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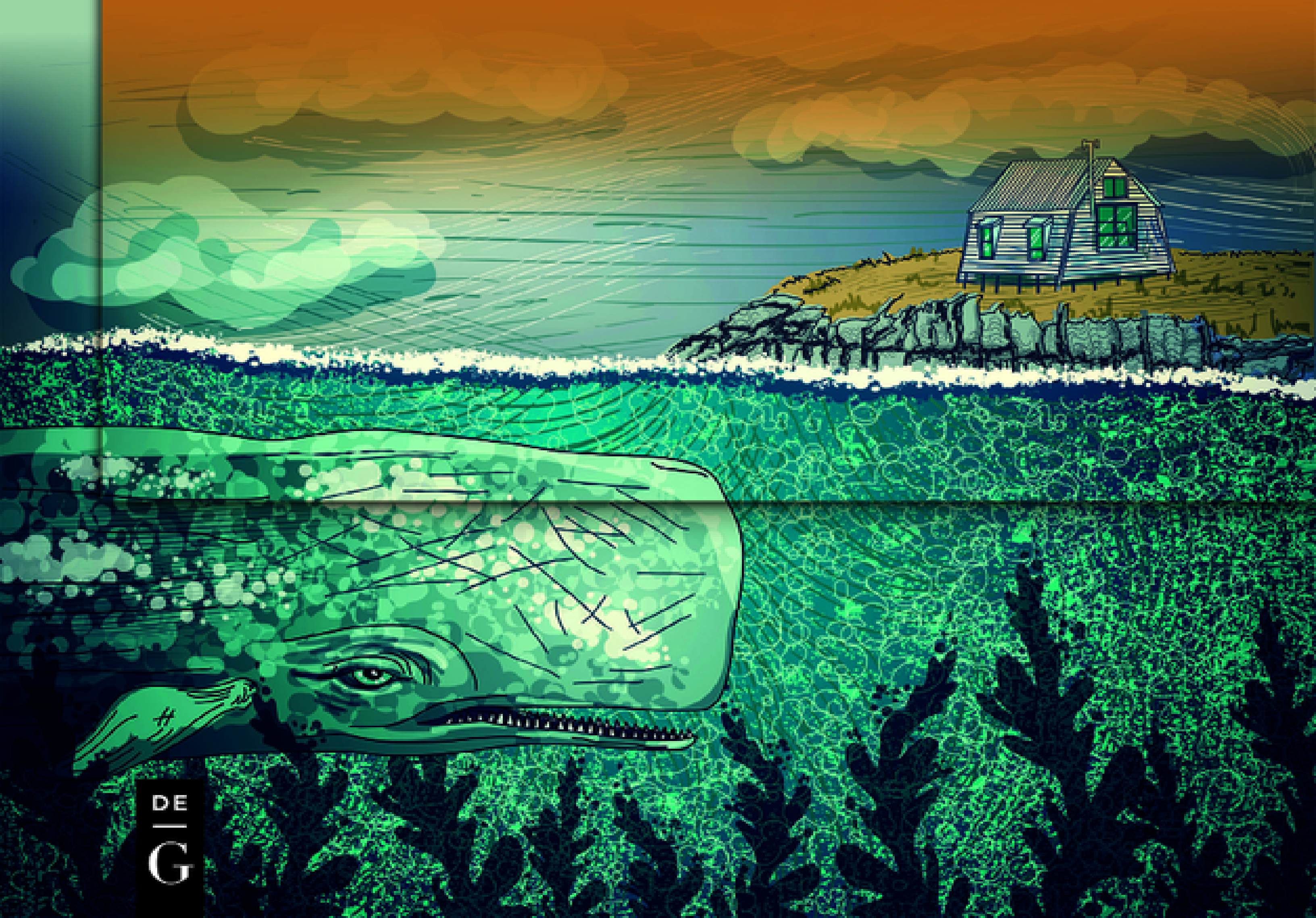
Nikolai Khots and Boris Khots

OBSERVABILITY AND MATHEMATICS MODELING

HILBERT, EUCLID, GAUSS-BOLYAI-LOBACHEVSKY, AND RIEMANN GEOMETRIES



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Übersicht

Contents

Nikolai Khots dedicates this book to his parents, Elena and Dmitriy Khots, with profound thanks for their love and support

Foreword 1

This book introduces a new approach to geometry, offering fresh interpretations of fundamental concepts such as points, lines, planes, and space. The central idea replaces the notion of infinity with a perspective based on "observers," forming the foundation of what the authors call the *Mathematics with Observers* theory.

Traditional mathematics—including arithmetic, linear algebra, calculus, geometry, differential geometry, algebra, and functional analysis—relies heavily on the concept of infinity. The authors propose incorporating observers into arithmetic, rendering it dependent on these observers. Consequently, other branches of mathematics built upon arithmetic also become observer-dependent. The term *Mathematics with Observers* encompasses both this modified arithmetic and the broader mathematical framework derived from it.

This book specifically examines geometry within the context of *Mathematics with Observers*.

The authors build on David Hilbert's classification of geometric properties, which include:

- Connection
- Order
- Parallels (Euclidean, Gauss–Bolyai–Lobachevsky, Riemann)
- Congruence
- Continuity

These properties are reevaluated through the lens of *Mathematics with Observers*.

Why is this approach crucial for contemporary mathematics and physics?

When we talk about lines, planes, and geometric bodies, we often describe them with precise definitions and characteristics. However, where exactly do these idealized entities exist? How do they manifest in the real world?

For example, polishing a metal plate will never produce a perfect plane because neither the tools nor the operations used are ideal. Due to the atomic structure of matter, achieving—or even approximating—an ideal plane is fundamentally impossible.

Similarly, what is a line? We might suggest that light travels in perfect straight lines, but light consists of discrete quanta and does not follow a continuous path. Thus classical geometry cannot claim to have unlimited applicability to real-world phenomena.

This limitation implies that physical space, as understood today, must conform to the frameworks of classical geometry—whether Euclidean or non-Euclidean. Similarly, calculations involving large systems or measurements of vast distances rely on the existing structure of the real number line.

Mathematics with Observers challenges these traditional foundations, rejecting the concept of infinity and redefining arithmetic and mathematics as a nested system of observable constructs.

This new framework allows for a reexamination of geometry and provides solutions to classical problems in both mathematics and physics.

Readers are encouraged to explore previously published works on *Mathematics with Observers* to gain a comprehensive understanding of this innovative perspective.

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Foreword 2

For many people, geometry is a challenging school subject. They learn, memorize some things, reason, and solve problems. Moving from the plane to three-dimensional space, they encounter the difficult term “stereometry” and the more straightforward “trigonometry.” Some formulas become etched in their memory for a lifetime. However, for most, their acquaintance with geometry ends there.

The purpose of this article is to discuss geometry as a science that plays a fundamental role in the system of knowledge. First and foremost, it’s essential to note that ancient mathematics was primarily geometry. As mathematics evolved, so did geometry. Its applications expanded, and new methods and techniques emerged.

Let’s list the names of outstanding geometers:

- Euclid
- Legendre

- Cauchy
- Gauss
- Lobachevsky
- Riemann
- Beltrami
- Bianchi
- Poincaré
- Grassmann
- Hilbert
- Cartan
- Alexandrov
- Efimov
- Pogorelov
- Pontryagin
- Chern

This list could continue indefinitely. Each of these individuals made remarkable contributions. If we were to delve into each one, it would fill an entire book. Their names are associated with scientific progress, which ultimately benefits all of humanity.

We shouldn’t forget other names like Descartes, Fermat, Pascal, and more. Each of these geometers interacted with other scholars, mathematicians, and specialists in related fields.

Geometry doesn't exist in isolation; it closely interacts with other mathematical disciplines.

In the 20th century, topology became closely connected to geometry, leading to the specialization known as 'geometry and topology.' Among the names mentioned earlier, only Euclid and Legendre have direct ties to school geometry. School geometry textbooks are based on Euclid's "Elements," adapted for French students by Legendre. It's worth noting that A. V. Pogorelov also authored a geometry textbook for schools, but the scientific contributions of other geometers to secondary education remain less known.

One distinctive feature of geometry is its abstraction. Let's compare the subjects of study in geometry, physics, and biology. Physics primarily investigates the inanimate, while biology focuses on living organisms. Many objects studied in physics and biology are tangible and observable.

Geometric concepts, however, are quite abstract. Initially, it seems straightforward—points, lines, triangles, rectangles, parallelograms, and circles. But then reasoning becomes more complex. Abstract thinking develops. Measurements of segments and angles emerge, followed by connections between these concepts and formulas. Thus, geometry explores metric relationships in geometric objects.

Despite this apparent detachment from nature, geometry is, in fact, the science closest to it. Metric relationships characterize many natural properties. Moreover, human scientific thinking is closely tied to geometric reasoning.

In ancient times, geometry was initially built empirically, accumulating facts. Eventually, it was observed that a finite number of fundamental assumptions led to all known geometric facts. The axiomatic method emerged, first for plane figures and later for three-dimensional space. This method allowed for the creation of theories of multidimensional spaces, including Hilbert's infinite-dimensional spaces and Banach spaces studied in functional analysis.

It is difficult to imagine that a system of axioms like this would arise in physics or biology. However, facts are accumulating in both fields. Scientists in physics and biology operate with specific concepts that allow them to convey meaningful information to each other. For instance, in physics, concepts like mass, energy, temperature, density, pressure, entropy, and time are used. These concepts evolve over time, and some physical ideas are formulated using geometric concepts.

Geometry has a significant impact on physics and mechanics. Analyzing a vast number of astronomical observations, Kepler determined that planets move around the Sun in ellipses. This insight allowed Newton to brilliantly confirm his law of gravitation. In other words, geometry helped establish the physical law of attraction between two bodies.

Another example related to gravity involves the work of famous geometers: Lobachevsky, Gauss, Riemann, Christoffel, and others. They developed a powerful geometric framework describing multidimensional spaces. Einstein used this framework to construct the general theory of relativity. Einstein proposed a system of gravitational equations expressed using the Riemann tensor. These equations are usually called Einstein's equations, which he proposed without a mathematical derivation. It was similar to how Newton proposed the law of universal gravitation, also without proof.

Einstein gave a lecture at Hilbert's, and under his influence, Hilbert derived these equations using the variational method by varying the integral of the scalar curvature over the entire space through changes in the space's metric.

Friedman discovered a solution to this system that describes the evolution of the entire universe. This triumph of geometric thinking was beyond imagination.

Suddenly, it became clear that the fundamental property of physical bodies—attraction to one another—could be elegantly described using geometric concepts such as curvature and curved space. Interestingly, even the concept of time, which ancient geometers avoided, became geometrically treated. It was envisioned as one of the coordinates, similar to spatial coordinates. In four-dimensional spacetime with three spatial coordinates (x, y, z) and one temporal coordinate (*t*), a metric was introduced. This metric allows us to measure the distance between two points ($x1, y1, z1$) and ($x2, y2, z2$) by integrating along the connecting curve. This approach is remarkably beautiful—a true triumph of geometry.

Further exploration into matter involved introducing additional coordinates. For example, Calabi-Yau spaces were constructed. In 2012, authors Shing-Tung Yau and Steve Nadis published the book “The Theory of Strings and Hidden Dimensions of the Universe.” Yau, a renowned geometer and student of Chern, provides insights from the heart of mathematical ideas and numerous contacts. The book features stories and photographs of famous geometers, primarily American ones like Chern, Calabi, Witten, Uhlenbeck, Yang, Mills, Donaldson, Nirenberg, and others. Russian mathematicians Pogorelov, Sobolev, and Perelman are also mentioned.

The second chapter of the book is titled ‘The Place of Geometry in the Cosmos,’ while the fourteenth chapter raises the question: ‘The End of Geometry?’ What leads the authors to consider the possible end of geometry? They point to hidden problems that may bring future challenges. Heisenberg’s uncertainty principle is a stumbling block. A new term emerges: ‘quantum geometry.’ Objects at the Planck scale don’t remain static; they constantly fluctuate, altering their parameters, including size and curvature. The authors express concerns about the fundamental incompatibility between quantum mechanics and general relativity, suggesting that geometry itself may be more derivative than fundamental. This implies that microscopic descriptions are more fundamental, while macroscopic properties are derived from them.

Geometry has evolved over millennia. The authors draw an interesting comparison: ‘If the great Euclid were present at a geometry seminar today, he would be bewildered by our discussions. In his time, geometry focused solely on three-dimensional space, and the concept of coordinates didn’t exist. Euclid would undoubtedly ask, ‘What is the physical meaning of these multidimensional spaces? How can we visualize them?’

He’d be surprised to learn that multidimensional spaces also contain regular polyhedra, akin to those he described in his work ‘Elements.’ While there are five regular polyhedra in three-dimensional space, there are six in four-dimensional space, including regular 120-cell and 600-cell polytopes. Euclid would likely inquire, ‘Who discovered this?’ The answer: Ludwig Schläfli, a Swiss scientist.

For higher dimensions (5D, 6D, etc.), only three regular polyhedra exist—analogous to the tetrahedron, cube, and octahedron. Euclid would conclude that 3D and 4D spaces are exceptional.

Regrettably, Euclid’s book ‘Elements’ was lost during historical upheavals, but fortunately, Arab scholars preserved it, and its contents are now studied worldwide—a triumph for the great mathematician.

Physicist D. Polchinski from Santa Barbara aptly paraphrases Mark Twain: ‘Reports of geometry’s demise are greatly exaggerated.’ He believes that geometry plays a vital role in discoveries and is part of something greater, not something ultimately discarded.

Albert Einstein constructed the general theory of relativity using Riemannian geometry and linked it to gravity. His equations describe the motion of the entire universe. Notably, solutions like the Friedman solution are remarkable achievements.

Einstein continued by seeking a unified field theory, believing it should harmonize the world. He famously said, ‘God is subtle, but not malicious,’ expressing hope that geometric relationships underlie the universe.

In the quest for unification, parameters emerge—some yielding theories of gravity, electromagnetism, weak interactions, or strong interactions. The unification of diverse theories may take different forms, but it remains a fruitful pursuit.

In this book, the authors explore geometry through the lens of *Mathematics with Observers*, a framework they introduced in their earlier work. This entirely novel perspective yields unexpected results that are not only essential for the evolution of geometry but also for advancing our understanding of the physical world, which relies heavily on geometric principles, particularly in physics.

Building on their previous publications, the authors expand the concepts of *Mathematics with Observers* to include geometric contexts. A key idea introduced is the concept of a sequence of Observers, each possessing its own arithmetic and capable of using only a finite set of numbers. Observers with larger numbers can utilize a broader range of numbers than those with smaller numbers.

This approach marks a significant departure from classical mathematics, aligning more closely with real-world limitations. For example, the memory capacity of computers, no matter how vast, is inherently finite. Consequently, this perspective eliminates the concept of continuous functions. In this framework, the classical theorem that guarantees a zero point for a continuous function on the interval $[a, b]$, when the function's values at the endpoints have opposite signs, no longer holds.

It is worth noting that even physicists, such as Lev Landau, have cautioned students studying mathematics to disregard existence theorems, famously stating, "*Do not pay attention to existence theorems. Mathematicians love to prove existence theorems*."

The innovative ideas and results presented in this book are poised to revolutionize modern geometry and lay the groundwork for transformative applications in various fields.

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1 Introduction

When we study a geometry in the ninth grade at school, we meet the beautiful set of understandable objects – points, lines, planes, spaces, plane and space figures and solids, with natural connections between them with natural logic of definitions and theorems. We understand and think that this beautiful set is real. However, when we become adults, we can ask ourselves – where and how does this set exist? Because we know the atomic structure of real nature and quantum nature of light, and we do not have uniformity and continuity.

It is possible to think that geometry is a mirror of our nature but simplified and approximate. However, this is a very naive statement as we can not think that something is a simplified representation of some matter if we even do not know this matter.

To better approach the reality, in this book, we consider geometry from Mathematics with Observers point of view. Mathematics with Observers was introduced by the authors based on the denial of “infinity” idea, going away from the existing images of natural and real numbers, replacing them to Observers-dependent sequences of finite sets and introducing

Observers-dependent arithmetic and logic. We consider in this book the basis of classic geometry from Mathematics with Observers point of view. As a basis for the analysis of our intuition of space, classic Mathematics considers four systems of things, called points, straight lines, planes, and spaces, connecting these elements in their mutual relations (see [→2]).

Here we consider the properties of connection, order, parallels (Euclid, Gauss–Bolyai–Lobachevsky, Riemann), congruence, continuity from Mathematics with Observers point of view.

We show that almost all classic geometry theorems are satisfied in Mathematics with Observers geometry with probabilities less than 1. For example, we proved the following theorem:

“In plane E_2W_n, there are a point *A* and a straight line *b* such that $A \notin b$, and we may have three different possible situations:

1. There is only one straight line *a* that contains point *A* and is parallel to line *b* (Euclidean geometry case);

2. There is more than one straight line *a* that contains point *A* and is parallel to line *b* (Gauss–Bolyai–Lobachevsky geometry case);

3. There is no straight line *a* that contains point *A* and is parallel to line *b* (Riemann geometry case).

This means that on the same plane, there are couples (point and straight line not containing this point) where Euclidean geometry works, other couples where Gauss–Bolyai–Lobachevsky geometry works, and other couples where Riemann geometry works.”

This means that classical geometry is not a limiting case of the Observers geometry, but only a particular case of it.

As a result, we prove that Mathematics with Observers gives a birth of a new geometry, and classical geometries become particular cases of this new geometry.

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2 Several definitions and statements of Mathematics with Observers

For references, see [→3] and [→1].

We call W_2 the set of all decimal fractions such that there are at most 2 digits in the integer part and 2 digits in the decimal part of the fraction. Visually, an element in W_2 looks like

$$\pm b_1 b_0 .\, a_1 a_2,$$

where $b_1, b_0, a_1, a_2 \in [0, 1, 2, \dots, 9]$.

We call W_3 the set of all decimal fractions such that there are at most 3 digits in the integer part and 3 digits in the decimal part of the fraction. Visually, an element in W_3 looks like

$$\pm b_2 b_1 b_0 .\, a_1 a_2 a_3,$$

where $b_2, b_1, b_0, a_1, a_2, a_3 \in [0, 1, 2, \dots, 9]$.

We call W_n the set of all decimal fractions such that there are at most *n* digits in the integer part and *n* digits in the decimal part of the fraction. Visually, an element in W_n looks like

$$\pm\underbrace{\_\ \dots\ \_}_{n}.\underbrace{\_\ \dots\ \_}_{n}.$$

We get $W_k \subset W_n$ if $k < n$.

We call a W_n-observer some system working within W_n. The set of W_n-observers is a finite well-ordered system ordered by *n*, and a W_n-observer sees what and how any W_k-observer with $k < n$ is doing in W_k. However, a W_k-observer is unaware of the existence of W_n-observers with $n > k$.

Note, for example, that a W_2-observer cannot see a full set W_2, whereas a W_3-observer sees what and how a W_2-observer is doing in W_2, but a W_3-observer cannot see a full set W_2. Only a W_m-observer ($m \geq 5$) can see a full set W_2.

Now we introduce arithmetic operations over numbers, elements of W_2. For $c = \pm c_0 .\, c_1 c_2$, $d = \pm d_0 .\, d_1 d_2 \in W_2$, we endow W_2 with the arithmetic $(+_2, -_2, \times_2, \div_2)$ from the W_2-observer point of view.

Definition 2.1.

Addition and subtraction

$$c \pm_2 d = c \pm d$$

if

$$c \pm d \in W_2,$$

and $c \pm_2 d$ is not defined if

$$c \pm d \notin W_2,$$

where $c \pm d$ is the classic arithmetic addition and subtraction.

Examples of addition and subtraction made by a W_2-observer in W_2:

$$\begin{aligned} 0.08 +_2 1.9 &= 1.98, \\ (-0.08) +_2 1.9 &= 1.82, \\ 80 +_2 44 &= \text{not defined}, \\ 20.36 -_2 0.87 &= 19.49, \\ 1.36 -_2 27.95 &= -26.59, \\ 2.36 -_2 (-99.95) &= \text{not defined}. \end{aligned}$$

Definition 2.2.

Multiplication

$$c \times_2 d = \pm(c_0 \bullet d_0.\, d_1 d_2 + 0.\, c_1 \bullet d_0.\, d_1 + 0.0 c_2 \bullet d_0),$$

where the sign ± is defined as usual in classic arithmetic, · means multiplication in classic arithmetic, and + means addition in classic arithmetic.

Examples of multiplication made by a W_2-observer in W_2:

$$\begin{aligned} 10 \times_2 9 &= 90, \\ (-3) \times_2 16 &= -48, \\ 15 \times_2 11 &= \text{not defined}, \\ 3.41 \times_2 2.64 &= 8.98, \\ 3.41 \times_2 (-2.64) &= -8.98, \\ 5.41 \times_2 22.64 &= \text{not defined}, \\ 98.41 \times_2 1.64 &= \text{not defined}, \\ 0.99 \times_2 0.09 &= 0. \end{aligned}$$

Definition 2.3.

Division

$$c \div_2 d = \begin{cases} r & \text{if } \exists\, r \in W_2 \;\; r \times_2 d = c, \\ \text{not defined} & \text{if no such } r \text{ exists.} \end{cases}$$

Examples of division in W_2 made by a W_2-observer in W_2:

$$80 \div_2 4 = 20,$$
$$2 \div_2 0.5 = \{4, 4.01, 4.02, 4.03, 4.04, 4.05, 4.06, 4.07, 4.08, 4.09\},$$

so that we get 10 different *r*,

$$2 \div_n 3 = \text{not defined},$$

since no *r* exists because

$$3 \times_2 0.66 = 1.98,$$
$$3 \times_2 0.67 = 2.01.$$

Now we introduce arithmetic operations over numbers, elements of W_3. For $c = \pm c_0.\, c_1 c_2 c_3$, $d = \pm d_0.\, d_1 d_2 d_3 \in W_3$, we endow W_3 with the arithmetic $(+_3, -_3, \times_3, \div_3)$ from the W_3-observer point of view.

Definition 2.4.

Addition and subtraction

$$c \pm_3 d = c \pm d$$

if

$$c \pm d \in W_3,$$

and $c \pm_3 d$ is not defined if

$$c \pm d \notin W_3,$$

where $c \pm d$ is the classic arithmetic addition and subtraction.

Examples of addition and subtraction in W_3 made by a W_3-observer in W_3:

$$0.008 +_3 1.09 = 1.098,$$
$$(-0.008) +_3 1.09 = 1.082,$$
$$800 +_3 440 = \text{not defined},$$
$$20.036 -_3 0.087 = 19.949,$$
$$1.036 -_3 27.095 = -26.59,$$
$$2.736 -_3 (-999.195) = \text{not defined}.$$

Definition 2.5.

Multiplication

$$c \times_3 d = \pm(c_0 \bullet d_0.\, d_1 d_2 d_3 + 0.\, c_1 \bullet d_0.\, d_1 d_2 + 0.0 c_2 \bullet d_0.\, d_1 + 0.00 c_3 \bullet d_0),$$

where sign ± is defined as usual in classic arithmetic, · means multiplication in classic arithmetic, and + means addition in classic arithmetic.

Examples of multiplication in W_3 made by a W_3-observer in W_3:

$$\begin{aligned} 100 \times_3 9 &= 900, \\ (-30) \times_3 14 &= -420, \\ 150 \times_3 10 &= \text{not defined}, \\ 3.415 \times_3 2.648 &= 9.036, \\ 3.415 \times_3 (-2.648) &= -9.036, \\ 15.412 \times_3 221.645 &= \text{not defined}, \\ 998.418 \times_3 1.645 &= \text{not defined}, \\ 0.999 \times_3 0.009 &= 0. \end{aligned}$$

Definition 2.6.

Division

$$c \div_3 d = \begin{cases} r & \text{if } \exists\, r \in W_3 \;\; r \times_3 d = c, \\ \text{not defined} & \text{if no such } r \text{ exists.} \end{cases}$$

Examples of division in W_3 made by a W_3-observer in W_3:

$$\begin{aligned} &600 \div_3 4 = 150, \\ &2 \div_3 0.5 = \{4, 4.001, 4.002, 4.003, 4.004, 4.005, 4.006, 4.007, 4.008, 4.009\}, \end{aligned}$$

so that we get 10 different *r*,

$$2 \div_n 3 = \text{not defined},$$

since no *r* exists because

$$\begin{aligned} 3 \times_3 0.666 &= 1.998, \\ 3 \times_3 0.667 &= 2.001. \end{aligned}$$

Generally, we now introduce arithmetic operations over numbers, elements of W_n. For $c = c_0.\, c_1 \ldots c_n$, $d = d_0.\, d_1 \ldots d_n \in W_n$, we endow W_n with the arithmetic $(+_n, -_n, \times_n, \div_n)$ from the W_n-observer point of view.

Definition 2.7.

Addition and subtraction

$$c \pm_n d = \begin{cases} c \pm d & \text{if } c \pm d \in W_n, \\ \text{not defined} & \text{if } c \pm d \notin W_n, \end{cases}$$

where $c \pm d$ is the standard addition and subtraction, and we write

$$\big((\ldots(f_1 +_n f_2)\ldots) +_n f_N\big) = \sum_{i=1}^{N}{}^{n} f_i$$

for $f_1, \ldots, f_N$ iff the contents of any parenthesis are in W_n, $f_1, \ldots, f_N \in W_n$.

Definition 2.8.

Multiplication

$$c \times_n d = \sum_{k=0}^{n}{}^{n} \sum_{m=0}^{n-k}{}^{n} 0.\underbrace{0\ldots0}_{k-1}c_k \cdot 0.\underbrace{0\ldots0}_{m-1}d_m,$$

where

$$c, d \geq 0,$$
$$c_0 \cdot d_0 \in W_n,$$
$$0.\underbrace{0\ldots0}_{k-1}c_k \;\cdot\; 0.\underbrace{0\ldots0}_{m-1}d_m$$

is the standard product, and $k = m = 0$ means that

$$0.\underbrace{0\ldots0}_{k-1}c_k = c_0$$

and

$$0.\underbrace{0\ldots0}_{m-1}d_m = d_0.$$

If either $c < 0$ or $d < 0$, then we compute

$$|c| \times_n |d|$$

and define

$$c \times_n d = \pm|c| \times_n |d|$$

where the sign ± is defined as usual. Note that if the content of at least one parenthesis (in the previous formula) is not in W_n, then $c \times_n d$ is not defined.

Definition 2.9.

Division

$$c \div_n d = \begin{cases} r & \text{if } \exists\, r \in W_n \;\; r \times_n d = c, \\ \text{not defined} & \text{if no such } r \text{ exists.} \end{cases}$$

Observers and arithmetic generate randomness and probability in Mathematics with Observers. Note that the probability of some event in W_n depends on the W_m-observer ($m \geq n$).

We have not classic arithmetic situations in Mathematics with Observers:

1. Additive associativity may fail.

For example, let $20, 90, -30 \in W_2$. Then $20 +_2 90 \notin W_2$, and hence

$$(20 +_2 90) +_2 (-30) \notin W_2$$

and

$$20 +_2 (90 -_2 30) = 80 \in W_2.$$

However, for $10, 20, 30 \in W_2$, we have

$$10 +_2 (20 +_2 30) = (10 +_2 20) +_2 30 = 60 \in W_2.$$

2. Multiplicative associativity may fail.

For example, let $50.12, 0.85, 0.61 \in W_2$. Then

$$50.12 \times_2 0.85 = 42.58; \quad (50.12 \times_2 0.85) \times_2 0.61 = 25.92,$$

whereas

$$0.85 \times_2 0.61 = 0.48; \quad 50.12 \times_2 0.48 = 24.04.$$

However, for $10, 2, 3 \in W_2$, we have

$$10 \times_2 2 = 20; \quad (10 \times_2 2) \times_2 3 = 60.00,$$
$$2 \times_2 3 = 6; \quad 10 \times_2 (2 \times_2 3) = 60.00.$$

3. Distributivity may fail.

For example, let $1.81, 0.74, 0.53 \in W_2$. Then

$$0.74 +_2 0.53 = 1.27; \quad 1.81 \times_2 1.27 = 2.24; \quad 1.81 \times_2 0.74 = 1.3; \quad 1.81 \times_2 0.53 = 0.93,$$

so that

$$1.81 \times_2 0.74 +_2 1.81 \times_2 0.53 = 2.23 \neq 2.24.$$

However, for $10, 2, 3 \in W_2$, we have

$$10 \times_2 (2 +_2 3) = 10 \times_2 2 +_2 10 \times_2 3 = 50.00.$$

We define the space $E_m W_n$ as follows. Consider the Cartesian product of *m* copies of W_n:

$$E_m W_n = \underbrace{W_n \times \cdots \times W_n}_{m}.$$

We call a "vector" any element from

$$E_m W_n : \mathbf{a} = (a_1, \ldots, a_m),$$
$$a_1, \ldots, a_m \in W_n.$$

If

$$\mathbf{a}, \mathbf{b} \in E_m W_n,$$
$$\mathbf{a} = (a_1, \ldots, a_m),$$
$$\mathbf{b} = (b_1, \ldots, b_m), \alpha \in W_n,$$

then we define

$$\mathbf{a} +_n \mathbf{b} = (a_1 +_n b_1, a_2 +_n b_2, \ldots, a_m +_n b_m)$$

if $a_1 +_n b_1, a_2 +_n b_2, \ldots, a_m +_n b_m \in W_n$ and

$$\alpha \times_n \mathbf{a} = (\alpha \times_n a_1, \ldots, \alpha \times_n a_m)$$

if $\alpha, \alpha \times_n a_1, \ldots, \alpha \times_n a_m \in W_n$.

We will use the following notations: **a**, **b**, **c** mean vectors, and *α*, *β* mean scalars.

Addition associativity in $E_m W_n$ does not exist.

There is no associativity of scalar multiplication.

There is no distributivity of scalar multiplication.

There is no distributivity of scalar multiplication for vector sums.

We define the scalar product of vectors

$$\mathbf{a} = (a_1, \ldots, a_m), \ \mathbf{b} = (b_1, \ldots, b_m) \in E_m W_n$$

as the following sum:

$$(\mathbf{a}, \mathbf{b}) = \big(\ldots \big((a_1 \times_n b_1 +_n a_2 \times_n b_2) +_n a_3 \times_n b_3\big) +_n \cdots +_n a_m \times_n b_m\big).$$

The scalar product in $E_m W_n$ is not distributive.

Scalar multiplication on scalar product in $E_m W_n$ is not associative.

The squared length of a vector **a** is

$$|\mathbf{a}|^2 = (\mathbf{a}, \mathbf{a}),$$

but the length itself is calculated as

$$\sqrt{|\mathbf{a}|^2} \in W_n$$

and does not always exists.

The space E_3W_n contains three standard vectors:

$$\begin{aligned} \mathbf{i} &= (1, 0, 0), \\ \mathbf{j} &= (0, 1, 0), \\ \mathbf{k} &= (0, 0, 1). \end{aligned}$$

We have

$$\begin{aligned} (\mathbf{i}, \mathbf{i}) &= (\mathbf{j}, \mathbf{j}) = (\mathbf{k}, \mathbf{k}) = 1, \\ |\mathbf{i}| &= |\mathbf{j}| = |\mathbf{k}| = 1, \\ (\mathbf{i}, \mathbf{j}) &= (\mathbf{i}, \mathbf{k}) = (\mathbf{k}, \mathbf{j}) = 0, \end{aligned}$$

i. e., **i**, **j**, **k** is an orthonormal basis in E_3W_n.

We define the vector product of vectors

$$\mathbf{a} = (a_1, a_2, a_3),\ \mathbf{b} = (b_1, b_2, b_3) \in E_3W_n$$

as the vector

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= (a_2 \times_n b_3 -_n a_3 \times_n b_2) \times_n \mathbf{i} -_n \\ &-_n (a_1 \times_n b_3 -_n a_3 \times_n b_1) \times_n \mathbf{j} +_n \\ &+_n (a_1 \times_n b_2 -_n a_2 \times_n b_1) \times_n \mathbf{k}. \end{aligned}$$

Note that:

The vector product in E_3W_n is not distributive;

Scalar multiplication on the vector product in E_3W_n is not associative;

The equality

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}, \mathbf{c}) \times_n \mathbf{b} -_n (\mathbf{a}, \mathbf{b}) \times_n \mathbf{c}$$

is incorrect in E_3W_n;

The equality

$$(\mathbf{a}, \mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}, \mathbf{c})$$

is incorrect in E_3W_n.

3 Observability and geometry: points, straight lines, planes, and spaces

Let us consider two Cartesian products of W_n:

$$E_2W_n = W_n \times W_n,$$
$$E_3W_n = W_n \times W_n \times W_n.$$

We call "point *A*" any element

$$(x, y), x, y \in W_n,$$

of E_2W_n or any element

$$(x, y, z), x, y, z \in W_n,$$

of E_3W_n.

We call (x, y) and (x, y, z) the coordinates of point $A \in E_2W_n$ and $\in E_3W_n$, respectively, and write

$$A(x, y) \in E_2W_n,$$
$$A(x, y, z) \in E_3W_n.$$

For E_3W_n, we have the standard basis:

$$\mathbf{e}_1 = \mathbf{i} = (1, 0, 0),$$
$$\mathbf{e}_2 = \mathbf{j} = (0, 1, 0),$$
$$\mathbf{e}_3 = \mathbf{k} = (0, 0, 1).$$

For any vector $\mathbf{A} = (x, y, z) \in E_3W_n$, we have

$$\mathbf{A} = x \times_n \mathbf{i} +_n y \times_n \mathbf{j} +_n z \times_n \mathbf{k}.$$

So the coordinates of "point *A*" in E_3W_n coincide with coordinates of the corresponding vector **A**, and for any two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, we have the vector

$$\mathbf{AB} = (x_2 -_n x_1, y_2 -_n y_1, z_2 -_n z_1).$$

For E_2W_n, we have the standard basis

$$\mathbf{e}_1 = \mathbf{i} = (1, 0),$$
$$\mathbf{e}_2 = \mathbf{j} = (0, 1),$$

and for any vector $\mathbf{A} = (x, y) \in E_2W_n$, we have

$$\mathbf{A} = x \times_n \mathbf{i} +_n y \times_n \mathbf{j}.$$

So the coordinates of “point *A*” in E_2W_n coincide with coordinates of the corresponding vector **A**, and for any two points $A(x_1, y_1)$ and $B(x_2, y_2)$ we have the vector

$$\mathbf{AB} = (x_2 -_n x_1, y_2 -_n y_1).$$

We call the “straight line $a \in E_2W_n$” the set of all points $A(x, y) \in E_2W_n$ satisfying the equation

$$a_1 \times_n x +_n a_2 \times_n y +_n a_3 = 0$$

for all

$$a_1, a_2, a_3, a_1 \times_n x, a_2 \times_n y, a_1 \times_n x +_n a_2 \times_n y \in W_n$$

such that $(a_1, a_2) \neq (0, 0)$. Two equations

$$a_1 \times_n x +_n a_2 \times_n y +_n a_3 = 0$$

and

$$b_1 \times_n x +_n b_2 \times_n y +_n b_3 = 0$$

define the same straight line if and only if the set of all points $A(x, y) \in E_2W_n$ satisfying the first equation and the set of all points $A(x, y) \in E_2W_n$ satisfying the second one coincide.

We call the “plane $\alpha \in E_3W_n$” the set of all points $A(x, y, z) \in E_3W_n$ satisfying the equation

$$a_1 \times_n x +_n a_2 \times_n y +_n a_3 \times_n z +_n a_4 = 0$$

for all

$$a_1, a_2, a_3, a_4, a_1 \times_n x, a_2 \times_n y, a_3 \times_n z, a_1 \times_n x +_n a_2 \times_n y, a_1 \times_n x +_n a_2 \times_n y +_n a_3 \times_n z \in$$

such that $(a_1, a_2, a_3) \neq (0, 0, 0)$. Two equations

$$a_1 \times_n x +_n a_2 \times_n y +_n a_3 \times_n z +_n a_4 = 0$$

and

$$b_1 \times_n x +_n b_2 \times_n y +_n b_3 \times_n z +_n b_4 = 0$$

define the same plane if and only if the set of all points $A(x, y, z) \in E_3W_n$ satisfying the first equation and the set of all points $A(x, y, z) \in E_3W_n$ satisfying the second one coincide.

We call the “straight line $a \in E_3W_n$” the set of all points $A(x, y, z) \in E_3W_n$ satisfying the system of equations

$$\begin{cases} a_1 \times_n x +_n a_2 \times_n y +_n a_3 \times_n z +_n a_4 = 0, \\ b_1 \times_n x +_n b_2 \times_n y +_n b_3 \times_n z +_n b_4 = 0 \end{cases}$$

for all $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in W_n$ such that $(a_1, a_2, a_3) \neq (0, 0, 0)$ and $(b_1, b_2, b_3) \neq (0, 0, 0)$, and satisfying the remaining plane conditions, provided that these two planes do not coincide. Two systems of equations

$$\begin{cases} a_1 \times_n x +_n a_2 \times_n y +_n a_3 \times_n z +_n a_4 = 0, \\ b_1 \times_n x +_n b_2 \times_n y +_n b_3 \times_n z +_n b_4 = 0 \end{cases}$$

and

$$\begin{cases} c_1 \times_n x +_n c_2 \times_n y +_n c_3 \times_n z +_n c_4 = 0, \\ d_1 \times_n x +_n d_2 \times_n y +_n d_3 \times_n z +_n d_4 = 0 \end{cases}$$

define the same straight line if and only if the set of all points $A(x, y, z) \in E_3W_n$ satisfying the first system of equations and the set of all points $A(x, y, z) \in E_3W_n$ satisfying the second one coincide.

Note that multiplication and addition in straight line and plane formulas are going up by corresponding pairs from left to right. We also assume that all these elements belong to W_n.

Theorem 3.1.
A straight line $a \in E_2W_n$ is a straight line $a \in E_3W_n$.

Proof.

The set of all points $A(x, y) \in E_2W_n$ satisfying the equation

$$a_1 \times_n x +_n a_2 \times_n y +_n a_3 = 0$$

coincides with the set of all points $A(x, y, z) \in E_3W_n$ satisfying the system of equations

$$\begin{cases} a_1 \times_n x +_n a_2 \times_n y +_n a_3 = 0, \\ z = 0. \end{cases}$$

We have to make several notes.

1. Let us consider two straight lines in E_2W_n:

$$a : x -_n y +_n 1 = 0,$$
$$b : 2 \times_n x -_n 2 \times_n y +_n 2 = 0.$$

In classical geometry, these two straight lines coincide, but in Mathematics with Observers, they do not. For example, if $n = 2$, then

$$A(50, 51) \in a,$$

but

$$A(50, 51) \notin b.$$

2. Let us consider

a:

$$\begin{cases} x -_n y +_n 1 = 0, \\ z = 0; \end{cases}$$

b:

$$\begin{cases} 2 \times_n x -_n 2 \times_n y +_n 2 = 0, \\ z = 0. \end{cases}$$

In classical geometry, these two straight lines coincide, but in Mathematics with Observers, they do not. For example, if $n = 2$, then

$$A(50, 51, 0) \in a,$$

but

$$A(50, 51, 0) \notin b.$$

3. Let us consider two planes in $E_3 W_n$:

α:

$$x +_n y +_n z -_n 3 = 0$$

and

β:

$$2 \times_n x +_n 2 \times_n y +_n 2 \times_n z -_n 6 = 0.$$

In classical geometry, these two planes coincide, but in Mathematics with Observers, they do not. For example, if $n = 2$, then

$$A(50, -50, 3) \in \alpha,$$

but

$$A(50, -50, 3) \notin \beta.$$

4. Let us consider ten different equations of straight lines in $E_2 W_2$:

$$a_1 : 99.99 \times_2 x = 1.98,$$
$$a_2 : 99.98 \times_2 x = 1.98,$$
$$a_3 : 99.97 \times_2 x = 1.98,$$
$$a_4 : 99.96 \times_2 x = 1.98,$$
$$a_5 : 99.95 \times_2 x = 1.98,$$
$$a_6 : 99.94 \times_2 x = 1.98,$$
$$a_7 : 99.93 \times_2 x = 1.98$$
$$a_8 : 99.92 \times_2 x = 1.98,$$
$$a_9 : 99.91 \times_2 x = 1.98,$$
$$a_{10} : 99.90 \times_2 x = 1.98.$$

All these equations describe the same straight line having the set of points $A(0.02, y)$ with any $y \in W_2$.

5. Let s consider ten different equations of planes in E_3W_2:

$$\alpha_1 : 99.99 \times_2 x = 1.98,$$
$$\alpha_2 : 99.98 \times_2 x = 1.98,$$
$$\alpha_3 : 99.97 \times_2 x = 1.98,$$
$$\alpha_4 : 99.96 \times_2 x = 1.98,$$
$$\alpha_5 : 99.95 \times_2 x = 1.98,$$
$$\alpha_6 : 99.94 \times_2 x = 1.98,$$
$$\alpha_7 : 99.93 \times_2 x = 1.98,$$
$$\alpha_8 : 99.92 \times_2 x = 1.98,$$
$$\alpha_9 : 99.91 \times_2 x = 1.98,$$
$$\alpha_{10} : 99.90 \times_2 x = 1.98.$$

All these equations describe the same plane having the set of points $A(0.02, y, z)$ with any $y, z \in W_2$.

6. Let us consider three straight lines in E_2W_2:

$$a : y = x,$$
$$b : 2 \times_2 y = 2 \times_2 x,$$
$$c : 0.1 \times_2 y = 0.1 \times_2 x.$$

We get

$$b \subset a \subset c$$

because

$$
\begin{aligned}
a &= \cup(x, x), \quad x \in W_2, \\
b &= \cup(x, x), \quad x \in [-49.99, -49.98, \ldots, -0.01, 0, 0.01, \ldots, 49.98, 49.99] \subset W_2, \\
c &= \cup(x, y), \quad x = x_0.\, x_1 x_2 \in W_2, \quad y \in [x_0.\, x_1 0, x_0.\, x_1 1, x_0.\, x_1 2, \ldots, x_0.\, x_1 9] \in W_2.
\end{aligned}
$$

Note that in classical geometry, these three straight lines coincide, but in Mathematics with Observers, they do not.

7. Let us consider two straight lines in E_2W_n:

$$
\begin{aligned}
a &: y = k_1 \times_n x +_n l_1 = f(x), \\
b &: y = k_2 \times_n x +_n l_2 = g(x).
\end{aligned}
$$

The functions *f* and *g* are single-valued functions. The superposition of these functions is:

$$
f\big(g(x)\big) = k_1 \times_n (k_2 \times_n x +_n l_2) +_n l_1,
$$

whereas in classical geometry,

$$
y = f\big(g(x)\big) = (k_1 \bullet k_2) \bullet x + (k_1 \bullet l_2 + l_1) = k \bullet x + l,
$$

where

$$
\begin{aligned}
k &= k_1 \bullet k_2, \\
l &= k_1 \bullet l_2 + l_1,
\end{aligned}
$$

and ·, + mean classic multiplication and addition, respectively.

This means that in classical geometry the superposition of functions representing straight lines is again a function representing a straight line.

Let us consider this situation in Mathematics with Observers.

8. Let $n = 2$ and consider two straight lines in E_2W_2:

$$
\begin{aligned}
a &: y = 2 \times_2 x = f(x), \\
b &: y = 3 \times_2 x = g(x).
\end{aligned}
$$

We get the sets *a* and *b* as subsets of E_2W_2:

$$
\begin{aligned}
a &= \big[(-49.99, -99.98), (-49.98, -99.96), \ldots, (-0.01, -0.02), (0, 0), (0.01, 0.02), \ldots \\
&\quad \ldots, (49.99, 99.98)\big], \\
b &= \big[(-33.33, -99.99), (-33.32, -99.96), \ldots, (-0.01, -0.03), (0, 0), (0.01, 0.03), \ldots \\
&\quad \ldots, (33.33, 99.99)\big].
\end{aligned}
$$

Of course, all these sets can see any W_m-observer with $m \geq 5$.

The functions *f* and *g* are single-valued functions. The superposition of these functions is

$$
f\big(g(x)\big) = 2 \times_2 (3 \times_2 x) = (2 \times_2 3) \times_2 x = 6 \times_2 x,
$$

because for any $x \in W_2$ and $r \in Z$ such that $r \times_2 x \in W_2$, the result of multiplication coincides with classic arithmetic multiplication.

We also get the set $f(g(x))$ as a subset of E_2W_2:

$$f\big(g(x)\big) = \big[(-16.66, -99.96), \dots, (-0.01, -0.06), (0, 0), (0.01, 0.06), \dots, (16.66, 99.96)\big].$$

This means that in this case the superposition of functions representing straight lines *a* and *b* is again a function representing a straight line *c*:

$$c : y = 6 \times_2 x.$$

9. Let again $n = 2$ and consider other two straight lines in E_2W_2:

$$a : y = 1.96 \times_2 x = f(x),$$
$$b : y = 2.87 \times_2 x = g(x).$$

The functions *f* and *g* are single-valued functions. The superposition of these functions is

$$f\big(g(x)\big) = 1.96 \times_2 (2.87 \times_2 x).$$

As we know,

$$\delta_3 = \alpha \times_n (\beta \times_n \gamma) -_n (\alpha \times_n \beta) \times_n \gamma, (\alpha, \beta, \gamma \in W_n)$$

is a random variable in W_n, and $\delta_3 = 0$ with probability $P < 1$. This means that in this case the superposition of functions representing straight lines *a* and *b* is not a function representing a straight line.

10. Let again $n = 2$ and consider the straight line in E_2W_2:

$$a : 0.01 \times_2 y = 0 = f(x).$$

We get the set *a* as a subset of E_2W_2:

$$a = \big[(x, -0.99), (x, -0.98), \dots, (x, -0.01), (x, 0), (x, 0.01), \dots, (x, 0.99)\big]$$

for all $x \in W_n$.

Of course, this set can see any W_m-observer with $m \geq 7$.

The function *f* is a multivalued function.

Let us consider the transformation of the parallel shift along the *y*-axis in E_2W_2, for example,

$$y \longrightarrow y -_2 1 = g(y).$$

The superposition of the functions *f* and *g* is

$$f\big(g(y)\big) = 0.01 \times_2 (y -_2 1) = 0,$$

$$y -_2 1 = -0.99, \qquad y = 0.01,$$

$$y -_2 1 = -0.98, \qquad y = 0.0,$$

$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdot$$

$$y -_2 1 = -0.01, \qquad y = 0.99,$$

and we get $y -_2 1 = 0,$ So we have $y = 1,$ So does the set $f(g(y))$

$$y -_2 1 = 0.01, \qquad y = 1.01,$$

$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdot$$

$$y -_2 1 = 0.98, \qquad y = 1.98,$$

$$y -_2 1 = 0.99. \qquad y = 1.99.$$

represent a straight line or not? Since

$$0.01 \times_2 (y -_2 1) = 0,$$

an answer to this question is positive only if the solution of the equation

$$0.01 \times_2 y = 0.01$$

coincides with solution of the equation

$$0.01 \times_2 (y -_2 1) = 0$$

considered above.

However, the equation

$$0.01 \times_2 y = 0.01$$

$$y = 1,$$

$$y = 1.01,$$

has the solutions $\cdots\cdots\cdot$ This means the set $f(g(y))$ does not represent a straight line,

$$y = 1.98,$$

$$y = 1.99.$$

that is, the straight line transformation of parallel shift along the *y*-axis in E_2W_2 may not represent a straight line. □

4 Observability and analysis of connections of points, straight lines, and planes

4.1 First property of connections

Let us consider two distinct points $\in E_2W_n$:

$$A(x_1, y_1), B(x_2, y_2).$$

Questions: Is there a straight line $AB = BA = a \in E_2W_n$ containing these points? Is this line uniquely defined?

Let us consider several examples.

1) Let us take $n = 2$ and $A(0, 2), B(1, 2) \in E_2W_2$. We are looking for a straight line a as a set of points (x, y) satisfying the equation

$$a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0.$$

We have

$$\begin{cases} a_1 \times_2 0 +_2 a_2 \times_2 2 +_2 a_3 = 0, \\ a_1 \times_2 1 +_2 a_2 \times_2 2 +_2 a_3 = 0. \end{cases}$$

We get

$$\begin{cases} a_1 = 0, \\ a_3 = -2 \times_2 a_2, \end{cases}$$

that is, the equation of a straight line a is

$$a_2 \times_2 y = 2 \times_2 a_2.$$

This means the following:

1. For each a_2 such that

$$1 \leq |a_2| \leq 49.99,$$

we get a straight line as the set of points $(x, 2)$ with all $x \in W_2$.

2. For each a_2 such that

$$0.1 \leq |a_2| \leq 0.99,$$

we get a straight line as the set of points

$$\big[(x, 2); (x, 2.01); (x, 2.02); \ldots; (x, 2.09)\big]$$

with all $x \in W_2$.

Note that here and everywhere further, by

$$[u, v, \ldots]$$

we denote the set of elements $u, v, \ldots$.

3. For each a_2 such that

$$0 < |a_2| \le 0.09,$$

we get a straight line as the set of points

$$\big[(x,2);(x,2.01);(x,2.02);\ldots;(x,2.09);(x,2.1);\ldots;(x,2.19);(x,2.2);\ldots;(x,2.99)\big]$$

with all $x \in W_2$.

So the points $A(0,2)$ and $B(1,2)$ determine three different straight lines. This means that two distinct points $\in E_2W_n$ may not uniquely determine a straight line containing these points.

1') Let us take E_3W_2 and the points $A(0,2,0)$ and $B(1,2,0)$.

We are looking for a straight line *a* as the set of points (x,y,z) satisfying the system of equations

$$\begin{cases} a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0, \\ z = 0. \end{cases}$$

We have

$$\begin{aligned} &a_1 \times_2 0 +_2 a_2 \times_2 2 +_2 a_3 = 0, \\ &a_1 \times_2 1 +_2 a_2 \times_2 2 +_2 a_3 = 0, \\ &z = 0. \end{aligned}$$

We get

$$\begin{aligned} &a_1 = 0, \\ &a_3 = -2 \times_2 a_2, \\ &z = 0, \end{aligned}$$

that is, the equation of a straight line *a* is

$$\begin{cases} a_2 \times_2 y = 2 \times_2 a_2, \\ z = 0, \end{cases}$$

which means the following:

1. For each a_2 such that

$$1 \le |a_2| \le 49.99,$$

we get a straight line as the set of points $(x, 2, 0)$ with all $x \in W_2$.

2. For each a_2 such that

$$0.1 \leq |a_2| \leq 0.99,$$

we get a straight line as the set of points

$$\left[(x, 2, 0); (x, 2.01, 0); (x, 2.02, 0); \ldots; (x, 2.09, 0)\right]$$

with all $x \in W_2$.

3. For each a_2 such that

$$0 < |a_2| \leq 0.09,$$

we get a straight line as the set of points

$$\left[(x, 2, 0); (x, 2.01, 0); (x, 2.02, 0); \ldots; (x, 2.09, 0); (x, 2.1, 0); \ldots; (x, 2.19, 0); \ldots; (x, 2.99, 0\right.$$

with all $x \in W_2$.

So the points $A(0, 2, 0)$ and $B(1, 2, 0)$ determine three different straight lines. This means that two distinct points $\in E_3W_n$ may not uniquely determine a straight line containing these points.

