Engineering Mechanics

Statics and Dynamics

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PREFACE

Engineering Mechanics, one of the oldest branches of physical science, is a subject of enormous importance. Although it is taught in the first year of engineering, its foundation is rooted in the two other fundamental subjects i.e., applied mathematics and physics. Basically, Engineering Mechanics is a subject that deals with the action of forces. It is broadly classified under Statics and Dynamics. Statics deals with the action of forces on the rigid bodies at rest whereas dynamics deals with motion characteristics of the bodies when subjected to force. The primary purpose of writing this book is to build basic concepts of engineering mechanics along with strong analytical and problem-solving abilities that would enhance the thinking capability of students.

The content of the book is so arranged that it would suit the standard syllabus of any university. The chapters of the book have been so written as to introduce the student to the underlying concepts, develop relevant theory, illustrate with examples and relate the subject to real world situations. The theory has been provided in a lucid manner with figurative description wherever possible to clearly explain the concepts. Simple problems are set forth first and are followed up with more difficult ones that would require application of concepts. Realistic problems have been included to ensure that the gap between theory and practice is narrowed down. Summary, objective type questions and theoretical questions are provided at the end of each chapter.

Though utmost care is taken to make this book error free, some mistakes/errors might have crept in by oversight. We request your omission and commission. Constructive criticism and noteworthy feedback will be gratefully acknowledged.

Prof. K. Shanker

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CHAPTER – 1

INTRODUCTION

LEARNING OBJECTIVES

After studying this chapter, you should be able to

- *Understand various branches of engineering mechanics, viz., statics, dynamics, kinetics and kinematics.*
- *List the SI units of physical quantities used in engineering mechanics*
- *State the basic laws used in engineering mechanics, viz., Newton's laws of motion, law of gravitation, law of transmissibility of force etc.,*
- *Classify various system of forces*

1.1 ENGINEERING MECHANICS

Mechanics may be defined as the branch of science that describes the behaviour of the bodies which are at rest or in motion under the action of forces. Engineering mechanics is the branch of engineering that applies the principles of mechanics to the practical engineering problems. It is generally based on Newton's laws and is often called Newtonian mechanics or classical mechanics. The basic principles of engineering mechanics are used in the study of subjects such as mechanics of solids, mechanics of fluids, space-craft, vibrations etc.

In engineering mechanics, it is idealized that a body as a continuum (continuous distribution of matter) and rigid that undergoes theoretically no deformation. But in real time, no solid body is perfectly rigid and every body undergoes a minute deformation under the action of some external forces. However, the deformation may be very small and may not be considered.

1.2 CLASSIFICATION OF ENGINEERING MECHANICS

Based on the state of the body considered for the study, engineering mechanics is divided into Statics and Dynamics.

1. Statics

It deals with the forces and their effects when they are acting on the stationary bodies.

2. Dynamics

It is concerned with the motion of bodies and how this motion and the forces causing the motion are correlative. Dynamics may be further divided into two parts

a. Kinematics
b. Kinetic

The problems of dynamics which deals without referring to the forces that causes the motion of the body is called as *kinematics* and if the problems deals with the forces that causing the motion is termed as *kinetics*. A simple model of classification is shown in the figure 1.1

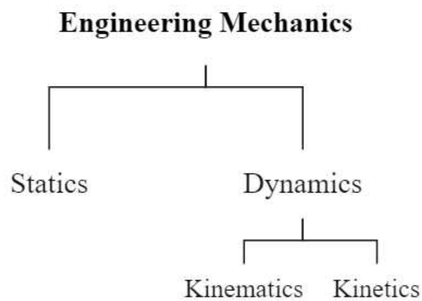


Fig. 1.1 Classification of Engineering Mechanics

1.3 BASIC CONCEPTS

The following are the basic terms which are used in the study of engineering mechanics.
Mass*:* The quantity of matter possessed by a body is called mass.

Weight*:* The force with which a body is attracted towards the centre of earth is called weight.

$$\text{Weight} = \text{Mass of the body} \times \text{Acceleration due to gravity}$$

$$\boxed{W = mg}$$

***Note:** The mass of a body is constant irrespective of its surrounding and location whereas, its weight may change due to the change in gravitational force.

Matter: It is a substance from which the bodies are made up of. It consists of atoms, molecules and occupies space, mass, volume etc.

Continuum: It is defined as continuous distribution of matter with no voids.

Rigid body: A rigid body is one in which the distance between any two arbitrary points will not change.

Time: Time is the measure of duration between successive events.

Space: It is the geometric region occupied by the bodies whose positions are described by linear and angular measurements relative to a coordinate system. If the coordinates involved are only in mutually perpendicular directions, they are known as cartesian coordination. If the coordinates involve angles as well as the distances, it is termed as Polar Coordinate System.

Particle: It is defined as a body or an object that has no size but mass which is concentrated at a point.

Length: Length measures the linear distances of an object.

Scalar: A physical quantity which requires only magnitude for its complete description is known as scalar. For example, distance, area, volume, mass, work, power, energy, time, density, speed, etc.

Vector: A physical quantity which requires both magnitude and direction for its complete description is known as vector. For example, force, displacement, velocity, acceleration, momentum, moment, couple, torque, impulse, weight, etc.

1.4 UNITS OF MEASUREMENT

Measurement of any physical quantity involves comparison with a certain basic, arbitrarily chosen, internationally accepted reference standard called unit. All the physical quantities which are linked with Engineering Mechanics are expressed in terms of three basic and fundamental quantities, i.e. length, mass and time. The units adopted to measure these fundamental quantities are called *fundamental units* which are given in the table 1.2. Units of other quantities like velocity, acceleration, momentum are written in terms of these basic units which are termed as *derived units*. Some of these derived units have been given a special name like newton, joule, watt etc. as shown in the table 1.3.

In earlier time, scientists have used different systems of units for measurement. Three such systems, the CGS, the FPS and the MKS system were in use extensively till recently. The base units for length, mass and time in these systems are given in the table 1.1.

Table 1.1: Fundamental quantities in different system of units

Fundamental quantities	System of units		
	M.K.S	**C.G.S**	**F.P.S**
Length	Metre	Centimetre	Foot
Mass	Kilogram	Gram	Pound
Time	Second	Second	Second

SI Units

The Systeme International d′ unites (in French) or International System of units (SI units) have gained the familiarity throughout the world. In SI system the fundamental units are metre (m) for length, kilogram (kg) for mass and second (s) for time as same as in MKS system. The difference between MKS and SI system arises mainly in selecting the unit of force. In MKS system of units, the unit of force is kilogram force (kgf), which is equal to a force capable of imparting an acceleration of 9.8 m/s^2 to a body of 1 kg mass.

Table 1.2: Fundamental Units

Physical Quantity	**SI units**	**Symbol**
Mass	Kilogram	Kg
Length	Meter	m
Time	Second	s

Table 1.3: Derived Units

Physical Quantity	Symbol	Derived unit
Force	N = kg m/s^2	Newton
Energy, Work, Heat	J = N m = kg m^2/s^2	Joule
Power	W = J/s = N m/s = kg m^2/s^2	Watt
Pressure, Stress	Pa = N/m^2 = kg /m s^2	Pascal
Frequency	Hz = s $^{-1}$	Hertz

1.5 FUNDAMENTAL LAWS OF ENGINEERING MECHANICS

Newton's first law

Every body continues in a state of rest or of uniform motion along a straight line unless it is acted upon by some external force to change that state. In a qualitative way, the tendency of undisturbed objects to stay at rest or to keep moving with the same velocity is called *inertia*.

Newton's second law

The rate of change of momentum of a body is directly proportional to the force applied on the body and the change of momentum takes place in the direction of the applied force. To measure the force quantitatively, the following relation is used.

$$F = ma$$

Newton's third law

This law states that when a body exerts a force on another body, the second body instantaneously exerts a force on to the first body. In simple words, this law can be stated as "to every action there is equal and opposite reaction"

Gravitational law of attraction

It states that everybody attracts any other body with a gravitational force whose magnitude is given by the following equation. In other words, the force of attraction between any two bodies is directly proportional to their masses (m_1, m_2) and inversely proportional to the square of the distance (d) between them.

$$F = G\,\frac{m_1 m_2}{d^2} \qquad \text{where G is constant of gravitation}$$

Parallelogram law of forces

If two forces acting at a point are represented in magnitude and direction, by the sides of a parallelogram, then their resultant force is represented, both in magnitude and direction, by the diagonal of the parallelogram drawn through that point.

Principle of transmissibility of forces

It states that the state of rest or state of motion of a rigid body will not change if force acting at a given point on a body is replaced by another force of same magnitude and direction but acting anywhere on the body along the line of action of the replaced force.

Fig. 1.2 Principle of transmissibility of forces

Polygon law of forces

If a number of coplanar concurrent forces acting on a body are represented in magnitude and direction by the sides of a polygon, taken in order, then their resultant can be represented by closing side of the polygon in magnitude and direction taken from first point to the last.

1.6 FORCE

Force may be defined as any action that changes or tends to change the state of rest or of uniform motion of a body. A force can produce push, pull, twist or rotation in a body. The unit of a force in SI system is Newton (N) and thus one Newton force produces an acceleration of 1 m/s^2 when applied to a body of mass 1 kg. And the unit of force in CGS system is dyne.

1.6.1 CHARACTERISTICS OF A FORCE

The properties which distinguish one force from other are called the characteristics of a force. To specify a force completely, the following four characteristics are necessary

i. Magnitude
ii. Point of application
iii. Direction
iv. Line of action

1.6.2 REPRESENTATION OF A FORCE

In the problems of mechanics, force is shown diagrammatically for a better visualization of the physical phenomenon. Usually, a force represented by an arrow with all four characteristics on it. In the figure, a block is resting on a ground. A force of 50 N is acting at a point A. Now, the force has the following four characteristics.

- Magnitude *50 N*
- Point of application at *A*
- Direction *Downward*
- Line of action is *vertical*

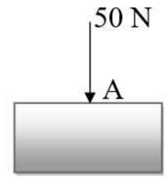


Fig. 1.3 Representation of a force

1.7 SYSTEM OF FORCES

When two or more forces are acting on a body, they constitute a system of forces. The system of forces can be classified based on the positions of lines of action of the forces. The classification of system of forces is given in the figure 1.2

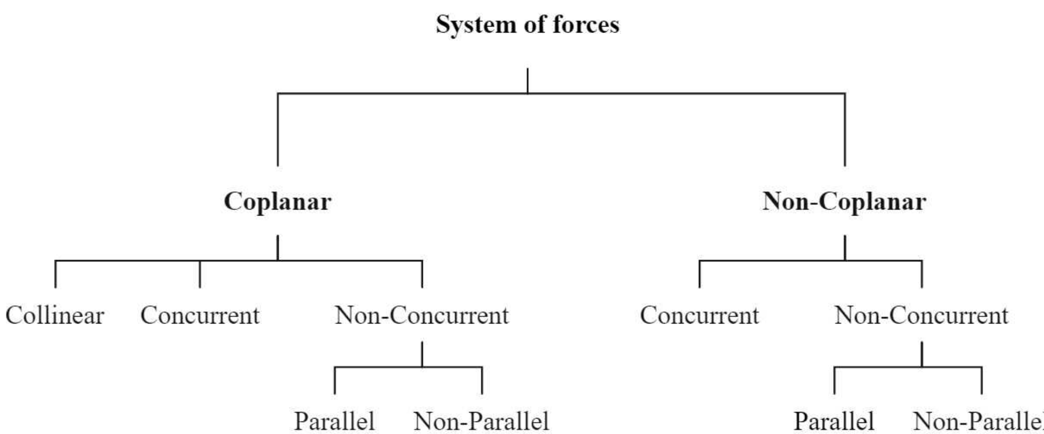


Fig. 1.4 Classification of System of Forces

1.7.1 COPLANAR FORCES

If the Lines of action of all forces lie in a same plane, they are called coplanar forces. Coplanar forces can be further categorised as

(i) Coplanar Collinear
(ii) Coplanar Concurrent and
(iii) Coplanar Non-concurrent.

(i) Coplanar Collinear

If the lines of action of all the forces are same, they are called as coplanar collinear forces.

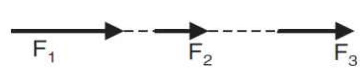


Fig. 1.5 Coplanar collinear forces

(ii) Coplanar Concurrent

In coplanar concurrent forces, the lines of action of forces lie in a same plane and meet at one point.

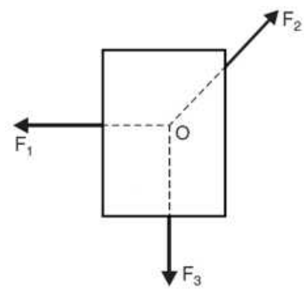
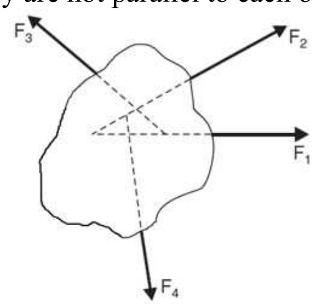


Fig. 1.6 Coplanar concurrent forces

(iii) Coplanar Non-concurrent

If the lines of action of forces lie in a same plane and do not pass through a single point they are called as Coplanar Non-concurrent forces. These coplanar non-concurrent forces can be divided into *coplanar parallel* and *coplanar non-parallel* force. If the lines of action of all forces lie in a same plane and are parallel to each other, they are called as coplanar parallel and if they are not parallel to each other are called *coplanar non-parallel* forces.



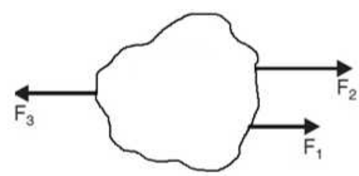


Fig. 1.7 Coplanar non-concurrent forces

1.7.2 NON-COPLANAR FORCES

In Non-Coplanar forces the lines of action of all forces do not lie in a same plane or in other words, their lines of action lie in the space. Non-Coplanar forces can be further categorised as
(i) Non- Coplanar Concurrent and
(ii) Non- Coplanar Non-concurrent.

(i) Non-Coplanar Concurrent

In this force system, the lines of action of forces pass through a single point but not lie in a same plane.

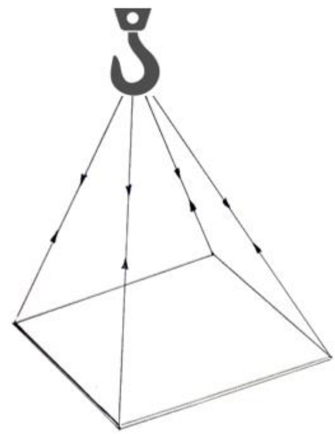


Fig. 1.8 Non-coplanar concurrent forces

(ii) Non-Coplanar Non-concurrent

If the lines of action of forces do not lie in a same plane and do not pass through a single point they are called as Non-Coplanar Non-concurrent forces. These Non-coplanar non-concurrent forces can be divided into *non-coplanar parallel* and *non-coplanar non-parallel* force. If the lines of action of all forces are parallel to each other but not lie in a same plane and, they are called as non-coplanar parallel and if they are not parallel to each other are called non-coplanar non-parallel forces.

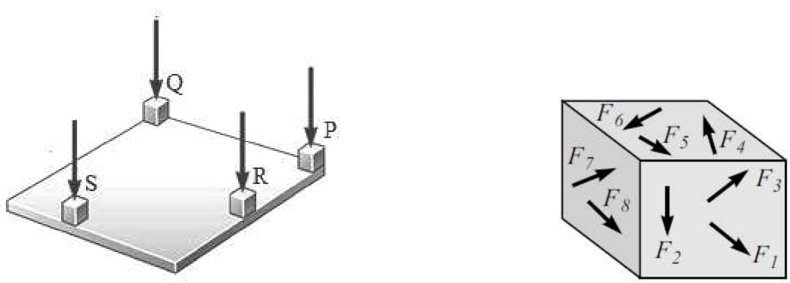


Fig. 1.9 Non-coplanar non-concurrent

SUMMARY

- The branch of science that describes the behavior of the bodies which are stationary and which are moving under the action of forces is called mechanics.
- Engineering mechanics is divided into statics and dynamics.
- Statics deals with the bodies at rest and dynamics deals with the bodies in motion
- Dynamics is divided into kinematics and kinetics.
- The study of motion of bodies without considering the forces is called kinematics and if the forces are considered is called kinetics

- The tendency of undisturbed objects to stay at rest or to keep moving with the same velocity is called inertia.
- An action that tends to change the state of rest or uniform motion of a body is called force.
- If the Lines of action of all forces lie in a same plane, they are called coplanar forces. Coplanar forces
- In Non-Coplanar forces the lines of action of all forces do not lie in a same plane or in other words, their lines of action lie in the space.

EXERCISES

I. Self-Assessment Questions

1. Define Engineering Mechanics.
2. Distinguish between Statics and Dynamics.
3. State the principle of transmissibility of forces.
4. What is the difference between kinetics and kinematics?
5. How do “mass” and “weight" differ from each other?
6. Define the term “Force”. What are its characteristics?
7. State and explain Newton’s laws of motion.
8. What is polygon law of forces?
9. Differentiate between coplanar and spatial force system.
10. Explain various types of coplanar and non-coplanar system of forces with neat sketches.

II. Multiple Choice Questions

1. The branch of mechanics which deals with motions without the reference of forces is known as
   (a) Statics (b) Kinematics
   (c) Kinetics (d) dynamics
2. The relative positions of any two particles of the body do not change even though forces are acting on it, then the body is said to be
   (a) Rigid Body (b) Body at Equilibrium
   (c) Body at Rest (d) Plastic Body
3. The property of the body that is conferred on it by the virtue of its inability to change its position is called
   (a) Motion (b) Rest
   (c) Equilibrium (d) Inertia
4. The agent that changes or tends to change the state of the body.
   (a) Work (b) Energy
   (c) Power (d) Force

5. One kg force is equal to
(a) 746 N (b) 8.91 N
(c) 9.81 N (d) 981 N
6. If several number of forces are acting at the same point, then they are called
(a) Concurrent forces (b) Coplanar forces
(c) Collinear forces (d) Non Concurrent forces
7. The forces whose line of action do not lie in the same plane but are meeting at one point are
(a) Coplanar-concurrent (b) Non coplanar non concurrent
(c) Non coplanar concurrent (d) Collinear
8. The subject 'Classical Mechanics' deals with the forces that obey
(a) Kepler's Laws Planetary Motion (b) Newton's Laws of Motion
(c) Einstein's Theory of Relativity (d) Weber's Laws magnetic forces
9. To specify the force, which of the following is not required
(a) Magnitude (b) Direction
(c) Point of action (d) Source of action
10. The 'unit force' concept is induced by
(a) Newton's First law of motion (b) Newton's Second law of motion
(c) Newton's Third law of motion (d) Newton's law of gravitation
11. A force may be applied anywhere along its line of action without altering the resultant effect of the force external to the rigid body on which it acts. This is
(a) Principle of transmissibility (b) Polygon law of forces
(c) Triangle law of forces (d) Parallelogram law of forces
12. If the line of action of several number of forces is same, then they are called
(a) Concurrent forces (b) Coplanar forces
(c) Collinear forces (d) Non-concurrent forces

Answers

1. b
2. a
3. d
4. d
5. c
6. a
7. c
8. b
9. d
10. b
11. a
12. c

CHAPTER – 2

RESULTANT OF COPLANAR FORCE SYSTEMS

LEARNING OBJECTIVES

After studying this chapter, you should be able to

- Determine the resultant of two force system using parallelogram law
- Resolve a force into a pair of perpendicular components
- Calculate the resultant of coplanar forces by the method of composition
- State and prove Varignon's theorem

2.1 INTRODUCTION

In the previous chapter, the fundamental laws and axioms of engineering mechanics was studied. The concept of force and various system of forces that are possible by the relative orientation have been illustrated. In this chapter, the concept of resultant and the methods to determine the resultant of coplanar forces systems will be discussed.

2.2 RESULTANT

When two or more forces acting on a body, there is a need to reduce these forces to a simplest single force without changing their effect. This single force is called resultant. It is used to express the effect of a system of forces on a body. Resultant is defined as a single force that produces the same effect as produced by the number of forces acting on the body. The process of finding the resultant is called the *composition of forces.*

Consider an example shown in the figure no. 2.1. An electric pole is to be lifted with the help of the cables. Two persons are lifting it by applying the forces of 5 N and 8 N separately as shown in the figure no. Now it is required to lift the same electric pole in the same direction by a single person. What is the amount of force that the single person should apply? These kinds of situations demand the magnitude, point of application and direction of that single force which is called as resultant. It is therefore necessary to compute the resultant for the given system of forces. Since all the forces have the same point of application in concurrent force system, the resultant also acts at the point of concurrency. Whereas as the point of application of the resultant will differ in non-concurrent force system.

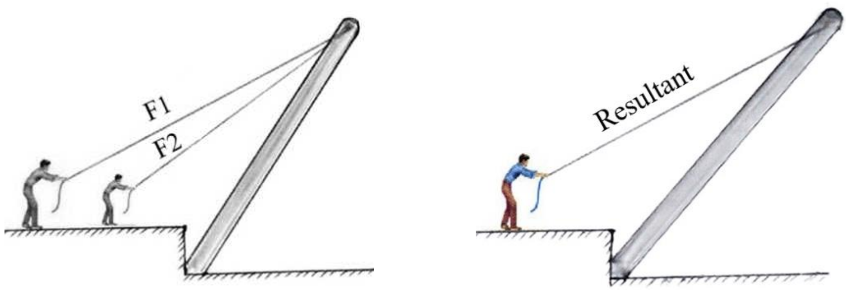


Fig. 2.1 Resultant example

2.2.1 RESULTANT OF COPLANAR FORCE SYSTEMS

As discussed in chapter 1, coplanar forces are those whose lines of action of all forces lie in a same plane. Based on the orientation of the forces, Coplanar forces are classified into three sub classes i.e., collinear, concurrent and non-concurrent.
The resultant of each sub class is studied separately as the following.

1. Resultant of coplanar concurrent forces
2. Resultant of coplanar non-concurrent forces

2.3 RESULTANT OF COPLANAR CONCURRENT FORCES

The resultant of coplanar concurrent force system can be analysed either by graphical method or analytical method. These two methods of finding resultant are discussed in the following sections.

2.3.1 GRAPHICAL METHOD

In graphical method each force is represented by its magnitude and direction to find the resultant. The different methods to find the resultant by graphical method are as follows

a. Triangle law
b. Parallelogram law
c. Polygon law

a. Triangle law

If two forces are acting at a point are represented in magnitude and direction by the two sides of a triangle then the length of the third side taken in the opposite direction gives the magnitude and direction of the resultant.

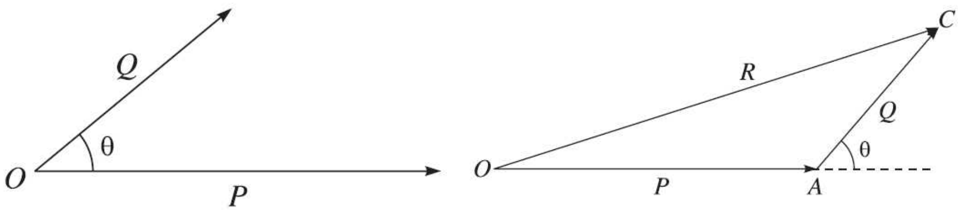


Fig. 2.2 Triangle law

b. Parallelogram law of forces

It states that "if two coplanar concurrent forces acting at a point can be represented in magnitude and direction, by the two adjacent sides of a parallelogram, then their resultant is represented by the diagonal of the parallelogram drawn through that point".

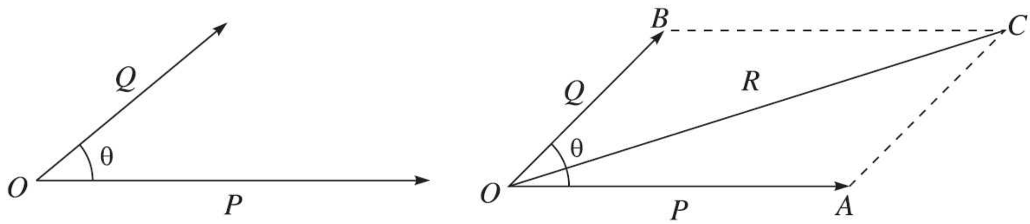


Fig. 2.3 Parallelogram law

c. Polygon law of forces

If number of forces are acting at a point simultaneously can be represented in magnitude and direction by the sides of a polygon, then the closing side of the polygon taken in opposite order gives the magnitude and direction of the resultant.

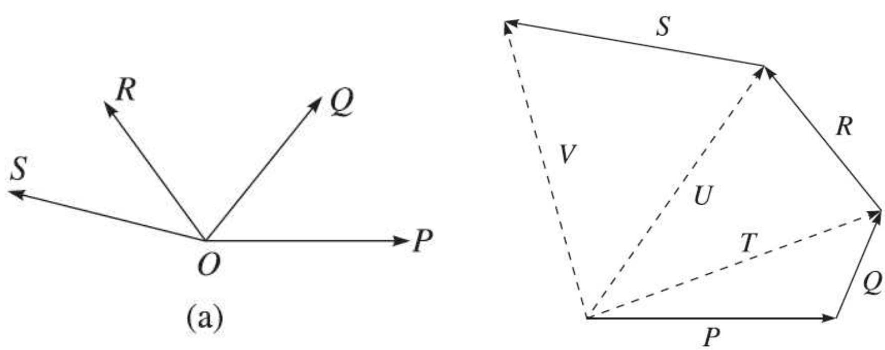


Fig. 2.4 Polygon law

2.3.2 ANALYTICAL METHOD

a. Two force system

Drawing of lines according to the given dimensions by using pencils, scales, protractors make graphical method more complicated. Hence, it is advantageous to solve the problems by analytical method. If there are only two coplanar concurrent forces acting on a body, then it is convenient to find the resultant analytically using parallelogram law.

Parallelogram law

If two forces F_1 and F_2 are acting at a point *O* making an angle θ between them, then the magnitude of the resultant R is given by

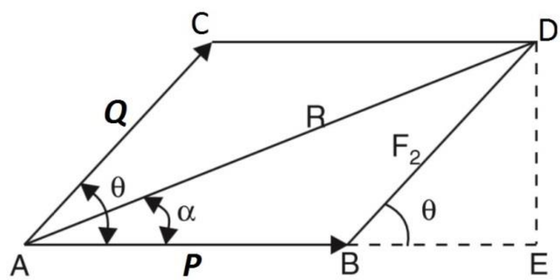
$$R=\sqrt{P^2+Q^2+\ 2PQ\ cos\ \theta}$$

The direction of the resultant with force P is given by

$$\alpha = \tan^{-1}\frac{Q\sin\theta}{P+Q\ cos\ \theta}$$

Proof:

Consider two forces P and Q acting on a particle with an angle θ between them as shown in the figure. By forming a parallelogram ABCD, while AB represents force P and AC represents force Q, then resultant is represented by the line AD.



$R = AD = \sqrt{AE^2+DE^2} = \sqrt{(AB+BE)^2+DE^2}$

We know AB = P

BE = BD cos θ = Q cos θ

DE = BD sin θ = Q sin θ

$R = \sqrt{(P+Q\cos\theta)^2+(Q\sin\theta)^2}$

$R = \sqrt{P^2+2PQ\ cos\ \theta+Q^2cos^2\theta+\ Q^2Sin^2\theta}$

$R= \sqrt{P^2+Q^2+\ 2PQ\ cos\ \theta}$

The inclination of the resultant is given by α

$\tan\alpha = \frac{DE}{AE} = \frac{DE}{AB+BE} = \frac{Q\sin\theta}{P+Q\ cos\ \theta}$

$\alpha = \tan^{-1}\frac{Q\sin\theta}{P+Q\ cos\ \theta}$

Special cases:

1. When θ = 90° $R = \sqrt{P^2+Q^2}$

2. When θ = 0° $R = \sqrt{P^2+Q^2+\ 2PQ} = P + Q$

3. When θ = 180° $R = \sqrt{P^2+Q^2-\ 2PQ} = P - Q$

4. When two forces are equal i.e., P = Q =F

Then R= $\sqrt{F^2 + F^2 + 2FF\cos\theta}$

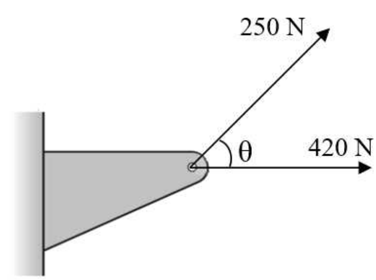
= $\sqrt{2F^2 + 2F^2\cos\theta}$

= $\sqrt{2F^2(1+\cos\theta)}$

= $\sqrt{2F^2 \times 2Cos^2\frac{\theta}{2}}$

= $\sqrt{4F^2Cos^2\frac{\theta}{2}}$ = 2F cos $\frac{\theta}{2}$

Example 2.1 Two forces having magnitude 420 N and 250 N are acting on the edge of a lever making an angle θ between them as shown in the figure. If the resultant of these forces is 600 N, determine the angle θ between the two forces. Also find the inclination of the resultant with the horizontal.



Solution Given two forces are 420 N and 250 N

Let P = 420 N and Q = 250 N

Resultant R = 600 N

We know R= $\sqrt{P^2 + Q^2 + 2PQ\cos\theta}$

600 = $\sqrt{420^2 + 250^2 + 2(420)(250)\cos\theta}$

Squaring on both sides

$600^2 = 420^2 + 250^2 + 2(420)(250)\cos\theta$

360000 = 238900 + 210000 cos θ

Cos θ = $\frac{121100}{210000}$

θ = $\cos^{-1}\frac{121100}{210000}$ = 54.78°

We also know α = $\tan^{-1}\frac{Q\sin\theta}{P+Q\cos\theta}$

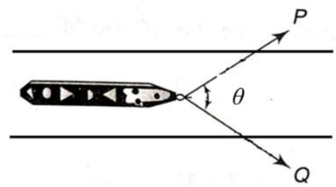
α = $\tan^{-1}\frac{250\sin 54.78}{420+250\cos 54.78}$ = 19.9°

Example 2.2 A boat which is in the middle of a canal is pulled by two persons with ropes by applying the forces P = 650 N and Q = 400 N. The angle between the two ropes is 60°. Find the magnitude of the resultant force and its inclination.

Solution Given S= 400 N, T = 650 N and θ = 60°

We know R = $\sqrt{P^2 + Q^2 + 2PQ\,cos\,\theta}$

= $\sqrt{650^2 + 400^2 + 2(650)(400)\,cos\,60}$ = 917.8 N



We also know α = $\tan^{-1}\frac{Q \sin \theta}{P+Q\,cos\,\theta}$

α = $\tan^{-1}\frac{400 \sin 60}{650+400\,cos\,60}$ = 22.17°

Example 2.3 The resultant of two concurrent forces is 120 N and the angle between the forces is 90°. The resultant makes an angle of 42° with one of the force. Find the magnitude of each force

Solution Given Resultant = 120 N

Angle between two forces θ = 90°

Angle of Resultant α = 45°

tan α = $\frac{Q \sin \theta}{P+Q\,cos\,\theta}$

tan 42° = $\frac{Q \sin 90}{P+Q\,cos\,90}$

tan 42° = $\frac{Q}{P}$

$\boxed{Q = 0.9P}$

When θ = 90° we have R = $\sqrt{P^2 + Q^2}$

120 = $\sqrt{P^2 + (0.9P)^2}$

$120^2 = P^2 + 0.81P^2$

14400 = $1.81P^2$

$P^2 = \frac{14400}{1.81} = 7955.8$

P = $\sqrt{7955.8}$ = 89.1 N

Q = 0.9 × 89.1 = 80.2 N

Example 2.4 The resultant of two forces one of which is 3 times the other is 360 N. When the direction of smaller force is reversed, the resultant is 250 N. Determine the two forces and the angle between them.

Solution Given one force is 3 times the other. If "F" is the magnitude of force, then

P = 3F and Q = F

Resultant = 360 N

R= $\sqrt{P^2 + Q^2 + 2PQ\, cos\, \theta}$

360 = $\sqrt{(3F)^2 + F^2 + 2(3F)(F)\, cos\, \theta}$

129600 = $9F^2 + F^2 + 6F^2\, cos\, \theta$

$10F^2 + 6F^2\, cos\, \theta$ = 129600 ______ (i)

When the direction of smaller force is reversed i.e., Q = -F, then Resultant = 250 N

250 = $\sqrt{(3F)^2 + (-F)^2 + 2(3F)(-F)\, cos\, \theta}$

$10F^2 - 6F^2\, cos\, \theta$ = 62500 ______ (ii)

Solving equations (i) and (ii) we get

$20F^2 = 192100$

F = 98 N

Then P = 3F = 3 × 98 = 294 N

and Q = F = 98 N

Substituting the values of P and Q in equation (i), we get

Cos θ = $\frac{33560}{57624}$ = 0.582

θ = $\cos^{-1}$ 0.582

θ = 54.3°

Example 2.5 Two forces equal to 4P and P respectively act on a particle. If first be doubled and the second increased by 20N the direction of the resultant is unaltered, find the value of **'P'?**

Solution P = 4P and Q = P

tan α = $\frac{P \sin \theta}{4P+P\, cos\, \theta}$ ________ (i)

Also given that if the P is double, then Q is increased by 20 N

P = 8P, Q = P + 20

tan α = $\frac{(P+20) \sin \theta}{8P+(P+20)\, cos\, \theta}$ ________ (ii)

Equating equation (i) and (ii)

$$\frac{P \sin \theta}{4P+P\, cos\, \theta} = \frac{(P+20) \sin \theta}{8P+(P+20)\, cos\, \theta}$$

$$\frac{P}{4P+P\cos\theta} = \frac{(P+20)}{8P+(P+20)\cos\theta}$$

$8P^2 + (P^2 + \cancel{20P}) \cos\theta = 4P^2 + 80P + (P^2 + 20P) \cancel{\cos\theta}$

$4P^2 = 80\ P$

$P = 20$

Example 2.6 Two equal forces act on a particle such that the square of their resultant is equal to the 2 times of their product. Find the angle between the forces.

Solution Given that two forces are equal i.e., $P = Q = F$ and

Square resultant is equal to two times of their product

$R^2 = 2F^2$ _______ (i)

When two forces are equal, we have

$R = 2F\cos\frac{\theta}{2}$

Squaring on both sides

$R^2 = 4F^2\cos^2\frac{\theta}{2}$ ________ (ii)

Equating equation (i) and (ii)

$4F^2\cos^2\frac{\theta}{2} = 2F^2$

$\cos^2\frac{\theta}{2} = \frac{2}{4} = \frac{1}{2} = 0.5$

$\cos\frac{\theta}{2} = \sqrt{0.5} = 0.707$

$\frac{\theta}{2} = \cos^{-1}(0.707) = 45°$

$\theta = 90°$

2.3.2.1 Resolution of a Force

Resolution of forces is the process of dividing (resolving) a single force into two forces (components) which have an equivalent effect as the given single force. When we compute the resultant for more than two forces, first we need to find the components for the given forces separately. Also, it is very convenient to resolve a force into a pair of perpendicular components. Which means; each force has to be resolved in horizontal (x-axis) and vertical (y-axis) directions.

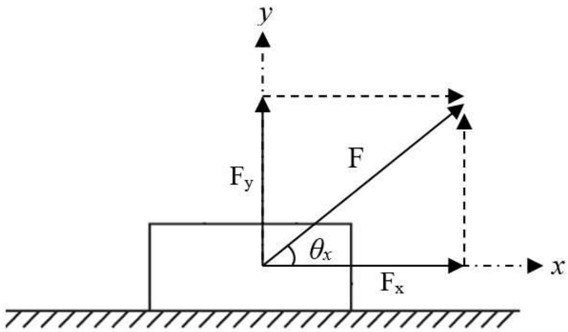


Fig. 2.5

Consider the figure where a single force having magnitude F is acting on the body with an angle θ_x with x-axis. Now, this force can be projected on to the x and y axes to obtain perpendicular components F_x and F_y. These are also called as rectangular components. The relation between these components and F is determined by the basic definitions of sine and cosine of the angle θ_x between F and the x-axis. We get following relation.

$$\cos \theta_x = F_x / F$$
$$\sin \theta_x = F_y / F$$

The above equations can be rewritten as

$$\boxed{\begin{array}{c} F_x = F \cos \theta_x \\ F_y = F \sin \theta_x \end{array}}$$

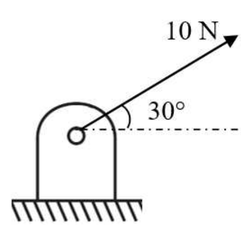
The components F_x and F_y are taken positive if they act in the positive directions of X and Y axes and negative if they act oppositely.

Note: If the inclination of the force F is considered with respect to y-axis i.e., θ_y then the components are given by $F_x = F \sin \theta_y$; $F_y = F \cos \theta_y$.

Example 2.7 A force of magnitude 10 N is acting on a bracket with an inclination of 30° with x-axis as shown in the figure. Find its x & y components.

Solution Given F = 10 N and $\theta_x = 30°$

We know



$$\boxed{F_x = F \cos \theta_x}$$

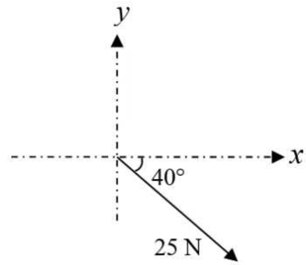
$$F_x = 10 \cos 30° = 5\sqrt{3} = 8.66 \text{ N}$$

$$\boxed{F_y = F \sin \theta_x}$$

$$F_x = 10 \sin 30° = 5 \text{ N}$$

Example 2.8 Find the components for the forces shown in the figures

Solution

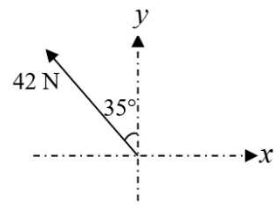


$F_x = F \cos \theta_x$

$F_x = 25 \cos 40° = 19.15$ N

$F_y = F \sin \theta_x$

$F_y =-25 \sin 40° = 16.06$ N

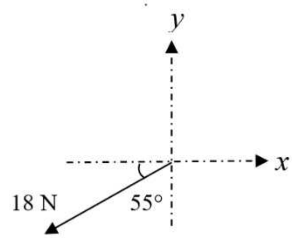


$F_x = F \cos \theta_x$

$F_x = 25 \cos 40° = 19.15$ N

$F_y = F \sin \theta_x$

$F_y = -25 \sin 40° = 16.06$ N



$F_x = F \cos \theta_x$

$F_x = -18 \cos 35° = 14.7$ N

$F_y = F \sin \theta_x$

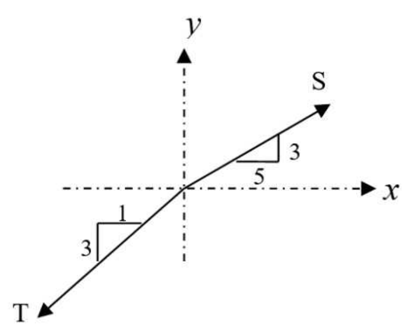
$F_y =-18 \sin 35° = 10.32$ N

Example 2.9 Determine the X and Y components of the forces S = 158 N and T = 110 N as shown in the figure

Solution

For force "S" $\tan\theta_1 = \frac{3}{5}$

$\theta_1 = \tan^{-1}\frac{3}{5} = 30.96°$



$F_x = 158 \cos 30.96° = 135.48$ N

$F_y = 158 \sin 30.96° = 81.28$ N

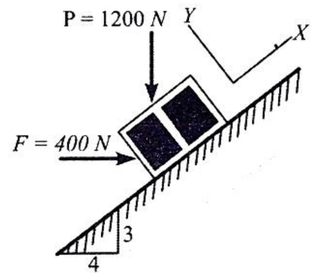
For force "T" $\tan\theta_2 = \frac{3}{1}$

$\theta_2 = \tan^{-1}\frac{3}{5} = 71.56°$

$F_x = 110 \cos 71.56° = 34.78$ N

$F_y = 110 \sin 71.56° = 104.35$ N

Example 2.10 The body on the incline shown in the figure is subjected to the vertical and horizontal forces shown. Find the components of each force along X-Y axis oriented parallel and perpendicular to the incline



Solution

$\theta = \tan^{-1}(3/4) = 36.87$

Angle of force "F" with the x-axis is 36.87°

We know $F_x = F \cos \theta$ and $F_y = F \sin \theta$

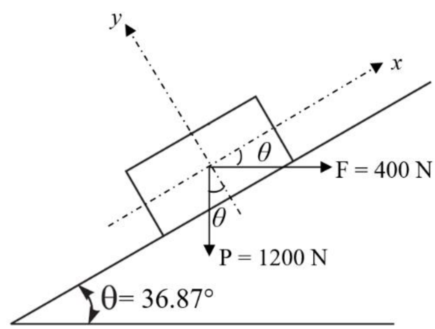
$F_x = 400 \cos 36.87° = 320$ N

$F_y = -400 \sin 36.87° = -320$ N

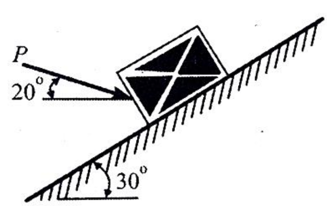
Angle of force "P" with the x-axis is 53.13°

$P_x = -1200 \cos 53.13° = -720$ N

$P_y = -1200 \sin 53.13° = -960$ N



Example 2.11 The body on the 30° incline in figure is acted upon by a force 'P' inclined at 20° with the horizontal. If 'P' is resolved into components parallel and perpendicular to the incline and the value of the parallel component is 300N, compute the value of the perpendicular component and of 'P'

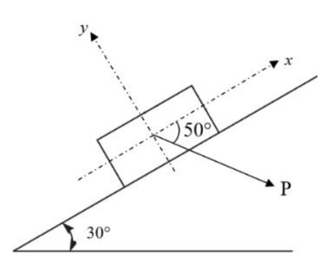


Solution Angle of 'P' with the incline will be 30+20=50°

$P_x = P \cos 50°$ i.e., $300 = P \cos 50°$

Therefore P = 466.72 N

Now $P_y = -P \sin 50 = -466.72 \times \sin 50 = -357.53$ N



RESULTANT OF COPLANAR CONCURRENT FORCES (Cont.)

b. System with more than two forces

When there are more than two forces are acting on a system, then its resultant can be calculated by following procedure

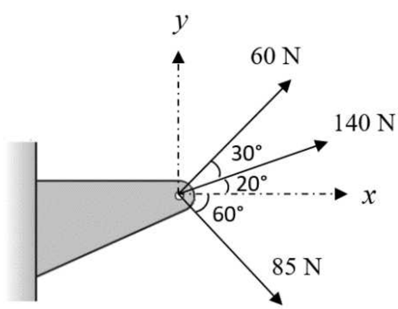
1. Resolve all the forces horizontally and find the algebraic sum of all the horizontal components (i.e., $\sum F_x$).
2. Resolve all the forces vertically and find the algebraic sum of all the vertical components (i.e., $\sum F_y$)
3. The resultant R of the given forces will be given by the equation:

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

4. The resultant force will be inclined at an angle θ, with the horizontal, such that

$$\tan\alpha = \frac{\Sigma F_y}{\Sigma F_x}$$

Example 2.12 Three forces are acting at a point on a hinge as shown in the figure. Determine the resultant of the system of forces



Solution $\Sigma F_x = 140 \cos 20° + 60 \cos 50° + 85 \cos 60°$

$\Sigma F_x = 212.62$ N

$\Sigma F_y = 140 \sin 20° + 60 \sin 50° - 85 \sin 60°$

$\Sigma F_y = 20.2$ N

Resultant $R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$

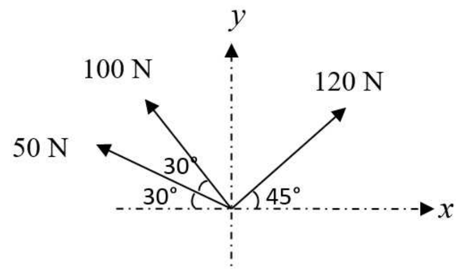
$R = \sqrt{(212.62)^2 + (20.2)^2}$

$R = 213.5$ N

$\tan\alpha = \frac{\Sigma F_y}{\Sigma F_x}$

$\alpha = \tan^{-1}\frac{20.2}{212.62} = 5.42°$

Example 2.13 A System of three forces acting at a point on a screw as shown in the figure. Determine the resultant.



Solution $\Sigma F_x = 120 \cos 45° - 50 \cos 30° - 100 \cos 60°$

$\Sigma F_x = -8.44$ N

$\Sigma F_y = 120 \sin 45° + 50 \sin 30° + 100 \sin 60°$

$\Sigma F_y = 196.45$ N

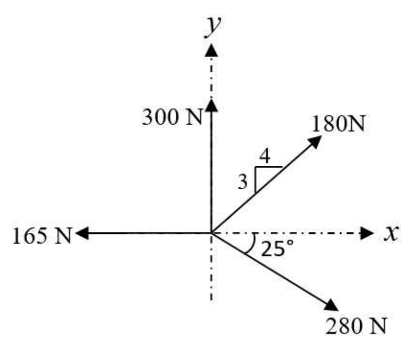
Resultant R $= \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$

$R = \sqrt{(-8.44)^2 + (196.45)^2}$

R = 196.63

$\alpha = \tan^{-1}\frac{196.45}{8.44} = 87.53°$

Example 2.14 Determine the resultant for the following force system



Solution In the figure the inclination for the force 220 N was not given. Instead, its slopes are given.

Now calculating the angle θ

$\tan \theta = \frac{4}{5}$

$\theta = \tan^{-1}\frac{4}{5} = 38.6°$

Σ F_x = 180 cos 45° + 300 cos 90° - 165 cos 0° +280 cos 25 °

Σ F_x = 127.27 + 0 – 165 + 253.76

Σ F_x = 216.03

Σ F_y = 180 sin 45° + 300 sin 90° - 165 sin 0° - 280 sin 25 °

Σ F_y = 127.27 + 300 – 0 -118.33

Σ F_y = 308.94 N

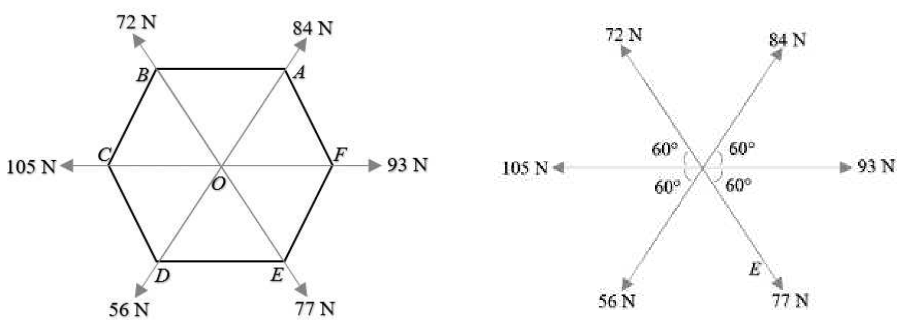
Resultant R $= \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$

R $= \sqrt{(216.03)^2 + (308.94)^2}$

R = 376.97 N

$\alpha = \tan^{-1}\frac{308.94}{216.03} = 55.03°$

Example 2.15 Six forces of magnitude 84 N, 72 N, 105 N, 56 N, 77 N and 93 N act from the centre of a regular hexagon towards its corners where two of its sides are horizontal as shown in the figure. Find the magnitude and direction of the resultant.



Solution Σ F_x = 93 + 84 cos 60° - 72 cos 60° - 105 - 56 cos 60° + 77cos 60°

Σ F_x = 4.5 N

Σ F_y = 84 sin 60° + 72 sin 60° - 56 sin 60° - 77 sin 60°

Σ F_y = 19.9 N

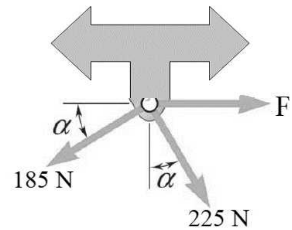
Resultant R $= \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$

R $= \sqrt{(4.5)^2 + (19.9)^2}$

R = 20.4 N

$\alpha = \tan^{-1}\frac{19.9}{4.5} = 77.25°$

Example 2.16 A hinge is subjected to the three forces as shown. Determine
(*a*) the required magnitude of the force F if the resultant of the three forces is to be vertical,
(*b*) the corresponding magnitude of the resultant. Take α = 32º



Solution Given that the resultant of three forces is vertical i.e., on y-axis.

When the resultant is acting along y-axis, $\sum F_x = 0$

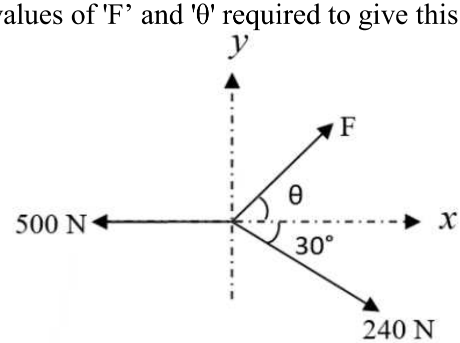
∴ F + 225 cos 58° -185 cos 32° = 0

F = 37.65

Also, when resultant is acting along y-axis, $\sum F_y = R$

R = - 225 sin 58° -185 sin 32° = - 288.8 N

Example 2.17 The force system shown in figure has a resultant of 200 N pointing up along the Y axis. Compute the values of 'F' and 'θ' required to give this resultant.



Solution Given, resultant R = 200 N

As the resultant is acting along y-axis, $\sum F_x = 0$

F cos θ – 500 + 240 cos 30° = 0

F cos θ =292.154 (i)

Also, when resultant is acting along y-axis, $\sum F_y = R$

F sin θ -240 sin 30°=200

F sin θ =320 (ii)

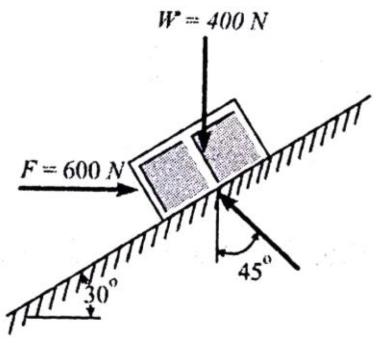
From (i) and (ii) $\tan\theta = \frac{320}{292.154}$

θ=47.6° and F = 433.3N

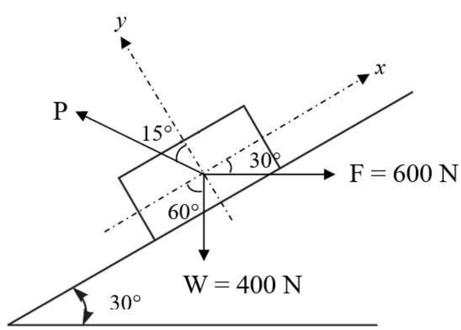
Note: If the resultant of system of forces lies on x-axis (Horizontally) then their sum of y-components is zero i.e, $\sum \mathbf{F_y = 0}$. In such cases R = $\sum F_x$

Also, if the resultant of system of forces lies on y-axis (vertically) then their sum of x components is zero i.e, $\sum \mathbf{F_y = 0}$. In such cases R = $\sum F_y$

Example 2.18 The block shown in figure is acted on by its weight W=400 N, A horizontal force F=600 N, and the pressure P exerted by the inclined plane. The resultant R of these forces is parallel to the incline. Determine P and R. Does the block move up or down the incline?



Solution Consider X and Y axes as shown in the figure



The inclination of forces F, W and P with x-axis are 30°, 60° and 75° respectively.

Given that the resultant of these forces is parallel to incline i.e., x-axis, then $\sum F_y = 0$

-400 sin 60°-600 sin 30°+ P sin 75° = 0

P = 669. 21 N

Also, when resultant is parallel to x-axis, $\sum F_x$ = R

R = 600 cos 30°- 400 cos 60°- 669.21 cos 75° = 146. 41 N

As 'R' is + ve means block moves up along the incline

2.4 RESULTANT OF COPLANAR NON-CONCURRENT FORCES

As all the forces in the concurrent force system pass through a single point, it is obvious that their resultant also passes through the same point. However, it is not true in the case of non-concurrent forces because, in non-concurrent forces the line of action of forces do not meet at a single point. Moreover, concurrent forces produce only translation effect where as non-concurrent forces produce both translation and rotational effect. For the problems dealing with the resultant of coplanar non-concurrent forces, same procedure is adopted as concurrent forces except, in addition to calculating the magnitude and direction of resultant we calculate the rotational effect of the forces and the point of application of the resultant. The measure of this rotational effect produced by a force on a body is known as *Moment of a force.*

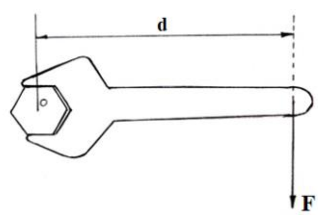
2.4.1 MOMENT OF A FORCE

The rotational effect produced by a force is called as Moment of a force.

or

The tendency of a force to rotate the body about a given point in called *Moment* of a force

Consider an example to understand the concept of moment. In the figure, a force F has been applied at end of the spanner which is acting perpendicular to its handle. This force has a tendency to rotate or turn the nut about its centre point. This rotational tendency is known as the moment "M" of the force "F". Moment is also referred as torque if it causes twisting.



The moment of a force about a point is given by the product of force and the perpendicular distance from the given point to the line of action of the force.

$$M = F \times d$$

Note: when d = 0, the moment is zero
The units of the moments are N - m

2.4.2 TYPES OF MOMENT

Depending upon the direction of the rotation of a body, the moment of a force is classified as

1. ***Clock wise moment***

The force that tries to rotate the body in the clockwise direction is called clock wise moment. The clock wise moment is considered as positive

2. Anti-clock wise moment

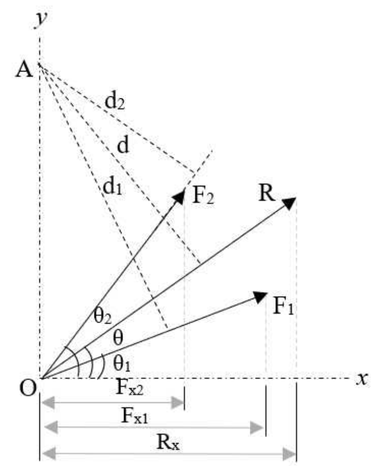
The force that tries to rotate the body in an anti-clockwise direction is called as anti-clock wise moment. The anti-clock wise moment is considered as negative

2.4.3 VARIGNON'S THEOREM

The moment of the resultant force about a point is equal to the sum of the moments of the components of the resultant about the same point. This theorem is also referred as *principle of moments*

$$R \times d = F_1 \times d_1 + F_2 \times d_2 + F_3 \times d_3 + \ldots\ldots$$

Proof:



Consider the resultant force R acting at a point O whose components are given by F_1 and F_2

Choose any point A as moment centre.

Moment of Resultant R about A

$Rd = R\,(AO \cos\theta) = AO\,(R \cos\theta)$

$Rd = AO\,R_x$ ____________ (i)

Moment of Force F_1 about A

$F_1 d_1 = F_1\,(AO \cos\theta_1) = AO\,(F_1 \cos\theta_1)$

$F_1 d_1 = AO\,F_{x1}$ ____________ (ii)

Moment of Force F_2 about A

$F_2 d_2 = F_2\,(AO \cos\theta_2) = AO\,(F_2 \cos\theta_2)$

$F_2 d_2 = AO\,F_{x2}$ _____________ (iii)

Adding equations (ii) and (iii)

$F_1d_1 + F_2d_2 = AO\ (F_{x1} + F_{x2})$ ______________ (iv)

But the sum of x components of the forces F_1 and F_2 is equal to the x components of resultant force R

i.e., $R_x = F_{x1} + F_{x2}$

$AO\ R_x = AO\ (F_{x1} + F_{x2})$

From equation (i) and (iv)

$\mathbf{Rd = F_1d_1 + F_2d_2}$

The theorem of varignon can be extended to a system of forces consisting more than two forces by the application of above method.

Note: *The moment of a force about a point is equal to twice the area of the triangle whose base is the line segment representing the force and the vertex is the point about which the moment is taken*

Example 2.19 Find the moment of a force 200 N about the point *O* acting on a beam of length 5 m vertically downward and upward as shown the figure (a) and (b)

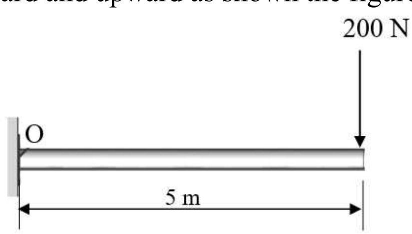


Figure (a)

Solution Figure (a)
Given; force = 200 N
Perpendicular distance from moment centre O to the force is 5 m
Moment M_O = Force x perpendicular distance
$= 200 \times 5 = 1000$ N-m

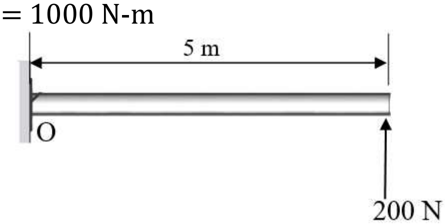


Figure (b)

Solution Figure (b)
Given; force = 200 N
Perpendicular distance from moment centre O to the force is 5 m
Moment M_O = Force x perpendicular distance
$= -200 \times 5 = -1000$ N-m

Example 2.20 A force 75 N is acting at the end of a flexible link at a distance 2 m from point O. determine the moment of a force about the point O for the figure (a) and (b)

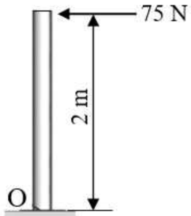


Figure (a)

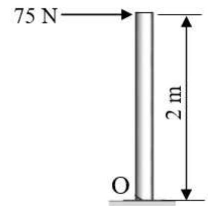


Figure (b)

Solution Figure (a)
Given: Force = 75 N
Perpendicular distance = 2 m
Moment M_O = Force x $\perp$ distance
$M_O = -150 \text{ x } 2 = -300$ N - m

Figure (b) Given: Force = 75 N
Perpendicular distance = 2 m
Moment M_O = Force x $\perp$ distance
$M_O = 150 \text{ x } 2 = 300$ N – m

Example 2.21 Determine the moment of a force about the point O as shown in the figure (a)

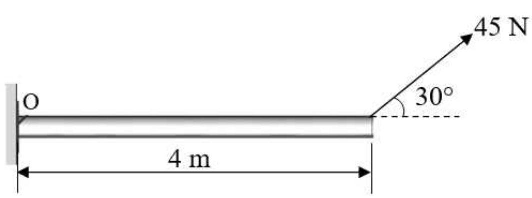


Figure (a)

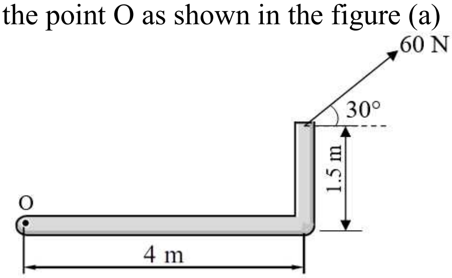


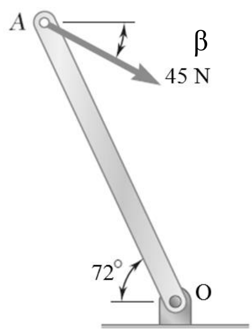
Figure (b)

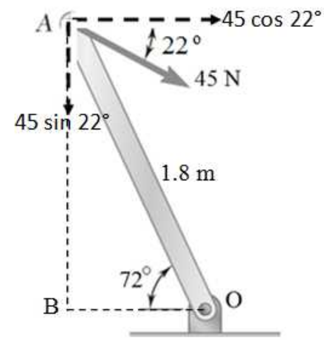
Solution
Consider figure a
Given
Force = 45 N
The vertical component of the force = 45 sin 30
Perpendicular distance = 4 m
Moment about O M_O = Force x $\perp$ distance
$M_O = -45 \sin 30 \text{ x } 4 = -90$ N-m
Consider figure b
Given; Force = 60 N
The horizontal component of the force = 60 cos 30 º
The vertical component of the force = 60 sin 30 º
Perpendicular distance from horizontal component to "O" = 1.5 m
Perpendicular distance from vertical component to "O" = 4 m

Moment about O M_O = Force x ⊥ distance

$$M_O = 60 \cos 30^\circ \times 1.5 - 60 \sin 30^\circ \times 4 = -42.05 \text{ N-m}$$

Example 2.22 A 45 N force is applied at the end of a link as shown. Determine the moment of the force about point O if the length OA = 1.8 m and β = 22º.





