

Arak M. Mathai  
Hans J. Haubold

# Modern Problems in Nuclear and Neutrino Astrophysics

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Arak M. Mathai and Hans J. Haubold

# **Modern Problems in Nuclear and Neutrino Astrophysics**

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A. M. Mathai

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# Contents

## 1 The Theory of Nuclear Energy Generation of Solar Type Stars

### 1.1 Classical Thermodynamics: Polytropic Gas Spheres in Convective Equilibrium

### 1.2 Quantum Theory of Atomic Structure: Opacity

### 1.3 Quantum Theory of Nuclear Structure: Nuclear Energy Generation Rates

### 1.4 Stellar Models with Nuclear Burning: Nuclear Reaction Rates

### 1.5 The Solar Neutrino Experiment: Solar Neutrino Emission Rates

### References

## 2 Mathematical Approaches to Nuclear Astrophysics

### 2.1 Introduction: Cosmic Nucleosynthesis of the Elements

### 2.2 The Thermonuclear Reaction Rate

### 2.3 Velocity Distribution Function and Nuclear Cross Section: Maxwell-Boltzmann Distribution Function

### 2.4 Nonresonant Neutron Capture Cross Section

### 2.5 Nonresonant Charged Particle Cross Section

### 2.6 Resonant Cross Section for Neutrons and Charged Particles

### 2.7 Parameterizations of Thermonuclear Reaction Rates

### 2.8 Nonresonant Reaction Rates

### 2.9 Resonant Reaction Rates

### 2.10 Series Representations for the Thermonuclear Functions: $G_{0,3}^{3,0}(\cdot)$

### 2.11 Series Representations for the Thermonuclear Functions: $G_{1,3}^{3,0}(\cdot)$

### References

### 3 The Solar Neutrino Problem

#### 3.1 Introduction: The Solar Neutrino Problem

#### 3.2 The Proton-Proton Chain

#### 3.3 An Analytic Model for the Central Region of the Sun

#### 3.4 Solar Thermonuclear Energy Generation: Energy Conservation and Solar Luminosity

#### 3.5 The Thermonuclear Reaction Rate

#### 3.6 The Neutrino Emission Rate

#### 3.7 The Integral for the Solar Nuclear Energy Generation and the Solar Neutrino Fluxes: The General Case of the Basic Integral

#### 3.8 The Solar Case of the Basic Integral

#### 3.9 Fox's H-Function

#### 3.10 Meijer's G-Function

#### 3.11 Analytic Results Connecting Solar Structure Parameters and Solar Neutrino Emission Rates

### References

## 4 Solar Nuclear and Neutrino Astrophysics Research, a Time Line

### 5 Nuclear Astrophysics, 2025 Update

#### 5.1 Explicit Evaluation of the Thermonuclear Reaction-Rate Probability Integrals

#### 5.2 Generalization of the Reaction-Rate Probability Integral

#### 5.3 An Extension Through Mathai's Pathway Idea

#### 5.4 The Pathway Extended Reaction-Rate Probability Integral

#### 5.5 Generalized Reaction-Rate Probability Integral in the Real Multivariate Case

#### 5.6 Real Matrix-Variate Case

[5.7 Most General Real Matrix-Variate Case](#)

[5.8 Generalized Reaction-Rate Integrals in the Complex Multivariate Case](#)

[5.9 Complex Matrix-Variate Case](#)

[5.10 Most General Form in the Complex Matrix-Variate Case](#)

[5.11 Reaction-Rate Probability Integral Through Optimization of Mathai Entropy](#)

[5.12 Multivariate Densities in the Real and Complex Domains](#)

[5.13 Matrix-Variate Densities in the Real and Complex Domains](#)

[5.14 Mellin Convolutions Involving Other Functions](#)

[5.15 Generalization to Real Matrix-Variate Case](#)

[5.16 Generalization to the Complex Matrix-Variate Case](#)

[5.17 Mellin Convolution of a Ratio](#)

[5.18 M-Convolution of a Ratio in the Real Matrix-Variate Case](#)

[5.19 Fractional Integral of the First Kind in the Complex Matrix-Variate Case](#)

[References](#)

[6 Neutrino Astrophysics, 2025 Update: The Entropic Approach to Solar Neutrinos](#)

[6.1 Solar Neutrinos: SuperKamiokande Data](#)

[6.2 Diffusion Entropy and Standard Deviation: Analysis](#)

[6.3 Probability Density Function and Differential Equation: Lévy Flights](#)

[6.4 Discussion](#)

[References](#)

[7 Neutrino Astrophysics, 2025 Update: Neutrino Masses and Mixings](#)

[7.1 Introduction](#)

[7.2 Modified Dirac and Majorana Neutrino Matrices and Their Distributions](#)

[7.3 Derivation of the Density of  \$U\$](#)

[7.4 Densities in Terms of the Eigenvalues](#)

[7.5 Exact Marginal Function of the Largest Eigenvalue  \$\lambda\_1\$  in \(7.19\)](#)

[7.6 Exact Marginal Function of  \$\lambda\_p\$ , the Smallest Eigenvalue](#)

[References](#)

[Author Index](#)

[Subject Index](#)

# 1. The Theory of Nuclear Energy Generation of Solar Type Stars

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## 1.1 Classical Thermodynamics: Polytropic Gas Spheres in Convective Equilibrium

Shortly after the discovery of the law of conservation of energy by Mayer (1942) and Helmholtz (1847) it was J. R. Mayer who raised the question for the origin of the radiative energy emitted by the Sun. J.R. Mayer's law of conservation of energy (first law of thermodynamics) took into account the energy due to heat. If  $dQ$  is the amount of heat energy which is absorbed by the system under question from its surroundings, then this law of conservation of energy is,

$$dQ = dU + dW, \quad (1.1)$$

where  $dU$  is the change in internal energy of the system when going from one equilibrium state to another, and  $dW$  is the amount of work done by the system on its surroundings. In 1848 J. R. Mayer stated that the source of the solar radiation energy should be the kinetic energy of infalling meteorites. At last a refinement of J. R. Mayer's idea we meet in the hypothesis of Helmholtz (1847) and Lord Kelvin (1861) who were able to show that the gravitational contraction of the Sun itself could be the significant source of radiated energy. However, already on the basis

of Lane's (1870) considerations of a stellar configuration in convective equilibrium taking into account internal gravitation and Ritter's (1878) results on the uniform expansion and contraction of gaseous configurations it has been shown that gravitational contraction cannot keep the Sun shining (HELMHOLTZ-KELVIN contraction time scale). This very important conclusion for the theory of internal structure of the Sun is closely related to the macro-structure of a gaseous sphere in stationary equilibrium described by the virial theorem of Clausius (1870) which is also called POINCARE's theorem:

$$2T + \Omega = 0 \quad (1.2)$$

where  $T$  is the total kinetic energy of particles, and  $\Omega$  is the total gravitational potential energy of the system. For an adiabatic process in a gaseous sphere in gravitational equilibrium it follows from (1.1) that the change  $dU$  in the internal energy of the sphere is given by

$$dU = -PdV, \quad (1.3)$$

where  $P$  is the gas pressure and  $dV$  is the change in the volume. Considering a perfect gas with  $PV = RT$  one has,

$$\frac{dT}{T} = -(\gamma - 1) \frac{dV}{V}, \quad (1.4)$$

where  $R$  is the gas constant and  $\gamma$  is the adiabatic index. Using (1.3) and (1.4) it holds for the perfect gas,

$$U = \frac{2T}{3(\gamma-1)}, \quad (1.5)$$

which can be written in the light of POINCARE's theorem (1.2),

$$U = -\frac{\Omega}{3(\gamma-1)}. \quad (1.6)$$

The result in (1.6) goes back to Ritter (1880a). The total energy  $E$  is then given by

$$E = U + \Omega = \frac{3\gamma-4}{3(\gamma-1)} \Omega. \quad (1.7)$$

From (1.7) it is evident that the gaseous sphere will be unstable against adiabatic pulsations for  $\gamma < 4/3$ . In his remarkable paper Ritter (1880a) also obtained the result that the period of oscillation of the gaseous sphere is inversely proportional to the square root of its mean density.

As a remark we should mention here that Maxwell (1860) in 1860 obtained the probability distribution of gas particles with velocities between  $v$  and  $v + dv$ ,

$$f(v)dv = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\frac{m}{kT}\right)^{\frac{3}{2}} v^2 \exp\left\{-\frac{mv^2}{2kT}\right\} dv, \quad (1.8)$$

where  $m$  is the mass of the particle under question. Already in Ritter's (1879) paper consequences of (1.8) played an important role for the development of convective equilibrium in a gaseous sphere under the influence of its own gravitation. Equation (1.8) follows also directly from the MAXWELL-BOLTZMANN solution L. Boltzmann's (1872) equation found by him in 1872.

For the first time the fundamental differential equation governing the structure of gaseous spheres was given by Ritter (1880) and Lane (1870),

$$\frac{1}{r^2} \frac{d}{dr} \left( \frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G \rho, \quad (1.9)$$

which can be derived by consideration of two fundamental equations of gravitational equilibrium,

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho, \quad \text{conservation of mass,} \quad (1.10)$$

$$\frac{dP}{dr} = -\frac{GM(r)\rho}{r^2}, \quad \text{hydrostatic equilibrium.} \quad (1.11)$$

A. Ritter studied the properties of gaseous spheres in convective equilibrium, where pressure  $P$  and matter density  $\rho$  obey the law

$$P = K\rho^{1+\frac{1}{n}}, \quad (1.12)$$

which are called polytropes of index  $n$ . Taking into account (1.12) and (1.9), we are led to the well-known LANE-EMDEN equation,

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) + \theta^n = 0, \quad (1.13)$$

where the solution satisfies the initial condition  $\theta(0) = 1$  and  $\theta'(0) = 0$ . The LANE-EMDEN equation (1.13) can be obtained from (1.9) and (1.12) by substituting  $\rho = \lambda\theta^n$ ,  $r = \alpha\xi$ , and  $\alpha = \left[ \frac{(n+1)K}{4\pi G} \lambda^{\frac{1}{n}-1} \right]^{1/2}$ .

Special cases of (1.13), namely, when  $n = 0, 1, 5$ , have explicit analytic solutions. The mathematical foundation for the study of the LANE-EMDEN equation (1.13) and of more general equations was made by Fowler (1914a, b, 1930, 1931) in a series of four papers during 1914–1931. However, the theory of polytropic stars governed by the second order nonlinear differential equation (1.13) culminated in Emden's (1907) book. The fundamental point of view adopted at that time was that the energy transport in the interior of a star would be by convective motion. But, already in 1894 Sampson (1894) introduced the concept of energy transfer by radiative rather than by convective processes. This concept had to wait upon further progress in thermodynamics. In 1905 Schuster (1905), E. Rutherford's predecessor in Manchester, applied the idea of radiative energy transfer of an atmosphere which also led to Schwarzschild's (1906) famous paper on the physical state of the Sun's atmosphere. It contains the concept of local thermodynamic equilibrium.

It is generally accepted in the literature that the publication of Emden's (1907) book containing the complete theory of polytropic stars marks the end of the first epoch in the study of the internal constitution of stellar configurations. In the following decades mainly the importance of radiative energy transfer in the interior of stars was realized and the theory of radiative transfer was developed in some detail.

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## 1.2 Quantum Theory of Atomic Structure: Opacity

Although E. Hertzsprung ([1905](#)) in 1905 had already recognized the distinction between giant and dwarf stars the full discovery of the HERTZSPRUNG-RUSSELL diagram had to wait until 1913. In 1911–1914 Hertzsprung ([1911](#), [1912](#)) and Russell ([1914](#)) were convinced as to the significance of their diagram for the study of the evolution of stars. Indeed the theory of the evolution of stars based on observations first became possible in connection with the study of the internal structure of the stars made by LANE, RITTER, and EMDEN as discussed above.

In Bohr ([1913](#)) laid the foundations of the quantum theory of atomic structure by establishing a link between the structure of the atom and PLANCK's quantum of action as given in his trilogy on the constitution of atoms and molecules. It is quite interesting that at about this time the idea was born that the up till now considered mechanical or radioactive energy source did not nearly suffice for supplying the radiation of the Sun. Taking into account that the luminosity of the Sun has not changed significantly since the formation of the Earth some  $10^9$  years ago, Perrin ([1920](#)) and Eddington ([1920](#)) in 1920 came to conjecture that in the interior of the Sun subatomic energy must be generated by the conversion of hydrogen into helium. As we shall see in the following the development of the theory of stellar nuclear energy generation is closely connected with the foundations of the theory of astrophysical plasmas. In 1923, for instance, Debye ([1923](#)) and E. HÜCKEL had shown that the electrons of a plasma move in such a way as to screen out the COULOMB field of a test charge for distances greater than the DEBYE-HÜCKEL length.

The combination of the theory of radiative equilibrium and BOHR's theory of atomic structure with LANE's, RITTER's, and EMDEN's results was realized by Eddington ([1916a](#), [b](#), [1918](#)) (1916–1918) and came up with a more refined theory of the internal structure of the stars. He was the first to apply the concept of radiative equilibrium to the interior of a star and made the assumption that heat is transferred inside a star by radiation whose flow controls the internal temperature distribution. The equations of equilibrium for a star in radiative equilibrium now consist of ([1.11](#)) taking into account the gas pressure  $P_g = kN_A\rho T/\mu$ , and the radiation pressure  $p_r = aT^4/3$ , the equation of radiative energy transport,

$$\frac{d}{dr} \left( \frac{1}{3} a T^4 \right) = - \frac{\chi \rho}{c} \frac{L(r)}{4\pi r^2}, \quad (1.14)$$

and the equation of conservation of energy,

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho \epsilon, \quad (1.15)$$

where  $\chi$  is the measure of the ability of a gas to absorb radiation (opacity), and  $\epsilon$  is the total amount of heat energy liberated per unit mass in unit time (energy generation rate). While the opacity  $\chi$  could be fixed by the physical theory of radiative transfer, the quantity  $\epsilon$  as the nuclear energy generation rate still remained uncertain at that time. The situation was surveyed at the end of the third decade of the century by Eddington's (1926) book. During the following decade much work was done on the derivation of detailed stellar models.

## 1.3 Quantum Theory of Nuclear Structure: Nuclear Energy Generation Rates

The decades after the publication of Eddington's (1926) book are characterized by the breakthrough of quantum mechanics into the physics of stellar interior. In Fowler (1926) made the fundamental discovery that the electron assembly in the white dwarfs must be degenerate in the sense of the FERMI-DIRAC statistics in the same way as shortly thereafter W. Pauli and A. Sommerfeld showed to be the case for electrons in metals. He derived the equation of state for degenerate matter,  $P = k\rho^{5/3}$ , where  $k$  is a constant. About this time Wentzel (1926), Kramers (1926), and Brillouin (1926) studied approximate solutions of the SCHRÖDINGER equation for a charged particle in the COULOMB field which later on were named as COULOMB wave functions. Charged particle interactions at low energies as expected in stellar interiors are dominated by the COULOMB barrier penetration factor first discussed in detail by Gamow (1928), Gurney and Condon (1928, 1929). Thus, the theoretical foundations for the stellar nuclear energy generation had been created. In Atkinson (1929) and F.G. Houtermans considered the transmutation of elements arising from proton captures by the help of simple physical considerations. Fowler

(1929) and Wilson (1929) developed the resonant penetration of charged particles leading to the distinction of nonresonant and resonant particle interactions in nuclear reactions. It is an important fact that Eddington (1926) in his book succeeded in deriving a relation between the mass  $M$ , luminosity  $L$ , and the opacity  $\chi$  from the considerations of a steady state of a star not knowing the dependence of the energy generation rate  $\epsilon$  on density  $\rho$  and temperature  $T$ .

However, contrary to EDDINGTON's point of view Milne (1930) suggested that the mass, luminosity, and opacity of a stellar model must be taken as independent parameters in the consideration of its steady state, and that the observed mass-luminosity relation should depend upon the intrinsic physics of the energy generating processes and not from deductions of steady state considerations only. During the years 1926–1939 important integral theorems were established by the analysis of the differential equations of the internal structure of stars (1.10), (1.11), and the total pressure as a sum of the gas kinetic pressure  $p_g$  and radiative pressure  $p_r$ . Those integral theorems permitted the estimation of mean values and values at the centre of the star for all relevant physical variables inside the star (density, pressure, temperature). May be the two most important theorems are due to Vogt (1926) and Russell (1927) as well as to Strömgren (1937). The VOGT-RUSSELL theorem is valid for gaseous stars in radiative equilibrium as well as in convective equilibrium and states that the four first-order ordinary differential equations (1.10), (1.11), (1.14), (1.15), under general assumptions for the three constitutive relations  $P$ ,  $\chi$ ,  $\epsilon$ , form a self-sufficient system for the problem of stellar structure if one takes into account, additionally, the boundary conditions:  $M(r) = 0$  at  $r = 0$ ;  $M(r) = M$ ,  $L(r) = L$ , and  $P(r) = 0$  at  $r = R$ . The theorem in question reads that the structure of a star is uniquely determined by its mass and chemical composition, if the pressure  $P$ , the opacity  $\chi$  and the rate of energy generation  $\epsilon$  are functions of the local values of density  $\rho$ , temperature  $T$ , and chemical composition  $X_i$  only. The main observational consequence of the VOGT-RUSSELL theorem was that different stars in the distribution of stars in the HERTZSPRUNG-RUSSELL diagram have different chemical composition. The theorem of STRÖMGREN states that for a star in radiative equilibrium with

negligible radiation pressure the mass-luminosity-radius relation has the form

$$L = \text{constant. } \frac{1}{\chi_0} \frac{M^{5+s}}{R^s} \mu^{7+s}, \quad (1.16)$$

if the rate of energy generation  $\epsilon$  and the opacity  $\chi$  obey the general power laws

$$\epsilon = \epsilon_0 \rho^\alpha T^\nu, \quad (1.17)$$

$$\chi = \chi_0 \rho T^{-3-s}, \quad (1.18)$$

where  $\alpha$ ,  $\nu$ ,  $s$ ,  $\epsilon_0$ , and  $\chi_0$  are arbitrary constants. The constant in Eq. (1.16) depends only on the exponents  $\alpha$ ,  $\nu$ , and  $s$ , respectively. Again, the value of STRÖMGREN's theorem (1.16) is to have the dependence of the luminosity  $L$  from  $M$  and  $R$  which allows a comparison of theory and observation. The special feature of STRÖMGREN's theorem is the more realistic dependence of  $L$  on the introduced physical assumptions about the rate of energy generation via  $\alpha$  and  $\nu$  and the opacity via  $s$ .

Besides the theory of the internal structure of ordinary stars (main sequence stars) Chandrasekhar (1931) realized in 1931 that with increasing relativistic degeneracy the radius of a white dwarf star tends to the limiting value zero at a finite limiting mass of  $1.4M_\odot$  with a slight dependence on chemical composition. The main result was that white dwarf stars are stable or can exist, only for masses  $M \leq 1.4M_\odot$  (chandrasekhar limiting mass). Then, the exact equation of state for a completely degenerate gas has been derived by Chandrasekhar (1935). Much of the material developed for physics of degenerate matter and white dwarf stars as well as the construction of stellar models by analytic methods was summarized in the monograph of Chandrasekhar (1939).

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## 1.4 Stellar Models with Nuclear Burning: Nuclear Reaction Rates

The decades after the publication of Chandrasekhar's (1939) monograph saw the working out of further details of the nuclear reactions and the determination of reaction rates on the basis of laboratory measurements. As an example for the elaboration of the nuclear reaction theory we refer to the analytic treatment of nuclear cross sections as given by Breit (1936) and Wigner's (1936) single resonance formula. The rapid advance of nuclear physics in the thirties of this century enable Weizsäcker (1937) and Bethe (1938) and Critchfield (1938) to work out the nuclear reactions that are possible at temperatures of about  $10^6$  to  $10^8$  K in the deep interior of the Sun (CNO cycle). Thus, the problem of stellar energy generation came to a solution. A second possibility for conversion of hydrogen into helium was offered by a reaction chain which Bethe (1939) had studied in 1939 (Proton-Proton chain). Thorough experimental investigations particularly those of Fowler (1984a), of reaction cross sections at low energies have contributed very significantly to our knowledge of nuclear energy generation in stars. One of the first results of the theory of stellar evolution taking into consideration hydrogen burning was that stars remain in the immediate vicinity of the main sequence until a considerable fraction ( $\approx 10\%$ ) of the hydrogen is burned. If no mixing takes place between burned and unburned material, the evolutionary track in the effective temperature-luminosity diagram then leads upward and to the right into the region of the red giants, a result obtained by Schönberg (1942) and S. Chandrasekhar as early as 1942. In connection with a more elaborate concept of astrophysical cross section factors given by Salpeter (1952a) (1952) the theory of nuclear burning stars made important advancement. In the deep interior of a star, suppose that a certain part of the hydrogen has been used up then the temperature rises to more than  $10^8$  K as a result of gravitational contraction. Öpik (1951) and Salpeter (1952b) remarked in 1951/1952 that helium burning sets in to give carbon in accordance with the triple- $\alpha$ -process. The growth of theoretical researches on stellar evolution, starting with the important paper by Hoyle (1955) and Schwarzschild (1955) denoted also a fundamental extension of the LANE-RITTER-EMDEN-EDDINGTON theory of the internal constitution of stars. The assumption hitherto made, that the material inside a star is continually mixed through its evolution, had to be given up in the face

of theoretical and observational results. The new concept of the theory of stellar evolution was born, considering the evolution of a star as successive stages of nuclear burning phases. These developments are described in the monograph of Schwarzschild (1958) and Hayashi ([1962](#)).

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## 1.5 The Solar Neutrino Experiment: Solar Neutrino Emission Rates

The development of large fast computers at the beginning of the 1960s has had a profound impact on the study of stellar structure. It was no longer necessary to use the techniques of integration by hand discussed in Schwarzschild's (1959) book. The first foundation of the elaborate numerical study of the internal structure and evolution of stars was the method of dividing the structure of a star into many concentric zones and then solving the differential equations of stellar structure in difference form at the boundaries of these zones, introduced by Henyey ([1959](#)). The second foundation of this study is Fowler's ([FowlerEtAl 1967](#); [HarrisEtAl 1983](#)) nuclear reaction rate systematics as published in a series of 'Handbücher der Kernastrophysik' (1967, 1975, 1983). Acting as a guide FOWLER's nuclear reaction rates are leading the computer from the main sequence stage of a star through the red giant stage up to the final stages of stellar evolution known as white dwarfs and supernovae. As far as the present scope of the theory of internal structure and evolution of stars is concerned a wide range of astrophysical problems emerged which are treated more or less independently from the classical theory of LANE-RITTER-EMDEN-EDDINGTON-CHANDRASEKHAR as described above briefly. To mention some: The theory of black holes and neutron stars, the emission of neutrinos and gravitational waves by stars, pulsating and oscillating stars, close binaries, population III stars, and so on. For an illustrative introduction to the recent history of the theory of the internal constitution and evolution of stars see Kippenhahn ([1984](#)) and Iben Jr. ([1985](#)).

In the following we are coming back once again to the classical theory of the internal structure of the Sun as established in the thirties. Surely, since the discovery of the nuclear reactions for the solar energy

generation by Weizsäcker (1937), Bethe (1938) and Chrichfield (1938), and Bethe (1939) it was known that additionally to the photon a second stable particle with no charge and a rest mass of approximately zero, that carries away energy in the course of nuclear reaction is emitted by the Sun: the neutrino. Further, two main characteristics of the neutrino were well known, namely, the weak interaction with matter and that neutrinos arise only in the energy generating regions of the stars and carry therefore unlike photons direct evidence of conditions in stellar cores. The first serious attempt to detect neutrinos emitted by the Sun goes back to Davis Jr. (1955) using a radio chemical neutrino detector based on the reaction  $^{37}Cl(\nu, e^-) ^{37}Ar$ . This radiochemical method was suggested by Pontecorvo (1946) in 1946. Note that at this time it was generally believed that neutrinos and antineutrinos were equivalent. After a long way of refinements of the  $Cl - Ar$ -experiment by R. DAVIS Jr. and associates the counting experiments valid for today's experimental and theoretical calculations of the solar neutrino flux started in 1968 by Daavis Jr. (1968), D.S. HARMER, and K.C. HOFFMAN. The net result of the solar neutrino experiment over more than 15 years is a disagreement between theory and observation of about a factor of four. Considering a standard model for the Sun and varying all of the parameters within the limits to be a plausible range of variation it has been shown by Bahcall (1982) that the disagreement still remains if not otherwise a contradiction with observational results appears.

As generally expected the neutrino flux from various nuclear reactions depends strongly on model parameters. But, for a first glance one can predict the number of solar neutrinos reaching the surface of the Earth per  $cm^2$  per s. This total number is approximately,

$$N_\nu = \frac{2L_\odot}{E} \approx 2 \times 10^{38} s^{-1}, \quad (1.19)$$

where  $L_\odot$  is the luminosity of the Sun and  $E$  is the energy liberated by the overall net nuclear reaction for the solar nuclear energy generation (Proton-Proton chain), not including the energy of the two emitted neutrinos. Thus, the solar neutrino flux on Earth is,

$$(1.20)$$

$$\phi_\nu = \frac{N_\nu}{4\pi(AU)^2} \approx 6.5 \times 10^{10} \nu \text{ cm}^{-2} \text{ s}^{-1}.$$

As will be shown in the following Chapters I and II the relation (1.19) can be derived by constructing a simple solar model solving the respective differential equations (1.10), (1.11), (1.15), taking into consideration the boundary conditions as discussed above, and assuming the equation of state for a perfect gas. For that we consider, instead of the phenomenological form of the rate of nuclear energy generation given in (1.17), the relation following from quantum statistical arguments as given by

$$\epsilon_{12} = \frac{1}{\rho} E_{12} r_{12}(\rho, T), \quad (1.21)$$

where  $r_{12}$  denotes the thermonuclear reaction rate between nuclear species 1 and 2,  $E_{12}$  is the energy liberated in the nuclear reaction under question. The thermonuclear reaction rate  $r_{12}$  in (1.21) can be derived from first principles including the distinction between resonant and nonresonant as well as MAXWELL-BOLTZMANNian and non-MAXWELL-BOLTZMANNian reaction rates. At last  $r_{12}$  will be a function not only of density  $\rho$ , temperature  $T$ , and chemical composition  $X_1$ , but also of subatomic quantities,

$$r_{12} = f(\rho, T, X_i, Z_1, Z_2, A_1, A_2, S_{12}) \quad (1.22)$$

where  $Z_i$  denotes the atomic number, and  $A_i$  is the mass number, and  $S_{12}$  is the nuclear cross section factor. Then, the nuclear output of the Sun ascribed to the nuclear reaction under question is according to (1.15) and (1.21),

$$\begin{aligned} L_{12}(R_\odot) &= \int_0^{R_\odot} dr \ 4\pi r^2 \rho(r) \epsilon_{12}(r) \\ &= \int_0^{R_\odot} dr \ 4\pi r^2 E_{12} r_{12}(\rho(r), T(r)). \end{aligned} \quad (1.23)$$

The total number of neutrinos due to the nuclear reaction between number species 1 and 2 in the Sun can then be written as

$$N_{12} = \frac{L_{12}(R_\odot)}{E_{12}} = 4\pi \int_0^{R_\odot} dr \ r^2 r_{12}(\rho(r), T(r)). \quad (1.24)$$

The number output of the Sun due to a reaction between nuclear species 1 and 2 as written in (1.22) is a function of stellar model parameters and subatomic quantities:

$$L_{12} = f(R, \rho, T, X_i, Z_1, Z_2, A_1, A_2, S_{12}). \quad (1.25)$$

What concerns the solar neutrino problem is that the relations (1.22) and (1.25) are equivalent to relations (1.17) and (1.16), but, in (1.22) and (1.25) the subatomic physics is taken into account. All cases of thermonuclear reaction rates (1.22) are derived in Chapter I, whereas the evaluation of (1.25) is given in Chap. 2 in detail.



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## 2. Mathematical Approaches to Nuclear Astrophysics

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### 2.1 Introduction: Cosmic Nucleosynthesis of the Elements

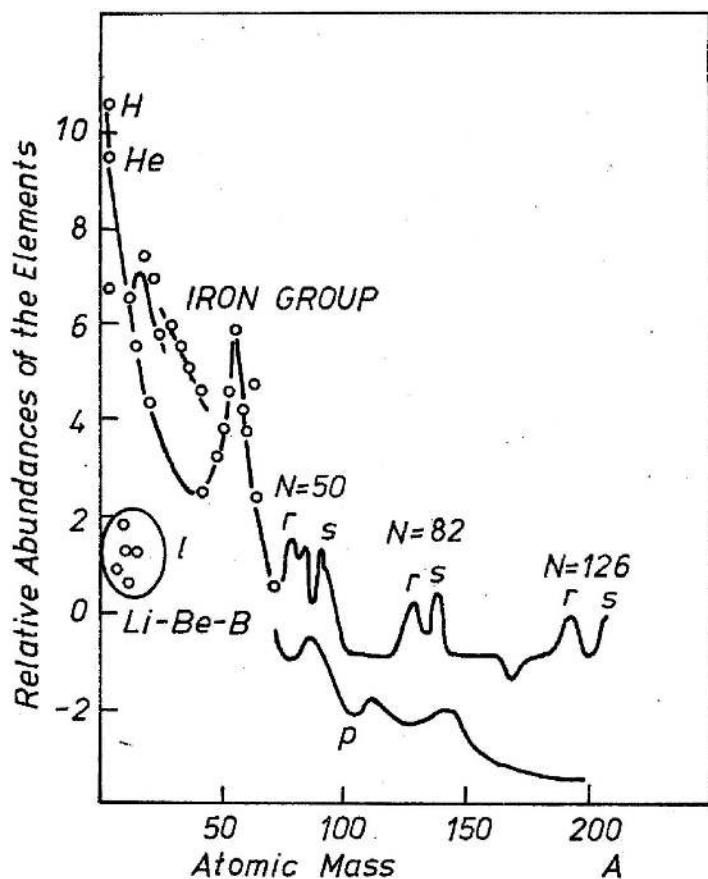
According to the current understanding of the evolution of the Universe the presently observed state of the Universe is the result of expansion from an extremely dense and extremely hot singular origin. Describing that evolution of the Universe by means of the standard cosmological model, based on the 'big-bang' hypothesis, cosmological nucleosynthesis occurs at the appropriate temperature in the course of expansion and goes on until the decreasing temperature stops nuclear reactions. No significant cosmological nucleosynthesis beyond helium-4 occurred due to the instability gaps at mass number 5 and mass number 8 as well as the constraints set by the present universal density and temperature. However, it is well-known that 'big-bang' nucleosynthesis results in the production of lithium-7 in amounts comparable to the solar system abundance of this nucleus. Some general features of the cosmological nucleosynthesis are listed in Table 2.1 and Figs. 2.1 and 2.2.

**Table 2.1** Basic facts of cosmic nucleosynthesis (cp. Burbidge et al. (1957), Fowler et al. (1983))

|  | Stage of nucleo-synthesis     | Temperature<br>Density<br>Time scale   | Main product of the nucleo-synthetic processes   | Site of nucleosynthesis   |
|--|-------------------------------|--|--|---|
| Cosmological nucleosynthesis starts three minutes after the "big bang"   |                               | $T \approx 10^9 \text{ K}$<br>$\rho \approx 1 \text{ g cm}^{-3}$   | $^1\text{H}$ , $^2\text{H}$ , $^3\text{He}$ , $^7\text{Li}$ , $^4\text{He}$  | Cosmology<br>"big bang"<br>                      |
| Stellar nucleosynthesis (hydrostatic) starts $10^9$ years after the "big bang"<br>Charged particle and neutron capture reactions | 1. H burning                  | $T \gtrsim 10^7 \text{ K}$<br>$\rho \approx 10^2 \text{ g cm}^{-3}$<br>$t \approx 10^{10} \text{ years}$         | Proton-proton chain in low mass stars<br>$^1\text{H} \rightarrow ^4\text{He}$<br>CNO cycles in high mass stars<br>$^1\text{H} \rightarrow ^4\text{He}$ $^{12}\text{C}$ , $^{16}\text{O} \rightarrow ^{14}\text{N}$   | Stellar physics<br>"Main sequence star"<br>      |
|  | 2. He burning s-process       | $T \gtrsim 10^8 \text{ K}$<br>$\rho \approx 10^3 \text{ g cm}^{-3}$<br>$t \approx 10^7 \text{ years}$            | Production of $^{12}\text{C}$ , $^{16}\text{O}$ via $(\alpha, \gamma)$ conversion of $^{14}\text{N}$ to $^{18}\text{O}$ , $^{21}\text{Ne}$ , $^{22}\text{Ne}$<br>$^{34}\text{He} \rightarrow ^{12}\text{C}$<br>to bridge the instability gaps at masses 5 and 8 production of neutrons by $(\alpha, n)$ reactions on $^{21}\text{Ne}$ , $^{22}\text{Ne}$ | "Red giant star"<br>                             |
|  | 3. C burning                  | $T \gtrsim 6 \times 10^8 \text{ K}$<br>$\rho \approx 10^5 \text{ g cm}^{-3}$<br>$t \approx 10^6 \text{ years}$   | $^{12}\text{C} + ^{12}\text{C} \rightarrow \alpha + ^{20}\text{Ne}$ , $p + ^{23}\text{Na}$ , $n + ^{24}\text{Mg}$<br>$16 \leq A \leq 28$   | "Onion skin star"<br>"Pre-supernova stage"<br> |
|  | 4. Ne burning transient stage | $T \gtrsim 10^9 \text{ K}$<br>$\rho \approx 10^6 \text{ g cm}^{-3}$  | $^{20}\text{Ne}(\gamma, \alpha) ^{16}\text{O}$ ,<br>$^{20}\text{Ne}(\alpha, \gamma) ^{24}\text{Mg}$<br>$^{24}\text{Mg}(\alpha, \gamma) ^{28}\text{Si}$   |   |
|  | 5. O burning                  | $T \gtrsim 1.5 \times 10^9 \text{ K}$<br>$\rho \approx 10^7 \text{ g cm}^{-3}$<br>$t \approx 10^5 \text{ years}$ | $^{16}\text{O} + ^{16}\text{O} \rightarrow \text{Si, S, Ar, Ca, ...}$<br>$16 \leq A \leq 28$   |   |
|  | 6. Si burning                 | $T \gtrsim 3 \dots 4 \times 10^9 \text{ K}$<br>$\rho \approx 10^8 \text{ cm}^{-3}$<br>$t \approx 1 \text{ s}$    | $\text{Si} \rightarrow \alpha, p, n \rightarrow \text{iron peak elements}$<br>$28 \leq A \leq 60$<br>inverse $(\alpha, \gamma)$ reactions:<br>$^{28}\text{Si}(\gamma, \alpha) ^{24}\text{Mg}$ $^{24}\text{Mg}(\gamma, \alpha) ^{20}\text{Ne}$ $(\gamma, \alpha) ^{16}\text{O}$<br>$^{16}\text{O}(\gamma, \alpha) ^{12}\text{C}(\gamma, 3\alpha)$         |   |

Table 2.1 (continued)

|  | Stage of nucleo-synthesis | Temperature<br>Density<br>Time scale   | Main product of the nucleo-synthetic processes  | Site of nucleosynthesis  |
|--|---------------------------|--|---|--|
| Stellar nucleo-synthesis (explosive) during stellar collapse to a neutron star or black hole | 7. s-process              | $T \gtrsim 5 \times 10^8 \text{ K}$<br>$\varrho \approx 10^2 \text{ g cm}^{-3}$<br>$t \approx 10^8 \dots 10^7 \text{ years}$ | long time scale; $\varphi_n \approx 10^5 \text{ cm}^{-3}$<br>$(n, \gamma)$ reactions on seed nuclei<br>$A \geq 60$                          | "Supernova"<br> |
|  | 8. r-process              | $T \gtrsim 10^{10} \text{ K}$<br>$\varrho \approx 10^9 \text{ g cm}^{-3}$<br>$t \approx 10 \dots 100 \text{ sec}$            | short time scale:<br>$\varphi_n \approx 10^{20} \text{ cm}^{-3}$ $(n, \gamma)$ reactions on seed nuclei<br>$A \geq 60$                      |  |
|  | 9. p-process              | $T \gtrsim 2 \times 10^9 \text{ K}$<br>$\varrho \approx 10^8 \text{ g cm}^{-3}$<br>$t \approx 10 \dots 100 \text{ sec}$      | Production of<br>heavy proton-rich nuclei<br>by $(\gamma, n)$ reactions   |  |
| Cosmic ray nucleo-synthesis (spallation)   | 10. l-process             |  | $^6\text{Li}, ^9\text{Be}, ^{10}\text{B}, ^{11}\text{B}$<br>Production of light elements by spallation of C, N, O by cosmic ray $\alpha, p$ | Cosmic ray physics $\alpha$<br>$\text{C} \xleftarrow{\alpha}$<br>$\text{O} \xleftarrow{\alpha}$    |



**Fig. 2.1** The standard abundance distribution of the elements in the Universe ( $\text{Si} \equiv 10^6$ ) (Burbidge et al. (1957))

Stars are born out of interstellar gas and are thought to be the site of stellar nucleosynthesis during their lifetime. It is now believed that most of the heavy elements are cooked in successive generations of stars. As can be seen in Table 2.1 and Figs. 2.1 and 2.2, within  $10^9$  years after the big-bang, stars are formed and stellar nucleosynthesis activities can take place in these stars. It was the study of the detailed plot of the number distribution of the cosmic elements as a function of atomic weight (cp. Figs. 2.1 and 2.2) and COULOMB barrier penetration considerations which led Burbidge et al. (1957) to postulate the basic nucleosynthetic processes in stars as shown in Table 2.1. Hydrogen burning, where four hydrogen nuclei are converted into one helium-4 nucleus, can occur via two different sets of nuclear reactions: the proton-proton chain and the CNO cycles. The most important feature of the CNO cycles is the conversion of carbon-12 and oxygen-16 into nitrogen-14. Helium burning bridges the mass number 5 and mass number 8 instability gaps via a three-particle reaction:  $3 \text{ helium-4} \rightarrow \text{carbon 12}$ . Helium burning includes the conversion of helium-4 into heavier elements such as carbon-12, oxygen-16, and neon-20. Helium burning is also considered as the site of the slow neutron capture processes (s-process). Carbon burning consists of the fusion of carbon nuclei themselves to build nucleides in the mass range  $16 \leq A \leq 28$ . Neon burning is a transient stage to photodisintegrate neon-20 into oxygen-16 and alpha particles. Again, oxygen burning produces elements in the mass range  $16 \leq A \leq 28$ , but at temperatures still higher than in carbon burning. In silicon burning a strong flux of neutrons, protons, and alphas are liberated mainly from silicon-28. These articles are captured by other nuclei to produce elements up to the iron peak elements. Thus, in these equilibrium processes between synthesis and disintegration, elements with the highest nuclear stability are produced. With the occurrence of silicon burning processes inside a star the so-called stationary stellar nucleosynthesis is finished. The core of the star consisting of iron peak elements will collapse never being stopped by nuclear energy generating reactions. During the collapse a shock wave is generated which propagates out through the star, adiabatically compressing, and thereby heating, the stellar matter through which it passes. The temperature in each nuclear burning shell of the star is raised suddenly and leads to explosive stellar nucleosynthesis in the various shells of the star. The final stages of this collapse and the explosive nucleosynthesis will be observed as supernova.

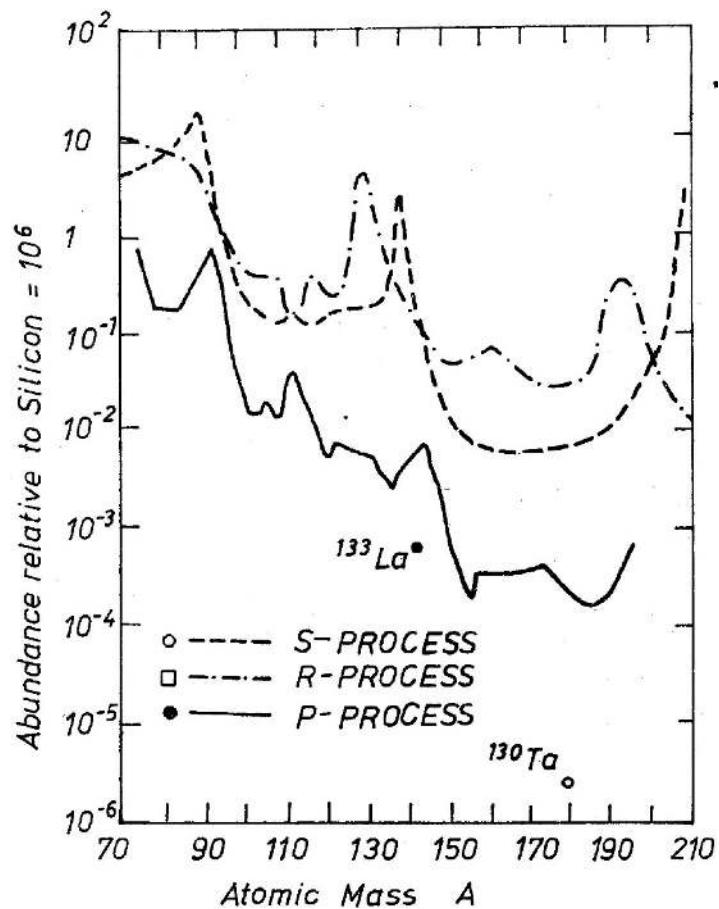


Fig. 2.2 Abundances of s, r, and p process elements

It is one of the most important features for stellar nucleosynthesis that the binding energy per nucleon decreases with increasing mass number for nucleides beyond the iron peak ( $A > 60$ ), and that these nuclei have large COULOMB barriers (cp. Figs. 2.1 and 2.2). Thus, they are not be formed by charged particle reactions. It is believed that most of these heavy elements are produced by neutron capture reactions which start with the iron group nuclei. These nuclei can be divided into two groups: if the flux of neutrons is weak, most chains of neutron captures will include only a few captures before the beta decay of the product nucleus. Because the neutron capture lifetime is slower than the beta decay lifetime, this kind of neutron capture is called s-process. In the opposite case, if there is a strong neutron flux, the neutron-rich elements will be formed by the rapid neutron capture process, where the neutron capture lifetime is much less than the beta decay lifetime (r-process). There are some very rare nuclei which are located on the proton-rich side of the valley of stability. It is believed that they are produced by the p-process in which photodisintegration occurs on s-process nuclei (Figs. 2.1 and 2.2).

Only to mention it, there are some very light, low abundance nuclei (hydrogen-2, helium-3, beryllium, lithium, bor). They are not produced in sufficient quantities in the cosmological nucleosynthesis and are immediately destroyed by thermonuclear reactions in the course of stellar nucleosynthesis. These nuclei are ascribed to l-processes: spallation reactions on carbon, nitrogen, and oxygen nuclei, by protons or alphas, in the high energy and low energy cosmic rays.

The present generally accepted framework of nuclear astrophysics can already be found in the classical paper by Burbidge et al. (1957) and most recently reviewed in the Nobel lecture of Fowler (1983). The laboratory approach to nuclear astrophysics, what concerns the element synthesizing nuclear reactions studied to a large extent in nuclear laboratories, is presented in the 'Handbücher der Astrophysik' by Fowler et al. (1967, 1975). The outcome of model computations with the best available nuclear input for the cosmological and stationary as well as explosive stellar nucleosynthesis is summarized in articles in 'Nucleosynthesis' edited by Biswas et al. (1980). The aim of the present chapter is to contribute to the problem of the parametrization of nuclear reaction rates as described in Fowler's paper (1983, Table I on p. 154 and Table II on P. 155; ep. also Bethe 1967). In this connection we try to continue the earlier investigations of Atkinson and Houtermans (1929), Gamow and Teller (1938), Salpeter (1952), Bahcall (1966), and Critchfield (1972) as recently considered in Haubold and John (1982) and Haubold and Mathai (1984). The examples for the analytic representation of nuclear reaction rates presented in the following sections should illustrate the field of nuclear astrophysics from the more mathematical point of view. They are worked out in the spirit of the remarks in Fowler's (1974) George Darwin lecture: "... each reaction rate, measured accurately or computed from good systematics, is a diagnostic tool in determining the astrophysical circumstances and sites of nucleosynthesis and energy generation."

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## 2.2 The Thermonuclear Reaction Rate

Before we shall come to the mathematical methods for the analytic representation of nuclear reaction rates it will be useful to discuss some basic nuclear astrophysical relations containing the thermonuclear reaction rate. Most of the formalism for the tabulations of the reaction rate of interacting particles under cosmological or stellar conditions has been presented by Fowler et al. (1967, 1975).

For the most common case, in which two particles in the initial channel (1 and 2) form two particles in the final channel (3 and 4),

$$1 + 2 \rightarrow 3 + 4 + E_{12}, \quad (2.1)$$

the reaction rate  $r_{12}$  is defined by (Fowler et al. 1967, 1975; Haubold and Mathai 1984)

$$r_{12} = \left(1 - \frac{1}{2}\delta_{12}\right)n_1n_2, \langle\sigma v\rangle_{12}, \quad [r_{12}] = \text{cm}^{-3}\text{s}^{-1}, \quad (2.2)$$

where  $n_1$  and  $n_2$  are the number densities of nuclei 1 and 2, respectively,  $\delta_{12}$  is the KRONECKER symbol,  $E_{12}$  is the energy release ( $E_{12} = (m_1 + m_2 - m_3 - m_4)c^2$ ),  $c$  denotes the velocity of light. The KRONECKER delta is introduced to avoid double counting in the reaction if 1 and 2 being identical. In (2.2) the quantity  $\langle\sigma v\rangle$  is the thermally averaged product of the cross section,  $\sigma$ , for the reaction, and relative velocity,  $v$ , of the interacting particles, 1 and 2, which is in fact the most important quantity for the analytic representation of the thermonuclear reaction rate (2.2) as we shall see in Sect. 2.3. For a gas of mean density,  $\rho$ , the number density,  $n_i$ , of particle,  $i$ , can be expressed in terms of its mass fraction,  $X_i$ , by the relation

$$(2.3)$$

$$n_i = \rho N_A \frac{X_i}{A_i}, i = 1, 2, 3, 4, [n_i] = \text{cm}^{-3},$$

where  $N_A$  stands for AVOGADRO's constant, and  $A_i$  is the atomic mass of particle  $i$  in atomic mass units (Note  $\sum_i X_i = 1$ ).

The mean lifetime,  $\tau_2(1)$ , of particle 1 for interaction with particle 2 can be given by

$$\begin{aligned} \lambda_2(1) &= \frac{1}{\tau_2(1)} = n_2 \langle \sigma v \rangle_{12} = \rho N_A \frac{X_2}{A_2} \langle \sigma v \rangle_{12} \\ &= -\frac{1}{n_1} \left( \frac{dn_1}{dt} \right)_2 = -\frac{1}{X_1} \left( \frac{dX_1}{dt} \right)_2, [\tau_2(1)] = \text{s}^{-1}, \end{aligned} \quad (2.4)$$

where  $\lambda_2(1)$  is the decay rate of 1 for interaction with 2. The definition of the mean lifetime of a nucleus for interaction with a nucleus in (2.4) shows directly the connection of the quantity  $\langle \sigma v \rangle$  with kinetic equations for the production and destruction of the respective nucleus (Haubold and Mathai 1984).

The energy generation rate,  $\epsilon_{12}$ , for the reaction (2.1) is defined by

$$\epsilon_{12} = \frac{1}{\rho} r_{12} E_{12}, [\epsilon_{12}] = \text{erg g}^{-1} \text{s}^{-1}, \quad (2.5)$$

where  $E_{12}$  is the energy given off in one single reaction (2.1).

By definition, the quantity  $\langle \sigma v \rangle$  in (2.2) arises from an integral over the respective cross section of the reaction, times relative velocity of the reacting particles, times the distribution function of the relative velocities of the particles:

$$\langle \sigma v \rangle = \int_0^\infty d^3v \sigma(v) vf(v) = \int_0^\infty dE \sigma(E) \left( \frac{2E}{\mu} \right)^{\frac{1}{2}} f(E), [\langle \sigma v \rangle] = \text{cm}^3 \text{s}^{-1}, \quad (2.6)$$

where  $d^3v = 4\pi v^2 dv$ ; the kinetic energy of the particles in the center-of-mass system is  $E = \mu v^2/2$ , the reduced mass of the particles is denoted by  $\mu = m_1 m_2 / (m_1 + m_2)$ , the reaction cross section is  $\sigma(v)$  and  $\sigma(E)$ , respectively. Equation (2.6) contains what we expected to be the basic quantities for the description of reactions between nuclear particles going on in the very hot and intermediate dense plasma at some stage of the evolution of the Universe and in the deep interior of stars: for quantummechanical reasons the cross section of the particle reaction and for reasons of statistical mechanics the distribution function of the velocities of the reacting particles.

The time reversal invariance of the strong, electromagnetic, and weak interactions leads to an important relation between the cross sections for the forward and backward nuclear reactions which should be noted here (cp. Blatt and Weisskopf 1959):

$$\frac{\sigma_{34}}{\sigma_{12}} = \frac{(1+\delta_{34})}{(1+\delta_{12})} \frac{g_1 g_2}{g_3 g_4} \frac{A_1 A_2 E_{12}}{A_3 A_4 E_{34}}, \quad (2.7)$$

which reads for the quantity  $\langle \sigma v \rangle$  in (2.6):

$$\frac{\langle \sigma v \rangle_{34}}{\langle \sigma v \rangle_{12}} = \frac{(1+\delta_{34}) g_1 g_2}{(1+\delta_{12}) g_3 g_4} \left( \frac{A_1 A_2}{A_3 A_4} \right)^{\frac{3}{2}} \exp \left\{ -\frac{E_{34} - E_{12}}{kT} \right\}, \quad (2.8)$$

where the  $g_i$ 's denote spin statistical weights  $g_i = (2s_i + 1)$ ,  $i = 1, 2, 3, 4$ . At high temperatures many nuclear states are populated and the  $g_i$ 's become partition functions in the expression (2.8). Equation (2.7) underlines the principle of microscopic reversibility, (2.8) represents the principle of detailed balance (see Haubold and Mathai [1984](#)). All relations given above are thought to be valid for reactions in a non-degenerate gas and for non-relativistic particle velocities.

## 2.3 Velocity Distribution Function and Nuclear Cross Section: Maxwell-Boltzmann Distribution Function

All the analytic expressions for astrophysically relevant nuclear reaction rates given in the tabulations of Fowler et al. ([1967](#), [1975](#)) underline the hypothesis that the distribution of the relative velocities of the reacting particles always remains Maxwell-Boltzmannian. Fixing the distribution function of the relative velocities of the particles as Maxwell-Boltzmannian has serious physical implications for the nuclear reaction rate theory (Haubold and Mathai [1984](#)). However, if we choose by the time the Maxwell-Boltzmannian approach to the nuclear reaction rate in (2.2) and (2.6) then the distribution of the relative velocities of the particles can be written in the following manner:

$$f(v)dv = \left(\frac{\mu}{2\pi kT}\right)^{\frac{3}{2}} \exp\left\{-\frac{\mu v^2}{2kT}\right\} 4\pi v^2 dv. \quad (2.9)$$

The function  $f(v)$  satisfies the normalization condition  $\int_0^\infty dv f(v) = 1$ . As given in (2.9) we can take it as the Maxwell-Boltzmannian relative kinetic energy spectrum for a non-degenerate, non-relativistic gas of particles (cp. Fig. 2.4)

$$f(E)dE = \frac{2}{\pi^{1/2}(kT)^{3/2}} \exp\left\{-\frac{E}{kT}\right\} E^{\frac{1}{2}} dE. \quad (2.10)$$

For a detailed discussion of the underlying physical assumptions for the application of (2.10) or the evaluation of reaction rates see Haubold and Mathai ([1984](#)).

## 2.4 Nonresonant Neutron Capture Cross Section

In the case of a nuclear reaction via neutron capture the solution of the respective SCHRÖDINGER equation is the plane wave which can be normalized to unit particle density at infinity. Then the cross section is determined by the ratio of the square of the absolute value of the wave function,  $\psi$ , for inelastic scattering by the current density of the particles,  $\mathbf{s}$

$$\sigma(v) = \frac{|\psi|^2}{|\mathbf{s}|} = \frac{|\psi|^2}{|\rho \mathbf{v}|} \sim \frac{1}{v}, \quad (2.11)$$

where  $\rho$  denotes the space density and  $\mathbf{v}$  the current velocity of the  $\psi$ -field, respectively. At low energy the  $s$ -wave interactions dominate. That is to say, the reactions take place chiefly through a particular angular momentum state  $l = 0$ . From (3.3) follows that  $\sigma v$  is a constant. Derivations from this occur at higher energies when other partial waves become

important and it is convenient to express the nonresonant slowly varying velocity dependence of the cross section,

$$\sigma(v) = \frac{T(v)}{v}, \quad (2.12)$$

as the first three terms of a MACLAURIN series in the relative velocity,  $v$ , of the neutron and the nucleus:

$$T(v) = T(0) + T'(0)v + \frac{1}{2}T''(0)v^2, \quad (2.13)$$

where  $[T(0)] = \text{cm}^3\text{s}^{-1}$ ,  $[T'(0)] = \text{cm}^2$ , and  $[T''(0)] = \text{cm}\text{s}$  (Fowler et al. 1967, 1975). In (2.13)  $T(0)$ ,  $T'(0)$ , and  $T''(0)$  are empirical constants measured in nuclear experiments, and the prime indicates differentiation with respect to  $v$ .

## 2.5 Nonresonant Charged Particle Cross Section

If two nuclei, of charges  $Z_1 e$  and  $Z_2 e$ , and masses  $m_1$  and  $m_2$ , collide with kinetic energy of relative motion,  $E = \mu v^2/2$ , then on the basis of non-relativistic quantum scattering theory one obtains, in spherical coordinates at  $r = 0$ , for the square of the wave function (Blatt and Weisskopf 1959):

$$|\psi(0)|^2 = |\Gamma(1 - i\eta(v))|^2 \exp(-\pi\eta(v)), \quad (2.14)$$

where

$$\eta(v) = \frac{Z_1 Z_2 e^2}{\hbar v} = \frac{Z_1 Z_2 c \alpha}{v}; \quad (2.15)$$

$\eta(v)$  is the Sommerfeld parameter,  $\hbar$  is the PLANCK quantum of action,  $\alpha$  is the Sommerfeld fine structure constant,  $\Gamma(\cdot)$  denotes the gamma function, and  $i^2 = -1$ . According to the elementary property of the gamma function we have from (2.14)

$$\begin{aligned} |\psi(0)|^2 &= \Gamma(1 + i\eta(v))\Gamma^*(1 + i\eta(v)) \exp(-\pi\eta(v)) \\ &= \Gamma(1 + i\eta(v))\Gamma(1 - i\eta(v)) \exp(-\pi\eta(v)) \\ &= \frac{\pi\eta(v)}{\sinh(\pi\eta(v))} \exp(-\pi\eta(v)) = \frac{2\pi\eta(v)}{\exp(2\pi\eta(v)) - 1}. \end{aligned} \quad (2.16)$$

For physical reasons the argument of the exponential function is always positive and thus the quantity in (2.16) is smaller than unity and for a strong COULOMB interaction, that means  $2\pi\eta(v) \gg 1$ , it becomes exponentially very small. This means that for strong repulsion and low energy collisions, the wave will only penetrate the potential barrier at the origin with a rapidly decreasing amplitude whose asymptotic form is

$$|\psi(0)|^2 \sim 2\pi\eta(v) \exp(-2\pi\eta(v)), \quad (2.17)$$

where the exponential factor is called Gamow factor and (2.17) is the barrier penetration factor. As in (2.11) the cross section for the charged particle reaction is obtained by

dividing the absolute square value of the wave function for inelastic scattering (2.17) by the probability current density which is proportional to the velocity of the  $\psi$ -field. Then, we obtain the overall velocity dependence of the nuclear reaction cross section at low energies for nonresonant charged particle interactions by

$$\sigma(v) \sim \eta^2(v) \exp(-2\pi\eta(v)). \quad (2.18)$$

It is convenient to factor out the energy dependence and express the cross section,  $\sigma(E)$ , by

$$\sigma(E) = \frac{S(E)}{E} \exp(-2\pi\eta(E)), \quad (2.19)$$

where

$$\eta(E) = \left(\frac{\mu}{2}\right)^{\frac{1}{2}} \frac{Z_1 Z_2 e^2}{\hbar E^{1/2}}. \quad (2.20)$$

Equation (2.19) defines the cross section factor,  $S(E)$ , representing the intrinsically nuclear parts of the probability for the occurrence of a nuclear reaction (cp. also Salpeter 1952). The cross section factor,  $S(E)$ , is often found to be constant or a slowly varying function of energy over a limited range of energy (Fowler et al. 1967, 1975). Far from a nuclear resonance  $S(E)$  may be conveniently expressed in terms of the power series expansion,

$$S(E) = S(0) + S'(0)E + \frac{1}{2}S''(0)E^2, \quad (2.21)$$

where  $[S(0)] = MeV$  barns,  $[S'(0)] = \text{barns}$ , and  $[S''(0)] = \text{barns } MeV^{-1}$ . The prime indicates differentiation with respect to  $E$ .

