe, y is the distance between the fringe and the central bright fringe, and D is the distance between the double slit and the screen

To get a destructive interference on the screen, the lights path difference from both slits must be odd multiple of a half wavelength $\lambda/2$, such that the two waves arriving at the screen differ in phase by 180°. This means that a condition for a dark fringe to be formed on the screen (a destructive interference) is,

$$d \sin\theta = \left(m + \frac{1}{2}\right)\lambda, \quad m = 0, \pm 1, \pm 2, \ldots \tag{19.2}$$

Figure 19.1 shows a setup of Young's double slit experiment. A picture of bright and dark fringes of the experiment is shown on the far right of the figure. From the figure, $\sin\theta \approx \tan\theta = y/D$ and the path difference is $d \sin\theta = dy/D$. Here, θ is the angle between the fringe and the central bright fringe, y is the distance on the screen between the fringe and the central bright fringe, and D is the distance between the double slit and the screen. Therefore, the bright and dark fringes satisfy,

$$y_{bright} = \frac{\lambda D}{d} m, \quad m = 0,\ \pm 1,\ \pm 2,\ \ldots \tag{19.3}$$

$$y_{dark} = \frac{\lambda D}{d}(m + \frac{1}{2}), \quad m = 0,\ \pm 1,\ \pm 2,\ \ldots \tag{19.4}$$

Average intensity of the interference pattern is,

$$I_{average} = I_0 \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right) \approx I_0 \cos^2\left(\frac{\pi d}{\lambda D} y\right). \tag{19.5}$$

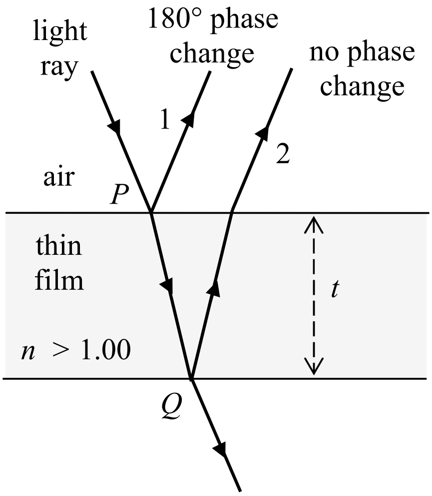


Fig. 19.2 Phase changes of light reflected from a thin film

(3) Light waves undergo 180° phase change as they are reflected from a medium of higher refractive index than the medium they are traveling, for example, at the air-glass interface. No phase change occurs for reflection of the light waves from a medium of lower refractive index, for example, at the glass-air interface.

Figure 19.2 shows a light ray from air (refractive index = 1.00) incident on a thin film with refractive index $n > 1.00$. A phase change of 180° occurs for reflected ray 1 at P, but no phase change occurs for reflected ray 2 at Q. This means that rays 1 and 2 differ in phase by 180°.

(4) The wavelength of light λ_n in a medium of refractive index n is,

$$\lambda_n = \frac{\lambda}{n}, \tag{19.6}$$

where λ is the wavelength of light in free space.

(5) The condition for constructive interference for a thin film of thickness t and refractive index n is

$$2nt = (m + \frac{1}{2})\lambda, \quad m = 0,\ 1,\ 2,\ ... \tag{19.7}$$

The condition for destructive interference is,

$$2nt = m\lambda, \quad m = 0,\ 1,\ 2,\ ... \tag{19.8}$$

(6) Newton's rings are concentric bright and dark rings formed when a convex lens is placed on a glass plate illuminated by light, as shown in Fig. 19.3. The radius of the bright ring is,

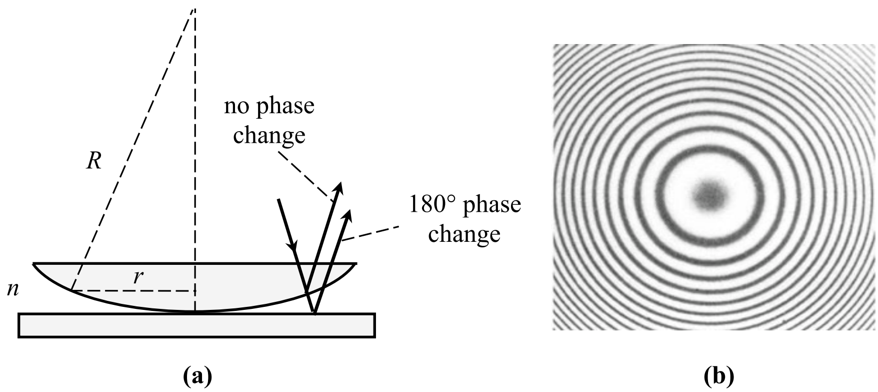


Fig. 19.3 Newton's rings experiment, **a** the setup, **b** observed rings

$$r = \sqrt{(m + \frac{1}{2})\frac{\lambda R}{n}}, \quad m = 0,\ 1,\ 2,\ \ldots \tag{19.9}$$

where R is the radius of curvature of the lens, λ is the wavelength of light, and n is the refractive index of the medium between the lens and the glass plate. Here, $m = 0$ corresponds to the first bright ring, $m = 1$ corresponds to the second bright ring, and so on.

The radius of the dark ring is,

$$r = \sqrt{m\frac{\lambda R}{n}}, \quad m = 0,\ 1,\ 2,\ \ldots \tag{19.10}$$

where $m = 0$ corresponds to the central dark spot, $m = 1$ corresponds to the first dark ring, $m = 2$ corresponds to the second dark ring, and so on.

Figure 19.3a shows the configuration of Newton's rings experiment in air where $n = 1.00$. Figure 19.3b is a picture of Newton's rings. The center spot where the lens touches the glass plate is dark. This is because the ray reflected from the bottom of the lens has no phase change while the one reflected from the plate has 180° phase change, and the interference of both rays is destructive giving a dark spot.

19.2 Problems and Solutions

Problem 19.1 In a Young's double slit experiment, the separation between slits is 0.09 mm and the screen is 1.0 m away from the slits. The third-order bright fringe is 2.0 cm from the central bright fringe. Calculate the wavelength of light of the experiment and the distance between the third dark fringe and the central bright fringe.

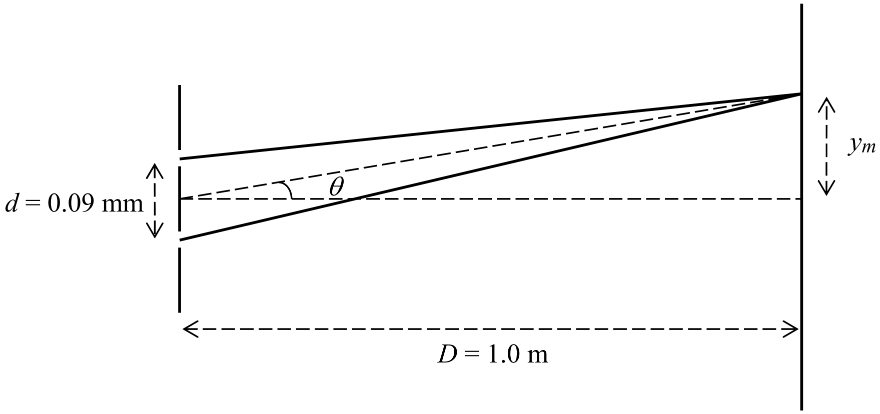


Fig. 19.4 Young's double slit experiment, Problem 19.1

Solution

Figure 19.4 shows Young's double slit experiment set up. Here, d is separation distance of slits, D is the distance between slits and screen, y_m is the location of m-th fringe, and θ is the angle between the central bright fringe and the m-th fringe.

A bright fringe is obtained when (Eq. 19.1),

$$d\sin\theta = m\lambda, \qquad m = 1,\ 2,\ 3,\ \ldots$$

or,

$$d\frac{y_m}{D} = m\lambda, \qquad m = 1,\ 2,\ 3,\ \ldots$$

This gives (Eq. 19.3),

$$y_m = m\frac{\lambda D}{d}, \qquad m = 1,\ 2,\ 3,\ \ldots$$

For the third bright fringe, we have,

$$y_3 = 3\frac{\lambda D}{d}.$$

Substituting the given numerical values, the wavelength of the light used in the experiment is,

$$2.0 \times 10^{-2}\ \text{m} = 3 \times \frac{\lambda(1.0\ \text{m})}{0.09 \times 10^{-3}\ \text{m}},$$
$$\lambda = 6.0 \times 10^{-7}\ \text{m}.$$

- wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)     5
(ratprint)  false
(%i4) solve(2e-2 = 3*lambda/0.09e-3, lambda)$ float(%);
(%o4) [lambda=6.0*10^-7]
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.

(%i4) Solve $2 \times 10^{-2} = 3 \times \frac{\lambda}{0.09 \times 10^{-3}}$ for λ.

For dark fringes (Eq. 19.4),

$$y_m = (m + \frac{1}{2})\frac{\lambda D}{d}, \qquad m = 0,\ 1,\ 2,\ \ldots$$

Here, $m = 0$ corresponds to first dark fringe, $m = 1$ to the second, and $m = 2$ to the third. The distance between the third dark fringe and the central bright fringe is obtained by $m = 2$,

$$y_2 = (2 + \frac{1}{2})\frac{(6.0 \times 10^{-7}\ \text{m})(1.0\ \text{m})}{0.09 \times 10^{-3}\ \text{m}} = 1.7 \times 10^{-2}\ \text{m}.$$

- wxMaxima codes:

```
(%i2) fpprintprec:5; y2:(2+1/2)*6e-7/0.09e-3;
(fpprintprec)     5
(y2)  0.016667
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and calculate y_2.

Problem 19.2 Interference pattern of a double slit separated by 0.25 mm is observed on a screen 1.0 m away. The double slit is illuminated by a monochromatic light of wavelength 589.8 nm. Calculate the separation distance between two adjacent bright fringes on the screen.

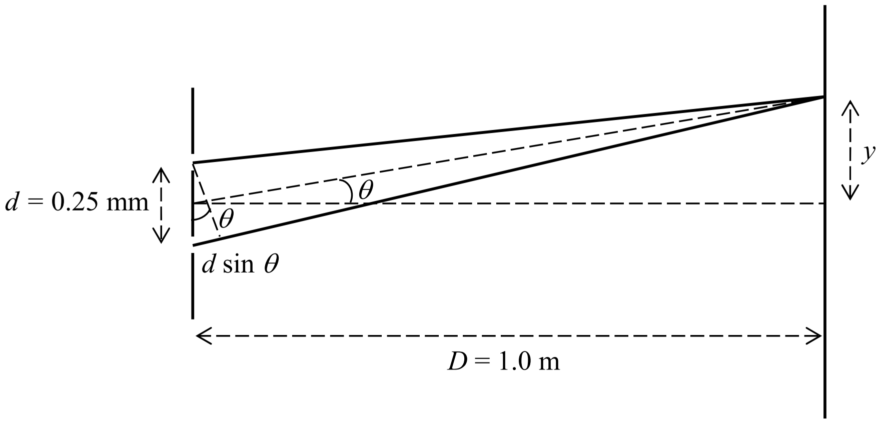


Fig. 19.5 Young's double slit experiment, Problem 19.2

Solution

Figure 19.5 shows the configuration of Young's double slit experiment. Here, y is the location of bright fringe from the central bright fringe, D is the slits-screen distance, d is the slits separation distance, θ is the angle between the bright fringe and the central bright fringe, and $d \sin\theta$ is the optical path difference of the rays from the two slits.

The optical path difference of the two rays from the slits is,

$$d \sin\theta = d\frac{y}{D}.$$

Constructive interference occurs if,

$$d\frac{y}{D} = m\lambda, \qquad m = 0,\ 1,\ 2,\ \ldots$$

This means that bright fringe is obtained at,

$$y_m = m\frac{\lambda D}{d}, \qquad m = 0,\ 1,\ 2,\ \ldots$$

Thus, the separation distance of two adjacent bright fringes is,

$$\begin{aligned} y_{m+1} - y_m &= (m+1)\frac{\lambda D}{d} - m\frac{\lambda D}{d} = \frac{\lambda D}{d} \\ &= \frac{(589.3 \times 10^{-9}\ \text{m})(1.0\ \text{m})}{0.25 \times 10^{-3}\ \text{m}} \\ &= 2.4 \times 10^{-3}\ \text{m}. \end{aligned}$$

- wxMaxima codes:

```
(%i4) fpprintprec:5; lambda:589.3e-9; D:1; d:0.25e-3;
(fpprintprec)     5
(lambda) 5.893*10^-7
(D)    1
(d)    2.5*10^-4
(%i5) separation_distance: lambda*D/d;
(separation_distance)    0.0023572
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of λ, D, and d.
(%i5) Calculate separation distance.

Problem 19.3 In Young's double slit experiment, light from a sodium vapor lamp (wavelength 589 nm) forms interference pattern with adjacent bright fringes separation of 0.35 cm. The distance of the double slit to the screen is 0.80 m. What is the separation distance of the double slit?

Solution

Figure 19.6 shows the setup of Young's double-slit experiment. In the figure y is the location of the bright fringe, D is the distance between the double slit and the screen, d is the separation distance of the double slit, θ is the angle between the bright fringe and the central bright fringe, and $d \sin\theta$ is the optical path difference of the rays of the two slits.

Optical path difference is,

$$d \sin\theta = d\frac{y}{D}.$$

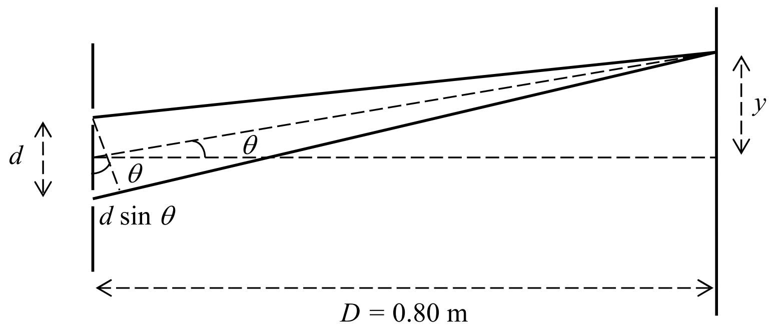


Fig. 19.6 Young's double slit experiment, Problem 19.3

Constructive interference occurs if optical path difference is zero or multiple of a wavelength λ of the light,

$$d\frac{y}{D} = m\lambda, \quad m = 0,\ 1,\ 2,\ ...$$

This means that a bright fringe is obtained at location,

$$y_m = m\frac{\lambda D}{d}.$$

Therefore, the distance between adjacent bright fringes is,

$$\begin{aligned}\Delta y = y_{m+1} - y_m &= \frac{(m+1)\lambda D}{d} - \frac{m\lambda D}{d} \\ &= \frac{\lambda D}{d}.\end{aligned}$$

Substituting the given numerical values into the equation enables the separation distance of the double slit to be calculated,

$$0.35 \times 10^{-2}\ \text{m} = \frac{(589 \times 10^{-9}\ \text{m})(0.80\ \text{m})}{d},$$
$$d = 1.3 \times 10^{-4}\ \text{m}.$$

- wxMaxima codes:

```
(%i5) fpprintprec:5; ratprint:false; lambda:589e-9; delta_y:0.35e-2; D:0.8;
(fpprintprec)     5
(ratprint)   false
(lambda)     5.89*10^-7
(delta_y)    0.0035
(D)    0.8
(%i7) solve(delta_y = lambda*D/d, d)$ float(%);
(%o7) [d=1.3463*10^-4]
```

Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of λ, Δy, and D.
(%i7) Solve $\Delta y = \lambda D/d$ for d.

Problem 19.4 The diameter of tenth bright ring changes from 1.40 cm to 1.27 cm when a liquid is filled between the lens and the glass plate in Newton's rings experiment. What is the refractive index of the liquid?