2) Let us continue to consider the same question in E_2W_2 and take other two points

$$A(99.99, 0), B(0, 98.88).$$

Again, we are looking for a straight line *a* as the set of points (x, y) satisfying the equation

$$a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0.$$

We have

$$\begin{cases} a_1 \times_2 (99.99) +_2 a_2 \times_2 (0) +_2 a_3 = 0, \\ a_1 \times_2 (0) +_2 a_2 \times_2 (98.88) +_2 a_3 = 0. \end{cases}$$

We must have

$$|a_1| \leq 1, \quad |a_2| \leq 1.01,$$

which means that

$$a_1 \times_2 (99.99) = a_2 \times_2 (98.88).$$

All possible positive a_1 form the set

$$\Phi = [0.01, 0.02, \ldots, 0.99, 1.00],$$

and we get

$$99.99 \times_2 \Phi = [0.99, 1.98, \ldots, 98.82, 99.99].$$

All possible positive a_2 form the set

$$\Psi = [0.01, 0.02, \ldots, 0.99, 1.00, 1.01],$$

and we get

$$98.88 \times_2 \Psi = [0.98, 1.96, \ldots, 97.74, 98.88, 99.86].$$

Direct calculations show that

$$99.99 \times_2 \Phi \cap 98.88 \times_2 \Psi = \Lambda,$$

where Λ here and further is the empty set.

So a straight line *a* containing the points $A(99.99, 0)$ and $B(0, 98.88)$ does not exist, that is, two distinct points $\in E_2 W_n$ may not determine a straight line containing these points.

2') Let us continue to consider same question in $E_3 W_2$ and take other two points

$$A(99.99, 0, 0), B(0, 98.88, 0).$$

Again, we are looking for a straight line *a* as the set of points (x, y, z) satisfying the system of equations

$$\begin{cases} a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0, \\ z = 0. \end{cases}$$

We have

$$\begin{aligned} &a_1 \times_2 (99.99) +_2 a_2 \times_2 (0) +_2 a_3 = 0, \\ &a_1 \times_2 (0) +_2 a_2 \times_2 (98.88) +_2 a_3 = 0, \\ &z = 0. \end{aligned}$$

We must have

$$|a_1| \le 1, \quad |a_2| \le 1.01,$$

which means that

$$a_1 \times_2 (99.99) = a_2 \times_2 (98.88).$$

All possible positive a_1 form the set

$$\Phi = [0.01, 0.02, \ldots, 0.99, 1.00],$$

and we get

$$99.99 \times_2 \Phi = [0.99, 1.98, \ldots, 98.82, 99.99].$$

All possible positive a_2 form the set

$$\Psi = [0.01, 0.02, \ldots, 0.99, 1.00, 1.01],$$

and we get

$$98.88 \times_2 \Psi = [0.98, 1.96, \ldots, 97.74, 98.88, 99.86].$$

Direct calculations show that

$$99.99 \times_2 \Phi \cap 98.88 \times_2 \Psi = \Lambda.$$

So a straight line *a* containing the points $A(99.99, 0, 0)$ and $B(0, 98.88, 0)$ does not exist, that is, two distinct points $\in E_3W_n$ may not determine a straight line containing these points.

3) Let us continue to consider same question in E_2W_2 and take other two points

$$A(99.99, 0), B(0, 98.37).$$

Again, we are looking for a straight line *a* as the set of points (x, y) satisfying the equation

$$a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0.$$

We have

$$\begin{cases} a_1 \times_2 (99.99) +_2 a_2 \times_2 (0) +_2 a_3 = 0, \\ a_1 \times_2 (0) +_2 a_2 \times_2 (98.37) +_2 a_3 = 0. \end{cases}$$

We must have

$$|a_1| \leq 1, \quad |a_2| \leq 1.01,$$

which means that

$$a_1 \times_2 (99.99) = a_2 \times_2 (98.37).$$

Again, all possible positive a_1 form the set

$$\Phi = [0.01, 0.02, \ldots, 0.99, 1.00],$$

and we get

$$99.99 \times_2 \Phi = [0.99, 1.98, \ldots, 98.82, 99.99].$$

Again, all possible positive a_2 form the set

$$\Psi = [0.01, 0.02, \ldots, 0.99, 1.00, 1.01],$$

and we get

$$98.37 \times_2 \Psi = [0.98, 1.96, \ldots, 97.29, 98.37, 99.35].$$

Direct calculations show that

$$99.99 \times_2 \Phi \cap 98.37 \times_2 \Psi = 61.92,$$

and we get only one point in intersection of these two sets, that is,

$$a_1 = 0.62; \quad a_2 = 0.63; \quad a_3 = -61.92.$$

So there is only one straight line *a* containing the points $A(99.99, 0)$ and $B(0, 98.37)$, that is, two distinct points $\in E_2W_n$ uniquely determine a straight line containing these points.

3') Let us continue to consider the same question in E_3W_2 and take other two points

$$A(99.99, 0, 0), B(0, 98.37, 0).$$

Again, we are looking for a straight line *a* as the set of points (x, y, z) satisfying the system of equations

$$\begin{cases} a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0, \\ z = 0. \end{cases}$$

We have

$$\begin{aligned} &a_1 \times_2 (99.99) +_2 a_2 \times_2 (0) +_2 a_3 = 0, \\ &a_1 \times_2 (0) +_2 a_2 \times_2 (98.37) +_2 a_3 = 0, \\ &z = 0. \end{aligned}$$

We must have

$$|a_1| \leq 1, \quad |a_2| \leq 1.01,$$

which means that

$$a_1 \times_2 (99.99) = a_2 \times_2 (98.37).$$

Again, all possible positive a_1 form the set

$$\Phi = [0.01, 0.02, \ldots, 0.99, 1.00],$$

and we get

$$99.99 \times_2 \Phi = [0.99, 1.98, \ldots, 98.82, 99.99].$$

Again, all possible positive a_2 form the set

$$\Psi = [0.01, 0.02, \ldots, 0.99, 1.00, 1.01],$$

and we get

$$98.37 \times_2 \Psi = [0.98, 1.96, \ldots, 97.29, 98.37, 99.35].$$

Direct calculations show that

$$99.99 \times_2 \Phi \cap 98.37 \times_2 \Psi = 61.92,$$

and we get only one point in intersection of these two sets, that is,

$$a_1 = 0.62; \quad a_2 = 0.63; \quad a_3 = -61.92.$$

So there is only one straight line *a* containing the points $A(99.99, 0, 0)$ and (0,98.37,0), that is, two distinct points $\in E_2W_n$ uniquely determine a straight line containing these points.

4) Let us take again $n = 2$ and $A(0, 0), B(1, 1) \in E_2W_2$. We are looking for a straight line *a* as the set of points $(x, y) \in E_2W_2$ satisfying the equation

$$a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0.$$

We have

$$\begin{cases} a_1 \times_2 0 +_2 a_2 \times_2 0 +_2 a_3 = 0, \\ a_1 \times_2 1 +_2 a_2 \times_2 1 +_2 a_3 = 0, \end{cases}$$

that is,

$$\begin{cases} a_3 = 0, \\ a_1 \times_2 1 +_2 a_2 \times_2 1 = 0, \end{cases}$$

that is,

$$\begin{cases} a_3 = 0, \\ a_1 = -a_2, \end{cases}$$

that is,

$$a : a_1 \times_2 x -_2 a_1 \times_2 y = 0.$$

Note that for any positive a_1, line *a* contains points *A* and *B*, but for two different values of a_1, the corresponding lines $a = a_{a_1}$ may be different too. Moreover, if

$$a_1^1 > a_1^2 > 0,$$

then

$$a_{a_1^1} \subset a_{a_1^2}.$$

Conclusion: Mathematics with Observers geometry in E_2W_n and in E_3W_n does not satisfy the first property of connection of classical geometry:

> “Two distinct points *A* and *B* always uniquely determine a straight line *a*: $AB = a$ or $BA = a$ ”.

The probability of correctness of this statement in Mathematics with Observers geometry is less than 1.

We get three different possibilities in Mathematics with Observers geometry in E_2W_n and E_3W_n:

1. Such a straight line exists and uniquely determined.
2. Such a straight line exists but is not uniquely determined.
3. Such a straight line does not exist.

So we proved the following:

Theorem 4.1.
In Mathematics with Observers geometry in plane E_2W_n, there are two distinct points A and B such that the straight line a containing these points does not exist.
Theorem 4.2.
In Mathematics with Observers geometry in plane E_2W_n, there are two distinct points A and B such that the straight line a containing these points exists and is uniquely determined.
Theorem 4.3.
In Mathematics with Observers geometry in plane E_2W_n, there are two distinct points A and B such that the straight line a containing these points exists and is not uniquely determined.
Theorem 4.4.
In Mathematics with Observers geometry in space E_3W_n, there are two distinct points A and B such that the straight line a containing these points does not exist.
Theorem 4.5.
In Mathematics with Observers geometry in space E_3W_n, there are two distinct points A and B such that the straight line a containing these points exists and is uniquely determined.
Theorem 4.6.
In Mathematics with Observers geometry in space E_3W_n, there are two distinct points A and B such that the straight line a containing these points exists and is not uniquely determined.

4.2 Second property of connections

Let us consider three distinct points $\in E_2W_n$:

$$A(x_1, y_1), B(x_2, y_2), C(x_3, y_3).$$

Let a straight line *a* be given by the equation

$$a_1 \times_n x +_n a_2 \times_n y +_n a_3 = 0$$

and contain points *A* and *B*, so that

$$\begin{cases} a_1 \times_n x_1 +_n a_2 \times_n y_1 +_n a_3 = 0, \\ a_1 \times_n x_2 +_n a_2 \times_n y_2 +_n a_3 = 0. \end{cases}$$

Let the same straight line *a* contain points *A* and *C*, so that

$$\begin{cases} a_1 \times_n x_1 +_n a_2 \times_n y_1 +_n a_3 = 0, \\ a_1 \times_n x_3 +_n a_2 \times_n y_3 +_n a_3 = 0. \end{cases}$$

Question: Does the straight line BC coincide with line *a*?

1) Let us continue to consider the same set E_2W_2 and take three points

$$A(0, 98.37), B(99.99, 0), C(99.91, 0).$$

There is unique straight line *a* containing points *A* and *B*, that is, $AB = a$, $BA = a$, and *a* is the set of all points $(x, y) \in E_2W_2$ satisfying the equation

$$0.62 \times_2 x +_2 0.63 \times_2 y -_2 61.92 = 0.$$

Direct calculations show that also point $C \in a$, $AC = a$ and $CA = a$.

Let us consider the straight line BC:

$$a_1 \times_n x +_n a_2 \times_n y +_n a_3 = 0.$$

We have

$$\begin{cases} a_1 \times_2 (99.99) +_2 a_2 \times_2 (0) +_2 a_3 = 0, \\ a_1 \times_2 (99.91) +_2 a_2 \times_2 (0) +_2 a_3 = 0, \end{cases}$$

and one of solutions of this system of equations is line a:

$$a_1 = 0.62; \quad a_2 = 0.63; \quad a_3 = -61.92.$$

However, we also have another set defined by the equation

$$a_1 \times_2 (99.99) = a_1 \times_2 (99.91).$$

The solution of this equation is

$$a_1 \in [0, \pm 0.01, \pm 0.02, \dots, \pm 0.98, \pm 0.99].$$

For $a_1 = 0$, we get

$$a_1 = 0,$$
$$a_3 = 0,$$
$$a_2 \times_n y = 0,$$

which means the following:

1. For each a_2 such that

$$1 \le |a_2| \le 99.99,$$

we get a straight line as the set of points $(x, 0)$ with all $x \in W_2$.

2. For each a_2 such that

$$0.1 \le |a_2| \le 0.99,$$

we get a straight line as the set of points

$$\big[(x, 0); (x, \pm 0.01); (x, \pm 0.02); \dots; (x, \pm 0.09)\big]$$

with all $x \in W_2$.

3. For each a_2 such that

$$0 < |a_2| \le 0.09,$$

we get a straight line as the set of points

$$\left[(x,0);(x,\pm 0.01);(x,\pm 0.02);\ldots;(x,\pm 0.09);(x,\pm 0.1);\ldots;(x,\pm 0.19);\ldots;(x,\pm 0.99)\right]$$
with all $x \in W_2$.

So the points $B(99.99,0)$ and $C(99.91,0)$ determine more than one different straight lines. This means that the straight line BC does not coincide with line *a*, but line *a* is only one of several other lines BC.

1') Let us consider set E_3W_2 and take three points

$$A(0,98.37,0), B(99.99,0,0), C(99.91,0,0).$$

There is a unique straight line *a* containing points *A* and *B*, that is, $AB = a$, $BA = a$, and *a* is the set of all points $(x,y,z) \in E_3W_2$ satisfying the system of equations

$$\begin{cases} 0.62 \times_2 x +_2 0.63 \times_2 y -_2 61.92 = 0, \\ z = 0. \end{cases}$$

Direct calculations show that point $C \in a$, $AC = a$, and $CA = a$.

Let us consider a straight line BC:

$$\begin{cases} a_1 \times_n x +_n a_2 \times_n y +_n a_3 = 0, \\ z = 0. \end{cases}$$

We have

$$\begin{aligned} &a_1 \times_2 (99.99) +_2 a_2 \times_2 (0) +_2 a_3 = 0, \\ &a_1 \times_2 (99.91) +_2 a_2 \times_2 (0) +_2 a_3 = 0, \\ &z = 0, \end{aligned}$$

and one of solutions of this system of equations is line *a*:

$$a_1 = 0.62; \quad a_2 = 0.63; \quad a_3 = -61.92.$$

However, we also have other sets defined by the equation

$$a_1 \times_2 (99.99) = a_1 \times_2 (99.91).$$

A solution of this equation is

$$a_1 \in [0, \pm 0.01, \pm 0.02, \ldots, \pm 0.98, \pm 0.99].$$

For $a_1 = 0$, we get

$$\begin{aligned} &a_1 = 0, \\ &a_3 = 0, \\ &a_2 \times_n y = 0, \\ &z = 0, \end{aligned}$$

which means the following:

1. For each a_2 such that

$$1 \leq |a_2| \leq 99.99,$$

we get a straight line as the set of points $(x, 0, 0)$ with all $x \in W_2$.

2. For each a_2 such that

$$0.1 \leq |a_2| \leq 0.99,$$

we get a straight line as the set of points

$$\left[(x,0,0);(x,\pm 0.01,0);(x,\pm 0.02,0);\ldots;(x,\pm 0.09,0)\right]$$

with all $x \in W_2$.

3. For each a_2 such that

$$0 < |a_2| \leq 0.09,$$

we get a straight line as the set of points

$$\left[(x,0,0);(x,\pm 0.01,0);\ldots;(x,\pm 0.09,0);(x,\pm 0.1,0);\ldots;(x,\pm 0.99,0)\right]$$

with all $x \in W_2$.

So the points $B(99.99, 0, 0)$ and $C(99.91, 0, 0)$ determine more than one different straight line. This means that the straight line BC does not coincide with line *a*, but line *a* is only one of several other lines BC.

2) Let us continue to consider the same set E_2W_2 and take three points

$$A(99.99, 0), B(0, 98.37), C(51.02, 48.16).$$

There is a unique straight line *a* containing points *A* and *B*, that is, $AB = a$, $BA = a$, and *a* is the set of all points $(x, y) \in E_2W_2$ satisfying the equation

$$0.62 \times_2 x +_2 0.63 \times_2 y -_2 61.92 = 0.$$

Direct calculations show that also point $C \in a$. Let us define the line AC and check whether $AC = a$ or not. We have

$$\begin{cases} a_1 \times_2 (99.99) +_2 a_2 \times_2 (0) +_2 a_3 = 0, \\ a_1 \times_2 (51.02) +_2 a_2 \times_2 (48.16) +_2 a_3 = 0. \end{cases}$$

One of the solutions is line *a*:

$$a_1 = 0.62; \quad a_2 = 0.63; \quad a_3 = -61.92.$$

Let us try to find other solutions if they exist. We get

$$a_1 \times_2 (99.99) = a_1 \times_2 (51.02) +_2 a_2 \times_2 (48.16),$$

that is,

$$a_1 \times_2 (99.99) -_n a_1 \times_2 (51.02) = a_2 \times_2 (48.16).$$

We must have

$$|a_1| \in [0, 0.01, 0.02, \ldots, 0.98, 0.99, 1],$$
$$|a_2| \in [0, 0.01, 0.02, \ldots, 0.98, 0.99, 1, 1.01, \ldots, 1.99, 2, 2.01, \ldots, 2.07].$$

Direct calculations show the following:

1. If

$$a_1 \in [\pm 0.01, \ldots, \pm 0.09],$$

then

$$a_1 = a_2,$$

and

$$a_1 = \pm 0.01,$$
$$a_2 = \pm 0.01,$$
$$a_3 = \mp 0.99,$$
$$\ldots$$
$$a_1 = \pm 0.09,$$
$$a_2 = \pm 0.09,$$
$$a_3 = \mp 8.91.$$

For example, let us consider the straight line *b*:

$$0.01 \times_n x +_n 0.01 \times_n y -_n 0.99 = 0.$$

Direct calculations show that the point $(30, 69) \in b$ but

$$0.62 \times_2 30 +_2 0.63 \times_2 69 -_2 61.92 \neq 0.$$

This means that *a* and *b* are different lines.

2. If

$$a_1 \in [\pm 0.61, \ldots, \pm 0.69],$$

then for positive a_1,

$$a_2 = a_1 -_n 0.01,$$

and for negative a_1,

$$a_2 = a_1 +_n 0.01,$$

and

$$a_1 = 0.61,$$
$$a_2 = 0.60,$$
$$a_3 = -60.93,$$
$$\dots$$
$$a_1 = 0.69,$$
$$a_2 = 0.68,$$
$$a_3 = -68.85.$$

For negative a_1,

$$a_2 = a_1 +_n 0.01,$$

and we have

$$a_1 = -0.61,$$
$$a_2 = -0.60,$$
$$a_3 = 60.93,$$
$$\dots$$
$$a_1 = -0.69,$$
$$a_2 = -0.68,$$
$$a_3 = 68.85.$$

For example, let us consider the straight line *c*:

$$0.61 \times_n x +_n 0.60 \times_n y -_n 60.93 = 0.$$

Direct calculations show that the point $(50.01, 49.99) \in c$ but

$$0.62 \times_2 50.01 +_2 0.63 \times_2 49.99 -_2 61.92 \neq 0.$$

This means that *a* and *c* are different lines.

2') Let us consider the set E_3W_2 and take three points

$$A(99.99, 0, 0), B(0, 98.37, 0), C(51.02, 48.16, 0).$$

There is a unique straight line *a* containing points *A* and *B*, i. e., $AB = a$, $BA = a$, and *a* is the set of all points $(x, y, z) \in E_3W_2$ satisfying the system of equations

$$\begin{cases} 0.62 \times_2 x +_2 0.63 \times_2 y -_2 61.92 = 0, \\ z = 0. \end{cases}$$

Direct calculations show that also point $C \in a$.

Let us define the line AC and check whether $AC = a$ or not. We have

$$\begin{aligned}
&a_1 \times_2 (99.99) +_2 a_2 \times_2 (0) +_2 a_3 = 0,\\
&a_1 \times_2 (51.02) +_2 a_2 \times_2 (48.16) +_2 a_3 = 0,\\
&z = 0.
\end{aligned}$$

One of the solutions is line *a*:

$$a_1 = 0.62; \quad a_2 = 0.63; \quad a_3 = -61.92.$$

Let us try to find other solutions if they exist. We get

$$a_1 \times_2 (99.99) = a_1 \times_2 (51.02) +_2 a_2 \times_2 (48.16),$$

that is,

$$a_1 \times_2 (99.99) -_n a_1 \times_2 (51.02) = a_2 \times_2 (48.16).$$

We must have

$$\begin{aligned}
&|a_1| \in [0, 0.01, 0.02, \ldots, 0.98, 0.99, 1],\\
&|a_2| \in [0, 0.01, 0.02, \ldots, 0.98, 0.99, 1, 1.01, \ldots, 1.99, 2, 2.01, \ldots, 2.07].
\end{aligned}$$

Direct calculations show the following:

1. If

$$a_1 \in [\pm 0.01, \ldots, \pm 0.09],$$

then

$$a_1 = a_2,$$

and

$$\begin{aligned}
a_1 &= \pm 0.01,\\
a_2 &= \pm 0.01,\\
a_3 &= \mp 0.99,\\
&\ldots\\
a_1 &= \pm 0.09,\\
a_2 &= \pm 0.09,\\
a_3 &= \mp 8.91.
\end{aligned}$$

For example, let us consider the straight line *b*:

$$\begin{cases} 0.01 \times_n x +_n 0.01 \times_n y -_n 0.99 = 0,\\ z = 0. \end{cases}$$

Direct calculations show that the point $(30, 69, 0) \in b$ but

$$0.62 \times_2 30 +_2 0.63 \times_2 69 -_2 61.92 \neq 0.$$

This means that *a* and *b* are different lines.

2. If

$$a_1 \in [\pm 0.61, \ldots, \pm 0.69],$$

then for positive a_1,

$$a_2 = a_1 -_n 0.01,$$

and

$$\begin{aligned} a_1 &= 0.61, \\ a_2 &= 0.60, \\ a_3 &= -60.93, \\ &\ldots \\ a_1 &= 0.69, \\ a_2 &= 0.68, \\ a_3 &= -68.85. \end{aligned}$$

For negative a_1,

$$a_2 = a_1 +_n 0.01,$$

and we have

$$\begin{aligned} a_1 &= -0.61, \\ a_2 &= -0.60, \\ a_3 &= 60.93, \\ &\ldots \\ a_1 &= -0.69, \\ a_2 &= -0.68, \\ a_3 &= 68.85. \end{aligned}$$

For example, let us consider the straight line *c*:

$$\begin{cases} 0.61 \times_n x +_n 0.60 \times_n y -_n 60.93 = 0, \\ z = 0. \end{cases}$$

Direct calculations show that the point $(50.01, 49.99, 0) \in c$ but

$$0.62 \times_2 50.01 +_2 0.63 \times_2 49.99 -_2 61.92 \neq 0.$$

This means that *a* and *c* are different lines.

3) Let us take again E_2W_2 and $A(0,2)$, $B(1,2)$, $C(2,2)$. There are three straight lines containing points $A(0,2)$ and $B(1,2)$:

The equation of line AB is

$$a_2 \times_2 y = 2 \times_2 a_2,$$

and we get the following:

1. For each a_2 such that

$$1 \leq |a_2| \leq 49.99,$$

we get a straight line as the set of points $(x, 2)$ with all $x \in W_2$.

2. For each a_2 such that

$$0.1 \leq |a_2| \leq 0.99,$$

we get straight line as the set of points

$$\left[(x,2);(x,2.01);(x,2.02);\ldots;(x,2.09)\right]$$

with all $x \in W_2$.

3. For each a_2 such that

$$0 < |a_2| \leq 0.09,$$

we get a straight line as the set of points

$$\left[(x,2);(x,2.01);(x,2.02);\ldots;(x,2.09);(x,2.1);\ldots;(x,2.19);(x,2.2);\ldots;(x,2.99)\right]$$

with all $x \in W_2$.

When we consider the straight lines containing points $A(0,2)$ and $C(2,2)$, we get

$$\begin{cases} a_1 \times_2 0 +_2 a_2 \times_2 2 +_2 a_3 = 0, \\ a_1 \times_2 2 +_2 a_2 \times_2 2 +_2 a_3 = 0, \end{cases}$$

and

$$\begin{cases} a_1 = 0, \\ a_3 = -2 \times_2 a_2, \end{cases}$$

that is, a straight line equation is

$$a_2 \times_2 y = 2 \times_2 a_2.$$

This means that points *A* and *C* define the same three straight lines as points *A* and *B*.

When we consider the straight lines containing points $B(1,2)$ and $C(2,2)$, we get

$$\begin{cases} a_1 \times_2 1 +_2 a_2 \times_2 2 +_2 a_3 = 0, \\ a_1 \times_2 2 +_2 a_2 \times_2 2 +_2 a_3 = 0, \end{cases}$$

and

$$\begin{cases} a_1 = 0, \\ a_3 = -2 \times_2 a_2, \end{cases}$$

that is, a straight line equation is

$$a_2 \times_2 y = 2 \times_2 a_2$$

So, in this case, we have the situation where three distinct points $\in E_2 W_n$,

$$A(0,2), B(1,2), C(2,2),$$

define the same set of three different straight lines in any pair combination.

3') Let us take $E_3 W_2$ and $A(0,2,0)$, $B(1,2,0)$, $C(2,2,0)$. There are three straight lines containing the points $A(0,2,0)$ and $B(1,2,0)$:

The equation of line AB is

$$\begin{cases} a_2 \times_2 y = 2 \times_2 a_2, \\ z = 0, \end{cases}$$

and we get the following:

1. For each a_2 such that

$$1 \le |a_2| \le 49.99,$$

we get a straight line as the set of points $(x, 2, 0)$ with all $x \in W_2$.

2. For each a_2 such that

$$0.1 \le |a_2| \le 0.99,$$

we get a straight line as the set of points

$$[(x,2,0); (x,2.01,0); (x,2.02,0); \ldots; (x,2.09,0)]$$

with all $x \in W_2$.

3. For each a_2 such that

$$0 < |a_2| \le 0.09,$$

we get a straight line as the set of points

$$[(x,2,0); (x,2.01,0); \ldots; (x,2.09,0); (x,2.1,0); \ldots; (x,2.19,0); (x,2.2,0); \ldots; (x,2.99,0$$

with all $x \in W_2$.

When we consider the straight lines containing the points $A(0,2,0)$ and $C(2,2,0)$, we get

$$a_1 \times_2 0 +_2 a_2 \times_2 2 +_2 a_3 = 0,$$
$$a_1 \times_2 2 +_2 a_2 \times_2 2 +_2 a_3 = 0,$$
$$z = 0,$$

and

$$\begin{cases} a_1 = 0, \\ a_3 = -2 \times_2 a_2, \end{cases}$$

that is, a straight line equation is

$$\begin{cases} a_2 \times_2 y = 2 \times_2 a_2, \\ z = 0, \end{cases}$$

which means that points *A* and *C* define the same three straight lines as points *A* and *B*.

When we consider the straight lines containing the points $B(1,2,0)$ and $C(2,2,0)$, we get

$$a_1 \times_2 1 +_2 a_2 \times_2 2 +_2 a_3 = 0,$$
$$a_1 \times_2 2 +_2 a_2 \times_2 2 +_2 a_3 = 0,$$
$$z = 0,$$

and

$$\begin{cases} a_1 = 0, \\ a_3 = -2 \times_2 a_2, \end{cases}$$

that is, a straight line equation is

$$\begin{cases} a_2 \times_2 y = 2 \times_2 a_2, \\ z = 0. \end{cases}$$

So, in this case, we have the situation where three distinct points $\in E_3 W_n$,

$$A(0,2,0), B(1,2,0), C(2,2,0),$$

define same set of three different straight lines in any pair combination.

So we have proved the following:

Theorem 4.7.
In Mathematics with Observers geometry in plane $E_2 W_n$, there are three distinct points A, B, C such that the straight line $a = AB$ containing these points exists and is uniquely determined, but there is more than one line $b = AC$, that is, $AB \neq AC$.

Theorem 4.8.
In Mathematics with Observers geometry in plane $E_2 W_n$, there are three distinct points A, B, C such that the straight line $a = AB = AC$ containing these points exists and is uniquely determined, but there is more than one line $b = BC$, i. e., $AB \neq BC$.

Theorem 4.9.

In Mathematics with Observers geometry in plane E_2W_n, there are three distinct points A, B, C such that there are more than one straight line AB containing these points, more than one straight line AC, and more than one straight line BC, but these three sets of straight lines coincide.

Theorem 4.10.

In Mathematics with Observers geometry in space E_3W_n, there are three distinct points A, B, C such that the straight line $a = AB$ containing these points exists and is uniquely determined, but there is more than one line $b = AC$, that is, $AB \neq AC$.

Theorem 4.11.

In Mathematics with Observers geometry in space E_3W_n, there are three distinct points A, B, C such that the straight line $a = AB = AC$ containing these points exists and is uniquely determined, but there is more than one line $b = BC$, that is, $AB \neq BC$.

Theorem 4.12.

In Mathematics with Observers geometry in space E_3W_n, there are three distinct points A, B, C such that there are more than one straight line AB containing these points, more than one straight line AC, and more than one straight line BC, but these three sets of straight lines coincide.

4.3 Third property of connections

Let us consider three distinct points $\in E_3W_n$:

$$A(x_1, y_1, z_1), B(x_2, y_2, z_2), C(x_3, y_3, z_3)$$

such that the vectors

$$\mathbf{AB} = (x_2 -_n x_1, y_2 -_n y_1, z_2 -_n z_1), \quad \mathbf{AC} = (x_3 -_n x_1, y_3 -_n y_1, z_3 -_n z_1)$$

are not parallel.

Questions: Is there a plane $ABC = \alpha \in E_3W_n$ containing these points? Is this plane uniquely defined?

We are looking for the “plane $ABC = \alpha \in E_3W_n$” as the set of all points $A(x, y, z) \in E_3W_n$ satisfying the equation

$$a_1 \times_n x +_n a_2 \times_n y +_n a_3 \times_n z +_n a_4 = 0$$

for all $a_1, a_2, a_3, a_4 \in W_n$ such that $(a_1, a_2, a_3) \neq (0, 0, 0)$.

1) Let us first consider three distinct points $\in E_3W_2$:

$$A(1, 0, 0), B(0, 1, 0), C(0, 0, 1).$$

In this case, the vectors

$$\mathbf{AB} = (-1, 1, 0), \quad \mathbf{AC} = (-1, 0, 1)$$

are not parallel. We get the system

$$a_1 \times_2 1 +_2 a_2 \times_2 0 +_2 a_3 \times_2 0 +_2 a_4 = 0,$$
$$a_1 \times_2 0 +_2 a_2 \times_2 1 +_2 a_3 \times_2 0 +_2 a_4 = 0,$$
$$a_1 \times_2 0 +_2 a_2 \times_2 0 +_2 a_3 \times_2 1 +_2 a_4 = 0,$$

and so

$$a_1 = a_2,$$
$$a_1 = a_3,$$
$$a_4 = -a_1.$$

So the equation of plane α in this case is

$$a_1 \times_2 x +_2 a_1 \times_2 y +_2 a_1 \times_2 z -_2 a_1 = 0.$$

For $a_1 = 1$, we get plane α_1 with equation

$$x +_2 y +_2 z -_2 1 = 0.$$

For $a_1 = 0.01$, we get plane α_2 with equation

$$0.01 \times_2 x +_2 0.01 \times_2 y +_2 0.01 \times_2 z -_2 0.01 = 0.$$

Let us take the point $D(0.2, 0.2, 0.6) \in E_3W_2$. We get

$$0.2 +_2 0.2 +_2 0.6 -_2 1 = 0.$$

So

$$D \in \alpha_1,$$

but

$$0.01 \times_2 0.2 +_2 0.01 \times_2 0.02 +_2 0.01 \times_2 0.06 -_2 0.01 \neq 0,$$

and thus

$$\alpha_1 \neq \alpha_2.$$

This means that three points

$$A(1,0,0), B(0,1,0), C(0,0,1) \in E_3W_2$$

not completely (not uniquely) determine the plane $ABC = \alpha$. Note that plane α_2 contains not only points A, B, C, but also other points, for example, the point $L(1,-1,1)$.

1A) Let us again consider three distinct points $\in E_3W_2$:

$$A(1,0,0), B(0,1,0), C(0,0,1).$$

We again get the system

$$a_1 \times_2 1 +_2 a_2 \times_2 0 +_2 a_3 \times_2 0 +_2 a_4 = 0,$$
$$a_1 \times_2 0 +_2 a_2 \times_2 1 +_2 a_3 \times_2 0 +_2 a_4 = 0,$$
$$a_1 \times_2 0 +_2 a_2 \times_2 0 +_2 a_3 \times_2 1 +_2 a_4 = 0,$$

and so

$$a_1 = a_2,$$
$$a_1 = a_3,$$
$$a_4 = -a_1.$$

So the equation of plane *α* in this case is

$$a_1 \times_2 x +_2 a_1 \times_2 y +_2 a_1 \times_2 z -_2 a_1 = 0.$$

Note that for any positive a_1, plane *α* contains points *A*, *B*, and *C*, but for two different values of a_1, the corresponding planes $\alpha = \alpha_{a_1}$ may be different too. Moreover, if

$$a_1^1 > a_1^2 > 0,$$

then

$$\alpha_{a_1^1} \subset \alpha_{a_1^2}.$$

If we consider all situations

$$a_1 \in [0.01, 0.09], [0.10, 0.19], \dots, [0.90, 0.99]. [1.00, 98.99], [99.00, 99.09], \dots, [99.90, 99.99],$$

then we get solutions for each case.

2) Let us continue to consider the same question in E_3W_2 and take other three distinct points

$$A(99.99, 0, 0), B(0, 98.88, 0), C(0, 0, 1).$$

In this case, the vectors

$$\mathbf{AB} = (-99.99, 98.88, 0), \quad \mathbf{AC} = (-99.99, 0, 1)$$

are not parallel.

Again, we are looking for the "plane $ABC = \alpha \in E_3W_2$" as the set of all points $A(x, y, z) \in E_3W_2$ satisfying the equation

$$a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 \times_2 z +_2 a_4 = 0$$

for all $a_1, a_2, a_3, a_4 \in W_2$ such that $(a_1, a_2, a_3) \neq (0, 0, 0)$. We get the system

$$a_1 \times_2 99.99 +_2 a_2 \times_2 0 +_2 a_3 \times_2 0 +_2 a_4 = 0,$$
$$a_1 \times_2 0 +_2 a_2 \times_2 98.88 +_2 a_3 \times_2 0 +_2 a_4 = 0,$$
$$a_1 \times_2 0 +_2 a_2 \times_2 0 +_2 a_3 \times_2 1 +_2 a_4 = 0,$$

or

$$a_1 \times_2 99.99 +_2 a_4 = 0,$$
$$a_2 \times_2 98.88 +_2 a_4 = 0,$$
$$a_3 \times_2 1 +_2 a_4 = 0.$$

We must have

$$|a_1| \le 1, \quad |a_2| \le 1.01,$$

and we have

$$a_1 \times_2 (99.99) = a_2 \times_2 (98.88).$$

All possible positive a_1 form the set

$$\Phi = [0.01, 0.02, \dots, 0.99, 1.00],$$

and we get

$$99.99 \times_2 \Phi = [0.99, 1.98, \dots, 98.82, 99.99].$$

All possible positive a_2 form the set

$$\Psi = [0.01, 0.02, \dots, 0.99, 1.00, 1.01],$$

and we get

$$98.88 \times_2 \Psi = [0.98, 1.96, \dots, 97.74, 98.88, 99.86].$$

Direct calculations show that

$$99.99 \times_2 \Phi \cap 98.88 \times_2 \Psi = \Lambda.$$

This means that for three distinct points

$$A(99.99, 0, 0), B(0, 98.88, 0), C(0, 0, 1) \in E_3W_2,$$

which form not parallel vectors

$$\mathbf{AB} = (-99.99, 98.88, 0), \quad \mathbf{AC} = (-99.99, 0, 1),$$

the plane $ABC \in E_3W_2$ does not exist.

3) Let us continue to consider the same question in E_3W_2 and take other three distinct points

$$A(99.99, 0, 0), B(0, 98.37, 0), C(0, 0, 1).$$

In this case the vectors

$$\mathbf{AB} = (-99.99, 98.37, 0), \quad \mathbf{AC} = (-99.99, 0, 1)$$

are not parallel.

Again, we are looking for the "plane $ABC = \alpha \in E_3W_2$" as the set of all points $A(x, y, z) \in E_3W_2$ satisfying the equation

$$a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 \times_2 z +_2 a_4 = 0$$

for all $a_1, a_2, a_3, a_4 \in W_2$ such that $(a_1, a_2, a_3) \neq (0, 0, 0)$. We get the system

$$a_1 \times_2 99.99 +_2 a_2 \times_2 0 +_2 a_3 \times_2 0 +_2 a_4 = 0,$$
$$a_1 \times_2 0 +_2 a_2 \times_2 98.37 +_2 a_3 \times_2 0 +_2 a_4 = 0,$$
$$a_1 \times_2 0 +_2 a_2 \times_2 0 +_2 a_3 \times_2 1 +_2 a_4 = 0,$$

or

$$a_1 \times_2 99.99 +_2 a_4 = 0,$$
$$a_2 \times_2 98.37 +_2 a_4 = 0,$$
$$a_3 +_2 a_4 = 0.$$

We must have

$$|a_1| \leq 1, \quad |a_2| \leq 1.01,$$

and thus

$$a_1 \times_2 (99.99) = a_2 \times_2 (98.37).$$

We get the following:

All possible positive a_1 form the set

$$\Phi = [0.01, 0.02, \ldots, 0.99, 1.00],$$

and we get

$$99.99 \times_2 \Phi = [0.99, 1.98, \ldots, 98.82, 99.99].$$

All possible positive a_2 form the set

$$\Psi = [0.01, 0.02, \ldots, 0.99, 1.00, 1.01],$$

and we get

$$98.37 \times_2 \Psi = [0.98, 1.96, \ldots, 97.29, 98.37, 99.35].$$

Direct calculations show that

$$99.99 \times_2 \Phi \cap 98.37 \times_2 \Psi = 61.92,$$

and we get only one point in intersection of these two sets, that is,

$$a_1 = 0.62; \quad a_2 = 0.63; \quad a_3 = 61.92; \quad a_4 = -61.92.$$

So there is only one plane α containing the points $A(99.99, 0, 0)$, $B(0, 98.37, 0)$, $C(0, 0, 1)$, that is, three distinct points $\in E_3W_2$ with not parallel vectors $\mathbf{AB}$ and $\mathbf{AC}$ uniquely determine a plane containing these points.

So we have proved the following:

Theorem 4.13.
In Mathematics with Observers geometry in space E_3W_n, there are three distinct points A, B, C with not parallel vectors $\mathbf{AB}$ and $\mathbf{AC}$ such that there is no plane containing these points.
Theorem 4.14.
In Mathematics with Observers geometry in space E_3W_n, there are three distinct points A, B, C with not parallel vectors $\mathbf{AB}$ and $\mathbf{AC}$ such that there is a unique plane containing these points.
Theorem 4.15.
In Mathematics with Observers geometry in space E_3W_n, there are three distinct points A, B, C with not parallel vectors $\mathbf{AB}$ and $\mathbf{AC}$ such that there is more than one plane containing these points.

4.4 Fourth property of connections

Let us consider any plane $\alpha \in E_3W_n$ and take any three distinct points in this plane:

$$A(x_1, y_1, z_1), B(x_2, y_2, z_2), C(x_3, y_3, z_3)$$

such that the vectors

$$\mathbf{AB} = (x_2 -_n x_1, y_2 -_n y_1, z_2 -_n z_1), \quad \mathbf{AC} = (x_3 -_n x_1, y_3 -_n y_1, z_3 -_n z_1)$$

are not parallel.

Question: Do these three points *A*, *B*, *C* of a plane *α* completely determine that plane?

1) Let us consider this question in E_3W_2 and take the plane *α*:

$$0.62 \times_2 x +_2 0.63 \times_2 y +_2 61.92 \times_2 z -_2 61.92 = 0.$$

As we have shown in Section →4.3, for three distinct points

$$A(99.99, 0, 0), B(0, 98.37, 0), C(0, 0, 1) \in \alpha,$$

the vectors

$$\mathbf{AB} = (-99.99, 98.37, 0), \quad \mathbf{AC} = (-99.99, 0, 1)$$

are not parallel. We also proved there that there is only one plane *α* containing the points $A(99.99, 0, 0)$, $B(0, 98.37, 0)$, $C(0, 0, 1)$, that is, three distinct points $\in E_3W_2$ with not parallel vectors $\mathbf{AB}$ and $\mathbf{AC}$ uniquely determine a plane containing these points.

2) Let us continue to consider this question in E_3W_2 and take a plane *α*:

$$x +_2 y +_2 z -_2 1 = 0.$$

As we have shown, three distinct points

$$A(1, 0, 0), B(0, 1, 0), C(0, 0, 1) \in \alpha$$

define the nonparallel vectors

$$\mathbf{AB} = (-1, 1, 0), \quad \mathbf{AC} = (-1, 0, 1).$$

We also proved there that these three points not completely (not uniquely) determine a plane *α*.

For example, plane *β*:

$$0.01 \times_2 x +_2 0.01 \times_2 y +_2 0.01 \times_2 z -_2 0.01 = 0$$

also contains the same three points *A*, *B*, *C*, but $\alpha \neq \beta$.

So we have proved the following:

Theorem 4.16.
In Mathematics with Observers geometry in space E_3W_n, there are a plane α and three distinct points $A, B, C \in \alpha$ with nonparallel vectors $\mathbf{AB}$ and $\mathbf{AC}$ such that only a unique plane α contains these points.

Theorem 4.17.
In Mathematics with Observers geometry in space E_3W_n, there are a plane α and three distinct points $A, B, C \in \alpha$ with nonparallel vectors $\mathbf{AB}$ and $\mathbf{AC}$ such that there is more than one plane containing these points.

4.5 Fifth property of connections

Let us consider two planes $\alpha \in E_3W_n$ and $\beta \in E_3W_n$, where *α* is the set of points (x, y, z) satisfying the equation

$$a_1 \times_n x +_n a_2 \times_n y +_n a_3 \times_n z +_n a_4 = 0,$$

and *β* is the set of points (x, y, z) satisfying the equation

$$b_1 \times_n x +_n b_2 \times_n y +_n b_3 \times_n z +_n b_4 = 0$$

for given $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in W_n$ such that $(a_1, a_2, a_3) \neq (0, 0, 0)$ and $(b_1, b_2, b_3) \neq (0, 0, 0)$.

Let the straight line $a \in E_3W_n$ be the set of all points $A(x, y, z) \in E_3W_n$ satisfying the system of equations

$$\begin{cases} a_1 \times_n x +_n a_2 \times_n y +_n a_3 \times_n z +_n a_4 = 0, \\ b_1 \times_n x +_n b_2 \times_n y +_n b_3 \times_n z +_n b_4 = 0. \end{cases}$$

Question: If two points *A*, *B* of a straight line *a* lie in a plane *γ*, then does every point of *a* lie in *γ*?

1) Let the plane $\alpha \in E_3W_2$ be the set of points (x, y, z) satisfying the equation

$$y -_2 1 = 0,$$

and let the plane $\beta \in E_3W_2$ be the set of points (x, y, z) satisfying the equation

$$z -_2 1 = 0$$

So the straight line $a \in E_3W_2$ is the set of all points $A \in E_3W_2$ with coordinates $(x, 1, 1)$, where *x* is any element $\in W_2$.

Let the plane $\gamma \in E_3W_2$ be the set of points (x, y, z) satisfying the equation

$$0.01 \times_2 x +_2 y -_2 z -_2 0.99 = 0.$$

Let us take two points *A*, *B* of a straight line *a*:

$$A(99.99, 1, 1), B(99.31, 1, 1).$$

For both points, we have

$$0.99 +_2 1 -_2 1 = 0.99.$$

So

$$A \in \gamma, \quad B \in \gamma.$$

Now let s take the third point *C* of a straight line *a*:

$$C(48.61, 1, 1).$$

Then we have

$$0.48 +_2 1 -_2 1 = 0.48 \neq 0.99.$$

This means that point *C* does not belong to plane *γ*. So, in this case, we get the negative answer for the question above.

2) It is clear that if we take any of two planes $\alpha \in E_3W_n$ or $\beta \in E_3W_n$ determining a straight line *a* as a plane *γ* (i. e., $\alpha = \gamma$ or $\beta = \gamma$) and take two points *A*, *B* of this straight line *a*, then every point *C* of *a* lies in *α* or *β*.

So, in this case, we get positive answer for the question above. Thus we have proved the following:

Theorem 4.18.
In Mathematics with Observers geometry in space E_3W_n, there are a plane γ and a straight line a with two distinct points $A, B \in a \cap \gamma$ such that any point $C \in a$ belongs to plane γ, that is, $a \in \gamma$.

Theorem 4.19.
In Mathematics with Observers geometry in space E_3W_n, there are a plane γ and a straight line a with two distinct points $A, B \in a \cap \gamma$ such that there is a point $C \in a$ that does not belong plane to γ, that is, $a \notin \gamma$.

4.6 Sixth property of connections

Let us consider two planes $\alpha \in E_3W_n$ and $\beta \in E_3W_n$, where α is the set of points (x, y, z) satisfying the equation

$$a_1 \times_n x +_n a_2 \times_n y +_n a_3 \times_n z +_n a_4 = 0,$$

and β is the set of points (x, y, z) satisfying the equation

$$b_1 \times_n x +_n b_2 \times_n y +_n b_3 \times_n z +_n b_4 = 0$$

for given $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in W_n$ such that $(a_1, a_2, a_3) \neq (0, 0, 0)$ and $(b_1, b_2, b_3) \neq (0, 0, 0)$.

Suppose these two planes have a common point $A(x_1, y_1, z_1)$, that is, point A satisfies the system of equations

$$\begin{cases} a_1 \times_n x +_n a_2 \times_n y +_n a_3 \times_n z +_n a_4 = 0, \\ b_1 \times_n x +_n b_2 \times_n y +_n b_3 \times_n z +_n b_4 = 0, \end{cases}$$

that is,

$$\begin{cases} a_1 \times_n x_1 +_n a_2 \times_n y_1 +_n a_3 \times_n z_1 +_n a_4 = 0, \\ b_1 \times_n x_1 +_n b_2 \times_n y_1 +_n b_3 \times_n z_1 +_n b_4 = 0. \end{cases}$$

Question: Does this system always have at least a second solution, point $B(x_2, y_2, z_2)$?

1) Let us consider two planes $\alpha \in E_3W_2$ and $\beta \in E_3W_2$, where α is the set of points (x, y, z) satisfying the equation

$$99.99 \times_2 x -_2 98.88 \times_2 y +_2 z = 0,$$

and β is the set of points (x, y, z) satisfying the equation

$$z = 0.$$

These two planes have a common point $A(0, 0, 0)$. Let us see whether there a second common point $B(x_2, y_2, z_2)$.

Point B has to satisfy the system of equations

$$\begin{cases} 99.99 \times_2 x -_2 98.88 \times_2 y +_2 z = 0, \\ z = 0, \end{cases}$$

and we have

$$99.99 \times_2 x -_2 98.88 \times_2 y = 0.$$

We must have

$$|x| \le 1, \quad |y| \le 1.01.$$

All possible positive *x* form the set

$$\Phi = [0.01, 0.02, \ldots, 0.99, 1.00],$$

and we get

$$99.99 \times_2 \Phi = [0.99, 1.98, \ldots, 98.82, 99.99].$$

All possible positive *y* form the set

$$\Psi = [0.01, 0.02, \ldots, 0.99, 1.00, 1.01],$$

and we get

$$98.88 \times_2 \Psi = [0.98, 1.96, \ldots, 97.74, 98.88, 99.86].$$

Direct calculations show that

$$99.99 \times_2 \Phi \cap 98.88 \times_2 \Psi = \Lambda.$$

So point *B* does not exist, that is, in this case, planes *α* and *β* have only one common point.

2) Let us take two planes $\alpha \in E_3W_2$ and $\beta \in E_3W_2$, where *α* is the set of points (x, y, z) satisfying the equation

$$x = 0,$$

and *β* is the set of points (x, y, z) satisfying the equation

$$z = 0.$$

These two planes have a common point $A(0, 0, 0)$, and the set of common points of planes *α* and *β* is $[B(0, y, 0)]$, where *y* is any element of W_n, that is, there is more than one in common point *B*.

So we have proved the following:

Theorem 4.20.
In Mathematics with Observers geometry in space E_3W_n, there are two planes α and β having only a unique common point.

Theorem 4.21.
In Mathematics with Observers geometry in space E_3W_n, there are two planes α and β having more than one common point.

4.7 Seventh property of connections

Questions: a) Does every straight line $\in E_2W_n$ contain at least two points?

b) Does every straight line $\in E_3W_n$ contain at least two points?

c) Does every plane $\in E_3W_n$ contain at least three points not lying in the same straight line?

d) Does the space E_3W_n contain at least four points not lying in any plane?

1) Let us consider the straight line $a \in E_2W_2$ with equation

$$99.99 \times_2 x -_2 98.88 \times_2 y = 0.$$

The point $A(0,0) \in a$. We must have

$$|x| \leq 1, \quad |y| \leq 1.01.$$

All possible positive *x* form the set

$$\Phi = [0.01, 0.02, \ldots, 0.99, 1.00],$$

and we get

$$99.99 \times_2 \Phi = [0.99, 1.98, \ldots, 98.82, 99.99].$$

All possible positive *y* form the set

$$\Psi = [0.01, 0.02, \ldots, 0.99, 1.00, 1.01],$$

and we get

$$98.88 \times_2 \Psi = [0.98, 1.96, \ldots, 97.74, 98.88, 99.86].$$

Direct calculations show that

$$99.99 \times_2 \Phi \cap 98.88 \times_2 \Psi = \Lambda.$$

So the straight line *a* contains only one point, that is, the answer to question a) in this case is negative.

1') Let us consider the straight line $a \in E_2W_2$ with equation

$$y = 0.$$

The point $A(0,0) \in a$, and also any point $B(x,0) \in a$ with $x \in W_2$. So the straight line *a* contains more than one point, that is, the answer to question a) in this case is positive.

So, we have proved the following:

Theorem 4.22.
In Mathematics with Observers geometry in plane E_2W_n, there is a straight line a having only one unique point.
Theorem 4.23.

In Mathematics with Observers geometry in plane E_2W_n there is a straight line a having more than one point.

2) Let us consider two planes $\alpha \in E_3W_2$ and $\beta \in E_3W_2$, where *α* is the set of points (x, y, z) satisfying the equation

$$99.99 \times_2 x -_2 98.88 \times_2 y +_2 z = 0,$$

and *β* is the set of points (x, y, z) satisfying the equation

$$z = 0.$$

The straight line $a \in E_3W_2$ is defined by the system of equations

$$\begin{cases} 99.99 \times_2 x -_2 98.88 \times_2 y +_2 z = 0, \\ z = 0, \end{cases}$$

and we have

$$99.99 \times_2 x -_2 98.88 \times_2 y = 0.$$

So we have only one point $A(0, 0, 0)$ as a solution of this system, that is, the straight line *a* contains only one point, so the answer to question b) is negative.

2’) Let us consider two planes $\alpha \in E_3W_2$ and $\beta \in E_3W_2$, where *α* is the set of points (x, y, z) satisfying the equation

$$y = 0,$$

and *β* is the set of points (x, y, z) satisfying the equation

$$z = 0.$$

The straight line $a \in E_3W_2$ is defined by the system of equations

$$\begin{cases} y = 0, \\ z = 0, \end{cases}$$

and we have

$$a : \left[A(x, 0, 0)\right]$$

for all $x \in W_2$. So we have more than two points *A* as solutions of this system, that is, the answer to question b) is positive.

So we have proved the following:

Theorem 4.24.
In Mathematics with Observers geometry in space E_3W_n, there is a straight line a having only one unique point.

Theorem 4.25.

In Mathematics with Observers geometry in space E_3W_n, there is a straight line a having more than one point.

3) Let us consider the plane $\alpha \in E_3W_2$ as the set of points (x, y, z) satisfying the equation

$$99.99 \times_2 x -_2 98.88 \times_2 y = 0.$$

The set of points satisfying this equation is

$$\left[A(0, 0, z)\right],$$

where *z* is any element $\in W_2$. So the set of all points $\in \alpha$ is the straight line $a \in E_3W_2$ defined by the system of equations

$$\begin{cases} x = 0, \\ y = 0, \end{cases}$$

that is, in this case the answer to question c) is negative.