---

## 2.6 Resonant Cross Section for Neutrons and Charged Particles

For a single resonance of energy,  $E_r$ , the cross section,  $\sigma(E)$ , of the nuclear reaction (2.1) can be represented as a function of energy in terms of the classical BREIT-WIGNER formula (Blatt and Weisskopf 1959),

$$\sigma(E) = \pi \lambda^2 \omega \frac{\Gamma_{12}\Gamma_{34}}{(E_r - E)^2 + (\Gamma/2)^2}, \quad (2.22)$$

where  $\lambda = h/(\mu v)$  is the reduced DE BROGLIE wavelength. The statistical factor  $\omega$  is defined by  $\omega = (2J + 1)/[(2J_1 + 1)(2J_2 + 1)]$ , where  $J$  is the angular momentum of the resonance state, and  $J_1$  and  $J_2$  are the angular momenta of particles 1 and 2, respectively. The total width,  $\Gamma$ , of the resonance state is given by  $\Gamma = h/\tau = \Gamma_{12} + \Gamma_{34} + \dots$ , where  $\tau$  is the effective lifetime of the state. The partial width,  $\Gamma_{12}$ , is the width for reemission of particles 1 and 2, and  $\Gamma_{34}$  is the width for emission of particles 3 and 4.

The partial width,  $\Gamma_{34}$ , for the absorption or emission of a certain particle by the compound nucleus, is a strong energy depending function and can be written (Blatt and Weisskopf 1959),

$$\Gamma_{34}(E) = \frac{2^{3/2} \mu^{1/2} R_0 D}{h} E^{1/2} P(E), \quad (2.23)$$

where  $R_0$  is the characteristic wavelength of nucleons inside the nucleus (of the order of  $10^{-13} \text{ cm}$ ),  $D$  is the average distance of levels, and  $P(E)$  denotes the barrier penetration factor. At low energy the s-wave ( $l = 0$ ) interactions dominate and the barrier penetration factor in (2.23) is given in (2.17), thus we have,

$$P(E) = 2\pi(\mu/2)^{1/2} \frac{Z_1 Z_2 e^2}{h E^{1/2}} \exp \left\{ -2\pi(\mu/2)^{1/2} \frac{Z_1 Z_2 e^2}{h E^{1/2}} \right\}. \quad (2.24)$$

For the total width,  $\Gamma$ , we consider an ad hoc linear energy dependence of the form

$$\Gamma(E) = \Gamma_0 + \Gamma_1 E, \quad (2.25)$$

where  $\Gamma_0$  and  $\Gamma_1$  are empirical constants measured in nuclear experiments;

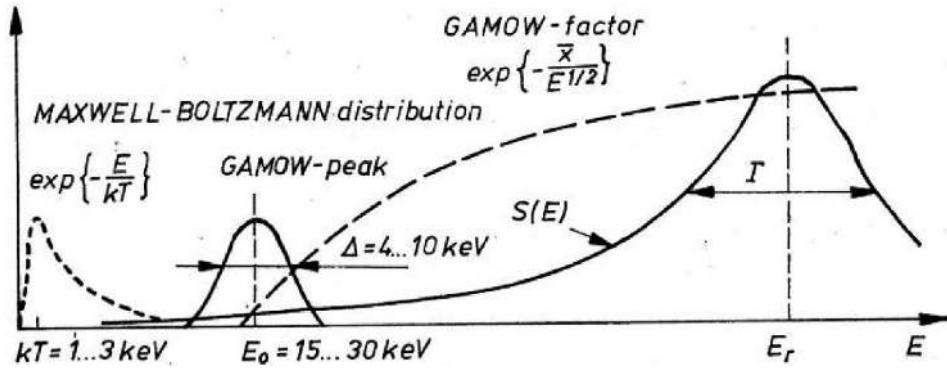
Inserting (2.22)–(2.24) and (2.25) we obtain the parameterized form of the BREIT-WIGNER one-resonance-level formula

$$\begin{aligned} \sigma(E) &= \left[ \frac{\pi^2 h^2}{2\mu E} \right] \left[ \frac{2^{3/2} \mu^{1/2} R_0 E^{1/2}}{h} \frac{2\pi \mu^{1/2} Z_1 Z_2 e^2}{2^{1/2} h E^{1/2}} \exp \left\{ -\frac{2\pi \mu^{1/2} Z_1 Z_2 e^2}{2^{1/2} h E^{1/2}} \right\} \right] \\ &\times \left[ \frac{\omega \Gamma_{12} D}{(E_r - E)^2 + ([\Gamma_0 + \Gamma_1 E]/2)^2} \right]. \end{aligned} \quad (2.26)$$

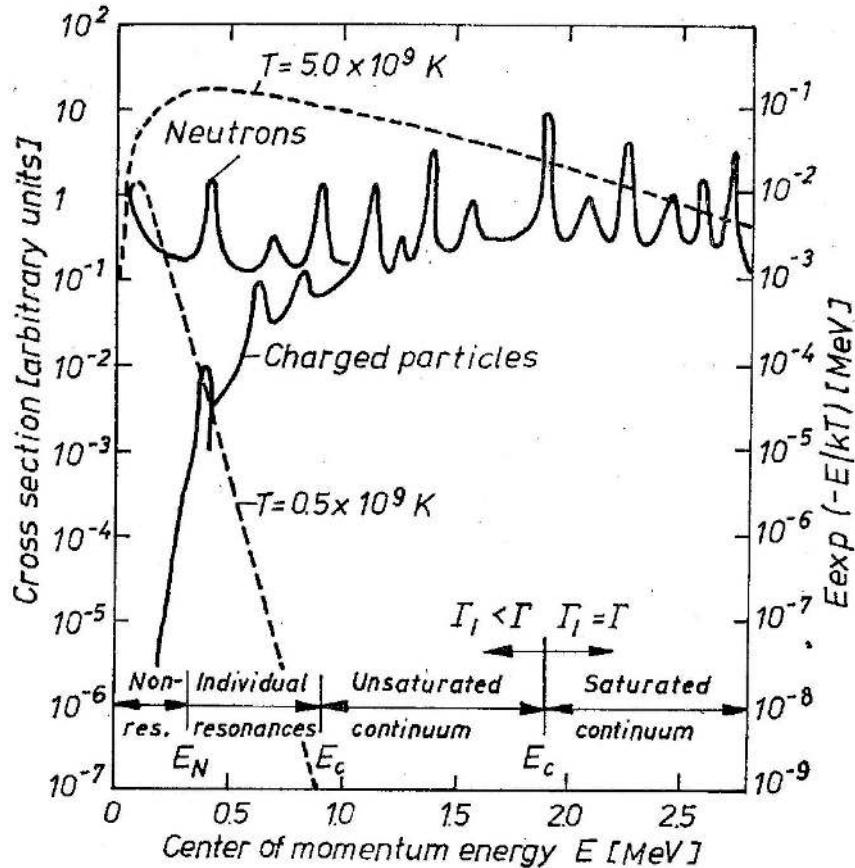
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## 2.7 Parameterizations of Thermonuclear Reaction Rates

In the following, we consider the form of the quantity  $\langle \sigma v \rangle$  in (2.6) of Chap. 2, taking into account the distribution function of the relative particle velocities as given in (2.9) and (2.10), and the nuclear cross section derived in (2.12), (2.19) and (2.26), respectively.



**Fig. 2.3** Schematic plot of the energy-dependent factors for the integral of thermonuclear reaction rates: Maxwell-Boltzmann distribution function, nonresonant nuclear cross section, and resonant nuclear cross section



**Fig. 2.4** Schematic plot of charged particle and neutron cross sections as a function of center-of-momentum energy, together with the Maxwell-Boltzmann distribution function (dashed lines) (Wagoner 1969)

Aside from the Lorentz factor due to resonance phenomena in nuclear reactions, the two dominant factors in the quantity  $\langle \sigma v \rangle$  are the COULOMB barrier (Gamow factor), which inhibits the reaction rate at low energies, and the tail of the distribution function of

relative velocities of the particles (Maxwell-Boltzmann factor) as shown in Fig. 2.3. Thus, the kernel of the integral of the quantity  $\langle \sigma v \rangle$  in (2.6) is a product of a rapidly rising cross section and a steeply falling distribution function, which gives a not quite asymmetrical peak, called the Gamow peak. For the treatment of nuclear reaction networks for cosmological or stellar nucleosynthetic calculations, these integrals have to be evaluated as far as possible in closed-form and the results represented in manageable analytic expressions (Fowler et al. 1967, 1975; Haubold and John 1982; Haubold and Mathai 1984).

To have a feeling for the integration problem of the quantity  $\langle \sigma v \rangle$  we refer to the schematic plot of typical charged particle and neutron cross sections, as a function of center-of-mass energy shown in Fig. 2.4 (Wagoner 1969).

Also, shown in Fig. 2.4 are two Maxwell-Boltzmann distribution functions of the relative particle velocities for temperatures near the limit of the region of interest (Wagoner 1969). Inserting the Maxwell distribution function (2.9) and (2.10), respectively, in (2.6), we obtain the following:

$$\langle \sigma v \rangle = \left( \frac{\mu}{2\pi kT} \right)^{\frac{3}{2}} \int_0^\infty dv \sigma(v) v^3 \exp \left\{ -\frac{\mu v^2}{2kT} \right\}, \quad (2.27)$$

$$\langle \sigma v \rangle = \left( \frac{8}{\pi\mu} \right)^{\frac{1}{2}} \frac{1}{(kT)^{\frac{3}{2}}} \int_0^\infty dE \sigma(E) E \exp \left\{ -\frac{E}{kT} \right\}. \quad (2.28)$$

The remaining factor in the kernel of the integrand of (2.27) and (2.28), respectively, is the nuclear cross section which has quite special energy characteristics depending on whether the reaction proceeds via a resonant or nonresonant mechanism.

According to Wagoner (1969) (see also Fowler et al. 1967, 1975) it is convenient to divide the quantity  $\langle \sigma v \rangle$  in (2.27) and (2.28) into four parts.

$$\langle \sigma v \rangle = \langle \sigma v \rangle_{nr} + (\Sigma_{res} \langle \sigma v \rangle_r) + \langle \sigma v \rangle_{uc} + \langle \sigma v \rangle_{sc}. \quad (2.29)$$

Splitting of  $\langle \sigma v \rangle$  in (2.29) in order of increasing energy as indicated in Fig. 2.4 corresponds to the successive synthesis of heavier elements out of lighter elements in stellar generations as discussed in Sect. 2.1 and outlined in Table 2.1. The nuclear reactions of the proton-proton chain are nonresonant ( $nr$ ), the nuclear reactions of the CNO cycles are dominated by one or a few resonances ( $r$ ), and the carbon burning, neon burning, oxygen burning, and silicon burning are characterized by strongly overlapping resonances (unsaturated continuum ( $uc$ ) and saturated continuum ( $sc$ )).

As already indicated, the quantity  $\langle \sigma v \rangle$  depends strongly on the temperature. In a highly evolved star going through explosive nucleosynthesis, the thermal distribution of energies has a sufficiently broad spectrum that a large number of compound nuclear states contribute to the effective average cross section of a considered reaction. Thus, the high temperature regions of the quantity  $\langle \sigma v \rangle$ , those are the last two terms on the right hand side of (4.3), must be treated by nuclear statistical models (e.g. HAUSER-FESHBACH model; c.f. Fowler 1974; Fowler et al. 1967, 1975). But, as shown by Fowler et al. (1967, 1975), even in the case of closely spaced or overlapping resonances, the formalism for thermonuclear reaction rates of nonresonant reactions, or reactions which proceed in the

wing of a single resonance, still remains valid. At least, in all cases mentioned the cross section factor  $S(E)$  is a smoothly varying function of energy as shown in Fig. 2.3. For that reason, in the following, we will concentrate our considerations on the closed-form evaluation of the first two terms for  $\langle \sigma v \rangle$  on the right hand side of (2.29). We shall consider the analytic representation of each of the first two terms on the right hand side in (2.29) separately.

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## 2.8 Nonresonant Reaction Rates

Here, we consider various situations such as neutral particles, charged particles etc.

### Case 2.1 The neutral particle case

Inserting (2.12) and (2.13) in (2.27) leads to the nonresonant neutron capture representation of  $\langle \sigma v \rangle$ ,

$$\langle \sigma v \rangle = \frac{1}{\pi^{\frac{1}{2}}} \sum_{\nu=0}^2 \frac{T^{(\nu)}(0)}{\nu!} 2^{\nu+1} \left( \frac{kT}{\mu} \right)^\nu \int_0^\infty dy y^{\nu+\frac{1}{2}} \exp \{-y\},$$

where  $y = \mu v^2 / (2kT)$ , and we obtain,

$$\langle \sigma v \rangle = \frac{1}{\pi^{\frac{1}{2}}} \sum_{\nu=0}^2 \frac{T^{(\nu)}(0)}{\nu!} 2^{\nu+1} \left( \frac{kT}{\mu} \right)^\nu \Gamma \left( \nu + \frac{3}{2} \right), \Re(\nu) > -\frac{3}{2}. \quad (2.30)$$

With (2.30) and (2.2), the closed-form representation of the nonresonant thermonuclear reaction rate in the neutral particle case is obtained (cf. also Haubold and John 1982).

### Case 2.2 The charged particle case with strict Maxwell-Boltzmann distribution function

Inserting (2.19) and (2.21) in (2.28) leads to the charged particle representation of  $\langle \sigma v \rangle$  if the reaction takes place far away from any resonance,

$$\langle \sigma v \rangle = \left( \frac{8}{\pi \mu} \right)^{\frac{1}{2}} \frac{1}{(kT)^{\frac{3}{2}}} \sum_{\nu=0}^2 \frac{S^{(\nu)}(0)}{\nu!} \int_0^\infty dE E^\nu \exp \left\{ -\frac{E}{kT} - 2\pi \eta(E) \right\},$$

where the Sommerfeld parameter  $\eta(E)$  is given in (2.20). With the substitution  $y = E / (kT)$  we obtain

$$\langle \sigma v \rangle = \left( \frac{8}{\pi \mu} \right)^{\frac{1}{2}} \sum_{\nu=0}^2 \frac{S^{(\nu)}(0)}{\nu!} \frac{1}{(kT)^{-\nu+\frac{1}{2}}} \int_0^\infty dy y^\nu e^{-y-zy^{-\frac{1}{2}}}, \quad (2.31)$$

and remove the parameter-dependent integral

$$N_{i_\nu}(z) = \int_0^\infty dy y^\nu e^{-y-zy^{-\frac{1}{2}}} \quad (2.32)$$

from (2.31). With (2.32) we write for the quantity  $\langle \sigma v \rangle$  in (2.31)

$$(2.33)$$

$$\langle \sigma v \rangle = \left( \frac{8}{\pi \mu} \right)^{\frac{1}{2}} \sum_{\nu=0}^2 \frac{S^{(\nu)}(0)}{\nu!} \frac{1}{(kT)^{-\nu+\frac{1}{2}}} N_{i_\nu}(z),$$

where

$$z = 2\pi \left( \frac{\mu}{2kT} \right)^{\frac{1}{2}} \frac{Z_1 Z_2 e^2}{h}. \quad (2.34)$$

With the intention to include more general energy dependent nuclear cross section factors,  $S(E)$ , than given in (2.21), we considered the following general collision probability integral (Haubold and Mathai 1984):

$$N_1(z; a, \rho, n, m) = a \int_0^\infty dt t^{-n\rho} e^{-at-zt^{-\frac{n}{m}}}. \quad (2.35)$$

For the analytic evaluation of the integral  $N_1(z; a, \rho, n, m)$  in (2.35), first we will give a general result and then get  $N_{i_\nu}(z)$  in (2.32) as a special case. For a detailed description of the mathematical method to tackle integrals of the type in (2.35), see Haubold and Mathai (1984). The following theorem is originally due to Saxena (1960) and a simple proof of the theorem by using random variables is given by Haubold and Mathai (1984):

### Theorem 2.1

$$\begin{aligned} N_1(z; a, \rho, n, m) &= a \int_0^\infty dt t^{-n\rho} e^{-at-zt^{-\frac{n}{m}}} \\ &= a^{n\rho} (2\pi)^{\frac{1}{2}(2-n-m)} m^{\frac{1}{2}} n^{\frac{1}{2}-n\rho} \\ &\quad \times G_{0,m+n}^{m+n,0} \left[ \frac{z^m a^n}{m^m n^n} \Big|_{0, \frac{1}{m}, \dots, \frac{m-1}{m}, \frac{1-n\rho}{n}, \dots, \frac{n-n\rho}{n}} \right], \end{aligned} \quad (2.36)$$

for  $\Re(a) > 0$  and  $\Re(z) > 0$ ;  $G_{0,m+n}^{m+n,0}(\cdot)$  denotes Meijer's G-function (see Mathai and Saxena 1973).

Consider the case  $m = 2, n = 1, a = 1$ , and  $\rho = -\nu$  in (2.36). Then, we have (cf. also Mathai 1971; Haubold and John 1982)

$$N_1(z; 1, -\nu, 1, 2) = \int_0^\infty dy y^\nu e^{-y-zy^{-\frac{1}{2}}} = \frac{1}{\pi^{\frac{1}{2}}} G_{0,3}^{3,0} \left[ \frac{z^2}{4} \Big|_{a, \frac{1}{2}, 1+\nu} \right], \quad (2.37)$$

where  $z$  is given in (2.34). With (2.37) it is known that the nonresonant thermonuclear reaction rate can be represented in closed-form by using the highly efficient theory of generalized hypergeometric functions as given in Mathai and Saxena (1973). In the case of the Maxwell-Boltzmann approach to the nonresonant thermonuclear reaction rate Meijer's G-function of the type  $G_{0,p}^{p,0}(\cdot)$  appears in the closed-form representation. For a critical review of the present status of the analytic evaluation of nonresonant thermonuclear reaction rates see Haubold and John (1982). Approximation consideration of  $N_{i_\nu}(z)$  in (2.37) are given in papers of Bahcall (1966) and Critchfield (1972). On the basis of the complex Mellin-Barnes integral representation of Meijer's G-function in (2.37), one can derive series representations of the integral  $N_{i_\nu}(z)$  for all five classes of

the parameters  $\nu, \nu \neq \pm \frac{\lambda}{2}, \lambda = 0, 1, 2, \dots; \nu$  a positive integer;  $\nu$  a negative integer;  $\nu$  a positive half integer;  $\nu$  a negative half integer; as given in Sect. 4.3. (Haubold and Mathai [1984](#)). Moreover, one finds series representations of  $N_{i_\nu}(z)$  which are termwise integrable over any finite range of the variable,  $z$ , given in [\(2.34\)](#) and which can be used for the numerical computation of  $\langle \sigma v \rangle$  in [\(2.33\)](#) also.

**Case 2.3** The charged particle case with depleted Maxwell-Boltzmann distribution function

In the following, we admit a depletion of the high energy tail of the Maxwell-Boltzmann distribution function of the relative kinetic energies of the reacting particles given in [\(2.10\)](#). For the discussion of the physical reasons for a depletion of the high energy tail of the Maxwell-Boltzmann distribution function, we refer to the paper of Haubold and John [\(1982\)](#) and Haubold and Mathai [\(1986c\)](#).

For the integral of  $\langle \sigma v \rangle$  in comparison with the strict Maxwell-Boltzmannian case [\(2.32\)](#) we have now

$$N_{i_\nu}(z; \delta) = \int_0^\infty dy y^\nu e^{-y - y^\delta - zy^{-\frac{1}{2}}}. \quad (2.38)$$

Consider the general integral

$$N_2(z; a, b, \delta, \nu, n, m) = \int_0^\infty dt t^\nu e^{-at - bt^\delta - zt^{-\frac{n}{m}}},$$

where  $z > 0, a > 0, b > 0, m, n = 1, 2, 3, \dots$ . Since

$$e^{-bt^\delta} = \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} t^{k\delta},$$

we have

$$N_2(z; a, b, \delta, \nu, n, m) = \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} \int_0^\infty dt t^{\nu+k\delta} e^{-at - zt^{-\frac{n}{m}}}. \quad (2.39)$$

According to the Theorem [2.1](#) (cf. also Haubold and Mathai [1984](#)) we have,

$$N_2(z; a, b, \delta, \nu, n, m) = \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} a^{-(\nu+k\delta+1)} (2\pi)^{\frac{1}{2}(2-n-m)} m^{\frac{1}{2}} n^{\frac{1}{2}+\nu+k\delta} \\ \times G_{0,m+n}^{m+n,0} \left[ \frac{z^m a^n}{m^m n^n} \mid_{0, \frac{1}{m}, \dots, \frac{m-1}{m}, \frac{1+\nu+k\delta}{n}, \dots, \frac{n+\nu+k\delta}{n}} \right].$$

Now, we consider [\(2.38\)](#) and have by putting  $a = 1, b = 1, n = 1, m = 2$

$$N_2(z; 1, 1, \delta, \nu, 1, 2) = \frac{1}{\pi^{\frac{1}{2}}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} G_{0,3}^{3,0} \left[ \frac{z^2}{4} \mid_{0, \frac{1}{2}, 1+\nu+k\delta} \right]. \quad (2.40)$$

For the numerical computation of [\(2.40\)](#) all the cases in Sect. 4.3 of Haubold and Mathai [\(1984\)](#) can apply with  $\nu$  replaced by  $\nu + k\delta, k = 0, 1, 2, \dots$ . Note that if  $\delta$  is an irrational number then  $\nu + k\delta$  cannot be a negative integer or negative half integer or

zero or positive integer or positive half integer unless  $\nu$  is also a suitable irrational number. Hence, in this case the poles of the integrand will be simple and the G-function can be easily evaluated as in the Case (i) of Sect. 4.3 of Haubold and Mathai (1984).

**Case 2.4** The charged particle case with modified Maxwell-Boltzmann distribution function

If there occurs a cut off of the high energy tail of the Maxwell-Boltzmann distribution function given in (2.10) we expect that the closed-form representation of the appropriate quantity  $\langle \sigma v \rangle$ , given by

$$N_{i_\nu}(z;d) = \int_0^d dy y^\nu e^{-y-zy^{-\frac{1}{2}}}, \quad (2.41)$$

will not lead to the type  $G_{0,p}^{p,0}(\cdot)$  of Meijer's G-function as we have had in the cases in (2.37) and (2.40), respectively. For the discussion of physical reasons for the cut off modification of the Maxwell-Boltzmann distribution function of the relative kinetic energy of the reacting particles, we refer to the papers of Haubold and John (1982) and Haubold and Mathai (1986a).

Again, we consider a more general form of the integral in (2.41) as

$$N_3(z;d, a, \rho, n, m) = \int_0^d dt t^{-n\rho} e^{-at-zt^{-\frac{n}{m}}}, \quad (2.42)$$

where  $a > 0, d > 0, z > 0$ . This can be evaluated by working out the density of a product of two independent real random variables by using two different techniques. For the detailed discussion of the mathematical methods to tackle integrals of the types as given in (2.42), we refer to Haubold and Mathai (1984, 1986a). The results contain the following theorem:

**Theorem 2.2** For  $z > 0, d > 0, a > 0, m$  and  $n$  positive integers,

$$\begin{aligned} N_3(z;d, a, \rho, n, m) &= \int_0^d dt t^{-n\rho} e^{-at-zt^{-\frac{n}{m}}} = \frac{m^{\frac{1}{2}}}{n} (2\pi)^{\frac{1-m}{2}} d^{-n\rho+1} \\ &\times \sum_{r=0}^{\infty} \frac{(-ad)^r}{r!} G_{n,m+n}^{m+n,0} \left[ \frac{z^m}{d^n m^m} \middle| \begin{matrix} -\rho + \frac{r+2}{n} + \frac{j-1}{n}, j=1, \dots, n \\ -\rho + \frac{r+1}{n} + \frac{j-1}{n}, j=1, \dots, n, \frac{j-1}{m}, j=1, \dots, m \end{matrix} \right] \end{aligned}$$

To obtain the special case realized in (2.41) we put  $n = 1, m = 2, a = 1, \rho = \nu$ , we get

$$N_3(z;d, 1, \nu, 1, 2) = \frac{1}{\pi^{\frac{1}{2}}} d^{-\nu+1} \sum_{r=0}^{\infty} \frac{(-d)^r}{r!} G_{1,3}^{3,0} \left[ \frac{z^2}{4d} \middle| \begin{matrix} -\nu+r+2 \\ -\nu+r+1, 0, \frac{1}{2} \end{matrix} \right]. \quad (2.43)$$

For computable series representation of  $N_{i_\nu}(z;d)$  in (2.43) for all classes of the parameters  $-\nu + r + 1$  see Sect. 4.4 of Haubold and Mathai (1986a).

## 2.9 Resonant Reaction Rates

Here also we will consider various situations.

**Case 2.5** The charged particle case with Maxwell-Boltzmann distribution function

We put (2.26) into (2.28) and obtain the representation of  $\langle \sigma v \rangle$  for resonant thermonuclear reactions (2.1):

$$\begin{aligned} \langle \sigma v \rangle &= (2\pi)^{\frac{5}{2}} \frac{Z_1 Z_2 e^2 R_0 \omega \Gamma_{12} D}{\mu^{\frac{1}{2}} (kT)^{\frac{3}{2}}} \int_0^\infty dE \exp \left\{ -\frac{E}{kT} - \frac{\bar{q}}{E^{\frac{1}{2}}} \right\} \\ &\quad \times \left[ (E_r - E)^2 + \left( \frac{1}{2}(\Gamma_0 + \Gamma_1 E_1) \right)^2 \right]^{-1} \end{aligned} \quad (2.44)$$

where  $\bar{q}$  is given by

$$\bar{q} = 2\pi \left( \frac{\mu}{2} \right)^{\frac{1}{2}} \frac{Z_1 Z_2 e^2}{h} = z(kT)^{\frac{1}{2}}, \quad (2.45)$$

where  $z$  is given in (2.34). From (2.44) we remove the integral

$$R = \int_0^\infty dE \exp \left\{ -\frac{E}{kT} - \frac{\bar{q}}{E^{\frac{1}{2}}} \right\} \left[ (E_r - E)^2 + \left( \frac{1}{2}(\Gamma_0 + \Gamma_1 E) \right)^2 \right]^{-1} \quad (2.46)$$

which may be written more conveniently as,

$$R = \left[ 1 + \left( \frac{\Gamma_1}{2} \right)^2 \right]^{-1} \int_0^\infty dE \exp \left\{ -\frac{E}{kT} - \frac{\bar{q}}{E^{\frac{1}{2}}} \right\} \left[ (\tilde{E}_r - E)^2 + \left( \frac{\tilde{\Gamma}}{2} \right)^2 \right]^{-1} \quad (2.47)$$

where  $\tilde{E}_r$  denotes a modified resonance energy,

$$\tilde{E}_r = \left[ E_r - \frac{\Gamma_0 \Gamma_1}{4} \right] \left[ 1 + \left( \frac{\Gamma_1}{2} \right)^2 \right]^{-1}, \quad (2.48)$$

with  $\tilde{\Gamma}$  as the modified total width,

$$\tilde{\Gamma} = [\Gamma_0 + E_r \Gamma_1] \left[ 1 + \left( \frac{\Gamma_1}{2} \right)^2 \right]^{-1}. \quad (2.49)$$

The form of the resonance denominator with  $\Gamma_0$  is conserved if one transforms  $\Gamma_0 \rightarrow \Gamma_0 + E \Gamma_1$ . Choosing the variable  $E = y[1 + (\frac{\Gamma_1}{2})^2]^{-\frac{1}{2}}$  leads to

$$\begin{aligned} R &= \int_0^\infty dy \exp \left\{ -y[kT(1 + (\Gamma_1/2)^2)]^{-1} - y^{-\frac{1}{2}} \bar{q} (1 + (\Gamma_1/2)^2)^{\frac{1}{2}} \right\} \\ &\quad \times [(E_r - y - \Gamma_0 \Gamma_1/4)^2 + ((\Gamma_0 + E_r \Gamma_1)/2)^2]^{-1}. \end{aligned}$$

With the notations

$$\begin{aligned} a &= [kT(1 + (\Gamma_1/2)^2)]^{-1}, \quad b = E_r - \frac{1}{4} \Gamma_0 \Gamma_1, \\ g &= \frac{1}{2}(\Gamma_0 + E_r \Gamma_1), \quad q = \bar{q}(1 + (\Gamma_1/2)^2)^{\frac{1}{2}}, \end{aligned}$$

we obtain

$$R(q, a, b, g) = \int_0^\infty dy \frac{\exp\{-ay - qy^{-\frac{1}{2}}\}}{(b-y)^2 + g^2}$$

and (2.44) can be written as,

$$\langle \sigma v \rangle = (2\pi)^{\frac{5}{2}} \frac{Z_1 Z_2 e^2 R_0 \omega \Gamma_{12} D}{\mu^{\frac{1}{2}} (kT)^{\frac{3}{2}}} [1 + (\Gamma_1/2)^2]^{-1} R(q, a, b, g)$$

(see Haubold and John 1979; Haubold and Mathai 1986b).

In order to comprehend cases in which the cross section factor or the partial width are additionally multiplied by energy-depending factors we consider the more general integral

$$R_1(q, a, b, g; \nu, n, m) = \int_0^\infty dt \frac{\exp\{-at - qt^{-\frac{n}{m}}\}}{(b-t)^2 + g^2}. \quad (2.50)$$

We may replace the denominator  $[(b-t)^2 + g^2]^{-1}$  in (2.50) by an equivalent integral for  $g^2 > 0$ . That is,

$$\frac{1}{(b-t)^2 + g^2} = \int_0^\infty dx \exp\{-[(b-t)^2 + g^2]x\}. \quad (2.51)$$

But, we can write

$$\begin{aligned} e^{-x(b-t)^2} &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k (b-t)^{2k} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k \sum_{k_1=0}^{2k} \binom{2k}{k_1} (-1)^{k_1} b^{2k-k_1} t^{k_1}, \end{aligned} \quad (2.52)$$

where, for example,  $\binom{m}{n} = \frac{m!}{n!(m-n)!}$ ,  $0! = 1$ . From (2.50) and (2.51) and (2.52) we have,

$$\begin{aligned} R_1(q, a, b, g; \nu, n, m) &= \int_0^\infty dx e^{-g^2 x} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k \sum_{k_1=0}^{2k} \binom{2k}{k_1} (-1)^{k_1} b^{2k-k_1} \\ &\quad \times \int_0^\infty dt t^{\nu+k_1} e^{-at - qt^{-\frac{n}{m}}}. \end{aligned} \quad (2.53)$$

Note that

$$\int_0^\infty dx x^k e^{-g^2 x} = \int_0^\infty dx x^{(k+1)-1} e^{-g^2 x} = \Gamma(k+1) (g^2)^{-(k+1)} = \frac{k!}{g^2 (g^2)^k}, \quad (2.54)$$

and according to Theorem 2.1, we have

$$(2.55)$$

$$\int_0^\infty dt t^{\nu+k_1} e^{-at-qt^{-\frac{n}{m}}} = a^{-(\nu+k_1+1)} (2\pi)^{\frac{1}{2}(2-n-m)} m^{\frac{1}{2}} n^{\frac{1}{2}+\nu+k_1} \\ \times G_{0,m+n}^{m+n,0} \left[ \frac{q^m a^n}{m^m n^n} \Big|_{0,\frac{1}{m},\dots,\frac{m-1}{m},\frac{1+\nu+k_1}{n},\dots,\frac{n+\nu+k_1}{n}} \right]$$

(cp. also Mathai [1971](#); Saxena [1960](#); Haubold and Mathai [1984](#)). Substituting (2.54) and (2.55) in (2.53) one has the following:

$$R_1(q, a, b, g; \nu, n, m) = \sum_{k=0}^{\infty} \frac{(-1)^k}{g^2 (g^2)^k} \sum_{k_1=0}^{2k} \binom{2k}{k_1} (-1)^{k_1} b^{2k-k_1} \\ \times a^{-(\nu+k_1+1)} (2\pi)^{\frac{1}{2}(2-n-m)} m^{\frac{1}{2}} n^{\frac{1}{2}+\nu+k_1} \\ \times G_{0,m+n}^{m+n,0} \left[ \frac{q^m a^n}{m^m n^n} \Big|_{0,\frac{1}{m},\dots,\frac{m-1}{m},\frac{1+\nu+k_1}{n},\dots,\frac{n+\nu+k_1}{n}} \right]. \quad (2.56)$$

Put  $n = 1, m = 2, \nu = 0$ , to get  $R(q, a, b, g)$ .

$$R_1(q, a, b, g; 0, 1, 2) = R(q, a, b, g) = \frac{1}{g^2 a \pi^{\frac{1}{2}}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(g^2)^k} \\ \times \sum_{k_1=0}^{2k} \binom{2k}{k_1} \frac{(-1)^{k_1}}{a^{k_1}} b^{2k-k_1} G_{0,3}^{3,0} \left[ \frac{q^2 a}{4} \Big|_{0,\frac{1}{2},1+k_1} \right] \quad (2.57)$$

for  $(b - v/a)^2/g^2 < 1$ , where  $v = (q^2 a/4)^{\frac{1}{3}}$  (Haubold and Mathai [1986b](#)).

Comparing the resonant result (2.57) with the nonresonant case (2.37) we observe that the former is an infinite sum over nonresonant contributions (Note that  $q^2 a = z^2$ ). The appearance of the G-function of the type  $G_{0,p}^{p,0}(\cdot)$  in (2.57) is due to the Maxwell-Boltzmannian approach to the resonant thermonuclear reaction rate. For the numerical computation of (2.57) all results obtained in Sect. 4.3 (Haubold and Mathai [1984](#)) are applicable.

**Case 2.6** The neutral particle case with Maxwell-Boltzmann distribution function

Considering neutron reactions, i.e.,  $q = 0$ , we are dealing with energy-independent partial width. In the case  $q = 0$ , this also means  $\Gamma_1 = 0$ , for (2.53) we obtain

$$R_1(0, \bar{a}, \bar{b}, \bar{g}; 0, 0, 0) = \int_0^\infty dx e^{-\bar{g}^2 x} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k \\ \times \sum_{k_1=0}^{2k} \binom{2k}{k_1} (-1)^{k_1} \bar{b}^{2k-k_1} \int_0^\infty dt t^{\nu+k_1} e^{-\bar{a}t}, \quad (2.58)$$

where

$$\bar{a} = \frac{1}{kT}, \quad \bar{b} = E_r, \quad \bar{g} = \frac{1}{2} \Gamma_0. \quad (2.59)$$

Taking into account (2.53) we get for (2.58)

$$R_1(0, \bar{a}, \bar{b}, \bar{g}; 0, 0, 0) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\bar{g}^2 (\bar{g}^2)^k} \sum_{k_1=0}^{2k} \binom{2k}{k_1} (-1)^{k_1} \bar{b}^{2k-k_1} \frac{(\nu+k_1)!}{\bar{a}(\bar{a})^{\nu+k_1}}. \quad (2.60)$$

With (2.60), the closed-form representation of the resonant thermonuclear reaction rate for neutron reactions is obtained.

**Case 2.7** The charged particle case with modified Maxwell-Boltzmann distribution function

In the following, we consider a modification of the Maxwell-Boltzmann distribution for resonant nuclear reactions. Instead of (2.50) we write

$$R_2(q, a, b, c, g; \nu, \delta, n, m) = \int_0^\infty dt t^\nu \frac{\exp\{-at-at^\delta-qt^{-\frac{n}{m}}\}}{(b-t)^2+g^2}. \quad (2.61)$$

Replacing the denominator of the kernel of the integral (2.61) as in the case (2.51) we obtain the following, denoting the sequence of parameters  $q, a, b, c, g; \nu, \delta, n, m$  by  $W$ , that is,  $W = \{q, a, b, c, g; \nu, \delta, n, m\}$ :

$$R_2(W) = \int_0^\infty dx e^{-g^2 x} \int_0^\infty dt t^\nu e^{-at-qt^{-\frac{n}{m}}-ct^\delta-x(b-t)^2}. \quad (2.62)$$

But

$$\begin{aligned} e^{-(ct^\delta+x(b-t)^2)} &= \sum_{k=0}^{\infty} (-1)^k \frac{[ct^\delta+x(b-t)^2]^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sum_{k_1=0}^k \binom{k}{k_1} c^{k-k_1} t^{\delta(k-k_1)} x^{k_1} (b-t)^{2k_1} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sum_{k_1=0}^k \binom{k}{k_1} c^{k-k_1} x^{k_1} \sum_{k_2=0}^{2k_1} \binom{2k_1}{k_2} \\ &\quad \times b^{2k_1-k_2} (-1)^{k_2} t^{k_2+\delta(k-k_1)}. \end{aligned} \quad (2.63)$$

Hence,

$$\begin{aligned} R_2(W) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sum_{k_1=0}^k \binom{k}{k_1} c^{k-k_1} \sum_{k_2=0}^{2k_1} \binom{2k_1}{k_2} b^{2k_1-k_2} (-1)^{k_2} \\ &\quad \times \int_0^\infty dx x^{k_1} e^{-g^2 x} \int_0^\infty dt t^{\nu+k_2+\delta(k-k_1)} e^{-at-qt^{-\frac{n}{m}}}, \end{aligned} \quad (2.64)$$

and due to the Theorem 2.1 we have

$$\begin{aligned} R_2(W) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sum_{k_1=0}^k \binom{k}{k_1} g^{-2(k_1+1)} k_1! c^{k-k_1} \sum_{k_2=0}^{2k_1} \binom{2k_1}{k_2} b^{2k_1-k_2} (-1)^{k_2} \\ &\quad \times a^{-(\nu+k_2+\delta(k-k_1)+1)} m^{\frac{1}{2}} n^{\nu+k_2+\delta(k-k_1)+\frac{1}{2}} \\ &\quad \times G_{0,m+n}^{m+n,0} \left[ \frac{q^m a^n}{m^m n^n} \mid_{0, \frac{1}{m}, \dots, \frac{m-1}{m}, \frac{1+\nu+k_2+\delta(k-k_1)}{n}, \dots, \frac{n+\nu+k_2+\delta(k-k_1)}{n}} \right]. \end{aligned}$$

That is,

(2.65)

$$\begin{aligned}
R_2(W) &= (2\pi)^{\frac{1}{2}(2-n-m)} m^{\frac{1}{2}} n^{\nu+\frac{1}{2}} a^{-(\nu+1)} g^{-1} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sum_{k_1=0}^k \binom{k}{k_1} g^{-k_1} k_1! c^{k-k_1} \\
&\quad \times \sum_{k_2=0}^{2k_1} \binom{2k_1}{k_2} b^{2k_1-k_2} (-1)^{k_2} a^{-(k_2+\delta(k-k_1))} n^{k_2+\delta(k-k_1)} \\
&\quad \times G_{0,m+n}^{m+n,0} \left[ \frac{q^m a^n}{m^m n^n} \Big|_{0, \frac{1}{m}, \dots, \frac{m-1}{m}, \frac{1+\nu+k_2+\delta(k-k_1)}{n}, \dots, \frac{n+\nu+k_2+\delta(k-k_1)}{n}} \right].
\end{aligned}$$

for  $|(b - \frac{n}{a} u^{\frac{1}{m+n}})^2/g^2| < 1$ , where  $u = q^m a^n / (m^m n^n)$ . Putting  $n = 1, m = 2, \nu = 0$  we obtain

$$\begin{aligned}
R_2(q, a, b, c, g; 0, \delta, 1, 2) &= \frac{1}{\pi^{\frac{1}{2}} a g} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sum_{k_1=0}^k \binom{k}{k_1} g^{-k_1} k_1! c^{k-k_1} \\
&\quad \times \sum_{k_2=0}^{2k_1} \binom{2k_1}{k_2} b^{2k_1-k_2} (-1)^{k_2} a^{-(k_2+\delta(k-k_1))} \\
&\quad \times G_{0,3}^{3,0} \left[ \frac{q^2 a}{4} \Big|_{0, \frac{1}{2}, 1+2+\delta(k-k_1)} \right],
\end{aligned} \tag{2.66}$$

for  $|(b - (q/2a)^{\frac{4}{3}})/g^2| < 1$ . Series representations for the numerical computation of the G-function contained in (2.66) are given in Sect. 2.10.

## 2.10 Series Representations for the Thermonuclear Functions: $G_{0,3}^{3,0}(\cdot)$

Now, we are going back to the collision probability integral containing as a part of the kernel a Maxwell-Boltzmann distribution function term. Consider the case  $m = 2, n = 1, a = 1$ , and  $\rho = -\nu$  in Eq. (2.35). Then, the closed-form representation of the thermonuclear reaction rate integral contains the Meijer's G-function of the following type: see for example (2.37),

$$\pi^{-\frac{1}{2}} G_{0,3}^{3,0} \left[ \frac{z^2}{4} \Big|_{0, \frac{1}{2}, 1+\nu} \right]. \tag{2.67}$$

In the following, we shall derive representations of (2.67) which will be suitable for the numerical evaluation. In the light of the results obtained by Critchfield (1972) we refer to (2.67) as the thermonuclear functions occurring in Eqs. (2.37), (2.40), (2.57) and (2.66), respectively.

**Case 2.8**  $\nu \neq \pm \frac{\lambda}{2}, \lambda = 0, 1, 2, \dots$

Then the poles of the Mellin-Barnes integral representation of (2.67) are simple and then the G-function has a simple series expansion. Consider the evaluation of the G-function in this case (Mathai and Saxena 1973):

$$G_{0,3}^{3,0} \left[ \frac{z^2}{4} \Big|_{0, \frac{1}{2}, 1+\nu} \right] = \frac{1}{2\pi i} \int ds \Gamma(s) \Gamma\left(\frac{1}{2} + s\right) \Gamma(1 + \nu + s) \left(\frac{z^2}{4}\right)^{-s}.$$

The poles are at  $s = 0, -1, -2, \dots; s = -\frac{1}{2}, \frac{1}{2} - 1, \dots; s = -1 - \nu, -1 - \nu - 1, \dots$ . The sum of the residues corresponding to the poles  $s = 0, -1, -2, \dots$ , denoted by  $S_1$ , is the following:

$$\begin{aligned} S_1 &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \Gamma\left(\frac{1}{2} - r\right) \Gamma(1 + \nu - r) \left(\frac{z^2}{4}\right)^r \\ &= \Gamma\left(\frac{1}{2}\right) \Gamma(1 + \nu) \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{1}{(1/2)_r (-\nu)_r} \left(\frac{z^2}{4}\right)^r \\ &= \Gamma\left(\frac{1}{2}\right) \Gamma(1 + \nu) {}_0F_2\left(-; \frac{1}{2}, -\nu; -\frac{z^2}{4}\right), \end{aligned} \quad (2.68)$$

where  ${}_pF_q(\cdot)$  denotes the generalized hypergeometric function. The sum of the residues corresponding to the poles  $s = -\frac{1}{2}, -\frac{1}{2} - 1, \dots$ , denoted by  $S_2$ , is the following:

$$\begin{aligned} S_2 &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \Gamma\left(-\frac{1}{2} - r\right) \Gamma\left(\frac{1}{2} + \nu - r\right) \left(\frac{z^2}{4}\right)^{\frac{1}{2}+r} \\ &= \Gamma\left(-\frac{1}{2}\right) \Gamma\left(\frac{1}{2} + \nu\right) \left(\frac{z^2}{4}\right)^{\frac{1}{2}} {}_0F_2\left(-; \frac{3}{2}, \frac{1}{2} - \nu; -\frac{z^2}{4}\right). \end{aligned} \quad (2.69)$$

Further on the sum of the residues corresponding to the poles  $s = -1 - \nu, -1 - \nu - 1, \dots$ , denoted by  $S_3$ , is the following:

$$\begin{aligned} S_3 &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \Gamma(-1 - \nu - r) \Gamma\left(-\frac{1}{2} - \nu - r\right) \left(\frac{z^2}{4}\right)^{1+\nu+r} \\ &= \Gamma(-1 - \nu) \Gamma\left(-\frac{1}{2} - \nu\right) \left(\frac{z^2}{4}\right)^{1+\nu} {}_0F_2\left(-; \nu + 2, \nu + \frac{3}{2}; -\frac{z^2}{4}\right). \end{aligned} \quad (2.70)$$

Now we have the following result:

**Theorem 2.3** From  $S_1, S_2$ , and  $S_3$  of (2.68), (2.69) and (2.70), respectively, we have

$$G_{0,3}^{3,0} \left[ \frac{z^2}{4} \Big|_{0, \frac{1}{2}, 1+\nu} \right] = S_1 + S_2 + S_3$$

for  $\nu \neq \pm \frac{\lambda}{2}, \lambda = 0, 1, 2, \dots$

Note that the series  $S_1, S_2$  and  $S_3$  are termwise integrable over any finite range and hence computations can be carried out by using these  ${}_0F_2(\cdot)$  functions (cf. Mathai and Saxena 1973).

**Case 2.9**  $\nu$  a positive integer

In this case the poles of the gammas in the integral representation of (2.67) are the following: The poles of  $\Gamma(s)$  are at  $s = 0, -1, -2, \dots, -\nu, -\nu - 1, \dots$ . The poles of  $\Gamma(1 + \nu + s)$  are at  $s = -\nu - 1, -\nu - 2, \dots$ . The poles of  $\Gamma(\frac{1}{2} + s)$  are at  $s = -\frac{1}{2}, -\frac{1}{2} - 1, \dots$ . Note that the poles at  $= -\frac{1}{2}, -\frac{1}{2} - 1, \dots$  and at  $s = 0, -1, \dots, -\nu$  are of order one each and the poles at  $s = -1 - \nu, -1 - \nu - 1, \dots$  are of order two each.

Again, for convenience, the sum of the residues will be denoted by  $S_j$ ,  $j = 1, 2, 3$ . The sum of the residues corresponding to the poles  $s = -\frac{1}{2}, -\frac{1}{2} - 1, \dots$ , denoted by  $S_1$ , is the following, by following through the same procedure as in the earlier case:

$$S_1 = \Gamma\left(-\frac{1}{2}\right)\Gamma\left(\frac{1}{2} + \nu\right)\left(\frac{z^2}{4}\right)^{\frac{1}{2}} {}_0F_2\left(-; \frac{3}{2}, \frac{1}{2} - \nu; -\frac{z^2}{4}\right). \quad (2.71)$$

For the sum of the residues corresponding to the poles at  $s = 0, -1, \dots, -\nu$ , denoted by  $S_2$ , we get the following:

$$S_2 = \Gamma(1 + \nu)\Gamma\left(\frac{1}{2}\right) \sum_{r=0}^{\nu} \frac{1}{(1/2)_r (-\nu)_r r!} \left(\frac{z^2}{4}\right)^r. \quad (2.72)$$

For poles of order two, we will have to use the general formula for evaluating the residues when the poles are of higher orders, and in the present case, the integrand has to be differentiated once. Let the integrand of the G-function in (2.67) be denoted by  $\Delta(s)$ . That is,

$$\Delta(s) = \Gamma(s)\Gamma(1 + \nu + s)\Gamma\left(\frac{1}{2} + s\right)\left(\frac{z^2}{4}\right)^{-s}.$$

Then, the sum of the residues corresponding to the poles  $s = -\nu - 1, \nu - 2, \dots$ , we have the following, denoted by  $S_3$ :

$$\begin{aligned} S_3 &= \sum_{r=0}^{\infty} \lim_{s \rightarrow -1-\nu-r} \frac{\partial}{\partial s} \{(s+1+\nu+r)^2 \Delta(s)\} \\ &= \sum_{r=0}^{\infty} \left(\frac{z^2}{4}\right)^{1+\nu+r} \lim_{s \rightarrow -1-\nu-r} \left\{ -\ln\left(\frac{z^2}{4}\right) + \frac{\partial}{\partial s} \right\} B(s) \\ B(s) &= \{(s+1+\nu+r)^2 \Gamma(1+\nu+s) \Gamma(s) \Gamma(s+1/2)\}. \end{aligned}$$

But, by inserting the factors  $(s+1+\nu+r-1)^2 \cdots (s+1+\nu)^2 (s+\nu) (s+\nu-1) \cdots s$  in the numerator and in the denominator, we can write  $B(s)$  as the following:

$$\begin{aligned} B(s) &= \frac{(s+1+\nu+r)^2 (s+\nu+r)^2 \cdots (s+1+\nu)^2 (s+\nu) \cdots s}{(s+\nu+r)^2 \cdots (s+1+\nu)^2 (s+\nu) \cdots s} \\ &\quad \times \Gamma(1+\nu+r+s) \Gamma(s) \Gamma\left(\frac{1}{2} + s\right) \\ &= \frac{\Gamma^2(s+r+\nu+2) \Gamma\left(\frac{1}{2} + s\right)}{(s+r+\nu)^2 \cdots (s+\nu+1)^2 (s+\nu) \cdots s} \end{aligned}$$

Now, by taking the limit we have the following, denoting the resulting quantity as  $B_r$ :

$$\begin{aligned} B_r &= \lim_{s \rightarrow -1-\nu-r} B(s) = \frac{\Gamma^2(1) \Gamma\left(\frac{1}{2} - 1 - \nu - r\right)}{(-1)^2 \cdots (-r)^2 (-r-1) \cdots (-r-\nu-1)} \\ &= \frac{(-1)^{1+\nu+2r} \Gamma^2(1) \Gamma\left(-\frac{1}{2} - \nu - r\right)}{r!(r+\nu+1)!} = \frac{(-1)^{1+\nu+r} \Gamma\left(-\frac{1}{2} - \nu\right)}{r!(\nu+r+1)! \left(\frac{3}{2} + \nu\right)_r}. \end{aligned} \quad (2.73)$$

Let

$$\begin{aligned}
A(s) &= \frac{\partial}{\partial s} \ln B(s) \\
&= 2\psi(s+r+\nu+2) + \psi\left(\frac{1}{2}+s\right) - \frac{2}{s+r+\nu} - \frac{2}{s+r+\nu-1} - \dots \\
&\quad - \frac{2}{s+\nu+1} - \frac{1}{s+\nu} - \dots - \frac{1}{s}.
\end{aligned}$$

Consider the limit and let

$$\begin{aligned}
A_r &= \lim_{s \rightarrow -1-r-\nu} A(s) \\
&= 2\psi(1) + \psi\left(-\frac{1}{2}-\nu-r\right) + \frac{2}{1} + \frac{2}{2} + \dots + \frac{2}{r} + \frac{1}{r+1} + \dots + \frac{1}{r+\nu+1} \quad (2.74) \\
&= \psi(r+1) + \psi(r+\nu+2) + \psi\left(-\frac{1}{2}-\nu-r\right),
\end{aligned}$$

where  $\psi(z)$  is the psi function or digamma function (cf. Mathai and Saxena 1973). Hence, for the sum of the residues corresponding to the poles  $s = -1 - \nu, -1 - \nu - 1, \dots$ , we obtain

$$S_3 = \sum_{r=0}^{\infty} \left(\frac{z^2}{4}\right)^{1+\nu+r} \left\{ -\ln\left(\frac{z^2}{4}\right) + A_r \right\} B_r, \quad (2.75)$$

where  $A_r$  and  $B_r$  are defined in (2.74) and (2.73) respectively. Hence, we have the following result:

**Theorem 2.4** For  $\nu$  a positive integer,

$$\begin{aligned}
\pi^{-\frac{1}{2}} G_{0,3}^{3,0} \left[ \frac{z^2}{4} \Big|_{0, \frac{1}{2}, 1+\nu} \right] &= \Gamma(1+\nu) \sum_{r=0}^{\nu} \frac{1}{(1/2)_r (-\nu)_r r!} \left(-\frac{z^2}{4}\right)^r \\
&\quad - 2\Gamma\left(\frac{1}{2}+\nu\right) \left(\frac{z^2}{4}\right)^{\frac{1}{2}} {}_0F_2\left(-; \frac{3}{2}, \frac{1}{2}-\nu; -\frac{z^2}{4}\right) \\
&\quad + \pi^{-\frac{1}{2}} \left(\frac{z^2}{4}\right)^{1+\nu} \sum_{r=0}^{\infty} \left(\frac{z^2}{4}\right)^r \left\{ -\ln\left(\frac{z^2}{4}\right) + A_r \right\} B_r,
\end{aligned}$$

where  $B_r$  and  $A_r$  are defined in (2.73) and (2.74) respectively.