Solution

In Newton's rings experiment, the radius of the $(m+1)$-th bright ring is (Eq. 19.9),

$$r_{m+1} = \sqrt{(m+\frac{1}{2})\frac{\lambda}{n}R},$$

where R is radius of curvature of the lens, λ is the wavelength of the light of the experiment, and n is the refractive index of the medium between the lens and the glass plate. For the tenth bright ring, we write,

$$\frac{1.40\text{ cm}}{2} = \sqrt{(9+\frac{1}{2})\frac{\lambda}{1.00}R},$$

because the index of refraction of air is $n = 1.00$. When the liquid is filled between the lens and glass plate, the tenth bright ring satisfies,

$$\frac{1.27\text{ cm}}{2} = \sqrt{(9+\frac{1}{2})\frac{\lambda}{n}R},$$

where n is the refractive index of the liquid. By squaring and dividing both equations, λ and R are cancelled out, and the refractive index of the liquid can be calculated,

$$\frac{(1.40\text{ cm})^2}{(1.27\text{ cm})^2} = \frac{9.5\lambda R}{9.5\lambda R/n},$$

$$n = \left(\frac{1.40\text{ cm}}{1.27\text{ cm}}\right)^2 = 1.22.$$

- wxMaxima codes:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)     5
(ratprint)  false
(%i4) solve(1.40^2/1.27^2 = (9.5*lambda*R) / (9.5*lambda*R/n), n)$
float(%);
(%o4) [n=1.2152]
```

Comments of the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.

(%i4) Solve $1.40^2/1.27^2 = (9.5\lambda R)/(9.5\lambda R/n)$ for n.

Problem 19.5 Diameters of the m-th and $(m + 10)$-th dark rings formed in Newton's rings experiment are 0.14 cm and 0.86 cm, respectively. When the space between the lens and the glass plate is filled with water, the diameters of the p-th and $(p + 10)$-th dark rings are 0.23 cm and 0.77 cm, respectively. Calculate the index of refraction of water.

Solution

For a Newton's rings experiment, the radii of the m-th and $(m + 10)$-th dark rings are (Eq. 19.10),

$$r_m = \sqrt{m\lambda R}, \tag{19.11}$$

$$r_{m+10} = \sqrt{(m + 10)\lambda R}, \tag{19.12}$$

where λ is the wavelength of light and R is radius of curvature of the lens. We have substituted index of refraction of air to be $n = 1.00$ in both equations.

When water fills the space between the lens and the glass plate, the p-th and $(p + 10)$-th radii of the dark rings are (Eq. 19.10),

$$r_p = \sqrt{p\frac{\lambda}{n}R}, \tag{19.13}$$

$$r_{p+10} = \sqrt{(p + 10)\frac{\lambda}{n}R}, \tag{19.14}$$

where n is the index of refraction of water.

Squaring and subtracting Eqs. (19.12) and (19.11) give,

$$r_{m+10}^2 - r_m^2 = 10\lambda R. \tag{19.15}$$

Squaring and subtracting Eqs. (19.14) and (19.13) give,

$$r_{p+10}^2 - r_p^2 = \frac{10\lambda R}{n}. \tag{19.16}$$

The index of refraction of water can be calculated from Eqs. (19.15) and (19.16),

$$n = \frac{r_{m+10}^2 - r_m^2}{r_{p+10}^2 - r_p^2} = \frac{d_{m+10}^2 - d_m^2}{d_{p+10}^2 - d_p^2} = \frac{(0.86\ \text{cm})^2 - (0.14\ \text{cm})^2}{(0.77\ \text{cm})^2 - (0.23\ \text{cm})^2} = 1.33.$$

- wxMaxima codes:

```
(%i1) fpprintprec: 5;
(fpprintprec)      5
(%i2) n: (0.86^2-0.14^2)/(0.77^2-0.23^2);
(n)   1.3333
```

Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i2) Calculate n.

Problem 19.6 A thin air wedge of angle θ shown in Fig. 19.7 is made from two glass plates. The air wedge is illuminated by a light of wavelength λ. Interference of light is formed while bright and dark fringes are observed. Show that the separation between adjacent bright fringes is $\lambda/(2\theta)$.

Solution 19.6

Figure 19.8 shows the air wedge and the locations of the m-th and $(m+1)$-th bright fringes. Here, x_m and x_{m+1} are the distances from the wedge edge, while d_m and d_{m+1} are the corresponding thicknesses of air.

Bright fringes are obtained if (Eq. 19.7),

$$
\begin{aligned}
2d_m &= (m+\frac{1}{2})\lambda,\\
2\theta x_m &= (m+\frac{1}{2})\lambda,\\
x_m &= (m+\frac{1}{2})\frac{\lambda}{2\theta},
\end{aligned}
$$

where the fact that $\theta = d_m/x_m$ was used as angle θ is small. The path difference between the rays reflected from the lower and upper glass plates is 2 dm.

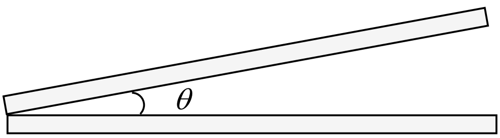


Fig. 19.7 Air wedge of Problem 19.6

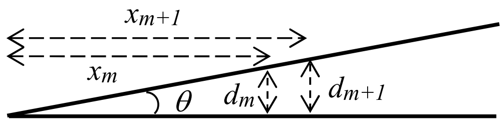


Fig. 19.8 Air wedge, Problem 19.6

For the adjacent bright fringe, we have,

$$2d_{m+1} = (m + 1 + \frac{1}{2})\lambda,$$
$$2\theta x_{m+1} = (m + 1 + \frac{1}{2})\lambda,$$
$$x_{m+1} = (m + 1 + \frac{1}{2})\frac{\lambda}{2\theta},$$

where the fact that $\theta = d_{m+1}/x_{m+1}$ was used as angle θ is small.

Therefore, the separation between adjacent bright fringes is,

$$x_{m+1} - x_m = (m + 1 + \frac{1}{2})\frac{\lambda}{2\theta} - (m + \frac{1}{2})\frac{\lambda}{2\theta}$$
$$= \frac{\lambda}{2\theta}.$$

Additional question: What is the separation between bright fringes if the wavelength of the light is 630 nm and the air wedge angle is 0.02°?

Answer: The separation is,

$$x_{m+1} - x_m = \frac{\lambda}{2\theta} = \frac{630 \times 10^{-9}\ \text{m}}{2(0.02\pi/180\ \text{rad})} = 9.0 \times 10^{-4}\ \text{m}.$$

- wxMaxima codes:

```
(%i2) fpprintprec:5; float(630e-9/(2*0.02*%pi/180));
(fpprintprec)     5
(%o2) 9.0241*10^-4
```

Comments on the codes:

(%i2) Set floating point print precision to 5 and calculate the separation distance.

Problem 19.7 An air wedge of angle 40 arc second is formed using two glass slides, Fig. 19.9. Bright and dark fringes are formed when the air wedge is illuminated by a monochromatic light and the separation distance between adjacent dark fringes is found to be 0.12 cm. Calculate the wavelength of the monochromatic light.

Solution 19.7

Figure 19.10 shows the air wedge, the wedge angle, θ, the distance of the dark fringe to the end of the wedge, x, and the thickness of air, d.

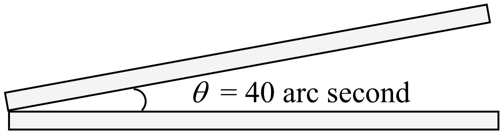


Fig. 19.9 Air wedge of Problem 19.7

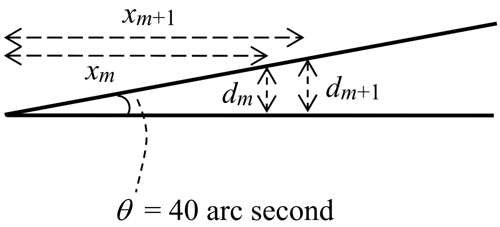


Fig. 19.10 Air wedge, Problem 19.7

We have,
1 degree = 60 arc minute = 60 × 60 arc second.
The angle is,

$$\theta = 40 \text{ arc second} = \frac{40}{60 \times 60} \text{ degree} = \frac{40}{60 \times 60} \times \frac{\pi}{180} \text{ rad}$$
$$= 1.939 \times 10^{-4} \text{ rad.}$$

A dark fringe is obtained if (Eq. 19.8),

$$2d_m = m\lambda. \tag{19.17}$$

An adjacent dark fringe is obtained if,

$$2d_{m+1} = (m+1)\lambda. \tag{19.18}$$

Angle θ is small, we have

$$\theta = \frac{d_m}{x_m} = \frac{d_{m+1}}{x_{m+1}}.$$

Using this equation, Eqs. (19.17) and (19.18) are written as,

$$2\theta x_m = m\lambda,$$
$$2\theta x_{m+1} = (m+1)\lambda.$$

From these two equations, the separation between adjacent dark fringes is,

$$\Delta x = x_{m+1} - x_m = \frac{\lambda}{2\theta}.$$

The wavelength of the light is,

$$\lambda = 2\theta\, \Delta x = 2(1.939 \times 10^{-4})(0.12 \times 10^{-2}\ \text{m})$$
$$= 4.7 \times 10^{-7}\ \text{m}.$$

- wxMaxima codes:

```
(%i3) fpprintprec:5; theta:float(40/3600*%pi/180); delta_x:0.12e-2;
(fpprintprec)     5
(theta)     1.9393*10^-4
(delta_x)   0.0012
(%i4) lambda: 2*theta*delta_x;
(lambda)    4.6542*10^-7
```

Comments on the codes:

(%i3) Set floating point print precision to 5, assign values of θ and Δx.
(%i4) Calculate λ.

Problem 19.8 In a Young's double slit experiment, the double slit separation is $d = 0.12$ mm, the distance between the double slit and the screen is $D = 110$ cm and the wavelength of light used is $\lambda = 546$ nm.

(a) Plot the average intensity of the interference pattern.
(b) Calculate the distance between the central maximum and the point where the intensity is 75% of the central maximum.
(c) Calculate the distance between adjacent bright fringes.

Solution

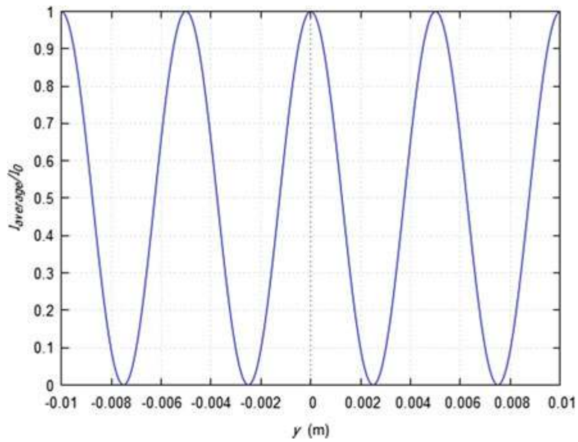
(a) Average intensity of interference pattern of Young's double-slit experiment is (Eq. 19.5),

$$I_{average} = I_0 \cos^2\left(\frac{\pi d}{\lambda D} y\right),$$

where I_0 is the intensity of the bright central maximum, d is separation distance of the slits, D is distance between the slits and the screen, y is distance from the central maximum on the screen, and λ is wavelength of light.

- Plot of $I_{average}/I_0$ against y for $-0.01 \leq y \leq 0.01$ m by wxMaxima:

```
(%i5) fpprintprec:5; d:0.12e-3; D:1.1; lambda:546e-9; I0:1;
(fpprintprec)     5
(d)   1.2*10^-4
(D)   1.1
(lambda)    5.46*10^-7
(I0)  1
(%i6) Iaverage: I0*cos(%pi*d*y/(lambda*D))^2;
(Iaverage)  cos(199.8*%pi*y)^2
(%i7) wxplot2d(Iaverage, [y,-0.01,0.01], grid2d, [xlabel,"{/Helvetica-
Italic y}  (m)"], [ylabel,"{/Helvetica-Italic I_{average}/I_0}"]);
```



Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of d, D, λ, and I_0.
(%i6) Define $I_{average}$ as a function of y.
(%i7) Plot $I_{average}$ against y for $-0.01 \leq y \leq 0.01$ m.

(b) The value of y such that the intensity is 75% of that of central maximum is calculated as follows,

$$
\begin{aligned}
I_{average} &= I_0 \cos^2\left(\frac{\pi d}{\lambda D} y\right), \\
0.75 I_0 &= I_0 \cos^2\left(\frac{\pi d}{\lambda D} y\right), \\
\cos\left(\frac{\pi d}{\lambda D} y\right) &= \sqrt{0.75} = 0.866, \\
\theta &= \frac{\pi d}{\lambda D} y = \cos^{-1}(0.866) = 0.524, \\
y &= \frac{\theta \lambda D}{\pi d} = \frac{0.524(546 \times 10^{-9}\ \text{m})(1.10\ \text{m})}{\pi(0.12 \times 10^{-3}\ \text{m})} = 8.3 \times 10^{-4}\ \text{m}.
\end{aligned}
$$

- wxMaxima codes:

```
(%i4) fpprintprec:5; d:0.12e-3; D:1.1; lambda:546e-9;
(fpprintprec)     5
(d)   1.2*10^-4
(D)   1.1
(lambda) 5.46*10^-7
(%i5) theta: acos(sqrt(0.75));
(theta) 0.5236
(%i7) y: theta*lambda*D/(%pi*d); float(%);
(y)   0.0026206/%pi
(%o7) 8.3417*10^-4
```

Comments on the codes:

(%i4)	Set floating point print precision to 5, assign values of d, D, and λ.
(%i5), (%i7)	Calculate θ and y.

- Alternative calculation:

```
(%i5) fpprintprec:5; ratprint:false; d:0.12e-3; D:1.1; lambda:546e-9;
(fpprintprec)     5
(ratprint)  false
(d)   1.2*10^-4
(D)   1.1
(lambda) 5.46*10^-7
(%i7) solve (cos(%pi*d*y/(lambda*D))=sqrt(0.75), y)$ float(%);
solve: using arc-trig functions to get a solution.
Some solutions will be lost.
(%o7) [y=8.3417*10^-4]
```

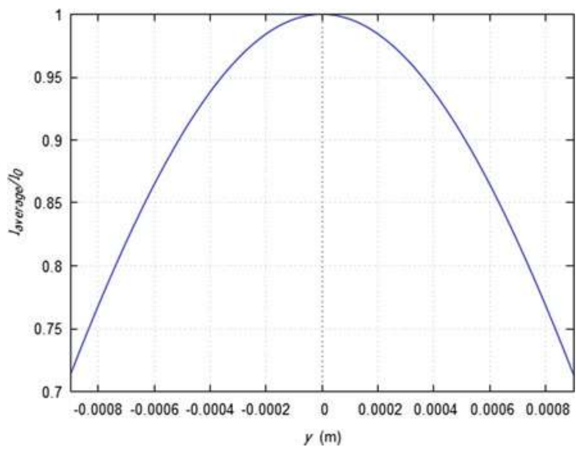
Comments on the codes:

(%i5) Set floating point print precision to 5, internal rational number print to false, assign values of values of d, D, and λ.

(%i7) Solve $\cos\left(\frac{\pi d}{\lambda D} y\right) = \sqrt{0.75}$ for y.

- Plot of $I_{average}/I_0$ against y for $-9 \times 10^{-4} \leq y \leq 9 \times 10^{-4}$ m by wxMaxima:

```
(%i5) fpprintprec:5; d:0.12e-3; D:1.1; lambda:546e-9; I0: 1;
(fpprintprec)     5
(d)   1.2*10^-4
(D)   1.1
(lambda) 5.46*10^-7
(I0)  1
(%i6) Iaverage: I0*cos(%pi*d*y/(lambda*D))^2;
(Iaverage)  cos(199.8*%pi*y)^2
(%i7) wxplot2d(Iaverage, [y,-9e-4,9e-4], grid2d, [xlabel,"{/Helvetica-
Italic y}  (m)"], [ylabel,"{/Helvetica-Italic I_{average}/I_0}"]);
```



Comments on the codes:

(%i5) Set floating point print precision to 5, assign values of d, D, λ, and I_0.

(%i6) Define $I_{average}$.

(%i7) Plot $I_{average}$ against y for $-9 \times 10^{-4} \leq y \leq 9 \times 10^{-4}$ m.

(c) The distance between adjacent bright fringes is,

$$\Delta y = y_{m+1} - y_m = \frac{(m+1)\lambda D}{d} - \frac{m\lambda D}{d} = \frac{\lambda D}{d}$$
$$= \frac{(546 \times 10^{-9}\ \text{m})(1.10\ \text{m})}{0.12 \times 10^{-3}\ \text{m}}$$
$$= 5.0 \times 10^{-3}\ \text{m}.$$

- wxMaxima codes:

```
(%i4) fpprintprec:5; d:0.12e-3; D:1.1; lambda:546e-9;
(fpprintprec)     5
(d)   1.2*10^-4
(D)   1.1
(lambda) 5.46*10^-7
(%i5) delta_y: lambda*D/d;
(delta_y) 0.005005
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of d, D, and λ.
(%i5) Calculate Δy.

Problem 19.9 Calculate the thickness of a soap film so that light of wavelength 600 nm incident on it is reflected constructively to get interference pattern. Index of refraction of soap film is 1.33.

Solution

Figure 19.11 shows the soap film of thickness, t, incident ray, and reflected rays.

Condition of constructive interference is (Eq. 19.7),

$$2nt = (m + \frac{1}{2})\lambda, \quad m = 0,\ 1,\ 2,\ ...$$

The thicknesses of the soap films are,

$$t = (m + \frac{1}{2})\frac{\lambda}{2n}, \quad m = 0,\ 1,\ 2,\ ...$$
$$= \frac{\lambda}{4n},\ \frac{3\lambda}{4n},\ \frac{5\lambda}{4n},\ ...$$
$$= \frac{600\ \text{nm}}{4(1.33)},\ \frac{3(600\ \text{nm})}{4(1.33)},\ \frac{5(600\ \text{nm})}{4(1.33)},\ ...$$
$$= 113\ \text{nm},\ 338\ \text{nm},\ 564\ \text{nm},\ ...$$

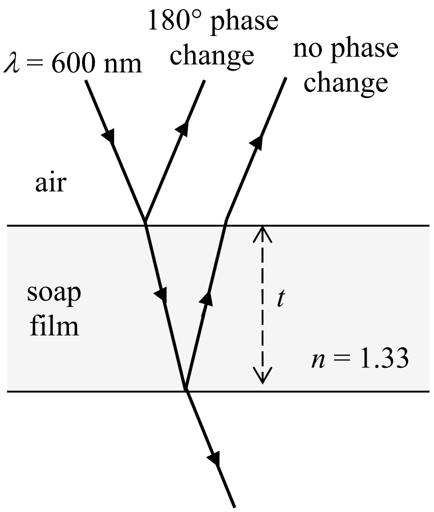


Fig. 19.11 Light interference by a soap film, Problem 19.9

- wxMaxima codes:

```
(%i3) fpprintprec:5; lambda:600e-9; n:1.33;
(fpprintprec)     5
(lambda) 6.0*10^-7
(n)    1.33
(%i4) t: lambda/(4*n);
(t)    1.1278*10^-7
(%i5) t: 3*lambda/(4*n);
(t)    3.3835*10^-7
(%i6) t: 5*lambda/(4*n);
(t)    5.6391*10^-7
```

Comments on the codes:

(%i3)	Set floating point print precision to 5, assign values of λ and n.
(%i4), (%i5), (%i6)	Calculate t.

Problem 19.10 Calculate the thickness of magnesium fluoride to be coated on glass so that a light of wavelength 500 nm incident on them is least reflected. Index of refraction of magnesium fluoride is $n_m = 1.38$ and that of glass is $n_g = 1.52$.

Solution

Figure 19.12 shows the magnesium fluoride layer, glass, incident ray, and reflected rays.