3') Let us consider the plane $\alpha \in E_3W_2$ as the set of points (x, y, z) satisfying the equation

$$x +_2 y +_2 z -_2 1 = 0.$$

Let us take three points satisfying this equation:

$$A(1, 0, 0), B(0, 1, 0), C(0, 0, 1).$$

Let us prove that these three points do not lie in the same straight line. Let $A, B, C \in a$, where line *a* is defined as the solution of the system of equations

$$\begin{cases} x +_2 y +_2 z -_2 1 = 0, \\ b_1 \times_2 x +_2 b_2 \times_2 y +_2 b_3 \times_2 z +_2 b_4 = 0. \end{cases}$$

We get

$$\begin{aligned} & x +_2 y +_2 z -_2 1 = 0, \\ & b_1 \times_2 1 +_2 b_2 \times_2 0 +_2 b_3 \times_2 0 +_2 b_4 = 0, \\ & b_1 \times_2 0 +_2 b_2 \times_2 1 +_2 b_3 \times_2 0 +_2 b_4 = 0, \\ & b_1 \times_2 0 +_2 b_2 \times_2 0 +_2 b_3 \times_2 1 +_2 b_4 = 0. \end{aligned}$$

We have

$$\begin{aligned} & x +_2 y +_2 z -_2 1 = 0, \\ & b_1 = b_2, \\ & b_1 = b_3, \\ & b_1 = -b_4, \end{aligned}$$

and line *a* is defined as the solution of the system of equations

$$\begin{cases} x +_2 y +_2 z -_2 1 = 0, \\ b_1 \times_2 x +_2 b_1 \times_2 y +_2 b_1 \times_2 z -_2 b_1 = 0. \end{cases}$$

If we call

$$b_1 \times_2 x +_2 b_1 \times_2 y +_2 b_1 \times_2 z -_2 b_1 = 0$$

plane *β*, then we get

$$\alpha \cap \beta = \alpha,$$

or

$$\alpha \cap \beta = \beta$$

depends on the coefficient b_1 (clearly, $b_1 \neq 0$). This means that points *A*, *B*, *C* do not lie in the same straight line. So in this case the answer to question c) is positive.

So we have proved the following:

Theorem 4.26.
In Mathematics with Observers geometry in space E_3W_n, there is a plane α such that any three distinct points $A, B, C \in \alpha$ lie in the same straight line.
Theorem 4.27.
In Mathematics with Observers geometry in space E_3W_n, there are a plane α and three distinct points $A, B, C \in \alpha$ not lying in a straight line.
4) Any plane $\alpha \in E_3W_n$ has the equation

$$a_1 \times_n x +_n a_2 \times_n y +_n a_3 \times_n z +_n a_4 = 0.$$

Let us take four points satisfying this equation:

$$A(0,0,0), B(1,0,0), C(0,1,0), D(0,0,1),$$

that is, we have

$$a_1 \times_n 0 +_n a_2 \times_n 0 +_n a_3 \times_n 0 +_n a_4 = 0,$$
$$a_1 \times_n 1 +_n a_2 \times_n 0 +_n a_3 \times_n 0 +_n a_4 = 0,$$
$$a_1 \times_n 0 +_n a_2 \times_n 1 +_n a_3 \times_n 0 +_n a_4 = 0.$$
$$a_1 \times_n 0 +_n a_2 \times_n 0 +_n a_3 \times_n 1 +_n a_4 = 0.$$

This means that

$$a_1 = 0,$$
$$a_2 = 0,$$
$$a_3 = 0,$$
$$a_4 = 0.$$

So plane *α* containing all points *A*, *B*, *C*, *D* must have the equation

$$0 \times_n x +_n 0 \times_n y +_n 0 \times_n z +_n 0 = 0,$$

but by definition we must have the condition

$$(a_1, a_2, a_3) \neq (0, 0, 0).$$

This means that the answer to question d) is positive.

So we have proved the following:

Theorem 4.28.
In Mathematics with Observers geometry the space E_3W_n contains at least four points A, B, C, D not lying in any plane α.

4.8 Point and line theorem

We have the following classical geometry theorem:

> "Two straight lines of a plane have either one point or no common point; two planes have no common point or a common straight line; a plane and a straight line not lying in it have no point or one common point."

We get the following questions:

a) Do two straight lines of a plane have either one point or no common point?

b) Do two planes have no common point or a common straight line?

c) Do a plane and a straight line not lying in it have no point or one common point?

Let us start with question a).

1) Let us take two straight lines $a, b \in E_2W_2$, where *a* satisfies the equation

$$3 \times_2 x -_2 y = 0,$$

and *b* satisfies the equation

$$y -_2 1 = 0.$$

Because the number 3 does not have an inverse number in W_2, we get

$$a \cap b = \Lambda$$

that is, the straight lines *a* and *b* have no common points. Note that these two lines are not parallel in the classical sense.

1') Let us take two straight lines $a, b \in E_3W_2$, where *a* has the system of equations

$$\begin{cases} 3 \times_2 x -_2 y = 0, \\ z = 0, \end{cases}$$

and *b* has the system of equations

$$\begin{cases} y -_2 1 = 0, \\ z = 0. \end{cases}$$

We get

$$a \cap b = \Lambda,$$

that is, the straight lines *a* and *b* have no common points.

2) Let us take two straight lines $a, b \in E_2W_2$, where *a* has the equation

$$0.08 \times_2 x +_2 0.03 \times_2 y -_2 0.11 = 0,$$

and *b* has the equation

$$x -_2 3.00 = 0.$$

On the interval $1 \leq x \leq 5$, line *a* contains only the set Ω of points $A(x, y) \in E_2W_2$:

$$\Omega = \big[A(x, y)\big] = \big[x \in [1.00, 1.01, \ldots, 1.99], y \in [1.00, 1.01, \ldots, 1.99]\big] \cup$$
$$\cup \big[x \in [4.00, 4.01, \ldots, 4.99], y \in [-7.00, -7.01, \ldots, -7.99]\big].$$

We get again

$$a \cap b = \Lambda,$$

that is, the straight lines *a* and *b* have no common points. Again, note that these two lines are not parallel in the classical sense.

2’) Let us take two straight lines $a, b \in E_3W_2$, where *a* has the system of equations

$$\begin{cases} 0.08 \times_2 x +_2 0.03 \times_2 y -_2 0.11 = 0, \\ z = 0, \end{cases}$$

and *b* has the system of equations

$$\begin{cases} x -_2 3.00 = 0, \\ z = 0. \end{cases}$$

We get again

$$a \cap b = \Lambda,$$

that is, the straight lines *a* and *b* have no common points.

3) Let us take two straight lines $a, b \in E_2W_2$, where *a* has the equation

$$3 \times_2 x -_2 y = 0,$$

and b has the equation

$$y = 0.$$

We get

$$a \cap b = A(0, 0),$$

that is, the straight lines a and b have one common point A.

3') Let us take two straight lines $a, b \in E_3W_2$, where a has the system of equations

$$\begin{cases} 3 \times_2 x -_2 y = 0, \\ z = 0, \end{cases}$$

and b has the system of equations

$$\begin{cases} y = 0, \\ z = 0. \end{cases}$$

We get

$$a \cap b = A(0, 0, 0),$$

that is, the straight lines a and b have one common point A.

4) Let us take two straight lines $a, b \in E_2W_2$, where a has the equation

$$x = 0,$$

and b has the equation

$$y = 0.$$

We get

$$a \cap b = A(0.00, 0.00),$$

that is, the straight lines a and b have one common point A.

4') Let us take two straight lines $a, b \in E_3W_2$, where a has the system of equations

$$\begin{cases} x = 0, \\ z = 0, \end{cases}$$

and b has the system of equations

$$\begin{cases} y = 0, \\ z = 0. \end{cases}$$

We get

$$a \cap b = A(0, 0, 0),$$

that is, the straight lines *a* and *b* have one point *A*.

5) Let us take two straight lines $a, b \in E_2W_2$, where *a* has the equation

$$0.01 \times_2 x +_2 0.01 \times_2 y = 0,$$

and *b* has the equation

$$y = 0.$$

We get

$$a \cap b = \big[(0,0), (\pm 0.01, 0), (\pm 0.02, 0), \ldots, (\pm 0.99, 0)\big],$$

that is, the straight lines *a* and *b* have two hundred common points (from the point of view of W_m-observer, $m \geq 3$).

5') Let us take two straight lines $a, b \in E_3W_2$, where *a* has the system of equations

$$\begin{cases} 0.01 \times_2 x +_2 0.01 \times_2 y = 0, \\ z = 0, \end{cases}$$

and *b* has the system of equations

$$\begin{cases} y = 0, \\ z = 0. \end{cases}$$

We get

$$a \cap b = \big[(0,0,0), (\pm 0.01, 0, 0), (\pm 0.02, 0, 0), \ldots, (\pm 0.99, 0, 0)\big],$$

that is, the straight lines *a* and *b* have two hundred common points (from the point of view of W_m-observer, $m \geq 3$).

So question a) has the negative answer: two distinct straight lines of a plane may have no point, one common point, or more than one common point.

So we have proved the following:

Theorem 4.29.
In Mathematics with Observers geometry in plane E_2W_n, there are two distinct straight lines a and b such that $a \cap b = \Lambda$.
Theorem 4.30.
In Mathematics with Observers geometry in plane E_2W_n, there are two distinct straight lines a and b such that $a \cap b$ contains only one point.
Theorem 4.31.
In Mathematics with Observers geometry in plane E_2W_n, there are two distinct straight lines a and b such that $a \cap b$ contains more than one point.
Theorem 4.32.

In Mathematics with Observers geometry in space E_3W_n, there are two distinct straight lines a and b such that $a \cap b = \Lambda$.

Theorem 4.33.

In Mathematics with Observers geometry in space E_3W_n, there are two distinct straight lines a and b such that $a \cap b$ contains only one point.

Theorem 4.34.

In Mathematics with Observers geometry in space E_3W_n, there are two distinct straight lines a and b such that $a \cap b$ contains more than one point.

Let us go to question b). Let us consider two planes $\alpha \in E_3W_n$ and $\beta \in E_3W_n$, where *α* is the set of points (x, y, z) satisfying the equation

$$a_1 \times_n x +_n a_2 \times_n y +_n a_3 \times_n z +_n a_4 = 0,$$

and *β* is the set of points (x, y, z) satisfying the equation

$$b_1 \times_n x +_n b_2 \times_n y +_n b_3 \times_n z +_n b_4 = 0$$

for given $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in W_n$ such that $(a_1, a_2, a_3) \neq (0, 0, 0)$ and $(b_1, b_2, b_3) \neq (0, 0, 0)$.

1) Let us take two planes $\alpha \in E_3W_2$ and $\beta \in E_3W_2$, where *α* is the set of points (x, y, z) satisfying the equation

$$x -_2 1 = 0,$$

and *β* is the set of points (x, y, z) satisfying the equation

$$x = 0.$$

We get

$$\alpha \cap \beta = \Lambda,$$

that is, the planes *α* and *β* have no common points.

2) Let us take two planes $\alpha \in E_3W_2$ and $\beta \in E_3W_2$, where *α* is the set of points (x, y, z) satisfying the equation

$$x = 0,$$

and *β* is the set of points (x, y, z) satisfying the equation

$$z = 0.$$

These two planes have a common point $A(0, 0, 0)$, and the set of common points of planes *α* and *β* is $[B(0, y, 0)]$, where *y* is any element of W_n, that is, there is a straight line $a \in E_3W_2$:

$$\begin{cases} x = 0, \\ z = 0. \end{cases}$$

This means that two planes $\alpha \in E_3W_2$ and $\beta \in E_3W_2$ have a common straight line *a*.

3) Let us consider two planes $\alpha \in E_3W_2$ and $\beta \in E_3W_2$, where *α* is the set of points (x, y, z) satisfying the equation

$$99.99 \times_2 x -_2 98.88 \times_2 y +_2 z = 0,$$

and *β* is the set of points (x, y, z) satisfying the equation

$$z = 0.$$

These two planes have a point $A(0, 0, 0)$ in common. Let us see whether there is a second common point $B(x_2, y_2, z_2)$.

Point *B* has to satisfy the system of equations

$$\begin{cases} 99.99 \times_2 x -_2 98.88 \times_2 y +_2 z = 0, \\ z = 0, \end{cases}$$

and we have

$$99.99 \times_2 x -_2 98.88 \times_2 y = 0.$$

We must have

$$|x| \leq 1, \quad |y| \leq 1.01.$$

All possible positive *x* form the set

$$\Phi = [0.01, 0.02, \ldots, 0.99, 1.00],$$

and we get

$$99.99 \times_2 \Phi = [0.99, 1.98, \ldots, 98.82, 99.99].$$

All possible positive *y* form the set

$$\Psi = [0.01, 0.02, \ldots, 0.99, 1.00, 1.01],$$

and we get

$$98.88 \times_2 \Psi = [0.98, 1.96, \ldots, 97.74, 98.88, 99.86].$$

Direct calculations show that

$$99.99 \times_2 \Phi \cap 98.88 \times_2 \Psi = \Lambda.$$

So point *B* does not exist, that is, in this case the planes *α* and *β* have only one common point.

4) Let us consider two planes $\alpha \in E_3W_2$ and $\beta \in E_3W_2$, where *α* is the set of points (x, y, z) satisfying the equation

$$99.99 \times_2 x -_2 98.37 \times_2 y +_2 z = 0,$$

and β is the set of points (x, y, z) satisfying the equation

$$z = 0.$$

These two planes have a in common point $A(0, 0, 0)$. Let us see whether there are other common points (x, y, z). These points have to satisfy the system of equations

$$\begin{cases} 99.99 \times_2 x -_2 98.37 \times_2 y +_2 z = 0, \\ z = 0, \end{cases}$$

and we have

$$99.99 \times_2 x -_2 98.37 \times_2 y = 0.$$

We must have

$$|x| \leq 1, \quad |y| \leq 1.01.$$

All possible nonzero *x* form the set

$$\Phi = [\pm 0.01, \pm 0.02, \ldots, \pm 0.99, \pm 1.00],$$

and we get

$$99.99 \times_2 \Phi = [\pm 0.99, \pm 1.98, \ldots, \pm 98.82, \pm 99.99].$$

All possible nonzero *y* form the set

$$\Psi = [\pm 0.01, \pm 0.02, \ldots, \pm 0.99, \pm 1.00, \pm 1.01],$$

and we get

$$98.88 \times_2 \Psi = [\pm 0.98, \pm 1.96, \ldots, \pm 97.29, \pm 98.37, \pm 99.35].$$

Direct calculations show that

$$99.99 \times_2 \Phi \cap 98.37 \times_2 \Psi = \big[(0.62, 0.63), (-0.62, -0.63)\big].$$

So in addition to point *A*, we have two points $B(0.62, 0.63, 0)$ and $C(-0.62, -0.63, 0)$ such that

$$\alpha \cap \beta = [A, B, C],$$

that is, these two planes have three common points.

So question b) has the negative answer: two planes may have no point, one point, or more than one common point or a straight line.

So we have proved the following:

Theorem 4.35.
In Mathematics with Observers geometry in space E_3W_n, there are two distinct planes α and β such that $\alpha \cap \beta = \Lambda$.

Theorem 4.36.
In Mathematics with Observers geometry in space E_3W_n*, there are two distinct planes α and β such that* $\alpha \cap \beta$ *contains only one point.*
Theorem 4.37.
In Mathematics with Observers geometry in space E_3W_n*, there are two distinct planes α and β such that* $\alpha \cap \beta$ *contains more than one point.*
Theorem 4.38.
In Mathematics with Observers geometry in space E_3W_n*, there are two distinct planes α and β such that* $\alpha \cap \beta$ *contains a straight line.*
Let us go to question c).

1) Let us consider the plane $\alpha \in E_3W_2$ with equation

$$x = 0$$

and the straight line $a \in E_3W_2$ with system of equations

$$\begin{cases} x -_2 1 = 0, \\ z = 0. \end{cases}$$

We get

$$\alpha \cap a = \Lambda,$$

that is, the straight line *a* does not lie in the plane *α*, and *α* and *a* have no common point.

2) Let us consider the plane $\alpha \in E_3W_2$ with equation

$$x = 0$$

and the straight line $a \in E_3W_2$ with system of equations

$$\begin{cases} y -_2 1 = 0, \\ z -_2 1 = 0. \end{cases}$$

We get

$$\alpha \cap a = A(0, 1, 1),$$

that is, the straight line *a* does not lie in the plane *α*, and *α* and *a* have one common point *A*.

3) Let the plane $\alpha \in E_3W_2$ be the set of points (x, y, z) satisfying the equation

$$y -_2 1 = 0,$$

and let $\beta \in E_3W_2$ be the set of points (x, y, z) satisfying the equation

$$z -_2 1 = 0.$$

Let the straight line $a \in E_3W_2$ be the set of all points $A \in E_3W_2$ defined by the system of equations

$$\begin{cases} y -_2 1 = 0, \\ z -_2 1 = 0. \end{cases}$$

It is the set of points in E_3W_2 with coordinates $(x, 1, 1)$, where *x* is any element of W_2.

Let the plane $\gamma \in E_3W_2$ be the set of points (x, y, z) satisfying the equation

$$0.01 \times_2 x +_2 y -_2 z -_2 0.99 = 0.$$

Let us take two points *A*, *B* of a straight line *a*:

$$A(99.99, 1, 1), B(99.31, 1, 1).$$

For both points, we have

$$0.99 +_2 1 -_2 1 = 0.99.$$

So

$$A \in \gamma, \quad B \in \gamma.$$

Now let us take a third point *C* of a straight line *a*:

$$C(48.61, 1, 1).$$

We have

$$0.48 +_2 1 -_2 1 = 0.48 \neq 0.99.$$

This means that point *C* does not belong to plane *γ*, that is, the straight line *a* does not lie in plane *γ*, and *γ* and *a* have at least two common points *A* and *B*.

So question c) has the negative answer: a plane and a straight line not lying in it may have no point, one point, or two or more common points.

So we have proved the following:

Theorem 4.39.
In Mathematics with Observers geometry in space E_3W_n, there are a plane α and a straight line a not lying in this plane such that $\alpha \cap a = \Lambda$.
Theorem 4.40.
In Mathematics with Observers geometry in space E_3W_n, there are a plane α and a straight line a not lying in this plane such that $\alpha \cap a$ contains only one point.
Theorem 4.41.
In Mathematics with Observers geometry in space E_3W_n, there are a plane α and a straight line a not lying in this plane such that $\alpha \cap a$ contains more than one point.
So the "point and line theorem" of classical geometry is incorrect in Mathematics with Observers geometry.

4.9 Line and plane theorem

Classical geometry contains the following theorem:

> “Through a straight line and a point not lying in it, or through two distinct straight lines having a common point, one and only one plane may be made to pass.”

We have the following questions:

a) Is it correct that through a straight line and a point not lying in it, one and only one plane may be made to pass?

b) Is it correct that through two distinct straight lines having a common point, one and only one plane may be made to pass?

Let us go to question a).

1) Let us take the straight line $a \in E_3W_2$ defined by the system of equations

$$\begin{cases} y = 0, \\ z = 0, \end{cases}$$

and the point

$$A(0, 99.99, -98.88) \in E_3W_2.$$

We are looking for the plane $\alpha \in E_3W_2$ as the set of points (x, y, z) satisfying the equation

$$a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 \times_2 z +_2 a_4 = 0,$$

where $a_1, a_2, a_3, a_4 \in W_2$ such that $(a_1, a_2, a_3) \neq (0, 0, 0)$, and the conditions

$$a \in \alpha, \quad A \in \alpha.$$

Line *a* is the set of points with coordinates $(x, 0, 0) \in E_3W_2$, where *x* is any element $\in W_2$. We must have

$$a_1 \times_2 x +_2 a_4 = 0.$$

This means that

$$\begin{cases} a_1 = 0, \\ a_4 = 0. \end{cases}$$

So we can rewrite the equation of plane *α* as

$$a_2 \times_2 y +_2 a_3 \times_2 z = 0.$$

Since $A \in \alpha$, we have

$$a_2 \times_2 99.99 -_2 a_3 \times_2 98.88 = 0.$$

We must have

$$|a_2| \leq 1, \quad |a_3| \leq 1.01.$$

All possible positive a_2 form the set

$$\Phi = [0.01, 0.02, \ldots, 0.99, 1.00],$$

and we get

$$99.99 \times_2 \Phi = [0.99, 1.98, \ldots, 98.82, 99.99].$$

All possible positive a_3 form the set

$$\Psi = [0.01, 0.02, \ldots, 0.99, 1.00, 1.01],$$

and we get

$$98.88 \times_2 \Psi = [0.98, 1.96, \ldots, 97.74, 98.88, 99.86].$$

As above, direct calculations show that

$$99.99 \times_2 \Phi \cap 98.88 \times_2 \Psi = \Lambda.$$

So plane α does not exist.

2) Let us take three distinct points in $E_3 W_2$:

$$A(1,0,0), B(0,1,0), C(0,0,1).$$

Let a straight line a contain points A and B. Because the vectors

$$\mathbf{AB} = (-1,1,0), \quad \mathbf{AC} = (-1,0,1)$$

are not parallel, point C does not belong to line a. Another way to prove this is as follows.

Line a has the system of equations

$$\begin{cases} b_1 \times_2 x +_2 b_2 \times_2 y +_2 b_3 = 0, \\ z = 0, \end{cases}$$

that is,

$$\begin{aligned} &b_1 \times_2 1 +_2 b_2 \times_2 0 +_2 b_3 = 0, \\ &b_1 \times_2 0 +_2 b_2 \times_2 1 +_2 b_3 = 0, \\ &z = 0, \end{aligned}$$

that is,

$$\begin{aligned} &b_1 = b_2, \\ &b_1 = -b_3, \\ &z = 0, \end{aligned}$$

that is, line *a* has the system of equations

$$\begin{cases} b_1 \times_2 x +_2 b_1 \times_2 y -_2 b_1 = 0, \\ z = 0, \end{cases}$$

and $C \notin a$.

Now we are looking for plane *α* containing line *a* and point *C* as the set of points $(x, y, z) \in E_3W_2$ satisfying the equation

$$a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 \times_2 z +_2 a_4 = 0.$$

We get the system

$$\begin{aligned} a_1 \times_2 1 +_2 a_2 \times_2 0 +_2 a_3 \times_2 0 +_2 a_4 = 0, \\ a_1 \times_2 0 +_2 a_2 \times_2 1 +_2 a_3 \times_2 0 +_2 a_4 = 0, \\ a_1 \times_2 0 +_2 a_2 \times_2 0 +_2 a_3 \times_2 1 +_2 a_4 = 0, \end{aligned}$$

and so

$$\begin{aligned} a_1 &= a_2, \\ a_1 &= a_3, \\ a_4 &= -a_1. \end{aligned}$$

So the equation of plane *α* in this case is

$$a_1 \times_2 x +_2 a_1 \times_2 y +_2 a_1 \times_2 z -_2 a_1 = 0.$$

For $a_1 = 1$, we get plane α_1 with equation

$$x +_2 y +_2 z -_2 1 = 0.$$

For $a_1 = 0.01$, we get plane α_2 with equation

$$0.01 \times_2 x +_2 0.01 \times_2 y +_2 0.01 \times_2 z -_2 0.01 = 0.$$

Let us take the point $D(0.2, 0.2, 0.6) \in E_3W_2$. We get

$$0.2 +_2 0.2 +_2 0.6 -_2 1 = 0.$$

So

$$D \in \alpha_1,$$

but

$$0.01 \times_2 0.2 +_2 0.01 \times_2 0.02 +_2 0.01 \times_2 0.06 -_2 0.01 \neq 0.$$

Thus

$$\alpha_1 \neq \alpha_2.$$

This means that a straight line and a point not lying in it not completely (not uniquely) determine a plane *α* that may be made to pass.

3) Let us take the straight line $a \in E_3W_2$ with system of equations

$$\begin{cases} y = 0, \\ z = 0, \end{cases}$$

and the point

$$A(0, 99.99, -98.37) \in E_3W_2.$$

We are looking for the plane $\alpha \in E_3W_2$ as the set of points (x, y, z) satisfying the equation

$$a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 \times_2 z +_2 a_4 = 0$$

with $a_1, a_2, a_3, a_4 \in W_2$ such that $(a_1, a_2, a_3) \neq (0, 0, 0)$ and conditions

$$a \in \alpha, \quad A \in \alpha.$$

Line *a* is the set of points with coordinates $(x, 0, 0) \in E_3W_2$, where *x* is any element $\in W_2$. We must have

$$a_1 \times_2 x +_2 a_4 = 0.$$

This means that

$$\begin{cases} a_1 = 0, \\ a_4 = 0. \end{cases}$$

So we can rewrite the equation of plane *α* as

$$a_2 \times_2 y +_2 a_3 \times_2 z = 0.$$

Since $A \in \alpha$, we have

$$a_2 \times_2 99.99 = a_3 \times_2 98.37.$$

We must have

$$|a_2| \leq 1, \quad |a_3| \leq 1.01.$$

All possible positive a_2 form the set

$$\Phi = [0.01, 0.02, \dots, 0.99, 1.00],$$

and we get

$$99.99 \times_2 \Phi = [0.99, 1.98, \dots, 98.82, 99.99].$$

All possible positive a_3 form the set

$$\Psi = [0.01, 0.02, \dots, 0.99, 1.00, 1.01],$$

and we get

$$98.37 \times_2 \Psi = [0.98, 1.96, \ldots, 97.29, 98.37, 99.35].$$

Direct calculations show that

$$99.99 \times_2 \Phi \cap 98.37 \times_2 \Psi = [61.92],$$

and we get only one point in intersection of these two sets, that is,

$$a_1 = 0; \quad a_2 = 0.62; \quad a_3 = 0.63 \quad a_4 = 0.$$

So in this case, there is only one plane *α* containing line *a* and point *A*. Thus question a) has the negative answer: Through a straight line and a point not lying in it, no plane, one plane, or more than one plane may be made to pass.

So we have proved the following:

Theorem 4.42.

In Mathematics with Observers geometry in space E_3W_n, there are a straight line a and a point A not lying in this line such that there is no plane α containing this line and point.

Theorem 4.43.

In Mathematics with Observers geometry in space E_3W_n there are a straight line a and a point A not lying in this line such that there is only one plane α containing this line and point.

Theorem 4.44.

In Mathematics with Observers geometry in space E_3W_n, there are a straight line a and a point A not lying in this line such that there is more than one plane α containing this line and point.

Let us go to question b).

1) Let us take three distinct points $\in E_3W_2$:

$$A(1, 0, 0), B(0, 99.99, 0), C(0, 0, 98.88).$$

Let the straight line *a* contain points *A* and *B*, that is, $a = AB$, and let the system of equations of this line be

$$\begin{cases} a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 \times_2 z +_2 a_4 = 0, \\ z = 0. \end{cases}$$

Since $A, B \in a$, we have

$$\begin{cases} a_1 +_2 a_4 = 0, \\ a_2 \times_2 99.99 +_2 a_4 = 0, \end{cases}$$

that is,

$$\begin{cases} a_1 = a_2 \times_2 99.99. \\ a_1 = -a_4. \end{cases}$$

So we can rewrite the system of equations of line *a* as

$$\begin{cases}(a_2 \times_2 99.99) \times_2 x +_2 a_2 \times_2 y -_2 a_2 \times_2 99.99 = 0,\\ z = 0.\end{cases}$$

We see that the point $C(0,0,98.88)$ does not belong to line *a* because its coordinate $z \neq 0$. We can make the same statement also because the vectors

$$\mathbf{AB} = (-1, 99.99, 0), \quad \mathbf{AC} = (-1, 0, 98.88)$$

are not parallel. Let the straight line *b* contain points *A* and *C*, that is, $b = AC$, and let the system of equations of this line be

$$\begin{cases}b_1 \times_2 x +_2 b_2 \times_2 y +_2 b_3 \times_2 z +_2 b_4 = 0,\\ y = 0.\end{cases}$$

Since $A, C \in b$, we have

$$\begin{cases}b_1 +_2 b_4 = 0,\\ b_3 \times_2 98.88 +_2 b_4 = 0,\end{cases}$$

that is,

$$\begin{cases}b_1 = b_3 \times_2 98.88,\\ b_1 = -b_4.\end{cases}$$

So we can rewrite the system of equations of line *b* as

$$\begin{cases}(b_3 \times_2 98.88) \times_2 x +_2 b_3 \times_2 z -_2 (b_3 \times_2 98.88) = 0,\\ y = 0.\end{cases}$$

These two distinct straight lines *a* and *b* have a common point *A*. We are looking for the plane *α* containing lines *a* and *b* as the set of points $(x, y, z) \in E_3W_2$ satisfying the equation

$$c_1 \times_2 x +_2 c_2 \times_2 y +_2 c_3 \times_2 z +_2 c_4 = 0.$$

We get the system

$$\begin{aligned}c_1 \times_2 1 +_2 c_2 \times_2 0 +_2 c_3 \times_2 0 +_2 c_4 = 0,\\ c_1 \times_2 0 +_2 c_2 \times_2 99.99 +_2 c_3 \times_2 0 +_2 c_4 = 0,\\ c_1 \times_2 0 +_2 c_2 \times_2 0 +_2 c_3 \times_2 98.88 +_2 c_4 = 0,\end{aligned}$$

and thus

$$\begin{aligned}c_1 &= c_2 \times_2 99.99,\\ c_1 &= c_3 \times_2 98.88,\\ c_4 &= -c_1.\end{aligned}$$

We must have

$$c_2 \times_2 99.99 = c_3 \times_2 98.88$$

and

$$|c_2| \leq 1, \quad |c_3| \leq 1.01.$$

All possible positive c_2 form the set

$$\Phi = [0.01, 0.02, \ldots, 0.99, 1.00],$$

and we get

$$99.99 \times_2 \Phi = [0.99, 1.98, \ldots, 98.82, 99.99].$$

All possible positive c_3 form the set

$$\Psi = [0.01, 0.02, \ldots, 0.99, 1.00, 1.01],$$

and we get

$$98.88 \times_2 \Psi = [0.98, 1.96, \ldots, 97.74, 98.88, 99.86].$$

Direct calculations show that

$$99.99 \times_2 \Phi \cap 98.88 \times_2 \Psi = \Lambda.$$

So plane α does not exist.

2) Let us take three distinct points $\in E_3W_2$:

$$A(1, 0, 0), B(0, 99.99, 0), C(0, 0, 98.37).$$

Let the straight line a contain points A and B, that is, $a = AB$, and let the system of equations of this line be

$$\begin{cases} a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 \times_2 z +_2 a_4 = 0, \\ z = 0. \end{cases}$$

Since $A, B \in a$, we have

$$\begin{cases} a_1 +_2 a_4 = 0, \\ a_2 \times_2 99.99 +_2 a_4 = 0, \end{cases}$$

that is,

$$\begin{cases} a_1 = a_2 \times_2 99.99, \\ a_1 = -a_4. \end{cases}$$

So we can rewrite the system of equations of line a as

$$\begin{cases} (a_2 \times_2 99.99) \times_2 x +_2 a_2 \times_2 y -_2 a_2 \times_2 99.99 = 0, \\ z = 0, \end{cases}$$

and we see that the point $C(0, 0, 98.37)$ does not belong to line a because its coordinate $z \neq 0$. We can make same statement also because the vectors

$$\mathbf{AB} = (-1, 99.99, 0), \quad \mathbf{AC} = (-1, 0, 98.37)$$

are not parallel.

Let the straight line *b* contain points *A* and *C*, that is, $b = AC$, and let the system of equations of this line be

$$\begin{cases} b_1 \times_2 x +_2 b_2 \times_2 y +_2 b_3 \times_2 z +_2 b_4 = 0, \\ y = 0. \end{cases}$$

Since $A, C \in b$, we have

$$\begin{cases} b_1 +_2 b_4 = 0, \\ b_3 \times_2 98.37 +_2 b_4 = 0, \end{cases}$$

that is,

$$\begin{cases} b_1 = b_3 \times_2 98.37, \\ b_1 = -b_4. \end{cases}$$

So we can rewrite the system of equations of line *b* as

$$\begin{cases} (b_3 \times_2 98.37) \times_2 x +_2 b_3 \times_2 z -_2 (b_3 \times_2 98.37) = 0, \\ y = 0. \end{cases}$$

These two distinct straight lines *a* and *b* have a common point *A*.

We are looking for the plane *α* containing lines *a* and *b* as the set of points $(x, y, z) \in E_3W_2$ satisfying the equation

$$c_1 \times_2 x +_2 c_2 \times_2 y +_2 c_3 \times_2 z +_2 c_4 = 0.$$

We get the system

$$\begin{aligned} &c_1 \times_2 1 +_2 c_2 \times_2 0 +_2 c_3 \times_2 0 +_2 c_4 = 0, \\ &c_1 \times_2 0 +_2 c_2 \times_2 99.99 +_2 c_3 \times_2 0 +_2 c_4 = 0, \\ &c_1 \times_2 0 +_2 c_2 \times_2 0 +_2 c_3 \times_2 98.37 +_2 c_4 = 0, \end{aligned}$$

and thus

$$\begin{aligned} &c_1 = c_2 \times_2 99.99, \\ &c_1 = c_3 \times_2 98.37, \\ &c_4 = -c_1. \end{aligned}$$

We must have

$$c_2 \times_2 99.99 = c_3 \times_2 98.37$$

and

$$|c_2| \leq 1, \quad |c_3| \leq 1.01.$$

All possible positive c_2 form the set

$$\Phi = [0.01, 0.02, \ldots, 0.99, 1.00],$$

and we get

$$99.99 \times_2 \Phi = [0.99, 1.98, \ldots, 98.82, 99.99].$$

All possible positive c_3 form the set

$$\Psi = [0.01, 0.02, \ldots, 0.99, 1.00, 1.01],$$

and we get

$$98.37 \times_2 \Psi = [0.98, 1.96, \ldots, 97.29, 98.37, 99.35].$$

Direct calculations show that

$$99.99 \times_2 \Phi \cap 98.88 \times_2 \Psi = 61.92,$$

and we get only one point in intersection of these two sets, that is,

$$\begin{aligned} c_1 &= 61.92, \\ c_2 &= 0.62, \\ c_3 &= 0.63, \\ c_4 &= -61.92. \end{aligned}$$

This means in that this case, through two distinct straight lines having one common point, only one plane may be made to pass.

3) Let us take the straight lines $a, b \in E_3W_2$, where *a* has the system of equations

$$\begin{cases} x = 0, \\ z = 0, \end{cases}$$

and *b* has the system of equations

$$\begin{cases} y = 0, \\ z = 0. \end{cases}$$

These two distinct straight lines have one common point $O(0, 0, 0)$. Let us consider the plane

$$\alpha : a_3 \times_2 z = 0,$$

where $a_3 \neq 0$. We have

$$a \subset \alpha, \quad b \subset \alpha$$

For $|a_3| \geq 1$

$$\alpha_1 = \big[(x, y, 0)\big],$$

where *x* and *y* are any elements $\in W_2$. For $|a_3| \in [0.1, 0.11, \ldots, 0.99]$,

$$\alpha_2 = \big[(x, y, 0), (x, y, \pm 0.01), \ldots, (x, y, \pm 0.09)\big],$$

where *x* and *y* are any elements $\in W_2$. For $|a_3| \in [0.01, 0.02, \ldots, 0.09]$,

$$\alpha_3 = \left[(x, y, 0), (x, y, \pm 0.01), \ldots, (x, y, \pm 0.99)\right],$$

where *x* and *y* are any elements $\in W_2$. This means that in this case, through two distinct straight lines having a common point, more than one plane may be made to pass.

So question b) has the negative answer: Through two straight lines having only one common point, no plane, one plane, or more than one plane may be made to pass.

So we have proved the following:

Theorem 4.45.
In Mathematics with Observers geometry in space E_3W_n, there are two distinct straight lines a and b having only one common point A such that there is no plane α containing these lines.
Theorem 4.46.
In Mathematics with Observers geometry in space E_3W_n, there are two distinct straight lines a and b having only one common point A such that there is only one plane α containing these lines.
Theorem 4.47.
In Mathematics with Observers geometry in space E_3W_n, there are two distinct straight lines a and b having only one common point A such that there is more than one plane α containing these lines.
So the “line and plane theorem” of classical geometry is incorrect in Mathematics with Observers geometry.

5 Observability and properties of points order analysis

We have to define what does the relation “between points of a straight line” mean in Mathematics with Observers geometry.

In the case of the so-called “simple” straight lines, for example, lines *a, b, c*:

$$a\in E_2W_n : x -_n y = 0,$$
$$b\in E_2W_n : y = 0,$$
$$c\in E_2W_n : x = 0,$$

we can define the relation “between points of straight line” in the following standard way. Take points

$$A, B \in a : A(1,1), B(2,2).$$

Then we say that the points

$$C(x,y) \in a : 1 < x < 2$$

lie between *A* and *B*. We can also say that the points

$$C'(x,y) \in a : 1 < y < 2$$

lie between *A* and *B*. These two definitions are equal in this case, and we can call say that a point $C''(x,y) \in a$ lies between *A* and *B* if at least one of the following conditions is satisfied:

$$1 < x < 2$$

or

$$1 < y < 2.$$

Take points

$$D, E \in b : D(1,0), E(2,0).$$

Then we say that the points

$$F(x,y) \in b : 1 < x < 2$$

lie between *D* and *E*.

Take points

$$I, J \in c : I(0,1), J(0,2).$$

Then we say that the points

$$K(x,y) \in c : 1 < y < 2$$

lie between *I* and *J*.

In case we have a "not simple" straight line, for example, line

$$d \in E_2W_2 : 0.01 \times_2 x -_2 0.01 \times_2 y = 0,$$

we can define the relation "between points of straight line" in the following "not simple" way. Take the points

$$L, M \in d : L(1,1), M(2,2).$$

Then we can say that a point $N(x,y) \in d$ lies between *L* and *M* if

$$\begin{cases} 1 < x < 2, \\ 1 < y < 2. \end{cases}$$

Take the points

$$O, P \in d : O(1.06, 1.89), P(2.11, 2.03).$$

Then we can say that a point $Q(x,y) \in d$ lies between *O* and *P* if

$$\begin{cases} 1.06 < x < 2.11, \\ 1.89 < y < 2.03. \end{cases}$$

Take the points

$$O', P' \in d : O'(2.06, 2.74), P'(2.11, 2.03).$$

Then we can say that a point $Q'(x,y) \in d$ lies between O' and P' if

$$\begin{cases} 2.06 < x < 2.11, \\ 2.03 < y < 2.74. \end{cases}$$

Take the points

$$O'', P'' \in d : O''(2.06, 2.74), P''(2.11, 2.74).$$

Then we can say that a point $Q''(x,y) \in d$ lies between O'' and P'' if

$$\begin{cases} 2.06 < x < 2.11, \\ y = 2.74. \end{cases}$$

Take the points

$$O''', P''' \in d : O'''(2.06, 2.74), P'''(2.06, 2.03).$$

Then we can say that a point $Q'''(x,y) \in d$ lies between O''' and P''' if

$$\begin{cases} x = 2.06, \\ 2.03 < y < 2.74. \end{cases}$$

Take the points

$$R, S \in d : R(1, 1.99), S(1.01, 1.03).$$

Then we can say that a point $T(x, y)$ lying between *R* and *S* does not exist.

Take the points

$$U, V \in d : U(1.34, 1.88), V(1.76, 1.89).$$

Then we can say that a point $W(x, y)$ lying between *U* and *V* does not exist.

To get one more possible logical situation, let us consider the line

$$e \in E_2W_2 : 0.08 \times_2 x +_2 0.03 \times_2 y -_2 0.11 = 0.$$

On the interval $1 \leq x \leq 5$, this line contains only the set Ω of points $A(x, y) \in E_2W_2$:

$$\Omega = \big[A(x, y)\big] = \big[x \in [1.00, 1.01, \ldots, 1.99], y \in [1.00, 1.01, \ldots, 1.99]\big] \cup \\ \cup \big[x \in [4.00, 4.01, \ldots, 4.99], y \in [-7.00, -7.01, \ldots, -7.99]\big].$$

Take the points

$$X, Y \in e : X(1.99, 1.88), Y(4.00, -7.25).$$

Then we can say that a point $Z(x, y)$ lying between *X* and *Y* does not exist.

Or take the points

$$X', Y' \in e : X'(1.35, 1.99), Y'(4.23, -7.00).$$

Then we can say that a point $Z'(x, y)$ lying between X' and Y' does not exist.

Now we give a general definition of the relation "between points of straight line".

First of all, we define the "closed interval" $[u, v]; u, v \in W_n$, as the set of all elements $w \in W_n$ satisfying the inequalities $u \leq w \leq v$ if $u \leq v$ or $v \leq w \leq u$ if $v \leq u$.

If from $[u, v]$ we remove the end points *u*, *v*, then we get the so-called "open interval" (u, v).

Definition 5.1.

Suppose we have a straight line $a \in E_2W_n$ and three distinct points

$$A(x_1, y_1), B(x_2, y_2), C(x_3, y_3) \in a.$$

We say that a point C lies between points A, B if one of the following conditions is satisfied:

1)

$$\begin{cases} x_3 \in (x_1, x_2), \\ y_3 \in (y_1, y_2); \end{cases}$$

2)

$$\begin{cases} x_1 = x_2 = x_3, \\ y_3 \in (y_1, y_2); \end{cases}$$

3)

$$\begin{cases} x_3 \in (x_1, x_2), \\ y_1 = y_2 = y_3. \end{cases}$$

Definition 5.2.

Suppose we have a straight line $a \in E_3W_n$ and three distinct points

$$A(x_1, y_1, z_1), B(x_2, y_2, z_2), C(x_3, y_3, z_3) \in a.$$

We say that a point C lies between points A, B if one of the following conditions is satisfied:

1)

$$\begin{matrix} x_3 \in (x_1, x_2), \\ y_3 \in (y_1, y_2). \\ z_3 \in (z_1, z_2); \end{matrix}$$

2)

$$\begin{matrix} x_1 = x_2 = x_3, \\ y_3 \in (y_1, y_2), \\ z_3 \in (z_1, z_2); \end{matrix}$$

3)

$$\begin{matrix} x_3 \in (x_1, x_2), \\ y_1 = y_2 = y_3, \\ z_3 \in (z_1, z_2); \end{matrix}$$

4)

$$\begin{matrix} x_1 = x_2 = x_3, \\ y_1 = y_2 = y_3, \\ z_3 \in (z_1, z_2); \end{matrix}$$

5)

$$\begin{aligned} x_3 &\in (x_1, x_2), \\ y_3 &\in (y_1, y_2), \\ z_1 &= z_2 = z_3; \end{aligned}$$

6)

$$\begin{aligned} x_1 &= x_2 = x_3, \\ y_3 &\in (y_1, y_2), \\ z_1 &= z_2 = z_3; \end{aligned}$$

7)

$$\begin{aligned} x_3 &\in (x_1, x_2), \\ y_1 &= y_2 = y_3, \\ z_1 &= z_2 = z_3. \end{aligned}$$

5.1 First property of points order

Let *A*, *B*, *C* be points of a straight line with *B* lying between *A* and *C*.

Question: Is *B* also lying between *C* and *A*?

In Mathematics with Observers geometry, we have the positive answer to this question because the definition of "between" is symmetric with respect to points *A* and *C*.

This means that we have the following:

Theorem 5.3.
In Mathematics with Observers geometry in plane E_2W_n, in any straight line a having at least three distinct points A, B, C such that B lies between A and C, point B also lies between C and A.
Theorem 5.4.
In Mathematics with Observers geometry in space E_3W_n, in any straight line a having at least three distinct points A, B, C such that B lies between A and C, point B also lies between C and A.

5.2 Second property of points order

Let us consider two distinct points in E_2W_n or E_3W_n lying in the same straight line:

$$A(x_1, y_1), C(x_2, y_2) \in a \in E_2W_n$$

or

$$A(x_1, y_1, z_1), C(x_2, y_2, z_2) \in b \in E_3W_n.$$

Question: Does there exist at least one point $B \in a$ or $B \in b$ lying between *A* and *C* and at least one point *D* such that *C* lies between *A* and *D*?

1) Let us take the straight line $a \in E_2W_2$ with equation

$$10 \times_2 x -_2 y = 0$$

and two points $A, C \in a$ with coordinates $A(0.99, 9.9)$, $C(1, 10)$. In this case, no point $B \in a$ lying between *A* and *C* exists. So in this case the answer to the question is negative.

1') Let us take the straight line $b \in E_3W_2$ with system of equations

$$\begin{cases} 10 \times_2 x -_2 y = 0, \\ z = 0, \end{cases}$$

and two points $A, C \in b$ with coordinates $A(0.99, 9.9, 0)$, $C(1, 10, 0)$. In this case, no point $B \in b$ lying between *A* and *C* exists. So in this case the answer to the question is negative.

2) Let us take the straight line $a \in E_2W_2$ with equation

$$99.99 \times_2 x -_2 y = 0$$

and two points $A, C \in a$ with coordinates $A(0, 0)$, $C(1, 99.99)$. In this case, no point $D \in a$ such that *C* lies between *A* and *D* exists. So in this case the answer to the question is negative.

2') Let us take the straight line $b \in E_3W_2$ with system of equations

$$\begin{cases} 99.99 \times_2 x -_2 y = 0, \\ z = 0, \end{cases}$$

and two points $A, C \in b$ with coordinates $A(0, 0, 0)$, $C(1, 99.99, 0)$. In this case, no point $D \in b$ such that *C* lies between *A* and *D* exists. So in this case the answer to the question is negative.

3) Let us take the straight line $a \in E_2W_2$ with equation

$$x -_2 y = 0$$

and two points $A, C \in a$ with coordinates $A(0, 0)$, $C(1, 1)$. If we take two points $B, D \in a$ with coordinates $B(0.5, 0.5)$, $D(1.5, 1.5)$, then we get point $B \in a$ lying between *A* and *C* and point *D* such that *C* lies between *A* and *D*. So in this case the answer to the question is positive.

3') Let us take the straight line $b \in E_3W_2$ with system of equations

$$\begin{cases} x -_2 y = 0, \\ z = 0, \end{cases}$$

and two points $A, C \in b$ with coordinates $A(0,0,0)$, $C(1,1,0)$. If we take two points $B, D \in b$ with coordinates $B(0.5, 0.5, 0)$, $D(1.5, 1.5, 0)$, then we get point $B \in b$ lying between *A* and *C* and point $D \in b$ such that *C* lies between *A* and *D*. So in this case the answer to the question is positive.

So we have proved the following:

Theorem 5.5.
In Mathematics with Observers geometry in plane E_2W_n, there are a straight line a and two distinct points $A, C \in a$ such that there is no point $B \in a$ lying between A and C.
Theorem 5.6.
In Mathematics with Observers geometry in plane E_2W_n, there are a straight line a and two distinct points $A, C \in a$ such that there is at least one point $B \in a$ lying between A and C.
Theorem 5.7.
In Mathematics with Observers geometry in plane E_2W_n there are a straight line a and two distinct points $A, C \in a$ such that there is no point $D \in a$ such that C lies between A and D.
Theorem 5.8.
In Mathematics with Observers geometry in plane E_2W_n, there are a straight line a and two distinct points $A, C \in a$ such that there is at least one point $D \in a$ such that C lies between A and D
Theorem 5.9.
In Mathematics with Observers geometry in space E_3W_n, there are a straight line a and two distinct points $A, C \in a$ such that there is no point $B \in a$ lying between A and C.
Theorem 5.10.
In Mathematics with Observers geometry in space E_3W_n, there are a straight line a and two distinct points $A, C \in a$ such that there is at least one point $B \in a$ lying between A and C.
Theorem 5.11.
In Mathematics with Observers geometry in space E_3W_n, there are a straight line a and two distinct points $A, C \in a$ such that there is no point $D \in a$ such that C lies between A and D.
Theorem 5.12.
In Mathematics with Observers geometry in space E_3W_n, there are a straight line a and two distinct points $A, C \in a$ such that there is at least one point $D \in a$ such that C lies between A and D

5.3 Third property of points order

Let us consider three distinct points $\in E_2W_n$ or $\in E_3W_n$ lying in the same straight line:

$$A(x_1, y_1), B(x_2, y_2), C(x_3, y_3) \in a \in E_2W_n$$

or

$$A(x_1,y_1,z_1),B(x_2,y_2,z_2),C(x_3,y_3,z_3)\in b\in E_3W_n.$$

Question: Is there one and only one of these points that lies between the other two?

1) Let us take the straight line $a\in E_2W_2$ with equation

$$a\in E_2W_2 : 0.01\times_2 x-_n 0.01\times_2 y=0$$

and three distinct points $A,B,C\in a$ with coordinates $A(0.96,0.96)$, $B(0.96,0.97)$, $C(0.97,0.97)$. In this case, none of these points lies between the other two.

So in this case the answer to the question is negative.

1') Let us take the straight line $b\in E_3W_2$ with system of equations

$$\begin{cases}0.01\times_2 x-_n 0.01\times_2 y=0,\\ z=0,\end{cases}$$

and three distinct points $A,B,C\in b$ with coordinates $A(0.96,0.96,0)$, $B(0.96,0.97,0)$, $C(0.97,0.97,0)$. In this case, none of these points lies between the other two.

So in this case the answer to this question is negative.

2) Let us take the straight line $a\in E_2W_2$ with equation

$$0.08\times_2 x+_2 0.03\times_2 y-_2 0.11=0$$

and three distinct points $A,B,C\in a$ with coordinates $A(1.99,1.88)$, $B(1.99,1.56)$, $C(4.00,-7.25)$. Direct calculations show that on the interval $1\le x\le 5$, this line contains only the set Ω of points $A(x,y)\in E_2W_2$:

$$\Omega=\big[A(x,y)\big]=\big[x\in[1.00,1.01,\dots,1.99],y\in[1.00,1.01,\dots,1.99]\big]\cup$$
$$\cup\big[x\in[4.00,4.01,\dots,4.99],y\in[-7.00,-7.01,\dots,-7.99]\big].$$

Then we can say that none of these points lies between the other two.

2') Let us take the straight line $b\in E_3W_2$ with system of equations

$$\begin{cases}0.08\times_2 x+_2 0.03\times_2 y-_2 0.11=0,\\ z=0,\end{cases}$$

and three distinct points $A,B,C\in b$ with coordinates $A(1.99,1.88,0)$, $B(1.99,1.56,0)$, $C(4.00,-7.25,0)$. Then we can say that none of these points lies between the other two.

3) Let us take the straight line $a \in E_2W_2$ with equation

$$x -_2 y = 0$$

and three points $A, B, C \in a$ with coordinates $A(0,0)$, $B(0.5, 0.5)$, $C(1,1)$. We get that point $B \in a$ lies between *A* and *C*, and *B* is only one of these points that lies between the other two. So in this case the answer to the question is positive.

3') Let us take the straight line $b \in E_2W_2$ with system of equations

$$\begin{cases} x -_2 y = 0, \\ z = 0, \end{cases}$$

and three points $A, B, C \in b$ with coordinates $A(0,0,0)$, $B(0.5, 0.5, 0)$, $C(1,1,0)$. We get that point $B \in b$ lies between *A* and *C*, and *B* is only one of these points that lies between the other two. So in this case the answer to the question is positive.

So we have proved the following:

Theorem 5.13.
In Mathematics with Observers geometry in plane E_2W_n, there are a straight line a and three distinct points $A, B, C \in a$ such that none of these points lies between the other two.
Theorem 5.14.
In Mathematics with Observers geometry in plane E_2W_n, there are a straight line a and three distinct points $A, B, C \in a$ such that is one and only one of these points that lies between the other two.
Theorem 5.15.
In Mathematics with Observers geometry in space E_3W_n, there are a straight line a and three distinct points $A, B, C \in a$ such that none of these points lies between the other two.
Theorem 5.16.
In Mathematics with Observers geometry in space E_3W_n, there are a straight line a and three distinct points $A, B, C \in a$ such that there is one and only one of these points that lies between the other two.