**Case 2.10**  $\nu$  a negative integer

Let  $\nu = -\mu, \mu = 2, 3, \dots$ . In this case, the poles of the G-function in (4.41) are  $s = -\frac{1}{2}, -\frac{1}{2} - 1, \dots$  of order 1 each;  $s = -1 - \nu, -1 - \nu - 1, \dots, 1$  or  $s = \mu - 1, \mu - 2, \dots, 1$  are of order one each, and the poles  $s = 0, -1, -2, \dots$  are of order two each. Let us again denote the sum of the residues by  $S_j, j = 1, 2, 3$ . The sum of the residues corresponding to the poles  $s = -\frac{1}{2}, -\frac{1}{2} - 1, \dots$  remains the same as before and it is equal to the following:

$$S_1 = \Gamma\left(-\frac{1}{2}\right) \Gamma\left(\frac{1}{2} + \nu\right) \left(\frac{z^2}{4}\right)^{\frac{1}{2}} {}_0F_2\left(-; \frac{3}{2}, \frac{1}{2}-\nu; -\frac{z^2}{4}\right). \quad (2.76)$$

The other set of poles of order 1 each are  $s = -1 - \nu - r, r = 0, 1, \dots, -\nu - 2$  and the sum of the residues, denoted by  $S_2$ , is the following:

(2.77)

$$\begin{aligned}
S_2 &= \sum_{r=0}^{-\nu-2} \lim_{s \rightarrow -1-\nu-r} (s+1+\nu+r)\Gamma(s+\nu+1)\Gamma(s)\Gamma\left(\frac{1}{2}+s\right)\left(\frac{z^2}{4}\right)^{-s} \\
&= \left(\frac{z^2}{4}\right)^{1+\nu} \sum_{r=0}^{-\nu-2} \left(\frac{z^2}{4}\right)^r \frac{(-1)^r}{r!} \Gamma(-\nu-1-r)\Gamma\left(-\frac{1}{2}-\nu-r\right) \\
&= \left(\frac{z^2}{4}\right)^{1+\nu} \Gamma(-\nu-1)\Gamma\left(-\frac{1}{2}-\nu\right) \sum_{r=0}^{-\nu-2} \frac{(-1)^r}{r!} \left(\frac{z^2}{4}\right)^r \frac{1}{(\nu+2)_r(\nu+3/2)_r}.
\end{aligned}$$

Poles of order 2 are at  $s = -r, r = 0, 1, 2, \dots$ . Consider the relation,

$$(s+r)^2 = \frac{(s+r)^2(s+r-1)^2 \cdots (s)^2(s-1) \cdots (s-\nu+1)}{(s+r-1)^2 \cdots s^2(s-1) \cdots (s-\nu+1)}.$$

Then

$$(s+r)^2\Gamma(s)\Gamma(s+\nu+1) = \frac{\Gamma^2(s+r+1)}{(s+r-1)^2 \cdots s^2(s-1) \cdots (s-\nu+1)}.$$

Let

$$\begin{aligned}
B'_r &= \lim_{s \rightarrow -r} B'(s) = \lim_{s \rightarrow -r} [(s+r)^2\Gamma(s)\Gamma(s+\nu+1)\Gamma(1+s)] \\
&= \frac{\Gamma^2(1)\Gamma\left(\frac{1}{2}-r\right)}{(-1)^2(-2)^2 \cdots (-r)^2(-r-1) \cdots (-r+\nu+1)} \\
&= \frac{\Gamma^2(1)\Gamma\left(\frac{1}{2}-r\right)(-1)^{\nu+1}}{r!(r-\nu-1)!}.
\end{aligned} \tag{2.78}$$

Let

$$\begin{aligned}
A'_r &= \lim_{s \rightarrow -r} A'(s) = \lim_{s \rightarrow -r} \left\{ \frac{\partial}{\partial s} \ln B'(s) \right\} \\
&= \psi(r+1) + \psi(r-\nu) + \psi\left(\frac{1}{2}-r\right).
\end{aligned} \tag{2.79}$$

Thus, the sum of the residues corresponding to the poles  $s = -r, r = 0, 1, \dots$ , by using steps similar to the ones employed before, and denoted by  $S_3$ , is the following:

$$S_3 = \sum_0^{\infty} \left(\frac{z^2}{4}\right)^r \left\{ -\ln \left(\frac{z^2}{4}\right) + A'_r \right\} B'_r, \tag{2.80}$$

where  $B'_r$  and  $A'_r$  are defined in (2.78) and (2.79) respectively. With this, we have the following theorem:

**Theorem 2.5** For  $\nu$  a negative integer,

$$\begin{aligned}
\pi^{-\frac{1}{2}}(S_1 + S_2 + S_3) &= \pi^{-\frac{1}{2}} G_{0,3}^{3,0} \left[ \frac{z^2}{4} \Big|_{0, \frac{1}{2}, 1+\nu} \right] \\
&= \pi^{-\frac{1}{2}} \left( \frac{z^2}{4} \right)^{1+\nu} \Gamma(-\nu - 1) \Gamma\left(-\frac{1}{2} - \nu\right) \\
&\quad \times \sum_{r=0}^{-\nu-2} \frac{(-1)^r}{r!} \left( \frac{z^2}{4} \right)^r \frac{1}{(\nu+2)_r (\nu+3/2)_r} \\
&\quad - 2\Gamma\left(\frac{1}{2} + \nu\right) \left( \frac{z^2}{4} \right)^{\frac{1}{2}} {}_0F_2 \left( -; \frac{3}{2}, \frac{1}{2} - \nu; - \frac{z^2}{4} \right) \\
&\quad + \pi^{-\frac{1}{2}} \sum_{r=0}^{\infty} \left( \frac{z^2}{4} \right)^r \left\{ -\ln\left(\frac{z^2}{4}\right) + A'_r \right\} B'_r,
\end{aligned}$$

where  $B'_r$  and  $A'_r$  are defined in (2.78) and (2.79), respectively.

**Case 2.11**  $\nu$  a positive half integer

Let  $\nu = m + \frac{1}{2}$ ,  $m = 0, 1, \dots$ . Then

$$\Gamma(s)\Gamma(s+\nu+1)\Gamma\left(\frac{1}{2}+s\right) = \Gamma(s)\Gamma\left(s+m+\frac{3}{2}\right)\Gamma\left(s+\frac{1}{2}\right)$$

and the poles of the Meijer's G-function in (2.67) are the following:  $s = -r$ ,  $r = 0, 1, \dots$  are of order 1 each;  $s = -\frac{1}{2} - r$ ,  $r = 0, 1, \dots, m$  are of order 1 each;  $s = -m - \frac{3}{2} - r$ ,  $r = 0, 1, \dots$  are of order 2 each. The sum of the residues corresponding to the poles  $s = -r$ ,  $r = 0, 1, \dots$  is the following, denoted again by  $S_1$ :

$$\begin{aligned}
S_1 &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \left( \frac{z^2}{4} \right)^r \Gamma\left(\frac{1}{2} - r\right) \Gamma\left(m + \frac{3}{2} - r\right) \\
&= \Gamma\left(\frac{1}{2}\right) \Gamma\left(m + \frac{3}{2}\right) \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{1}{(1/2)_r (-m-1/2)_r} \left( \frac{z^2}{4} \right)^r \\
&= \Gamma\left(\frac{1}{2}\right) \Gamma\left(m + \frac{3}{2}\right) {}_0F_2 \left( -; \frac{1}{2}, -m - \frac{1}{2}; - \frac{z^2}{4} \right).
\end{aligned} \tag{2.81}$$

Evidently, the sum of the residues corresponding to the poles  $s = -\frac{1}{2} - r$ ,  $r = 0, 1, \dots, m$ , denoted by  $S_2$ , is the following:

$$\begin{aligned}
S_2 &= \sum_{r=0}^m \frac{(-1)^r}{r!} \left( \frac{z^2}{4} \right)^{\frac{1}{2}+r} \Gamma\left(-\frac{1}{2} - r\right) \Gamma(1 + m - r) \\
&= \Gamma\left(-\frac{1}{2}\right) \Gamma(1 + m) \left( \frac{z^2}{4} \right)^{\frac{1}{2}} \sum_{r=0}^m \frac{(-1)^r}{r!} \left( \frac{z^2}{4} \right)^r \frac{1}{(3/2)_r (-m)_r}.
\end{aligned} \tag{2.82}$$

Let

$$C(s) = \left( s + m + \frac{3}{2} + r \right)^2 \Gamma\left(\frac{1}{2} + s\right) \Gamma\left(\frac{3}{2} + m + s\right) \Gamma(s).$$

As done in the previous cases, insert the following factors in the numerator and in the denominator.

$$\left(s + m + \frac{3}{2} + r - 1\right)^2 \cdots \left(s + m + \frac{3}{2}\right)^2 \left(s + m + \frac{3}{2} - 1\right) \cdots \left(s + \frac{1}{2}\right)$$

and write

$$C(s) = \frac{\Gamma^2(s + m + \frac{3}{2} + r + 1)\Gamma(s)}{\left(s + m + \frac{3}{2} + r - 1\right)^2 \cdots \left(s + m + \frac{3}{2}\right)^2 \left(s + m + \frac{3}{2} - 1\right) \cdots \left(s + \frac{1}{2}\right)}.$$

Now take the limit and denote the result as  $C_r$ , that is,

$$\begin{aligned} C_r &= \lim_{s \rightarrow -m - \frac{3}{2} - r} C(s) \\ &= \frac{\Gamma^2(1)\Gamma(-m - \frac{3}{2} - r)}{(-1)^2 \cdots (-r)^2 (-r-1) \cdots (-m-1-r)} \\ &= \frac{(-1)^{m+1+r}\Gamma(-m - \frac{3}{2})}{r!(m+1+r)!(m+5/2)_r}. \end{aligned} \quad (2.83)$$

Let  $D(s) = \frac{\partial}{\partial s} \ln C(s)$  and

$$D_r = \lim_{s \rightarrow -m - \frac{3}{2} - r} D(s) = \psi(m+1) + \psi(m+r+2) + \psi(-m - \frac{3}{2} - r). \quad (2.84)$$

Hence, the sum of the residues corresponding to the poles of order 2 each, denoted by  $S_3$ , is the following:

$$S_3 = \sum_{r=0}^{\infty} \left(\frac{z^2}{4}\right)^{m+\frac{3}{2}+r} \left\{ -\ln\left(\frac{z^2}{4}\right) + D_r \right\} C_r. \quad (2.85)$$

Therefore, we have the following theorem:

**Theorem 2.6** For  $\nu$  a positive half integer, namely  $\nu = m + \frac{1}{2}, m = 0, 1, \dots$

$$\begin{aligned} \pi^{-\frac{1}{2}}(S_1 + S_2 + S_3) &= \pi^{-\frac{1}{2}} G_{0,3}^{3,0} \left[ \frac{z^2}{4} \mid_{0, \frac{1}{2}, 1+\nu} \right] \\ &= \Gamma(m + \frac{3}{2}) {}_0F_2 \left( -; \frac{1}{2}, -m - \frac{1}{2}; -\frac{z^2}{2} \right) \\ &\quad - 2\Gamma(1+m) \left(\frac{z^2}{4}\right)^{\frac{1}{2}} \sum_{r=0}^m \frac{(-1)^r}{r!} \left(\frac{z^2}{4}\right)^r \frac{1}{(3/2)_r (-m)_r} \\ &\quad \pi^{-\frac{1}{2}} \left(\frac{z^2}{4}\right)^{m+\frac{3}{2}} \sum_{r=0}^{\infty} \left(\frac{z^2}{4}\right)^r \left\{ -\ln\left(\frac{z^2}{4}\right) + D_r \right\} C_r, \end{aligned}$$

where  $C_r$  and  $D_4$  are defined in (2.83) and (2.84), respectively.

**Case 2.12**  $\nu$  a negative half integer

Let  $\nu = -m - \frac{1}{2}, m = 0, 1, 2, \dots$  In this case,

$$\Gamma(s)\Gamma\left(s + \frac{1}{2}\right)\Gamma(s + \nu + 1) = \Gamma(s)\Gamma\left(s + \frac{1}{2}\right)\Gamma\left(s + \frac{1}{2} - m\right).$$

The poles at  $s = -r, r = 0, 1, 2, \dots$  are of order 1 each; the poles at  $s = m - \frac{1}{2} - r, r = 0, 1, \dots, m - 1$  are of order 1 each and the poles at  $s = -\frac{1}{2} - r, r = 0, 1, 2, \dots$  are of order 2 each. Again, we will denote the sum of the residues by  $S_j, j = 1, 2, 3$ . The sum of the residues corresponding to the poles  $s = -r, r = 0, 1, \dots$ , denoted by  $S_1$ , is the following:

$$\begin{aligned} S_1 &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \left(\frac{z^2}{4}\right)^r \Gamma\left(\frac{1}{2} - r\right) \Gamma\left(\frac{1}{2} - m - r\right) \\ &= \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2} - m\right) \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \left(\frac{z^2}{4}\right)^r \frac{1}{(1/2)_r (m+1/2)_r} \\ &= \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2} - m\right) {}_0F_2\left(-; \frac{1}{2}, m + \frac{1}{2}; -\frac{z^2}{4}\right). \end{aligned} \quad (2.86)$$

For the sum of the residues at the poles  $s = m - \frac{1}{2} - r, r = 0, 1, \dots, m - 1$ , we have the following, denoted by  $S_2$ :

$$\begin{aligned} S_2 &= \sum_{r=0}^{m-1} \frac{(-1)^r}{r!} \left(\frac{z^2}{4}\right)^{r-m+\frac{1}{2}} \Gamma(m - r) \Gamma\left(m - \frac{1}{2} - r\right) \\ &= \left(\frac{z^2}{4}\right)^{-m+\frac{1}{2}} \Gamma(m) \Gamma\left(m - \frac{1}{2}\right) \sum_{r=0}^{m-1} \frac{(-1)^r}{r!} \left(\frac{z^2}{4}\right)^r \frac{1}{(-m+1)_r (-m+3/2)_r}. \end{aligned} \quad (2.87)$$

Let

$$C'(s) = \left(s + \frac{1}{2} + r\right)^2 \Gamma\left(s + \frac{1}{2}\right) \Gamma\left(s + \frac{1}{2} - m\right) \Gamma(s).$$

As done in the previous cases, insert the following factors in the numerator and in the denominator

$$\left(s + \frac{1}{2} + r - 1\right)^2 \cdots \left(s + \frac{1}{2}\right)^2 \left(s - \frac{1}{2}\right) \cdots \left(s + \frac{1}{2} - m\right)$$

and write

$$C's(s) = \frac{\Gamma^2\left(s + \frac{1}{2} + r + 1\right)}{\left(s + \frac{1}{2} + r - 1\right)^2 \cdots \left(s + \frac{1}{2}\right)^2 \left(s - \frac{1}{2}\right) \cdots \left(s + \frac{1}{2} - m\right)}.$$

Now, take the limit of the logarithmic derivative of  $C'(s)$  and write

$$D'_r = \lim_{s \rightarrow -\frac{1}{2} - r} \frac{\partial}{\partial s} \ln C'(s) = \psi(1) + \psi(m + r + 1) + \psi\left(-\frac{1}{2} - r\right). \quad (2.88)$$

Let

$$\begin{aligned} C'_r &= \lim_{s \rightarrow -\frac{1}{2} - r} C'(s) = \frac{\Gamma^2(1)\Gamma(-\frac{1}{2} - r)}{(-1)^2(-2)^2 \cdots (-r)^2(-r-1) \cdots (-r-m)} \\ &= \frac{(-1)^{m+r}\Gamma(-\frac{1}{2})}{r!(m+r)!(3/2)_r}. \end{aligned} \quad (2.89)$$

Now, we can write the sum of the residues corresponding to the poles at  $s = -\frac{1}{2} - r, r = 0, 1, \dots$  as the following, denoted by  $S_3$ :

$$S_3 = \sum_{r=0}^{\infty} \left(\frac{z^2}{4}\right)^{r+\frac{1}{2}} \left\{ -\ln\left(\frac{z^2}{4}\right) + D'_r \right\} C'_r, \quad (2.90)$$

where  $D'_r$  and  $C'_r$  are given in (2.89) and (2.90), respectively. Hence, we have the following theorem:

**Theorem 2.7** For  $\nu$  a negative half integer, namely  $\nu = -m - \frac{1}{2}, m = 0, 1, \dots$

$$\begin{aligned} \pi^{-\frac{1}{2}}(S_1 + S_2 + S_3) &= \pi^{-\frac{1}{2}} G_{0,3}^{3,0} \left[ \frac{z^2}{4} \Big|_{0, \frac{1}{2}, 1+\nu} \right] \\ &= \Gamma\left(\frac{1}{2} - m\right) {}_0F_2\left(-; \frac{1}{2}, m + \frac{1}{2}; -\frac{z^2}{4}\right) \\ &\quad + \pi^{-\frac{1}{2}} \left(\frac{z^2}{4}\right)^{-m+\frac{1}{2}} \Gamma(m)\Gamma(m - \frac{1}{2}) \sum_{r=0}^{m-1} \frac{(-1)^r}{r!} \left(\frac{z^2}{4}\right)^r \\ &\quad \times \frac{1}{(-m+1)_r(-m+3/2)_r} \\ &\quad + \pi^{-\frac{1}{2}} \left(\frac{z^2}{4}\right)^{\frac{1}{2}} \sum_{r=0}^{\infty} \left(\frac{z^2}{4}\right)^r \left\{ -\ln\left(\frac{z^2}{4}\right) + D'_r \right\} C'_r, \end{aligned}$$

where  $D'_r$  and  $C'_r$  are given in (2.88) and (2.89), respectively.

## 2.11 Series Representations for the Thermonuclear Functions: $G_{1,3}^{3,0}(\cdot)$

In view of the integral of the nonresonant nuclear reaction rate with modified Maxwell-Boltzmann distribution in (2.32) we have to consider the special case  $n = 1, m = 2$ , and  $a = 1$  in (2.33):

$$\begin{aligned} &\pi^{-\frac{1}{2}} d^{-\rho+1} \sum_{r=0}^{\infty} \frac{(-d)^r}{r!} G_{1,3}^{3,0} \left[ \frac{z^2}{4d} \Big|_{-\rho+r+1, 0, \frac{1}{2}}^{-\rho+r+2} \right] \\ &= \pi^{-\frac{1}{2}} d^{-\rho+1} \sum_{r=0}^{\infty} \frac{(-d)^r}{r!} \frac{1}{2\pi i} \int_{t-i\infty}^{t+i\infty} ds \frac{\Gamma(-\rho+r+1+s)\Gamma(s)\Gamma(\frac{1}{2}+s)}{\Gamma(-\rho+r+2+s)} \left(\frac{z^2}{4d}\right)^{-s} \\ &= \pi^{-\frac{1}{2}} d^{-\rho+1} \sum_{r=0}^{\infty} \frac{(-d)^r}{r!} \frac{1}{2\pi i} \int_{t-i\infty}^{t+i\infty} ds \frac{\Gamma(s)\Gamma(\frac{1}{2}+s)}{(-\rho+r+1+s)} \left(\frac{z^2}{4d}\right)^{-s}. \end{aligned} \quad (2.91)$$

For the Mellin-Barnes integral representation of Meijer's G-function see the details in Mathai and Saxena (1973).

**Case 2.13**  $-\rho + r + 1 \neq \frac{\lambda}{2}, \lambda = 0, 1, 2, \dots$

Note that the poles of the integrand in (2.91) are at  $s = \rho - r - 1, s = 0, -1, -2, \dots$  and  $s = -\frac{1}{2}, -\frac{1}{2} - 1, \dots$ . Hence if  $-\rho + r + 1 \neq \frac{\lambda}{2}, \lambda = 0, 1, 2, \dots$  all the poles are simple and the following are the residues:

$$\begin{aligned} \text{at } s = \rho - r - 1, \text{ is } & \Gamma(\rho - r - 1) \Gamma\left(\rho - r - \frac{1}{2}\right) \left(\frac{z^2}{4d}\right)^{-\rho+r+1} \\ \text{at } s = -\nu, \text{ is } & \frac{(-1)^\nu \Gamma(\frac{1}{2} - r)}{\nu!(-\rho+r+1-\nu)} \left(\frac{z^2}{4d}\right)^\nu \\ \text{at } s = -\frac{1}{2} - \nu, \text{ is } & \frac{(-1)^\nu \Gamma(-\frac{1}{2} - \nu)}{\nu!(-\rho+r+\frac{1}{2}-\nu)} \left(\frac{z^2}{4d}\right)^{\frac{1}{2}+\nu}. \end{aligned}$$

Hence, we have the following result:

**Theorem 2.8** For  $z > 0, d > 0, -\rho + r + 1 \neq \frac{\lambda}{2}, \lambda = 0, 1, 2, \dots$

$$\begin{aligned} & \pi^{-\frac{1}{2}} d^{-\rho+1} \sum_{r=0}^{\infty} \frac{(-d)^r}{r!} G_{1,3}^{3,0} \left[ \frac{z^2}{4d} \mid_{-\rho+r+1, 0, \frac{1}{2}}^{-\rho+r+2} \right] \\ &= \pi^{-\frac{1}{2}} d^{-\rho+1} \sum_{r=0}^{\infty} \frac{(-d)^r}{r!} \left\{ \Gamma(\rho - r - 1) \Gamma(\rho - r - 1/2) \left(\frac{z^2}{4d}\right)^{-\rho+r+1} \right. \\ & \quad \left. + \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \Gamma(\frac{1}{2} - \nu)}{\nu!(-\rho+r+1-\nu)} \left(\frac{z^2}{4d}\right)^\nu + \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \Gamma(-\frac{1}{2} - \nu)}{\nu!(-\rho+r-\nu+1/2)} \left(\frac{z^2}{4d}\right)^{\frac{1}{2}+\nu} \right\}. \end{aligned}$$

**Case 2.14**  $-\rho + r - 1 = \mu, \mu = 0, 1, 2, \dots$

If  $-\rho + r - 1 = \mu, \mu = 0, 1, 2, \dots$  then at  $s = -\mu$  there is a pole of order two in the integrand of (2.91). The remaining poles are simple. The residue at  $s = -\mu, \mu = 0, 1, 2, \dots$ , denoted by  $S_1$ , is therefore the following:

$$\begin{aligned} S_1 &= \lim_{s \rightarrow -\mu} \frac{\partial}{\partial s} \left\{ \frac{(s+\mu)^2 C(s)}{(s+\mu)} \right\}, C(s) = \Gamma(s) \Gamma\left(\frac{1}{2} + s\right) \left(\frac{z^2}{4d}\right)^{-s}, \\ &= \left(\frac{z^2}{4d}\right)^\mu \left\{ -\ln \left(\frac{z^2}{4d}\right) + A \right\} B, \text{ where} \\ B &= \lim_{s \rightarrow -\mu} \left\{ \frac{\Gamma(s+\mu+1) \Gamma(s+1/2)}{(s+\mu-1) \cdots (s)} \right\} = \frac{(-1)^\mu}{\mu!} \Gamma\left(\frac{1}{2} - \mu\right), \end{aligned} \tag{2.92}$$

and

$$\begin{aligned} A &= \lim_{s \rightarrow -\mu} \frac{\partial}{\partial s} \ln \left\{ \frac{\Gamma(s+\mu+1) \Gamma(s+1/2)}{(s+\mu-1) \cdots (s)} \right\} \\ &= \psi(\mu + 1) + \psi\left(\frac{1}{2} - \mu\right) \end{aligned} \tag{2.93}$$

where  $\psi(z)$  is the psi function  $\psi(z) = \frac{\partial}{\partial z} \ln \Gamma(z)$  (Mathai and Saxena 1973). Thus, we can give the following result:

**Theorem 2.9** For  $z > 0, d > 0, -\rho + r + 1 = \mu, \mu = 0, 1, 2, \dots$

$$\begin{aligned} & \pi^{-\frac{1}{2}} d^{-\rho+1} \sum_{r=0}^{\infty} \frac{(-d)^r}{r!} G_{1,3}^{3,0} \left[ \frac{z^2}{4d} \mid {}_{-\rho+r+1,0,\frac{1}{2}}^{-\rho+r+2} \right] \\ &= \pi^{-\frac{1}{2}} d^{-\rho+1} \sum_{r=0}^{\infty} \frac{(-d)^r}{r!} \left\{ \sum_{\nu=0, \neq \mu}^{\infty} \frac{(-1)^r \Gamma(-\nu+1/2)}{\nu!(-\rho+r+1-\nu)} \left( \frac{z^2}{4d} \right)^{\nu} \right. \\ & \quad \left. + \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu} \Gamma(-\nu-1/2)}{\nu!(-\rho+r-\nu+1/2)} \left( \frac{z^2}{4d} \right)^{\frac{1}{2}+\nu} + \left( \frac{z^2}{4d} \right)^{\mu} \left\{ -\ln \left( \frac{z^2}{4d} \right) + A \right\} B \right\}, \end{aligned} \quad (2.94)$$

where  $A$  and  $B$  are given in (2.93) and (2.92), respectively.

**Case 2.15**  $-\rho + r - 1 = \frac{\lambda}{2}, \lambda = 1, 3, 5, \dots$

If  $-\rho + r - 1 = \frac{\lambda}{2}, \lambda = 1, 3, 5, \dots$  then there is a pole of order two at  $s = -\frac{\lambda}{2}$ . The residue at  $s = -\frac{\lambda}{2}$  reduces to the following, denoted by  $S_1$ :

$$\begin{aligned} S_1 &= \lim_{s \rightarrow -\lambda/2} \frac{\partial}{\partial s} \left\{ \frac{(s+\lambda/2)^2 \Gamma(s) \Gamma(s+1/2)}{(s+\lambda/2)} \left( \frac{z^2}{4d} \right)^{-s} \right\} \\ &= \left( \frac{z^2}{4d} \right)^{\lambda/2} \left\{ -\ln \left( \frac{z^2}{4d} \right) + A_1 \right\} B_1 \end{aligned}$$

where

$$B_1 = \frac{(-1)^{\lambda_1}}{\lambda_1!} \Gamma\left(-\frac{\lambda}{2}\right), \quad \frac{\lambda}{2} = \lambda_1 + \frac{1}{2}, \quad (2.95)$$

and

$$A_1 = \psi(\lambda_1 + 1) + \psi\left(-\frac{\lambda}{2}\right). \quad (2.96)$$

**Theorem 2.10** For  $z > 0, d > 0, -\rho + r + 1 = \frac{\lambda}{2}, \lambda = 1, 3, 5, \dots, \frac{\lambda}{2} = \lambda_1 + \frac{1}{2}$

$$\begin{aligned} & \pi^{-\frac{1}{2}} d^{-\rho+1} \sum_{r=0}^{\infty} \frac{(-d)^r}{r!} G_{1,3}^{3,0} \left[ \frac{z^2}{4d} \mid {}_{-\rho+r+1,0,\frac{1}{2}}^{-\rho+r+2} \right] \\ &= \pi^{-\frac{1}{2}} d^{-\rho+1} \sum_{r=0}^{\infty} \frac{(-d)^r}{r!} \left\{ \sum_{\nu=0, \neq \lambda_1}^{\infty} \frac{(-1)^{\nu} \Gamma(-\nu-1/2)}{\nu!(-\rho-\nu+r+1/2)} \left( \frac{z^2}{4d} \right)^{\frac{1}{2}+\nu} \right. \\ & \quad \left. + \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu} \Gamma(-\nu+1/2)}{\nu!(-\rho+r-\nu+1)} \left( \frac{z^2}{4d} \right)^{\nu} + \left( \frac{z^2}{4d} \right)^{\frac{\lambda}{2}} \left\{ -\ln \left( \frac{z^2}{4d} \right) + A_1 \right\} B_1 \right\}, \end{aligned}$$

where  $B_1$  and  $A_1$  are given in (2.95) and (2.96), respectively.

Note that in all the Theorems 2.8, 2.9, and 2.10, the inner series are nothing but a hypergeometric series of the type  ${}_1F_2(z)$  and hence convergent for all  $z$ . Then, the outer sum in these theorems is dominated by a  ${}_1F_1(z)$  which is also convergent for all values of the variable  $z$ . Hence, the series forms on the right side in Theorems 2.8, 2.9, and 2.10 are convergent (cp. Haubold and Mathai 1986a).



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### 3. The Solar Neutrino Problem

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#### 3.1 Introduction: The Solar Neutrino Problem

“...some say the solar neutrino problem is still with us. Others say there never was a problem. My purpose is to present this ambiguous situation to you in such a way that you can make your own judgement.” (Fowler 1977). It is not the aim of the present chapter to add one more suggestion to the long list of attempts for the solution of the so-called solar neutrino problem (for an updated list cf. Haxton 1984).

It is the merit of R. DAVIS, Jr. and his associates to have developed a radiochemical method of detecting the solar neutrino via the reaction  $\nu_e + {}^{37}Cl \rightarrow {}^{37}Ar + e^-$  over a period of about 30 years (Bahcall and Davis 1982). This detection method is sensitive primarily to the high energy neutrinos from the decay  ${}^8B \rightarrow {}^8Be^* + e^+ + \nu_e$ , where  ${}^8B$  is produced in the proton-proton chain with the overall net result:  $4p + e^- \rightarrow \alpha + e^+ + 2\nu_e + 26.73\text{MeV}$ . The branching ratios of the proton-proton chain are very sensitive to the solar temperature and require a detailed model of the evolution and internal structure of the Sun (Haubold and Gerth 1983). The observations yield, as an average of the runs from 1970 to 1983, a rate  $\sum_i (\phi_i \sigma_i)_{obs} = (2.1 \pm 0.3)SNU$  ( $1\text{ SNU} = 10^{-36}$  captures  $s^{-1}$  per  ${}^{37}Cl$  atom (Davis et al. 1984). The rate predicted by the standard solar model is  $\sum_i (\phi_i \sigma_i)_{theor} = (7.6 \pm 3.3)SNU$  ( $3\sigma$ ) (Bahcall et al. 1982). The disagreement between standard theory and observations by a factor of four constitutes what has come to be known as the solar neutrino problem in the sense questioned by FOWLER as mentioned above. Over more than 15 years a large number of more or less fundamental solutions, modifying either the neutrino physics, nuclear physics or the solar model construction, have been proposed. The present chapter is an attempt to construct a simple analytic model to study the internal structure and the neutrino emission of the Sun in its present stage of evolution.

In Sect. 3.2, the nuclear reactions of the proton-proton chain and its branching are considered and the total nuclear energy generation rate of the proton-proton chain under the assumption of near-statistical equilibrium between the reactions will be derived. In Sect. 3.3, an analytic model for the central nuclear burning region of the Sun is constructed by taking into account the assumptions of mass conservation, hydrostatic equilibrium, and energy conservation. These calculations assume a non-linear density distribution for the solar model and employ the equation of state of the perfect gas. What concerns the method of solar model construction is that in some sense we are going back to the method of Chandrasekhar (1939) and Hayashi et al. (1962). In Sect. 3.4, the basic integral of the solar nuclear energy generation and solar neutrino emission is derived taking into account the closed-form representation of the nonresonant thermonuclear reaction rate (Haubold and Mathai 1984). That integral is the analytic equivalent to the rough estimation of the solar neutrino flux by  $N_\nu \approx 2L_\odot/E$ , where  $L_\odot$  is the luminosity of the Sun and  $E$  is the energy liberated by the overall net reaction as mentioned above, not including the energy of the two neutrinos ( $N_\nu \approx 2 \times 10^{38} s^{-1}$ ). In Sect. 3.5, the integral of the nuclear energy generation and solar neutrino emission is evaluated analytically in closed-form by means of modern results of the integration theory of generalized hypergeometric functions (cp. Mathai and Saxena 1973, 1978). In bringing together the results of the analytic evaluations described in Sects. 3.3, 3.4, 3.5, respectively, analytic results for the solar neutrino emission rates are represented in Sect. 3.10.

**Table 3.1** Basic facts for the proton-proton chain (adapted from KAVANAGH (1982), BAHCALL et al. (1982), and HAXTON (1984))

| (i)  | (ii)                                   | (**) (iii)            | (**) (iv)                          | (**) (v)              | (**) (vi)                  | (**) (vii)                 |
|--|--|-----------------------|------------------------------------|-----------------------|----------------------------|----------------------------|
| <b>Reactions, except for reaction energies not including positron annihilation but not including neutrino energy</b> | Termination and locally contained heat | Branching percentages | $S_{11}(0)$ [keV b]                | $\tau_D(0)$ [years]   | $E_{\text{cut-max}}$ [keV] | $E_{\nu,\text{max}}$ [MeV] |
| (0) ${}^1\text{H} + \text{e}^- + {}^1\text{H} \rightarrow {}^2\text{H} + \nu_e$                                      | PPI                                    | 0.25%                 |                                    |                       |                            | 1.44 (mono)                |
| (1) ${}^1\text{H} + {}^1\text{H} \rightarrow {}^2\text{H} + \text{e}^+ + \nu_e + 1.19 \text{ MeV}$                   |  | 99.75%                | $S_{11}(0) = 3.88 \times 10^{-22}$ | $5.8 \times 10^9$     |                            | 0.42                       |
| (2) ${}^2\text{H} + {}^1\text{H} \rightarrow {}^3\text{He} + \gamma + 5.49 \text{ MeV}$                              | $Q_{\text{PPI}} = 26.2 \text{ MeV}$    |                       | $S_{12}(0) 2.5 \times 10^{-4}$     | $3.2 \times 10^{-15}$ | 15                         |                            |
| (3) ${}^3\text{He} + {}^3\text{He} \rightarrow {}^4\text{He} + {}^2\text{H} + 12.85 \text{ MeV}$                     | 40%                                    |                       | $S_{33}(0) 4.7 \times 10^3$        | $1.5 \times 10^5$     | 33                         |                            |
| (4) ${}^3\text{He} + {}^4\text{He} \rightarrow {}^7\text{Be} + \gamma + 1.54 \text{ MeV}$                            |  | 31%                   | $S_{34}(0) 0.52$                   | $6.5 \times 10^7$     | 107                        |                            |
| (5) ${}^7\text{Be} + \text{e}^- \rightarrow {}^7\text{Li} + \nu_e + \gamma + 0.05 \text{ MeV}$                       | PPII                                   | 99.7%                 |                                    | 0.2                   |                            | 0.86 (90%) (mono)          |
| (6) ${}^7\text{Li} + {}^1\text{H} \rightarrow {}^4\text{He} + {}^4\text{He} + 17.34 \text{ MeV}$                     | $Q_{\text{PPII}} = 25.7 \text{ MeV}$   |                       |                                    | $2 \times 10^{-4}$    |                            | 0.38 (10%)                 |
| (7) ${}^7\text{Be} + {}^1\text{H} \rightarrow {}^8\text{B} + \gamma + 0.14 \text{ MeV}$                              | PPIII                                  | 0.3%                  | $S_{17}(0) 2.9 \times 10^{-2}$     | 71                    | 117                        |                            |
| (8) ${}^8\text{B} \rightarrow {}^8\text{Be} + \text{e}^+ + \nu_e + 7.7 \text{ MeV}$                                  | $Q_{\text{PPIII}} = 19.1 \text{ MeV}$  |                       |                                    | $3 \times 10^{-8}$    |                            | 14.06                      |
| (9) ${}^8\text{Be} \rightarrow {}^4\text{He} + {}^4\text{He} + 3.0 \text{ MeV}$                                      |  |                       |                                    | $10^{-19}$            |                            |                            |

## 3.2 The Proton-Proton Chain

The calculation of the rate of energy generation and neutrino production from the proton-proton chain is not so simple as in the case of the CNO cycles, for example, because of the three possible modes of termination of the chain (cf. Table 3.1, column (i) and (ii), respectively). The locally liberated heat,  $Q$ , varies from  $Q_{\text{ppIII}} = 19.1 \text{ MeV}$  to  $Q_{\text{ppI}} = 26.2 \text{ MeV}$  because of different neutrino energy losses. The individual exoergic reaction energies shown in Table 3.1 (column (i)) include positron annihilation but do not include neutrino energy. The rate of energy generation is not simply proportional to the rate of the first reaction (1) in the proton-proton chain, the  ${}^1\text{H}(p, e^+ \nu_e) {}^2\text{H}$  reaction, but depends also on density, temperature and the abundances of the various nuclei that enter into the various reactions. The temperature of the deep interior of the Sun is high enough, that all reactions in each mode of termination can occur in time scales short compared to the age of the Sun. The only exception is the proton-proton reaction itself (cf. Table 3.1). Thus, one can assume that a near-statistical equilibrium among the intermediate products of the modes will be established, in which the reaction rates of creation and destruction of each nucleus are approximately equal. Under those conditions, the calculation of energy generation and neutrino production for the full chain as shown in Table 3.1 is much simplified (Fowler 1977; Kavanagh 1982; Bahcall et al. 1982). The branching of the various reactions of the proton-proton chain is shown in columns (ii) and (iii), respectively, in Table 3.1. For the principal mode (PPI) terminating with reaction (3) we have the rate of energy generation per unit mass,

$$S_{\text{PPI}} = \frac{1}{\rho} Q_{\text{PPI}} \frac{1}{2} n_3 n_3 < \sigma v >_{33}, \quad (3.1)$$

where  $n_3$  is the number density of  ${}^3\text{He}$  nuclei and  $< \sigma v >_{33}$  is the thermally averaged product of the cross section  $\sigma$  for the reaction and relative velocity  $v$  of the interacting particles. The mode PPI is accompanied by neutrino production in the reactions (1),  ${}^1\text{H}(p, e^+ \nu_e) {}^2\text{H}$ , and (0),  ${}^1\text{H}(e^- p, \nu_e) {}^2\text{H}$ , respectively. In the mode PPII under the assumption of near-statistical equilibrium, the net rate of creation of  ${}^4\text{He}$  nuclei is equal to the rate of any of the reactions occurring in the mode PPII (the same holds for PPIII). Choosing the first reactions of each of the modes PPII and PPIII, we get for the rates of energy generation per unit mass

$$(3.2)$$

$$\epsilon_{PPII} = \frac{1}{\rho} Q_{PPII} n_7 p_{7e^-},$$

and

$$\epsilon_{PPIII} = \frac{1}{\rho} Q_{PPIII} n_7 n_1 < \sigma v >_{71}, \quad (3.3)$$

where  $n_7$  and  $n_1$  are the number densities of  $^7Be$  and  $^1H$ , respectively, and  $p_{7e^-}$  is the probability per unit time per nucleus of electron capture,  $< \sigma v >_{71}$  is the thermally averaged product of the cross section  $\sigma$  for the reaction (7) and relative velocity  $v$  of the interacting particles. The neutrino producing reactions of PPI and PPIII are the reactions (5) and (8), respectively.

For PPI the condition of near-statistical equilibrium means that

$$\frac{1}{2} n_1 n_1 < \sigma v >_{11} - 2 \left( \frac{1}{2} \right) n_3 n_3 < \sigma v >_{33} - n_3 n_4 < \sigma v >_{34} = 0, \quad (3.4)$$

where  $n_4$  is the number density of  $^4He$  nuclei, the factors  $(1/2)$  account for the fact that a reaction between identical particles always involves two of these particles. The factor 2 in (3.57) accounts for the fact that two  $^3He$  nuclei are consumed in each reaction (3). If the reaction  $^3He(\alpha, \gamma) ^7Be$  would be negligible one simply states  $\frac{1}{2} n_3 n_3 < \sigma v >_{33} = \left( \frac{1}{2} \right) \frac{1}{2} n_1 n_1 < \sigma v >_{11}$  which is correct if only the PPI mode operates. Similarly, for the reactions following the formation of  $^7Be$  we have

$$n_3 n_4 < \sigma v >_{34} - n_7 p_{7e^-} - n_7 n_1 < \sigma v >_{71} = 0. \quad (3.5)$$

For the total rate of energy generation per unit mass from all three modes PPI, PPII, PPIII we find

$$\epsilon_{PP} = \epsilon_{PPI} + \epsilon_{PPII} + \epsilon_{PPIII}, \quad (3.6)$$

which we can write with (3.1), (3.2), (3.3), and by the help of (3.5),

$$\begin{aligned} \epsilon_{PP} = & \frac{1}{\rho} [Q_{PPI} \frac{1}{2} n_3 n_3 < \sigma v >_{33} + Q_{PPII} n_3 n_4 < \sigma v >_{34} \\ & + (Q_{PPIII} - Q_{PPII}) n_7 n_1 < \sigma v >_{71}]. \end{aligned} \quad (3.7)$$

Equation (3.7) is the total nuclear energy generation rate for the proton-proton chain expressed in terms of thermonuclear reaction rates of the reactions (3), (4), and (7) shown in Table 3.1. Taking into account (3.57) we can also write  $\epsilon_{PP}$  in (3.7) in terms of the reaction (1),  $^1H(p, e^+ \nu_e) ^2H$ , which determines the overall thermonuclear energy generation of the proton-proton chain, the reaction (4),  $^3He(\alpha, \gamma) ^7Be$ , which is the branching reaction between the modes PPI and PPII/PPIII in the proton-proton chain, and the reaction (7),  $^7Be(p, \gamma) ^8B$ , which is the first reaction of the mode PPIII containing the high energy neutrino producing reaction of the decay  $^8B \rightarrow ^8B^* + e^+ + \nu_e$ . As we know, the neutrinos from the decay of  $^8B$  dominate the capture rate in the solar neutrino experiment of DAVIS and associates (Fowler 1977; Bahcall et al. 1982; Bahcall and Davis 1982).

What concerns the neutrino producing reactions in the proton-proton chain is that the beta decay reactions (1) and (8) in Table 3.1 produce neutrinos in an allowed FERMI distribution with maximum energy  $E_{\nu_{\max}}$  as shown in column (vii) of Table 3.1. However, the flux of  $^8B$  neutrinos deviates from an allowed FERMI distribution because the  $^8Be$  final state populated in this decay is a broad resonance where  $E_{\nu_{\max}}$  is computed for the center of this resonance (Fowler 1977; BAHCALL et al. 1982). The electron capture reactions (0) and (5) produce line sources of neutrinos of maximum energy  $E_{\nu_{\max}}$  also shown in Table 3.1 column (vii). The electron capture of  $^7Be$  leads partially to the ground state of  $^7Li$  (90%) and partially to an excited state (10%) of  $^7Li$  (Fowler 1977; Bahcall et al. 1982).

One of the most serious problems in the theory of solar nuclear energy generation and solar neutrino production is the dependence of both on the cross sections of the nuclear reactions involved. It is quite interesting that the first reaction of the proton-proton chain,  $^1H(p, e^+ \nu_e) ^2H$  has never been directly observed in the nuclear laboratory because of its extremely small cross section. This reaction can occur only if the two protons are brought together by a nuclear collision. During the extremely short time

of encounter between the protons one or both must have a chance to beta decay to become a neutron, a positron, and an electron neutrino. The neutron can then be captured by the second proton to form a deuteron which is a very rare event. May be this reaction will never be observed in the laboratory. However, the present state of the theory of beta decay makes it possible to estimate the cross section and thus the cross section factor with high precision. Practically, the same holds for the reaction (0) shown in Table 3.1 (ep. Bahcall et al. 1982). For all other nuclear reactions shown in Table 3.1, the nuclear cross section can be measured in the nuclear laboratory, but, at least to large energies in comparison to the order of the central temperature of the Sun ( $T_c \approx 15.5 \times 10^6 K$ ,  $kT_c \approx 1.34 \text{ keV}$ ; cf. Table 3.1). It is generally accepted that the necessary extrapolation of the experimental cross sections measured for energies up till  $E_{1ab,\text{max}}$  as shown in column (vi) of Table 3.1 to solar energies is well understood theoretically (Fowler 1977; Kavanagh 1982; Bahcall et al. 1982; Haxton 1984). If one removes the strong energy dependence of the nuclear cross section due to the COULOMB barrier and displays the experimental data in the form of the astrophysical nuclear cross section factor  $S(E)$ , where  $E$  is the center-of-mass energy, one obtains the extrapolated  $S$ -factors at zero energy as shown in column (iv) of Table 3.1 (Bahcall et al. 1982).

**Table 3.2** Basic parameters for the solar model (adopted from BAHCALL et al. (1982))

| Solar parameter(*)                                 |                 | Value                                    |
|--|-----------------|--|
| Luminosity   | $L_\odot$       | $3.86 \times 10^{33} \text{ erg s}^{-1}$ |
| Mass   | $M_\odot$       | $1.99 \times 10^{33} \text{ g}$          |
| Radius   | $R_\odot$       | $6.96 \times 10^{10} \text{ cm}$         |
| Central hydrogen abundance by mass                 | $X_c$           | 0.355                                    |
| Central helium abundance by mass                   | $Y_c$           | 0.6222                                   |
| Central heavy elements abundances by mass          | $Z_c$           | 0.0228                                   |
| mean molecular weight for central solar conditions | $\mu_c$         | 0.8417                                   |
| <i>Standard solar model parameters(*)</i>          |                 |  |
| Central density                                    | $\rho_{c\odot}$ | $156 \text{ g}^{-3}$                     |
| Central pressure                                   | $P_{c\odot}$    | $2.39 \times 10^{17} \text{ dyn}^{-2}$   |
| Central temperature                                | $T_{c\odot}$    | $15.5 \times 10^6 \text{ K}$             |
| <i>Analytic solar model parameters</i>             |                 |  |
| Central density                                    | $\rho_{c\odot}$ | $3.52 \text{ g}^{-3}$                    |
| Central pressure                                   | $P_{c\odot}$    | $1.31 \times 10^{17} \text{ dyn}^{-2}$   |
| Central temperature                                | $T_{c\odot}$    | $9.7 \times 10^6 \text{ K}$              |

### 3.3 An Analytic Model for the Central Region of the Sun

The equations of the internal structure of the Sun—mass conservations, hydrostatic equilibrium, energy conservation, and energy transport - form a system of nonlinear differential equations, three relations which characterize specifically the behaviour of the interior of the Sun, and particular conditions for the boundaries. A complete solution of that boundary value problem can only be obtained by numerical integration techniques. However, the aim of the present Section is to construct an analytic solar model by separating the condition of hydrostatic equilibrium from the consideration of the energy transport inside the solar material (cp. Haubold and Mathai 1984; Haubold and Mathai 1986). As it is well-known from the computation of detailed standard solar models, the temperature of the Sun increases towards the center, but does not attain the value necessary for thermonuclear energy generation until about  $r = 0.2R_\odot$  ( $M = 0.35M_\odot$ ,  $L = 0.95L_\odot$ ; cf. Table 3.2; Bahcall et al. 1982). Thus, throughout the region from  $r = R_\odot$  to  $r = 0.2R_\odot$ , we can adopt a constant value of the luminosity  $L$ . For the treatment of the energy generation by nuclear reactions in the deep interior of the Sun, we take into consideration the hydrostatic equilibrium and the energy conservation which determine the changes in the solar central

conditions. We regard the density distribution  $\rho(r)$  of our stellar model as an arbitrary but specified function of the distance parameter  $r$ ,

$$\rho(r) = \rho_0 \left[ 1 - \left( \frac{r}{R_\odot} \right)^\delta \right], \delta > 0 \quad (3.8)$$

which is capable to reproduce the solar density distribution in the central region by choosing the free parameter  $\delta$ ;  $\rho(r = 0) = \rho_c$  denotes the central stellar matter density,  $\rho(R_\odot) = 0$ . For reasons of symmetry of the solar model we take  $\delta = 2$ . From the equation of mass conservation,

$$\frac{d}{dr} M(r) = 4\pi r^2 \rho(r), \quad (3.9)$$

we obtain the mass distribution with (3.8):

$$M(r) = 4\pi \rho_c \int_0^r dt t^2 \left[ 1 - \left( \frac{t}{R_\odot} \right)^\delta \right] = \frac{4\pi}{3} \rho_c r^3 \left[ 1 - \frac{3}{(\delta+3)} \left( \frac{r}{R_\odot} \right)^\delta \right]. \quad (3.10)$$

Note that from (3.10) one can get the central density, that is,

$$\rho(r = 0) = \rho_c = \frac{3}{4\pi} \frac{(\delta+3)}{\delta} \frac{M_\odot}{R_\odot^3}.$$

Putting  $\delta = 2$  we have the solar case for (7.3),

$$M(r) = \frac{5}{2} M_\odot \left( \frac{r}{R_\odot} \right)^3 \left[ 1 - \frac{3}{5} \left( \frac{r}{R_\odot} \right)^2 \right], \rho_{c\odot} = \frac{15}{8\pi} \frac{M_\odot}{R_\odot^3}, \quad (3.11)$$

the respective central density  $\rho_{c\odot}$  for the analytic solar model is given in Table 3.2. The equation of hydrostatic equilibrium between the total pressure per unit volume and the gravity per unit volume

$$\frac{d}{dr} P(r) = -\frac{GM(r)\rho(r)}{r^2}, \quad (3.12)$$

has the following solution with (3.8) and (3.10):

$$\begin{aligned} P(r) &= P(0) - \int_0^r dt \frac{GM(t)\rho(t)}{t^2} \\ &= \frac{4\pi G}{3} \rho_c^2 R_\odot^2 \left\{ \xi - \frac{1}{2} \left( \frac{r}{R_\odot} \right)^2 + \frac{(\delta+6)}{(\delta+2)(\delta+3)} \left( \frac{r}{R_\odot} \right)^{\delta+2} \right. \\ &\quad \left. - \frac{3}{2(\delta+1)(\delta+3)} \left( \frac{r}{R_\odot} \right)^{2\delta+2} \right\}, \end{aligned} \quad (3.13)$$

where we took into account the boundary conditions,

$$P(r = R_\odot) = 0; P(r = 0) = P_c = \frac{3G}{4\pi} \xi \frac{(\delta+3)^2}{\delta^2} \frac{M_\odot^2}{R_\odot^4};$$

$G$  is the gravitational constant, and

$$\xi = \frac{1}{2} - \frac{(\delta+6)}{(\delta+2)(\delta+3)} + \frac{3}{2(\delta+1)(\delta+3)}. \quad (3.14)$$

For our analytic solar model we obtain with  $\delta = 2$  for (3.13):

$$(3.15)$$

$$\begin{aligned} P(r) &= 5P_c \left\{ \frac{1}{5} - \frac{1}{2} \left( \frac{r}{R_\odot} \right)^2 + \frac{2}{5} \left( \frac{r}{R_\odot} \right)^4 - \frac{1}{10} \left( \frac{r}{R_\odot} \right)^6 \right\}, \\ P_{c\odot} &= \frac{15}{16\pi} G \frac{M_\odot^2}{R_\odot^4}, \end{aligned}$$

the respective value for the central pressure is given in Table 3.2.

Thus, (3.9), (3.12), and the boundary conditions mentioned above determine the pressure distribution (3.13), corresponding to the given density distribution (3.8), for which hydrostatic equilibrium will be obtained. In the range of temperatures of the interior of the Sun and because we restrict ourselves to a star of solar mass we can make the restriction that the radiation pressure inside the solar material is negligible (Chandrasekhar 1939). According to the perfect gas law, in the simplest form suggested by the kinetic theory of gases, we have the pressure

$$P(r) = \frac{kN_A}{\mu} \rho(r) T(r). \quad (3.16)$$

In (3.16),  $\mu$  is the mean molecular weight,  $N_A$  is AVOGADRO's constant, and  $k$  is the BOLTZMANN constant. With (3.16) the temperature distribution  $T(r)$  corresponding to hydrostatic equilibrium is determined by (3.8) and (3.13):

$$\begin{aligned} T(r) &= \frac{\mu}{kN_A} \frac{P(r)}{\rho(r)} = \frac{4\pi G}{3kN_A} \mu \rho_c R_\odot^2 \left[ 1 - \left( \frac{r}{R_\odot} \right)^\delta \right]^{-1} \left\{ \xi - \frac{1}{2} \left( \frac{r}{R_\odot} \right)^2 \right. \\ &\quad \left. + \frac{(\delta+6)}{(\delta+2)(\delta+3)} \left( \frac{r}{R_\odot} \right)^{\delta+2} - \frac{3}{2(\delta+1)(\delta+3)} \left( \frac{r}{R_\odot} \right)^{2\delta+2} \right\}, \end{aligned} \quad (3.17)$$

and the central temperature is given by

$$T(r=0) = T_c = \frac{G}{kN_A} \xi \frac{(\delta+3)}{\delta} \mu \frac{M_\odot}{R_\odot},$$

where  $\xi$  is given by (3.14).

In the case  $\delta = 2$  of our analytic solar model we have for (3.17),

$$\begin{aligned} T(r) &= 5T_c \left[ 1 - \left( \frac{r}{R_\odot} \right)^2 \right]^{-1} \left\{ \frac{1}{5} - \frac{1}{2} \left( \frac{r}{R_\odot} \right)^2 + \frac{2}{5} \left( \frac{r}{R_\odot} \right)^4 - \frac{1}{10} \left( \frac{r}{R_\odot} \right)^6 \right\}, \\ T_{c\odot} &= \frac{1}{2} \frac{G}{kN_A} \mu \frac{M_\odot}{R_\odot}, \end{aligned} \quad (3.18)$$

and the corresponding central temperature  $T_{c\odot}$  is given in Table 3.2.

## 3.4 Solar Thermonuclear Energy Generation: Energy Conservation and Solar Luminosity

The analytic solar model constructed in Sect. 3.7 specified the distribution of mass and pressure for a given density distribution and required the temperature distribution to give hydrostatic equilibrium. This temperature distribution will in general not lead to thermal equilibrium. However, aside from the question of energy transport the solar model will be in complete thermal equilibrium only if the equation of energy conservation is satisfied:

$$\frac{d}{dr} L(r) = 4\pi r^2 \rho(r) \epsilon(r). \quad (3.19)$$

Equation (3.19) states that the net increase in the rate of energy flow from the inside to the outside of a spherical shell of the Sun is equal to the rate of energy production within the shell. In (3.19),  $L(r)$  represents the energy flux through the sphere with radius  $r$ ,  $\epsilon(r)$  is the rate of thermonuclear energy

generation per unit mass and includes the tiny energy losses via solar neutrinos. The net outflow of energy per second,  $L(r)$ , through the sphere of radius  $r$  is determined, in the case of radiative transfer in solar matter, by the local values of the opacity and the temperature gradient. However, with our assumed density distribution (3.8) the energy transport equation can be satisfied at only one point of the Sun. This serious assumption is justified by the aim of the present chapter to investigate the physical conditions in the central nuclear burning region of the Sun where neutrinos are generated. Note that the luminosity,

$$L(R_\odot) = \int_0^{R_\odot} dr \ 4\pi r^2 \rho(r) \epsilon(r), \quad (3.20)$$

is not completely insensitive to the energy generation rate but, basically determined by the mass of the star (Chandrasekhar 1939). The quantity  $L(r)$  remains constant and will be equal to its value at the surface of the Sun,  $L(R_\odot)$ , as long as one remains outside the central region where nuclear energy generation takes place. If we are concerned with only one specific reaction  $1 + 2 \rightarrow 3 + 4$ , then we have the “internal luminosity” going back to the specific nuclear reaction in question:

$$L_{12}(R_\odot) = \int_0^{R_\odot} dr \ 4\pi r^2 \rho(r) \epsilon_{12}(r). \quad (3.21)$$

The energy generation rate per unit mass  $\epsilon_{12}(r)$  is, beside the equation of state (3.16) (the opacity is not specified in our solar model by assumption), one of the three material equations of our solar model. Let  $E_{12}$  denote the amount of energy given off in a single reaction of the proton-proton chain written in the standard notation  $1 + 2 \rightarrow 3 + 4$ , where 1 and 2 denote the incoming particles, and 3 and 4 denote the outgoing particles.

In the more or less developed analytic theory of the internal structure of the Sun one refers in general to the definition of the thermonuclear energy generation rate in the following form:

$$\epsilon(r) = \epsilon_0(\rho_0, T_0) \left( \frac{\rho(r)}{\rho_0} \right)^\alpha \left( \frac{T(r)}{T_0} \right)^\beta, \alpha, \beta \text{ real}. \quad (3.22)$$

In (3.22) the thermonuclear energy generation rate is expressed in terms of powers of the density and temperature, where the subscript 0 designates central conditions,  $\alpha$  and  $\beta$  are constants, and  $\epsilon_0$  contains the chemical composition (Chandrasekhar 1939; Hayashi, Hōshi, and Sugimoto 1962; Haubold and Mathai 1984, 1986). In general, the representation (3.22) of  $\epsilon(r)$  does not describe the energy generation rate of a specific reaction, but, the total nuclear energy generation rate of a chain of nuclear reactions like the proton-proton chain.

In the following we do not refer to the representation (3.22), but, take into consideration the definition of the nuclear energy generation rate,

$$\epsilon_{12}(r) = \frac{1}{\rho(r)} E_{12} r_{12}(\rho(r), T(r)), \quad (3.23)$$

containing the thermonuclear reaction rate  $r_{12}(\rho(r), T(r))$  whose theory can be formulated on the basis of physical principles (cf. e.g., Parker et al. 1964; Haubold and Mathai 1984; Haubold and John 1981).

### 3.5 The Thermonuclear Reaction Rate

All reactions involved in the proton-proton chain are nonresonant. In the calculations of the energy generation and neutrino emission via the proton-proton chain described in Sect. 3.2 we will adopt the following definition of the nonresonant nuclear cross section (Salpeter 1952; Parker et al. 1964; Haubold and Mathai 1984):

$$\sigma(E) = \frac{S(E)}{E} \exp \{-2\pi\eta(E)\}, \quad (3.24)$$

where  $\eta(E)$  is the SOMMERFELD parameter, given by,

$$(3.25)$$

$$\eta(E) = \left(\frac{\mu^*}{2}\right)^{\frac{1}{2}} \frac{Z_1 Z_2 e^2}{h E^{1/2}},$$

where  $Z_1$  and  $Z_2$  are the charges of the interacting particles,  $E$  is the center-of-mass energy,  $e$  is the quantum of electric charge,  $h$  is PLANCK's quantum of action,  $\mu^* = m_1 m_2 / (m_1 + m_2) = A_1 A_2 / (A_1 + A_2) N_A$  is the reduced mass,  $A_1$  and  $A_2$  are the atomic mass numbers of the particles. As mentioned in Sect. 3.2, the cross section for the nuclear reactions that occur in the proton-proton chain cannot generally be measured at the energies of interest for solar conditions (cp. Table 3.1 column (vi)). Hence for the cross section factor  $S(E)$  in (3.24), extrapolations from higher energy measurements must be used to obtain a zero energy intercept,  $S(0)$ , and average values for the first and second derivatives at low energies,  $S'(0)$  and  $S''(0)$ . Therefore,  $S(E)$  is expanded by using a MACLAURIN series in the light of the weak dependence of  $S(E)$  on the relative kinetic energy of the particles (Bahcall 1966; Critchfield 1972; Haubold and Mathai 1984)

$$S(E) = S(0) + S'(0)E + S''(0)E^2. \quad (3.26)$$

All the relevant cross section factors for the proton-proton chain evaluated at zero energy,  $S(0)$ , are shown in Table 1 column (iv) (adopted from Bahcall et al. 1982).