Reflection of light ray at air-magnesium fluoride interface results in 180° phase change, so is reflection at magnesium fluoride-glass interface, because both reflections are from higher index of refraction materials. As a result both reflected rays are

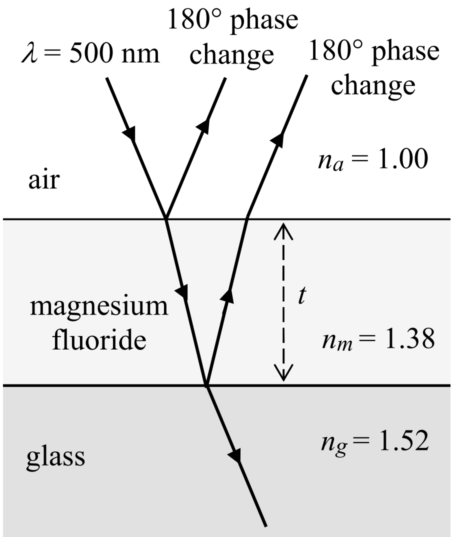


Fig. 19.12 Light interference by a magnesium fluoride coating, Problem 19.10

in phase. Because we want both reflected rays to be out of phase, the path difference $2t$ is one-half of a wavelength. The wavelength of light in magnesium fluoride is λ/n_m. To get destructive interference,

$$2t = \frac{1}{2}\frac{\lambda}{n_m}.$$

The thickness of the magnesium fluoride layer is,

$$t = \frac{\lambda}{4n_m} = \frac{500\text{ nm}}{4(1.38)} = 90.6\text{ nm}.$$

- wxMaxima codes:

```
(%i3) fpprintprec:5; lambda:500; n_m:1.38;
(fpprintprec)     5
(lambda) 500
(n_m) 1.38
(%i4) t: lambda/(4*n_m);
(t)   90.58
```

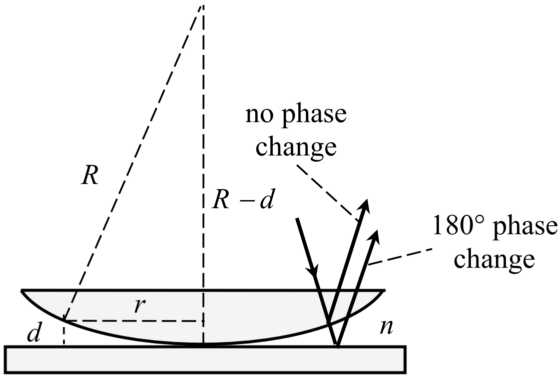


Fig. 19.13 Newton's rings experiment, Problem 19.11

Comments on the codes:

(%i3) Set floating point print precision to 5, assign values of λ and n_m.
(%i4) Calculate t.

Problem 19.11 For a Newton's rings experiment, show that the radius of bright ring is,

$$r = \sqrt{(m + \frac{1}{2})\frac{\lambda R}{n}}, \quad m = 0,\ 1,\ 2,\ ...$$

where R is the radius of curvature of the lens, λ is wavelength of light, and n is index of refraction of the medium between the lens and glass plate.

Solution

Figure 19.13 shows the setup of Newton's rings experiment. Here, R is the radius of curvature of the lens, r is radius of a bright ring, and d is thickness of the medium with refractive index n.

Using the Pythagoras' theorem,

$$R^2 = r^2 + (R - d)^2 = r^2 + R^2 - 2Rd + d^2,$$
$$2Rd = r^2 + d^2.$$

Because $r^2 >> d^2$, we write,

$$2Rd = r^2,$$
$$2d = \frac{r^2}{R}.$$

From the figure, $2d$ is the path difference of upward reflected rays from the lens and the glass plate. The ray reflected upward from the lens has no phase change, while the ray reflected upward from the glass plate has a 180° phase change. To get constructive interference,

$$2d = \frac{r^2}{R} = (m + \frac{1}{2})\frac{\lambda}{n},$$
$$r = \sqrt{(m + \frac{1}{2})\frac{\lambda R}{n}}, \quad m = 0,\ 1,\ 2,\ ...$$

where λ/n is the wavelength of light in the medium with refractive index n.

- wxMaxima codes:

```
(%i1) solve(r^2/R = (m + 1/2)*lambda/n, r);
(%o1) [r=-sqrt((2*R*m*lambda)/n+(R*lambda)/n)/sqrt(2),
       r=sqrt((2*R*m*lambda)/n+(R*lambda)/n)/sqrt(2)]
(%i2) radcan(%);
(%o2) [r=-(sqrt(R)*sqrt(2*m+1)*sqrt(lambda))/(sqrt(2)*sqrt(n)),
       r=(sqrt(R)*sqrt(2*m+1)*sqrt(lambda))/(sqrt(2)*sqrt(n))]
```

Comments on the codes:

(%i1) Solve $\frac{r^2}{R} = (m + \frac{1}{2})\frac{\lambda}{n}$ for r.
(%i2) Simplify the output.
(%o2) The solutions.

Problem 19.12 In a Newton's rings experiment, the wavelength of light used is 6700 Å and the 20-th dark ring is 11 mm in radius. Calculate,

(a) the thickness of air at the point
(b) the radius of curvature of the lens.

Solution

(a) Figure 19.14 shows the lens, glass plate, and geometry of the Newton's rings experiment.

The air thickness between lens surface and glass surface changes by $\lambda/2$ when we move from a dark ring to adjacent dark ring. Thus, thickness of 20-th dark ring is,

$$d = 20 \times \frac{\lambda}{2} = 20 \times \frac{6700 \times 10^{-10}\ \text{m}}{2} = 6.7 \times 10^{-6}\ \text{m}.$$

(b) From Fig. 19.14 and the Pythagoras' theorem,

$$R^2 = r^2 + (R - d)^2 = r^2 + R^2 - 2Rd + d^2,$$
$$2Rd = r^2 + d^2.$$

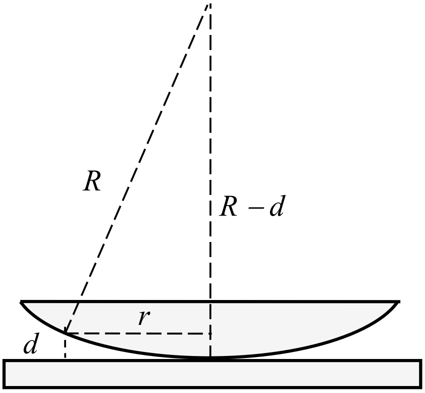


Fig. 19.14 Newton's rings experiment, Problem 19.12

As $r^2 >> d^2$, we write

$$2Rd = r^2,$$
$$R = \frac{r^2}{2d}.$$

The radius of curvature of the lens is,

$$R = \frac{r^2}{2d} = \frac{(11 \times 10^{-3}\ \mathrm{m})^2}{2(6.7 \times 10^{-6}\ \mathrm{m})} = 9.0\ \mathrm{m}.$$

- wxMaxima codes:

```
(%i3) fpprintprec:5; lambda:6700e-10; r:11e-3;
(fpprintprec)     5
(lambda) 6.7*10^-7
(r)   0.011
(%i4) d: 20*(lambda/2);
(d)   6.7*10^-6
(%i5) R: r^2/(2*d);
(R)   9.0299
```

Comments on the codes:

(%i3) Set floating point print precision to 5, assign values of λ and r.

(%i4), (%i5) Calculate d and R.

19.3 Summary

- Interference effects show the wave nature of light.
- Interference of light is an effect of superposition of light waves at a point. Persistent interference pattern exists if,
  - (a) wave sources are coherence (that is, the phase difference of sources is constant),
  - (b) the sources are monochromatic (that is, the same wavelength) and
  - (c) linear superposition principle is obeyed.
- Examples of interference effects of light are bright and dark rings in Newton's ring experiment, bright and dark fringes of Young' double-slit experiment, and bright and dark fringes in air wedge.

19.4 Exercises

Exercise 19.1 In Young's double slit experiment, the separation between slits is 5.0 $\times$ 10^{-5} m and the screen is 2.0 m away from the slits. The third-order bright fringe is 6.2 cm from the central bright fringe. Determine the wavelength of the light.

(Answer: 5.2 $\times$ 10^{-7} m)

Exercise 19.2 Light of wavelength 500 nm is incident on a layer of oil whose index of refraction is 1.46. What is the minimum thickness of the layer so that the reflected lights interfere constructively?

(Answer: 8.56 $\times$ 10^{-8} m)

Exercise 19.3 In Young's double-slit experiment, light of wavelength 500 nm illuminates two slits that are separated by 1.0 mm. The screen is 5.0 m away. Calculate the separation between adjacent bright fringes on the screen.

(Answer: 2.5 $\times$ 10^{-3} m)

Exercise 19.4 Laser light of wavelength 630 nm in Young's double-slit experiment produces an interference pattern in which the adjacent bright fringes are separated by 8.4 mm. A second light produces an interference pattern in which the adjacent bright fringes are separated by 7.5 mm. What is the wavelength of this second light?

(Answer: 560 nm)

Exercise 19.5 Calculate the thickness of magnesium fluoride to be coated on glass so that a light of wavelength 400 nm incident on them is least reflected. Index of refraction of magnesium fluoride is 1.38 and that of glass is 1.52.

(Answer: 72.5 nm)

Chapter 20
Diffraction of Light

Abstract Problems on diffraction of light are solved in this last chapter. Diffraction is bending or spreading of light at aperture or obstacle. Problems on diffraction by a single slit and diffraction by a grating and its resolving power are discussed. Solutions obtained by analysis and computer calculation of wxMaxima are presented.

20.1 Basic Concepts and Formulae

(1) When light waves encounter an aperture or an obstacle, the waves spread out as they travel and undergo interference. This is called diffraction. Diffraction of light is due to interference of continuous distribution of coherence sources of light.

(2) The Fraunhofer diffraction pattern of light by a single slit of width a on a screen consists of a bright central region and an alternating dark and bright regions is shown in Fig. 20.1.

The angle, θ, of the dark fringe is given by,

$$\sin\theta = m\frac{\lambda}{a}, \quad m = \pm 1,\ \pm 2,\ \ldots \tag{20.1}$$

where λ is the wavelength of light, a is width of the slit, and m is order number.

(3) The intensity of light, I, on the screen, varies with angle, θ, according to,

$$I = I_0\left[\frac{\sin(\beta/2)}{\beta/2}\right]^2, \quad \text{where } \beta = \frac{2\pi a \sin\theta}{\lambda}, \tag{20.2}$$

and I_0 is the intensity at $\theta = 0$, as shown in Figure 20.2.

(4) Rayleigh criterion states that two images formed by an aperture are just resolved if the central maximum diffraction pattern of one image falls on the first minimum of the other.

W. M. S. Wan Hassan et al., *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-43165-4_20

Figure 20.3 shows intensity patterns of two slits that are just resolved according to the Rayleigh criterion. The intensity patterns are drawn separately in (a), while the intensity pattern of both is shown in (b).

The limiting resolving angle for a diffraction by a slit of width, a, is,

$$\theta_{min} = \frac{\lambda}{a}. \tag{20.3}$$

The limiting resolving angle for a circular aperture of diameter, D, is,

$$\theta_{min} = 1.22\frac{\lambda}{D}. \tag{20.4}$$

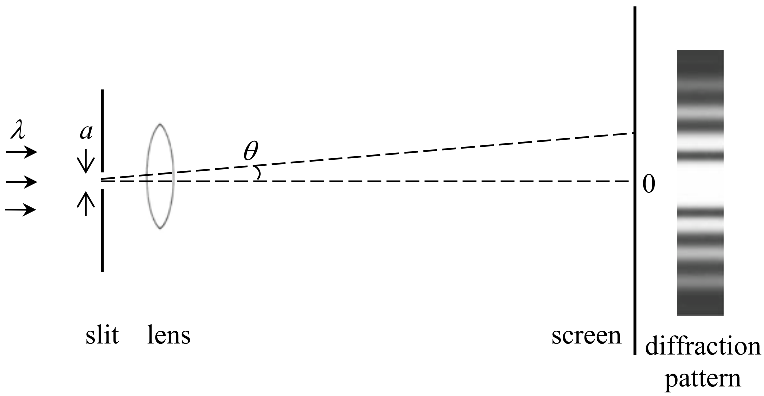
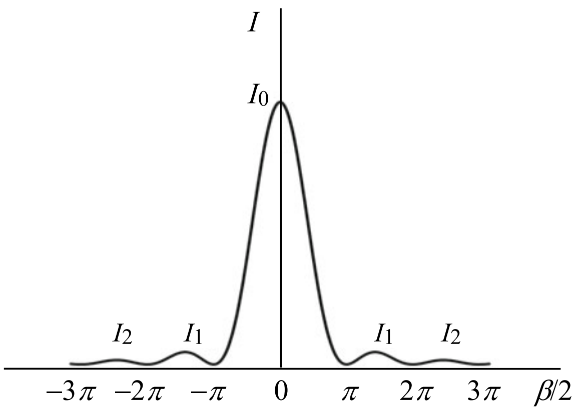


Fig. 20.1 Single slit diffraction. Light of wavelength λ is incident on a narrow slit of width a. Diffraction pattern is observed on a screen. The angle of the dark fringe is θ

Fig. 20.2 Intensity of light I against $\beta/2$ of a single slit diffraction



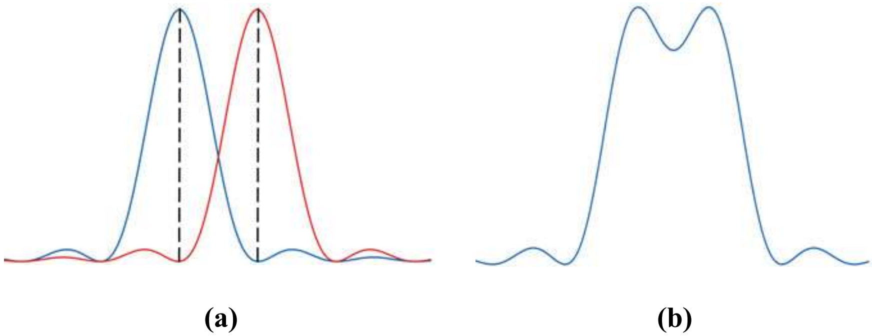


Fig. 20.3 Intensity patterns of two slits that are just resolved according to the Rayleigh criterion. The intensity patterns are shown separately in (**a**), while the intensity pattern as observed on the screen is shown in (**b**)

(5) A diffraction grating consists of packed identical slits. Condition for maximum intensity (bright fringe) is,

$$d \sin\theta = m\lambda, \quad m = 0, 1, 2, \ldots \tag{20.5}$$

where d is the distance between slits, θ is diffraction angle, λ is wavelength of light, and m is order number of the diffraction pattern. Zeroth-order maximum is at angle, $\theta = 0$; first-order maximum corresponding to $m = 1$, is at angle, θ, satisfying $\sin\theta = \lambda/d$; second-order maximum corresponding to $m = 2$, is at angle, θ, satisfying $\sin\theta = 2\lambda/d$; and so on.

Figure 20.4 shows the diffraction of a monochrome light by a diffraction grating.

From Eq. (20.5) and Fig. 20.4, one writes,

$$\theta_m = \sin^{-1}\left(\frac{m\lambda}{d}\right), \tag{20.6}$$

$$y_m \approx \frac{m\lambda D}{d}. \tag{20.7}$$

Resolving power, R, of a diffraction grating at m-th order diffraction is,

$$R = Nm = \frac{\lambda}{\Delta\lambda}, \tag{20.8}$$

where N is the number of lines of the diffraction grating, $\Delta\lambda$ is wavelength separation of two monochromatic light waves that are barely distinguishable and λ is their mean wavelength.

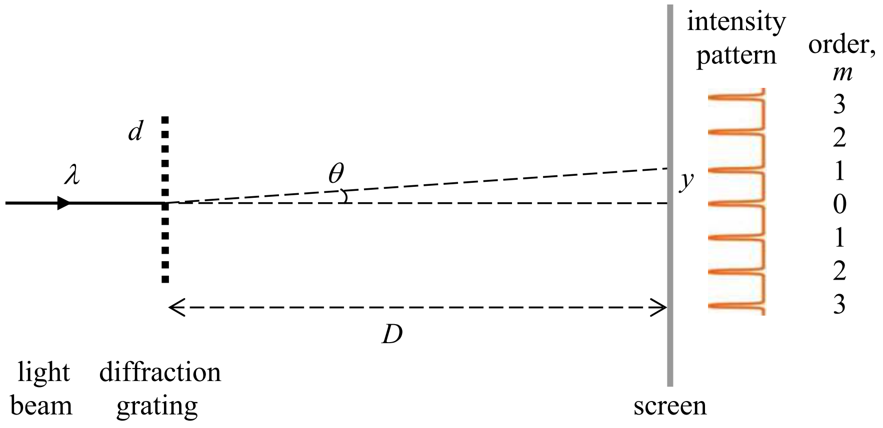


Fig. 20.4 Diffraction of a monochrome light by a diffraction grating. Light of wavelength λ is incident on a diffraction grating with slit separation *d*. Bright fringe is observed on a screen a distance *D* away, at angle *θ* or a distance *y* from the central maximum

20.2 Problems and Solutions

Problem 20.1 A plane wave of monochromatic light ($\lambda = 5900$ Å) is incident on a slit of width, $a = 0.04$ mm. A converging lens ($f = +70$ cm) is placed behind the slit to focus the light on a screen. What is the separation between the first and the second minima?

Solution

Figure 20.5 shows the slit, lens, screen, and geometry of the single slit diffraction. Also shown on the far right is the diffraction pattern. Here, *a* is the width of the slit and *θ* is the diffraction angle.