5.4 Fourth property of points order

Let us consider four distinct points in E_2W_n or E_3W_n lying in the same straight line:

$$A(x_1, y_1), B(x_2, y_2), C(x_3, y_3), D(x_4, y_4) \in a \in E_2W_n$$

or

$$A(x_1, y_1, z_1), B(x_2, y_2, z_2), C(x_3, y_3, z_3), D(x_4, y_4, z_4) \in b \in E_3W_n.$$

Question: Is it always possible to arrange these four points so that *B* will lie between *A* and *C* and also between *A* and *D* and, moreover, so that *C* will lie between *A* and *D* and also between *B* and *D*?

1) Let us take the straight line $a \in E_2W_2$ with equation

$$a \in E_2W_2 : 0.01 \times_2 x -_n 0.01 \times_2 y = 0$$

and four distinct points $A, B, C, D \in a$ with coordinates $A(0.96, 0.96)$, $B(0.96, 0.97)$, $C(0.97, 0.97)$, $D(0.97, 0.98)$.

In this case, it is not possible to arrange these four points so that *B* will lie between *A* and *C* and also between *A* and *D* and, moreover, so that *C* will lie between *A* and *D* and also between *B* and *D*. So in this case the answer to the question is negative.

1') Let us take the straight line $b \in E_3W_2$ with system of equations

$$\begin{cases} 0.01 \times_2 x -_n 0.01 \times_2 y = 0, \\ z = 0, \end{cases}$$

and four distinct points $A, B, C, D \in b$ with coordinates $A(0.96, 0.96, 0)$, $B(0.96, 0.97, 0)$, $C(0.97, 0.97, 0)$, $D(0.97, 0.98, 0)$.

In this case, it is not possible to arrange these four points so that *B* will lie between *A* and *C* and also between *A* and *D* and, moreover, so that *C* will lie between *A* and *D* and also between *B* and *D*. So in this case the answer to the question is negative.

2) Let us take the straight line $a \in E_2W_2$ with equation

$$x -_2 y = 0$$

and four points $A, B, C, D \in a$ with coordinates $A(0, 0)$, $B(0.5, 0.5)$, $C(1, 1)$, $D(1.5, 1.5)$. These four points are already arranged so that *B* lies between *A* and *C* and also between *A* and *D* and, moreover, so that *C* lies between *A* and *D* and also between *B* and *D*. So in this case the answer to the question is positive.

2') Let us take the straight line $b \in E_3W_2$ with system of equations

$$\begin{cases} x -_2 y = 0, \\ z = 0, \end{cases}$$

and four points $A, B, C, D \in b$ with coordinates $A(0, 0, 0)$, $B(0.5, 0.5, 0)$, $C(1, 1, 0)$, $D(1.5, 1.5, 0)$. These four points are already arranged so that *B* lies between *A* and *C* and also between *A* and *D* and, moreover, so that *C* lies between *A* and *D* and also between *B* and *D*. So in this case the answer to the question is positive.

So we have proved the following:

Theorem 5.17.
In Mathematics with Observers geometry in plane E_2W_n, there are a straight line a and four distinct points $A, B, C, D \in a$ such that it is impossible to arrange these four points so that B will lie between A and C and also between A and D and, moreover, so that C will lie between A and D and also between B and D.

Theorem 5.18.
In Mathematics with Observers geometry in plane E_2W_n, there are a straight line a and four distinct points $A, B, C, D \in a$ such that it is possible to arrange these four points so that B will lie between A and C and also between A and D and, moreover, so that C will lie between A and D and also between B and D.

Theorem 5.19.
In Mathematics with Observers geometry in space E_3W_n, there are a straight line a and four distinct points $A, B, C, D \in a$ such that it is impossible to arrange these four points so that B will lie between A and C and also between A and D and, moreover, so that C will lie between A and D and also between B and D.

Theorem 5.20.
In Mathematics with Observers geometry in space E_3W_n, there are a straight line a and four distinct points $A, B, C, D \in a$ such that it is possible to arrange these four points so that B will lie between A and C and also between A and D and, moreover, so that C will lie between A and D and also between B and D.

5.5 Fifth property of points order

Let us consider three distinct points $\in E_2W_n$ not lying in the same straight line:

$$A(x_1, y_1), B(x_2, y_2), C(x_3, y_3).$$

Let $a \in E_2W_n$ be a straight line not passing through any of the points *A*, *B*, *C*.

Question: Is the following statement correct or not in Mathematics with Observers geometry: If line *a* passes through a point of the segment AB, then it also passes through either a point of the segment BC or a point of the segment AC?

1) Let us consider three distinct points $\in E_2W_n$:

$$A(0, 0), B(0, 3), C(1, 3).$$

These points do not lie in the same straight line: Points *A*, *B* lie in the straight line

$$x = 0,$$

points *A*, *C* lie in the straight line

$$3 \times_n x -_n y = 0,$$

and points *B*, *C* lie in the straight line

$$y -_n 3 = 0.$$

Let us take the straight line *a* with equation

$$y -_n 1 = 0.$$

It passes through a point $D(0, 1)$ of the segment AB, has no common points with the segment BC, and has no common points with the segment AC. So in this case the answer to the question is negative.

2) Let's consider three distinct points $\in E_2W_n$:

$$A(0, 0), B(0, 6), C(2, 6).$$

These points do not lie in the same straight line: Points *A*, *B* lie in the straight line

$$x = 0,$$

points *A*, *C* lie in the straight line

$$3 \times_n x -_n y = 0,$$

and points *B*, *C* lie in the straight line

$$y -_n 6 = 0.$$

Let us take the straight line *a* with equation

$$y -_n 3 = 0.$$

It passes through a point $D(0, 3)$ of the segment AB, has no common points with the segment BC, and passes through a point $F(1, 3)$ of the segment AC. So in this case the answer to the question is positive.

So we have proved the following:

Theorem 5.21.
In Mathematics with Observers geometry in plane E_2W_n, there are three distinct points A, B, C not lying in the same straight line and a straight line a not passing through any of these points such that line a passes through a point of the segment AB and does not pass through either a point of the segment BC or a point of the segment AC.
Theorem 5.22.

In Mathematics with Observers geometry in plane E_2W_n, there are three distinct points A, B, C not lying in the same straight line and a straight line a not passing through any of these points such that line a passes through a point of the segment AB and passes through either a point of the segment BC or a point of the segment AC.

5.6 Number of points theorem

Classical geometry states that

> "Between any two points of a straight line, there always exists an unlimited number of points."

Question: Is the following statement correct or not in Mathematics with Observers geometry: Between any two points of a straight line, there always exists an unlimited number of points?

1) Let us take the straight line $a \in E_2W_2$ with equation

$$99.99 \times_2 x -_2 y = 0$$

and two points $A, C \in a$ with coordinates $A(0.62, 61.92)$, $C(0.63, 62.91)$. In this case, there is no point $B \in a$ such that *B* lies between *A* and *C*. So in this case the answer to the question is negative.

1') Let us take the straight line $a \in E_3W_2$ with equation

$$\begin{cases} 99.99 \times_2 x -_2 y = 0, \\ z = 0, \end{cases}$$

and two points $A, C \in a$ with coordinates $A(0.62, 61.92, 0)$, $C(0.63, 62.91, 0)$. In this case, there is no point $B \in a$ such that *B* lies between *A* and *C*. So in this case the answer to the question is negative.

2) Let us take any straight lines $a \in E_2W_2$ and $b \in E_3W_2$. From the point of view of W_9-observer, the space E_2W_2 contains no more than $4 \times_9 10^8$ points, and from the point of view of W_{13}-observer, the space E_3W_2 contains no more than $8 \times_{13} 10^{12}$ points. So in this case the answer to the question is negative.

3) The set W_n has exactly $2 \times_m 10^{2 \times_m n} -_m 1$ elements from the W_m-observer point of view, $m \geq 2 \times_n n +_n 1$.

The set E_2W_n has exactly $4 \times_m 10^{4\times_m n} -_m 1$ points from the W_m-observer point of view, $m \geq 2 \times_n 2 \times_n n +_n 1$.

The set E_3W_n has exactly $8 \times_m 10^{6\times_m n} -_m 1$ points from the W_m-observer point of view, $m \geq 2 \times_n 3 \times_n n +_n 1$.

So we have proved the following:

Theorem 5.23.
In Mathematics with Observers geometry in plane E_2W_n, there are a straight line a and two distinct points $A, B \in a$ such that there are no points between A and B.
Theorem 5.24.
In Mathematics with Observers geometry in space E_3W_n, there are a straight line a and two distinct points $A, B \in a$ such that there are no points between A and B.
Theorem 5.25.
In Mathematics with Observers geometry the plane E_2W_n (and so on any straight line in this plane) contains no more than $4 \times_m 10^{4\times_m n} -_m 1$ points from the W_m-observer point of view, $m \geq 2 \times_n 2 \times_n n +_n 1$.
Theorem 5.26.
In Mathematics with Observers geometry the space E_3W_n (and so on any straight line in this space) contains no more than $4 \times_m 10^{4\times_m n} -_m 1$ points from the W_m-observer point of view, $m \geq 2 \times_n 2 \times_n n +_n 1$.

5.7 Line and regions theorem

Classical geometry states the following:

> "Every straight line *a* that lies in a plane *α* divides the remaining points of this plane into two regions having the following properties: Every point *A* of the one region determines for each point *B* of the other region a segment AB containing a point of the straight line *a*. On the other hand, any two points *A*, A' of the same region determine a segment AA' containing no point of *a*."

Let us first consider the statement

> "Every straight line *a* that lies in a plane *α* divides the remaining points of this plane into two regions."

Question: Is this statement correct in Mathematics with Observers?

Let us take the straight line $a \in E_2W_n$:

$$a : a_1 \times_n x +_2 a_2 \times_n y +_2 a_3 = 0$$

for all $a_1, a_2, a_3, a_1 \times_n x, a_2 \times_n y, a_1 \times_n x +_n a_2 \times_n y \in W_n$ such that $(a_1, a_2) \neq (0, 0)$ and define the following regions $R_1^a, R_2^a \subset E_2W_n$:

R_1^a:

$$\begin{cases} a_1 \times_n x +_n a_2 \times_n y +_n a_3 \in W_n, \\ a_1 \times_n x +_n a_2 \times_n y +_n a_3 > 0; \end{cases}$$

R_2^a:

$$\begin{cases} a_1 \times_n x +_n a_2 \times_n y +_n a_3 \in W_n, \\ a_1 \times_n x +_n a_2 \times_n y +_n a_3 < 0. \end{cases}$$

Consider the set $R^a \subset E_2W_n$:

$$R^a = R_1^a \cup a \cup R_2^a.$$

Question: $R^a = E_2W_n$ or $R_3^a = E_2W_n \setminus R^a \neq \Lambda$?

Here sign "\" means the operation "minus" for sets (for sets *A* and *B*, $A \setminus B$ is the set of all elements of *A* not belonging to *B*), and Λ means the empty set.

Let's consider several examples.

Example 1.

Let

$$a : y = 0.$$

Then

$$R_1^a : y > 0,$$
$$R_2^a : y < 0,$$
$$R_3^a = \Lambda.$$

This means that we have the positive answer in this case.

Example 2.

Let

$$a : 3 \times_n x = 0.$$

Then

$$R_1^a : 3 \times_n x > 0,$$
$$R_2^a : 3 \times_n x < 0,$$
$$R_3^a = (x, y), x \in [-99\ldots 9.99\ldots 9, -33\ldots 3.33\ldots 34] \cup$$
$$\cup [33\ldots 3.33\ldots 34, 99\ldots 9.99\ldots 9], y \in W_n.$$

This means that we have the negative answer in this case.

Example 3.

Let

$$a : 99\ldots 9.99\ldots 9 \times_n x +_n 99\ldots 9.99\ldots 9 \times_n y = 0.$$

We get that

$$99\ldots 9.99\ldots 9 \times_n x, 99\ldots 9.99\ldots 9 \times_n y \in W_n$$

if and only if the points $(x, y) \in a$ satisfy the system

Q:

$$\begin{cases} -1 \leq x \leq 1, \\ -1 \leq y \leq 1. \end{cases}$$

The set $Q \subset E_2 W_n$ is the square with center (0,0) and side 2. Then

R_1^a:

$$\begin{cases} 99\ldots 9.99\ldots 9 \times_n x +_n 99\ldots 9.99\ldots 9 \times_n y \in W_n, \\ 99\ldots 9.99\ldots 9 \times_n x +_n 99\ldots 9.99\ldots 9 \times_n y > 0, \end{cases}$$

and

R_2^a:

$$\begin{cases} 99\ldots 9.99\ldots 9 \times_n x +_n 99\ldots 9.99\ldots 9 \times_n y \in W_n, \\ 99\ldots 9.99\ldots 9 \times_n x +_n 99\ldots 9.99\ldots 9 \times_n y < 0. \end{cases}$$

Note that

$$R^a = R_1^a \cup a \cup R_2^a \subset Q,$$

but

$$Q \setminus R^a \neq \Lambda,$$

because, for example,

$$(x,y) = (0.99\ldots9, 0.99\ldots9) \not\subset R^a.$$

This means that

$$R_3^a = E_2W_n \setminus R^a = (E_2W_n \setminus Q) \cup \big(Q \setminus R^a\big).$$

So we have the negative answer in this case.

Example 4.

Let

$$a : y -_n 99\ldots9.99\ldots9 = 0.$$

Then

$$R_1^a : y -_n 99\ldots9.99\ldots9 > 0.$$

So

$$R_1^a = \Lambda,$$
$$R_2^a = E_2W_n \setminus a,$$
$$R_3^a = \Lambda.$$

This means that we have the negative answer in this case.

So we have proved the following:

Theorem 5.27.
In Observer's geometry in the plane E_2W_n, there is a straight line a transforming the remaining points of this plane into the region R_1^a or R_2^a, where

$$a : a_1 \times_n x +_2 a_2 \times_n y +_2 a_3 = 0;$$

R_1^a:

$$\begin{cases} a_1 \times_n x +_n a_2 \times_n y +_n a_3 \in W_n, \\ a_1 \times_n x +_n a_2 \times_n y +_n a_3 > 0; \end{cases}$$

R_2^a:

$$\begin{cases} a_1 \times_n x +_n a_2 \times_n y +_n a_3 \in W_n, \\ a_1 \times_n x +_n a_2 \times_n y +_n a_3 < 0. \end{cases}$$

Theorem 5.28.
In Observer's geometry in the plane E_2W_n, there is a straight line a dividing the remaining points of this plane into two regions R_1^a and R_2^a, where

$$a : a_1 \times_n x +_2 a_2 \times_n y +_2 a_3 = 0;$$

R_1^a:

$$\begin{cases} a_1 \times_n x +_n a_2 \times_n y +_n a_3 \in W_n, \\ a_1 \times_n x +_n a_2 \times_n y +_n a_3 > 0; \end{cases}$$

R_2^a:

$$\begin{cases} a_1 \times_n x +_n a_2 \times_n y +_n a_3 \in W_n, \\ a_1 \times_n x +_n a_2 \times_n y +_n a_3 < 0. \end{cases}$$

Theorem 5.29.

In Observer's geometry in the plane E_2W_n, there is a straight line a dividing the remaining points of this plane into three regions R_1^a, R_2^a, and R_3^a, where

$$a : a_1 \times_n x +_2 a_2 \times_n y +_2 a_3 = 0;$$

R_1^a:

$$\begin{cases} a_1 \times_n x +_n a_2 \times_n y +_n a_3 \in W_n, \\ a_1 \times_n x +_n a_2 \times_n y +_n a_3 > 0; \end{cases}$$

R_2^a:

$$\begin{cases} a_1 \times_n x +_n a_2 \times_n y +_n a_3 \in W_n, \\ a_1 \times_n x +_n a_2 \times_n y +_n a_3 < 0; \end{cases}$$

$$R_3^a = E_2W_n \setminus \left(R_1^a \cup a \cup R_2^a\right).$$

Questions:

a) Is the following statement correct in Mathematics with Observers geometry: For $R_3^a = \Lambda$, every point *A* of the one region determines with each point *B* of the other region a segment AB containing a point of the straight line *a*?

b) Is the following statement correct in Mathematics with Observers geometry: For $R_3^a = \Lambda$, any two points *A*, A' of the same region determine a segment AA' containing no point of *a*?

Let us start with question a).

1) Let a plane $\alpha \in E_3W_2$ have the equation

$$z = 0.$$

So we are in E_2W_2. Let the straight line *a* have the equation

$$y = -1,$$

and let R_1^a, R_2^a be two regions of plane *α*:

$$R_1^a = \left[(x, y)\right], (x, y) \in E_2W_2, y > -1,$$
$$R_2^a = \left[(x, y)\right], (x, y) \in E_2W_2, y < -1,$$

where *x* is any element $\in W_2$. So

$$\alpha = R_1^a \cup a \cup R_2^a.$$

Let us take two points

$$A(99.99, 0) \in R_1^a, \quad B(0, -98.88) \in R_2^a.$$

We looking for a straight line *b* as the set of points (x, y) satisfying the equation

$$a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0$$

and containing points *A*, *B*. We have

$$\begin{cases} a_1 \times_2 (99.99) +_2 a_2 \times_2 (0) +_2 a_3 = 0, \\ a_1 \times_2 (0) -_2 a_2 \times_2 (98.88) +_2 a_3 = 0. \end{cases}$$

We must have

$$|a_1| \leq 1, \quad |a_2| \leq 1.01.$$

This means that

$$a_1 \times_2 (99.99) = -a_2 \times_2 (98.88).$$

All possible positive a_1 form the set

$$\Phi = [0.01, 0.02, \ldots, 0.99, 1.00],$$

and we get

$$99.99 \times_2 \Phi = [0.99, 1.98, \ldots, 98.82, 99.99].$$

All possible negative a_2 form the set

$$\Psi = [-0.01, -0.02, \ldots, -0.99, -1.00, -1.01],$$

and we get

$$-98.88 \times_2 \Psi = [0.98, 1.96, \ldots, 97.74, 98.88, 99.86]$$

Direct calculations show that

$$99.99 \times_2 \Phi \cap -98.88 \times_2 \Psi = \Lambda.$$

So the straight line *b* containing the points $A(99.99, 0)$, $B(0, -98.88)$ does not exist, that is, a segment AB does not exist. So in this case the answer to the question is negative.

2) Let us again take the plane $\alpha \in E_3W_2$ with equation

$$z = 0.$$

So we are again in E_2W_2. Let again the equation of straight line *a* be

$$y = -1,$$

and let the regions R_1^a and R_2^a of plane *a* be

$$R_1^a = \big[(x,y)\big], (x,y) \in E_2W_2, y > -1,$$
$$R_2^a = \big[(x,y)\big], (x,y) \in E_2W_2, y < -1,$$
where *x* is any element $\in W_2$. So

$$\alpha = R_1^a \cup a \cup R_2^a.$$

Let us take other two points

$$A(99.99, 0), B(0, -98.37).$$

Again, we look for a straight line *b* as the set of points (x, y) satisfying the equation

$$a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0$$

and containing points *A*, *B*. We have

$$\begin{cases} a_1 \times_2 (99.99) +_2 a_2 \times_2 (0) +_2 a_3 = 0, \\ a_1 \times_2 (0) -_2 a_2 \times_2 (98.37) +_2 a_3 = 0. \end{cases}$$

We must have

$$|a_1| \leq 1, \quad |a_2| \leq 1.01.$$

This means that

$$a_1 \times_2 (99.99) = -a_2 \times_2 (98.37).$$

Again, all possible positive a_1 form the set

$$\Phi = [0.01, 0.02, \ldots, 0.99, 1.00],$$

and we get

$$99.99 \times_2 \Phi = [0.99, 1.98, \ldots, 98.82, 99.99].$$

All possible negative a_2 form the set

$$\Psi = [0.01, 0.02, \ldots, 0.99, 1.00, 1.01],$$

and we get

$$-98.37 \times_2 \Psi = [0.98, 1.96, \ldots, 97.29, 98.37, 99.35].$$

Direct calculations show that

$$99.99 \times_2 \Phi \cap -98.37 \times_2 \Psi = 61.92,$$

and we get only one point in intersection of these two sets, that is,

$$a_1 = 0.62; \quad a_2 = -0.63; \quad a_3 = -61.92.$$

So there is only one straight line *b* containing points $A(99.99, 0)$ and $B(0, -98.37)$, that is, a segment AB exists. Let us now see whether line *a* intersects the segment AB or not? We get the system of equations

$$\begin{cases} 0.62 \times_2 x -_2 0.63 \times_2 y -_2 61.92 = 0, \\ y +_2 1 = 0. \end{cases}$$

We have

$$0.62 \times_2 x = 61.29,$$

but direct calculation shows that

$$0.62 \times_2 98.89 = 61.24$$

and

$$0.62 \times_2 98.90 = 61.30.$$

This means that

$$a \cap b = \Lambda,$$

that is, the segment AB does not contain a point of the straight line *a*. So in this case the answer to the question is negative.

3) Let us again take the plane $\alpha \in E_3W_2$ with equation

$$z = 0.$$

So we are again in E_2W_2. Let now the straight line *a* equation be

$$y = -0.33,$$

and let two regions R_1^a, R_2^a of plane *a* be

$$R_1^a = \left[(x, y)\right], (x, y) \in E_2W_2, y > -0.33,$$
$$R_2^a = \left[(x, y)\right], (x, y) \in E_2W_2, y < -0.33,$$

where *x* is any element of W_2. So

$$\alpha = R_1^a \cup a \cup R_2^a.$$

Let us take two points

$$A(99.99, 0), B(0, -98.37).$$

Again, we look for a straight line *b* as the set of points (x, y) satisfying the equation

$$a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0$$

and containing points *A*, *B*. We have

$$\begin{cases} a_1 \times_2 (99.99) +_2 a_2 \times_2 (0) +_2 a_3 = 0, \\ a_1 \times_2 (0) -_2 a_2 \times_2 (98.37) +_2 a_3 = 0. \end{cases}$$

We must have

$$|a_1| \leq 1, \quad |a_2| \leq 1.01.$$

This means that

$$a_1 \times_2 (99.99) = -a_2 \times_2 (98.37).$$

Again, all possible positive a_1 form the set

$$\Phi = [0.01, 0.02, \ldots, 0.99, 1.00],$$

and we get

$$99.99 \times_2 \Phi = [0.99, 1.98, \ldots, 98.82, 99.99].$$

All possible negative a_2 form the set

$$\Psi = [-0.01, -0.02, \ldots, -0.99, -1.00, -1.01],$$

and we get

$$-98.37 \times_2 \Psi = [0.98, 1.96, \ldots, 97.29, 98.37, 99.35].$$

Direct calculations show that

$$99.99 \times_2 \Phi \cap -98.37 \times_2 \Psi = 61.92,$$

and we get only one point in intersection of these two sets, that is,

$$a_1 = 0.62; \quad a_2 = -0.63; \quad a_3 = -61.92.$$

So there is only one straight line *b* containing points $A(99.99, 0)$, $B(0, -98.37)$, that is, a segment AB exists. Let us now see whether line *a* intersects the segment AB or not. We get the system of equations

$$\begin{cases} 0.62 \times_2 x -_2 0.63 \times_2 y -_2 61.92 = 0, \\ y +_2 0.33 = 0. \end{cases}$$

We have

$$0.62 \times_2 x = 61.74,$$

and direct calculations show

$$
\begin{aligned}
0.62 \times_2 99.60 &= 61.74,\\
0.62 \times_2 99.61 &= 61.74,\\
0.62 \times_2 99.62 &= 61.74,\\
0.62 \times_2 99.63 &= 61.74,\\
0.62 \times_2 99.64 &= 61.74,\\
0.62 \times_2 99.65 &= 61.74,\\
0.62 \times_2 99.66 &= 61.74,\\
0.62 \times_2 99.67 &= 61.74,\\
0.62 \times_2 99.68 &= 61.74,\\
0.62 \times_2 99.69 &= 61.74.
\end{aligned}
$$

This means that

$$a \cap b = \big[(99.60, -0.33), (99.61, -0.33), \ldots, (99.69, -0.33)\big],$$

that is, the segment AB contains ten points of the straight line *a*. So in this case the answer to the question is positive.

4) Let us again take the plane $\alpha \in E_3W_2$ with equation

$$z = 0.$$

So we are again in E_2W_2. Let again the straight line *a* equation be

$$y = -1,$$

and let two regions R_1^a, R_2^a of plane *a* be

$$
\begin{aligned}
R_1^a &= \big[(x, y)\big], (x, y) \in E_2W_2, y > -1,\\
R_2^a &= \big[(x, y)\big], (x, y) \in E_2W_2, y < -1,
\end{aligned}
$$

where *x* is any element of W_2. So

$$\alpha = R_1^a \cup a \cup R_2^a.$$

Let us take other two points

$$A(0, 1), B(0, -2).$$

Again, we look for a straight line *b* as the set of points (x, y) satisfying the equation

$$a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0$$

and containing points *A*, *B*. We have

$$
\begin{cases}
a_1 \times_2 (0) +_2 a_2 \times_2 (1) +_2 a_3 = 0,\\
a_1 \times_2 (0) -_2 a_2 \times_2 (2) +_2 a_3 = 0,
\end{cases}
$$

that is,

$$\begin{cases} a_2 +_2 a_3 = 0, \\ -a_2 \times_2 2 +_2 a_3 = 0, \end{cases}$$

that is,

$$\begin{cases} a_2 = 0, \\ a_3 = 0, \end{cases}$$

and line *b* has the equation

$$a_1 \times_2 x = 0.$$

So we have three distinct straight lines:

1. For each a_1 such that

$$1 \le |a_1| \le 99.99,$$

we get a straight line *c* as the set of points $(0, y)$ with any $y \in W_2$.

2. For each a_1 such that

$$0.1 \le |a_1| \le 0.99,$$

we get a straight line *d* with the set of points

$$\big[(0, y); (\pm 0.01, y); (\pm 0.02, y); \ldots; (\pm 0.09, y)\big]$$

with all $y \in W_2$.

3. For each a_1 such that

$$0 < |a_1| \le 0.09,$$

we get a straight line *e* as the set of points

$$\big[(0, y); (\pm 0.01, y); (\pm 0.02, y); \ldots; (\pm 0.09, y); \ldots; (\pm 0.99, y)\big]$$

with all $y \in W_2$, and we have

$$c \subset d \subset e.$$

If we take

$$a_1 = 1,$$

then we get

$$b = c,$$

and line *b* has the equation

$$x = 0.$$

In this case,

$$a \cap b = (0, -1),$$

that is, the segment AB exists and contains one point of the straight line *a*. So in this case the answer to the question is positive.

So we have proved the following:

Theorem 5.30.
In Observer's geometry in the plane E_2W_n, there are a straight line a with $R_3^a = \Lambda$ and point A of the region R_1^a and point B of the region R_2^a such that the segment AB contains no point of the straight line a.
Theorem 5.31.
In Observer's geometry in the plane E_2W_n, there are a straight line a with $R_3^a = \Lambda$ and point A of the region R_1^a and point B of the region R_2^a such that the segment AB contains exactly one point of the straight line a.
Theorem 5.32.
In Observer's geometry in the plane E_2W_n, there are a straight line a with $R_3^a = \Lambda$ and point A of the region R_1^a and point B of the region R_2^a such that the segment AB contains more than one point of the straight line a.
Now let us go to question b).

1) Let the plane $\alpha \in E_3W_2$ have the equation

$$z = 0.$$

So we are in E_2W_2. Let the equation of a straight line $a \in E_2W_2$ be

$$y = 0,$$

and two regions R_1^a, R_2^a of plane *a* be

$$R_1^a = \left[(x, y)\right], (x, y) \in E_2W_2, y > 0,$$
$$R_2^a = \left[(x, y)\right], (x, y) \in E_2W_2, y < 0,$$

where *x* is any element $\in W_2$. So

$$\alpha = R_1^a \cup a \cup R_2^a.$$

Let us take two points $A, A' \in R_1$:

$$A(0.09, 0.19), A'(0.21, 0.43) \in R_1.$$

We get a straight line *b* as the set of points (x, y) satisfying the equation

$$2 \times_2 x -_2 y +_2 0.01 = 0$$

and containing points *A*, A'. The segment AA' contains the points

$$\left[(0.09, 0.19), (0.10, 0.21), \ldots, (0.20, 0.41), (0.21, 0.43)\right]$$

but contains no point of *a* and region R_2^a. So in this case the answer to the question is positive.

We have another straight line *c* as the set of points (x, y) satisfying the equation

$$0.01 \times_2 x +_2 0.01 \times_2 y = 0$$

and containing points *A*, A'. The segment AA' contains the points

$$\left[(0.09, 0), (0.09, \pm 0.01), \ldots, (0.09, \pm 0.99), (0.10, 0), (0.10, \pm 0.01), \ldots, (0.10, \pm 0.99), \ldots \right.$$
$$\left. \ldots, (0.21, 0), (0.21, \pm 0.01), \ldots, (0.21, \pm 0.99)\right]$$

and contains many points of line *a* and regions R_1, R_2. So in this case the answer to the question is negative.

2) Let the plane $\alpha \in E_3W_2$ have the equation

$$z = 0.$$

So we are in E_2W_2. Let the equation of a straight line *a* be

$$y = 1,$$

and let two regions R_1^a, R_2^a of plane *a* be

$$R_1^a = \left[(x, y)\right], (x, y) \in E_2W_2, y > 1,$$
$$R_2^a = \left[(x, y)\right], (x, y) \in E_2W_2, y < 1,$$

where *x* is any element $\in W_2$. So

$$\alpha = R_1^a \cup a \cup R_2^a$$

Let's take two points

$$A(99.99, 0), B(0, -98.88) \in R_2.$$

We look for a straight line *b* as the set of points (x, y) satisfying the equation

$$a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0$$

and containing points *A*, *B*. We have

$$\begin{cases} a_1 \times_2 (99.99) +_2 a_2 \times_2 (0) +_2 a_3 = 0, \\ a_1 \times_2 (0) -_2 a_2 \times_2 (98.88) +_2 a_3 = 0. \end{cases}$$

We must have

$$|a_1| \leq 1, \quad |a_2| \leq 1.01.$$

This means that

$$a_1 \times_2 (99.99) = -a_2 \times_2 (98.88).$$

All possible positive a_1 form the set

$$\Phi = [0.01, 0.02, \ldots, 0.99, 1.00],$$

and we get

$$99.99 \times_2 \Phi = [0.99, 1.98, \ldots, 98.82, 99.99].$$

All possible negative a_2 form the set

$$\Psi = [-0.01, -0.02, \ldots, -0.99, -1.00, -1.01],$$

and we get

$$-98.88 \times_2 \Psi = [0.98, 1.96, \ldots, 97.74, 98.88, 99.86].$$

Direct calculations show that

$$99.99 \times_2 \Phi \cap -98.88 \times_2 \Psi = \Lambda.$$

So a straight line *b* containing the points $A(99.99, 0)$, $B(0, -98.88)$ does not exist, that is, the segment AB does not exist. So in this case the answer to the question is negative.

So we have proved the following:

Theorem 5.33.
In Observer's geometry in the plane E_2W_n, there are a straight line a with $R_3^a = \Lambda$ and points A, A' of the region R_1^a such that the segment AA' contains no point of the straight line a.
Theorem 5.34.
In Observer's geometry in the plane E_2W_n, there are a straight line a with $R_3^a = \Lambda$ and points A, A' of the region R_1^a such that the segment AA' contains exactly one point of the straight line a.
Theorem 5.35.
In Observer's geometry in the plane E_2W_n, there are a straight line a with $R_3^a = \Lambda$ and points A, A' of the region R_1^a such that the segment AA' contains more than one point of the straight line a.

5.8 Polygon and regions theorem

Classical geometry calls a system of segments $AB, BC, CD, \ldots, KL$ without self-intersections (except points $A, B, \ldots, L$) a broken line joining *A* with *L* or shortly a broken line $ABCDE \ldots KL$. If the point *A* coincides with *L*, then the broken line is called a polygon. The segments $AB, BC, CD, \ldots, KA$ are called the sides of the polygon, and the points

$A, B, C, D, \ldots, K$ are called the vertices. Polygons having $3, 4, 5, \ldots, n$ vertices are called, respectively, triangles, quadrangles, pentagons, ..., *n*-gons.

Classical geometry states:

> "Every polygon whose vertices all lie in a plane *α* divides the points of this plane not belonging to the broken line into two regions, an interior and an exterior, having the following properties:
>
> **a)** If *A* is a point of the interior region (interior point) and *B* is a point of the exterior region (exterior point), then any broken line joining *A* and *B* must have at least one common point with the polygon.
> **b)** If, on the other hand, *A*, A' are two points of the interior and *B*, B' are two points of the exterior region, then there are always a broken line joining *A* with A' and a broken line joining *B* with B' without a common point with the polygon.
> **c)** There exist in the plane *α* that lie entirely outside the given polygon, but there are no straight lines that lie entirely within it".

Question: Are statements a), b), and c) correct in Mathematics with Observers geometry?

First, let us consider the statement "Every polygon whose vertices all lie in a plane *α* divides the points of this plane not belonging to the broken line into two regions, an interior and an exterior" and check it.

1) Let's consider the polygon in E_2W_n with four vertices

$$A(2,1), B(-2,1), C(-2,-1), D(2,-1)$$

and four sides, segments

$$AB, BC, CD, DA$$

lying on the corresponding straight lines *a*, *b*, *c*, *d* with equations

$$\begin{aligned} a &: y -_n 1 = 0, \\ b &: x +_n 2 = 0, \\ c &: y +_n 1 = 0, \\ d &: x -_n 2 = 0. \end{aligned}$$

The interior region R^{inter} of this polygon is the set of points $(x, y) \in E_2W_n$ satisfying the system

$$y -_n 1 < 0,$$
$$x +_n 2 > 0,$$
$$y +_n 1 > 0,$$
$$x -_n 2 < 0.$$

The exterior region R^{exter} of this polygon is the set of points

$$(x, y) \in E_2W_n \setminus \left(R^{\text{inter}} \cup AB \cup BC \cup CD \cup DA\right),$$

and we see that in this case,

$$R^{\text{inter}} \neq \Lambda,$$
$$R^{\text{exter}} \neq \Lambda.$$

2) Let us consider the polygon in E_2W_2 with four vertices

$$A(99.99, 99.99), B(-99.99, 99.99), C(-99.99, -99.99), D(99.99, -99.99)$$

and four sides, the segments

$$AB, BC, CD, DA$$

lying on the corresponding straight lines *a*, *b*, *c*, *d* with equations

$$a : y -_2 99.99 = 0,$$
$$b : x +_2 99.99 = 0,$$
$$c : y +_2 99.99 = 0,$$
$$d : x -_2 99.99 = 0.$$

The interior region R^{inter} of this polygon is the set of points $(x, y) \in E_2W_2$ satisfying the system

$$y -_2 99.99 < 0,$$
$$x +_2 99.99 > 0,$$
$$y +_2 99.99 > 0,$$
$$x -_2 99.99 < 0.$$

The exterior region R^{exter} of this polygon is the set of points

$$(x, y) \in E_2W_2 \setminus \left(R^{\text{inter}} \cup AB \cup BC \cup CD \cup DA\right),$$

and we see that

$$E_2W_2 = \left(R^{\text{inter}} \cup AB \cup BC \cup CD \cup DA\right),$$

that is,

$$R^{\text{exter}} = \Lambda.$$

This means that this polygon with vertices in a plane *α* does not divide the points of this plane not belonging to the broken line into two regions, an interior and an exterior, that is, in this case, we have only one region, the interior region.

3) Let us consider the polygon $\in E_2W_2$ with four vertices

$$A(0.00, 0.00), B(0.00, 9.00), C(0.01, 9.00), D(0.01, 0.00)$$

and four sides, the segments

$$AB, BC, CD, DA$$

lying on the corresponding straight lines *a*, *b*, *c*, *d* with equations

$$\begin{aligned} &a : x = 0, \\ &b : y -_2 9.00 = 0, \\ &c : x -_2 0.01 = 0, \\ &d : y = 0. \end{aligned}$$

The interior region R^{inter} of this polygon is the set of points $(x, y) \in E_2W_2$ satisfying the system

$$\begin{aligned} &x > 0, \\ &y -_2 9.00 < 0, \\ &x -_2 0.01 < 0, \\ &y > 0. \end{aligned}$$

The exterior region R^{exter} of this polygon is the set of points

$$(x, y) \in E_2W_2 \setminus \left(R^{\text{inter}} \cup AB \cup BC \cup CD \cup DA\right),$$

and we see that

$$\begin{aligned} &R^{\text{inter}} = \Lambda, \\ &E_2W_2 = R^{\text{exter}} \cup AB \cup BC \cup CD \cup DA. \end{aligned}$$

This means that this polygon with vertices in a plane *α* does not divide the points of this plane not belonging to the broken line into two regions, an interior and an exterior, that is, in this case, we have only one region, the exterior region.

So we have proved the following:

Theorem 5.36.
In Mathematics with Observers geometry in the plane E_2W_n, there is a polygon with all vertices in a plane α that divides the points of this plane not belonging to the broken line into two regions, an interior and an exterior.

Theorem 5.37.
In Mathematics with Observers geometry in the plane E_2W_n*, there is a polygon with all vertices in a plane a that pushes the points of this plane not belonging to the broken line into one region, an interior.*
Theorem 5.38.
In Mathematics with Observers geometry in the plane E_2W_n*, there is a polygon with all vertices in a plane a that pushes the points of this plane not belonging to the broken line into one region, an exterior.*
Let us go now to the general case of two regions and questions: Are statements a), b), and c) correct in Mathematics with Observers geometry?

Let us start with question a).

1) Let us consider the polygon $\in E_2W_n$ with four vertices

$$A(2,1), B(-2,1), C(-2,-1), D(2,-1)$$

and four sides, the segments

$$AB, BC, CD, DA$$

lying on the corresponding straight lines *a*, *b*, *c*, *d* with equations

$$\begin{aligned} a &: y -_n 1 = 0, \\ b &: x +_n 2 = 0, \\ c &: y +_n 1 = 0, \\ d &: x -_n 2 = 0. \end{aligned}$$

The interior region R^{inter} of this polygon is the set of points $(x,y) \in E_2W_n$ satisfying the system

$$\begin{aligned} y -_n 1 &< 0, \\ x +_n 2 &> 0, \\ y +_n 1 &> 0, \\ x -_n 2 &< 0. \end{aligned}$$

The exterior region R^{exter} of this polygon is the set of points

$$(x,y) \in E_2W_n \setminus \left(R^{\text{inter}} \cup AB \cup BC \cup CD \cup DA\right).$$

Let us take the points $E(0,0)$ of the interior region (interior point) and $F(3,9)$ of the exterior region (exterior point), and take the straight line *e* with equation

$$e : 3 \times_n x -_n y = 0,$$

and on this line, we have the segment EF with

$$E, F \in e$$

and

$$e \cap AB = \Lambda,$$
$$e \cap BC = \Lambda,$$
$$e \cap CD = \Lambda,$$
$$e \cap DA = \Lambda,$$

that is, line *e* has no common points with the polygon, that is, the segment EF has no common points with the polygon. So in this case the answer to the question is negative.

2) Let us again consider the polygon $\in E_2W_n$ with four vertices

$$A(2,1), B(-2,1), C(-2,-1), D(2,-1)$$

and four sides, the segments

$$AB, BC, CD, DA$$

lying on the corresponding straight lines *a*, *b*, *c*, *d* with equations

$$a : y -_n 1 = 0,$$
$$b : x +_n 2 = 0,$$
$$c : y +_n 1 = 0,$$
$$d : x -_n 2 = 0.$$

Let us take the points $E(0,0)$ of the interior region (interior point) and $F(3,6)$ of the exterior region (exterior point) and take the straight line *e* with equation

$$e : 2 \times_n x -_n y = 0.$$

We have

$$E, F \in e$$

and

$$e \cap AB = (0.5, 1),$$
$$e \cap BC = \Lambda,$$
$$e \cap CD = (-0.5, -1),$$
$$e \cap DA = \Lambda,$$

and so

$$EF \cap AB = (0.5, 1),$$

that is, the segment EF has one common point with the polygon. So in this case the answer to the question is positive.

So we have proved the following:

Theorem 5.39.
In Mathematics with Observers geometry in the plane E_2W_n, there are a polygon, a point A of the interior region (interior point), a point B of the exterior region (exterior point), and a broken line joining A and B such that these line and polygon have no common point.
Theorem 5.40.
In Mathematics with Observers geometry in the plane E_2W_n, there are a polygon, a point A of the interior region (interior point), a point B of the exterior region (exterior point), and a broken line joining A and B such that these line and polygon have at least one common point.
Let us go to question b).

1) Let us consider the polygon $\in E_2W_n$ with four vertices

$$A(2,-1), B(-2,-1), C(-2,0), D(2,0)$$

and four sides, the segments

$$AB, BC, CD, DA$$

lying on the corresponding straight lines *a*, *b*, *c*, *d* with equations

$$\begin{aligned} a &: y +_n 1 = 0, \\ b &: x +_n 2 = 0, \\ c &: y = 0, \\ d &: x -_n 2 = 0. \end{aligned}$$

Let us take two points (for example, we take $n = 2$)

$$E(0.09, 0.19), E'(0.21, 0.43)$$

of the exterior region (exterior points). We get a straight line *e* as the set of points (x, y) satisfying the equation

$$2 \times_2 x -_2 y +_2 0.01 = 0$$

containing points *E*, E'. The segment EE' contains the points

$$\big[(0.09, 0.19), (0.10, 0.21), \ldots, (0.20, 0.41), (0.21, 0.43)\big]$$

and points of the exterior region, but contains no points of polygon and interior region. So in this case the answer to the question is positive.

2) Let us consider the polygon $\in E_2W_n$ with four vertices

$$A(2,-1), B(-2,-1), C(-2,0), D(2,0)$$

and four sides, the segments

$$AB, BC, CD, DA$$

lying on the corresponding straight lines *a*, *b*, *c*, *d* with equations

$$a : y +_n 1 = 0,$$
$$b : x +_n 2 = 0,$$
$$c : y = 0,$$
$$d : x -_n 2 = 0.$$

Let us take two points (for example, we take $n = 2$)

$$F(-1, -0.5), F'(1, -0.5)$$

of the interior region (interior points). We get a straight line *f* as the set of points (x, y) satisfying the equation

$$y +_2 0.5 = 0$$

containing points *F*, F'. The segment FF' contains the points

$$[(-1, -0.5), (-0.99, -0.5), \ldots, (0.99, -0.5), (1, -0.5)]$$

and points of the interior region, but contains no point of the polygon and exterior region. So in this case answer for question is positive.

3) Let us take, for example, $n = 2$. Let us consider the polygon $\in E_2W_2$ with four vertices

$$A(-99.99, -98.99), B(-99.99, 98.99), C(99.99, 98.99), D(99.99, -98.99)$$

and four sides, the segments

$$AB, BC, CD, DA$$

lying on the corresponding straight lines *a*, *b*, *c*, *d* with equations

$$a : x +_2 99.99 = 0,$$
$$b : y -_2 98.99 = 0,$$
$$c : x -_2 99.99 = 0,$$
$$d : y +_2 98.99 = 0.$$

Let us take two points

$$G(0, 99.00), G'(0, -99.00)$$

of the exterior region (exterior points). We get a straight line *g* as the set of points (x, y) satisfying the equation

$$x = 0$$

containing points *G*, G'. The segment GG' contains the points

$$\left[(0,99.00),(0,98.99),(0,98.98),\ldots,(0,-98.98),(0,-98.99),(0,-99.00)\right]$$
points of the exterior region, and points of the polygon and interior region. We have the same situation not only for line g, but also for any broken line connecting points G and G'. So in this case the answer to this question is negative.

4) Let us again take, for example, $n = 2$. Let us consider the polygon $\in E_2W_2$ with twelve vertices

$$A_1(-1.00,-9.00), A_2(-1.00,0.00), A_3(0.00,0.00), A_4(0.00,5.00),$$
$$A_5(-1.00,5.00), A_6(-1.00,9.00), A_7(1.00,9.00), A_8(1.00,5.00),$$
$$A_9(0.01,5.00), A_{10}(0.01,0.00), A_{11}(1.00,0.00), A_{12}(1.00,-9.00)$$
and twelve sides, the segments

$$A_1A_2, A_2A_3, A_3A_4, A_4A_5, A_5A_6, A_6A_7, A_7A_8, A_8A_9, A_9A_{10}, A_{10}A_{11}, A_{11}A_{12}, A_{12}A_1$$
lying on the corresponding straight lines

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}$$
with equations

$$a_1 : x +_2 1.00 = 0,$$
$$a_2 : y = 0,$$
$$a_3 : x = 0,$$
$$a_4 : y -_2 5.00 = 0,$$
$$a_5 : x +_2 1.00 = 0,$$
$$a_6 : y -_2 9.00 = 0,$$
$$a_7 : x -_2 1.00 = 0,$$
$$a_8 : y -_2 5.00 = 0,$$
$$a_9 : x -_2 0.01 = 0,$$
$$a_{10} : y = 0,$$
$$a_{11} : x -_2 1.00 = 0,$$
$$a_{12} : y +_2 9.00 = 0.$$
Let us take two points

$$H(0.00,-2.00), H'(0.00,6.00)$$
of the exterior region (exterior points). We get a straight line h as the set of points (x, y) satisfying the equation

$$x = 0$$
containing points H, H'. The segment HH' contains the segment A_3A_4 of the polygon.

We have the same situation not only for line *h*, but also for any broken line connecting points *H* and H'. So in this case the answer to the question is negative.

So we have proved the following:

Theorem 5.41.
In Mathematics with Observers geometry in the plane E_2W_n, there are a polygon, two points B, B' of the exterior region (exterior points), and a broken line joining B and B' such that these line, polygon, and interior region have no common point.
Theorem 5.42.
In Mathematics with Observers geometry in the plane E_2W_n there is a polygon such that if A, A' are two points of the interior region and B, B' are two points of the exterior region, then there are a broken line joining A with A' and a broken line joining B with B' without a common point with the polygon.
Theorem 5.43.
In Mathematics with Observers geometry in the plane E_2W_n, there are a polygon and two points A, A' of the interior such that a broken line joining A with A' without a common point with the polygon does not exist.
Theorem 5.44.
In Mathematics with Observers geometry in the plane E_2W_n, there are a polygon and two points B, B' of the exterior such that a broken line joining B with B' without a common point with the polygon does not exist.
Let us go to question c).

1) Let us consider the polygon $\in E_2W_2$ with four vertices

$$A(99.99, 99.99), B(-99.99, 99.99), C(-99.99, -99.99), D(99.99, -99.99)$$
and four sides, the segments

$$AB, BC, CD, DA$$
lying on the corresponding straight lines *a*, *b*, *c*, *d* with equations

$$\begin{aligned}a &: y -_n 99.99 = 0,\\ b &: x +_n 99.99 = 0,\\ c &: y +_n 99.99 = 0,\\ d &: x -_n 99.99 = 0.\end{aligned}$$
The interior region R^{inter} of this polygon is the set of points $(x, y) \in E_2W_n$ satisfying the system

$$y -_n 99.99 < 0,$$
$$x +_n 99.99 > 0,$$
$$y +_n 99.99 > 0,$$
$$x -_n 99.99 < 0.$$

The exterior region R^{exter} of this polygon is the set of points

$$(x, y) \in E_2W_n \setminus \left(R^{\text{inter}} \cup AB \cup BC \cup CD \cup DA\right).$$

We see that there are no straight lines in E_2W_2 that lie entirely outside this polygon, but there are many straight lines that lie entirely within it. For example, taking line *e* with equation

$$e : 2 \times_2 x -_2 y = 0,$$

we get

$$e \cap a = \Lambda,$$
$$e \cap b = \Lambda,$$
$$e \cap c = \Lambda,$$
$$e \cap d = \Lambda,$$

because for any point $(x, y) \in e$,

$$x \in [-49.99, -49.98, \ldots, 49.99]$$

and

$$y \in [-99.98, -99.97, \ldots, 99.98].$$

So in this case the answer to this question is negative.

2) Let us consider the polygon in E_2W_n with four vertices

$$A(2, 1), B(-2, 1), C(-2, -1), D(2, -1)$$

and four sides, the segments

$$AB, BC, CD, DA$$

lying on the corresponding straight lines *a*, *b*, *c*, *d* with equations

$$a : y -_n 1 = 0,$$
$$b : x +_n 2 = 0,$$
$$c : y +_n 1 = 0,$$
$$d : x -_n 2 = 0.$$

Let us take the straight line *e* with equation

$$e : y = 3.$$

This line lies entirely outside the given polygon. So in this case the answer to this question is positive.

3) Let us consider the polygon $\in E_2W_2$ with four vertices

$$A(2,2), B(-2,2), C(-2,-2), D(2,-2)$$

and four sides, the segments

$$AB, BC, CD, DA$$

lying on the corresponding straight lines *a*, *b*, *c*, *d* with equations

$$\begin{aligned} a &: y -_n 2 = 0, \\ b &: x +_n 2 = 0, \\ c &: y +_n 2 = 0, \\ d &: x -_n 2 = 0. \end{aligned}$$

Let us take the straight line *e* with equation

$$e : 99.99 \times_2 x +_2 99.99 \times_2 y = 0,$$

and we get

$$\begin{aligned} e \cap a &= \Lambda, \\ e \cap b &= \Lambda, \\ e \cap c &= \Lambda, \\ e \cap d &= \Lambda, \end{aligned}$$

because for all points $(x, y) \in e$, we have

$$x, y \in [-1, -0.99, \ldots, 0.99, 1].$$

So this line lies entirely inside the given polygon. So in this case the answer to this question is negative.

So we have proved the following:

Theorem 5.45.
In Mathematics with Observers geometry in the plane E_2W_n, there are a polygon and a straight line such that this line lies entirely outside this polygon, that is, belongs to the exterior region of this polygon.

Theorem 5.46.
In Mathematics with Observers geometry in the plane E_2W_n, there is a polygon such that there is no straight line lying entirely outside this polygon.

Theorem 5.47.

In Mathematics with Observers geometry in the plane E_2W_n, there is a polygon such that there is a straight line that lies entirely inside this polygon, that is, belongs to the interior region of this polygon.

5.9 Plane and regions theorem

Classical geometry states:

> "Every plane *α* divides the remaining points of the space into two regions having the following properties:
>
> **a)** Every point *A* of the region determines with each point *B* of the other region, the segment AB, within which lies a point of *α*.
>
> **b)** On the other hand, any two points *A*, A' lying within the same region determine the segment AA' containing no point of *α*."

Let us first consider the statement:

> "Every plane *α* divides the remaining points of E_3W_n into two regions."

Let us take the plane $\alpha \in E_3W_n$:

$$\alpha : a_1 \times_n x +_n a_2 \times_n y +_n a_3 \times_n z +_n a_4 = 0$$

for any

$$a_1, a_2, a_3, a_4, a_1 \times_n x, a_2 \times_n y, a_3 \times_n z, a_1 \times_n x +_n a_2 \times_n y +_n a_3 \times_n z \in W_n$$

such that $(a_1, a_2, a_3) \neq (0, 0, 0)$ and define the regions $R_1^\alpha, R_2^\alpha \subset E_3W_n$:

R_1^α:

$$\begin{cases} a_1 \times_n x +_n a_2 \times_n y +_n a_3 \times_n z +_n a_4 \in W_n, \\ a_1 \times_n x +_n a_2 \times_n y +_n a_3 \times_n z +_n a_4 > 0, \end{cases}$$

and

R_2^α:

$$\begin{cases} a_1 \times_n x +_n a_2 \times_n y +_n a_3 \times_n z +_n a_4 \in W_n, \\ a_1 \times_n x +_n a_2 \times_n y +_n a_3 \times_n z +_n a_4 < 0, \end{cases}$$

and consider the set $R^\alpha \subset E_3W_n$:

$$R^\alpha = R_1^\alpha \cup \alpha \cup R_2^\alpha$$

Question: Do we have $R^{\alpha} = E_3 W_n$ or $R_3^a = E_3 W_n \setminus R^{\alpha} \neq \Lambda$?