Solution: Given OA = 1.8 m and β = 22º

From the triangle AOB,

$\sin 72^\circ = \frac{AB}{1.8}$

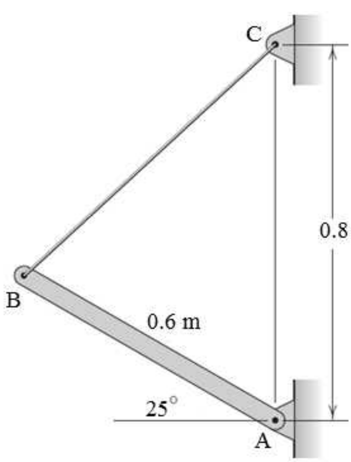
AB = 1.71

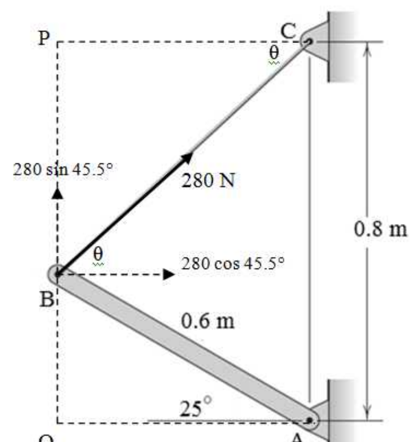
$\cos 72^\circ = \frac{OB}{1.8}$

OB = 0.55

$M_O = 45 \cos 22^\circ \times 1.71 - 45 \sin 22^\circ \times 0.55 = 62.07$ N-m

Example 2.23 A tension of 280 N is developed in the cable BC. Determine the moment of this tension about point O as the tension is applied at point B of the flexible link.





Solution Consider the triangle OAB

sin 25º = $\frac{OB}{0.6}$

OB = 0.25

cos 25º = $\frac{OA}{0.6}$

OA = 0.54
PB = CA – OB = 0.8 – 0.25 = 0.55

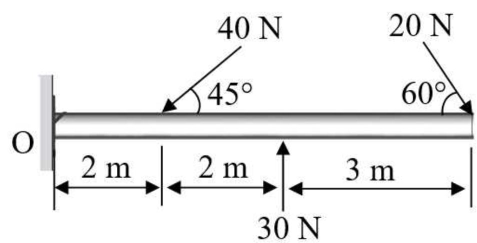
PC = OA = 0.54

tan θ = $\frac{PB}{PC} = \frac{0.55}{0.54}$

θ = 45.5°

M_A = 280 cos 45.5° x 0.25 + 280 sin 45.5° x 0.54 = 156.9 N-m

Example 2.24 Determine the magnitude and direction for the system of coplanar non-concurrent forces acting on a beam as shown in the figure



Solution Σ F_x = - 40 cos 45° - 20 cos 60° = - 18.28 N

Σ F_y = 40 sin 45° + 30 - 20 sin 60° = -15.6

Resultant R = $\sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$

= $\sqrt{(-18.28)^2 + (-15.6)^2}$ = 24.03 N

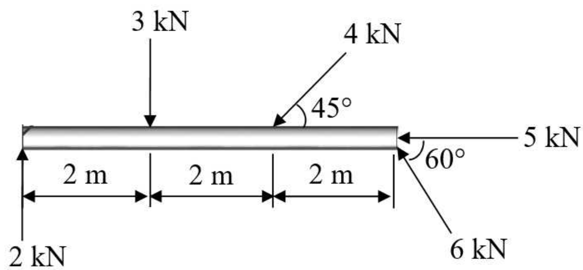
Σ M_O = 40 sin 45° x 2 – 30 x 4 + 20 sin 60° x 7

Σ M_O = 57.81 N-m

x-intercept; $\frac{\Sigma M_O}{\Sigma F_y} = \frac{57.81}{15.6}$ = 3.7

α = $\tan^{-1}\frac{\Sigma F_y}{\Sigma F_x} = \tan^{-1}\frac{15.6}{18.28}$ = 40.47°

Example 2.25 Determine the magnitude and direction of the resultant for the system of forces shown in the figure



Solution $\Sigma F_x = -4\cos 45° - 6\cos 60° = -5.82$ kN

$\Sigma F_y = 2 - 3 - 4\sin 45° + 6\sin 60° = 1.36$ kN

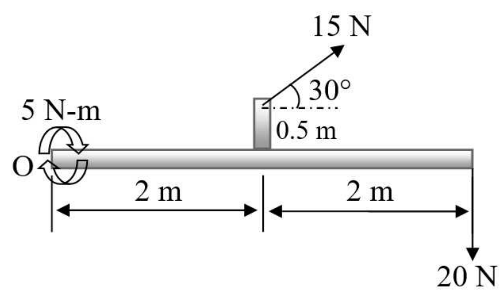
Resultant = $\sqrt{(-5.82)^2 + (1.36)^2} = 5.98$ kN

$\Sigma M_O = 3 \text{x} 2 + 4\sin 45° \text{ x } 4 - 6\sin 60° \text{ x } 6 = -13$ kN-m

x-intercept; $\frac{\Sigma M_O}{\Sigma F_y} = \frac{13}{1.36} = 9.55$ m

$\alpha = \tan^{-1}\frac{\Sigma F_y}{\Sigma F_x} = \tan^{-1}\frac{1.36}{-5.82} = 13.15°$

Example 2.26 A bracket is subjected to two forces and a couple as shown. Fint the resultant.



Solution $\Sigma F_x = 15\cos 30° = 12.99$ N

$\Sigma F_y = 15\sin 30° - 20 = -12.5$ N

Resultant = $\sqrt{(12.99)^2 + (-12.5)^2} = 18.02$ N

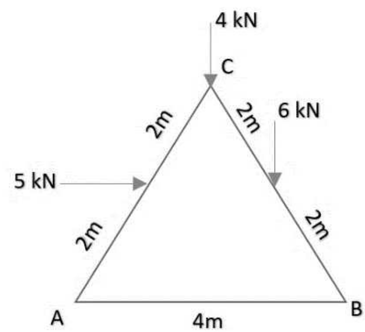
$\Sigma M_O = 15\cos 30° \text{ x } 0.5 - 15\sin 30° \text{ x } 15 + 20 \text{ x } 4 + 5$

$\Sigma M_O = -21$ N-m

x-intercept; $\frac{\Sigma M_O}{\Sigma F_y} = \frac{21}{12.5} = 1.68$ m

$\alpha = \tan^{-1}\frac{\Sigma F_y}{\Sigma F_x} = \tan^{-1}\frac{12.5}{12.99} = 43.89°$

Example 2.27 Three forces of magnitude 4 kN, 5 kN and 6 kN are acting on an equilateral triangle as shown in the figure. Compute the magnitude and direction of resultant.



Solution $\Sigma F_x = 5$ kN

$\Sigma F_y = -4-6 = -10$ kN

Resultant = $\sqrt{(5)^2 + (-10)^2} = 11.18$ kN

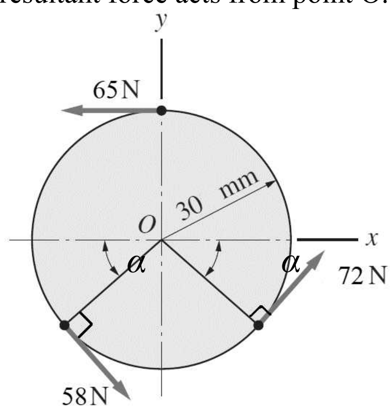
$\Sigma M_A = 5 \times \sqrt{3} + 4 \times 2 + 6 \times 3 = 34.66$ kN-m

x-intercept; $\frac{\Sigma M_A}{\Sigma F_y} = \frac{34.66}{10} = 3.46$ m

y-intercept; $\frac{\Sigma M_A}{\Sigma F_x} = \frac{34.66}{5} = 6.932$ m

$\alpha = \tan^{-1}\frac{\Sigma F_y}{\Sigma F_x} = \tan^{-1}\frac{10}{5} = 63.43°$

Example 2.28 Determine the resultant of the three forces shown in the figure if $\alpha = 40°$. Also, find the distance at which the resultant force acts from point O.



Solution $\Sigma F_x = 72 \cos 50° + 58 \cos 50° - 65 = 18.56$ N

$\Sigma F_y = 72 \sin 50° - 58 \sin 50° = 10.72$ N

Resultant R = $\sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$

$R = \sqrt{(18.56)^2 + (10.72)^2} = 21.43$ N

$\Sigma M_O = -72 \times 30 - 58 \times 30 - 65 \times 30 = -5850$ N-mm

x-intercept; $\frac{\Sigma M_O}{\Sigma F_y} = \frac{5850}{10.72} = 545.7$ mm

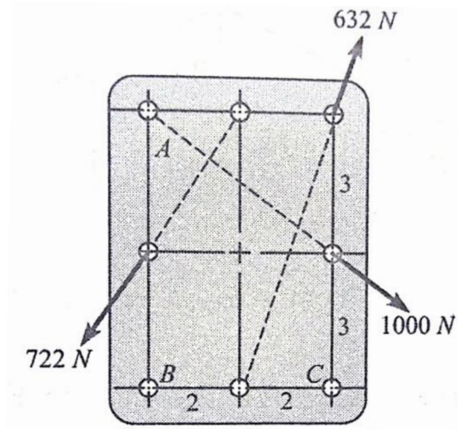
y-intercept; $\frac{\Sigma M_O}{\Sigma F_x} = \frac{5850}{18.56} = 315.19$ mm

From Varignon's theorem

$\Sigma M_O = R \times d$

$d = \frac{\Sigma M_O}{R} = \frac{5850}{21.43} = 272.98$ mm

Example 2.29 Compute the resultant of three forces acting on the plate shown in figure. Locate its intersection with AB and BC.



Solution Let θ_1 be the inclination of force 632 N, θ_2 be the inclination of force 722 N and θ_3 be the inclination of force 1000 N. Then,

$\theta_1 = \tan^{-1} \frac{6}{2} = 71.565°$

$\theta_2 = \tan^{-1} \frac{3}{2} = 56.3099°$

$\theta_3 = \tan^{-1} \frac{3}{4} = 36.87°$

$\Sigma F_x = 632 \cos 71.565° - 722 \cos 56.3099° + 1000 \cos 36.87° = 599.36N$

$\Sigma F_y = 632 \sin 71.565° - 722 \sin 56.3099°_2 - 1000 \sin 36.87° = -601.17N$

$R = \sqrt{(599.36)^2 + (-601.17)^2} = 848.907N$

$\theta = tan^{-1} \left[\frac{601.17}{599.36}\right] = (45.086)°$

Resultant acts along IV quadrant as ΣF_x is +ve and ΣF_y is –ve

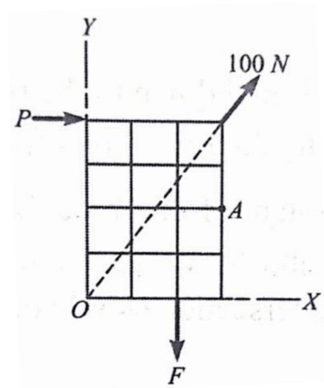
Now to its location its intersection with AB and BC

$\Sigma M_B = -632 \sin 71.565° \times 2 - 722 \cos 56.3099° \times 3 + 1000 \cos 36.87° \times 6 = 2399.34$ N-m

x-intercept; $\frac{\Sigma M_B}{\Sigma F_y} = \frac{2399.34}{601.1} = 3.99$ m

y-intercept; $\frac{\Sigma M_B}{\Sigma F_x} = \frac{2399.34}{599.36} = 4$ m

Example 2.30 The three forces shown on grid produce a horizontal resultant force through point A. Find the magnitude and sense of P and F



Solution Let θ be the inclination of the force 100 N. Then,

$\theta = tan^{-1}\left(\frac{4}{3}\right) = 53.13°$

As the resultant is horizontal; Σ Fy=0

100 sin $\theta_1 - F = 0$

F=80N

As the resultant is passing through A

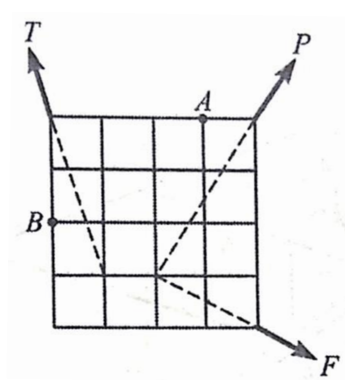
$\Sigma M_A = 0$

P×2+100 cos $\theta_1 \times 2 - 80 \times 1 = 0$

P=-20N

'-ve ' sign indicates that P is acting towards left

Example 2.31 The three forces shown in figure are required to cause a horizontal resultant acting through point A if T=316 N determine the values of P and F



Solution Let θ_1 be the inclination of force F, θ_2 be the inclination of force P and θ_3 be the inclination of force T. Then,

$\theta_1 = tan^{-1}\left(\frac{1}{2}\right) = (26.56)°$

$\theta_2 = tan^{-1}\left(\frac{3}{2}\right) = (56.31)°$

$\theta_3 = tan^{-1}\left(\frac{3}{1}\right) = (71.56)°$

As 'R' is horizontal, $\Sigma F_y = 0$

P sin θ_2+T sin θ_3-F sin $\theta_1 = 0$

0.832P -0.447F =-299.775 ……….(i)

As 'R'is passing through point A

$\Sigma M_A = 0$

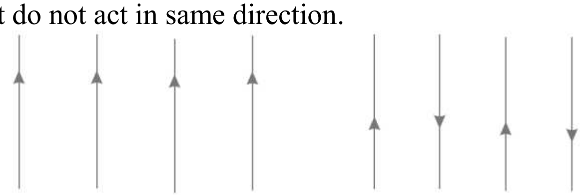
(T sin $\theta_3 \times 3$)- P sin $\theta_2 \times 1 - F\cos\theta_1 \times 3$-F sin 1 ×1 =0

-0.832P -3.13 F =-899.325 ……….(ii)

Solving (i) and (ii) we get F =335.225N and P=-180.2 N

2.5 PARALLEL FORCES AND COUPLES

If the line of action of all forces acting on a system are parallel to each other, they are known as parallel forces. The parallel forces can be further classified as like parallel forces and unlike parallel forces. If the line of action of all the forces are parallel to each other and act in the same direction, they form like parallel forces. In unlike parallel forces, the line of action of forces are parallel but do not act in same direction.



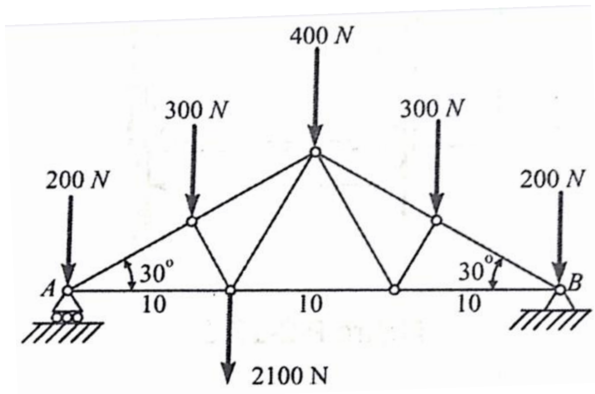
2.5.1 COUPLE

If two unlike parallel forces equal in magnitude acting on a body are said to form a couple.

2.5.2 Characteristics of couple

1. The algebraic sum of forces (resultant), constituting the couple, is zero
2. A couple can produce only rotary motion but not translatory.
3. The algebraic sum of the moment of the forces, constituting the couple about any point is the same and equal to the moment of the couple.
4. A couple does not have moment centre, like moment of force.

Example 3.32 Determine the amount and position of the resultant of the loads acting on the truss shown in figure



Solution ΣF_y = -200-300-400-300-200-2100 =- 3500 N

Resultant force R=3500N (acting downward)

$\Sigma M_A =$ R x d

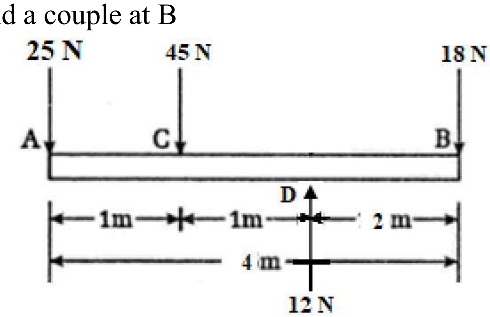
300 ×7.5+400×15+300×22.5 +200×30+2100 × 10 =3500 × d

4200=3500 × d

d = 12 m to the right of 'A

Example 3.33 Four unlike parallel forces are acting on a beam as shown in the figure. Reduce these forces to

i. A single force
ii. A single force and a couple at A
iii. A single force and a couple at B



Solution

i. Resultant force R = ΣF_y

R = -25-45+12-18 = -76 N

From Varignon's theorem

$\Sigma M_A =$ R x d

$45 \times 1 - 12 \times 2 + 18 \times 4 = 76 \times d$

$d = 1.22$ m

ii. If a single force has to act at A, it has to be a downward force of 76 N at A together with a clockwise couple of 1.22 x 76 =92.92 N-m

iii. If a single force has to act at A, it has to be a downward force of 76 N at B together with a counter clockwise couple of (4 – 1.22) x 76 =211.28 N-m

SUMMARY

- Parallelogram law states that if two coplanar concurrent forces acting at a point can be represented in magnitude and direction, by the two adjacent sides of a parallelogram, then their resultant is represented by the diagonal of the parallelogram drawn through that point.
- Resultant is a single force that produces the same effect as produced by the number of forces acting on the body.
- The process of finding the resultant is called the *composition of forces.*
- *Resolution of forces* is the process of dividing (resolving) a single force into two forces (components) which have an equivalent effect as the given single force.
- The tendency of a force to rotate the body about a given point in called *Moment* of a force
- The moment of a force about a point is given by the product of force and the perpendicular distance from the point to the line of action of the force.
- If two unlike parallel forces equal in magnitude acting on a body are said to form a couple.

EXERCISES

I. SELF-ASSESSMENT QUESTIONS

1. Define the term Resultant.
2. State the Parallelogram law of forces.
3. State the Polygon law of forces.
4. What is composition of forces?
5. What is resolution of forces?
6. Differentiate between composition and resolution of forces.
7. Give the statement of Varignon’s theorem.
8. Distinguish between Moment and Couple.

II. Multiple Choice Questions

1. The process of finding out the resultant force is called_________ of forces
   a) Composition
   b) Analysis
   c) Genesis
   d) Resolution
2. The resultant of two forces P and Q acting at an angle θ is ________
   a) $\sqrt{P^2 + Q^2 + 2PQSin\theta}$
   b) $\sqrt{P^2 + Q^2 + 2PQCos\theta}$
   c) $\sqrt{P^2 + Q^2 - 2PQCos\theta}$
   d) $\sqrt{P^2 + Q^2 + 2PQTan\theta}$
3. The resultant of two equal forces P making an angle θ, is given by
   a) 2P sin θ/2
   b) 2P cos θ/2
   c) 2P tan θ/2
   d) 2P cot θ/2
4. Parallelogram law is applicable to determine the resultant if number of forces acting in a concurrent coplanar force system are
   a) Three
   b) Four
   c) Two
   d) None of these
5. The equal and opposite force of resultant is called
   a) Resolver
   b) Equilibrant
   c) Equivalent
   d) Frictional Force
6. Two forces P and Q are acting at a point at right angles to each other, then the magnitude of their resultant is given by
   a) $(P + Q)^2$
   b) P + Q
   c) $P^2 + Q^2$
   d) $\sqrt{P^2 + Q^2}$
7. The moment of a force about a point is
   a) A vector quantity
   b) A scalar quantity
   c) Zero
   d) None of these

8. A couple consists of
    a) two like parallel forces of same magnitude.
    b) two like parallel forces of different magnitudes.
    c) two unlike parallel forces of same magnitude.
    d) two unlike parallel forces of different magnitude
9. ____________ is the process of resolving a single force into two forces (components) which have an equivalent effect as the given single force.
    a) Composition
    b) Analysis
    c) Genesis
    d) Resolution
10. If the resultant of a system of coplanar concurrent forces lies on x-axis (Horizontally) then their $\sum F_y$ is equal to
    a) One
    b) Zero
    c) Resultant
    d) Sum of forces

Answer

1. a
2. b
3. b
4. c
5. b
6. d
7. a
8. c
9. d
10. b

CHAPTER – 3

EQUILIBRIUM OF COPLANAR FORCE SYSTEMS

LEARNING OBJECTIVES

After studying this chapter, you should be able to

- *State the necessary and sufficient conditions of equilibrium for coplanar force system.*
- *Construct the Free Body Diagram for a given body*
- *Prove and apply Lami's theorem*
- *Apply the conditions of equilibrium for solving static problems*

3.1 INTRODUCTION

In the previous two chapters, different systems of forces have been discussed and the method of finding their resultant is explained. In this chapter, we shall look at a case that arises when the resultant and moment of a force becomes zero. If the resultant of a system of forces is zero, the body will remain at rest or move with constant velocity, and if the moment is also zero, then there will not be any rotational motion. A body with such conditions is said to be in equilibrium and this chapter deals with the problems concerned with equilibrium conditions.

3.2 EQUILIBRIUM

Equilibrium is a stable condition of a body, in which the forces acting on a body has a zero resultant. Thus, a body will be in equilibrium if the resultant force acting on it is zero. In other words, if a body is at rest or moving with uniform velocity under the action of number of forces, then the body is said to be in a state of equilibrium.

A hockey puck sliding with constant speed in a straight line on a frictionless horizontal surface and a book on the table are the examples of equilibrium. In statics, we are concerned largely with objects that are not moving in any way, either in translation or in rotation in the reference frame from which we observe them. Such objects are said to be in static equilibrium. Examples of static equilibrium are shown in figure 3.1(a) and (b)

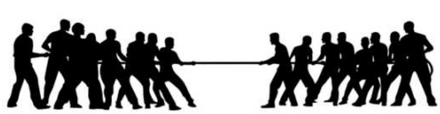


Fig. 3.1 (a): Static Equilibrium

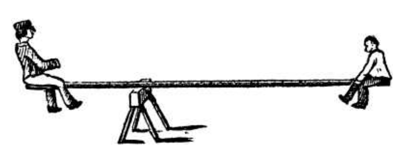


Fig. 3.1 (b): Static Equilibriu

3.2.1 EQUILIBRANT

Equilibrant is a single force which bring the system to equilibrium. Thus, equilibrant is equal in magnitude, opposite in direction and collinear to resultant force. Sometimes the resultant of force system is not equal to zero. That means the body is not in equilibrium. The force which is required to keep the body in equilibrium is known equilibrant.

3.3 EQUILIBRIUM CONDITIONS FOR COPLANAR CONCURRENT FORCES

If a body is in equilibrium under the action of coplanar concurrent forces there will be no translation movement. In such cases, the algebraic sum of all the external forces acting on the body should be zero i.e., their resultant is zero.
We know

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

To make resultant equal to zero, the algebraic sum of x and y-components (i.e., ΣF_x and ΣF_y) of all forces should be zero.
Mathematically,

$$\Sigma F_x = 0$$
$$\Sigma F_y = 0$$

The above equations are the conditions of equilibrium for coplanar concurrent force system.

3.4 FREE BODY DIAGRAM

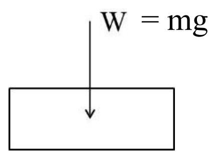
The representation of a body by all forms of forces acting on it when the body is detached from all contact surfaces is called Free Body Diagram (FBD). The first step in the study of equilibrium of bodies is to draw the free body diagram. While drawing FBD we shall imagine that we remove all the supports and replace them by reactions which they exert on the body.

Before going into the procedure for constructing free body diagram, it is necessary to be familiar with the various types of forces. The following three forces have an important role in drawing FBD.

i. Self-weight
ii. Reactions
iii. Applied forces

i. Self-weight

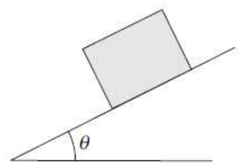
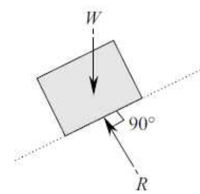
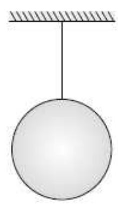
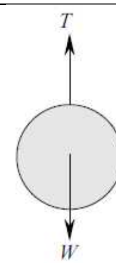
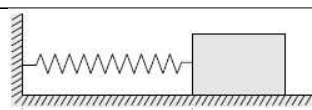
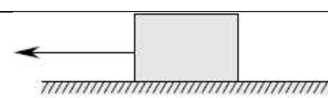
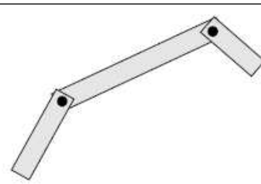
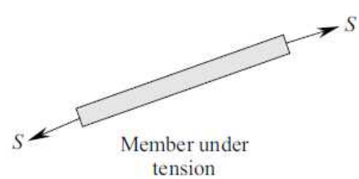
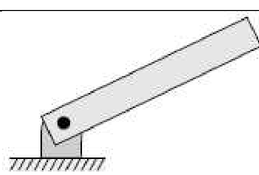
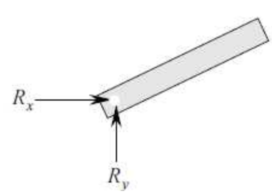
Self-weight is the force with which a body is attracted towards the centre of earth. Hence every body on the earth has self-weight and its direction is always vertical and downwards which acts through the centre of gravity of the body.



ii. Reactions

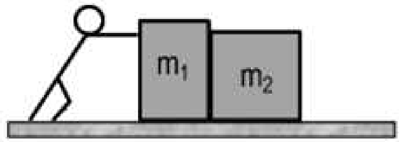
Reaction forces are genereated, if the movement of a body is restricted. There are many types of reactions that act upon a body due to its attachment with its surroundings. These reactions and their free body diagram is shown in table 3.1

Table 3.1: Types of reactions and their free body diagram

	Type of Reaction	***Figure***	***Free Body diagram***
a	Normal reaction		
b	Tensile pull		
c	Restoring force in a spring		
d	Tension or compression members		
e	Support reactions		

iii. Applied forces

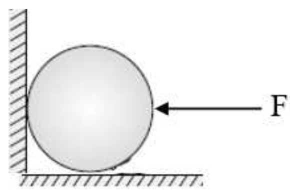
These are the forces applied externally on a body. For example, if a block had been pushed by a person, here the applied force is the force applied on the block by the person.



The general procedure for constructing a free-body diagram is as follows.

1. Draw the body as it is, by removing all the contact surfaces.
2. Indicate the self-weight of the body, which acts vertically downward and acts through the centre of gravity of the body.
3. Draw all the applied forces that are acting on the body.
4. Count how many point of contacts that the body has. Indicate the reactions at the point of contacts in a direction opposite and perpendicular to the contact surface.

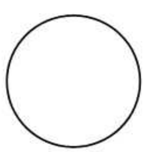
Example 3.1 A sphere of radius “r” is placed on the floor and pushed against a vertical wall with a force F as shown in the figure. Draw the free body diagram.



Solution

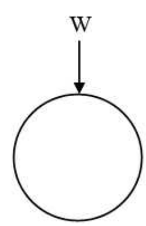
Step 1

Draw the body as it is by removing all the contact surfaces



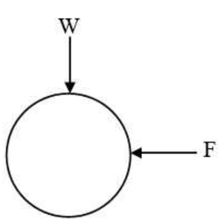
Step 2

Indicate the self-weight of the body, which acts vertically downward and passes through the centre of gravity of the body.



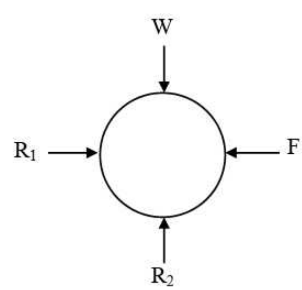
Step 3

Draw all the applied forces that are acting on the body

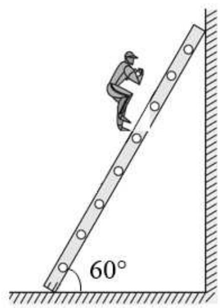


Step 4

Indicate the reactions at the point of contacts in a direction opposite and perpendicular to the contact surface.



Example 3.2 A man weighing 65 N is standing on a ladder resting on smooth floor and against a wall. The weight of the ladder is 15 N. Draw the free body diagram



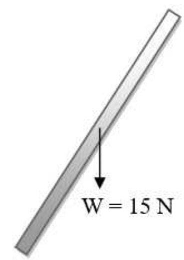
Solution

Step 1

Draw the body as it is by removing all the contact surfaces

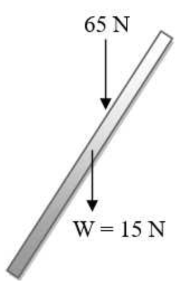
Step 2

Indicate the self-weight of the body, which acts vertically downward and passes through the centre of gravity of the body.



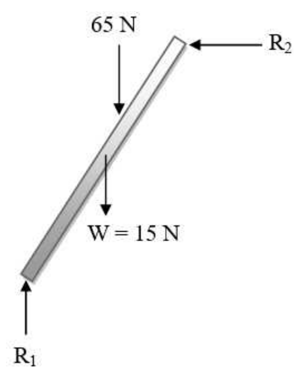
Step 3

Draw all the applied forces that are acting on the body



Step 4

Indicate the reactions at the point of contacts in a direction opposite and perpendicular to the contact surface.

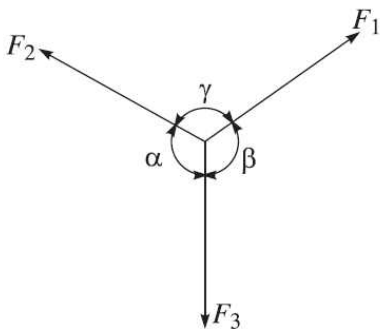


3.5 LAMI'S THEOREM

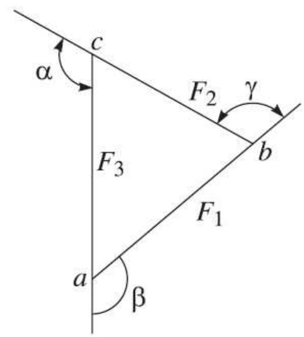
When a body is in equilibrium under the action of three forces, each force is proportional to the sine of the angle between other two forces. According to the triangle law, if three forces acting on a body are represented by the sides of a triangle as a closed triangle, then the body will be in equilibrium.

Let F_1, F_2, F_3 are the three forces are acting with the angles α, β and γ between them as shown in the figure. Then according to the Lami's theorem

$$\frac{F_1}{Sin\ \alpha} = \frac{F_1}{Sin\ \beta} = \frac{F_1}{Sin\ \gamma}$$



Proof From the above figure, three forces can be represented one after the other in magnitude and direction as shown in the figure. As the body is in equilibrium, the head of the last force shoud conicide with the tail of the first force.

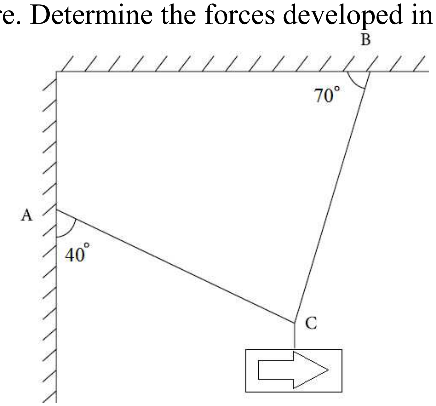


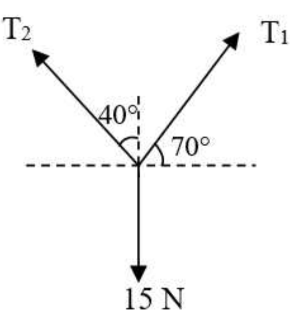
Applying sine rule for the triangle

$$\frac{F_1}{Sin\ (180-\alpha)} = \frac{F_1}{Sin\ (180-\beta)} = \frac{F_1}{Sin\ (180-\gamma)}$$

$$\frac{F_1}{Sin\ \alpha} = \frac{F_1}{Sin\ \beta} = \frac{F_1}{Sin\ \gamma}$$

Example 3.3 A Sign board of weight 15N is hung with the help of two cables as shown in the figure. Determine the forces developed in the cable?





Solution Let T_1 and T_2 be the tensions developed in the cables BC and AC respectively.

Applying the equilibrium conditions for the system of forces shown in the figure, we obtain

$\sum F_x = 0$

$T_1 \cos 70° - T_2 \cos 50° = 0$

$T_1 = 1.88\ T_2$ ---- (i)

$\sum F_y = 0$

$T_1 \sin 70° + T_2 \sin 50° - 15 = 0$

$T_1 \sin 70° + T_2 \sin 50° = 15$ ---- (ii)

Substituting Eq. (i) in Eq. (ii) we get

$1.88\, T_2 \sin 70° + T_2 \sin 50° = 15$

$T_2 = 5.92$ N

From Eq. (i)

$T_1 = 11.13$ N

As the system is in equilibrium under the action of only three forces, this problem can also be solved using Lami's theorem.

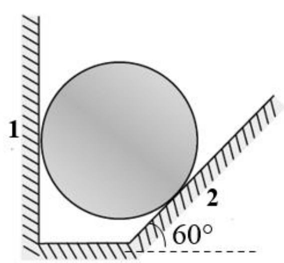
Applying Lami's theorem for the system of forces shown in the figure, we get

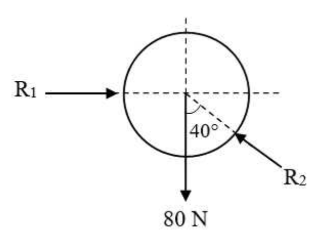
$$\frac{T_1}{\sin 140°} = \frac{T_2}{\sin 160°} = \frac{15}{\sin 60°}$$

$$T_1 = \frac{15}{\sin 60°} \times \sin 140° = 11.13 \text{ N}$$

$$T_2 = \frac{15}{\sin 60°} \times \sin 160° = 5.92 \text{ N}$$

Example 3.4 A sphere of 80 N is kept within an inclined plane and the vertical wall as shown in the figure. Assuming the surfaces are smooth, determine the reactions developed at the points 1 and 2.





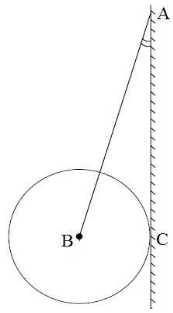
Solution Applying Lami's theorem for the system of forces shown in the FBD, we get

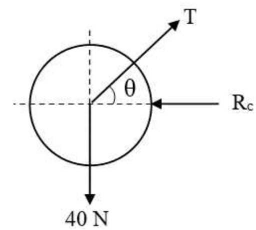
$$\frac{R_1}{\sin 140°} = \frac{R_2}{\sin 90°} = \frac{80}{\sin 130°}$$

$$R_1 = \frac{80}{\sin 130°} \times \sin 140° = 67.12 \text{ N}$$

$$R_2 = \frac{80}{\sin 130°} \times \sin 90° = 104.4 \text{ N}$$

Example 3.5 A ball of weight 40 N and radius 7 cm is attached by a string AB to a wall as shown in figure. Determine the tension in the string and reaction at point C if AC = 10cm.





Solution: Given Radius (BC) = 7 cm and AC = 10 cm

If θ is the angle made by the string w. r. t. x-axis, then

$$\tan\theta = \frac{AC}{BC} = \frac{10}{7}$$

$$\theta = \tan^{-1}\frac{10}{7} = 55°$$

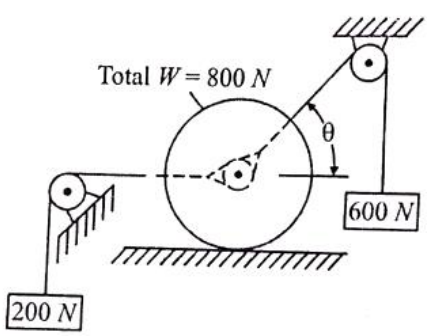
Let T be the tension developed in the string AB and R_C be the reaction at point C. Applying Lami's theorem for the system of forces shown in the figure, we get

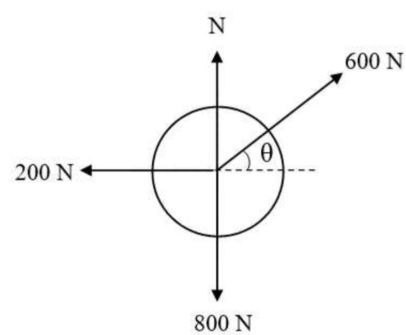
$$\frac{R_C}{\sin 145°} = \frac{T}{\sin 90°} = \frac{40}{\sin 125°}$$

$$R_1 = \frac{40}{\sin 125°} \times \sin 145° = 28\text{N}$$

$$R_2 = \frac{40}{\sin 125°} \times \sin 90° = 48.83\text{ N}$$

Example 3.6 Cords are looped around a small spacer separating two cylinders each weighting 400N and pass over frictionless pulleys to weights of 200 N and 600 N. Determine the angle θ and the normal reaction "N" between the cylinders and the smooth horizontal surface





Solution Considering the free body diagram of sphere

ΣFx= 0

600 cos θ - 200 –N sin 15 =0

600 cos θ = 0.259 N +200 ………….. (i)

$\Sigma Fy = 0$

$600 \sin \theta = 800 - N \cos 15 = 0$

$600 \sin \theta = 800 - 0.966\, N$ ………………(ii)

Squaring and adding of (i) & (ii), we get

$(600)^2 = (0.259N - 200)^2 + (800 - 0.966N)^2$

$(600)^2 = N^2 + (200)^2 + (800)^2 - 1442N$

$N^2 - 1442N + 320000 = 0$

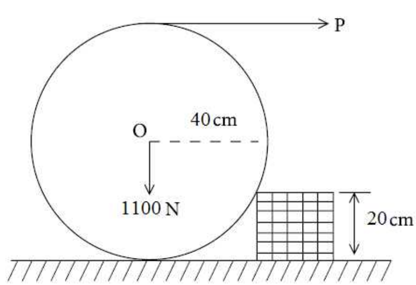
$N^2 = 1168.03$ N

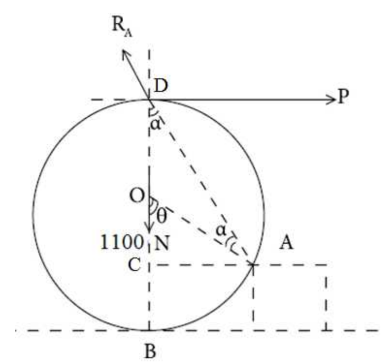
N = 273.96 N

Substituting N value in equation (i) we get

$\theta = 63.15°$

Example 3.7 A roller of weight 1100 N, radius 40 cm is required to be pulled over a brick of height 20 cm as shown in the Fig. by a horizontal pull applied at the end of a string wound round the circumference of the roller. Determine the value of P which will just tend to turn the roller over the brick.





Solution From the free body diagram, OM = 40-20 = 20 cm

OA = 40 cm

Let COA = θ

$\cos \theta = \frac{OM}{OA} = \frac{20}{40} = 0.5$

θ = 60º

DOA = 180 – θ = 180 – 60 = 120º

We know OD = OA= 40 cm

Let ODA = DAO = α

From the ΔODA, α + α + 120º = 180º

Therefore α = 30º

Considering the equilibrium of forces shown in the figure

$\sum F_x = 0$

$P - R_A \sin\alpha = 0$

$P = R_A \sin\alpha$

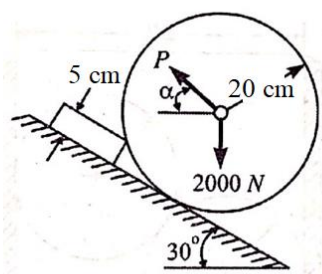
$\sum F_y = 0$

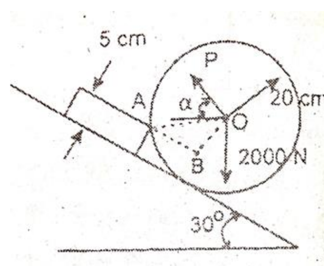
$R_A \cos\alpha - 1100 = 0$

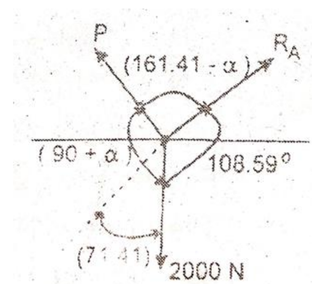
$R_A = \frac{1100}{\cos 30} = 1270.1$ N

Therefore P = 1270.1 sin 30 = 635.05 N

Example 3.8 Determine the amount and direction of the smallest force "P" required to start the wheel in fig. over the block?







Solution Consider ΔAOB

OA= 20 cm; OB = 20-5=15 cm

$\cos\theta = \frac{15}{20}$

$\theta = \cos^{-1}(\frac{15}{20}) = (41.41)°$

Applying Lami's theorem

$$\frac{P}{\sin 108.59} = \frac{2000}{\sin(161.41-\alpha)} = \frac{R_A}{\sin(90+\alpha)}$$

$$P = \frac{1895.65}{\sin(161.45-\alpha)}$$

For "P" to be min;

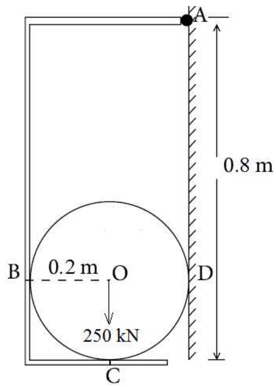
$\sin(161.41-\alpha) = \max = 1$

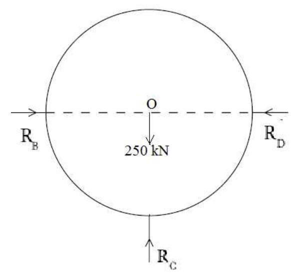
$161.41 - \alpha = 90°$

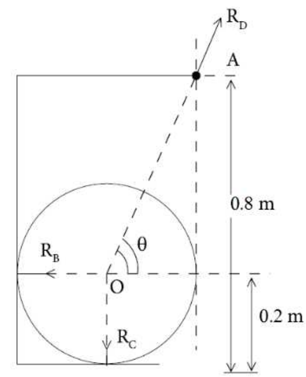
$\alpha = (71.41)° and\ P\ min = 1895.65\ N$

Also $R_A = \frac{2000 \times \sin(90+\alpha)}{\sin(161.41-\alpha)} = 637.58N$

Example 3.9 A 250 kN cylinder is supported by a frame ABC which is hinged to the wall at A as shown in the figure. Determine the reactions at the points A, B, C and D. Neglect the weight of the frame.







Solution Considering the free body diagram of the cylinder

$\sum F_x = 0$; $R_B = R_D$

$\sum F_y = 0$; $R_C = 250$

Considering the free body diagram of the frame

$\tan\theta = \frac{0.6}{0.2}$

$\theta = 71.56°$

$\sum F_x = 0$; $R_B = R_A \cos\theta$

$\sum F_y = 0$; $R_C = R_A \sin\theta$

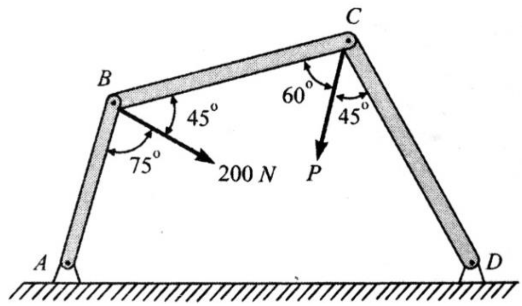
$250 = R_A \sin\theta$

$R_A = \frac{250}{\sin 71.56} = 263.53$ kN

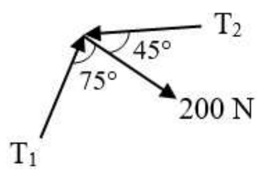
$R_B = 263.53 \cos 71.56 = 83.35$ kN

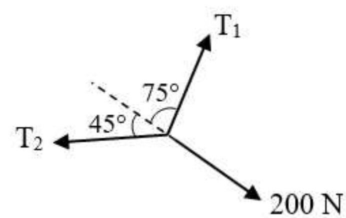
$R_D = 83.35$ kN (since $R_B = R_D$)

Example 3.10 Three bars pinned together at B and C and supported by hinges at A and D as shown in the figure form a four-link mechanism. Determine the value of P which is required to hold the system in equilibrium.



Solution The free body diagram of the system at joint B is shown in the figure





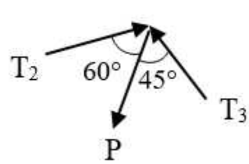
Applying Lami's equation for fig, we get

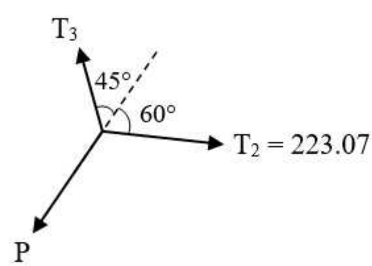
$$\frac{T_1}{\sin 135°} = \frac{T_2}{\sin 105°} = \frac{200}{\sin 120°}$$

$$T_1 = \frac{200}{\sin 120°} \times \sin 135° = 163.2\text{N}$$

$$T_2 = \frac{200}{\sin 120°} \times \sin 105° = 223.07\text{N}$$

The free body diagram of the system at joint A is shown in the figure





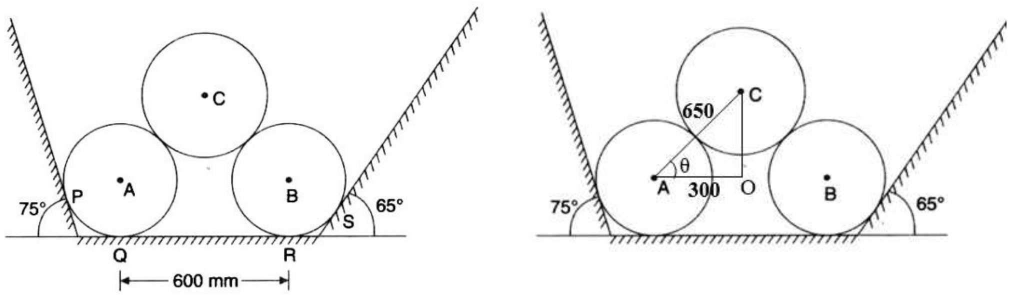
Applying Lami's equation for fig, we get

$$\frac{P}{\sin 105°} = \frac{T_3}{\sin 120°} = \frac{223.07}{\sin 135°}$$

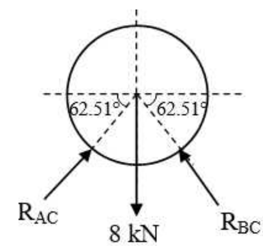
$$P = 304.7\text{N}$$

Example 3.11 Three spheres A, B, and C of diameters 500 mm, 500 mm and 800 mm, respectively are placed in a trench with smooth side walls and floor as shown in the figure. The center-to-center distance of spheres A and B is 600 mm. The cylinders A, B and C weigh

4 kN, 4 kN and 8 kN respectively. Determine the reactions developed at contact points P, Q, R and S.



Solution Considering the right-angle triangle AOC in the fig., we get



$$\cos\theta = \frac{AO}{AC} = \frac{300}{650}$$

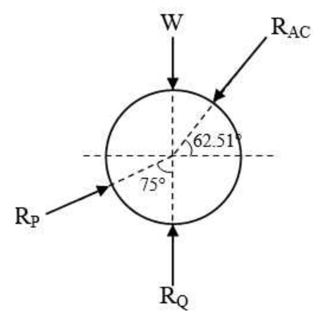
$\theta = 62.51°$

Applying Lami's theorem for the system of forces of sphere "C"

$$\frac{R_{AC}}{\sin 152.51°} = \frac{R_{BC}}{\sin 152.51°} = \frac{8}{\sin 54.98°}$$

$R_{AC} = 4.509$ N $R_{BC} = 4.509$ N

Consider the free body diagram of sphere "A"



$\sum F_x = 0$

$R_P \cos 15° - 4.509 \cos 62.5° = 0$

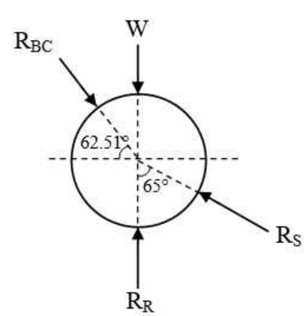
$R_P = 2.15$ N

$\sum F_y = 0$

$R_Q - 4 + 2.15 \sin 15° - 4.509 \sin 62.51° = 0$

$R_Q = 7.44$ N

Consider the free body diagram of sphere "B"



$\sum F_x = 0$

$4.509 \cos 62.51° - R_s \cos 25° = 0$

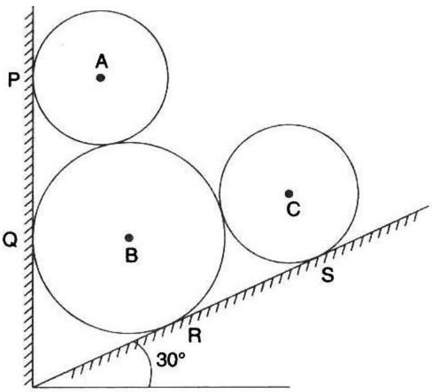
$R_s = 2.29$ N

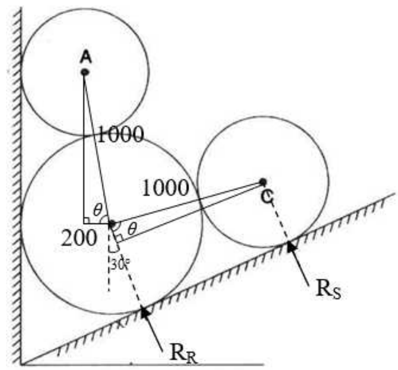
$\sum F_y = 0$

$R_R + 2.296 \sin 25° - 4 - 4.509 \sin 62.51° = 0$

$R_R = 7.03$ N

Example 3.12 Three smooth spheres A, B, and C weighing 300 N, 600 N and 300 N respectively and having diameters 800 mm, 1200 mm and 800 mm respectively are placed in a trench as shown in figure. Determine the reactions developed at contact points P, Q, R and S.





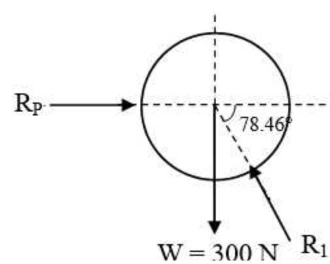
Solution: Angle of Reaction between sphere A and sphere B

$\cos\theta = \frac{200}{1000}$

$\theta = 78.46°$

(i) Sphere "A"

The free body diagram of sphere A is shown in the figure. The forces in the figure can also be arranged as shown in figure.



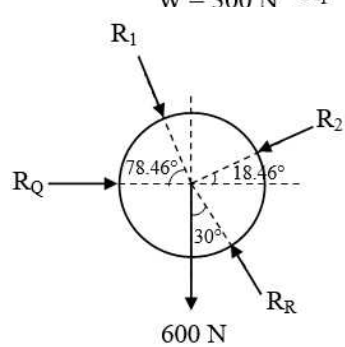
Applying Lami's equation for sphere A

$\frac{R_1}{\sin 90°} = \frac{R_P}{\sin 168.49°} = \frac{300}{\sin 101.54°}$

$R_1 = \frac{300}{\sin 101.54°} \times \sin 90° = 306.18$

$R_P = \frac{300}{\sin 101.54°} \times \sin 168.49° = 61.25$

Consider the FBD of sphere "B"



$\sum F_x = 0$

$R_{O_2} - R_R \cos 60° - R_2 \cos 18.46° + 306.18 \cos 78.46° = 0 \quad \ldots\ldots(1)$

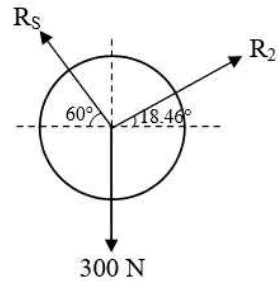
$\sum F_y = 0$

$R_R \sin 60° - R_2 \sin 18.46° - 600 - 306.18 \sin 78.46° = 0 \quad \ldots\ldots(2)$

In the above equation "two" variables are unknown.

So, solving sphere "C" We get R_S & R_2 value

Consider the FBD of sphere "C"



$$\frac{R_2}{\sin 150°} = \frac{R_S}{\sin(108.46°)} = \frac{300}{\sin(101.54°)}$$

$R_2 = 153.094$ N

$R_S = 290.4$ N

Substitute R_2 value in eqn (2)

We get $R_R = 1095.2$

Substitute R_R and R_2 value in eqn (1)

We get $R_Q = 631.5$

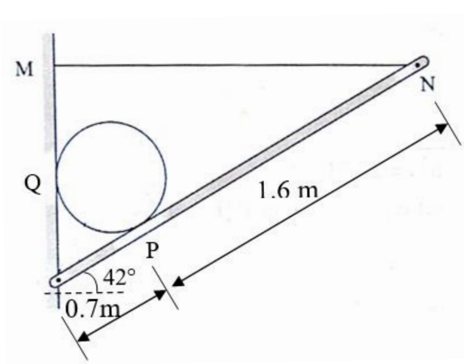
3.6 EQUILIBRIUM CONDITIONS FOR COPLANAR NON-CONCURRENT FORCES

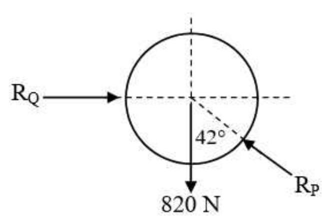
A body will be equilibrium under the action of coplanar non-concurrent forces if there is no translation and rotation of a body. That is the algebraic sum of all the external forces acting on the body should be zero and also the algebraic sum of moments of forces about any fixed point should be zero.
Mathematically,

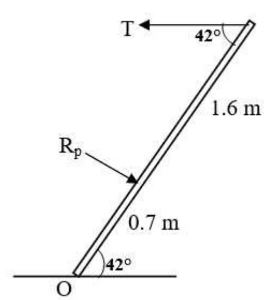
$$\Sigma F_x = 0$$
$$\Sigma F_y = 0$$
$$\Sigma M_a = 0$$

The above equations are the conditions of equilibrium for coplanar non-concurrent force system.

Example 3.13 A circular log of weight 820 N is supported by a bracket as shown in the figure. Determine the reactions at P and Q and the tension in the cable.







Solution Consider the FBD of sphere

$$\frac{R_P}{\sin 90} = \frac{R_Q}{\sin 138} = \frac{820}{\sin 132}$$

$$R_P = \frac{820}{\sin 132} \times \sin 90 = 1103.41 \text{ N}$$

$$R_Q = \frac{820}{\sin 132} \times \sin 138 = 738.33 \text{ N}$$

Consider the FBD of member MPN

$$\sum M_M = 0$$

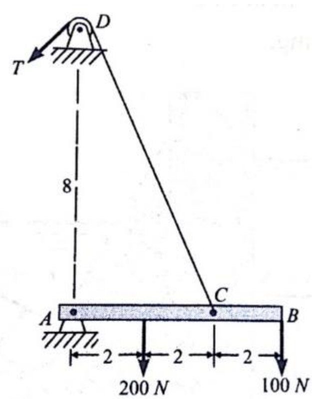
R_P x 0.7 – T cos 48 x 2.3 = 0

T = 501.87 N

Example 3.14 A beam AB is supported in a horizontal position by a hinge A and a cable which runs from C over a small pulley at D as shown in fig. Compute the tension T in the cable and the horizontal and vertical components of the reaction at A. Neglect the weight of the beam and the size of pulley at D.

Solution tan θ = $\frac{8}{4}$

θ = $\tan^{-1}(2)$ = 63.43°



Applying conditions of equilibrium

ΣM_A= 0 + (200×2) - (T sin 63.43°×4) + (100×6) = 0

T=279.508 N

ΣF_x = 0

R_{AH} =T cos 63.43°

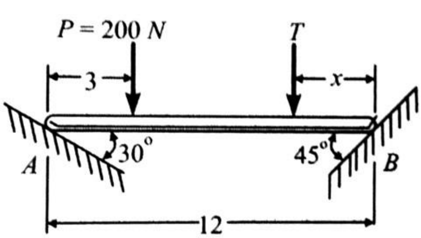
R_{AH} =125.02 N

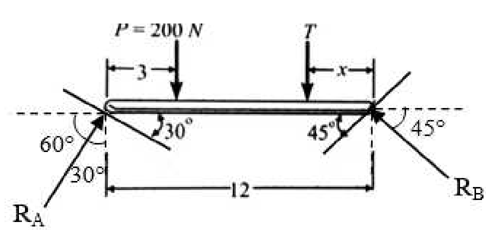
ΣF_y=0

R_{AB}-200+T sin 63.43° -100 = 0

R_{AV} = 50.01N

Example 3.15 A 12 m bar of negligible weight rests in a horizontal position on the smooth inclines as shown in the figure 5. Compute the distance *x* at which load T=100 N should be placed from point B to keep the bar horizontal





Solution $\sum F_x = 0$

$R_A \cos 60 - R_B \cos 45 = 0$

$0.5R_A = 0.707R_B$

$R_A = 1.414R_B$

$\sum F_y = 0$

$R_A \sin 60 - 200 - 100 + R_B \sin 45 = 0$

Substituting R_A value in above eqn

$(1.414R_B \times 0.866) - 200 - 100 + (R_B \times 0.707) = 0$

$R_B = 155.32 \text{ N}$

$R_A = 219.622 \text{ N}$

Taking moment about point A

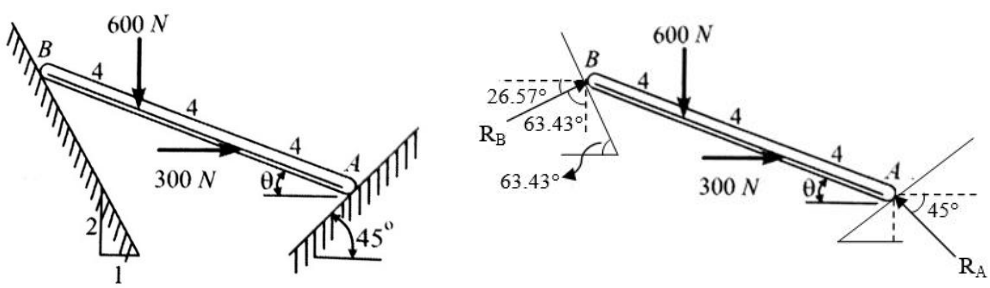
$\sum M_A = 0$

$(200 \times 3) + [T \times (12 - X)] - [R_B \sin 45 \times 12] = 0$

$600 + 100(12 \times x) - 155.32 \times 0.707 \times 12 = 0$

$X = 4.82 \text{ m}$

Example 3.16 Bar AB of negligible weight is subjected to a vertical force of 600 N and a horizontal force of 300N applied as shown in the figure 6. Find the angle Θ at which equilibrium exists. Assume smooth inclined surfaces.



Solution $\sum F_x = 0$

$R_B \cos 26.57 + 300 - R_A \cos 45 = 0$

$0.894R_B - 0.707R_A = -300 \quad \ldots\ldots \text{(i)}$

$\sum F_y = 0$

$R_B \sin 26.57 - 600 + R_A \sin 45 = 0$

$0.447R_B + 0.707R_A = 600 \quad \ldots\ldots \text{(ii)}$

Solving eqn (i) and (ii) $1.341R_B = 300 \quad \rightarrow \boxed{R_B = 223.678\text{ N}}$

Taking moment about point A: $\sum M_A = 0$

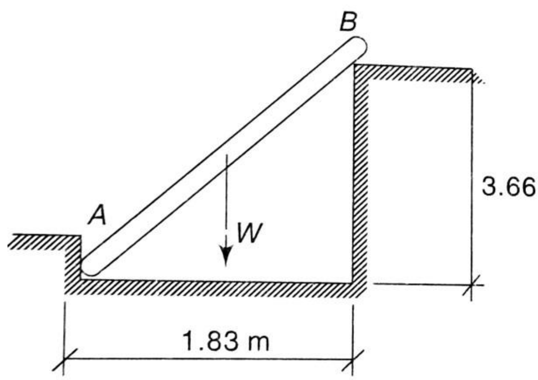
$300 \times 4\sin\theta - 600 \times 8\cos\theta + R_B\cos 26.57 \times 12\sin\theta + R_B\sin 26.57 \times 12\cos\theta$

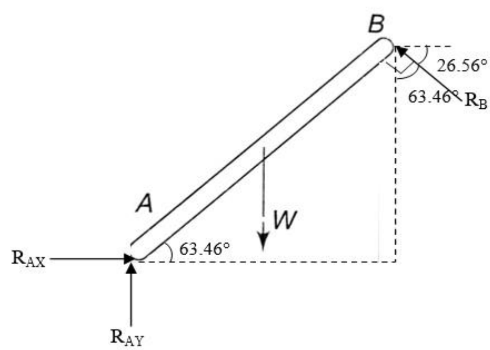
$1200\sin\theta - 4800\cos\theta + 2400.66\sin\theta + 1200.58\cos\theta = 0$

$3600.66\sin\theta = 3599.42\cos\theta$

$\boxed{\theta = 45°}$

Example 3.17 A 667.5 N man stands on the middle rung of a 222.5 N ladder, shown in figure below. Assuming the end B rests on the corner of a wall and a stop at A to prevent slipping, find the reactions at A and B.