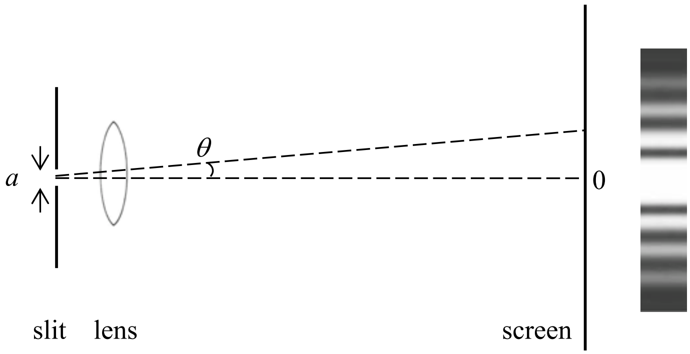


Fig. 20.5 Single slit diffraction experiment, *a* is slit width and *θ* is angle of dark fringe, Problem 20.1

For this diffraction, the minimum (dark) is obtained if (Eq. 20.1),

$$a \ \sin\theta = m\lambda, \qquad m = \pm 1, \ \pm 2, \ ...$$

As θ is a small angle, $\sin\theta \approx \theta$, one writes,

$$a\theta_m = m\lambda,$$
$$\theta_m = m\frac{\lambda}{a}, \quad m = \pm 1, \ \pm 2, \ ...$$

For the first minimum (first dark fringe), the diffraction angle is,

$$\theta_1 = \frac{\lambda}{a} = \frac{5900 \times 10^{-10} \text{ m}}{0.04 \times 10^{-3} \text{ m}} = 1.5 \times 10^{-2} \text{ rad.}$$

For the second minimum, the diffraction angle is,

$$\theta_2 = 2 \times \frac{\lambda}{a} = 2 \times \frac{5900 \times 10^{-10} \text{ m}}{0.04 \times 10^{-3} \text{ m}} = 2.9 \times 10^{-2} \text{ rad.}$$

The angular difference of the two minima is,

$$\Delta\theta = \theta_2 - \theta_1 = 1.5 \times 10^{-2} \text{ rad.}$$

The separation of the two minima is

$$\Delta y = f \cdot \Delta\theta = 0.70 \text{ m} \times 1.5 \times 10^{-2} = 0.01 \text{ m.}$$

- wxMaxima codes:

```
(%i4) fpprintprec:5; lambda:5900e-10; a:0.04e-3; f:70e-2;
(fpprintprec)     5
(lambda)    5.9*10^-7
(a)   4.0*10^-5
(f)   0.7
(%i5) theta1: lambda/a;
(theta1) 0.01475
(%i6) theta2: 2*lambda/a;
(theta2) 0.0295
(%i7) delta_theta: theta2-theta1;
(delta_theta) 0.01475
(%i8) delta_y: f*delta_theta;
(delta_y) 0.010325
```

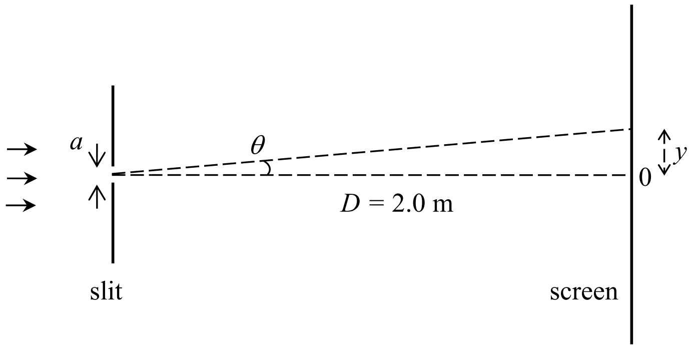


Fig. 20.6 Single slit diffraction experiment, a is slit width, θ is angle of dark fringe, y is on-screen distance of dark fringe, and D is slit-screen distance, Problem 20.2

Comments on the codes:

(%i4)	Set floating point print precision to 5, assign values of λ, a, and f.
(%i5), (%i6)	Calculate θ_1 and θ_2.
(%i7), (%i8)	Calculate $\Delta\theta$ and Δy.

Problem 20.2 A light of wavelength 580 nm is shined on a slit of width 0.30 mm. A screen is positioned 2.0 m away from the slit. Determine,

(a) the location of the first dark fringe,
(b) the width of the central bright fringe,
(c) the width of the first bright fringe.

Solution

(a) Figure 20.6 shows the slit, lens, and geometry of the problem. Here, a is the width of the slit, θ is angle of diffraction, and D is distance between the slit and the screen.

The first dark fringe satisfies $a \sin\theta_1 = \lambda$ (Eq. 20.1). This means that,

$$\sin\theta_1 = \pm\frac{\lambda}{a} = \pm\frac{y_1}{D},$$

$$y_1 = \pm\frac{D\lambda}{a} = \pm\frac{(2.0 \text{ m})(580 \times 10^{-9} \text{ m})}{0.30 \times 10^{-3} \text{ m}} = \pm 3.9 \times 10^{-3} \text{ m}.$$

(b) The width of the central bright fringe is two times y_1,

$$2y_1 = 7.7 \times 10^{-3} \text{ m}.$$

(c) The first-order bright fringe is located between the first and second dark fringes, that is, between y_1 and y_2. Calculate y_2,

$$y_2 = \frac{2D\lambda}{a} = \frac{2(2.0\ \text{m})(580 \times 10^{-9}\ \text{m})}{0.30 \times 10^{-3}\ \text{m}} = 7.7 \times 10^{-3}\ \text{m}.$$

The width of the first-order bright fringe is,

$$y_2 - y_1 = 7.7 \times 10^{-3}\ \text{m} - 3.9 \times 10^{-3}\ \text{m} = 3.9 \times 10^{-3}\ \text{m}.$$

- wxMaxima codes:

```
(%i4) fpprintprec:5; lambda:580e-9; a:0.3e-3; D:2;
(fpprintprec)     5
(lambda)    5.8*10^-7
(a)   3.0*10^-4
(D)   2
(%i5) y1: D*lambda/a;
(y1)  0.0038667
(%i6) width_of_central_bright_fringe: 2*y1;
(width_of_central_bright_fringe) 0.0077333
(%i7) y2: 2*D*lambda/a;
(y2)  0.0077333
(%i8) y2-y1;
(%o8) 0.0038667
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of λ, a, and D.
(%i5) Calculate y_1.
(%i6) Calculate the width of the central bright fringe.
(%i7) Calculate y_2.
(%i8) Calculate the width of first-order bright fringe.

Figure 20.7 shows the intensity of the diffraction pattern. Diffraction angle of the first dark fringe is θ_1 and the location of the fringe is y_1. Diffraction angle of the second dark fringe is θ_2 and the location of the second dark fringe is y_2. The width of central bright fringe is $2y_1$ and the width of the first bright fringe is $y_2 - y_1$.

Problem 20.3 Figure 20.8 shows a curve of intensity, I, against $\beta/2$ of a single slit diffraction. The intensity is given by,

$I = I_0\left[\frac{\sin(\beta/2)}{\beta/2}\right]^2$, where $\beta = \frac{2\pi a \sin\theta}{\lambda}$,

and I_0 is the maximum intensity of the central bright fringe. Calculate the intensity ratio of first- and second-order maxima (I_1 and I_2) to that of central maximum, I_0, that is, calculate I_1/I_0 and I_2/I_0.

Solution

The first-order intensity maximum, I_1, is located approximately in the middle of $\beta/2 = \pi$ and $\beta/2 = 2\pi$, that is, at $\beta/2 = 3\pi/2$. The intensity ratio of the first maximum

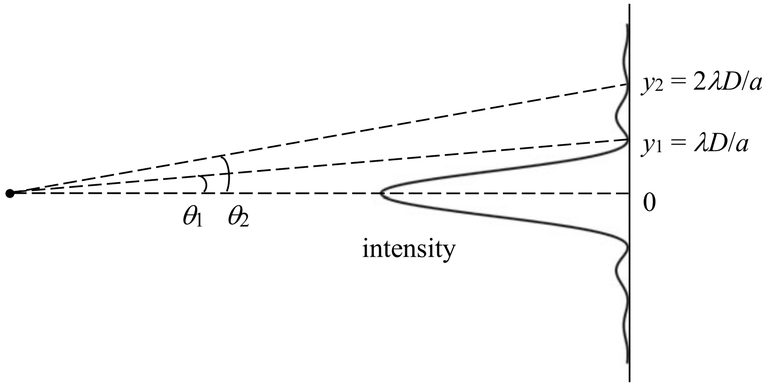
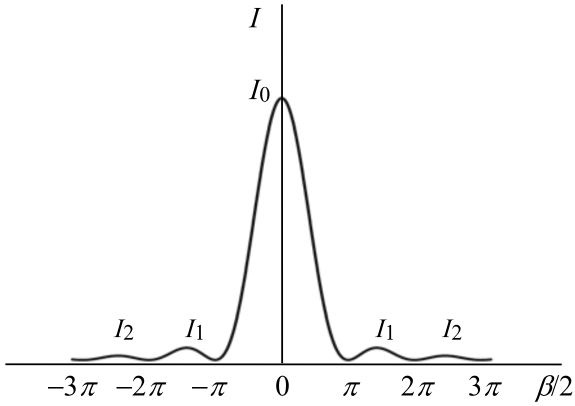


Fig. 20.7 Intensity of the diffraction pattern

Fig. 20.8 Intensity I against $\beta/2$ of a single slit diffraction



to that of the central maximum is,

$$\frac{I_1}{I_0} = \left[\frac{\sin(3\pi/2)}{3\pi/2}\right]^2 = 0.045.$$

The second-order intensity maximum, I_2, is located approximately in the middle of $\beta/2 = 2\pi$ and $\beta/2 = 3\pi$, that is, at $\beta/2 = 5\pi/2$. The intensity ratio of the second maximum to that of the central maximum is,

$$\frac{I_2}{I_0} = \left[\frac{\sin(5\pi/2)}{5\pi/2}\right]^2 = 0.016.$$

This means that, I_1 and I_2 are approximately 4.5% and 1.6% of I_0, respectively.

- wxMaxima codes:

```
(%i1) fpprintprec: 5;
(fpprintprec)     5
(%i3) I1_over_I0: (sin(3*%pi/2)/(3*%pi/2))^2; float(%);
(I1_over_I0) 4/(9*%pi^2)
(%o3) 0.045032
(%i5) I2_over_I0: (sin(5*%pi/2)/(5*%pi/2))^2; float(%);
(I2_over_I0) 4/(25*%pi^2)
(%o5) 0.016211
```

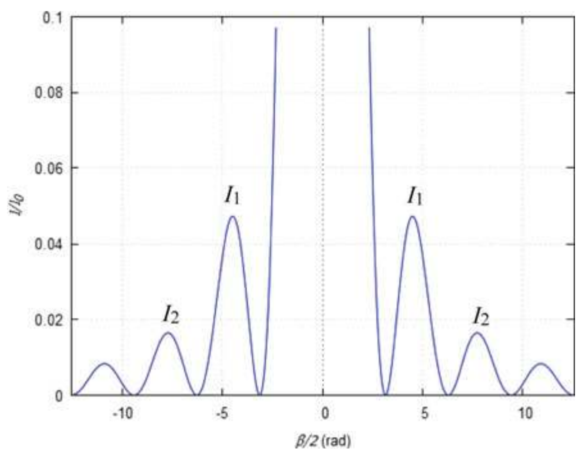
Comments on the codes:

(%i1) Set floating point print precision to 5.
(%i3), (%i5) Calculate I_1/I_0 and I_2/I_0.

Further question: Plot the intensity, I, against $\beta/2$ of a single slit diffraction to check the results.

- Plot of I against $\beta/2$ for $-4\pi \le \beta/2 \le 4\pi$ rad by wxMaxima:

```
(%i2) I0:1; I:I0*(sin(betaovertwo)/betaovertwo)^2;
(I0)  1
(I)   sin(betaovertwo)^2/betaovertwo^2
(%i3) wxplot2d(I, [betaovertwo, -4*%pi, 4*%pi], [y,0, 0.1], grid2d,
[xlabel,"{/Symbol-Italic b/2} (rad)"], [ylabel,"{/Helvetica-Italic
I/I_0}"]);
```



Comments on the codes:

(%i2) Assign $I_0 = 1$ and define $I = I_0\left[\frac{\sin(\beta/2)}{\beta/2}\right]^2$.
(%i3) Plot I against $\beta/2$ for -4π rad $\leq \beta/2 \leq 4\pi$ rad.

Problem 20.4 In a single slit diffraction experiment, a light of wavelength 580 nm is incident on a slit of width 0.30 mm. A screen is located 2.0 m away from the slit. By setting the intensity of central maximum as $I_0 = 1.00$, plot the curve of

(a) intensity, I, versus angle of diffraction, θ, in radian,
(b) intensity, I, versus angle of diffraction, θ, in degree,
(c) intensity, I, versus distance on the screen, y.

Solution

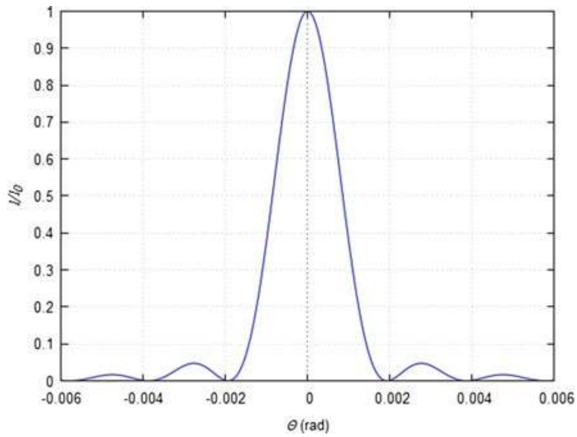
(a) Intensity, I, at angle of diffraction, θ, is given by,

$$I = I_0\left[\frac{\sin(\beta/2)}{\beta/2}\right]^2, \text{ where } \beta = \frac{2\pi a \sin\theta}{\lambda}.$$

To plot the curve by wxMaxima, first, assign the values of wavelength, λ, slit width, a, slit-screen distance, D, and intensity of the central maximum, I_0. Next, define β in terms of θ (radian) and I in terms of β. Lastly, plot I against θ (radian) using the wxplot2d function.

- Plot by wxMaxima:

```
(%i5) fpprintprec:5; lambda:580e-9; a:0.3e-3; D:2; I0:1;
(fpprintprec)     5
(lambda)    5.8*10^-7
(a)   3.0*10^-4
(D)   2
(I0)  1
(%i6) beta: 2*%pi*a*sin(theta)/lambda;
(beta) 1034.5*%pi*sin(theta)
(%i7) I: I0*sin(beta/2)^2/(beta/2)^2;
(I) (3.7378*10^-6*sin(517.24*%pi*sin(theta))^2)/(%pi^2*sin(theta)^2)
(%i8) wxplot2d(I, [theta,-0.006,0.006], grid2d, [xlabel,"{/Symbol-Italic Q}
(rad)"], [ylabel,"{/Helvetica-Italic I/I_0}"]);
```



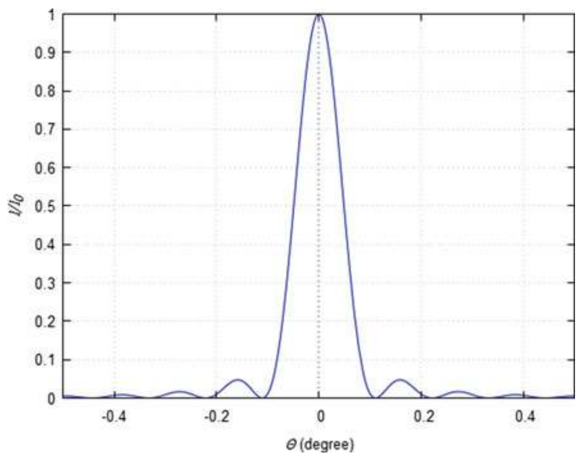
Comments on the codes:

(%i5)	Set floating point print precision to 5, assign values of λ, a, D, and I_0.
(%i6), (%i7)	Define β and I.
(%i8)	Plot I against θ for $-0.006 \leq \theta \leq 0.006$ rad.

(b) Assign the values of wavelength, λ, slit width, a, slit-screen distance, D, and intensity of the central maximum, I_0. Define β in terms of θ (degree) and I in terms of β. Plot I against θ (degree) using the wxplot2d function.

- Plot by wxMaxima:

```
(%i5) fpprintprec:5; lambda:580e-9; a:0.3e-3; D:2; I0:1;
(fpprintprec)     5
(lambda)    5.8*10^-7
(a)   3.0*10^-4
(D)   2
(I0)  1
(%i6) beta: 2*%pi*a*sin(degree*%pi/180)/lambda;
(beta) 1034.5*%pi*sin((%pi*degree)/180)
(%i7) I: I0*sin(beta/2)^2/(beta/2)^2;
(I) (3.7378*10^-6*sin(517.24*%pi*sin((%pi*degree)/180))^2)
/(%pi^2*sin((%pi*degree)/180)^2)
(%i8) wxplot2d(I, [degree,-0.5,0.5], grid2d, [xlabel,"{/Symbol-Italic Q}
(degree)"], [ylabel,"{/Helvetica-Italic I/I_0}"]);
```



Comments on the codes:

(%i5)	Set floating point print precision to 5, assign values of λ, *a*, *D*, and I_0.
(%i6), (%i7)	Define β and *I*.
(%i8)	Plot *I* against θ for $-0.5° \le \theta \le 0.5°$.

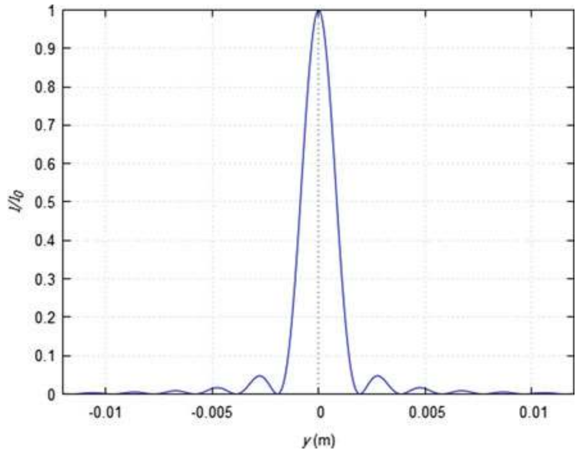
(c) The intensity, *I*, at position, *y*, is

$$I = I_0\left[\frac{\sin(\beta/2)}{\beta/2}\right]^2, \text{ where } \beta = \frac{2\pi a \sin\theta}{\lambda} = \frac{2\pi a y}{\lambda D},$$

because $\sin\theta = y/D$. To plot *I* against *y*, assign the values of wavelength, λ, slit width, *a*, slit-screen distance, *D*, and intensity of the central maximum, I_0. Next, define β in terms of *y* and *I* in terms of β. Lastly, plot *I* against *y* using the wxplot2d function.

- Plot by wxMaxima:

```
(%i5) fpprintprec:5; lambda:580e-9; a:0.3e-3; D:2; I0:1;
(fpprintprec)     5
(lambda)    5.8*10^-7
(a)   3.0*10^-4
(D)   2
(I0)   1
(%i6) beta: 2*%pi*a*y/lambda;
(beta) 1034.5*%pi*y
(%i7) I: I0*sin(beta/2)^2/(beta/2)^2;
(I) (3.7378*10^-6*sin(517.24*%pi*y)^2)/(%pi^2*y^2)
(%i8) wxplot2d(I, [y,-0.012,0.012], grid2d, [xlabel,"{/Helvetica-Italic y}
(m)"], [ylabel,"{/Helvetica-Italic I/I_0}"]);
```



Comments on the codes:

(%i5)	Set floating point print precision to 5, assign values of λ, a, D, and I_0.
(%i6), (%i7)	Define β and I.
(%i8)	Plot I against y for $-0.012 \leq y \leq 0.012$ m.

Problem 20.5 A light of wavelength, λ = 580 nm, is incident to a slit of width, a_1 = 29 μm = 50λ. The screen is located a distance, D = 0.8 m, away from the slit. A diffraction pattern is observed on the screen. The experiment is repeated using different slits of width, a_2 = 58 μm = 100λ, and a_3 = 87 μm = 150λ. How do the diffraction patterns change?

Solution

This problem is solved by plotting the intensities of the three diffraction patterns from slits of different widths. We plot these three curves of intensity I against diffraction angle θ (degree),

$$I_1 = I_0\left[\frac{\sin(\beta/2)}{\beta/2}\right]^2, \text{ where } \beta = \frac{2\pi a_1 \sin\theta}{\lambda},$$

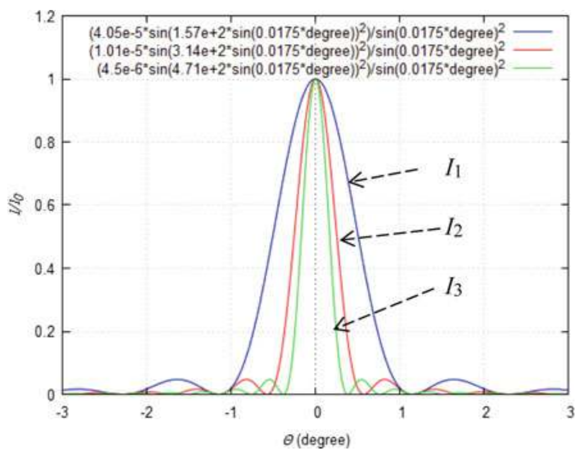
$$I_2 = I_0\left[\frac{\sin(\beta/2)}{\beta/2}\right]^2, \text{ where } \beta = \frac{2\pi a_2 \sin\theta}{\lambda},$$

$$I_3 = I_0\left[\frac{\sin(\beta/2)}{\beta/2}\right]^2, \text{ where } \beta = \frac{2\pi a_3 \sin\theta}{\lambda}.$$

Assign the values of wavelength, λ, slit widths, a_1, a_2, and a_3, slit-screen distance, D, and intensity of the central maximum, I_0. Define β in terms of θ (degree) and I in terms of β. Plot I against θ (degree) using the wxplot2d function.

- Plot by wxMaxima:

```
(%i7) fpprintprec:3; lambda:580e-9; a1:50*lambda; a2:100*lambda;
a3:150*lambda; D:0.8; I0:1;
(fpprintprec)     3
(lambda)    5.8*10^-7
(a1)  2.9*10^-5
(a2)  5.8*10^-5
(a3)  8.7*10^-5
(D)   0.8
(I0)  1
(%i8) beta: float(2*%pi*a1*sin(degree*%pi/180)/lambda);
(beta) 3.14*10^2*sin(0.0175*degree)
(%i9) I1: I0*sin(beta/2)^2/(beta/2)^2;
(I1)  (4.05*10^-5*sin(1.57*10^2*sin(0.0175*degree))^2)/sin(0.0175*degree)^2
(%i10) beta: float(2*%pi*a2*sin(degree*%pi/180)/lambda);
(beta) 6.28*10^2*sin(0.0175*degree)
(%i11) I2: I0*sin(beta/2)^2/(beta/2)^2;
(I2)  (1.01*10^-5*sin(3.14*10^2*sin(0.0175*degree))^2)/sin(0.0175*degree)^2
(%i12) beta: float(2*%pi*a3*sin(degree*%pi/180)/lambda);
(beta) 9.42*10^2*sin(0.0175*degree)
(%i13) I3: I0*sin(beta/2)^2/(beta/2)^2;
(I3)  (4.5*10^-6*sin(4.71*10^2*sin(0.0175*degree))^2)/sin(0.0175*degree)^2
(%i14) wxplot2d([I1, I2, I3], [degree,-3,3], [y,0,1.2], grid2d,
[xlabel,"{/Symbol-Italic Q} (degree)"], [ylabel,"{/Helvetica-Italic
I/I_0}"]);
```



Comments on the codes:

(%i7)	Set floating point print precision to 5, assign values of λ, a_1, a_2, a_3, D, and I_0.
(%i8), (%i9)	Define β and I_1.
(%i10), (%i11)	Define β and I_2.
(%i12), (%i13)	Define β and I_3.
(%i14)	Plot I_1, I_2, and I_3 against θ for $-3° \le \theta \le 3°$.

The diffraction fringe widths decrease as the slit widths increase. This means that narrow slit gives wide diffraction. Table 20.1 gives the angular and linear widths of the central bright fringe (central maxima) of the three experiments. The angular and linear widths of the central maxima are calculated as $2\lambda/a \times 180/\pi$ and $2\lambda D/a$, respectively.

Problem 20.6 Calculate separation distance of two points on the moon that are just resolved by the Palomar Mountain telescope. Diameter of the telescope aperture is 5.0 m, earth-moon distance is 3.86×10^5 km, and $\lambda = 5500$ Å.

Solution

For circular aperture of the telescope, the Rayleigh resolving criterion is (Eq. 20.4),

$$\theta_{min} = 1.22\frac{\lambda}{D}.$$

The resolving angle is,

$$\theta_{min} = 1.22 \times \frac{5500 \times 10^{-10}\ \text{m}}{5.0\ \text{m}} = 1.3 \times 10^{-7}\ \text{rad}.$$

The separation distance so that two points on the moon can be resolved is,

$$\begin{aligned}\Delta x &= d \cdot \theta_{min} = 3.86 \times 10^5\ \text{km} \times 1.3 \times 10^{-7}\ \text{rad} = 0.052\ \text{km} \\ &= 52\ \text{m}.\end{aligned}$$

Table 20.1 Angular and linear widths of the central maxima of a single slit diffraction

Slit width a (μm)	Angular width (degree)	Linear width (mm)
29	2.3	32
58	1.1	16
87	0.76	11

- wxMaxima codes:

```
(%i4) fpprintprec:5; lambda:5500e-10; D:5; d:3.86e8;
(fpprintprec)     5
(lambda) 5.5*10^-7
(D)    5
(d)    3.86*10^8
(%i5) theta_min: 1.22*lambda/D;
(theta_min) 1.342*10^-7
(%i6) delta_x: d*theta_min;
(delta_x) 51.801
```

Comments on the codes:

(%i4)	Set floating point print precision to 5, assign values of λ, D, and d.
(%i5), (%i6)	Calculate θ_{min} and Δx.

Problem 20.7

(a) Estimate the limiting resolving angle of human eyes. The diameter of the pupil is 2.0 mm, index of refraction of the eye is 1.33, and the wavelength of light in air is 550 nm.
(b) What is the spatial resolution of the eye at 25 cm away?

Solution

(a) The wavelength of light in human eye is (Eq. 4.3),

$$\lambda = \frac{\lambda_0}{n} = \frac{550 \text{ nm}}{1.33} = 414 \text{ nm}.$$

The limiting resolving angle of the eye is (Eq. 20.4),

$$\theta_{min} = 1.22\frac{\lambda}{D} = 1.22 \times \frac{414 \times 10^{-9} \text{ m}}{2.0 \times 10^{-3} \text{ m}} = 2.5 \times 10^{-4} \text{ rad}.$$

(b) At a distance of 25 cm from the eye, spatial resolution of the eye is,

$$\Delta x = d \cdot \theta_{min} = (25 \times 10^{-2} \text{ m})(2.5 \times 10^{-4} \text{ rad}) = 6.3 \times 10^{-5} \text{ m}.$$

This means that, at 25 cm away, two points that are less than $6.3 \times 10-5$ m apart cannot be resolved by the eye.

- wxMaxima codes:

```
(%i5) fpprintprec:5; lambda0:550e-9; n:1.33; D:2e-3; d:25e-2;
(fpprintprec)     5
(lambda0)   5.5*10^-7
(n)   1.33
(D)   0.002
(d)   0.25
(%i6) lambda: lambda0/n;
(lambda) 4.1353*10^-7
(%i7) theta_min: 1.22*lambda/D;
(theta_min) 2.5226*10^-4
(%i8) delta_x: d*theta_min;
(delta_x) 6.3064*10^-5
```

Comments on the codes:

(%i5)	Set floating point print precision to 5, assign values of λ_0, n, D, and d.
(%i6), (%i7)	Calculate λ and θ_{min}, part (a).
(%i8)	Calculate Δx, part (b).

Problem 20.8 A microscope uses light of sodium lamp of wavelength 589 nm to probe subjects. The aperture of the objective is 1.0 cm in diameter. Calculate the limiting resolving angle.

Solution

The limiting resolving angle of the microscope is (Eq. 20.4),

$$\theta_{min} = 1.22\frac{\lambda}{D} = 1.22 \times \frac{589 \times 10^{-9}\ \text{m}}{1.0 \times 10^{-2}\ \text{m}} = 7.2 \times 10^{-5}\ \text{rad}.$$

This means that two points subtending less than 7.2×10^{-5} rad at the objective of the microscope cannot be resolved.

- wxMaxima codes:

```
(%i3) fpprintprec:5; lambda:589e-9; D:1e-2;
(fpprintprec)     5
(lambda) 5.89*10^-7
(D)   0.01
(%i4) theta_min: 1.22*lambda/D;
(theta_min) 7.1858*10^-5
```

Comments on the codes:

(%i3) Set floating point print precision to 5 and assign value of λ.
(%i4) Calculate θ_{min}.

Problem 20.9 A helium neon laser light of wavelength 632.8 nm is incident to a diffraction grating that has 7000 lines per cm. At what angles do maximum intensities be observed?

Solution

There are 7000 lines or slits in one cm, so the width of a slit is,

$$d = \frac{1.0}{7000}\ \text{cm} = \frac{1.0 \times 10^{-2}}{7000}\ \text{m} = 1.429 \times 10^{-6}\ \text{m}.$$

For a diffraction grating, to get maximum intensities (bright fringes) (Eq. 20.5),
$d \sin\theta = m\lambda, \quad m = 0, 1, 2, \ldots$
The first-order maximum, $m = 1$,

$$\sin\theta_1 = \frac{\lambda}{d} = \frac{632.8 \times 10^{-9}\ \text{m}}{1.429 \times 10^{-6}\ \text{m}} = 0.316,$$
$$\theta_1 = 0.321\ \text{rad} = 18.4^\circ.$$

The second-order maximum, $m = 2$,

$$\sin\theta_2 = \frac{2\lambda}{d} = \frac{2 \times 632.8 \times 10^{-9}\ \text{m}}{1.429 \times 10^{-6}\ \text{m}} = 0.633,$$
$$\theta_2 = 0.685\ \text{rad} = 39.3^\circ.$$

The third-order maximum, $m = 3$,

$$\sin\theta_3 = \frac{3\lambda}{d} = \frac{3 \times 632.8 \times 10^{-9}\ \text{m}}{1.429 \times 10^{-6}\ \text{m}} = 0.949,$$
$$\theta_3 = 1.25\ \text{rad} = 71.7^\circ.$$

For $m = 4$, calculation gives

$$\sin\theta_4 = \frac{4\lambda}{d} = \frac{4 \times 632.8 \times 10^{-9}\ \text{m}}{1.429 \times 10^{-6}\ \text{m}} = 1.27.$$

This cannot be because it is greater than 1. This means that the diffraction patterns that can be observed by this laser light are first- , second- , and third-order maxima.

- wxMaxima codes:

```
(%i3) fpprintprec:5; lambda:632.8e-9; d:1/5000*1e-2;
(fpprintprec)     5
(lambda) 6.328*10^-7
(d)   2.0*10^-6
(%i6) sintheta1:lambda/d; theta1:asin(sintheta1);
theta1_deg:float(theta1*180/%pi);
(sintheta1) 0.3164
(theta1) 0.32193
(theta1_deg) 18.445
(%i9) sintheta2:2*lambda/d; theta2:asin(sintheta2);
theta2_deg:float(theta2*180/%pi);
(sintheta2) 0.6328
(theta2) 0.68516
(theta2_deg) 39.257
(%i12) sintheta3:3*lambda/d; theta3:asin(sintheta3);
theta3_deg:float(theta3*180/%pi);
(sintheta3) 0.9492
(theta3) 1.2507
(theta3_deg) 71.659
(%i13) sintheta4: 4*lambda/d;
(sintheta4) 1.2656
```

Comments on the codes:

(%i3) Set floating point print precision to 5, assign values of λ and d.
(%i6) Calculate θ_1 and convert the angle to degree.
(%i9) Calculate θ_2 and convert the angle to degree.
(%i12) Calculate θ_3 and convert the angle to degree.

Problem 20.10 The first-order spectrum lines are obtained at 30° when a light is incident to a diffraction grating with 6000 lines per cm. What is the wavelength of the light?

Solution

For a diffraction grating, a condition to get maximum intensity (bright bands) is (Eq. 20.5),

$d \sin\theta = m\lambda, \quad m = 0, 1, 2, \ldots$

For this problem,

$$d \sin\theta = m\lambda,$$
$$\left(\frac{1.0 \times 10^{-2}\ \text{m}}{6000}\right) \sin\left(30 \times \frac{\pi}{180}\right) = (1)\lambda,$$
$$\lambda = 8.3 \times 10^{-7}\ \text{m}.$$

The wavelength of the light is 8.3×10^{-7} m.

- wxMaxima codes:

```
(%i4) fpprintprec:5; d:1/6000*1e-2; theta:float(30/180*%pi); m:1;
(fpprintprec)     5
(d)    1.6667*10^-6
(theta) 0.5236
(m)    1
(%i5) lambda: d*sin(theta);
(lambda) 8.3333*10^-7
```

Comments on the codes:

(%i4) Set floating point print precision to 5, assign values of d, θ, and m.
(%i5) Calculate λ.

20.3 Summary

- In a single slit diffraction, the condition for destructive interference is

$$a \sin\theta = m\lambda, \quad m = \pm 1,\ \pm 2,\ \pm 3,\ \ldots$$

  where a is the width of the slit and θ is diffraction angle. The intensity at a point on the screen is given by $I_\theta = I_0\left[\frac{\sin(\beta/2)}{\beta/2}\right]^2$, where $\beta = \frac{2\pi a \sin\theta}{\lambda}$.
- The condition for intensity maxima for a diffraction grating whose slits are separated by a distance d is

$$d \sin\theta = m\lambda, \quad m = 0,\ \pm 1,\ \pm 2,\ \pm 3,\ \ldots$$

  where θ is the diffraction angle and m is order number.

20.4 Exercises

Exercise 20.1 In a single slit diffraction experiment, a light of wavelength 600 nm is incident on a slit of width 1.90 μm. What are the diffraction angles of the first and second dark fringes?

(Answer: $\theta_1 = 18.4°$, $\theta_2 = 39.2°$)

Exercise 20.2 In a single slit diffraction experiment, a light of wavelength 610 nm is incident on a slit of width 3.1×10^{-5} m, and diffraction pattern is formed on a screen located 2.5 m away from the slit. Calculate the distance from the central maximum to the first and second minima on the screen.

(Answer: $y_1 = 0.049$ m, $y_2 = 0.098$ m)

Exercise 20.3 An astronomical telescope has a diameter of 5.60 m. Calculate the maximum angle of resolution for this telescope at a wavelength of 600 nm.

(Answer: 1.31×10^{-7} rad)

Exercise 20.4 A beam of light of wavelength 540 nm is incident normally on a diffraction grating with a slit spacing of 1.70×10^{-6} m. What are the angles for the first- and second-order maxima?

(Answer: $\theta_1 = 18.5°$, $\theta_2 = 39.4°$)

Exercise 20.5 A diffraction grating just resolves the wavelengths 610.0 and 610.2 nm in the first order. What is the number of slits in the grating?

(Answer: 3050)

Appendix A
Introduction to wxMaxima

wxMaxima is an open computer algebra system software that can be installed on *Microsoft Windows* operating system, as well as on *Linux* and *OS X* operating systems. On mobiles, an apps *Maxima On Android* is available and can easily be installed. Other popular computer algebra systems are *Maple* and *Mathematica*. wxMaxima is a document based interface for the computer algebra system called *Maxima*. *Maxima* was developed from the *Macsyma* project since 1982 by the Department of Energy of the USA. This means wxMaxima gives menu and dialogue for various commands, plots, and animations of *Maxima*.

wxMaxima is distributed under *GNU General Public License*.

wxMaxima can be downloaded and installed on your pc from:

https://sourceforge.net/projects/maxima/files/Maxima-Windows/.

A manual of wxMaxima can be read from:

http://maxima.sourceforge.net/docs/manual/en/maxima.html.

A short and useful tutorial to start using wxMaxima can be obtained from:

http://Math-blog.com/2007/06/04/A-10-min-tutorial-for-solving-math-problems-with-maxima/.

In this book wxMaxima version 5.43.0 on *Microsoft Windows* was used. The installer file was maxima-clisp-sbcl-5.43.0-win64.exe. Older or newer versions wxMaxima would give minor changes in output display, but the calculation output should be almost the same.

Using wxMaxima

Figure A.1 shows wxMaxima window when it is started. On top of the window are *File*, *Edit*, *View*, *Cell*, *Maxima*, *Equations*, *Algebra*, *Calculus*, *Simplify*, *List*, *Plot*, *Numeric*, and *Help* menus. Under these menus, are other menus in icons. Discussion on using these menus is not done in this appendix

To give a command, type the command, type; to end it, and simultaneously press <shift> and <enter> keys to execute it. wxMaxima will display its response or output.

W. M. S. Wan Hassan et al., *Physics—Problems, Solutions, and Computer Calculations*,
https://doi.org/10.1007/978-3-031-43165-4

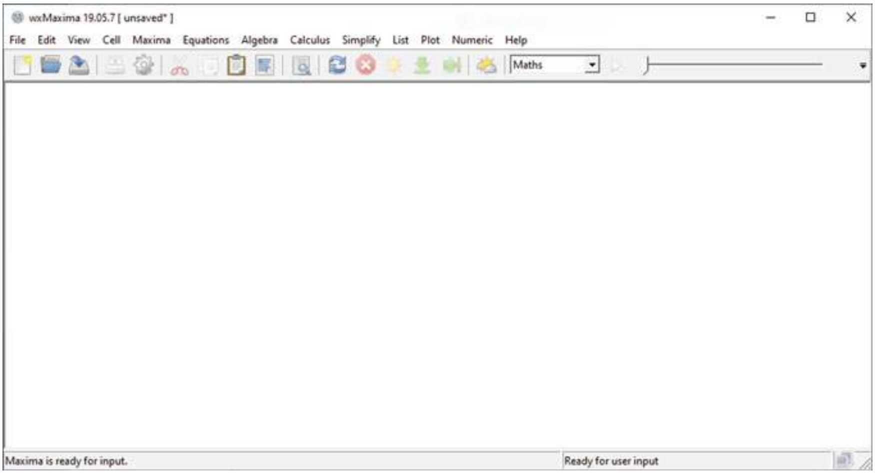


Fig. A.1 WxMaxima window

To exit wxMaxima, press <ctrl-Q> , click *File* menu and choose *Exit*, or click the cross icon on the top right of the wxMaxima window.

For example, you are to calculate (3 – 0.5) × 6.54^3 and plot the line $y = 2x + 3$ for $-5 \leq x \leq 5$. Type (3–0.5)*6.54^3; and simultaneously press <shift> and <enter> . wxMaxima will display its result of calculation. Next, type y: 2*x + 3; and simultaneously press <shift> and <enter> keys to define the line. Lastly, type wxplot2d (y, [x, − 5,5]); and simultaneously press <shift> and <enter> again to instruct wxMaxima plot the line. Figure A.2 shows the wxMaxima window after three commands were executed.

This appendix briefly discusses how to do the calculations as in this book. These are small set of calculations that wxMaxima can perform. Readers must study various sources about wxMaxima or Maxima from the internet for further applications.

Simple Calculations

To calculate 2 × 5, type 2*5; and simultaneously press <shift> and <enter> keys. For division, addition, and subtraction, use /, + , − . The result of calculating 2 × 5, 2 ÷ 5, 2 + 5, and 2 − 5 is as follows,