The MAXWELL-BOLTZMANN averaged thermonuclear reaction rate of a nonresonant nuclear reaction  $r_{12}$  in (3.23) can now be written (HAUBOLD and MATHAI 1984):

$$\begin{aligned} r_{12} &= \left(1 - \frac{1}{2} \delta_{12}\right) n_1 n_2 \left(\frac{8}{\pi \mu^*}\right)^{\frac{1}{2}} \frac{1}{(kT)^{3/2}} \sum_{\nu=0}^2 \frac{S^\nu(0)}{\nu!} \\ &\times \int_0^\infty dE E^\nu \exp\left\{-\left(\frac{E}{kT} + 2\pi\eta(E)\right)\right\}, \end{aligned} \quad (3.27)$$

where  $\delta_{12}$  is the KRONECKER symbol,  $n_1$  and  $n_2$  are the particle number densities. The number densities are more explicitly written as

$$n_i(r) = N_A \frac{X_i}{A_i} \rho(r), \quad (3.28)$$

where  $X_i$  is the atomic abundance by mass of the nuclei of type  $i$ . In connection with the explicit form of the thermonuclear reaction rate in (3.27) it is convenient to introduce the quantity  $\langle \sigma v \rangle_{12}$ , defined by the relation

$$r_{12} = \left(1 - \frac{1}{2} \delta_{12}\right) n_1 n_2 \langle \sigma v \rangle_{12}, \quad (3.29)$$

where  $\langle \sigma v \rangle_{12}$  is the thermally averaged product of the cross section  $\sigma$  for the reaction and relative velocity  $v$  of the interacting particles. The mean lifetime,  $r_2(1)$ , of nucleus 1 for interaction with nucleus 2 is given as follows:

$$\lambda_2(1) = \frac{1}{\tau_2(1)} = n_2 \langle \sigma v \rangle_{12} = \rho N_A \frac{X_2}{A_2} \langle \sigma v \rangle_{12}, \quad (3.30)$$

where  $\lambda_2(1)$  is the decay rate of 1 for interaction with 2. The mean lifetime,  $\tau_2(1)$  defined in (3.30), for the reactions of the proton-proton chain are given in Table 3.1 column (v) for central physical conditions of the standard model of the Sun (cp. Kavanagh 1982).

After defining the basic operations (3.23), (3.29) and (3.30) for the description of the dynamics of nuclear reactions in the deep interior of the Sun we find the final representation of the thermonuclear reaction rate by substituting (3.28) in (3.27) and writing  $y = E/(kT)$ :

$$\begin{aligned} r_{12}(r) &= \left(1 - \frac{1}{2} \delta_{12}\right) N_A^{\frac{5}{2}} \frac{X_1 X_2 (A_1 + A_2)^{\frac{1}{2}}}{(A_1 A_2)^{3/2}} \frac{8^{\frac{1}{2}}}{\pi} \rho^2(r) \\ &\times \sum_{\nu=0}^2 (kT(r))^{\nu - \frac{1}{2}} \frac{S^\nu(0)}{\nu!} N_{i_\nu}(z(r)), \end{aligned} \quad (3.31)$$

where

$$N_{i_\nu}(z(r)) = \int_0^\infty dy y^\nu e^{-y-z(r)y^{-\frac{1}{2}}} = \frac{1}{\pi^{\frac{1}{2}}} G_{0,3}^{3,0} \left[ \frac{z^2(r)}{4} \Big|_{1+\nu, \frac{1}{2}, 0} \right], \quad (3.32)$$

$$z(r) = 2\pi \left( \frac{\mu^*}{2kT(r)} \right)^{\frac{1}{2}} \frac{Z_1 Z_2 e^2}{h}. \quad (3.33)$$

Equation (3.32) is the basic closed-form representation of the nonresonant thermonuclear reaction rate integral by means of MEIJER's G-function (cf. Mathai and Saxana 1973; Haubold and Mathai 1984). This closed-form representation is appropriate to perform analytical operations and from (3.32) approximate expressions easily follow for small and large values of the characteristic parameter  $z(r)$  in (3.33), that is the COULOMB barrier energy divided by thermal energy, which appears in the argument of the G-function (Haubold and John 1981; Haubold and Mathai 1984).

For large values of  $z(r)$  we get the asymptotic representation for the G-function in (3.32) (cf. Mathai and Saxena 1973)

$$G_{0,3}^{3,0} \left[ \frac{z^2(r)}{4} \Big|_{1+\nu, \frac{1}{2}, 0} \right] \rightarrow 2 \left( \frac{\pi}{3} \right)^{\frac{1}{2}} \left( \frac{z(r)}{2} \right)^{\frac{2\nu+1}{3}} \exp \left\{ -3 \left( \frac{z(r)}{2} \right)^{\frac{2}{3}} \right\}. \quad (3.34)$$

This relation reproduces results from the well-known papers of Salpeter (1952, for  $\nu = 0$ ) and Bahcall (1966, for  $\nu = 0, 1, 2$ ). Inserting (8.16) for  $N_{i_\nu}(z(r))$  in (3.31) we get

$$\begin{aligned} r_{12}(r) &= \left(1 - \frac{1}{2} \delta_{12}\right) N_A^{\frac{5}{2}} \frac{X_1 X_2 (A_1 + A_2)^{\frac{1}{2}}}{(A_1 A_2)^{3/2}} \frac{2^{\frac{5}{2}}}{3^{1/2} \pi} \rho^2(r) \\ &= \exp \left\{ -3 \left( \frac{z(r)}{2} \right)^{\frac{2}{3}} \right\} \sum_{\nu=0}^2 (kT(r))^{\nu - \frac{1}{2}} \frac{S^{(\nu)}(0)}{\nu!} \left( \frac{z(r)}{2} \right)^{\frac{2\nu+1}{2}}. \end{aligned} \quad (3.35)$$

Once again, it is obvious that if  $S(E)$  in (3.26) is nearly constant most of the value of (3.35) comes from values of  $E$  near zero. The first approximation for well-behaved cross section factors is to treat  $S(E)$  as a constant,  $S(0)$ , defined as the value at  $E = 0$ . Hence, for  $\nu = 0$  we obtain

$$\begin{aligned} r_{12}(r) &= \left(1 - \frac{1}{2} \delta_{12}\right) N_A^{\frac{5}{2}} \frac{X_1 X_2 (A_1 + A_2)^{\frac{1}{2}}}{(A_1 A_2)^{3/2}} \frac{2^{\frac{5}{2}}}{3^{1/2} \pi} \rho^2(r) \\ &\quad \times \frac{S(0)}{(kT(r))^{1/2}} \left( \frac{z(r)}{2} \right)^{\frac{1}{2}} \exp \left\{ -3 \left( \frac{z(r)}{2} \right)^{\frac{2}{3}} \right\}. \end{aligned} \quad (3.36)$$

that reproduces the asymptotic representation of the thermonuclear reaction rate underlying the fundamental papers of Fowler (1984; cf. also Parker et al. 1964; Critchfield 1972; Haubold and Mathai 1984).

### 3.6 The Neutrino Emission Rate

In the following evaluations we do not take into account the asymptotic form (3.36) of the thermonuclear reaction rate but the closed-form representation (3.31) with (3.32) for  $\nu = 0$ , and write (3.21) in terms of  $r_{12}(\rho(r), T(r))$ ,

$$L_{12}(R_\odot) = \int_0^{R_\odot} dr 4\pi r^2 E_{12} r_{12}(\rho(r), T(r)). \quad (3.37)$$

If we divide the “internal luminosity”  $L_{12}(R_\odot)$  by the amount of energy  $E_{12}$ , then we get the total number of particles per second  $N_{12}$  liberated in the reaction  $1 + 2 \rightarrow 3 + 4$  in question, that is,

$$(3.38)$$

$$\begin{aligned}
\frac{L_{12}(R_\odot)}{E_{12}} &= N_{12} = 4\pi \int_0^{R_\odot} dr \ r^2 \ r_{12}(\rho(r), T(r)) \\
&= 4\pi \left(1 - \frac{1}{2} \delta_{12}\right) N_A^{\frac{5}{2}} \frac{X_1 X_2 (A_1 A_2)^{\frac{1}{2}}}{(A_1 + A_2)^{3/2}} \frac{8^{\frac{1}{2}}}{\pi} S_{12} R_\odot^3 \\
&\quad \times \int_0^1 dx \ x^2 \rho^2(R_\odot x) (kT(R_\odot x))^{-\frac{1}{2}} \int_0^\infty dy \ e^{-y - z y^{-\frac{1}{2}} (kT(R_\odot x))^{-\frac{1}{2}}},
\end{aligned}$$

where  $x = r/R_\odot$  and

$$z = 2\pi \left(\frac{\mu^*}{2}\right)^{\frac{1}{2}} \frac{Z_1 Z_2 e^2}{h} = E_G^{\frac{1}{2}},$$

where  $E_G$  is the GAMOW energy. The flux of solar neutrinos at the Earth due to the reaction  $1 + 2 \rightarrow 3 + 4$  in the Sun can then be written

$$\phi_{12} = \frac{N_{12}}{4\pi(AU)^2} = \frac{L_{12}(R_\odot)}{4\pi(AU)^2 E_{12}}, \quad (3.39)$$

where  $AU$  abbreviates the mean distance between the Earth and the Sun (Astronomical Unit  $= 1.496 \times 10^{13} \text{ cm}$ ) and  $N_{12}$  is also called the neutrino emission rate.

### 3.7 The Integral for the Solar Nuclear Energy Generation and the Solar Neutrino Fluxes: The General Case of the Basic Integral

Consider the following integral,

$$Q_\delta = \int_0^R dr \ r^p [\rho(r)]^q [kT(r)]^{-t} \int_0^\infty dy \ e^{-ay - z[kT(r)]^{-b} y^{-\frac{n}{m}}}, \quad (3.40)$$

for the case (3.8) of the density distribution and (3.17) of the temperature distribution, respectively;  $\delta > 0, p > 0, t > 0, a > 0, q > 0, z > 0$ . For convenience the inner integral in (3.40) will be rewritten in terms of the MELLIN-BARNES type integral by using a result in Haubold and Mathai (1984) which will be stated here as a lemma (cf. also Haubold and Mathai 1987).

**Lemma 3.1** For  $\alpha > 0, \beta > 0, u > 0, \Re(1 - \gamma + s) > 0$ ,

$$\int_0^\infty dt \ t^{-\gamma} e^{-\alpha t - ut^{-\beta}} = \frac{\alpha^{\gamma-1}}{\beta} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \ \Gamma(s/\beta) \Gamma(1 - \gamma + s) \left(\alpha u^{\frac{1}{\beta}}\right)^{-s}, \quad (3.41)$$

where  $i^2 = -1, c > \Re(\gamma - 1)$ , and  $\Re(\cdot)$  denotes the real part of  $(\cdot)$ .

When  $\beta = n/m, n, m = 1, 2, \dots$  one has  $\Gamma(s/\beta) = \Gamma(ms/n)$ . If  $s/n$  is replaced by  $s'$  and if  $\Gamma(ms')$  and  $\Gamma(1 - \gamma + ns')$  are expanded by using the multiplication formula for gamma functions and by using the resulting quantity if the right hand side of (3.41) is written as a MEIJER's G-function then the result in (3.41) agrees with the result established by Saxena (1960) with the help of transform calculus. A proof of (3.41) by using statistical techniques is given in Haubold and Mathai (1984) and hence the proof won't be repeated here.

From (3.41) we may note that the right hand side of (3.41) is a H-function, see for example Mathai and Saxena (1978), and that it exists for all values of  $\alpha u^{1/\beta} > 0$  and the inner integral in (3.40) for the case (3.8) is dominated by a beta type integral. Hence, by substituting (3.41) in (3.40) and by rewriting (3.40) one has the following:

$$\begin{aligned}
Q_\delta &= a^{-1} (m/n) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \ \Gamma(ms/n) \Gamma(1 + s) \left(\alpha z^{1/\beta}\right)^{-s} \\
&\quad \times \int_0^\infty dr \ r^p [\rho(r)]^q [kT(r)]^{\frac{bs}{\beta} - t},
\end{aligned} \quad (3.42)$$

where  $\beta = n/m$ .

Now the inner integral in (3.42) for the density distribution (3.8) and the temperature distribution (3.17) reduces to the following, denoting the integral by  $I_1$ :

$$\begin{aligned} I_1 &= \int_0^R dr r^p [\rho(r)]^q [kT(r)]^{\frac{bs}{\beta}-t} = R^{p+1} \int_0^1 dx x^p [\rho(Rx)]^q [kT(Rx)]^{\frac{bs}{\beta}-t} \\ &= R^{p+1} \rho_c^q \left[ \frac{4\pi\mu G}{3N_A k} \rho_c R^2 \right]^{\frac{bs}{\beta}-t} \int_0^1 dx x^p (1-x^\delta)^{q+t-\frac{bs}{\beta}} \\ &\quad \times \left\{ \xi - \frac{1}{2} x^2 + \frac{(\delta+6)}{(\delta+2)(\delta+3)} x^{\delta+2} - \frac{3}{2(\delta+1)(\delta+3)} x^{2\delta+2} \right\}^{\frac{bs}{\beta}-t}, \end{aligned} \quad (3.43)$$

and where  $\xi$  is given by (3.14). Making the transformation  $y = x^\delta$  and taking out  $\xi$  one has the following:

$$I_1 = \frac{R^{p+1} \rho_c^q}{\delta} \left[ \frac{4\pi\mu G}{3N_A k} \xi \rho_c R^2 \right]^{\frac{bs}{\beta}-t} \int_0^1 dy y^{\frac{p+1}{\delta}} (1-y)^{q+t-\frac{bs}{\beta}} [1-u(y)]^{\frac{bs}{\beta}-t}, \quad (3.44)$$

where

$$u(y) = \xi^{-1} y^{\frac{2}{\delta}} \left\{ \frac{1}{2} - \frac{(\delta+6)}{(\delta+2)(\delta+3)} y + \frac{3}{2(\delta+1)(\delta+3)} y^2 \right\}, \quad (3.45)$$

where  $\xi$  is given in (3.14).

### 3.8 The Solar Case of the Basic Integral

In the case  $\delta = 2$ ,  $[1 - u(y)]$  of (3.44) reduces to the following form:

$$\begin{aligned} 1 - u(y) &= 1 - \xi^{-1} y \left( \frac{1}{2} - \frac{2}{5} y + \frac{1}{10} y^2 \right), \xi = \frac{1}{5}, \\ &= 1 - \frac{5}{2} y + 2y^2 - \frac{1}{2} y^3 \\ &= (1-y)^2 (1 - \frac{1}{2} y). \end{aligned}$$

Hence (3.44) reduces to the following form, denoted by  $I_2$ :

$$\begin{aligned} I_2 &= \int_0^{R_\odot} dr r^p [\rho(r)]^q [kT(r)]^{\frac{bs}{\beta}-t} = \frac{1}{2} \rho_c^q R_\odot^{p+1} \left[ \frac{4\pi\mu G}{15N_A k} \rho_c R_\odot^2 \right]^{\frac{bs}{\beta}-t} \\ &\quad \times \int_0^1 dy y^{\frac{p+1}{2}-1} (1-y)^{q+\frac{bs}{\beta}-t} (1 - \frac{1}{2} y)^{\frac{bs}{\beta}-t}. \end{aligned} \quad (3.46)$$

But

$$\left( 1 - \frac{1}{2} y \right)^{-(t-\frac{bs}{\beta})} = \sum_{m_1=0}^{\infty} \frac{(t-bs/\beta)_{m_1}}{m_1!} \left( \frac{1}{2} \right)^{m_1} y^{m_1}.$$

Hence

$$\begin{aligned} &\int_0^1 dy y^{\frac{p+1}{2}-1} (1-y)^{q+\frac{bs}{\beta}-t} (1 - \frac{1}{2} y)^{\frac{bs}{\beta}-t} \\ &= \sum_{m_1=0}^{\infty} \frac{(t-bs/\beta)_{m_1}}{m_1!} \left( \frac{1}{2} \right)^{m_1} \frac{\Gamma(m_1+(p+1)/2) \Gamma(1+q-t+bs/\beta)}{\Gamma(1+q-t+m_1+(p+1)/2+bs/\beta)} \end{aligned}$$

by using a beta integral. Thus, we have for  $\delta = 2$ , noting that

(3.47)

$$\begin{aligned} \left(t - \frac{bs}{\beta}\right)_{m_1} &= \frac{\Gamma(t+m_1-bs/\beta)}{\Gamma(t-bs/\beta)}, \\ I_2 &= \frac{1}{2} \rho_c^q R_\odot^{p+1} \left[ \frac{4\pi\mu G}{15N_A k} \rho_c R_\odot^2 \right]^{\frac{bs}{\beta}-t} \Gamma((p+1)/2) \sum_{m_1=0}^{\infty} \frac{((p+1)/2)_{m_1}}{m_1!} \left(\frac{1}{2}\right)^{m_1} \\ &\quad \times \frac{\Gamma(t+m_1-bs/\beta)\Gamma(q-t+1+bs/\beta)}{\Gamma(t-bs/\beta)\Gamma(1+q-t+m_1+(p+1)/2+bs/\beta)}. \end{aligned}$$

Substituting (3.47) in (3.42) we have for  $\delta = 2, \beta = n/m$ ,

$$\begin{aligned} Q_2 &= a^{-1} \left(\frac{m}{n}\right) \frac{1}{2} R_\odot^{p+1} \rho_c^q \left[ \frac{4\pi\mu G}{15N_A k} \rho_c R_\odot^2 \right]^{-t} \\ &\quad \times \Gamma\left(\frac{p+1}{2}\right) \sum_{m_1=0}^{\infty} \frac{[(p+1)/2]_{m_1} (1/2)_{m_1}}{m_1!} \\ &\quad \times \frac{1}{2\pi i} \int_{L_1} ds \frac{\Gamma(ms/n)\Gamma(1+s)\Gamma(1+q-t+bsm/n)\Gamma(t+m_1-bsm/n)}{\Gamma(t-bsm/n)\Gamma(1+q-t+m_1+(p+1)/2+bsm/n)} d^{-s} \end{aligned} \quad (3.48)$$

where  $L_1$  is a suitable contour, and

$$d = az^{\frac{m}{n}} \left( \frac{15N_A k}{4\pi\mu\rho_c R_\odot^2 G} \right)^{\frac{mb}{n}}. \quad (3.49)$$

Note that the integral in (3.48) is nothing but a H-function, see Mathai and Saxena (1978). Hence, we have for  $\delta = 2$ ,

$$\begin{aligned} Q_2 &= a^{-1} \left(\frac{m}{n}\right) \frac{1}{2} R_\odot^{p+1} \rho_c^q \left[ \frac{4\pi\mu G}{15N_A k} \rho_c R_\odot^2 \right]^{-t} \\ &\quad \times \Gamma\left(\frac{p+1}{2}\right) \sum_{m_1=0}^{\infty} \frac{\left(\frac{p+1}{2}\right)_{m_1} \left(\frac{1}{2}\right)_{m_1}}{m_1!} \\ &\quad \times H_{2,4}^{3,1}[d|_{(0,m/n),(1,1),(q-t+1,bsm/n),(1-t,bsm/n)}^{(1-t-m_1,1),(1+q-t+m_1+(p+1)/2,bsm/n)}]. \end{aligned} \quad (3.50)$$

It is easy to note that the H-function exists for all values of  $d > 0$  and that it behaves like the following, see Mathai and Saxena (1973). For small values of  $d$  it behaves like,

$|d|^\alpha, \alpha = \min \{0, 1, \frac{n}{bm}(q-t+1)\} = 0$  for  $q \geq t$ . For large values of  $d$  it behaves like  $|d|^\beta$  where  $\beta = \max \left\{ \frac{n}{mb}(-t-m_1) \right\} = -\frac{nt}{mb}$ . Hence, using this fact we have the following result, noting that

$$\sum_{m_1=0}^{\infty} \frac{[(p+1)/2]_{m_1} (1/2)_{m_1}}{m_1!} = \left(1 - \frac{1}{2}\right)^{-\frac{p+1}{2}} = 2^{\frac{p+1}{2}}.$$

## 3.9 Fox's H-Function

**Definition**

$$\begin{aligned} H_{p,q}^{m,n}(z) &= H_{p,q}^{m,n}[z|_{(b_q, B_q)}^{(a_p, A_p)}] = H_{p,q}^{m,n}[z|_{(b_1, B_1), \dots, (b_q, B_q)}^{(a_1, A_1), \dots, (a_p, A_p)}] = \frac{1}{2\pi i} \int_L ds z^s \psi(s), \\ \psi(s) &= \frac{\{\prod_{j=1}^m \Gamma(b_j - B_j s)\} \{\prod_{j=1}^n \Gamma(1 - a_j + A_j s)\}}{\{\prod_{j=m+1}^q \Gamma(1 - b_j + B_j s)\} \{\prod_{j=n+1}^p \Gamma(a_j - A_j s)\}} \end{aligned}$$

where  $0 \leq n \leq p, 1 \leq m \leq q; A_j, j = 1, \dots, p$  and  $B_j, j = 1, \dots, q$  are real positive numbers,  
 $a_j, j = 1, \dots, p$  and  $b_j, j = 1, \dots, q$  are complex numbers such that  $A_j(b_h + \nu) \neq B_h(b_j - \lambda - 1)$ , for  
 $\nu, \lambda = 0, 1, 2, \dots; h = 1, \dots, m; j = 1, \dots, n$ .  $L$  is a contour separating the points  
 $s = (b_j + \nu)/B_j, j = 1, \dots, m; \nu = 0, 1, \dots$  which are the poles of  $\Gamma(b_j - B_j s), j = 1, \dots, m$ , from the  
points  $s = (a_j - \nu - 1)/A_j, j = 1, \dots, n; \nu = 0, 1, \dots$ , which are the poles of

$\Gamma(1 - a_j + A_j s)$ ,  $j = 1, \dots, n$ . The H-function is an analytic function of  $z$  and makes sense if the following existence conditions are satisfied: (i) for all  $z \neq 0$  with  $\mu > 0$ , and (ii) for  $0 < |z| < \beta^{-1}$  with  $\mu > 0$ ;

$$\mu = \sum_{j=1}^q B_j - \sum_{j=1}^p A_j, \quad \beta = \left\{ \prod_{j=1}^p A_j^{A_j} \right\} \left\{ \prod_{j=1}^q B_j^{-B_j} \right\}.$$

Asymptotic expansions: Orders for small and large values:

$$H_{p,q}^{m,n}(z) \rightarrow O(|z|^c), \text{ for small values of } z$$

for  $\mu \geq 0$  and  $c = \min \{ \Re(b_j/B_j), j = 1, \dots, m \}$ ; and

$$H_{p,q}^{m,n}(z) \rightarrow O(|z|^d) \text{ for large values of } z$$

for  $\mu \geq 0$ ,  $|\arg z| < \alpha\pi/2$ , and  $d = \max \{ \Re((a_j - 1)/A_j), j = 1, \dots, n \}$ , where

$$\begin{aligned} \alpha &= \sum_{j=1}^n A_j - \sum_{j=n+1}^p A_j + \sum_{j=1}^m B_j - \sum_{j=m+1}^q B_j; \\ \beta &= \left\{ \prod_{j=1}^p A_j^{A_j} \right\} \left\{ B_j^{-B_j} \right\}; \\ \gamma &= \sum_{j=1}^q b_j - \sum_{j=1}^p a_j + \frac{p}{2} - \frac{q}{2}; \\ \lambda &= \sum_{j=1}^m B_j - \sum_{j=m+1}^q B_j - \sum_{j=1}^p A_j; \\ \mu &= \sum_{j=1}^q B_j - \sum_{j=1}^p A_j. \end{aligned}$$

### 3.10 Meijer's G-Function

This G-function is a special case of the H-function. When  $A_j = 1$ ,  $j = 1, \dots, p$  and  $B_j = 1$ ,  $j = 1, \dots, q$ , one has

$$H_{p,q}^{m,n}[z] \Big|_{(b_j, 1), j=1, \dots, q}^{(a_j, 1), j=1, \dots, p} = G_{p,q}^{m,n}[z] \Big|_{b_j, j=1, \dots, q}^{a_j, j=1, \dots, p}.$$

The definition as well as the asymptotic expansions for the G-function can be derived from those of the H-function.

**Theorem 3.1** For  $\delta = 2$ ,  $Q_2$  is approximated to the following:

$$Q_2 \approx a^{-1} \left( \frac{m}{n} \right) \frac{1}{2} R_{\odot}^{p+1} \rho_c^q \left[ \frac{4\pi\mu G}{15N_A k} \rho_c R_{\odot}^2 \right]^{-t} \Gamma \left( \frac{p+1}{2} \right) 2^{\frac{p+1}{2}}$$

for small values of  $d$ , and

$$Q_2 \approx a^{-1} \left( \frac{m}{n} \right) \frac{1}{2} R_{\odot}^{p+1} \rho_c^q \left[ \frac{4\pi\mu G}{15N_A k} \rho_c R_{\odot}^2 \right]^{-t} \Gamma \left( \frac{p+1}{2} \right) 2^{\frac{p+1}{2}} d^{-\frac{nt}{mb}}$$

for large values of  $d$ , where  $d$  is defined in (3.49) (cf. also Haubold and Mathai 1987).

### 3.11 Analytic Results Connecting Solar Structure Parameters and Solar Neutrino Emission Rates

Neglecting the nuclear energy generation of the Sun via the CNO cycle, because it provides a negligible contribution to the total energy output of the Sun, we conclude from Eqs. (3.6), (3.7) and (3.20) that

$$\begin{aligned} L(R_\odot) &= 4\pi \int_0^{R_\odot} dr \ r^2 \rho(r) (\epsilon_{PPI} + \epsilon_{PPII} + \epsilon_{PPIII}) \\ &= 4\pi \frac{Q_{PPI}}{2} \int_0^{R_\odot} dr \ r^2 n_3 n_3 <\sigma v>_{33} + 4\pi Q_{PPII} \int_0^{R_\odot} dr \ r^2 n_3 n_4 <\sigma v>_{34} \\ &\quad + 4\pi (Q_{PPIII} - Q_{PPII}) \int_0^{R_\odot} dr \ r^2 n_7 n_1 <\sigma v>_{71}, \end{aligned} \quad (3.51)$$

or, we may write, with Eq. (3.1)

$$\begin{aligned} L(R_\odot) &= 4\pi \frac{Q_{PPI}}{4} \int_0^{R_\odot} dr \ r^2 n_1 n_1 <\sigma v>_{11} \\ &\quad + 4\pi (Q_{PPII} - \frac{1}{2} Q_{PPI}) \int_0^{R_\odot} dr \ r^2 n_3 n_4 <\sigma v>_{34} \\ &\quad + 4\pi (Q_{PPIII} - Q_{PPII}) \int_0^{R_\odot} dr \ r^2 n_7 n_1 <\sigma v>_{71}. \end{aligned} \quad (3.52)$$

According to (3.38) and (3.6) also we obtain the neutrino fluxes of the nuclear reactions of the proton-proton chain in terms of the same quantities  $<\sigma v>$  as included in (3.52).

$$\phi_{11} = \phi(pp) = \frac{N_{11}}{4\pi(AU)^2} = \frac{1}{2(AU)^2} \int_0^{R_\odot} dr \ r^2 n_1 n_1 <\sigma v>_{11}, \quad (3.53)$$

$$\begin{aligned} \phi_{71} &= \phi({}^7Be) = \frac{N_{34} - N_{71}}{4\pi(AU)^2} \\ &= \frac{1}{(AU)^2} \left\{ \int_0^{R_\odot} dr \ r^2 n_3 n_4 <\sigma v>_{34} - \int_0^{R_\odot} dr \ r^2 n_7 n_1 <\sigma v>_{71} \right\}, \end{aligned} \quad (3.54)$$

$$\phi_8 = \phi({}^8B) = \frac{N_{71}}{4\pi(AU)^2} = \frac{1}{(AU)^2} \int_0^{R_\odot} dr \ r^2 n_7 n_1 <\sigma v>_{71}. \quad (3.55)$$

With the representations of the nuclear output of the Sun by (3.52) and the respective neutrino fluxes by (3.53), (3.54), and (3.55) we traced back the computation of these quantities to the evaluation of the basic integral in (3.38). For the solar model discussed in Sect. 3.3 and the closed form representation of the nonresonant thermonuclear reaction rate obtained in Sect. 3.4 we get the neutrino emission rate in (3.38) and (3.50) for  $a = 1, b = \frac{1}{2}, n = 1, m = 2, p = 2, q = 2$ , and  $t = \frac{1}{2}$ .

$$N_{12} = 4\pi \left(1 - \frac{1}{2} \delta_{12}\right) N_A^{\frac{5}{2}} \frac{X_1 X_2 (A_1 + A_2)^{\frac{1}{2}}}{(A_1 A_2)^{3/2}} \frac{8^{\frac{1}{2}}}{\pi} S_{12}(0) Q_2, \quad (3.56)$$

where

$$Q_2 = \left(\frac{\pi}{4}\right)^{\frac{1}{2}} \frac{R_\odot^3 \rho_{\odot}^2}{T_{\odot}^{1/2}} \sum_{m_1=0}^{\infty} \frac{(3.2)_{m_1} (1/2)^{m_1}}{m_1!} H_{2,4}^{3,1} \left[ \frac{z^2}{T_c} \Big|_{(0,2),(1,1),(5/2,1),(1/2,1)}^{(1/2-m_1,1),(4+m_1,1)} \right] \quad (3.57)$$



**Fig. 3.1** Astrophysical Observatory Potsdam (AOP), Germany: Preparations for this book took into account the fact that the AOP celebrated its 150th Anniversary of the establishment of the Astrophysikalisches Observatorium Potsdam on 1st July 1874. AOP was the world's first observatory to emphasize explicitly the research area of astrophysics. Albert A. Michelson, studying under Hermann von Helmholtz at the Berlin University, developed his interferometer and performed the first time the famous Michelson experiment at AOP (cellar of the east dome, right hand-side of the photo) in 1881 supported by the then director of AOP, Herman C. Vogel. The authors are taking the opportunity to thank Dr. Reiner John (1942–2007, Astronomical Observatory Potsdam-Babelsberg) and Dr. Ewald Gerth (1934–2022, Astrophysical Observatory Potsdam-Teufelsberg) for many years of exciting and productive cooperation on the issues of nuclear and neutrino astrophysics as described in this book. <https://www.scirp.org/journal/paperinformation?paperid=134139>

where  $z$  is given in (3.9). The value of the central matter density of the solar model is given in (3.8) and that for the central temperature in (3.18). According to the Theorem 3.1, the approximated value for  $Q_2$  in (3.57) can now be written as

$$Q_2 \approx (2\pi)^{\frac{1}{2}} \frac{R_{\odot}^3 \rho_{c\odot}^2}{T_c^{1/2}} \quad (3.58)$$

for small values of  $d = z^2/T_c$  given in (9.10), and

$$Q_2 \approx (2\pi)^{\frac{1}{2}} R_{\odot}^3 \rho_{c\odot}^2 \frac{1}{z} \quad (3.59)$$

for large values of  $d = z^2/T_c$ , where  $z$  is given in (3.6). Inserting (3.58) into (3.56) we obtain finally the analytic representation of the relationship of the solar structure parameters and a solar neutrino emission rate for small values of the characteristic parameter  $d = z^2/T_c$  in (3.57).

$$N_{12} \approx \left(1 - \frac{1}{2}\delta_{12}\right) 16\pi^{\frac{1}{2}} N_A^{\frac{5}{2}} \frac{(A_1+A_2)^{\frac{1}{2}}}{(A_1 A_2)^{3/2}} S_{12}(0) X_1 X_2 \frac{R_{\odot}^3 \rho_{c\odot}^2}{T_{c\odot}^{1/2}}. \quad (3.60)$$

Inserting (3.59) into (3.56) we get finally the neutrino emission rate for large values of the characteristic parameter  $d = z^2/T_c$  in (3.57) (Fig. 3.1).

$$N_{12} \approx \left(1 - \frac{1}{2}\delta_{12}\right) \left(\frac{27}{\pi}\right)^{\frac{1}{2}} \frac{N_A^3 h}{e^2} \frac{(A_1+A_2)}{(A_1 A_2)^2 Z_1 Z_2} S_{12}(0) X_1 X_2 R_{\odot}^3 \rho_{c\odot}^2. \quad (3.61)$$

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## 4. Solar Nuclear and Neutrino Astrophysics Research, a Time Line

Arak M. Mathai<sup>1</sup> and Hans J. Haubold<sup>2</sup>✉

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1930 Wolfgang Pauli hypothesizes the existence of neutrinos to account for the beta decay energy conservation crisis.

1933 Enrico Fermi writes down the correct theory for beta decay, incorporating the neutrino.

1956 Fred Reines and Clyde Cowan discover (electron anti-) neutrinos using a nuclear reactor.

The possible existence of neutrino oscillations, which is a consequence of neutrino masses and mixing, would be experimental evidence of elementary particle physics beyond the standard model.

1957 Bruno Pontecorvo proposes neutrino-antineutrino oscillations and this is the first time neutrino oscillations are hypothesized. B. Pontecorvo, Neutrino experiments and the problem of conservation of leptonic charge, Sov. Phys. JETP 26, 984 (1968).

1962 Ziro Maki, Masami Nakagawa and Sakata introduce neutrino flavour mixing and flavour oscillations. Z. Maki, M. Nakagawa, and S. Sakata, Remarks on the unified model of elementary particles, Prog. Theor. Phys. 28, 870 (1962).

1962 Muon neutrinos are discovered by Leon Lederman, Mel Schwartz, Jack Steinberger and colleagues at Brookhaven National Laboratories, and it is confirmed that they are different from electron neutrinos.

In the 1960s, the first experiment to detect solar neutrinos was Raymond Davis's Homestake Experiment, in which he observed a deficit in

the flux of solar neutrinos with respect to the prediction of the Standard Solar Model, using a chlorine-based detector. In detecting solar neutrinos in the Homestake Experiment, it became clear that the number detected was much lower than that predicted by models of the solar interior. The problem could be solved by revising the model for the internal structure of the Sun (solar physics), the assumed mechanisms of thermonuclear reactions (nuclear physics), or the properties of neutrinos and understanding the limits of the detection mechanisms (neutrino physics). Most neutrinos passing through the Earth emanate from the Sun. About 65 billion solar neutrinos per second pass through every square centimetre perpendicular to the direction of the Sun in the region of the Earth.

Observations of solar neutrinos were first made by the Homestake experiment using a radiochemical method, and then followed by real-time measurement with KAMIOKANDE-II and other radiochemical experiments using gallium by SAGE and GALLEX/GNO:

R. Davis, Jr., D. S. Harmer, and K. C. Hoffman, Search for neutrinos from the Sun, *Phys. Rev. Lett.* 20, 1205 (1968).

K. S. Hirata, T. Kajita, T. Kifune, K. Kihara, M. Nakahata, K. Nakamura et al. (The Kamiokande-II Collaboration), Observation of  $^8B$  solar neutrinos in the Kamiokande-II detector, *Phys. Rev. Lett.* 63, 16 (1989).

A. I. Abazov, O. L. Anosov, E. L. Faizov, V. N. Gavrin, A. V. Kalikhov, T. V. Knodel et al., Search for neutrinos from Sun using the reaction  $^{71}Ga$  (electron-neutrino e-)  $^{71}Ge$ , *Phys. Rev. Lett.* 67, 3332 (1991).

P. Anselmann et al. (The GALLEX Collaboration), Solar neutrinos observed by GALLEX at Gran Sasso, *Phys. Lett. B* 285, 376 (1992).

M. Altmann et al. (The GNO Collaboration), GNO solar neutrino observations: Results for GNO I, *Phys. Lett. B* 490, 16 (2000).

1963 R. F. Stein and A. G. W. Cameron (Eds.), *Stellar Evolution, Proceedings of an international conference, November 13–15, 1963*, Plenum Press 1966.

1964 John Bahcall and Raymond Davis discuss the feasibility of measuring neutrinos from the Sun and made the case for the Homestake Mine experiment.

1968 Raymond Davis and colleagues get first radiochemical solar neutrino results using cleaning fluid in the Homestake Mine in North Dakota, leading to the observed deficit subsequently known as the “solar neutrino problem”.

1972 F. Reines (Ed.), *Cosmology, Fusion @ Other Matters, George Gamow Memorial Volume*, Colorado Associated University Press 1972.



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2 September 1982

Dr. H.J. Haubold  
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German Democratic Republic

Dear Dr. Haubold,

Thanks for your letter dated 14 August 1982 which reached me here just now. I am enclosing two copies of the corrected bibliography of our book.

I do not have ready references for integrals involving functions of the type  $G_{p,0}^{0,p}$ . One way of tackling this type of integral is to replace the G-function by its Mellin - Barnes representation and then interchange the integral. The conditions are to be checked. In our book we did not work out the conditions in detail. Luke :The Special Functions and Their Approximations:, (Academic Press) Vol.I & II have lots of conditions worked out in detail.

Another way of tackling is to expand the gammas in the G-function by using any one of the asymptotic expansions and then integrate term by term which will also yield computable series representation for the integral.

The postal address of our university has changed. Please use the address given below for fast delivery of mail coming to our university.

If you would like to visit our university and spend a few days here then I can arrange some funds to meet part of your expenses. If you are interested please let me know about the items that you will need such as letters etc from me.

With best wishes  
yours sincerely

*A. M. Mathai*  
Dr. A.M. Mathai  
Professor

Postal address: 805 Sherbrooke Street West, Montreal, PQ, Canada H3A 2K6

**Fig. 4.1** <https://www.growkudos.com/projects/a-m-mathai-centre-for-mathematical-and-statistical-sciences-nurturing-the-love-for-mathematics>

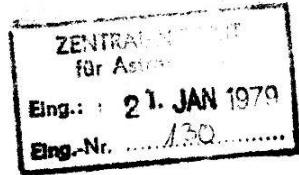
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January 3, 1979



Dr. H. J. Haubold  
Zentralinstitut für Astrophysik der Akademie der  
Wissenschaften der DDR  
Potsdam, East Germany (DDR)

Dear Dr. Haubold,

Thank you for sending me the corrigendum to your paper with Dr. R. W. John. It has occurred to me that in any future work you may wish to refer to C. L. Critchfield's Chapter 11, pg. 186 in Cosmology, Fusion & Other Matters, Editor: F. Reines, Colorado Associated University Press 1972. Critchfield's Eq. (17), corrected for typographical errors is commonly used in many of our calculations when his  $\xi$  (our  $\tau$ ) drops as low as unity but this rarely occurs.

Sincerely yours,

William A. Fowler  
Institute Professor of Physics

WAF:mb

**Fig. 4.2** <https://www.nobelprize.org/prizes/physics/1983/fowler/facts/>

May 22, 1985

Dear Dr. Haubold:

Thank you for your recent letter and for the reprints. The preprint for "Fortschritte" has not arrived at this time.

The only "simple" derivatives I have found for bipolar functions are those in eqs (9.9) and (9.10) of my lecture notes. In generic terms

$$(P^2 u)^2 = [P^2 u + (psK_8 + psK_1)^2][P^2 u + (psK_8 - psK_1)^2]$$

where  $p, q, r$  is any permutation of  $s, d, n$ .

I am enclosing copies of page-proofs that I have of Gamow's autobiography My World Line which Viking Press published in 1970. They tell about the early days of nuclear astrophysics.

In 1938 Bengt Stömgren was at the Carnegie Institution of Washington (Department of Terrestrial Magnetism) and they and the George Washington University devoted their annual conference on theoretical physics to stellar problems. As an authority on nuclear physics Bethe was invited to attend although Hans did not take astrophysics seriously at that time. Some months prior to the meeting I (as graduate student at G.W.U.) had suggested that a deuteron could be formed in the collision between protons in stars in accordance with the Gamow-Teller selection rule for beta-decay. They (Gamow and Teller were my supervisors) told me to forget it because von

(over)

Fig. 4.3 [https://en.wikipedia.org/wiki/Charles\\_Critchfield](https://en.wikipedia.org/wiki/Charles_Critchfield)

Weizsäcker had decided that the idea was not worth pursuing.

During the conference Bethe made that same suggestion whereupon Tamm and Teller told him that they had discouraged me from working on it. Hans then proposed that he and I develop the theory and publish together. That we did, independently, he in Cornell and I in Washington. A few months later, <sup>(after conference)</sup> Hans wrote to us saying that he thought we should collaborate only on the p-p reaction. We could discern that he had discovered something although he did not tell us what it was. It was the carbon-cycle, of course. Our results on the p-p reaction agreed, even to the detail that we both forgot the factor two that allows for either proton to become a neutron. After publication Robert Oppenheimer wrote to Hans reminding him that we had ignored the "matrix element of isotopic spin". I hope you will find these notes of interest.

Currently I am writing up my notes on elliptic functions and extending them into transformation theory (and perhaps to <sup>the</sup> elliptic modular functions) with the thought of possible publication. The theorists at LANL have urged me to do that and I enjoy it.

With very best regards,

Sincerely yours,

Charles L. Bratchfield

1973 A. M. Mathai and R. K. Saxena, Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences, Lecture Notes in Mathematics, Vol. 348, Springer, Berlin-Heidelberg-New York 1973.

In 1974, conversation of HJH with Hans-Juergen Treder makes us aware that the solar neutrino problem is crucial for discovering new physics. Treder's advice was simple: "Ich glaube, dass die Werke der grossen Meister eine staendige Quelle von Ideen und neuen Zielstellungen fuer die aktuelle wissenschaftliche Arbeit sind". A similar advice was ones given by Albert Einstein (Figs. [4.1](#), [4.2](#) and [4.3](#)).

With this advice in mind the topic solar neutrino radiation made us turning to the papers of the Proceedings of the first Solvay Council, held in 1911 focusing on The Theory of Radiation and Quantum. Planck's quantum of action was discovered earlier but quantum mechanics not yet developed, and many questions discussed during the Council were destined to lead to new fundamental physics (or mathematics). Among them Einstein's comment that neither Herr Boltzmann nor Herr Planck has given a definition of  $W$  (probability in Boltzmann's entropy  $S = k \ln W$ ) and Poincare's closing question if it was still possible to represent basic physical laws in terms of ordinary differential equations. We got stuck at this point comparing physics of solar neutrino radiation and physics of black body radiation because we did not have an idea of Planck's "Staubkoernchen" (Pascual Jordan) for neutrino radiation.

1975 A. M. Mathai and P. N. Rathie, Basic Concepts in Information Theory and Statistics: Axiomatic Foundations and Applications, John Wiley & Sons, New Delhi 1975.

Already since 1975 we pursued the analysis of Homestake experiment data by Fourier analysis, wavelet analysis, and Lomb-Spargel periodograms prospectively to discover possible periodic variations in the publicly available data sets of the experiment. This analysis was jointly undertaken in cooperation with Kunitomo Sakurai (Kanagawa University, Yokohama, Japan). We shared the belief that the discovery of periodic variation of the solar neutrino flux may lead to new solar physics or new neutrino physics.

A summary of basic knowledge about solar physics, nuclear physics, and neutrino physics, having the Solar Neutrino Problem in mind, was provided in the first 18 pages of the 1988 edition of our book titled "Modern Problems in Nuclear and Neutrino Astrophysics". Back in 1974 we hoped that such a survey may provide us a way to contribute to the solution of the Solar Neutrino Problem.

1978 A. M. Mathai and R. K. Saxena, The H-function with Applications in Statistics and Other Disciplines, John Wiley and Sons, New Delhi 1978.

1979 we initiated an exchange of information with Raymond Davis Brookhaven National Laboratory asking him to provide us on a regular basis the data from his Homestake chlorine solar neutrino experiment. This cooperation by exchange of letters and visits to Raymond Davis workplace at the University of Pennsylvania was active until 2002 (Figs. [4.4](#) and [4.5](#)).



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March 17, 1983

Academie der Wissenschaften der DDR  
Zentralinstitut für Astrophysik  
1002 Berlin-Buch  
DDR-Lichtenbergstrasse 17a

Dr. H. J. Haubold  
Zentralinstitut für Astrophysik  
DDR-1500  
Potsdam, East Germany

Dear Professor Haubold:

Sorry for the long delay in answering your request for data from our chlorine solar neutrino experiment. I have enclosed copies of our latest reports given at meetings. In addition I am including a copy of our latest data plot and table. The columns in the data Table have the following meanings.

1. Date of the beginning of the exposure.
2. Date of the end of the exposure.
3. The date corresponding to the time at which one-half of the atoms that are collected would be produced assuming a uniform production rate.
4. The most likely value of the  $^{37}\text{Ar}$  production rate in 615 metric tons of  $\text{C}_2\text{Cl}_4$  in atoms per day.
5. The lower limit for the  $^{37}\text{Ar}$  production rate in the same units, corresponds to a  $1\sigma$  error.
6. The upper limit for the  $^{37}\text{Ar}$  production rate in the same units, corresponding to a  $1\sigma$  error.

This table includes all of the data analyzed. The remainder of the experimental runs are still being counted, so I do not have the results for 1982. Above columns 4, 5, and 6 are the combined values, however this average does not include runs nos. 71 and 72. However if these are included there will be very little difference.

If I can be of further help please let me know. Values for 1982 will come out slowly as the counting measurements are completed. If you are interested in the 1982 results please send me a note later on.

Sincerely yours,

*Raymond Davis Jr.*  
Raymond Davis Jr.

jd  
enc.

**Fig. 4.4** <https://www.nobelprize.org/prizes/physics/2002/davis/biographical/>

KANAGAWA UNIVERSITY  
INSTITUTE OF PHYSICS  
ROKKAKUBASHI, YOKOHAMA, 221, JAPAN

September 30, 1989

Prof. H.J. Haubold  
Room S-3260B  
Outer Space Division  
United Nations  
New York, N.Y. 10017  
U.S.A.

Dear Professor Haubold:

It is nice to receive a letter from you I have known for many years since our first meeting in India.

In the letter not dated, you request a copy of my paper entitled "Solar Neutrino Problem as Viewed from the Active Phenomena on the Sun." This is printed in the Proceedings of XVI-th INS Internatl. Symp., held in March 1988. No reprint was made of it, so herewith enclosed is the Xeroxed copy taken from the proceedings. Also enclosed herewith is a sheet of the notice for our new books on cosmic ray astrophysics and neutrino astrophysics. I guess you are interested to see them.

I am now planning to attend the 21st ICRC to be held in Australia in this coming January and to present a paper on the solar neutrinos, in which it is shown that a chaotic process is taking place inside the sun.

I am looking forward hearing from you about your current status some day.

With best wishes.



Kunitomo Sakurai  
Professor of Physics

*Fig. 4.5* <https://www.walshmedicalmedia.com/open-access/kunitomo-sakurai-solar-neutrino-problems-how-they-were-solved-2332-2519-1000132.pdf>

At the same time, we also took up a problem that was discussed in the 1970s concerning the closed-form evaluation of thermonuclear reaction

rates described in publications of Bethe, Salpeter, and Fowler and explicitly discussed by Charles Critchfield. Critchfield's paper can be considered, in principle, as a follow-up research result to his paper published in cooperation with Hans Bethe in 1939 that led to the Nobel Prize for Bethe in 1967 for his work on the production of energy in stars.

1980s The Kamioka experiment is built in a zinc mine in Japan.

1983 William A. Fowler receives the Nobel Prize in Physics for his theoretical and experimental studies of the nuclear reactions of importance in the formation of the chemical elements in the universe: W.A. Fowler, Experimental and theoretical nuclear astrophysics: the quest for the origin of the elements, *Reviewes of Modern Physics* 56 (1984) 149–179.

1985 The “atmospheric neutrino anomaly” is observed by IMB and Kamiokande.

1986 T. Pinch, *Confronting Nature: The Sociology of Solar-Neutrino Detection*, D. Reidel Publishing Company Dordrecht.

1986 Kamiokande group makes first directional counting observation solar of solar neutrinos and confirms deficit.

1987 The Kamiokande and IMB experiments detect burst of neutrinos from Supernova 1987A in the Large Magellanic Cloud, a satellite galaxy of the Milky Way, heralding the birth of neutrino astronomy, and setting many limits on neutrino properties, such as mass.

1987 conversation of HJH with Hans A. Bethe and F. Reines, meeting at the Michelson-Morley Conference 1987 in Cleveland. Our research work started focusing on the Big Bang model, including Big Bang nucleosynthesis and predictions of the cosmic microwave background radiation.

S. Gottloeber, HJH, J. P. Muecket, and V. Mueller, *Early Evolution of the Universe and Formation of Structure*, Akademie-Verlag, Berlin 1990, de Gruyter 2024).

The following is the generalized form of the basic reaction-rate probability integral in the real scalar positive variable case:

$$I_{a,b}(\delta, \rho) = \int_0^{\infty} v^{\gamma-1} e^{-av^{\delta}-bv^{-\rho}} dv$$

for  $a > 0, b > 0, \delta > 0, \rho > 0, \Re(\gamma) > 0$ . Before the integral is evaluated, let us consider some particular cases. For  $\delta = 1, \rho = 1$ , it is the basic Bessel integral. For  $\delta = 1, \rho = 1$ , the integrand, normalized, is the inverse Gaussian density. For  $\delta = 1, \rho = \frac{1}{2}$ , it is the reaction-rate probability integral in nuclear reaction-rate theory. For  $\delta = 1$  and general  $\rho$ , it is Krätsel integral

and Krätsel transform is associated with it. The integral is also known in the literature by different names such as generalized gamma integral, ultra gamma integral and super gamma integral.

The closed-form representation of the integral can be written in terms of a H-function, namely,

$$g(u) = \frac{1}{\rho \delta a^{\gamma/\delta}} H_{0,2}^{2,0} \left[ a^{\frac{1}{\delta}} u \Big|_{(0, \frac{1}{\rho}), (\frac{\gamma}{\delta}, \frac{1}{\delta})} \right], u = b^{\frac{1}{\rho}}$$

where the  $c$  in the contour is such that  $c > 0, \delta > 0, \rho > 0, \Re(\gamma) > 0$ .

November 27, 1989

Dr. Hans Joachim Haubold  
United Nations  
United Nations, NY 10017

Dear Dr. Haubold:

This is in response to your letter of November 14, 1989. I am sorry to have been so long in responding but I have been away.

Since you have seen our first chapter in the Gamow memorial volume, you are already aware that Gamow made a very simple calculation, using the Jeans' criterion, to obtain a value for the diameter of a typical galaxy based on fundamental constants. Robert Herman and I corrected some minor errors in his formulation which we also published in *Nature*, and which is referred to in the memorial volume.

We later attempted to carry the discussion based on the Jeans' criterion a bit further in a paper we wrote in 1967, but which we never published. Perhaps it will be of some interest to you, and I therefore enclose a copy.

Considerations such as your mass scale formula have been made by several investigators, starting, I believe, with Chandrasekhar, as early as 1937. You will find such matters quite adequately reviewed in the book The Anthropic Cosmological Principle by Barrow and Tipler (Oxford University Press 1988 - paperback edition). On page 234 a mass scale formula is given with the exponent zeta instead of your exponent ( $n/2$ ). This formula does give a typical mass figure for a protogalaxy.

If you have read my paper with Gamow on the fundamental constants or my American Scientist article on the subject, you will realize that I believe the Dirac LNH is really based on dimensional considerations in which he used the age of the universe as a characteristic time. It follows then that to form dimensionless ratios one must have some other dimensional constant also vary with the epoch. We now know that the variation of G with the epoch is observationally unacceptable (work of Canuto, Goldberg, Shapiro, etc.). One can use other characteristic times, such as the time of decoupling, which is a number unique to a particular cosmological model. This yields Dirac-like ratios, without the problem of a dependence of anything on the epoch.

I have no problem in using the results of dimensional analysis in the absence of a theory, but feel that one has to be careful not to read too much into the results. For example, over the years people seem to have forgotten the origin of the Planck time, so that it achieves enormous importance in analyses of the early universe.

I do not want to analyze the reprint you sent me in detail, but

*Fig. 4.6* [https://en.wikipedia.org/wiki/Ralph\\_Alpher](https://en.wikipedia.org/wiki/Ralph_Alpher)

do note with some surprise that you have not used the temperature-time relation determined by the dominant radiation field, which seems to me to be appropriate for the temperature in the expanding universe during the time prior to radiation decoupling. (See the paper by Alpher and Herman in Physics Today.) This will surely affect your result.

Sincerely,

*Ralph A. Alpher*

Ralph A. Alpher  
Department of Physics  
Union College  
Schenectady, NY 12308

CC: Dr. Robert Herman

1991 SAGE (in Russia) and GALLEX (in Italy) confirm the solar neutrino deficit in radiochemical experiments.

1995 H. J. Haubold and A. M. Mathai, A Heuristic Remark on the Periodic Variation in the Number of Solar Neutrinos Detected on Earth, *Astrophysics and Space Science* 228, 113–134.

1995 With Reiner John, Ewald Gerth, under the guidance of Arak M. Mathai, we stopped analyzing the data of Davis's Homestake experiment and turned our attention to the operation of the SuperKamiokande experiment and the analysis of publicly made available solar neutrino data.

1996 SuperKamiokande, the largest particle detector ever, begins searching for neutrino interactions on 1 April at the site of the Kamioka experiment, with a Japan-US team.

1998 Analysing more than 500 d of data, the SuperKamiokande team reports evidence of oscillations in atmospheric neutrinos implying that neutrinos have nonzero mass, thus suggesting physics beyond the Standard Model of Particle Physics.

2001 The Sudbury Neutrino Observatory (SNO) reported observation of neutral currents from solar neutrinos, along with charged currents and elastic scatters, providing convincing evidence that neutrino oscillations are the cause of the solar neutrino deficit.

An initial indication of solar neutrino oscillations was obtained from the difference between the 8B solar neutrino fluxes as measured in the elastic-scattering channel at SuperKamiokande and the charged-current channel at the Sudbury Neutrino Observatory in 2001. Solar neutrino oscillation was subsequently established by including neutral-current measurements from

SNO. Solar neutrino oscillations were confirmed using reactor antineutrinos by KamLAND (Fig. 4.6):

S. Fukuda et al. (The SuperKamiokande Collaboration), Solar  ${}^8B$  and hep neutrino measurements from 1258 d of SuperKamiokande data, *Phys. Rev. Lett.* 86, 5651 (2001).

Q. R. Ahmad et al. (The SNO Collaboration), Measurement of the rate of  $\nu_e + d \rightarrow p + p + e^-$  interactions produced by  ${}^8B$  solar neutrinos at the Sudbury Neutrino Observatory, *Phys. Rev. Lett.* 87, 071301 (2001).

Q. R. Ahmad et al. (The SNO Collaboration), Direct evidence for neutrino flavor transformation from neutral current interactions in the Sudbury Neutrino Observatory, *Phys. Rev. Lett.* 89, 011301 (2002).

K. Eguchi et al. (The KamLAND Collaboration), First results from Kamland: Evidence for reactor antineutrino disappearance, *Phys. Rev. Lett.* 90, 021802 (2003).