Solution $\sum F_x = 0; \qquad R_{AX} - R_B\cos 26.56 = 0$

$R_{AX} = R_B\cos 26.56 \quad \ldots\ldots \text{(i)}$

$\sum F_y = 0; \quad R_{Ay} - 667.5 - 222.5 + R_B\sin 26.56 = 0$

$R_{Ay} = 890 - R_B\sin 26.56 \quad \ldots\ldots \text{(ii)}$

Taking moment about A

$\sum M_A = 0$

$667.5 \times 0.915 + 222.5 \times 0.915 - R_B \times 4.092 = 0$

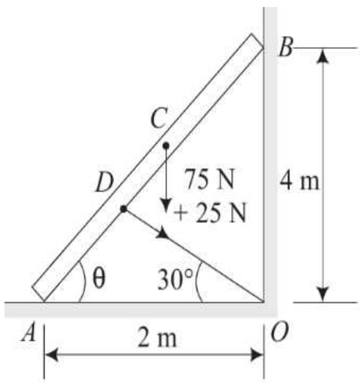
$R_B = \frac{814.35}{4.092} = 199\text{N}$

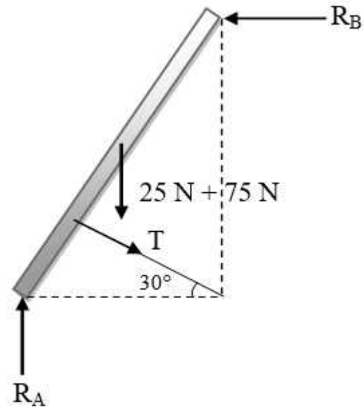
Sub R_B in eqn (i) and (ii)

$R_{Ax} = 179\text{ N}$

$R_{Ay} = 801\text{ N}$

Example 3.18 A man weighing 75 N stands on the middle rung of a 25 N ladder resting on smooth floor and against a wall. The ladder is prevented from slipping by a string OD. Find the tension in string & reactions at A and B as shown in the figure





Solution $\tan\theta = \frac{OB}{OA} = \frac{4}{2} = 2$

$\theta = 63.43°$

$\sum F_x = 0$

$T\cos 30° - R_B = 0$

$R_B = T\cos 30°$

$\sum f_y = 0$

$R_A - 100 - T\sin 30° = 0$

$R_A = T\sin 30° + 100$

Taking moment about A

$\sum M_A = 0$

$100\,[1] + T\sin 30\,[2] - R_B[4] = 0$

$4T\cos 30° - 2T\sin 30° = 100$

$T = \frac{100}{2.464} = 40.58\text{N}$

$R_A = T\sin 30° + 100 = 40.58(\sin 30°) + 100 = 120.29\text{N}$

$R_B = T\cos 30° = 40.58(\cos 30°) = 35.14\text{ N}$

SUMMARY

- Equilibrium is a stable condition of a body, in which the forces acting on a body has a zero resultant.
- Equilibrant is a single force which bring the system to equilibrium.
- The representation of a body by all forms of forces acting on it when the body is detached from all contact surfaces is called Free Body Diagram
- Lami's theorem states that, when a body is in equilibrium under the action of three forces, each force is proportional to the sine of the angle between other two forces.

EXERCISES

I. Self-Assessment Questions

1. Define the term equilibrium.
2. State the equilibrium conditions for coplanar concurrent forces.
3. What is free body diagram? Explain with an example.
4. Draw the free body diagram for a ball floating on water.
5. What is the difference between resultant and equilibriant?
6. State and prove Lami's theorem.
7. State the equilibrium conditions for coplanar non-concurrent forces.

II. Multiple Choice Questions

1. The minimum number of forces to act on the body to keep the body under equilibrium,
   a) 1
   b) 2
   c) 3
   d) 4
2. According to Lami's Theorem, the three forces.
   a) Must be equal.
   b) Must be at 120^0 to each other.
   c) Must be equal and act at 120^0 to each other
   d) Need not be equal and need not act at 120^0 to each other
3. An electric light fixture weighing 15N hangs from points C, by two wires AC and BC. The wire AC is inclined at 60^0 to the roof and wire BC at 45^0 with wall. Then the tension in AC is
   a) 7.76N
   b) 10.98N
   c) 7.5N
   d) 12N
4. If the resultant of a number of forces acting on a body is_______, then the body will be in equilibrium.

a) One
b) Zero
c) Positive
d) Negative

5. In a Free Body Diagram, which of the following is not shown
   a) Forces
   b) Reactions
   c) Contact Surfaces
   d) Weight
6. Which of the following forces is always taken downwards?
   (a) Normal Reaction (b) Weight (c) Friction (d) Centripetal

Answers

1. b
2. d
3. b
4. b
5. c
6. b

CHAPTER – 4

FRICTION

LEARNING OBJECTIVES

After studying this chapter, you should be able to

- *Explain the concept of friction, limiting friction and coefficient of friction*
- *Understand the useful and harmful effects of friction*
- *Distinguish between angle of repose and angle of friction*
- *Evaluate the problems involving blocks, ladders and wedges*

4.1 FRICTION

Friction is a force of restriction of movement or motion of a body when it slides over another body. Whenever the surfaces of two bodies are in contact and slide one over the other, their motion will be always restricted by a force. This force is called frictional force. This is due to the interlocking of irregularities between the contact surfaces. Even though the surfaces appear as smooth and soft, there is always some roughness which can be seen at micro scale as shown in the figure 4.1. The interlocking of these minutely projecting particles results in friction.

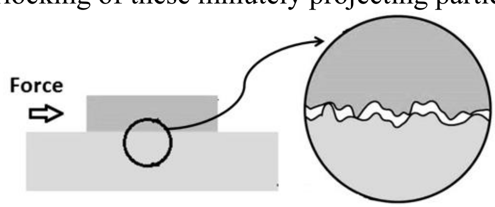


Fig. 4.1 Irregularities between the surfaces

4.2 LIMITING FRICTION

The friction exerted when a body is about to move or it is just on the point of moving over another body is called as *limiting friction.* It can also be stated as the maximum value of frictional force when a body just tends to move.

Consider a block A as shown in the figure 4.2. Whenever a force (P) is applied to the block, force of friction (F) is developed between the block and the surface which acts opposite to that in which the body tends to move. As long as the force "P" is small the block will not start moving. At this stage P = F. When the magnitude of "P" is gradually increased, the force of friction "F" also increases gradually but upon a limit, beyond which there is no increase in the magnitude of "F". At this condition, the magnitude of the frictional force when the block is about to start is known as *Limiting Friction.*

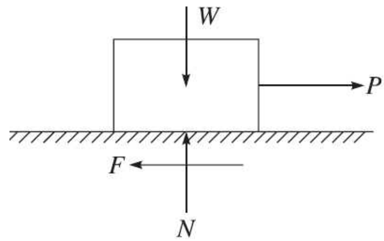


Fig. 4.2

4.3 DRY FRICTION OR COULOMB FRICTION

If the surfaces of two bodies in contact are unlubricated, the friction developed between them is known as dry friction. This is also refereed as coulomb friction, since C.A. Coulomb studied its characteristics in 1781.

4.3.1 TYPES OF DRY FRICTION

1. ***Static friction***

Static friction is the friction that exists when the body remains at rest. This condition arises when the applied force is less than the limiting friction. Static friction may have any value between zero and limiting friction.

2. ***Dynamic friction***

Dynamic friction is the friction that exists when the body starts moving over another body. This condition arises when the applied force exceeds the limiting friction. The magnitude of dynamic friction is found to be less than limiting friction. Based on the nature of the movement of the body, dynamic friction is divided into two types. They are,

a. *Sliding friction:* It is the friction experienced when a body slides over another body.
b. *Rolling friction:* It is the friction experienced when a body rolls over another body.

4.4 USEFUL AND HARMFUL EFFECTS OF FRICTION

1. Friction is necessary for starting, moving and stopping of vehicle. Even walking of man depends on the friction between the shoe and the ground.
2. Friction is harmful in machines. Friction between the brake shoe and drum generates heat as a result the temperature of both brake shoe and drum are increased.

4.5 COEFFICIENT OF FRICTION (C.O.F)

It is defined as the ratio of the limiting friction to the normal reaction. It is denoted by the letter μ.

$$\text{Coefficient of friction } \mu = \frac{F}{N}$$

$$\text{(or)} \qquad F = \mu N$$

Where F is the limiting friction and N is the normal reaction between the contact surfaces. The value of this coefficient of friction (c.o.f) depends on the type and nature of the materials in contact. A low value of coefficient of friction indicates that the force required to move an object is less that the object which has high coefficient of friction. For example, ice on stainless steel has a low coefficient of friction, rubber on wood.

There are essentially two kinds of coefficients; static and kinetic. ***Coefficient of static friction (μ_s)*** describes the ratio of the limiting friction to the normal reaction when the body is not moving. ***Coefficient of dynamic friction (μ_d)*** describes the ratio of the limiting friction to the normal reaction when the body is moving. Coefficient of dynamic friction is less than about 20% to 25 % than the coefficient of static friction.

Table 4.1: Approximate value of coefficient of static friction

Metal on metal	0.15 – 0.20
Wood on wood	0.25 – 0.50
Stone on stone	0.40 – 0.70
Metal on wood	0.20 – 0.60
Metal on stone	0.30 – 0.70

4.6 LAWS OF DRY FRICTION or COULOMB'S LAWS OF FRICTION

1. The frictional force acts in a direction opposite to that in which the body tends to move.
2. The magnitude of frictional force and applied force are same until the limiting value is reached.
3. The magnitude of the limiting friction bears a constant ratio to the normal reaction between the two contacting surfaces.
4. The frictional force is independent of contact area between two surfaces
5. The force of friction depends upon the nature of the surfaces in contact.

4.7 ANGLE OF FRICTION

The frictional force and normal reaction can be combined into a resultant force as shown in the figure 4.3. The angle which the resultant makes with the normal reaction is called angle of friction denoted by θ.

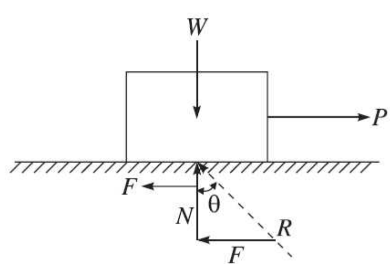


Fig. 4.3

From the figure

$$\tan\theta = \frac{F}{N}$$

(Where F is frictional force and N is normal reaction)

If the frictional force is the limiting friction, then the angle made by the resultant with the normal reaction is called angle of limiting friction denoted by α. Then

$$\tan\alpha = \frac{F}{N} = \mu$$

(Where F is limiting friction and N is normal reaction)

4.8 ANGLE OF REPOSE

It is the maximum inclination of the plane on which a body can rest without sliding down.

Consider a block resting on a plane inclined at θ as shown in the figure 4.4. As long as the magnitude of θ is small, the block will be at static equilibrium i.e., at rest. When θ is increased gradually, at a certain stage the block starts moving down the plane. This angle θ when the block is about to start moving down is called as angle of repose.

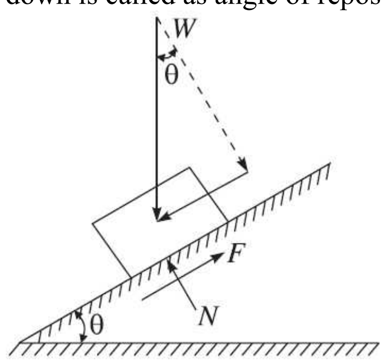


Fig. 4.4

If θ is the initial angle at which the block is at rest, then

$$\tan\theta = \frac{F}{N}$$

If φ is the angle at which the block is about to move, frictional force will be limiting friction and thence

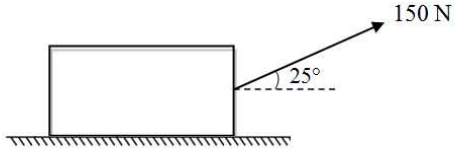
$$\tan\varphi = \frac{F}{N} = \mu$$

$$= \tan\alpha$$

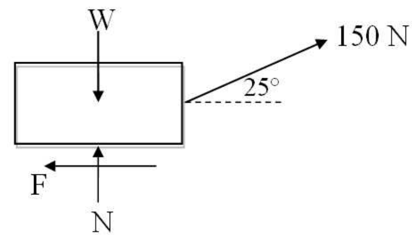
$$\varphi = \alpha$$

Thus, angle of repose is same as angle of limiting friction.

Example 4.1 A block shown in Figure is just moved by a force of 150 N. The weight of the block is 600 N. Determine the coefficient of static friction between the block and the floor.



Solution Considering the free body diagram of the block



$\sum F_y = 0$

$N - W + 150 \sin 25° = 0$

$N = 600 + 150 \sin 25° = 663.4$ N

$\sum F_x = 0$

$150 \cos 25° - F = 0$

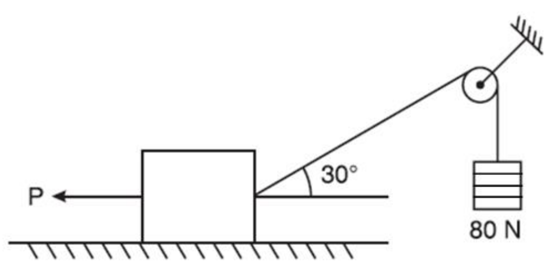
$F = 135.9$ N

We know, $\mu = \frac{F}{N}$

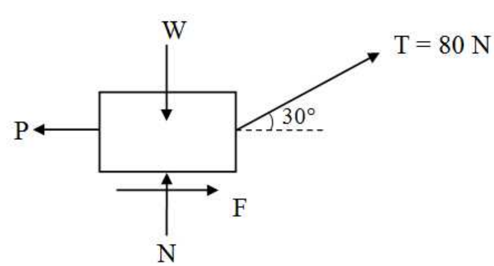
$\mu = \frac{135.9}{663.4}$

$\mu = 0.2$

Example 4.2 A block weighing 280 N is attached to a string which passes over a frictionless pulley and supports a weight of 80 N as shown in figure. The coefficient of friction between the block A and the horizontal plane is 0.32. Determine the value of P if motion is impending towards the left.



Solution Considering the free body diagram of the block



$\sum F_y = 0$

$N - W + 80 \sin 30° = 0$

$N = 280 + 80 \sin 30° = 320$ N

We know, $F = \mu N = 0.32 \times 320$

$F = 102.4$ N

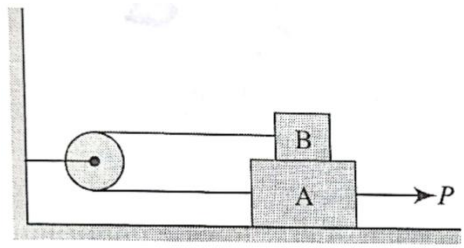
$\sum F_x = 0$

$80 \cos 30° + F - P = 0$

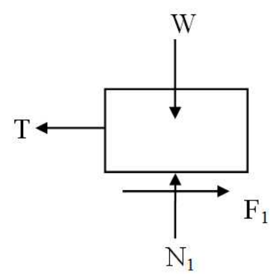
$P = 80 \cos 30° + 102.4$

P = 171.68 N

Example 4.3 Block A weighing 520 N is resting on a rough floor supporting block B weighing 260 N as shown in the figure. Block A and block B are connected with a string passing over a smooth pulley. Determine the magnitude of force P at impending motion and the tension induced in the string, if the coefficient of friction for all contact surfaces is 0.28.



Solution Consider the FBD of block B



$\sum F_y = 0$

$N_1 - W = 0$

$N_1 = 260$ N

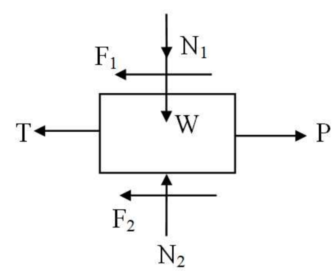
We know, $F_1 = \mu N = 0.28 \times 260 = 72.8$ N

$\sum F_x = 0$

$F_1 - T = 0$

$T = 72.8$ N

Consider the FBD of block A



$\sum F_y = 0$

$N_2 - N_1 - W = 0$

$N_2 = N_1 + W = 260 + 520 = 780$ N

$F_2 = \mu N_2$

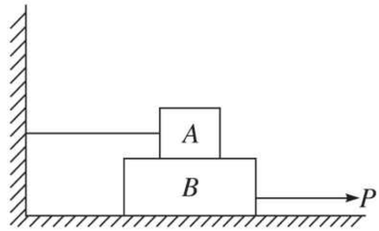
$F_2 = 0.28 \times 780 = 218.4$ N

$\sum F_x = 0$

$P - T - F_1 - F_2 = 0$

$P = 72.8 + 72.8 + 218.4 = 364$ N

Example 4.4 Block A of weight 600 rests over block B weighing 1200 N as shown in the figure. Block A is attached to a vertical wall by a horizontal string. If the coefficient of friction between *A* and *B* is 0.24 and between *B* and the floor is 0.32, what should be the value of *P* to move the block B?



Solution Consider the FBD of block A

$\sum F_y = 0$

$N_1 = 600$ N

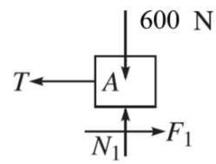
We know, $F = \mu N$

$F_1 = 0.24 \times 600 = 144$ N

$\sum F_x = 0$

$F_1 = T$

$T = 144$ N



Consider the FBD of block B

$\sum F_y = 0$

$N_2 - N_1 - 1200 = 0$

$N_2 = 600 + 1200 = 1800$ N

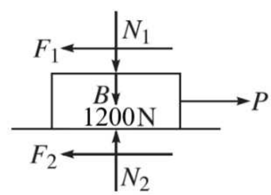
We know, $F = \mu N$

$F_2 = 0.32 \times 1800 = 576$ N

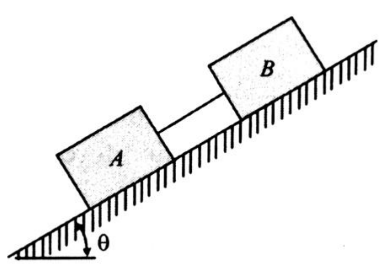
$\sum F_x = 0$

$P - F_1 - F_2 = 0$

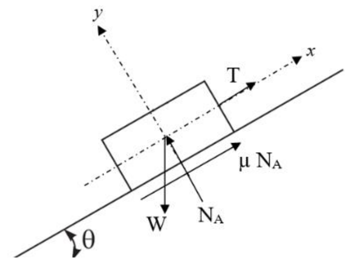
$P = 144 + 576 = 720$ N



Example 4.5 Bodies A and B are joined by a cord parallel to the inclined plane shown in figure. Under body 'A' which weights 200 N; μ=0.2 while μ=0.5 under body B which weights 300 N. Determine the angle ϴ at which motion impends. What is then the tension in the cord?



Solution Consider FBD of block A



$\Sigma F_x = 0$

$T + 0.2\ N_A - 200 \sin\theta = 0$ ----(1)

$\Sigma F_y = 0$

$N_A - 200 \cos\theta = 0$

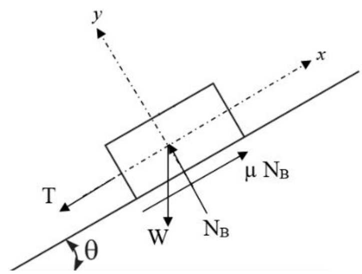
$N_A = 200 \cos\theta$

Substituting N_A value in the equation 1, we get

$T + 0.2\ (200 \cos\theta) - 200 \sin\theta = 0$

$T = 200 \sin\theta - 40 \cos\theta$ ---(2)

Consider FBD of block 'B'



$\Sigma F_x = 0$

$0.5\ N_B - 300 \sin\theta - T = 0$ ---(3)

$\Sigma F_y = 0$

$\therefore N_B - 300 \cos\theta = 0$

$\therefore N_B = 300 \cos\theta$

Substituting N_B value in the equation 3, we get

$T = 0.5\ (300 \cos\theta) - 300 \sin\theta$ ----(4)

Equating (2) and (4)

$200 \sin\theta - 40 \cos\theta = 150 \cos\theta - 300 \sin\theta$

$500 \sin\theta = 190 \cos\theta$

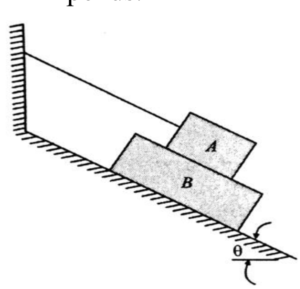
$\tan\theta\ 190/500$

$\therefore \theta = (20.806)^0$

Substitute θ value in equation 2, we get

$T = 200 \sin 20.806 - 40 \cos 20.806 = 33.65$ N.

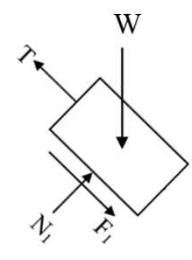
Example 4.6 Block A in figure weights 120 N, block B weighs 200 N and the cord is parallel to the incline. If coefficient of friction for all surface in contact is 0.25, determine the angle θ of the incline for which motion of B impends.



Solution Consider FBD of block A

$\Sigma F_y = 0$

$N_1 = 120 \cos\theta$ ---(1)



$\Sigma F_x = 0$

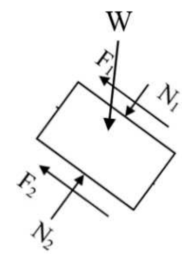
$120 \cos\theta + F_1 = T$

Using $F_1 = 0.25\ N_1 = 0.25 \times 120 \cos\theta$

$F_1 = 30 \cos\theta$

$120 \sin\theta + 30 \cos\theta = T$ ---(2)

Consider FBD of block B



$\Sigma F_y = 0$

$N_2 - 200 \cos\theta - N_2 = 0$

$N_2 - 200 \cos\theta - 120 \cos\theta = 0$

$N_2 = 320 \cos\theta$ ---(3)

$\Sigma F_x = 0$

$200 \sin\theta - F_1 - F_2 = 0$

$200 \sin\theta - 30 \cos\theta - 0.25 N_2 = 0$

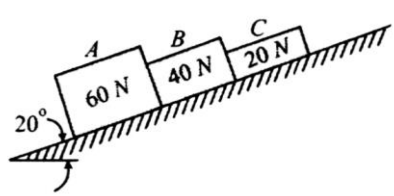
$200 \sin\theta - 30 \cos\theta - 0.25\ (320 \cos\theta) = 0$

$200 \sin\theta = 110 \cos\theta$

$\tan\theta = 110/200$

$\theta = 28.81^0$

Example 4.7 The three blocks with weights as shown in figure are placed on a 20^0 incline so that they are in contact with each other and at rest. Determine which, if any, of the blocks will move and the friction force acting under each. Assume that under blocks ‘A’ and ‘C’, the coefficients of friction are $\mu_s = 0.50$ and $\mu_k = 0.40$ while under B they are $\mu_s = 0.30$ and $\mu_k = 0.20$.



Solution Consider F.B.D of block ‘C’

As $\mu_s = 0.5$

$\varphi \therefore \tan\varphi = 0.5$

$\therefore \varphi = 26.56^0$ as $\varphi > 20^0$

$\therefore$ Block ‘C’ is at rest and not producing any force on block ‘B’.

∴ Frictional force=20 sin 20=6.84 N.

Consider F.B.D of block 'B'

As μ_s= 0.3

tan φ = 0.3

φ =16.7^0 as φ < 20^0

∴ Block 'B' is in motion.

∴ It is producing some force on 'A'

ΣF_y =0

N_B = 40 cos 20 = 37.587 N

Now, frictional force=0.3 N_B = 11.276N

To calculate force acting on 'A',

Net force acting alone the plane = 40 sin 20 – F.F

= 13.681-11.276 = 2.405 N.

F.B.D of block 'A'

μ_s=0.5

ΣF_y = 0

N_A= 60 cos 20 = 56.3815 N

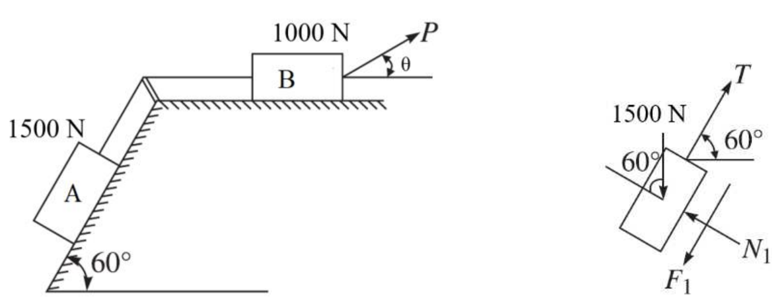
∴ Limiting F.F =0.5 × N_A= 28.1907 N

Forcing acting down the plane

= 60 sin 20 + 2.405 = 22.926 N

As 220.906 < 28.1907 => The Body 'A' remains at rest and F.F acting under block 'A' will be 22.926 N.

Example 4.8 Referring to the figure below determine the least value of the force 'P' to cause motion to impend rightward. Assume the coefficient of friction under the blocks to be 0.2 and the pulley to be frictionless.



Solution: Given μ=0.20

Consider FBD of Block A

$$\sum F_y=0$$

N_1-1500 cos 60°=0

$$N_1=1500\times\frac{1}{2}=750N$$

$F_1=0.2N_A=0.2\times750=150N$

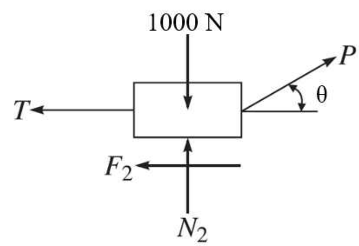
$$\sum F_x=0$$

$$T-F_1-1500\sin 60=0$$

$$T=F_1+1500\sin 60$$

$$=150+1299=1499N$$

Consider FBD of Block B



$$\sum F_y=0$$

$$N_2-10000+P\sin\theta=0$$

$$N_2=1000-P\sin\theta$$

$$\sum F_x=0$$

$$-T+P\cos\theta-F_2=0$$

$$F_2=\ \mu N_2=0.2\ (1000-P\sin\theta)$$

$$\therefore -T+P\cos\theta-0.2\ (1000-P\sin\theta)=0$$

$$-1449+P\cos\theta+0.2P\sin\theta-200=0$$

$$p(\cos\theta+0.2\sin\theta)=0$$

$$p=\frac{1649}{\cos\theta+0.2\sin\theta}$$

For value of P to be least, the denominator should have max. value.

$$\frac{d}{d\theta}(\cos\theta+0.2\sin\theta)=0$$

$$\therefore \qquad -\sin\theta+0.2\cos\theta=0$$

$$\therefore \qquad \frac{\sin\theta}{\cos\theta}=0.2=\tan\theta$$

$$\therefore \qquad \theta=\tan^{-1}0.2=11°20'$$

Hence least value of P is

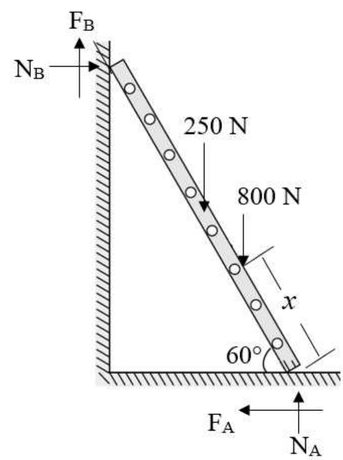
$$P_{least}=\frac{1649}{\cos(11°20')+1\ 0.2\sin(11°20')}$$

$$=\frac{1649}{-98+.059}=1611.67N$$

Example 4.9 A ladder 5m long and of 250N weight is placed against a vertical wall in a where its inclination to the vertical is 30. A man weighting 800N climbs the ladder. At what position will he induce slipping? The co-efficient of friction for both the contact surface of the ladder viz. with the wall and the floor is 0.2

Solution Length of ladder = 3m

Coefficient of friction on all surfaces μ = 0.2



$F_A = 0.2N_A$

$F_B = 0.2N_B$

To find ‘x’ to impend slipping

$\sum F_x = 0$

$N_B - F_A = 0$

$N_B = 0.2N_A$

or $\quad N_A = \frac{N_B}{0.2}$

$\sum F_y = 0$

$F_B + N_A - W - 800 = 0$

$0.2N_B + N_A = 800 + 250 = 1050$

$N_B\left(0.2 + \frac{1}{0.2}\right) = 1050$

$\frac{1.04}{0.2}N_B = 1050$

$N_B = \frac{1050 \times .2}{1.04} = 201.92N$

$N_A = \frac{201.92}{.2} = 1009.61N$

$\sum M_B = 0$

$-0.2N_A \times 2.60 + N_A \times 1.50$

$-250 \times 1.3 - 800(3 - x) = 0$

$-0.2 \times 1009.61 \times 2.60 + 1009.61 \times 1.5$

$= +250 \times 1.3 + 800 \times 3 - 800x$

$-525 + 1514.41 = -800x$

$-3250 + 1514.41 = -800x$

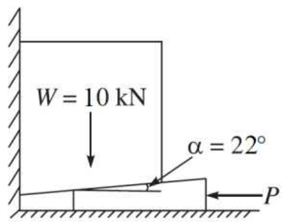
$+1735.60 = f800x$

$x = 2.17m$

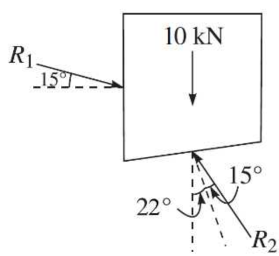
4.9 WEDGES

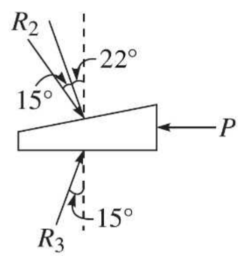
A wedge is a simple device which is used to produce small adjustments in the position of bodies and also used to raise heavy loads. Wedges are generally made of wood or metal having two non-parallel surfaces. The weight of the wedge is very small compared to the weight lifted. Hence, in all problems the self-weight of wedge is neglected. Wedges largely depend on friction to function. For the convenience of analysis, it is always advantageous to consider a single resultant force instead of normal reaction and frictional force.

Example 4.10 Determine force P required to start the movement of the wedge as shown in Fig. The angle of friction for all surfaces of contact is 15°.



Solution





Applying Lami's theorem to the system of forces on block

$$\frac{R_1}{\sin 143} = \frac{R_2}{\sin 75} = \frac{10}{\sin 142}$$

$R_1 = 9.77$ kN

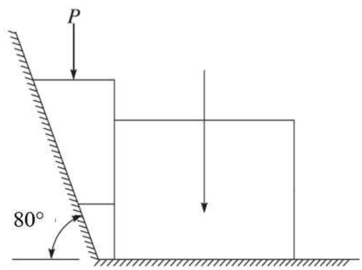
$R_2 = 15.68$ kN

Applying Lami's theorem to system of forces on the wedge

$$\frac{P}{\sin 128} = \frac{R_2}{\sin 105}$$

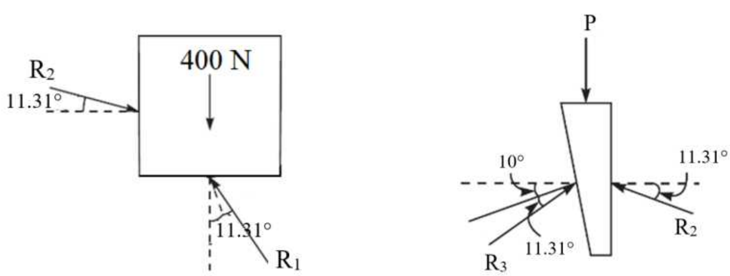
P = 12.79 kN

Example 4.11 A block of weight 400N is to be moved by a force P acting on the weightless wedge as shown in the figure. If the coefficient of friction for all contact surfaces is 0.2, find the value of P.



Solution: Given coefficient of friction is 0.2

$$\therefore \qquad \emptyset = \tan^{-1}(0.2) = 11.31°$$



Applying Lami's theorem to the system of forces on block

$$\frac{R_1}{\sin 78.69} = \frac{R_2}{\sin 168.69} = \frac{400}{\sin 112.62}$$

$$R_1 = 424.92 \text{ N}$$

$$R_2 = 85 \text{ N}$$

From FBD of wedge

$$\frac{P}{\sin 147.38} = \frac{R_2}{\sin 111.31}$$

$$P = 49.19 \text{ N}$$

4.10 SCREW JACK

A screw jack is a device which is used to lift or hold heavy loads. It consists of a square threaded screw which rotates in a nut. The top surface of the screw supports the load which is to be lifted or lowered. The friction between the external threads of a screw and the internal threads of a nut helps in raising or lowering the load.

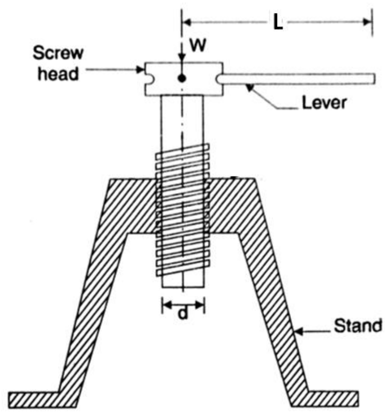
4.10.1 TERMS IN SCREW JACK

1. ***Lead of the screw:*** It is the axial distance moved by the screw when it makes one complete revolution.
2. ***Pitch:*** It is the axial distance measured between two consecutive threads.

If the screw is single threaded, then lead of the screw is equal to the pitch.

4.10.2 FORCE REQUIRED TO LIFT A LOAD

Consider a screw jack consisting of square threaded screw and a base.



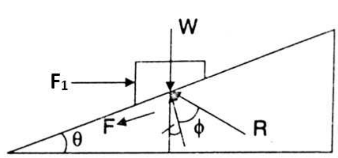
Let, L = the length of the lever
d = mean diameter of the screw
W = load to be lifted
F = force applied at the end of the liver
F_1 = equivalent force at the screw.
If the screw rotates one complete revolution, it raises a height equal to the pitch. This can be treated with that of inclined plane as shown in the figure

$$\tan \theta = \frac{p}{\pi d}$$

Forces F_1 and F are related by the following relation

$$F \times L = F_1 \times \frac{d}{2}$$

$$F_1 = \frac{2FL}{d}$$



Writing the equation of equilibrium
$\Sigma Fx = 0$

$$F_1 - R \sin(\theta + \varphi) = 0$$

$$F_1 = R \sin(\theta + \varphi)$$

$$\Sigma F_y = 0$$

$$W - R\cos(\theta + \varphi) = 0$$

$$W = R\cos(\theta + \varphi)$$

$$\frac{F1}{W} = \frac{R\sin(\theta - \varphi)}{R\cos(\theta - \varphi)} = \tan(\theta + \varphi)$$

$$F_1 = W\tan(\theta+\varphi)$$

$$\text{But,} \quad F_1 = \frac{2FL}{d}$$

$$\frac{2FL}{d} = W\tan(\theta+\varphi)$$

$$F = \frac{d}{2L}W\tan(\theta+\varphi)$$

$$F = \frac{d}{2L}W\tan(\theta+\varphi)$$

$$F = \frac{d}{2L}W\frac{\tan\theta + \tan\varphi}{1-\tan\theta\tan\varphi}$$

Efficiency of the screw jack = $\frac{Ideal\ effort}{Actual\ effort}$

Ideal effort is one when friction is zero.

F under ideal condition is $F = \frac{d}{2L}W\frac{\tan\theta + 0}{1-\tan\theta \times 0}$

Since $\mu = 0$, then $\tan\varphi = 0$.

F under actual condition $\frac{d}{2L}W\tan(\theta+\varphi)$

Efficiency = $\frac{\tan\theta}{\tan\theta + \varphi}$

4.10.3 CONDITION FOR SELF-LOCKING

In a screw jack, if there is no friction between the nut and screw, the load automatically lowers after the removal of applied force. The term self-locking indicates that the frictional force between the screw and nut is enough to support the load i.e., the removal of applied force will not unwind the screw to lower the load.

Example 4.12 The pitch of a screw jack is 10 mm and length of lever is 500 mm. The mean distance of threads is 60 mm. Find the force required at the end of lever if 30 kN load is (i) Lifted (ii) Lowered. The coefficient of friction is 0.10.

Solution Given pitch =10 mm

Lever length L = 500mm

Mean radius r =30mm

Load on screw jack W = 30kN

Now, $\tan\theta = \frac{p}{\pi d} = \frac{10}{60\pi} = 0.053$

$\theta = \tan^{-1}(0.053) = 3.307°$

$\tan\varphi = 0.10$

$\therefore \quad \varphi = \tan^{-1}(0.1) = 5.71°$

$\therefore \tan(\theta+\varphi) = \tan(3.307+5.71)$

$= 0.1538$

$\tan(\varphi-\theta) = \tan(5.71-3.307) = 0.0467$

Force in lifting load:

We know, F = $\frac{d}{2L}$W tan (θ+φ)

$= \frac{60}{2\times500} \times 30{,}000 \times 0.1538$

$= 276.8$ N

Force in lowering Load:

F = $\frac{d}{2L}$W tan (θ−φ)

$\frac{60}{2\times500} \times 30{,}000 \times 0.0467$

$= 84$ N

Example 4.13 A square threaded screw jack 75 mm mean diameter and 15 mm pitch is required to lift a load of 500 N. The coefficient of friction is 0.075. Find force to be applied if lever arm is of 400 mm long. Determine if the jack is self-locking.

Solution Given Pitch = 15 mm

Mean diameter = 75 mm

Load to be lifted W = 500 N

Coefficient of friction μ= $\tan\varphi = 0.075$

Length of the lever L = 400 mm

$\tan\theta = \frac{P}{\pi d} = \frac{15}{75\pi} = 0.0636$

$\theta = \tan^{-1}(0.0636) = 3.64°$

$\tan\varphi = 0.075$

$\varphi = \tan^{-1}(0.075) = 4.289°$

$\tan(\theta + \varphi) = \tan(3.64 + 4.289) = 0.139$

Force applied F = $\frac{d}{2L}$ W tan (θ + φ)

$= \frac{75}{2\times400} 500 \times\ 0.139 = 6.529\text{N}$

Efficiency = $\frac{\tan\theta}{\tan\theta + \varphi}$

$= \frac{0.0636}{0.139} \times 100$

$\eta = 45.75\%$

The jack is self-locking since efficiency is less than 50%

Example 4.14 Outside diameter of a square threaded spindle of screw jack is 40 mm. The screw pitch is 10 mm. If the coefficient of friction between the screw and the nut is 0.15, neglecting friction between the nut and collar, determine

i. Force to be applied at the end of 450 mm lever to raise a load of 2000 N
ii. The efficiency of screw jack.
iii. Force required to be applied at pitch radius to lower the same load of 2000 N.
iv. Efficiency while lowering the load.
v. What should be the pitch for the maximum efficiency of the screw and what should be the value of maximum efficiency.

Solution Given Outer diameter = 40 mm

Pitch = 10 mm

Inner diameter = $(40 - 10) \ = 30\text{mm}$

Mean diameter (d) = $\frac{40+30}{2}$ = 35 mm

$\mu = 0.15 = \tan\varphi$

$\varphi = \tan^{-1}(0.15) = 8.55°$

$\tan\theta = \frac{\text{p}}{\pi\text{d}} = \frac{10}{\pi\times35} = 0.09096$

$\theta = \tan^{-1}(0.09096) = 5.197°$

$tan\ (\theta + \varphi) = \tan(8.53 + 5.197)$

$= \tan 13.727 = 0.244$

i. ***Force required to raise the load***

Length of the lever L = 450 mm

Load W = 2000 N

F = $\frac{d}{2L}$W tan (θ+φ)

$$= \frac{35}{2 \times 450} \times 2000 \times 0.244$$

$$= 18.977\ N$$

ii. Efficiency of screw jack

$$\eta = \frac{\tan\theta}{\tan(\theta + \varphi)} = \frac{0.09096}{0.244} \times 10$$

$$= 37.28\%$$

iii. Force required to lower the same load

$$F = \frac{d}{2L} W \tan(\theta - \varphi)$$

$$= \frac{35}{2 \times 450} \times 2000 \times \tan(5.197 - 8.55)$$

= 4.55 N

iv. Efficiency while lowering

$$= \frac{\tan\theta}{\tan(\theta - \varphi)}$$

$$= \frac{0.9096}{-0.58}$$

This will be greater than 100 i.e., no effect required

v. ***Max. efficiency*** $= \frac{1 - \sin\varphi}{1 + \sin\varphi}$

$$= \frac{1 - \sin 8.53}{1 + \sin 8.53} = \frac{-0.148}{+0.148}$$

$$= 74.21\%$$

$$\alpha = 45 - \frac{\varphi}{2} = 45 - \frac{8.53}{2}$$

$$= 40.735 > \varphi$$

Hence screw jack will no long remain self-locking.

$$\text{max. pitch} = \tan\theta \times \pi d$$

$$= \tan 40.735 \times \pi \times 33$$

$$= 94.67\text{mm}$$

SUMMARY

- Friction is a force of restriction of movement or motion of a body when it slides over another body.
- The friction exerted when a body is about to move or it is just on the point of moving over another body is called as *limiting friction.*
- If the surfaces of two bodies in contact are unlubricated, the friction developed between them is known as dry friction.
- Static friction is the friction that exists when the body remains at rest.
- Dynamic friction is the friction that exists when the body starts moving over another body.
- Coefficient of friction is defined as the ratio of the limiting friction to the normal reaction. It is denoted by the letter μ.
- A wedge is a simple device which is used to produce small adjustments in the position of bodies and also used to raise heavy loads.
- A screw jack is a device which is used to lift or hold heavy loads.

EXERCISES

I. Self-Assessment Questions

1. Define the term "Friction"
2. What is limiting friction?
3. Differentiate between static friction and dynamics friction.
4. Explain the useful and harmful effects of friction.
5. Define "coefficient of friction"
6. State the laws of dry friction.
7. Show that angle of repose is equal to angle of limiting friction.
8. What are wedges? What is its used?
9. Explain the working of a screw jack.

II. Multiple Choice Questions

1. The angle of response(α) holds the following relation with the angle of friction(φ) in case of limiting equilibrium
   a) $\alpha = \varphi$
   b) $\alpha = 2\varphi$
   c) $\alpha = \varphi/2$
   d) $\alpha = \varphi^2$
2. Angle of friction in case of body sliding by its own accord will be
   a) Zero
   b) Equal to angle of inclination
   c) At a right angle

d) None of above

3. The coefficient of friction depends on
   a) Area of contact
   b) Shape of surfaces
   c) Strength of surfaces
   d) Nature of surface
4. _____________ friction is the friction experienced by a body when it is in motion
   a) Static
   b) Dynamic
   c) Kinetic
   d) Sliding
5. Dynamic friction as compared to static friction is
   a) Same
   b) More
   c) Less
   d) Has no correlation
6. Frictional force encountered after commencement of motion is called
   a) Post friction
   b) Limiting friction
   c) Dynamic friction
   d) Frictional resistance
7. The efficiency of a screw jack may be increased by
   a) Increasing its pitch
   b) Decreasing its pitch
   c) Increasing the load to be lifted
   d) Decreasing the load to be lifted

Answers

1. a
2. b
3. d
4. b
5. c
6. c
7. a

CHAPTER – 5

CENTROID AND CENTRE OF GRAVITY

LEARNING OBJECTIVES

After studying this chapter, you should be able to

- *Understand the importance of centroid and centre of gravity*
- *Differentiate between centroid and centre of gravity*
- *Determine the centroid for curves and surfaces*
- *Identify the location of centre of gravity for composite bodies*

5.1 CENTROID

The centroid is a point where the whole area of a plane figure is assumed to be concentrated. It is sometimes referred to as centre of area or geometric centre of the region. The term centroid is related to geometrical shapes where mass and weight are not involved. If the shapes are in symmetric in nature then the centroid lies on the axis of symmetry. An axis passing through centroid is known as centroidal axis which is used to find Moment of Inertia.

5.2 IMPORTANCE OF CENTROIDS

In order to produce uniform stress, line of action of load must pass through centroid. In bending of beams the neutral axis passes through centroid of section. In order to have stability, the centriod should be located carefully. For example, in rotating shafts if the axis has some eccentricity, then it can tear apart the machine, this is just a simple example. In practical application centroid is very important like in designing ships etc. For calculating area moment of inertia, again centriod is important.

5.3 METHODS OF FINDING CENTROID

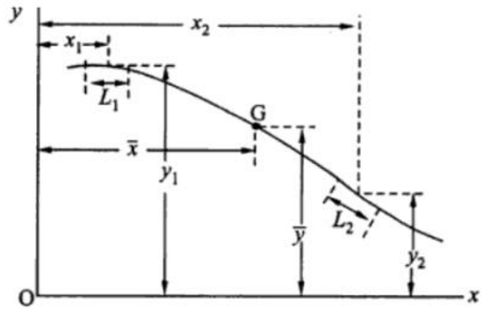
The centroid of curves or areas may be found by using any one of the following methods

1. By geometrical considerations
2. By method of moments
3. By method of integration

5.4 CENTROID OF A CURVE

a. By method of moments

Centroid of a curve is a point in which the total length of the curve is acting.
Consider a curve of total length L. Let us divide the curve into small segments L_1, L_2….. The centroids of these segments are c_1, c_2….. lying at distance (x_1,y_1), (x_2,y_2),…. as shown in the figure



Let c (x, y) be the distance of centroid, then taking the first moments about the reference axes we have

$$l_1x_1+l_2x_2++l_3x_3+\ldots\ldots\ldots$$
$$l_1y_1+l_2y_2+l_3y_3+\ldots\ldots\ldots.$$

Applying the principle of moments, we have
(The moment of the total length of the line is equal to the sum of the moments of individual segments)

$$Lx = l_1x_1+l_2x_2++l_3x_3+\ldots\ldots\ldots$$
$$x = \frac{l1x1+l2x2+l3x3+\cdots\ldots\ldots}{L}$$

Similarly

$$Ly = l_1y_1+l_2y_2+l_3y_3+\ldots\ldots\ldots.$$
$$y = \frac{l1y1+l2y2+l3y3+\cdots\ldots\ldots}{L}$$

Where $L = l_1 + l_2 + l_3$ ……

b. By method of integration

The above equations can be written as

$$L\bar{x} = \sum l_i x_i$$
$$L\bar{y} = \sum l_i y_i$$

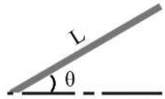
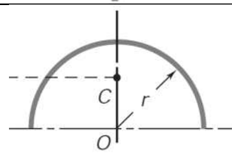
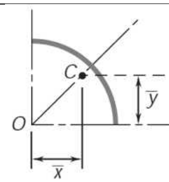
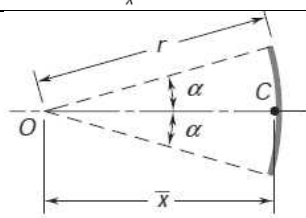
If the small curves are large in number, then the summations in the above equations can be replaced by integration. Integration is the process of summing up infinitesimal quantities, it is equivalent to a finite summation. If dL represents the small length instead of l, the above equation can be written as

$$L\bar{x} = \int x\,dL$$

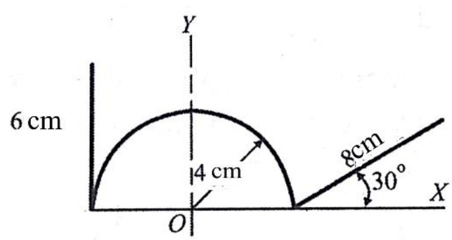
Similarly

$$L\bar{y} = \int y\,dL$$

Table 5.1: Table Centroid of simple curves

Shape		Length	x	y
Straight line		L	(L/2) cos θ	(L/2) sin θ
Straight line		L	L/2	0
Straight line		L	0	L/2
Semicircular arc		πR	$\frac{2R}{\pi}$	0
Quarter circle arc		$\frac{\pi R}{2}$	$\frac{2R}{\pi}$	$\frac{2R}{\pi}$
Sector of a circle		2αR	$\frac{Rsin\alpha}{\alpha}$	0

Example 5.1 A homogeneous wire of uniform cross section is bent into the shape shown in the figure. Determine the co-ordinates of the centroid.



Solution ***Vertical line***

Length (L_1) = 6 cm

x_1 = - 4cm

y_1= 3cm

Semi-circular arc

Length (L_2) = π R = $\pi \times 4$ = 12.56 cm

$x_2 = 0$

$y_2 = \frac{2R}{\pi} = \frac{2 \times 4}{\pi} = \frac{8}{\pi} = 2.54$ cm

Inclined Line

Length (L_3) = 8 cm

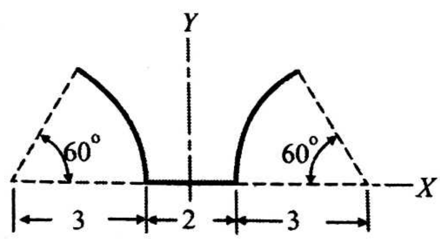
$x_3 = 4 + 4\cos30° = 7.46$ cm

$y_3 = 4\sin30° = 2$ cm

$\bar{x} = \frac{L_1x_1+L_2x_2+L_3x_3}{L_1+L_2+L_3} = \frac{6(-4)+12.56(0)+8(7.46)}{6+12.56+8} = 1.344$ cm

$\bar{y} = \frac{L_1y_1+L_2y_2+L_3y_3}{L_1+L_2+L_3} = \frac{6(3)+12.56(2.54)+8(2)}{6+12.56+8} = 2.485$ cm

Example 5.2 Locate the centre of gravity of the bend wire shown in the figure. The wire is homogenous and of uniform cross section



Solution Given figure is symmetric about y-axis

Arc

Length (L_1) = $2\alpha R$ = 3.14 cm

$y_1 = \frac{R\sin\alpha}{\alpha} = 1.43$ cm

Horizontal Line

Length (L_2) = 2 cm

$y_2 = 0$

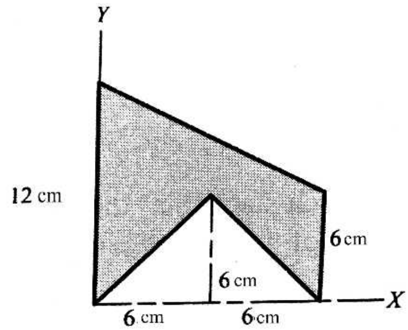
Arc

Length (L_3) = 3.14 cm

$y_3 = 4\sin30° = 1.43$ cm

$\bar{y} = \frac{L_1y_1+L_2y_2+L_3y_3}{L_1+L_2+L_3} = \frac{3.14(1.43)+2(0)+3.14(1.43)}{3.14+2+3.14} = 1.0845$ cm

Example 5.3 Determine the centroid of the lines that form the boundary of the area shown in the figure



Solution ***Vertical line***

Length (L_1) = 12 cm

$x_1 = 0$; $y_1 = 6$ cm

Inclined Line

Length (L_2) = 8.485 cm

$x_2 = 3$ cm; $y_2 = 3$ cm

Inclined Line

Length (L_3) = 8.485 cm

$x_3 = 9$ cm; $y_3 = 3$ cm

Vertical Line

Length (L_4) = 6 cm

$x_4 = 12$ cm; $y_4 = 3$ cm

Inclined Line

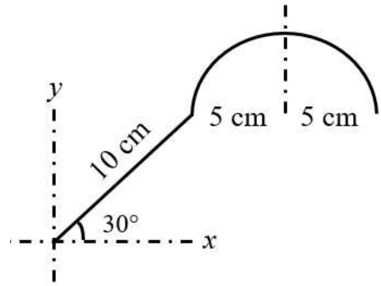
Length (L_5) = 13.416 cm

$x_5 = 6$ cm; $y_5 = 9$ cm

$$\bar{x} = \frac{L_1x_1+L_2x_2+L_3x_3+L_4x_4+L_5x_5}{L_1+L_2+L_3+L_4+L_5} = \frac{12(0)+8.485(3)+8.485(9)+6(12)+13.416(6)}{12+8.485+8.485+6+13.416} = 5.256 \text{ cm}$$

$$\bar{y} = \frac{L_1y_1+L_2y_2+L_3y_3+L_4y_4+L_5y_5}{L_1+L_2+L_3+L_4+L_5} = \frac{12(6)+8.485(3)+8.485(3)+6(3)+13.416(9)}{12+8.485+8.485+6+13.416} = 5.4076 \text{ cm}$$

Example 5.4 Locate the centroid of a bent wire as shown in the figure



Solution Length (L_1) = 10 cm

$x_1 = 5 \cos 30° = 4.33$ cm

$y_1 = 5 \sin 30° = 2.5$ cm

Length (L_2) = $\pi R = \pi \times 5 = 15.7$ cm

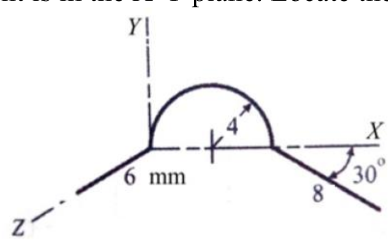
$x_2 = 10 \cos 30° + 5 = 13.66$ cm

$y_2 = 10 \sin 30° + \frac{2R}{\pi} = 8.18$ cm

$\overline{x} = \frac{L_1 x_1 + L_2 x_2}{L_1 + L_2} = \frac{10(4.3.)+15.7(13.66)}{10+15.7} = 10.0179$ cm

$\overline{y} = \frac{L_1 y_1 + L_2 y_2}{L_1 + L_2} = \frac{10(2.5)+15.7(8.18)}{10+15.7} = 5.9698$ cm

Example 5.5 A uniform wire is bent into the shape as shown in the figure 3. The straight segments lie in the X-Z plane and the line of 80 mm length makes an angle of 30° with the X-axis. The semi-circular segment is in the X-Y plane. Locate the centre of gravity of the wire.



Solution ***Straight line on Z-axis***

Length (L_1) = 6 cm

$x_1 = 0$

$y_1 = 0$

$z_1 = 3$ cm

Semi-circular arc

Length (L_2) = $\pi R = \pi \times 4 = 12.56$ cm

$x_2 = 4$ cm

$y_2 = \frac{2R}{\pi} = \frac{2\times 4}{\pi} = 2.54$ cm

$z_2 = 0$

Inclined line

Length (L_3) = 8 cm

$x_3 = 8 + 4\cos 30° = 11.46$ cm

$y_3 = 0$

$z_3 = 4\sin 30° = 2$

$\bar{x} = \frac{L_1x_1+L_2x_2+L_3x_3}{L_1+L_2+L_3} = \frac{6(0)+12.56(4)+8(11.46)}{6+12.56+8} = 5.33$ cm

$\bar{y} = \frac{L_1y_1+L_2y_2+L_3y_3}{L_1+L_2+L_3} = \frac{6(0)+12.56(2.54)+8(0)}{6+12.56+8} = 1.201$ cm

5.5 CENTROID OF AREAS

a. By method of moments

Consider a plane area as shown in the figure. Divide the area into small areas a_1, a_2, a_3 …….. whose centroid is to be determined. Let (x_1, y_1), (x_2, y_2), (x_3, y_3), ... be the co-ordinates of the centroids of these small areas from a fixed axes *OX* and *OY* as shown.

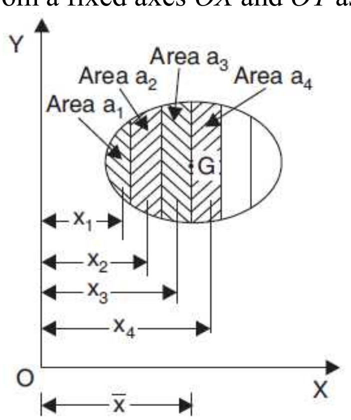


Fig.

Let C be the centroid of total area A whose distance form OY is $\bar{x}$

From the principle of moments, the moments of all small areas about the axis OY must be equal to the moment of total area about the same axis. i.e.,

$$A\bar{x} = a_1x_1 + a_2x_2 + a_3x_3 \ldots..$$

$$\bar{x} = \frac{a_1x_1 + a_2x_2 + a_3x_3}{A}$$

Similarly

$$A\bar{y} = a_1y_1 + a_2y_2 + a_3y_3 \ldots..$$

$$\bar{y} = \frac{a_1y_1 + a_2y_2 + a_3y_3}{A}$$

Where $A = a_1 + a_2 + a_3 \ldots\ldots..$

b. By method of integration

The above equations can be written as

$$A\bar{x} = \sum a_i x_i$$

$$A\bar{y} = \sum a_i y_i$$

Note: The above equations are called ***moment of area****. Hence the moment of area is defined as the product of area and perpendicular distance from the centre of area to the axis of moments.*

If the small areas are large in number, then the summations in the above equations can be replaced by integration. Integration is the process of summing up infinitesimal quantities, it is equivalent to a finite summation. If dA represents the small area instead of a, the above equations can be written as

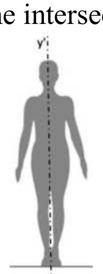
$$A\bar{x} = \int x\,dA$$

Similarly

$$A\bar{y} = \int y\,dA$$

5.6 AXIS OF SYMMETRY

Axis of symmetry is a line through which shape on each side is a mirror image. For example, figure no shows the view of a human body which is symmetric about y-axis. That means, if the figure is folded about y-axis, each point on one side will coincide with the other side. In real environment most of the objects are symmetric in nature. Axis of symmetry has an important role in determining the centroid. If an area or a line is symmetric about x or y- axis then the centroid lies on the axis of symmetry. If it is symmetry about both x and y axes, then their centroid lies at the intersection of the axes.



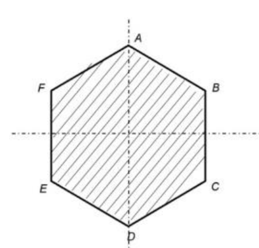
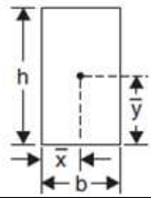
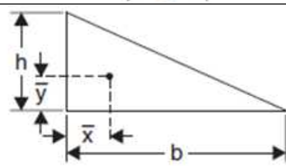
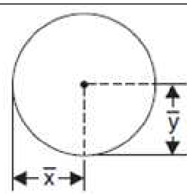
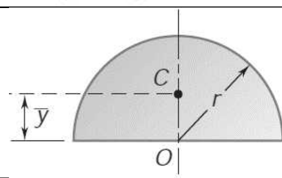
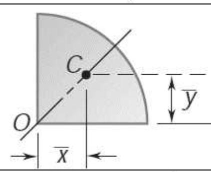
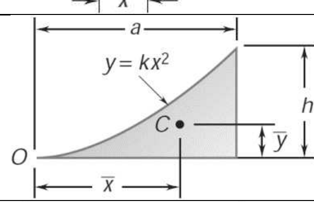
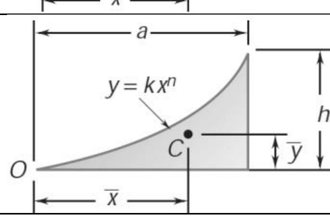
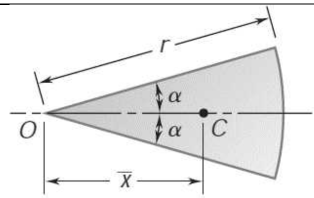
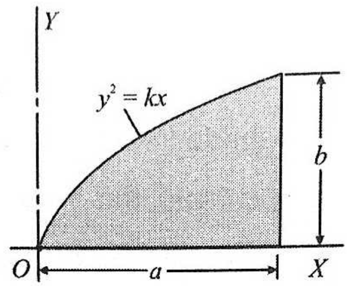


Table 5.2: Centroids of plane geometrical shapes

Shape		Area	x	y
Rectangle		$b \times h$	$\frac{b}{2}$	$\frac{h}{2}$
Triangle		$\frac{1}{2} \times b \times h$	$\frac{b}{3}$	$\frac{h}{3}$
Circle		$\pi\, r^2$	$\frac{d}{2}$	$\frac{d}{2}$
Semi-circle		$\frac{\pi\, r^2}{2}$	-	$\frac{4\, r}{3\, \pi}$
Quarter circle		$\frac{\pi\, r^2}{4}$	$\frac{4\, r}{3\, \pi}$	$\frac{4\, r}{3\, \pi}$
Parabolic spandrel		$\frac{ah}{3}$	$\frac{3\, a}{4}$	$\frac{3\, h}{10}$
General spandrel		$\frac{ah}{n+1}$	$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$

Circular sector		αr^2	$\frac{2r\sin\alpha}{3\alpha}$	0

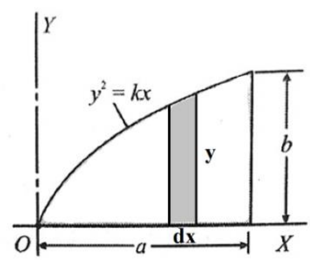
Example 5.6 Determine the centroid of the area shown in fig. which is bounded by the x-axis the line x = a and the parabola $y^2 = kx$.



Solution Given $y^2 = kx$

$\therefore \quad b^2 = k.a \qquad \therefore \quad b = \sqrt{k.a}$

Consider a strip parallel to Y axis as shown.



Elemental Area, $dA = y.dx$

Total area $A = \int dA = \int_0^a y.dx = \int_0^a \sqrt{k.x}dx$

$= \sqrt{k}\left[\frac{x^{3/2}}{3/2}\right]_0^a = \sqrt{k}.a^{3/2} \times \frac{2}{3}$

Now $A\bar{x} = \int xdA$

$\therefore \sqrt{k}.a^{3/2} \times \frac{2}{3}\bar{x} = \int_0^a x.\sqrt{k}\ x.dx = \sqrt{k}\left[\frac{x^{5/2}}{5/2}\right]_0^a = \sqrt{k}.a^{5/2} \times \frac{2}{5}$

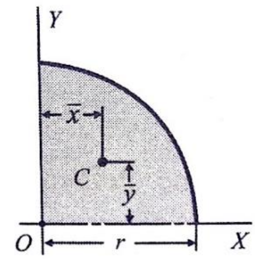
$\therefore \ \bar{x} = \frac{3}{5}a$

Now, $A\bar{y} = \int_0^a ydA$

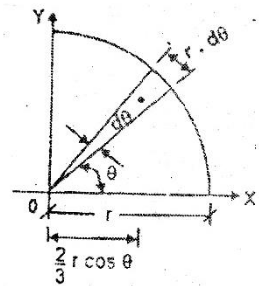
$\therefore \sqrt{k}.a^{3/2}.\frac{2}{3}\bar{y} = \int_0^a \left(\frac{y}{2}\right).y.dx = \frac{1}{2}\int_0^a y^2dx = \frac{k}{2}\left[\frac{x^2}{2}\right]_0^a = \frac{k}{4}.a^2$

$\therefore \ \bar{y} = \sqrt{k}.\sqrt{a} \times \frac{3}{8} \qquad \bar{y} = \frac{3}{8}b$

Example 5.7 Determine the centroid of the quarter circle shown in fig. whose radius is r.