Let us consider several examples.

Example 1.

Let

$$\alpha : y = 0.$$

Then

$$\begin{aligned} &R_1^{\alpha} : y > 0, \\ &R_2^{\alpha} : y < 0, \\ &R_3^{\alpha} = \Lambda, \end{aligned}$$

which means the positive answer in this case.

Example 2.

Let

$$\alpha : 3 \times_n x = 0.$$

Then

$$\begin{aligned} &R_1^{\alpha} : 3 \times_n x > 0, \\ &R_2^{\alpha} : 3 \times_n x < 0, \\ &R_3^{\alpha} = (x, y, z), x \in [-99 \ldots 9.99 \ldots 9, -33 \ldots 3.33 \ldots 34] \cup \\ &\qquad \cup [33 \ldots 3.33 \ldots 34, 99 \ldots 9.99 \ldots 9], y, z \in W_n. \end{aligned}$$

This means the negative answer in this case.

Example 3.

Let

$$\alpha : 99 \ldots 9.99 \ldots 9 \times_n x +_n 99 \ldots 9.99 \ldots 9 \times_n y = 0.$$

We get that

$$99 \ldots 9.99 \ldots 9 \times_n x, 99 \ldots 9.99 \ldots 9 \times_n y, z \in W_n$$

if and only if the points $(x, y, z) \in \alpha$ satisfy the system

Q:

$$\begin{cases} -1 \leq x \leq 1, \\ -1 \leq y \leq 1, \end{cases}$$

with arbitrary *z*. The set $Q \subset E_3W_n$ is the square on the (x, y)-plane with center (0,0) and side 2. Then

R_1^α:

$$\begin{cases} 99\ldots9.99\ldots9 \times_n x +_n 99\ldots9.99\ldots9 \times_n y \in W_n, \\ 99\ldots9.99\ldots9 \times_n x +_n 99\ldots9.99\ldots9 \times_n y > 0, \end{cases}$$

and

R_2^α:

$$\begin{cases} 99\ldots9.99\ldots9 \times_n x +_n 99\ldots9.99\ldots9 \times_n y \in W_n, \\ 99\ldots9.99\ldots9 \times_n x +_n 99\ldots9.99\ldots9 \times_n y < 0. \end{cases}$$

Note that

$$R^\alpha = R_1^\alpha \cup \alpha \cup R_2^\alpha \subset Q,$$

but

$$Q \setminus R^\alpha \neq \Lambda,$$

because, for example,

$$(x, y, z) = (0.99\ldots9, 0.99\ldots9, z) \not\subset R^\alpha.$$

This means that

$$R_3^\alpha = E_3W_n \setminus R^\alpha = (E_3W_n \setminus Q) \cup \big(Q \setminus R^\alpha\big),$$

and thus the answer is negative in this case.

Example 4.

Let

$$\alpha : 99\ldots9.99\ldots9 \times_n x +_n 99\ldots9.99\ldots9 \times_n y +_n 99\ldots9.99\ldots9 \times_n z = 0.$$

We get that

$$99\ldots9.99\ldots9 \times_n x, 99\ldots9.99\ldots9 \times_n y, 99\ldots9.99\ldots9 \times_n z \in W_n$$

if and only if the points $(x, y, z) \in \alpha$ satisfy the system

P:

$$
\begin{aligned}
-1 \le x \le 1,\\
-1 \le y \le 1,\\
-1 \le z \le 1.
\end{aligned}
$$

The set $P \subset E_3W_n$ is the cube with center (0,0,0) and side =2. Then

R_1^α:

$$
\begin{cases}
99\ldots9.99\ldots9 \times_n x +_n 99\ldots9.99\ldots9 \times_n y +_n 99\ldots9.99\ldots9 \times_n z \in W_n,\\
99\ldots9.99\ldots9 \times_n x +_n 99\ldots9.99\ldots9 \times_n y +_n 99\ldots9.99\ldots9 \times_n z > 0,
\end{cases}
$$

and

R_2^α:

$$
\begin{cases}
99\ldots9.99\ldots9 \times_n x +_n 99\ldots9.99\ldots9 \times_n y +_n 99\ldots9.99\ldots9 \times_n z \in W_n,\\
99\ldots9.99\ldots9 \times_n x +_n 99\ldots9.99\ldots9 \times_n y +_n 99\ldots9.99\ldots9 \times_n z < 0.
\end{cases}
$$

Note that

$$R^\alpha = R_1^\alpha \cup \alpha \cup R_2^\alpha \subset P,$$

but

$$P \setminus R^\alpha \neq \Lambda,$$

because, for example,

$$(x, y, z) = (0.99\ldots9, 0.99\ldots9, 0.99\ldots9) \not\subset R.$$

This means that

$$R_3^\alpha = E_3W_n \setminus R^\alpha = (E_3W_n \setminus P) \cup \big(P \setminus R^\alpha\big),$$

and thus the answer is negative in this case.

Question: For $R_3^\alpha = \Lambda$, are statements a) and b) correct in Mathematics with Observers geometry?

Let us start with question a).

1) Let a plane $\alpha \in E_3W_2$ have the equation

$$y = -1,$$

and let R_1^α, R_2^α be two regions of the space E_3W_2:

$$
\begin{aligned}
R_1^\alpha &= \big[(x, y, z)\big], (x, y, z) \in E_3W_2, y > -1,\\
R_2^\alpha &= \big[(x, y, z)\big], (x, y, z) \in E_3W_2, y < -1,
\end{aligned}
$$

and so

$$E_3W_2 = R_1^\alpha \cup \alpha \cup R_2^\alpha$$

Let us take two points

$$A(99.99, 0, 0) \in R_1^\alpha, \quad B(0, -98.88, 0) \in R_2^\alpha.$$

We looking for a straight line *a* as the set of points (x, y, z) satisfying the system of equations

$$\begin{cases} a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0, \\ z = 0, \end{cases}$$

and containing points *A*, *B*. We have

$$\begin{cases} a_1 \times_2 (99.99) +_2 a_2 \times_2 (0) +_2 a_3 = 0, \\ a_1 \times_2 (0) -_2 a_2 \times_2 (98.88) +_2 a_3 = 0. \end{cases}$$

We must have

$$|a_1| \leq 1, \quad |a_2| \leq 1.01.$$

This means that

$$a_1 \times_2 (99.99) = -a_2 \times_2 (98.88).$$

All possible positive a_1 form the set

$$\Phi = [0.01, 0.02, \ldots, 0.99, 1.00],$$

and we get

$$99.99 \times_2 \Phi = [0.99, 1.98, \ldots, 98.82, 99.99].$$

All possible negative a_2 form the set

$$\Psi = [-0.01, -0.02, \ldots, -0.99, -1.00, -1.01],$$

and we get

$$-98.88 \times_2 \Psi = [0.98, 1.96, \ldots, 97.74, 98.88, 99.86].$$

Direct calculations show that

$$99.99 \times_2 \Phi \cap -98.88 \times_2 \Psi = \Lambda.$$

So a straight line *a* containing the points $A(99.99, 0, 0)$, $B(0, -98.88, 0)$ does not exist, that is, a segment AB does not exist. So in this case the answer to this question is negative.

2) Let us again take the plane $\alpha \in E_3W_2$ with equation

$$y = -1$$

and two regions R_1^α, R_2^α of the space E_3W_2:

$$R_1^\alpha = \big[(x,y,z)\big], (x,y,z) \in E_3W_2, y > -1,$$
$$R_2^\alpha = \big[(x,y,z)\big], (x,y,z) \in E_3W_2, y < -1.$$

So

$$E_3W_2 = R_1^\alpha \cup \alpha \cup R_2^\alpha.$$

Let us take other two points

$$A(99.99, 0, 0) \in R_1, \quad B(0, -98.37, 0) \in R_2.$$

Again, we look for a straight line *a* as th set of points (x, y, z) satisfying the system of equations

$$\begin{cases} a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0, \\ z = 0, \end{cases}$$

and containing points *A*, *B*. We have

$$\begin{cases} a_1 \times_2 (99.99) +_2 a_2 \times_2 (0) +_2 a_3 = 0, \\ a_1 \times_2 (0) -_2 a_2 \times_2 (98.37) +_2 a_3 = 0. \end{cases}$$

We must have

$$|a_1| \leq 1, \quad |a_2| \leq 1.01.$$

This means that

$$a_1 \times_2 (99.99) = -a_2 \times_2 (98.37).$$

Again, all possible positive a_1 form the set

$$\Phi = [0.01, 0.02, \ldots, 0.99, 1.00],$$

and we get

$$99.99 \times_2 \Phi = [0.99, 1.98, \ldots, 98.82, 99.99].$$

All possible negative a_2 form the set

$$\Psi = [0.01, 0.02, \ldots, 0.99, 1.00, 1.01],$$

and we get

$$-98.37 \times_2 \Psi = [0.98, 1.96, \ldots, 97.29, 98.37, 99.35].$$

Direct calculations show that

$$99.99 \times_2 \Phi \cap -98.37 \times_2 \Psi = 61.92,$$

and we get only one point in intersection of these two sets, that is,

$$a_1 = 0.62; \quad a_2 = -0.63; \quad a_3 = -61.92.$$

So there is only one straight line *a* containing points $A(99.99, 0)$, $B(0, -98.37)$, that is, a segment AB exists.

Let us now see if the plane *α* intersects the segment AB. We get the system of equations

$$0.62 \times_2 x -_2 0.63 \times_2 y -_2 61.92 = 0,$$
$$z = 0,$$
$$y +_2 1 = 0.$$

We have

$$0.62 \times_2 x = 61.29,$$

but direct calculation shows that

$$0.62 \times_2 98.89 = 61.24$$

and

$$0.62 \times_2 98.90 = 61.30.$$

This means that

$$\alpha \cap a = \Lambda,$$

that is, the segment AB does not contain a point of the plane *α*. So in this case the answer to the question is negative.

3) Let us again take a plane $\alpha \in E_3W_2$ with equation

$$y +_2 0.33 = 0$$

and two regions R_1^α, R_2^α of the space E_3W_2:

$$R_1^\alpha = \big[(x, y, z)\big], (x, y, z) \in E_3W_2, y > -0.33,$$
$$R_2^\alpha = \big[(x, y, z)\big], (x, y, z) \in E_3W_2, y < -0.33.$$

So

$$E_3W_2 = R_1^\alpha \cup \alpha \cup R_2^\alpha.$$

Let us take the same two points

$$A(99.99, 0, 0) \in R_1^\alpha, \quad B(0, -98.37, 0) \in R_2^\alpha.$$

Again, we look for a straight line *a* as the set of points (x, y, z) satisfying the system of equations

$$\begin{cases} a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0, \\ z = 0, \end{cases}$$

and containing points *A*, *B*. We have

$$\begin{cases} a_1 \times_2 (99.99) +_2 a_2 \times_2 (0) +_2 a_3 = 0, \\ a_1 \times_2 (0) -_2 a_2 \times_2 (98.37) +_2 a_3 = 0. \end{cases}$$

We must have

$$|a_1| \leq 1, \quad |a_2| \leq 1.01.$$

This means that

$$a_1 \times_2 (99.99) = -a_2 \times_2 (98.37).$$

Again, all possible positive a_1 form the set

$$\Phi = [0.01, 0.02, \ldots, 0.99, 1.00],$$

and we get

$$99.99 \times_2 \Phi = [0.99, 1.98, \ldots, 98.82, 99.99].$$

All possible negative a_2 form the set

$$\Psi = [-0.01, -0.02, \ldots, -0.99, -1.00, -1.01],$$

and we get

$$-98.37 \times_2 \Psi = [0.98, 1.96, \ldots, 97.29, 98.37, 99.35].$$

Direct calculations show that

$$99.99 \times_2 \Phi \cap -98.37 \times_2 \Psi = 61.92,$$

and we get only one point in intersection of these two sets, that is,

$$a_1 = 0.62; \quad a_2 = -0.63; \quad a_3 = -61.92.$$

So there is only one straight line *a* containing points $A(99.99, 0, 0)$, $B(0, -98.37, 0)$, that is, a segment AB exists. Let us now see if the plane *α* intersects the segment AB. We get the system of equations

$$\begin{aligned} &0.62 \times_2 x -_2 0.63 \times_2 y -_2 61.92 = 0, \\ &z = 0, \\ &y +_2 0.33 = 0. \end{aligned}$$

We have

$$0.62 \times_2 x = 61.74,$$

and a direct calculation shows that

$$\begin{aligned}0.62 \times_2 99.60 &= 61.74,\\ 0.62 \times_2 99.61 &= 61.74,\\ 0.62 \times_2 99.62 &= 61.74,\\ 0.62 \times_2 99.63 &= 61.74,\\ 0.62 \times_2 99.64 &= 61.74,\\ 0.62 \times_2 99.65 &= 61.74,\\ 0.62 \times_2 99.66 &= 61.74,\\ 0.62 \times_2 99.67 &= 61.74,\\ 0.62 \times_2 99.68 &= 61.74,\\ 0.62 \times_2 99.69 &= 61.74.\end{aligned}$$

This means that

$$\alpha \cap a = \left[(99.60, -0.33, 0), (99.61, -0.33, 0), \ldots, (99.69, -0.33, 0)\right],$$

that is, the segment AB contains ten points of the plane α. So in this case the answer to this question is positive.

So we have proved the following:

Theorem 5.48.
In Mathematics with Observers geometry in the space E_3W_n, there are a plane α that divides the remaining points of the space into two regions R_1^α, R_2^α (so $R_3^\alpha = \Lambda$) and two points $A \in R_1^\alpha$, $B \in R_2^\alpha$ such that the segment AB contains no point of α.

Theorem 5.49.
In Mathematics with Observers geometry in the space E_3W_n, there are a plane α that divides the remaining points of space into two regions R_1^α, R_2^α (so $R_3^\alpha = \Lambda$) and two points $A \in R_1^\alpha$, $B \in R_2^\alpha$ such that a segment AB contains exactly one point of α.

Theorem 5.50.
In Mathematics with Observers geometry in the space E_3W_n, there are a plane α that divides the remaining points of space into two regions R_1^α, R_2^α (so $R_3^\alpha = \Lambda$) and two points $A \in R_1^\alpha$, $B \in R_2^\alpha$ such that the segment AB contains more than one point of α.

Now let us go to question b).

1) Let us take the plane $\alpha \in E_3W_2$ with equation

$$y = 0$$

and two regions R_1^α, R_2^α of the space E_3W_2:

$$\begin{aligned}R_1^\alpha &= \left[(x, y, z)\right], (x, y, z) \in E_3W_2, y > 0,\\ R_2^\alpha &= \left[(x, y, z)\right], (x, y, z) \in E_3W_2, y < 0.\end{aligned}$$

So

$$E_3W_2 = R_1^\alpha \cup \alpha \cup R_2^\alpha.$$

Let us take two points $A, A' \in R_1^\alpha$:

$$A(0.09, 0.19, 0), A'(0.21, 0.43, 0) \in R_1^\alpha.$$

We get a straight line *a* as the set of points (x, y, z) satisfying the system of equations

$$\begin{cases} a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0, \\ z = 0, \end{cases}$$

containing points *A*, A'. The segment AA' contains the points

$$\big[(0.09, 0.19, 0), (0.10, 0.21, 0), \ldots, (0.20, 0.41, 0), (0.21, 0.43, 0)\big]$$

and contains no point of *α* and region R_2^α. So in this case the answer to the question is positive.

2) We have another straight line *b* as the set of points (x, y, z) satisfying the system of equations

$$\begin{cases} 0.01 \times_2 x +_2 0.01 \times_2 y = 0, \\ z = 0, \end{cases}$$

and containing points *A*, A'. The segment AA' contains the points

$$\big[(0.09, 0, 0), (0.09, \pm 0.01, 0), \ldots, (0.09, \pm 0.99, 0), (0.10, 0, 0), (0.10, \pm 0.01, 0), \ldots$$
$$\ldots, (0.10, \pm 0.99, 0), \ldots$$
$$\ldots, (0.21, 0, 0), (0.21, \pm 0.01, 0), \ldots, (0.21, \pm 0.99, 0)\big]$$

and many points of the plane *α* and regions R_1^α, R_2^α. So in this case the answer to the question is negative.

3) Let us take the plane $\alpha \in E_3W_2$ with equation

$$y = 1$$

and two regions R_1^α, R_2^α of the space E_3W_2:

$$R_1^\alpha = \big[(x, y, z)\big], (x, y, z) \in E_3W_2, y > 1,$$
$$R_2^\alpha = \big[(x, y, z)\big], (x, y, z) \in E_3W_2, y < 1.$$

So

$$E_3W_2 = R_1^\alpha \cup \alpha \cup R_2^\alpha.$$

Let us take two points

$$A(99.99, 0, 0), A'(0, -98.88, 0) \in R_2.$$

We look for a straight line *a* as the set of points (x, y, z) satisfying the system of equations

$$\begin{cases} a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0, \\ z = 0, \end{cases}$$

and containing points *A*, A'. We have

$$\begin{cases} a_1 \times_2 (99.99) +_2 a_2 \times_2 (0) +_2 a_3 = 0, \\ a_1 \times_2 (0) -_2 a_2 \times_2 (98.88) +_2 a_3 = 0. \end{cases}$$

We must have

$$|a_1| \leq 1, \quad |a_2| \leq 1.01.$$

This means that

$$a_1 \times_2 (99.99) = -a_2 \times_2 (98.88).$$

All possible positive a_1 form the set

$$\Phi = [0.01, 0.02, \ldots, 0.99, 1.00],$$

and we get

$$99.99 \times_2 \Phi = [0.99, 1.98, \ldots, 98.82, 99.99].$$

All possible negative a_2 form the set

$$\Psi = [-0.01, -0.02, \ldots, -0.99, -1.00, -1.01],$$

and we get

$$-98.88 \times_2 \Psi = [0.98, 1.96, \ldots, 97.74, 98.88, 99.86].$$

Direct calculations show that

$$99.99 \times_2 \Phi \cap -98.88 \times_2 \Psi = \Lambda.$$

So the straight line *a* containing points $A(99.99, 0, 0)$ and $A'(0, -98.88, 0)$ does not exist, that is, a segment AA' does not exist. So in this case the answer to the question is negative.

So we have proved the following:

Theorem 5.51.
In Mathematics with Observers geometry in the space E_3W_n, there are a plane α that divides the remaining points of the space into two regions R_1^α, R_2^α (so $R_3^\alpha = \Lambda$) and two points $A, A' \in R_1^\alpha$ such that the segment AA' contains points of R_1^α and no point of $\alpha \cup R_2^\alpha$.

Theorem 5.52.
In Mathematics with Observers geometry in the space E_3W_n, there are a plane α that divides the remaining points of space into two regions R_1^α, R_2^α (so $R_3^\alpha = \Lambda$) and two points $A, A' \in R_1^\alpha$ such that the segment AA' contains points of R_1^α, points of α, and points of R_2^α.

Theorem 5.53.

In Mathematics with Observers geometry in the space E_3W_n, there are a plane α that divides the remaining points of space into two regions R_1^α, R_2^α (so $R_3^\alpha = \Lambda$) and two points $A, A' \in R_1^\alpha$ such that the segment AA' does not exist, that is, the points A, A' do not lie together on any straight line.

6 Observability and properties of parallel straight lines

In classical geometry, two lines *a*, *b* lying in a plane *α* are called parallel if

$$a \cap b = \Lambda.$$

In Mathematics with Observers geometry we have to strengthen this definition in the following way: A straight line *a* that lies in a plane *α* divides the remaining points of this plane into three regions, R_1^a, R_2^a, and R_3^a. Also, a straight line *b* that lies in a plane *α* divides the remaining points of this plane into three regions, R_1^b, R_2^b, and R_3^b.

We call straight lines $a, b \subset \alpha$ parallel and write $a \parallel b$ if

$$\begin{cases} b \subset (R_1^a \cup R_3^a), \\ b \cap R_1^a \neq \Lambda, \end{cases}$$

or

$$\begin{cases} b \subset (R_2^a \cup R_3^a), \\ b \cap R_2^a \neq \Lambda, \end{cases}$$

and

$$a \cap b = \Lambda.$$

We call straight lines $b, a \subset \alpha$ parallel and write $b \parallel a$ if

$$\begin{cases} a \subset (R_1^b \cup R_3^b), \\ a \cap R_1^b \neq \Lambda, \end{cases}$$

or

$$\begin{cases} a \subset (R_2^b \cup R_3^b), \\ a \cap R_2^b \neq \Lambda, \end{cases}$$

and

$$b \cap a = \Lambda.$$

So we define the parallelism of straight lines not symmetrically: $a \parallel b$ and $b \parallel a$. Here we put additional conditions because, for example, we get the following situations:

1. Let $n = 2$ and consider two straight lines *a*, *b* in E_2W_2:

$$\begin{aligned} &a : 99.99 \times_2 x -_2 99.99 \times_2 y = 0, \\ &b : y = 2. \end{aligned}$$

Note that

$$a \subset \big[(x,y), x \in [-1,1], y \in [-1,1]\big].$$

We have

$$\begin{aligned}
&R_2^b = \big[(x,y), x \in W_2, y \in W_2, y < 2\big],\\
&a \subset R_2^b,\\
&b \cap a = \Lambda,
\end{aligned}$$

that is, $b \parallel a$. However,

$$\big(R_1^a \cup a \cup R_2^a\big) \cap b = \Lambda,$$

that is,

$$a \cap b = \Lambda,$$

we get $b \nparallel a$. This means that in this case, the relations $a \parallel b$ and $b \parallel a$ are not symmetric in Observer's geometry.

2. Let $n = 2$ and consider two straight lines *a*, *b* in E_2W_2:

$$\begin{aligned}
&a : y = 0,\\
&b : y = 2.
\end{aligned}$$

We have

$$\begin{aligned}
&R_2^b = \big[(x,y), x \in W_2, y \in W_2, y < 2\big],\\
&a \subset R_2^b,\\
&b \cap a = \Lambda,
\end{aligned}$$

that is, $b \parallel a$. Also,

$$\begin{aligned}
&R_1^a = \big[(x,y), x \in W_2, y \in W_2, y > 0\big],\\
&b \subset R_1^a,\\
&a \cap b = \Lambda,
\end{aligned}$$

so we get $a \parallel b$. This means that in this case, the relations $a \parallel b$ and $b \parallel a$ are symmetric in Observer's geometry.

3. Let us first consider first three straight lines $a, b, c \in E_2W_2$:

$$\begin{aligned}
&a : y = 2,\\
&b : x = 2,\\
&c : 99.99 \times_2 x -_2 99.99 \times_2 y = 0.
\end{aligned}$$

We have

$$
\begin{aligned}
&R_2^a = \big[(x, y), x \in W_2, y \in W_2, y < 2\big],\\
&c \subset R_2^a,\\
&a \cap c = \Lambda,
\end{aligned}
$$

that is, $a \parallel c$. Also,

$$
\begin{aligned}
&R_2^b = \big[(x, y), x \in W_2, y \in W_2, x < 2\big],\\
&c \subset R_2^b,\\
&b \cap c = \Lambda,
\end{aligned}
$$

that is, $b \parallel c$. However,

$$a \cap b = (2, 2) \neq \Lambda,$$

that is, $a \nparallel b$. So in this case the relations $a \parallel c$ and $b \parallel c$ ($a \neq b$) do not mean the relation $a \parallel b$, that is we do not have parallelism transitivity.

Note that we have the same situation in classical Gauss–Bolyai–Lobachevsky geometry.

4. Let us consider three straight lines $a, b, c \in E_2W_2$:

$$
\begin{aligned}
&a : y = 2,\\
&b : y = 1,\\
&c : y = 0.
\end{aligned}
$$

We have

$$
\begin{aligned}
&R_2^a = \big[(x, y), x \in W_2, y \in W_2, y < 2\big],\\
&c \subset R_2^a,\\
&a \cap c = \Lambda,
\end{aligned}
$$

that is, $a \parallel c$. Also,

$$
\begin{aligned}
&R_2^b = \big[(x, y), x \in W_2, y \in W_2, y < 1\big],\\
&c \subset R_2^b,\\
&b \cap c = \Lambda,
\end{aligned}
$$

that is, $b \parallel c$, and

$$
\begin{aligned}
&b \subset R_2^a,\\
&a \cap b = \Lambda,
\end{aligned}
$$

that is, $a \parallel b$.

So in this case the relations $a \parallel c$ and $b \parallel c$ ($a \neq b$) mean the relation $a \parallel b$, that is, we have parallelism transitivity.

Note that we have the same situation in classical Euclidean geometry.

5. Let us take two straight lines $a, b \in E_2W_2$, *a* with equation

$$3 \times_2 x -_2 y = 0,$$

and *b* with equation

$$y -_2 1 = 0.$$

Because the number 3 does not have an inverse number in W_2, we get

$$a \cap b = \Lambda,$$

that is, the straight lines *a* and *b* have no common points but are not parallel in common and Mathematics with Observers geometry senses. To see this, let us consider

$$R_1^b = \left[(x, y), (x, y) \in E_2W_2, y > 1\right],$$
$$R_2^b = \left[(x, y), (x, y) \in E_2W_2, y < 1\right],$$

and

$$R_3^b = \Lambda.$$

We get

$$\begin{cases} a \cap R_1^b \neq \Lambda, \\ a \cap R_2^b \neq \Lambda, \end{cases}$$

that is, $a \nparallel b$.

6. Let us take two straight lines $a, b \in E_2W_2$, *a* with equation

$$y = 0,$$

and *b* with equation

$$y -_2 1 = 0.$$

The straight line *a* divides the remaining points of E_2W_2 into two regions R_1^a and R_2^a with

$$R_1^a = \left[(x, y), (x, y) \in E_2W_2, y > 0\right],$$
$$R_2^a = \left[(x, y), (x, y) \in E_2W_2, y < 0\right].$$

So

$$E_2W_2 = R_1^a \cup a \cup R_2^a.$$

Also, the straight line *b* divides the remaining points of E_2W_2 into two regions R_1^b and R_2^b with

$$R_1^b = \big[(x,y),(x,y) \in E_2W_2, y > 1\big],$$
$$R_2^b = \big[(x,y),(x,y) \in E_2W_2, y < 1\big].$$

So

$$E_2W_2 = R_1^b \cup a \cup R_2^b.$$

Lines $b \subset R_1^a$ and $a \subset R_2^b$, and

$$a \cap b = \Lambda,$$

that is, the straight lines *a* and *b* are parallel in common and Mathematics with Observers geometry senses.

6.1 Parallel lines theorem

Classical Euclidean geometry states:

> "If two straight lines *a*, *b* of a plane do not meet a third straight line *c* of the same plane, then they do not meet each other."

Question: Is this statement correct in Observer's geometry?

1) Three straight lines in E_2W_2

$$b : -0.01 \times_2 x +_2 y -_2 1 = 0,$$
$$c : 0.01 \times_2 x +_2 y -_2 1 = 0,$$

and

$$d : y -_2 1 = 0$$

do not meet the fourth straight line

$$a : y = 0,$$

but

$$b \cap c \cap d = (0,1) \in E_2W_2.$$

So the answer to the question is negative.

2) Two straight lines in E_2W_2

$$a : 3 \times_2 x -_2 y = 0$$

and

$$b : 7 \times_2 x -_2 y = 0$$

do not meet the third straight line

$$c : y -_2 1 = 0,$$

but

$$a \cap b = (0, 0) \in E_2W_2$$

So the answer to this question in this case also is negative.

3) Two straight lines in E_2W_2

$$a : 3 \times_2 x -_2 y = 0$$

and

$$b : 3 \times_2 x -_2 y -_2 3 = 0$$

do not meet the third straight line

$$c : y -_2 1 = 0,$$

and *a* and *b* are parallel. Note that in this case, $a \nparallel c$ and $b \nparallel c$. So in this case the answer to the question is positive.

So we have proved the following:

Theorem 6.1.
In Mathematics with Observers geometry in the plane E_2W_n, there are three straight lines a, b, c such that $a \cap c = \Lambda$, $b \cap c = \Lambda$, and $a \cap b = \Lambda$.

Theorem 6.2.
In Mathematics with Observers geometry in the plane E_2W_n, there are three straight lines a, b, c such that $a \cap c = \Lambda$, $b \cap c = \Lambda$, and $a \cap b \neq \Lambda$.

6.2 Euclid's axiom

Question. Is the following statement correct in Mathematics with Observers geometry: In a plane *α*, through any point *A* lying outside of a straight line *a*, there can be drawn a unique straight line parallel to line *a*?

1) Let plane $\alpha \in E_3W_2$ have the equation

$$z = 0.$$

So we are in E_2W_2. Let straight line *a* have the equation

$$y = 0,$$

and let two regions R_1^a and R_2^a of plane *a* be

$$R_1^a = \big[(x,y)\big], (x,y) \in E_2W_2, y > 0,$$
$$R_2^a = \big[(x,y)\big], (x,y) \in E_2W_2, y < 0,$$

where *x* is any element of W_2. So

$$E_2W_2 = R_1^a \cup a \cup R_2^a.$$

Let us take two points

$$A(-99.99, 0.01), B(0,1) \in R_1.$$

We look for a straight line *b* as the set of points (x,y) satisfying the equation

$$a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0$$

and containing points *A*, *B*. We have

$$\begin{cases} -a_1 \times_2 (99.99) +_2 a_2 \times_2 (0.01) +_2 a_3 = 0, \\ a_1 \times_2 (0) +_2 a_2 \times_2 (1) +_2 a_3 = 0, \end{cases}$$

that is,

$$\begin{cases} -a_1 \times_2 (99.99) +_2 a_2 \times_2 (0.01) -_2 a_2 = 0, \\ a_3 = -a_2. \end{cases}$$

For

$$a_2 = 1,$$

we get

$$a_1 = -0.01,$$

and the equation of line *b* is

$$b : -0.01 \times_2 x +_2 y -_2 1 = 0.$$

We have

$$b \subset R_1^a$$

and

$$b \cap a = \Lambda.$$

We have

$$R_1^b = \big[(x,y)\big], (x,y) \in E_2W_2, -0.01 \times_2 x +_2 y -_2 1 > 0,$$
$$R_2^b = \big[(x,y)\big], (x,y) \in E_2W_2, -0.01 \times_2 x +_2 y -_2 1 < 0,$$

and

$$a \subset R_2^b.$$

So lines *a*, *b* are parallel, $a \parallel b$. Also, line *c* with equation

$$0.01 \times_2 x +_2 y -_2 1 = 0$$

is parallel to line *a* and contains the points *B* and $A'(99.99, 0.01)$. Consider straight line *d* with equation

$$y = 1.$$

We see that lines *a*, *d* are parallel and line *d* contains point *B*. This means that in the plane

$$\alpha : z = 0,$$

through point *B* lying outside straight line *a*, we can draw at least three distinct straight lines *b*, *c*, *d* parallel to line *a*. So in this case the answer to the question is negative.

2) Let us continue to consider case 1) without situation with

$$a_2 = 1.$$

So we have line *e* with equation

$$a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0$$

with conditions

$$\begin{cases} -a_1 \times_2 (99.99) +_2 a_2 \times_2 (0.01) -_2 a_2 = 0, \\ a_3 = -a_2. \end{cases}$$

We must have

$$|a_1| \leq 1.$$

All possible positive a_1 form the set

$$\Phi = [0.01, 0.02, \ldots, 0.99, 1.00],$$

and we get

$$99.99 \times_2 \Phi = [0.99, 1.98, \ldots, 98.82, 99.99].$$

If

$$|a_2| < 1,$$

then we get

$$\begin{cases} -a_1 \times_2 (99.99) -_2 a_2 = 0, \\ a_3 = -a_2, \end{cases}$$

and possible situations are

$$\begin{cases} a_1 = -0.01, \\ a_2 = 0.99, \\ a_3 = -0.99, \end{cases}$$

or

$$\begin{cases} a_1 = 0.01, \\ a_2 = -0.99, \\ a_3 = 0.99. \end{cases}$$

These two solutions define the same line, that is, the equation of line *e* is

$$-0.01 \times_2 x +_2 0.99 \times_2 y -_2 0.99 = 0.$$

We have

$$A, B \in e,$$

but

$$\begin{cases} x = -99.99, \\ 0.99 \times_2 y = 0, \end{cases}$$

that is, line *e* contains the points

$$\big[(-99.99, 0), (-99.99, \pm 0.01), \dots, (-99.99, \pm 0.09)\big],$$

and

$$e \cap a \neq \Lambda,$$
$$e \cap R_1^a \neq \Lambda,$$
$$e \cap R_2^a \neq \Lambda.$$

So lines *a*, *e* are not parallel. Thus we cannot assume that

$$|a_2| < 1.$$

3) Let us consider the last case where

$$|a_2| > 1.$$

We look for a straight line *f* with equation

$$f : a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0$$

and containing points *A*, *B*. If we take negative a_1 such that

$$|a_1| \in \Phi \setminus [0.01, 1] = [0.02, \dots, 0.99]$$

and so

$$99.99 \times_2 [0.02, \dots, 0.99] = [1.98, \dots, 98.82],$$

that is,

$$a_1 \times_2 (99.99) \in [-1.98, \dots, -98.82],$$

then from the system

$$\begin{cases} -a_1 \times_2 (99.99) +_2 a_2 \times_2 (0.01) -_2 a_2 = 0, \\ a_3 = -a_2, \end{cases}$$

we get the following equation for a_2:

$$a_2 \times_2 (0.01) -_2 a_2 = a_1 \times_2 (99.99).$$

Solution a_2 of this equation exists not for all a_1. For example, for

$$a_1 = -0.02,$$

we get

$$a_2 = 1.99$$

and

$$f : -0.02 \times_2 x +_2 1.99 \times_2 y -_2 1.99 = 0.$$

We have

$$R_1^f = \big[(x, y)\big], (x, y) \in E_2W_2, -0.02 \times_2 x +_2 1.99 \times_2 y -_2 1.99 > 0,$$
$$R_2^f = \big[(x, y)\big], (x, y) \in E_2W_2, -0.02 \times_2 x +_2 1.99 \times_2 y -_2 1.99 < 0,$$

line $f \subset R_1^a$ contains points *A*, *B*, line $a \subset R_2^f$, which means that lines *a*, *f* are parallel, $a \parallel f$. As in 1) of this section, we have the lines

$$g : 0.02 \times_2 x +_2 1.99 \times_2 y -_2 1.99 = 0$$

and

$$h : y -_2 1 = 0$$

containing point *B* and parallel to line *a*. For all cases where line *f* exists, we have that line $f \subset R_1^a$ contains points *A*, *B* and that line $a \subset R_2^f$, which means that lines *a* and *f* are parallel, $a \parallel f$.

4) Let plane $\alpha \in E_3W_2$ have the equation

$$z = 0.$$

So we are in E_2W_2. Let the equation of straight line *a* be

$$y = 0,$$

and let two regions R_1^a, R_2^a of plane *a* be

$$R_1^a = \left[(x, y)\right], (x, y) \in E_2 W_2, y > 0,$$
$$R_2^a = \left[(x, y)\right], (x, y) \in E_2 W_2, y < 0,$$

where *x* is any element $\in W_2$. So

$$E_2 W_2 = R_1^a \cup a \cup R_2^a.$$

Let us take two points

$$A(-99.99, 0.01), B'(0, u) \in R_1^a,$$

where

$$0 < u < 1.$$

We look for a straight line *b* as the set of points (x, y) satisfying the equation

$$a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0$$

and containing points *A*, B'. We have

$$\begin{cases} -a_1 \times_2 (99.99) +_2 a_2 \times_2 (0.01) +_2 a_3 = 0, \\ a_1 \times_2 (0) +_2 a_2 \times_2 (u) +_2 a_3 = 0, \end{cases}$$

that is,

$$\begin{cases} -a_1 \times_2 (99.99) +_2 a_2 \times_2 (0.01) -_2 a_2 \times_2 (u) = 0, \\ a_3 = -a_2 \times_2 (u). \end{cases}$$

As above, for negative a_1,

$$-a_1 \times_2 (99.99) \in [0.99, 1.98, \ldots, 98.82, 99.99].$$

We must have

$$a_2 > 0.$$

If

$$a_2 < 1,$$

then

$$a_2 \times_2 0.01 = 0$$

and

$$a_2 \times_2 u < 0.99.$$

If

$$a_2 = 1,$$

then

$$a_2 \times_2 0.01 = 0.01,$$
$$a_2 \times_2 u = u,$$

and

$$0.99 +_2 0.01 -_2 u \neq 0.$$

5) Let us now consider the case

$$a_2 > 1.$$

We have

$$\begin{cases} -a_1 \times_2 (99.99) +_2 a_2 \times_2 (0.01) -_2 a_2 \times_2 (u) = 0, \\ a_3 = -a_2 \times_2 (u). \end{cases}$$

For

$$u = 0.01,$$

a_2 does not exist, and thus line *b* does not exist. This means that there is only one line

$$y -_2 0.01 = 0$$

containing point B' and parallel to line *a*.

6) Let us continue with the case

$$a_2 > 1.$$

For

$$u = 0.02,$$

we get

$$\begin{cases} a_1 = -0.01, \\ a_2 = 99.99, \\ a_3 = -1.98, \end{cases}$$

and line *b* contains points *A*, B' and has the equation

$$-0.01 \times_2 x +_2 99.99 \times_2 y -_2 1.98 = 0.$$

We get

$$R_1^b = \big[(x,y)\big], (x,y) \in E_2W_2, -0.01 \times_2 x +_2 99.99 \times_2 y -_2 1.98 > 0,$$
$$R_2^b = \big[(x,y)\big], (x,y) \in E_2W_2, -0.01 \times_2 x +_2 99.99 \times_2 y -_2 1.98 < 0,$$

and

$b \subset R_1^a$, $a \subset R_2^b$, $a \cap b = \Lambda$, and $a \parallel b$.

For

$$u = 0.1,$$

we get

$$\begin{cases} a_1 = -0.09, \\ a_2 = 99.00, \\ a_3 = -9.90, \end{cases}$$

and line *b* with equation

$$-0.09 \times_2 x +_2 99.00 \times_2 y -_2 9.90 = 0$$

contains points *A*, B' and is parallel to line *a*.

7) Let us take two points

$$A(-99.99, 0.01), B'(0, u) \in R_1^a,$$

where

$$u > 1.$$

We look for a straight line *b* as the set of points (x, y) satisfying the equation

$$a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0$$

and containing points *A*, B'. We have

$$\begin{cases} -a_1 \times_2 (99.99) +_2 a_2 \times_2 (0.01) +_2 a_3 = 0, \\ a_1 \times_2 (0) +_2 a_2 \times_2 (u) +_2 a_3 = 0, \end{cases}$$

that is,

$$\begin{cases} -a_1 \times_2 (99.99) +_2 a_2 \times_2 (0.01) -_2 a_2 \times_2 (u) = 0, \\ a_3 = -a_2 \times_2 (u). \end{cases}$$

As above, for negative a_1,

$$-a_1 \times_2 (99.99) \in [0.99, 1.98, \ldots, 98.82, 99.99].$$

We must have

$$a_2 > 0.$$

For example, if

$$u = 1.25,$$

then we get

$$a_1 = -0.12, \quad a_2 = 9.69, \quad a_3 = -12.06,$$

and line *b* with equation

$$-0.12 \times_2 x +_2 9.69 \times_2 y -_2 12.06 = 0$$

contains points *A*, B' and is parallel to line *a*. Also, lines

$$c : 0.12 \times_2 x +_2 9.69 \times_2 y -_2 12.06 = 0$$

and

$$d : y -_2 1.25 = 0$$

contain point B' and are parallel to line *a*.

8) For point $B'(0, u)$ with arbitrary

$$u > 0,$$

there is straight line *b* with equation

$$b : y -_2 u = 0$$

containing point B' and parallel to line *a*.

So we have proved the following:

Theorem 6.3.
In Mathematics with Observers geometry in the plane E_2W_n, there are a straight line a that divides the remaining points of plane into two regions R_1^a and R_2^a (so $R_3^a = \Lambda$) and two points $A, B \in R_1^a$ such that there is no straight line b containing points A, B and parallel to a.
Theorem 6.4.
In Mathematics with Observers geometry in the plane E_2W_n, there are a straight line a that divides the remaining points of plane into two regions R_1^a and R_2^a (so $R_3^a = \Lambda$) and two points $A, B \in R_1^a$ such that there is unique straight line b containing points A, B and parallel to a.
Theorem 6.5.
In Mathematics with Observers geometry in the plane E_2W_n, there are a straight line a that divides the remaining points of plane into two regions R_1^a and R_2^a (so $R_3^a = \Lambda$) and two points $A, B \in R_1^a$ such that there is more than one straight line b containing points A, B and parallel to a.

6.3 Gauss–Bolyai–Lobachevsky axiom

Question. Is the following statement correct in Mathematics with Observers geometry: In a plane *α*, through any point *A* lying outside a straight line *a*, there can be drawn more than one straight line parallel to line *a*?

1) Let plane $\alpha \in E_3W_2$ have the equation

$$z = 0.$$

So we are in E_2W_2.

Let straight line *a* have the equation

$$y = 0,$$

and let two regions R_1^a, R_2^a of plane *α* be

$$R_1^a = \big[(x, y), (x, y) \in E_2W_2, y > 0\big]$$

and

$$R_2^a = \big[(x, y), (x, y) \in E_2W_2, y < 0\big],$$

where *x* is any element $\in W_2$. So

$$E_2W_2 = R_1^a \cup a \cup R_2^a.$$

Let us take two points

$$A(-99.99, 0.01), B(0, 1) \in R_1^a.$$

We look for a straight line *b* as the set of points (x, y) satisfying the equation

$$a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0$$

and containing points *A*, *B*. We have

$$\begin{cases} -a_1 \times_2 (99.99) +_2 a_2 \times_2 (0.01) +_2 a_3 = 0, \\ a_1 \times_2 (0) +_2 a_2 \times_2 (1) +_2 a_3 = 0, \end{cases}$$

that is,

$$\begin{cases} -a_1 \times_2 (99.99) +_2 a_2 \times_2 (0.01) +_2 a_3 = 0, \\ a_3 = -a_2. \end{cases}$$

For

$$a_2 = 1,$$

we get

$$a_1 = -0.01,$$

and the equation of line *b* is

$$b : -0.01 \times_2 x +_2 y -_2 1 = 0.$$

We have

$$b \subset R_1^a$$

and

$$b \cap a = \Lambda.$$

We have

$$R_1^b = \left[(x, y)\right], (x, y) \in E_2W_2, -0.01 \times_2 x +_2 y -_2 1 > 0,$$
$$R_2^b = \left[(x, y)\right], (x, y) \in E_2W_2, -0.01 \times_2 x +_2 y -_2 1 < 0,$$

and

$$a \subset R_2^b.$$

So line *a* is parallel to line *b*, and line *b* is parallel to line *a*, that is, $a \parallel b$ and $b \parallel a$.

Note that we can write

$$b : y = 0.01 \times_2 x +_2 1.$$

We get

$$b = S_{-99} \cup S_{-98} \cup \cdots \cup S_0 \cup S_{0.99} \cup S_{1.99} \cup \cdots \cup S_{98.99} \cup S_{99.99},$$

where

$$S_{-99} = \left[(x, 0.01), -99.99 \le x \le 99.00\right],$$
$$S_{-98} = \left[(x, 0.02), -98.99 \le x \le 98.00\right],$$
$$\ldots$$
$$S_0 = \left[(x, 1.00), -0.99 \le x \le 0.00\right],$$
$$S_{0.99} = \left[(x, 1.00), 0.00 \le x \le 0.99\right],$$
$$S_{1.99} = \left[(x, 1.01), 1.00 \le x \le 1.99\right]$$
$$\ldots$$
$$S_{98.99} = \left[(x, 1.98), 98.00 \le x \le 98.99\right],$$
$$S_{99.99} = \left[(x, 1.99), 99.00 \le x \le 99.99\right].$$

Also, line *c* with equation

$$0.01 \times_2 x +_2 y -_2 1 = 0$$

is parallel to line *a* and contains points *B* and $A'(99.99, 0.01)$.

Note that we can write

$$c: y = -0.01 \times_2 x +_2 1.$$

We get

$$b = T_{-99} \cup T_{-98} \cup \cdots \cup T_0 \cup T_{0.99} \cup T_{1.99} \cup \cdots \cup T_{98.99} \cup T_{99.99},$$

where

$$\begin{aligned}
T_{-99} &= \big[(x, 1.99), -99.99 \le x \le 99.00\big], \\
T_{-98} &= \big[(x, 1.98), -98.99 \le x \le 98.00\big], \\
&\dots \\
T_0 &= \big[(x, 1.00), -0.99 \le x \le 0.00\big], \\
T_{0.99} &= \big[(x, 1.00), 0.00 \le x \le 0.99\big], \\
T_{1.99} &= \big[(x, 0.99), 1.00 \le x \le 1.99\big], \\
&\dots \\
T_{98.99} &= \big[(x, 0.02), 98.00 \le x \le 98.99\big], \\
T_{99.99} &= \big[(x, 0.01), 99.00 \le x \le 99.99\big].
\end{aligned}$$

So line *a* is parallel to line *c*, and line *c* is parallel to line *a*, that is, $a \parallel c$ and $c \parallel a$. Moreover,

$$b \cap c = \big[(x, 1)\big], -0.99 \le x \le 0.99$$

and

$$B \in (b \cap c),$$

that is, $b \nparallel c$.

Consider straight line *d* with equation

$$y = 1.$$

We see that lines *a* and *d* are parallel and that line *d* contains point *B*. This means that in the plane

$$\alpha : z = 0,$$

through point *B* lying outside straight line *a*, we can draw at least three distinct straight lines *b*, *c*, *d* parallel to line *a*.

1') Let again plane $\alpha \in E_3W_2$ have the equation

$$z = 0.$$

So we are in E_2W_2. Let the equation of straight line *a* be

$$y = -1,$$

and let two regions R_1^a and R_2^a of plane *a* be

$$R_1^a = \big[(x, y), (x, y) \in E_2W_2, y > -1\big]$$

and

$$R_2^a = \big[(x, y), (x, y) \in E_2W_2, y < -1\big],$$

where *x* is any element $\in W_2$. So

$$E_2W_2 = R_1^a \cup a \cup R_2^a.$$

Let us take two points

$$A(-99.99, 0.00), B(0, 99.99) \in R_1^a.$$

We look for a straight line *b* as the set of points (x, y) satisfying the equation

$$a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0$$

and containing points *A*, *B*. We have

$$\begin{cases} -a_1 \times_2 (99.99) +_2 a_2 \times_2 (0.00) +_2 a_3 = 0, \\ a_1 \times_2 (0) +_2 a_2 \times_2 (99.99) +_2 a_3 = 0, \end{cases}$$

that is,

$$\begin{cases} -a_1 \times_2 (99.99) +_2 a_3 = 0, \\ a_2 \times_2 (99.99) +_2 a_3 = 0. \end{cases}$$

For

$$a_2 = 1,$$

we get

$$a_1 = -1$$

and

$$a_3 = -99.99,$$

and the equation of line *b* is

$$b : y = x +_2 99.99 = 0.$$

We have

$$b \subset R_1^a$$

and

$$b \cap a = \Lambda.$$

Also, we have

$$R_1^b = \left[(x,y),(x,y) \in E_2W_2, y = x +_2 99.99 > 0\right]$$

and

$$R_2^b = \left[(x,y),(x,y) \in E_2W_2, y = x +_2 99.99 < 0\right],$$

where *x* is any element such that $-99.99 \leq x \leq 0$, and

$$a \cap R_1^b = \Lambda,$$
$$a \cap R_2^b \neq \Lambda.$$

So line *a* is parallel to line *b*, and line *b* is parallel to line *a*, that is, $a \parallel b$ and $b \parallel a$. Also, line *c* with equation

$$c : y = -x +_2 99.99 = 0$$

is parallel to line *a* and contains points *B* and $A'(99.99, 0.00)$. So line *a* is parallel to line *c*, and line *c* is parallel to line *a*, that is, $a \parallel c$ and $c \parallel a$. Moreover,

$$b \cap c = (0, 99.99) = B,$$

that is, $b \nparallel c$.

If we consider straight line *d* with equation

$$y = 99.99,$$

we see that lines *a* and *d* are parallel and that line *d* contains point *B*. This means that in the plane

$$\alpha : z = 0,$$

through point *B* lying outside straight line *a*, we can draw at least three distinct straight lines *b*, *c*, *d* parallel to line *a*.

So we have proved the following:

Theorem 6.6.
In Mathematics with Observers geometry in plane E_2W_n, there are a straight line a that divides the remaining points of the plane into two regions R_1^a and R_2^a (so $R_3^a = \Lambda$) and a point $A \in R_1^a$ such that there are is than one straight line b containing point A and parallel to a.

2) Let us take two straight lines $a, c \in E_2W_2$:

$$a : 99.99 \times_2 x -_2 99.99 \times_2 y = 0,$$
$$c : y +_2 2 = 0.$$

Line *a* is the set of points $(x, x) \in E_2W_2$ with $x \in [-1, -0.99, -0.98, \ldots, 0.99, 1]$ and divides E_2W_2 into three regions R_1^a, R_2^a, R_3^a:

$$R_1^a = \left[(x, y), (x, y) \in E_2W_2, 99.99 \times_2 x -_2 99.99 \times_2 y > 0\right],$$
$$R_2^a = \left[(x, y), (x, y) \in E_2W_2, 99.99 \times_2 x -_2 99.99 \times_2 y < 0\right],$$

and

$$R_1^a \cup a \cup R_2^a \subset Q,$$

where

$$Q = \left[(x, y), -1 \leq x \leq 1, 1 \leq y \leq 1\right],$$
$$R_3^a = E_2W_2 \setminus \left(R_1^a \cup a \cup R_2^a\right)$$

Line *c* divides E_2W_2 into two regions R_1^c, R_2^c:

$$R_1^c = \left[(x, y), (x, y) \in E_2W_2, y +_2 2 > 0\right],$$
$$R_2^c = \left[(x, y), (x, y) \in E_2W_2, y +_2 2 < 0\right],$$

where *x* is any element $\in W_2$, and

$$E_2W_2 = R_1^c \cup c \cup R_2^c.$$

We get

$$a \subset R_1^c,$$

but

$$c \subset R_3^a.$$

So $c \parallel a$, but $a \nparallel c$.

3) Let us take three straight lines $a, b, c \in E_2W_2$:

$$a : 99.99 \times_2 x -_2 99.99 \times_2 y = 0,$$
$$b : 99.99 \times_2 x +_2 99.99 \times_2 y = 0,$$
$$c : y +_2 2 = 0.$$

Line *a* is the set of points $(x, x) \in E_2W_2$ with $x \in [-1, -0.99, -0.98, \ldots, 0.99, 1]$, line *b* is the set of points $(x, -x) \in E_2W_2$ with $x \in [-1, -0.99, -0.98, \ldots, 0.99, 1]$, and

$$a \cap b = \left[A(0, 0)\right].$$

Line *c* divides E_2W_2 into two regions R_1^c, R_2^c:

$$R_1^c = \left[(x,y),(x,y) \in E_2W_2, y +_2 2 > 0\right],$$
$$R_2^c = \left[(x,y),(x,y) \in E_2W_2, y +_2 2 < 0\right],$$
where *x* is any element $\in W_2$, and

$$E_2W_2 = R_1^c \cup c \cup R_2^c.$$

We get

$$a \subset R_1^c,$$
$$b \subset R_1^c,$$
but line *a* is the set of points $(x,x) \in E_2W_2$ with $x \in [-1,-0.99,-0.98,\ldots,0.99,1]$.