2002 Raymond Davis and Masatoshi Koshiba won shares of the Nobel Prize in Physics for their roles in the detection of neutrinos from the Sun and Supernova 1987A: R. Davis, Jr., A half-century with solar neutrinos, *Reviewes of Modern Physics* 75 (2003) 985–994.

2010 N. Scafetta, Fractal and Diffusion Entropy Analysis of Time Series: Theory, concepts, applications and computer codes for studying fractal noises and Levy Walk signals, VDM Verlag Dr. Mueller, Saarbruecken 2010.

2014 Analysis of Solar Neutrino Data from SuperKamiokande I and II, *Entropy* 16, 1414–1425.

In physics one is used to be confronted with either difficult experimental results or ambitious theoretical considerations in one way or another. The time series of data from solar neutrino experiments like Homestake (US) and SuperKamiokande (Japan) appear to be noisy and irregular. The authors have analysed solar neutrino data from the two experiments since 1974 by applying mathematically rigorous Fourier analysis, wavelet analysis, Lomb-Scargle periodograms. In the case of the SuperKamiokande solar neutrino data we made the decision to study the data with Standard Deviation Analysis (SDA) and Diffusion Entropy Analysis (DEA). The purpose is to study the scaling exponent of a complex time series that may manifest long-range correlations and fractal statistics. Scaling analysis (SDA, Hurst analysis, Detrended Fluctuation Analysis, etc.) relays on the assumption that physical data are characterised by fractal Brownian memory. Such methods of scaling analysis are based on the evaluation of the variance of a diffusion process. DEA does NOT relay on the assumption of fractal Brownian memory. DEA evaluates the scaling

exponent of the probability density function (pdf) through the Shannon entropy of the diffusion process generated by those fluctuating data. SDA and similar methodologies evaluate only the scaling of the variance and NOT the pdf scaling. One can call “H” (Hurst) the scaling exponent detected by means of the variance-based methods (SDA) and “delta” the scaling exponent detected by DEA. Our results show that “H” and “delta” are not identical. This means that the SuperKamiokande data are not characterized by fractal Gaussian statistics. It further means that the data are characterized by Levy statistics. Additionally, we found that “delta”  $< 1$  which clearly indicated Levy flights and super-diffusion. This research result, by analysing SuperKamiokande data, did even allow us to identify the fractal differential equation that is guiding the now called solar neutrino probability density function (pdf). This pdf we identified in terms of a Fox H-function  $H[1,1;2,2]$ . The solar neutrino pdf was discovered by analysing SuperKamiokande experiment data. The opposite (and most welcome!) approach to the SuperKamiokande experiment has been published in papers titled

Probability Density Function for Neutrino Masses and Mixings, Phys. Rev. D94, 115004 (2016) and

Mellin Transform Approach to Rephasing Invariants, Phys. Rev. D102, 036001 (2020).

The authors discover the solar neutrino probability density in terms of Fox’ H-function by assuming the probability density function:

$$p(x, t) = \frac{1}{t^\delta} f\left(\frac{x}{t^\delta}\right), \quad (4.1)$$

where  $\delta$  denotes the scaling exponent of the pdf. In the variance based methods, scaling is studied by direct evaluation of the time behavior of the variance of the diffusion process. If the variance scales, one would have

$$\sigma_x^2(t) \sim t^{2H}, \quad (4.2)$$

where  $\sigma_x^2(t)$  is the variance of the diffusion process and where  $H$  is the Hurst exponent. To evaluate the Shannon entropy of the diffusion process at time  $t$ , defined  $S(t)$  as

$$S(t) = - \int_{-\infty}^{+\infty} dx p(x, t) \ln p(x, t) \quad (4.3)$$

and with the previous  $p(x, t)$  one has

The solution of the generalized diffusion equation

$$\frac{\partial p(x,t)}{\partial t} = K^\alpha \leftarrow_\infty D_x^\alpha p(x,t), \quad (4.4)$$

where  $\leftarrow_\infty D_x^\alpha$  is the fractional Weyl operator, is

$$\begin{aligned} p(x,t) &= \frac{1}{\alpha} \frac{1}{t^{1/\alpha}} \frac{t^{1/\alpha}}{|x|} H_{2,2}^{1,1} \left[ \frac{|x|}{Kt^{1/\alpha}} \middle| \begin{matrix} (1, \frac{1}{\alpha}), (1, \frac{1}{2}) \\ (1, 1), (1, \frac{1}{2}) \end{matrix} \right] \\ &= \frac{1}{\alpha|x|} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(1+\frac{s}{\alpha})\Gamma(-\frac{s}{2})}{\Gamma(-s)\Gamma(1+\frac{s}{2})} \left( \frac{|x|}{Kt^{1/\alpha}} \right)^{-s} ds \end{aligned}$$

2018 A. M. Mathai and H. J. Haubold, Erdelyi-Kober Fractional Calculus: From a Statistical Perspective, Inspired by Solar Neutrino Physics, Springer Briefs in Mathematical Physics 31, Springer, Singapore.

Since these discoveries, Borexino and KamLAND experiments have measured the neutrino fluxes from different solar nuclear fusion processes, such as pp, pep,  $^7Be$ , and Carbon-Nitrogen-Oxygen cycle:

M. Agostini et al. (The Borexino Collaboration), First simultaneous precision spectroscopy of pp,  $^7Be$ , and pep solar neutrinos with Borexino phase-II, Phys. Rev. D 100, 082004 (2019).

M. Agostini et al. (The BOREXINO Collaboration), Comprehensive measurement of pp-chain solar neutrinos, Nature 562, 505 (2018).

A. Gando et al. (The KamLAND Collaboration),  $^7Be$  solar neutrino measurement with KamLAND, Phys. Rev. C 92, 055808 (2015).

M. Agostini et al. (The BOREXINO Collaboration), Experimental evidence of neutrinos produced in the CNO fusion cycle in the Sun, Nature (London) 587, 577 (2020).

All measurements to date are naturally explained by neutrino flavor change due to neutrino oscillations with matter effects predicted by Mikheyev, Smirnov, and Wolfenstein, termed the MSW effect: higher energy neutrinos undergo adiabatic conversion from the electron flavor state to the second mass eigenstate. While neutrino oscillations and MSW effect is consistent with all current solar neutrino measurements, two distinctive predictions are yet to be observed: the characteristic energy dependence of the solar neutrino electron-flavor survival probability  $P_{ee}(E_\nu)$  distortion due to the MSW effect in the Sun and the day/night flux asymmetry induced by the matter effect in the Earth. One of the interests of solar neutrino experiments is to determine the neutrino oscillation parameters of  $\Delta m_{21}^2$  and  $\sin^2 \Delta_{12}$ . Independent of solar neutrino measurements, the

KamLAND experiment used reactor anti-neutrinos to measure the same oscillation parameters, assuming CPT symmetry holds:

S. P. Mikheyev and A. Y. Smirnov, Resonance amplification of oscillations in matter and spectroscopy of solar neutrinos, Sov. J. Nucl. Phys. 42, 913 (1985).

L. Wolfenstein, Neutrino oscillations in matter, Phys. Rev. D 17, 2369 (1978).

A. J. Baltz and J. Weneser, Effect of transmission through the Earth on neutrino oscillations, Phys. Rev. D 35, 528 (1987).

J. Bouchez, M. Cribier, J. Rich, M. Spiro, D. Vignaud, and W. Hampel, Matter effects for solar neutrino oscillations, Z. Phys. C 32, 499 (1986).

E. D. Carlson, Terrestrially enhanced neutrino oscillations, Phys. Rev. D 34, 1454 (1986).

M. Cribier, W. Hampel, J. Rich, and D. Vignaud, MSW regeneration of solar  $\nu_e$  in the Earth, Phys. Lett. B 182, 89 (1986).

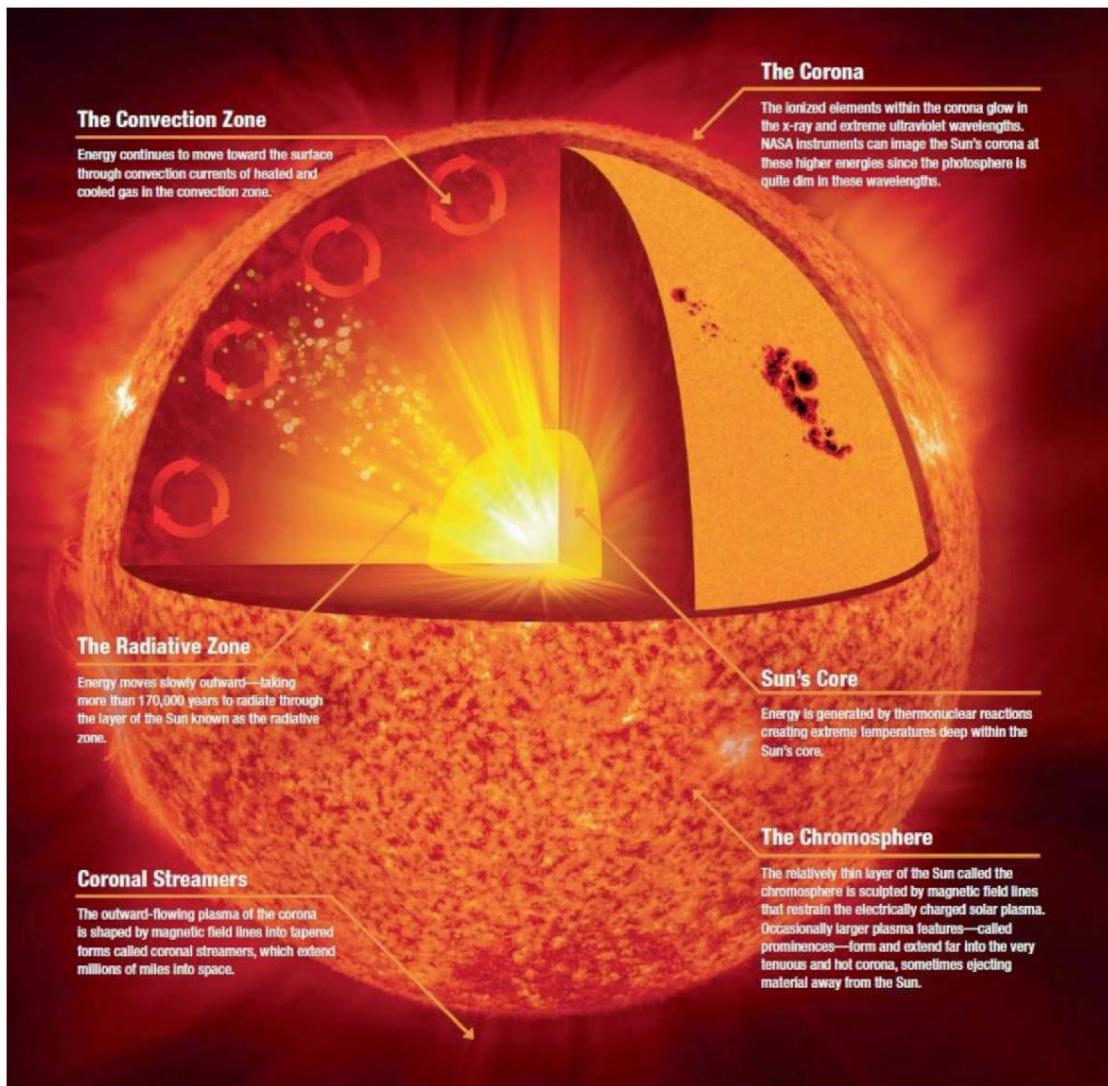
S. T. Petcov, Diffractive like (or parametric resonance like?) enhancement of the Earth (day night) effect for solar neutrinos crossing the Earth core, Phys. Lett. B 434, 321 (1998).

P. Bakhti and A. Y. Smirnov, Oscillation tomography of the Earth with solar neutrinos and future experiments, Phys. Rev. D 101, 123031 (2020).

A. Gando, Y. Gando, H. Hanakago, H. Ikeda, K. Inoue, K. Ishidoshiro et al. (The KamLAND Collaboration), Reactor on-off antineutrino measurement with KamLAND, Phys. Rev. D 88, 033001 (2013).

Many subsequent radiochemical and water Cherenkov detectors confirmed the deficit, but neutrino oscillation was not conclusively identified as the source of the deficit until the Sudbury Neutrino Observatory provided clear evidence of neutrino flavour change in 2001. Solar neutrinos have energies below 20 MeV and travel an astronomical unit between the source in the Sun and detector on the Earth. At energies above 5 MeV, solar neutrino oscillations actually take place in the Sun through a resonance known as the MSW effect, a different process from the vacuum oscillation. The transition between the low energy regime (the MSW effect is negligible) and the high energy regime (the oscillation probability is determined by matter effects) lies in the region of about 2 MeV for the solar neutrinos. The MSW effect is important at the very large electron densities of the Sun where electron neutrinos are produced. The high-energy neutrinos seen, for example, in Sudbury Neutrino Observatory and in SuperKamiokande, are produced mainly as the higher mass

eigenstate in matter and remain as such as the density of solar material changes. When neutrinos go through the MSW resonance the neutrinos have the maximal probability to change their nature, but it happens that this probability is negligibly small this is sometimes called propagation in the adiabatic regime.



**Fig. 4.7** Nuclear Astrophysics: The diagram shows the internal structure of the Sun. The interior of the Sun is a ball of swirling hot plasma that is held together by a balance of forces between gravity and pressure. The dense gases inside the Sun are so massive that they create a strong gravitational pull, which helps to keep solar material from escaping. As a counter force, the expanding hot gases create a large amount of pressure pushing outward toward the Sun's surface. The push and pull between gravity and pressure create conditions that maintain the three interior regions of the Sun: the core, the radiative zone, and the convective zone. The core is the centre of the Sun and extends about a quarter of the way to the surface. About half of the Sun's mass is within the core. Even though the core is made of gas, it is 10 times denser than lead. It is also the hottest region of the Sun, about 15 million degrees Celsius (27 million degrees Fahrenheit). The Sun's core is the only place in our solar system where the temperature and density conditions are high enough for nuclear fusion reaction to occur naturally. The nuclear

reactions within the core fuse hydrogen atoms into helium atoms, releasing extremely large amounts of energy in the process. Some of the energy that is created in the core travels to the surface of the Sun through the Sun's atmosphere, and out into space, enough of it reaching Earth's surface to sustain life. An in terms of physics, very important by-product of the nuclear reactions are the neutrinos

The Chap. 5 “Nuclear Astrophysics, 2025 Update”, first, gives the explicit evaluation of the basic reaction-rate probability integral and its representations in computable series form. Then, a more general form of the reaction-rate probability integral, which can be called the generalized Bessel integral, is examined. Here also, computable series forms are given. Connection of this model to Krätzel integral, Krätzel transform, inverse Gaussian density etc. is established. Next, Mathai's pathway extension of the reaction-rate probability integral is explored. The discussion so far is for real scalar variable situations. A multivariate extension or a  $p$ -variate model for the generalized reaction-rate integral is explored. This is then extended to the corresponding integral in the complex domain. Next, a real  $p \times q$  matrix-variate integral is discussed, which can be considered as the real matrix-variate extension of the reaction-rate probability integral. Then, the matrix-variate integral is extended to the complex domain. All the above topics can be considered as Mellin convolutions and M-convolutions of a product involving generalized gamma functions as the basic functions. Next, other functions are incorporated into a Mellin convolution of a product. It is shown that when a type-1 beta form of the function is considered, one can reach fractional integral of the second kind. This idea is extended to real and complex matrix-variate cases. Then, a Mellin convolution of a ratio is examined and its connections to fractional integral of the first kind are established. These ideas are extended to matrix-variate cases in the real and complex domains. The purpose of this chapter is to discuss mathematical aspects connected with generalized reaction-rate probability integral so that physicists and others can explore the possibility of the corresponding new physics or communication theory.

The Chap. 6 “Neutrino Astrophysics, 2025 Update: The Entropic Approach to Solar Neutrinos” is utilizing data from the SuperKamiokande solar neutrino detection experiment and analyses them by diffusion entropy analysis and standard deviation analysis. The main result of analysis indicates that solar neutrinos are subject to Lévy flights (anomalous diffusion, super-diffusion). Subsequently the chapter derives the probability density function and the governing fractional diffusion equation (fractional Fokker-Planck Equation) for solar neutrino Lévy flights.

## The conclusion is Does SuperKamiokande Observe Lévy Flights of Solar Neutrinos?

The Chap. 7 “Neutrino Astrophysics, 2025 Update: Neutrino Masses and Mixings” is addressing one of the highest priorities in fundamental particle physics concerning the discovery of non-zero neutrino masses that remains one of the very few hints regarding the nature of physics beyond the Standard Model of Elementary Particle Interactions. The questions about neutrino properties still remain unanswered, such as the their absolute masses, the ordering of the mass states and the charge-parity violating phase. The neutrino sector of the seesaw-modified Standard Model is investigated under the anarchy principle. The anarchy principle leading to the seesaw ensemble is studied analytically with tools of random matrix theory. The probability density function is obtained (Fig. 4.7).



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## 5. Nuclear Astrophysics, 2025 Update

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### 5.1 Explicit Evaluation of the Thermonuclear Reaction-Rate Probability Integrals

For the sake of ready reference, the basic materials will be restated here. Let  $x_1 > 0$  and  $x_2 > 0$  be two real scalar positive variables with the associated functions  $f_1(x_1)$  and  $f_2(x_2)$  respectively. Let the joint function of  $x_1$  and  $x_2$  be  $f_1(x_1)f_2(x_2)$ , the product. If  $x_1 > 0$  and  $x_2 > 0$  are real scalar random variables with the densities  $f_1(x_1)$  and  $f_2(x_2)$ , then we say that  $x_1$  and  $x_2$  are statistically independently distributed when we take the joint density as  $f_1(x_1)f_2(x_2)$ , the product. Let  $u = x_1x_2$  the product. Consider the transformation  $u = x_1x_2$ ,  $v = x_2$ . Then, we can see that the wedge product of differentials are connected by the relation  $dx_1 \wedge dx_2 = \frac{1}{v}du \wedge dv$  and then the marginal function of  $u$ , denoted by  $g(u)$ , is given by the following:

$$g(u) = \int_0^\infty \frac{1}{v} f_1\left(\frac{u}{v}\right) f_2(v) dv = \int_0^\infty \frac{1}{v} f_1(v) f_2\left(\frac{u}{v}\right) dv \quad (5.1)$$

where the second form of the integral is obtained by taking the transformation as  $u = x_1x_2$  and  $v = x_1$ . Now, consider the evaluation of the following integral denoted by  $I_{a,b}$ :

$$I_{a,b} = \int_0^\infty v^{\gamma-1} e^{-av-bv^{-\frac{1}{2}}} dv. \quad (5.2)$$

This (5.2) is the basic reaction-rate probability integral in nuclear reaction-rate theory (Mathai and Haubold 1988). Let us denote the Mellin transform of a real-valued scalar function  $f(x)$  of the real scalar variable  $x$ , with Mellin parameter  $s$ , as  $M_f(s)$ , that is,  $M_f(s) = \int_0^\infty x^{s-1} f(x) dx$  whenever the integral is convergent. In order to evaluate the integral in (5.2) by using (5.1), let us take

$$f_1(x_1) = e^{-x_1^{\frac{1}{2}}}, x_1 > 0 \Rightarrow M_{f_1}(s) = \int_0^\infty x_1^{s-1} e^{-x_1^{\frac{1}{2}}} dx_1 = 2\Gamma(2s), \Re(s) > 0$$

where  $\Re(\cdot)$  denotes the real part of  $(\cdot)$ , and let

$$\begin{aligned} f_2(x_2) &= x_2^\gamma e^{-ax_2}, x_2 > 0, a > 0 \Rightarrow \\ M_{f_2}(s) &= \int_0^\infty x_2^{\gamma+s-1} e^{-ax_2} dx_2 = \Gamma(\gamma+s)a^{-(\gamma+s)}, \Re(s) > -\Re(\gamma). \end{aligned} \quad (5.3)$$

Note that

$$f_1\left(\frac{u}{v}\right) = e^{-(\frac{u}{v})^{\frac{1}{2}}} = e^{-bv^{-\frac{1}{2}}} \Rightarrow b = u^{\frac{1}{2}} \text{ or } u = b^2$$

and  $\frac{1}{v}f_2(v) = \frac{1}{v}v^\gamma e^{-av} = v^{\gamma-1}e^{-av}$ . Note that, by taking the Mellin transform of  $g$  in (5.1), we have  $M_g(s) = M_{f_1}(s)M_{f_2}(s)$ , which is the Mellin convolution of a product property also. This property is established easily.

$$\begin{aligned} M_g(s) &= \int_0^\infty u^{s-1} g(u) du = \int_{u=0}^\infty \left[ \int_{v=0}^\infty \frac{1}{v} f_1\left(\frac{u}{v}\right) f_2(v) dv \right] du \\ &= \int_0^\infty \int_0^\infty x_1^{s-1} f_1(x_1) x_2^{s-1} f_2(x_2) dx_1 \wedge dx_2 = M_{f_1}(s)M_{f_2}(s) \end{aligned} \quad (5.4)$$

where we have reused the same transformation  $u = x_1 x_2$  with the associated Jacobian  $dx_1 \wedge dx_2 = \frac{1}{v} du \wedge dv$  at the second line of the above derivation. Now, from (5.3) and (5.4), we have

$$M_g(s) = M_{f_1}(s)M_{f_2}(s) = 2\Gamma(2s)a^{-(\gamma+s)}\Gamma(\gamma+s), \Re(s) > 0, \Re(s) > -\Re(\gamma). \quad (5.5)$$

Then, by taking the inverse Mellin transform of (11.5) we have  $g$ . That is,

$$g(u) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2}{a^\gamma} \Gamma(2s)\Gamma(\gamma+s)(au)^{-s} ds \quad (5.6)$$

where  $i = \sqrt{(-1)}$  and  $c$  in the contour is any real number  $> 0$ . This, (5.6) can be written as a H-function. That is,

$$g(u) = \frac{2}{a^\gamma} H_{0.2}^{2,0} [au]_{(0.2),(\gamma,1)}, u \geq 0. \quad (5.7)$$

For the theory and applications of the H-function, see Mathai et al. (2010). By using the duplication formula for the gamma function, see for example, Mathai (1993), one can write

$$2\Gamma(2s) = \pi^{-\frac{1}{2}} 2^{2s} \Gamma(s) \Gamma\left(s + \frac{1}{2}\right). \quad (5.8)$$

Now, the integrand in (5.6) can be written as

$$\frac{1}{a^\gamma \pi^{\frac{1}{2}}} \Gamma(s) \Gamma\left(s + \frac{1}{2}\right) \Gamma(\gamma + s) \left(\frac{a}{4} u\right)^{-s}.$$

which enables us to write the H-function in terms of a G-function, that is,

$$g(u) = \frac{1}{a^\gamma \pi^{\frac{1}{2}}} G_{0,3}^{3,0} \left[ \frac{au}{4} \middle|_{0, \frac{1}{2}, \gamma} \right], u \geq 0. \quad (5.9)$$

For the theory and applications of the G-function, see, for example, Mathai (1993). Note that in the gamma product  $\Gamma(s) \Gamma(s + \frac{1}{2}) \Gamma(\gamma + s)$  the poles are simple as long as  $\gamma$  is not an integer or half-integer. In this case, one can write the G-function representation in (5.9) in terms of simple hypergeometric series. Note that the poles of  $\Gamma(s)$  are at  $s = -\nu, \nu = 0, 1, \dots$ . The poles of  $\Gamma(\frac{1}{2} + s)$  are at  $s = -\frac{1}{2} - \nu, \nu = 0, 1, \dots$  and the poles of  $\Gamma(\gamma + s)$  are at  $s = -\gamma - \nu, \nu = 0, 1, \dots$ . Hence, for  $\gamma \neq \frac{\nu}{2}, \nu = 0, 1, \dots$  the sum of the residues at the poles of  $\Gamma(s)$  is the following, denoted by  $S_1$ :

$$\begin{aligned} S_1 &= \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} \Gamma\left(\frac{1}{2} - \nu\right) \Gamma(\gamma - \nu) \left(\frac{a}{4} u\right)^\nu \\ &= \Gamma\left(\frac{1}{2}\right) \Gamma(\gamma) \sum_{\nu=0}^{\infty} \frac{1}{(1-\gamma)_\nu (\frac{1}{2})_\nu} \left(\frac{a}{4} u\right)^\nu \\ &= \Gamma\left(\frac{1}{2}\right) \Gamma(\gamma) {}_0F_2\left( \begin{matrix} \cdot \\ 1 - \gamma, \frac{1}{2} \end{matrix} ; -\frac{a}{4} u \right), u = b^2, \end{aligned} \quad (5.10)$$

where, for example,  $(a)_m = a(a+1)\cdots(a+m-1)$ ,  $a \neq 0$ ,  $(a)_0 = 1$  is the Pochhammer symbol. For writing (5.10) in terms of the hypergeometric series  ${}_0F_2$ , we have used basically two properties:

$$\begin{aligned} \lim_{z \rightarrow -\nu} (z + \nu) \Gamma(z) &= \lim_{z \rightarrow -\nu} (z + \nu) \frac{(z + \nu - 1) \cdots z}{(z + \nu - 1) \cdots z} \Gamma(z) \\ &= \lim_{z \rightarrow -\nu} \frac{\Gamma(z + \nu + 1)}{[(z + \nu - 1) \cdots z]} = \frac{(-1)^\nu}{\nu!} \Gamma(1) = \frac{(-1)^\nu}{\nu!}. \end{aligned} \quad (5.11)$$

$$\begin{aligned} \Gamma(z) &= (z - 1)(z - 2) \cdots (z - \nu) \Gamma(z - \nu) \Rightarrow \\ \Gamma(z - \nu) &= \frac{\Gamma(z)}{(z - 1)(z - 2) \cdots (z - \nu)} \\ &= \frac{(-1)^\nu}{(1 - z)(1 - z + 1) \cdots (1 - z + \nu - 1)} = \frac{(-1)^\nu}{(1 - z)_\nu} \end{aligned} \quad (5.12)$$

whenever  $\Gamma(z)$  and  $\Gamma(z - \nu)$  are defined. Now, evaluating the sum of the residues at the poles of  $\Gamma(s + \frac{1}{2})$ , that is at  $s = -\frac{1}{2} - \nu, \nu = 0, 1, \dots$ , we have the following, denoted by  $S_2$ :

(5.13)

$$\begin{aligned}
S_2 &= \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{\nu!} (au/4)^{\frac{1}{2}} \Gamma\left(-\frac{1}{2} - \nu\right) \Gamma\left(\gamma - \frac{1}{2} - \nu\right) (au/4)^{\nu} \\
&= (au/4)^{\frac{1}{2}} \Gamma\left(-\frac{1}{2}\right) \Gamma\left(\gamma - \frac{1}{2}\right) {}_0F_2\left(\begin{array}{l}; \frac{3}{2}, \frac{3}{2} - \gamma; -au/4\end{array}\right), u = b^2.
\end{aligned}$$

The sum of the residues at the poles of  $\Gamma(s + \gamma)$  or at the points  $s = -\gamma - \nu, \nu = 0, 1, \dots$ , is the following, denoted by  $S_3$ , by using steps parallel to the ones used in (5.10) and (5.13):

$$\begin{aligned}
S_3 &= \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{\nu!} \left(\frac{au}{4}\right)^{\gamma} \Gamma(-\gamma - \nu) \Gamma\left(\frac{1}{2} - \gamma - \nu\right) \left(\frac{au}{4}\right)^{\nu} \\
&= \Gamma(-\gamma) \Gamma\left(\frac{1}{2} - \gamma\right) \left(\frac{au}{4}\right)^{\gamma} {}_0F_2\left(\begin{array}{l}; 1 + \gamma, \frac{1}{2} + \gamma; -\frac{au}{4}\end{array}\right), u = b^2.
\end{aligned} \tag{5.14}$$

From (5.10), (5.13) and (5.14), we have the following series form for  $I_{a,b}$ , which will be stated as a theorem.

**Theorem 5.1** For  $a > 0, b > 0, \Re(\gamma) > 0$ , the reaction-rate probability integral

$$\begin{aligned}
I_{a,b} &= \int_0^{\infty} x^{\gamma-1} e^{-ax-bx^{-\frac{1}{2}}} dx = \frac{2}{a^{\gamma}} H_{0,2}^{2,0} \left[ ab^2 \Big|_{(0,2),(\gamma,1)} \right], u \geq 0 \\
&= a^{-\gamma} \pi^{-\frac{1}{2}} G_{0,3}^{3,0} \left[ \frac{ab^2}{4} \Big|_{0, \frac{1}{2}, \gamma} \right], u \geq 0 \text{ and for } \gamma \neq \frac{\nu}{2}, \nu = 0, 1, \dots \\
I_{a,b} &= \Gamma\left(\frac{1}{2}\right) \Gamma(\gamma) {}_0F_2\left(\begin{array}{l}; 1 - \gamma, \frac{1}{2}; -\frac{ab^2}{4}\end{array}\right) \\
&\quad + \Gamma\left(-\frac{1}{2}\right) \Gamma\left(\gamma - \frac{1}{2}\right) \left(\frac{ab^2}{4}\right)^{\frac{1}{2}} {}_0F_2\left(\begin{array}{l}; \frac{3}{2}, \frac{3}{2} - \gamma; -\frac{ab^2}{4}\end{array}\right) \\
&\quad + \Gamma(-\gamma) \Gamma\left(\frac{1}{2} - \gamma\right) \left(\frac{ab^2}{4}\right)^{\gamma} {}_0F_2\left(\begin{array}{l}; 1 + \gamma, \frac{1}{2} + \gamma; -\frac{ab^2}{4}\end{array}\right), u \geq 0.
\end{aligned}$$

**Note 5.1** Theorem 5.1 can also be established by using statistical procedures. In this case, consider two real scalar positive random variables  $x_1 > 0, x_2 > 0$ , statistically independently distributed with densities  $f_1(x_1)$  and  $f_2(x_2)$  respectively. Due to independence, their joint density will be the product  $f_1(x_1)f_2(x_2)$ . Let  $u = x_1 x_2$ . Consider the  $(s-1)$ th moment of  $u$ , that is  $E[u^{s-1}]$  where  $E[\cdot]$  means the expected value of  $[\cdot]$ . Due to independence,

$$\begin{aligned}
E(u^{s-1}) &= E(x_1^{s-1}) E(x_2^{s-1}), E(u^{s-1}) = \int_0^{\infty} u^{s-1} g(u) du = M_g(s) \\
E(x_1^{s-1}) &= \int_0^{\infty} x_1^{s-1} f_1(x_1) dx_1 = M_{f_1}(s) \\
E(x_2^{s-1}) &= \int_0^{\infty} x_2^{s-1} f_2(x_2) dx_2 = M_{f_2}(s) \Rightarrow \\
M_g(s) &= M_{f_1}(s) M_{f_2}(s)
\end{aligned}$$

where  $g(u)$  is the density of  $u$ , which is to be determined. We can take the same functions  $f_1(x_1)$  and  $f_2(x_2)$  as in Sect. 5.1 but in this case, multiply by the respective

normalizing constants to construct densities out of those  $f_1$  and  $f_2$ . These, normalizing constants get canceled in the equation  $M_g(s) = M_{f_1}(s)M_{f_2}(s)$  and hence the inverse, namely  $g(u)$  will be one and the same as obtained in Sect. 5.1. The original derivation of the reaction-rate probability integral in 1984, which was reported in Mathai and Haubold (1988), was done through statistical distribution theory as described in this Note 5.1. When  $\gamma$  is an integer or a half-integer, then some of the poles can be of order 1 and the remaining of order 2. In this case, one can obtain computable series forms involving gamma function, psi function and logarithm, details may be seen from Mathai (1993) where the techniques of deriving the residues from poles of all types of orders are given.

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## 5.2 Generalization of the Reaction-Rate Probability Integral

A generalization of the integral  $I_{a,b}$  is the following integral:

$$I_{a,b}(\delta, \rho) = \int_0^\infty v^{\gamma-1} e^{-av^\delta - bv^{-\rho}} dv \quad (5.15)$$

for  $a > 0, b > 0, \delta > 0, \rho > 0, \Re(\gamma) > 0$ . For the evaluation of the integral (5.15) also, we can proceed exactly as in the case of the integral  $I_{a,b}$  of Sect. 5.1. Before the integral is evaluated, let us consider some particular cases. For  $\delta = 1, \rho = 1$ , (5.15) is the basic Bessel integral. For  $\delta = 1, \rho = 1$ , the integrand in (5.15), normalized, is the inverse Gaussian density. For  $\delta = 1, \rho = \frac{1}{2}$ , it is the reaction-rate probability integral in nuclear reaction-rate theory. For  $\delta = 1$  and general  $\rho$ , it is Krätsel integral and Krätsel transform is associated with it. Mathai (2012) has created a statistical density out of (5.15) and studied its properties. We may call (5.15) as a generalized Bessel integral. (5.15) is also known in the literature by different names such as generalized gamma integral, ultra gamma integral and super gamma integral. From the series representations, it is obvious that (5.15) does not belong to the gamma family of functions, see also Mathai (2016). For the evaluation of (5.15), consider

$$f_1(x_1) = e^{-x_1^\rho} \Rightarrow M_{f_1}(s) = \int_0^\infty x_1^{s-1} e^{-x_1^\rho} dx_1 = \frac{1}{\rho} \Gamma\left(\frac{s}{\rho}\right), \Re(s) > 0, \rho > 0 \quad (5.16)$$

$$\begin{aligned} f_2(x_2) &= x_2^\gamma e^{-ax_2^\delta} \Rightarrow M_{f_2}(s) = \int_0^\infty x_2^{\gamma+s-1} e^{-ax_2^\delta} dx_2 \\ &= \frac{1}{\delta} \Gamma\left(\frac{\gamma+s}{\delta}\right) a^{-\left(\frac{\gamma+s}{\delta}\right)}, \Re(s) > -\Re(\gamma), \delta > 0, a > 0, \Re(\gamma) > 0 \end{aligned} \quad (5.17)$$

where  $M_{f_1}(s)$  and  $M_{f_2}(s)$  are the Mellin transforms of  $f_1$  and  $f_2$  respectively, with Mellin parameter  $s$ . Let  $u = x_1 x_2$  and  $v = x_2$  with the joint function  $f_1(x_1) f_2(x_2)$ , the product. Let  $g(u)$  be the function associated with  $u$ . Then,

$$(5.18)$$

$$g(u) = \int_0^\infty \frac{1}{v} f_1\left(\frac{u}{v}\right) f_2(v) dv = \int_0^\infty \frac{1}{v} f_1(v) f_2\left(\frac{u}{v}\right) dv$$

where the second part integral is obtained by taking  $x_1 = v$  in the transformation  $u = x_1 x_2$ . Now, for  $\Re(s) > 0, \Re(s) > -\Re(\gamma), \Re(\gamma) > 0 \Rightarrow \Re(s) > 0$ ,

$$\begin{aligned} M_g(s) &= M_{f_1}(s) M_{f_2}(s) = \frac{1}{\rho \delta a^{\gamma/\delta}} \Gamma\left(\frac{s}{\rho}\right) \Gamma\left(\frac{\gamma+s}{\delta}\right) a^{-\frac{s}{\delta}} \\ &= \frac{1}{\rho \delta a^{\gamma/\delta}} \Gamma\left(\frac{s}{\rho}\right) \Gamma\left(\frac{\gamma}{\delta} + \frac{s}{\delta}\right) a^{-\frac{s}{\delta}}. \end{aligned} \quad (5.19)$$

Therefore, from the inverse Mellin transform,

$$g(u) = \frac{1}{\rho \delta a^{\gamma/\delta}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma\left(\frac{s}{\rho}\right) \Gamma\left(\frac{\gamma}{\delta} + \frac{s}{\delta}\right) \left(a^{\frac{1}{\delta}} u\right)^{-s} ds, u = b^{\frac{1}{\rho}}, i = \sqrt{(-1)}. \quad (5.20)$$

As shown in Sect. 5.1, this (5.20) can be written in terms of a H-function, namely,

$$g(u) = \frac{1}{\rho \delta a^{\gamma/\delta}} H_{0,2}^{2,0} \left[ a^{\frac{1}{\delta}} u \Big|_{(0, \frac{1}{\rho}), (\frac{\gamma}{\delta}, \frac{1}{\delta})} \right], u = b^{\frac{1}{\rho}} \quad (5.21)$$

where the  $c$  in the contour is such that  $c > 0, \delta > 0, \rho > 0, \Re(\gamma) > 0$ . Consider the special case  $\rho = \delta$ . In this case, one can write the H-function in (5.21) in terms of a G-function. Replace  $\frac{s}{\delta}$  by  $s$ , thereby  $s$  by  $\delta s$  and  $ds$  by  $\delta ds$ . Then, we have the following:

$$\begin{aligned} g(u) &= \frac{1}{\delta a^{\gamma/\delta}} \frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} \Gamma(s) \Gamma\left(\frac{\gamma}{\delta} + s\right) (au^\delta)^{-s} ds, c_1 > 0 \\ &= \frac{1}{\delta a^{\gamma/\delta}} G_{0,2}^{2,0} \left[ au^\delta \Big|_{0, \frac{\gamma}{\delta}} \right], u \geq 0, u = b^{\frac{1}{\delta}}, \rho = \delta \end{aligned} \quad (5.22)$$

where  $G(\cdot)$  is the G-function as explained in Sect. 11 or see Mathai (1993). Then, for  $\frac{\gamma}{\delta} \neq \nu, \nu = 0, 1, \dots$  the poles of the integrand in (5.22) are simple and then proceeding as in Sect. 5.1 we have the following result:

**Theorem 5.2** For the  $g(u)$  defined in (vii) above, the following is the explicit computable series form when  $\frac{\gamma}{\delta} \neq \nu, \nu = 0, 1, \dots$ :

$$\begin{aligned} g(u) &= \frac{1}{\delta a^{\gamma/\delta}} \left[ \Gamma\left(\frac{\gamma}{\delta}\right) {}_0F_1\left( \ ; 1 - \frac{\gamma}{\delta}; au^\delta \right) \right. \\ &\quad \left. + \Gamma\left(-\frac{\gamma}{\delta}\right) (au^\delta)^{\frac{\gamma}{\delta}} {}_0F_1\left( \ ; 1 + \frac{\gamma}{\delta}; au^\delta \right), u = b^{\frac{1}{\delta}} \right] \end{aligned} \quad (5.23)$$

where  ${}_0F_1$  is the classical Bessel series. The details of the derivation are omitted because the derivation is parallel to that in Sect. 5.1.

### 5.3 An Extension Through Mathai's Pathway Idea

In a physical system the ideal behavior of a variable  $x$ , under observation, may be governed by the function  $e^{-bx^{-\rho}}$ ,  $b > 0$ ,  $\rho > 0$ . But, in a practical situation the behavior may be  $e^{-bx^{-\rho}}$  or in the neighborhood of this ideal behavior. The ideal function and its neighborhoods are covered by the function

$$(1 + b(q-1)x^{-\rho})^{-\frac{1}{q-1}} \rightarrow e^{-bx^{-\rho}} \text{ for } q \geq 1. \quad (5.24)$$

For  $q < 1$ ,  $q-1 = -(1-q)$ ,  $q < 1$  and then

$$(1 - b(1-q)x^{-\rho})^{\frac{1}{1-q}} \rightarrow e^{-bx^{-\rho}} \text{ when } 1 - b(1-q)x^{-\rho} > 0 \text{ and } q \leq 1. \quad (5.25)$$

Thus, for  $-\infty < q < 1$ , the binomial form in (5.25), for  $1 < q < \infty$ , the binomial form in (5.24), and for  $q \rightarrow 1$ , the exponential form  $e^{-bx^{-\rho}}$ , are all reached through the parameter  $q$ , and either from the binomial form in (5.24) or from the binomial form in (5.25). From these observations, Mathai (2005) proposed a rectangular matrix-variate model, known as Mathai's pathway model. If the pathway model is taken for a real scalar positive variable, then it will be either the model in (5.24), which can switch into (5.25) or vice-versa. Consider the following integral, denoted by  $I_{a,b}(\delta, \rho; q)$ , where the pathway idea is incorporated:

$$\begin{aligned} I_{a,b}(\delta, \rho; q) &= \int_0^\infty v^{\gamma-1} e^{-av^\delta} [1 + b(q-1)v^{-\rho}]^{-\frac{1}{q-1}} dv \\ &= \int_0^{(b(1-q))^{1/\rho}} v^{\gamma-1} e^{-av^\delta} [1 - b(1-q)v^{-\rho}]^{\frac{1}{1-q}}, q < 1, 1 - b(1-q)v^{-\rho} > 0 \\ &= I_{a,b}(\delta, \rho; 1) = I_{a,b}(\delta, \rho) \text{ for } q \rightarrow 1. \end{aligned}$$

Again, we will evaluate the integral  $I_{a,b}(\delta, \rho; q)$  by using the Mellin convolution of a product property. Let us consider the case  $q > 1$  first.

**Case 5.1**  $q > 1$ . Let  $f_1(x_1) = [1 + (q-1)x_1^\rho]^{-\frac{1}{q-1}}$  and  $f_2(x_2) = x_2^\gamma e^{-ax_2^\delta}$ . Then,

$$\begin{aligned} M_{f_1}(s) &= \int_0^\infty x_1^{s-1} [1 + (q-1)x_1^\rho]^{-\frac{1}{q-1}} dx_1, b = u^\rho \\ &= \frac{1}{\rho} [(q-1)]^{-\frac{s}{\rho}} \frac{\Gamma(\frac{s}{\rho}) \Gamma(\frac{1}{q-1} - \frac{s}{\rho})}{\Gamma(\frac{1}{q-1})} \end{aligned} \quad (5.26)$$

for  $q > 1$ ,  $\rho > 0$ ,  $\Re(s) > 0$ ,  $\Re(s) < \frac{\rho}{q-1}$  where the integral is evaluated with the help of a real scalar type-2 beta integral.

$$M_{f_2}(s) = \int_0^\infty x_2^{\gamma+s-1} e^{-ax_2^\delta} dx_2 = \frac{1}{\delta} a^{-\frac{\gamma+s}{\delta}} \Gamma\left(\frac{\gamma+s}{\delta}\right) \quad (5.27)$$

for  $\Re(s) > -\Re(\gamma)$ ,  $\Re(\gamma) > 0$ ,  $a > 0$ ,  $\delta > 0$ . Then, from the Mellin convolution of a product property, the Mellin transform of  $g$ , the function corresponding to  $u = x_1 x_2$ , is the following:

$$M_g(s) = \frac{(a^{\frac{1}{\delta}}(q-1)^{\frac{1}{\rho}})^{-s}}{\delta \rho a^{\gamma/\delta} \Gamma(\frac{1}{q-1})} \Gamma\left(\frac{s}{\rho}\right) \Gamma\left(\frac{\gamma+s}{\delta}\right) \Gamma\left(\frac{1}{q-1} - \frac{s}{\rho}\right) \quad (5.28)$$

for  $\delta > 0$ ,  $\rho > 0$ ,  $a > 0$ ,  $q > 1$ ,  $0 < \Re(s) < \frac{\rho}{q-1}$ ,  $b = u^\rho$ . Hence, from the inverse

Mellin transform, the function  $g(u)$  is available as the following, where

$$c^{-1} = \rho \delta a^{\gamma/\delta} \Gamma\left(\frac{1}{q-1}\right):$$

$$\begin{aligned} g(u) &= c \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma\left(\frac{s}{\rho}\right) \Gamma\left(\frac{\gamma+s}{\delta}\right) \Gamma\left(\frac{1}{q-1} - \frac{s}{\rho}\right) [a^{\frac{1}{\delta}}(q-1)^{\frac{1}{\rho}} u]^{-s} ds \\ &= c H_{1,2}^{2,1} \left[ a^{\frac{1}{\delta}}(q-1)^{\frac{1}{\rho}} u \Big|_{(0, \frac{1}{\rho}), (\frac{\gamma}{\delta}, \frac{1}{\delta})}^{(1-\frac{1}{q-1}, \frac{1}{\rho})} \right], u \geq 0, u = b^{\frac{1}{\rho}}. \end{aligned} \quad (5.29)$$

But, for  $\rho = \delta$ , we may replace  $\frac{s}{\delta}$  by  $s$  thereby  $s$  by  $\delta s$  and  $ds$  by  $\delta ds$ . Then,  $g(u)$  of (5.29) can be written in terms of a G-function as the following:

$$g(u) = \frac{1}{\delta a^{\gamma/\delta} \Gamma(\frac{1}{q-1})} G_{1,2}^{2,1} [ab(q-1) \Big|_{0, \frac{\gamma}{\delta}}^{1-\frac{1}{q-1}}], u \geq 0, \rho = \delta, u = b^{\frac{1}{\rho}}, q > 1. \quad (5.30)$$

That is,  $g(u)$  of (5.30) is given by the following, for  $c_1^{-1} = \delta a^{\gamma/\delta} \Gamma(\frac{1}{q-1})$ ,  $q > 1$ :

$$g(u) = c_1 \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \Gamma\left(\frac{\gamma}{\delta} + s\right) \Gamma\left(\frac{1}{q-1} - s\right) [ab(q-1)]^{-s} ds \quad (5.31)$$

where the  $c$  in the contour is such that  $0 < c < \frac{\rho}{q-1}$ . This G-function in (5.31) can be evaluated as the sum of the residues at the poles of  $\Gamma(s) \Gamma(\frac{\gamma}{\delta} + s)$ . The continuation part gives a divergent series. The poles of  $\Gamma(s) \Gamma(\frac{\gamma}{\delta} + s)$  are simple if

$\frac{\gamma}{\delta} \neq \nu$ ,  $\nu = 0, 1, \dots$ . Hence, in this special case we will write down the computable series form for the function  $g(u)$ . The steps are parallel to those used in Sects. 5.1 and 5.2 and hence the final result will be stated here as a theorem.

**Theorem 5.3** For  $\delta = \rho$ ,  $q > 1$ ,  $\frac{\gamma}{\delta} \neq \nu$ ,  $\nu = 0, 1, \dots$ ,  $g(u)$  of (5.31) above is the following, where  $c_1$  is given in (5.31) above and  $u = b^{\frac{1}{\rho}}$ ,  $\rho = \delta$ ,  $q > 1$ :

$$\begin{aligned} g(u) &= c_1 \left[ \Gamma\left(\frac{1}{q-1}\right) \Gamma\left(\frac{\gamma}{\delta}\right) {}_1F_1\left(\frac{1}{q-1}; 1 - \frac{\gamma}{\delta}; ab(q-1)\right) \right. \\ &\quad \left. + \Gamma\left(\frac{1}{q-1} + \frac{\gamma}{\delta}\right) \Gamma\left(-\frac{\gamma}{\delta}\right) (ab(q-1))^{\frac{\gamma}{\delta}} {}_1F_1\left(\frac{1}{q-1} + \frac{\gamma}{\delta}; 1 + \frac{\gamma}{\delta}; ab(q-1)\right) \right]. \end{aligned} \quad (5.32)$$

Observe that the confluent hypergeometric series  ${}_1F_1$  is convergent for all values of  $ab(q - 1)$ .

**Case 5.2**  $q < 1$ . Let  $f_1(x_1) = [1 - (1 - q)x_1^\rho]^{\frac{1}{1-q}}$ ,  $q < 1$  and  $f_2(x_2) = x_2^\gamma e^{-ax_2^\delta}$ .

Then, their Mellin transforms are the following, where the computational steps are parallel to those in the case  $q > 1$  and hence the details are not given here:

$$M_{f_1}(s) = \frac{1}{\rho} \Gamma\left(\frac{s}{\rho}\right) (1 - q)^{-\frac{s}{\rho}} \frac{\Gamma(\frac{1}{1-q} + 1)}{\Gamma(\frac{1}{1-q} + 1 + \frac{s}{\rho})}$$

for  $\Re(s) > 0, \rho > 0, q < 1$ .

$$M_{f_2}(s) = \frac{1}{\delta} \Gamma\left(\frac{\gamma + s}{\delta}\right) a^{-\frac{\gamma+s}{\delta}}$$

for  $\Re(s) > -\Re(\gamma) > 0, \Re(\gamma) > 0, a > 0, \delta > 0$ . Therefore, the Mellin transform of  $u = x_1 x_2$ , when the joint function is  $f_1(x_1)f_2(x_2)$ , is given by the following:

$$M_g(s) = \frac{\Gamma(\frac{1}{1-q} + 1)}{\rho \delta a^{\gamma/\delta}} \frac{\Gamma(\frac{s}{\rho}) \Gamma(\frac{\gamma+s}{\delta})}{\Gamma(1 + \frac{1}{1-q} + \frac{s}{\rho})} (a^{\frac{1}{\delta}} (1 - q)^{\frac{1}{\rho}})^{-s}$$

for  $\delta > 0, \rho > 0, q < 1, \Re(\gamma) > 0, a > 0, \Re(s) > 0, u^\rho = b$ . Therefore, from the inverse Mellin transform, the function  $g(u)$  for  $u^\rho = b$  is the following, for

$$c_2 = \frac{\Gamma(1 + \frac{1}{1-q})}{\delta \rho a^{\gamma/\delta}} :$$

$$\begin{aligned} g(u) &= c_2 \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\frac{s}{\rho}) \Gamma(\frac{\gamma+s}{\delta})}{\Gamma(1 + \frac{1}{1-q} + \frac{s}{\rho})} (a^{\frac{1}{\delta}} (b(1 - q))^{\frac{1}{\rho}})^{-s} ds \\ &= c_2 H_{0,2}^{2,0} \left[ a^{\frac{1}{\delta}} (b(1 - q))^{\frac{1}{\rho}} \Big|_{(0, \frac{1}{\rho}), (\frac{\gamma}{\delta}, \frac{1}{\delta})}^{(1 + \frac{1}{1-q}, \frac{1}{\rho})} \right], q < 1 \end{aligned} \quad (5.33)$$

for  $a > 0, \delta > 0, \rho > 0, \Re(\gamma) > 0, b > 0, q < 1$ . As in the case for  $q > 1$ , here also we can represent  $g(u)$  in terms of a G-function for the case  $\rho = \delta$ . Let  $\rho = \delta$ . Now, replace  $\frac{s}{\delta}$  by  $s$  thereby  $s$  by  $\delta s$  and  $ds$  by  $\delta ds$ . Then,  $c_2$  of (5.33) changes to  $c'_2$  where  $c'_2$  is the same  $c_2$  with  $\rho$  removed from the denominator because this is canceled from the  $\delta$  coming from  $\delta ds$ . That is,

$$\begin{aligned} g(u) &= c'_2 \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s) \Gamma(\frac{\gamma+s}{\delta})}{\Gamma(1 + \frac{1}{1-q} + s)} [ab(1 - q)]^{-s} ds \\ &= c'_2 G_{0,2}^{2,0} [ab(1 - q)]_{0, \frac{\gamma}{\delta}}^{1 + \frac{1}{1-q}}, \rho = \delta, q < 1 \end{aligned} \quad (5.34)$$

where the  $c$  in the contour is any real number  $> 0$ . Note that for  $\frac{\gamma}{\delta} \neq \nu, \nu = 0, 1, \dots$  the poles of the integrand in (5.34) are simple. Hence, in this case, proceeding as in the case of  $q > 1$ , we have computable series form, which will be stated as a theorem.

**Theorem 5.4** For  $q < 1, \rho = \delta, \frac{\gamma}{\delta} \neq \nu, \nu = 0, 1, \dots, g(u)$  in (5.34) is given by the following series forms:

$$\begin{aligned} g(u) &= \frac{\Gamma(1+\frac{1}{1-q})}{\delta a^{\gamma/\delta}} \left[ \frac{\Gamma(\frac{\gamma}{\delta})}{\Gamma(1+\frac{1}{1-q})} {}_1F_1\left(-\frac{1}{1-q}; 1 - \frac{\gamma}{\delta}; -ab(1-q)\right) \right. \\ &\quad \left. + \frac{\Gamma(-\frac{\gamma}{\delta})}{\Gamma(1+\frac{1}{1-q}-\frac{\gamma}{\delta})} [ab(1-q)]^{\frac{\gamma}{\delta}} {}_1F_1\left(\frac{\gamma}{\delta} - \frac{1}{1-q}; 1 + \frac{\gamma}{\delta}; -ab(1-q)\right) \right] \end{aligned} \quad (5.35)$$

for  $a > 0, \delta > 0, a > 0, b > 0, q < 1, \Re(\gamma) > 0, u^\rho = b, \rho = \delta$ .

**Note 5.2** Note that

$$\lim_{q_1 \rightarrow 1} [1 + a(q_1 - 1)x_2^\delta]^{-\frac{1}{q_1-1}} = e^{-ax_2^\delta}.$$

Hence, one may consider an extended integral  $I_{a,b}(\delta, \rho)$  as the following, denoted by  $I_{a,b}(\delta, \rho; q_1, q_2)$ , where

$$I_{a,b}(\delta, \rho; q_1, q_2) = \int_0^\infty v^{\gamma-1} [1 + a(q_1 - 1)v^\delta]^{-\frac{1}{q_1-1}} [1 + b(q_2 - 1)v^{-\rho}]^{-\frac{1}{q_2-1}} dv$$

for  $q_1 > 1$  or  $q_1 < 1$  or  $q_1 \rightarrow 1$  and then independently  $q_2 > 1, q_2 < 1, q_2 \rightarrow 1$ . Thus, there are nine situations in the above integral. Such an integral called “a versatile integral” was evaluated in Mathai and Haubold (2019). The above versatile integral can also be handled by using the procedures in Sects. 5.1, 5.2, 5.3. Hence, this will not be discussed in detail here.

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## 5.4 The Pathway Extended Reaction-Rate Probability Integral

The following are three different extended forms of the basic reaction-rate probability integral in the real scalar positive variable case:

$$I_{a,b}(q_1) = \int_0^\infty v^{\gamma-1} [1 + a(q_1 - 1)v]^{-\frac{1}{q_1-1}} e^{-bv^{-\frac{1}{2}}} dv, q_1 > 1 \quad (5.36)$$

$$I_{a,b}(q_2) = \int_0^\infty v^{\gamma-1} e^{-av} [1 + b(q_2 - 1)v^{-\frac{1}{2}}]^{-\frac{1}{q_2-1}} dv, q_2 > 1 \quad (5.37)$$

$$\begin{aligned}
I_{a,b}(q_1, q_2) &= \int_0^\infty v^{\gamma-1} [1 + a(q_1 - 1)v]^{-\frac{1}{q_1-1}} \\
&\quad \times [1 + b(q_2 - 1)v^{-\frac{1}{2}}]^{-\frac{1}{q_2-1}} dv, q_j > 1, j = 1, 2.
\end{aligned} \tag{5.38}$$

Observe that in the first two integrals, (5.36), (5.37), there are three situations each, namely  $q_j > 1$ ,  $q_j < 1$ ,  $q_j \rightarrow 1$ . In the third case (5.28) there are nine situations, and each situation is an extended form of the reaction-rate integral except the original basic integral. The techniques used in Sects. 5.1, 5.2, 5.3s can be applied to derive the H-function format or G-function format or series forms for the left side in each of (5.36)-(5.38). For the sake of illustration of the techniques, we will evaluate one of the above integrals.