Solution Consider strip as shown in fig.



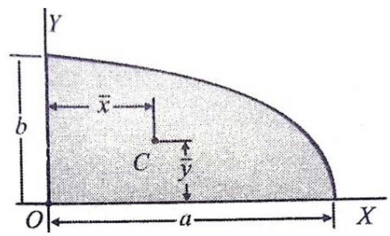
Area of elemental strip $= \frac{1}{2}(\mathrm{r.d\theta}).\mathrm{r} = \frac{\mathrm{d\theta}}{2} \times \mathrm{r}^2$

$\mathrm{A}.\bar{\mathrm{x}} = \int_0^{\pi/2} \mathrm{x.dA} \left(\frac{\pi}{4}\mathrm{r}^2\right).\bar{\mathrm{x}} = \int_0^{\pi/2} \mathrm{r}\, \frac{2}{3}\mathrm{r}\cos\theta.\mathrm{r}^2$

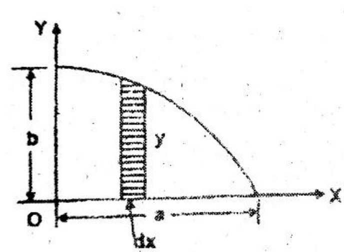
$= \frac{\mathrm{r}^3}{3}\int_0^{\pi/2} \cos\theta \mathrm{d}\theta = \frac{\mathrm{r}^3}{3}[\sin\theta]_0^{\pi/2} = \frac{\mathrm{r}^3}{3}$

Similarly, $\bar{\mathrm{y}} = \frac{4\mathrm{r}}{3\pi} = 0.424\mathrm{r}$; $\quad \bar{\mathrm{x}} = \frac{4\mathrm{r}}{3\pi} = 0.424\mathrm{r}$

Example 5.8 Determine the centroid of the quadrant of the ellipse shown in figure. The equation of the ellipse is $\frac{\mathrm{x}^2}{\mathrm{a}^2} + \frac{\mathrm{y}^2}{\mathrm{b}^2} = 1$



Solution Here $\mathrm{y} = \sqrt{\mathrm{b}^2 - \frac{\mathrm{b}^2}{\mathrm{a}^2}.\mathrm{x}^2}$



Consider elemental strip as shown in fig.

Area of strip $= \mathrm{dA} = \mathrm{y.dx}$

Total area $\mathrm{A} = \int \mathrm{dA} = \int_0^{\mathrm{a}} \mathrm{ydx}$

$= \int_0^{\mathrm{a}} \sqrt{\mathrm{b}^2 - \frac{\mathrm{b}^2}{\mathrm{a}^2}.\mathrm{x}^2 \mathrm{dx}}$

$= \frac{\mathrm{b}}{\mathrm{a}}\int_0^{\mathrm{a}} \sqrt{\mathrm{a}^2 - \mathrm{x}^2}.\mathrm{dx}$

$= \frac{\mathrm{b}}{\mathrm{a}}\left[\frac{\mathrm{x}}{2}\sqrt{\mathrm{a}^2 - \mathrm{x}^2} + \frac{\mathrm{a}^2}{2}\sin\left(\frac{\mathrm{x}}{2}\right)\right]_0^{\mathrm{a}} = \frac{\mathrm{b}}{\mathrm{a}}\left[\frac{\mathrm{a}^2}{2} \times \frac{\pi}{2}\right] = \frac{\pi}{4}\mathrm{ab}$

Now $A\bar{x} = \int_0^a x dA \left(\frac{\pi}{4}ab\right)\bar{x} = \int_0^a \sqrt{b^2 - \frac{b^2}{a^2}.x^2 dx}$

$= \frac{b}{a}\int_0^a x\sqrt{a^2 - x^2}.dx$

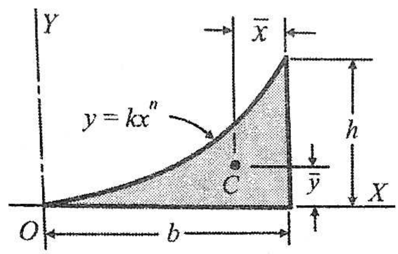
Assume $a^2 - x^2 = t^2$

$\therefore \quad -2xdx = 2tdt \quad \therefore \quad xdx = -tdt$

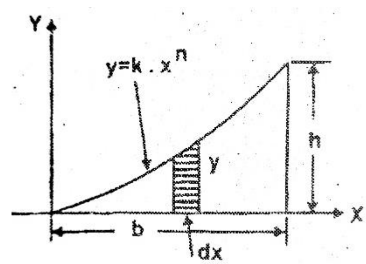
$\therefore \left(\frac{\pi}{4}ab\right)\bar{x} = \frac{b}{a}\int_a^0 -tdt = \frac{b}{a}\int_a^0 t^2 dt = \frac{b}{a}\left[\frac{t^3}{3}\right]_0^a = \frac{ba^2}{3}$

$\bar{x} = \frac{4a}{3\pi}$ Similarly $\bar{y} = \frac{4b}{3\pi}$

Example 5.9 Compute the area of the spandrel in fig. bounded by the X-axis, the line x = b, and the curve $y = kx^n$ where $n \geq 0$. What is the location of centroid from the line x = b?



Solution $y = k.x^n \quad \therefore h = k.b^n$



Consider strip as shown in fig.

Area of strip $dA = y.dx$

Total area $= A = \int dA = \int_0^a y.dx$

$= \int_0^b k.x^n dx = k.\left[\frac{x^{n+1}}{n+1}\right]_0^b = \frac{k.b^{n+1}}{(n+1)} = \frac{bh}{(n+1)}$

Now, $A.\bar{x} = \int xdA$

$\frac{bh}{(n+1)}\bar{x} = \int_0^b xydx = \int_0^b xkx^n dx$

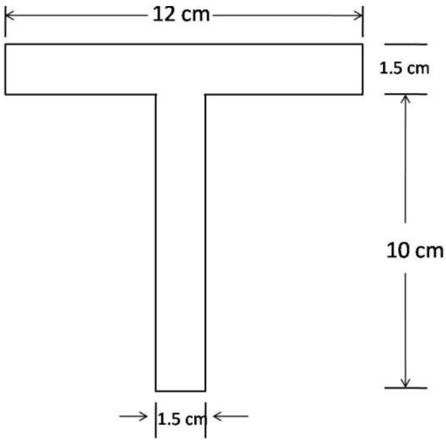
$= \int_0^b kx^{n+1}dx = k\left[\frac{x^{n+2}}{n+2}\right]_0^b = \frac{k.b^{n+2}}{(n+2)} = \frac{k.b^n b^2}{(n+2)} = \frac{hb^2}{(n+2)}$

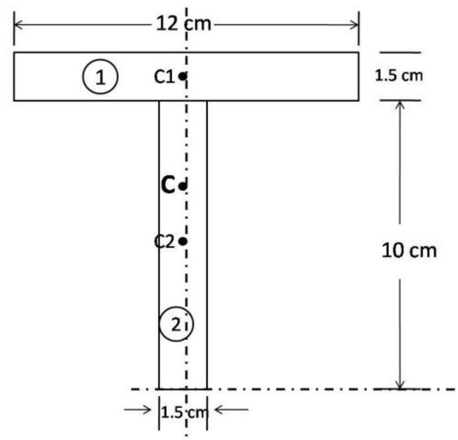
$\therefore \quad \bar{x} = \frac{(n+1)}{(n+2)}b$

The position of centroid from the line x = b can be calculated as follows

$\bar{x}\text{ (from line x = b)} = b - \frac{(n+1)}{(n+2)}b = \frac{(n+2)b-(n+1)b}{(n+2)} = \frac{b}{(n+2)}$

Example 5.10 Find the centroid for the T section shown below





Solution The figure is symmetry about y-axis

The figure can be divided as

i. Rectangle of dimensions 12 x 1.5
ii. Rectangle of dimensions 1.5 x 10

Rectangle 1

Area (a_1) = 12×1.5=18 cm^2

y_1 = 10 + (1.5/2) =10.75 cm

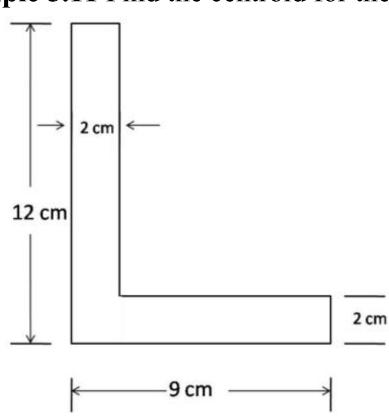
Rectangle 2

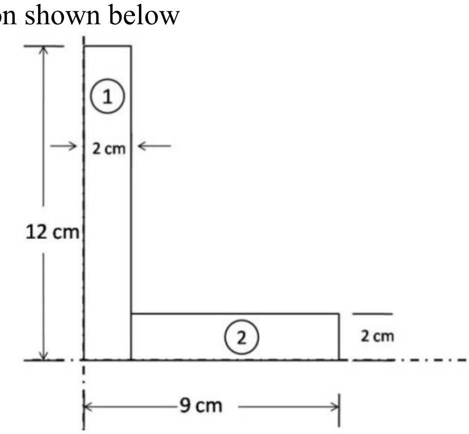
Area (a_2) = 10×1.5=15 cm^2

y_2 = (10/2) =5 cm

$$\bar{y}=\frac{a_1y_1+a_2y_2}{a_1+a_2}=\frac{18\times 10.75+15\times 5}{18+15}=8.136 \text{ cm}$$

Example 5.11 Find the centroid for the L section shown below





Solution The figure can be divided as

i. Rectangle of dimensions 12 x 2
ii. Rectangle of dimensions 7 x 2

Rectangle 1

Area (a_1) = 12×2 = 24 cm^2

x_1 = (2/2) = 1 cm

y_1 = (12/2) = 6 cm

Rectangle 2

Area (a_2) = (9-2)×2 = 14 cm^2

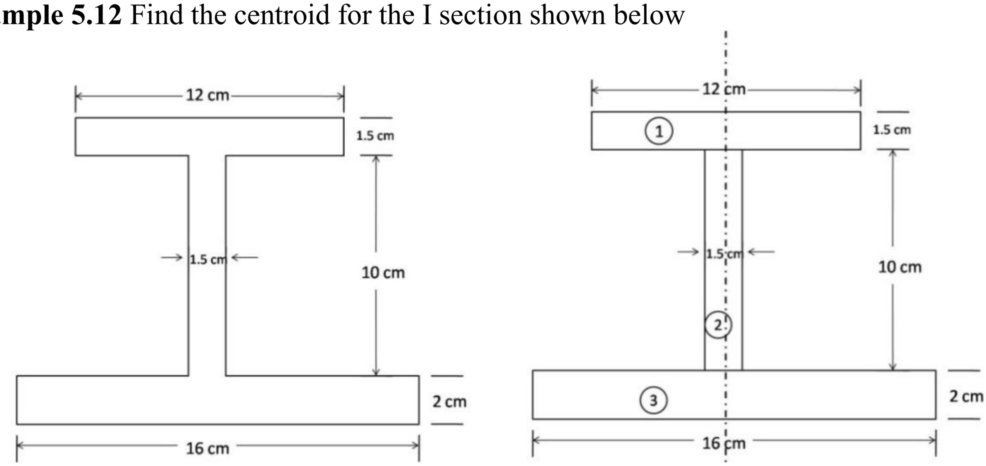
x_2 = 2+(7/2) = 2+3.5 = 5.5 cm

y_2 = (2/2) = 1 cm

$\overline{x} = \frac{a_1x_1+a_2x_2}{a_1+a_2} = \frac{24\times1+14\times5.5}{24+14} = 2.657$ cm

$\overline{y} = \frac{a_1y_1+a_2y_2}{a_1+a_2} = \frac{24\times6+14\times1}{24+14} = 4.157$ cm

Example 5.12 Find the centroid for the I section shown below



Solution The figure can be divided as

i. Rectangle of dimensions 12 x 1.5
ii. Rectangle of dimensions 10 x 1.5
iii. Rectangle of dimensions 16 x 2

Rectangle 1

Area (a_1) = 12×1.5 = 18 cm^2

y_1 = 2+10+(1.5/2) =12.75 cm

Rectangle 2

Area (a_2) = 1.5×10 = 15 cm^2

y_2 = 2+(10/2) =7 cm

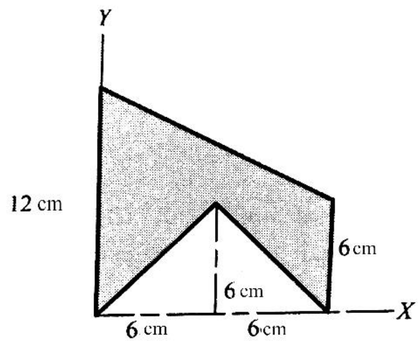
Rectangle 3

Area (a_3) = 16×2 = 32 cm^2

y_3 = (2/2) = 1 cm

$\bar{y} = \frac{a_1y_1+a_2y_2+a_3y_3}{a_1+a_2+a_3} = \frac{18\times12.75+15\times7+32\times1}{18+15+32}$ = 5.638 cm

Example 5.13 Locate the centroid for the shaded area shown in the figure



Solution The figure can be divided as

i. Right angle triangle of base 12 cm and height 6 cm
ii. Rectangle of size 12 cm x 6 cm
iii. Isosceles triangle of base 12 cm and height 6 cm

Right angle triangle

Area (a_1) = $\frac{1}{2} \times b \times h = \frac{1}{2} \times 12 \times 6$ = 36 cm^2

$x_1 = \frac{b}{3} = \frac{12}{3}$ = 4 cm

$y_1 = 6 + \frac{h}{3} = 6 + \frac{6}{3}$ = 8 cm

Rectangle

Area (a_2) = $b \times h$ = 12 x 6 = 72 cm^2

$x_2 = \frac{12}{2}$ = 6 cm

$y_2 = \frac{6}{2}$ = 3 cm

Isosceles triangle

Area (a_3) = $\frac{1}{2} \times b \times h = \frac{1}{2} \times 12 \times 6$ = 36 cm^2

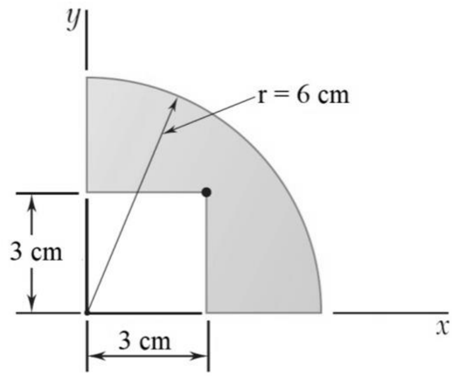
$x_3 = 6$ cm

$y_3 = \frac{6}{3} = 2$ cm

$\bar{x} = \frac{a_1x_1+a_2x_2-a_3x_3}{a_1+a_2-a_3} = \frac{36(4)+72(6)-36(6)}{36+72-36} = \frac{360}{72} = 5$ cm

$\bar{y} = \frac{a_1y_1+a_2y_2-a_3y_3}{a_1+a_2-a_3} = \frac{36(8)+72(3)-36(2)}{36+72-36} = \frac{432}{72} = 6$ cm

Example 5.14 Locate the centroid for the shaded area shown in the figure



Solution The figure can be divided as

i. Quarter circle of radius 6 cm
ii. Square of dimensions 3 x 3

Quarter circle

Area (a_1) = $\frac{\pi\, r^2}{4} = \frac{\pi\, 6^2}{4} = 28.27$ cm^2

$x_1 = \frac{4\,r}{3\,\pi} = \frac{4\,(6)}{3\,\pi} = 2.54$ cm

$y_1 = \frac{4\,r}{3\,\pi} = \frac{4\,(6)}{3\,\pi} = 2.54$ cm

Square

Area (a_2) = 3 x 3 = 9 cm^2

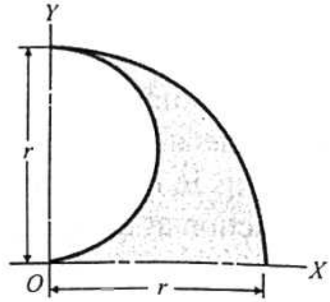
$x_2 = 1.5$ cm

$y_2 = 1.5$ cm

$\bar{x} = \frac{a_1x_1-a_2x_2}{a_1-a_2} = \frac{28.27\times2.54-9\times1.5}{28.27-9} = 3.02$ cm

$\bar{y} = \frac{a_1y_1-\,a_2y_2}{a_1-a_2} = \frac{28.27\times2.54-9\times1.5}{28.27-9} = 3.02$ cm

Example 5.15 Locate the centroid of the shaded area in the figure created by cutting semicircle of diameter r from a quarter circle of radius r



Solution The figure can be divided as

i. Quarter circle of radius r
ii. Semicircle of diameter r

Quarter circle

Area (a_1) = $\frac{\pi r^2}{4}$

$x_1 = \frac{4r}{3\pi}$

$y_1 = \frac{4r}{3\pi}$

Semi circle

Area (a_2) = $\frac{\pi r^2}{2} = \frac{\pi\left(\frac{r}{2}\right)^2}{2} = \frac{\pi r^2}{8}$

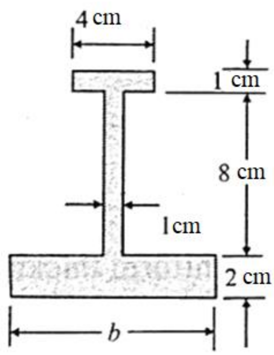
$x_2 = \frac{4r}{3\pi} = \frac{4\left(\frac{r}{2}\right)}{3\pi} = \frac{4r}{6\pi}$

$y_2 = \frac{r}{2}$

$$\bar{x} = \frac{a_1x_1 - a_2x_2}{a_1 + a_2} = \frac{\frac{\pi r^2}{4}\left(\frac{4r}{3\pi}\right) - \frac{\pi r^2}{8}\left(\frac{4r}{6\pi}\right)}{\frac{\pi r^2}{4} - \frac{\pi r^2}{8}} = 0.6366 \text{ r}$$

$$\bar{y} = \frac{a_1y_1 - y_2}{a_1 + a_2} = \frac{\frac{\pi r^2}{4}\left(\frac{4r}{3\pi}\right) - \frac{\pi r^2}{8}\left(\frac{r}{2}\right)}{\frac{\pi r^2}{4} - \frac{\pi r^2}{8}} = 0.3488 \text{ r}$$

Example 5.16 In order to fully utilize the different values of the compressive stress Sc and the tensile stress St in the cast iron beams, it is desirable to locate the centroidal axis so that the ratio of its distance to the top of a section to its distance from the bottom equals Sc/St. Using the section shown in the figure, find the dimension b to satisfy this criterion of Sc/St = 3



Solution The given figure is symmetric about y-axis
The figure can be divided as

i. Rectangle of size $2 \times b$ cm
ii. Rectangle of size 1×8 cm
iii. Rectangle of size 4×1 cm

Rectangle 1

Area (a_1) = 2b cm^2

y_1 = 1 cm

Rectangle 2

Area (a_1) = 1×8 = 8 cm^2

y_2 = 2+4=6 cm

Rectangle 3

Area (a_1) = 4x1 = 4 cm^2

y_3 = 2+8+0.5 = 10.5 cm

$\bar{y} = \frac{a_1y_1+a_2y_2-a_3y_3}{a_1+a_2-a_3} = \frac{2b(1)+8(6)+4(10.5)}{2b+8+4} = \frac{90+2b}{12+2b}$

Distance of the centroid from top = $11 - \bar{y}$

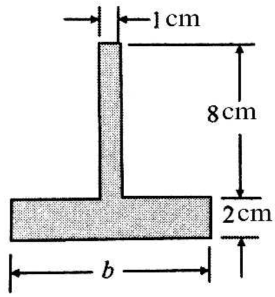
$$= 11 - \frac{90+2b}{12+2b}$$

Given condition: $\frac{distance\ of\ centroid\ from\ top}{distance\ of\ centroid\ from\ bottom} = 3$

$$\frac{11 - \frac{90+\ 2b}{12+2b}}{\frac{90+\ 2b}{12+2b}} = 3$$

b = 16.2857 cm

Example 5.17 Determine the dimension b that will locate the centroidal axis at 40 mm above the base of the section shown in the figure



Solution The given figure is symmetric about y-axis
The figure can be divided as

i. Rectangle of dimensions b × 2 cm
ii. Rectangle of dimensions 1 × 8 cm

Rectangle 1

Area (a_1) = 2b

$y_1 = 1$

Rectangle 2

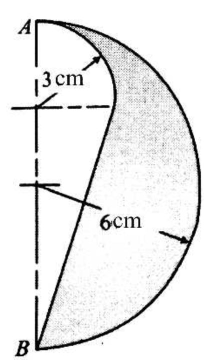
Area (a_2) = 8 ×1 = 8 cm^2

$y_2 = 2 + 4 = 6$ cm

$\bar{y} = \frac{a_1y_1+a_2y_2}{a_1+a_2} = \frac{2b(1)+8(6)}{2b+8} = 4$

b = 2.67 cm

Example 5.18 The surface of a plate of uniform thickness is given by the shaded area in the figure. If this plate is suspended from a hinge at A, what angle will line AB make with the vertical?



Solution Figure can be divided into

i. Semi-circle of radius 6 cm
ii. Quarter circle of radius 3 cm
iii. Right angle triangle

Semi-circle

Area (a_1) = $\frac{\pi r^2}{2} = \frac{\pi 6^2}{2}$ = 56.55 cm^2

$x_1 = \frac{4r}{3\pi} = \frac{4(6)}{3\pi}$ = 2.546 cm

y_1 = 6 cm

Quarter circle

Area (a_2) = $\frac{\pi 3^2}{4}$ = 7.068 cm^2

$x_2 = \frac{4(3)}{3\pi}$ = 1.273 cm

$y_2 = 9 + \frac{4(3)}{3\pi}$ = 10.273 cm

Right angle triangle

Area (a_3) = $\frac{1}{2} \times 3 \times 9$ = 13.5 cm^2

$x_3 = \frac{3}{3}$ = 1 cm

$y_3 = 9 - \frac{9}{3}$ = 6 cm

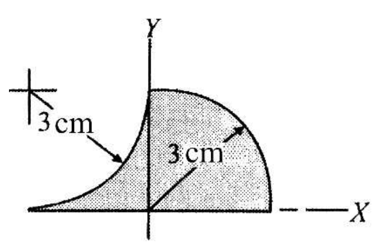
$\bar{x} = \frac{a_1x_1 - a_2x_2 - a_3x_3}{a_1 - a_2 - a_3} = \frac{56.55\ (2.546) - 7.068\ (1.273) - 13.5\ (1)}{56.55 - 7.068 - 13.5}$ = 3.376 cm

$\bar{y} = \frac{a_1y_1 - a_2y_2 - a_3y_3}{a_1 - a_2 - a_3} = \frac{56.55\ (6) - 7.068\ (10.273) - 13.5\ (6)}{56.55 - 7.068 - 13.5}$ = 5.161 cm

$\tan\theta = \frac{3.376}{6.839}$

θ = 26.27°

Example 5.19 With respect to the given axis, locate the centroid of the shaded area shown in the figure



Solution Figure can be divided as

i. Square of size 3 x 3 cm
ii. Quarter circle of radius 3 cm
iii. Quarter circle of radius 3 cm

Square

Area (a_1) = 3 x 3 = 9 cm^2

x_1 = -1.5 cm; y_1 = 1.5 cm

Quarter circle

Area (a_2) = $\frac{\pi\, 3^2}{4}$ = 7.07 cm^2

$x_2 = -(3 - \frac{4\,(3)}{3\,\pi}) = -1.73$ cm

$y_2 = 3 - \frac{4\,(3)}{3\,\pi} = 1.73$ cm

Quarter circle

Area (a_3) = $\frac{\pi\, 3^2}{4}$ = 7.07 cm^2

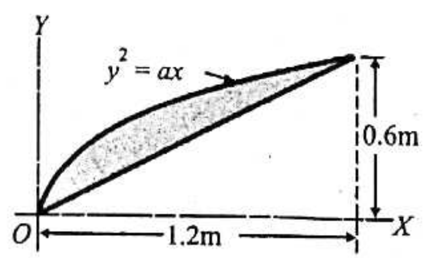
$x_3 = \frac{4\,(3)}{3\,\pi} = 1.27$ cm

$y_3 = \frac{4\,(3)}{3\,\pi} = 1.27$ cm

$$x = \frac{a_1x_1 - a_2x_2 + x_3}{a_1 - a_2 + a_3} = \frac{9(-1.5) - 7.07(-1.73) + 7.07(1.27)}{9 - 7.07 + 7.07} = 0.857 \text{ cm}$$

$$y = \frac{a_1y_1 - a_2y_2 + a_3y_3}{a_1 - a_2 + a_3} = \frac{9(1.5) - 7.07(1.73) + 7.07(1.27)}{9 - 7.07 + 7.07} = 1.14 \text{ cm}$$

Example 5.20 Locate the centroid of the shaded area enclosed by the curve y^2=ax and the straight line shown in the figure



Solution Divide the figures into

i. The curve y^2=ax i.e., y $\sqrt{a}\, x^{\frac{1}{2}}$ similar to y = k.x^n
ii. Right angle triangle base 12 m and height 6 m

For curve

Area= $\frac{1}{n+1}$ bh

= $\frac{1}{0.5+1}$ x 12 x 6 = 48 m^2

x_1 = 12 - $\frac{b}{n+2}$ = 7.2 m

y_1 = $\frac{n+1}{4n+2}$ x h = 2.25 m

Right angle triangle

Area = $\frac{1}{2} \times 12 \times 6$ = 36 m^2

x_2 = 12 - $\frac{12}{3}$ = 8 m

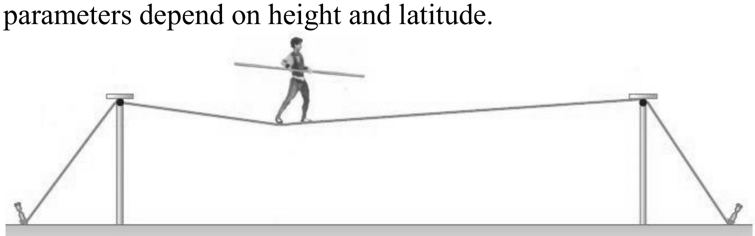
y_2 = $\frac{6}{3}$ = 2 m

x = $\frac{a_1x_1 - a_2x_2}{a_1 - a_2}$ = $\frac{48(7.2)-36(8)}{48-36}$ = 4.8 cm

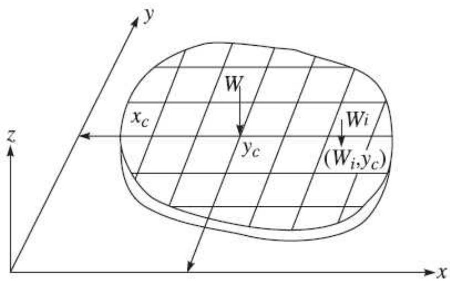
y = $\frac{a_1y_1 - a_2y_2}{a_1 - a_2}$ = $\frac{48(2.25)-36(2)}{48-36}$ = 3 cm

5.7 CENTRE OF GRAVITY

It is defined as the point through which the whole weight of a body may be assumed to act irrespective of the orientation of the body. In simple terms, the resultant weight of a body always acts at the point of centre of gravity. The point of centre of gravity and centre of mass is same for normal bodies under normal conditions. If the body considered for the study is of very large size and at high altitude, the point of centre of gravity and centre of mass is different because some parameters depend on height and latitude.



Consider a flat plate having thickness t as shown in the figure. Let the plate be divided into small parts each having a weight w1, w2, ….. The resultant of these forces is the total weight W of the plate acting at a point $(\bar{x}, \bar{y})$



Taking moment about Y axis

$$W\bar{x} = w_1x_1 + w_2x_2 + w_3x_3 \ldots\ldots = \sum w_ix_i$$

$$\bar{x} = \frac{\sum w_ix_i}{W}$$

Similarly

$$W\bar{y} = w_1y_1 + w_2y_2 + w_3y_3 \ldots\ldots = \sum w_ix_i$$

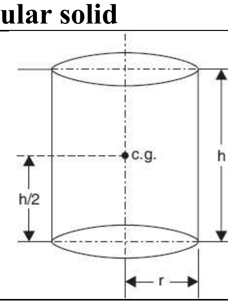
$$\bar{y} = \frac{\sum w_iy_i}{W}$$

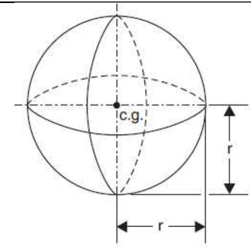
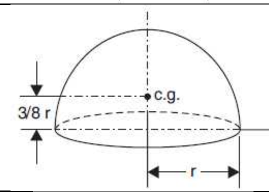
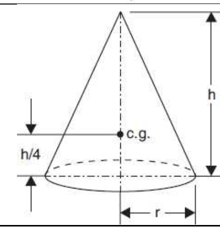
5.7.1 DIFFERENCE BETWEEN CENTROID AND CENTRE OF GRAVITY

	Centroid	Centre of gravity
1	Centroid is a point in a plane area such that the moment of area about any axis through that point is zero.	Centre of gravity is a point through which the resultant gravitational force acts irrespective of orientation of the body
2	Centroid applies to plane area only	Centre of gravity applies to bodies with mass and weight

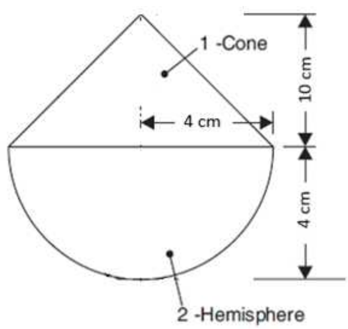
5.7.2 CENTRE OF GRAVITY OF SOLID BODIES

Table 5.3 Centroid of gravity of regular solids

Shape	**Volume**	**Regular solid**
Cylinder	$\pi r^2 h$	

Sphere	$\frac{4}{3}\pi r^3$	
Hemisphere	$\frac{2}{3}\pi r^3$	
Right circular cone	$\frac{1}{3}\pi r^2 h$	

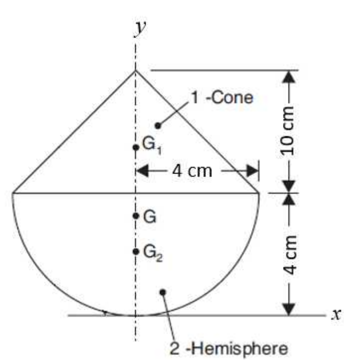
Example 5.21 A right circular cone of height 10 cm and radius 4 cm is placed on a solid hemisphere of diameter 8 cm as shown in the figure. Determine the position of centre of gravity.



Solution The body can be divided into

i. Cone

ii. Hemisphere



Cone

Volume (v_1) = $\frac{1}{3}\pi r^2 h$

= $\frac{1}{3} \times \pi \times 4^2 \times 10$ = 251.3 cm^3

$y_1 = 4 + \frac{10}{3}$ = 5.33 cm

Hemisphere

Volume (v_2) = $\frac{1}{2} \times \frac{4}{3} \times \pi \times 4^3$ = 134.04 cm^3

$y_2 = 4 - \frac{3}{8} \times 4 = 2.5$ cm

$y = \frac{v_1 y_1 + v_2 y_2}{v_1 + v_2} = \frac{251.3(5.33)+134.04(2.5)}{251.3+134.04} = 4.34$ cm

SUMMARY

- The centroid is a point where the whole area of the figure is assumed to be concentrated.
- If an area is symmetric in nature then the centroid lies on the axis of symmetry.
- Centre of gravity of a body is a point through which the resultant gravitational force acts irrespective of orientation of the body.
- The point of centre of gravity and centre of mass is same for normal bodies under normal conditions.
- If the body considered for the study is of very large size and at high altitude, the point of centre of gravity and centre of mass is different because some parameters depend on height and latitude.

EXERCISES

I. Self-Assessment Questions

1. Define the terms "centroid" and "centre of gravity"
2. How axis of symmetry is helpful in determining the centroid?
3. Differentiate between centroid and centre of gravity

II. Multiple Choice Questions

1. The distance between centroid and centroidal axes is
   a) Unity
   b) Zero
   c) Positive
   d) Negative
2. If first moment w. r. t. any axis is zero, then the axis must be
   a) Centroidal axis
   b) X – axis
   c) Y – Axis
   d) Z – Axis
3. Centroid of a triangle of base 'b' and height 'h' lies on the axis at a height of
   a) h/4 from base
   b) h/3 from base
   c) h/2 from base
   d) 2h/3 from base

4. Centroid of circular sector of radius R and subtending an angle 2α is
   a) $R \sin\alpha / \alpha$
   b) $2R \sin\alpha / \alpha$
   c) $(2R/3) \sin\alpha/\alpha$
   d) $(R/3) \sin\alpha/\alpha$
5. The centre of gravity of hemisphere lies at a distance of ______ units from its base measured along the vertical radius
   a) 3r/8
   b) 3/8r
   c) 8r/3
   d) 8/3r
6. The centre of gravity of right circular cone of diameter ‘d’ and height ‘h’ lies at a distance of ______ units from the base measured along the vertical radius
   a) h/2
   b) h/3
   c) h/4
   d) h/6
7. The point through which the whole weight of the body acts is called
   a) Centre of Gravity b) Moment of inertia.
   c) Both A and B d) None of the above
8. A circular hole of radius(r) is cut out from a circular disc of a radius (2r) in such a way that the diagonal of the hole is the radius of the disc, the C.G of the section lies at
   a) Centre of disc b) Centre of Hole
   c) Somewhere in the Disc d) Somewhere in the Hole.
9. If an area is symmetric about both x and y axis, then the centroid lies at
   a. X axis
   b. Y axis
   c. Intersection of x and y axis
   d. Anywhere on the area

Answers

1. b
2. a
3. b
4. c
5. a
6. c
7. a
8. c
9. c

CHAPTER – 6

AREA MOMENT OF INERTIA

LEARNING OBJECTIVES

After studying this chapter, you should be able to

- *Understand the concept of area moment of inertia*
- *State and prove parallel axis and perpendicular axis theorem*
- *Determine the area moment of inertia for given areas and evaluate radius of gyration*
- *Find the product of inertia for given shapes*

6.1 INTRODUCTION

According to the Newton's first law, a body continues in its state of rest or uniform motion unless some external agency acts on it. The property by virtue of which any body opposes any change in its present state is called inertia. The word inertia is used to describe the tendency of a body to be at rest or continue with a constant velocity. It is not only used in mechanics, but also in other areas, to indicate resistance to any change. This property is more useful in solving the problems related to theory of machines, mechanics of solids, fluid mechanics and dynamics.

Resistance to change (inertia) in the cross-sectional area of a bar is considered as an important property in engineering mechanics. This property is referred as area moment of inertia. It is a quantitative measure of the relative distribution of area for a given plane figure with respect to a reference axis. This property is used to predict resistance of plane geometric area to bending and deflection.

6.2 AREA MOMENT OF INERTIA

Consider a plane area A as shown in the figure. If r is the distance of elemental area dA form the axis XX′, then the moment of inertia of the area is defined as

$$I_{xx'} = \int r^2 dA$$

The term r dA is moment of area and the term r^2 dA is called moment of moment of area or simply second moment of area. Area moment of inertia is also called as *second moment of area*

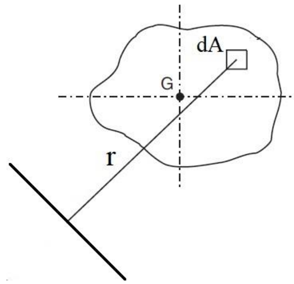


Fig. 6.1

The moment of inertia of the given area with respect to x and y axis is given as

$$\text{Ixx} = \int y^2 dA$$

$$\text{Iyy} = \int x^2 dA$$

The choice of reference axis has a significant impact in calculating moment of inertia. For example, in the above figure, the area A will have different moment of inertial with respect to x and y axes.

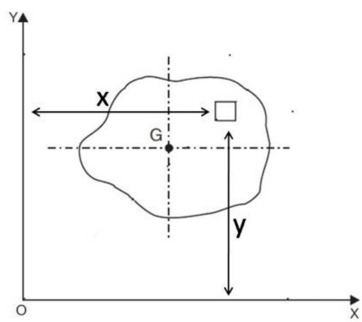


Fig. 6.2

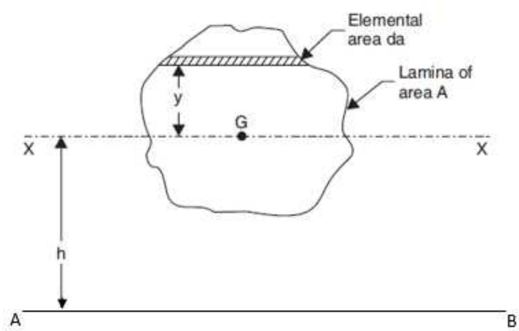
The expression of area moment of inertia is involved in many formulas relating to strength of beams, columns, deflection of beams etc., The moments of inertia of areas are always positive quantities and is a fourth dimensional term. Hence the units of area moment of inertia are mm^4 or m^4.

6.3 PARALLEL AXIS THEOREM (Transfer Formula)

Parallel axis theorem is used to transfer the centroidal moment of inertia to any other parallel axis.

This theorem states that "moment of inertia (I) about any non-centroidal axis (AB) is equal to sum of moment of inertia about centroidal axis (I_g) parallel to AB and area (A) times the square of the distance (y) between I_{AB} and I_{GG}"

$$I_{AB} = I_g + Ay^2$$



Proof

$$IAB = \int (h + y)^2 \, dA$$

$$IAB = \int (h^2 + 2hy + y^2)^2 \, dA$$

$$IAB = \int h^2 \, dA + 2y \int h \, dA + y^2 \int dA$$

$$I_{AB} = Igg + Ay^2$$

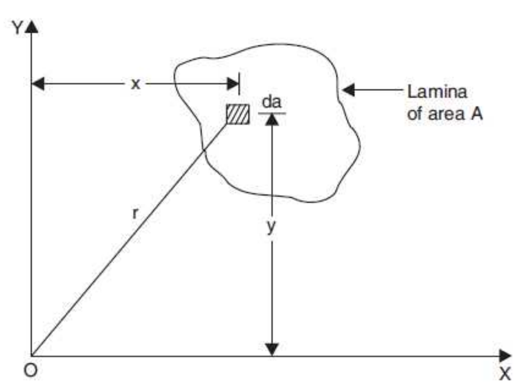
Note: Centroidal moment of inertia is the least possible moment of inertia of the area

6.4 PERPENDICULAR AXIS THEOREM

The moment of inertia of a given area about an axis perpendicular to the plane area passing through a point O is equal to the sum of moment of inertia about any two mutually perpendicular axes passing through that point O, the axes being in the plane of the area.

It is also termed as polar moment of inertia

$$\mathbf{I_{ZZ} = I_{XX} + I_{YY}}$$



Proof

$$Izz = \int r^2 \, dA$$

$$= \int (x^2 + y^2) \, dA$$

$$= \int x^2 \, dA + \int y^2 \, dA$$

$$I_{ZZ} = I_{XX} + I_{YY}$$

6.5 POLAR MOMENT OF INERTIA

The moment of inertia of an area about an axis perpendicular to a plane of the area is called polar moment of inertia and is denoted by J

$$J = I_{zz} = I_{xx} + I_{yy}$$

6.6 RADIUS OF GYRATION

Radius of gyration is usually denoted by the symbol k and is given by the relation

$$k = \sqrt{\frac{I}{A}}$$

or $$I = Ak^2$$

6.7 MOMENT OF INERTIA OF COMPOSITE AREAS

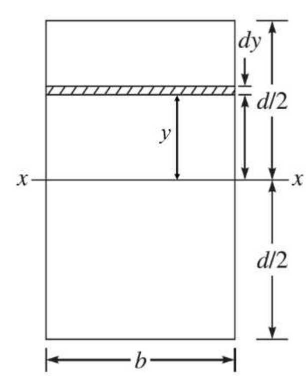
6.7.1 MOMENT OF INERTIA OF A RECTANGLE ABOUT THE CENTROIDAL AXIS

Consider an elemental strip of width *dy* at a distance *y* from the axis

Elemental area dA =b dy

Moment of inertia of the elemental area about the x-axis is y^2 (b dy)

Moment of inertia of the total area about the x-axis is



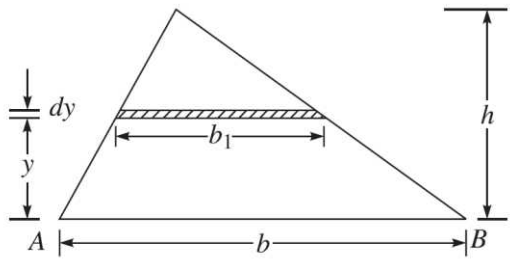
$$Ixx = \int_{-d/2}^{d/2} y^2 \, b \, dy$$

$$= b\left[\frac{y^3}{3}\right]_{-d/2}^{d/2}$$

$$Ixx = \frac{bd^3}{12}$$

6.7.2 MOMENT OF INERTIA OF A TRIANGLE ABOUT ITS BASE

Let us consider an elemental strip which is at a distance y from the base AB as shown in the figure.



Let dy be the thickness of the elemental strip and dA its area.
Now, width of this strip is given by

$$\frac{b_1}{b} = \frac{h-y}{h}$$

$$b_1 = \left[1 - \frac{y}{h}\right] b$$

The moment of inertia about the base

$$I_{AB} = \int y^2 dA$$

$$= \int_0^h y^2\, b_1\, dy$$

$$= \int_0^h y^2 \left[1 - \frac{y}{h}\right] b\, dy$$

$$= b\int_0^h y^2\, dy - \frac{b}{h}\int_0^h y^3\, dy$$

$$= b\left[\frac{y^3}{3}\right]_0^h - \frac{b}{h}\left[\frac{y^4}{4}\right]_0^h$$

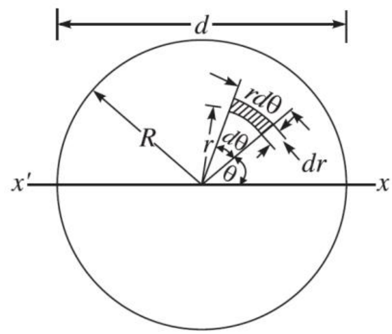
$$= \mathrm{b}\left[\frac{h^3}{3}\right] - \frac{b}{h}\left[\frac{h^4}{4}\right]$$

$$= \mathrm{b}h^3\left[\frac{1}{3}\right] - \left[\frac{1}{4}\right]$$

$$= \frac{bh^3}{12}$$

6.7.3 Moment of Inertia of Circle about its Diametral Axis

Consider an element of radius r and thickness dr located at an angle θ.



Now, Area of the elemental strip is

$$dA = (r d\theta)(dr)$$

The moment of inertia of elemental strip about the diametral axis is

$$dI_x = y^2 dA$$

$$= (r \sin\theta)^2 \times r d\theta\, dr$$

Now the moment of inertia of circle about x-x is given by

$$I_{xx} = \int_0^R \int_0^{2\pi} r^3 \sin^2\theta \, dr \, d\theta$$

$$= \int_0^{2\pi} \sin^2\theta \left(\frac{r^4}{4}\right)_0^R d\theta$$

$$= \frac{R^4}{4} \int_0^{2\pi} \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{R^4}{8} \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

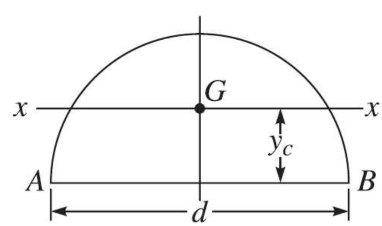
$$= \frac{R^4}{8} [2\pi - 0 + 0 - 0]$$

$$I_{xx} = \frac{\pi R^4}{4} \quad \text{or} \quad \frac{\pi d^4}{64} \qquad \because R = \frac{d}{2}$$

6.7.4 Moment of Inertia of Semi Circle

(a) About the base axis a – b

Consider an element of radius r and thickness dr located at an angle θ.



Area of the elemental strip is

$$dA = (r d\theta)(dr)$$

The moment of inertia of elemental strip about the base axis a-b is

$$dI_x = y^2 dA$$

$$= (r\sin\theta)^2 \times r d\theta\, dr$$

Now the moment of inertia of semi-circle about the base axis a-b is

$$I_{ab} = \int_0^R \int_0^\pi r^3 \sin^2\theta\, dr\, d\theta$$

$$= \int_0^\pi \sin^2\theta \left(\frac{r^4}{4}\right)_0^R d\theta$$

$$= \frac{R^4}{4}\int_0^\pi \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{R^4}{8}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^\pi$$

$$= \frac{R^4}{8}[\pi - 0]$$

$$I_{ab} = \frac{\pi R^4}{8} \quad \text{or} \quad \frac{\pi d^4}{128} \qquad \because R = \frac{d}{2}$$

(b) About centroidal x – axis

By parallel axis theorem, we know that

$$I_{ab} = I_{gg} + Ay^2$$

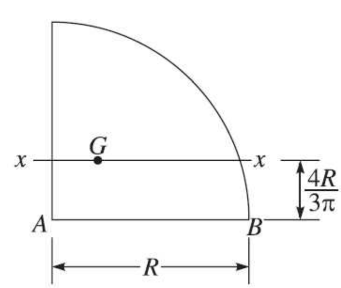
$$\text{Where } A = \frac{\pi R^2}{2} \text{ and } y = \frac{4R}{3\pi}$$

$$\therefore \qquad I_{gg} = I_{ab} - Ay^2 = \frac{\pi R^4}{8} - \left(\frac{\pi R^2}{2}\right) \times \left(\frac{4R}{3\pi}\right)^2$$

$$\therefore \qquad I_{gg} = I_{xx} = 0.11r^4$$

6.7.5 MOMENT OF INERTIA MOMENT OF INERTIA OF QUARTER CIRCLE

(a) About the base AB



Consider an element of radius r and thickness dr located at an angle θ.

Area of the elemental strip is

$$dA = (r d\theta)\, dr$$

The moment of inertia of elemental strip about the base axis a-b is

$$dI_x = y^2 dA$$

$$= (r \sin\theta)^2 \times r d\theta\, dr$$

Now the moment of inertia of quarter circle about the base axis a-b is

$$I_{ab} = \int_0^R \int_0^{\pi/2} r^3 \sin^2\theta\, dr\, d\theta$$

$$= \int_0^{\pi/2} \sin^2\theta \left[\frac{r^4}{4}\right]_0^R d\theta$$

$$= \frac{R^4}{4} \int_0^{\pi/2} \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{R^4}{4}\left[\frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\pi/2}\right]$$

$$= \frac{R^4}{4 \times 2}\left[\pi/2 - \frac{1}{2}(0)\right]$$

$$\therefore \qquad I_{ab} = \frac{\pi R^4}{16} \text{ or } \frac{\pi d^4}{256}$$

(b) About centroidal x – axis

By parallel axis theorem, we know that

$$I_{ab} = I_{gg} + Ay^2$$

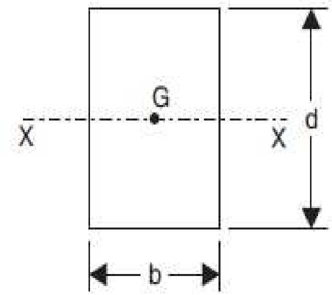
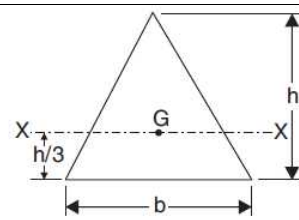
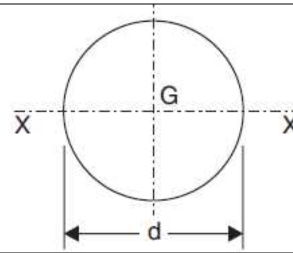
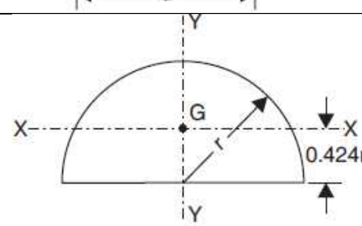
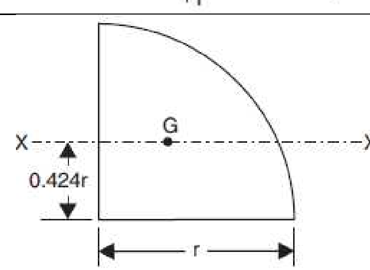
Where $A = \frac{\pi R^2}{4}$ and $y = \frac{4R}{3\pi}$

$$\therefore \qquad I_{gg} = I_{ab} - Ay^2$$

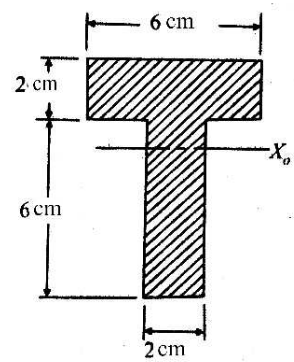
$$I_{gg} = \frac{\pi R^4}{16} - \left(\frac{\pi R^2}{4}\right) \times \left(\frac{4R}{3\pi}\right)^2$$

$$= r^4\left[\frac{\pi}{16} - \frac{4}{9\pi}\right] = 0.055\, R^4$$

Table 6.1: Moment of Inertia of Standard shapes

Shape	Axis	Moment of inertia	
Rectangle	i. Centroidal x-x ii. Centroidal y-y iii. Base ab	$\frac{bd^3}{12}$ $\frac{db^3}{12}$ $\frac{bd^3}{3}$	
Triangle	i. Centroidal x-x ii. Base ab	$\frac{bh^3}{36}$ $\frac{bh^3}{12}$	
Circle	i. Diametral axis	$\frac{\pi d^4}{64}\,or\,\frac{\pi R^4}{4}$	
Semi – circle	i. Base ab ii. Centroidal axis	$\frac{\pi d^4}{128}$ 0.0068598 d^4 = 0.11 R^4	
Quarter Circle	i. Base ab ii. Centroidal axis	$\frac{\pi d^4}{256}$ 0.00343 d^4 = 0.0055 R^4	

Example 6.1 Determine the moment of inertia of the T-section shown in the figure about its centroidal x axis.



Solution This problem can be solved by finding the centroid first and then finding moment of inertia

Centroid

The figure is symmetrical about y-axis

The figure can be divided as

i. Rectangle 1
ii. Rectangle 2

Rectangle 1

Area (a_1) = 6 x 2 = 12 cm^2

Centroid (y_1) = 3 cm

Rectangle 2

Area (a_2) = 2 x 6 = 12 cm^2

Centroid (y_2) = 6+1 = 7 cm

Now $\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{12(3)+12(7)}{12+12} = \frac{120}{24} = 5$ cm

Moment of Inertia

For rectangle 1

$I_{XX} = I_{GG} + Ay^2$

$= \frac{bd^3}{12} + 12\ (5\text{-}3)^2$

$= \frac{2x6^3}{12} + 12\ (2)^2 = 84\ cm^4$

For rectangle 2

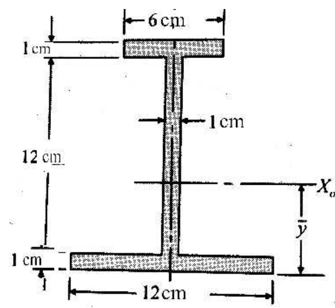
$I_{XX} = I_{GG} + Ay^2$

$= \frac{bd^3}{12} + 12\ (7\text{-}5)^2$

$= \frac{6 \times 2^3}{12} + 12\ (2)^2 = 52\ \text{cm}^4$

Moment of inertial of total figure = 84 + 52 = 136 cm^4

Example 6.2 Determine the moment of inertial of the area shown in the figure with respect to its centroidal axes.



Solution First let us find the centroid and then we will find moment of inertia

Centroid

$a_1 = 12\ \text{cm}^2$
$a_2 = 12\ \text{cm}^2$
$a_3 = 6\ \text{cm}^2$
$y_1 = 0.5$ cm
$y_2 = 1+6 = 7$ cm
$y_3 = 1+12+0.5 = 13.5$ cm

$$y = \frac{a_1y_1+a_2y_2+a_3y_3}{a_1+a_2+a_3} = \frac{12(0.5)+12(7)+6(13.5)}{12+12+6} = \frac{171}{30} = 5.7 \text{ cm}$$

Moment of Inertia

For rectangle 1

$I_{XX} = I_{GG} + Ay^2$

$= \frac{bd^3}{12} + 12\ (5.7\text{-}0.5)^2$

$= \frac{12x1^3}{12} + 12\ (5.7\text{-}0.5)^2 = 325.48\ \text{cm}^4$

For rectangle 2

$I_{XX} = I_{GG} + Ay^2$

$= \frac{bd^3}{12} + 12\ (7\text{-}5.7)^2$

$= \frac{1 \times 12^3}{12} + 12\ (7\text{-}5.7)^2 = 164.28\ \text{cm}^4$

For rectangle 3

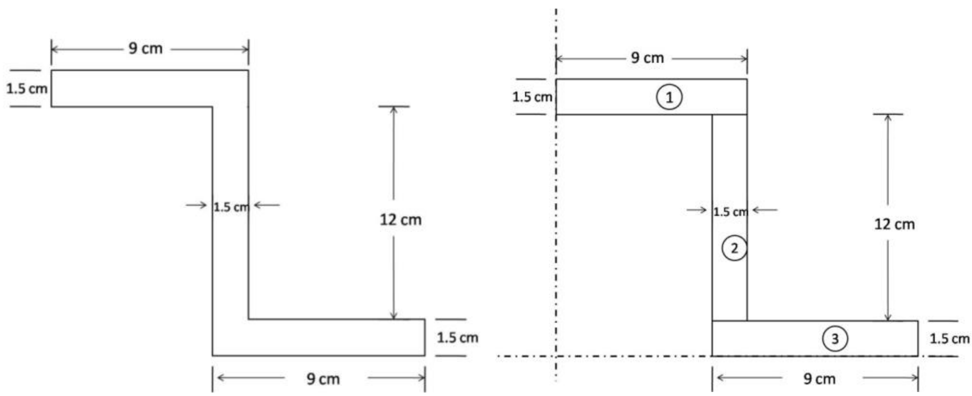
$I_{XX} = I_{GG}+Ay^2$

$= \frac{bd^3}{12} +6\ (13.5\text{-}5.7)^2$

$= \frac{6 \times 1^3}{12} +6\ (13.5\text{-}5.7)^2 = 365.54\ cm^4$

Moment of inertial of total figure = 325.48+164.28+365.54 = 855.3 cm^4

Example 6.3 Determine the moment of inertia for the Z section shown below about centroidal axes



Solution Centroid

Rectangle 1

Area (a_1) = 9×1.5 = 13.5 cm^2

x_1 = (9/2) = 4.5 cm

y_1 = 1.5+12 + (1.5/2) = 14.25 cm

Rectangle 2

Area (a_2) = 12×1.5 = 18 cm^2

x_2 = (9-1.5)+(1.5/2) = 8.25 cm

y_2 = 1.5+ (12/2) = 7.5 cm

Rectangle 3

Area (a_3) = 9×1.5 = 13.5 cm^2

x_3 = (9-1.5)+(9/2) = 12 cm

y_3 = (1.5/2) =0.75 cm

$\bar{x} = \frac{a_1x_1+a_2x_2+a_3x_3}{a_1+a_2+a_3} = \frac{13.5\times4.5+18\times8.25+13.5\times12}{13.5+18+13.5}$ =8.25 cm

$\bar{y}= \frac{a_1y_1+a_2y_2+a_3y_3}{a_1+a_2+a_3} = \frac{13.5\times14.25+18\times7.5+13.5\times0.75}{13.5+18+13.5} = 7.5$ cm

Moment of Inertia about centroidal x-axis

For rectangle 1

$I_{XX} = I_{GG}+Ay^2$

$= \frac{bd^3}{12} +13.5\ (14.25 - 7.5)^2$

$= \frac{9\times1.5^3}{12} +13.5\ (14.25 - 7.5)^2 = 617.625\ cm^4$

For rectangle 2

$I_{XX} = I_{GG}+Ay^2$

$= \frac{bd^3}{12} +18\ (7.5-7.5)^2$

$= \frac{12\times1.5^3}{12} +18\ (7.5-7.5)^2 = 3.375\ cm^4$

For rectangle 3

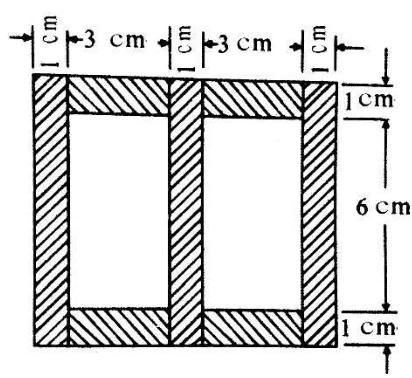
$I_{XX} = I_{GG}+Ay^2$

$= \frac{bd^3}{12} +13.5\ (7.5-0.5)^2$

$= \frac{9\times1.5^3}{12} +13.5\ (7.5-0.75)^2 = 617.625\ cm^4$

Moment of inertial of total figure = 617.625 + 3.375 + 617.625 = 1238.625 cm^4

Example 6.4 Blocks of wood are glued together to form the shaded section shown in the figure. Find the moment of inertia of the shaded section about its horizontal and vertical centroidal axes.



Solutions I_{XX} = (Ixx of rectangle 1) – 2(Ixx of rectangle 2)

$I_{XX} = \frac{9\times8^3}{12} - 2 \times \frac{3\times6}{12} = 276\ cm^4$

I_{YY} of rectangle 1

$I_{YY} = \frac{8 \times 9^3}{12} = 486$ cm^4

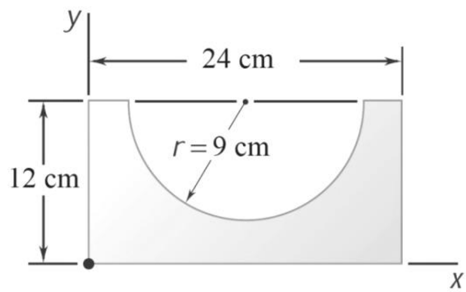
I_{YY} of rectangle 2

$I_{YY} = \frac{6 \times 3^3}{12} + (6x3)\,(1.5\text{-}0.5)^2 = 85.5$ cm^4

I_{YY} of total figure

$= 486 - 2(85.5) = 315$ cm^4

Example 6.5 Determine the moment of inertia of the shaded area with respect to the *x* axis



Solution MI of total figure = MI of rectangle – MI of quarter circle

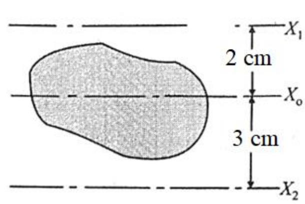
i. MI of rectangle = $\frac{bd^3}{3} = \frac{24 \times 12^3}{3} = 12096$ cm^4
ii. MI of quartile circle = 0.0068598 (d)^4 + A (y)^2

$$= 0.0068598\,(9)^4 + \frac{\pi \times 4.5^2}{2}\left[12 - \frac{4 \times 4.5}{3\pi}\right]^2$$

$$= 3283.47 \text{ cm}^4$$

MI of total figure = 12096 – 3283.47 = 8812.53 cm^4

Example 6.6 The shaded section show in the figure has an area of 5 cm^2 and a moment of inertia of 30 cm^4 about axis x_1. Find the moment of inertia about x_2



Solution Moment of inertial about X_o

Using parallel axis theorem

$I_{AB} = I_{xx} + AY^2$

$30 = I_{xx} + 5(2)^2$

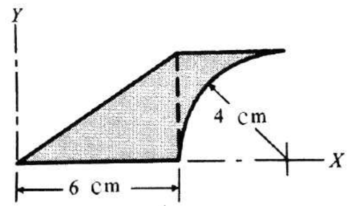
$Ixx = 10\ cm^4$

Moment of inertial about X_2

$I_{AB} = Ixx + AY^2$

$I_{AB} = 10 + 5(3)^2 = 55cm^4$

Example 6.7 Find the moment of inertia about the indicated Y axis for the shaded area shown in the figure.