Line *a* divides E_2W_2 into three regions R_1^a, R_2^a, R_3^a:

$$R_1^a = \left[(x,y),(x,y) \in E_2W_2, 99.99 \times_2 x -_2 99.99 \times_2 y > 0\right],$$
$$R_2^a = \left[(x,y),(x,y) \in E_2W_2, 99.99 \times_2 x -_2 99.99 \times_2 y < 0\right],$$
$$R_3^a = E_2W_2 \setminus \left(R_1^a \cup a \cup R_2^a\right),$$
and line *b* is the set of points $(x,-x) \in E_2W_2$ with $x \in [-1,-0.99,-0.98,\ldots,0.99,1]$.

Line *b* divides E_2W_2 into three regions R_1^b, R_2^b, R_3^b:

$$R_1^b = \left[(x,y),(x,y) \in E_2W_2, 99.99 \times_2 x +_2 99.99 \times_2 y > 0\right],$$
$$R_2^b = \left[(x,y),(x,y) \in E_2W_2, 99.99 \times_2 x +_2 99.99 \times_2 y < 0\right],$$
$$R_3^b = E_2W_2 \setminus \left(R_1^b \cup b \cup R_2^b\right),$$
and

$$R_1^a \cup a \cup R_2^a \cup R_1^b \cup b \cup R_2^b \subset Q,$$

where

$$Q = \left[(x,y), -1 \leq x \leq 1, -1 \leq y \leq 1\right].$$

We get

$$a \subset R_1^c,$$
$$b \subset R_1^c,$$

but

$$c \subset R_3^a,$$
$$c \subset R_3^b.$$
So $c \parallel a$ and $c \parallel b$, but $a \nparallel c$ and $b \nparallel c$.

So we have proved the following:

Theorem 6.7.
In Mathematics with Observers geometry in a plane α, through any point A lying outside a straight line c, there can be drawn more than one straight line that has no common points with c, belongs to one region R_1^c*, and is not parallel to c.*
Note that if we take points $B(0.5, 0.5) \in a$ and $D(0.5, -0.5) \in b$ and consider the vectors

$$\mathbf{AB} = (0.5, 0.5) \in E_2W_2,$$
$$\mathbf{AD} = (0.5, -0.5) \in E_2W_2,$$

then we get

$$(\mathbf{AB}, \mathbf{AD}) = 0.$$

So we have proved the following:

Theorem 6.8.
In Mathematics with Observers geometry in a plane α, there are a straight line c and a point A lying outside c such that through this point A, there can be drawn at least two straight lines a and b that are perpendicular to each other, have no common points with c, belong to one region R_1^c*, and are not parallel to c.*
4) Let us take six straight lines $a, b, c, d, e, f \in E_2W_2$:

$$a : 99.99 \times_2 x -_2 99.99 \times_2 y = 0,$$
$$b : 99.99 \times_2 x +_2 99.99 \times_2 y = 0,$$
$$c : y +_2 2 = 0,$$
$$d : x +_2 2 = 0,$$
$$e : y -_2 2 = 0,$$
$$f : x -_2 2 = 0.$$

Line *a* is the set of points $(x, x) \in E_2W_2$ with $x \in [-1, -0.99, -0.98, \dots, 0.99, 1]$, line *b* is the set of points $(x, -x) \in E_2W_2$ with $x \in [-1, -0.99, -0.98, \dots, 0.99, 1]$, and

$$a \cap b = [A(0, 0)].$$

Line *c* divides E_2W_2 into two regions R_1^c, R_2^c:

$$R_1^c = [(x, y)], (x, y) \in E_2W_2, y +_2 2 > 0,$$
$$R_2^c = [(x, y)], (x, y) \in E_2W_2, y +_2 2 < 0,$$

where *x* is any element $\in W_2$, and

$$E_2W_2 = R_1^c \cup c \cup R_2^c.$$

Line *d* divides E_2W_2 into two regions R_1^d, R_2^d:

$$R_1^d = \big[(x,y)\big], (x,y) \in E_2W_2, x +_2 2 > 0,$$
$$R_2^d = \big[(x,y)\big], (x,y) \in E_2W_2, x +_2 2 < 0,$$
where *y* is any element $\in W_2$, and

$$E_2W_2 = R_1^d \cup d \cup R_2^d.$$

Line *e* divides E_2W_2 into two regions R_1^e, R_2^e:

$$R_1^e = \big[(x,y)\big], (x,y) \in E_2W_2, y -_2 2 > 0,$$
$$R_2^e = \big[(x,y)\big], (x,y) \in E_2W_2, y -_2 2 < 0,$$
where *x* is any element $\in W_2$, and

$$E_2W_2 = R_1^e \cup e \cup R_2^e.$$

Line *f* divides E_2W_2 into two regions R_1^f, R_2^f:

$$R_1^f = \big[(x,y)\big], (x,y) \in E_2W_2, x -_2 2 > 0,$$
$$R_2^f = \big[(x,y)\big], (x,y) \in E_2W_2, x -_2 2 < 0,$$
where *y* is any element $\in W_2$, and

$$E_2W_2 = R_1^f \cup f \cup R_2^f.$$

We get

$$c \perp d, \quad c \perp f, \quad e \perp d, \quad e \perp f, \quad c \parallel e, \quad d \parallel f, \quad e \parallel c, \quad f \parallel d,$$

and

$$R_1^a \cup a \cup R_2^a \cup R_1^b \cup b \cup R_2^b \subset Q,$$

where

$$Q = \big[(x,y), -1 \le x \le 1, -1 \le y \le 1\big].$$

We have

$$a \subset R_1^c, \quad a \subset R_1^d, \quad a \subset R_2^e, \quad a \subset R_2^f,$$
$$b \subset R_1^c, \quad b \subset R_1^d, \quad b \subset R_2^e, \quad b \subset R_2^f,$$
$c \subset R_3^a$, $d \subset R_3^a$, $e \subset R_3^a$, $f \subset R_3^a$, and $c \subset R_3^b$, $d \subset R_3^b$, $e \subset R_3^b$, $f \subset R_3^b$.

So we have proved the following:

Theorem 6.9.
In Mathematics with Observers geometry in a plane α, there are straight lines c, d, e, f such that
$$c \perp d, \quad c \perp f, \quad d \perp e, \quad f \perp e$$

and a point A lying outside these straight lines such that through this point A, there can be drawn at least two straight lines a and b that perpendicular to each other, have no common points with c, d, e, f, and, moreover,

$$a\subset R_1^c, \quad a \subset R_1^d, \quad a \subset R_2^e, \quad a \subset R_2^f,$$

$$b\subset R_1^c, \quad b \subset R_1^d, \quad b \subset R_2^e, \quad b \subset R_2^f,$$

$c \subset R_3^a$, $d \subset R_3^a$, $e \subset R_3^a$, $f \subset R_3^a$, $c \subset R_3^b$, $d \subset R_3^b$, $e \subset R_3^b$, $f \subset R_3^b$, *and* $a \nparallel c$, $a \nparallel d$, $a \nparallel e$, $a \nparallel f$, $b \nparallel c$, $b \nparallel d$, $b \nparallel e$, $b \nparallel f$, *but* $c \parallel a$, $d \parallel a$, $e \parallel a$, $f \parallel a$, $c \parallel b$, $d \parallel b$, $e \parallel b$, $f \parallel b$.

5) As we saw above, we have the following: In a plane *α*, there are a straight line *b* and a point *A* lying outside of this straight line such that through this point *A*, there can be drawn at least two straight lines *a* and *c* that parallel to line *b*, and we have $a \parallel b$, $b \parallel a$, $c \parallel b$, $b \parallel c$.

So we have the following:

Theorem 6.10.
In Mathematics with Observers geometry in a plane α, there are a straight line b and a point A lying outside this straight line such that through this point A, there can be drawn at least two straight lines a and c that are parallel to line b.

6.4 Riemann axiom

Question. Is the following statement correct in Mathematics with Observers geometry: In a plane *α*, through any point *A* lying outside a straight line *a*, there can be drawn no straight line parallel to line *a*?

Let us consider a classical Riemannian geometry model for E_3W_2: a straight line $a \in E_3W_2$ is an intersection of plane $\alpha \in E_3W_2$ with the unit sphere, where plane *α* contains the origin, the point (0,0,0). In Euclid's three-dimensional space, such two distinct straight lines intersect in two points.

1) Let us take two planes

$$\alpha : x +_2 y -_2 4 \times_2 z = 0$$

and

$$\beta : y -_2 4 \times_2 z = 0$$

and the unit sphere with equation

$$\text{Sph} : x \times_2 x +_2 y \times_2 y +_2 z \times_2 z = 1.$$

We get

$$\alpha \cap \beta \cap \mathrm{Sph} = \Lambda,$$

because we have

$$\begin{cases} x = 0, \\ y -_2 4 \times_2 z = 0, \\ 17 \times_2 (z \times_2 z) = 1, \end{cases}$$

and

$$17^{-1}$$

does not exist because

$$17 \times_2 0.05 = 0.85,$$
$$17 \times_2 0.06 = 1.02.$$

So, unlike in the classical case, two lines do not intersect.

2) Let us take two planes

$$\alpha : x = 0$$

and

$$\beta : y = 0$$

and the unit sphere with equation

$$\mathrm{Sph} : x \times_2 x +_2 y \times_2 y +_2 z \times_2 z = 1.$$

We get

$$\alpha \cap \beta \cap \mathrm{Sph} = \big[(0,0,-1),(0,0,1)\big].$$

So, as in the classical case, two lines intersect in two points.

3) Let us take two planes

$$\alpha : 0.01 \times_2 x -_2 0.02 \times_2 y -_2 0.22 \times_2 z = 0$$

and

$$\beta : z = 0$$

and the unit sphere with equation

$$\mathrm{Sph} : x \times_2 x +_2 y \times_2 y +_2 z \times_2 z = 1.$$

We get

$$
\begin{aligned}
\alpha\cap\beta\cap\text{Sph}= \big[&(\pm 0.60,\pm 0.80,0),\dots\\
&\dots(\pm 0.69,\pm 0.80,0),(\pm 0.60,\pm 0.81,0),\dots,(\pm 0.69,\pm 0.81,0),\dots\\
&\dots,(\pm 0.60,\pm 0.89,0),\dots,(\pm 0.69,\pm 0.89,0),(\pm 0.80,\pm 0.60,0),\dots\\
&\dots,(\pm 0.89,\pm 0.60,0),(\pm 0.80,\pm 0.61,0),\dots,(\pm 0.89,\pm 0.61,0),\dots\\
&\dots,(\pm 0.80,\pm 0.69,0),\dots,(\pm 0.89,\pm 0.69,0)\big],
\end{aligned}
$$

that is, the total number of common points here is 800 (from the point of view of W_m-observer with $m \geq 13$). So, unlike in the classical case, we have 800 points in the intersection of two lines (from the point of view of W_m-observer with $m \geq 13$).

So we have proved the following:

Theorem 6.11.
In Mathematics with Observers geometry in the interpretation of classical Riemann model for the space E_3W_n (i. e., straight line $a \in E_3W_2$ is the intersection of a plane $\alpha \in E_3W_2$ with the unit sphere, where plane a contains the origin, point (0,0,0)*), there are two straight lines a and b such that $a \cap b = \Lambda$.*
Theorem 6.12.
In Mathematics with Observers geometry in the interpretation of classical Riemann model for the space E_3W_n, there are two straight lines a and b such that $a \cap b \neq \Lambda$ contains exactly two points.
Theorem 6.13.
In Mathematics with Observers geometry in the interpretation of classical Riemann model for the space E_3W_n, there are two straight lines a and b such that $a \cap b \neq \Lambda$ contains more than two points.

6.5 Observability and geometry: the main parallel lines theorem

Let us first prove the following theorem.

Theorem 6.14.
In the plane E_2W_n, there are a point A and a straight line b not containing this point such that any straight line a containing point A and line b are not parallel: $a \nparallel b$ and $b \nparallel a$.
Let us first consider the case $n = 2$, that is, we are in E_2W_2. Let us consider the class of straight lines with equations

$$y = k \times_2 x +_2 c$$

or

$$0 = x +_2 c,$$

where all elements are in W_2. Let $A(-99.99, 99.99) \in E_2W_2$ and consider the straight line *b*:

$$b : x -_2 y = 0.$$

Then

$$A \notin b.$$

We look for a straight line *a* containing point *A*. We have

$$a : y = k \times_2 x +_2 c$$

or

$$a : 0 = x +_2 c$$

for all

$$k, c, x, y, k \times_2 x, k \times_2 x +_2 c \in W_2.$$

Let us consider the first case. We must have

$$99.99 = k \times_2 (-99.99) +_2 c.$$

We get

$$\begin{cases} |k| \leq 1, \\ c = 99.99 +_2 k \times_2 99.99, \end{cases}$$

and thus

$$-1 \leq k \leq 0.$$

So line *a* has the equation

$$a : y = k \times_2 x +_2 (99.99 +_2 k \times_2 99.99).$$

If

$$k = -1,$$

then we get

$$a : y = -x$$

and

$$a \cap b = (0, 0),$$

that is, $a \nparallel b$ and $b \nparallel a$.

If

$$k = 0,$$

then we get

$$a : y = 99.99$$

and

$$a \cap b = (99.99, 99.99),$$

that is, $a \nparallel b$ and $b \nparallel a$.

If

$$-1 < k < 0,$$

then we get

$$a : y = k \times_2 x +_2 (99.99 +_2 k \times_2 99.99)$$

and

$$A(-99.99, 99.99) \in a.$$

For $x = 99.99$ and $k = -0.99$, on line a, we get

$$y = -0.99 \times_2 99.99 +_2 (99.99 -_2 0.99 \times_2 99.99) = -97.65.$$

For $x = 99.99$ and $k = -0.01$, on line a, we get

$$y = -0.01 \times_2 99.99 +_2 (99.99 -_2 0.01 \times_2 99.99) = 98.01.$$

For $x = 99.99$ and $-1 < k < 0$, on line a, we get

$$y \in [-97.65, 98.01],$$

that is,

$$y < 99.99.$$

This means that

$$a \cap R_1^b \neq \Lambda$$

and

$$a \cap R_2^b \neq \Lambda.$$

Also, we have

$$b \cap R_1^a \neq \Lambda$$

and

$$b \cap R_2^a \neq \Lambda.$$

This means that straight lines *a* and *b* are not parallel, $a \nparallel b$, and straight lines *b* and *a* are not parallel, $b \nparallel a$.

Let us now consider the second case:

$$a : 0 = x +_2 c.$$

Clearly, $x, c \in W_2$.

We look for a straight line *a* containing point *A*. We have

$$0 = -99.99 +_2 c$$

or

$$c = 99.99,$$

that is,

$$a : x = -99.99.$$

So

$$a \cap b = B(-99.99, -99.99).$$

This means that $a \nparallel b$ and $b \nparallel a$.

So we have proved the theorem for $n = 2$ and the class of straight lines with equations

$$y = k \times_2 x +_2 c$$

or

$$0 = x +_2 c.$$

The theorem is still correct for this class of straight lines for all *n*, and the proof is practically the same.

Now let us consider the general case of any straight lines. First, let us take $n = 2$. Let again consider the point $A(-99.99, 99.99) \in E_2 W_2$. We look for a straight line *a* containing this point. We have

$$a : a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0$$

for all

$$a_1, a_2, a_3, x, y, a_1 \times_2 x, a_2 \times_2 y, a_1 \times_2 x +_2 a_2 \times_2 y \in W_2$$

such that $(a_1, a_2) \neq (0, 0)$ and

$$a_1 \times_2 (-99.99) +_2 a_2 \times_2 99.99 +_2 a_3 = 0.$$

We get

$$\begin{cases} |a_1| \leq 1, \\ |a_2| \leq 1. \end{cases}$$

Also, if

$$a_1 = a_2,$$

then

$$a_3 = 0.$$

However, if

$$a_1 = -a_2 = 1$$

or

$$a_1 = -a_2 = -1,$$

then line a does not exist, because in this case,

$$a_1 \times_2 99.99 -_2 a_2 \times_2 99.99 \notin W_2.$$

So the equation of line a is

$$a : a_1 \times_2 x +_2 a_2 \times_2 y +_2 (a_1 \times_2 99.99 -_2 a_2 \times_2 99.99) = 0$$

for any

$$a_1, a_2, a_3, x, y, a_1 \times_2 x, a_2 \times_2 y, a_1 \times_2 x +_2 a_2 \times_2 y, a_1 \times_2 99.99 -_2 a_2 \times_2 99.99 \in W_2$$

such that $(a_1, a_2) \neq (0, 0)$ and

$$\begin{cases} |a_1| \leq 1, \\ |a_2| \leq 1. \end{cases}$$

Note that for line a, we have that or any x, y, a_1, a_2,

$$|a_1| \times_2 |x| \leq |a_1| \times_2 99.99$$

and

$$|a_2| \times_2 |y| \leq |a_2| \times_2 99.99.$$

Moreover, inequalities become equalities only in the cases

$$\begin{cases} |a_1| = 1, \\ x = \pm 99.99, \end{cases}$$

or

$$\begin{cases} 0.10 \le |a_1| \le 0.99, \\ x = \pm 99.9\star, \end{cases}$$

or

$$\begin{cases} 0.01 \le |a_1| \le 0.09, \\ x = \pm 99.\star\star, \end{cases}$$

where $\star$ is any digit 0,1,...,9.

The same situation takes place for a_2:

$$\begin{cases} |a_2| = 1, \\ y = \pm 99.99, \end{cases}$$

or

$$\begin{cases} 0.10 \le |a_2| \le 0.99, \\ y = \pm 99.9\star, \end{cases}$$

or

$$\begin{cases} 0.01 \le |a_2| \le 0.09, \\ y = \pm 99.\star\star. \end{cases}$$

Now rewrite the equation of line *a* as

$$a : a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_1 \times_2 99.99 -_2 a_2 \times_2 99.99 = 0$$

and let us assume that a_1 and a_2 have opposite signs, for example,

$$\begin{cases} -1 < a_1 < 0, \\ 0 < a_2 < 1. \end{cases}$$

Line *a* exists in this case only if

$$\begin{cases} -0.99 \le a_1 \le -0.10, \\ 0.01 \le a_2 \le 0.09 \\ -a_1 \times_2 99.99 +_2 a_2 \times_2 99.99 \in W_2, \\ x = -99.9\star, \\ y = 99.\star\star, \end{cases}$$

or

$$\begin{cases} -0.99 \le a_1 \le -0.10, \\ 0.10 \le a_2 \le 0.99, \\ -a_1 \times_2 99.99 +_2 a_2 \times_2 99.99 \in W_2, \\ x = -99.9\star, \\ y = 99.9\star, \end{cases}$$

or

$$\begin{cases} -0.09 \le a_1 \le -0.01, \\ 0.01 \le a_2 \le 0.09, \\ -a_1 \times_2 99.99 +_2 a_2 \times_2 99.99 \in W_2, \\ x = -99.\star\star, \\ y = 99.\star\star, \end{cases}$$

or

$$\begin{cases} -0.09 \le a_1 \le -0.01, \\ 0.10 \le a_2 \le 0.99, \\ -a_1 \times_2 99.99 +_2 a_2 \times_2 99.99 \in W_2, \\ x = -99.\star\star, \\ y = 99.9\star. \end{cases}$$

We get the same result when the coefficients a_1 and a_2 of line *a* have opposite signs:

$$\begin{cases} -1 < a_2 < 0, \\ 0 < a_1 < 1. \end{cases}$$

Let us now consider straight line *b*:

$$b : x -_2 y = 0.$$

Question: Is there straight line *a* parallel to line *b*?

Note that $A \in R_3^b$, where $R_3^b \subset E_2W_2$, $R_3^b = E_2W_2 \setminus (R_1^b \cup b \cup R_2^b)$,

R_1^b:

$$\begin{cases} x -_2 y \in W_n, \\ x -_2 y > 0, \end{cases}$$

R_2^b:

$$\begin{cases} x -_2 y \in W_n, \\ x -_2 y < 0. \end{cases}$$

As we proved above, in the case where the coefficients a_1 and a_2 of line *a* have opposite signs,

$$\begin{cases} -1 < a_1 < 0, \\ 0 < a_2 < 1, \end{cases}$$

or

$$\begin{cases} -1 < a_2 < 0, \\ 0 < a_1 < 1, \end{cases}$$

we get

$$a \subset R_3^b,$$

and so $a \not\parallel b$.

Let us now consider two extreme cases.

Case 1. $a_1 = 0$, and hence

$$a : a_2 \times_2 y -_2 a_2 \times_2 99.99 = 0.$$

With any a_2 such that

$$\begin{cases} |a_2| \leq 1, \\ a_2 \neq 0, \end{cases}$$

we get

$$(99.99, 99.99) \in a \cap b.$$

Case 2. $a_2 = 0$, and hence

$$a : a_1 \times_2 x +_2 a_1 \times_2 99.99 = 0.$$

With any a_1 such that

$$\begin{cases} |a_1| \leq 1, \\ a_1 \neq 0, \end{cases}$$

we get

$$(-99.99, -99.99) \in a \cap b.$$

This means that in both extreme cases, $a \not\parallel b$.

Let us now consider the “central” case for line *a*:

$$\begin{cases} a_1, a_2 \in W_n, \\ 0 < |a_1| < 1, \\ 0 < |a_2| < 1, \\ a_1 = a_2, \\ a_1 \times_2 x +_2 a_1 \times_2 y = 0. \end{cases}$$

This means that $O(0,0) \in a$, that is, $O(0,0) \in a \cap b$, that is, $a \nparallel b$.

So in the general case, line *a* satisfies the system

$$\begin{cases} a_1, a_2, a_1 \times_2 x, a_2 \times_2 y, a_1 \times_2 x +_2 a_2 \times_2 y, a_1 \times_2 99.99 -_2 a_2 \times_2 99.99 \in W_2, \\ 0 < |a_1| \leq 1, \\ 0 < |a_2| \leq 1, \\ a_1 \times_2 a_1 +_2 a_2 \times_2 a_2 < 2, \\ a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_1 \times_2 99.99 -_2 a_2 \times_2 99.99 = 0. \end{cases}$$

Let us now consider several cases with specific values of a_1 and a_2.

S1.

$$\begin{cases} a_1 = 1, \\ a_2 = 0.50. \end{cases}$$

We get the equation of line *a* in this case:

$$a : x +_2 0.50 \times_2 y +_2 99.99 -_2 0.50 \times_2 99.99 = 0,$$

that is,

$$a : x +_2 0.50 \times_2 y +_2 50.04 = 0.$$

We look for the intersection $a \cap b$, that is, the situation with $x = y$. For $x = -33.36$, we get

$$0.50 \times_2 y = -16.68.$$

Such *y* does not exist because

$$-0.50 \times_2 33.3\star = -16.65$$

and

$$-0.50 \times_2 33.4\star = -16.70$$

for any digit ⋆∈(0,1,...,9).

For $x = -33.39$, we get

$$0.50 \times_2 y = -16.65$$

and

$$-0.50 \times_2 33.3\star = -16.65$$

for any digit ⋆∈(0,1,...,9).

This means that

$$B(-33.39, -33.39) \in a \cap b,$$

that is, in this specific case, $a \nparallel b$.

S2.

$$\begin{cases} a_1 = 0.83, \\ a_2 = 0.54. \end{cases}$$

We get the equation of line *a* in this case:

$$a : 0.83 \times_2 x +_2 0.54 \times_2 y +_2 0.83 \times_2 99.99 -_2 0.54 \times_2 99.99 = 0,$$

that is,

$$a : 0.83 \times_2 x +_2 0.54 \times_2 y +_2 28.98 = 0.$$

Direct calculation shows that $C(-17.65, -26.75) \in a$ because

$$0.83 \times_2 (-17.65) +_2 0.54 \times_2 (-26.75) +_2 28.98 = -14.59 -_2 14.39 +_2 28.98 = 0.$$

Also, we get

$$C \in R_1^b.$$

Now we look for the intersection $a \cap b$, that is, the situation with $x = y$, if this intersection exists. For $x = y = -21.1\star$, we get

$$0.83 \times_2 (-21.1\star) +_2 0.54 \times_2 (-21.1\star) +_2 28.98 = -17.51 -_2 11.39 +_2 28.98 = 0.08,$$

and for $x = y = -21.2\star$, we get

$$0.83 \times_2 (-21.2\star) +_2 0.54 \times_2 (-21.2\star) +_2 28.98 = -17.59 -_2 11.44 +_2 28.98 = -0.05,$$

that is,

$$a \cap b = \Lambda.$$

Now we look for another point $D \in a$ that belongs to R_2^b. Direct calculation shows that $D(-34.7\star, -0.4\star) \in a$ because

$$0.83 \times_2 (-34.7\star) +_2 0.54 \times_2 (-0.4\star) +_2 28.98 = -28.78 -_2 0.20 +_2 28.98 = 0,$$

and we get

$$D \in R_2^b.$$

This means that $a \nparallel b$.

Let us now consider a common situation with line a satisfying the system

$$\begin{cases} a_1, a_2, x, y, a_1 \times_2 x, a_2 \times_2 y, a_1 \times_2 x +_2 a_2 \times_2 y, a_1 \times_2 99.99 -_2 a_2 \times_2 99.99 \in W_2, \\ 0 < a_1 < 1, \\ 0 < a_2 < 1, \\ a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_1 \times_2 99.99 -_2 a_2 \times_2 99.99 = 0. \end{cases}$$

Let us take the limiting values of a_1 and a_2. If $a_1 = 0.99$ and $a_2 = 0.01$, we get

$$a : 0.99 \times_2 x +_2 0.01 \times_2 y +_2 0.99 \times_2 99.99 -_2 0.01 \times_2 99.99 = 0,$$

that is, we have points on line a:

$$(-99.9\star, 99.\star\star) \in a.$$

If $a_1 = 0.01$ and $a_2 = 0.99$, then we get

$$a : 0.01 \times_2 x +_2 0.99 \times_2 y +_2 0.01 \times_2 99.99 -_2 0.99 \times_2 99.99 = 0,$$

that is, we have the same points on line a:

$$(-99.9\star, 99.\star\star) \in a.$$

If

$$(99.9\star, 99.9\star) \in a,$$

then

$$a_1 = a_2,$$

and if

$$(99.\star\star, 99.\star\star) \in a,$$

then

$$a_1 = a_2 \in [0.01, 0.02, \dots, 0.09].$$

Now we look for possible coefficients a_1 and a_2 of the equation of line a in the case

$$(0, -99.99) \in a.$$

This means that

$$a_1 \times_2 0 -_2 a_2 \times_2 99.99 +_2 a_1 \times_2 99.99 -_2 a_2 \times_2 99.99 = 0,$$

that is,

$$a_1 \times_2 99.99 -_2 a_2 \times_2 99.99 -_2 a_2 \times_2 99.99 = 0.$$

We get by direct calculation the following pairs a_1, a_2:

$$\begin{cases} a_1 = 0.02, \\ a_2 = 0.01, \end{cases}$$
$$\begin{cases} a_1 = 0.04, \\ a_2 = 0.02, \end{cases}$$
$$\begin{cases} a_1 = 0.06, \\ a_2 = 0.03, \end{cases}$$
$$\begin{cases} a_1 = 0.08, \\ a_2 = 0.04, \end{cases}$$
$$\begin{cases} a_1 = 0.2, \\ a_2 = 0.1, \end{cases}$$
$$\begin{cases} a_1 = 0.22, \\ a_2 = 0.11, \end{cases}$$
$$\begin{cases} a_1 = 0.24, \\ a_2 = 0.12, \end{cases}$$
$$\begin{cases} a_1 = 0.26, \\ a_2 = 0.13, \end{cases}$$

$$\begin{cases} a_1 = 0.28, \\ a_2 = 0.14, \end{cases}$$

$$\begin{cases} a_1 = 0.4, \\ a_2 = 0.2, \end{cases}$$

$$\begin{cases} a_1 = 0.42, \\ a_2 = 0.21, \end{cases}$$

$$\begin{cases} a_1 = 0.44, \\ a_2 = 0.22, \end{cases}$$

$$\begin{cases} a_1 = 0.46, \\ a_2 = 0.23, \end{cases}$$

$$\begin{cases} a_1 = 0.48, \\ a_2 = 0.24, \end{cases}$$

$$\begin{cases} a_1 = 0.6, \\ a_2 = 0.3, \end{cases}$$

$$\begin{cases} a_1 = 0.62, \\ a_2 = 0.31, \end{cases}$$

$$\begin{cases} a_1 = 0.64, \\ a_2 = 0.32, \end{cases}$$

$$\begin{cases} a_1 = 0.66, \\ a_2 = 0.33, \end{cases}$$

$$\begin{cases} a_1 = 0.68, \\ a_2 = 0.34, \end{cases}$$

$$\begin{cases} a_1 = 0.8, \\ a_2 = 0.4, \end{cases}$$

$$\begin{cases} a_1 = 0.82, \\ a_2 = 0.41, \end{cases}$$

$$\begin{cases} a_1 = 0.84, \\ a_2 = 0.42, \end{cases}$$

$$\begin{cases} a_1 = 0.86, \\ a_2 = 0.43, \end{cases}$$

$$\begin{cases} a_1 = 0.88, \\ a_2 = 0.44. \end{cases}$$

Let us consider the first case

$$\begin{cases} a_1 = 0.02, \\ a_2 = 0.01. \end{cases}$$

The equation of line a is

$$0.02 \times_2 x +_2 0.01 \times_2 y +_2 0.02 \times_2 99.99 -_2 0.01 \times_2 99.99 = 0,$$

that is,

$$a^1 : 0.02 \times_2 x +_2 0.01 \times_2 y +_2 0.99 = 0.$$

Note that $(-33.00, -33.00) \in (a^1 \cap b)$. This means that $a^1 \nparallel b$. Note that $(-33.\star\star, -33.\star\star) \in a^1$.

Let us consider the next case

$$\begin{cases} a_1 = 0.04, \\ a_2 = 0.02. \end{cases}$$

The equation of line a is

$$0.04 \times_2 x +_2 0.02 \times_2 y +_2 0.04 \times_2 99.99 -_2 0.02 \times_2 99.99 = 0,$$

that is,

$$a^2 : 0.04 \times_2 x +_2 0.02 \times_2 y +_2 1.98 = 0.$$

Note that $(-33.00, -33.00) \in (a^2 \cap b)$. This means that $a^2 \nparallel b$. Note that $(-33.\star\star, -33.\star\star) \in a^2$.

Let us consider the next case

$$\begin{cases} a_1 = 0.06, \\ a_2 = 0.03. \end{cases}$$

The equation of line a is

$$0.06 \times_2 x +_2 0.03 \times_2 y +_2 0.06 \times_2 99.99 -_2 0.03 \times_2 99.99 = 0,$$

that is,

$$a^3 : 0.06 \times_2 x +_2 0.03 \times_2 y +_2 2.97 = 0.$$

Note that $(-33.00, -33.00) \in (a^3 \cap b)$. This means that $a^3 \nparallel b$. Note that $(-33.\star\star, -33.\star\star) \in a^3$.

Let us consider the next case

$$\begin{cases} a_1 = 0.08, \\ a_2 = 0.04. \end{cases}$$

The equation of line a is

$$0.08 \times_2 x +_2 0.04 \times_2 y +_2 0.08 \times_2 99.99 -_2 0.04 \times_2 99.99 = 0,$$

that is,

$$a^4 : 0.08 \times_2 x +_2 0.04 \times_2 y +_2 3.96 = 0$$

Note that $(-33.00, -33.00) \in (a^4 \cap b)$. This means that $a^4 \nparallel b$. Note that $(-33.\star\star, -33.\star\star) \in a^4$. Also, note that lines a^1, a^2, a^3, a^4 coincide:

$$a^1 = a^2 = a^3 = a^4 = a^{(1)}.$$

Let us consider the next case

$$\begin{cases} a_1 = 0.2, \\ a_2 = 0.1. \end{cases}$$

The equation of line a is

$$0.2 \times_2 x +_2 0.1 \times_2 y +_2 0.2 \times_2 99.99 -_2 0.1 \times_2 99.99 = 0,$$

that is,

$$a^5 : 0.2 \times_2 x +_2 0.1 \times_2 y +_2 9.99 = 0.$$

Note that $(-33.30, -33.30) \in (a^5 \cap b)$. This means that $a^5 \nparallel b$. Note that $(-33.3\star, -33.3\star) \in a^5$. Also, note that $a^5 \subset a^{(1)}$ and $a^5 \neq a^{(1)}$ because $(-33.00, -33.00) \notin a^5$.

Let us consider the next case

$$\begin{cases} a_1 = 0.22, \\ a_2 = 0.11. \end{cases}$$

The equation of line a is

$$0.22 \times_2 x +_2 0.11 \times_2 y +_2 0.22 \times_2 99.99 -_2 0.11 \times_2 99.99 = 0,$$

that is,

$$a^6 : 0.22 \times_2 x +_2 0.11 \times_2 y +_2 10.98 = 0.$$

Note that $(-33.30, -33.30) \in (a^6 \cap b)$. This means that $a^6 \nparallel b$. Note that $(-33.3\star, -33.3\star) \in a^6$. Also, note that $a^6 \subset a^{(1)}$ and $a^6 \neq a^{(1)}$ because $(-33.00, -33.00) \notin a^6$.

Let us consider the next case

$$\begin{cases} a_1 = 0.24, \\ a_2 = 0.12. \end{cases}$$

The equation of line a is

$$0.24 \times_2 x +_2 0.12 \times_2 y +_2 0.24 \times_2 99.99 -_2 0.12 \times_2 99.99 = 0,$$

that is,

$$a^7 : 0.24 \times_2 x +_2 0.12 \times_2 y +_2 11.97 = 0.$$

Note that $(-33.30, -33.30) \in (a^7 \cap b)$. This means that $a^7 \nparallel b$. Note that $(-33.3\star, -33.3\star) \in a^7$. Also, note that $a^7 \subset a^{(1)}$ and $a^7 \neq a^{(1)}$ because $(-33.00, -33.00) \notin a^7$.

Let us consider the next case

$$\begin{cases} a_1 = 0.26, \\ a_2 = 0.13. \end{cases}$$

The equation of line a is

$$0.26 \times_2 x +_2 0.13 \times_2 y +_2 0.26 \times_2 99.99 -_2 0.13 \times_2 99.99 = 0,$$

that is,

$$a^8 : 0.26 \times_2 x +_2 0.13 \times_2 y +_2 12.96 = 0.$$

Note that $(-33.30, -33.30) \in (a^8 \cap b)$. This means that $a^8 \nparallel b$. Note that $(-33.3\star, -33.3\star) \in a^8$. Also, note that $a^8 \subset a^{(1)}$ and $a^8 \neq a^{(1)}$ because $(-33.00, -33.00) \notin a^8$.

Let us consider the next case

$$\begin{cases} a_1 = 0.28, \\ a_2 = 0.14. \end{cases}$$

The equation of line a is

$$0.28 \times_2 x +_2 0.14 \times_2 y +_2 0.28 \times_2 99.99 -_2 0.14 \times_2 99.99 = 0,$$

that is,

$$a^9 : 0.28 \times_2 x +_2 0.14 \times_2 y +_2 13.95 = 0.$$

Note that $(-33.30, -33.30) \in (a^9 \cap b)$. This means that $a^9 \nparallel b$. Note that $(-33.3\star, -33.3\star) \in a^9$. Also, note that $a^9 \subset a^{(1)}$ and $a^9 \neq a^{(1)}$ because $(-33.00, -33.00) \notin a^9$.

Let us consider the next case

$$\begin{cases} a_1 = 0.4, \\ a_2 = 0.2. \end{cases}$$

The equation of line a is

$$0.4 \times_2 x +_2 0.2 \times_2 y +_2 0.4 \times_2 99.99 -_2 0.2 \times_2 99.99 = 0,$$

that is,

$$a^{10} : 0.4 \times_2 x +_2 0.2 \times_2 y +_2 19.98 = 0.$$

Note that $(-33.30, -33.30) \in (a^{10} \cap b)$. This means that $a^{10} \nparallel b$. Note that $(-33.3\star, -33.3\star) \in a^{10}$. Also, note that $a^{10} \subset a^{(1)}$ and $a^{10} \neq a^{(1)}$ because $(-33.00, -33.00) \notin a^{10}$.

Let us consider the next case

$$\begin{cases} a_1 = 0.42, \\ a_2 = 0.21. \end{cases}$$

The equation of line a is

$$0.42 \times_2 x +_2 0.21 \times_2 y +_2 0.42 \times_2 99.99 -_2 0.21 \times_2 99.99 = 0,$$

that is,

$$a^{11} : 0.42 \times_2 x +_2 0.21 \times_2 y +_2 20.97 = 0.$$

Note that $(-33.30, -33.30) \in (a^{11} \cap b)$. This means that $a^{11} \nparallel b$. Note that $(-33.3\star, -33.3\star) \in a^{11}$. Also, note that $a^{11} \subset a^{(1)}$ and $a^{11} \neq a^{(1)}$ because $(-33.00, -33.00) \notin a^{11}$.

Let us consider the next case

$$\begin{cases} a_1 = 0.44, \\ a_2 = 0.22. \end{cases}$$

The equation of line a is

$$0.44 \times_2 x +_2 0.22 \times_2 y +_2 0.44 \times_2 99.99 -_2 0.22 \times_2 99.99 = 0,$$

that is,

$$a^{12} : 0.44 \times_2 x +_2 0.22 \times_2 y +_2 21.96 = 0.$$

Note that $(-33.30, -33.30) \in (a^{12} \cap b)$. This means that $a^{12} \nparallel b$. Note that $(-33.3\star, -33.3\star) \in a^{12}$. Also, note that $a^{12} \subset a^{(1)}$ and $a^{12} \neq a^{(1)}$ because $(-33.00, -33.00) \notin a^{12}$.

Let us consider the next case

$$\begin{cases} a_1 = 0.46, \\ a_2 = 0.23. \end{cases}$$

The equation of line a is

$$0.46 \times_2 x +_2 0.23 \times_2 y +_2 0.46 \times_2 99.99 -_2 0.23 \times_2 99.99 = 0,$$

that is,

$$a^{13} : 0.46 \times_2 x +_2 0.23 \times_2 y +_2 22.95 = 0.$$

Note that $(-33.30, -33.30) \in (a^{13} \cap b)$. This means that $a^{13} \nparallel b$. Note that $(-33.3\star, -33.3\star) \in a^{13}$. Also, note that $a^{13} \subset a^{(1)}$ and $a^{13} \neq a^{(1)}$ because $(-33.00, -33.00) \notin a^{13}$.

Let us consider the next case

$$\begin{cases} a_1 = 0.48, \\ a_2 = 0.24. \end{cases}$$

The equation of line a is

$$0.48 \times_2 x +_2 0.24 \times_2 y +_2 0.48 \times_2 99.99 -_2 0.24 \times_2 99.99 = 0,$$

that is,

$$a^{14} : 0.48 \times_2 x +_2 0.24 \times_2 y +_2 23.94 = 0.$$

Note that $(-33.30, -33.30) \in (a^{14} \cap b)$. This means that $a^{14} \nparallel b$. Note that $(-33.3\star, -33.3\star) \in a^{14}$. Also, note that $a^{14} \subset a^{(1)}$ and $a^{14} \neq a^{(1)}$ because $(-33.00, -33.00) \notin a^{14}$.

Let us consider the next case

$$\begin{cases} a_1 = 0.6, \\ a_2 = 0.3. \end{cases}$$

The equation of line a is

$$0.6 \times_2 x +_2 0.3 \times_2 y +_2 0.6 \times_2 99.99 -_2 0.3 \times_2 99.99 = 0,$$

that is,

$$a^{15} : 0.6 \times_2 x +_2 0.3 \times_2 y +_2 29.97 = 0.$$

Note that $(-33.30, -33.30) \in (a^{15} \cap b)$. This means $a^{15} \nparallel b$. Note that $(-33.3\star, -33.3\star) \in a^{15}$. Also, note that $a^{15} \subset a^{(1)}$ and $a^{15} \neq a^{(1)}$ because

$(-33.00, -33.00) \notin a^{15}$.

Let us consider the next case

$$\begin{cases} a_1 = 0.62, \\ a_2 = 0.31. \end{cases}$$

The equation of line a is

$$0.62 \times_2 x +_2 0.31 \times_2 y +_2 0.62 \times_2 99.99 -_2 0.31 \times_2 99.99 = 0,$$

that is,

$$a^{16} : 0.62 \times_2 x +_2 0.31 \times_2 y +_2 30.96 = 0.$$

Note that $(-33.30, -33.30) \in (a^{16} \cap b)$. This means that $a^{16} \nparallel b$. Note that $(-33.3\star, -33.3\star) \in a^{16}$. Also, note that $a^{16} \subset a^{(1)}$ and $a^{16} \neq a^{(1)}$ because $(-33.00, -33.00) \notin a^{16}$.

Let us consider the next case

$$\begin{cases} a_1 = 0.64, \\ a_2 = 0.32. \end{cases}$$

The equation of line a is

$$0.64 \times_2 x +_2 0.32 \times_2 y +_2 0.64 \times_2 99.99 -_2 0.32 \times_2 99.99 = 0,$$

that is,

$$a^{17} : 0.64 \times_2 x +_2 0.32 \times_2 y +_2 31.95 = 0.$$

Note that $(-33.30, -33.30) \in (a^{17} \cap b)$. This means that $a^{17} \nparallel b$. Note that $(-33.3\star, -33.3\star) \in a^{17}$. Also, note that $a^{17} \subset a^{(1)}$ and $a^{17} \neq a^{(1)}$ because $(-33.00, -33.00) \notin a^{17}$.

Let us consider the next case

$$\begin{cases} a_1 = 0.66, \\ a_2 = 0.33. \end{cases}$$

The equation of line a is

$$0.66 \times_2 x +_2 0.33 \times_2 y +_2 0.66 \times_2 99.99 -_2 0.33 \times_2 99.99 = 0,$$

that is,

$$a^{18} : 0.66 \times_2 x +_2 0.33 \times_2 y +_2 32.94 = 0.$$

Note that $(-33.30, -33.30) \in (a^{18} \cap b)$. This means that $a^{18} \nparallel b$. Note that $(-33.3\star, -33.3\star) \in a^{18}$. Also, note that $a^{18} \subset a^{(1)}$ and $a^{18} \neq a^{(1)}$ because $(-33.00, -33.00) \notin a^{18}$.

Let us consider the next case

$$\begin{cases} a_1 = 0.68, \\ a_2 = 0.34. \end{cases}$$

The equation of line a is

$$0.68 \times_2 x +_2 0.34 \times_2 y +_2 0.68 \times_2 99.99 -_2 0.34 \times_2 99.99 = 0,$$

that is,

$$a^{19} : 0.68 \times_2 x +_2 0.34 \times_2 y +_2 33.93 = 0.$$

Note that $(-33.30, -33.30) \in (a^{19} \cap b)$. This means that $a^{19} \nparallel b$. Note that $(-33.3\star, -33.3\star) \in a^{19}$. Also, note that $a^{19} \subset a^{(1)}$ and $a^{19} \neq a^{(1)}$ because $(-33.00, -33.00) \notin a^{19}$.

Let us consider the next case

$$\begin{cases} a_1 = 0.8, \\ a_2 = 0.4. \end{cases}$$

The equation of line a is

$$0.8 \times_2 x +_2 0.4 \times_2 y +_2 0.8 \times_2 99.99 -_2 0.4 \times_2 99.99 = 0,$$

that is,

$$a^{20} : 0.8 \times_2 x +_2 0.4 \times_2 y +_2 39.96 = 0.$$

Note that $(-33.30, -33.30) \in (a^{20} \cap b)$. This means that $a^{20} \nparallel b$. Note that $(-33.3\star, -33.3\star) \in a^{20}$. Also, note that $a^{20} \subset a^{(1)}$ and $a^{20} \neq a^{(1)}$ because $(-33.00, -33.00) \notin a^{20}$.

Let us consider the next case

$$\begin{cases} a_1 = 0.82, \\ a_2 = 0.41. \end{cases}$$

The equation of line a is

$$0.82 \times_2 x +_2 0.41 \times_2 y +_2 0.82 \times_2 99.99 -_2 0.41 \times_2 99.99 = 0,$$

that is,

$$a^{21} : 0.82 \times_2 x +_2 0.41 \times_2 y +_2 40.95 = 0$$

Note that $(-33.30, -33.30) \in (a^{21} \cap b)$. This means that $a^{21} \nparallel b$. Note that $(-33.3\star, -33.3\star) \in a^{21}$. Also, note that $a^{21} \subset a^{(1)}$ and $a^{21} \neq a^{(1)}$ because $(-33.00, -33.00) \notin a^{21}$.

Let us consider the next case

$$\begin{cases} a_1 = 0.84, \\ a_2 = 0.42. \end{cases}$$

The equation of line *a* is

$$0.84 \times_2 x +_2 0.42 \times_2 y +_2 0.84 \times_2 99.99 -_2 0.42 \times_2 99.99 = 0,$$

that is,

$$a^{22} : 0.84 \times_2 x +_2 0.42 \times_2 y +_2 41.94 = 0.$$

Note that $(-33.30, -33.30) \in (a^{22} \cap b)$. This means that $a^{22} \nparallel b$. Note that $(-33.3\star, -33.3\star) \in a^{22}$. Also, note we have $a^{22} \subset a^{(1)}$ and $a^{22} \neq a^{(1)}$ because $(-33.00, -33.00) \notin a^{22}$.

Let us consider the next case

$$\begin{cases} a_1 = 0.86, \\ a_2 = 0.43. \end{cases}$$

The equation of line *a* is

$$0.86 \times_2 x +_2 0.43 \times_2 y +_2 0.86 \times_2 99.99 -_2 0.43 \times_2 99.99 = 0,$$

that is,

$$a^{23} : 0.86 \times_2 x +_2 0.43 \times_2 y +_2 42.93 = 0.$$

Note that $(-33.30, -33.30) \in (a^{23} \cap b)$. This means that $a^{23} \nparallel b$. Note that $(-33.3\star, -33.3\star) \in a^{23}$. Also, note that $a^{23} \subset a^{(1)}$ and $a^{23} \neq a^{(1)}$ because $(-33.00, -33.00) \notin a^{23}$.

Let us consider the next case

$$\begin{cases} a_1 = 0.88, \\ a_2 = 0.44. \end{cases}$$

The equation of line *a* is

$$0.88 \times_2 x +_2 0.44 \times_2 y +_2 0.88 \times_2 99.99 -_2 0.44 \times_2 99.99 = 0,$$

that is,

$$a^{24} : 0.88 \times_2 x +_2 0.44 \times_2 y +_2 43.92 = 0.$$

Note that $(-33.30, -33.30) \in (a^{24} \cap b)$. This means that $a^{24} \nparallel b$. Note that $(-33.3\star, -33.3\star) \in a^{24}$. Also, note that $a^{24} \subset a^{(1)}$ and $a^{24} \neq a^{(1)}$ because $(-33.00, -33.00) \notin a^{24}$.

Note that lines

$$a^5, a^6, a^7, a^8, a^9, a^{10}, \ldots, a^{20}, a^{21}, a^{22}, a^{23}, a^{24}$$

coincide,

$$a^5 = a^6 = a^7 = a^8 = a^9 = a^{10} = \cdots = a^{20} = a^{21} = a^{22} = a^{23} = a^{24} = a^{(2)},$$

and

$$a^{(2)} \subset a^{(1)}, \quad a^{(2)} \neq a^{(1)}.$$

Now we look for possible coefficients a_1, a_2 in the equation of line a in the case

$$(99.99, 0) \in a.$$

This means

$$a_1 \times_2 99.99 +_2 a_2 \times_2 0 +_2 a_1 \times_2 99.99 -_2 a_2 \times_2 99.99 = 0,$$

that is,

$$a_1 \times_2 99.99 +_2 a_1 \times_2 99.99 -_2 a_2 \times_2 99.99 = 0.$$

By direct calculation we get the following pairs a_1, a_2:

$$\begin{cases} a_2 = 0.02, \\ a_1 = 0.01, \end{cases}$$
$$\begin{cases} a_2 = 0.04, \\ a_1 = 0.02, \end{cases}$$
$$\begin{cases} a_2 = 0.06, \\ a_1 = 0.03, \end{cases}$$
$$\begin{cases} a_2 = 0.08, \\ a_1 = 0.04, \end{cases}$$
$$\begin{cases} a_2 = 0.2, \\ a_1 = 0.1, \end{cases}$$
$$\begin{cases} a_2 = 0.22, \\ a_1 = 0.11, \end{cases}$$

$$\begin{cases} a_2 = 0.24, \\ a_1 = 0.12, \end{cases}$$
$$\begin{cases} a_2 = 0.26, \\ a_1 = 0.13, \end{cases}$$
$$\begin{cases} a_2 = 0.28, \\ a_1 = 0.14, \end{cases}$$
$$\begin{cases} a_2 = 0.4, \\ a_1 = 0.2, \end{cases}$$
$$\begin{cases} a_2 = 0.42, \\ a_1 = 0.21, \end{cases}$$
$$\begin{cases} a_2 = 0.44, \\ a_1 = 0.22, \end{cases}$$
$$\begin{cases} a_2 = 0.46, \\ a_1 = 0.23, \end{cases}$$
$$\begin{cases} a_2 = 0.48, \\ a_1 = 0.24, \end{cases}$$
$$\begin{cases} a_2 = 0.6, \\ a_1 = 0.3, \end{cases}$$
$$\begin{cases} a_2 = 0.62, \\ a_1 = 0.31, \end{cases}$$
$$\begin{cases} a_2 = 0.64, \\ a_1 = 0.32, \end{cases}$$
$$\begin{cases} a_2 = 0.66, \\ a_1 = 0.33, \end{cases}$$
$$\begin{cases} a_2 = 0.68, \\ a_1 = 0.34, \end{cases}$$
$$\begin{cases} a_2 = 0.8, \\ a_1 = 0.4, \end{cases}$$
$$\begin{cases} a_2 = 0.82, \\ a_1 = 0.41, \end{cases}$$
$$\begin{cases} a_2 = 0.84, \\ a_1 = 0.42, \end{cases}$$
$$\begin{cases} a_2 = 0.86, \\ a_1 = 0.43, \end{cases}$$
$$\begin{cases} a_2 = 0.88, \\ a_1 = 0.44. \end{cases}$$

The first four cases give us the same straight line *a* with equation

$$a^{(3)}:0.01\times_2 x+_2 0.02\times_2 y+_2 0.01\times_2 99.99-_2 0.02\times_2 99.99=0,$$

that is,

$$a^{(3)}:0.01\times_2 x+_2 0.02\times_2 y-_2 0.99=0.$$

Note that $(33.00,33.00)\in(a^{(3)}\cap b)$. This means that $a^{(3)}\nparallel b$. Note that $(33.\star\star,33.\star\star)\in a^{(3)}$.

The remaining twenty cases give us the same straight line *a* (which differ from the first four cases) with equation

$$a^{(4)}:0.1\times_2 x+_2 0.2\times_2 y+_2 0.1\times_2 99.99-_2 0.2\times_2 99.99=0,$$

that is,

$$a^{(4)}:0.1\times_2 x+_2 0.2\times_2 y-_2 9.99=0.$$

Note that $(33.30,33.30)\in(a^{(4)}\cap b)$. This means that $a^{(4)}\nparallel b$. Note that $(33.3\star,33.3\star)\in a^{(4)}$ and $a^{(4)}\subset a^{(3)}$, $a^{(4)}\neq a^{(3)}$ because $(33.00,33.00)\notin a^{(4)}$.