### Case 5.3 Evaluation of the first integral, $q_1 > 1$

The integral to be evaluated is the following:

$$I_{a,b}(q_1) = \int_0^\infty v^{\gamma-1} [1 + a(q_1 - 1)v]^{-\frac{1}{q_1-1}} e^{-bv^{-\frac{1}{2}}} dv, q_1 > 1.$$

Let  $f_1(x_1) = e^{-x_1^{\frac{1}{2}}}$  and  $f_2(x_2) = x_2^\gamma [1 + a(q_1 - 1)x_2]^{-\frac{1}{q_1-1}}$ . Then,  $f_1(\frac{u}{v}) = e^{-(\frac{u}{v})^{\frac{1}{2}}} = e^{-bv^{-\frac{1}{2}}}$ ,  $b = u^{\frac{1}{2}} \Rightarrow u = b^2$ .

$$\begin{aligned}
M_{f_1}(s) &= \int_0^\infty x_1^{s-1} e^{-x_1^{\frac{1}{2}}} dx_1 \\
&= 2 \int_0^\infty y^{2s-1} e^{-y} dy \\
&= 2\Gamma(2s) = \pi^{-\frac{1}{2}} 2^{2s} \Gamma(s) \Gamma(s + \frac{1}{2}),
\end{aligned} \tag{5.39}$$

for  $\Re(s) > 0$ . Observe that  $\Gamma(2s)$  is expanded by using the duplication formula for gamma functions, which was already illustrated before.

$$\begin{aligned}
M_{f_2}(s) &= \int_0^\infty x_2^{\gamma+s-1} [1 + a(q_1 - 1)x_2]^{-\frac{1}{q_1-1}} dx_2 \\
&= [a(q_1 - 1)]^{-(\gamma+s)} \frac{\Gamma(\gamma+s) \Gamma(\frac{1}{q_1-1} - (\gamma+s))}{\Gamma(\frac{1}{q_1-1})}
\end{aligned}$$

for  $\Re(s) > -\Re(\gamma)$ ,  $\Re(s) < \frac{1}{q_1-1} - \Re(\gamma)$ ,  $q_1 > 1$ ,  $a > 0$ ,  $\Re(\gamma) > 0$ . Then, from the Mellin convolution property, the Mellin transform of  $g$  is the following:

$$\begin{aligned}
M_g(s) &= \frac{[a(q_1-1)]^{-s} 4^s}{[a(q_1-1)]^\gamma \pi^{\frac{1}{2}} \Gamma(\frac{1}{q_1-1})} \\
&\quad \times \Gamma(s) \Gamma(s + \frac{1}{2}) \Gamma(\gamma + s) \Gamma\left(\frac{1}{q_1-1} - \gamma - s\right)
\end{aligned}$$

for  $q_1 > 1, a > 0, \Re(\gamma) > 0, 0 < \Re(s) < \frac{1}{q_1-1}$ . Then, from the inverse Mellin transform  $g(u)$ , which is the integral to be evaluated, is the following:

$$\begin{aligned}
g(u) &= c_1 \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \Gamma\left(s + \frac{1}{2}\right) \Gamma(\gamma + s) \\
&\quad \times \Gamma\left(\frac{1}{q_1-1} - \gamma - s\right) \left[\frac{a(q_1-1)u}{4}\right]^{-s} ds \\
&= c_1 G_{1,3}^{3,1} \left[ \frac{a(q_1-1)u}{4} \Big|_{0, \frac{1}{2}, \gamma}^{1+\gamma-\frac{1}{q_1-1}} \right] \\
c_1^{-1} &= \pi^{\frac{1}{2}} [a(q_1-1)]^\gamma \Gamma\left(\frac{1}{q_1-1}\right), q_1 > 1, a > 0, u = b^2, b > 0.
\end{aligned} \tag{5.40}$$

Observe that one can evaluate (5.40) as the sum of the residues at the poles of  $\Gamma(s)\Gamma(s + \frac{1}{2})\Gamma(\gamma + s)$  because the continuation part is divergent. The poles are simple when  $\gamma$  is not an integer or half-integer. We will obtain the series form for this case. The method is the same as the ones used in Sects. 5.1–5.3. Hence, the final result will be written as a theorem.

**Theorem 5.5** For  $\gamma \neq \frac{\nu}{2}, \nu = 0, 1, \dots, g(u)$  of (5.40) is the following for  $u = b^2, a > 0, b > 0, q_1 > 1$ :

$$\begin{aligned}
g(u) &= \int_0^\infty v^{\gamma-1} [1 + a(q_1-1)v]^{-\frac{1}{q_1-1}} e^{-bv^{-\frac{1}{2}}} dv \\
&= \frac{1}{\pi^{\frac{1}{2}} [a(q_1-1)]^\gamma \Gamma(\frac{1}{q_1-1})} \left[ \Gamma\left(\frac{1}{2}\right) \Gamma(\gamma) \Gamma\left(\frac{1}{q_1-1} - \gamma\right) \right. \\
&\quad \times {}_1F_2\left(\frac{1}{q_1-1} - \gamma; \frac{1}{2}, 1 - \gamma; -\frac{a(q_1-1)b^2}{4}\right) \\
&\quad + \Gamma\left(-\frac{1}{2}\right) \Gamma\left(\gamma - \frac{1}{2}\right) \Gamma\left(\frac{1}{q_1-1} - \gamma + \frac{1}{2}\right) \left[\frac{a(q_1-1)b^2}{4}\right]^{\frac{1}{2}} \\
&\quad \times {}_1F_2\left(\frac{1}{q_1-1} - \gamma + \frac{1}{2}; \frac{3}{2}, \frac{3}{2} - \gamma; -\frac{a(q_1-1)b^2}{4}\right) \\
&\quad + \Gamma(-\gamma) \Gamma\left(\frac{1}{2} - \gamma\right) \Gamma\left(\frac{1}{q_1-1}\right) \left[\frac{s(q_1-1)b^2}{4}\right]^\gamma \\
&\quad \left. \times {}_1F_2\left(\frac{1}{q_1-1}; 1 + \gamma, \frac{1}{2} + \gamma; -\frac{a(q_1-1)b^2}{4}\right) \right] \tag{5.41}
\end{aligned}$$

**Case 5.4** First integral,  $q_1 < 1$

Consider

$$I_{a,b}(q_1) = \int_0^{\frac{1}{a(1-q_1)}} v^{\gamma-1} [1 - a(1-q_1)v]^{\frac{1}{1-q_1}} e^{-bv^{-\frac{1}{2}}} dv$$

for  $q_1 < 1, a > 0, b > 0, 1 - a(1 - q_1)v > 0$ . We use the same technique as before. Here  $f_1(x_1)$  remains the same with  $M_{f_1}(s) = \pi^{-\frac{1}{2}}\Gamma(s)\Gamma(s + \frac{1}{2})4^s, \Re(s) > 0$ . But,  $f_2(x_2)$  will be different;

$$\begin{aligned} f_2(x_2) &= x_2^\gamma [1 - a(1 - q_1)x_1]^{\frac{1}{1-q_1}} \Rightarrow \\ M_{f_2}(s) &= [a(1 - q_1)]^{-(\gamma+s)} \frac{\Gamma(\gamma+s)\Gamma(1+\frac{1}{1-q_1})}{\Gamma(1+\frac{1}{1-q_1}+s)} \end{aligned}$$

for  $\Re(s) > -\Re(\gamma), \Re(\gamma) > 0$ . From the Mellin convolution of the product property, the Mellin transform of  $g(u)$  is the following:

$$\begin{aligned} M_g(s) &= \frac{\Gamma(1+\frac{1}{1-q_1})}{\pi^{\frac{1}{2}}[a(1-q_1)]^\gamma} \left[ \frac{a(1-q_1)}{4} \right]^{-s} \\ &\times \frac{\Gamma(s)\Gamma(s+\frac{1}{2})\Gamma(\gamma+s)}{\Gamma(1+\frac{1}{1-q_1}+s)} \end{aligned}$$

for  $a > 0, q_1 < 1, \Re(s) > 0, \Re(\gamma) > 0$ . From the inverse Mellin transform, we have  $g(u)$  as the following:

$$\begin{aligned} g(u) &= c_1 \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\Gamma(s+\frac{1}{2})\Gamma(\gamma+s)}{\Gamma(1+\frac{1}{1-q_1}+s)} \left[ \frac{a(1-q_1)b^2}{4} \right]^{-s} ds \\ &= c_1 G_{1,3}^{3,0} \left[ \frac{a(1-q_1)b^2}{4} \Big|_{0,\frac{1}{2},\gamma}^{1+\frac{1}{1-q_1}} \right], u = b^2 \\ c_1 &= \frac{\Gamma(1+\frac{1}{1-q_1})}{\pi^{\frac{1}{2}}[a(1-q_1)]^\gamma} \end{aligned} \tag{5.42}$$

where  $c$  in the contour is any real number  $> 0, a > 0, b > 0, \Re(\gamma) > 0, q_1 < 1$ . Note that, as in the previous case, we can obtain a series from for  $g(u)$  when  $\gamma \neq \frac{\nu}{2}, \nu = 0, 1, \dots$  in which case the poles of  $\Gamma(s)\Gamma(s + \frac{1}{2})\Gamma(\gamma)$  are simple. The result will be stated as a theorem.

**Theorem 5.6** For  $\gamma \neq \frac{\nu}{2}, \nu = 0, 1, \dots$ , the  $g(u)$  for case (2) in (5.42) above, is the following, where  $c_1$  is given in (5.42) above:

(5.43)

$$\begin{aligned}
g(u) = & c_1 \left[ \frac{\Gamma(\frac{1}{2})\Gamma(\gamma)}{\Gamma(1+\frac{1}{1-q_1})} {}_1F_2 \left( -\frac{1}{1-q_1}; \frac{1}{2}, 1-\gamma; \frac{a(1-q_1)b^2}{4} \right) \right. \\
& + \frac{\Gamma(-\frac{1}{2})\Gamma(\gamma-\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{1-q_1})} \left[ \frac{a(1-q_1)b^2}{4} \right]^{\frac{1}{2}} \\
& \times {}_1F_2 \left( \frac{1}{2} - \frac{1}{1-q_1}; \frac{3}{2}, \frac{3}{2} - \gamma; \frac{a(1-q_1)b^2}{4} \right) \\
& + \frac{\Gamma(-\gamma)\Gamma(\frac{1}{2}-\gamma)}{\Gamma(1+\frac{1}{1-q_1}-\gamma)} \left[ \frac{a(1-q_1)b^2}{4} \right]^{\gamma} \\
& \left. \times {}_1F_2 \left( \gamma - \frac{1}{1-q_1}; 1+\gamma, \frac{1}{2}+\gamma; \frac{a(1-q_1)b^2}{4} \right) \right], q_1 < 1
\end{aligned}$$

**Note 5.3** If the integrals considered in Sects. 5.1–5.3 are extended to the whole real line, then it can be achieved by replacing  $v$  in the integral by  $|v|$ , the absolute value of  $v$ . For example, the generalized reaction-rate integral will then be of the following form:

$$\int_{-\infty}^{\infty} |x|^{\gamma-1} e^{-|x|^{\delta}-b|x|^{-\rho}} dx \quad (5.44)$$

for  $a > 0, b > 0, \delta > 0, \rho > 0, \Re(\gamma) > 0$ . Corresponding changes may be made in all other integrals such as  $I_{a,b}$ ,  $I_{a,b}(q_1)$ ,  $I_{a,b}(q_2)$ ,  $I_{a,b}(q_1, q_2)$ . Mellin convolution of the product of Maxwell-Boltzmann and Raleigh densities and their extended forms to the whole line are covered in (5.44).

## 5.5 Generalized Reaction-Rate Probability Integral in the Real Multivariate Case

There may not be any corresponding physics or reaction-rate probability integral or Krätsel integral yet in what we are going to discuss in this section. These results may be motivating factors for developing corresponding physics or communication theory or engineering problems later on. Let  $X$  be a  $p \times 1$  real vector with  $X' = [x_1, \dots, x_p]$ , where a prime denotes the transpose and the  $x_j$ 's are distinct (functionally independent) real scalar variables,  $-\infty < x_j < \infty, j = 1, \dots, p$ . Then,  $y = X'X = x_1^2 + \dots + x_p^2$  is an isotropic quantity, in the sense that  $y$  remains invariant under the rotation of the axes of coordinates or under orthonormal transformations, that is, if  $Z = AX, AA' = I_p, A'A = I_p$ , then  $Z'Z = X'X$ . Such isotropic variables appear in different disciplines.  $X'X$  is associated with isotropic random points in  $p$ -dimensional Euclidean space in geometrical probability problems, see Mathai (1999). Also,  $X'X$  is associated with spherically symmetric distributions in statistical distribution theory and related areas. Let us consider the integral, denoted by  $\mathbf{I}_{a,b}(\delta, \rho)$  with  $I$  in bold,

$$\mathbf{I}_{a,b}(\delta, \rho) = \int_X (X'X)^\gamma e^{-a(X'X)^\delta - b(X'X)^{-\rho}} dX \quad (5.45)$$

for  $a > 0, b > 0, \delta > 0, \rho > 0$ .

The following general notations will be used in this and coming sections. Real scalar variables, whether they are mathematical variables or random variables, will be denoted by lower-case letters such as  $x, y$ . Vector (a  $p \times 1$  or  $1 \times p$  matrix)/matrix variables will be denoted by capital letters such as  $X, Y$ , whether the variables are mathematical or random. Scalar constants will be denoted by  $a, b$  etc. and vector/matrix constants by  $A, B$  etc. Variables in the complex domain will be written with a tilde such as  $\tilde{x}, \tilde{y}, \tilde{X}, \tilde{Y}$ . No tilde will be used on constants. The determinant of a  $p \times p$  matrix  $A$  will be written as  $|A|$  or as  $\det(A)$ . When  $A$  is in the complex domain, then  $|A| = a + ib, i = \sqrt{(-1)}$ ,  $a, b$  are real scalars. Then, the absolute value of the determinant is written as  $|\det(A)| = \sqrt{a^2 + b^2}$ . Also,  $A^*$  will denote the conjugate transpose of  $A$ , denoting the conjugate by  $A^c$  and transpose by  $A'$ . For the  $p \times q$  matrix  $X = (x_{ij})$ , where the elements  $x_{ij}$ 's are distinct real scalar variables, the wedge product of differentials will be denoted as  $dX = \wedge_{i=1}^p \wedge_{j=1}^q dx_{ij}$ . If the  $p \times p$  matrix  $Y = (y_{ij}) = Y'$  (real symmetric), then  $dY = \wedge_{i \geq j} dy_{ij} = \wedge_{i \leq j} dy_{ij}$ . For two real scalar variables  $x$  and  $y$  with differentials  $dx$  and  $dy$ , the wedge ( $\wedge$ ) product of the differentials is defined as  $dx \wedge dy = -dy \wedge dx \Rightarrow dx \wedge dx = 0, dy \wedge dy = 0$ . When the  $p \times q$  matrix  $\tilde{X} = (\tilde{x}_{jk})$  is in the complex domain, then one can write  $\tilde{X} = X_1 + iX_2, i = \sqrt{(-1)}$ ,  $X_1, X_2$  are  $p \times q$  real matrices. Then,  $d\tilde{X}$  will be defined as  $d\tilde{X} = dX_1 \wedge dX_2$ . In the coming sections, we will be considering only real-valued scalar functions, the argument may be scalar/vector/matrix in the real or complex domain. A statistical density will be defined as a real-valued scalar function  $f(X)$  of  $X$  such that  $f(X) \geq 0$  in the domain of  $X$  and  $\int_X f(X) dX = 1$  where  $X$  may be scalar/vector/matrix or a sequence of matrices in the real or complex domain but  $f(X)$  has to be a real-valued scalar function. In the following discussions, a result on Jacobian of matrix transformation is going to be frequently used. This will be stated as a lemma here without proof. For the proof and for other related materials, see Mathai (1997).

**Lemma 5.1** Let the  $p \times q, p \leq q$  matrix  $X = (x_{jk})$  of rank  $p$  in the real domain. Let  $Y = XX'$ . Then,  $Y > O$  (real positive definite). Then, going through a transformation involving a lower triangular matrix with positive diagonal elements and a unique semi-orthonormal matrix and after integrating out the differential element corresponding to the semi-orthonormal matrix, the following connection is obtained between the differential elements  $dX$  and  $dY$ :

$$dX = \frac{\pi^{\frac{pq}{2}}}{\Gamma_p(\frac{q}{2})} |Y|^{\frac{q}{2} - \frac{p+1}{2}} dY$$

where, for example,  $\Gamma_p(\alpha)$  is the real matrix-variate gamma function defined by the following:

$$\begin{aligned}\Gamma_p(\alpha) &= \pi^{\frac{p(p-1)}{4}} \Gamma(\alpha) \Gamma\left(\alpha - \frac{1}{2}\right) \cdots \Gamma\left(\alpha - \frac{p-1}{2}\right), \Re(\alpha) > \frac{p-1}{2} \\ &= \int_{Z>O} |Z|^{\alpha - \frac{p+1}{2}} e^{-\text{tr}(Z)} dZ, \Re(\alpha) > \frac{p-1}{2}\end{aligned}$$

where  $\text{tr}(\cdot)$  means the trace of  $(\cdot)$  and  $Z$  is a  $p \times p$  real positive definite matrix. Let  $\tilde{X}$  be a  $p \times q, p \leq q$  matrix of rank  $p$  in the complex domain with distinct complex scalar variables as elements. Let  $\tilde{Y} = \tilde{X}\tilde{X}^* > O$  (Hermitian positive definite), where  $\tilde{X}^*$  is the conjugate transpose of  $\tilde{X}$ . Then, going through a transformation involving a lower triangular matrix with real and positive diagonal elements and a unique semi-unitary matrix and then integrating out the differential element corresponding to the semi-unitary matrix, one has the following connection:

$$d\tilde{X} = \frac{\pi^{pq}}{\tilde{\Gamma}_p(q)} |\det(\tilde{Y})|^{q-p} d\tilde{Y}$$

where, for example, the complex matrix-variate gamma function is defined as the following:

$$\begin{aligned}\tilde{\Gamma}_p(\alpha) &= \pi^{\frac{p(p-1)}{2}} \Gamma(\alpha) \Gamma(\alpha - 1) \cdots \Gamma(\alpha - p + 1), \Re(\alpha) > p - 1 \\ &= \int_{\tilde{Z}>O} |\det(\tilde{Z})|^{\alpha-p} e^{-\text{tr}(\tilde{Z})} d\tilde{Z}\end{aligned}$$

where  $\tilde{Z}$  is a  $p \times p$  Hermitian positive definite matrix.

Let us continue the evaluation of our integral in (5.45). For a  $1 \times p$  matrix we can apply Lemma 5.1, to  $y = X'X$  considering  $X'$  be that  $1 \times p$  matrix. Then,

$$dX = \frac{\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})} y^{\frac{p}{2}-1} dy. \quad (5.46)$$

Then,

$$\mathbf{I}_{a,b}(\delta, \rho) = \frac{\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})} \int_0^\infty y^{\gamma + \frac{p}{2} - 1} e^{-ay^\delta - by^{-\rho}} dy. \quad (5.47)$$

Now, comparing with our integral in earlier sections, the only change is  $\gamma$  there is replaced by  $\gamma + \frac{p}{2}$ . Hence, the explicit evaluation is available from the results in earlier sections and hence further discussion is omitted. Now, for  $X$  a  $p \times 1$  vector in the real domain, consider the following extended integrals, denoted by  $I_1, I_2, I_3$  for  $q_1 > 1, q_2 > 1$ :

$$I_1 = \int_X [X'X]^\gamma [1 + a(q_1 - 1)(X'X)^\delta]^{-\frac{1}{q_1-1}} e^{-b(X'X)^{-\rho}} dX \quad (5.48)$$

$$I_2 = \int_X [X'X]^\gamma e^{-a(X'X)^\delta} [1 + b(q_2 - 1)(X'X)^{-\rho}]^{-\frac{1}{q_2-1}} dX \quad (5.49)$$

$$\begin{aligned} I_3 &= \int_X [X'X]^\gamma [1 + a(q_1 - 1)(X'X)^\delta]^{-\frac{1}{q_1-1}} \\ &\quad \times [1 + b(q_2 - 1)(X'X)^{-\rho}]^{-\frac{1}{q_2-1}} dX. \end{aligned} \quad (5.50)$$

Consider again,  $y = X'X$  and from (5.46) above, the differential element is available. Then, in (5.48)–(5.50) we obtain a multiplicative factor  $\frac{\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})}$ , the parameter  $\gamma$  becomes  $\gamma + \frac{p}{2} - 1$  and  $X'X$  is replaced by  $y$ . Then, (5.48)–(5.50) become the corresponding integrals of earlier sections and hence further discussion is omitted.

## 5.6 Real Matrix-Variate Case

Let  $X = (x_{jk})$  be a  $p \times q, p \leq q$  matrix of rank  $p$  in the real domain with  $pq$  distinct real scalar variables as the elements  $x_{jk}$ 's. Then,

$$\text{tr}(XX') = \sum_{j=1}^p \sum_{k=1}^q x_{jk}^2 \quad (5.51)$$

because for any real matrix  $A = (a_{jk})$ ,  $\text{tr}(AA') = \text{tr}(A'A) =$  the sum of squares of all elements in  $A$ . Now, think of (5.46) as coming from a  $1 \times pq$  vector  $U$  so that  $UU'$  is the sum of squares in (5.46). Hence, we may use the result from the previous section on real vector-variate case. Here, one has  $pq$  variables instead of  $p$  variables in the previous section. Now,

$$dX = \frac{\pi^{\frac{pq}{2}}}{\Gamma(\frac{pq}{2})} y^{\frac{pq}{2}-1} dy, y = \text{tr}(XX'). \quad (5.52)$$

For a  $p \times q, p \leq q$  matrix of rank  $p$  in the real domain, let

$$M_{a,b}(\delta, \rho) = \int_X [\text{tr}(XX')]^\gamma e^{-a[\text{tr}(XX')^\delta] - b[\text{tr}(XX')]^{-\rho}} dX \quad (5.53)$$

$$= \frac{\pi^{\frac{pq}{2}}}{\Gamma(\frac{pq}{2})} \int_0^\infty y^{\gamma + \frac{pq}{2} - 1} e^{-ay^\delta - by^{-\rho}} dy \quad (5.54)$$

for  $y = \text{tr}(XX'), a > 0, b > 0, \delta > 0, \rho > 0$ . Now, the integral in (5.54) is available from the corresponding real scalar variable case with  $\gamma$  replaced by  $\gamma + \frac{pq}{2}$  and hence further discussion is omitted. If there is a multiplicative factor in terms of a

determinant such as  $|XX'|^\gamma$  and trace factor  $[\text{tr}(XX')]^\eta$ , then the integral in (5.53) becomes quite general. Can we evaluate such an extended Bessel integral or reaction-rate probability integral. This will be explored next.

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## 5.7 Most General Real Matrix-Variate Case

Consider again a  $p \times q, p \leq q$  real matrix of rank  $p$  with  $pq$  distinct real scalar variables as elements. For  $\delta > 0, \rho > 0, a > 0, b > 0, \Re(\gamma) > \frac{p-1}{2}, \Re(\eta) > 0$ , consider the integral

$$M_{a,b}(\delta, \rho; \gamma, \eta) = \int_X |XX'|^\gamma [\text{tr}(XX')]^\eta \times e^{-a[\text{tr}(XX')]^\delta - b[\text{tr}(XX')]^{-\rho}} dX. \quad (5.55)$$

Note that, even though  $X$  has  $pq$  real scalar variables as elements,  $Y = XX'$  is symmetric and positive definite with  $p(p+1)/2$  distinct real scalar variables in  $Y$ . Let  $k = p(p+1)/2$ . One can go from the differential element  $dX$  to the differential element  $dT$  where  $T$  is a lower triangular matrix with positive diagonal elements by going from  $X$  to  $Y = XX'$  and then from  $Y$  to  $T$  or one can go directly from  $X$  to  $T$  by using another result from Mathai (1997), which will be stated here without proof.

**Lemma 5.2** Let  $X = (x_{jk})$  be a  $p \times q, p \leq q$  matrix of rank  $p$  in the real domain with  $pq$  distinct real scalar variables as elements  $x_{jk}$ 's. Let  $T$  be a lower triangular matrix with positive diagonal elements  $t_{jj}$ 's. Let  $U$  be a  $p \times q$  unique semi-orthonormal matrix,  $UU' = I_p$ . Let  $X = TU$ . A unique choice of  $U$  can be made by putting the condition that the first nonzero element in every row is positive. Then,

$$X = TU, UU' = I_p \Rightarrow dX = \frac{\pi^{\frac{pq}{2}}}{\Gamma_p(\frac{q}{2})} \left\{ \prod_{j=1}^p t_{jj}^{q-j} \right\} dT.$$

In the corresponding complex case,  $\tilde{X}$  is a  $p \times q, p \leq q$  matrix of rank  $p$  in the complex domain,  $\tilde{T}$  is a lower triangular matrix in the complex domain with real and positive diagonal elements  $t_{jj}$ 's and  $\tilde{U}$  is a unique semi-unitary matrix  $\tilde{U}\tilde{U}^* = I_p$ . Then,

$$\tilde{X} = \tilde{T}\tilde{U}, \tilde{U}\tilde{U}^* = I_p \Rightarrow d\tilde{X} = \frac{\pi^{pq}}{\tilde{\Gamma}_p(q)} \left\{ \prod_{j=1}^p t_{jj}^{2(q-j)+1} \right\} d\tilde{T}.$$

Now, continuing with the evaluation of our integral in (5.55), observe that

$$|XX'| = \prod_{j=1}^p t_{jj}^2, \text{tr}(XX') = \text{tr}(TT') = \sum_{j \geq r} t_{jr}^2 \quad (5.56)$$

in the real case, and in the complex case,  $\tilde{t}_{jr} = t_{jr1} + it_{jr2}$ ,  $i = \sqrt{(-1)}$ ,  $t_{jr1}, t_{jr2}$  are real scalar variables and then

$$|\tilde{t}_{jr}|^2 = t_{jr1}^2 + t_{jr2}^2. \quad (5.57)$$

Let the total number of  $t_{jr}$ 's be  $k$ . Then, in the real case  $k = p(p+1)/2$  and in the complex case  $k = p^2$ . The sum of squares of all the real elements in  $T$  or  $\tilde{T}$  can be made into a single scalar quantity  $r^2$  where  $r$  is the polar radius, through a general polar coordinate transformation. This transformation and the associated Jacobian will be stated as a lemma without proof. For the proof and other details, see Mathai (1997).

**Lemma 5.3** Let  $x_1, \dots, x_k$  be real scalar variables,  $-\infty < x_j < \infty$ ,  $j = 1, \dots, k$ . Consider the following transformation to  $r$  and  $\theta_j$ 's:

$$\begin{aligned} x_1 &= r \sin \theta_1 \\ x_2 &= r \cos \theta_1 \sin \theta_2 \\ x_j &= r \cos \theta_1 \cdots \cos \theta_{j-1} \sin \theta_j, j = 1, \dots, k-2, -\frac{\pi}{2} < \theta_j \leq \frac{\pi}{2} \\ x_{k-1} &= r \cos \theta_1 \cdots \cos \theta_{k-1}, -\pi < \theta_{k-1} \leq \pi. \end{aligned}$$

Then,

$$dx_1 \wedge \cdots \wedge dx_k = r^{k-1} \left\{ \prod_{j=1}^{k-1} |\cos \theta_j|^{k-1-j} \right\} dr \wedge d\theta_1 \wedge \cdots \wedge d\theta_{k-1}.$$

Now, let us apply Lemma 5.3 to the  $t_{jk}$ 's in (5.56). Then, in the real case,  $|XX|^\gamma = |TT'|^\gamma = \{\prod_{j=1}^k (t_{jj}^2)^\gamma\}$ . The Jacobian part from Lemma 5.2 gives  $\prod_{j=1}^k t_{jj}^{q-j}$  and therefore

$$|XX'|^\gamma dX = |TT'|^\gamma dT = \prod_{j=1}^k (t_{jj}^2)^{\gamma + \frac{q}{2} - \frac{j}{2}} dT$$

and hence the  $r^2$  coming from this product is

$$\prod_{j=1}^p (r^2)^{\gamma + \frac{q}{2} - \frac{j}{2}} = (r^2)^{p(\gamma + \frac{q}{2}) - p(p+1)/4}. \quad (5.58)$$

The total number of  $r^2$  coming from  $t_{jk}$ 's is  $p(\gamma + \frac{q}{2}) - \frac{p(p+1)}{4} + \eta + \frac{p(p+1)}{4} - \frac{1}{2} = p(\gamma + \frac{q}{2}) + \eta - \frac{1}{2}$ . The product of  $\theta_j$ 's

coming from the factor containing  $|TT'|$  and the Jacobian part is integrated out in Mathai (2003) and the result, denoted by  $I_\theta$ , is the following:

$$I_\theta = \frac{2\Gamma_p(\gamma + \frac{q}{2})}{\Gamma(p(\gamma + \frac{q}{2}))}, \Re(\gamma) > -\frac{q}{2} + \frac{p-1}{2}. \quad (5.59)$$

Then, the integral in (5.55) is the following for  $u = \text{tr}XX'$ :

$$\begin{aligned} M_{a,b}(\delta, \rho; \gamma, \eta) &= \int_X |XX'|^\gamma u^\eta e^{-au^\delta - bu^{-\rho}} dX \\ &= \frac{2\Gamma_p(\gamma + \frac{q}{2})}{\Gamma(p(\gamma + \frac{q}{2}))} \int_{r=0}^{\infty} (r^2)^{p(\gamma + \frac{q}{2}) + \eta - \frac{1}{2}} \\ &\quad \times e^{-a(r^2)^\delta - b(r^2)^{-\rho}} dr \\ &= \frac{\Gamma_p(\gamma + \frac{q}{2})}{\Gamma(p(\gamma + \frac{q}{2}))} \int_{y=0}^{\infty} y^{p(\gamma + \frac{q}{2}) + \eta - 1} \\ &\quad \times e^{-ay^\delta - by^{-\rho}} dy. \end{aligned} \quad (5.60)$$

Now, the integral in (5.55) is available from the previous section by evaluating the integral over  $y$  in (5.60) above.

## 5.8 Generalized Reaction-Rate Integrals in the Complex Multivariate Case

Now, we look into the extension of reaction-rate integral and general Bessel integral to the complex domain. First we consider the vector case and then the matrix-variate case. Let  $\tilde{X}$  be a  $p \times 1$  vector in the complex domain with  $p$  distinct scalar complex variables as elements  $\tilde{x}_j$ 's. Then,

$$\tilde{X}^* \tilde{X} = |\tilde{x}_1|^2 + \cdots + |\tilde{x}_p|^2 = (x_{11}^2 + x_{12}^2) + \cdots + (x_{p1}^2 + x_{p2}^2)$$

where  $\tilde{x}_j = x_{j1} + ix_{j2}$ ,  $i = \sqrt{(-1)}$ ,  $x_{j1}, x_{j2}$  are real scalar variables. Consider the transformation  $y = \tilde{X}^* \tilde{X}$ . Note that  $y$  here is real also and  $y$  contains sum of squares of  $2p$  real scalar variables as opposed to  $p$  variables in the corresponding real case. Apply Lemma 5.1 to  $y$ . Then,

$$d\tilde{X} = \frac{\pi^{\frac{2p}{2}}}{\Gamma(\frac{2p}{2})} y^{\frac{2p}{2}-1} dy = \frac{\pi^p}{\Gamma(p)} y^{p-1} dy. \quad (5.61)$$

Consider the following integral, even though it is real-valued, in order to distinguish it from the corresponding real case, we will use a tilde.

$$(5.62)$$

$$\begin{aligned}\tilde{I}_{a,b}(\delta, \rho) &= \int_{\tilde{X}} [\tilde{X}^* \tilde{X}]^\gamma e^{-a(\tilde{X}^* \tilde{X})^\delta - b(\tilde{X}^* \tilde{X})^{-\rho}} d\tilde{X} \\ &= \frac{\pi^p}{\Gamma(p)} \int_{y=0}^{\infty} y^{\gamma+p-1} e^{-ay^\delta - by^{-\rho}} dy.\end{aligned}$$

Now, the integral in (5.62) can be evaluated by using the techniques in Sects. 5.1–5.3. As in the real case, we can consider the pathway extended models also, which can all be evaluated by making the transformation in (5.61) above and then using the techniques from Sects. 5.1–5.3. Hence, we will only list the pathway forms here. For simplicity, we will write them by using the symbol  $y = \tilde{X}^* \tilde{X}$  and denoting them by  $\tilde{I}_1, \tilde{I}_2, \tilde{I}_3$  for  $q_1 > 1, q_2 > 1$ :

$$\tilde{I}_1 = \int_{\tilde{X}} y^{\gamma+p-1} [1 + a(q_1 - 1)y^\delta]^{-\frac{1}{q_1-1}} e^{-by^{-\rho}} d\tilde{X} \quad (5.63)$$

$$\tilde{I}_2 = \int_{\tilde{X}} y^{\gamma+p-1} e^{-ay^\delta} [1 + b(q_2 - 1)y^{-\rho}]^{\frac{1}{q_2-1}} d\tilde{X} \quad (5.64)$$

$$\tilde{I}_3 = \int_{\tilde{X}} y^{\gamma+p-1} [1 + a(q_1 - 1)y^\delta]^{-\frac{1}{q_1-1}} [1 + b(q_2 - 1)y^{-\rho}]^{-\frac{1}{q_2-1}} d\tilde{X}. \quad (5.65)$$

Observe that there are three situations each in (5.63) and (5.64), that is  $q_1 > 1, < 1, \rightarrow 1$  and  $q_2 > 1, < 1, \rightarrow 1$  and there are nine situations in (5.65).

## 5.9 Complex Matrix-Variate Case

Let  $\tilde{X}$  be a  $p \times q, p \leq q$  matrix of rank  $p$  in the complex domain with distinct  $pq$  complex scalar variables as elements. Then,  $\tilde{Y} = \tilde{X} \tilde{X}^*$  is Hermitian positive definite. Note that

$$\text{tr}(\tilde{X} \tilde{X}^*) = \sum_{j=1}^p \sum_{k=1}^q |\tilde{x}_{jk}|^2 = \sum_{j=1}^p \sum_{k=1}^q (x_{jk1}^2 + x_{jk2}^2) \quad (5.66)$$

where  $\tilde{x}_{jk} = x_{jk1} + ix_{jk2}, i = \sqrt{(-1)}$ ,  $x_{jk1}, x_{jk2}$  are real scalar variables. Thus, there are  $2pq$  sum of squares of real scalar variables in (5.66) above, as opposed to  $pq$  sum of squares in the corresponding real case. Consider the transformation  $y = \text{tr}(\tilde{X} \tilde{X}^*)$ . Then, we can apply Lemma 5.1 on a  $1 \times 2pq$  real vector and then we have the following result:

$$y = \text{tr}(\tilde{X} \tilde{X}^*) \Rightarrow d\tilde{X} = \frac{\pi^{\frac{2pq}{2}}}{\Gamma(\frac{2pq}{2})} y^{\frac{2pq}{2}-1} dy = \frac{\pi^{pq}}{\Gamma(pq)} y^{pq-1} dy. \quad (5.67)$$

Now, for the  $p \times q, p \leq q$  matrix of rank  $p$  in the complex domain, consider the evaluation of the following integral:

$$(5.68)$$

$$\begin{aligned}\tilde{\mathbf{I}}_1 &= \int_{\tilde{X}} [\operatorname{tr}(\tilde{X}\tilde{X}^*)]^\gamma e^{-(\operatorname{tr}(\tilde{X}\tilde{X}^*))^\delta - b(\operatorname{tr}(\tilde{X}\tilde{X}^*))^{-\rho}} d\tilde{X} \\ &= \frac{\pi^{pq}}{\Gamma(pq)} \int_{y=0}^{\infty} y^{\gamma+pq-1} e^{-ay^\delta - by^{-\rho}} dy.\end{aligned}$$

Now, the integral in (5.68) can be evaluated by using the techniques from Sects. 5.1–5.3.

## 5.10 Most General Form in the Complex Matrix-Variate Case

Let  $\tilde{X} = (\tilde{x}_{jk})$  be a  $p \times q, p \leq q$  matrix of rank  $p$  in the complex domain with  $pq$  distinct scalar complex variables as elements. Then  $\tilde{X}\tilde{X}^* = \tilde{Y} > O$  (Hermitian positive definite). Consider the integral

$$\begin{aligned}\tilde{M}_1 &= \int_{\tilde{X}} |\det(\tilde{X}\tilde{X}^*)|^\gamma [\operatorname{tr}(\tilde{X}\tilde{X}^*)]^\eta \\ &\quad \times e^{-a(\operatorname{tr}(\tilde{X}\tilde{X}^*))^\delta - b(\operatorname{tr}(\tilde{X}\tilde{X}^*))^{-\rho}} d\tilde{X}.\end{aligned}\tag{5.69}$$

Consider the transformation  $\tilde{X} = \tilde{T}\tilde{U}$ , where  $\tilde{T}$  is a  $p \times p$  lower triangular matrix with real and positive diagonal elements and  $\tilde{U}$  is a  $p \times q$  unique semi-unitary matrix,  $\tilde{U}\tilde{U}^* = I_p$ . Then, after integrating out the differential element corresponding to  $\tilde{U}$ , we have the following result, see also Lemma 5.2:

$$\tilde{X} = \tilde{T}\tilde{U} \Rightarrow d\tilde{X} = \frac{\pi^{pq}}{\Gamma_p(q)} \left\{ \prod_{j=1}^p t_{jj}^{2(q-j)+1} \right\} d\tilde{T}.\tag{5.70}$$

Note that

$$\operatorname{tr}(\tilde{T}\tilde{T}^*) = \sum_{j=1}^p t_{jj}^2 + \sum_{j>k} |\tilde{t}_{jk}|^2\tag{5.71}$$

where  $|\tilde{t}_{jk}|^2 = t_{jk1}^2 + t_{jk2}^2$ ,  $\tilde{t}_{jk} = t_{jk1} + it_{jk2}$ ,  $i = \sqrt{(-1)}$ ,  $t_{jk1}, t_{jk2}$  are real scalar variables. Thus, there is a total of  $p + 2\frac{p(p-1)}{2} = p^2$  sum of squares of real scalar variables in  $\operatorname{tr}(\tilde{T}\tilde{T}^*)$ . In the notation of Lemma 5.3,  $k = p^2$  in the complex case and  $k = p(p+1)/2$  in the real case. Then, under the general polar coordinate transformation in Lemma 5.3 we have the following, denoting the complex scalar variables as  $\tilde{x}_1, \dots, \tilde{x}_{p^2}$ :

$$d\tilde{x}_1 \wedge \dots \wedge d\tilde{x}_{p^2} = r^{p^2-1} \left\{ \prod_{j=1}^{p^2-1} |\cos \theta_j|^{p^2-j-1} \right\} dr \wedge d\theta_1 \wedge \dots \wedge d\theta_{p^2-1}.\tag{5.72}$$

and  $\text{tr}(\tilde{T}\tilde{T}^*) = r^2$ . Sine and cosine of  $\theta_j$ 's are coming from the determinant  $|\det(\tilde{T}\tilde{T}^*)|$  and the integral over this function of  $\theta_j$ 's is already evaluated in Mathai (2003) for the real and complex cases. For the complex case, it is the following, denoted by  $\tilde{I}_\theta$ :

$$\tilde{I}_\theta = \frac{2\tilde{\Gamma}_p(\gamma+q)}{\Gamma(p(\gamma+q))}, \Re(\gamma) > -q + p - 1. \quad (5.73)$$

In the complex case, combining all factors of  $r^2$  we have  $(r^2)^{p(\gamma+q)+\eta-\frac{1}{2}}$ . Now, the integral in (5.69) reduces to the following:

$$\begin{aligned} \int_{\tilde{X}} |\det(\tilde{X}\tilde{X}^*)|^\gamma & [ \text{tr}(\tilde{X}\tilde{X}) ]^\eta e^{-a(\text{tr}(\tilde{X}\tilde{X}^*))^\delta - b(\text{tr}(\tilde{X}\tilde{X}^*))^{-\rho}} d\tilde{X} \\ &= \frac{2\tilde{\Gamma}_p(\gamma+q)}{\Gamma(p(\gamma+q))} \int_r (r^2)^{p(\gamma+q)+\eta-\frac{1}{2}} e^{-a(r^2)^\delta - b(r^2)^{-\rho}} dr \\ &= \frac{\tilde{\Gamma}_p(\gamma+q)}{\Gamma(p(\gamma+q))} \int_{u=0}^{\infty} u^{p(\gamma+q)+\eta-1} e^{-au^\delta - bu^{-\rho}} du. \end{aligned} \quad (5.74)$$

This is the most general form of the integral in the complex case. The evaluation of the integral over  $u$  can be done by using the techniques in Sects. 5.1–5.3.

## 5.11 Reaction-Rate Probability Integral Through Optimization of Mathai Entropy

Basic measure of uncertainty in a scheme (a set of mutually exclusive and totally exhaustive events along with the corresponding probabilities) is Shannon entropy defined as the following, where the first expression is for a discrete distribution and the second item is for a density function  $f(x)$  of a real scalar random variable  $x$ , both denoted by  $S(f)$ :

$$\begin{aligned} S(f) &= -c \sum_{j=1}^k p_j \ln p_j, p_j > 0, j = 1, \dots, k, p_1 + \dots + p_k = 1, \quad S(f) \\ &= -c \int_x f(x) \ln f(x) dx \end{aligned} \quad (5.75)$$

where  $c$  is a constant. This basic Shannon entropy is generalized in various directions. One  $\alpha$ -generalized entropy is Havrda-Charvat entropy denoted by  $H_\alpha(f)$  where

$$H_\alpha(f) = \frac{\int_x [f(x)]^\alpha dx - 1}{2^{1-\alpha} - 1}, \alpha \neq 1. \quad (5.76)$$

A variant of  $H_\alpha(f)$  is Tsallis entropy  $T_\alpha(f)$  given by

$$T_\alpha(f) = \frac{\int_x [f(x)]^\alpha dx - 1}{1-\alpha}, \alpha \neq 1. \quad (5.77)$$

Mathai entropy is also a variant of  $H_\alpha(f)$ , denoted by  $M_\alpha(f)$  and defined as

$$M_\alpha(f) = \frac{\int_X [f(X)]^{1+\frac{a-\alpha}{\eta}} dX - 1}{\alpha - a}, \alpha \neq a, \eta > 0 \quad (5.78)$$

where  $a$  is a fixed anchoring point,  $\alpha$  is the parameter of interest and the deviation of  $\alpha$  from  $a$  is measured in  $\eta$  units. Here,  $f(X)$  is defined as a real-valued scalar function of  $X$  such that  $f(X) \geq 0$  for all  $X$  in the domain of  $X$  and  $\int_X f(X) dX = 1$ . It is called a density function where  $X$  may be a scalar/vector/matrix or a sequence of matrices in the real or complex domain. Thus, (5.78) is a very general concept. It can also be taken as an expected value, namely

$$M_\alpha(f) = E \left[ \frac{[f(X)]^{\frac{a-\alpha}{\eta}} - 1}{\alpha - a} \right] = \frac{E[[f(X)]^{\frac{a-\alpha}{\eta}} - 1]}{\alpha - a}, \alpha \neq a, \eta > 0.$$

Note that when  $\alpha \rightarrow a$ ,  $M_\alpha(f) \rightarrow S(f)$  = Shannon entropy and hence  $M_\alpha(f)$  is an  $\alpha$ -generalized Shannon entropy. Similarly,  $H_\alpha(f) \rightarrow S(f)$  and  $T_\alpha(f) \rightarrow S(f)$  when  $\alpha \rightarrow 1$ . Thus,  $H_\alpha(f)$  and  $T_\alpha(f)$  are also  $\alpha$ -generalized Shannon entropy. Let us consider the optimization of Mathai entropy  $M_\alpha(f)$  under the following two moment-type constraints, where we take  $x$  as a real scalar variable to start with:

$$(i) : E[x^\gamma e^{-b_1 x^{-\rho}}]^{\frac{a-\alpha}{\eta}} = \text{given} ; \quad (ii) : E[(x^\gamma e^{-b_1 x^{-\rho}})^{\frac{a-\alpha}{\eta}} x^\delta] = \text{given} \quad (5.79)$$

where  $b_1 > 0, \rho > 0, \delta > 0, \Re(\gamma) > 0$  and  $E[\cdot]$  denotes the expected value of  $[\cdot]$ . If we use calculus of variation to optimize  $M_\alpha(f)$  under the constraints in (5.79) above, then the Euler equation is the following, where  $\lambda_1$  and  $\lambda_2$  are the Lagrangian multipliers:

$$\begin{aligned} \frac{\partial}{\partial f} [f^{1+\frac{a-\alpha}{\eta}} - \lambda_1 (x^\gamma e^{-b_1 x^{-\rho}})^{\frac{a-\alpha}{\eta}} f - \lambda_2 [(x^\gamma e^{-b_1 x^{-\rho}})^{\frac{a-\alpha}{\eta}} x^\delta f]] &= 0 \Rightarrow \\ f^{\frac{a-\alpha}{\eta}} &= \lambda_3 [x^\gamma e^{-b_1 x^{-\rho}}]^{\frac{a-\alpha}{\eta}} [1 + \lambda_4 x^\delta] \Rightarrow \\ f &= \lambda_5 x^\gamma e^{-b_1 x^{-\rho}} [1 + \lambda_6 x^\delta]^{\frac{\eta}{a-\alpha}} \end{aligned}$$

where  $\lambda_3$  to  $\lambda_6$  are some constants. Let us consider the case  $\alpha < a$ . Then, the exponent  $\frac{\eta}{a-\alpha} > 0$  since  $\eta > 0$ . Then, if  $f$  has to remain as a density function for all possible values of  $a, \alpha, \eta, \alpha < a, \eta > 0$ , then  $\lambda_6$  has to be negative. Let  $\lambda_6 = -b(a - \alpha)$ ,  $b > 0, \alpha < a$  and let  $c_1$  be corresponding normalizing constant and let the resulting density be denoted as

$$f_1(x) = c_1 x^\gamma e^{-b_1 x^{-\rho}} [1 - b(a - \alpha) x^\delta]^{\frac{\eta}{a-\alpha}}, \alpha < a, b_1 > 0, b > 0, \eta > 0 \quad (5.80)$$

under the condition  $1 - b(a - \alpha)x^\delta > 0, \delta > 0, \rho > 0, \Re(\gamma) > 0$ . When  $\alpha > a$ , then  $a - \alpha = -(\alpha - a)$  and then  $f_1(x)$  switches into the function

$$f_2(x) = c_2 x^\gamma e^{-b_1 x^{-\rho}} [1 + b(\alpha - a)x^\delta]^{-\frac{\eta}{\alpha - a}}, \alpha > a \quad (5.81)$$

for  $b > 0, b_1 > 0, \delta > 0, \rho > 0, \eta > 0, \Re(\gamma) > 0$ . When  $\alpha \rightarrow a$ , then both  $f_1(x)$  and  $f_2(x)$  go to

$$f_3(x) = c_3 x^\gamma e^{-b_1 x^{-\rho}} e^{-b\eta x^\delta} \quad (5.82)$$

for  $b_1 > 0, b > 0, \delta > 0, \rho > 0, \eta > 0, \Re(\gamma) > 0$ , where  $c_1, c_2, c_3$  are the respective normalizing constants.

## 5.12 Multivariate Densities in the Real and Complex Domains

Let  $X$  be a  $p \times 1$  vector of real scalar variables  $x_j$ 's,  $X' = [x_1, \dots, x_p]$  so that  $u = X'X = x_1^2 + \dots + x_p^2$ . If  $\tilde{X}$  is a  $p \times 1$  vector in the complex domain with  $\tilde{X}' = [\tilde{x}_1, \dots, \tilde{x}_p]$ ,  $\tilde{x}_j = x_{j1} + ix_{j2}$ ,  $i = \sqrt{(-1)}$ ,  $x_{j1}, x_{j2}$  are real scalar quantities,  $j = 1, \dots, k$ , then, let  $\tilde{u} = \tilde{X}^* \tilde{X}$  where  $\tilde{X}^*$  is the conjugate transpose of  $\tilde{X}$ . Then,  $\tilde{u} = (x_{11}^2 + x_{12}^2) + \dots + (x_{p1}^2 + x_{p2}^2)$  = sum of squares of  $2p$  real scalar variables when  $\tilde{X}$  is in the complex domain. Let  $X$  in  $f(X)$  of  $M_\alpha(f)$  be a  $p \times 1$  vector in the real domain. Let us replace  $x$  in the constraints (i) and (ii) in (5.79) by  $u$ , then  $f_j(x)$  of (5.80)–(5.82) change to corresponding densities in terms of  $X$ , where for example  $f_1(x)$  in the real scalar case changes to

$$f_1(X) = C_1 u^\gamma e^{-b_1 u^{-\rho}} [1 - b(a - \alpha)u^\delta]^{-\frac{\eta}{\alpha - a}}, \alpha < a \quad (5.83)$$

for

$1 - b(a - \alpha)u^\delta > 0, b > 0, b_1 > 0, \eta > 0, \delta > 0, \rho > 0, \Re(\gamma) > 0, \alpha < a, u = X'X$ . Then,  $f_1(X)$  of (5.83) above is a real multivariate extension of the pathway density for the case  $\alpha < a$ . We can obtain  $f_2(X)$  for  $\alpha > a$  and  $f_3(X)$  for  $\alpha \rightarrow a$ . Now, let us replace  $x$  in the constraints in (i) and (ii) of (5.79) by  $\tilde{u} = \tilde{X}^* \tilde{X}$  where  $\tilde{X}$  is a  $p \times 1$  vector in the complex domain. Then, we will end up with densities corresponding to (5.80)–(5.82), where for example the density for the case  $\alpha > a$ , denoted by  $f_2(\tilde{X})$ , is the following:

$$f_2(\tilde{X}) = \tilde{C}_2 \tilde{u}^\gamma e^{-b_1 \tilde{u}^{-\rho}} [1 + b(\alpha - a)\tilde{u}^\delta]^{-\frac{\eta}{\alpha - a}}, \alpha > a \quad (5.84)$$

for  $\delta > 0, \rho > 0, b > 0, b_1 > 0, \Re(\gamma) > 0, \eta > 0$ .

## 5.13 Matrix-Variate Densities in the Real and Complex Domains

Let  $X$  be a  $p \times q, p \leq q$  matrix of rank  $p$  in the real domain. Let  $u = \text{tr}(XX')$ . We use the same notation  $u$  for convenience. Observe that  $XX' > O$  (positive definite). Let us replace  $x$  in the constraints in (i) and (ii) of (5.79) by  $u = \text{tr}(XX')$ . Then, optimization of Mathai entropy  $M_\alpha(f)$  produces three densities denoted by  $f_j(X), j = 1, 2, 3$  where  $X$  is now a  $p \times q, p \leq q$  matrix of rank  $p$  in the real domain. For example,  $f_2(X)$  in this case will be the following, again denoting the normalizing constant by  $C_2$ :

$$f_2(X) = C_2 u^\gamma e^{-b_1 u^{-\rho}} [1 + b(\alpha - a)u^\delta]^{-\frac{\eta}{\alpha-a}}, \alpha > a \quad (5.85)$$

for  $b > 0, b_1 > 0, \delta > 0, \rho > 0, \eta > 0, \Re(\gamma) > 0, u = \text{tr}(XX')$ . Let the  $p \times q, p \leq q$  matrix  $\tilde{X}$  of rank  $p$  be in the complex domain. Let  $\tilde{u} = \text{tr}(\tilde{X}\tilde{X}^*)$  where  $\tilde{X}^*$  is the conjugate transpose of  $\tilde{X}$ . Observe that  $\tilde{u}$  here is also real and we use the same notation  $\tilde{u}$  for convenience. Now, replace  $x$  of the constraints in (i) and (ii) of (5.79) by this  $\tilde{u}$  and then optimize Mathai entropy  $M_\alpha(f)$ . Then, we end up with three densities corresponding to (5.80)–(5.82), where for example,  $f_3(\tilde{X})$  is the following:

$$f_3(\tilde{X}) = \tilde{C}_3 \tilde{u}^\gamma e^{-b_1 \tilde{u}^{-\rho} - b\eta \tilde{u}^\delta}, \quad (5.86)$$

for  $\delta > 0, \rho > 0, b > 0, b_1 > 0, \eta > 0, \Re(\gamma) > 0, \tilde{u} = \text{tr}(\tilde{X}\tilde{X}^*)$ . This,  $f_3(\tilde{X})$  is the matrix-variate density in the complex domain corresponding to (5.82) in the real scalar case. In a similar manner, one can consider various constraints for the matrix-variate case and obtain all the densities considered in Sects. 5.1–5.4 through optimization of Mathai entropy. In order to limit the size of the manuscript, we stop the discussion of the construction of densities through optimization of entropies.

## 5.14 Mellin Convolutions Involving Other Functions

In Sects. 5.1–5.7 we have considered reaction-rate probability integral, its generalizations and its extensions. These can also be looked upon as basic Bessel integral, its generalizations and its pathway extensions. They can also be considered as Mellin convolution of a product involving gamma and generalized gamma functions and generalized gamma densities as basic functions. We did not look into Mellin convolution of a ratio involving generalized gamma functions because our interest was to look into generalizations of the basic reaction-rate probability integral. In the present section, we will examine the situation if one of the functions involved is a type-1 beta function type. In terms of statistical densities, we are going to examine Mellin convolution of a product when one of the densities involved is a

type-1 beta density. Let  $x_1 > 0, x_2 > 0$  be two real scalar variables with the associated functions  $f_1(x_1)$  and  $f_2(x_2)$  respectively and with the joint function  $f_1(x_1)f_2(x_2)$ , the product, where let

$$f_1(x_1) = c \frac{1}{\Gamma(\alpha)} x_1^\gamma (1 - x_1)^{\alpha-1}, 0 \leq x_1 \leq 1 \quad (5.87)$$

for  $\Re(\alpha) > 0, \Re(\gamma) > -1$  with  $c = \Gamma(\gamma + 1 + \alpha)/\Gamma(\gamma + 1)$  if  $f_1(x_1)$  is a statistical density. This density, is the type-1 real scalar beta density with the parameters  $(\gamma + 1, \alpha)$  under the usual notation. Let the second function  $f_2(x_2) = f(x_2)$  an arbitrary function or arbitrary density if we are considering statistical densities. Let  $u = x_1 x_2$ . Let the function corresponding to  $u$  be  $g(u)$ . Then, from the Mellin convolution of a product property

$$\begin{aligned} g(u) &= \int_v \frac{1}{v} f_1\left(\frac{u}{v}\right) f_2(v) dv = c \frac{1}{\Gamma(\alpha)} \left(\frac{u}{v}\right)^\gamma \left(1 - \frac{u}{v}\right)^{\alpha-1} f(v) dv \\ &= c \frac{u^\gamma}{\Gamma(\alpha)} \int_{v>u>0} v^{-\gamma-\alpha} (v - u)^{\alpha-1} f(v) dv = c K_{2,\gamma,u}^{-\alpha}(f) \end{aligned} \quad (5.88)$$

where  $K_{2,\gamma,u}^{-\alpha}(f)$  is Erdélyi-Kober fractional integral of the second kind of order  $\alpha$  and parameter  $\gamma$  in the real scalar variable case. Thus, a simple change of the basic functions to a type-1 beta form and a general function for  $f_2(x_2)$ , leads to a fractional integral of the second kind. Mathai (2009) shows that if we take the first function  $f(x_1)$  as

$$f_1(x_1) = \frac{\phi(x_1)}{\Gamma(\alpha)} (1 - x_1)^{\alpha-1}, \Re(\alpha) > 0$$

and  $f_2(x_2) = f(x_2)$  as above, then by specializing  $\phi(x_1)$ , one can obtain all the various definitions of fractional integrals of the second kind of order  $\alpha$  introduced by various authors from time to time, in the real scalar case situation. For example, if  $\phi(x_1) = x_1^\gamma$  one has Erdélyi-Kober fractional integral. If  $\phi(x_1) = x_1^{-\alpha}$ , one had Weyl fractional integral of the second kind. In this case, if  $v$  is bounded above by a constant  $b, v \leq b$ , then it becomes Riemann-Liouville fractional integral of the second kind and so on. From the derivation of (5.88), observe that this fractional integral of the second kind can also be obtained as a constant multiple of a density of a product of real scalar positive random variables when one of the densities is a type-1 beta density with the parameters  $(\gamma + 1, \alpha)$  and the other is an arbitrary density.