Solution Divide the figure into basic shapes

i. Right angle triangle
ii. Square
iii. Quarter circle

Moment of inertia of Right angle triangle about Y

$$I_{YY} = I_{XX} = I_{GG} + Ay^2$$

$$= \frac{bh^3}{36} + \left[\frac{1}{2} \times 6 \times 4\right] (4)^2$$

$$= \frac{4 \times 6^3}{36} + \left[\frac{1}{2} \times 6 \times 4\right] (4)^2 = 216\ cm^4$$

Moment of inertia of square about Y

$$I_{YY} = I_{GG} + Ay^2$$

$$= \frac{bd^3}{12} + (4x4)\ (8)^2$$

$$= \frac{4 \times 4^3}{12} + (4x4)\ (8)^2 = 1045.33\ cm^4$$

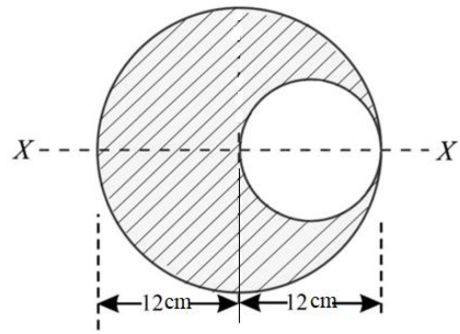
Moment of inertia of quarter circle about Y

$$I_{YY} = I_{GG} + Ay^2$$

$$= 0.0055\ R^4 + \frac{\pi \times 4^2}{4}\left[10 - \frac{4 \times 4}{3\pi}\right]^2 = 880.612\ cm^4$$

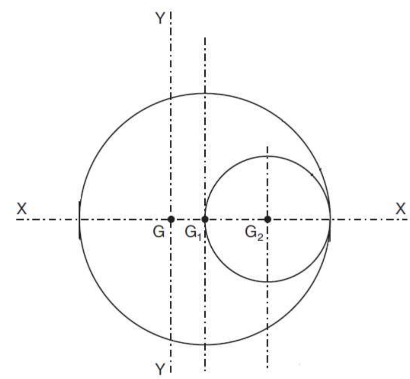
MI of total figure about given y-axis = 216 + 1045.33 – 880.612 = 380.718 cm^4

Example 6.8 A circular hole of diameter 12 cm is punched out from a circular plate of radius 12 cm as shown in the Figure. Find the moment of inertia of the shaded area about both the centroidal axes.



Solution First let us find the centroid and then we will find moment of inertia

Centroid



$a_1 = 452.39$ cm^2; $a_2 = 113$ cm^2

$x_1 = 12 = 12$ cm; $x_2 = 12+6 = 18$ cm

$$\bar{x} = \frac{a_1x_1 - a_2x_2}{a_1 - a_2} = \frac{452.39(12) - 113(18)}{452.39 - 113} = 10 \text{ cm}$$

Moment of Inertia

$$I_{XX} = \frac{\pi \times d^4}{64} - \frac{\pi \times d^4}{64}$$

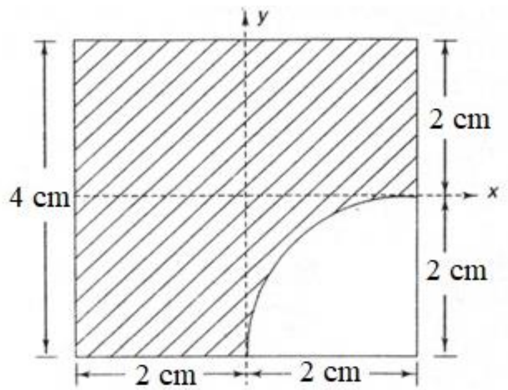
$$= \frac{\pi \times 24^4}{64} - \frac{\pi \times 12^4}{64} = 15268.14 \text{ cm}^4$$

$$I_{YY} = \left[\frac{\pi\, d^4}{64} + A\,(y)^2\right] - \left[\frac{\pi\, d^4}{64} + A\,(y)^2\right]$$

$$= \left[\frac{\pi \times 24^4}{64} + \pi \times (12)^2\,(12 - 10)^2\right] - \left[\frac{\pi\, 12^4}{64} + \pi \times (6)^2\,(18 - 10)^2\right]$$

$= 18095.57 - 8256.10 = 9839.46$ cm^4

Example 6.9 A corner of radius 2 cm is cut off from a square plate of 4 cm side as shown in the figure. Find the moment of inertia of the remaining plate about the given x-axis.



Solution Moment of inertia of total figure about x-axis = Moment of inertia of square – moment of inertia of quarter circle

MI of Square

$I_{xx} = \frac{bd^3}{12} = \frac{4 \times 4^3}{12} = 21.33$ cm^4

MI of Quarter circle

$I_{xx} = 0.0055\ R^4 + A\ (y)^2$

$= 0.0055\ (2)^4 + \frac{\pi \times 2^2}{4}\left[2 - \frac{4 \times 2}{3\pi}\right]^2$

$= 4.25$ cm^4

Moment of inertia of total figure = 21.33 – 4.25 = 17.08 cm^4

6.7 PRODUCT OF INERTIA

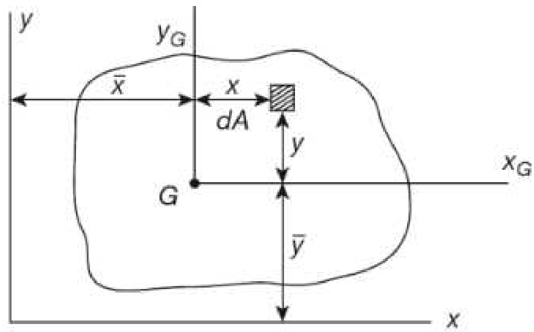
The product of inertia of an area is defined with reference to a pair of perpendicular axes that lie in a same plane of area. The product of inertia of an area, which is denoted by I_{xy} is given by

$$I_{xy} = \int xy\, dA$$

The term $\int xy\, dA$ is known as product of inertia, since it is obtained by multiplying the element of area by the product of two distances. The unit of Product of inertia is same as area moment of inertia i.e, mm^4 or m^4. Unlike the moments of inertia I_x and I_y, the product of inertia I_{xy} can be positive, negative, or zero. When an area A is symmetric about x axis or y axis or both, then the product of inertia I_{xy} of A will be zero.

6.8 TRANSFER FORMULA FOR PRODUCT OF INERTIA

Consider a plane area as shown in the figure



Form the definition of product of inertia we have

$$I_{X_G Y_G} = \int xy\, dA$$

Also

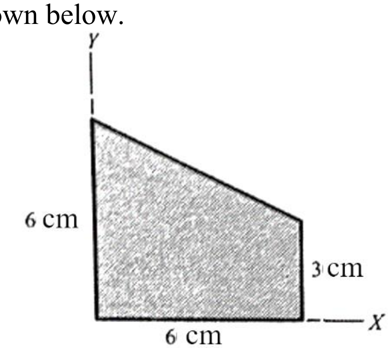
$$I_{XY} = \int (\bar{x} + x)(\bar{y} + y) dA$$

$$\int (xy + x\bar{y} + \bar{x}\, y + \bar{x}\, \bar{y})dA$$

$$= \int xy\, dA + \int x\bar{y}\, dA + \int \bar{x}\, y\ dA + \int \bar{x}\, \bar{y}\ \ dA$$

$$I_{XY} = I_{X_G Y_G} + \text{A}\, \bar{x}\, \bar{y}$$

Example 6.10 Find the product of inertia with respect to the specified XY axes for the shaded area for the figure shown below.



Solution The figure can be divided as

- Right angle triangle of base 6 cm and height 3 cm
- Rectangle of size 6x3

For figure 1

$P_{xy} = P_{xy} + A\ \bar{x}\bar{y}$

$= \frac{-b^2h^2}{72} + A.\ \bar{x}\bar{y}$

$= \frac{-6^2 \times 3^2}{72} + \left[\frac{1}{2} \times 6 \times 3\right] \times 2 \times (3 + 1)$

= -4.5 +72 = 67.5 cm^4

ii. For figure 2

$P_{xy} = P_{xy} + A\ \bar{x}\bar{y}$

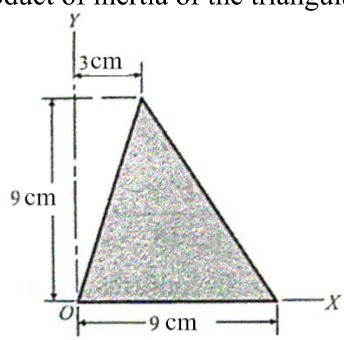
= 0 + (6 x 3) x 3 x 1.5

= 81 cm^4

Total $P_{xy} = P_{xy}(1) + P_{xy}(2)$

= 67.5 + 81 = 148.5 cm^4

Example 6.11 Compute the product of inertia of the triangular area shown in the figure



Solution The figure can be divided as

- Right angle triangle of base 3 cm and height 9 cm
- Right angle triangle of base 6 cm and height 9 cm

i. For figure 1

$P_{xy} = P_{xy} + A\ \bar{x}\bar{y}$

$= \frac{b^2h^2}{72} + A\ \bar{x}\bar{y}$

$= \frac{9^2 \times 3^2}{72} + \left[\frac{1}{2} \times 9 \times 3\right] \times 2 \times (3)$

= 10.125 + 81 = 91.125 cm^4

ii. For figure 2

$P_{xy} = P_{xy} + A\bar{x}\bar{y}$

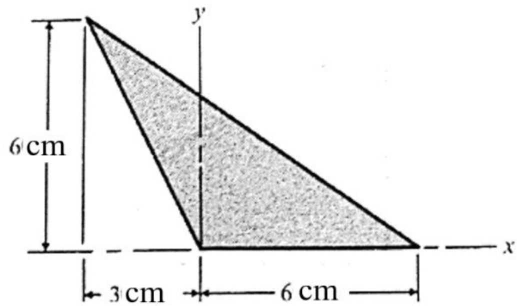
$= \frac{-b^2h^2}{72} + A\ \bar{x}\bar{y}$

$= \frac{-6^2 \times 9^2}{72} + \left[\frac{1}{2} \times 6 \times 8\right] \times 5 \times (3)$

= -40.5 + 405 = 364.5 cm^4

Total 91.125 + 364.5 = 455.625 cm^4

Example 6.12 Determine the product of inertia of the shaded triangular area shown in the figure with respect to the given X and Y axes



Solution The figure can be divided as

- Right angle triangle of base 9 cm and height 6 cm
- Right angle triangle of base 3 cm and height 6 cm

Product of inertia for figure 1

$P_{xy} = P_{xy} + A\bar{x}\bar{y}$

$= \frac{-b^2h^2}{72} + A\ 0\bar{y}$

$= \frac{-3^2 \times 6^2}{72} = -40.5$

Product of inertia for figure 2

$P_{xy} = P_{xy} + A\ \bar{x}\bar{y}$

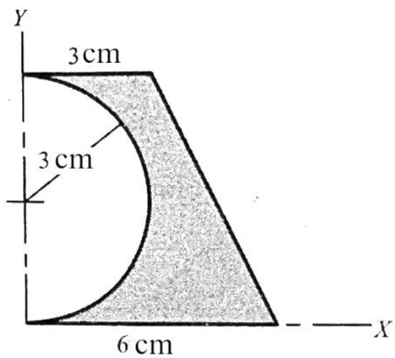
$= \frac{-b^2h^2}{72} + A\ \bar{x}\bar{y}$

$= \frac{-6^2 \times 3^2}{72} + \left[\frac{1}{2} \times 6 \times 3\right] \times (-2) \times (2)$

$= -4.5 - 36 = -40.5$

Final = – 40.5 – (–40.5) = 0

Example 6.13 Find the product of inertia of the shaded area about the specified X and Y axes



Solution The figure can be divided as

- Rectangle of size 3 x 6 cm
- Right angle triangle of base 3 cm and height 6 cm
- Semi-circle of radium 3 cm

i. For figure 1

$P_{xy} = P_{xy} + A\,\bar{x}\bar{y}$

$= 0 + (6 \times 3) \times 1.5 \times 3$

$= 81 cm^4$

ii. For figure 2

$P_{xy} = P_{xy} + A\,\bar{x}\bar{y}$

$= \frac{-b^2h^2}{72} + A\,\bar{x}\bar{y}$

$= \frac{-3^2 \times 6^2}{72} + \left[\frac{1}{2} \times 3 \times 6\right] \times 2 \times (3+1)$

$= 67.5 cm^4$

iii. For figure 3

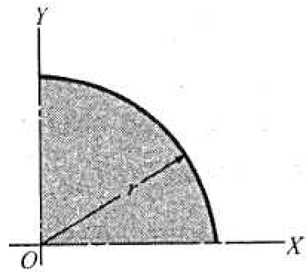
$P_{xy} = P_{xy} + A\,\bar{x}\bar{y}$

$= 0 + \left[\frac{\pi \times 6^2}{2}\right] \times \left(\frac{4 \times 3}{3\pi}\right) \times (3)$

$= 54 cm^4$

Final PI = 81+67.5-54 = 94.5 cm^4

Example 6.14 Determine the product of inertia of the quarter circular area shown in figure with respect to the given X and Y axes



Solution Consider an elemental area dA as shown in the figure

$$dA = (r\,d\theta) \times dr$$

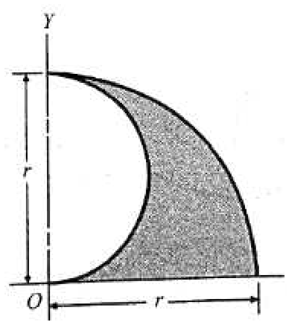
$$P_{xy} = \int_0^R \int_0^{\pi/2} (r\,d\theta\,dr)\,r cos\theta\,r sin\theta$$

$$= \int_0^R r^3\,dr \int_0^{\pi/2} \sin\theta\,cos\theta$$

$$= \frac{R^4}{4} \times \frac{1}{2}$$

$$= \frac{R^4}{8}$$

Example 6.15 Use the result of the previous problem to determine the product of inertia of the shaded area shown in the figure with respect to given x and y axes



Solution The figure can be divided as

- Quarter circle of radius r
- Semi-circle of radius r/2

i. P_{xy} for figure 1

$P_{xy} = \frac{r^4}{8}$

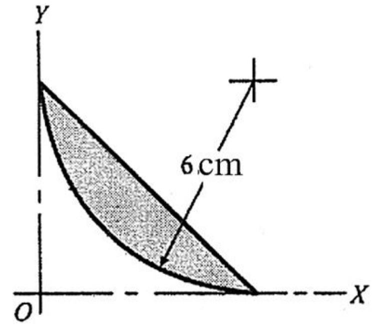
ii. P_{xy} for figure 2

$P_{xy} = P_{xy} + A\,\bar{x}\bar{y}$

$$= 0 + \left[\frac{\pi\times(r/2)^2}{2}\right]\times\left(\frac{4\times(r/2)}{3\pi}\right)\times\left(\frac{r}{2}\right)=\frac{r^4}{24}$$

Final $P_{xy}=\frac{r^4}{8}-\frac{r^4}{24}=\frac{r^4}{12}$

Example 6.15 Compute the product of inertia of the shaded area shown in the figure with respect to specified X and Y axes.



Solution The figure can be divided as

- Quarter circle of radius 6 cm
- Right angle triangle of base 6 cm and height 6 cm

i. For figure 1

$P_{xy} = P_{xy} + A\,\bar{x}\bar{y}$

$$\frac{r^4}{8} = P_{xy} + \left[\frac{\pi\times(r)^2}{4}\right]\times\left(\frac{-4\,r}{3\pi}\right)\times\left(\frac{-4r}{3\pi}\right)$$

$$\therefore Pxy + \frac{r^4}{8} - r^4\,(0.14147) = -0.01647\,r^4$$

Now $P_{xy} = P_{xy} + A\,\bar{x}\bar{y}$

$$= -0.01647\,r^4 + \left[\frac{\pi\times(r)^2}{4}\right]\times\left(r-\frac{4\,r}{3\pi}\right)\times\left(r-\frac{4r}{3\pi}\right)$$

$= 0.24373\ r^4 = 0.24373 \times 6^4 = 315.877\ cm^4$

ii. For figure 2

$P_{xy} = P_{xy} + A\,\bar{x}\bar{y}$

$$= \frac{-b^2h^2}{72} + A\,\bar{x}\bar{y}$$

$$= \frac{-6^2\times6^2}{72} + \left[\frac{1}{2}\times6\times6\right]\times(4)\times(4)$$

$= -18 + 288 = 270\ cm^4$

Final $P_{xy}= 315.877 - 270 = 45.877\ cm^4$

SUMMARY

- Area moment of inertia is a quantitative measure of the relative distribution of area for a given plane figure with respect to a reference axis.
- Parallel axis theorem is used to transfer the centroidal moment of inertia to any other parallel axis
- Parallel axis theorem states that "moment of inertia (I) about any non-centroidal axis (AB) is equal to sum of moment of inertia about centroidal axis (I_g) parallel to AB and area (A) times the square of the distance (y) between I_{AB} and I_{GG}"
- The moment of inertia of an area about an axis perpendicular to a plane of the area is called polar moment of inertia
- The product of inertia of an area is defined with reference to a pair of perpendicular axes that lie in a same plane of area.

EXERCISES

I. Self-Assessment Questions

1. Define second moment of area
2. State and prove parallel axis theorem for area moment of inertia
3. Define the terms radius of gyration and polar moment of inertia
4. Prove that $I_{zz} = I_{xx} + I_{yy}$
5. Define product of inertia of an area.

II. Multiple Choice Questions

1. The unit of area moment of inertia is ____
   a. mm^2
   b. mm^3
   c. mm^4
   d. Kg
2. The sign of moment of inertia of an area is always ____
   a. Positive
   b. Negative
   c. can be positive or negative
   d. zero
3. Area moment of inertia is also called as
   a. First moment of area
   b. Second moment of area
   c. Moment of area
   d. Moment of a force

4. Area moment of inertia of a triangle of base b and height h about the base is
    a. $\frac{hb^3}{12}$
    b. $\frac{bh^3}{12}$
    c. $\frac{bh^3}{36}$
    d. $\frac{hb^3}{24}$
5. Area moment of inertia of a plane area is concerned with
    a. Bending
    b. Rotation
    c. Hardness
    d. Brittleness
6. Moment of inertia of an area about an axis perpendicular to the plane of area is called
    a. Radius of gyration
    b. Product of inertia
    c. Polar moment of inertia
    d. Second moment of area
7. In area moment of inertia, parallel axis theorem provides the following relation
    a. $I_{AB} = I_{xx} + I_{yy}$
    b. $I_{AB} = Ak^2$
    c. $I_{AB} = I_{gg} + Md^2$
    d. $I_{AB} = I_{gg} + Ad^2$
8. Radius of gyration of an area is given by
    a. $K = \sqrt{\frac{I}{A}}$
    b. $K = \sqrt{\frac{I}{M}}$
    c. $K = \frac{I}{A}$
    d. $I = A^2k$
9. Product of inertia is defined by
    a. $\int x\, dA$
    b. $\int x^2 dA$
    c. $\int x\, y\, dA$
    d. $\int x^3 dA$
10. Unit of product of inertia is
    a. mm^2
    b. mm^3
    c. mm^4
    d. mm
11. The product of inertia (I_{xy}) can be
    a. Positive

b. Negative
c. Zero
d. Any of the above

Answers

1. c
2. a
3. b
4. b
5. a
6. c
7. d
8. a
9. c
10. c
11. d

CHAPTER – 7

MASS MOMENT OF INERTIA

LEARNING OBJECTIVES

After studying this chapter, you should be able to

- *Define mass moment of inertia*
- *Differentiate between area moment of inertia and mass moment of inertia*
- *Derive the expression for mass moment of inertia for different bodies*
- *Calculate the mass moment of inertia of a given solid object*

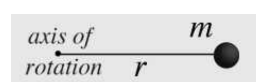
7.1 INTRODUCTION

Mass moment of inertia is a quantity that determines the ability of a solid object to resist change in rotational speed about a specific axis. It not only depends on the mass but also the distribution of mass around the axis which is considered for the study. This property helps in determining the torque required for a desired angular acceleration about a rotational axis. Mass moment of inertia holds the same relation to angular acceleration as in case mass has with linear acceleration. The higher the Mass Moment of Inertia the smaller the angular acceleration about that axis for a given torque. For a point mass, the mass moment of inertia is the mass time the square of perpendicular distance to the rotational axis.

7.2 MASS MOMENT OF INERTIA: Definition

Mass moment of Inertia of a body having a mass m is defined as the product of the mass and the square of the distance between the mass centre of the body and the axis which is considered.

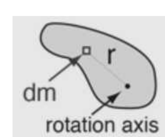
Since the moment of inertia of a point mass is defined by



$$I = mr^2$$

Consider an infinitesimal mass element dm of a body having mass m. This mass element dm is called a differential element of mass. The moment of inertia of this element dm is given by

$$dI = r^2\, dm$$



The "d" preceding any quantity denotes a vanishingly small or differential amount of it

Then the moment of inertia of the total body to cover the entire mass is given by

$$I = \int dI = \int r^2 dm$$

Where,

- dm is elemental mass of the body
- r is distance from the axis

Note: The units of mass moment of inertia are kg-m^2

Example: Usually, a fly wheel is designed to have a large mass moment of inertia about its axis of rotation.

The following figure shows a pottery wheel which has larger mass moment of inertia

Fig. 7.1 Pottery wheel

7.3 AREA AND MASS MOMENT OF INERTIA: DIFFERENCES

Area moment of Inertia	*Mass moment of inertia*
This property is related to plane areas	This property is related to three-dimensional objects
Distribution of area with respect to some reference axis	Distribution of mass with respect to some reference axis
It is a measure of resistance to bending	It is a measure of an object's resistance to change in rotation direction
Units: mm^4 or m^4	Units: kg-m^2

7.4 RADIUS OF GYRATION

Radius of gyration may be defined as the distance from the axis of rotation at which, if the whole mass of the body is concentrated, its moment of inertia about the axis is the same as that with the actual distribution of mass.

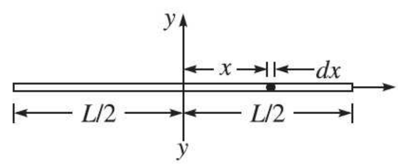
$$I = MK^2$$

$$K = \sqrt{\frac{I}{M}}$$

7.5 MASS MOMENT OF INERTIA OF HOMOGENEOUS BODIES

7.5.1 MASS MOMENT OF INERTIA OF A UNIFORM ROD

Let m be the mass of the rod per unit length
Then mass of the element $dm = m\ dx$.



$$\mathrm{I} = \int_{-L/2}^{L/2} \mathrm{x}^2\,\mathrm{dm} = \int_{-L/2}^{L/2} \mathrm{x}^2\,\mathrm{mdx}$$

$$= m\left[\frac{x^3}{3}\right]_{-L/2}^{L/2}$$

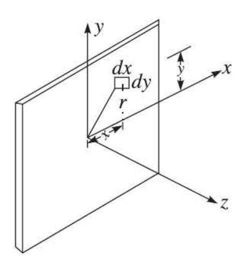
$$= \frac{mL^3}{12}$$

$$I = \frac{ML^2}{12}$$

7.5.2 MOMENT OF INERTIA OF PLATE

Let's consider a plate of width b, depth d, thickness t and mass M. Consider an elemental strip of width *dy* at a distance *y* from *x*-axis as shown

Elemental mass dm = ρ b t dy; where ρ is the unit mass of the material



$$\mathrm{I_{xx}} = \int_{-d/2}^{+d/2} \mathrm{y}^2\,\mathrm{dm}$$

$$= \int_{-d/2}^{+d/2} \mathrm{y}^2 \rho\,\mathrm{b}\,\mathrm{t}\,\mathrm{dy}$$

$$= \rho\,\mathrm{b}\,\mathrm{t}\left[\frac{y^3}{3}\right]_{-d/2}^{d/2}$$

$$= \frac{\rho\,\mathrm{b}\,\mathrm{t}\,\mathrm{d}^3}{12}$$

But mass of the plate M = ρ b t d

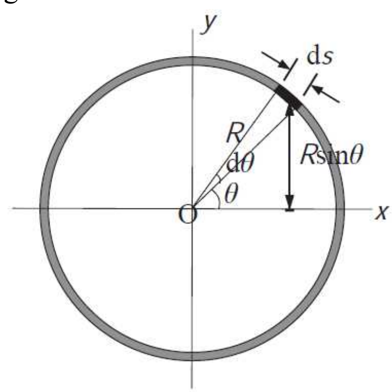
$$I_{xx} = \frac{Md^2}{12}$$

Similarly, moment of inertia of the plate about *y*-axis i s

$$I_{yy} = \frac{Mb^2}{12}$$

7.5.3 MOMENT OF INERTIA OF CIRCULAR RING

Consider a ring of radius R and mass m as shown in the figure.
Mass per unit length of ring is $\rho = {}^{m}/_{2\pi R}$.



Consider an elemental length $ds = R\,d\theta$ at angle θ to the x-axis.

The mass of elemental length $dm = \rho ds = \rho R\,d\theta$.

Moment of inertia of elemental mass about x-axis is

$$dI_x = y^2 dm$$

$$dI_x = (R\sin\theta)^2 \rho R\,d\theta$$

Total moment of inertia $\quad I_x = \int_{-\pi/2}^{\pi/2} \rho R^3 \sin^2\theta\,d\theta$

$$I_x = \rho R^3 \int_{-\pi/2}^{\pi/2} sin^2\theta\,d\theta = \rho R^3 \int_{-\pi/2}^{\pi/2} \left[\frac{1-cos\,2\theta}{2}\right] d\theta$$

$$I_x = \rho\pi R^3$$

$$I_x = \left(\frac{m}{2\pi R}\right)\pi R^3 = \frac{1}{2} mR^2$$

Moment of inertia of the ring about diametral axis is

$$I_x = \frac{1}{2} mR^2$$

Similarly, $I_y = \frac{1}{2} mR^2$

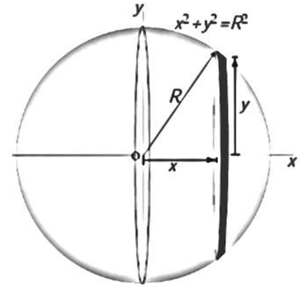
7.5.4 MASS MOMENT OF INERTIA OF A HOMOGENEOUS SPHERE

Consider a differential element of thickness dx at a distance of x from y-axis.

From geometry,

$$R^2 = x^2 + y^2$$

$$\therefore y^2 = R^2 - x^2$$



The mass moment of inertia of element,

$$d(I_{xx}) = \frac{1}{2}y^2 dm$$

$$= \frac{1}{2}(R^2 - x^2)dm$$

$$= \frac{1}{2}(R^2 - x^2)(\rho\pi y^2 dx)$$

$$= \frac{1}{2}(R^2 - x^2)\rho\pi(R^2 - x^2)\,dx$$

$$\frac{1}{2}\rho\pi(R^2 - x^2)^2\,dx$$

Hence mass moment of inertia of sphere,

$$I_{xx} = \frac{1}{2}\rho\pi\int_{-R}^{+R}(R^2 + x^2)^2\,dx$$

$$= \frac{1}{2}\rho\pi \times \frac{2 \times 8R^5}{15}$$

$$= \rho.\left(\frac{4}{3}\pi R^3\right).\frac{2}{5}R^2$$

$$= \frac{2}{5}mR^2$$

$$I_{xx} = I_{yy} = I_{ZZ} = \frac{2}{5}mR^2$$

Example 7.1 Two circular discs of radius 10 cm and thickness 4 cm made of steel are attached to the two ends of an aluminum rod of radius 4 cm and length 60 cm as shown in the figure. Find the moment of inertia about the axis of rotation. Assume the density of aluminum as 2710 kg/m^3 and steel as 7860 kg/m^3

Solution

i. *Circular disc*

For circular disc $I_{zz} = \frac{Mr^2}{2}$

Mass of the circular disc (M) = ρ x v

= ρ x $\pi r^2 h$

= 7860 x π $(0.1)^2$x 0.04 = 9.877 kg

ii. Cylinder

For rod $I_{zz} = \frac{Mr^2}{2}$

Mass of the rod (M) = ρ x v = 2710 x π $(0.04)^2$x 0.6 = 8.173 kg

Moment of inertia of resulting solid = 2 ($I_{zz\,(circular\,disc)}$) + $I_{zz\,(cylinder)}$

$$= 2\left[\frac{Mr^2}{2}\right] + \frac{Mr^2}{2} = 2\left[\frac{9.877\,(0.1)^2}{2}\right] + \frac{8.173(0.04)^2}{2}$$

= 0.105 kg-m^2

Example 7.2 A hallow sphere of 40 cm outer diameter and 30 cm inter diameter is made of cast iron. Determine the moment of inertia and radius of gyration of the hallow sphere with respect to diametral axis. Take density of cast iron as 7860 kg/m^3

Solution

i. Larger sphere

Mass of the sphere (M) = ρ x v

$= 7860 \times \frac{4}{3}\pi r^3 = 7860 \times \frac{4}{3}\pi\,(20 \times 10^{-2})^3$

= 263.3 kg

ii. Smaller sphere

Mass of the sphere (M) = ρ x v

$= 7860 \times \frac{4}{3}\pi r^3 = 7860 \times \frac{4}{3}\pi\,(15 \times 10^{-2})^3$

= 111.11 kg

For sphere $I = \frac{2}{5}MR^2$

Moment of inertia of the resulting solid = I $_{(larger\ sphere)}$ – I $_{(Smaller\ sphere)}$

$I = \frac{2}{5}MR^2 - \frac{2}{5}MR^2$

$$= \frac{2}{5} \times 263.3 \times 0.2^2 - \frac{2}{5} \times 111.11 \times 0.15^2$$

= 3.21 kg-m^2

Radius of gyration $k = \sqrt{\frac{3.21}{263.3-111.11}} = 0.14\,m$

Example 7.3 A right circular cone made of steel has a height of 50 cm and a base diameter of 60 cm. A hole 15 cm deep and 20 cm diameter is drilled from the centre of the base of the cone and filled with lead. Determine the mass moment of inertia of the resulting solid with respect to its geometric axis. Assume the density of lead as 11400 kg/m^3 and steel as 7860 kg/m^3

Solution Moment of inertia of the resulting solid = I $_{(cone)}$ – I $_{(steel\ cylinder)}$ + I $_{(lead\ cylinder)}$

i. Cone

Mass of the cone (M) = ρ x v

$= 7860 \times \frac{1}{3}\pi r^2 h = 7860 \times \frac{1}{3}\pi \times (0.3)^2 \times 0.5$

= 370.39 kg

Mass Moment of Inertia of cone = $\frac{3}{10}MR^2$

$$= \frac{3}{10} \times 370.39 \times 0.3^2$$

= 10 kg – m^2

ii. Steel Cylinder (hole)

Mass of the cylinder (M) = ρ x v

= 7860 x πr^2h

$= 7860 \times \pi\ (0.1)^2 \times 0.15$

= 37.03 kg

Mass Moment of Inertia of cylinder $= \frac{MR^2}{2} = \frac{37.03 \times 0.1^2}{2}$

= 0.185 kg-m^2

iii. Lead Cylinder

Mass of the cylinder (M) = ρ x v

= 11400 x πr^2h

$= 11400 \times \pi\ (0.1)^2 \times 0.15$

= 53.7 kg

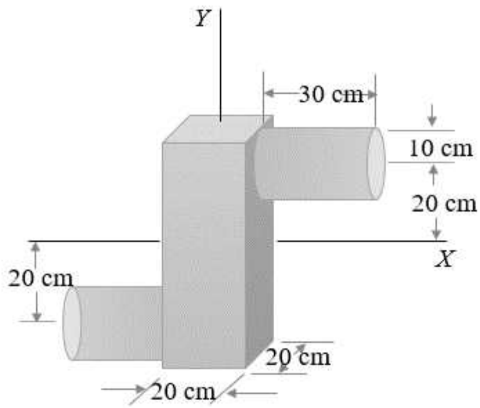
Mass Moment of Inertia of cylinder = $\frac{MR^2}{2} = \frac{53.7 \times 0.1^2}{2}$

= 0.268 kg-m^2

Moment of inertia of the resulting solid = I $_{(cone)}$ – I $_{(steel\ cylinder)}$ + I $_{(lead\ cylinder)}$

= 10 – 0.185 + 0.268= 10.08 kg-m^2

Example 7.4 A steel forging consists of a rectangular prism and two cylinders of diameter 20 cm. and length 30 cm as shown in the figure. Determine the moments of inertia of the forging with respect to the given x-axis, knowing that the density of steel is 7700 kg/m^3.



Solution Moment of inertia of the resulting solid = I $_{\text{(rectangular prism)}}$ + 2 x I $_{\text{(cylinder)}}$

i. Prism

Mass = 7700 x 0.2 x 0.2 x 0.6 = 184.8 kg

Moment of inertia = $\frac{1}{12}$ M $(0.6^2 + 0.2^2)$

$$= \frac{1}{12} \text{ x } 184.8 \text{ x } (0.6^2 + 0.2^2)$$

$$= 6.16 \text{ kg-m}^2$$

ii. Cylinder

Mass = 7700 x $\pi\ (0.1)^2\ x$ 0.3 = 72.5 kg

Moment of inertia $I_{xx} = I_{gg} + Md^2$

$$= \frac{MR^2}{2} + 72.5\ (0.2)^2$$

$$= \frac{72.5\ x\ 0.1^2}{2} + 72.5\ (0.2)^2$$

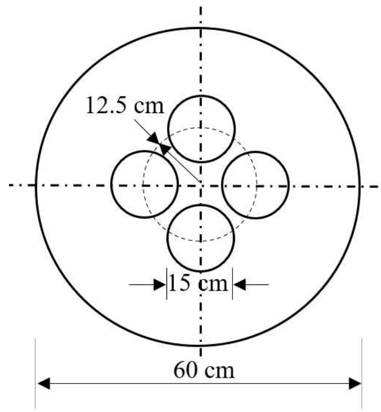
$$= 3.26 \text{ kg-m}^2$$

Moment of inertia of the resulting solid = I $_{\text{(rectangular prism)}}$ + 2 x I $_{\text{(cylinder)}}$

$$= 6.16 + 2(3.26)$$

$$= 12.68 \text{ kg m}^2$$

Example 7.5 The right circular cylinder shown in the figure is made of steel of density 7860 kg/m^3. It is 0.15 m long and has diameter 0.6 m. Four holes each 0.15 in diameter and equally spaced around a circle of 0.25 m diameter are drilled from the cylinder. Compute the mass moment of inertia about geometric axis of the body



Mass of the large cylinder (M) = ρ x v

= 7860 x $\pi r^2 h$

= 7860 x $\pi \times 0.3^2 \times 0.15$

= 333.35 kg

Mass of the small cylinder (M) = ρ x v

= 7860 x $\pi r^2 h$

= 7860 x $\pi \times 0.075^2 \times 0.15$

= 20. 83 kg

Moment of inertia of the resulting solid = $I_{(larger\ cylinder)} - 4 \times I_{(hole)}$

Moment of inertia of larger cylinder about geometric axis is

$$= \frac{MR^2}{2} = \frac{333.35 \times 0.3^2}{2} = 15\ kg$$

Moment of inertia of hole about geometric axis is

$$I_{xx} = I_g + Md^2$$

$$= \frac{MR^2}{2} + Md^2$$

$$= \frac{20.83 \times 0.075^2}{2} + 20.83 \times 0.125^2$$

0.384 kg – m^2

Moment of inertia of the resulting solid = $I_{(larger\ cylinder)} - 4 \times I_{(hole)}$

= 15 – 4 (0.384)

= 13.46 kg-m^2

SUMMARY

- Mass moment of inertia is a quantity that determines the ability of a solid object to resist change in rotational speed about a specific axis.
- Mass moment of inertia is defined as the product of the mass and the square of the distance between the mass centre of the body and the axis which is considered.
- Radius of gyration may be defined as the distance from the axis of rotation at which, if the whole mass of the body is concentrated, its moment of inertia about the axis is the same as that with the actual distribution of mass.

EXERCISES

I. SELF-ASSESSMENT QUESTIONS

1. Define mass moment of inertia
2. Distinguish between area moment of inertia and mass moment of inertia
3. Define radius of gyration
4. Explain mass moment of inertia with real time applications

II. MULTIPLE CHOICE QUESTIONS

1. Mass moment of inertia of a circular ring about x axis is
   $a)\ \frac{ML^2}{12}$ $b)\ \frac{1}{2}\mathrm{mR}^2$ $c)\ \frac{2}{5}\mathrm{mR}^2$ $d)\ \frac{\mathrm{Mb}^2}{12}$
2. Mass moment of inertia of a sphere about diametral axis is
   $a)\ \frac{ML^2}{12}$ $b)\ \frac{1}{2}\mathrm{mR}^2$ $c)\ \frac{2}{5}\mathrm{mR}^2$ $d)\ \frac{\mathrm{Mb}^2}{12}$
3. Mass moment of inertia of a uniform rod is
   $a)\ \frac{ML^2}{12}$ $b)\ \frac{1}{2}\mathrm{mR}^2$ $c)\ \frac{2}{5}\mathrm{mR}^2$ $d)\ \frac{\mathrm{Mb}^2}{12}$

Answers

1. b
2. c
3. a

CHAPTER – 8

INTRODUCTION TO DYNAMICS

LEARNING OBJECTIVES

After studying this chapter, you should be able to

- Differentiate between kinematics and kinetics
- Classify different types of motion
- Understand the difference between absolute motion and relative motion
- Distinguish between particle dynamics and rigid body dynamics

8.1 INTRODUCTION

In the previous chapters, we studied the problems under static conditions. From now, we shall deal with problems under dynamic conditions i.e., problems related to motion. As defined in chapter-1, Dynamics is the part of engineering mechanics which involves the motion of bodies. Dynamics further divided into kinematics and kinetics.

8.2 KINEMATICS AND KINETICS

The problems of dynamics dealing with the analysis of motion without considering the forces that causes the motion is called kinematics. In kinematics we study the relationships between displacement, velocity, acceleration and time of a given motion. A moving car, water flowing in a river and a freely falling body are the examples of kinematics. In all these cases we are only concerned in describing their movement (ex. Position, velocity, displacement etc.,) without considering the forces. Figure 8.1 shows some examples of kinematics

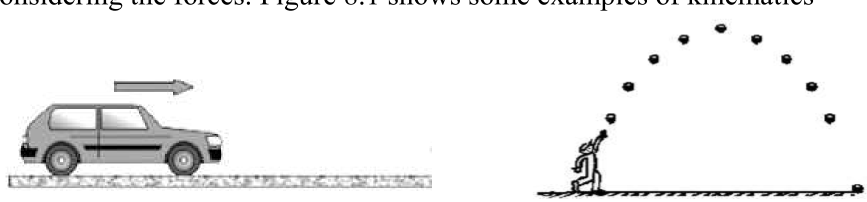


Fig. 8.1 Example of Kinematics

Kinetics, on the other hand, is the study of motion of bodies by considering the forces causing the motion. In kinetics, we study the relationships between forces, mass and motion of the bodies. So kinetic analysis helps in predicting the motion caused by a given force or in determining the force necessary to give required motion. A bullet fired from a gun, impact of two objects, truck moving on an inclined road are the examples of kinetics. Figure 8.2 shows some examples of kinetics

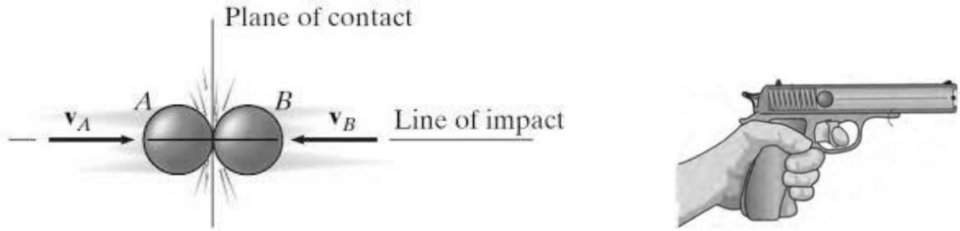


Fig. 8.2 Example of Kinetics

8.3 MOTION

A body is said to be in motion if it changes its position with respect to a reference point. A moving body can have three types of motions i.e., translational, rotational or general plane motion. The classification of motion is shown in figure 8.3.

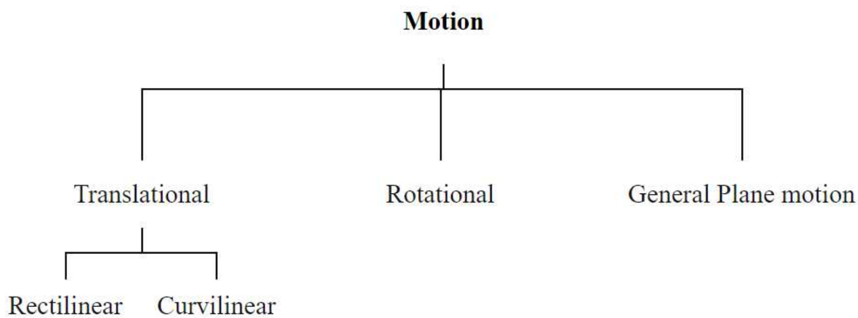


Fig. 8.3 Classification of Motion

8.3.1 TRANSLATIONAL MOTION

The motion of a body in which a body shifts from one point to another is called translational motion. In this type of motion, all particles of a body move uniformly in the same line or direction.

a. Rectilinear

If the path followed by the particles is a straight line as shown in the figure 8.4 (a), then it is known as rectilinear motion. Rectilinear motion is also known as linear motion

b. Curvilinear

If the path followed by the particles is a curve as shown in the figure 8.4(b), then such a motion is called curvilinear.

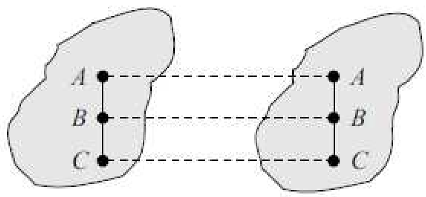


Fig. 8.4 (a) Rectilinear Motion

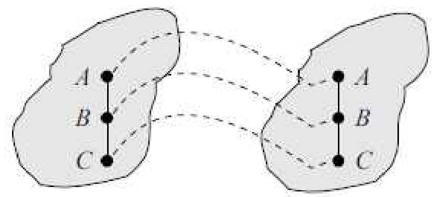


Fig 8.4 (b) Curvilinear Motion

8.3.2 ROTATIONAL

In rotational motion, all particles of a rigid body moves in a concentric circle. Also, the particles maintain parallel paths in the plane of rotation.

8.3.3 GENERAL PLANE MOTION

It is a combination of translational and rotational motion

8.4 ABSOLUTE MOTION AND RELATIVE MOTION

A motion of a particle or a rigid body can be described with respect to a reference frame, which may be fixed or moving. If the motion of a particle is described with respect to a fixed reference frame, then such motion is termed as absolute motion. The term absolute is sometimes combined with the kinematic variables (e.g., absolute acceleration) to signify the fixed nature of the reference frame. On the other hand, relative motion describes the motion of a particle with respect to a moving reference frame (coordinate system).

8.5 PARTICLE AND RIGID BODY DYNAMICS

It is always convenient to identify the type of body before analyzing its motion. In most cases, certain assumptions are made regarding the type of body and simplified models are developed. As engineering mechanics is concerned, we may consider bodies as particle or rigid bodies.

8.5.1 PARTICLE DYNAMICS

A particle is defined as a point mass that has no physical dimension. In dynamics we treat the body as a particle when the motion is in translation and the dimensions of the body are not significant. The assumption of particle can also be used when the rotational motion of a body is comparatively small.

8.5.2 RIGID BODY DYNAMICS

A rigid body is defined as a body that has a shape and physical dimensions. The particles on the rigid body are fixed in position relative to each other. That is, the distance between any two points on the body does not change under the action of forces. This implies that the shape of a rigid body will not change under the application of forces. In dynamics, we treat the body as rigid when the size and shape of the body is utmost important in analyzing the motion. In rigid body dynamics, the body may undergo both translation and rotational motion.

SUMMARY

- Dynamics deals with the bodies in motion. Dynamics further divided into kinematics and kinetics
- Kinematics deals with the analysis of motion without considering the forces
- In kinematics, we study the relationships between displacement, velocity, acceleration and time of a given motion.
- Kinetic is the study of motion of bodies by considering the forces.
- A body is said to be in motion if it is changing its position with respect to a reference point.
- There are three types of motion, translational, rotational and general plane motion.
- The motion of a particle with respect to a fixed reference frame is known as absolute motion and with respect to a moving reference frame is known as relative motion.
- A body is treated as a particle when the motion is in translation and the dimensions of the body are not significant.
- A body is treated as rigid when the size and shape of the body is utmost important in analyzing the motion

EXERCISES

I. SELF-ASSESSMENT QUESTIONS

1. Define dynamics
2. Distinguish between statics and dynamics.
3. Explain the terms kinematics and kinetics. Give some real time examples for each.
4. Differentiate between kinematics and kinetics.
5. What do you understand by particle dynamics and rigid body dynamics?
6. Explain the types of motion with suitable examples.
7. Distinguish between
    a. Rectilinear motion and curvilinear motion
    b. Rotational motion and general plane motion
8. When do you consider the body as a particle in dynamics?
9. What is relative motion and absolute motion?

II. MULTIPLE CHOICE QUESTIONS

1. The branch of mechanics that deals with motion of bodies without consideration of forces is called
    (a) Kinematics

(b) Statics
(c) Kinetics
(d) None of the above

2. The relationships between forces, mass and motion of the bodies is studied under
(a) Statics
(b) Kinematics
(c) Kinetics
(d) None of the above

3. The motion along a straight line path is called
(a) Relative motion
(b) Absolute motion
(c) Curvilinear motion
(d) Rectilinear motion

4. The motion of a particle along a curved path is called
(a) Relative motion
(b) Absolute motion
(c) Rectilinear motion
(d) Curvilinear motion

5. In rectilinear motion, all the particles in the body have same
(a) Acceleration
(b) Velocity
(c) Displacement
(d) All the above

6. A rigid body can be idealized as a particle when
(a) The size of body is small
(b) No rotational motion involved
(c) No translational motion involved
(d) The body is at rest

7. If the motion is described w. r. t a system of reference axes fixed on earth then the motion is called
(a) Absolute motion
(b) Relative motion
(c) Rectilinear motion
(d) Curvilinear motion

8. If the motion of a particle is described with reference to moving axes, the motion is referred to as
(a) Absolute motion
(b) Relative motion
(c) Rectilinear motion
(d) Curvilinear motion

Answers

1. a
2. c
3. d
4. d
5. d
6. b
7. a
8. b

CHAPTER – 9

RECTILINEAR MOTION

LEARNING OBJECTIVES

After studying this chapter, you should be able to

- *Describe rectilinear motion and the terms involved with it*
- *Apply the equations of motion to solve the problems of uniform acceleration.*
- *Solve the problems related to acceleration due to gravity*
- *Evaluate the problems of motion with varying acceleration*
- *Construct the motion curves for various problems*

9.1 INTRODUCTION

Kinematics of particle refers to the study of geometry of motion of a particle without taking forces and mass into consideration. The geometry of motion of particle is defined by specifying its position, speed, velocity and its acceleration. The word "particle" in the chapter title refers that all the objects considered for the study are treated as particles (a point mass that has no dimension). This assumption is true only if the object moves in translational motion without rotating. That is, if all the parts (particles) of an object moves exactly the same way, then it can be treated as a particle. As all the parts move exactly the same way, the motion parameters like velocity, acceleration and displacement of each particle will be same as that of the other. Thus, considering a particle instead of whole body and describing its motion, we describe the motion of the entire body.

Note: We also consider a body as a particle when the dimensions of the body are not considered and not involved in solving the problems.

9.2 RECTILINEAR MOTION

A particle moving along a straight line is said to be in rectilinear motion. Initially, we study this simple type of motion because engineers have to analyze linear motions in most cases such as motion of car on a straight road or the motion of a freely falling body. Motion is mathematically described by the terms distance, displacement, speed, velocity, acceleration etc., the definition of these terms are give below.

Position: Position refers to the location of a particle relative to some reference point.

Distance: It is the actual path travelled by the body

Displacement: It is the minimum distance travelled by a body from a starting point to the end point along a straight line.

Note

i. Distance is a scalar quantity whereas displacement is a vector quantity
ii. The magnitude of distance and displacement may or may not be equal.
iii. The magnitude of the displacement for a course of motion may be zero but the corresponding distance will not be zero.
iv. The units for distance and displacement are meters

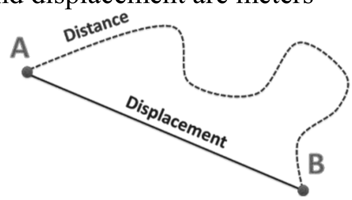


Fig. 9.1 Path indicating distance and displacement

Speed: Speed describes how fast an object is moving. The speed at any instant is called the instantaneous speed. The speedometer of vehicles gives the reading of instantaneous speed.
Average speed: It is the total distance covered divided by time

$$\text{Average speed} = \frac{Total\ distance}{time}$$

Fig. 9.2 Speedometer showing instantaneous speed

Velocity: Velocity is the speed in a given direction. It is defined as the rate of change of displacement with respect to time. Based on the variation in displacement and time, velocity can be average velocity, instantaneous velocity and uniform velocity.
Average velocity: It is the ratio of displacement Δx to the time interval Δt

$$\text{Average Velocity} = \frac{displacement}{time} = \frac{\Delta x}{\Delta t}$$

Instantaneous velocity: The velocity of a particle at any instant 't' is obtained from the average velocity by shrinking the time interval Δt closer and closer to 0. As Δt dwindles, the average velocity approaches a limiting value, which is the velocity at that instant

$$\text{Instantaneous velocity } v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

Note

i. Speed is a scalar quantity whereas velocity is a vector quantity.
ii. Units for speed and velocity are m/s
iii. The magnitude of velocity is known as speed of the particle

Acceleration The rate of change of velocity with respect to time is known as acceleration.

Average acceleration: It is the ratio of change in velocity Δv to the time interval Δt in which the change occurs

$$Average\ Acceleration = \frac{\Delta v}{\Delta t}$$

Instantaneous acceleration: The acceleration a of the particle at any instant 't' is obtained from the average acceleration by choosing smaller and smaller values for Δt and Δv

$$Instantaneous\ aceleration\ a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

Note

i. The acceleration can be zero, positive or negative.
ii. Positive acceleration is termed as acceleration whereas negative acceleration is called deceleration or retardation
iii. The units of acceleration is m/s^2

9.3 MOTION WITH UNIFORM VELOCITY

If the velocity of a particle remains unchanged throughout its travel then it is called as uniform velocity motion. In such case, acceleration is equal to zero.

Displacement = velocity x time

9.4 MOTION WITH UNIFORM ACCELERATION

Let us consider a body which accelerates with uniform acceleration 'a' from initial velocity 'u' to final 'v'. The body covers a distance 's' during time 't'. Then the relation between motion parameters are given by the three equations as shown in the table no. 1

Table 9.1: Equations of motion

Relation	*Equation*
Relation between v - t	$v = u + at$
Relation between s - t	$s = ut + ½ at^2$
Relation between v - s	$v^2 – u^2 = 2as$

Example 9.1 A car travelled at a speed of 13.2 m/s for 12 min, then at 17.66 m/s, for 20 min and finally at 22 m/s for 8 m. what is its average speed over this interval.

Solution We know Displacement = velocity x time

$s_1 = v_1t_1$; $s_2 = v_2t_2$; $s_3 = v_3t_3$

$s = s_1 + s_2 + s_3$

Total distance = [13.2 x (12 x 60)] + [17.6 x (20 x 60)] + [22 x (8 x 60)]

= 9504 + 21120 + 10560 = 41184 m

$$\text{Average speed} = \frac{\text{Total distance}}{\text{Total time}}$$

Average speed = $\dfrac{41184}{(12 + 20 + 8) \times 60} = 17.16\, m/s$

Example 9.2 Bike B which is travelling at an average speed of 17.6 m/s overtakes bike A which is at rest at fuel filling station. After 10 min. bike A starts moving at an average speed of 22 m/s. How long will it take for Bike A to overtake Bike B?

Solution Time required for bike A to overtake bike B is "t" sec.

Distance covered by car A in 't' sec = 22 x t

Same distance covered by car B = 17.6 x (10 x 60 + t)

22t = 17.6 (600+t)

t= 2400 seconds = 40 minutes.

Example 9.3 A 1500 cc diesel car accelerates uniformly from rest to 90 kmph in 15 sec. Find the acceleration and displacement during this time.

Solution Given u = 0, t = 15 sec, v = 90 kmph = $\frac{90\times1000}{60\times60}$ = 25 m/s

i. Acceleration

We know v = u + at

a = $\frac{v-u}{t}$ = $\frac{25-0}{15}$ = 1.6 m/s^2

ii. Displacement

We know $v^2 - u^2 = 2as$

s = $\frac{v^2-u^2}{2a}$ = $\frac{25^2-0}{2\times1.6}$ = 195.3 m

Example 9.4 A bike changes its speed uniformly form 110 kmph to 45 kmph in a distance of 387 m. What is the acceleration of the bike?

Solution Given u = 110 kmph = $\frac{110\times5}{18}$ = 30.5 m/s

v = 45 kmph = $\frac{45\times5}{18}$ = 12.5 m/s

Distance s = 387 m

We know $v^2 - u^2 = 2as$

a = $\frac{v^2-u^2}{2s}$ = $\frac{12.5^2-30.5^2}{2\times387}$ = - 1 m/s^2

Example 9.5 A car which travels along a straight road has to reach a speed of 125 kmph. Its initial speed is 80 kmph and it accelerated 2.5 m/s^2 uniformly. (i) How long the car take to reach the given speed? (ii) What is the distance travelled during this time?

Solution Given u = 80 kmph = $\frac{80\times5}{18}$ = 22.22 m/s

v = 125 kmph = $\frac{125\times5}{18}$ = 34.72 m/s

a= 2.5 m/s^2

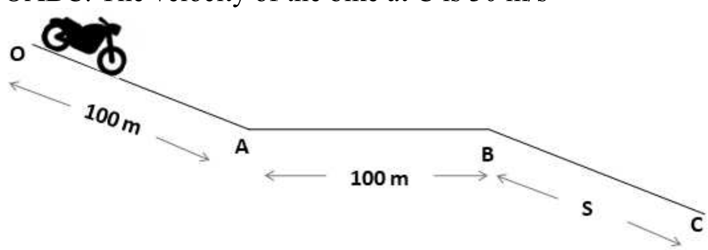
i. We know v = u + at

t = $\frac{v-u}{a} = \frac{34.72-22.22}{2.5}$ = 5 sec

ii. We know $v^2 - u^2 = 2as$

s = $\frac{v^2-u^2}{2a} = \frac{34.72^2-22.22^2}{2\times2.5}$ = 142.35 m

Example 9.6 A bike coming from a hill station begins its motion at point O from rest. The bike travels with uniform acceleration of 2 m/s^2 along the path OA and BC and with uniform velocity along the path AB. Find (i) the distance of the path BC and (ii) the time required to travel total path OABC. The velocity of the bike at C is 30 m/s



Solution

i. Distance of the path BC

Consider the motion from O →A

Given: u = 0, s = 100 m, a = 2 m/s^2, and v = ?

We know $v^2 - u^2 = 2as$

$v^2 - 0 = 2 \text{ x } 2 \text{ x } 100$

v = 20 m/s

Consider the motion from B →C

u = 20 m/s, v = 30 m/s, a = 2 m/s^2 and s = ?

We have $v^2 - u^2 = 2as$

$33^2 - 20^2 = 2 \text{ x } 2 \text{ x } s$

s = 125 m

ii. Time required to travel total path

Motion from O →A

We have v = u + at

$$t_1 = \frac{v-u}{a} = \frac{20-0}{2} = 10 \text{ sec}$$

Motion from B →C

$$t_3 = \frac{v-u}{a} = \frac{30-20}{2} = 5 \text{ sec}$$

Motion from A →B

We know $v = \frac{s}{t}$

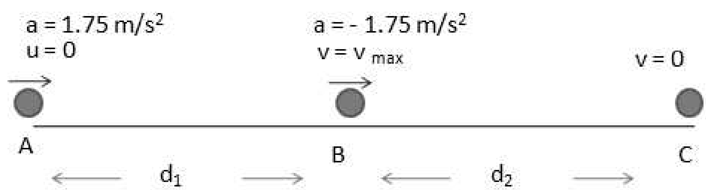
$$t_2 = \frac{s}{v} = \frac{100}{20} = 5 \text{ sec}$$

Time required to travel total path OABC is $t = t_1 + t_2 + t_3$

t = 10 + 5 + 5 = 20 sec

Example 9.7 An automobile starts from rest attains a maximum speed at some point and then halts at the ends of a road which is 2 km long. It can accelerate or decelerate at 1.75 m/s^2. What is the time required to cover 2 km?

Solution



Motion from A →B

Given $u = 0$, $v = v_{max}$, $a = 1.75$ m/s^2,

$s = d_1$ and $t = t_1$

We have $v = u + at$

$v_{max} = 0 + 1.75 \times t_1$

$$t_1 = \frac{v_{max}}{1.75}$$

We have $s = ut + ½\, at^2$

$d_1 = 0\,(t_1) + ½\,(1.75)\,(t_1^2)$

$$d_1 = 0.875\left[\frac{v_{max}}{1.75}\right]^2$$

Motion from B →C

Given $u = v_{max}$, $v = 0$, $a = -1.75$ m/s^2,

$s = d_2$ and $t = t_2$

We have $v = u + at$

$0 = v_{max} + 1.75 \times t_2$

$t_2 = \frac{v_{max}}{1.75}$

Also we have $s = ut + ½ at^2$

$$d_2 = v_{max}\left[\frac{v_{max}}{1.75}\right] + \frac{1}{2} \times (-1.75)\left[\frac{v_{max}}{1.75}\right]^2$$

$$d_2 = v_{max}\left[\frac{v_{max}}{1.75}\right] - 0.875\left[\frac{v_{max}}{1.75}\right]^2$$

We know total displacement $s = d_1 + d_2$

$$2000 = 0.875\left[\frac{v_{max}}{1.75}\right]^2 + \left[v_{max}\left[\frac{v_{max}}{1.75}\right] - 0.875\left[\frac{v_{max}}{1.75}\right]^2\right]$$

$$2000 = \left[\frac{v^2{}_{max}}{1.75}\right]$$

$v^2{}_{max} = 3500$

$V_{max} = 59.16$ m/s

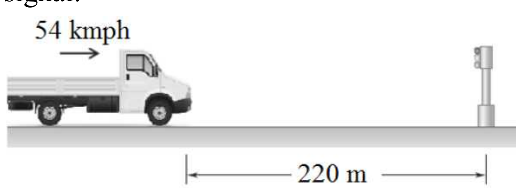
Total time $t = t_1 + t_2$

$t = \frac{v_{max}}{1.75} + \frac{v_{max}}{1.75}$

$= \frac{59.16}{1.75} + \frac{59.16}{1.75}$

= 67.61sec

Example 9.8 A truck driver travelling at a speed of 54 kmph observes a traffic signal light 220 m ahead of him turning red. The red light is timed to stay for 22 seconds. If the driver wishes to pass the signal without stopping, find (i) the required uniform deceleration (ii) the speed of the truck as it passes the signal.



Solution Given: $u = 54$ kmph $= \frac{54 \times 5}{18} = 15$ m/s

$t = 22$ sec and $s = 220$ m

We know $s = ut + ½ at^2$

$220 = 15 \times 22 + ½ (a) 22^2$

$a = -0.45$ m/s^2

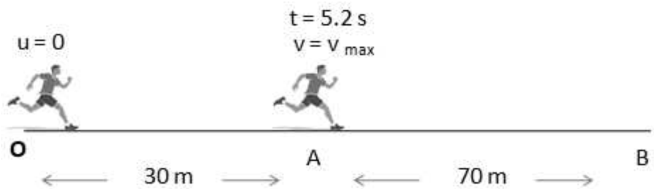
We also know $v = u + at$

$v = 15 – 0.45 \times 22$

$v = 5.1$ m/s

Example 9.9 An athlete in a 100 m race accelerates uniformly upto a distance of 30 m and then he runs the remaining distance with constant velocity. If the athlete takes 5.2 sec to cover 30 m, find (i) his acceleration (ii) final velocity (iii) time for race

Solution



i. **Acceleration**

Consider the motion from O →A

Given $u = 0$, $v = v_{max}$, $t_1 = 5.2$ sec and $s = 30$ m, $a = 2.22$ m/s^2

We know $s = ut + ½ at^2$

$30 = ½ a (5.2)^2$

$30 = 13.52 a$

$a = 2.22$ m/s^2

ii. Final velocity

We know $v^2 – u^2 = 2as$

$V^2_{max} – 0 = 2 \times 2.22 \times 30$

$V^2_{max} = 133.2$

$V_{max} = 11.54$ m/s

iii. Time for race

Time for A →B

$$t = \frac{s}{v} = \frac{70}{11.54} = 6.06 \text{ sec}$$

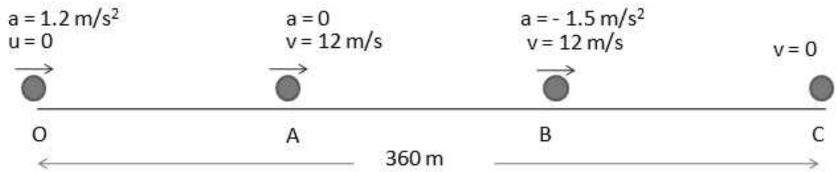
total time t = t1 + t2

= 5.2 + 6.06

= 11.26 s

Example 9.10 An automobiles starting from rest speeds up to 12m/s with a constant acceleration of $1.2m/s^2$, runs at the speed for a time, and finally comes to rest with a decleration of $1.5m/s^2$. If the total distance travelled is 360 m, find the total time required.