Let us get back to the common situation with line *a* satisfying the system

$$\begin{cases} a_1,a_2,x,y,a_1\times_2 x,a_2\times_2 y,a_1\times_2 x+_2 a_2\times_2 y,a_1\times_2 99.99-_2 a_2\times_2 99.99\in W_2,\\ 0<a_1<1,\\ 0<a_2<1,\\ a_1\neq a_2,\\ (a_1\times_2 x+_2 a_2\times_2 y)+_2(a_1\times_2 99.99-_2 a_2\times_2 99.99)=0. \end{cases}$$

Case 1. $a_1,a_2\in[0.01,0.02,\dots,0.09]$. Then

$$\begin{aligned} a_1\times_2 99.99&\in[0.99,1.98,2.97,3.96,4.95,5.94,6.93,7.92,8.91],\\ a_2\times_2 99.99&\in[0.99,1.98,2.97,3.96,4.95,5.94,6.93,7.92,8.91], \end{aligned}$$

and

$$a_1\times_2 99.99-_2 a_2\times_2 99.99\in[\pm0.99,\pm1.98,\pm2.97,\pm3.96,\pm4.95,\pm5.94,\pm6.93,\pm7.92].$$

For any $x\in W_2$,

$$\begin{aligned} a_1\times_2 x\in{}&[0.00,\pm0.01,\pm0.02,\dots,\pm0.99]\cup[0.00,\pm0.02,\pm0.04,\dots,\pm1.98]\cup\\ &\cup[0.00,\pm0.03,\pm0.06,\dots,\pm2.97]\cup\dots\cup[0.00,\pm0.09,\pm0.18,\dots,\pm8.91] \end{aligned}$$

and for any $y\in W_2$,

$$a_2 \times_2 y \in [0.00, \pm 0.01, \pm 0.02, \dots, \pm 0.99] \cup [0.00, \pm 0.02, \pm 0.04, \dots, \pm 1.98] \cup$$
$$\cup [0.00, \pm 0.03, \pm 0.06, \dots, \pm 2.97] \cup \dots \cup [0.00, \pm 0.09, \pm 0.18, \dots, \pm 8.91].$$

Now let us prove that the lines

$$a : (a_1 \times_2 x +_2 a_2 \times_2 y) +_2 (a_1 \times_2 99.99 -_2 a_2 \times_2 99.99) = 0$$

and

$$b : x -_2 y = 0$$

are not parallel for $a_1, a_2 \in [0.01, 0.02, \dots, 0.09]$.

First, let us choose $a_1 = 0.05$ and $a_2 = 0.02$. In this case,

$$a : (0.05 \times_2 x +_2 0.02 \times_2 y) +_2 2.97 = 0,$$

and we get two sets of points $B(-59.\star\star, -2.\star\star) \subset a$ and $C(-39.\star\star, -51.\star\star) \subset a$. We also have $B \subset R_2^b$, $C \subset R_1^b$, that is, $a \nparallel b$.

Let us make another choice: $a_1 = 0.09$, $a_2 = 0.04$. In this case,

$$a : (0.09 \times_2 x +_2 0.04 \times_2 y) +_2 4.95 = 0,$$

and we get two sets of points $D(-11.\star\star, -99.\star\star) \subset a$ and $E(-51.\star\star, -9.\star\star) \subset a$. We also have $D \subset R_1^b$ and $E \subset R_2^b$, that is, $a \nparallel b$.

The general case can be proved in the same way.

Case 2. Let $a_1 \in [0.01, 0.02, \dots, 0.09]$ and $a_2 \in [0.10, 0.11, \dots, 0.19]$. Then

$$a_1 \times_2 99.99 \in [0.99, 1.98, 2.97, 3.96, 4.95, 5.94, 6.93, 7.92, 8.91]$$

and

$$a_2 \times_2 99.99 \in [9.99, 10.98, 11.97, 12.96, 13.95, 14.94, 15.93, 16.92, 17.91, 18.90].$$

For any $x \in W_2$,

$$a_1 \times_2 x \in [0.00, \pm 0.01, \pm 0.02, \dots, \pm 0.99] \cup [0.00, \pm 0.02, \pm 0.04, \dots, \pm 1.98] \cup$$
$$\cup [0.00, \pm 0.03, \pm 0.06, \dots, \pm 2.97] \cup \dots \cup [0.00, \pm 0.09, \pm 0.18, \dots, \pm 8.91]$$

Also, for any $y \in W_2$,

$$a_2 \times_2 y \in [0.00, \pm 0.01, \pm 0.02, \dots, \pm 0.09] \cup [\pm 0.10, \pm 0.11, \dots, \pm 0.19] \cup \dots$$
$$\cup [\pm 9.90, \pm 9.91, \dots, \pm 9.99] \cup \dots .$$

Now let us prove that the lines

$$a : (a_1 \times_2 x +_2 a_2 \times_2 y) +_2 (a_1 \times_2 99.99 -_2 a_2 \times_2 99.99) = 0$$

and

$$b : x -_2 y = 0$$

are not parallel for $a_1 \in [0.01, 0.02, \ldots, 0.09]$ and $a_2 \in [0.10, 0.11, \ldots, 0.19]$.

First, let us choose $a_1 = 0.05$ and $a_2 = 0.19$. In this case,

$$a : (0.05 \times_2 x +_2 0.19 \times_2 y) -_2 13.95 = 0,$$

and we get two sets of points $F(47.\star\star, 61.1\star) \subset a$ and $G(94.\star\star, 51.1\star) \subset a$. We also have $F \subset R_2^b$ and $G \subset R_1^b$, that is, $a \nparallel b$.

The general case can be proved in the same way.

Case 3. Let $a_1, a_2 \in [0.10, 0.11, \ldots, 0.19]$. Then

$$a_1 \times_2 99.99 \in [9.99, 10.98, 11.97, 12.96, 13.95, 14.94, 15.93, 16.92, 17.91, 18.90]$$

and

$$a_2 \times_2 99.99 \in [9.99, 10.98, 11.97, 12.96, 13.95, 14.94, 15.93, 16.92, 17.91, 18.90].$$

For any $x \in W_2$,

$$a_1 \times_2 x \in [0.00, \pm 0.01, \pm 0.02, \ldots, \pm 0.09] \cup [\pm 0.10, \pm 0.11, \ldots, \pm 0.19] \cup \ldots \cup [\pm 9.90, \pm 9.91, \ldots, \pm 9.99] \ldots .$$

Also, for any $y \in W_2$,

$$a_2 \times_2 y \in [0.00, \pm 0.01, \pm 0.02, \ldots, \pm 0.09] \cup [\pm 0.10, \pm 0.11, \ldots, \pm 0.19] \cup \ldots \cup [\pm 9.90, \pm 9.91, \ldots, \pm 9.99] \ldots .$$

Now let us prove that lines *a*, *b* are not parallel for $a_1, a_2 \in [0.10, 0.11, \ldots, 0.19]$. First, let us choose $a_1 = 0.12$ and $a_2 = 0.19$. In this case,

$$a : (0.12 \times_2 x +_2 0.19 \times_2 y) -_2 6.93 = 0,$$

and we get two sets of points $H(50.8\star, 4.9\star) \subset a$ and $I(10.2\star, 30.1\star) \subset a$. We also have $H \subset R_1^b$ and $I \subset R_2^b$, that is, $a \nparallel b$.

The general case can be proved in the same way.

Case 4. Let $a_1 \in [0.01, 0.02, \ldots, 0.09]$ and $a_2 \in [0.20, 0.21, \ldots, 0.99]$. Then

$$a_1 \times_2 99.99 \in [0.99, 1.98, 2.97, 3.96, 4.95, 5.94, 6.93, 7.92, 8.91]$$

and

$$a_2 \times_2 99.99 \in [19.98, 20.97, 21.96, \ldots, 97.83, 98.82].$$

For any $x \in W_2$,

$$a_1 \times_2 x \in [0.00, \pm 0.01, \pm 0.02, \ldots, \pm 0.99] \cup [0.00, \pm 0.02, \pm 0.04, \ldots, \pm 1.98] \cup$$
$$\cup [0.00, \pm 0.03, \pm 0.06, \ldots, \pm 2.97] \cup \cdots \cup [0.00, \pm 0.09, \pm 0.18, \ldots, \pm 8.91].$$

Also, for any $y \in W_2$,

$$a_2 \times_2 y \in [0.00, \pm 0.01, \pm 0.02, \ldots, \pm 0.09] \cup [\pm 0.10, \pm 0.11, \ldots, \pm 0.19] \cup \ldots$$
$$\cup [\pm 9.90, \pm 9.91, \ldots, \pm 9.99] \cup \ldots .$$

Now let us prove that lines *a*, *b* are not parallel for $a_1 \in [0.01, 0.02, \ldots, 0.09]$ and $a_2 \in [0.20, 0.21, \ldots, 0.99]$.

First, let us choose $a_1 = 0.07$ and $a_2 = 0.91$. In this case,

$$a : (0.07 \times_2 x +_2 0.91 \times_2 y) -_2 83.97 = 0,$$

and we get two sets of points $J(18.\star\star, 90.9\star) \subset a$ and $K(96.\star\star, 84.9\star) \subset a$. We also have $J \subset R_2^b$ and $K \subset R_1^b$, that is, $a \nparallel b$.

The general case can be proved in the same way.

Case 5. Let $a_1, a_2 \in [0.20, 0.21, \ldots, 0.99]$. Then

$$a_1, a_2 \times_2 99.99 \in [19.98, 20.97, 21.96, \ldots, 97.83, 98.82].$$

For any $x \in W_2$,

$$a_1 \times_2 x \in [0.00, \pm 0.01, \pm 0.02, \ldots, \pm 0.09] \cup [\pm 0.10, \pm 0.11, \ldots, \pm 0.19] \cup \ldots$$
$$\cup [\pm 9.90, \pm 9.91, \ldots, \pm 9.99] \cup \ldots .$$

Now let us prove that lines *a*, *b* are not parallel for $a_1, a_2 \in [0.20, 0.21, \ldots, 0.99]$. First, let us choose $a_1 = 0.27$ and $a_2 = 0.91$. In this case,

$$a : (0.27 \times_2 x +_2 0.91 \times_2 y) -_2 63.99 = 0,$$

and we get two sets of points $L(51.4\star, 55.1\star) \subset a$ and $M(61.1\star, 52.2\star) \subset a$. We also we have $L \subset R_2^b$ and $M \subset R_1^b$, that is, $a \nparallel b$.

The general case can be proved in the same way, as well as the general statement of the theorem for any *n*.

Let us now prove now the following:

Theorem 6.15.

In the plane E_2W_n, there are a point A and a straight line b not containing this point such that there is only one straight line a containing point A and parallel to line b: $a \parallel b$.

Let us consider the situation with $n = 2$, that is, we are in E_2W_2. Let $A(0, 0.01) \in E_2W_2$. Let straight line *b* have the equation

$$b : y = 0.$$

So we get

$$R_1^b : \big[(x, y) \in E_2W_2 : y > 0\big],$$
$$R_2^b : \big[(x, y) \in E_2W_2 : y < 0\big],$$
$$R_3^b = \Lambda.$$

We looking for a straight line *a* containing point *A* and parallel to straight line *b*. Because $A \in R_1^b$ and we must have $a \parallel b$, we get $a \subset R_1^b$, that is, any point $(x, y) \in a$ must have $y > 0$. This means that $y \geq 0.01$. So we have such straight line *a*:

$$a : y = 0.01,$$
$$A \in a, a \parallel b.$$

Let us consider another straight line *a* in E_2W_2:

$$a : 0.01 \times_2 y = 0 = f(x).$$

We get the set *a* as a subset of E_2W_2:

$$a = \big[(x, -0.99), (x, -0.98), \ldots, (x, -0.01), (x, 0), (x, 0.01), \ldots, (x, 0.99)\big]$$

for any $x \in W_n$. Clearly, fully this set may be seen by any W_m-observer with $m \geq 7$. The function *f* is multivalued.

We get

$$A \in a,$$

but

$$a \nparallel b$$

because

$$a \supset b, \quad a \neq b.$$

Let us consider the transformation of parallel shift along the *y*-axis in E_2W_2:

$$y \longrightarrow y -_2 1 = g(y).$$

The superposition of the functions *f* and *g* is

$$f(g(y)) = 0.01 \times_2 (y -_2 1) = 0,$$

and we get

$$y -_2 1 = -0.99,$$

$$y -_2 1 = -0.98,$$

$$\cdots\cdots\cdots\cdots$$

$$y -_2 1 = -0.01,$$

$$y -_2 1 = 0,$$

$$y -_2 1 = 0.01,$$

$$\cdots\cdots\cdots\cdots$$

$$y -_2 1 = 0.98,$$

$$y -_2 1 = 0.99.$$

So we have

$$y = 0.01,$$

$$y = 0.02,$$

$$\cdots\cdots$$

$$y = 0.99,$$

$$y = 1,$$

$$y = 1.01,$$

$$\cdots\cdots$$

$$y = 1.98,$$

$$y = 1.99.$$

So we have to find out whether the set $f(g(y))$ represents the straight line

$$0.01 \times_2 (y -_2 1) = 0.$$

The answer is positive only if the solution of the equation

$$0.01 \times_2 y = 0.01$$

coincides with that of the equation

$$0.01 \times_2 (y -_2 1) = 0,$$

which is considered above. However, the solution of the former is

$$y = 1,$$

$$y = 1.01,$$

$$\cdots\cdots$$

$$y = 1.98,$$

$$y = 1.99.$$

This means that the set $f(g(y))$ does not represent a straight line, that is, the straight line transformation of parallel shift along the *y*-axis in E_2W_2 may not represent a straight line.

So we continue the search of straight line *a* such that

$$A \in a$$

and

$$a \parallel b.$$

Let us continue to consider the situation with $n = 2$, that is, we are in E_2W_2. Let us first take $A(0, 0.10) \in E_2W_2$. We look for a straight line *a* containing this point. We have

$$a : a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0$$

for any $a_1, a_2, a_3, x, y, a_1 \times_2 x, a_2 \times_2 y, a_1 \times_2 x +_2 a_2 \times_2 y \in W_2$ such that $(a_1, a_2) \neq (0, 0)$ and

$$a_1 \times_2 0 +_2 a_2 \times_2 0.10 +_2 a_3 = 0.$$

This means that

$$a_3 = -a_2 \times_2 0.10,$$

and the equation of line *a* is

$$a : a_1 \times_2 x +_2 a_2 \times_2 y -_2 a_2 \times_2 0.10 = 0.$$

Because generally two straight lines *c* and *d* with equations

$$c : c_1 \times_2 x +_2 c_2 \times_2 y +_2 c_3 = 0$$

and

$$d : -c_1 \times_2 x -_2 c_2 \times_2 y -_2 c_3 = 0$$

coincide, without loss of generality, we may assume that $a_2 > 0$ in the equation of line *a*. Moreover, we must have $0.10 \leq a_2$, because if $0 < a_2 < 0.10$, then

$$a_2 \times_2 0.10 = 0,$$

and the equation of line *a* becomes

$$a : a_1 \times_2 x +_2 a_2 \times_2 y = 0,$$

that is, $O(0,0) \in a$, and $a \nparallel b$.

In the case $a_2 = 1$ the equation of line *a* becomes

$$a : a_1 \times_2 x +_2 y -_2 0.10 = 0.$$

For $a_1 = 0$, we get

$$a : y -_2 0.10 = 0,$$

and in this case, $a \parallel b$ because

$$\begin{aligned} R_1^a &: \left[(x,y) \in E_2W_2 : y -_2 0.10 > 0\right], \\ R_2^a &: \left[(x,y) \in E_2W_2 : y -_2 0.10 < 0\right], \\ R_3^a &= \Lambda, \end{aligned}$$

and $b \subset R_2^a$.

Let us rename this straight line *a* as straight line *c*. In the case $a_2 = 0.10$ the equation of line *a* becomes

$$a : a_1 \times_2 x +_2 0.10 \times_2 y -_2 0.01 = 0.$$

For $a_1 = 0$, we get

$$a : 0.10 \times_2 y -_2 0.01 = 0,$$

and in this case, $a \parallel b$ because

$$R_1^a: \big[(x,y)\in E_2W_2: 0.10\times_2 y -_2 0.01 > 0\big],$$
$$R_2^a: \big[(x,y)\in E_2W_2: 0.10\times_2 y -_2 0.01 < 0\big],$$
$$R_3^a = \Lambda,$$
and $b \subset R_2^a$. This means that

$$a = a^1 \cup a^2 \cup a^3 \cup a^4 \cup a^5 \cup a^6 \cup a^7 \cup a^8 \cup a^9 \cup a^{10},$$

where straight lines a^1, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10} have the equations

$$a^1: y = 0.10,$$
$$a^2: y = 0.11,$$
$$a^3: y = 0.12,$$
$$a^4: y = 0.13,$$
$$a^5: y = 0.14,$$
$$a^6: y = 0.15,$$
$$a^7: y = 0.16,$$
$$a^8: y = 0.17,$$
$$a^9: y = 0.18,$$
$$a^{10}: y = 0.19.$$

Let us rename straight line *a* as straight line *d*. We get

$$c \subset d$$

and

$$c \neq d.$$

This means that for the chosen point *A* and straight line *b*, there are at least two distinct straight lines *a* containing point *A* that are parallel to line *b*.

Let us now get take $A(0, 0.01) \in E_2W_2$. We look for a straight line *a* containing this point. We have

$$a: a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0$$

for any $a_1, a_2, a_3, x, y, a_1 \times_2 x, a_2 \times_2 y, a_1 \times_2 x +_2 a_2 \times_2 y \in W_2$ such that $(a_1, a_2) \neq (0,0)$ and

$$a_1 \times_2 0 +_2 a_2 \times_2 0.01 +_2 a_3 = 0.$$

This means that

$$a_3 = -a_2 \times_2 0.01,$$

and thus the equation of line *a* is

$$a : a_1 \times_2 x +_2 a_2 \times_2 y -_2 a_2 \times_2 0.01 = 0.$$

Let line *b* have the equation

$$b : y = 0.$$

So we get

$$R_1^b : \left[(x, y) \in E_2W_2 : y > 0\right],$$
$$R_2^b : \left[(x, y) \in E_2W_2 : y < 0\right],$$
$$R_3^b = \Lambda.$$

Because $A \in R_1^b$ and we must to have $a \parallel b$, we get $a \subset R_1^b$, that is, $y > 0$ for any point $(x, y) \in a$. This means that $y \geq 0.01$.

Because generally two straight lines *c* and *d* with equations

$$c : c_1 \times_2 x +_2 c_2 \times_2 y +_2 c_3 = 0$$

and

$$d : -c_1 \times_2 x -_2 c_2 \times_2 y -_2 c_3 = 0$$

coincide, without loss of generality, we may assume that $a_2 > 0$ in the equation of line *a*. Moreover, we must have $1 \leq a_2$, because if $0 < a_2 < 1$, then

$$a_2 \times_2 0.01 = 0,$$

and the equation of line *a* becomes

$$a : a_1 \times_2 x +_2 a_2 \times_2 y = 0,$$

that is, $O(0, 0) \in a$, and $a \nparallel b$.

In the case $a_2 = 1$ the equation of line *a* becomes

$$a : a_1 \times_2 x +_2 y -_2 0.01 = 0.$$

For $a_1 = 0$, we get

$$a : y -_2 0.01 = 0,$$

and $a \parallel b$ in this case, because

$$R_1^a : \left[(x, y) \in E_2W_2 : y -_2 0.01 > 0\right],$$
$$R_2^a : \left[(x, y) \in E_2W_2 : y -_2 0.01 < 0\right],$$
$$R_3^a = \Lambda,$$

and $b \subset R_2^a$.

Assuming that $a_1 \neq 0$, we consider four limiting cases: $a_1 = 0.01$, $a_1 = -0.01$, $a_1 = 99.99$, $a_1 = -99.99$.

For $a_1 = 0.01$, we get

$$a : 0.01 \times_2 x +_2 y -_2 0.01 = 0.$$

For $x = 99.99$, on this line, we get $y = -0.98$, that is, in this case, $a \nparallel b$.

For $a_1 = -0.01$, we get

$$a : -0.01 \times_2 x +_2 y -_2 0.01 = 0.$$

For $x = -99.99$, on this line, we get $y = -0.98$, that is, in this case, $a \nparallel b$.

For $a_1 = 99.99$, we get

$$a : 99.99 \times_2 x +_2 y -_2 0.01 = 0.$$

For $x = 0.01$, on this line, we get $y = -0.98$, that is, in this case, $a \nparallel b$.

For $a_1 = -99.99$, we get

$$a : -99.99 \times_2 x +_2 y -_2 0.01 = 0.$$

For $x = -0.01$, on this line, we get $y = -0.98$, that is, in this case, $a \nparallel b$.

So we must have $a_1 = 0$, that is,

$$a : y -_2 0.01 = 0,$$

and in this case, $a \parallel b$.

In the other limiting case $a_2 = 99.99$ the equation of line a becomes

$$a : a_1 \times_2 x +_2 99.99 \times_2 y -_2 0.99 = 0.$$

For $a_1 = 0$, we get

$$a : 99.99 \times_2 y -_2 0.99 = 0.$$

This line coincides with line

$$a : y -_2 0.01 = 0,$$

and in this case, $a \parallel b$.

Assuming that $a_1 \neq 0$, we consider four limiting cases: $a_1 = 0.01$, $a_1 = -0.01$, $a_1 = 99.99$, $a_1 = -99.99$.

For $a_1 = 0.01$, we get

$$a : 0.01 \times_2 x +_2 99.99 \times_2 y -_2 0.99 = 0.$$

For $x = 99.99$, on this line, we get $y = 0$, that is, in this case, $a \nparallel b$.

For $a_1 = -0.01$, we get

$$a : -0.01 \times_2 x +_2 99.99 \times_2 y -_2 0.99 = 0.$$

For $x = -99.99$, on this line, we get $y = 0$, that is, in this case, $a \nparallel b$.

For $a_1 = 99.99$, we get

$$a : 99.99 \times_2 x +_2 99.99 \times_2 y -_2 0.99 = 0.$$

For $x = 0.01$, on this line, we get $y = 0$, that is, in this case, $a \nparallel b$.

For $a_1 = -99.99$, we get

$$a : -99.99 \times_2 x +_2 99.99 \times_2 y -_2 0.99 = 0.$$

For $x = -0.01$, on this line, we get $y = 0$, that is, in this case, $a \nparallel b$.

So we must have $a_1 = 0$, that is,

$$a : 99.99 \times_2 y -_2 0.99 = 0.$$

This line coincides with line

$$a : y -_2 0.01 = 0,$$

and in this case, $a \parallel b$.

Note this is the same line as that in the first limiting case $a_1 = 1$. So the theorem is proved for $n = 2$. The general statement of the theorem for any *n* may be proved in the same way.

Let us make a very important note. Let us consider the equation

$$0.1 \times_2 (y -_2 0.1) = 0.$$

Its solution is the set

$$y -_2 0.1 = [0.00, \pm 0.01, \pm 0.02, \ldots, \pm 0.09],$$

that is,

$$y = [0.10, 0.10 \pm 0.01, 0.10 \pm 0.02, \ldots, 0.10 \pm 0.09].$$

The solution of this equation is the following set of straight lines:

$$\begin{aligned} y &= 0.19, \\ y &= 0.18, \\ &\ldots \\ y &= 0.10, \\ y &= 0.09, \\ &\ldots \\ y &= 0.01. \end{aligned}$$

So if

$$0.1 \times_2 (y -_2 0.1) = 0$$

is the equation of a straight line, then we would have had at least two different lines containing point *A* and parallel to line *b*. However,

$$0.1 \times_2 (y -_2 0.1) = 0$$

is not an equation of a straight line.

Let us now prove the following:

Theorem 6.16.

In the plane E_2W_n, there are a point A and a straight line b such that $A \notin b$ and there is more than one straight line that contains the point A and is parallel to line b: $a \parallel b$.

Let us first consider the case $n = 2$, that is, we are in $\alpha = E_2W_2$.

1) Let the equation of straight line *a* be

$$y = -0.01,$$

and let two regions R_1^a, R_2^a of plane *a* be

$$R_1^a = \left[(x, y), (x, y) \in E_2W_2, y > -0.01\right]$$

and

$$R_2^a = \left[(x, y), (x, y) \in E_2W_2, y < -0.01\right],$$

where *x* is any element $\in W_2$. So

$$E_2W_2 = R_1^a \cup a \cup R_2^a.$$

Let us take two points

$$A(-99.99, 0.00), B(0, 99.99) \in R_1^a.$$

We look for a straight line *b* as the set of points (x, y) satisfying the equation

$$a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0$$

and containing points *A*, *B*. We have

$$\begin{cases} -a_1 \times_2 (99.99) +_2 a_2 \times_2 (0.00) +_2 a_3 = 0, \\ a_1 \times_2 (0) +_2 a_2 \times_2 (99.99) +_2 a_3 = 0, \end{cases}$$

that is,

$$\begin{cases} -a_1 \times_2 (99.99) +_2 a_3 = 0, \\ a_2 \times_2 (99.99) +_2 a_3 = 0. \end{cases}$$

For

$$a_2 = 1,$$

we get

$$a_1 = -1$$

and

$$a_3 = -99.99,$$

and the equation of line *b* is

$$b : y = x +_2 99.99 = 0.$$

We have

$$b \subset R_1^a,$$

so

$$b \cap R_2^a = \Lambda,$$

and

$$b \cap a = \Lambda$$

We have

$$R_1^b = \left[(x, y), (x, y) \in E_2W_2, y = x +_2 99.99 > 0\right],$$
$$R_2^b = \left[(x, y), (x, y) \in E_2W_2, y = x +_2 99.99 < 0\right],$$

with any $-99.99 \leq x \leq 0$, and

$$a \cap R_1^b = \Lambda,$$
$$a \cap R_2^b \neq \Lambda.$$

So line *a* is parallel to line *b*, and line *b* is parallel to line *a*, that is, $a \parallel b$ and $b \parallel a$.

Also, line *c* with equation

$$c : y = -x +_2 99.99 = 0$$

is parallel to line *a* and contains points *B* and $A'(99.99, 0.00)$. So line *a* is parallel to line *c*, line *c* is parallel to line *a*, that is, $a \parallel c$ and $c \parallel a$. Moreover, $b \cap c = (0, 99.99) = B$, that is, $b \nparallel c$. This means that in a plane $\alpha = E_2W_2$, through point *B* lying outside of a straight line *a*, there can be drawn at least two distinct straight lines *b* and *c* that are parallel to line *a*.

1') Let $A(0, 1) \in E_2W_2$, and let *b* be a straight line with equation

$$b : y = 0.$$

Then

$$R_1^b : \big[(x, y) \in E_2W_2 : y > 0\big],$$
$$R_2^b : \big[(x, y) \in E_2W_2 : y < 0\big],$$
$$R_3^b = \Lambda,$$

and $A \notin b$.

Let us consider two straight lines

$$a^1 : 0.01 \times_2 x -_2 y +_2 1 = 0$$

and

$$a^2 : y -_2 1 = 0.$$

We can see that $A \in a^1$, $A \in a^2$, and $a^1 \subset R_1^b$, $a^2 \subset R_1^b$. Also, we have

$$R_1^{a^1} : \big[(x, y) \in E_2W_2 : 0.01 \times_2 x -_2 y +_2 1 > 0\big],$$
$$R_2^{a^1} : \big[(x, y) \in E_2W_2 : 0.01 \times_2 x -_2 y +_2 1 < 0\big],$$
$$R_3^{a^1} = \Lambda,$$
$$R_1^{a^2} : \big[(x, y) \in E_2W_2 : y > 1\big]$$
$$R_2^{a^2} : \big[(x, y) \in E_2W_2 : y < 1\big],$$
$$R_3^{a^2} = \Lambda,$$

and

$$b \subset R_1^{a^1},$$
$$b \subset R_2^{a^2},$$
$$b \cap a^1 = \Lambda,$$
$$b \cap a^2 = \Lambda.$$

This means that $a^1 \parallel b$, $a^2 \parallel b$, $b \parallel a^1$, and $b \parallel a^2$, and we have proved the theorem for $n = 2$.

The general statement of this theorem for any *n* can be proved in the same way.

So we have proved the following main theorem on parallel lines in Mathematics with Observers geometry.

Theorem 6.17.
In the plane E_2W_n, there are a point A and a straight line b such that $A \notin b$ and we have possible three different situations:

1. *There is only one straight line a that contains point A and is parallel to line b (Euclidean geometry case).*

2. *There is more than one straight line a that contains point A and is parallel to line b (Gauss–Bolyai–Lobachevsky geometry case).*

3. *Any straight line a containing point A is not parallel to line b (Riemann geometry case).*

This means that the same plane has couples (a point and a straight line not containing this point) where Euclidean geometry works, couples where Gauss–Bolyai–Lobachevsky geometry works, and couples where Riemann geometry works.

7 Observability and properties of congruence analysis

Let us now consider the definition of segments congruence in Mathematics with Observers geometry. For E_2W_n, let us first consider four points

$$A(x_1,y_1), B(x_2,y_2), A'(x_3,y_3), B'(x_4,y_4) \in E_2W_n.$$

Points *A*, *B* or A', B' do not necessarily lie on existing straight lines. For these points, we have the corresponding vectors

$$\mathbf{a}=(x_1,y_1), \mathbf{b}=(x_2,y_2), \mathbf{a}'=(x_3,y_3), \mathbf{b}'=(x_4,y_4) \in E_2W_n.$$

Let us consider two vectors in E_2W_n:

$$\begin{gathered}\mathbf{AB}= \mathbf{b} -_n \mathbf{a} = (x_2 -_n x_1, y_2 -_n y_1),\\ \mathbf{A'B'}= \mathbf{b}' -_n \mathbf{a}' = (x_4 -_n x_3, y_4 -_n y_3)\end{gathered}$$

and the scalar products

$$\begin{gathered}(\mathbf{AB},\mathbf{AB})= (x_2 -_n x_1) \times_n (x_2 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_2 -_n y_1),\\ \left(\mathbf{A'B'},\mathbf{A'B'}\right)= (x_4 -_n x_3) \times_n (x_4 -_n x_3) +_n (y_4 -_n y_3) \times_n (y_4 -_n y_3)\end{gathered}$$

if

$$\begin{aligned}&x_2 -_n x_1 \in W_n,\\ &y_2 -_n y_1 \in W_n,\\ &x_4 -_n x_3 \in W_n,\\ &y_4 -_n y_3 \in W_n,\\ &(x_2 -_n x_1) \times_n (x_2 -_n x_1) \in W_n,\\ &(x_4 -_n x_3) \times_n (x_4 -_n x_3) \in W_n,\\ &(y_2 -_n y_1) \times_n (y_2 -_n y_1) \in W_n,\\ &(y_4 -_n y_3) \times_n (y_4 -_n y_3) \in W_n,\\ &(x_2 -_n x_1) \times_n (x_2 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_2 -_n y_1) \in W_n,\\ &(x_4 -_n x_3) \times_n (x_4 -_n x_3) +_n (y_4 -_n y_3) \times_n (y_4 -_n y_3) \in W_n.\end{aligned}$$

In Mathematics with Observers geometry, we say that segment AB is congruent to segment $A'B'$, denoted $AB \equiv A'B'$, if

$$(\mathbf{AB},\mathbf{AB}) = \left(\mathbf{A'B'},\mathbf{A'B'}\right) > 0.$$

Now let us go to the situation where points *A*, *B* and points A', B' are points of existing straight lines.

For E_2W_n, let straight line *a* have the equation

$$a_1 \times_n x +_n a_2 \times_n y +_n a_3 = 0$$

and points

$$A(x_1, y_1), B(x_2, y_2) \in a.$$

Let the equation of straight line a' be

$$b_1 \times_n x +_n b_2 \times_n y +_n b_3 = 0,$$

and let

$$A'(x_3, y_3), B'(x_4, y_4) \in a'.$$

Let us consider two vectors in E_2W_n,

$$\mathbf{AB} = (x_2 -_n x_1, y_2 -_n y_1)$$

and

$$\mathbf{A'B'} = (x_4 -_n x_3, y_4 -_n y_3)$$

and scalar products

$$(\mathbf{AB}, \mathbf{AB}) = (x_2 -_n x_1) \times_n (x_2 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_2 -_n y_1)$$

and

$$(\mathbf{A'B'}, \mathbf{A'B'}) = (x_4 -_n x_3) \times_n (x_4 -_n x_3) +_n (y_4 -_n y_3) \times_n (y_4 -_n y_3)$$

if

$$\begin{aligned}
&x_2 -_n x_1 \in W_n, \\
&y_2 -_n y_1 \in W_n, \\
&x_4 -_n x_3 \in W_n \\
&y_4 -_n y_3 \in W_n, \\
&(x_2 -_n x_1) \times_n (x_2 -_n x_1) \in W_n, \\
&(x_4 -_n x_3) \times_n (x_4 -_n x_3) \in W_n, \\
&(y_2 -_n y_1) \times_n (y_2 -_n y_1) \in W_n, \\
&(y_4 -_n y_3) \times_n (y_4 -_n y_3) \in W_n, \\
&(x_2 -_n x_1) \times_n (x_2 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_2 -_n y_1) \in W_n, \\
&(x_4 -_n x_3) \times_n (x_4 -_n x_3) +_n (y_4 -_n y_3) \times_n (y_4 -_n y_3) \in W_n.
\end{aligned}$$

In Mathematics with Observers geometry, we say that segment AB of straight line a is congruent to segment $A'B'$ of straight line a', denoted $AB \equiv A'B'$, if

$$(\mathbf{AB}, \mathbf{AB}) = (\mathbf{A'B'}, \mathbf{A'B'}) > 0.$$

For E_3W_n, first, let us consider four points

$$A(x_1, y_1, z_1), B(x_2, y_2, z_2), A'(x_3, y_3, z_3), B'(x_4, y_4, z_4) \in E_3W_n.$$

Points *A*, *B* or points A', B' are not necessarily points of existing straight lines. For these points, we have the corresponding vectors

$$\mathbf{a} = (x_1, y_1, z_1), \mathbf{b} = (x_2, y_2, z_2), \mathbf{a}' = (x_3, y_3, z_3), \mathbf{b}' = (x_4, y_4, z_4) \in E_3 W_n.$$

Let us consider two vectors $\in E_3 W_n$,

$$\mathbf{AB} = \mathbf{b} -_n \mathbf{a} = (x_2 -_n x_1, y_2 -_n y_1, z_2 -_n z_1)$$

and

$$\mathbf{A'B'} = \mathbf{b}' -_n \mathbf{a}' = (x_4 -_n x_3, y_4 -_n y_3, z_4 -_n z_3)$$

and scalar products

$$(\mathbf{AB}, \mathbf{AB}) = (x_2 -_n x_1) \times_n (x_2 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_2 -_n y_1) +_n (z_2 -_n z_1) \times_n (z_2$$

and

$$(\mathbf{A'B'}, \mathbf{A'B'}) = (x_4 -_n x_3) \times_n (x_4 -_n x_3) +_n (y_4 -_n y_3) \times_n (y_4 -_n y_3) +_n (z_4 -_n z_3) \times_n ($$

if

$x_2 -_n x_1 \in W_n,$
$y_2 -_n y_1 \in W_n,$
$z_2 -_n z_1 \in W_n,$
$x_4 -_n x_3 \in W_n,$
$y_4 -_n y_3 \in W_n,$
$z_4 -_n z_3 \in W_n,$
$(x_2 -_n x_1) \times_n (x_2 -_n x_1) \in W_n,$
$(x_4 -_n x_3) \times_n (x_4 -_n x_3) \in W_n,$
$(y_2 -_n y_1) \times_n (y_2 -_n y_1) \in W_n,$
$(y_4 -_n y_3) \times_n (y_4 -_n y_3) \in W_n,$
$(z_2 -_n z_1) \times_n (z_2 -_n z_1) \in W_n,$
$(z_4 -_n z_3) \times_n (z_4 -_n z_3) \in W_n,$
$(x_2 -_n x_1) \times_n (x_2 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_2 -_n y_1) +_n (z_2 -_n z_1) \times_n (z_2 -_n z_1) \in W_n,$
$(x_4 -_n x_3) \times_n (x_4 -_n x_3) +_n (y_4 -_n y_3) \times_n (y_4 -_n y_3) +_n (z_4 -_n z_3) \times_n (z_4 -_n z_3) \in W_n.$

In Mathematics with Observers geometry, we say that segment AB is congruent to segment $A'B'$, denoted $AB \equiv A'B'$, if

$$(\mathbf{AB}, \mathbf{AB}) = (\mathbf{A'B'}, \mathbf{A'B'}) > 0.$$

Now let us go to the situation where points *A*, *B* and points A', B' are points of existing straight lines. For $E_3 W_n$, let straight line *a* have the system of equations

$$\begin{cases} a_1 \times_n x +_n a_2 \times_n y +_n a_3 \times_n z +_n a_4 = 0, \\ b_1 \times_n x +_n b_2 \times_n y +_n b_3 \times_n z +_n b_4 = 0, \end{cases}$$

and let

$$A(x_1, y_1, z_1), B(x_2, y_2, z_2) \in a.$$

Let straight line a' have the equation

$$\begin{cases} c_1 \times_n x +_n c_2 \times_n y +_n c_3 \times_n z +_n c_4 = 0, \\ d_1 \times_n x +_n d_2 \times_n y +_n d_3 \times_n z +_n d_4 = 0, \end{cases}$$

and let

$$A'(x_3, y_3, z_3), B'(x_4, y_4, z_4) \in a'.$$

Let us consider two vectors in $E_3 W_n$,

$$\mathbf{AB} = (x_2 -_n x_1, y_2 -_n y_1, z_2 -_n z_1)$$

and

$$\mathbf{A'B'} = (x_4 -_n x_3, y_4 -_n y_3, z_4 -_n z_3)$$

and scalar products

$$(\mathbf{AB}, \mathbf{AB}) = (x_2 -_n x_1) \times_n (x_2 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_2 -_n y_1) +_n (z_2 -_n z_1) \times_n (z_2$$

and

$$\left(\mathbf{A'B'}, \mathbf{A'B'}\right) = (x_4 -_n x_3) \times_n (x_4 -_n x_3) +_n (y_4 -_n y_3) \times_n (y_4 -_n y_3) +_n (z_4 -_n z_3) \times_n ($$

if

$$\begin{aligned}
&x_2 -_n x_1 \in W_n, \\
&y_2 -_n y_1 \in W_n, \\
&z_2 -_n z_1 \in W_n, \\
&x_4 -_n x_3 \in W_n, \\
&y_4 -_n y_3 \in W_n, \\
&z_4 -_n z_3 \in W_n, \\
&(x_2 -_n x_1) \times_n (x_2 -_n x_1) \in W_n, \\
&(x_4 -_n x_3) \times_n (x_4 -_n x_3) \in W_n, \\
&(y_2 -_n y_1) \times_n (y_2 -_n y_1) \in W_n, \\
&(y_4 -_n y_3) \times_n (y_4 -_n y_3) \in W_n, \\
&(z_2 -_n z_1) \times_n (z_2 -_n z_1) \in W_n, \\
&(z_4 -_n z_3) \times_n (z_4 -_n z_3) \in W_n, \\
&(x_2 -_n x_1) \times_n (x_2 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_2 -_n y_1) +_n (z_2 -_n z_1) \times_n (z_2 -_n z_1) \in W_n, \\
&(x_4 -_n x_3) \times_n (x_4 -_n x_3) +_n (y_4 -_n y_3) \times_n (y_4 -_n y_3) +_n (z_4 -_n z_3) \times_n (z_4 -_n z_3) \in W_n.
\end{aligned}$$

In Mathematics with Observers geometry, we say that segment AB of straight line *a* is congruent to segment $A'B'$ of straight line a', denoted $AB \equiv A'B'$ if

$$(\mathbf{AB}, \mathbf{AB}) = \left(\mathbf{A'B'}, \mathbf{A'B'}\right) > 0.$$

Let now consider the definition of congruence of angles in Mathematics with Observers geometry. Let us first define the angle formed by three points *A*, *B*, *C* in E_2W_n or E_3W_n.

For E_2W_n, let us first consider three points

$$A(x_1, y_1), B(x_2, y_2), C(x_3, y_3) \in E_2W_n.$$

Points *A*, *B*, points *A*, *C*, or points *B*, *C* are not necessarily points of existing straight lines. For these points, we have the corresponding vectors

$$\mathbf{a} = (x_1, y_1), \mathbf{b} = (x_2, y_2), \mathbf{c} = (x_3, y_3) \in E_2W_n.$$

Let us consider three vectors in E_2W_n,

$$\begin{aligned}
&\mathbf{AB} = \mathbf{b} -_n \mathbf{a} = (x_2 -_n x_1, y_2 -_n y_1),\\
&\mathbf{BA} = -\mathbf{AB},\\
&\mathbf{AC} = \mathbf{c} -_n \mathbf{a} = (x_3 -_n x_1, y_3 -_n y_1),\\
&\mathbf{CA} = -\mathbf{AC},\\
&\mathbf{BC} = \mathbf{c} -_n \mathbf{b} = (x_3 -_n x_2, y_3 -_n y_2),\\
&\mathbf{CB} = -\mathbf{BC}.
\end{aligned}$$

The system formed by two vectors $\mathbf{AB}$, $\mathbf{AC}$ we call an angle $\angle BAC$. The system formed by two vectors $\mathbf{BA}$, $\mathbf{BC}$ we call an angle $\angle ABC$. The system formed by two vectors $\mathbf{CA}$, $\mathbf{CB}$ we call an angle $\angle ACB$.

Let us consider the scalar products

$$\begin{aligned}
&(\mathbf{AB}, \mathbf{AB}) = (x_2 -_n x_1) \times_n (x_2 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_2 -_n y_1),\\
&(\mathbf{AC}, \mathbf{AC}) = (x_3 -_n x_1) \times_n (x_3 -_n x_1) +_n (y_3 -_n y_1) \times_n (y_3 -_n y_1),\\
&(\mathbf{AB}, \mathbf{AC}) = (x_2 -_n x_1) \times_n (x_3 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_3 -_n y_1).
\end{aligned}$$

We now assume that

$$
\begin{aligned}
&x_2 -_n x_1 \in W_n,\\
&y_2 -_n y_1 \in W_n,\\
&x_3 -_n x_1 \in W_n,\\
&y_3 -_n y_1 \in W_n,\\
&x_3 -_n x_2 \in W_n,\\
&y_3 -_n y_2 \in W_n,\\
&(x_2 -_n x_1) \times_n (x_2 -_n x_1) \in W_n,\\
&(y_2 -_n y_1) \times_n (y_2 -_n y_1) \in W_n,\\
&(x_3 -_n x_1) \times_n (x_3 -_n x_1) \in W_n,\\
&(y_3 -_n y_1) \times_n (y_3 -_n y_1) \in W_n,\\
&(x_2 -_n x_1) \times_n (x_2 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_2 -_n y_1) \in W_n,\\
&(x_3 -_n x_1) \times_n (x_3 -_n x_1) +_n (y_3 -_n y_1) \times_n (y_3 -_n y_1) \in W_n,\\
&(x_2 -_n x_1) \times_n (x_3 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_3 -_n y_1) \in W_n.
\end{aligned}
$$

Let us consider three other points

$$A'\left(x_1', y_1'\right), B'\left(x_2', y_2'\right), C'\left(x_3', y_3'\right) \in E_2W_n.$$

For these points, we have the corresponding vectors

$$\mathbf{a}' = \left(x_1', y_1'\right), \mathbf{b}' = \left(x_2', y_2'\right), \mathbf{c}' = \left(x_3', y_3'\right) \in E_2W_n$$

Let us consider three vectors in E_2W_n,

$$
\begin{aligned}
&\mathbf{A}'\mathbf{B}' = \mathbf{b}' -_n \mathbf{a}' = \left(x_2' -_n x_1', y_2' -_n y_1'\right),\\
&\mathbf{B}'\mathbf{A}' = -\mathbf{A}'\mathbf{B}',\\
&\mathbf{A}'\mathbf{C}' = \mathbf{c}' -_n \mathbf{a}' = \left(x_3' -_n x_1', y_3' -_n y_1'\right),\\
&\mathbf{C}'\mathbf{A}' = -\mathbf{A}'\mathbf{C}',\\
&\mathbf{B}'\mathbf{C}' = \mathbf{c}' -_n \mathbf{b}' = \left(x_3' -_n x_2', y_3' -_n y_2'\right),\\
&\mathbf{C}'\mathbf{B}' = -\mathbf{B}'\mathbf{C}'.
\end{aligned}
$$

The system formed by two vectors $\mathbf{A}'\mathbf{B}'$, $\mathbf{A}'\mathbf{C}'$ is called the angle $\angle B'A'C'$. The system formed by two vectors $\mathbf{B}'\mathbf{A}'$, $\mathbf{B}'\mathbf{C}'$ is called the angle $\angle A'B'C'$. The system formed by two vectors $\mathbf{C}'\mathbf{A}'$, $\mathbf{C}'\mathbf{B}'$ is called the angle $\angle A'C'B'$.

Let us consider the scalar products

$$
\begin{aligned}
&\left(\mathbf{A}'\mathbf{B}', \mathbf{A}'\mathbf{B}'\right) = \left(x_2' -_n x_1'\right) \times_n \left(x_2' -_n x_1'\right) +_n \left(y_2' -_n y_1'\right) \times_n \left(y_2' -_n y_1'\right),\\
&\left(\mathbf{A}'\mathbf{C}', \mathbf{A}'\mathbf{C}'\right) = \left(x_3' -_n x_1'\right) \times_n \left(x_3' -_n x_1'\right) +_n \left(y_3' -_n y_1'\right) \times_n \left(y_3' -_n y_1'\right),\\
&\left(\mathbf{A}'\mathbf{B}', \mathbf{A}'\mathbf{C}'\right) = \left(x_2' -_n x_1'\right) \times_n \left(x_3' -_n x_1'\right) +_n \left(y_2' -_n y_1'\right) \times_n \left(y_3' -_n y_1'\right).
\end{aligned}
$$

We assume that

$$
\begin{aligned}
&x_2' -_n x_1' \in W_n,\\
&y_2' -_n y_1' \in W_n,\\
&x_3' -_n x_1' \in W_n,\\
&y_3' -_n y_1' \in W_n,\\
&x_3' -_n x_2' \in W_n,\\
&y_3' -_n y_2' \in W_n,\\
&\left(x_2' -_n x_1'\right) \times_n \left(x_2' -_n x_1'\right) \in W_n,\\
&\left(x_3' -_n x_1'\right) \times_n \left(x_3' -_n x_1'\right) \in W_n,\\
&\left(y_2' -_n y_1'\right) \times_n \left(y_2' -_n y_1'\right) \in W_n,\\
&\left(y_3' -_n y_1'\right) \times_n \left(y_3' -_n y_1'\right) \in W_n,\\
&\left(x_2' -_n x_1'\right) \times_n \left(x_2' -_n x_1'\right) +_n \left(y_2' -_n y_1'\right) \times_n \left(y_2' -_n y_1'\right) \in W_n,\\
&\left(x_3' -_n x_1'\right) \times_n \left(x_3' -_n x_1'\right) +_n \left(y_3' -_n y_1'\right) \times_n \left(y_3' -_n y_1'\right) \in W_n,\\
&\left(x_2' -_n x_1'\right) \times_n \left(x_3' -_n x_1'\right) +_n \left(y_2' -_n y_1'\right) \times_n \left(y_3' -_n y_1'\right) \in W_n.
\end{aligned}
$$

In this case, we say that $\angle BAC$ is congruent to $\angle B'A'C'$ and write

$$\angle BAC \equiv \angle B'A'C'$$

if the following conditions are satisfied:

$$
\begin{aligned}
(\mathbf{AB}, \mathbf{AB})&> 0,\\
\left(\mathbf{A'B'}, \mathbf{A'B'}\right)&> 0,\\
(\mathbf{AC}, \mathbf{AC})&> 0,\\
\left(\mathbf{A'C'}, \mathbf{A'C'}\right)&> 0,\\
(\mathbf{AB}, \mathbf{AB})&= \left(\mathbf{A'B'}, \mathbf{A'B'}\right),\\
(\mathbf{AC}, \mathbf{AC})&= \left(\mathbf{A'C'}, \mathbf{A'C'}\right),\\
(\mathbf{AB}, \mathbf{AC})&= \left(\mathbf{A'B'}, \mathbf{A'C'}\right).
\end{aligned}
$$

Let us now go to E_3W_n and consider three points

$$A(x_1, y_1, z_1), B(x_2, y_2, z_2), C(x_3, y_3, z_3) \in E_3W_n.$$

Points *A*, *B*, points *A*, *C*, or points *B*, *C* are not necessarily points of the existing straight lines. For all these points, we have the corresponding vectors

$$\mathbf{a} = (x_1, y_1, z_1), \mathbf{b} = (x_2, y_2, z_2), \mathbf{c} = (x_3, y_3, z_3) \in E_2W_n.$$

Let us consider three vectors $\in E_3W_n$,

$$\begin{aligned}
&\mathbf{AB} = \mathbf{b} -_n \mathbf{a} = (x_2 -_n x_1, y_2 -_n y_1, z_2 -_n z_1),\\
&\mathbf{BA} = -\mathbf{AB},\\
&\mathbf{AC} = \mathbf{c} -_n \mathbf{a} = (x_3 -_n x_1, y_3 -_n y_1, z_3 -_n z_1),\\
&\mathbf{CA} = -\mathbf{AC},\\
&\mathbf{BC} = \mathbf{c} -_n \mathbf{b} = (x_3 -_n x_2, y_3 -_n y_2, z_3 -_n z_2),\\
&\mathbf{CB} = -\mathbf{BC}.
\end{aligned}$$

The system formed by two vectors $\mathbf{AB}$, $\mathbf{AC}$ is called the angle $\angle BAC$. The system formed by two vectors $\mathbf{BA}$, $\mathbf{BC}$ is called the angle $\angle ABC$. The system formed by two vectors $\mathbf{CA}$, $\mathbf{CB}$ is called the angle $\angle ACB$.

Let us consider the scalar products

$$\begin{aligned}
&(\mathbf{AB}, \mathbf{AB}) = (x_2 -_n x_1) \times_n (x_2 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_2 -_n y_1) +_n (z_2 -_n z_1) \times_n (z_2 -\\
&(\mathbf{AC}, \mathbf{AC}) = (x_3 -_n x_1) \times_n (x_3 -_n x_1) +_n (y_3 -_n y_1) \times_n (y_3 -_n y_1) +_n (z_3 -_n z_1) \times_n (z_3 -\\
&(\mathbf{AB}, \mathbf{AC}) = (x_2 -_n x_1) \times_n (x_3 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_3 -_n y_1) +_n (z_2 -_n z_1) \times_n (z_3 -
\end{aligned}$$

We now assume now that

$$\begin{aligned}
&x_2 -_n x_1 \in W_n,\\
&y_2 -_n y_1 \in W_n,\\
&z_2 -_n z_1 \in W_n,\\
&x_3 -_n x_1 \in W_n,\\
&y_3 -_n y_1 \in W_n,\\
&z_3 -_n z_1 \in W_n,\\
&x_3 -_n x_2 \in W_n,\\
&y_3 -_n y_2 \in W_n,\\
&z_3 -_n z_2 \in W_n,\\
&(x_2 -_n x_1) \times_n (x_2 -_n x_1) \in W_n,\\
&(y_2 -_n y_1) \times_n (y_2 -_n y_1) \in W_n,\\
&(z_2 -_n z_1) \times_n (z_2 -_n z_1) \in W_n,\\
&(x_3 -_n x_1) \times_n (x_3 -_n x_1) \in W_n,\\
&(y_3 -_n y_1) \times_n (y_3 -_n y_1) \in W_n,\\
&(z_3 -_n z_1) \times_n (z_3 -_n z_1) \in W_n,\\
&(x_2 -_n x_1) \times_n (x_2 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_2 -_n y_1) +_n (z_2 -_n z_1) \times_n (z_2 -_n z_1) \in W_n,\\
&(x_3 -_n x_1) \times_n (x_3 -_n x_1) +_n (y_3 -_n y_1) \times_n (y_3 -_n y_1) +_n (z_3 -_n z_1) \times_n (z_3 -_n z_1) \in W_n,\\
&(x_2 -_n x_1) \times_n (x_3 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_3 -_n y_1) +_n (z_2 -_n z_1) \times_n (z_3 -_n z_1) \in W_n.
\end{aligned}$$

Let us consider three other points

$$A'(x_1', y_1', z_1'), B'(x_2', y_2', z_2'), C'(x_3', y_3', z_3') \in E_3 W_n.$$

For these points, we have the corresponding vectors

$$\mathbf{a}' = \left(x_1', y_1', z_1'\right), \mathbf{b}' = \left(x_2', y_2', z_2'\right), \mathbf{c}' = \left(x_3', y_3', z_3'\right) \in E_3 W_n.$$

Let us consider three vectors in $E_3 W_n$:

$$\begin{aligned}
\mathbf{A}'\mathbf{B}' &= \mathbf{b}' -_n \mathbf{a}' = \left(x_2' -_n x_1', y_2' -_n y_1', z_2' -_n z_1'\right),\\
\mathbf{B}'\mathbf{A}' &= -\mathbf{A}'\mathbf{B}',\\
\mathbf{A}'\mathbf{C}' &= \mathbf{c}' -_n \mathbf{a}' = \left(x_3' -_n x_1', y_3' -_n y_1', z_3' -_n z_1'\right),\\
\mathbf{C}'\mathbf{A}' &= -\mathbf{A}'\mathbf{C}',\\
\mathbf{B}'\mathbf{C}' &= \mathbf{c}' -_n \mathbf{b}' = \left(x_3' -_n x_2', y_3' -_n y_2', z_3' -_n z_2'\right),\\
\mathbf{C}'\mathbf{B}' &= -\mathbf{B}'\mathbf{C}'.
\end{aligned}$$

The system formed by two vectors $\mathbf{A}'\mathbf{B}'$, $\mathbf{A}'\mathbf{C}'$ is called the angle $\angle B'A'C'$. The system formed by two vectors $\mathbf{B}'\mathbf{A}'$, $\mathbf{B}'\mathbf{C}'$ is called the angle $\angle A'B'C'$. The system formed by two vectors $\mathbf{C}'\mathbf{A}'$, $\mathbf{C}'\mathbf{B}'$ is called the angle $\angle A'C'B'$.