## 5.15 Generalization to Real Matrix-Variate Case

Let  $X_1 > O$  and  $X_2 > O$  be two  $p \times p$  real positive definite matrices with distinct elements as real scalar variables. Let the associated functions be the real-valued scalar functions  $f_1(X_1)$  and  $f_2(X_2)$  respectively and let the joint function be

$f_1(X_1)f_2(X_2)$ . If  $X_1$  and  $X_2$  are matrix-variate real random variables with the associated densities  $f_1(X_1)$  and  $f_2(X_2)$  then, when the joint function is the product  $f_1(X_1)f_2(X_2)$ , we say that  $X_1$  and  $X_2$  are statistically independently distributed.

Consider the symmetric product  $U = X_2^{\frac{1}{2}} X_1 X_2^{\frac{1}{2}}$  and let  $X_2 = V$  where  $X_2^{\frac{1}{2}}$  is the symmetric positive definite square root of the positive definite matrix  $X_2 > O$ . From Mathai (1997) we have the Jacobian and we can show that

$dX_1 \wedge dX_2 = |V|^{-\frac{p+1}{2}} dU \wedge dV$ . This involves the transformation of a symmetric matrix to a symmetric matrix and we have not listed the corresponding Jacobian as a lemma in order to save space. Then, the function corresponding to  $U$ , again denoted by  $g(U)$ , is the following:

$$g(U) = \int_V |V|^{-\frac{p+1}{2}} f_1(V^{-\frac{1}{2}}UV^{-\frac{1}{2}})^\gamma f_2(V) dV. \quad (5.89)$$

Let

$$f_1(X_1) = \frac{1}{\Gamma_p(\alpha)} |X_1|^\gamma |I - X_1|^{\alpha - \frac{p+1}{2}}, \quad O < X_1 < I, \quad \Re(\alpha) > \frac{p-1}{2} \quad (5.90)$$

and let  $f_2(X_2) = f(X_2)$  where  $f$  is an arbitrary function,  $O < X_1 < I$  means  $X_1 > O, I - X_1 > O$  (both positive definite), and  $f_1(X_1) = 0$  elsewhere. As observed in the real scalar case, if we multiply by the constant

$\Gamma_p(\gamma + \frac{p+1}{2} + \alpha)/\Gamma_p(\gamma + \frac{p+1}{2})$ , then  $f_1(X_1)$  becomes a real matrix-variate type-1 beta density. Now, from (5.89) and (5.90),

$$\begin{aligned} g(U) &= \frac{1}{\Gamma_p(\alpha)} \int_V |V|^{-\frac{p+1}{2}} |(V^{-\frac{1}{2}}UV^{-\frac{1}{2}})|^\gamma |I - V^{-\frac{1}{2}}UV^{-\frac{1}{2}}|^{\alpha - \frac{p+1}{2}} f(V) dV \\ &= \frac{|U|^\gamma}{\Gamma_p(\alpha)} \int_{V-U>O} |V|^{-\alpha-\gamma} |V - U|^{\alpha - \frac{p+1}{2}} f(V) dV \\ &= K_{2,\gamma,U}^{-\alpha}(f) \end{aligned} \quad (5.91)$$

where  $K_{2,\gamma,U}^{-\alpha}(f)$  is Erdélyi-Kober fractional integral of the second kind of order  $\alpha$  and parameter  $\gamma$  in the real matrix-variate case. Since (5.91) is identical with Erdélyi-Kober fractional integral of the second kind when  $p = 1$  or in the real scalar variable case, Mathai (2015) defined (5.91) as Erdélyi-Kober fractional integral of the second kind in the real matrix-variate case. Again, in the real matrix-variate case also, if we take  $f_1(X_1) = \frac{1}{\Gamma_p(\alpha)} \phi(X_1) |I - X_1|^{\alpha - \frac{p+1}{2}}$  for  $O < X_1 < I$ , then we can show that by specializing  $\phi(X_1)$  one can obtain the real matrix-variate versions of the various definitions of fractional integral of the second kind introduced by various authors for the real scalar variable case. These, matrix-variate versions are not defined by anyone so far.

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## 5.16 Generalization to the Complex Matrix-Variate Case

Let  $\tilde{X}_1 > O$  and  $\tilde{X}_2 > O$  be two  $p \times p$  Hermitian positive definite matrices in the complex domain with the associated functions  $f_1(\tilde{X}_1)$  and  $f_2(\tilde{X}_2)$  respectively and with the joint function  $f_1(\tilde{X}_1)f_2(\tilde{X}_2)$  where  $f_1$  and  $f_2$  are real-valued scalar functions of the matrix argument in the complex domain. Let  $\tilde{U} = \tilde{X}_2^{\frac{1}{2}} \tilde{X}_1 \tilde{X}_2^{\frac{1}{2}}$  and  $\tilde{V} = \tilde{X}_2$ , where  $\tilde{X}_2^{\frac{1}{2}}$  is the Hermitian positive definite square root of the Hermitian positive definite matrix  $\tilde{X}_2$ . Again, from Mathai (1997) we can show that  $d\tilde{X}_1 \wedge d\tilde{X}_2 = |\det(\tilde{V})|^{-p} d\tilde{U} \wedge d\tilde{V}$ . Let

$$f_1(\tilde{X}_1) = \frac{1}{\tilde{\Gamma}_p(\alpha)} |\det(\tilde{X}_1)|^\gamma |\det(I - \tilde{X}_1)|^{\alpha-p}, O < \tilde{X}_1 < I, \Re(\alpha) > p - 1$$

and  $f_1(\tilde{X}_1) = 0$  elsewhere, and let  $f_2(\tilde{X}_2) = f(\tilde{X}_2)$  an arbitrary function. Then,

$$\begin{aligned} g(\tilde{U}) &= \frac{1}{\tilde{\Gamma}_p(\alpha)} \int_{\tilde{V}} |\det(\tilde{V})|^{-p} |\det(\tilde{V}^{-\frac{1}{2}} \tilde{U} \tilde{V}^{-\frac{1}{2}})|^\gamma \\ &\quad \times |\det(I - \tilde{V}^{-\frac{1}{2}} \tilde{U} \tilde{V}^{-\frac{1}{2}})|^{\alpha-p} f(\tilde{V}) d\tilde{V}. \end{aligned} \quad (5.92)$$

$$= \frac{|\det(\tilde{U})|^\gamma}{\tilde{\Gamma}_p(\alpha)} \int_{\tilde{V} - \tilde{U} > O} |\det(\tilde{V})|^{-\gamma-\alpha} |\det(\tilde{V} - \tilde{U})|^{\alpha-p} f(\tilde{V}) d\tilde{V} \quad (5.93)$$

$$= \tilde{K}_{2,\gamma,\tilde{U}}^{-\alpha}(f) \quad (5.94)$$

where  $\tilde{K}_{2,\gamma,\tilde{U}}^{-\alpha}(f)$  is Erdélyi-Kober fractional integral of the second kind of order  $\alpha$  and parameter  $\gamma$  in the complex matrix-variate case. Mathai (2013) has defined like this when he extended fractional calculus to the complex matrix-variate case. In the real scalar case (5.94) will agree with Erdélyi-Kober fractional integral of the second kind.

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## 5.17 Mellin Convolution of a Ratio

Let us start with real scalar variables first. Let  $x_1 > 0$  and  $x_2 > 0$  be two real scalar positive variables. We can define a product in two different ways  $u = x_1 x_2$ ,  $v = x_2$  or  $v = x_1$ . We can define a ratio in four different ways  $u_1 = \frac{x_1}{x_2}$  with  $v = x_1$  or  $v = x_2$  and  $u_1 = \frac{x_2}{x_1}$  with  $v = x_1$  or  $v = x_2$ . All these different representations can

lead to different items in different disciplines as can be seen from the following illustration. Let  $u_1 = \frac{x_2}{x_1}$  with  $v = x_2$  so that

$$x_2 = v, x_1 = \frac{v}{u_1} \Rightarrow dx_1 \wedge dx_2 = -\frac{v}{u_1^2} du_1 \wedge dv.$$

Let the function corresponding to  $u_1$  be denoted by  $g_1(u_1)$ . Then,

$$g_1(u_1) = \int_v \frac{v}{u_1^2} f_1\left(\frac{v}{u_1}\right) f_2(v) dv. \quad (5.95)$$

Let

$$f_1(x_1) = \frac{x_1^{\gamma-1}}{\Gamma(\alpha)} (1-x_1)^{\alpha-1}, \quad 0 < x_1 < 1, \quad \Re(\alpha) > 0, \quad \Re(\gamma) > 0$$

and  $f_2(x_2) = f(x_2)$  an arbitrary function. Then,

$$\begin{aligned} g_1(u_1) &= \frac{1}{\Gamma(\alpha)} \int_v \frac{v}{u_1^2} \left(\frac{v}{u_1}\right)^{\gamma-1} (1-\frac{v}{u_1})^{\alpha-1} f(v) dv \\ &= \frac{u_1^{-\alpha-\gamma}}{\Gamma(\alpha)} \int_{0 < v < u_1} v^\gamma (u_1 - v)^{\alpha-1} f(v) dv \\ &= K_{1,\gamma,u_1}^{-\alpha}(f) \end{aligned} \quad (5.96)$$

where  $K_{1,\gamma,u_1}^{-\alpha}(f)$  is Erdélyi-Kober fractional integral of the first kind of order  $\alpha$  and parameter  $\gamma$  for the real scalar variable case. By multiplying  $f_1$  with a constant, one has a real scalar type-1 beta density and if  $f(x_2)$  is an arbitrary density, then the fractional integral of the first kind in (ii) is a constant multiple of the density of a ratio of two independently distributed real scalar positive random variables. This connection was pointed out by Mathai (2009).

## 5.18 M-Convolution of a Ratio in the Real Matrix-Variate Case

Let  $X_1 > O$  and  $X_2 > O$  be two  $p \times p$  real positive definite matrices and let the associated functions be  $f_1(X_1)$  and  $f_2(X_2)$  respectively and let the joint function be  $f_1(X_1)f_2(X_2)$ . Let  $U_1 = X_2^{\frac{1}{2}} X_1^{-1} X_2^{\frac{1}{2}}$  be a symmetric ratio of the matrices  $X_1$  and  $X_2$ . Let  $V = X_2$ . Then, from Mathai (1997), we can show that

$$dX_1 \wedge dX_2 = |U_1|^{-(p+1)} |V|^{\frac{p+1}{2}} dU_1 \wedge dV.$$

Let

$$f_1(X_1) = \frac{1}{\Gamma_p(\alpha)} |X_1|^{\gamma - \frac{p+1}{2}} |I - X_1|^{\alpha - \frac{p+1}{2}}, O < X_1 < I,$$

for  $\Re(\alpha) > \frac{p-1}{2}$ ,  $\Re(\gamma) > \frac{p-1}{2}$  and let  $f_2(X_2) = f(X_2)$  an arbitrary function. Then,

$$\begin{aligned} g_1(U_1) &= \int_V |U_1|^{-(p+1)} |V|^{\frac{p+1}{2}} f_1(V^{\frac{1}{2}} U_1^{-1} V^{\frac{1}{2}}) f(V) dV \\ &= \frac{1}{\Gamma_p(\alpha)} \int_V |U_1|^{-(p+1)} |V|^{\frac{p+1}{2}} |V^{\frac{1}{2}} U_1^{-1} V^{\frac{1}{2}}|^{\gamma - \frac{p+1}{2}} \\ &\quad \times |I - V^{\frac{1}{2}} U_1^{-1} V^{\frac{1}{2}}|^{\alpha - \frac{p+1}{2}} f(V) dV \\ &= \frac{|U_1|^{-\gamma-\alpha}}{\Gamma_p(\alpha)} \int_{U_1 - V > O} |V|^\gamma |U_1 - V|^{\alpha - \frac{p+1}{2}} f(V) dV \\ &= K_{1,\gamma,U_1}^{-\alpha}(f) \end{aligned} \tag{5.97}$$

where  $K_{1,\gamma,U_1}^{-\alpha}(f)$  is Erdélyi-Kober fractional integral of the first kind of order  $\alpha$  and parameter  $\gamma$  in the real matrix-variate case. This was established by Mathai in one of the series of papers published in Linear Algebra and Its Applications in 2013–2016, extending fractional calculus to the matrix-variate cases in the real and complex domain.

## 5.19 Fractional Integral of the First Kind in the Complex Matrix-Variate Case

Let  $\tilde{X}_1 > O$  and  $\tilde{X}_2 > O$  be  $p \times p$  Hermitian positive definite matrices in the complex domain with the associated functions  $f_1(\tilde{X}_1)$  and  $f_2(\tilde{X}_2)$  respectively and with the joint function  $f_1(\tilde{X}_1)f_2(\tilde{X}_2)$  where  $f_1$  and  $f_2$  are real-valued scalar functions of the matrices  $\tilde{X}_1$  and  $\tilde{X}_2$  in the complex domain. Consider the symmetric ratio  $\tilde{U}_1 = \tilde{X}_2^{\frac{1}{2}} \tilde{X}_1^{-1} \tilde{X}_2^{\frac{1}{2}}$  and  $\tilde{V} = \tilde{X}_2$  where  $\tilde{X}_2^{\frac{1}{2}}$  is the Hermitian positive definite square root of the Hermitian positive definite matrix  $\tilde{X}_2 > O$ . From Mathai (1997), it can be shown that  $d\tilde{X}_1 \wedge d\tilde{X}_2 = |\det(\tilde{U}_1)|^{-2p} |\det(\tilde{V})|^p d\tilde{U}_1 \wedge d\tilde{V}$ . Let

$$f_1(\tilde{X}_1) = \frac{1}{\tilde{\Gamma}_p(\alpha)} |\det(\tilde{X}_1)|^{\gamma-p} |\det(I - \tilde{X}_1)|^{\alpha-p}, O < \tilde{X}_1 < I$$

for  $\Re(\alpha) > p-1$ ,  $\Re(\gamma) > p-1$  and  $O < \tilde{X}_1 < I$  means  $\tilde{X}_1 > O$ ,  $I - \tilde{X}_1 > O$  (both Hermitian positive definite). Let  $f_2(\tilde{X}_2) = f(\tilde{X}_2)$ , an arbitrary function. Then for  $\Re(\alpha) > p-1$ , the function corresponding to  $\tilde{U}_1$ , denoted by  $g_1(\tilde{U}_1)$ , is the following:

(5.98)

$$\begin{aligned}
g_1(\tilde{U}_1) &= \frac{1}{\tilde{\Gamma}_p(\alpha)} \int_{\tilde{V}} |\det(\tilde{V})|^p |\det(\tilde{U}_1)|^{-2p} |\det(\tilde{V}^{\frac{1}{2}} \tilde{U}_1^{-1} \tilde{V}^{\frac{1}{2}})|^{\gamma-p} \\
&\quad \times |\det(I - \tilde{V}^{\frac{1}{2}} \tilde{U}_1^{-1} \tilde{V}^{\frac{1}{2}})|^{\alpha-p} f(\tilde{V}) dV \\
&= \frac{|\det(\tilde{U}_1)|^{-\gamma-\alpha}}{\tilde{\Gamma}_p(\alpha)} \int_{(\tilde{U}_1 - \tilde{V}) > O} |\det(\tilde{V})|^{\gamma} |\det(\tilde{U}_1 - \tilde{V})|^{\alpha-p} f(\tilde{V}) dV \\
&= \tilde{K}_{1,\gamma,\tilde{U}_1}^{-\alpha}(f)
\end{aligned}$$

where  $\tilde{K}_{1,\gamma,\tilde{U}_1}^{-\alpha}(f)$  is Erdélyi-Kober fractional integral of the first kind of order  $\alpha$  and parameter  $\gamma$  in the complex domain as defined by Mathai (2013). A brief overview of the development of fractional calculus in the complex domain and for functions of matrix argument is given in the Springer Brief of Mathai and Haubold (2018). Hence, further discussion of fractional integrals and fractional derivatives is not attempted here.



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## 6. Neutrino Astrophysics, 2025 Update: The Entropic Approach to Solar Neutrinos

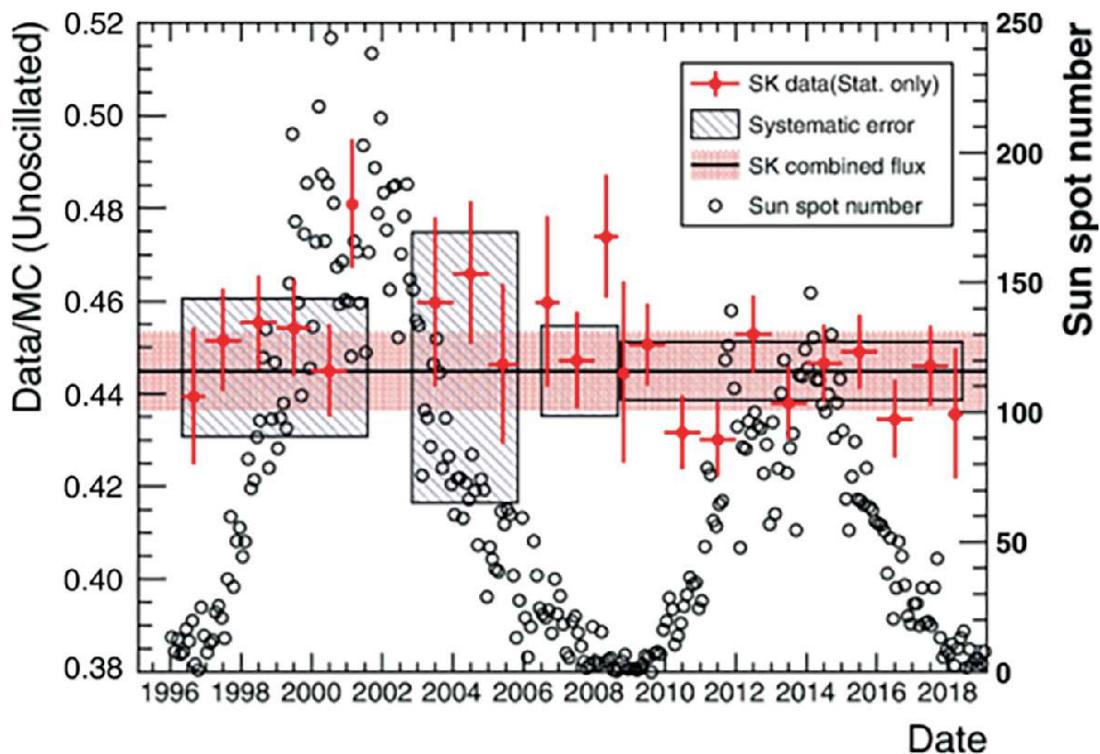
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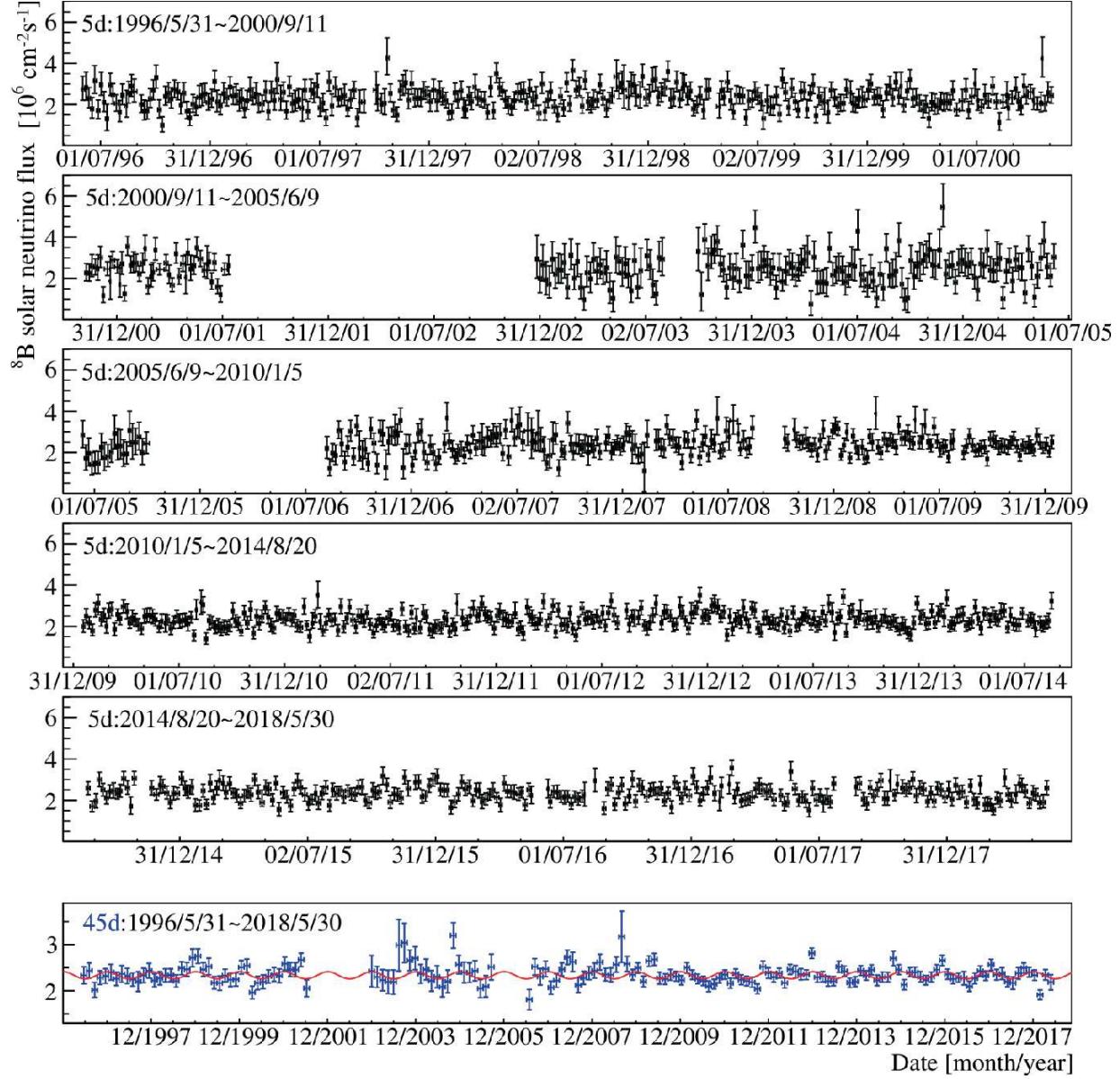
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### 6.1 Solar Neutrinos: SuperKamiokande Data

Over the past 50 years, radio-chemical and real-time solar neutrino experiments have proven to be sensitive tools to test both astrophysical and elementary particle physics models and principles (Sakurai [2018](#); Orebí Gann et al. [2021](#)). Solar neutrino detectors (radio-chemical: Homestake, GALLEX + GNO, SAGE, real-time: SuperKamiokande, SNO, Borexino) have demonstrated that the Sun is powered by thermonuclear fusion reactions. Today fluxes, particularly from the pp-chain have been measured: *pp*,  $^7Be$ , *pep*,  $^8B$ , and, *hep*. Experiments with solar neutrinos and reactor anti-neutrinos (KamLAND) have confirmed that solar neutrinos undergo flavor oscillations (Mikheyev–Smirnov–Wolfenstein (MSW) model). Results from solar neutrino experiments are consistent with the Mikheyev–Smirnov–Wolfenstein Large Mixing Angle (MSW-LMA) model, which predicts a transition from vacuum-dominated to matter-enhanced oscillations, resulting in an energy dependent electron neutrino survival probability.



**Fig. 6.1** Yearly solar neutrino flux measured by SuperKamiokande. The redfilled circle points show the SuperKamiokande data with statistical uncertainty and the gray striped area show the systematic uncertainty for each phase. The horizontal black solid line (red shaded area) shows the combined value of measured flux (its combined uncertainty). The black-blank circle points show the sunspot numbers from 1996 to 2018 (Abe et al. [2024b](#))



**Fig. 6.2** Measured  $^{8}\text{B}$  solar neutrino fluxes for 5-day (top five panels, black data points) and 45-day (bottom panel, blue data points) intervals without  $1/R^2$  correction. The errors in the 5-day (the 45-day) plot are asymmetric (symmetric) errors of the average fluxes. The solid-red curve in the 45-day plot is the expected sinusoidal solar neutrino flux based on the elliptical orbit of the Earth (Abe et al. 2024a)

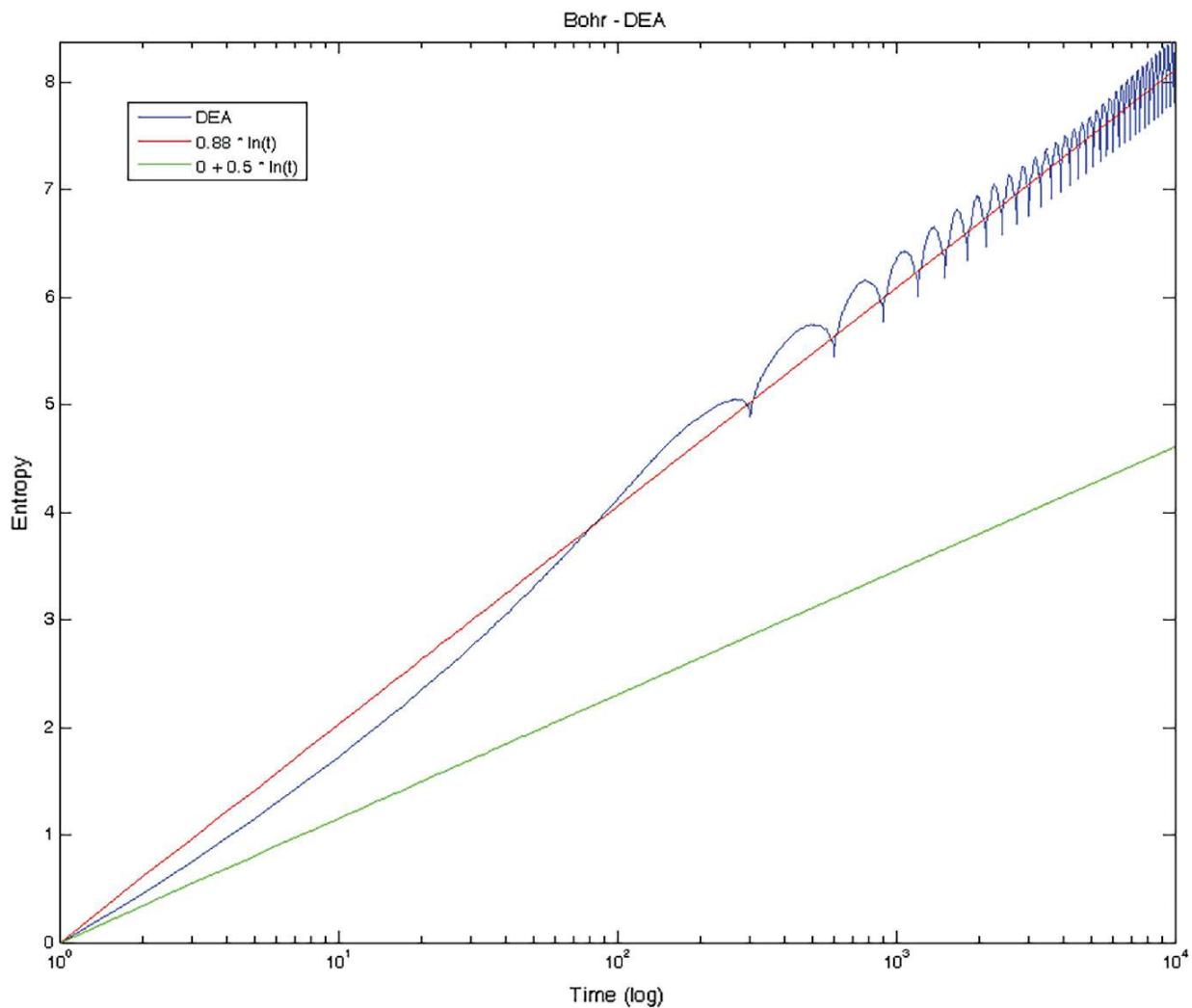
## 6.2 Diffusion Entropy and Standard Deviation: Analysis

For all radio-chemical and real-time solar neutrino experiments, periodic variation in the detected solar neutrino fluxes have been reported, based mainly on Fourier and wavelet analysis methods (standard deviation analysis). Other attempts to analyze the same data sets, particularly

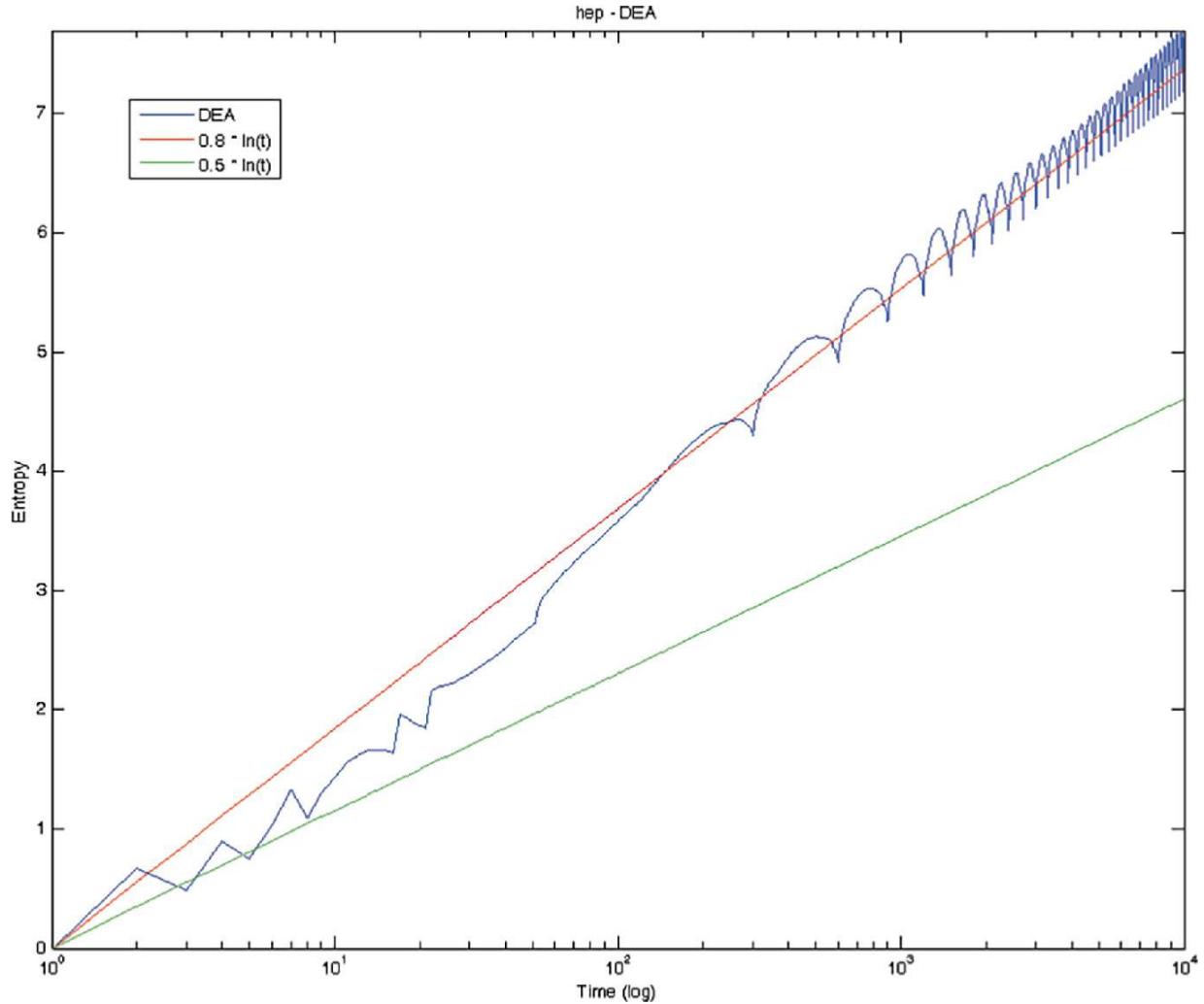
undertaken by the experimental collaborations of real-time solar neutrino experiments themselves, have failed to find evidence for such variations of the solar neutrino flux over time (Abe et al. 2024a). Periodicities in the solar neutrino fluxes, if confirmed, could provide evidence for new solar, nuclear, or neutrino physics beyond the commonly accepted physics of vacuum-dominated and matter-enhanced oscillations of massive neutrinos (MSW model) that is, after 50 years of solar neutrino experiment and theory, considered to be the ultimate solution to the solar neutrino problem (Fig. 6.1).

Specifically, subsequent to the analysis made by the SuperKamiokande collaboration, the SNO experiment collaboration has painstakingly searched for evidence of time variability at periods ranging from 10 years down to 10 min. SNO has found no indications for any time variability of the  $^8B$  flux at any timescale, including in the frequency window in which  $g$ -mode oscillations of the solar core might be expected to occur. Despite large efforts to utilize helio-seismology and helio-neutrinospectroscopy, at present time there is no conclusive evidence in terms of physics for time variability of the solar neutrino fluxes from any solar neutrino experiment. If such a variability over time would be discovered, a mechanism for a chronometer for solar variability could be proposed based on relations between properties of thermonuclear fusion and  $g$ -modes (Buldgen et al. 2024; Sturrock et al. 2021).

All above findings encouraged the conclusion that Fourier and wavelet analysis, which are based upon the analysis of the variance of the respective time series (standard deviation analysis: SDA) should be complemented by the utilization of diffusion entropy analysis (DEA), which measures the scaling of the probability density function (pdf) of the diffusion process generated by the time series thought of as the physical source of fluctuations (Scafetta 2010). For this analysis, we have used the publicly available data of SuperKamiokande-I (1996-05-31-2001-07-15) and SuperKamiokande-II (2002-12-10-2005-10-06) (see Fig. 6.2) (Yoo et al. 2003; Cravens et al. 2008; Abe et al. 2024b). Such an analysis does not reveal periodic variations of the solar neutrino fluxes but shows how the pdf scaling exponent departs in the non-Gaussian case from the Hurst exponent. Figures 6.3 and 6.4 show the scaling exponents (DEA) for the SuperKamiokande I and II data. The respective Hurst exponents for SDA are visible in Figs. 6.5 and 6.6 (Mathai and Haubold 2018). SuperKamiokande is sensitive mostly to neutrinos from the  $^8B$  and  $hep$  branch of the  $pp$  nuclear fusion chain in solar burning. Above approximately 4 MeV the detector can pick-out the scattering of solar neutrinos off atomic electrons which produces Cherenkov radiation in the detector. The  $^8B$  and rarer  $hep$  neutrinos have a spectrum which ends near 20 MeV.



**Fig. 6.3** The Diffusion Entropy Analysis (DEA) of the  ${}^8B$  solar neutrino data from the SuperKamiokande I and II experiment



**Fig. 6.4** The Diffusion Entropy Analysis (DEA) of the *hep* solar neutrino data from the SuperKamiokande I and II experiment

Assuming that the solar neutrino signal is governed by a probability density function with scaling given by the asymptotic time evolution of a pdf of  $x$ , obeying the property (Scafetta 2010; Culbreth et al. 2023)

$$p(x, t) = \frac{1}{t^\delta} f\left(\frac{x}{t^\delta}\right), \quad (6.1)$$

where  $\delta$  denotes the scaling exponent of the pdf. In the variance based methods, scaling is studied by direct evaluation of the time behavior of the variance of the diffusion process. If the variance scales, one would have

$$\sigma_x^2(t) \sim t^{2H}, \quad (6.2)$$

where  $\sigma_x^2(t)$  is the variance of the diffusion process and where  $H$  is the Hurst exponent. To evaluate the Shannon entropy of the diffusion process at time  $t$ , defined  $S(t)$  as

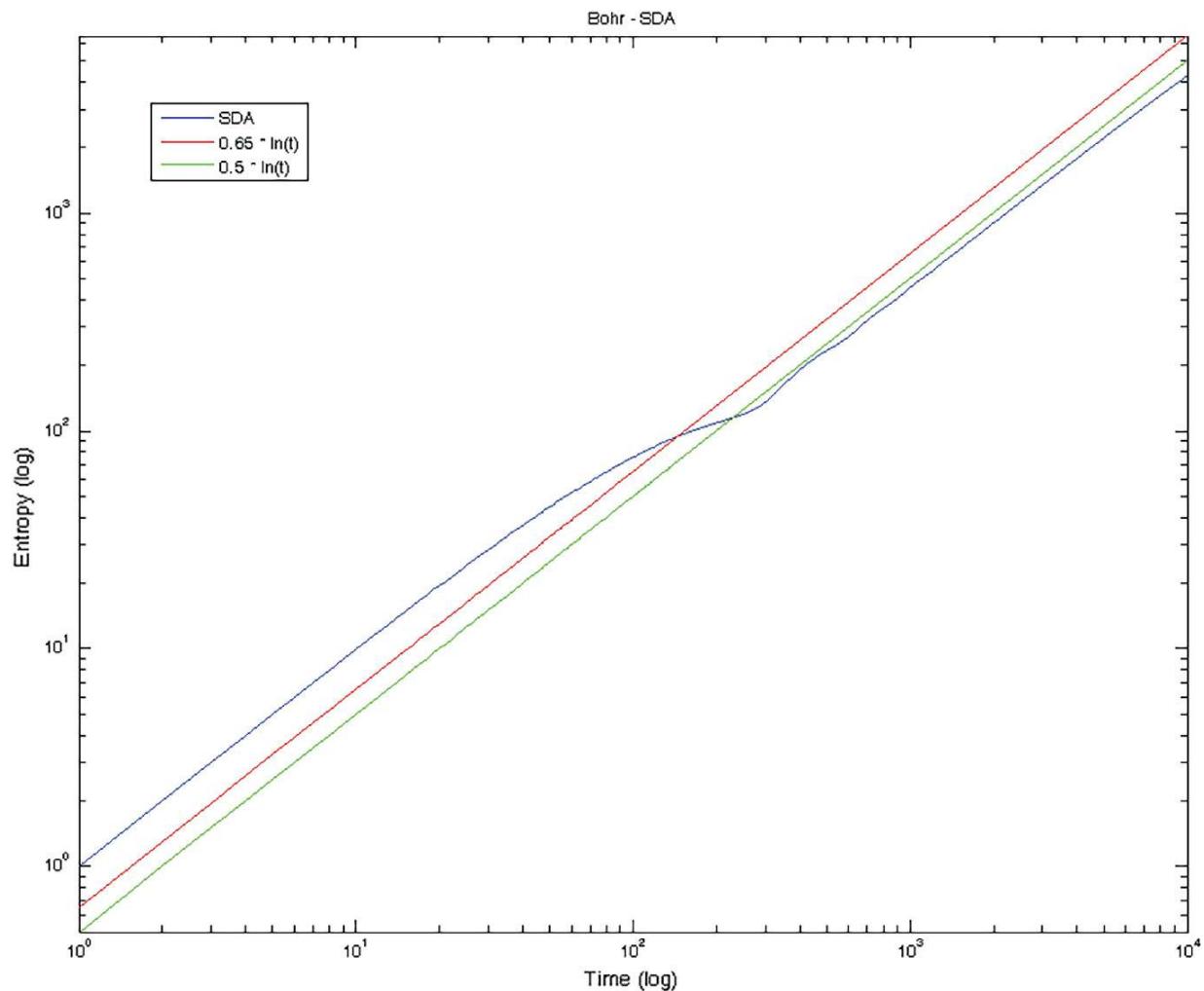
$$S(t) = - \int_{-\infty}^{+\infty} dx \ p(x, t) \ln p(x, t) \quad (6.3)$$

and with the previous  $p(x, t)$  one has

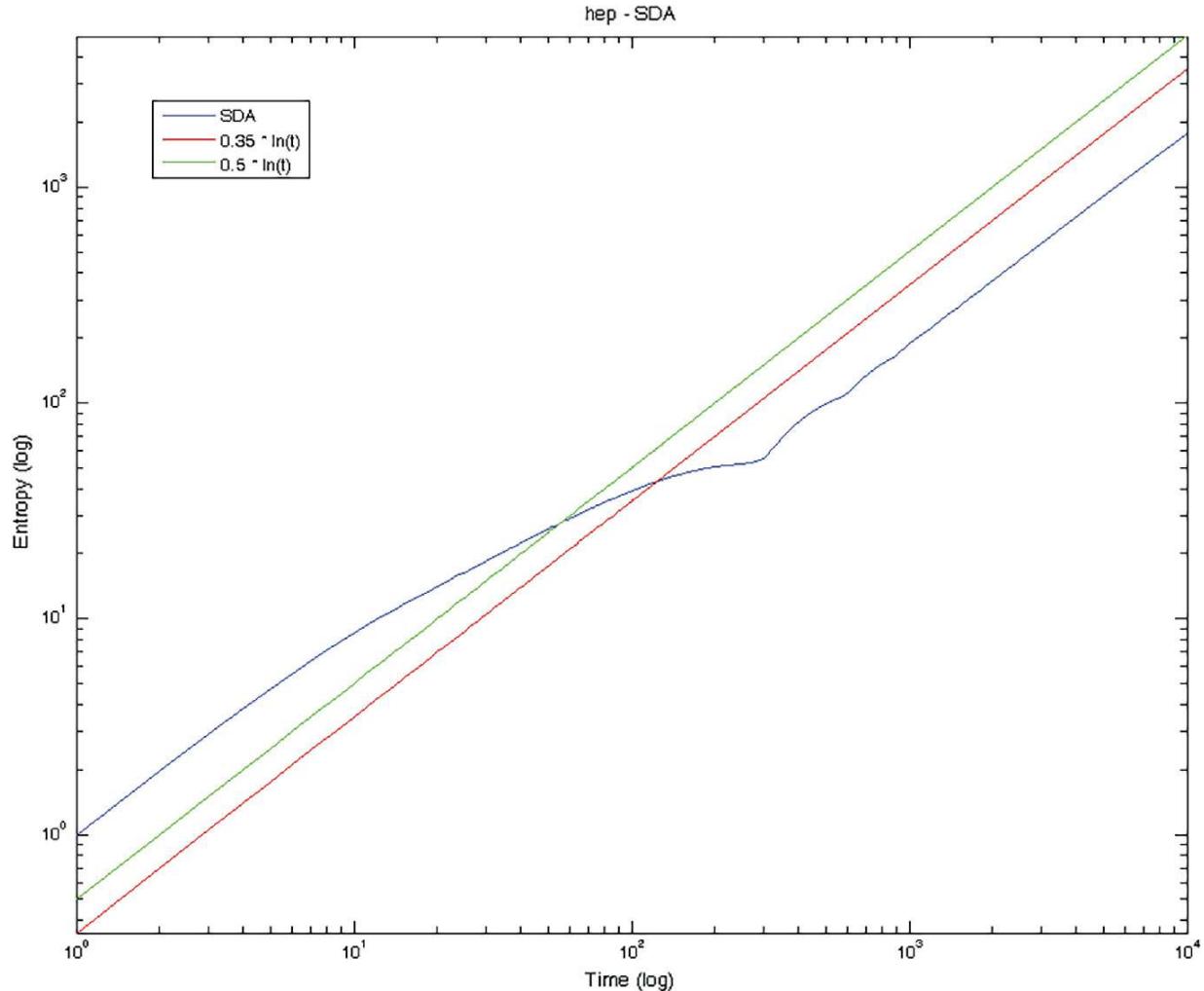
$$S(t) = A + \delta \ln (t), \quad A = - \int_{-\infty}^{+\infty} dy f(y) \ln f(y). \quad (6.4)$$

The scaling exponent  $\delta$  is the slope of the entropy against the logarithmic time scale. The slope is visible in Figs. 6.3 and 6.4 for the SuperKamiokande data measured for  ${}^8B$  and *hep*. The Hurst exponents (SDA) are  $H = 0.66$  and  $H = 0.36$  for  ${}^8B$  and *hep*, respectively, see Figs. 6.5 and 6.6 (Mathai and Haubold 2018). The *pdf* scaling exponents (DEA) are  $\delta = 0.88$  and  $\delta = 0.80$  for  ${}^8B$  and *hep*, respectively, as shown in Figs. 6.3 and 6.4. The values for both SDA and DEA indicate a deviation from Gaussian behavior which would require that  $H = \delta = 0.5$ .

A test computation for the application of SDA and DEA to data that are known to exhibit non-Gaussian behavior have been published by Haubold et al. (2012) and Tsallis (2024). In this test computation, SDA and DEA, applied to the magnetic field strength fluctuations recorded by the Voyager-I spacecraft in the heliosphere clearly revealed the scaling behavior of such fluctuations as previously already discovered by non-extensive statistical mechanics considerations that lead to the determination of the non-extensivity  $q$ -triplet.



**Fig. 6.5** The Standard Deviation Analysis (SDA) of the  $^8B$  solar neutrino data from the SuperKamiokande I and II experiment



**Fig. 6.6** The Standard Deviation Analysis (SDA) of the *hep* solar neutrino data from the SuperKamiokande I and II experiment

## 6.3 Probability Density Function and Differential Equation: Lévy Flights

We consider a diffusion process generated by a waiting time pdf with a finite characteristic time  $T$  that can be modeled with a Poissonian distribution, and a jump length pdf  $\lambda(x)$  given by a Lévy distribution with index  $0 < \alpha < 2$  (Metzler and Klafter [2000](#)). The Fourier transform of  $\lambda(x)$  is

$$\hat{\lambda}(k) = \exp(-\sigma^\alpha |k|^\alpha) \sim 1 - \sigma^\alpha |k|^\alpha. \quad (6.5)$$

Then  $\lambda(x)$  has the asymptotic behavior given by

$$(6.6)$$

$$\lambda(x) \sim A_\alpha \sigma^\alpha |x|^{-1-\alpha} = A_\alpha \sigma^{1-\mu} |x|^{-\mu}$$

for  $|x| \gg \sigma$  and  $\mu = 1 + \alpha$ . Substituting the asymptotic expansion of the jump length pdf  $\hat{\lambda}(k)$  in the Fourier space and the waiting time pdf of

$$\Phi(t) = \frac{1}{\tau} \exp\left(-\frac{t}{\tau}\right) \quad (6.7)$$

where  $\tau = T < \infty$  is the characteristic waiting time in the Laplace space into

$$\hat{p}(k, s) = \frac{1 - \hat{\Phi}(s)}{s} \frac{\hat{p}_0(k)}{1 - \hat{\Phi}(s)\hat{\lambda}(k)}, \quad (6.8)$$

where  $\hat{p}_0(k)$  is the Fourier transform of the initial condition  $p(x, 0)$ , we obtain the following jump pdf in the Fourier-Laplace space

$$\hat{p}(k, s) = \frac{1}{s + K^\alpha |k|^\alpha}, \quad (6.9)$$

where  $K^\alpha = \sigma^\alpha / \tau$  is the generalized diffusion constant. Equation (6.9) is the solution of the generalized diffusion equation (Hilfer [2018](#))

$$\frac{\partial p(x, t)}{\partial t} = K^\alpha {}_{-\infty}D_x^\alpha p(x, t), \quad (6.10)$$

where  ${}_{-\infty}D_x^\alpha$  is the fractional Weyl operator. Upon Laplace inversion of Eq. (6.9), we get the characteristic function of the jump pdf

$$\hat{p}(k, t) = \exp(-K^\alpha t |k|^\alpha). \quad (6.11)$$

Equation (6.11) is the characteristic function of a centered and symmetric Lévy distribution. The Fourier inversion of (6.7) can be analytically obtained by making use of the Fox function (Mathai et al. [2010](#); Mathai and Haubold [2018](#))

$$\begin{aligned} p(x, t) &= \frac{1}{\alpha} \frac{1}{t^{1/\alpha}} \frac{t^{1/\alpha}}{|x|} H_{2,2}^{1,1} \left[ \frac{|x|}{Kt^{1/\alpha}} \middle| \begin{smallmatrix} (1, \frac{1}{\alpha}), (1, \frac{1}{2}) \\ (1, 1), (1, \frac{1}{2}) \end{smallmatrix} \right] \\ &= \frac{1}{\alpha|x|} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(1+\frac{s}{\alpha})\Gamma(-\frac{s}{2})}{\Gamma(-s)\Gamma(1+\frac{s}{2})} \left( \frac{|x|}{Kt^{1/\alpha}} \right)^{-s} ds, \end{aligned}$$

where  $c$  in the contour is such that  $-\alpha < c < 0$ . Replacing  $\frac{s}{\alpha}$  by  $s$ , observing that  $\alpha$  coming from  $s$   $ds$  is canceled with  $\alpha$  sitting outside, we have the following:

$$p(x, t) = \frac{1}{|x|} \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{\Gamma(1+s)\Gamma(-\frac{\alpha}{2}s)}{\Gamma(-\alpha s)\Gamma(1+\frac{\alpha}{2}s)} \left( \frac{|x|}{Kt^{1/\alpha}} \right)^{-\alpha s} ds, -1 < c' < 0.$$

By using the duplication formula for gamma functions, we have

$$\Gamma(-\alpha s) = \Gamma\left(2\left(-\frac{\alpha}{2}s\right)\right) = 2^{-\alpha s-1}\pi^{-\frac{1}{2}}\Gamma\left(-\frac{\alpha}{2}s\right)\Gamma\left(\frac{1}{2} - \frac{\alpha}{2}s\right)$$

so that one  $\Gamma(-\frac{\alpha}{2}s)$  is canceled. Then,

$$p(x, t) = \frac{2\pi^{\frac{1}{2}}}{|x|} \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{\Gamma(1+s)}{\Gamma\left(\frac{1}{2} - \frac{\alpha}{2}s\right)\Gamma\left(1 + \frac{\alpha}{2}s\right)} \left( \frac{|x|}{2Kt^{1/\alpha}} \right)^{-\alpha s} ds.$$

Evaluating the H-function as the sum of the residues at the poles of  $\Gamma(1+s)$ , which are at  $s = -1 - \nu, \nu = 0, 1, \dots$ , we have the following series:

$$\begin{aligned} p(x, t) &= \frac{2\pi^{\frac{1}{2}}}{|x|} \left( \frac{|x|^\alpha}{(2K)^\alpha t} \right) \\ &\times \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} \frac{1}{\Gamma\left(\frac{1+\alpha}{2} + \frac{\alpha}{2}\nu\right)\Gamma\left(1 - \frac{\alpha}{2}(1+\nu)\right)} \left( \frac{|x|^\alpha}{(2K)^\alpha t} \right)^\nu, \alpha \neq 1, \alpha \neq 2. \end{aligned}$$

## 6.4 Discussion

The first solar neutrino experiment led by Raymond Davis Jr. showed a deficit of neutrinos relative to the solar model prediction, referred to as the solar neutrino problem since the 1970s. The Kamiokande experiment led by Masatoshi Koshiba successfully observed solar neutrinos, as first reported in 1980s. The solar neutrino problem was solved due to neutrino oscillations by comparing the SuperKamiokande and Sudbury Neutrino Observatory results. While recent decades have offered tremendous advances in solar neutrinos across the fields of solar physics (Bulgen et al. 2024; Yang and Tian 2024), nuclear physics (Bertulani et al. 2022; Hwang et al. 2023), and neutrino physics (Sturrock et al. 2021; Slad 2024), many lingering mysteries remain.

This chapter takes advantage of publicly available solar neutrino SuperKamiokande data and analyses them by applying diffusion entropy analysis and standard deviation analysis. The result is a scaling exponent  $\delta < 1$  indicating anomalous diffusion of solar neutrinos in terms of Lévy flights. Based on this result the chapter develops the probability density

function for neutrino flights and derives the respective differential equation in terms of a fractional Fokker-Planck equation. Accordingly, the closed form analytic representation of the neutrino power density function is given as a Fox H-function that can be used for further numerical exercises for the benefit of solar neutrino physics.

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## 7. Neutrino Astrophysics, 2025 Update: Neutrino Masses and Mixings

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### 7.1 Introduction

The Standard Model (SM) of Particle Physics is the pinnacle of the understanding of neutrino physics (Deppisch 2019; Oberauer et al. 2020). It comes with a plethora of parameters, the masses and the flavour mixings, that are seemingly not fixed by any known fundamental principle. In the SM, the neutrino spectrum is simple: all neutrinos are massless. Neutrino oscillations, where neutrinos seemingly change flavour in flight, cannot be accommodated in the SM due to the mass of the neutrinos. Neutrino oscillations thus imply massive neutrino eigenstates, and the SM must be extended. Moreover, neutrino oscillation experimental data suggest that the neutrino spectrum is not hierarchical, with three massive light neutrinos and a mixing matrix exhibiting near-maximal mixing (Deppisch 2019; Oberauer et al. 2020).

To make sense of the neutrino sector, it was argued that the light neutrino mass matrix could be generated randomly from a more fundamental Dirac neutrino mass matrix and a more fundamental Majorana neutrino mass matrix with random elements distributed according to a Gaussian ensemble, a principle dubbed the anarchy principle (Haba and Murayama 2001). These more fundamental neutrino mass matrices would come from the extended SM where the seesaw mechanism occurs (Yanagida 1979). It was argued that the probability density function (PDF) for the mixing angles and phases is the appropriate Haar measure of the symmetry group, implying near-maximal mixings (Fortin et al. 2016; Haba et al. 2023; Fortin et al. 2020). We have shown that the PDF can be obtained either by analysing data by diffusion entropy analysis (done for the case of solar neutrinos from observations emanating from SuperKamiokande) (Scafetta 2010; Mathai and Haubold 2018; Haubold and Mathai 2024) or as proceeding in this chapter. Then, the anarchy principle was analysed mostly numerically, reaching conclusions, for example about the preferred normal hierarchy of the neutrino masses.

Although several numerical results have been obtained, few analytical results on the seesaw ensemble, which is derived from the anarchy principle, exist. That it is the case even though random matrix theory is a well-studied subject in mathematics is surprising. It is therefore clear that a thorough analytical investigation of the seesaw ensemble is possible.

It is an open issue to investigate analytically the seesaw ensemble derived from the anarchy principle with the help of the usual tools of random matrix theory. The seesaw ensemble PDF can be obtained from  $N \times N$  fundamental Dirac and Majorana neutrino mass matrices with real or complex elements. The joint PDF for the singular (eigen) values in the complex (real) case can be derived and it can be shown that the group variables decouple straightforwardly as in the usual Gaussian ensembles.

The following notations will be used in this chapter: Real scalar ( $1 \times 1$  matrix) variables, whether random or mathematical, will be denoted by lower-case letters such as  $x, y, z$ . Real vector ( $1 \times p$  or  $p \times 1, p > 1$ , matrix), matrix ( $p \times q$ ) variables will be denoted by capital letters such as  $X, Y$ . Variables in the complex domain will be indicated with a tilde such as  $\tilde{x}, \tilde{y}, \tilde{X}, \tilde{Y}$ . Constants will be denoted by  $a, b$  etc. for scalars and  $A, B$  etc. for vectors/matrices. No tilde will be used on constants. For a  $p \times p$  matrix  $B$ ,  $|B|$  or  $\det(B)$  will denote the determinant of the matrix  $B$ . When  $B$  is in the complex domain, one can write  $|B| = a + ib, i = \sqrt{(-1)}$ ,  $a, b$  are real scalars. Then, the absolute value of the determinant will be denoted by  $|\det(B)| = |\det(BB^*)|^{\frac{1}{2}} = \sqrt{(a^2 + b^2)}$  where  $B^*$  means the conjugate transpose of  $B$ , that is  $B^* = (B')^c = (B^c)'$  where a prime denotes the transpose and  $c$  in the exponent denotes the conjugate. If  $X = (x_{jk})$  is a  $p \times q$  matrix in the real domain, where the  $x_{jk}$ 's are distinct real scalar variables, then the wedge product of their differentials will be denoted as  $dX = \wedge_{j=1}^p \wedge_{k=1}^q dx_{jk}$ . If  $X = X'$  (real symmetric), then  $dX = \wedge_{j \geq k} dx_{jk}$ . For two real scalar variables  $x_1$  and  $x_2$ , the wedge product is defined as  $dx_1 \wedge dx_2 = -dx_2 \wedge dx_1$  so that  $dx_j \wedge dx_j = 0, j = 1, 2$ . Also,  $\int_X f(X) dX$  will denote the real-valued scalar function  $f(X)$  of  $X$  is integrated over  $X$ . If the  $p \times q$  matrix  $\tilde{X}$  is in the complex domain, then  $\tilde{X} = X_1 + iX_2, i = \sqrt{(-1)}$ ,  $X_1, X_2$  are real  $p \times q$  matrices, then  $d\tilde{X} = dX_1 \wedge dX_2$ . Other notations will be explained whenever they occur for the first time.