Solution



Motion from O to A

Given: u=0, v=12m/s and $a=1.2m/s^2$

Let, Time $=t_1$

Displacement$=s_1$

We know, $v = u + at$

$12 = 0 + 1.2t_1$

$t_1=10sec$,

Also $s = ut + \frac{1}{2} at^2$

$s_1 = 0 + \frac{1}{2} \times (1.2) \times (10)^2$

$s_1 = 60$ m

Motion from A to B

Given u=12m/s, v=12m/s and a=0

Time$=t_2$

Displacement$=s_2$

We know Displacement = Velocity x time

s_2 = Velocity × time

$s_2=12t_2$

Motion from B to C

Given: u=12m/s, v=0 and $a= - 1.5 m/s^2$

Let, Time$=t_3$

Displacement$=s_3$

$v = u + at$

$0 = 12 - 1.5t_3$

$t_3 = 8$ sec

Also, $S_3 = 12 \times 8 - \frac{1}{2} \times 1.58^2 = 48$ mts

Now $S_1 + S_2 + S_3 = 360$m

$60 + 12t_2 + 48 = 360$

$t_2 = 21$sec

Total time $= t_1 + t_2 + t_3 = 10 + 21 + 8 = 39$sec

Example 9.11 An auto A is moving at 6 m/s and accelerating at 1.5 m/s^2 to overtake an auto B which is 115.2 m ahead. If auto B is moving at 18 m/s and decelerating at 0.9 m/s^2, how soon will A pass B?

Solution For auto A

Given: u = 6m/s and a = 1.5m/s^2

Displacement =S time =t sec

For A $S = ut + 1/2at^2$

$S = 6t + (1.5/2)\, t_2$ ---- (1)

For auto B

Given u = 18m/s a = – 0.9m/s^2

Displacement = S - 115.2

time = t sec

$(S - 115.2) = 18t - (0.9/2)\, t_2$

$S = 18t - 0.45t^2 + 115.2$ ---- (2)

From (1) and (2) i.e performing (2) – (1)

$12t - 1.2t^2 + 115.2 = 0$

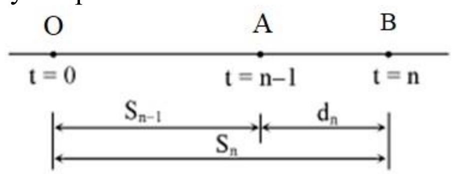
$1.2t^2 - 12t - 115.2 = 0$

Solving we get,

t = 16sec or t = – 6sec

9.5 DISTANCE COVERED IN THE NTH SECOND OF MOTION

A particle travelling along a straight line has initial velocity 'u' and acceleration 'a'. Initially the particle is at point O. In (n-1) second it moves to point A and after n seconds it reaches point B. Distance travelled by the particle in nth second of its motion is given by the path OB.



Let s_n be the total distance travelled in *n* seconds, s_{n-1} be the total distance travelled in *(n-1)* seconds.

If d_n is the distance travelled in nth second, then from the figure it is clear that

$$d_n = \text{Displacement up to n seconds} - \text{displacement up to (n-1) seconds}$$

$$\text{i.e, } d_n = s_n - s_{n-1}$$

We know $s = ut + \frac{1}{2} at^2$

$d_n = [un + \frac{1}{2} an^2] - [u(n-1) + \frac{1}{2} a(n-1)^2]$

$\quad = [un + \frac{1}{2} an^2] - [un - u + \frac{1}{2} a (n^2 + 1 - 2n)]$

$\quad = u + \frac{a}{2}(2n - 1)$

Example 9.12 A car moving with constant acceleration travels 7.2 m during the 10th sec of its motion and 5.4 m during the 12th sec of its motion. Find its initial velocity.

Solution We know $d_n = u + \frac{a}{2} (2n - 1)$

As per given data

$7.2 = u + \frac{a}{2}[2(10) - 1]$

7.2 = u+9.5a --------- (i)

Also, $5.4 = u + \frac{a}{2}[2(12) - 1]$

5.4 = u+11.5a ---------------- (ii)

Solving (1) and (2) We get

$a = - 0.9 \text{ m/s}^2$ and

$u = 15.75$ m/s

9.6 ACCELERATION DUE TO GRAVITY

The equations of motion in table no 9.1 also applies to the problems of acceleration due to gravity except that here 'a' is replaced with 'g'. g is used to denote acceleration due to gravity whose magnitude is given by 9.81 m/s^2. The value of g is considered as negative when the object is thrown upward and positive when it is dropped from a certain height.

Example 9.13 A ball is thrown up along a vertical axis with an initial speed of 15 m/s

(i) How long does the ball take to reach its maximum height?
(ii) What is the maximum height reached by the ball?
(iii) How much time the ball would take to reach 4.5 m height?

Solution Given u = 15 m/s, v = 0, a = -9.8

i. Time to reach maximum height

We know v = u + at

$$t = \frac{v-u}{a} = \frac{0-15^2}{-9.8} = 1.53 \text{ sec}$$

ii. Maximum height reached

We know $v^2 - u^2 = 2as$

$$s = \frac{v^2-u^2}{2a} = \frac{0-15^2}{2\times(9.8)} = 11.4 \text{ m}$$

iii. Time to reach 4.5 m

We know $s = ut + ½ at^2$

$4.5 = 15 \text{ x } t + ½ (-9.8) t^2$

$4.5 = 15t – 4.9t^2$

$4.9t^2 – 15t + 4.5 = 0$

t = 2.72 sec

t = 0.33 sec

Example 9.14 A stone is thrown vertically upward and returns to earth in 5 sec. How does it go?

Solution Time required for upward motion = t_1 sec.

Time required for downward motion = t_2 sec

$t_1 + t_2 = 5$ sec ----(i)

For upward motion

$v = u – g\, t_1$

$0 = u – 9.81\, t_1$

$u = 9.81\, t_1$

Now $s = ut_1 - \frac{1}{2} gt_1^2$

$= (9.81\ t_1)\ t_1 - \frac{1}{2} \times 9.81\ t_1^2 = 4.905\ t_1^2$ ---- (ii)

For downward motion

$V = u + gt^2$

$V = 0 + 9.81\ t_2^2$

$S = 0 + 1/2\ 9.81\ t_2^2$

$S = 4.905\ t_2^2$ ---- (iii)

Equating (ii) and (iii) we get $t_1 = t_2$ put in (i), we get

$t_1 = 2.5$ sec

$t_2 = 2.5$ sec

$S = 4.905 \times (2.5)^2 = 30.656$ m

Example 9.15 A boy drops a ball standing on a building which is 24 m tall. At the same time another ball thrown upward with initial velocity 12 m/s from the ground. Determine time and height where both the ball cross.

Solution Ball – 1

Given $u = 0$, $s = 24 - h$ and $a = 9.81$ m/s^2

We know $s = ut + \frac{1}{2} at^2$

$24 - h = 0 \times t + \frac{1}{2} (9.81)\ t^2$ ----(i)

Ball – 2

$u = 12$, $s = h$ and $a = -9.81$

We know $s = ut + \frac{1}{2} at^2$

$h = 12\ t - \frac{1}{2} (9.81)\ t^2$ ----(ii)

Adding equation (i) and (ii)

$12\ t = 24$

$t = 2$ sec

$h = 12\ (2) - \frac{1}{2} (9.81)\ (2)^2$

$h = 4.38$ m

Example 9.16 A stone is thrown up vertically in the air with a velocity 36 m/s. Three seconds later, another stone is thrown along the same path. What initial velocity must the second stone should have in order to pass the first ball 30 m from the ground?

Solution For 1st ball

$S = ut - \frac{1}{2} gt^2$

$30 = 36t - 4.905t^2$

$4.905\ t^2 - 36t + 30 = 0$

Solving above quadratic equation we get

$t = 6.3809$ sec or $t = 0.9585$ sec

Select $t = 6.3809$ sec because it is more than 3 sec

For 2nd ball

$S = ut - \frac{1}{2} gt^2$

But here time = $t - 3$

$30 = u\ (t\text{-}3) - 4.905\ (t\text{-}3)^2$

Substitute $t = 6.3809$ sec

$u = 25.4567$ m/s.

Example 9.17 A stone is thrown into a well and five seconds later a splash sound is heard. If the velocity of sound is 336 m/s, what is the depth, of the well?

Solution If t = time taken to reach the stone up to water level

$5 - t$ = time required by the sound to travel up.

For stone $u = 0$

$g = 9.81$

$S = ut + \frac{1}{2} gt^2$

$S = 4.905t^2$ ---- (1)

For sound, Distance = Velocity x time

$S = 336\ (S - t)$ ---- (2)

Equating (1) and (2), we get $4.905t^2 = 336\ (5 - t)$

Solving above equation $t = 4.68$ sec $= -73.18$ sec

Select $t = 4.68$ sec

$s = 4.905(4.68)^2 = 107.48$ m

Example 9.18 A stone is dropped in to a well with no initial velocity and 4.5 sec later the splash is heard. Then a second stone is thrown downward in to the well with an initial velocity v_0 and the splash is heard in 4 sec. If the velocity of sound is constant at 336 m/s, determine the initial velocity of the second stone?

Solution For 1st stone

$S = ut + ½ gt^2$

$S = 4.905\ t^2$ ---- (1)

For sound $S= 336\ (4.5 – t)$ ---- (2)

Equating (1) and (2)

$4.905\ t^2 + 336t - 1512 = 0$

By solving the above equation, we get

$t = 4.2378$ sec.

Depth of well $= 4.905 \times (4.2378)^2 = 88.088$ m

For 2nd stone sound

$h = (4 – t) \times 336$

$88.088 = (4 – t) \times 336$

$t = 3.7378$ sec

For stone $h = ut + ½ gt^2$

$88.088 = u\ (3.7378) + ½ \times 9.81\ (3.7378)$

$u = 5.23$ m/s.

Example 9.19 A balloon rises from the ground with a constant acceleration of $0.9 m/s^2$. Five seconds later, a stone is thrown vertically up from the launching site. What must be the minimum initial velocity of the stone for it to just touch the balloon? Note that the balloon and the stone have the same velocity at contact.

Solution Let the time required for the stone to touch balloon = t sec

Time required for the balloon = (t+5) sec

For balloon $S = ut+ ½ at^2$

$S=0+ ½ \times 0.9(t+5)^2=0.45(t+5)^2$ ----(1)

For stone $S = ut+ ½ at^2$

$S= ut- ½ \times 9.81t^2$

$S= ut - 4.905\ t^2$ ----(2)

Equating (1) and (2) we get

$0.45(t+5)^2 = ut-4.905\ t^2$ ----(3)

For balloon $v = u + at$

$v = 0 + 0.9(t+5)$

$v = 0.9\ (t+5)$ ----(4)

For stone $v = u + gt$

$V = u - 9.81t$ ----(5)

Equating (4) and (5)

$0.9(t+5) = u-9.81t$

$u = 10.71t + 4.5$

$u = (10.71\times1.45) + 4.5 = 20.03$ m/s

9.7 MOTION WITH VARYING ACCELERATION

A particle may not maintain a constant acceleration during its entire motion. This implies that, the rate at which the velocity is changing is not uniform. Such kind of motions is termed as motion with varying acceleration. The motion of a body under variable acceleration can be analysed by using differential and integral equations of motion.

Example 9.20 The straight-line motion of a particle is defined by $a=3+\frac{1}{2}t$ when t=0, s = 2m, and v = -4m/s. Find s at t = 6sec.

Solution Given $a=3+\frac{1}{2}t$

$a = \frac{dv}{dt} = 3+\frac{1}{2}t$

$dv = (3 + \frac{1}{2}t)\ dt$

Integrating on both sides

$v = 3t + \frac{1}{2}\frac{t^2}{2} + C$

At t = 0; v = - 4m/s C = -4

$v=3t + \frac{t^2}{4} - 4$

Also, $v = \frac{ds}{dt} = 3t + \frac{t^2}{4} - 4$

$ds=[3t+\frac{t^2}{4}-4]\ dt$

Integrating on both sides

$s=3.\frac{t^2}{2}+\frac{1}{4}.\frac{t^3}{3}-4t+C_1$

At t = 0; s = 2m $C_1=2$

$s = \frac{t^3}{12}+\frac{3t^2}{2}-4t+2$

Substitute t = 6 sec

$s = \frac{6^3}{12} + \frac{3\times6^2}{2} - 4(6) + 2 = 50m$

Example 9.21 The rectilinear motion of the particle is expressed by the relation $a = t^3 - 3t^2 + 5$ where "a" is the acceleration in m/s^2 and "t" is the time in seconds. The velocity and displacement of the particle at t = 2 sec is 4.44 and 17.52 respectively. Find the displacement, velocity and acceleration at time t = 4 sec.

Solution Given $a = t^3 - 3t^2 + 5$

We know $\frac{dv}{dt} = a$

$\therefore \frac{dv}{dt} = t^3 - 3t^2 + 5$

$dv = (t^3 - 3t^2 + 5)\, dt$

Integrating both sides

$\int dv = \int (t^3 - 3t^2 + 5)\, dt$

$v = \frac{t^4}{4} - \frac{3t^3}{3} + 5t + c_1$

Given that, at t = 2 s; v = 4.44 m/s.

$4.44 = \frac{2^4}{4} - \frac{3(2)^3}{3} + 5(2) + c_1$

$c_1 = -1.56$

$v = \frac{t^4}{4} - \frac{3t^3}{3} + 5t - 1.56$ ----- (i)

We also know that $\frac{ds}{dt} = v$

$\therefore \frac{ds}{dt} = \frac{t^4}{4} - \frac{3t^3}{3} + 5t - 1.56$

$ds = (\frac{t^4}{4} - \frac{3t^3}{3} + 5t - 1.56)\, dt$

Integrating on both sides

$\int ds = \int (\frac{t^4}{4} - \frac{3t^3}{3} + 5t - 1.56)\, dt$

$s = \frac{t^5}{20} - \frac{t^4}{4} + \frac{5t^2}{2} - 1.56t + c_2$

We know at t= 2, s = 17.52 m

$17.52 = \frac{(2)^5}{20} - \frac{(2)^4}{4} + \frac{5(2)^2}{2} - 1.56(2) + c_2$

$c_2 = 13.04$

$s = \frac{t^5}{20} - \frac{t^4}{4} + \frac{5t^2}{2} - 1.56t + 13.04$ ----- (ii)

At t = 4, from equation (i) and (ii)

$s = \frac{(4)^5}{20} - \frac{(4)^4}{4} + \frac{5(4)^2}{2} - 1.56(4) + 13.04 = 34$ m

$v = \frac{2^4}{4} - \frac{3(2)^3}{3} + 5(2) - 1.56 = 4.44$ m/s

$a = 4^3 - 3(4)^2 + 5 = 21$

Example 9.22 The rectilinear motion of a particle is governed by the equation s = r sin ωt where r and ω are constants. Show that the acceleration is $a = -\omega^2 s$

Solution Given s = r sin ω t

We know $v = \frac{ds}{dt}$

$v = \frac{d}{dt}$ (r sin ω t)

v = r ω cos ω t

Also $a = \frac{dv}{dt}$

$= \frac{d}{dt}$ (r ω cos ω t)

= -r ω . ω sin ω t

$= -w^2$(r sin ω t)

$a = -\omega^2 s$

Example 9.23 The motion of a particle along a straight line is defined by $s = \frac{1}{3}t^3 - 36t$

(a) Find the average acceleration during the fourth second

(b) When particle reverse its direction, what is acceleration.

Solution Given $s = \frac{1}{3}t^3 - 36t$

We know, $v = \frac{ds}{dt}$

$= \frac{d}{dt}\left[\frac{1}{3}t^3 - 36t\right]$

$= t^2 - 36$

Also, $a = \frac{dv}{dt}$

$a = \frac{d}{dt}(t^2 - 36) = 2t$

Case (a)

Acceleration after 3 sec= 2×3=6m/s^2

Acceleration after 4 sec=2×4=8m/s^2

Average acceleration during 4th sec= (6+8)/2= 7 m/s^2

Case (b)

When the particle reverse its direction, its velocity=0

t^3-36 = 0

t=6sec

Now a=2t=2×6=12m/s^2

Example 9.24 The velocity of a particle moving along the X axis is defined by v=kx^3-4x^2+6x where v is in m/s, x is in m and k is constant. If k=1; compute the value of the acceleration when x=2m.

Solution Given v=kx^3-4x^2+6x

Now, $a = \frac{dv}{dt} = \frac{dv}{dx}(\frac{dx}{dt})$

$\frac{dv}{dx}$ = k. 3x^2 – 8x + 6

Also $\frac{dx}{dt}$ =v

a = [k.3x^2-8x+6] × [k.x^3-4x^2+6x]

Substitute k=1 and x=2

a= 8m/s^2

Example 9.25 The motion of a particle in the rectilinear motion is defined by a =$6\sqrt{v}$ where a is in m/s^2 and v in m/s. When t= 2s, v=36m/s and s=30m, determine the value of s at t=3sec.

Solution Given a= $6\sqrt{v}$

We know $a = \frac{dv}{dt}$

$\frac{dv}{dt} = 6\sqrt{v}$

$\frac{dv}{\sqrt{v}} = 6.dt$

Integrating on both sides

$\frac{v^{1/2}}{1/2} = 6t + C$

$2\sqrt{v} = 6t + C$ at t=2sec v=36m/s

$2\sqrt{36} = 6 \times 2 + C$ C=0

$2\sqrt{v}=6t$

$\sqrt{v}=3t$

$v=9t^2$

We know $v = \frac{ds}{dt}$

$\frac{ds}{dt} = 9t^2$

$ds = 9t^2.dt$ -------(1)

Integrating on both sides

$s = 0(t^3/3) + c_1$

At t=2sec, s = 30 m

$30=9(23/3) + c_1$ $c_1=6$

$s=3t^3+6$

Now put t=3sec

$s=3\times33+6$

s=87mts

Example 9.26 The rectilinear motion of a particle is governed by $a= -8s^{-2}$ where 'a' is in m/s^2 and s in m when t=1 s, s=4m and v=22m/s. Determine the acceleration of the particle at t=2sec.

Solution Given: a=f(s) i.e., acceleration = function of 's'

Start with a ds = v.dv

i.e -8s-2 .ds = v.dv

Integrating on both sides

$\frac{-8s^{-2}}{-s} = \frac{v^2}{2} + C$

$\frac{8}{s} = \frac{v^2}{2} + C$

When s = 4m, v = 2m/s

$\frac{8}{4} = \frac{2^2}{2} + C; \therefore C=0$

$\frac{8}{s} = \frac{v^2}{2}$

$v = \frac{4}{\sqrt{s}}$

Also, v = $\frac{ds}{dt}$

$\frac{4}{\sqrt{s}} = \frac{ds}{dt}$

4 dt = $\sqrt{s}$ ds

Integrating on both sides

4t = $\frac{s^{3/2}}{\frac{3}{2}}$ + C1

When t=1sec; s = 4 m   C1= -1.33

4t = $\frac{2}{3}$.s$^{3/2}$-1.33

(4t+1.33) $\frac{3}{2}$ = S$^{3/2}$

s = (6t+2)3/2

Now differentiate w.r.t ‘t’

v = $\frac{ds}{dt}$ =$\frac{2}{3}$ (6t+2)$^{-1/3}$×6

v = 4(6t+2)$^{-1/3}$

Again, differentiate w.r.t ‘t’

a = $\frac{dv}{dt}$ =4×(-$\frac{1}{3}$)(6t+2)$^{-4/3}$× 6

a = -8(6t+2)$^{-4/3}$

Put t=2sec

a=-8(6×2+2)$^{-4/3}$=-0.2371 m/s^2

Example 9.27 The rectilinear motion of a particle is governed by a=12t-6t^2. It starts from rest when t=0. Determine its velocity when it returns to its starting position.

Solution Starts from rest means at t=0; V=0 & S=0

a= 12t-vt^2

dv=(12t-6t^2) dt

Integrating on both sides

v=$\frac{12t^2}{2}$- $\frac{6t^3}{3}$+C

At t=0; V=0   C=0

v=6t^2-2t^3 ----(1)

As $v=\frac{ds}{dt}$

$ds=(6t^2-2t^3)\ dt$

Integrating on both sides

$s=\frac{6t^3}{3}-\frac{2t^4}{4}+C_1$

At t=0; S=0 C1=0

$s=2t^3-t^4/2$ ---- (2)

Return to its starting position means s=0

$0=2t^3-t^4/2$

t=4sec

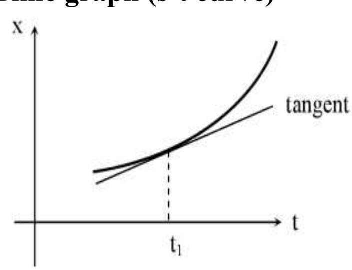
Putting in equation (1)

v=6(4)2-2(4)3=-32m/s

9.8 MOTION CURVES

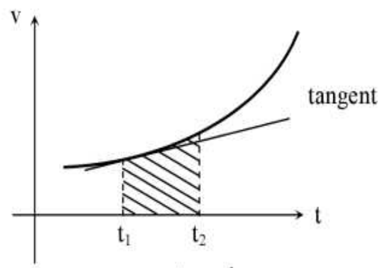
Motion curves are the graphical representation of displacement, velocity and acceleration with time. This method is used when the motion has distinct phases. Motion curves also provide a way of using experimental data to draw a-t, v-t or s-t curves if any one of them is known.

i. Displacement v/s Time graph (s-t curve)



$v = \frac{ds}{dt}$ = slope of the tangent
Therefore, Velocity at any instant = slope of tangent to s-t curve at that instant

ii. Velocity v/s Time graph (v-t curve)



v – t graph

a = $\frac{dv}{dt}$ = slope of the tangent

Therefore, acceleration at any instant = slope of tangent to v-t curve at that instant

We know, $\frac{ds}{dt}$ = v

ds = v dt

Integrating on both sides

$$\int ds = \int v\,dt$$

If the particles position is s_1 at time t_1 and s_2 at time t_2, then

$$\int_{s_1}^{s_2} ds = \int_{t_1}^{t_2} v\,dt$$

$s_2 - s_1 = \int_{t_1}^{t_2} v\,dt$

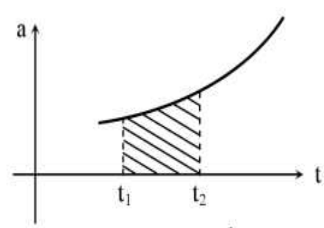
Therefore, Change in displacement = Area under v-t curve

Note: Area measured under the velocity and time graph from t_1 to t_2 provides change in displacement during that time interval as shown by hatching area in figure

iii. Acceleration v/s Time graph (a-t curve)

The slope of a-t curve is a jerk

Jerk = $\frac{da}{dt}$



a – t graph

We know, $\frac{dv}{dt}$ = a

dv = v dt

Integrating on both sides

$$\int dv = \int a\,dt$$

If the particles velocity is v_1 at time t_1 and v_2 at time t_2, then

$$\int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} a\, dt$$

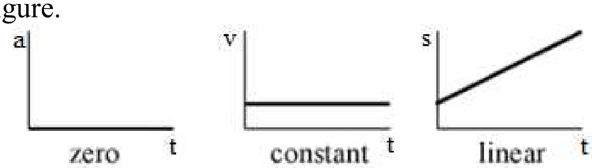
$v_2 - v_1 = \int_{t_1}^{t_2} a\, dt$

Therefore, Change in velocity = Area under a-t curve

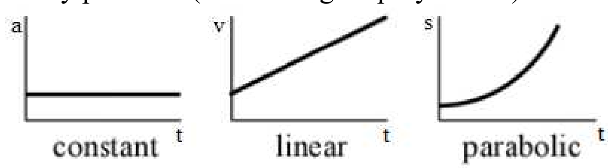
Note: Area measured under the acceleration and time graph from t_1 to t_2 provides the change in velocity during that time interval as shown by hatching area in figure

Some noteworthy points

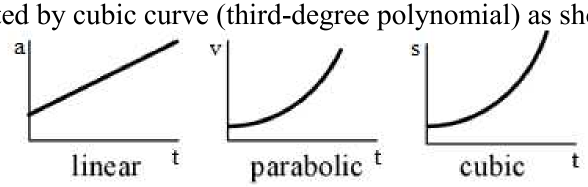
i. If the velocity is constant (acceleration is zero), it will be represented by horizontal line. Then the displacement s will be represented by oblique straight line as shown in the figure.



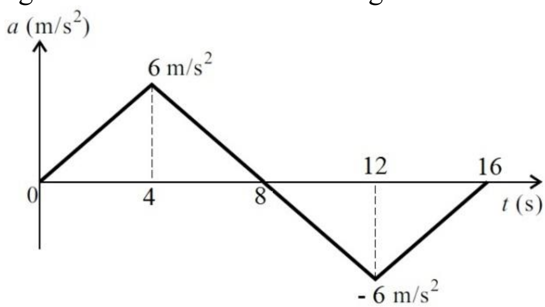
ii. If the acceleration is constant (other than zero), it will be represented by horizontal line. Then velocity v will be represented oblique straight line and s will be represented by parabola (second-degree polynomial) as shown in the figure



iii. If the acceleration is varying, it will be represented by oblique straight line. Then the velocity v is represented by parabolic curve and displacement s will be represented by cubic curve (third-degree polynomial) as shown in the figure



Example 9.28 The acceleration-time diagram for the linear motion is shown in Figure. Construct v-t and s-t diagrams for the motion assuming that the motion starts from rest.



Solution

i. Velocity – time diagram

We know, change in velocity = area under a-t diagram

At t = 4 sec

$v_4 - v_0 = \frac{1}{2} \times 4 \times 6 = 12$

$v_4 = 12$ m/s ($\because v_0 = 0$)

At t = 8 sec

$v_8 - v_4 = \frac{1}{2} \times 4 \times 6 = 12$

$v_8 = v_4 + 12 = 12 + 12 = 24$ m/s

At t = 12 sec

$v_{12} - v_8 = \frac{1}{2} \times 4 \times -6 = -12$

$v_{12} = v_8 - 12 = 24 - 12 = 12$ m/s

At t = 16 sec

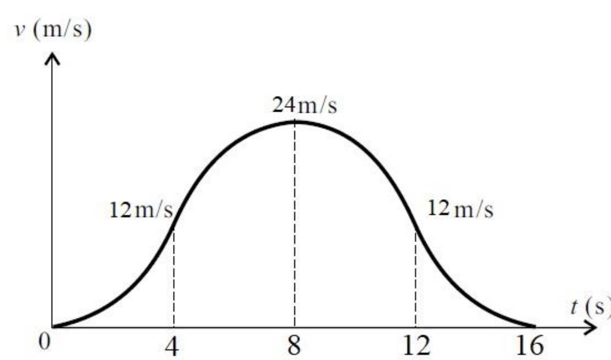
$v_{16} - v_{12} = \frac{1}{2} \times 4 \times -6 = -12$

$v_{16} = v_{12} - 12 = 12 - 12 = 0$ m/s

Now, let us plot the v-t diagram with the values obtained.

The orientation of the curves for v-t and s-t diagrams for different time intervals are shown below

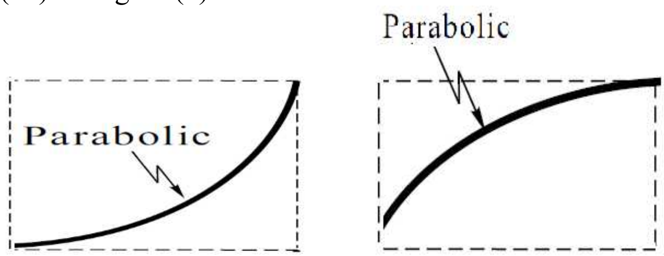
	0 – 4 sec	4 – 8 sec	8 – 12 sec	12 – 16 sec
a – t	Linear (Increasing)	Linear (Decreasing)	Linear (Decreasing)	Linear (Increasing)
v – t	Parabolic (anticlockwise)	Parabolic (clockwise)	Parabolic (clockwise)	Parabolic (anticlockwise)
s – t	Cubic	cubic	cubic	cubic



ii. Displacement – time diagram

We know, change in displacement = area under v – t diagram

Note: To calculate the area under a parabolic curve shown in the figure we have 1/3 (bh) for figure (a) and 2/3 (bh) for figure (b)



At t = 4 sec

$s_4 - s_0 = \frac{1}{3} \times 4 \times 6 = 8$

$s_4 = 8$ m $\quad (\because s_0 = 0)$

At t = 8 sec

$s_8 - s_4 = (4 \text{ x } 12) + [\frac{2}{3} \times 4 \times 6] = 64$

$s_8 = s_4 + 64 = 8 + 64 = 72$ m

At t = 12 sec

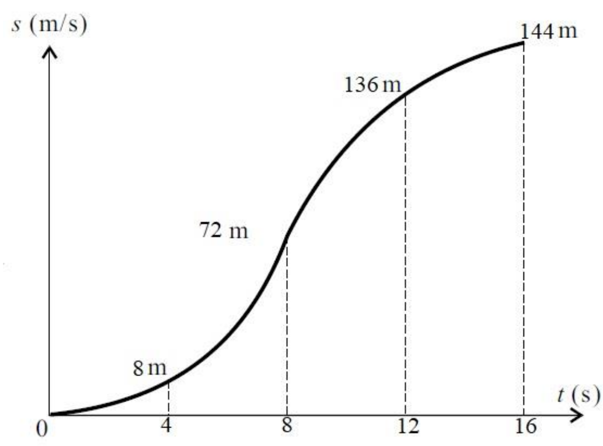
$s_{12} - s_8 = (4 \text{ x } 12) + [\frac{2}{3} \times 4 \times 6] = 64$

$s_{12} = s_8 + 72 = 64 + 72 = 136$ m

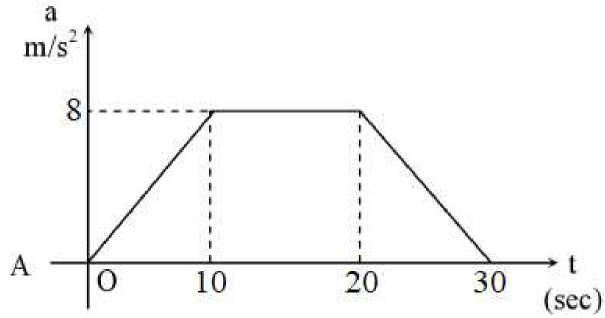
At t = 16 sec

$s_{16} - s_{12} = \frac{1}{3} \times 4 \times 6 = 8$

$s_{16} = s_{12} + 8 = 136 + 8 = 144$ m



Example 9.29 The acceleration-time curve of a particle moving in rectilinear motion is shown in the figure. Draw v-t and s-t curve.



Solution

i. Velocity – time diagram

We know, change in velocity = area under a-t diagram

At t = 10 sec

$v_{10} - v_0 = \frac{1}{2} \times 10 \times 8 = 40$

$v_{10} = 40$ m/s $(\because v_0 = 0)$

At t = 20 sec

$v_{20} - v_{10} = 10 \times 8 = 80$

$v_{20} = v_{10} + 80 = 40 + 80 = 120$ m/s

At t = 30 sec

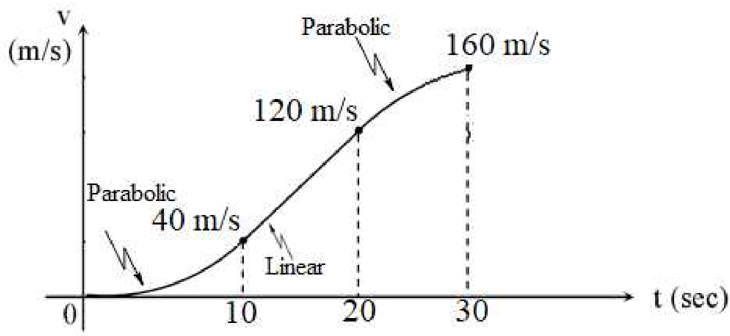
$v_{30} - v_{20} = \frac{1}{2} \times 10 \times 8 = 40$

$v_{30} = v_{20} + 40 = 120 + 40 = 160$ m/s

Now, let us plot the v-t diagram with the values obtained.

The orientation of the curves for v-t and s-t diagrams for different time intervals are shown below

	0 – 10 sec	10 – 20 sec	20 – 30 sec
a – t	Linear (Increasing)	Constant	Linear (Decreasing)
v – t	Parabolic (anticlockwise)	Linear	Parabolic (clockwise)
s – t	Cubic	Parabolic	cubic



i. **Displacement – time diagram**

We know, change in displacement = area under v – t diagram

At t = 10 sec

$s_{10} - s_0 = \frac{1}{3} \times 10 \times 40 = 133.33$

$s_{10} = 133.33$ m ($\because s_0 = 0$)

At t = 20 sec

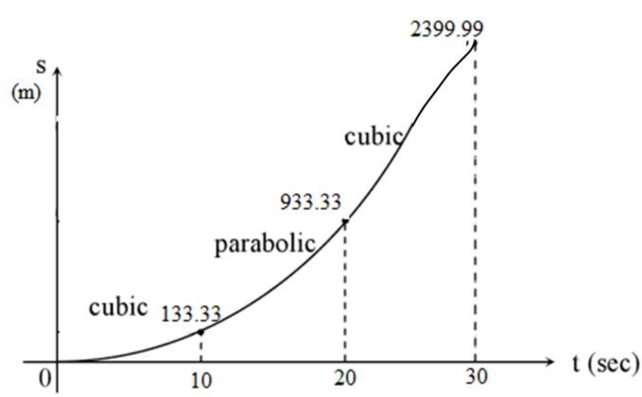
$s_{20} - s_{10} = (10 \times 40) + [\frac{1}{2} \times 10 \times 80] = 800$

$s_{20} = s_{10} + 800 = 133.33 + 800 = 933.33$ m

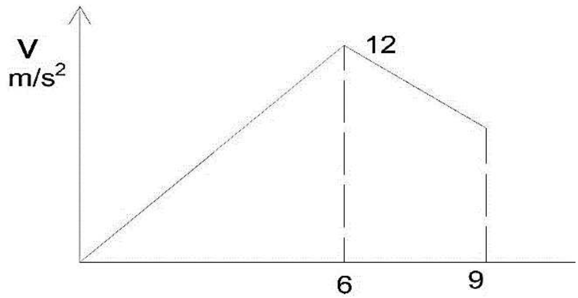
At t = 30 sec

$s_{30} - s_{20} = (10 \times 120) + [\frac{2}{3} \times 10 \times 40] = 1466.66$

$s_{30} = s_{20} + 1466.66 = 933.33 + 1466.66 = 2399.99$ m



Example 9.30 The motion of a particle starting from rest is governed by a-t curve shown in the figure. Sketch the v-t and s-t curves. Determine the displacement at t =9 sec.



Solution

i. **Velocity-time diagram**

We know, change in velocity = area under a-t diagram

At t = 6 sec

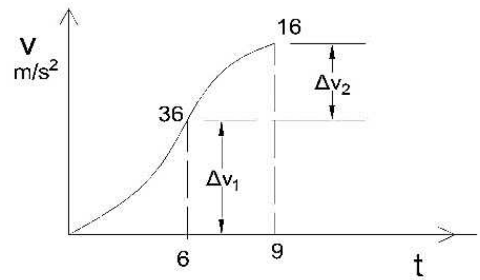
$v_6 - v_0 = \frac{1}{2} \times 6 \times 12 = 36$

$v_6 = 36$ m/s $\quad (\because v_0 = 0)$

At t = 9 sec

$v_9 - v_6 = (3 \times 8) + [\frac{1}{2} \times 3 \times 4] = 30$

$v_9 = v_6 + 30 = 36 + 30 = 66$ m/s



ii. **Displacement – time diagram**

At t = 6 sec

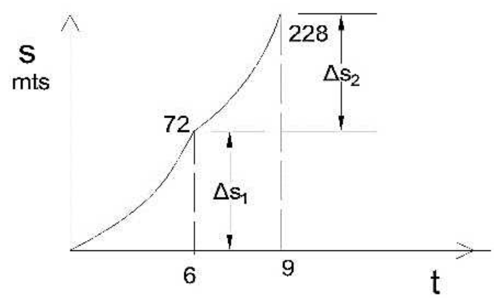
$s_6 - s_0 = \left[\frac{1}{3} \text{ x } 6 \text{ x } 36\right]$

$s_6 = 72$ m

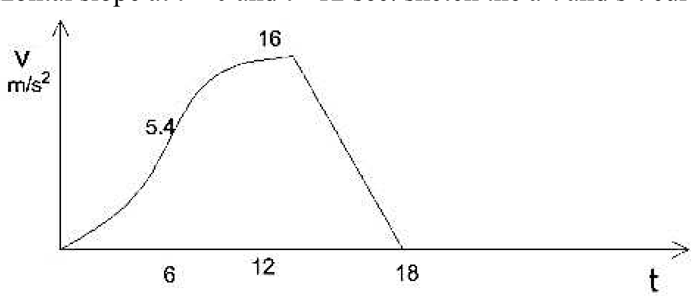
At t = 9 sec

$s_9 - s_6 = (3 \text{ x } 36) + \left[\frac{2}{3} \text{ x } 3 \text{ x } 30\right] = 168$

$s_9 = s_6 + 168 = 72 + 168 = 240$ m



Example 9.31 The curved portion of the v-t curve shown in figure are seconds-degree parabolas with horizontal slope at t = 0 and t= 12 sec. sketch the a-t and s-t curves if s_o is zero.



Solution As the v- curve is having horizontal slope at t = 0 and t = 12 sec hence value of acceleration at time will be zero.

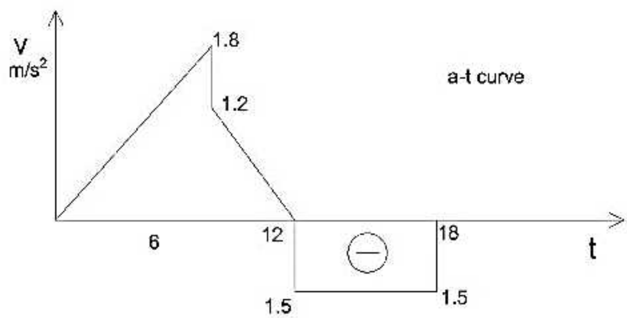
Now, $\Delta V_1 = (Area)_{a-1}$

Curve between 0 to 6 sec

$5.4-0 = \frac{1}{2}$ x 6 x a a = 1.8 m/s^2

Now: $\Delta V_2 = (Area)_{a-1}$ curve between 12 to 18 sec

0-9 = ax 6 a= -1.5 m /s^2



For s-t curve $\Delta S = (Area)_{a-1}$ curve or

$V_1 (\Delta t) + (Area)_{a-1} \times t_2$

$\Delta S_1 = \frac{1}{3}$ x 6 x 5.4

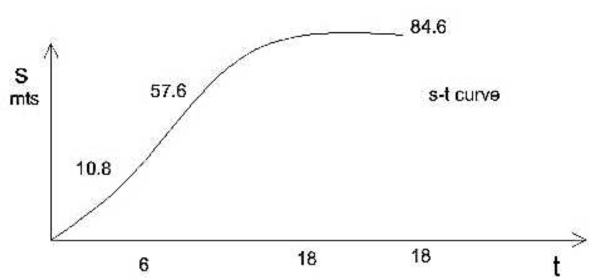
S_2 -0 = 10.8 S_2 =10.8 m/s

$\Delta S_2 = (5.4 \text{ x } 6) + \frac{2}{3} \text{ x } 6 \text{ x } (9 - 5.4)$

S_3 – 10.8 = 46.8 S_3 = 57.6 mts

$\Delta S_3 = \frac{1}{2}$ x 6 x 9

S_4 – 57.6 = 27 S_4 = 84.6 mts



SUMMARY

- A particle moving along a straight line is said to be in rectilinear motion
- Distance is the actual path travelled whereas displacement is the minimum distance travelled from a starting point to the end point along a straight line.
- Speed describes how fast an object is moving and velocity is the speed in a given direction.
- Average speed is the total distance covered divided by time
- Average velocity is the ratio of displacement Δx to the time interval Δt

$$Instantaneous\ velocity\ v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

- If the velocity of a particle remains unchanged throughout its travel then it is called as uniform velocity motion
- The rate of change of velocity with respect to time is known as acceleration.
- The rate at which the velocity is changing is uniform, it is referred as uniform acceleration
- Motion curves are the graphical representation of displacement, velocity and acceleration with time.

EXERCISES

I. Self-Assessment Questions

1. Define the following terms
   a. Velocity
   b. Acceleration
   c. Displacement
2. What are motion curves? What is its importance?
3. Distinguish between average acceleration and instantaneous acceleration.
4. What is the difference between uniform motion and uniformly accelerated motion?
5. What is rectilinear motion?
6. Define instantaneous velocity and average velocity.
7. Explain free-fall acceleration.
8. What are the differences encountered while solving problems of acceleration due to gravity?

II. Multiple Choice Questions

1. If the velocity of a particle is constant, then the acceleration will be
   a. Negative
   b. Positive
   c. Zero
   d. All the above

2. If a particle has uniformly accelerated motion, which of the following is correct
    a. Velocity is constant
    b. Acceleration is constant
    c. Position is constant
    d. Time is constant
3. The negative value of acceleration indicates
    a. Decrease in velocity
    b. Increase in velocity
    c. Constant velocity
    d. Velocity is zero
4. A stone is thrown up along a vertical axis with a velocity u. The total time taken to come back to the point of release is
    a. $\frac{2u}{g}$ b. $\frac{u^2}{2g}$ c. $\frac{u^2}{g}$ d $\frac{u}{g}$
5. A stone is thrown up along a vertical axis with a velocity u. The maximum height reached by it is
    a. $\frac{u^2}{g}$ b. $\frac{u}{g}$ c. $\frac{2u}{g}$ d $\frac{u^2}{2g}$
6. Area under v-t curve represents
    a. Change in acceleration
    b. Change in velocity
    c. Change in displacement
    d. Change in time
7. Area under a-t curve represents
    a. Change in acceleration
    b. Change in velocity
    c. Change in displacement
    d. Change in time
8. The slope of the velocity – time curve represents
    a. velocity
    b. displacement
    c. acceleration
    d. frequency of travel
9. The velocity – time graph of a uniform accelerated body is a
    a. circle
    b. straight line
    c. parabola
    d. hyperbola
10. The distance traveled by a particle in n^{th} second is
    a. $S_n = u + a(2n - 1)/2$
    b. $S_n = u + a(2n + 1)/2$
    c. $S_n = u + a(n-1)/2$
    d. $S_n = ut + ½ at^2$
11. The position of a body with respect to time is given by $x = 3t^3 - 6t^2 + 15t + 7$. At t=0

the acceleration is

a. 3
b. -12
c. 6
d. 7

12. A car accelerates uniformly from rest and acquires a speed of 54 km/hr in 15 seconds. The acceleration is
(a) 1 m/s^2
(b) 2 m/s^2
(c) 4 m/s^2
(d) 3 m/s^2

Answers

1. c
2. b
3. a
4. a
5. d
6. c
7. b
8. c
9. b
10. a
11. b
12. a

CHAPTER – 10

CURVILINEAR MOTION

LEARNING OBJECTIVES

After studying this chapter, you should be able to
- *Define the position, velocity and acceleration of a particle in curvilinear motion*
- *Understand various coordinate systems in curvilinear motion*
- *Apply different coordinate systems to solve the problems of curvilinear motion*
- *Solve the problems of 3D curvilinear motion*

10.1 INTRODUCTION

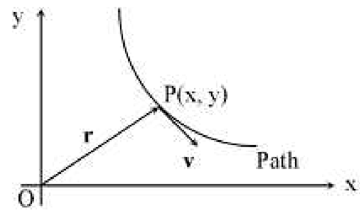
If a particle undergoes translational motion along a curved path then the motion is said to be curvilinear motion. This kind of motion may be two-dimensional or three-dimensional. If the path lies in a plane it is termed as two-dimensional (plane curvilinear) motion otherwise it is three-dimensional (space curvilinear) motion. In this chapter, we shall discuss the concept and problems of plane curvilinear motion as well as space curvilinear motion.

Fig. 10.1 Example of curvilinear motion

10.2 POSITION, VELOCITY AND ACCELERATION

In curvilinear motion we describe the motion of a particle by stating its position, velocity and acceleration same as in rectilinear motion. All these quantities are defined in vector form as given below.

Position vector: The line OP which is drawn from the origin O to the particle P is called position vector.



$$\bar{r} = x\,i + y\,j$$

$$OP = |\bar{r}| = \sqrt{x^2 + y^2}$$

Velocity vector: It is the rate of change of position vector with respect to time. The vector joining P and P' is the change in position vector Δr during the time interval Δt

$$average\ velociy\ v = \frac{\Delta \bar{r}}{\Delta t}$$

$$\bar{V} = \frac{d\bar{r}}{dt}$$

In curvilinear motion, velocity of particle is always tangent to the curved path at every instant
Speed: Magnitude of velocity vector
Acceleration: It is the rate of change of velocity vector with respect to time

$$\bar{a} = \frac{d\bar{V}}{dt}$$

10.3 COORDINATE SYSTEM FOR CURVILINEAR MOTION

When a particle is moving along a curve, its motion can be described by different co-ordinate systems. In curvilinear motion, particle changes its direction at every instant. For this reason, the curvilinear motion analysis can be carried out by adopting different coordinate system. The different types of coordinate systems (to name few) are rectangular, normal and tangential, polar and cylindrical. In this chapter, we study the curvilinear motion by considering following coordinate systems.

i. Rectangular coordinate system
ii. Normal and tangential coordinate system (Path coordinate)
iii. Polar coordinate system
iv. Cylindrical coordinates (3D – Coordinate system)

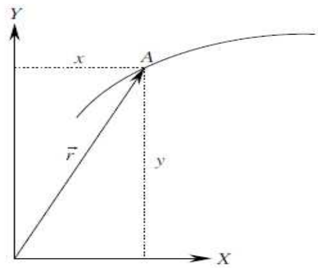
10.4 RECTANGULAR COORDINATES

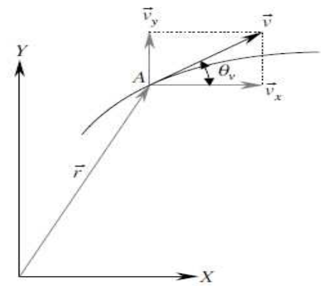
In rectangular coordinate system, velocity and acceleration can be resolved into two mutually perpendicular components parallel to x and y-axis. These components can be combined to obtain their resultant.

Rectangular components of velocity

Let us consider a particle moving in x-y plane as shown in the figure. Let its position at an instant of time be A. The position vector $\bar{r}$ of the particle is given by

$$\bar{r} = x\,i + y\,j$$





Then velocity vector can be obtained by differentiating above equation with respect to time

$$\bar{v} = \frac{d\bar{r}}{dt}$$

$$\bar{v} = v_x\, i + v_y\, j \qquad \because \frac{dx}{dt} = v_x;\ \frac{dy}{dt} = v_y$$

Where v_x and v_y are x and y components of velocity

Magnitude $$v = \sqrt{{v_x}^2 + {v_y}^2}$$

Direction $$\theta = tan^{-1}\frac{v_y}{v_x}$$

Rectangular components of acceleration

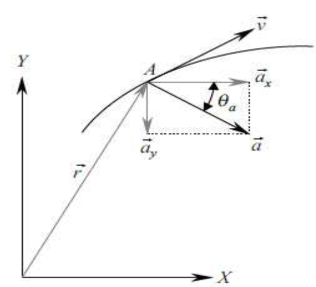
The acceleration vector can be obtained by differentiating the equation of velocity vector with respect to time

$$\bar{a} = \frac{d\bar{v}}{dt}$$

$$\frac{dv_x}{dt} i + \frac{dv_y}{dt} j$$

$$\frac{d^2x}{dt^2} i + \frac{d^2y}{dt^2} j$$

$$\bar{a} = a_x\, i + a_y\, j$$



Where a_x and a_y are x and y components of acceleration

Magnitude $$a = \sqrt{{a_x}^2 + {a_y}^2}$$

Direction $$\theta = tan^{-1}\frac{a_y}{a_x}$$

Note: The velocity of particle is always tangential to curved path whereas the acceleration does not bear any relationship with the direction of motion.

Example 10.1 The motion of a particle moving along a curved path is given by the equation $y^2 = \frac{4}{3}x$ where x and y are in meters. Also, its position in 'x' direction w. r. t. t is given by the relation, $x = t^2$. Determine the position, velocity and acceleration of the particle along y-direction when particle has position x = 6 m.

Solution Given $y^2 = \frac{4}{3}x$ and $x = t^2$

$y^2 = \frac{4}{3}t^2$

$y = 1.15t$

At x = 6 m the time will be

$6 = t^2$

$t = 2.45$ sec

The position of particle at x = 6 is

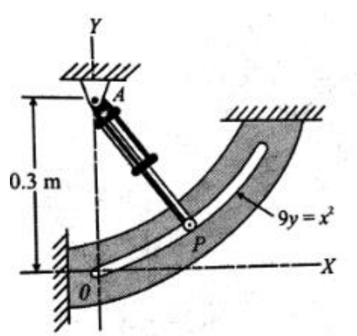
$y^2 = \frac{4}{3} \times 6$

$y = 2.83$ m

$v_y = \frac{dy}{dt} = 1.15$ m/s

$a_y = \frac{dv_y}{dt} = 0$

Example 10.2 The telescoping rod shown in fig. forces the pin P to move along the fixed path $9y = x^2$ where x and y are in cm. At any time t, the x coordinate of P is given by x $= t^2 - 5t$. Determine the y coordinate of velocity and acceleration of P at x = 6 cm.



Solution Given $x = t^2 - 5t$ at x = 6 cm, t = ?

$t^2 - 5t = 6$

$t^2 - 5t - 6 = 0$

$t = 6$ or $t = -1$

It is given that $9y = x^2$

$9y = (t^2 - 5t^2)^2$

$9y = t^4 - 10\,t^3 + 25\,t^2$ ---- (i)

Differentiate equation (i) w. r. t. time

$9\ \frac{dy}{dt} = 4\,t^3 - 30\,t^2 + 50\,t$

i.e., $9\,v_y = 4\,t^3 - 30\,t^2 + 50\,t$ ---- (ii)

The y-coordinate of velocity at t = 6 sec is

$9\,v_y = 4\,(6)^3 - 30\,(6)^2 + 50\,(6)$

$v_y = 9.33$ cm/sec.

Differentiate equation (ii) w. r. t. time

$9\ \frac{dv_y}{dt} = 12\,t^2 - 60\,t + 50$

i.e., $9\,a_y = 12\,t^2 - 60\,t + 50$

The y-coordinate of acceleration at t = 6 sec is

$9\,a_y = 12\,(6)^2 - 60\,(6) + 50$

$a_y = 13.56$ cm/sec^2

Example 10.3 A particle moves in the X –Y plane so that its x coordinate is defined by $x = 5\,t^3 - 105\,t$ where x is in cm and t is in seconds. When t = 2 s the total acceleration is 75 m/s^2. If the y component of acceleration is constant and the particles starts from rest at the origin when t = 0, determine its velocity when t = 4 sec.

Solution Given $x = 5\,t^3 - 105\,t$

Differentiate w. r. t. time

$\frac{dx}{dt} = 15\,t^2 - 105$

$v_x = 15\,t^2 - 105$

Differentiate once again w. r. t. time

$\frac{dv_x}{dt} = 30\,t$

i.e., $a_x = 30\,t$

At t = 2; $a_x = 60$ cm/sec^2

We know $a = \sqrt{a_x^2 + a_y^2}$

$75 = \sqrt{60^2 + a_y^2}$

$a_y = 45$ cm/sec^2

Now, $a_y = \frac{dv_y}{dt}$

$dv_y = 45 \times dt$

Integrating on both sides

$v_y = 45\ t + C$

At $t = 9$; $v_y = 0$
$C = 0$.

$v_y = 45\ t$

Now at $t = 4$ sec.

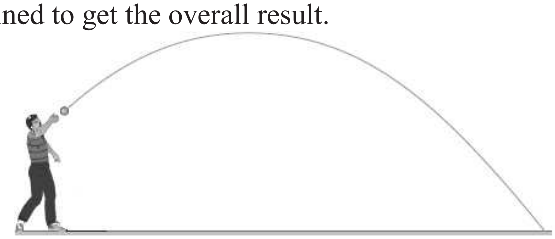
$v_x = 15\ (4)^2 - 105 = 135$ cm/sec.

And $v_y = 45\ (4) = 180$ cm/sec.

Total velocity $v = \sqrt{135^2 + 180^2} = 225$ cm/sec

10.5 PROJECTILE MOTION

Any object, when projected upwards (not vertical) with some angle, traces a path in the air and strikes the ground at some point other than the point of projection. The object travels in both horizontal and vertical direction and traces curvilinear path. This projected object which moving with the combined effect of horizontal and vertical forces is called a projectile. As the projectile moves in both the directions, it has vertical component and horizontal component. The problems of projectiles can be analyzed separately in horizontal and vertical directions and then can be combined to get the overall result.



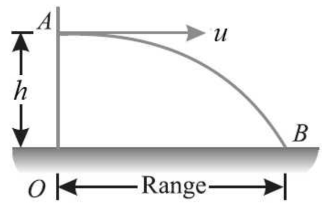
Figure

IMPORTANT TERMS

i. The path traced by a projectile is known as ***trajectory***
ii. The velocity with which the projectile is projected is known as ***velocity of projection***
iii. The angle at which a projectile is projected is known as ***angle of projection***
iv. The time taken by a projectile to return back to the ground is known as ***time of flight***
v. The distance between the point of projection and the point of strike is known as ***range***

10.6 MOTION OF A PROJECTILE WHEN PROJECTED HORIZONTALLY

Consider an object thrown horizontally with velocity u from the point A which is at a height h from the horizontal plane Ox. Let it hit the horizontal plane at B after a time t as shown in the figure



Initial velocity along x-axis and y-axis is u and zero respectively. Similarly acceleration along OB and OA is zero and g.

Vertical motion

$$s = u\,t + \tfrac{1}{2}at^2$$
$$h = 0 + \tfrac{1}{2}at^2$$

Time of flight

$$t = \sqrt{\frac{2h}{g}}$$

Horizontal motion

The distance travelled by the projectile along oy is

$$s = u\,t + \tfrac{1}{2}at^2$$
$$\text{Range} = OB = ut + 0$$

Substituting for t,

$$Range\ = u\sqrt{\frac{2h}{g}}$$

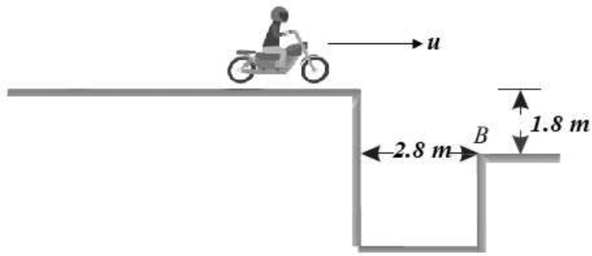
Example 10.4 A person standing on a building which is 10 m height, throws a stone with a horizontal velocity 7 m/s. Calculate the range of the stone

Solution Given h = 10 m, velocity u = 7 m/s and Range = ?

We know,

$$Range\ = u\sqrt{\frac{2h}{g}} = 7\sqrt{\frac{2\ \times 10}{9.81}} = 9.99\ m$$

Example 10.5 A motor cyclist wants to jump over a ditch as shown in the figure. Calculate the minimum velocity necessary for jumping.



Solution Given range = 2.8 m, h = 1.8 m and u =?
We know,

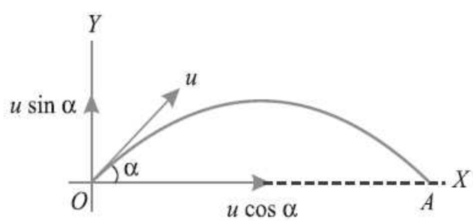
$$Range = u\sqrt{\frac{2h}{g}}$$

$$2.8 = u\sqrt{\frac{2 \times 1.8}{g}}$$

u = 4.62 m/s

10.7 MOTION OF A PROJECTILE ON A LEVEL GROUND

Consider the motion of a projectile which is projected from point O at an angle α to the horizontal, with an initial u as shown in the figure. The particle moves with the combined effects of vertical as well as horizontal directions. The acceleration of projectile motion is uniform along vertical downward direction and is zero in horizontal direction.



Horizontal motion
The horizontal components of the velocity u will be *u cos α* and this will be constant throughout the motion as there is no acceleration along the horizontal direction.

Vertical motion
The vertical components of the velocity u will be *u sin α* and this will change with time because of acceleration due to gravity in the vertical downward direction

Equation of trajectory
Let P(x, y) represent the position of projectile after t seconds. Considering the vertical motion under gravity,
We know $s = ut + \frac{1}{2} at^2$

$$y = (u \sin \alpha)\, t - \tfrac{1}{2} gt^2$$

Considering horizontal motion, as we know in horizontal motion the acceleration is zero, we have

$$\text{Displacement} = \text{velocity x time}$$
$$x = (u \cos \alpha)\, t$$
$$t = \frac{x}{u \cos \alpha}$$

Substituting this value in eq we get

$$y = u\, sin\, \alpha \frac{x}{u \cos \alpha} - \frac{1}{2} g \left(\frac{x}{u \cos \alpha}\right)^2$$
$$y = x\, tan\, \alpha - \frac{1}{2} g \frac{gx^2}{u^2 cos^2 \alpha}$$

The above equation is in the form of $y = ax - bx^2$ which is the equation of parabola. Hence the equation of trajectory of a projectile is a *parabola.*

Maximum height reached

When the particle reaches maximum height, the vertical component of the velocity (v_y) is zero. Considering vertical motion

Initial velocity is u sin α, final velocity is 0 and acceleration due to gravity is -g

We have $v^2 - u^2 = 2as$

$$0 - (u \sin \alpha)^2 = -2gh$$
$$h = \frac{u^2 sin^2 \alpha}{2g}$$

Time taken to reach maximum height

Using first equation of motion v = u + at, when projectile reaches maximum height

$$0 = u \sin \alpha - gt$$
$$t = \frac{u \sin \alpha}{g}$$

Since the time of ascent is equal to the time of descent, then the time of flight (total time taken to reach maximum height and to return back to the ground) is given by

$$t = \frac{2u \sin \alpha}{g}$$

Horizontal range

It is the distance between the point of projection and the point where the particle the strikes the ground.

Considering the horizontal motion, the projection moves with uniform velocity u cos α.

$$R = (u \cos \alpha)\, t$$
$$= u\, cos\, \alpha \frac{2u \sin \alpha}{g}$$
$$R = \frac{u^2 \sin 2\alpha}{g}$$

Example 10.6 A ball is hit from an apartment balcony which strikes the ground at a distance 108 m horizontally from the apartment. If the initial velocity of the ball is 30 m/s at 53.1° to the horizontal, how high is the balcony above the ground?

Solution $u_x = u \cos \theta = 30 \cos 53.1 = 18$ m/s

$u_y = u \sin \theta = 30 \sin 53.1 = 24$ m/s

Horizontal distance $x = u_x t$

$108 = 18 t$

$t = 6$ sec

Vertical distance

$y = u_y t - ½ gt^2$

$y = 24(6) - 4.905 (6)^2 = 32.58$ m

Example 10.7 A ball is released with the initial velocity of 48.3 m/s upward at 30° to the horizontal from a tower which is 19.32 m above a level ground. What horizontal distance will the ball travel before hitting the ground?

Solution $u_x = u \cos \theta = 48.3 \cos 30 = 41.83$ m/s

$u_y = u \sin \theta = 48.3 \sin 30 = 24.15$ m/s

Vertical distance $y = u_y t - ½ gt^2$

$-19.32 = 24.15t - 4.905 t^2$

$4.905t^2 – 24.14t + 19.32 = 0$

$t = 5.624$ sec or – 0.7 sec

Neglect negative value

Horizontal distance $x = u_x t$

$= 41.83 \times 5.624 = 235.24$ m.

Example 10.8 Projectile is fired with an initial velocity of v_0 m/s upward at an angle θ with the horizontal. Find the horizontal distance covered before the projectile returns to its original level. Also determine the maximum height attained by the projectile.

Solution $v^2 = u^2 + 2as$

For y direction $v_y^2 = u_y^2 – 2gh$

At B $v_y = 0$

$0 = (u \sin \theta)^2 – 2g\, h_{max}$

$$h_{max} = \frac{u^2 sin^2\theta}{2g}$$

For displacement in y direction

$y = u_y t - ½ gt^2$

The time of flight from O to C is t sec

$0 = u \sin \theta \times t - ½ gt^2$

$\frac{g}{2} = t^2 = u \sin \theta \times t$

$t = \frac{2u\,sin^2\theta}{g}$

Now, $x = u_x\, t$

$= \cos \theta \times \frac{2u\,sin^2\theta}{g} = \frac{u^2 2sin\theta cos\theta}{g}$

Range $= \frac{u^2 sin^2\theta}{g}$

Example 10.9 A ball is thrown so that it just clears a 7.5 m wall 30 m away. If it left the hand 1.5 m above the ground and at an angle of 60° to the horizontal, what was the initial velocity of the ball?