```
(%i1) 2*5;
(%o1) 10
(%i2) 2/5;
(%o2) 2/5
(%i3) 2+5;
(%o3) 7
(%i4) 2-5;
(%o4) -3
```

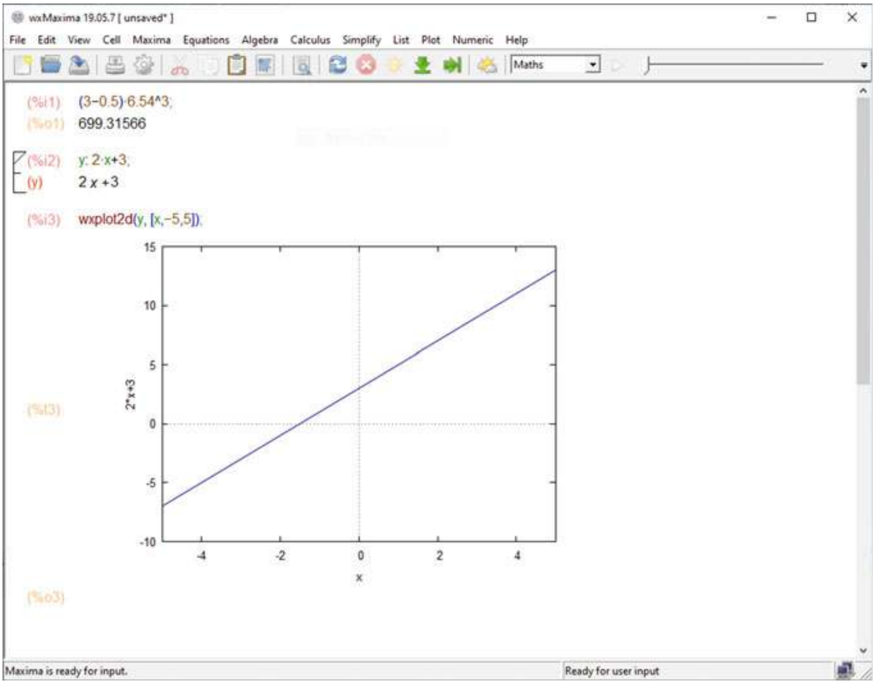


Fig. A.2 wxMaxima input and output in the window

wxMaxima tags the input typed by user as (%i—) and the output as (%o—). These tags could be used for further calculation. The last output is tagged as %. Typing %; will give −3, typing %o1; will give 10, and typing %o3; will give 7.

```
(%i5) %;
(%o5) -3
(%i6) %o1;
(%o6) 10
(%i7) %o3;
(%o7) 7
```

To hide the output type $ instead of ; at the end of a command, followed by simultaneously pressing <shift> and <enter> keys. wxMaxima executes the command and will not display the output. Other command to hide the output is *ratprint:false;*. This command will suppress display of output related to internal rational number calculation of wxMaxima. Command to limit number of digits of numerical value that is displayed is *fpprintprec,* floating point print precision. For example, the command *fpprintprec*:5; will display only 5 significant digits.

To calculate $\sqrt{3}$, type sqrt(3); and simultaneously press <shift> and <enter> keys. To get its decimal value type float(%); and simultaneously press <shift> and <enter> keys again, the result is,

```
(%i1) sqrt(3);
(%o1) sqrt(3)
(%i2) float(%);
(%o2) 1.732050807568877
(%i3) fpprintprec:5;
(fpprintprec)     5
(%i4) float(%o2);
(%o4) 1.7321
```

To input 2^{10}, type 2^10; to input 3×10^8, type 3e8; and to input 1.6×10^{-19}, type 1.6e-19. To calculate $\frac{2^{10}}{3\times10^8} \times 1.6 \times 10^{-19}$ for example, we only have to type 2^10/3e8*1.6e-19; and the result is,

```
(%i1) 2^10/3e8*1.6e-19;
(%o1) 5.461333333333334*10^-25
```

Let us say you want to change 2^{10} to 2^9 in the calculation. Use the computer mouse to go to 2^10, and do the editing to replace 10 with 9 using <del> or <backspace> and 9, followed by simultaneous press of <shift> and <enter> keys. The results is,

```
(%i2) 2^9/3e8*1.6e-19;
(%o2) 2.730666666666667*10^-25
```

This way of editing and recalculation is very useful to correct typos and to recalculate. We do not have to retype the whole input, just correct the typos and simultaneously press <shift> and <enter> keys. This is an advantage of using a software or an apps as opposed to using a calculator to do calculations.

Restart

To start a new calculation, click *Maxima* menu at top of the window, and choose *Restart Maxima*. This will clear the wxMaxima memory and we can start a new session of the calculation. We will always *Restart Maxima* for a new calculation.

Assignment

To assign a value of 3 to m, i.e. $m = 3$, type m: 3;. To assign $a = 11$, type a: 11;. Do not forget to simultaneously press <shift> and <enter> for each command. Thereafter m and a are always in the memory of the computer and can be used for further calculation. To calculate $F = ma$ for example, type F: m*a; simultaneously press <shift> and <enter> .

```
(%i2) m: 3; a: 11;
(m)   3
(a)   11
(%i3) F: m*a;
(F)   33
```

We frequently use this method in solving, calculating, and checking physics problems in this book.

Substitution

The values of m and a in the previous example, can also be substituted into the formula $F = ma$ by function *subst* as follows:

```
(%i1) F: m*a;
(F)   a*m
(%i2) subst([m=3, a=11], F);
(%o2) 33
```

or more simply,

```
(%i1) F: subst([m=3, a=11], m*a);
(F)   33
```

Function Definition

To define a function use :=. For example, define $g(x) = 2x^2 + x - 3$ and calculate $g(4)$. This is performed by wxMaxima as follows,

```
(%i1) g(x) := 2*x^2 + x - 3;
(%o1) g(x):=2*x^2+x-3
(%i2) g(4);
(%o2) 33
```

Define

Another way to define a function is by predefined function *define*. Arguments of *define* are the function and its definition. The previous example can be realized as follows:

```
(%i1) definition: 2*x^2 + x - 3;
(definition)      2*x^2+x-3
(%i2) define(g(x), definition);
(%o2) g(x):=2*x^2+x-3
(%i3) g(4);
(%o3) 33
```

Solving an Equation

To solve an equation, use wxMaxima built-in function called *solve*. For example, solve $12x + 3 = 5$, i.e. find x that satisfies the equation. Type solve(12*x + 3 = 5,x);, simultaneously press <shift> and <enter> keys, and wxMaxima gives $x = 1/6$ as a solution,

```
(%i1) solve(12*x+3=5,x);
(%o1) [x=1/6]
```

To use *solve* two arguments are needed, the first is the equation "12*x + 3 = 5" and the second is the unknown variable to be found "x". The *solve* built-in function is frequently used in this book.

The output of *solve* is a *list* as indicated by the square bracket [...]. In this example, x is not yet assigned the value of 1/6. To pick the value 1/6 from a *list*, the right-hand side *rhs*(...) built-in function is useful. Thus, to solve the equation and assign the solution as x, the codes are,

```
(%i1) solve(12*x + 3 = 5, x);
(%o1) [x=1/6]
(%i2) x: rhs(%o1[1]);
(x)   1/6
```

To solve a quadratic equation $5y^2 + y - 6 = 0$, type solve(5*y^2 + y–6 = 0,y); and simultaneously press <shift> and <enter> keys. The result is,

```
(%i1) solve( 5*y^2 + y - 6 = 0, y);
(%o1) [y=-6/5,y=1]
```

Therefore, the solutions of the quadratic equation are $y = -6/5$ and $y = 1$.

To solve the quadratic equation and assign the solutions as y_1 and y_2, the codes are,

```
(%i1) solve( 5*y^2 + y - 6 = 0, y);
(%o1) [y=-6/5,y=1]
(%i3) y1: rhs(%o1[1]); y2: rhs(%o1[2]);
(y1)  -6/5
(y2)  1
```

Solving Simultaneous Equations

To solve simultaneous equations,

$$2x + 5y = 11,$$
$$x - 4y = 7,$$

type solve([2*x + 5*y = 11, x-4*y = 7], [x,y]); and simultaneously press <shift> and <enter> keys. The result is,

```
(%i1) fpprintprec:5;
(fpprintprec)     5
(%i3) solve( [2*x+5*y=11, x-4*y=7], [x,y]); float(%);
(%o2) [[x=79/13,y=-3/13]]
(%o3) [[x=6.0769,y=-0.23077]]
```

Here, the first argument of *solve* is a *list* of two equations [2*x+5*y = 11, x-4*y = 7], and the second argument is a *list* of two variables [x,y] that are to be calculated. The solutions of the system of equations are $x = 6.07\ldots$ and $y = -0.230\ldots$

To assign x and y the values of the solutions, the codes can be as follows,

```
(%i1) fpprintprec:5;
(fpprintprec)     5
(%i3) solve( [2*x+5*y=11, x-4*y=7], [x,y]); float(%);
(%o2) [[x=79/13,y=-3/13]]
(%o3) [[x=6.0769,y=-0.23077]]
(%i5) x: rhs(%o3[1][1]); y: rhs(%o3[1][2]);
(x)   6.0769
(y)   -0.23077
```

As another example, solve the following system of equations,

$3x + y - z = 0,$

$2x - 3y + z = 1,$

$2x + y + 2z = 7.$

Type solve([3*x + y–z = 0, 2*x-3*y + z = 1, 2*x + y + 2*z = 7],[x,y,z]); and simultaneously press <shift> and <enter> keys. The result is,

```
(%i1) fpprintprec:5;
(fpprintprec)     5
(%i3) solve([3*x+y-z=0,2*x-3*y+z=1,2*x+y+2*z=7],[x,y,z]); float(%);
(%o2) [[x=17/31,y=27/31,z=78/31]]
(%o3) [[x=0.54839,y=0.87097,z=2.5161]]
```

Solutions of the system of equations are $x = 0.54\ldots$, $y = 0.87\ldots$, and $z = 2.51\ldots$.

To assign x, y, and z the values of the solutions, the codes can be as follows,

```
(%i1) fpprintprec:5;
(fpprintprec)     5
(%i3) solve([3*x+y-z=0,2*x-3*y+z=1,2*x+y+2*z=7],[x,y,z]); float(%);
(%o2) [[x=17/31,y=27/31,z=78/31]]
(%o3) [[x=0.54839,y=0.87097,z=2.5161]]
(%i4) x: rhs(%o3[1][1]);
(x)   0.54839
(%i5) y: rhs(%o3[1][2]);
(y)   0.87097
(%i6) z: rhs(%o3[1][3]);
(z)   2.5161
```

Angle

Angles are in radian. This means that cos(60) is cosine of 60 rad, and is not cosine of 60°. If cosine of 60° is needed, we type cos(60/180*%pi). We convert angle in degree to angle in rad in the argument of the built in function *cos*. In wxMaxima π is typed as %pi.

```
(%i1) cos(60);
(%o1) cos(60)
(%i2) float(%);
(%o2) -0.9524129804151563
(%i3) cos(60/180*%pi);
(%o3) 1/2
(%i4) float(%);
(%o4) 0.5
```

The inverse cosine, inverse sine, and inverse tangent functions, i.e. $\cos^{-1}$, $\sin^{-1}$, and $\tan^{-1}$ called *acos*, *asin*, and *atan* in wxMaxima will give angles in radians. If angles in degrees are needed, conversion must be made by multiplying the angle in radian by 180 and division by %pi.

```
(%i1) acos(1/2);
(%o1) %pi/3
(%i2) float(%);
(%o2) 1.047197551196598
(%i3) float(%*180/%pi);
(%o3) 60.0
```

The codes show that $\cos^{-1}(1/2) = 1.047 \text{ rad} \equiv 60°$.

Logarithm

The built in function *log*(…) is the natural logarithm (logarithm of base *e*). Thus,

$$\log e = \ln e = 1,$$
$$\log(e \times e) = \ln e^2 = 2,$$
$$\log 10 = \ln 10 = 2.3026,$$

as in the following codes,

```
(%i1) fpprintprec:5;
(fpprintprec)     5
(%i2) log(%e);
(%o2) 1
(%i3) log(%e*%e);
(%o3) 2
(%i5) log(10); float(%);
(%o4) log(10)
(%o5) 2.3026
```

Here, the Euler's number $e = 2.718\ldots$ is typed as %e in wxMaxima.

To get logarithm of base 10, you divide log(x) by log(10), because,

$$\log_{10} x = \frac{\log_e x}{\log_e 10} = \frac{\ln x}{\ln 10}.$$

The codes below show that $\log_{10}(0.2) = -0.69\ldots$, $\log_{10}(1) = 0$, $\log_{10}(2) = 0.30\ldots$, $\log_{10}(4) = 0.60\ldots$, $\log_{10}(10) = 1$, and $\log_{10}(151) = 2.1\ldots$.

```
(%i1) fpprintprec:5;
(fpprintprec)     5
(%i2) float(log(0.2)/log(10));
(%o2) -0.69897
(%i3) float(log(1)/log(10));
(%o3) 0.0
(%i4) float(log(2)/log(10));
(%o4) 0.30103
(%i5) float(log(4)/log(10));
(%o5) 0.60206
(%i6) float(log(10)/log(10));
(%o6) 1.0
(%i7) float(log(151)/log(10));
(%o7) 2.179
```

If one defines log10(x) as log(x)/log(10) at the beginning, then the defined function can be used repeatedly. The above codes become,

```
(%i1) log10(x):= float( log(x)/log(10) );
(%o1) log10(x):=float(log(x)/log(10))
(%i2) fpprintprec:5;
(fpprintprec)     5
(%i3) log10(0.2);
(%o3) -0.69897
(%i4) log10(1);
(%o4) 0.0
(%i5) log10(2);
(%o5) 0.30103
(%i6) log10(4);
(%o6) 0.60206
(%i7) log10(10);
(%o7) 1.0
(%i8) log10(151);
(%o8) 2.179
```

Differentiation

To differentiate a mathematical expression, use wxMaxima built-in function *diff*. For example given $y = 2\ \sin(5x + \pi/4)$, differentiate y with respect to x, i.e. find dy/dx. We type diff(2*sin(5*x + %pi/4), x); and simultaneously press <shift> and <enter> keys. Alternatively, we can first define y followed by the differentiation with respect to x, that is, we input y:2*sin(5*x + %pi/4); followed by diff(y,x);. The result is,

```
(%i1) diff(2*sin(5*x + %pi/4), x);
(%o1) 10*cos(5*x+%pi/4)
(%i2) y:  2*sin(5*x + %pi/4);
(y)   2*sin(5*x+%pi/4)
(%i3) diff(y, x);
(%o3) 10*cos(5*x+%pi/4)
```

%pi is a predefined constant π. Other predefined constants are Euler's number %e, Euler–Mascheroni constant %gamma, and golden ratio %phi. Their values can be checked as follows,

```
(%i4) float(%pi); float(%e); float(%gamma); float(%phi);
(%o1) 3.141592653589793
(%o2) 2.718281828459045
(%o3) 0.5772156649015329
(%o4) 1.618033988749895
```

Integration

To integrate, use built-in function *integrate*. For example, calculate $\int_1^4 5x^2 dx$. Key in integrate(5*x^2, x, 1, 4); and simultaneously press <shift> and <enter> keys. The result is,

```
(%i1) integrate(5*x^2, x, 1, 4);
(%o1) 105
```

This means *integrate* needs four arguments: function, variable, lower limit, and upper limit.

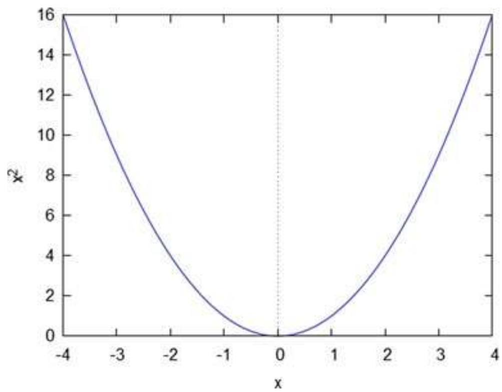
Another built-in function to do definite integration is *romberg*. This is a numerical integration built-in function. To calculate the same problem above, key in romberg(5*x^2, x, 1, 4); and simultaneously press <shift> and <enter> keys. The result is,