This chapter is organized as follows: Sect. 7.2 gives a mathematical introduction to the models. Section 7.3 derives the distribution of the light neutrino mass matrix in explicit computable form. Section 7.4 is providing the densities in terms of the eigenvalues, including the cases of the largest eigenvalue and the smallest eigenvalue.

## 7.2 Modified Dirac and Majorana Neutrino Matrices and Their Distributions

Let  $X$  and  $\tilde{X}$  be  $p \times n, p \leq n$  matrix of rank  $p$  in the real and complex domains respectively. If  $X$  and  $\tilde{X}$  are  $p \times n$  matrix-variate random variables in the real and complex domains respectively, then this matrix  $X$  corresponds to the  $N \times N$  Dirac matrix  $M_D$  considered in Fortin et al. (2016); Haba et al. (2023); Fortin et al. (2020). Let  $Y > O, \tilde{Y} > O$  be  $n \times n$  real positive definite and Hermitian positive definite matrices in the real and complex domains respectively, where  $(\cdot) > O$  denotes the matrix  $(\cdot)$  is real positive definite or Hermitian positive definite. If  $Y$  and  $\tilde{Y}$  are  $n \times n$  matrix-variate random variables in the real and complex domains, then this  $Y$  corresponds to the  $N \times N$  Majorana matrix  $M_R$  in Fortin et al. (2016); Haba et al. (2023); Fortin et al. (2020). Let  $U = -XY^{-1}X$  where  $U$  corresponds to the light neutrino mass matrix  $M_\nu$  in Fortin et al. (2016); Haba et al. (2023); Fortin et al. (2020). Due to our assumption of  $X$  being a full rank matrix, the rows of  $X$  are linearly independent so that a singular distribution for any column of  $X$  is avoided. The columns of  $X$  are  $p \times 1$  which corresponds to a  $p$ -vector in multivariate statistical analysis. Also if the columns of  $X$  are iid (independently and identically distributed) then  $X$  can represent a sample matrix of a sample of size  $n$  from a  $p$ -variate population. When  $p = n$ ,  $X$  will be a  $n \times n$  square matrix as considered in Fortin et al. (2016); Haba et al. (2023); Fortin et al. (2020). Let  $X$  have a  $p \times n$  matrix-variate Gaussian density in the standard form, denoted by  $f_1(X)$ , where

$$f_1(X) dX = c_1 e^{-\text{tr}(XX')} dX, \quad f_1(\tilde{X}) d\tilde{X} = \tilde{c}_1 e^{-\text{tr}(\tilde{X}\tilde{X}^*)} d\tilde{X} \quad (7.1)$$

where  $c_1$  is the normalizing constant,  $c_1 = \pi^{pn/2}$ . If a real scalar scaling constant  $b_1 > 0$  is introduced in the exponent in (7.1) and write the exponent as  $-b_1 \text{tr}(XX')$ , then the normalizing constant changes to  $(b_1 \pi)^{pn/2}$ . In the corresponding complex case,  $\tilde{c}_1 = \pi^{pq}$  and if a real scaling factor  $b_1 > 0$  is present, then  $\tilde{c}_1 = (b_1 \pi)^{pq}$  respectively. If a location parameter matrix is to be included, then replace  $X$  in (2.1) by  $X - M$  and  $\tilde{X}$  by  $\tilde{X} - \tilde{M}$  where the  $M$  and  $\tilde{M}$  matrices can act as the mean value or expected value of  $X$  and expected value of  $\tilde{X}$  respectively, where  $M = E[X]$ ,  $\tilde{M} = E[\tilde{X}]$  where  $E[(\cdot)]$  denotes the expected value of  $(\cdot)$ . There is not going to be any change in the normalizing constants  $c_1$  and  $\tilde{c}_1$ . If scaling matrices are also to be inserted, then replace  $XX'$  by  $A(X - M)B(X - M)'$  and  $\tilde{X}\tilde{X}^*$  by  $A(\tilde{X} - \tilde{M})B(\tilde{X} - \tilde{M})^*$  respectively, where  $A > O$  is  $p \times p$  and  $B > O$  is  $n \times n$  real positive definite constant matrices in the real case, and  $A = A^* > O$  and  $B = B^* > O$  (Hermitian positive definite matrices) in the complex domain. The normalizing constants will change to  $c_1 = \pi^{pq/2}|A|^{\frac{p}{2}}|B|^{\frac{p}{2}}$  and  $\tilde{c}_1$  changes to  $\tilde{c}_1 = \pi^{pq}|\det(A)|^n|\det(B)|^p$  respectively. These changes are taking place due to Lemma 2.1 given below. When  $XX'$  is changed to  $AXBX'$ , one can give interpretations in terms of the covariance matrices of the columns and rows of  $X$ . For example, the inverse of  $A$  can act as the covariance matrix of each  $p \times 1$  column vector of  $X$  and similarly the inverse of  $B$  can act as the covariance matrix of each row of  $X$  when  $X$  is a sample matrix from some  $p$ -variate population. The above are some of the advantages of considering the  $p \times n$  matrix  $X$  and inserting location parameter matrix and scaling matrices in  $XX'$ . Corresponding interpretations can be given in the complex case also.

**Lemma 7.1** Let  $X = (x_{jk})$  be a  $p \times q$  matrix in the real domain with  $pq$  distinct real scalar variables as elements  $x_{jk}$ 's. Let  $A$  be a  $p \times p$  and  $B$  be a  $q \times q$  real nonsingular constant matrices. Consider the linear transformation  $Y = AXB$ . Then,

$$Y = AXB, |A| \neq 0, |B| \neq 0, \Rightarrow dY = |A|^q |B|^p dX$$

In the corresponding complex domain,  $\tilde{Y} = A\tilde{X}B$  where  $A$  and  $B$  are nonsingular and they may be in the real or complex domain, then

$$\tilde{Y} = A\tilde{X}B, |A| \neq 0, |B| \neq 0, \Rightarrow d\tilde{Y} = |\det(A)|^{2q} |\det(B)|^{2p} d\tilde{X}.$$

In the discussion to follow, we will need the Jacobian from a symmetric transformation, for example,  $p = q$ ,  $B = A'$  in the real case in Lemma 7.1. In the symmetric case, the Jacobian will not be available from Lemma 7.1.

**Lemma 7.2** Consider the  $p \times p$  matrices  $X = X'$ ,  $A$  and  $\tilde{X} = \tilde{X}^*$ ,  $A$  in the real and complex domains respectively where  $A$  is a nonsingular constant matrix and in the complex case  $A$  could be real or complex. Note that we assume  $X$  is symmetric in the real domain and  $\tilde{X}$  is Hermitian in the complex domain. Then,

$$Y = AXA', |A| \neq 0, \Rightarrow dY = |A|^{p+1} dX, \tilde{Y} = A\tilde{X}A^* \Rightarrow d\tilde{Y} = |\det(A)|^{2p} d\tilde{X}.$$

If  $X$  is skew symmetric then the exponent  $p + 1$  changes to  $p - 1$  in the Jacobian part in Lemma 7.2. If  $\tilde{X}$  is skew Hermitian, there is no change,  $2p$  in the exponent will remain as  $2p$ .

Let the  $n \times n$  matrix  $Y$ , corresponding to the Majorana neutrino mass matrix  $M_R$  in Fortin et al. (2016); Haba et al. (2023); Fortin et al. (2020) have the following density:

$$f_2(Y)dY = c_2|Y|^{\alpha-\frac{n+1}{2}}e^{-\text{tr}(Y)}dY, \quad f_2(\tilde{Y})d\tilde{Y} = \tilde{c}_2|\det(\tilde{Y})|^{\alpha-n}e^{-\text{tr}\tilde{Y}}d\tilde{Y} \quad (7.2)$$

where  $Y = Y' > O$  (real positive definite),  $\tilde{Y} = \tilde{Y}^* > O$  (Hermitian positive definite). Here  $\alpha$  is a free parameter which may be given some physical interpretations. The  $n \times n$  positive definite matrix  $Y$  can always be written as  $Y_1 Y_1'$  where  $Y_1$  is a  $n \times n_1, n \leq n_1$  matrix of rank  $n$  where  $n_1$  can be equal to  $n$  or  $Y_1$  can be a square matrix also. Further,  $Y_1$  can be a square root of  $Y$  where a square root can be uniquely defined when  $Y$  is positive definite or  $\tilde{Y}$  is Hermitian positive definite. Thus, the exponents in the densities of  $Y$  and  $\tilde{Y}$  can have exactly the same structures as in the corresponding densities in Fortin et al. (2016); Haba et al. (2023); Fortin et al. (2020). But in Fortin et al. (2016); Haba et al. (2023); Fortin et al. (2020) the densities are Gaussian forms and in (7.2) above, the densities are real and complex matrix-variate Gamma densities for  $Y$  and  $\tilde{Y}$ . The normalizing constants  $c_2$  and  $\tilde{c}_2$  are the following:

$$c_2 = \frac{1}{\Gamma_n(\alpha)}, \quad \Re(\alpha) > \frac{n-1}{2}, \quad \tilde{c}_2 = \frac{1}{\tilde{\Gamma}_n(\alpha)}, \quad \Re(\alpha) > n-1$$

where  $\Re(\cdot)$  means the real part of  $(\cdot)$  and  $\Gamma_n(\alpha)$  and  $\tilde{\Gamma}_n(\alpha)$  are the real and complex matrix-variate gamma functions defined as the following:

$$\begin{aligned} \Gamma_n(\alpha) &= \begin{cases} \pi^{\frac{n(n-1)}{4}} \Gamma(\alpha) \Gamma(\alpha - \frac{1}{2}) \cdots \Gamma(\alpha - \frac{n-1}{2}), \Re(\alpha) > \frac{n-1}{2} \\ \int_{Z > O} |Z|^{\alpha - \frac{n+1}{2}} e^{-\text{tr}(Z)} dZ \end{cases} \\ \tilde{\Gamma}_n(\alpha) &= \begin{cases} \pi^{\frac{n(n-1)}{2}} \Gamma(\alpha) \Gamma(\alpha - 1) \cdots \Gamma(\alpha - n + 1), \Re(\alpha) > n - 1 \\ \int_{\tilde{Z} > O} |\det(\tilde{Z})|^{\alpha-n} e^{-\text{tr}(\tilde{Z})} d\tilde{Z}, \Re(\alpha) > n - 1. \end{cases} \end{aligned} \quad (7.3)$$

Let  $Y^{\frac{1}{2}}$  be the positive definite square root of the positive definite matrix  $Y > O$ . Then, if we consider  $X$  of (7.1) to be scaled by  $XY^{-\frac{1}{2}}$  then this scaling has the effect of making the rows of  $X$  correlation-free if  $Y$  is the correlation matrix of each row of  $X$ . Thus, in a physical situation if the rows are likely to be correlated then they can be made correlation free by scaling with the proper scaling matrix, namely the square root of the inverse of the correlation matrix. In the scaled  $XX'$ , namely  $AXBX'$  considered above,  $B^{-1}$  corresponds to  $Y$  in the present discussion. When scaled with the proper scaling matrix  $XX'$  goes to

$(XY^{-\frac{1}{2}})(XY^{-\frac{1}{2}})' = XY^{-1}X'$  and one has similar changes in the complex case also. Hence, our light neutrino mass matrix  $U = XY^{-1}X$  has proper interpretations in terms of scaling models, making rows correlation free etc.

Our interest is to derive the density of  $U$ . For this purpose, we need either the assumption that  $X$  and  $Y$  are independently distributed, in that case the joint density of  $X$  and  $Y$  is  $f_1(X)f_2(Y)$ , the product, or we have to assume that  $f_1(X)$  is a conditional density, in the sense, for every given  $Y$ , one has the density of  $X$  a matrix-variate Gaussian as in (7.1) and  $f_2(Y)$  is then the marginal density of  $Y$  and again the joint density will be the product  $f_1(X)f_2(Y)$ . We will assume  $f_1(X)$  being a conditional density of  $X$  for every given  $Y$  and derive the density of  $U = XY^{-1}X'$  and  $\tilde{U} = \tilde{X}\tilde{Y}^{-1}\tilde{X}^*$ .

### 7.3 Derivation of the Density of $U$

In our notation, the light neutrino mass matrix is  $U = -XY^{-1}X'$  in the real case and  $\tilde{U} = -\tilde{X}\tilde{Y}^{-1}\tilde{X}^*$  in the complex case. Let us consider the real case first. Ignoring the sign,  $U$  can be written as  $U = (XY^{-\frac{1}{2}})(XY^{-\frac{1}{2}})'$  since  $Y$  is symmetric real positive definite. Let  $Z = XY^{-\frac{1}{2}} \Rightarrow X = ZY^{\frac{1}{2}}$  and  $dX = |Y|^{\frac{p}{2}}dZ$ , for fixed  $Y$ , by using Lemma 7.1, and  $U = ZZ'$ . The joint density of  $X$  and  $Y$ , is the conditional density of  $X$ , given  $Y$ , times the marginal density of  $Y$ . That is, denoting the joint density by  $f(X, Y)$ , we have the following:

$$\begin{aligned} f(X, Y)dX \wedge dY &= c_1 c_2 |Y|^{\alpha - \frac{n+1}{2}} e^{-\text{tr}(ZYZ')} e^{-\text{tr}(Y)} |Y|^{\frac{p}{2}} dZ \wedge dY \\ &= c_1 c_2 |Y|^{\alpha + \frac{p}{2} - \frac{n+1}{2}} e^{-\text{tr}[Y(I_n + Z'Z)]} dZ \wedge dY. \end{aligned}$$

But

$$\text{tr}(Y + ZYZ') = \text{tr}(Y(I_n + Z'Z)) = \text{tr}[(I_n + Z'Z)^{\frac{1}{2}} Y (I_n + Z'Z)^{\frac{1}{2}}]$$

Even though  $Z'Z$  is singular and positive semi-definite, due to the presence of the identity matrix  $I = I_n$ , we may take  $I_n + Z'Z$  to be positive definite and hence one may consider the positive definite square root of  $I_n + Z'Z$ . Now, we can integrate out  $Y$  by using a real matrix-variate gamma of (7.3). That is,

$$\int_{Y>0} |Y|^{\frac{p}{2} + \alpha - \frac{n+1}{2}} e^{-\text{tr}[(I_n + Z'Z)^{\frac{1}{2}} Y (I_n + Z'Z)^{\frac{1}{2}}]} dY = \Gamma_n(\alpha + \frac{p}{2}) |I_n + Z'Z|^{-(\alpha + \frac{p}{2})}, \Re(\alpha) > \frac{n-1}{2}.$$

But we can write  $|I_n + Z'Z|$  in terms of the  $p \times p$  real positive definite matrix  $ZZ'$ . Consider the expansion of the following determinant in two different ways in terms of its submatrices, denoting the determinant by  $\eta$ :

$$\begin{aligned} \eta &= \begin{vmatrix} I_p & -Z \\ Z' & I_n \end{vmatrix} = |I_p| |I_n - Z'I_p^{-1}(-Z)| = |I_p| |I_n + Z'Z| = |I_n + Z'Z| \\ \eta &= |I_n| |I_p - (-Z)I_n^{-1}Z'| = |I_n| |I_p + ZZ'| = |I_p + ZZ'| \Rightarrow |I_n + Z'Z| = |I_p + ZZ'|. \end{aligned} \tag{7.4}$$

Hence, the density of  $Z$ , denoted by  $g(Z)$ , is the following:

$$g(Z)dZ = c_1 c_2 \Gamma_n(\alpha + \frac{p}{2}) |I_p + ZZ'|^{-(\alpha + \frac{p}{2})} dZ. \tag{7.5}$$

Going through steps parallel to the real case, one has the corresponding result in the complex case, denoted by  $\tilde{g}(\tilde{Z})$  as the following:

$$\tilde{g}(\tilde{Z})d\tilde{Z} = \tilde{c}_1 \tilde{c}_2 \tilde{\Gamma}_n(\alpha + p) |\det(I + \tilde{Z}\tilde{Z}^*)|^{-(\alpha + p)} d\tilde{Z}. \tag{7.6}$$

Our matrix is  $U = ZZ'$ . We can go from the density of  $Z$  to the density of  $U = ZZ'$  by using the following result from Mathai (1997) which will be stated as a lemma. This result is available in Chap. 5 as Lemma 5.1. For the sake of ready reference, this result is given as the next lemma.

**Lemma 7.2** Let  $X = (x_{jk})$  be a  $p \times q$ ,  $p \leq q$  matrix of rank  $p$  in the real domain where the  $x_{jk}$ 's are distinct real scalar variables. Let  $S = XX'$ . Then, going through a transformation involving a lower triangular matrix with positive diagonal elements and a unique semi-

orthonormal matrix and then integrating out the differential element corresponding to the semi-orthonormal matrix, we have the following relationship between  $dX$  and  $dS$ :

$$dX = \frac{\pi^{\frac{pq}{2}}}{\Gamma_p(\frac{q}{2})} |S|^{\frac{q}{2} - \frac{p+1}{2}} dS.$$

In the corresponding complex case, let  $\tilde{X}$  be  $p \times q, p \leq q$  matrix in the complex domain with distinct scalar complex variables as elements. Let  $\tilde{S} = \tilde{X}\tilde{X}^*$ . Then, going through a transformation involving a lower triangular matrix with real and positive diagonal elements and a unique semi-unitary matrix and then integrating out the differential element corresponding to the semi-unitary matrix, we have the following connection:

$$d\tilde{X} = \frac{\pi^{pq}}{\tilde{\Gamma}_p(q)} |\det(\tilde{S})|^{q-p} d\tilde{S}.$$

With the help of Lemma 7.2, we can go to the density of  $Z$  in (7.5) to the density of  $U = ZZ'$ , denoted by  $g_1(U)$ . Since the variable is changed from a  $p \times n$  matrix to a  $p \times p$  matrix, the normalizing constant will change. Hence we may write

$$g_1(U) dU = c |U|^{\frac{n}{2} - \frac{p+1}{2}} |I + U|^{-(\alpha + \frac{p}{2})} dU \quad (7.7)$$

for  $\Re(\alpha) > \frac{n-1}{2}, n > p - 1$ , where  $c$  is the corresponding normalizing constant. This  $g_1(U)$  is a real matrix-variate type 2 beta density with the parameters  $(\frac{n}{2}, \alpha + \frac{p-n}{2})$ . Hence, the normalizing constants, denoted by  $c$  in the real case and  $\tilde{c}$  in the complex case, are the following:

$$c = \frac{\Gamma_p(\frac{n}{2})\Gamma_p(\alpha + \frac{p}{2} - \frac{n}{2})}{\Gamma_p(\alpha + \frac{p}{2})}, \Re(\alpha) > \frac{n-1}{2}, \tilde{c} = \frac{\tilde{\Gamma}_p(n)\tilde{\Gamma}_p(\alpha + p - n)}{\tilde{\Gamma}_p(\alpha + p)}, \Re(\alpha) > n - 1, \quad (7.8)$$

evaluated from real and complex  $p \times p$  matrix-variate type 2 beta densities respectively. Using steps parallel to the real case, we have the corresponding density  $\tilde{g}_1(\tilde{U})$  in the complex case as the following, for  $\Re(\alpha) > n - 1$ :

$$\tilde{g}_1(\tilde{U}) d\tilde{U} = \tilde{c} |\det(\tilde{U})|^{\alpha-p} |\det(I + \tilde{U})|^{-(\alpha+n)} d\tilde{U} \quad (7.9)$$

where  $\tilde{c}$  is given in (7.8).

## 7.4 Densities in Terms of the Eigenvalues

From (7.7) the  $p \times p$  real positive definite matrix  $U$  has a real matrix-variate type 2 beta density with the parameters  $(\frac{n}{2}, \alpha + \frac{p}{2} - \frac{n}{2})$  with  $\Re(\alpha) > \frac{n}{2} - \frac{p}{2} + \frac{p-1}{2} = \frac{n-1}{2}$ , that is, with the density

$$g_1(U) dU = \frac{\Gamma_p(\alpha + \frac{p}{2})}{\Gamma_p(\frac{n}{2})\Gamma_p(\alpha + \frac{p}{2} - \frac{n}{2})} |U|^{\frac{n}{2} - \frac{p+1}{2}} |I + U|^{-(\alpha + \frac{p}{2})} dU \quad (7.10)$$

for  $\Re(\alpha) > \frac{n-1}{2}$ . The corresponding density in the complex domain is the following where  $\tilde{U}$  is Hermitian positive definite and  $|\det(\tilde{U})|$  means the absolute value of the determinant of  $\tilde{U}$ :

$$\tilde{g}_1(\tilde{U})d\tilde{U} = \frac{\tilde{\Gamma}_p(\alpha+p)}{\tilde{\Gamma}_p(n)\tilde{\Gamma}_p(\alpha+p-n)} |\det(\tilde{U})|^{n-p} |\det(I + \tilde{U})|^{-(\alpha+p)} d\tilde{U} \quad (7.11)$$

for  $\Re(\alpha) > n - 1$ . We can convert  $U$  and  $\tilde{U}$  and write the densities in terms of their eigenvalues. If  $\mu_j$  is an eigenvalue of  $U$ , then  $0 < \mu_j < \infty, j = 1, \dots, p$ . Similar is the case for the eigenvalues of  $\tilde{U}$ . For convenience, let us convert  $U$  and  $\tilde{U}$  to the corresponding type 1 beta form. Consider the transformation

$$V = (I + U)^{-\frac{1}{2}} U (I + U)^{-\frac{1}{2}}, \tilde{V} = (I + \tilde{U})^{-\frac{1}{2}} \tilde{U} (I + \tilde{U})^{-\frac{1}{2}}.$$

Then,  $V$  and  $\tilde{V}$  will be  $p \times p$  matrix-variate type 1 beta with the same parameters, see Mathai (1997); Mathai et al. (2022). Let the densities of  $V$  and  $\tilde{V}$  be denoted by  $g_2(V)$  and  $\tilde{g}_2(\tilde{V})$  respectively. Then,

$$g_2(V)dV = \frac{\Gamma_p(\alpha+\frac{p}{2})}{\Gamma_p(\frac{n}{2})\Gamma_p(\alpha+\frac{p}{2}-\frac{n}{2})} |V|^{\frac{n}{2}-\frac{p+1}{2}} |I - V|^{\alpha+\frac{p}{2}-\frac{n}{2}-\frac{p+1}{2}} dV \quad (7.12)$$

and

$$\tilde{g}_2(\tilde{V})d\tilde{V} = \frac{\tilde{\Gamma}_p(\alpha+p)}{\tilde{\Gamma}_p(n)\tilde{\Gamma}_p(\alpha+p-n)} |\det(\tilde{V})|^{n-p} |\det(I - \tilde{V})|^{\alpha-n} d\tilde{V} \quad (7.13)$$

for  $\Re(\alpha) > \frac{n-1}{2}, n - 1$  respectively in the real and the corresponding complex case. If  $\lambda_j$  is an eigenvalue of  $V$ , then  $\lambda_j = \frac{\mu_j}{(1+\mu_j)} \Rightarrow \mu_j = \frac{\lambda_j}{(1-\lambda_j)}, 0 < \lambda_j < 1, 0 < \mu_j < \infty, j = 1, \dots, p$ . Let  $Q$  be a  $p \times p$  unique orthonormal matrix,  $QQ' = I, Q'Q = I$  such that  $Q'VQ = \text{diag}(\lambda_1, \dots, \lambda_p)$  with  $1 > \lambda_1 > \lambda_2 > \dots > \lambda_p > 0$ . Correspondingly, let  $\tilde{Q}$  be a unique unitary matrix,  $\tilde{Q}\tilde{Q}^* = I, \tilde{Q}^*\tilde{Q} = I$  such that  $\tilde{Q}^*\tilde{V}\tilde{Q} = \text{diag}(\lambda_1, \dots, \lambda_p)$ , where  $\tilde{Q}^*$  means the conjugate transpose of  $\tilde{Q}$ . When  $\lambda_j$ 's are real scalar variables we can assume  $Pr\{\lambda_i = \lambda_j, i \neq j\} = 0$  almost surely. Hence, without loss of generality we assume that the  $\lambda_j$ 's are distinct,  $1 > \lambda_1 > \dots > \lambda_p > 0$ . Observe that the eigenvalues of Hermitian matrices are also real and hence the eigenvalues of both  $V$  and  $\tilde{V}$  will be real and we will denote them by the same symbols  $\lambda_j$ 's. Also,

$Q'VQ = D = \text{diag}(\lambda_1, \dots, \lambda_p) \Rightarrow V = QDQ', |V| = \lambda_1 \cdots \lambda_p, |I - V| = \prod_{j=1}^p (1 - \lambda_j)$  and when  $V$  is transformed to its eigenvalues a factor  $\prod_{i < j} (\lambda_i - \lambda_j)$  comes in, both in the real and complex cases, see Mathai (1997); Mathai et al. (2022). If the differential elements corresponding to  $Q$  and  $\tilde{Q}$  are denoted by  $dG$  and  $d\tilde{G}$  respectively, then from Mathai (1997); Mathai et al. (2022),  $G = Q'(\text{d}Q), \tilde{G} = \tilde{Q}(\text{d}\tilde{Q})$  where, for example,  $(\text{d}Q)$  is the matrix of differentials in  $Q$  and the integrals over  $dG$  and  $d\tilde{G}$  are the following results which will be written as a lemma, see Mathai (1997); Mathai et al. (2022):

**Lemma 7.3** For the  $G, dG, \tilde{G}, d\tilde{G}$  as defined above, we have

$$\int dG = \frac{\pi^{\frac{p^2}{2}}}{\Gamma_p(\frac{p+1}{2})}, \int d\tilde{G} = \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)}.$$

Let us verify this lemma for  $p = 2, 3$ . For a  $p \times p$  real positive definite matrix  $X$  we have

$$\int_{X>0} |X|^{\alpha-\frac{p+1}{2}} e^{-\text{tr}(X)} dX = \Gamma_p(\alpha), \Re(\alpha) > \frac{p-1}{2}$$

from the real matrix-variate gamma integral. In the complex case, let the  $p \times p$  matrix  $\tilde{X}$  be Hermitian positive definite. Then, from the complex matrix-variate gamma integral we have

$$\int_{\tilde{X}>0} |\det(\tilde{X})|^{\alpha-p} e^{-\text{tr}(\tilde{X})} d\tilde{X} = \tilde{\Gamma}_p(\alpha), \Re(\alpha) > p-1.$$

Consider the integrals in the real and complex cases when  $\alpha = \frac{p+1}{2}$  in the real case and  $\alpha = p$  in the complex case. Then,

$$\int_{X>0} e^{-\text{tr}(X)} dX = \Gamma_p(\frac{p+1}{2}), \int_{\tilde{X}>0} e^{-\text{tr}(\tilde{X})} d\tilde{X} = \tilde{\Gamma}_p(p).$$

If we go through a unique orthonormal transformation involving an orthonormal matrix  $Q$  then in the real case

$$\Gamma_p(\frac{p+1}{2}) = \int_D \{ \prod_{i < j} (\lambda_i - \lambda_j) \} e^{-\text{tr}(D)} dD \int_Q dG, \int_Q dG = \frac{\pi^{\frac{p^2}{2}}}{\Gamma_p(\frac{p}{2})}$$

and in the corresponding complex case

$$\tilde{\Gamma}_p(p) = \int_D \{ \prod_{i < j} (\lambda_i - \lambda_j)^2 \} e^{-\text{tr}(D)} dD \int_{\tilde{Q}} d\tilde{G} = \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)}.$$

Then, in the real case, for  $p = 2$ ,  $\Gamma_p(\frac{p+1}{2}) = \Gamma_2(\frac{3}{2}) = \pi/2$ .  $\frac{\pi^{\frac{p^2}{2}}}{\Gamma_p(\frac{p}{2})} = \frac{\pi^2}{\Gamma_2(1)} = \pi$ . Now,  $\Gamma_p(\frac{p+1}{2})$  divided by the right side quantity  $\pi$  gives  $\frac{1}{2}$ . Now, consider the integral over  $D$  for  $p = 2$  in the real case. Let  $u_1 = \lambda_1 - \lambda_2$ .

$$\begin{aligned} \int_D (\lambda_1 - \lambda_2) e^{-(\lambda_1 + \lambda_2)} dD &= \int_{u_1=0}^{\infty} u_1 e^{-u_1} du_1 \int_{\lambda_2=0}^{\infty} e^{-2\lambda_2} d\lambda_2 \\ &= \frac{1}{2}. \end{aligned}$$

Hence, for  $p = 2$  in the real case, Lemma 4.1 is verified. Now, for  $p = 3$  in the real case, the left side quantity  $\Gamma_p(\frac{p+1}{2}) = \Gamma_3(2) = \frac{\pi^2}{2} \cdot \frac{\pi^{\frac{p^2}{2}}}{\Gamma_p(\frac{p}{2})} = \frac{\pi^{9/2}}{\Gamma_3(\frac{3}{2})} = 2\pi^2$ . Then,

$\Gamma_p(\frac{p+1}{2}) / [\pi^{\frac{p^2}{2}} / \Gamma_p(\frac{p}{2})] = \frac{\pi^2}{2} / (2\pi^2) = \frac{1}{4}$ . Now, consider the integral over  $D$ . Let  $u_1 = \lambda_1 - \lambda_2, u_2 = \lambda_2 - \lambda_3, u_3 = \lambda_3$ . Then,

$$\begin{aligned}
& \int_D (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3) e^{-(\lambda_1 + \lambda_2 + \lambda_3)} dD \\
&= \int_0^\infty \int_0^\infty \int_0^\infty u_1(u_1 + u_2)u_2 e^{-(u_1 + 2u_2 + 3u_3)} du_1 \wedge du_2 \wedge du_3 \\
&= \frac{1}{3} \left[ \frac{2}{4} + \frac{2}{8} \right] = \frac{1}{4}.
\end{aligned}$$

Hence, the result for  $p = 3$  in the real case is verified. Now, consider the complex case. The left side quantity is  $\tilde{\Gamma}_p(p) = \tilde{\Gamma}_2(2) = \pi^{\frac{p(p-1)}{2}} \Gamma(2)\Gamma(1) = \pi$  for  $p = 2$  and  $2\pi^3$  for  $p = 3$ . The right side quantity  $[\pi^{p(p-1)} / \tilde{\Gamma}_p(p)] = \pi$  for  $p = 2$  and  $\frac{\pi^3}{2}$  for  $p = 3$ . Now, the left side quantity divided by the right side quantity gives the following:  $\pi/\pi = 1$  for  $p = 2$  and  $(2\pi^3)/(\frac{\pi^3}{2}) = 4$  for  $p = 3$ . Now, consider the integral over  $D$  in the complex case. As before, let  $u_1 = \lambda_1 - \lambda_2, u_2 = \lambda_2 - \lambda_3$ . Then, for  $p = 2$ ,

$$\begin{aligned}
& \int_D (\lambda_1 - \lambda_2)^2 e^{-(u_1 + 2\lambda_2)} du_1 \wedge d\lambda_2 \\
&= \int_0^\infty \int_0^\infty u_1^2 e^{-(u_1 + 2\lambda_2)} du_1 \wedge d\lambda_2 = 1.
\end{aligned}$$

Thus, the result for  $p = 2$  is verified. Now, consider  $p = 3$ .

$$\begin{aligned}
& \int_0^\infty \int_0^\infty \int_0^\infty u_1^2(u_1 + u_2)^2 u_2^2 e^{-(u_1 + 2u_2 + 3\lambda_3)} du_1 \wedge du_2 \wedge d\lambda_3 \\
&= \int_0^\infty e^{-3\lambda_3} d\lambda_3 \left[ \int_0^\infty \int_0^\infty (u_1^4 u_2^2 + 2u_1^3 u_2^3 + u_1^2 u_2^4) e^{-u_1 - 2u_2} du_1 \wedge du_2 \right] \\
&= 4.
\end{aligned}$$

Hence, the result for  $p = 3$  in the complex case is verified.

The joint density of  $\lambda_1, \dots, \lambda_p$  is the following, denoted by  $g_3(D)$  in the real case and  $\tilde{g}_3(D)$  in the complex case:

$$\begin{aligned}
g_3(D) dD &= \frac{\Gamma_p(\alpha + \frac{p}{2})}{\Gamma_p(\frac{n}{2}) \Gamma_p(\alpha + \frac{p}{2} - \frac{n}{2})} \frac{\pi^{\frac{p^2}{2}}}{\Gamma_p(\frac{p}{2})} \left\{ \prod_{i < j} (\lambda_i - \lambda_j) \right\} \\
&\times \left\{ \prod_{j=1}^p \lambda_j^{\frac{n}{2} - \frac{p+1}{2}} \right\} \left\{ \prod_{j=1}^p (1 - \lambda_j)^{\alpha - \frac{n-1}{2}} \right\} dD
\end{aligned} \tag{7.14}$$

$$\begin{aligned}
\tilde{g}_3(D) dD &= \frac{\tilde{\Gamma}_p(\alpha + p)}{\tilde{\Gamma}_p(n) \tilde{\Gamma}_p(\alpha + p - n)} \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)} \left\{ \prod_{i < j} (\lambda_i - \lambda_j)^2 \right\} \\
&\times \left\{ \prod_{j=1}^p \lambda_j^{n-p} \right\} \left\{ \prod_{j=1}^p (1 - \lambda_j)^{\alpha - n} \right\} dD.
\end{aligned} \tag{7.15}$$

We can write  $\prod_{i < j} (\lambda_i - \lambda_j)$  in the real case and  $\prod_{i < j} (\lambda_i - \lambda_j)^2$  in the complex case, in the  $p \times p$  matrix case, in terms of Vandermonde's determinant.

$$\prod_{i < j} (\lambda_i - \lambda_j) = \begin{vmatrix} \lambda_1^{p-1} & \lambda_1^{p-2} & \cdots & \lambda_1 & 1 \\ \lambda_2^{p-1} & \lambda_2^{p-2} & \cdots & \lambda_2 & 1 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_p^{p-1} & \lambda_p^{p-2} & \cdots & \lambda_p & 1 \end{vmatrix} = A = (a_{ij}), a_{ij} = \lambda_i^{p-j}$$

for all  $i$  and  $j$ . Let us use the general expansion for a determinant. Then, for  $K = (k_1, \dots, k_p)$  where  $k_1, \dots, k_p$  is a given permutation of  $1, 2, \dots, p$ , we have the following:

$$|A| = \sum_K (-1)^{\rho_K} a_{1k_1} \cdots a_{pk_p} = \sum_K (-1)^{\rho_K} \lambda_1^{p-k_1} \lambda_2^{p-k_2} \cdots \lambda_p^{p-k_p}.$$

Here,  $\rho_K$  is the number of transpositions needed to bring  $(k_1, \dots, k_p)$  to the natural order  $(1, 2, \dots, p)$ . Then, if  $\rho_K$  is odd then we have  $-1$  and if  $\rho_K$  is even then we have  $+1$  as the coefficient. For example, for  $p = 3$  the possible permutations are

$(1, 2, 3), (1, 3, 2), (2, 3, 1), (2, 1, 3), (3, 1, 2), (3, 2, 1)$ . There are  $3! = 6$  terms. In the general case there are  $p!$  terms. For example, for the sequence  $(1, 2, 3)$  we have

$k_1 = 1, k_2 = 2, k_3 = 3 \Rightarrow \rho_K = 0$  and the corresponding sign is  $+1$ . For  $(1, 3, 2)$  we have  $k_1 = 1, k_2 = 3, k_3 = 2$ . Here one transposition is needed to bring to the natural order  $(1, 2, 3)$  and hence  $\rho_K = 1$  and the corresponding sign is  $-1$ , and so on. In the complex case,

$$\prod_{i < j} (\lambda_i - \lambda_j)^2 = |A|^2 = |AA'| = |A'A|.$$

Let  $A' = [\beta_1, \beta_2, \dots, \beta_p]$ , where  $\beta_j$  is the  $j$ -th column of  $A'$ ,

$\beta'_j = [\lambda_j^{p-1}, \lambda_j^{p-2}, \dots, \lambda_j, 1], j = 1, \dots, p$ . Let  $A'A = B = (b_{ij}), b_{ij} = \sum_{r=1}^p \lambda_r^{2p-(i+j)}$ . Then,

$$\prod_{i < j} (\lambda_i - \lambda_j)^2 = |B| = |A'A| = \sum_K (-1)^{\rho_K} b_{1k_1} b_{2k_2} \cdots b_{pk_p}$$

where

$$\begin{aligned} b_{1k_1} &= \lambda_1^{2p-(1+k_1)} + \lambda_2^{2p-(1+k_1)} + \cdots + \lambda_p^{2p-(1+k_1)} \\ b_{2k_2} &= \lambda_1^{2p-(2+k_2)} + \lambda_2^{2p-(2+k_2)} + \cdots + \lambda_p^{2p-(2+k_2)} \\ \vdots &= \vdots \\ b_{pk_p} &= \lambda_1^{2p-(p+k_p)} + \lambda_2^{2p-(p+k_p)} + \cdots + \lambda_p^{2p-(p+k_p)} \end{aligned}$$

Let

$$b_{1k_1} b_{2k_2} \cdots b_{pk_p} = \sum_{r_1, \dots, r_p} \lambda_1^{r_1} \cdots \lambda_p^{r_p}. \quad (7.16)$$

In the real case, the joint density of  $\lambda_1, \dots, \lambda_p$  is the following:

(7.17)

$$\begin{aligned}
g_3(D) \mathrm{d}D &= \frac{\Gamma_p(\alpha + \frac{p}{2})}{\Gamma_p(\frac{n}{2})\Gamma_p(\alpha + \frac{p}{2} - \frac{n}{2})} \frac{\pi^{\frac{p^2}{2}}}{\Gamma_p(\frac{p}{2})} \left\{ \prod_{j=1}^p \lambda_j^{\frac{n}{2} - \frac{p+1}{2}} \right\} \left\{ \prod_{j=1}^p (1 - \lambda_j)^{\alpha - \frac{n-1}{2}} \right\} \\
&\times \left( \sum_K (-1)^{\rho_K} \lambda_1^{p-k_1} \lambda_2^{p-k_2} \cdots \lambda_p^{p-k_p} \right) \mathrm{d}D \\
&= \frac{\Gamma_p(\alpha + \frac{p}{2})}{\Gamma_p(\frac{n}{2})\Gamma_p(\alpha + \frac{p}{2} - \frac{n}{2})} \frac{\pi^{\frac{p^2}{2}}}{\Gamma_p(\frac{p}{2})} \sum_K (-1)^{\rho_K} \lambda_1^{m_1} \cdots \lambda_p^{m_p} \left\{ \prod_{j=1}^p (1 - \lambda_j)^{\alpha - \frac{n-1}{2}} \right\} \mathrm{d}D
\end{aligned}$$

where,  $\Re(\alpha) > \frac{n-1}{2}$  and  $m_j = \frac{n}{2} - \frac{p+1}{2} + p - k_j$ .

In the complex case,

$$\tilde{g}_3(D) \mathrm{d}D = \frac{\tilde{\Gamma}_p(\alpha + p)}{\tilde{\Gamma}_p(n)\tilde{\Gamma}_p(\alpha + p - n)} \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)} \sum_K (-1)^{\rho_K} \lambda_1^{m_1} \cdots \lambda_p^{m_p} \left\{ \prod_{j=1}^p (1 - \lambda_j)^{\alpha - n} \right\} \mathrm{d}D \quad (7.18)$$

where  $\Re(\alpha) > n - 1$  and  $m_j = n - p + r_j$  where  $r_j$  is defined in (7.16). Hence, in (7.17) and (7.18) we will use the same notation  $m_j$  as the exponent of  $\lambda_j$ ,  $j = 1, \dots, p$  with the understanding that in the real case  $m_j = \frac{n}{2} - \frac{p+1}{2} + p - k_j$  and in the complex case  $m_j = n - p + r_j$ . Further, for simplicity, we may write the joint density of the eigenvalues in the real and complex cases as the following:

$$g_3(D) \mathrm{d}D = c \sum_K (-1)^{\rho_K} \lambda_1^{m_1} \cdots \lambda_p^{m_p} (1 - \lambda_1)^\gamma \cdots (1 - \lambda_p)^\gamma \mathrm{d}D \quad (7.19)$$

$$\tilde{g}_3(D) \mathrm{d}D = \tilde{c} \sum_K (-1)^{\rho_K} \lambda_1^{m_1} \cdots \lambda_p^{m_p} (1 - \lambda_1)^\gamma \cdots (1 - \lambda_p)^\gamma \mathrm{d}D \quad (7.20)$$

where

$$m_j = \frac{n}{2} - \frac{p+1}{2} + p - k_j, \gamma = \alpha - \frac{n-1}{2}, c = \frac{\Gamma_p(\alpha + \frac{p}{2})}{\Gamma_p(\frac{n}{2})\Gamma_p(\alpha + \frac{p}{2} - \frac{n}{2})} \frac{\pi^{\frac{p^2}{2}}}{\Gamma_p(\frac{p}{2})}, j = 1, \dots, p$$

in the real case, and

$$m_j = n - p + r_j, \gamma = \alpha - n, \tilde{c} = \frac{\tilde{\Gamma}_p(\alpha + p)}{\tilde{\Gamma}_p(n)\tilde{\Gamma}_p(\alpha + p - n)} \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)}, j = 1, \dots, p$$

in the complex case, where  $r_j$  is defined in (7.16). Since we have written the joint density of the eigenvalues, both in the real and complex cases, by using the same format, we can use the same procedure to obtain the densities of the largest eigenvalue, smallest eigenvalue etc. Integration over  $\lambda_1, \dots, \lambda_{p-1}$  is needed to obtain the density of the smallest eigenvalue  $\lambda_p$ . Similarly, integration over  $\lambda_p, \dots, \lambda_2$  is needed to obtain the density of the largest eigenvalue  $\lambda_1$ . In the complex case,  $m_j, j = 1, \dots, p$  are always positive integers. Hence, integration by parts can get rid off the factor  $\lambda_j^{m_j}$ . But, if the  $m_j$  is large or moderately large then the final expression, even though a finite sum, will be messy. Similarly, when  $m_j$  or  $\gamma$  in the real or complex case is a

positive integer, then integration by parts will eliminate the corresponding factor either  $\lambda_j^{m_j}$  or  $(1 - \lambda_j)^\gamma$ . But the expressions may become messy when the parameter  $m_j$  or  $\gamma$  is large or moderately large. Hence, we will consider series expansions which will be valid for the real and complex cases. When the parameters are positive integers, then these series will terminate into a finite sum. Since  $0 < \lambda_j < 1$  the series will converge fast even if it is an infinite series.

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## 7.5 Exact Marginal Function of the Largest Eigenvalue $\lambda_1$ in (7.19)

For both the real and complex cases, whether  $\gamma$  is a positive integer or not, let us expand  $(1 - \lambda_j)^\gamma$  to obtain a convenient representation. Note that

$$(1 - \lambda_p)^\gamma = \sum_{t_p=0}^{\infty} \frac{(-\gamma)_{t_p}}{t_p!} \lambda_p^{t_p}$$

where, for example, the notation  $(a)_m = a(a + 1) \cdots (a + m - 1)$ ,  $(a)_0 = 1$ ,  $a \neq 0$  is the Pochhammer symbol. This will be a finite sum when  $\gamma$  is a positive integer. Now, we start integrating from  $\lambda_p$  onward.

$$\int_{\lambda_p=0}^{\lambda_{p-1}} \lambda_p^{m_p} (1 - \lambda_p)^\gamma d\lambda_p = \sum_{t_p=0}^{\infty} \frac{(-\gamma)_{t_p}}{t_p!} \frac{1}{m_p + t_p + 1} \lambda_{p-1}^{m_p + t_p + 1}.$$

Now, multiply this with  $\lambda_{p-1}^{m_{p-1}} (1 - \lambda_{p-1})^\gamma$  and integrate  $\lambda_{p-1}$  from 0 to  $\lambda_{p-2}$ , and so on. The final form, denoted by  $f_1(\lambda_1)$  is the marginal function corresponding to the largest eigenvalue  $\lambda_1$ . That is,

$$\begin{aligned} f_1(\lambda_1) &= \sum_{t_p=0}^{\infty} \frac{(-\gamma)_{t_p}}{t_p!} \frac{1}{m_p + t_p + 1} \sum_{t_{p-1}=0}^{\infty} \frac{(-\gamma)_{t_{p-1}}}{t_{p-1}!} \frac{1}{m_p + m_{p-1} + t_p + t_{p-1} + 2} \cdots \\ &\quad \times \sum_{t_2=0}^{\infty} \frac{(-\gamma)_{t_2}}{t_2!} \frac{1}{m_p + \cdots + m_2 + t_p + \cdots + t_2 + (p-1)} \lambda_1^{m_p + \cdots + m_1 + t_p + \cdots + t_{p-1} + (p-1)} (1 - t_1)^\gamma, \end{aligned}$$

for  $0 \leq \lambda_1 \leq 1$  and zero elsewhere. Now, incorporating the remaining factors from (7.17) and (7.18), we have the marginal density of  $\lambda_1$ .

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## 7.6 Exact Marginal Function of $\lambda_p$ , the Smallest Eigenvalue

In the complex case,  $m_j$  is a positive integer for all  $j$ , and in the real case,  $m_j$  is either a positive integer or a half-integer. Since we are integrating out, starting from  $\lambda_1$ ,  $\lambda_2 < \lambda_1 < 1$ , we may write, for convenience,

$$\lambda_1^{m_1} = [1 - (1 - \lambda_1)]^{m_1} = \sum_{t_1=0}^{\infty} \frac{(-m_1)_{t_1}}{t_1!} (1 - \lambda_1)^{t_1}.$$

Then,

$$\begin{aligned}
\int_{\lambda_1=\lambda_2}^1 \lambda_1^{m_1} (1-\lambda_1)^\gamma d\lambda_1 &= \sum_{t_1=0}^{\infty} \frac{(-m_1)_{t_1}}{t_1!} \int_{\lambda_1=\lambda_2}^1 (1-\lambda_1)^{\gamma+t_1} d\lambda_1 \\
&= \sum_{t_1=0}^{\infty} \frac{(-m_1)_{t_1}}{t_1!} \frac{1}{\gamma+t_1+1} \lambda_2^{\gamma+t_1+1}.
\end{aligned}$$

Now, multiply by  $\lambda_2^{m_2} (1-\lambda_2)^\gamma$  and integrate out  $\lambda_2$ , and so on. Final result, denoted by  $f_p(\lambda_p)$ , is the following:

$$\begin{aligned}
f_p(\lambda_p) &= \sum_{t_1=0}^{\infty} \frac{(-m_1)_{t_1}}{t_1!} \frac{1}{\gamma+t_1+1} \sum_{t_2=0}^{\infty} \frac{(-m_2)_{t_2}}{t_2!} \frac{1}{2\gamma+t_1+t_2+2} \cdots \\
&\quad \times \sum_{t_{p-1}=0}^{\infty} \frac{(-m_{p-1})_{t_{p-1}}}{t_{p-1}!} \frac{1}{(p-1)\gamma+t_1+\cdots+t_{p-1}+(p-1)} \\
&\quad \times \lambda_p^{m_p} (1-\lambda_p)^{p\gamma+t_1+\cdots+t_{p-1}+(p-1)}, \quad 0 \leq \lambda_p \leq 1
\end{aligned}$$

and this multiplied by the remaining factors from (7.17) and (7.18) will be the marginal density of the smallest eigenvalue  $\lambda_p$ . Observe that if one wishes to compute the marginal density of  $\lambda_s$  for any  $s$ , then integrate out  $\lambda_1, \dots, \lambda_{s-1}, \lambda_p, \dots, \lambda_{s+1}$ . Then, multiply by the remaining factors from (7.17) and (7.18). By integrating out  $\lambda_1, \dots, \lambda_{s-1}$ , one obtains the joint marginal function of  $\lambda_s, \dots, \lambda_p$ , and so on.



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# Author Index

## A

Abe, K. 129  
Alpher, R.A. 85  
Atkinson, R.D'E. 5, 19

## B

Bahcall, J.N. 8, 19, 29, 51–53, 55, 56, 61, 63, 76  
Barnes, C.A. 30, 38, 47, 64  
Bemmerer, D. 127  
Bertulani, C.A. 135  
Bethe, H.A. 7, 8, 19, 81–83  
Biswas, S. 19  
Blatt, J.M. 21, 23, 24  
Bohr, N. 4  
Boltzmann, L. 58, 78  
Breit, G. 7, 24, 25  
Buldgen, G. 128, 135  
Burbidge, E.M. 15–17, 19  
Burbidge, G.R. 15–17, 19

## C

Caughlan, G.R. 19–21, 23, 24, 26, 27  
Chandrasekhar, S. 6–8, 52, 58, 60  
Cheoun, M.-K. 135  
Clausius, R. 1  
Clayton, D.D. 62  
Cleveland, B.T. 83  
Condon, E.U. 5  
Cravens, J.P. 129  
Critchfield, C.L. 7, 19, 30, 37, 61, 63, 81  
Culbreth, G. 130

## D

Davis Jr., R. 8, 51, 55, 76, 80, 81, 84, 135  
De Broglie, M.L. 24

Debye, P. 4

## **E**

Eddington, A.S. 4

Emden, R. 3

## **F**

Fowler, R.H. 3, 5, 19–21, 23, 24, 26, 27, 51, 52, 55, 63, 81, 82

Fowler, W.A. 7, 19–21, 23, 24, 26, 27, 51, 52, 55, 63, 81, 82

## **G**

Gamow, G. 5, 19, 23, 25, 26, 64, 76

Gerth, E. 51, 71, 84

Gurney, R.W. 5

## **H**

Hall, F.W. 135

Harmer, D.S. 8, 76

Harris, M.J. 8

Haubold, A. 19–22, 26, 28–35, 49, 51, 52, 56, 60–65, 69, 93, 97, 103, 124, 129, 131, 134, 135

Haubold, H.J. 19–22, 26, 28–35, 49, 51, 52, 56, 60–65, 69, 83 87, 93, 103, 124, 129, 131, 134, 135

Haxton, W.C. 51, 53, 55

Hayashi, C. 7, 52, 60

Helmholtz, H. 71

Henyey, L.G. 7

Heo, K. 135

Hertzsprung, E. 4

Hilfer, R. 134

Hoffman, K.C. 8, 76

Houtermans, F.G. 5, 19

Hoyle, F. 7, 15, 17, 19

Hückel, E. 4

Huebner, W.F. 63

Hwang, E. 135

## **I**

Iben Jr., I. 8

## **J**

Jang, D. 135  
John, R.W. 19, 26, 28–31, 33, 60, 63, 71, 76, 78, 80, 84

## **K**

Kavanagh, R.W. 52, 53, 55, 62  
Kippenhahn, R. 8  
Klafter, J. 132  
Ko, H. 135  
Kramers, H.A. 5  
Kumar, D. 135

## **L**

Lane, J.H. 1, 2  
Le Levier, R. 7  
Lenee, R.D. 7  
Lubow, S.H. 63

## **M**

Mathai, A.M. 19–22, 26, 29–35, 38–40, 47–49, 52, 56, 60–65, 67, 69, 76, 78, 80, 83, 84, 87, 89, 93–95, 97, 99, 103, 107, 108, 110–112, 115–124, 129, 131, 134  
Maxwell, J.C. 2, 26  
Mayer, R.J. 1  
Metzler, R. 132  
Milne, E.A. 5

## **O**

Öpik, E.J. 7  
Orebi Gann, G.D. 127

## **P**

Parker, P.D. 60, 61, 63  
Perrin, J. 4  
Piatibratova, O. 135  
Pontecorvo, B. 8, 75  
Provost, S.B. 124

## **R**

Ramadurai, S. 19

Ritter, A. 1, 2

Rowley, J.K. 55

Russell, H.N. 4

## **S**

Sakurai, K. 78, 127

Salpeter, E.E. 7, 19, 24, 61, 63, 81

Sampson, R.A. 3

Saxena, R.K. 29, 34, 38–40, 47, 48, 52, 63, 65, 67, 76, 80

Scafetta, N. 84, 128, 130

Schönberg, M. 7

Scholkmann, F. 128

Schuster, A. 3

Schwarzschild, K. 3

Schwarzschild, M. 7

Serenelli, A. 127

Slad, L.M. 135

Strömgren, R. 6

Sturrock, P.A. 128, 135

Sugimoto, D. 7, 52, 60

## **T**

Teller, E. 19

Thomson, W. (Lord Kelvin) 1

Tian, Z. 135

Tsallis, C. 116, 131

## **U**

Ulrich, R.K. 63

## **V**

Vahla, M.N.S. 19

Vogt, H. 5

## **W**

Wagoner, R.V. 26, 27

Weisskopf, V.F. 21, 23, 24

Weizsäcker, C.F.v. 7, 8

Wentzel, G. 5

Wigner, E. 7, 25

Wilson, A.H. 5

## **Y**

Yang, W. 135

Yoo, J. 128

## **Z**

Zimmerman, B.A. 19–21, 23, 24, 26, 27

Zuber, K. 127

# Subject Index

## C

Coulomb barrier 5, 15, 18, 25, 56, 63

Cross section factor 7, 9, 24, 27, 29, 33, 55, 56, 61, 63

## D

Diffusion entropy analysis 84, 86, 89, 128, 130, 131, 135, 137

## E

Equation of state 5, 6, 9, 52, 60

Erdélyi-Kober fractional integral 120–124

Euler transform 117

## H

Hep neutrinos 84

Homestake experiment 75, 76, 78, 84

## K

Kamiokande experiment 135

Krätsel transform 83, 89, 97

## M

Mathai entropy 116

Meijer function 29–31, 37, 43, 47, 62, 65, 69

MSW effect 88, 89

## N

Neutrino, astrophysics 71, 79, 89, 137

Neutrino, oscillations 2, 75, 84, 88, 135, 137

Neutrino, astrophysics 75

Neutrinos, solar 7–10, 51, 52, 55, 59, 64, 69, 71, 75–80, 83–89, 127–133, 135, 137

Neutrinos, variation with time 127

Nuclear, astrophysics 15, 19, 89, 90

Nuclear, reactions 5–8, 15, 19, 21–27, 35, 46, 51, 55, 56, 60–62, 70, 82, 83, 90, 93, 97

Nuclear, astrophysics 15, 93, 94, 96, 98, 100, 102, 104, 106, 108, 110, 112, 114, 116, 118, 120, 122, 124

## P

Pathway model 99

Pep neutrinos 87, 127

PP chain 7, 8, 87, 127

PP neutrinos 87

Probability density function 86, 87, 89, 90, 128, 130, 132, 135, 137

Product, real scalar case 120

## R

Ratio, real scalar case 120

Reaction rates 6, 8–10, 19, 20, 22, 25–29, 32, 35, 37, 52, 55, 60–63, 70, 81

## S

Solar neutrino problem 51

Solar neutrinos 8, 10, 51, 52, 55, 59, 64, 69, 71, 75–80, 83–89, 127–137

Standard deviation analysis 86, 89, 127, 128, 132, 133, 135

Stellar energy generation 7, 19, 20, 51, 52, 54

Sum, real scalar case 118

Sun 1, 3, 4, 7–9, 51, 52, 55–60, 62, 64, 69, 70, 76, 84, 88–90, 127

## W

Weyl fractional integral, second kind 120