Solution $u_x = u \cos 60 = 0.5\, u$

$u_y = u \sin 60 = 0.866\, u$

From O to A, x = 30m; y = 7.5-1.5=6m

Horizontal distance $x = u_x\, t$

$30 = 0.5\, u \times t$

$u \times t = 60$

Vertical distance

$y = u_y\, t - ½ gt^2$

$6 = 0.866u \times t - ½\, 4.905\, t^2$

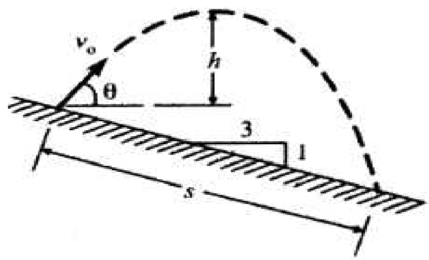
Substituting u x t = 60 is the above equation

$6 = 0.866(60) - ½\, 4.905\, t^2$

t = 3.061 sec

$u = \frac{60}{3.061} = 19.601 sec$

Example 10.10 In figure, a ball thrown down the incline strikes it at a distance s = 76.35 m. if the ball rises to a maximum height h = 19.32 m above the point of release, compute its initial velocity and inclination θ.



Solution $\alpha = \tan^{-1}\left(\frac{1}{3}\right) = 18.43°$

From O to D

x = 76.35 cos α = 72.434 m

y = 76.35 sin α = 2.138 m

$h\ max = \frac{(U \sin\theta)^2}{2\,g}$

$19.32 \times 2 \times 9.81 = (U \sin\theta)^2$

U sin θ = 19.47 m/s.

From O to D, $x = u_x\, t$

72.434 = u cos θ x t ------ (i)

$y = u_y\, t - \frac{1}{2}\, g\, t^2$

$24.138 = u \sin\theta \times t - 4.905\, t^2$

As u sin θ = 19.47

$24.138 = 19.47 \times t - 4.905\, t^2$

t = 4.96 sec

Put in (i) $u \cos\theta = \frac{72.434}{4.96} = 14.6$ m/s

Take ratio $\frac{U \sin\theta}{U \cos\theta} = \frac{17.47}{4.6}$

θ = 53.13° and u = 24.33 m/s.

Example 10.11 A particle has an initial velocity of 30m/s up to the right at 30° with the horizontal. The components of acceleration are constant at $a_x = -1.2$ m/s^2 and $a_y = -6$ m/s^2. Compute the horizontal distance covered until the particles reach a point 18m below its original elevation.

Solution $u_x = u \cos\theta = 30 \cos 30 = 0.5\, u$

$u_y = u \sin\theta = 30 \sin 30 = 15$ m/s

$a_t = -1.2$ m/s^2 and $a_y = -6$ m/s^2

Vertical displacement, $y = u_y\, t - 1/2\, a_y\, t^2$

$-18 = 15\, t - 1/2 \times 6 \times t^2$

$3t^2 - 15\, t - 18 = 0$

$t = 6$ sec.

Horizontal displacement $x = u_x\, t + ½\, a\, x\, t^2$

$= 25.98 \times 6 – ½ \times 6^2\, t^2 = 134.28$ mts.

Example 10.12 A particle moves along the path $y = x^2 – 4x + 100$, starting with an initial velocity of $v_o = (4\hat{\imath} – 16\hat{\jmath})$ m/s. If v_x is constant, determine v_y and a_y at x= 16 m.

Solution $y^2 = x^2 - 4x + 100$

At x = 16m; $y = 16^2 – 4 \times 16 + 100 = 292$ m.

$u_x = 4$ m/s. as velocity in 'x' direction is constant

$u_y = -16$ m/s. Value of $a\, x = 0$

Horizontal displacement $x = u \times t$

$16 = 4 \times t$

$t = 4$ sec.

Now, $v_y = u_y + a\, y\, t$ and $y = u_y\, t + ½\, ay\, t^2$

$292 = -16\, (4) + ½\, ay\, (4)^2$

$a_y = 44.5$ m/s^2

and now, $v_y = u_y + a\, y\, t = -16 + 44.5\, (4) = 162$ m/s.

Example 10.13 If the velocity of the particle is defined by $v = (2t + 1)\, \hat{\imath} + 3\hat{\jmath}$ m/s and its position vector at t = 1sec is $r = 4\hat{\imath} + 3\hat{\jmath}$ m, determine the path of the particle in terms of its x and y coordinates.

Solution $v = (2t + 1)\, \hat{\imath} + 3\hat{\jmath}$ m/s

$u_x = 2t + 1$

As $u_x = \frac{dx}{dt}$

$dx = (2t + 1)\, dt$

Integrating on both sides

$x = t^2 + t + C_1$ ----(i)

$u_y = 3$ m/s

$u_y = \frac{dy}{dt}$

$dy = 3.\ dt$

Integrating on both

$y = 3t + C_2$ ----(ii)

at t = 1 sec; Position vector $r = 4\hat{\imath} + 3\hat{\jmath}$

i.e., x = 4m and y = 3m

Substituting in equation (i) and (ii) we get

$4 = 1 + 1 + C_1$ and $3 = 3(1) + C_2$

$C_1 = 2$ and $C_2 = 0$

$x = t^2 + t + 2$ and $y = 3t$

$t = \frac{y}{3}$

Put in equation of 'x'

$x = (\frac{y}{3})^2 + \frac{y}{3} + 2$

Example 10.14 A rocket is released from a jet fighter flying horizontal at 330 m/s at an altitude of 2400 m above its target. The rocket thrust gives it a constant horizontal acceleration of 0.6g. Determine the angle between the horizontal and the line is sight to the target.

Solution $y = u_y\, t - ½\, g\, t^2$

$2400 = 0 + ½\,(9.81)\, t^2$

$t = 22.12$ sec

Now, $x = u_x \text{ x } t + ½\, a \text{ x } t^2 = (330 \text{ x } 22.12) + ½ \text{ x } (0.6 \text{ x } 9.81)\,(22.12)^2$

$= 8739.62$ mts

Now, $\theta = \tan^{-1}(\frac{2400}{8739.62}) = 15.35°$

Example 10.15 If the initial velocity of an object is 12 m/s, determine the horizontal distance it can cover without rising more than 3 m.

Solution $h_{max} = \frac{(u \sin\theta)^2}{2\,g}$

$3 = \frac{(12 \sin\theta)^2}{2\,g}$

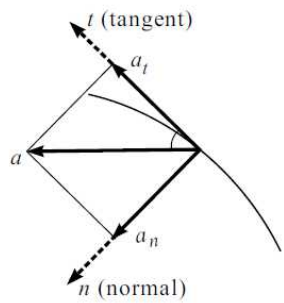
$\theta = 39.74°$

Now $x = \frac{u^2 \sin\theta}{2\,g} = \frac{12^2 \text{ x} \sin(2 \text{ x } 39.74)}{g} = 14.43$ m.

10.8 NORMAL AND TANGENTIAL COORDINATES

Normal and tangential (n-t) coordinates also know as path coordinates describes the motion of a particle moving along a curved path in terms of components that are normal and tangential to its path.

Let us consider t and n axes as shown in the figure. The t-axis is tangent to the curve at the instant considered and is positive along the direction of particle's movement. n-axis which is perpendicular to t-axis and is taken positive towards the centre of curvature of the curve.



Tangential component: Tangential component of acceleration a_t is equal to rate of change of speed of the particle.

$$a_t = \frac{d\bar{v}}{dt}$$

If the magnitude of a_t is constant, the equations of motion shown in the table no 9.1 (equations of uniformly accelerated motion) can be used taking the acceleration equal to a_t. Further, consider $a_t = 0$, if the particle travels at a constant speed and in such cases the acceleration reduces to its normal component.

Normal component: Normal component of acceleration a_n is equal to the square of its speed divided by the radius of curvature of the curve.

$$a_n = \frac{v^2}{\rho}$$

Where, v - velocity
ρ – Radius of curvature

The acceleration's normal component always points towards the center of the curvature of the path.

The magnitude of acceleration $a = \sqrt{(a_n)^2 + (a_t)^2}$

Where
a_n - normal acceleration
a_t - tangential acceleration

Direction $\theta = tan^{-1}\frac{a_n}{a_t}$

Radius of curvature

The symbol ρ is used to indicate the *radius of curvature* of the curve. If the equation of the path is known, its radius of curvature can be calculated using the formula

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\left|\frac{d^2x}{dy^2}\right|}$$

The inverse of ρ, (i.e., $1/\rho$), is known as the *curvature* of the path.

Note

1. When the particle is accelerating, then the direction of tangential component is same as direction of velocity vector. If the particle is decelerating, then the direction of tangential component is opposite to the direction of velocity vector.
2. The normal component of acceleration is always directed towards the centre of curvature

Example 10.16 A particle moves a path of 40 m radius so that its arc distance from a fixed point on the path is given by $s=4t^3 - 10t$ where s is in m and t is in seconds. Compute the total acceleration at the end of 2 sec.

Solution $s = 4t^3 - 10t$

$v = \frac{ds}{dt} = 12t^2 - 10$

$a = \frac{dv}{dt} = 24t$

When t = 2 sec

$a_t = 48$ m/s^2 and

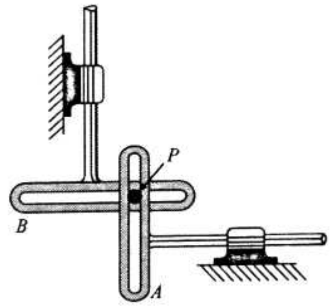
$v = 12(2)^2 - 10 = 38$ m/s

$a_t = \frac{v^2}{\rho} = \frac{(38)^2}{40} = 36.1$ m/s^2

Total acceleration $a = \sqrt{{a_t}^2 + {a_n}^2}$

$$a = \sqrt{48^2 + 36.1^2} = 60.06 \text{ m/s}^2$$

Example 10.17 The pin P moves along a curved path which is determined by the motion of two slotted links A and B. at the instant shown in the figure A has velocity of 12 m/s and acceleration of 19 cm/s both to the right, while B has a velocity of 16 m/s and an acceleration of 5 cm/s both vertically downward. Find the radius of curvature of the path if P at this instant and sketch how the path curves



Solution Form the figure

$a_n = 10 \sin \theta - 5 \cos \theta$

$= 10 \sin 53.13 - 5 \cos 53.13 = 5 \text{ cm/s}^2$

Now an $= \frac{v^2}{\rho}$

r= $\frac{20^2}{5}$ = 80 cm

Example 10.18 A stone is thrown with an initial velocity of 30 m/s upward at 60° to the horizontal. Compute the radius of curvature of its path at the position where it is 15 m horizontally from its initial position

Solution $u_x = 30 \cos 60 = 15$ m/s

Now $v_x = u_x = 15$ m/s

$u_y = 30 \sin 60 = 25.98$ m/s

And $v_y = u_y - gt$

sx = u x t = 25.98 – 9.81 (1)

15 = 15 t

t = 1 sec

From fig $a_n = 9.81 \cos \theta$

= 9.81 cos θ 47.151 = 6.671 m/s2

Also an $= \frac{v^2}{\rho}$

r = $\frac{22.056^2}{6.671}$ = 72.917 m

Example 10.19 A particle has an initial velocity of 60m/s to the right at a slope of 0.75. The components of acceleration are constant at ax = -3.6 m/s^2 and ay = - 6 m/s^2. Compute the radius of curvature at the start and at the top of the path

Solution Radius of curvature at start

$u_x = u \cos \theta$

$v = \tan^{-1}(0.75) = 36.87$

$= 60 \cos 36.87 = 48$ m/s

$u_y = u \sin \theta$

$= 60 \sin 36.87 = 36$ m/s

From figure

$a_n = a_y \cos \theta - a_y \sin \theta$

$= 6 \cos 36.87 - 3.6 \sin 36.87 = 2.64$ m/s^2

We know $a_n = \frac{v^2}{\rho}$;

$\rho = \frac{v^2}{a_n}$

$r = \frac{60^2}{2.64} = 1363.64$ m

Radius of curvature at top of path

At the top of the path $v_y = 0$

Now $v_y = u_y + a_y t$

$0 = 36 - 6(t)$

$t = 6$ sec

As $v_x = u_x + at$

$= 48 - 3.6\ (6) = 26.4$ m/s

From figure $a_n = 6$m/s^2

Now $a_n = \frac{v^2}{\rho}$

$\rho = \frac{26.4^2}{6} = 116.16$ m

Example 10.20 A particle moves counter clockwise on a circular path of 400 m radius. It starts from a position which is horizontally to the right of the center of the path and moves so that $s = 10t^2 + 20t$ where s is the arc distance in m and t in seconds. Compute the horizontal and vertical components of acceleration at the end of 3 sec.

Solution $s = 10t^2 + 20t$

$v = \frac{ds}{dt} = 20t + 20$

After t = 3 sec; v = 80m/s

$a = \frac{dv}{dt} = 20$ m/s^2

Also $a_n = \frac{v^2}{\rho} = \frac{80^2}{400}$ = 16 m/s

Angular velocity $\omega = \frac{v}{\rho} = \frac{80}{400}$ = 0.2 rad/sec

Angular displacement θ after 3 sec = ω t = 0.2 x 3 = 0.6 radians = 34.38 degree

Form the figure

$a_x = - a_n \cos\theta - a_t \sin\theta$

= 16 cos 34.38 20 sin 34.38 = - 24.5 m/s^2

And $a_y = - a_t \cos\theta - a_n \sin\theta$

= 20 cos 34.38 – 16 sin 34.38 = 7.47 m/s^2

Example 10.21 The velocity of a particle is defined by $v_x = 100 - t^{3/2}$ and $v_y = 100 + 10t + 2t^2$ where v is in m/s and t is in seconds. Determine the radius of curvature at the top of its path.

Solution Given $v_x = 100 - t^{3/2}$ and v_y = 100 + 10 t + 2t2

Differentiating w r t time

$a_x = \frac{dv}{dt} = -\frac{3}{2} t^{1/2}$ and $a_y = \frac{dv}{dt} = 10 - 4t$

At top of path $v_y = 0$

$100 + 10t - 2t^2 = 0$

$2t^2 - 10t - 100 = 0$

t = 10 sec or t = -5 sec

Select t = 10 sec

$v_x = 100 - 10^{3/2} = 68.38$

$a_x = \frac{3}{2}(10)^{1/2}$ = 4.47 m/s^2

$a_y = 10 - 4(10) = -30$

From figure $a_n = \frac{v^2}{\rho}$

$\rho = \frac{68.38^2}{30}$ = 155.86 m

Example 10.22 Using the data of the preceding problem, determine the radius of curvature of the path of the particle at t = 12 sec

Solution $v_x = 100 - t^{3/2}$ and

$v_y = 100 + 10t + 2t^2$

and $a_x = -\frac{3}{2} t^{½}$ and $a_y = 10 - 4t$

At t = 12 sec

$v_x = 100 - (12)^{3/2} = 58.43$ m/s^2

$v_y = 100 + 10(12) - 2(12)^3 = -38$ m/s^2

Also $a_x = -\frac{3}{2}(12)^{½} = -5.192$ m/s^2

And $a_y = 10 - 4(12) = -38$ m/s^2

From the figure

$a_n = a_x$ sing $\theta + a_y \cos\theta$

$= 5.198 \sin 49.33 + 38 \cos 49.33 = 28.706$ m/s2

Now an $= \frac{v^2}{\rho}$

$\rho = \frac{89.65^2}{28.706} = 280$ m

Example 10.23 The rectangular components of acceleration for a particle are $a_x = 3t$ and $a_y = 30 - 10t$ where a is in m/s^2. If the particle starts from rest at the origin, find the radius of curvature of the path at t = 2 sec

Solution Given $a_x = 3t$ and $a_y = 30 + 10t$

$a_x = \frac{dv_x}{dt}$

$a_y = \frac{dv_y}{dt}$

At t = 0

v_x and v_y both are zero

$C_1 = 0$ and $C_2 = 0$

$v_x = 1.5\ t^2$ and

$v_y = 30t-5t^2$

At t = 2 sec

$v_x = 1.5\ (2)^2 = 6$ m/s and

$v_y = 30 - 5(2)^2 = 40$ m/s

Also $a_x = 3(2) = 6$ m/s^2

$a_y = 30 - 10(2) = 10$ m/s^2

From figure

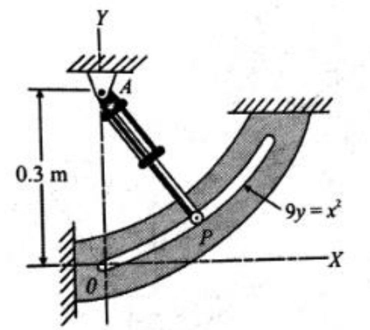
$a_n = a_x \sin\theta - a_y \cos\theta$

$= 6 \sin 81.47 - 10 \cos 81.47 = 4.45$ m/s^2

Now an $= \frac{v^2}{\rho}$

$\rho = \frac{40.447^2}{4.45} = 367.6$ m

Example 10.24 The telescoping rod shown in the figure forces the pin P to move along the fixed path $9y = x^2$ where x and y are in cm. At any time t the coordinate of P is given by $x = t^2 - 5t$. At what rate is the speed of P changing at t=6sec. Solve without using the value of a_y



Solution Given $x = t^2 - 5t$ and $9y = x^2$

$y = \frac{(t^2-5t)^2}{9}$

Position vector $r = (t^2-5t)\,\hat{\imath} + \frac{(t^2-5t)^2}{9}\,\hat{\jmath}$

Differentiating w. r. t. 't'

$\frac{d\bar{r}}{dt} = \bar{v} = (2t-5)\,\hat{\imath} + \frac{2}{9}(t^2-5t)(2t-5)\,\hat{\jmath}$

$= (2t-5)\,\hat{\imath} + \frac{2}{9}(2t^3-15t^2+25t)\,\hat{\jmath}$

Differentiating again w. r. t. 't'

$\frac{d^2\bar{r}}{dt^2} = \frac{d\bar{v}}{dt} = \bar{a} = 2\,\hat{\imath} + \frac{2}{9}(6t^2-30t^2+25)\,\hat{\jmath}$

At t = 6 sec

$\bar{a} = 2\,\hat{\imath} + 13.55\,\hat{\jmath}$

Acceleration $a = \sqrt{2^2 + 13.55^2} = 13.7$ m/s^2

Example 10.25 The position vector of a particle is defined by $r = (2t^3 - 12t)\,\hat{\imath} + 6t^2\hat{\jmath}$ where r is in m and t is in seconds. find the principal radius of curvature of the path of the particle at t= 2 sec

Solution Given $x = 2t^3 - 12t$ and $y = 6t^2$

Differentiating w. r. t. 't'

$\frac{dx}{dt} = v_x = 6t^2 - 12$

$\frac{dy}{dt} = v_y = 12t$

Differentiating once again w. r. t. ‘t’

$\frac{dv_x}{dt} = a_x = 12t$

$\frac{dv_y}{dt} = a_y = 12$

At t = 2 sec

$v_x = 6(2)^2 - 12 = 12$ m/s

$v_y = 12(2) = 24$ m/s

$a_x = 12(2) = 24$ m/s^2

$a_y = 12$ m/s^2

From the fig. $a_n = 24 \sin\theta - 12 \cos\theta$

$= 24 \sin 63.43 - 12 \cos 63.43 = 16.1$ m/s^2

We know $a_n = \frac{v^2}{\rho}$

$\rho = \frac{26.833^2}{16.1} = 44.72$ m

Example 10.26 The position vector of a particle is given by $r = 2t^2\hat{\imath} + 10t\hat{\jmath} + 1/3\ t^3\hat{k}$ where r is in m and t is in seconds. Determine the normal and tangential components of acceleration and the principal radius of curvature of the path of the particle at t = 3 sec

Solution Given $\bar{r} = 2t^2\hat{\imath} + 10t\,j + t^3\hat{k}$

Differentiating w. r. t. ‘t’

$\frac{d\bar{r}}{dt} = \bar{v} = 4t\hat{\imath} + 10\,j + t^2\hat{k}$

Differentiating again w. r. t. ‘t’

$\frac{d\bar{v}}{dt} = \bar{a} = 4\hat{\imath} + 0\,j + 2t\,\hat{k}$

At t = 3 sec

$\bar{v} = 12\hat{\imath} + 10\,j + 9\hat{k}$

Magnitude of v = $\sqrt{325}$ m/s

$\bar{a} = 4\hat{\imath} + 0\,j + 6\hat{k}$

Magnitude of a = $\sqrt{25}$ m/s^2

Taking cross product

$$\bar{v} \times \bar{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 12 & 10 & 9 \\ 4 & 0 & 6 \end{vmatrix} = 60\hat{\imath} - 36\,j - 40\hat{k}$$

Its magnitude $|\bar{v} \times \bar{a}| = \sqrt{60^2 + 36^2 + 40^2 =} \sqrt{6496}$

r = $\frac{v^3}{|\bar{v} \times \bar{a}|} = \frac{(\sqrt{325})^3}{\sqrt{6496}}$ = 72.69 m

$a_n = \frac{v^2}{\rho} = \frac{(\sqrt{325})^2}{72.69}$

a = $\sqrt{{a_t}^2 + {a_n}^2}$

$\sqrt{25} = \sqrt{{a_t}^2 + (4.47)^2}$

a_t = 5.658 m/s^2

Example 10.27 The position of a vector of a particle is given by $r = 9t\hat{\imath} - \frac{2}{3}t^3 j - 3t^2\hat{k}$ where r is in m and t is in seconds. Determine the normal and tangential components of acceleration and the principal radius of curvature of the path of the particle at t = 2 sec

Solution Given $\bar{r} = 9t\hat{\imath} - \frac{2}{3}t^3 j - 3t^2\hat{k}$

Differentiating w. r. t. 't'

$\frac{d\bar{r}}{dt} = \bar{v} = 9\hat{\imath} + 2t^2\, j - 6t\,\hat{k}$

Differentiating again w. r. t. 't'

$\frac{d\bar{v}}{dt} = \bar{a} = 0\hat{\imath} - 4t\, j - 6\,\hat{k}$

At t = 2 sec

$\bar{v} = 9\,\hat{\imath} - 8\,j - 12\hat{k}$

Magnitude of v = 17 m/s

$\bar{a} = 0\hat{\imath} - 8\,j - 6\hat{k}$

Magnitude of a = 10 m/s^2

Taking cross product

$$\bar{v} \times \bar{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 9 & -8 & 12 \\ 0 & -8 & 6 \end{vmatrix} = 48\hat{\imath} - 54\,j - 72\hat{k}$$

Its magnitude $|\bar{v} \times \bar{a}| = \sqrt{48^2 + 54^2 + 72^2 =}\ 102$

r = $\frac{v}{|\bar{v} \times \bar{a}|} = \frac{(17)^3}{102}$ = 48.16 m

Example 10.28 A particle starting from rest moves with the acceleration $\bar{a} = 4t\hat{\imath} - 3t^2 j - 6\hat{k}$ m/s^2. Determine the principal radius of curvature of its path at t = 2 sec

Solution $\bar{a} = 4t\hat{\imath} - 3t^2 j - 6\hat{k}$

$\bar{a} = \frac{d\bar{v}}{dt}$

$d\bar{v}$ = ($4t\hat{\imath} - 3t^2 j - 6\hat{k}$) dt

Integrating on both sides

$\bar{v} = 2t^2\hat{\imath} - t^3 j - 6t\hat{k} + C$

At t= 0, v = 0, therefore C = 0

$\bar{v} = 2t^2\hat{\imath} - t^3 j - 6t\hat{k}$

At t = 2 sec

$\bar{v} = 8\hat{\imath} - 8 j - 12\hat{k}$

Magnitude of v = $\sqrt{272}$ m

$\bar{a} = 8\hat{\imath} - 12 j - 6\hat{k}$

Taking cross product $\bar{v} \times \bar{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 8 & -8 & 12 \\ 8 & -12 & 6 \end{vmatrix} = 96\hat{\imath} + 48 j - 32\hat{k}$

Its magnitude $|\bar{v} \times \bar{a}| = \sqrt{96^2 + 48^2 + 32^2} = 112$

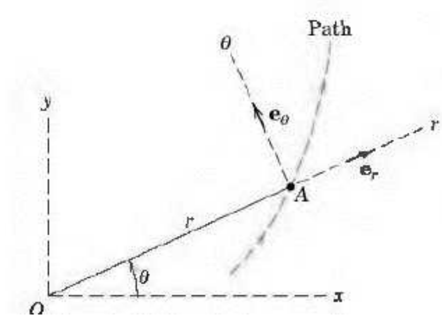
r = $\frac{v^3}{|\bar{v} \times \bar{a}|} = \frac{(\sqrt{272})^3}{112}$ = 40.05 m

10.9 POLAR (R-θ) COORDINATES

If the motion of a particle is specified in terms of the radial distance R and an angular measurement θ, such types of problems can be analyzed using polar coordinates. The polar coordinate system can also be referred as radial and transverse components. Let us consider Particle A which is moving in a curved path as shown in the figure is described by the polar coordinate r and θ. X-axis is taken as a reference line for measuring θ. The position vector r of the particle has the magnitude as given below

$$\mathbf{r} = re_r$$

Where e_r is the unit vector



Velocity and acceleration vectors can be obtained as

$$\bar{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\bar{a} = [\ddot{r} - r(\dot{\theta})^2]\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta}]\,\hat{e}_\theta$$

Where

$v_r = \dot{r}$

$v_\theta = r\dot{\theta}$

$v = \sqrt{{v_r}^2 + {v_\theta}^2}$

$a_r = \ddot{r} - r(\dot{\theta})^2$

$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$

$a = \sqrt{{a_r}^2 + {a_\theta}^2}$

Example 10.29 The polar coordinates of a particle is given by $\theta = 0.4\,t + 0.08t^3$ and $r = 0.5 + 0.09t^2$, where θ is in radians, r is in meters and t is in seconds. Calculate the magnitudes of velocity and acceleration of the particle at t = 2 sec.

Solution Given $r = 0.5 + 0.09t^2$

$\dot{r} = 0.18\,t$

$\ddot{r} = 0.18$

At t = 2 sec

$r = 0.5 + 0.09(2)^2 = 0.86$ m

$\dot{r} = 0.18\,(2) = 0.36$ m/s

$\ddot{r} = 0.18$ m/s^2

Given: $\theta = 0.4\,t + 0.08t^3$

$\dot{\theta} = 0.4 + 0.24\,t^2$

$\ddot{\theta} = 0.48\,t$

At t = 2 sec

$\theta = 0.4\,(2) + 0.08(2)^3 = 1.44$ rad

$\dot{\theta} = 0.4 + 0.24\,(2)^2 = 1.36$ rad/s

$\ddot{\theta}$ = 0.48 (2) = 0.96 rad/s^2

The velocity components can be obtained using the earlier equations for t = 3 s

$v_r = \dot{r}$ = 0.36 m/s

$v_\theta = r\dot{\theta}$ = 0.86 (1.36) = 1.16

$v = \sqrt{{v_r}^2 + {v_\theta}^2} = \sqrt{0.36^2 + 1.16^2}$ = 1.214 m/s

The acceleration components can be obtained using the earlier equations for t = 3 s

$a_r = \ddot{r} - r(\dot{\theta})^2$ = 0.18 – 0.86(1.36)^2 = - 1.41 m/s^2

$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$ = 0.86 (0.96) + 2 (0.36)(1.36) = 1.8 m/s^2

$a = \sqrt{{a_r}^2 + {a_\theta}^2} = \sqrt{(-1.41)^2 + (1.8)^2}$ = 2.28 m/s^2

Example 10.30 The motion of a particle is defined by the parametric equation r = 4t^2 and θ = 2t where r is in cm and t is in seconds. Find the total acceleration of the particle at t = 2 sec.

Solution Given r = 4t^2 and θ = 2t

$\dot{r}$ = 8 t and $\dot{\theta}$ = 2

$\ddot{r}$ = 8 and $\ddot{\theta}$ = 0

At t= 2 sec

r = 16 m	and	θ = 4 rad
$\dot{r}$ = 16 m/s	and	$\dot{\theta}$ = 2 rad/sec
$\ddot{r}$ = 8 m/s^2	and	$\ddot{\theta}$ = 0

$a_r = \ddot{r} - r(\dot{\theta})^2$ = 8 – 16 (2)^2 = - 56 m/s^2

$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$ = 16(0) + 2(16)(2) = 64 m/s^2

$a = \sqrt{{a_r}^2 + {a_\theta}^2} = \sqrt{(-56)^2 + (64)^2}$ = 85.01 m/s^2

Example 10.31 The polar coordinates of a particle moving on a plane curve are given by r = t^3 – 3t +10 and r = 2θ, where r is in cm, θ is in radians and t is in seconds. Determine the acceleration of the particle at t = 3 sec.

Solution Given r = t^3 – 3t +10 and $\theta = \frac{r}{2} = \frac{t^3}{2} - \frac{3}{2}t + 5$

$\dot{r}$ = 3t^2 - 3 and $\dot{\theta} = \frac{3t^2}{2} - \frac{3}{2}$

$\ddot{r}$ = 6t and $\ddot{\theta}$ = 3t

At t= 2 sec

r = $(2)^3 - 3(2) + 10 = 12$ cm

$\dot{r} = 3(2)^2 - 3 = 9$ cm/sec

$\ddot{r} = 6(2) = 12$ cm/sec^2

$\theta = \frac{(2)^3}{2} - \frac{3}{2}(2) + 5 = \frac{12}{2} = 6$ rad

$\dot{\theta} = \frac{3(2)^2}{2} - \frac{3}{2} = 4.5$ rad/sec

$\ddot{\theta} = 3(2) = 6$ rad/sec^2

$a_r = \ddot{r} - r(\dot{\theta})^2 = 12 - 12\,(4.5)^2 = -231$ cm/s^2

$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 12(6) + 2(9)(4.5) = 153$ cm/s^2

$a = \sqrt{a_r{}^2 + a_\theta{}^2} = \sqrt{(-231)^2 + (153)^2} = 277.07$ m/s^2

Example 10.32 In a certain plane motion, it is known that when the radial distance from a fixed origin is r = 50 cm, the r – θ components of velocity are $v_r = 150$ cm/s and $v_\theta = 200$ cm/s, while the components of acceleration are $a_r = -150$ cm/s^2 and $a_\theta = 250$ cm/s^2. At this instant, find the angular acceleration α (i.e, $\ddot{\theta}$) of the positive vector and the component of acceleration normal to the path.

Solution Given r = 50 cm. $v_r = 10$ cm/s and $v_\theta = 200$ cm/s

$a_r = -150$ cm/s^2 and $a_\theta = 250$ cm/s^2

$v_r = \dot{r} = 150$ cm/sec

$v_\theta = r\dot{\theta}$

$200 = 50\,\dot{\theta}$

$\dot{\theta} = 4$ rad/sec

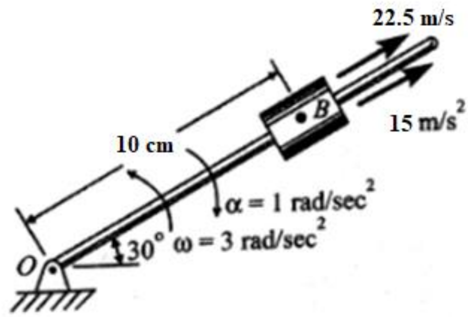
$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$

$250 = 50\,\ddot{\theta} + 2(150)(4)$

$\ddot{\theta} = -19$ rad/sec^2

Angular acceleration α = - 19 rad/sec^2

Example 10.33 At the position shown in the figure, the block B is sliding outward along the straight rod with the given values of velocity and acceleration relative to the rod simultaneously, the rod has the given values of angular velocity ω and angular acceleration α. Find the total acceleration of the block. What is the component of this acceleration normal to the path described by the block?



Solution Given vr = 22.5 cm/sec = $\dot{r}$

r = 10 cm

ar = 15 m/sec^2

Angular acceleration = a = $\ddot{\theta}$ = - 1 rad/sec^2

Angular velocity = $\dot{\theta}$ = 3 rad/sec

$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$ = 10(-1) + 2(22.5)(3) = 125 cm/s^2

$a = \sqrt{{a_r}^2 + {a_\theta}^2} = \sqrt{(15)^2 + (125)^2}$ = 125.89 cm/s^2

10.10 SPACE CURVILINEAR (3D CURVILINEAR MOTION)

Cylindrical coordinates (r-θ-z)

The coordinate systems which have been dealt so far are used to describe the motion of a particle in the *x–y* plane (i.e., two-dimensional). Cylindrical coordinate system is an extension of polar coordinates, where the motion of a particle is described in x-y-z plane (i.e, three-dimensional motion). In cylindrical coordinate system, the three coordinates are R and θ (which are polar coordinates) and axial coordinate z. This axial coordinate is same as the z coordinate in rectangular coordinate system. In cylindrical coordinates, the position vector **r** is the sum of the position vector expression in polar coordinates and the *z* component which is given as follows

$$\mathbf{r} = re_r + ze_z$$

$$v = \frac{dr}{dt} = v_r e_r + v_\theta e_\theta + v_z e_z$$

$$a = \frac{dv}{dt} = a_r e_r + a_\theta e_\theta + a_z e_z$$

Where

$v_r = \dot{r}$

$v_\theta = r\dot{\theta}$

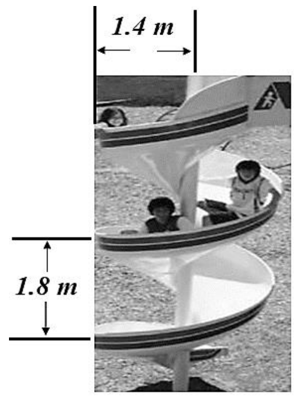
$v_z = \dot{z}$

$a_r = \ddot{r} - r(\dot{\theta})^2$

$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$

$a_z = \ddot{z}$

Example 10.34 A boy slides down the spiral slider with a constant speed of v = 0.8 m/s as shown in the figure. Determine the magnitude of its acceleration. The slider descends a vertical distance of 1.8 m for every full revolution. The mean radius of the ramp is r = 1.4 m.



Solution The inclination angle of the ramp $\varphi = \tan^{-1}\frac{L}{2\pi r}$

$= \tan^{-1}\frac{1.8}{2\pi(1.4)} = 11.56°$

i. Velocity

Form the figure

$V_\theta = 0.8 \cos 11.56 = 0.78$ m/s

$V_z = 0.8 \sin 11.56 = 0.16$ m/s

$v_\theta = r\dot{\theta}$

$0.78 = 1.4\,\dot{\theta}$

$\dot{\theta} = 0.56$ rad/s

ii. Acceleration

As r = 1.4 m is constant, $\dot{r} = \ddot{r} = 0$

Moreover, $\dot{\theta}$ is constant, then $\ddot{\theta} = 0$

Now,

$a_r = \ddot{r} - r(\dot{\theta})^2 = 0 - 1.4\,(0.56)^2 = -0.44$ m/s^2

$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 1.4\,(0) + 2(0)(0.56) = 0$

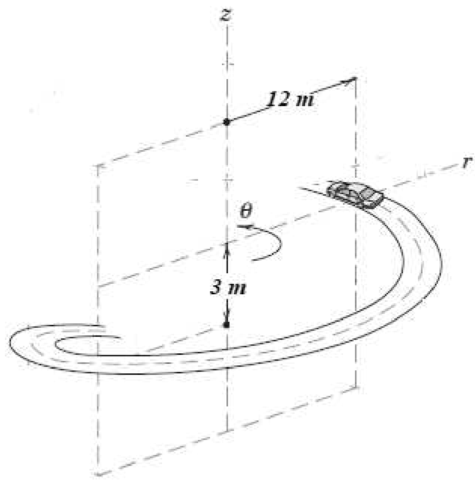
As v_z is constant, a_z is zero.

So the magnitude of boys acceleration is

$a = \sqrt{{a_r}^2 + {a_\theta}^2 + {a_z}^2}$

$= \sqrt{-0.44^2 + 0^2 + 0^2} = 0.44$ m/s^2

Example 10.35 The roller coaster is travelling along a track which is in the form of a cylindrical helix of 12 m radius rising 3 m for each half turn as shown in the figure. At the position shown the roller coaster has a speed of 7.2 m/s, which is decreasing at the rate of 1.3 m/s^2. Determine the r, θ and z components of the acceleration of the car.



Solution The inclination angle of the ramp $\varphi = \tan^{-1} \frac{L}{2\pi r}$

$= \tan^{-1} \frac{6}{2\pi(12)} = 4.55°$

Given: r = 12 m and v = 7.2 m/s

$\dot{r} = \ddot{r} = 0$

$v_z = v \sin \varphi = 7.2 \sin 4.55 = 0.57$ m/s

$v_\theta = v \cos \varphi = 7.2 \cos 4.55 = 7.18$ m/s

We know $v_\theta = r\dot{\theta}$

$\dot{\theta} = \frac{v_\theta}{r} = \frac{7.18}{12} = 0.598$ rad/s

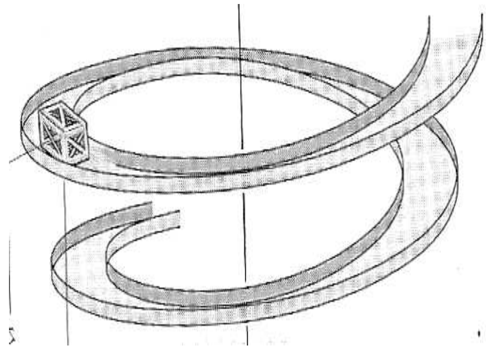
Also given $a_{\theta z} = 1.3$ m/s^2

$a_z = -1.3 \sin 4.55 = -0.103$ m/s^2

$a_\theta = -1.3 \cos 4.55 = -1.295$ m/s^2

$a_r = \ddot{r} - r(\dot{\theta})^2 = 0 - 12\,(0.598)^2 = -4.29$ m/s^2

Example 10.36 A trunk slides down the spiral staircase which is defined by r = 0.3 m, $\theta = 0.6t^3$ rad, and $z = (3 - 0.3t^2)$ m, where *t* is in seconds. Determine the magnitudes of the velocity and acceleration of the box at the instant $\theta = 2\pi$ rad.



Solution Given r = 0.3 m

$\dot{r} = \ddot{r} = 0$

$\theta = 0.6t^3$ rad

$\dot{\theta} = 1.8\ t^2$ rad/s

$\ddot{\theta} = 3.6$ t rad/s^2

$z = (3 - 0.3t^2)$ m

$\dot{z} = -\ 0.6t$ m/s

$\ddot{z} = -0.6$ m/s^2

When $\theta = 2\pi$ rad

$\theta = 0.6t^3$

$2\pi = 0.6t^3$

t = 2.19 sec

So, at t = 2.19 sec

$\dot{\theta} = 1.8\ (2.19)^2 = 8.63$ rad/s

$\ddot{\theta} = 3.6\ (2.19) = 7.88$ rad/s^2

$\dot{z} = -\ 0.6\ (2.19) = -1.314$ m/s

$\ddot{z} = -0.6$ m/s^2

Velocity

$v_r = \dot{r} = 0$

$v_\theta = r\dot{\theta} = 0.3(8.63) = 2.589$ m/s

$v_z = \dot{z}$ = -1.314 m/s

$v = \sqrt{{v_r}^2 + {v_\theta}^2 + {v_z}^2} = \sqrt{0^2 + 2.589^2 + (-1.314)^2}$ = 2.9 m/s

Acceleration

$a_r = \ddot{r} - r(\dot{\theta})^2$ = 0 – 0.3(8.63)^2 = - 22.34 m/s^2

$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$ = 0.3 (7.88) + 2 (0) (8.63) = 2.36 m/s^2

$a_z = \ddot{z}$ = -0.6 m/s^2

$a = \sqrt{{a_r}^2 + {a_\theta}^2 + {a_z}^2}$

$= \sqrt{(-22.34)^2 + 2.36^2 + (-0.6)^2}$ = 22.47 m/s^2

SUMMARY

- The translational motion of a particle along a curved path is known as curvilinear motion.
- In curvilinear motion, velocity of particle is always tangent to the curved path at every instant, while the acceleration of particle is not so.
- The curvilinear motion of a particle can be described by using different co-ordinate systems.
- In rectangular coordinate system, velocity and acceleration is resolved into two mutually perpendicular components parallel to x and y-axis.
- The path traced by the particle when projected upward (not vertical) is called projectile
- In normal and tangential coordinate system, the acceleration is resolved into set of rectangular components coinciding with the normal and tangent to the path.
- Normal-tangential (n-t) coordinates also called as path coordinates
- In polar coordinates, the motion of a particle is specified by radial distance R and an angular measurement θ
- Cylindrical coordinate system is an extension of polar coordinates, where the motion of a particle is described in x-y-z plane (i.e, three-dimensional motion).
- In cylindrical coordinate system, the three coordinates are *R* and *θ* (which are polar coordinates) and axial coordinate z.

EXERCISES

I. SELF-ASSESSMENT QUESTIONS

1. Define curvilinear motion of a particle
2. Explain coordinate systems used for analyzing curvilinear motion.
3. Define normal and tangential acceleration of a particle in n-t coordinates system.
4. Compare rectangular and path coordinates in curvilinear motion.

5. Define the terms, *trajectory*, *range* and *time of flight* used in projectile motion.
6. How the motion of a particle is defined in polar coordinates?
7. Explain the coordinate system used for solving 3D curvilinear motion.
8. List out the assumptions made in projectile motion.

II. MULTIPLE CHOICE QUESTIONS

1. The normal component of acceleration is zero when motion is
   a. rectilinear
   b. curvilinear
   c. relative
   d. absolute
2. In curvilinear motion, direction of acceleration will _________.
   a. remain constant
   b. be always tangential
   c. be always normal
   d. change at every instant
3. In curvilinear motion velocity of a particle is always ________ to the curved path at every instant.
   a. normal
   b. tangential
   c. perpendicular
   d. zero
4. The motion of bodies along a curved path is called
   a. Absolute motion
   b. Relative motion
   c. Rectilinear motion
   d. Curvilinear motion
5. The time of flight of projectile motion having a velocity u and angle α is given by
   a. $\frac{u\sin\alpha}{g}$ b. $\frac{u^2 sin^2\alpha}{2g}$ c. $\frac{2u\sin\alpha}{g}$ d. $\frac{u^2\sin 2\alpha}{g}$
6. In projectile motion, time taken to reach maximum height is given by
   a. $\frac{u\sin\alpha}{g}$ b. $\frac{u^2 sin^2\alpha}{2g}$ c. $\frac{2u\sin\alpha}{g}$ d. $\frac{u^2\sin 2\alpha}{g}$
7. In a projectile motion, the maximum range is obtained when the angle of inclination is
   a. 30° b. 45° c. 60° d. 90°
8. The equation of projectile motion of a particle is a
   a. parabola b. spiral c. hyperbola d. ellipse

Answer

1. a
2. d
3. b
4. d
5. c

6. a
7. b
8. a

CHAPTER – 11

RELATIVE AND CONSTRAINED MOTION

LEARNING OBJECTIVES

After studying this chapter, you should be able to

- *Describe the relative motion of a particle*
- *Solve the problems of relative motion along a straight path and inclined path*
- *Determine the velocity and acceleration of a particle under constrained motion*

11.1 RELATIVE MOTION

Relative motion describes the motion of a particle with respect to a moving reference frame. In many cases, it is essential to analyze the motion of two or more particles relative to each other. Let us consider two trains A & B moving in parallel lines in the same direction with equal speed. The passenger sitting in train A doesn't feel the motion of train B (as both the trains are in same speed). If train B moves little faster than train A, then the passenger in train A feels a slight motion. If two trains move in opposite direction, the passenger in train A feels that speed is very high. In such situations, it is required to relate their accelerations, velocities and positions.

11.2 RELATIVE MOTION ALONG A STRAIGHT PATH

Let us consider two particles A and B which are positioned on a straight path by s_A and s_B as shown in the figure 11.1. These positions are given w. r. t. a point O. If we want to describe the relative position of A w r t B, then we need consider the origin at B that gives $s_{A/B}$

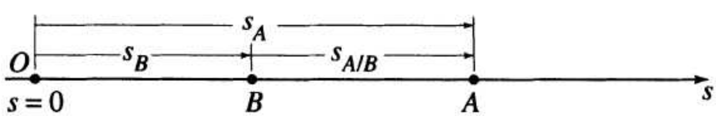


Fig. 11.1

Note: The absolute positions of A and B are given by observing from point O whereas the relative position $s_{A/B}$ is given by observing from point B.

From the figure

$$OA = OB + BA$$

$$s_A = s_B + s_{A/B}$$

Differentiating the above equation, we get

$$\frac{ds_A}{dt} = \frac{ds_B}{dt} + \frac{ds_{A/B}}{dt}$$

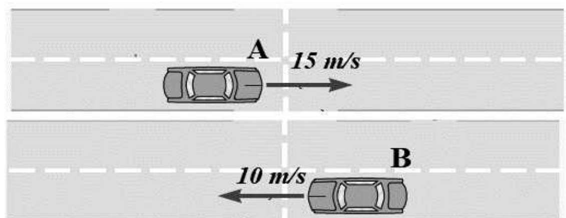
$$v_A = v_B + v_{A/B}$$

Again differentiating the above equation, we get

$$\frac{dv_A}{dt} = \frac{dv_B}{dt} + \frac{dv_{A/B}}{dt}$$

$$a_A = a_B + a_{A/B}$$

Example 11.1 On a four-way straight road, two cars are travelling in a direction opposite to each other as shown in the figure. Car A is travelling at a speed 15 m/s which is decreasing at a rate of 1.2 m/s^2. At the same instant, car B is travelling at a speed 10 m/s which is increasing at a rate of 0.8 m/s^2. Determine the relative velocity and relative acceleration of car A w r t to car B.



Solution Velocity of car A relative to car B

We know, $v_A = v_B + v_{A/B}$

$$v_{A/B} = v_A - v_B$$

$$= 15 – (-10) = 25 \text{ m/s}$$

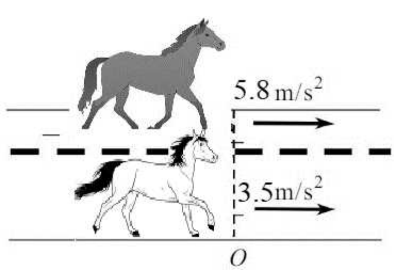
Acceleration of car A relative to car B

We know, $a_A = a_B + a_{A/B}$

$$a_{A/B} = a_A – a_B$$

$$= - 1.2 – (-0.8) = -0.4 \text{ m/s}^2$$

Example 11.2 Two horses, white and brown are competing in a race. Both the horses start from rest at the same instant from point O and move along a straight path as shown in the figure. White horse moves with an acceleration of 3.5 m/s^2 and brown horse move with an acceleration of 5.8 m/s^2. Find the relative position, velocity and acceleration of brown horse w. r. t. white horse 4 seconds from the start.



Solution ***White horse (A)***

Given: $u = 0$, $a = 3.5\ m/s^2$ $t = 4$

$v = u + at$

$= 0 + 3.5 \times 4 = 14\ m/s$

$s = ut + ½\ at^2$

$= 0 + ½ \times 3.5 \times 4^2 = 28\ m$

Brown horse (B)

Given: $u = 0$, $a = 5.8\ m/s^2$ $t = 4$

$v = u + at$

$= 0 + 5.8 \times 4 = 23.2\ m/s$

$s = ut + ½\ at^2$

$= 0 + ½ \times 5.8 \times 4^2 = 46.4\ m$

Position of brown horse relative to white horse

$s_{brown/while} = s_{brown} - s_{white}$

$= 46.4 - 28 = 18.6\ m$

Velocity of brown horse relative to white horse

$v_{brown/while} = v_{brown} - v_{white}$

$= 23.2 - 14 = 9.2\ m/s$

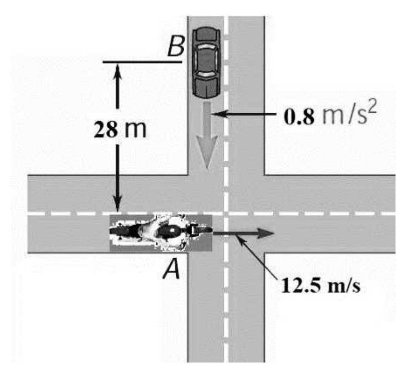
Acceleration of brown horse relative to white horse

$a_{brown/while} = a_{brown} - a_{white}$

$= 5.8 - 3.5 = 2.3\ m/s^2$

11.3 RELATIVE MOTION ALONG RIGHT-ANGLED PATH

Example 11.3 A bike is traveling east at the constant speed of 12.5 m/s. As bike crosses the intersection point as shown in the figure, a car starts from rest 28 m north of the intersection and moves towards south with a constant acceleration of 0.8 m/s^2. Determine the position, velocity, and acceleration of car relative to bike 3 s after bike crosses the intersection.



Solution ***Motion of bike***

Given: $v_A = 12.5\ m/s$

As A is moving with constant speed, its acceleration is zero

$\therefore a_A = 0$,

Also, for constant speed we know

Displacement = Velocity x time

i.e., $s_A = 12.5\ t$

At t = 3 s

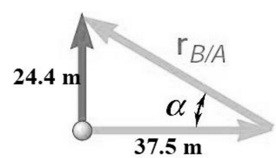
$s_A = 12.5 \times 3 = 37.5$ m

Motion of car

$a_B = -\ 0.8\ \text{m/s}^2$

$v_B = u_B + at = 0 + 0.8t$

$s_B = ut + ½\ at^2 = 0 + ½\ o.8\ t^2$



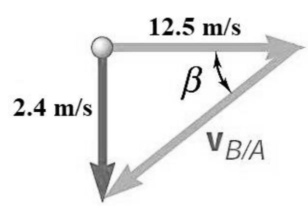
For t = 3 sec

$v_B = -\ 0.8(3) = -\ 2.4$ m/s and

$s_B = 28 - ½\ o.8\ (3)^2 = 24.4$ m

Position of car relative to bike

Magnitude of relative position is given by



$s_{B/A} = \sqrt{{s_A}^2 + {s_B}^2}$

$s_{B/A} = \sqrt{24.4^2 + 37.5^2} = 44.74$ m

$\alpha = tan^{-1}\frac{24.4}{37.5} = 33.05°$

Velocity, of car relative to bike

Magnitude of relative velocity is given by

$v_{B/A} = \sqrt{{v_A}^2 + {v_B}^2}$

$v_{B/A} = \sqrt{12.5^2 + 2.4^2} = 12.72$ m/s

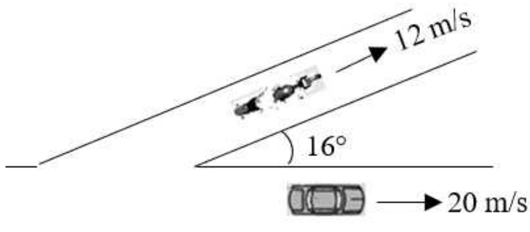
$\beta = tan^{-1}\frac{2.4}{12.5} = 10.86°$

Acceleration of car relative to bike

$a_{B/A} = -\ 0.8\ \text{m/s}^2$

11.4 RELATIVE MOTION ALONG AN INCLINED PATH

Example 11.4 A car and a bike are travelling on two difference inclinations of road as shown in the figure. The velocity of car and bike at the instant shown are 20 m/s and 12 m/s respectively. Determine the velocity of bike w. r. t. car.



Solution Given v_A = 12 m/s, v_B = 20 m/s

The magnitude of relative velocity at an inclined motion can be determined by using law of cosines

$v_{A/B} = \sqrt{v_A{}^2 + v_B{}^2 - 2v_A v_B \cos\ \theta}$

$v_{A/B} = \sqrt{12^2 + 20^2 - 2(12)(20)\cos\ 16}$

$v_{A/B}$ = 9.08 m/s

The angle α can be determined by using law of sines

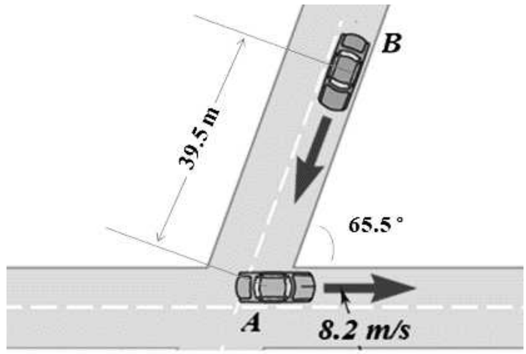
$$\frac{v_A}{sin\alpha} = \frac{v_{A/B}}{\sin\theta}$$

$$\frac{12}{sin\alpha} = \frac{9.08}{\sin 16}$$

α = 21.36°

Example 11.5 Cars A and B are at a distance 39.5 m. Car A moves with constant velocity of 8.2 m/s whereas car B starts from rest and move with an acceleration of 0.9 m/s^2. Determine

i. Position of car B relative to car A
ii. Velocity of car B relative to car A
iii. Acceleration of car B relative to car A 6 seconds after car A passes the intersection



Solution ***Car A*** Given u_A =8.2 m/s

As it is moving with constant velocity, ∴ $a_A = 0$

At t = 6 sec

$s_A = u \times t = 8.2 \times 6 = 49.2$ m

Car B

Given: $u_B = 0$, $a_B = 0.9$ m/s^2

Distance travelled in 6 sec is

$s = ut + ½ at^2$

$= 0 + ½ \times 0.9 \times 6^2 = 16.2$ m

Initial position of car B from O is 39.5 m

After 6 sec, position of car B relative to O is

$s_B = 39.5 – 16.2 = 23.3$ m

$v_B = u_B + a_Bt = 0 + 0.9 \times 6 = 5.4$ m/s

i. ***Position of car B relative to car A***

We know, $s_{B/A} = \sqrt{{s_A}^2 + {s_B}^2 - 2s_As_B \cos\ \theta}$

$s_{B/A} = \sqrt{49.2^2 + 23.3^2 - 2(49.2)(23.3) \cos\ 65.5}$

$s_{B/A} = 44.\ 8$ m

From law of sines, we know

$$\frac{s_B}{sin\alpha} = \frac{s_{B/A}}{\sin\theta}$$

$$\frac{23.3}{sin\alpha} = \frac{44.8}{\sin 65.5}$$

$\alpha = 28.24°$

ii. Velocity of car B relative to car A

We know, $v_{B/A} = \sqrt{v_A{}^2 + v_B{}^2 - 2v_A v_B \cos\ \theta}$

$v_{B/A} = \sqrt{8.2^2 + 5.4^2 - 2(8.2)(5.4)\cos\,(180 - 65.5)}$

$v_{B/A} = 11.5$ m/s

$$\frac{v_B}{sin\alpha} = \frac{v_{B/A}}{\sin\theta}$$

$$\frac{5.4}{sin\alpha} = \frac{11.5}{\sin(180 - 65.5)}$$

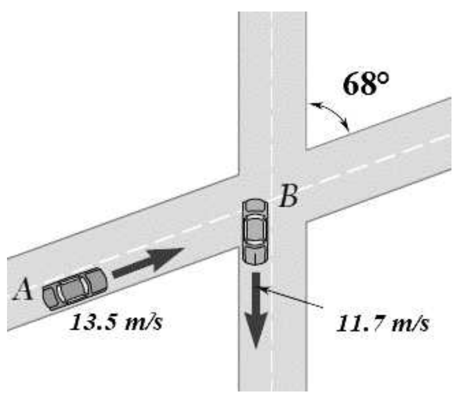
$\alpha = 25.29°$

iii. Acceleration of car B relative to car A

$a_{B/A} = a_B - a_A = 0.9 - 0 = 0.9$ m/s^2

Example 11.6 A truck is traveling towards south at a constant speed of 11.7 m/s. Four seconds after truck passes through the intersection shown in the figure a car passes through the same intersection at a constant velocity of 13.5 m/s. Determine

a) Velocity of truck relative to car
b) Change in position of truck relative to car during a 5 s interval
c) Distance between two vehicles 3 s after car has passed through the intersection



Solution

a. Velocity of truck relative to car

$v_A = 13.5$ m/s

$v_B = 11.7$ m/s

We know, $v_{B/A} = \sqrt{v_A{}^2 + v_B{}^2 - 2v_A v_B \cos\ \theta}$

$v_{B/A} = \sqrt{13.5^2 + 11.7^2 - 2(13.5)(11.7)\cos\ 112}$

$v_{B/A} = 20.9$ m/s

$$\frac{v_B}{sin\alpha} = \frac{v_{B/A}}{\sin\theta}$$

$$\frac{11.7}{sin\alpha} = \frac{20.9}{\sin 112}$$

$\alpha = 31.26°$

b. *Change in position of truck relative to car during a 5 s interval*

Displacement = $v_{B/A}$ x t = 20.9 x 5 = 104.5 m

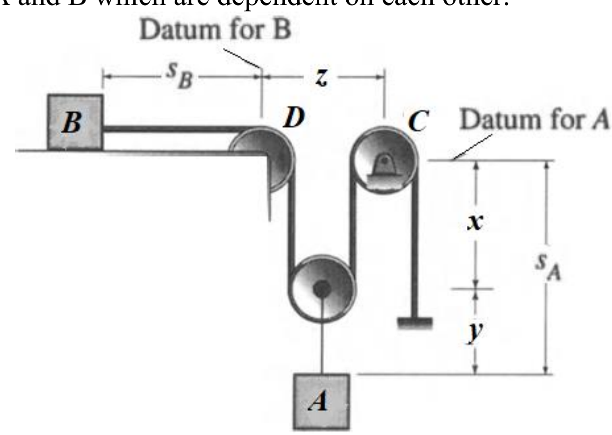
c. *Distance between two vehicles 3 s after car has passed through the intersection*

v_A = 13.5 m/s

$s_A = v_A$ x t = 13.5 x 5 = 67.5 m

11.5 CONSTRAINED MOTION

In some cases the motion of bodies are dependent due to the constraints imposed by interconnecting members. When the bodies are constrained to move, then their velocities, accelerations and displacements are not independent of each other. Figure shows that the motions of blocks A and B which are dependent on each other.



The length of the cable passing over the pulleys is assumed to be constant. The block A and block B are positioned with respect to the fixed datum passing through the centers of pulley C and D respectively. At any time t, s_A and s_B are the positions of block A and block B while x, y, z and r are the constants. The total length of the cable can be written as

$$L = 2(s_A - y) + x + 2\pi r + \frac{\pi r}{2} + s_B$$

We know x, y, r and L are constant, the above equation can be reduced to

$$2s_A + s_B = \text{constant}$$

On differentiating the above equation, we get

$$2\frac{ds_A}{dt} + \frac{ds_B}{dt} = 0$$

$$2v_A + v_B = 0$$

Again differentiating, we get

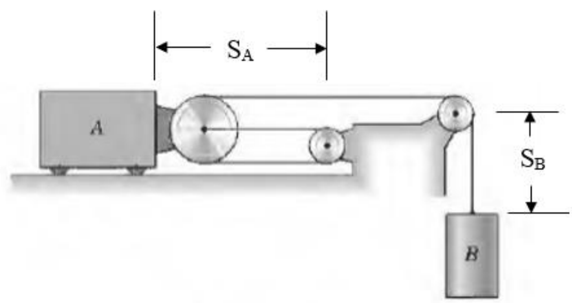
$$2\frac{dv_A}{dt} + \frac{dv_B}{dt} = 0$$

$$2a_A + a_B = 0$$

Thus, the velocities and accelerations of blocks A and B are related to each other. It is also clear that the position changes of blocks A and B are related in the same manner. i.e.,

$$2\Delta s_A + \Delta s_B = 0$$

Example 11.7 If the block A has a velocity of 1.6 m/s to the right, determine the velocity of block B



Solution Length of the cable L = $2s_A + s_B$ + constant

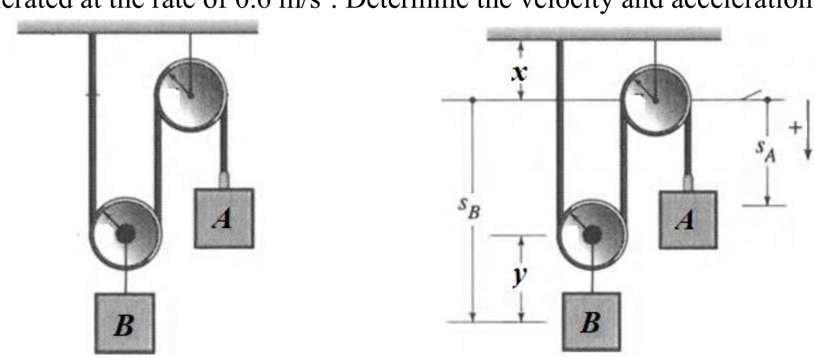
On differentiating, we get

$$3v_A + v_B = 0$$

$$v_B = -3v_A = -3(-1.6)$$

$$= 4.6 \text{ m/s (down)}$$

Example 11.8 Block B shown in the figure is moving downward at the rate of 3.5 m/s and is being decelerated at the rate of 0.6 m/s^2. Determine the velocity and acceleration of block A



Solution Length of the cable L = sA + 2(sB – y) + x + 2πr
We know x, y, r, and L are constants, then above equations can be written as

$$s_A + 2s_B = 0$$

On differentiating, we get

$$v_A + 2v_B = 0$$

$$v_A = - 2v_B$$

Given $v_B = 3.5$ m/s

$$\therefore v_A = - 2\ (3.5) = - 7 \text{ m/s}$$

-ve sign indicates that the velocity of A is directed upward.