```
(%i1) romberg(5*x^2, x, 1, 4);
(%o1) 105.0
```

Two-Dimensional Plot

To plot in 2D, use the built in function *wxplot2d*. For example, plot the curve $y = x^2$ for $-4 \leq x \leq 4$. Key in wxplot2d(x^2, [x,-4,4]); and simultaneously press <shift> and <enter> keys. The result is,

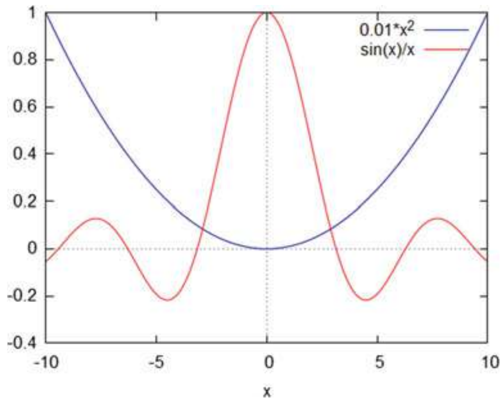
```
(%i1) wxplot2d( x^2, [x,-4,4] );
```



The command *wxplot2d* needs two arguments. First, the expression of the curve. Second, a *list* consisting of the variable, the lower, and upper limits in square brackets.

Another example, plot two curves $y = 0.01x^2$ and $z = \sin(x)/x$, for $-10 \leq x \leq 10$. We define the two functions and use the definitions in the *wxplot2d* command. The result is,

```
(%i2) y: 0.01*x^2;    z: sin(x)/x;
(y)    0.01*x^2
(z)    sin(x)/x
(%i3) wxplot2d([y,z], [x,-10,10] );
```



Vector

For vector calculation, *vect* package or module has to be loaded by the command *load*("*vect*");. A vector is defined as a *list* in square bracket [...]. The operator for dot (scalar) product is the dot · and for vector (cross) product is ~ followed by *express*(%). For example, given vectors, $\boldsymbol{A} = 4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ and $\boldsymbol{B} = 5\mathbf{i} + 6\mathbf{j} + 7\mathbf{k}$, calculate $\boldsymbol{A}\cdot\boldsymbol{B}$, magnitude of A and B, the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$, $\boldsymbol{A} \times \boldsymbol{B}$, and magnitude of $\boldsymbol{A} \times \boldsymbol{B}$. The wxMaxima calculation is as follows,

```
(%i1) fpprintprec:5;
(fpprintprec)     5
(%i2) load("vect");
(%o2) "C:/maxima-5.43.0/share/maxima/5.43.0/share/vector/vect.mac"
(%i4) A:[4,3,2]; B:[5,6,7];
(A)   [4,3,2]
(B)   [5,6,7]
(%i5) A.B;
(%o5) 52
(%i7) magnitudeA: sqrt(A.A); float(%);
(magnitudeA) sqrt(29)
(%o7) 5.3852
(%i9) magnitudeB: sqrt(B.B); float(%);
(magnitudeB) sqrt(110)
(%o9) 10.488
(%i11) acos(A.B/(magnitudeA*magnitudeB)); float(%);
(%o10) acos(52/(sqrt(29)*sqrt(110)))
(%o11) 0.40098
(%i12) float(%*180/%pi);
(%o12) 22.975
(%i14) A~B; express(%);
(%o13) [4,3,2]~[5,6,7]
(%o14) [9,-18,9]
(%i16) magnitudeAxB: sqrt(%.%); float(%);
(magnitudeAxB) 9*sqrt(6)
(%o16) 22.045
(%i18) asin(magnitudeAxB/(magnitudeA*magnitudeB)); float(%);
(%o17) asin((9*sqrt(6))/(sqrt(29)*sqrt(110)))
(%o18) 0.40098
(%i19) float(%*180/%pi);
(%o19) 22.975
```

The calculation gives, $\boldsymbol{A}\cdot\boldsymbol{B} = 52$, $A = 5.385$, $B = 10.49$, angle between $\boldsymbol{A}$ and $\boldsymbol{B}$ is 23°, $\boldsymbol{A} \times \boldsymbol{B} = 9\mathbf{i} - 18\mathbf{j} + 9\mathbf{k}$, $|\boldsymbol{A} \times \boldsymbol{B}| = 22.05$, angle between $\boldsymbol{A}$ and $\boldsymbol{B}$ is 23°.

Statistics

For statistics, the variable values are entered as a *list*. The built-in functions *mean*, *var*, and *std* can be called to calculate mean, variance, and standard deviation of the variable. For example, calculate the mean, variance, and standard deviation of 20.4, 62.5, 61.3, 44.2, 11.1, and 23.7. The result is,

```
(%i1) fpprintprec:5;
(fpprintprec)     5
(%i2) x: [20.4, 62.5, 61.3, 44.2, 11.1, 23.7];
(x)   [20.4,62.5,61.3,44.2,11.1,23.7]
(%i3) mean(x);
(%o3) 37.2
(%i4) var(x);
(%o4) 402.6
(%i5) std(x);
(%o5) 20.065
```

The wxMaxima calculation says that the mean of x is 37.2, the variance of x is 402.6, and the standard deviation of x is 20.06.

Table. A.1 List of (x, y) values

x	y
0	–0.8
2	–0.7
4	–0.2
6	0.2
8	0.1
10	0.6
12	0.7

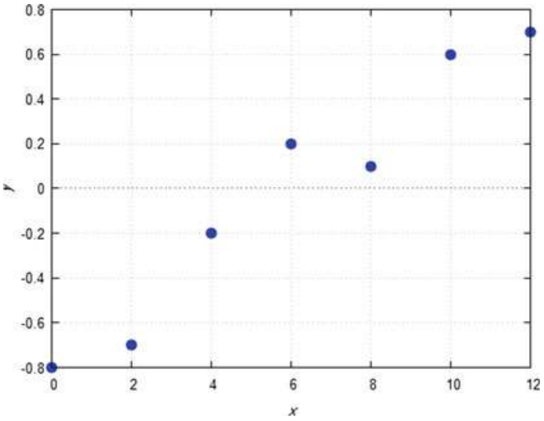
Linear Least Square Fitting

Determine the best line $y = mx + c$ by the linear least square method of (x, y) data in Table A.1.

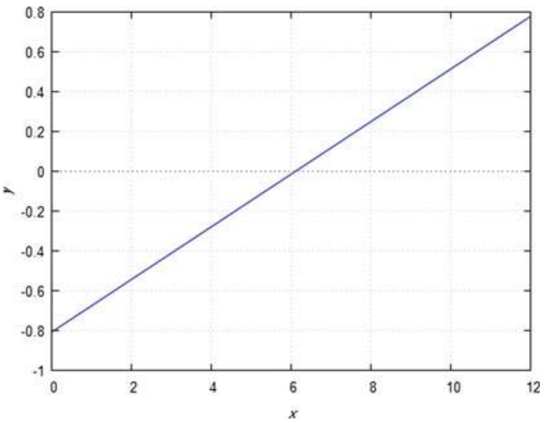
To do least square line fit by wxMaxima, the (x, y) data are entered as *matrix*, then load the *lsquares* routine by *load*("*lsquares*"); command, lastly the predefined command *lsquares_estimates* is called.

wxMaxima codes:

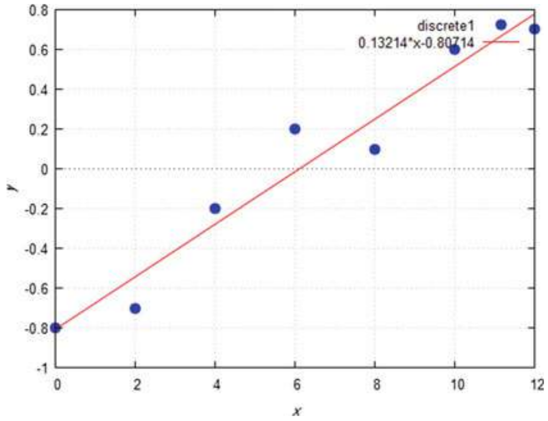
```
(%i1) fpprintprec:5;
(fpprintprec)     5
(%i2) data: matrix([0,-0.8],[2,-0.7],[4,-0.2],
      [6,0.2],[8,0.1],[10,0.6],[12,0.7]);
(data)      matrix(
            [0,   -0.8],
            [2,   -0.7],
            [4,   -0.2],
            [6,   0.2],
            [8,   0.1],
            [10,  0.6],
            [12,  0.7])
(%i3) load("lsquares");
(%o3) "C:/maxima-5.43.0/share/maxima/5.43.0/share/lsquares/lsquares.mac"
(%i5) lsquares_estimates(data,[x,y],y=m*x+c,[m,c])$ float(%);
(%o5) [[m=0.13214,c=-0.80714]]
(%i7) m: rhs(%o5[1][1]); c: rhs(%o5[1][2]);
(m)   0.13214
(c)   -0.80714
(%i8) xy: [[0,-0.8],[2,-0.7],[4,-0.2],[6,0.2],[8,0.1],[10,0.6],[12,0.7]];
(xy)  [[0,-0.8],[2,-0.7],[4,-0.2],[6,0.2],[8,0.1],[10,0.6],[12,0.7]]
(%i9) wxplot2d([discrete, xy],[style, points], grid2d,
[xlabel,"{/Helvetica-Italic x}"], [ylabel,"{/Helvetica-Italic y}"]);
```




```
(%i10) y: m*x + c;
(y)   0.13214*x-0.80714
(%i11) wxplot2d(y,[x,0,12], grid2d, [xlabel,"{/Helvetica-Italic x}"],
[ylabel,"{/Helvetica-Italic y}"]);
```




```
(%i12) wxplot2d([[discrete,xy],y],[x,0,12],[style,[points],[lines]],
grid2d,[xlabel,"{/Helvetica-Italic x}"], [ylabel,"{/Helvetica-Italic y}"]);
```



Comments on the codes:

(%i1)	Set floating point print precision to 5.
(%i2)	Assign "data" as matrix of (x, y) values.
(%i3)	Load "lssquares" routine.
(%i5)	Calculate *m* and *c* by the least square fit.
(%o4), (%o5)	The results.
(%i7)	Assign values of *m* and *c*.
(%i8), (%i9)	Assign xy as data points and plot the points.
(%i10), (%i11)	Assign line *y* and plot the line.
(%i12)	Plot the data point and the line.

The calculation by wxMaxima says that the line fit has the slope $m = 0.13$ and the y axis intercept $c = -0.81$. The line fit is $y = 0.13x - 0.81$. We plot the data points, the fitted line, and the data points with the fitted line in three separate plots.

Simplify

To simplify an expression use *ratsimp*(*expression*); or *radcan*(*expression*);. For example, the following codes show that,

$$bx + b\left(\frac{a}{b} - x\right) + a = 2a.$$

```
(%i1) b*x + b*(a/b - x) + a;
(%o1) b*x+b*(a/b-x)+a
(%i2) ratsimp(%);
(%o2) 2*a
(%i3) radcan(%o1);
(%o3) 2*a
```

Example 1, calculate $\int_0^L \frac{dx}{(L+a-x)^2}$.

```
(%i2) assume(L>0); assume(a>0);
(%o1) [L>0]
(%o2) [a>0]
(%i3) integrate(1/(L+a-x)^2,x,0,L);
(%o3) 1/a-1/(a+L)
(%i4) ratsimp(%);
(%o4) L/(a^2+L*a)
```

This means that,

$$\int_0^L \frac{dx}{(L+a-x)^2} = \frac{1}{a} - \frac{1}{a+L} = \frac{L}{a^2+aL}.$$

Example 2, calculate $\int_0^L \frac{dx}{(L+a-x)^{3/2}}$.

```
(%i2) assume(L>0); assume(a>0);
(%o1) [L>0]
(%o2) [a>0]
(%i3) integrate(1/(L+a-x)^(3/2), x,0,L);
(%o3) 2/sqrt(a)-2/sqrt(a+L)
(%i4) radcan(%);
(%o4) (2*sqrt(a+L)-2*sqrt(a))/(sqrt(a)*sqrt(a+L))
```

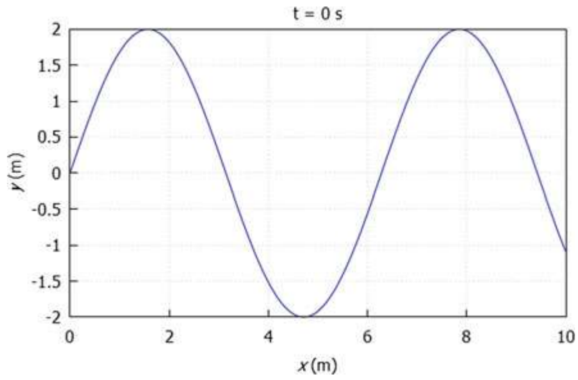
This means that,

$$\int_0^L \frac{dx}{(L+a-x)^{3/2}} = \frac{2}{\sqrt{a}} - \frac{2}{\sqrt{a+L}} = \frac{2\sqrt{a+L}-2\sqrt{a}}{\sqrt{a}\sqrt{a+L}}.$$

Animation

A simple animation of a harmonic wave $y(x, t) = 2\sin(x - 10t)$ travelling to the right is as follows:

```
(%i1) with_slider_draw(
        t,[0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1],
        title=concat("t = ",t," s"),
        grid=true,
        xlabel="{/Helvetica-Italic x} (m)",
        ylabel="{/Helvetica-Italic y} (m)",
        explicit(2*sin(x - 10*t),x,0,10));
```



To run the animation, right click the graphic and choose *Start Animation.*

Solving Second-Order Ordinary Differential Equation

To solve a second-order ordinary differential equation (ODE), use predefined functions *ode*2 and *ic*2. Function *ode*2 solves the ODE and gives a general solution of the ODE. Its format is *ode*2(ODE, dependent variable, independent variable). Function *ic*2 sets the initial conditions and gives a particular solution of the ODE. Its format is *ic*2(output of *ode*2, independent variable value, dependent variable value, first derivative of dependent variable value).

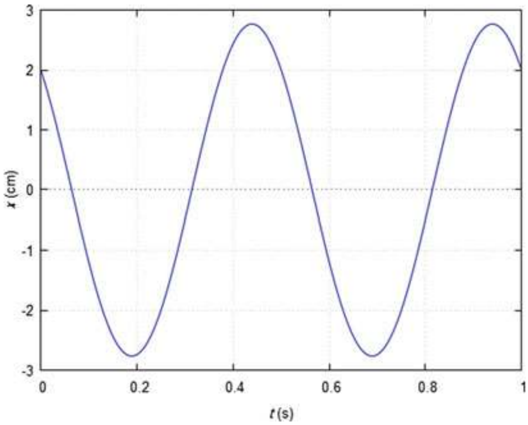
As an example, solve

$$\frac{d^2x}{dt^2} + \omega^2 x = 0,$$

where $\omega = 4\pi$ rad/s, x is displacement (dependent variable) in cm, and t is time (independent variable) in second. The initial conditions at $t = 0$ s, is $x = 2$ cm, and $dx/dt = -24$ cm/s.

The codes are:

```
(%i2) fpprintprec:5; omega:4*%pi;
(fpprintprec)     5
(omega) 4*%pi
(%i3) soln: ode2('diff(x,t,2) + omega^2*x = 0, x, t);
(soln) x=%k1*sin(4*%pi*t)+%k2*cos(4*%pi*t)
(%i5) ic2(soln, t=0, x=2, 'diff(x,t)=-24); float(%);
(%o4) x=2*cos(4*%pi*t)-(6*sin(4*%pi*t))/%pi
(%o5) x=2.0*cos(12.566*t)-1.9099*sin(12.566*t)
(%i6) x: rhs(%);
(x)   2.0*cos(12.566*t)-1.9099*sin(12.566*t)
(%i7) wxplot2d(x, [t,0,1], grid2d,[xlabel,"{/Helvetica-Italic t} (s)"],
[ylabel,"{/Helvetica-Italic x} (cm)"]);
```



Comments on the codes:

(%i2)	Set floating point print precision to 5 and assign value of ω.
(%i3)	Get a general solution of $d^2x/dt^2 + \omega^2 x = 0$.
(soln)	A general solution of the ODE is x = constant 1 $\times$ $\sin(4\pi t)$ + constant 2 $\times$ $\cos(4\pi t)$.
(%i5)	Set the initial conditions and get a particular solution.
(%o4), (%o5)	The particular solution.
(%i6)	Assign x.
(%i7)	Plot x against t for $0 \le t \le 1$ s.

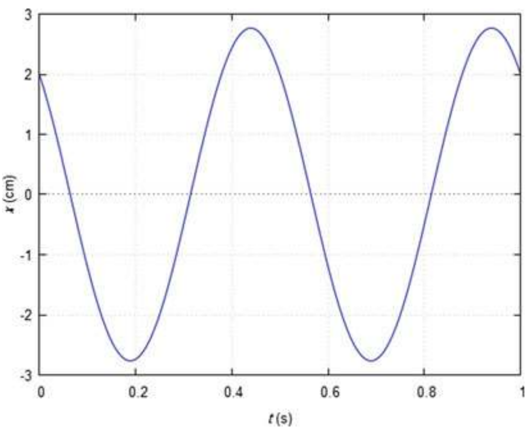
The codes say that the solution of the ODE is, (%o4) or (%o5),

$$x = 2\cos(4\pi t) - \frac{6}{\pi}\sin 4\pi t)$$
$$= 2\cos(12.566t) - 1.9099\sin(12.566t).$$

Another way to solve the second-order ordinary differential equation by wxMaxima is to use predefined functions *atvalue* and *desolve*. The arguments of *desolve* are the ODE and the function we want a solution. The arguments of *atvalue* are the initial conditions: the dependent variable, the independent variable value, and the dependent variable value. We solve again the above problem using predefined functions *atvalue* and *desolve*.

◆ wsMaxima codes:

```
(%i2) fpprintprec:5; omega:4*%pi;
(fpprintprec)      5
(omega) 4*%pi
(%i3) equation: 'diff(x(t),t,2)+omega^2*x(t)=0;
(equation)   'diff(x(t),t,2)+16*%pi^2*x(t)=0
(%i5) atvalue(x(t), t=0, 2); atvalue(diff(x(t),t), t=0, -24);
(%o4) 2
(%o5) -24
(%i7) desolve(equation, x(t)); float(%);
(%o6) x(t)=2*cos(4*%pi*t)-(6*sin(4*%pi*t))/%pi
(%o7) x(t)=2.0*cos(12.566*t)-1.9099*sin(12.566*t)
(%i8) define(x(t), rhs(%));
(%o8) x(t):=2.0*cos(12.566*t)-1.9099*sin(12.566*t)
(%i9) wxplot2d(x(t), [t,0,1], grid2d,[xlabel,"{/Helvetica-Italic t} (s)"],
[ylabel,"{/Helvetica-Italic x} (cm)"]);
```



Comments on the codes:

(%i2)	Set floating point print precision to 5 and assign value of ω.
(%i3)	Define the differential equation.
(%i5)	Set the initial conditions.
(%i7)	Solve the differential equation and get a particular solution.
(%o6), (%o7)	The solution.
(%i8)	Define $x(t)$;
(%i9)	Plot $x(t)$ for $0 \le t \le 1$ s.

Solving First-Order Ordinary Differential Equation

To solve a first-order ordinary differential equation (ODE), use predefined functions *ode*2 and *ic*1. Function *ode*2 solves the ODE and gives a general solution of the

ODE. Its format is *ode*2(ODE, dependent variable, independent variable). Function *ic*1 sets the initial condition and gives a particular solution of the ODE. Its format is *ic*1(output of *ode*2, independent variable value, dependent variable value).

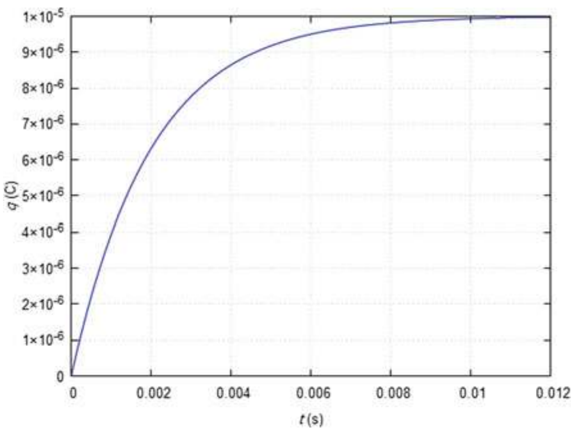
As an example, solve a direct current *RC* circuit equation,

$$R\frac{dq}{dt} + \frac{q}{C} = E,$$

where resistance $R = 2000\ \Omega$, capacitance $C = 1 \times 10^{-6}$ F, emf $\varepsilon = 10$ V, charge q (dependent variable) is in coulomb and time t (independent variable) is in second. The initial condition is, at $t = 0$ s, $q = 0$ C.

The codes are:

```
(%i2) fpprintprec:5; ratprint:false;
(fpprintprec)     5
(ratprint)  false
(%i3)soln: ode2(R*'diff(q,t) + q/C =emf    , q, t);
(soln) q=%e^(-t/(C*R))*(C*emf*%e^(t/(C*R))+%c)
(%i4) ic1(soln, t=0, q=0);
(%o4) q=%e^(-t/(C*R))*(C*emf*%e^(t/(C*R))-C*emf)
(%i5) q: rhs(%);
(q)%e^(-t/(C*R))*(C*emf *%e^(t/(C*R))-C*emf)
(%i8)R:2000; C:1e-6; emf      :10;
(R)   2000
(C)   10.0*10^-7
(emf) 10
(%i9) wxplot2d(q, [t,0,12e-3], grid2d, [xlabel,"{/Helvetica-Italic t}
(s)"], [ylabel,"{/Helvetica-Italic q} (C)"]);
```



Comments on the codes:

(%i2) Set floating point print precision to 5 and internal rational number print to false.
(%i3) Solve ODE $R\frac{dq}{dt} + \frac{q}{C} = \varepsilon$ and get a general solution.
(soln) A general solution is $q = e^{-t/(RC)}\left(C\varepsilon e^{t/(RC)} + \text{ constant }\right)$
(%i4) Set the initial condition and get a particular solution.
(%o4) The particular solution.
(%i5) Assign the solution to q.
(%i8) Assign values of R, C, and ε.
(%i9) Plot q against t for $0 \leq t \leq 12 \times 10^{-3}$ s.

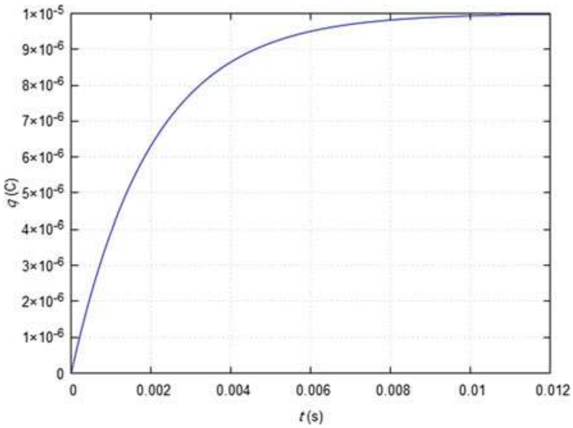
The codes say that the solution of the ODE is, (%o4),

$$\begin{aligned} q &= e^{-t/(RC)}(C\varepsilon\ e^{t/(RC)} - C\varepsilon) \\ &= C\varepsilon\ (1 - e^{-t/(RC)}). \end{aligned}$$

Another way to solve first-order ordinary differential equation by wxMaxima is to use predefined functions *atvalue* and *desolve*. The arguments of *desolve* are the ODE and the function we want a solution. The arguments of *atvalue* are the initial conditions: the dependent variable, the independent variable value, and the dependent variable value. We solve again the above problem using predefined functions *atvalue* and *desolve*.

◆ wsMaxima codes:

```
(%i1) fpprintprec:5;
(fpprintprec)     5
(%i2)equation: R*'diff(q(t),t)+ q(t)/C = emf    ;
(equation)R*('diff(q(t),t,1))+q(t)/C=     emf
(%i3) atvalue(q(t), t=0, 0);
(%o3) 0
(%i5) desolve(equation, q(t)); float(%);
(%o4)q(t)=C*emf-C*emf   *%e^(-t/(C*R))
(%o5)q(t)=C*      emf-(1.0*C*emf)/2.7183^(t/(C*R))
(%i6) define(q(t), rhs(%));
(%o6)q(t):=C*      emf-(1.0*C*emf)/2.7183^(t/(C*R))
(%i9)R:2000; C:1e-6; emf      :10;
(R)   2000
(C)   10.0*10^-7
(emf) 10
(%i10) wxplot2d(q(t), [t,0,12e-3], grid2d,[xlabel,"{/Helvetica-Italic t}
(s)"], [ylabel,"{/Helvetica-Italic q} (C)"]);
```



Comments on the codes:

(%i1)	Set floating point print precision to 5.
(%i2)	Define the differential equation.
(%i3)	Set the initial condition.
(%i5)	Solve the differential equation and get a particular solution.
(%o4), (%o5)	The solution.
(%i6)	Assign values of R, C, and ε.
(%i9)	Define $q(t)$.
(%i10)	Plot $q(t)$ for $0 \leq t \leq 12 \times 10^{-3}$ s.

We end this *Introduction to wxMaxima* here. For further use and application of wxMaxima the readers are required to look for internet sources and the *Help* menu at the top right of the wxMaxima window.

Appendix B
Physical Constants

Symbol	Value	Physical quantity
c	3.0×10^8 m s^{-1}	Speed of light
μ_0	$4\pi \times 10^{-7}$ H m^{-1}	Permeability of free space, Permeability constant
ε_0	8.85×10^{-12} F m^{-1}	Permittivity of free space, Permittivity constant
e	-1.6022×10^{-19} C	Electron charge
h	6.63×10^{-34} J s	Planck constant
m_e	9.11×10^{-31} kg 5.48×10^{-4} u	Mass of electron
m_p	1.673×10^{-27} kg 1.007825 u	Mass of proton
m_d	3.34×10^{-27} kg 2.014102 u	Mass of deuteron
R	8.31 J K^{-1} mol^{-1}	Molar gas constant, Universal gas constant
R_H	1.097×10^7 m^{-1}	Rydberg constant
N_A	6.02×10^{23} mol^{-1}	Avogadro number
$k_B = \frac{R}{N_A}$	1.38×10^{-23} J K^{-1}	Boltzmann constant
G	6.67×10^{-11} N m^2 kg^{-2}	Universal gravitational constant
g	9.8 m s^{-2}	Acceleration of gravity, Gravitational acceleration
$k = \frac{1}{4\pi\varepsilon_0}$	9×10^9 N m^2 C^{-2}	Electrostatic constant
$k_m = \frac{\mu_0}{4\pi}$	10^{-7} Wb A$^{-1\ \mathrm{m}-1}$	Magnetic constant

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Appendix C
Conversion Factors

Quantity	Conversion
Length	1 m = 39.37 inch = 3.28 feet = 100 cm
	1 feet = 30.48 cm
	1 inch = 2.54 cm
	1 mile = 5280 feet = 1.609 km
	1 Å = 10^{-8} cm = 10^{-10} m
	1 μm = 10^{-4} cm = 10^{-6} m 1 nm = 10^{-7} cm = 10^{-9} m
Mass	1 slug = 14.59 kg = 32.2 lb
	1 g = 10^{-3} kg = 6.85×10^{-5} slug
	1 u = 1.66×10^{-27} kg = 931.5 meV/c^2
Time	1 year = 365 day = 3.16×10^7 s
	1 day = 24 h = 1.44×10^3 min = 8.64×10^4 s 1 h = 60 min = 3600 s
Area	1 cm^2 = 0.155 $inch^2$
	1 $inch^2$ = 6.452 cm^2
	1 m^2 = 10.76 $feet^2$
	1 $feet^2$ = 144 $inch^2$ = 0.0929 m^2 1 hectare = 10^4 m^2 = 2.471 acre 1 acre = 4047 m^2 = 0.4047 hectare = 4840 $yard^2$ = 43,560 $feet^2$
Volume	1 m^3 = 10^6 cm^3 = 10^3 dm^3 = 10^3 L 1 dm^3 = 1 L 1 cm^3 = 1 mL
	1 m^3 = 35.3 $feet^3$ = 6.1×10^4 $inch^3$
	1 $feet^3$ = 2.83×10^{-2} m^3 = 28.32 L

(continued)

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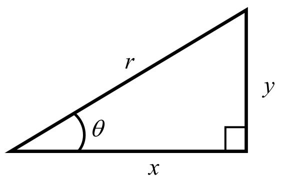
(continued)

Quantity	Conversion
Speed	1 mile hour^{-1} = 1.47 feet s^{-1} = 0.447 m s^{-1} = 1.609 km hour^{-1}
	1 m s^{-1} = 100 cm s^{-1} = 3.281 feet s^{-1} = 3.6 km hour^{-1} = 2.237 mile hour^{-1}
	1 km hour^{-1} = 0.278 m s^{-1} = 0.621 mile hour^{-1} = 0.911 feet s^{-1} 1 mile minute^{-1} = 60 mile hour^{-1} = 88 feet s^{-1}
Acceleration	1 m s^{-2} = 3.28 feet s^{-2} = 100 cm s^{-2}
	1 feet s^{-2} = 0.3048 m s^{-2} = 30.48 cm s^{-2}
Force	1 N = 10^5 dyne = 0.2247 lb
Pressure	1 N m^{-2} = 1 Pa = 10 dyne cm^{-2} = 1.45 × 10^{-4} lb inch^{-2}
	1 atm = 1.013 × 10^5 Pa
Energy and power	1 J = 10^7 erg = 0.239 cal = 0.738 feet lb
	1 eV = 1.6 × 10^{-19} J = 1.6 × 10^{-12} erg
	1 cal = 4.18 J
	1 horse power = 745 W = 550 feet lb s^{-1}
Magnetic field	1 T = 10^4 gauss 1 T = 1 Wb m^{-2}

Appendix D
Mathematical Formulae

Roots of a quadratic equation
$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$
Trigonometric identity

$\sin\theta = \frac{y}{r}\csc\theta = \frac{1}{\sin\theta} = \frac{r}{y}$	
$\cos\theta = \frac{x}{r}\ \sec\theta = \frac{1}{\cos\theta} = \frac{r}{x}$	
$\tan\theta = \frac{y}{x}\cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}$	

$\sin^2\theta + \cos^2\theta = 1$	$x^2 + y^2 = r^2$	$1 + \tan^2\theta = \sec^2\theta$
$\sin 2\theta = 2\sin\theta\cos\theta$	$\sin(-\theta) = -\sin\theta$	$\sin^2\theta = \frac{1-\cos 2\theta}{2}$
$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$	$\cos(-\theta) = \cos\theta$	$\cos^2\theta = \frac{1+\cos 2\theta}{2}$
$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$	$\tan(-\theta) = -\tan\theta$	$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$
$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$	$\sin\theta + \sin\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$	
$\sin\theta\sin\phi = \frac{1}{2}[\cos(\theta-\phi) - \cos(\theta+\phi)]$	$\sin\theta - \sin\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$	
$\cos\theta\cos\phi = \frac{1}{2}[\cos(\theta-\phi) + \cos(\theta+\phi)]$	$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$	
$\sin\theta\cos\phi = \frac{1}{2}[\sin(\theta+\phi) + \sin(\theta-\phi)]$	$\cos\theta - \cos\phi = -2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$	

(continued)

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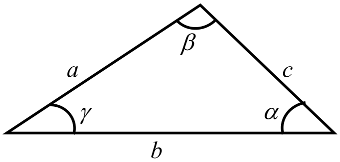
(continued)

Roots of a quadratic equation	
$\cos\theta\sin\phi = \frac{1}{2}[\sin(\theta+\phi) - \sin(\theta-\phi)]$	$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$
Cosine rule $c^2 = a^2 + b^2 - 2ab\cos\gamma$	
Sine rule $\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$	

Series expansion	
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$	$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$	$(1-x)^n = 1 - nx + \frac{n(n-1)x^2}{2!} - \frac{n(n-1)(n-2)x^3}{3!} + \ldots$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$	$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$
$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ldots$	$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$	$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \ldots$
$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \ldots$	$\sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} + \ldots$

Differentiation	Integration
$\frac{d}{dx}\sin x = \cos x$	$\int \sin x\ dx = -\cos x$
$\frac{d}{dx}\cos x = -\sin x$	$\int \cos x\ dx = \sin x$
$\frac{d}{dx}e^x = e^x$	$\int e^x\ dx = e^x$
$\frac{d}{dx}x^n = nx^{n-1}$	$\int x^n\ dx = \frac{x^{n+1}}{n+1}$
$\frac{d}{dx}\ln(x + \sqrt{x^2+a^2}) = \frac{1}{\sqrt{x^2+a^2}}$	$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x + \sqrt{x^2+a^2})$
$\frac{d}{dx}\ln(x + \sqrt{x^2-a^2}) = \frac{1}{\sqrt{x^2-a^2}}$	$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln(x + \sqrt{x^2-a^2})$
$\frac{d}{dx}\frac{1}{(x^2+a^2)^{1/2}} = -\frac{x}{(x^2+a^2)^{3/2}}$	$\int \frac{x\ dx}{(x^2+a^2)^{3/2}} = -\frac{1}{(x^2+a^2)^{1/2}}$
$\frac{d}{dx}\frac{x}{a^2(x^2+a^2)^{1/2}} = \frac{1}{(x^2+a^2)^{3/2}}$	$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2(x^2+a^2)^{1/2}}$
$\frac{d}{dx}(a^2-x^2)^{3/2} = -3x(a^2-x^2)^{1/2}$	$\int x(a^2-x^2)^{1/2}dx = -\frac{1}{3}(a^2-x^2)^{3/2}$

Dot (scalar) product of two vectors
Given two vectors, $\boldsymbol{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$, $\boldsymbol{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\,\mathbf{k}$, then, $\boldsymbol{A} \cdot \boldsymbol{B} = A_x\,B_x + A_y\,B_y + A_z\,B_z$, also the magnitudes of A and B are, $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$, $B = \sqrt{B_x^2 + B_y^2 + B_z^2}$. Also, $\boldsymbol{A}\cdot\boldsymbol{B} = AB\cos\theta$, where θ is the small angle between $\boldsymbol{A}$ and $\boldsymbol{B}$.

Cross (vector) product of two vectors

Given two vectors,
$\boldsymbol{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$,
$\boldsymbol{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\,\mathbf{k}$,
then,

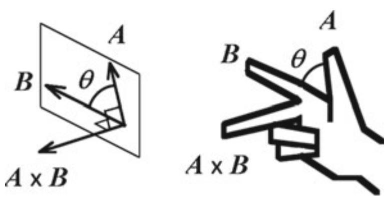
$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_yB_z - A_zB_y)\mathbf{i} - (A_xB_z - A_zB_x)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}.$$

Also,
$|\boldsymbol{A} \times \boldsymbol{B}| = AB\sin\theta$,
where θ is the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$, and,

$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$,

$B = \sqrt{B_x^2 + B_y^2 + B_z^2}$.

The right hand rule



Cramer's rule
The solutions for $a_1x + b_1y = c_1,$ $a_2x + b_2y = c_2,$ are $x = \dfrac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1},$ $y = \dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$

Logarithm	
$\log_b x = \frac{\log_{10} x}{\log_{10} b} = \frac{\log_e x}{\log_e b}$	$x = b^a \Rightarrow \log_b x = a$
$\log_x x = \log_{10} 10 = \log_e e = \log_2 2 = 1$	$1000 = 10^3 \Rightarrow \log_{10} 1000 = 3$
$2 = e^{0.693} \Rightarrow \log_e 2 = 0.693$	$512 = 2^8 \Rightarrow \log_2 512 = 8$

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