Again differentiating the equation, we get

$$a_A + 2a_B = 0$$

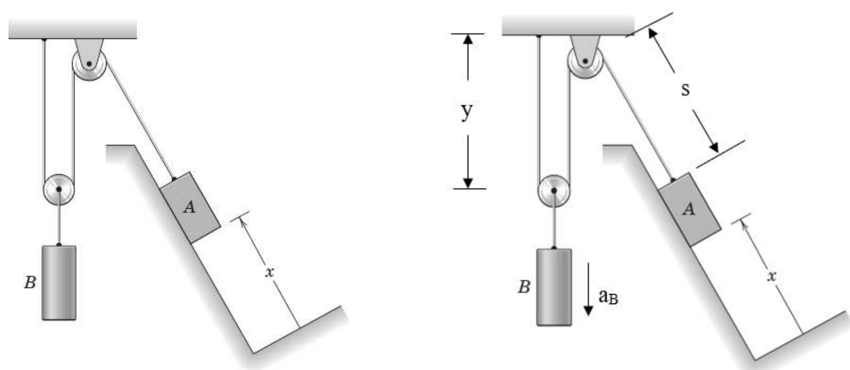
$$a_A = - 2a_B$$

Given $a_B = - 0.6$ m/s^2

$$a_A = - 2(-0.6) = 1.2 \text{ m/s}^2$$

Positive sign indicates that the block A is being accelerated downward

Example 11.9 If the velocity of block A up the incline is increasing at the rate of 0.036 m/s each second, determine the acceleration of block B.



Solution Length of the cable $L = 2s_B + s_A + 2\pi r$
We know r and L are constants, then above equations can be written as

$$s_A + 2s_B = 0$$

On differentiating, we get

$$v_A + 2v_B = 0$$

Again differentiating the equation, we get

$$a_A + 2a_B = 0$$

$$a_B = -\frac{a_A}{2}$$

Given $a_A = 0.036$ m/s^2

$$\therefore a_B = -\frac{0.036}{2} = -0.018 \text{ m/s}^2$$

SUMMARY

- Relative motion describes the motion of a particle with respect to a moving reference frame.
- In many cases, it is essential to analyze the motion of two or more particles relative to each other.
- In some cases, the motion of bodies are dependent due to the constraints imposed by interconnecting members.
- When the bodies are constrained to move, then their velocities, accelerations and displacements are not independent of each other.

EXERCISES

I. Self-Assessment Questions

1. Describe the relative motion of a particle.
2. Differentiate between absolute motion and relative motion of a particle.
3. What do you understand my constrained motion?
4. Define displacement, velocity and acceleration of a particle under constrained motion.

CHAPTER – 12

D'ALEMBERT'S PRINCIPLE

LEARNING OBJECTIVES

After studying this chapter, you should be able to

- *State the Newton's second law of motion*
- *Explain the concept of D'Alembert's principle*
- *Apply D'Alembert's principle to solve kinetic problems*

12.1 INTRODUCTION

So far, we studied the kinematics of particle, where forces and masses of the body are not taken into consideration. From this chapter we begin the second part of dynamics (i.e., kinetics) where forces and masses are taken into consideration. This chapter deals with solving kinetic problems in plane motion using D' Alembert's principle. This principle transforms dynamic problem into a static equilibrium problem by introducing an inertia force.

12.2 NEWTON'S SECOND LAW OF MOTION

According to Newton's second law "the rate of change of momentum is directly proportional to the impressed force and takes place in the direction in which the force acts." We know that the momentum is the product of mass and velocity of the body whose direction is same as the velocity, then according to the definition,

$$\mathrm{F}\ \alpha\ \frac{mv - mu}{t}$$

$$\mathrm{F}\ \alpha\ m\frac{v - u}{t}$$

$$\mathrm{F}\ \alpha\ ma$$

It is evident that the proportionality constant is one, we can write the above equation as

$$\mathrm{F} = \mathrm{ma}$$

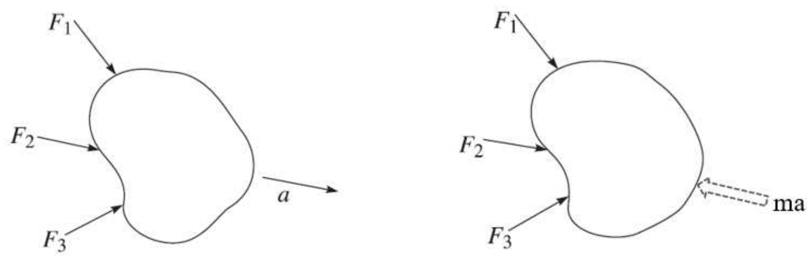
12.3 D' ALEMBERT'S PRINCIPLE

French mathematician D'Alembert in 1743 developed a method to convert a dynamic problem into an equivalent static problem. This is achieved by applying a reverse effective force to a moving body and can be analyzed using equations of static equilibrium. D' Alembert viewed

Newton second law from different perspective. The equation F=ma may be written as F – ma = 0. The term '– ma' is called as inertia force or reverse effective force. If a body is acted upon by a system of forces instead of a single force F, then this F can be reduced to resultant force R. Mathematically,

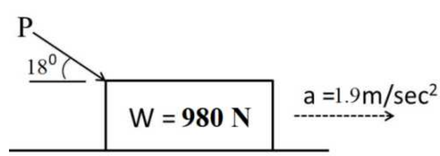
$$R - ma = 0$$

Consider a body subjected to system of forces which causes the body to move with an acceleration a in the direction of the resultant. Now, applying a force equal to ma in the opposite direction of acceleration as shown in the figure makes the whole body into equilibrium.

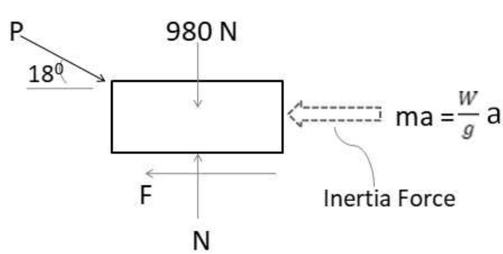


Figure

Example 12.1 A block having a weight 980 N is resting on a plane surface as shown in the figure. What should be the value of force P, to move the block with acceleration of 1.9 m/s^2 to the right side? Assume coefficient of friction between the block and plane is 0.24



Solution



Considering the F. B. D of the block

$\sum F_y = 0$

$N - 980 - P \sin 18° = 0$

$N = 980 + P \sin 18°$

We know, $F = \mu N = 0.2\ (980 + P \sin 18°)$

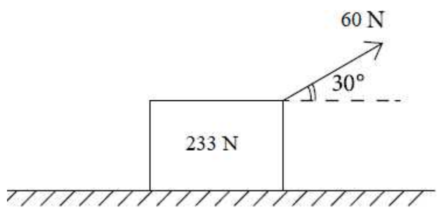
$\sum F_x = 0$

$P \cos 18° - F - ma = 0$

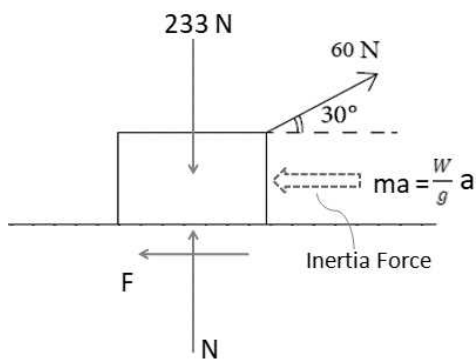
$P \cos 18° - 0.2\ (980 + P \sin 18°) - \left(\frac{980}{9.81}\right) 1.9 = 0$

$P = 380.9$ N

Example 12.2 A block of weight 100 N is lying on a rough horizontal surface, whose coefficient of friction is 0.22. It is being pulled by a force of 60 N as shown in the figure. Determine the acceleration of the block. Also find the distance travelled by the block after 4 seconds.



Solution



Considering the F. B. D of the block

$\sum F_y = 0$

$N - 100 + 60 \sin 30° = 0$

$N = 70$ N

We know, $F = \mu N = 0.22\ (70) = 15.4$

$\sum F_x = 0$

$60 \cos 30° - F - ma = 0$

$60 \cos 30° - 15.4 - \left(\frac{100}{9.81}\right) a = 0$

$a = 3.58\ m/s^2$

Distance travelled by the block in 4 sec is

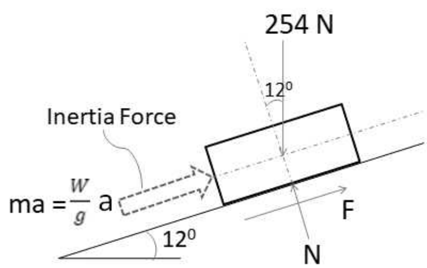
$s = ut + ½ at^2$

$= 0 + ½ (3.58) (4)^2$

$= 28.64$ m

Example 12.3 A block of weight 254 N is placed on a rough inclined surface having inclination 12° to the horizontal. It is pushed down the plane with an initial velocity of 18.4 m/s. What is the distance travelled and time taken by the block when it comes to rest? Assume the coefficient of friction as 0.33

Solution



Free body diagram of the block

Considering F. B. D of the block

$\sum F_y = 0$

$N - 254 \sin 78° = 0$

$N = 248.45$ N

We know, $F = \mu N = 0.33 (248.45) = 82$ N

$\sum F_x = 0$

$F + ma - 254 \cos 78° = 0$

$82 + \frac{254}{9.81} a = 52.8$

$a = - 1.12$ m/s^2 (Retardation)

Given u = 18.4 m/s, v = 0 and

We know $a = - 1.12$ m/s^2

Using the equation $v = u + at$

$0 = 18.4 - 1.12t$

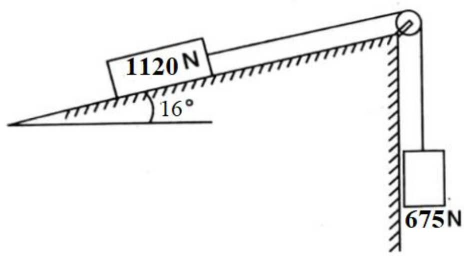
$t = 16.42$ sec.

$s = ut + ½ at^2$

s = 18.4 x 16.42 + ½ (-1.12) x $(16.42)^2$

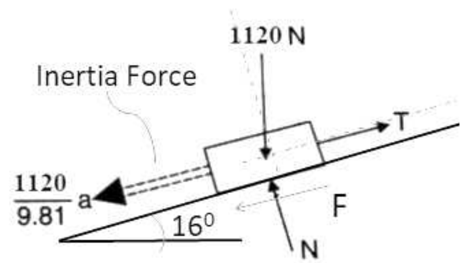
s = 151.1 m

Example 12.4 Block A of weight 1120 is placed on a rough inclined plane which is pulled up by means of rope passing over a pulley as shown in the figure. The other end of the rope is fixed to another block of weight 675 N. Determine (i) Tension in the rope (ii) acceleration of the block A (iii) Distance moved by block A in 2 sec. Assume coefficient of friction between the block A and the plane is 0.22



Solution

Considering the F. B. D of the block A



$\sum F_y = 0$

N – 1120 sin 74° = 0

N = 1076.6

We know, F = μN = 0.22 (1076.6) = 236.85

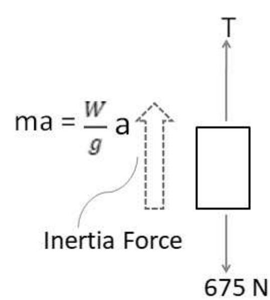
$\sum F_x = 0$

T – F – 1120 cos 74° – ma = 0

T – 236.85 – 1120 cos 74° – $\frac{1120}{9.81}$a = 0

114.16 a – T = – 71.86 --------------- (i)

Considering the F. B. D of the block B



$\sum F_y = 0$

T + ma – 675 = 0

T + $\frac{675}{9.81}$a – 675 = 0

T + 68.8 a = 675 ------------------- (ii)

Adding equation (i) and (ii), we get

a (114.16 + 68.8) = 675 – 71.86

a = 3.29 m/s^2

Substituting 'a' value in equation (ii)

$T = 675 - 68.8\ (3.29) = 448.64$ N

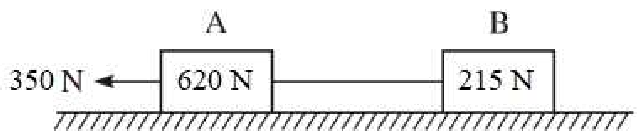
Distance moved by block A in 2 sec is

$s = ut + \frac{1}{2} at^2$

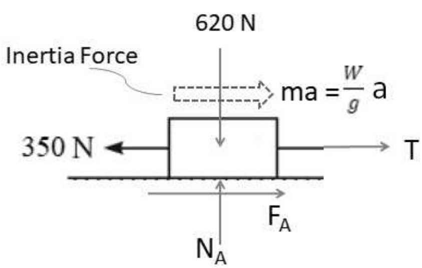
$s = 0 + \frac{1}{2} (3.29)(2)^2$

$s = 6.58$ m

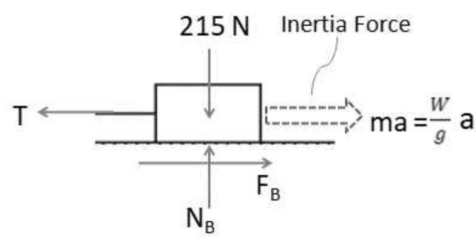
Example 12.5 Block A and block B of weight 620 N and 215 N respectively are connected by a rope and move along a horizontal plane under the action of 350 N force applied to the block A as shown in the figure. Apply D' Alembert's principle to find the acceleration of two bodies and tension in the rope. Assume the coefficient of friction between the blocks and the surface as 0.28



Solution



Free body diagram of the block A



Free body diagram of the block B

Considering the F. B. D of block B

$\sum F_y = 0$

$N_B - 215 = 0$

$N_B = 215$ N

We know, $F_B = \mu N = 0.28\ (215) = 60.2$ N

$\sum F_x = 0$

$F_B + ma - T = 0$

$60.2 + \frac{215}{9.81} a - T = 0$

$T - \frac{215}{9.81} a = 60.2$ ---------- (i)

Considering the F. B. D of block A

$\sum F_y = 0$

$N_A - 620 = 0$

$N_A = 620$ N

We know, $F_A = \mu N = 0.28\,(620) = 173.6$ N

$\sum F_x = 0$

$T + F_A + ma - 350 = 0$

$T + \frac{620}{9.81}\,a = 176.4$ ----------- (ii)

Subtracting equation (i) from (ii)

$a\left(\frac{620}{9.81} + \frac{215}{9.81}\right) = 176.4 - 60.2$

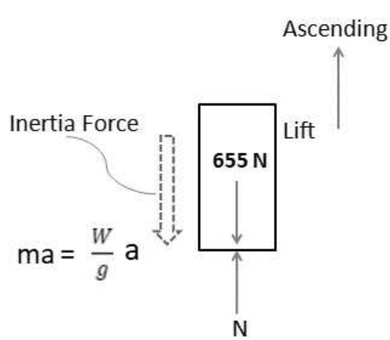
$a = 1.36\ m/s^2$

Substituting 'a' value in equation (ii), we get

$T = 90.45$ N

Example 12.6 A passenger of weight 655 N enters into a lift. The acceleration of the lift during ascending or descending is 1.25 m/s^2. Determine the reaction of the floor of the lift on the passenger during ascending and descending of the lift.

Solution *When the lift is ascending*



Considering F. B. D. of the lift

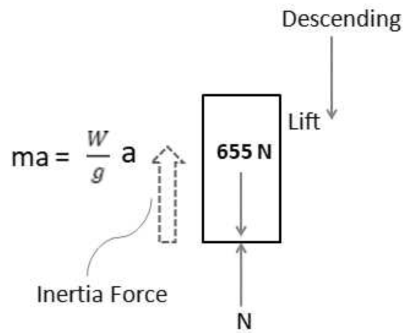
$\sum F_y = 0$

$N - 655 - \frac{655}{9.81} \times 1.25 = 0$

$N = 655 + \frac{655}{9.81} \times 1.25$

$N = 738.4$ N

When the lift is descending



Considering F. B. D. of the lift

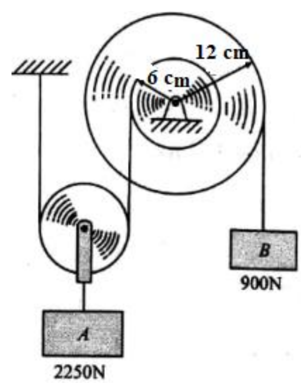
$\sum F_y = 0$

$N - 655 + \frac{655}{9.81} x1.25 = 0$

$N = 655 - \frac{655}{9.81} x1.25$

N = 571.5 N

Example 12.7 Determine the acceleration of body A in fig. assuming the pulleys to be friction less and of negligible weight.



Solution From pulley taking ΣM about point O = 0

$\therefore T_1 \times 12 = T_2 \times 6$

$\therefore T_2 = 2T_1$ ------(1)

To find direction of motion of body A and B. Assume body A in equilibrium

$\therefore 2T_2 = 2250 \therefore T_2 = 1125N$

$\therefore T_1 = 562.5N$

As $900 > T_1$

∴ Body B moves down and hence A moves up.

Using kinematic relation; $2T_2 \times S_A = T_1 \times S_B$

We know $T_2 = 2T_1$

∴ $4S_A = S_B$

Diff. two times w.r.t 't' $\therefore 4a_A = a_B$ ----(2)

For body 'A' Applying inertia force opposite to motion,

$2T_2 - 2250 - \frac{2250}{9.81} \times a_A = 0$

$\therefore 4T_1 - \frac{2250}{9.81} \times a_A = 2250$ ----(3)

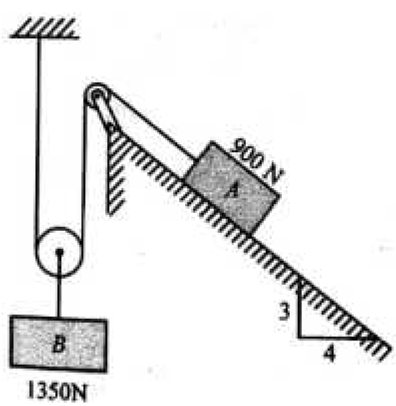
For body 'B' $T_1 + \frac{900}{9.81} \times a_B - 900 = 0$

Using (2) we get $T_1 - \frac{3600}{9.81} \times a_A = 900$ ---- (4)

Solving (3) and (4) $T_1 = 608.108$N

$a_A = 0.7954$ m/s^2 and $a_B = 3.18$m/s^2

Example 12. 8 Find the acceleration of each body in the system shown in fig.



Solution Assume body B is in equilibrium

∴2T=1350

∴T=675N

For block 'A'

Frictional force=μN=0.5×720=360N

(1) If we assume A is moving up

∴ Frictional force acts downward as

T < (540+360) N

∴ A cannot move up.

(2)If we assume A is moving down.

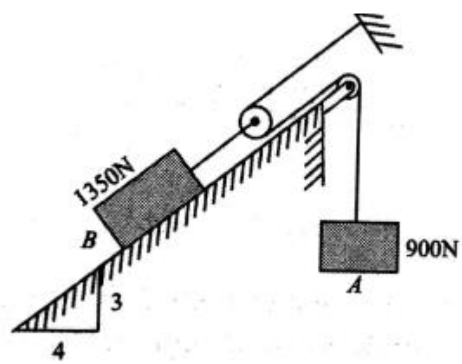
∴ Frictional force acts up.

As 540 < (T+360)

∴ A cannot move down.

So, the body A is in equilibrium. Hence the complete system is in equilibrium.

Example 12.9 Referring to figure, compute the acceleration of body B and the tension in the cord supporting body A.



Solution W_A =900N;

W_B=1350N. Assume A at rest

∴T=900N ∴2T=1800N

For block 'B';

Here value of 2T > (810+216)

∴ Body B moves up the plane and therefore 'A' moves down.

Using kinematic relation: 2T.S_B=T.S_A

$\therefore 2a_B=a_A$ ----(1)

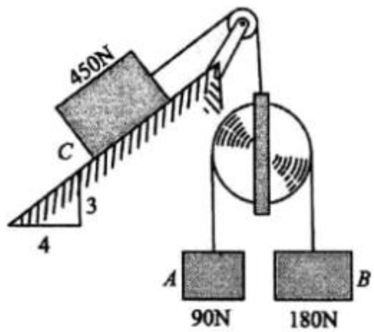
Applying inertia force opposite to the motion

For 'A': $900+T+\frac{1350}{9.81}\times a_A=0$ ----(2)

For 'B': $2T-810-216-\frac{1800}{9.81}\times a_B=0$ ----(3)

Solving (1), (2) and (3) a_B=1.534m/$s^2$and T=618.5N

Example 12.10 Determine the acceleration of each body in fig .Assuming the pulleys to be frictionless and of negligible weight. The inclined plane is smooth**.**



Solution In case of bodies A and B acceleration are same and it can be concluded that B is moving down and A is up.

$T_1 - 90 - \frac{90}{9.81}\ a = 0$

$T_1 - 9.174\ a = 90$ ----(1)

$T_1 + ma - 180 = 0$

$T_1 + 18.439\ a = 180$ ----(2)

Solving (1) and (2)

T_1=120N; a = 3.27 m/s^2.

Free Body Diagram of (C)

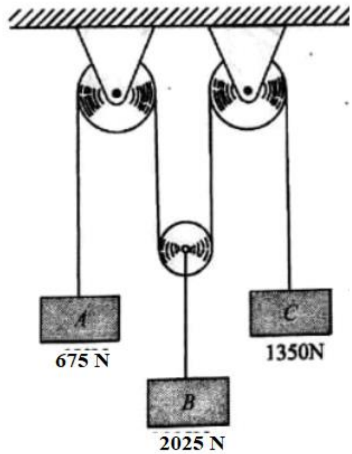
As downward force 270 >2 T_1 i.e. 240N

'C' is moving down the plane

$\frac{450}{9.81} \times a + 2T_1 - 270 = 0$

a = 0.654m/s^2

Example 12.11 Determine the tension in the curd supporting body C in figure. The pulleys are frictionless and of negligible weight.



Solution To determine motion of a system assume body A is at Rest

Therefore, T_1= 675 N and

Hence $2T_1$ =1350 N

As W_B = 2025 N > $2T_1$

i.e.,1350N (B moves down)

Also, W_C = 1350 N>T_1

i.e., 675 N (C moves Down)

Hence, body A moves Up.

From kinematic relation:

$T_1S_A - 2T_1S_B - T_1S_C = 0$

Therefore, $a_A - 2a_B - a_C = 0$

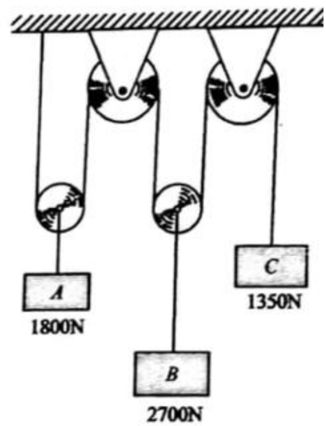
From A: $T_1 - 675 - 68.807\ a_A = 0$

From B: $2T_1 - 2025 + 206.422\ a_B = 0$

From C: $T_1 - 1350 + 137.615\ a_C = 0$

From above equations; T_1=952.954 N.

Example 12.12 The pulleys in figure are frictionless and negligible weight. Determine the Tension in the Cable supporting body C.



Solution We can find the direction of motion as explained in earlier problem. Body B and C moving down and A is moving Up.

From kinematic relation $2TS_A - 2TS_B - TS_C = 0$

Therefore, $2a_A - 2a_B - a_C = 0$ --------- (1)

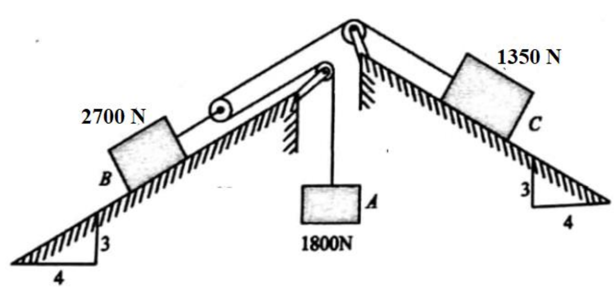
From A: $2T - 183.486\ a_A = 1800$ -------------(2)

From B: $2T_1 + 275.23\ a_B = 2700$-------------(3)

From C: $T + 137.615\ a_C = 1350$--------------(4)

Solving (1), (2), (3) and (4) we get T= 1125 N

Example 12.13 In the system of connected bodies in figure, the coefficient of kinetic friction is 0.2 under bodies B and C. Determine the acceleration of each body and the tension in the cord supporting A.



Solution Assume A is at rest

Therefore, T = 1800 N, and 2T = 3600 N

As 2T i.e. 3600> 2160+576 i.e. 2736, B moves up

Also 2700> 1800+720 i.e. 2520, C moves down

So, A also moves down

From kinetic relation: 2 $TS_B - STS_A - TS_C = 0$

Therefore, $2a_B = 2a_A + a_C = 0$

From A: $T - 183.486a_A = 1800$ -------------(2)

From B: $2T + 366.972a_B = 2736$-------------(3)

From C: $T + 458.716a_C = 1980$--------------(4)

Solving the above equations: T = 1537.06 N

$a_A = 1.27 m/s^2$ $a_B = 1.08 m/s^2$ $a_C = 0.9 m/s^2$

SUMMARY

- Newton's second law states that the rate of change of momentum is directly proportional to the impressed force and takes place in the direction in which the force acts.
- Momentum is the product of mass and velocity of the body whose direction is same as the velocity
- French mathematician D'Alembert in 1743 developed a method to convert a dynamic problem into an equivalent static problem.
- According to D'Alembert, the equation F=ma may be written as F – ma = 0.
- The term '– ma' is called as inertia force or reverse effective force.

EXERCISES

I. Self-Assessment Questions

1. State Newton's second law of motion.
2. Define the term momentum
3. State D'Alembert's principle.

II. Multiple Choice Questions

1. The physical quantity, which is a measure of inertia, is
   (a) mass (b) velocity (c) acceleration (d) force
2. The acceleration of a block sliding down an inclined plane is
   (a) same as acceleration due to gravity
   (b) less than acceleration due to gravity
   (c) greater than acceleration due to gravity

(d) uniformly increasing

3. Inertia force becomes zero, when body moves with
   a. uniform velocity b. uniform acceleration
   c. uniform deceleration d. none of these
4. If a lift accelerates in upward direction then the reaction of its floor on the passengers will be
   a. zero
   b. less than the reaction when lift is stationary
   c. greater than the reaction when lift is stationary
   d. none of these

Answers

1. a
2. b
3. a
4. c

CHAPTER – 13

WORK ENERGY METHOD

LEARNING OBJECTIVES

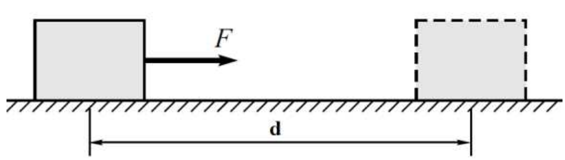
After studying this chapter, you should be able to
- Define work, energy and power
- State and derive work energy equation
- Determine the work done by a force in different cases
- Apply the work energy equation for connected system

13.1 INTRODUCTION

In the last chapter, D'Alembert's principle is used to solve the kinetic problems involving force and acceleration. In this chapter, another approach, called work-energy approach, is used to solve kinetic problems. This method is beneficial over D'Alembert's method when the problem involves velocities, rather than acceleration. Initially the terms work, energy and power are discussed then the work-energy equation is derived. Further, kinetic problems were solved using work-energy equation.

13.2 WORK

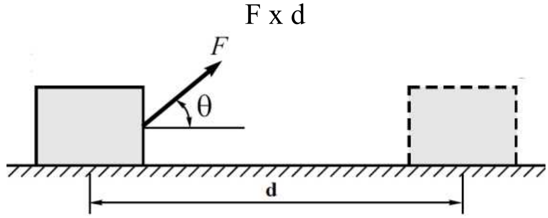
Work is defined as the product of force and distance. Here, the distance should be in the direction of the force.



If F is force acting on the body which moves a distance d in the direction of force as shown in the figure 13.1, then the work done on the body is given by

Work done = force x distance

F x d



If the distance moved and the force acting are not in the same direction as shown in the figure 13.2, then the force can be resolved into a component in the direction of motion. In this case, work done is give by

$$W = F \cos \theta \times d$$

If the force is perpendicular to the distance, then the work done will be zero. The unit of work done is Nm. If 1 N force acting on a body which makes the body to move through 1 m distance, then the work done is 1 Nm. This is also called as one Joule, denoted by J. Hence one Joule (J) is the work done by the force of 1 N when applied on a body whose distance is 1 m.

13.3 ENERGY

Energy is defined as the capacity to do work. Energy is the product of power and time. There are many forms of energies such as heat energy, mechanical energy, electrical energy and chemical energy. In engineering mechanics, we are interested in mechanical energy only. The mechanical energy may be further categorized into potential energy and kinetic energy.

i. **Potential energy**

The energy possessed by the body by virtue of its position is known as potential energy

ii. **Kinetic energy**

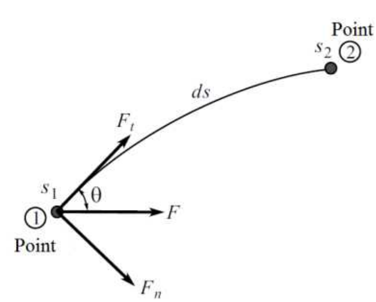
The energy possessed by the body by virtue of its motion is known as kinetic energy

13.4 POWER

Power is defined as the rate of doing work. Hence, it can be obtained by dividing the total work done by time. Unit of power is Watt (W) and is defined as one Joule of work done in one second.

13.5 PRINCIPLE OF WORK AND ENERGY

Consider a particle of mass *m* moving along any arbitrary path as shown in the figure. Let force **F** act on it. **F** can be resolved in two components, tangential to the path *Ft* and normal to the path *Fn*.



Work done on the particle by force **F** is

$$dW = F_t \, ds$$

Where *ds* is measured along the path
Using Newton's second law

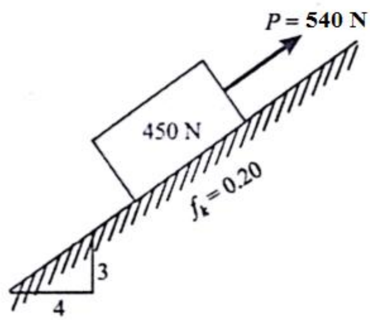
$$dW = m\frac{dv}{dt}ds = m\,dv\frac{ds}{dt} = m\,dv\,v$$

Work done when speed of particle increases from $v1$ to $v2$.

$$W = \int_{s_1}^{s_2} F_t\,ds = \int_{v_1}^{v_2} mv\,dv = \frac{1}{2}mv_2{}^2 - \frac{1}{2}mv_1{}^2$$

The work done by force F is equal to change in kinetic energy of the body. This is principle of work and energy or work energy theorem.

Example 13.1 After the block in figure has moved 3m from rest, the constant force P is removed. Find the velocity of the block when it returns to its initial position.



Solution Velocity of body after moving 3m is

Work done=change in KE.

(540-270-72)×3=$\frac{1}{2}$× $\frac{450}{9.81}$×(V^2-0)

V=5.089 m/s

At this stage force P is removed. The body still moves up till the velocity becomes zero.

(-270-72)×S=$\frac{1}{2}$×$\frac{450}{9.81}$ × (0-5.089^2)

S=1.74 mts

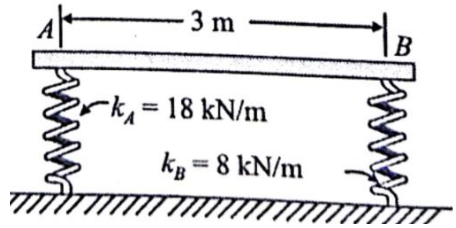
Total distance travelled upward =3+1.74=4.74 m

To come back to original position the body has to move 4.74m. down

(270-72)×4.74= $\frac{1}{2}$× $\frac{450}{9.81}$ (V^2-0)

V=6.4m/s

Example 13.2 The rigid horizontal bar shown in fig. is supported by two springs. A 1350 N weight is then placed upon the bar at A. A sudden blow projects the weight towards B with an initial velocity of 1.8 m/s. What is its velocity when it reaches B? Neglect friction and weight of the bar?



Solution When body is at A, then spring at A will compress by

S_A = 1350/(18 x 10^3) = 0.075 m

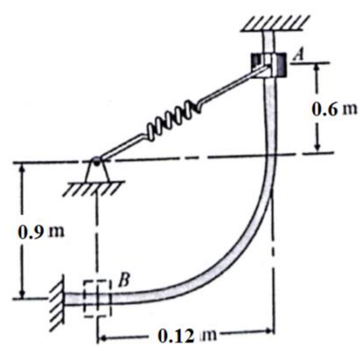
Work done = 1350 x 0.075 = 101.25 N-m

Now Work done = change in K.E

$101.25 = \frac{1}{2} \times (1350/9.81)(V^2 - 18^2)$

V= 2.17m/s

Example 13.3 A 450N weight moves along the smooth rigid from A to B as shown in fig. Find its velocity at B if it starts from rest at A. The free velocity of the spring is 0.6 m and its modulus is 0.9 kN/m.



Solution Length of spring at A = 1.3416 mts

SA = 1.3416-0.6 =.7416 mts

Length of spring at B = 0.9 m

SB= 0.9 -0.6 = 0.3m

Now Work done = Change in K.E. + Change in energy stored in spring.

$450 \times 1.5 = (450 / 2\times 9.81)(V^2-0) + \frac{1}{2} k(S_B{}^2 - S_A{}^2)$

$= (450\times V^2)/(2\times 9.81) + ½ (0.9 \times 10^3)(0.3^2-0.7416^2)$

V = 6.2 m/s.

Example 13.4 A Weight of 90 N is swung in a vertical circle at the end of 0.9 m cord. At the lowest position of the weight, the tension in the cord is 360N. How high above the lowest position will the weight rise on the circular path.

Solution $Fy = 0$

$T - 90 – 10.1936V^2 = 0$

$360-90 = 10.1936\ V^2$

V= 5.146 m/s

$M= V^2/r = 90/9.81 = V^2 / 0.9 = V^2 / 0.9 = 10.1936\ V^2$

The final velocity of body will be zero 'h' is the rise of the body

Work done = Change in K.E.

$-90 \times h = \frac{1}{2} \times (90/9.81)\ (0- 5.146^2)$

H = 1.35mts.

Example 13.5 A weight W is attached to one end of a stiff rod of length L and negligible weight that is hinged to horizontal axis at the other end. The rod is released from rest in a horizontal position and allowed to swing freely in a vertical arc. Through what angel must it swing to cause a tension in it of 1.5W?

Solution $T = W/g\ x\ V^2/r + W\ Cos(90-0)$

$1.5\ W= W/g\ x\ V^2/L + W\ Sin\theta$

$V^2 = (1.5 – Sin\theta)\ gL$

The weight moves 'h' m from A to B

Work done = Change in K.E.

$W\ x\ h = W/2g\ (V^2 – 0) = W/2g\ (1.5 – Sin\theta)\ g\ L$

$3\ Sin\theta = 1.5$

$\theta = 30^0$

Example 13.6 The rod in the previous problem is displaced an angle θ from its lowest position and released from rest. Find θ so that the tension in the rod at the lowest position is four times that just after release.

Solution $\sum F_y = 0$ $T_1 – W- W/g\ x\ V^2 /L =0$

$T_1 = W + W/g\ x\ V^2/L$ -------(1)

Body moves 'h' m vertically downward

Work done = change in K.E.

W x h = ½ x W/g (V^2 – 0)

W x (L-L Cos θ) = W/g (V^2)

V^2 = 2g (1- Cos θ) ------------ (2)

But T = W Cos θ

Now T_1 = 4T --------- given

T_1 = 4W Cos θ ------------------ (3)

Solving (1) (2) (3)

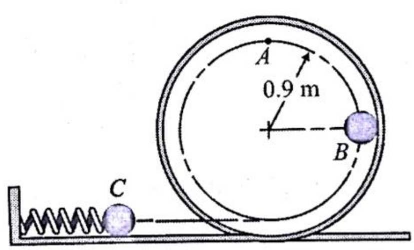
4w cos θ = W + W/g x 2g L(1- Cos θ)/L

4 Cos θ = 1 + 2(1- cos θ)

6 Cos θ= 3

$\theta = 60^0$

Example 13.7 In the given fig. by how much should the spring be compressed so that it will cause 4.5 N pellet to travel completely around the frictionless vertical loop ? What force is exerted by the track upon the pellet when it is at position B? k= 0.9 kN/m.



Solution Initial position of ball is at A velocity at A

W= W/g x V_A^2 /0.9

V_A = 2.97 m/s

Distance travelled for A to C

Vertically = 2 x 0.9 = 1.8m

Work done = Change in K.E

4.5 x 1.8 = $\frac{1}{2}$× (4.5 /9.81) (V^2_C –V^2_A)

V_C= 6.64m/s

Now spring will compress by 'S' m

At that stage V =0

Energy stored in spring = Change in K.E.

$\frac{1}{2}$ k S^2 = ½ x w/g (V^2_C – 0)

$\frac{1}{2}\times$ 0.9 x10^3 x S^2 = $\frac{1}{2}\times$ 4.5/9.81 (6.64^2 -0)

S^2 = 0.0225 m

S = 0.15 m

Distance travelled from C to B = 0.9 m

4.5 x 0.9 = $\frac{1}{2}\times$ 4.5/9.81 (V^2_B –V^2_C)

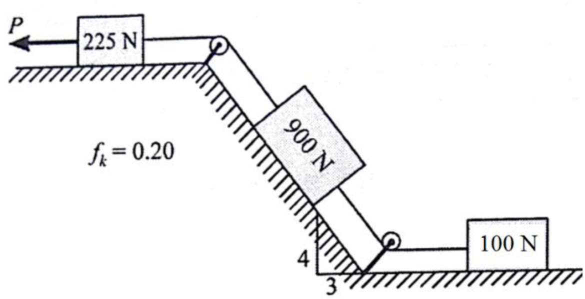
V^2_B = 5.14 m/s

Force exerted by track

N = W/g x V^2_B/r = 4.5/9.81 x $(5.14)^2$/0.9 = 13.46

13.6 WORK-ENERGY APPLIED TO CONNECTED SYSTEMS

Example 13.8 Determine the constant force P that will give the system of bodies shown in figure a velocity of 3 m/s, after moving 4.5 m from rest.



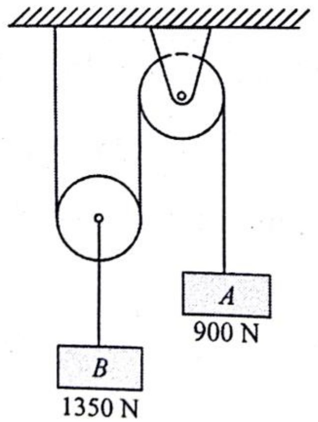
Solution Here velocity of all the bodies are same

Total Work done = Change in K.E

(P- 45 – 108 -720 -20) x 4.5 = $\frac{1}{2}\times$ ($\frac{(225+900+100)}{9.81}$) ($3^2$- 0)

Solving P = 1017.87 N

Example 13.9 Through what distance will body A in figure move in changing its velocity from 1.8 m/s to 3.6 m/s ? Assume the pulleys to be frictionless and of negligible weight.



Solution Assume 'A' at rest

T = 900 N

2T = 1800N > 1350N

Block 'B' moves up and hence block 'A' moves down

Using kinematic relation 2T S_B = T S_A ---------------------------(1)

Diff. w. r. t time $2V_B = V_A$

Applying work energy equation

Total Work done = Change in K.E.

$(-1350 \times S_B) + (900 \times S_A) = \frac{1}{2}\times \frac{(1350)}{9.81}(V_B^2 - U_B^2) + \frac{1}{2}\times \frac{(900)}{9.81}(V_A^2 - U_A^2)$

Given U_A = 1.8 m/s

V_A = 3.6 m/s

U_B = 1.8/2 = 0.9 m/s

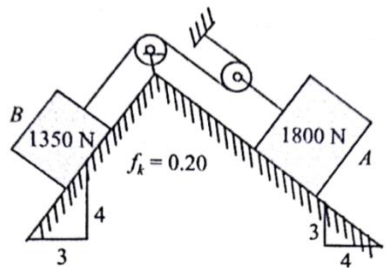
V_B = 3.6/2 = 1.8 m/s

Using (1)

$(-1350 \times S_A/2) + 900\ S_A = \frac{1}{2}\frac{(1350)}{9.81}(1.8^2 - 0.9^2) + \frac{1}{2}\frac{(900)}{9.81}(3.6^2 - 1.8^2)$

Solving S_A = 2.725mts.

Example 13.10 In what distance will body A of figure attain a velocity of 3 m/s., starting from rest?



Solution As explained in previous chapter the motion of the bodies can be determined

Body A is moving up and body B is moving down

From Kinematic relation : $T .S_B = 2TS_A$ --------------------(1)

$V_B = 2\ V_A$ --------------------- (2)

From Kinematic relation

$(1080 - 162)\ S_B + (-1080 - 288)\ S_A = \frac{1}{2} \times \frac{(1350)}{9.81} (V_B^2 - U_B^2) + \frac{1}{2} \times \frac{(1800)}{9.81} (V_A^2 - U_A^2)$

Given $U_A = 0$, $V_A = 3$ m/s

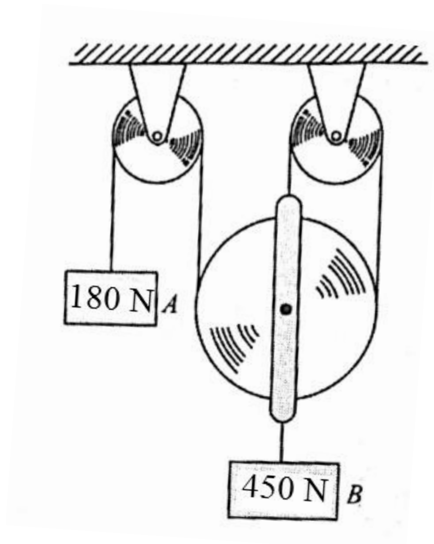
$V_B = 6$ m/s and $U_B = 0$

Using (1) & (2)

$918 \times 2\ S_A - 1368\ S_A = 3302.75$

$S_A = 7.057$ m

Example 13.11 Assume the pulleys in fig. to be frictionless and of negligible weight . Find the velocity of body B after it has moved 3 m. from rest. Then use this result to determine the acceleration body B.



Solution Direction of motion of bodies; Body 'B' moves up and body 'A' moves down.

From Kinematic relation $T\,S_A = 3\,T\,S_B$

$S_A = 3\,S_B$ -------------------------------- (1)

$V_A = 3V_B$ -------------------------------- (2)

From work energy equation

$180 \times S_A - 450\,S_B = \frac{1}{2} \times \frac{(180)}{9.81}(V_A^2 - U_A^2) + \frac{1}{2} \times \frac{(450)}{9.81}(V_B^2 - U_B^2)$

Here $U_A = 0$ & $U_B = 0$ as bodies are starting from rest.

Also $S_A = 9$m $S_B = 3$m

$180 \times 9 - 450 \times 3 = 9.174\,(3V_B)^2 + 22.936\,(V_B)^2$

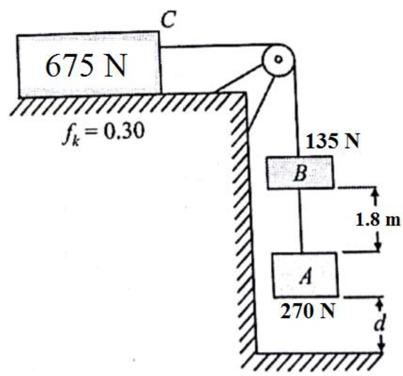
$V_B = 1.6$ m/s

Now.., $V_B^2 = U_B^2 + 2a_B S_B$

$(1.6)^2 = 0 + 2a_B \times 3$

$a_B = 0.427 \text{m/s}^2$

Example 13.12 The system shown in fig. is connected by flexible inextensible cords. The system starts from rest, find the distance d between 'A' and the ground so that the system comes to rest with body 'B' just touching A



Solution In this case the displacements abd velocities of all the bodies are same. Assume the system is moving with velocity 'V' when the body 'A' is touching the ground.

Work energy equation

$(270 + 135 - 202.5)\ d = \frac{(270+135+675)}{2 \ x\ 9.81} (V^2 - U^2)$

Here U =0

205.5 d = 55.046 V^2 --------------------------- (1)

As body 'A' is touching the ground its velocity is zero. But body B and C should move 1.8 m to touch body A so that final velocity of the system is zero. Here initial velocity of B and C is 'V'

$(135 - 202.5)\ x\ 1.8 = \frac{1}{2} \times \frac{(135+675)}{9.81} (0 - V^2)$

V =1.7155 m/s

Put in (1) d= 0.8m.

SUMMARY

- Work is defined as the product of force and distance
- If F is force acting on the body which moves a distance d in the direction of force then work done is equal to force times the distance
- Energy is defined as the capacity to do work.
- The energy possessed by the body by virtue of its position is known as potential energy
- The energy possessed by the body by virtue of its motion is known as kinetic energy
- Power is defined as the rate of doing work.
- The work done by force F is equal to change in kinetic energy of the body. This is principle of work and energy or work energy theorem.

EXERCISES

I. Self-Assessment Questions

1. Define the terms “work”, “energy” and power.
2. Derive work-energy equation.

II. Multiple Choice Questions

1. Work and energy principle is useful where problems are mainly concerned with
   a. acceleration b. time
   c. displacement d. angular velocity
2. Principle of work and energy relates the parameters
   a. Force, displacement and time
   b. Force, displacement and velocity
   c. Force, displacement and acceleration
   d. Force, time, velocity and mass
3. Energy may be defined as
   (a) Power of doing work
   (b) Rate of doing work
   (c) Work done in unit time
   (d) Capacity of doing work
4. Unit of power is
   a. horsepower b. joule/sec
   c. watt d. all of these
5. The kinetic energy of a particle of mass m and velocity v is quantified by
   (a) mv (b) mv^2 (c) $\frac{1}{2} mv^2$ (d) $2mv^2$
6. The unit of work in S.I. system is
   (a) lb-ft (b) N-m (c) joule (d) both (b) and (c)

Answers

1. c
2. b
3. d
4. d
5. c
6. b

CHAPTER – 14

IMPULSE AND MOMENTUM

LEARNING OBJECTIVES

After studying this chapter, you should be able to

- *Define impulse and momentum*
- *State the principle of impulse-momentum and apply it to solve problems.*
- *Apply the principle of conservation of linear momentum to solve problems*

14.1 INTRODUCTION

In the previous chapter we have solved the problems involving force, velocity and displacement using work energy method. This method cannot be useful in solving the problems involving force, time and velocity. In this chapter, the Impulse Momentum method is dealt which is useful for solving the problems involving force, time and velocity.

14.2 IMPULSE MOMENTUM

The principle of Impulse momentum states that the total impulse acting on a body in a given time interval is equal to the change in momentum of the body.
We saw in the previous chapter that the Newton's second law can be represented by the equation

$$R = ma$$
$$= m\,dv/dt$$
$$Rdt = mdv$$

Integrating on both sides

$$\int Rdt = \int mdv$$

If initial velocity is u and after time interval t it becomes v, then

$$\int_0^t Rdt = m[v]_u^v = mv - mu$$

The term mv – mu gives the change in linear momentum of body where as the term $\int_0^t Rdt$ is called impulse. The units of impulse are N – Sec and it is a vector quantity. The above equation can be written as

Impulse = Final momentum – Initial momentum

14.3 PRINCIPLE OF CONSERVATION OF MOMENTUM

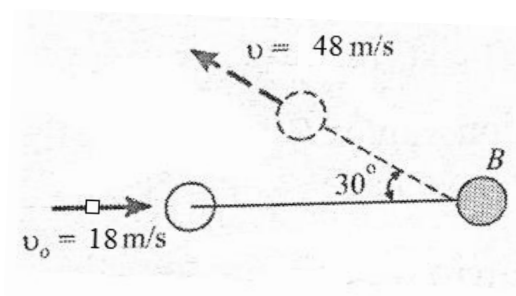
Total momentum of the system remains unchanged if the resultant force R is zero. Such situation arises in many cases because the force system consists of only action and reaction on the elements of the system. The resultant force is zero, only when entire system is considered, but not when the free body of each element of the system is considered. If a man jumps off a boat, the action of the man is equal and opposite to the reaction of the boat. Hence, the resultant is zero in the system. Therefore:

Total initial momentum = Total final momentum

$$\sum m_i v_i = \sum m_i u_i$$

Note: This principle applies to the whole system and not for the individual elements of the system

Example 14.1 A 1N baseball is thrown with a velocity of 18 m/s toward a batter. After being struck by the bat B, the ball has a velocity of 48 m/s directed as shown in figure. Find the average force exerted on the ball if impact lasts for 0.02 sec.



Solution u= 18 m/s,

V= -48 cos 30 = -41.57 m/s

m= w/g = 1/9.81

t= 0.02sec

We know F x t = m (V - u)

F x 0.02 = 1/9.81 (-41.57-18)

F= -303.615 N

Example 14.2 A 40 ton rail road car moving at 0.88 m/s along a level track strikes and is coupled to a stationary 60 ton car in 0.5 sec. Determine average force acting in each car during the coupling.

Solution

m_1= 40ton = 40× 10^3kg

m_2= 60ton = 60× 10^3kg

t=0.5 sec

u_1=0.88m/s

u_2=0

$V_1=V_2=V_C$ (common velocity)

Law of conservation of momentum

$M_1\ u_1+m_2\ u_2=m_1\ v_1+m_2\ v_2$

$40\times 10^3 \times 0.88+0 = (40\times 10^3+60\ 10^3)\ Vc$

V_c=0.352m/s

For body no (2)

Impulse – momentum equation

Force× time = m (v-u)

$F\times 0.5 = 60\times 10^3(0.352-0)$

F=42240 N=42.24 kN

Example 14.3 During an Apollo mission to the moon, the lunar excursion module (LEM) is separated from the command module by an average thrust P lasting for 1 sec. As a result, the LEM has a velocity 20 m/s slower than the command module. Determine P assuming the mass of LEM and the command module to the respectively 120 kg-sec^2/m and 30 kg –sec^2/m.

Solution

Initial velocity of LEM and CM =$u_1=u_2=u_c$

Final velocity of CM=v_2

Final velocity of LEM= v_2-20

Law of conservation of momentum

$m_1\ u_1+m_2\ u_2=m_1\ v_1+m_2\ v_2$

$(120+300)U_c=120\ (v_2-20\ )+300V_2$

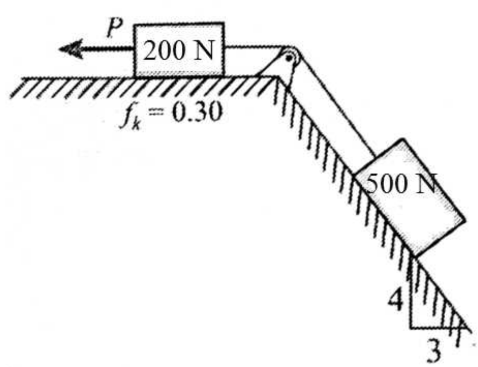
$U_c=V_2-5.7143$

Impulse momentum equation F× t=m(v-u)

$P \times 1=120[(v_2-20)-(v_2-5.7143)]$

P=-1714.286N.

Example 14.4 The system shown in figure is moving rightward at a velocity of 4.5 m/s when a constant horizontal force P is applied as shown. Determine the value of P that will give the system a leftward velocity 9m/s in a time interval of 10 sec.



Solution Two fractional forces are shown, drawn solid for rightward motion and drawn dotted for leftward motion.

For rightward motion, initial velocity = 4.5m/s

Final velocity =0

Time = t_1

For leftward motion, initial velocity=0

Final velocity =9m/s time=t_2

Also $t_1+t_2=10$sec ---- (1)

For rightward motion, impulse momentum equation

Force× time = mass (v-u)

$(400-90-60-P)t_1=[(500+200)/g](0-4.5)$

$(250-P)t_1=-321.101$ ---- (2)

For leftward motion, $(P-60-90-400)\ t_2 = [(500+200)/g]\ (9-0)$

$(P-550)t_2=642.202$ ----(3)

Trial and error:

Trail	Assume ‘P’	t_1(sec.)	t_2(sec.)	Calculated ‘P’
01	600	0.917	9.083	620.707
02	620.707	0.866	9.134	620.31
03	620.31	0.867	9.133	620.31

P = 620.31N

Example 14.5 A 100N shell is fired from a 2,00,000 N cannon with a velocity of 600m/s. Find the modules of nest spring that will limit the recoil of the cannon to 0.9 m.

Solution

$m_1 = 1000/g$ kg $m_2 = 2,00,000/g$ kg

$m_1 = 0$: $u_2 = 0$

$V_1 = 600$ms: $V_2 = ?$

Law of conservation of momentum,

$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$

$0+0 = (1000/g) \times 600 + (2,00,000/g) V_2$ $V_2 = -3$m/s

Energy stored in the spring = change in K.E

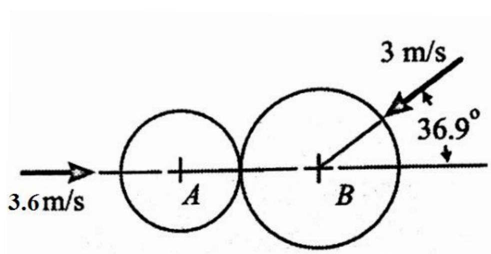
$\frac{1}{2} K.(\delta\ 1)^2 = \frac{1}{2} m(V^2 - u^2)$

$\frac{1}{2} K(0.9)^2 = 1/2 \times (2,00,000/g)(0-3^2)$

$K = 226.526 \times 10^3$ N/m

=226.526 kN/m

Example 14.6 Just before they collide, two disks on a horizontal surface have the velocities shown in figure. Knowing that 90 N disk 'A' rebounds to the left with a velocity of 1.8 m/s, determine the rebound velocity of the 135 N disk 'B'. Assuming the colliding surfaces are smooth.



Solution

Law of conservation of momentum,

For 'x' direction: $m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$

$90/g \times 3.6 + [135/g] (-3\cos 36.9) = 90/g(-1.8) + 135/g\ (V_2)x$

$V_2 = 1.201$ m/s

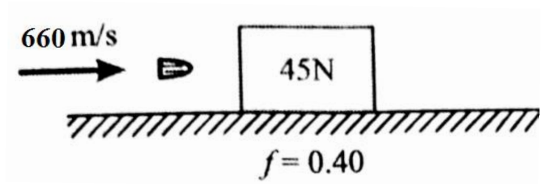
For 'y' direction $0 + 135/g\ (-3\sin 36.9) = 0 + 135/g(V_2)y$

$(V_2)\ y = -1.8012$ m/s

Final velocity of disk 'B'

$V_B = 2.165$ m/s.

Example 14.7 A bullet weighing 0.3N and moving at 660m/s penetrates the 45 N body and emerges with a velocity 180 m/s as shown in fig. How far and how long does the body then move?



Solution $m_1 = \frac{0.3}{g}$ kg $\quad m_2 = \frac{45}{g}$ kg

$u_1 = 660$ m/s: $\quad u_2 = 0$

$V_1 = 180$ m/s; $\quad V_2 = ?$

Law of conservation of momentum,

$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$

$(0.3/g) \times 660 + 0 = (0.3/g) \times 180 + 45/g \times v_2$

$V_2 = 3.2$ m/s.

Work energy equation $-F \times S = \frac{1}{2} m (v^2 - u^2)$

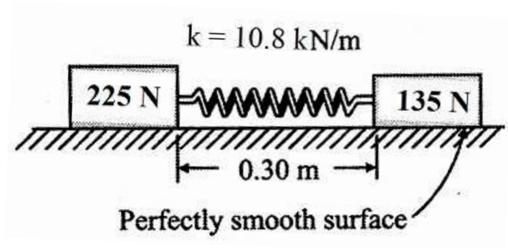
$-18 \times S = \frac{1}{2} \times \frac{45}{g} (0 - 3.2)^2 \qquad S = 1.305$ mts

Impulse – momentum equation $-F \times t = m(v-u)$

$$-18 \times t = \frac{45}{g}(0 - 3.2)$$

$t = 0.8155$ sec

Example 14.8 The spring shown in fig .has free length of 0.3 m. It is compressed to half its length and the blocks are suddenly released from rest ,determine velocity of each block when the spring is again 0.3 m long.



Solution when spring is compressed, energy is stored in it.

Energy stored in spring $=\frac{1}{2}K\,(\delta l\,)^2$

Where K = spring constant (stiffness of spring)

= it is forced required per unit deformation

Energy$=\frac{1}{2}\times(10.8\times 10^3)\times(0.15)^2$ $\delta l=(0.3)/2=0.15$

$=121.5=\frac{1}{2}m_1u_1^2+\frac{1}{2}m_2\,u_2^2=\frac{1}{2}\left(\frac{225}{g}\right)u_1^2\times+\frac{1}{2}\times(\frac{135}{g})\,u_2^2$

When the spring comes to original position , the initial velocity of both the blocks will be zero.

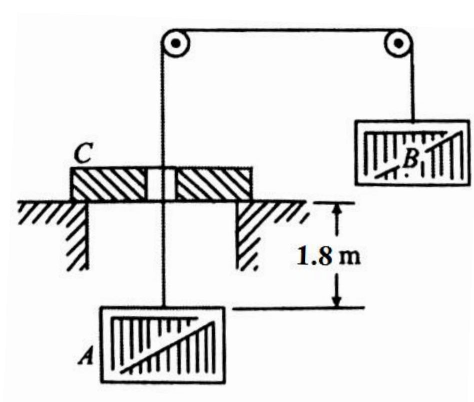
Law of conservation of momentum :$m_1\,u_1+m_2\,u_2=0$

$(225/g\,)\times u_1+(135/g)\times u_2=0$ $u_1=-0.6u_2$ put in(1)

$121.5=\frac{1}{2}\times\left(\frac{225}{g}\right)(-0.6u_2)^2+\frac{1}{2}\times\left(\frac{135}{g}\right)\times u_2^2$

$U_2=3.32M/s$ $u_1=-2m/s$

Example 14.9 After body ‘A’ shown in figure moves 1.8 m from rest, it picks up body C. How much further does ‘A’ move before reversing its direction? Assume $W_A=45N$: $W_8=90N$; $Wc=67.5N$.



Solution Here the velocities of ‘A’ and ‘B’ are same.

Let the velocity of body ‘A’ just before it touches the body ‘C’ be V_A

$V_A=V_B=V$

From work energy equation

$V=3.431\text{m/s}$

The block ‘A’ is touching block ‘C’ with velocity 3.431m/s. After touching block ‘C’ the velocity changes and the whole system moves with a different velocity say ‘V_1’ until velocity becomes zero.

Again W-E equation

$(-45 - 67.5 + 90) \times S = \frac{[45+67.5+90]}{2g} (0^2-V_1^2)$

$-22.5\times S=-10.321\ V_1^2$

$-22.5\ S =10.321\ V_1^2$

Law of conservation of momentum

$m_A\ u_A + m_B\ u_B + m_C\ u_C = (m_A+m_B+m_C)\ V_1$

$\left[\frac{45}{g} + \frac{90}{g}\right] \times\ 3.431 + 0 = \frac{[45+67.5+90]}{2g}\ \times V_1$

$V_1=2.287\ \text{m/s}$

Put in (1) $S=2.4\text{mts}$

SUMMARY

- The principle of Impulse momentum states that the total impulse acting on a body in a given time interval is equal to the change in momentum of the body.
- The units of impulse are N – Sec and it is a vector quantity.
- The principle of conservation of momentum may be stated as, the momentum is conserved in a system in which resultant force is zero.
- Conservation of momentum applies to entire system and not to individual elements of the system.

EXERCISES

I. Self-Assessment Questions

1. Define impulse of a force.
2. Derive the mathematical expression for the impulse-momentum equation.

3. Describe the effect of impulsive forces upon the motion of bodies?
4. Distinguish between impulsive force and impulse of a force

II. Multiple Choice Questions

1. Impulse of a force acting on a body is equal to
   (a) Momentum of the body (b) Change in momentum of the body
   (c) Rate of change in momentum of the body (d) Product of momentum and time
2. Unit of impulse of a force is
   (a) N (b) N.m (c) N/s (d) N.s
3. Impulse of a force between time limits is equal to area under ____________ graph between the time limits.
   (a) Force & time (b) Acceleration & time (c) Velocity & time (d) Displacement &time
4. Impulse momentum equation relates
   (a) Force, velocity and displacement (b) Force, velocity and time
   (c) Force, displacement and time (d) Force and acceleration
5. Principle of impulse and momentum is derived from
   a. Newton's first law of motion b. Newton's second law of motion
   c. Newton's third law of motion d. none of these

Answers

1. b
2. d
3. a
4. b
5. b