Let us consider the scalar products

$$\begin{aligned}
\left(\mathbf{A}'\mathbf{B}', \mathbf{A}'\mathbf{B}'\right) &= \left(x_2' -_n x_1'\right) \times_n \left(x_2' -_n x_1'\right) +_n \left(y_2' -_n y_1'\right) \times_n \left(y_2' -_n y_1'\right) +_n \left(z_2' -_n z_1'\right.\\
\left(\mathbf{A}'\mathbf{C}', \mathbf{A}'\mathbf{C}'\right) &= \left(x_3' -_n x_1'\right) \times_n \left(x_3' -_n x_1'\right) +_n \left(y_3' -_n y_1'\right) \times_n \left(y_3' -_n y_1'\right) +_n \left(z_3' -_n z_1'\right.\\
\left(\mathbf{A}'\mathbf{B}', \mathbf{A}'\mathbf{C}'\right) &= \left(x_2' -_n x_1'\right) \times_n \left(x_3' -_n x_1'\right) +_n \left(y_2' -_n y_1'\right) \times_n \left(y_3' -_n y_1'\right) +_n \left(z_2' -_n z_1'\right.
\end{aligned}$$

We assume that

$$x_2' -_n x_1' \in W_n,$$
$$y_2' -_n y_1' \in W_n,$$
$$z_2' -_n z_1' \in W_n,$$
$$x_3' -_n x_1' \in W_n,$$
$$y_3' -_n y_1' \in W_n,$$
$$z_3' -_n z_1' \in W_n,$$
$$x_3' -_n x_2' \in W_n,$$
$$y_3' -_n y_2' \in W_n,$$
$$z_3' -_n z_2' \in W_n,$$
$$\left(x_2' -_n x_1'\right) \times_n \left(x_2' -_n x_1'\right) \in W_n,$$
$$\left(x_3' -_n x_1'\right) \times_n \left(x_3' -_n x_1'\right) \in W_n,$$
$$\left(y_2' -_n y_1'\right) \times_n \left(y_2' -_n y_1'\right) \in W_n,$$
$$\left(y_3' -_n y_1'\right) \times_n \left(y_3' -_n y_1'\right) \in W_n,$$
$$\left(z_2' -_n z_1'\right) \times_n \left(z_2' -_n z_1'\right) \in W_n,$$
$$\left(z_3' -_n z_1'\right) \times_n \left(z_3' -_n z_1'\right) \in W_n,$$
$$\left(x_2' -_n x_1'\right) \times_n \left(x_2' -_n x_1'\right) +_n \left(y_2' -_n y_1'\right) \times_n \left(y_2' -_n y_1'\right) +_n \left(z_2' -_n z_1'\right) \times_n \left(z_2' -_n z_1'\right) \in W_n$$
$$\left(x_3' -_n x_1'\right) \times_n \left(x_3' -_n x_1'\right) +_n \left(y_3' -_n y_1'\right) \times_n \left(y_3' -_n y_1'\right) +_n \left(z_3' -_n z_1'\right) \times_n \left(z_3' -_n z_1'\right) \in W_n$$
$$\left(x_2' -_n x_1'\right) \times_n \left(x_3' -_n x_1'\right) +_n \left(y_2' -_n y_1'\right) \times_n \left(y_3' -_n y_1'\right) +_n \left(z_2' -_n z_1'\right) \times_n \left(z_3' -_n z_1'\right) \in W_n$$

In this case, we say that $\angle BAC$ is congruent to $\angle B'A'C'$, denoted $\angle BAC \equiv \angle B'A'C'$, if the following conditions are satisfied:

$$(\mathbf{AB}, \mathbf{AB}) > 0,$$
$$\left(\mathbf{A'B'}, \mathbf{A'B'}\right) > 0,$$
$$(\mathbf{AC}, \mathbf{AC}) > 0,$$
$$\left(\mathbf{A'C'}, \mathbf{A'C'}\right) > 0,$$
$$(\mathbf{AB}, \mathbf{AB}) = \left(\mathbf{A'B'}, \mathbf{A'B'}\right),$$
$$(\mathbf{AC}, \mathbf{AC}) = \left(\mathbf{A'C'}, \mathbf{A'C'}\right),$$
$$(\mathbf{AB}, \mathbf{AC}) = \left(\mathbf{A'B'}, \mathbf{A'C'}\right).$$

Now we have to define the angle between two straight lines. By classical geometry definition we have the following definition:

If *A*, A', *O*, *B* are four points of a straight line *a* in E_2W_n or E_3W_n, where *O* lies between *A* and *B* but not between *A* and A', then this means the following: The points *A*, A' are situated on the line *a* upon one and the same side of the point *O*, and the points *A*, *B* are situated on the straight line *a* upon different sides of the point *O*. All the points of *a* that lie upon the same side of *O*, when taken together, are called the half-ray emanating from *O*.

Let us now consider two distinct straight lines *a*, *b* in $E_2 W_n$ or $E_3 W_n$ having at least one point *O* in their intersection: $O \in a \cap b$. Note that generally there may be more than one such point. By classical geometry definition we have the following: Let *h*, *k* be any two distinct half-rays $h \subset a$ and $k \subset b$ emanating from the point *O*. The system formed by these two half-rays *h*, *k* is called an angle and denoted by $\angle(h, k)$ or $\angle(k, h)$.

The half-rays *h* and *k* are called the sides of the angle, and the point *O* is called the vertex of the angle. Now we can go to the definition of congruence of angles in Mathematics with Observers geometry. Let us start with $E_2 W_n$.

Let $a, b \in E_2 W_n$ be two straight lines

$$a : a_1 \times_n x +_n a_2 \times_n y +_n a_3 = 0$$

and

$$b : b_1 \times_n x +_n b_2 \times_n y +_n b_3 = 0$$

having common point $O(x_0, y_0)$. Let *h*, *k* be two distinct half-rays $h \subset a$ and $k \subset b$ emanating from the point *O*. So we get $\angle(h, k)$.

Suppose we also have two straight lines in $E_2 W_n$,

$$a' : a_1' \times_n x +_n a_2' \times_n y +_n a_3' = 0$$

and

$$b' : b_1' \times_n x +_n b_2' \times_n y +_n b_3' = 0$$

having common point $O'(x_0', y_0')$. Let h', k' be any two distinct half-rays $h' \subset a'$ and $k' \subset b'$ emanating from the point O'. So we get $\angle(h', k')$. Let

$$O(x_0, y_0), O'\left(x_0', y_0'\right)$$

and suppose we have four points

$$\begin{gathered} A(x_1, y_1) \in h, \\ B(x_2, y_2) \in k, \\ A'\left(x_1', y_1'\right) \in h', \\ B'\left(x_2', y_2'\right) \in k' \end{gathered}$$

and the corresponding vectors

$$\begin{aligned}\mathbf{OA}&= (x_1 -_n x_0, y_1 -_n y_0),\\ \mathbf{OB}&= (x_2 -_n x_0, y_2 -_n y_0),\\ \mathbf{O'A'}&= \left(x_1' -_n x_0', y_1' -_n y_0'\right),\\ \mathbf{O'B'}&= \left(x_2' -_n x_0', y_2' -_n y_0'\right),\end{aligned}$$

such that

$$\begin{aligned}(\mathbf{OA}, \mathbf{OA})&> 0,\\ (\mathbf{OB}, \mathbf{OB})&> 0,\\ \left(\mathbf{O'A'}, \mathbf{O'A'}\right)&> 0,\\ \left(\mathbf{O'B'}, \mathbf{O'B'}\right)&> 0,\end{aligned}$$

and

$$\begin{aligned}(\mathbf{OA}, \mathbf{OA})&= \left(\mathbf{O'A'}, \mathbf{O'A'}\right),\\ (\mathbf{OB}, \mathbf{OB})&= \left(\mathbf{O'B'}, \mathbf{O'B'}\right).\end{aligned}$$

Let us take the scalar products

$$\begin{aligned}(\mathbf{OA}, \mathbf{OB})&= (x_1 -_n x_0) \times_n (x_2 -_n x_0) +_n (y_1 -_n y_0) \times_n (y_2 -_n y_0),\\ \left(\mathbf{O'A'}, \mathbf{O'B'}\right)&= \left(x_1' -_n x_0'\right) \times_n \left(x_2' -_n x_0'\right) +_n \left(y_1' -_n y_0'\right) \times_n \left(y_2' -_n y_0'\right).\end{aligned}$$

If

$$(\mathbf{OA}, \mathbf{OB}) = \left(\mathbf{O'A'}, \mathbf{O'B'}\right),$$

then we say that

$$\angle AOB = \angle(Ah, Bk)$$

is congruent to

$$\angle A'O'B' = \angle\left(A'h', B'k'\right)$$

and write

$$\angle(Ah, Bk) \equiv \angle\left(A'h', B'k'\right).$$

We assume here that all elements participating in the previous equalities belong to W_n. This means that in Mathematics with Observers geometry, we do not define the congruence of angles $\angle(h, k)$ and $\angle(h', k')$. We can define it only in the case where for any points *A*, *B*, A', B' satisfying the above conditions, we have

$$\angle(Ah, Bk) \equiv \angle\left(A'h', B'k'\right).$$

Then

$$\angle(h, k) \equiv \angle\left(h', k'\right).$$

Let us now go to E_3W_n. Let *u*, *v* be two straight lines lying in plane *α*,

$$u:\begin{cases} a_1 \times_n x +_n a_2 \times_n y +_n a_3 \times_n z +_n a_4 = 0, \\ b_1 \times_n x +_n b_2 \times_n y +_n b_3 \times_n z +_n b_4 = 0, \end{cases}$$

and

$$v:\begin{cases} c_1 \times_n x +_n c_2 \times_n y +_n c_3 \times_n z +_n c_4 = 0, \\ d_1 \times_n x +_n d_2 \times_n y +_n d_3 \times_n z +_n d_4 = 0, \end{cases}$$

having common point $O(x_0, y_0, z_0)$.

Let *h*, *k* be two distinct half-rays $h \subset u$ and $k \subset v$ emanating from the point *O*. So we get $\angle(h, k)$. Suppose we also have two straight lines u', v' lying in plane α',

$$u':\begin{cases} a_1' \times_n x +_n a_2' \times_n y +_n a_3' \times_n z +_n a_4' = 0, \\ b_1' \times_n x +_n b_2' \times_n y +_n b_3' \times_n z +_n b_4' = 0, \end{cases}$$

and

$$v':\begin{cases} c_1' \times_n x +_n c_2' \times_n y +_n c_3' \times_n z +_n c_4' = 0, \\ d_1' \times_n x +_n d_2' \times_n y +_n d_3' \times_n z +_n d_4' = 0, \end{cases}$$

having common point $O'(x_0', y_0', z_0')$.

Let h', k' are any two distinct half-rays $h' \subset a'$ and $k' \subset b'$ emanating from the point O'. So we get $\angle(h', k')$. Suppose we also have four points

$$\begin{gathered} A(x_1, y_1, z_1) \in h, \\ B(x_2, y_2, z_2) \in k, \\ A'(x_1', y_1', z_1') \in h', \\ B'(x_2', y_2', z_2') \in k' \end{gathered}$$

and the corresponding vectors

$$\begin{aligned} \mathbf{OA} &= (x_1 -_n x_0, y_1 -_n y_0, z_1 -_n z_0), \\ \mathbf{OB} &= (x_2 -_n x_0, y_2 -_n y_0, z_2 -_n z_0), \\ \mathbf{O'A'} &= (x_1' -_n x_0', y_1' -_n y_0', z_1' -_n z_0'), \\ \mathbf{O'B'} &= (x_2' -_n x_0', y_2' -_n y_0', z_2' -_n z_0') \end{aligned}$$

such that

$$(\mathbf{OA},\mathbf{OA}) > 0,$$
$$(\mathbf{OB},\mathbf{OB}) > 0,$$
$$(\mathbf{O'A'},\mathbf{O'A'}) > 0,$$
$$(\mathbf{O'B'},\mathbf{O'B'}) > 0,$$

and

$$(\mathbf{OA},\mathbf{OA}) = (\mathbf{O'A'},\mathbf{O'A'}),$$
$$(\mathbf{OB},\mathbf{OB}) = (\mathbf{O'B'},\mathbf{O'B'}).$$

Let us take the scalar products

$$(\mathbf{OA},\mathbf{OB}) = (x_1 -_n x_0) \times_n (x_2 -_n x_0) +_n (y_1 -_n y_0) \times_n (y_2 -_n y_0) +_n (z_1 -_n z$$
$$(\mathbf{O'A'},\mathbf{O'B'}) = (x'_1 -_n x'_0) \times_n (x'_2 -_n x'_0) +_n (y'_1 -_n y'_0) \times_n (y'_2 -_n y'_0) +_n (z'_1 -_n$$

If we have

$$(\mathbf{OA},\mathbf{OB}) = (\mathbf{O'A'},\mathbf{O'B'}),$$

then we say that

$$\angle AOB = \angle(Ah, Bk)$$

is congruent to

$$\angle A'O'B' = \angle(A'h', B'k')$$

and write

$$\angle(Ah, Bk) \equiv \angle(A'h', B'k').$$

We assume here that all elements participating in the previous equalities belong to W_n. This means that in Mathematics with Observers geometry, we do not define the congruence of angles $\angle(h,k)$ and $\angle(h',k')$. We can define it only in the case where for any points *A*, *B*, A', B' satisfying the above conditions, we have

$$\angle(Ah, Bk) \equiv \angle(A'h', B'k').$$

Then

$$\angle(h,k) \equiv \angle(h',k').$$

7.1 First property of congruence

In classical Euclidean geometry, we have the following statement:

> If *A*, *B* are two points on a straight line *a* and if A' is a point upon the same or another straight line a', then, upon a given side of A' on the straight line a', there always exists a unique one point B' such that the segment AB (or BA) is congruent to the segment $A'B'$.

Question: Is this statement correct in Mathematics with Observers geometry?

1) Let us take in E_2W_2 straight line *a* with equation

$$x = 0$$

and points

$$A(0,0), B(0,1) \in a$$

Let straight line a' have the equation

$$x -_2 y = 0$$

and point

$$A'(0,0) \in a'.$$

Any point $B' \in a'$ has the coordinates $B'(x,x)$. We have

$$(\mathbf{AB}, \mathbf{AB}) = 1$$

and

$$\left(\mathbf{A'B'}, \mathbf{A'B'}\right) = 2 \times_2 (x \times_2 x).$$

We get

$$2 \times_2 (0.79 \times_2 0.79) = 0.98,$$
$$2 \times_2 (0.80 \times_2 0.80) = 1.28.$$

This means that point $B' \in a'$ such that

$$(\mathbf{AB}, \mathbf{AB}) = \left(\mathbf{A'B'}, \mathbf{A'B'}\right)$$

does not exist, that is, the answer to the question in this case is negative.

2) Let us take in E_2W_2 straight line *a* with equation

$$x = 0$$

and points

$$A(0,0), B(0,1) \in a$$

Let straight line a' have the equation

$$y = 0$$

and point

$$A'(0,0) \in a'$$

Let us take the point

$$B'(1,0) \in a'.$$

We have

$$(\mathbf{AB}, \mathbf{AB}) = \left(\mathbf{A}'\mathbf{B}', \mathbf{A}'\mathbf{B}'\right) = 1$$

and

$$AB \equiv A'B',$$

that is, the answer to the question in this case is positive.

3) Let us take in E_2W_2 straight line *a* with equation

$$x = 0$$

and points

$$A(0,0), B(0,1) \in a.$$

Let straight line a' have the equation

$$1.4 \times_2 x -_2 y = 0$$

and the point

$$A'(0,0) \in a'.$$

Let us take two points

$$B'(0.60, 0.84), B''(0.61, 0.85) \in a'.$$

We have

$$(\mathbf{AB}, \mathbf{AB}) = 1,$$
$$\left(\mathbf{A}'\mathbf{B}', \mathbf{A}'\mathbf{B}'\right) = 0.60 \times_2 0.60 +_2 0.84 \times_2 0.84 = 1$$

and

$$\left(\mathbf{A}'\mathbf{B}'', \mathbf{A}'\mathbf{B}''\right) = 0.61 \times_2 0.61 +_2 0.85 \times_2 0.85 = 1.$$

This means that a point $B' \in a'$ such that

$$(\mathbf{AB}, \mathbf{AB}) = \left(\mathbf{A}'\mathbf{B}', \mathbf{A}'\mathbf{B}'\right)$$

exists but is not unique, and

$$AB \equiv A'B',$$
$$AB \equiv A'B'',$$

that is, the answer to the question in this case is negative.

So we have proved the following:

Theorem 7.1.

In Mathematics with Observers geometry in the plane E_2W_n, there are two points A, B on a straight line a and a point A' upon the same or another straight line a' such that upon a given side of A' on the straight line a', there is no point B' such that the segment AB (or BA) is congruent to the segment $A'B'$.

Theorem 7.2.
In Mathematics with Observers geometry in the plane E_2W_n, there are two points A, B on a straight line a and a point A' upon the same or another straight line a' such that upon a given side of A' on the straight line a', there is only one point B' such that the segment AB (or BA) is congruent to the segment $A'B'$.

Theorem 7.3.
In Mathematics with Observers geometry in the plane E_2W_n, there are two points A, B on a straight line a and a point A' upon the same or another straight line a' such that upon a given side of A' on the straight line a', there is more than one point B' such that the segment AB (or BA) is congruent to the segment $A'B'$.

7.2 Second property of congruence

In classical Euclidean geometry, we have the following statement:

> If a segment AB is congruent to the segment $A'B'$ and also to the segment $A''B''$, then the segment $A'B'$ is congruent to the segment $A''B''$.

Question: Is this statement correct in Observer's Mathematics geometry?

We must have

$$(\mathbf{AB}, \mathbf{AB}) = \left(\mathbf{A'B'}, \mathbf{A'B'}\right),$$
$$(\mathbf{AB}, \mathbf{AB}) = \left(\mathbf{A''B''}, \mathbf{A''B''}\right).$$

So

$$\left(\mathbf{A'B'}, \mathbf{A'B'}\right) = \left(\mathbf{A''B''}, \mathbf{A''B''}\right).$$

This means that the answer to the question is positive.

So we have proved the following:

Theorem 7.4.
In Mathematics with Observers geometry, if a segment AB is congruent to the segment $A'B'$ and also to the segment $A''B''$, then the segment $A'B'$ is congruent to the segment $A''B''$.

7.3 Third property of congruence

In classical Euclidean geometry we have the following statement:

Let AB and BC be two segments of a straight line a that have no common points aside from the point B, and, furthermore, let $A'B'$ and $B'C'$ be two segments of the same or another straight line a' having, likewise, no common point other than B'. If

$$AB \equiv A'B'$$

and

$$BC \equiv B'C',$$

then

$$AC \equiv A'C'.$$

Question: Is this statement correct in Mathematics with Observers geometry?

1) Let us take in E_2W_2 straight line a with equation

$$x = 0$$

and points

$$A(0,0), B(0,1), C(0,2) \in a.$$

Let straight line a' have the equation

$$y = 0$$

and points

$$A'(0,0), B'(1,0), C'(2,0) \in a'.$$

We have

$$(\mathbf{AB}, \mathbf{AB}) = (\mathbf{A'B'}, \mathbf{A'B'}) = 1,$$
$$(\mathbf{BC}, \mathbf{BC}) = (\mathbf{B'C'}, \mathbf{B'C'}) = 1.$$

So

$$AB \equiv A'B',$$
$$BC \equiv B'C',$$

and also

$$(\mathbf{AC}, \mathbf{AC}) = (\mathbf{A'C'}, \mathbf{A'C'}) = 4.$$

So

$$AC \equiv A'C'$$

that is, the answer to the question in this case is positive.

2) Let us take in E_2W_2 straight line a with equation

$$x = 0$$

and points

$$A(0,0), B(0,1), C(0,2) \in a.$$

Let s straight line a' have the equation

$$1.4 \times_2 x -_2 y = 0$$

and points

$$A'(0,0), B'(0.60, 0.84), C(1.20, 1.68) \in a'.$$

We have

$$(\mathbf{AB}, \mathbf{AB}) = 1,$$
$$(\mathbf{A'B'}, \mathbf{A'B'}) = 0.60 \times_2 0.60 +_2 0.84 \times_2 0.84 = 1,$$

and

$$(\mathbf{BC}, \mathbf{BC}) = 1,$$
$$(\mathbf{B'C'}, \mathbf{B'C'}) = (1.20 -_2 0.60) \times_2 (1.20 -_2 0.60) +_2 (1.68 -_2 0.84) \times_2 (1.68 -_2 0.84) = 1.$$

We get

$$(\mathbf{AC}, \mathbf{AC}) = 4,$$

but

$$(\mathbf{A'C'}, \mathbf{A'C'}) = 1.20 \times_2 1.20 +_2 1.68 \times_2 1.68 = 4.16.$$

This means that

$$(\mathbf{AC}, \mathbf{AC}) \neq (\mathbf{A'C'}, \mathbf{A'C'})$$

and

$$AC \not\equiv A'C',$$

that is, the answer to the question in this case is negative.

3) Let us take in E_2W_2 straight line *a* with equation

$$x = 0$$

and points

$$A(0,0), B(0, 1.46), C(0, 2.92) \in a.$$

Let straight line a' have the equation

$$x -_2 y = 0$$

and points

$$A'(0,0), B'(1.02, 1.02), C'(2.04, 2.04) \in a'.$$

We have

$$(\mathbf{AB},\mathbf{AB}) = 2.08,$$
$$\left(\mathbf{A'B'},\mathbf{A'B'}\right) = 1.02 \times_2 1.02 +_2 1.02 \times_2 1.02 = 2.08$$

and

$$(\mathbf{BC},\mathbf{BC}) = 2.08,$$
$$\left(\mathbf{B'C'},\mathbf{B'C'}\right) = (2.04 -_2 1.02) \times_2 (2.04 -_2 1.02) +_2 (2.04 -_2 1.02) \times_2 (2.04 -_2 1.02) = 2.0$$

So

$$AB \equiv A'B',$$
$$BC \equiv B'C'.$$

We get

$$(\mathbf{AC},\mathbf{AC}) = 8.49,$$

but

$$\left(\mathbf{A'C'},\mathbf{A'C'}\right) = 2.04 \times_2 2.04 +_2 2.04 \times_2 2.04 = 8.32.$$

This means that

$$AC \not\equiv A'C',$$

that is, the answer to this question in this case is negative.

So we have proved the following:

Theorem 7.5.
In Mathematics with Observers geometry in the plane E_2W_n, there are two segments AB and BC on a straight line a that have no common points aside from the point B, and there are two segments $A'B'$ and $B'C'$ of the same or another straight line a' having no common point other than B' with $AB \equiv A'B'$ and $BC \equiv B'C'$ such that $AC \not\equiv A'C'$.

Theorem 7.6.
In Mathematics with Observers geometry in the plane E_2W_n, there are two segments AB and BC on a straight line a that have no points in common aside from the point B, and there are two segments $A'B'$ and $B'C'$ of the same or another straight line a' having no common point other than B' with $AB \equiv A'B'$ and $BC \equiv B'C'$ such that $AC \equiv A'C'$.

7.4 Angle in Observer's geometry. First statement

Let us first consider the following classical geometry statement:

> The half-rays *h* and *k*, taken together with the point *O*, divide the remaining points of the plane into two regions having the following property: If *A* is a point of one region and *B* a

point of the other, then every broken line joining *A* and *B* either passes through *O* or has a common point with one of the half-rays *h* and *k*.

Question: Is this statement correct in Mathematics with Observers geometry?

1) Let us take in the plane E_2W_n two straight lines

$$a : x = 0$$

and

$$b : y = 0$$

and two half-rays $h \subset a$ and $k \subset b$, taken together with the point $O(0, 0)$,

$$h : y > 0$$

and

$$k : x > 0.$$

Then the interior region $R_i^{\angle(h,k)}$ of $\angle(h, k)$ is

$$R_i^{\angle(h,k)} = R_1^a \cap R_1^b,$$

and the exterior region $R_o^{\angle(h,k)}$ of $\angle(h, k)$ is

$$R_o^{\angle(h,k)} = E_2W_n \setminus \left(R_i^{\angle(h,k)} \cup h \cup k\right).$$

So half-rays *h* and *k*, taken together with the point *O*, in this case divide the remaining points of the plane *α* into two regions.

If $a. b \subset \alpha$ are two lines on a plane and $h \subset a$ and $k \subset b$, then two regions may be $R_1^a \cap R_1^b$, $R_2^a \cap R_1^b$, $R_1^a \cap R_2^b$, or $R_2^a \cap R_2^b$ and may not cover full plane *α* in the case where $R_3^a \neq \Lambda$ or $R_3^b \neq \Lambda$.

2) Let us take in the plane E_2W_n two straight lines

$$a : x +_n 99 \ldots 9.99 \ldots 9 = 0$$

and

$$b : y -_n 99 \ldots 9.99 \ldots 9 = 0$$

and two half-rays $h \subset a$ and $k \subset b$, taken together with the point $O(-99 \ldots 9.99 \ldots 9, 99 \ldots 9.99 \ldots 9)$:

$$h : y < 99 \ldots 9.99 \ldots 9,$$
$$k : x > -99 \ldots 9.99 \ldots 9.$$

Then the interior region $R_i^{\angle(h,k)}$ of $\angle(h, k)$ is

$$R_i^{\angle(h,k)} = R_1^a \cap R_2^b,$$

and the exterior region $R_o^{\angle(h,k)}$ of $\angle(h,k)$ is

$$R_o^{\angle(h,k)} = E_2W_n \setminus \left(R_i^{\angle(h,k)} \cup h \cup k\right) = \Lambda.$$

So half-rays *h* and *k*, taken together with the point *O*, in this case transform the remaining points of the plane *a* into one region.

3) Let's take two straight lines $a, b \subset E_2W_n$ with equations

$$a : y = 0$$

and

$$b : 3 \times_n x -_n y = 0.$$

Let *h* be a half-ray of the straight line *a* emanating from the point $O(0,0)$, and let *k* be a half-ray of the straight line *b* emanating from the same point $O(0,0)$. Let the interior region $R_i^{\angle(h,k)}$ of $\angle(h,k)$ be

$$R_i^{\angle(h,k)} = R_1^a \cap R_1^b,$$

and let the exterior region $R_o^{\angle(h,k)}$ of $\angle(h,k)$ be

$$R_o^{\angle(h,k)} = E_2W_n \setminus \left(R_i^{\angle(h,k)} \cup h \cup k\right).$$

Let us take the points

$$A(1,1) \in R_i^{\angle(h,k)}$$

and

$$B(-1,1) \in R_o^{\angle(h,k)}.$$

Then segment AB does not intersect half-rays *h*, *k*:

$$AB \cap h = \Lambda$$

because $a \cap c = \Lambda$, where

$$c : y = 1,$$

and

$$AB \cap k = \Lambda$$

because

$$3 \times_n 0.33\ldots33 = 0.99\ldots99,$$
$$3 \times_n 0.33\ldots34 = 1.00\ldots02.$$

So in this case the answer to the question is negative.

4) Let us take two straight lines $a, b \subset E_2W_n$ with equations

$$a : y = 0$$

and

$$b : x = 0.$$

Let *h* be a half-ray of the straight line *a* emanating from the point $O(0, 0)$, and let *k* be a half-ray of the straight line *b* emanating from the same point $O(0, 0)$. Let the interior region $R_i^{\angle(h,k)}$ be given by the system

$$\begin{cases} y > 0, \\ x > 0, \end{cases}$$

and let the exterior region $R_o^{\angle(h,k)}$ be given by the equality

$$R_o^{\angle(h,k)} = E_2W_n \setminus \left(R_i^{\angle(h,k)} \cup h \cup k\right).$$

Let us take the points

$$A(1, 1) \in R_i^{\angle(h,k)}$$

and

$$B(-1, 1) \in R_o^{\angle(h,k)}.$$

Then the segment AB intersects the half-ray *k* in point $C(0, 1)$:

$$AB \cap K = C.$$

This means that in this case the answer to the question is positive.

So we have proved the following:

Theorem 7.7.
In Mathematics with Observers geometry in the plane E_2W_n, there are two half-rays h and k, taken together with the point O, such that the remaining points of the plane are transformed into one region.
Theorem 7.8.
In Mathematics with Observers geometry in the plane E_2W_n, there are two half-rays h and k, taken together with the point O, such that the remaining points of the plane are divided into two regions.
Theorem 7.9.
In Mathematics with Observers geometry in the plane E_2W_n, there are two half-rays h and k, taken together with the point O, such that the remaining points of the plane are divided into three regions.
Theorem 7.10.
In Mathematics with Observers geometry in the plane E_2W_n, there are two half-rays h and k, taken together with the point O, that divide the remaining points of the plane a into two regions, and point A of one region and point B of the other such that there is a broken line joining A and B that neither passes through O nor has a common point with one of the half-rays h, k.
Theorem 7.11.

In Mathematics with Observers geometry in the plane E_2W_n, there are two half-rays h and k, taken together with the point O, that divide the remaining points of the plane a into two regions, and point A of one region and point B of the other such that there is a broken line joining A and B that either passes through O or has a common point with one of the half-rays h, k.

7.5 Angle in Observer's geometry. Second statement

Let us consider another classical geometry statement:

> If points *A*, A' both lie within the same region, then it is always possible to join these two points by a broken line that neither passes through *O* nor has a common point with either of the half-rays *h*, *k*.

Question: Is this statement correct in Mathematics with Observers geometry?

1) Let's take two straight lines $a, b \subset E_2W_2$ with equations

$$a : 99.99 \times_2 x -_2 98.37 \times_2 y = 0$$

and

$$b : 98.37 \times_2 x -_2 99.99 \times_2 y = 0.$$

Let *h* be a half-ray of the straight line *a* emanating from the point $O(0,0)$, and let *k* be a half-ray of the straight line *b* emanating from the same point $O(0,0)$.

The interior region $R_i^{\angle(h,k)}$ is given by the system

$$\begin{cases} 99.99 \times_2 x -_2 98.37 \times_2 y > 0, \\ 98.37 \times_2 x -_2 99.99 \times_2 y < 0. \end{cases}$$

This means that

$$\begin{aligned} R_i^{\angle(h,k)} = \big[&(0.01, 0.01), (0.02, 0.02), \ldots \\ &\ldots (0.62, 0.62), (0.63, 0.63), (0.63, 0.64), (0.64, 0.63), (0.64, 0.64), (0.64, 0.65) \ldots \\ &\ldots, (0.89, 0.88), (0.89, 0.89), (0.89, 0.90), \ldots \\ &\ldots (0.99, 0.98), (0.99, 0.99), (0.99, 1.00), (1.00, 1.00)\big]. \end{aligned}$$

Let us consider two points

$$A(0.01, 0.01), A'(0.62, 0.62).$$

Straight line *a* containing these points is

$$a : x -_2 y = 0,$$

and the segment

$$AA' \subset R_i^{\angle(h,k)}.$$

So in this case the answer to the question is positive.

2) Let us consider the same lines as in the previous case and other two points

$$A(0.89, 0.88), A'(1.00, 1.00).$$

Straight line *b* containing these points is

$$b : a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0,$$

and we have the system of equations

$$\begin{cases} a_1 \times_2 0.89 +_2 a_2 \times_2 0.88 +_2 a_3 = 0, \\ a_1 \times_2 1 +_2 a_2 \times_2 1 +_2 a_3 = 0, \end{cases}$$

that is,

$$\begin{cases} a_1 \times_2 0.89 +_2 a_2 \times_2 0.88 = a_1 +_2 a_2, \\ a_1 +_2 a_2 +_2 a_3 = 0. \end{cases}$$

So

$$\begin{aligned} &a_1 \in [\pm 0.01, \pm 0.02, \ldots, \pm 0.09], \\ &a_2 = -a_1, \\ &a_3 = 0, \end{aligned}$$

and the segment

$$AA' \not\subset R_i^{\angle(h,k)}.$$

So in this case the answer to the question is negative.

3) Let us take two straight lines $a, b \subset E_2W_2$ with equations

$$a : x = 0$$

and

$$b : y = 0.$$

Let *h* be a half-ray of the straight line *a* emanating from the point $O(0,0)$, and let *k* be a half-ray of the straight line *b* emanating from the same point $O(0,0)$. The exterior region $R_o^{\angle(h,k)}$ is given by

$$R_o^{\angle(h,k)} = E_2W_n \setminus \left(R_i^{\angle(h,k)} \cup h \cup k\right).$$

Let us consider two points

$$A(-2,3), A'(2,-3) \in R_o^{\angle(h,k)}.$$

It is possible to join these two points by a broken line ACA' that neither passes through *O* nor has a common point with interior region $R_i^{\angle(h,k)}$ and either of the half-rays *h*, *k*, where

$$C(-2,-3),$$

and $R_o^{\angle(h,k)}$ is the set of all points in E_2W_2 satisfying the system of inequalities

$$\begin{cases} x > 0, \\ y > 0. \end{cases}$$

So in this case the answer to this question is positive.

4) Let us take two straight lines $a, b \subset E_2W_2$ with equations

$$a : y = 0$$

and

$$b : 0.01 \times_2 x -_2 y +_2 0.99 = 0.$$

Let *h* be a half-ray of the straight line *a* emanating from the point $O(-99.99, 0)$, and let *k* be a half-ray of the straight line *b* emanating from the same point $O(-99.99, 0)$. In this case, we have

$$h = a$$

and

$$k = b.$$

The exterior region $R_o^{\angle(h,k)}$ is given by

$$R_o^{\angle(h,k)} = R_2^a \cup R_2^b,$$

where

$$R_2^a = \left[(x,y) \in E_2W_2 : y < 0\right],$$
$$R_2^b = \left[(x,y) \in E_2W_2 : 0.01 \times_2 x -_2 y +_2 0.99 < 0\right],$$

and the interior region $R_i^{\angle(h,k)}$ is given by the system of inequalities

$$\begin{cases} y > 0, \\ 0.01 \times_2 x -_2 y +_2 0.99 > 0. \end{cases}$$

Let us consider two points

$$A(x_1, y_1), A'(x_2, y_2) \in R_o^{\angle(h,k)},$$

where

$$\begin{cases} y_1 < 0, \\ 0.01 \times_2 x_2 -_2 y_2 +_2 0.99 < 0. \end{cases}$$

Then it is impossible to join these two points by a broken line that neither passes through *O* nor has a common point with the interior region R_1 and either of the half-rays *h*, *k*.

If we consider the line

$$c : 3 \times_2 x -_2 y -_2 1.00 = 0,$$

then

$$c \cap h = \Lambda,$$
$$c \cap k = \Lambda,$$

but

$$c \cap R_i^{\angle(h,k)} \neq \Lambda.$$

This means that in this case the answer to the question is negative.

So we have proved the following:

Theorem 7.12.
In Mathematics with Observers geometry in the plane E_2W_n, there are two half-rays h and k, taken together with the point O, that divide the remaining points of the plane a into two regions, and points A, A' of the same region such that there is a broken line joining A and A' that neither passes through O nor has a common point with either of the half-rays h, k.

Theorem 7.13.
In Mathematics with Observers geometry in the plane E_2W_n, there are two half-rays h and k, taken together with the point O, that divide the remaining points of the plane a into two regions, and points A, A' of the same region such that there is a broken line joining A and A' that either passes through O or has a common point with one of the half-rays h, k.

7.6 Fourth property of congruence

Classical geometry states:

> Let an angle $\angle(h, k)$ be given in a plane *α*, and let a straight line a' be given in a plane α'. Suppose also that in α' a definite side of the straight line a' is assigned. Let h' be a half-ray of the straight line a' emanating from a point O' of this line. Then in the plane α', there is a unique half-ray k' such that the angle $\angle(h, k)$, or $\angle(k, h)$, is congruent to the angle $\angle(h', k')$ and at the same time, all interior points of the angle $\angle(h', k')$ lie upon the given side of a', that is,
>
> $$\angle(h, k) \equiv \angle(h', k').$$
>
> Every angle is congruent to itself, that is,
>
> $$\angle(h, k) \equiv \angle(h, k)$$

or

$$\angle(h,k) \equiv \angle(k,h).$$

Question: Is this statement correct in Mathematics with Observers geometry?

1) Let two straight lines in E_2W_2,

$$a : 99.99 \times_2 x -_2 98.37 \times_2 y = 0$$

and

$$b : 99.99 \times_2 x -_2 96.37 \times_2 y = 0,$$

have a common point $O(0,0)$ and two distinct half-rays

$$h \subset a,$$
$$h : \left[(x,y) \in a : x > 0\right],$$

and

$$k \subset b,$$
$$k : \left[(x,y) \in b : x > 0\right],$$

emanating from the point *O*. So we get $\angle(h,k)$ and $\angle(k,h)$.

Direct calculations give us the following results:

$$h = C(0.62, 0.63),$$

that is, *h* contains only one point, and

$$k = \left[E(0.28, 0.29), F(0.54, 0.56), G(0.80, 0.83)\right],$$

that is, *k* contains only three points.

We have the vectors

$$\mathbf{OC} = (0.62, 0.63),$$
$$\mathbf{OE} = (0.28, 0.29),$$
$$\mathbf{OF} = (0.54, 0.56),$$
$$\mathbf{OG} = (0.80, 0.83),$$

and we get

$$(\mathbf{OC}, \mathbf{OC}) = 0.72 > 0,$$
$$(\mathbf{OE}, \mathbf{OE}) = 0.08 > 0,$$
$$(\mathbf{OF}, \mathbf{OF}) = 0.50 > 0,$$
$$(\mathbf{OG}, \mathbf{OG}) = 1.28 > 0.$$

So

$$(\mathbf{OC},\mathbf{OC}) \neq (\mathbf{OE},\mathbf{OE}),$$
$$(\mathbf{OC},\mathbf{OC}) \neq (\mathbf{OF},\mathbf{OF}),$$
$$(\mathbf{OC},\mathbf{OC}) \neq (\mathbf{OG},\mathbf{OG}),$$

which means that

$$\angle(h,k) \equiv \angle(h,k)$$

and

$$\angle(Ch,Ek) \not\equiv \angle(Ek,Ch),$$
$$\angle(Ch,Fk) \not\equiv \angle(Fk,Ch),$$
$$\angle(Ch,Gk) \not\equiv \angle(Gk,Ch),$$

that is,

$$\angle(h,k) \not\equiv \angle(k,h).$$

So in this case the answer to the question is negative.

2) Let two straight lines in E_2W_2,

$$a: 99.99 \times_2 x -_2 98.37 \times_2 y = 0$$

and

$$b: y = 0,$$

have a common point $O(0,0)$ and two distinct half-rays

$$h \subset a,$$
$$h: \big[(x,y) \in a : x > 0\big],$$

and

$$k \subset b,$$
$$k: \big[(x,y) \in b : x > 0\big],$$

emanating from the point *O*. So we get $\angle(h,k)$.

Now let us take a straight line in E_2W_2,

$$a': 99.99 \times_2 x -_2 96.37 \times_2 y = 0,$$

and any other straight line in E_2W_2,

$$b': a_1 \times_2 x -_2 a_2 \times_2 y = 0,$$

having common point $O(0,0)$ with line a'. We get two distinct half-rays

$$h' \subset a',$$
$$h: \big[(x,y) \in a : x > 0\big],$$

and

$$k' \subset b',$$
$$k' : \left[(x,y) \in b' : x > 0\right].$$

Direct calculations give us the following results:

$$h = C(0.62, 0.63),$$

that is, *h* contains only one point, and

$$h' = \left[E(0.28, 0.29), F(0.54, 0.56), G(0.80, 0.83)\right],$$

that is, h' contains only three points.

We have the vectors

$$\mathbf{OC} = (0.62, 0.63),$$
$$\mathbf{OE} = (0.28, 0.29),$$
$$\mathbf{OF} = (0.54, 0.56),$$
$$\mathbf{OG} = (0.80, 0.83),$$

and we get

$$(\mathbf{OC}, \mathbf{OC}) = 0.72 > 0,$$
$$(\mathbf{OE}, \mathbf{OE}) = 0.08 > 0,$$
$$(\mathbf{OF}, \mathbf{OF}) = 0.50 > 0,$$
$$(\mathbf{OG}, \mathbf{OG}) = 1.28 > 0.$$

So

$$(\mathbf{OC}, \mathbf{OC}) \neq (\mathbf{OE}, \mathbf{OE}),$$
$$(\mathbf{OC}, \mathbf{OC}) \neq (\mathbf{OF}, \mathbf{OF}),$$
$$(\mathbf{OC}, \mathbf{OC}) \neq (\mathbf{OG}, \mathbf{OG}).$$

This means that for any line b' and any points $D \in k$ and $K, L, M \in b'$,

$$\angle(Ch, Dk) \not\equiv \angle\left(Eh', Kk'\right),$$
$$\angle(Ch, Dk) \not\equiv \angle\left(Fh', Lk'\right),$$
$$\angle(Ch, Dk) \not\equiv \angle\left(Gh', Mk'\right).$$

So in this case the answer to the question is negative.

3) Let two straight lines in $E_2 W_n$,

$$a : y = 0$$

and

$$b : x = 0,$$

have a common point $O(0,0)$ and two distinct half-rays

$$h \subset a,$$
$$h : \left[(x,y) \in a : x > 0\right],$$

and

$$k \subset b,$$
$$k : \left[(x,y) \in b : y > 0\right],$$

emanating from the point O. So we get $\angle(h,k)$.

Let us consider straight lines in E_2W_n,

$$a' : y = 1$$

and

$$b' : x = 1,$$

having a common point $O'(1,1)$ with line a'. Let h', k' are any two distinct half-rays

$$h' \subset a',$$
$$h' : \left[(x,y) \in a' : x > 1\right],$$

and

$$k' \subset b',$$
$$k' : \left[(x,y) \in b' : y > 1\right],$$

emanating from the point O'. So we get $\angle(h',k')$.

Let us take points

$$A(1,0) \in h,$$
$$B(0,1) \in k,$$
$$A'(2,1) \in h',$$
$$B'(1,2) \in k'$$

and the corresponding vectors

$$\mathbf{OA} = (1,0),$$
$$\mathbf{OB} = (0,1),$$
$$\mathbf{O'A'} = (1,0),$$
$$\mathbf{O'B'} = (0,1).$$

We get

$$(\mathbf{OA},\mathbf{OA})= 1 > 0,$$
$$(\mathbf{OB},\mathbf{OB})= 1 > 0,$$
$$\left(\mathbf{O'A'},\mathbf{O'A'}\right)= 1 > 0,$$
$$\left(\mathbf{O'B'},\mathbf{O'B'}\right)= 1 > 0,$$

and thus

$$OA\equiv O'A',$$
$$OB\equiv O'B'.$$

We have the scalar products

$$(\mathbf{OA},\mathbf{OB})= 1\times_n 0 +_n 0\times_n 1 = 0,$$
$$\left(\mathbf{O'A'},\mathbf{O'B'}\right)= 1\times_n 0 +_n 0\times_n 1 = 0,$$

and their equality

$$(\mathbf{OA},\mathbf{OB}) = \left(\mathbf{O'A'},\mathbf{O'B'}\right) = 0,$$

that is,

$$\angle(Ah,Bk)\equiv\angle\left(A'h',B'k'\right).$$

Moreover, if we take points

$$A(x_0,0)\in h,\quad x_0\geq 0.1,$$
$$B(0,y_0)\in k,\quad y_0\geq 0.1,$$
$$A'(x_0+_n 1,1)\in h',$$
$$B'(1,y_0+_n 1)\in k',$$

and the corresponding vectors

$$\mathbf{OA}= (x_0,0),$$
$$\mathbf{OB}= (0,y_0),$$
$$\mathbf{O'A'}= (x_0,0),$$
$$\mathbf{O'B'}= (0,y_0),$$

then we get

$$(\mathbf{OA},\mathbf{OA})= x_0\times_n x_0 > 0,$$
$$(\mathbf{OB},\mathbf{OB})= y_0\times_n y_0 > 0,$$
$$\left(\mathbf{O'A'},\mathbf{O'A'}\right)= x_0\times_n x_0 > 0,$$
$$\left(\mathbf{O'B'},\mathbf{O'B'}\right)= y_0\times_n y_0 > 0.$$

Note that for the remaining possible points of h, k, h', k',

$$
\begin{aligned}
A''(x_0, 0) &\in h, \quad 0 < x_0 < 0.1, \\
B''(0, y_0) &\in k, \quad 0 < y_0 < 0.1, \\
A'''(x_0 +_n 1, 1) &\in h', \\
B'''(1, y_0 +_n 1) &\in k',
\end{aligned}
$$

we get

$$
\begin{aligned}
\left(\mathbf{OA''}, \mathbf{OA''}\right) &= x_0 \times_n x_0 = 0, \\
\left(\mathbf{OB''}, \mathbf{OB''}\right) &= y_0 \times_n y_0 = 0, \\
\left(\mathbf{O'A'''}, \mathbf{O'A'''}\right) &= x_0 \times_n x_0 = 0, \\
\left(\mathbf{O'B'''}, \mathbf{O'B'''}\right) &= y_0 \times_n y_0 = 0.
\end{aligned}
$$

This means that

$$
\begin{aligned}
OA &\equiv O'A', \\
OB &\equiv O'B'.
\end{aligned}
$$

We have the scalar products

$$
\begin{aligned}
(\mathbf{OA}, \mathbf{OB}) &= x_0 \times_n 0 +_n 0 \times_n y_0 = 0, \\
\left(\mathbf{O'A'}, \mathbf{O'B'}\right) &= x_0 \times_n 0 +_n 0 \times_n y_0 = 0,
\end{aligned}
$$

and their equality

$$
(\mathbf{OA}, \mathbf{OB}) = \left(\mathbf{O'A'}, \mathbf{O'B'}\right) = 0,
$$

that is,

$$
\angle(Ah, Bk) \equiv \angle\left(A'h', B'k'\right),
$$

which means that

$$
\angle(h, k) \equiv \angle\left(h', k'\right).
$$

This means that in this case the answer to the question is positive.

4) Let take three straight lines $a, b, c \subset E_2W_2$:

$$
\begin{aligned}
&a : 2 \times_2 x -_2 y = 0, \\
&b : x -_2 y = 0, \\
&c : 0.50 \times_2 x -_2 y = 0.
\end{aligned}
$$

These lines have a unique common point $O(0, 0)$,

$$
a \cap b \cap c = O.
$$

Denote by *h* the half-ray of the straight line *a* emanating from a point *O*,

$$
h = \left[(x, y) \in a, x > 0\right],
$$

by *k* the half-ray of the straight line *b* emanating from same point *O*,

$$k = \big[(x, y) \in b, x > 0\big],$$

and by *l* the half-ray of the straight line *c* emanating from the same point *O*,

$$l = \big[(x, y) \in c, x > 0\big].$$

Let us take the points

$$A(1, 2) \in h, \quad C(1, 1) \in k, \quad B(2, 1) \in l.$$

Then we have

$$\begin{aligned} \mathbf{OA} &= (1, 2), \\ \mathbf{OC} &= (1, 1), \\ \mathbf{OB} &= (2, 1), \end{aligned}$$

and we get

$$\begin{aligned} (\mathbf{OA}, \mathbf{OA}) &= 1 \times_2 1 +_2 2 \times_2 2 = 5, \\ (\mathbf{OB}, \mathbf{OB}) &= 2 \times_2 2 +_2 1 \times_2 1 = 5, \\ (\mathbf{OA}, \mathbf{OC}) &= 1 \times_2 1 +_2 2 \times_2 1 = 3, \\ (\mathbf{OB}, \mathbf{OC}) &= 2 \times_2 1 +_2 1 \times_2 1 = 3, \\ (\mathbf{OA}, \mathbf{OA}) &= (\mathbf{OB}, \mathbf{OB}), \\ (\mathbf{OA}, \mathbf{OC}) &= (\mathbf{OB}, \mathbf{OC}), \end{aligned}$$

that is,

$$\begin{aligned} OA &\equiv OB, \\ OC &\equiv OC. \end{aligned}$$

This means that the points $A \in h$, $C \in k$, $B \in l$ satisfy the definition of congruence of angles and

$$\angle(Ah, Ck) \equiv \angle(Bl, Ck).$$

Let us now take the other points

$$D(2.02, 4.04) \in h, \quad F(2.29, 2.29) \in k, \quad E(4.05, 2.00) \in l.$$

We have

$$\begin{aligned} \mathbf{OD} &= (2.02, 4.04), \\ \mathbf{OF} &= (2.29, 2.29), \\ \mathbf{OE} &= (4.05, 2.00), \end{aligned}$$

and we get

$$(\mathbf{OD},\mathbf{OD}) = 2.02 \times_2 2.02 +_2 4.04 \times_2 4.04 = 20.40,$$
$$(\mathbf{OE},\mathbf{OE}) = 4.05 \times_2 4.05 +_2 2.00 \times_2 2.00 = 20.40,$$
$$(\mathbf{OD},\mathbf{OF}) = 2.02 \times_2 2.29 +_2 4.04 \times_2 2.29 = 13.86,$$
$$(\mathbf{OE},\mathbf{OF}) = 4.05 \times_2 2.29 +_2 2.00 \times_2 2.29 = 13.84,$$
$$(\mathbf{OD},\mathbf{OD}) = (\mathbf{OE},\mathbf{OE}),$$
$$(\mathbf{OD},\mathbf{OF}) \neq (\mathbf{OE},\mathbf{OF}),$$

that is,

$$OD \equiv OE,$$
$$OF \equiv OF,$$

and

$$(\mathbf{OD},\mathbf{OF}) \neq (\mathbf{OE},\mathbf{OF}).$$

So

$$\angle(Dh, Fk) \not\equiv \angle(El, Fk).$$

This means that in this case the answer to the question is negative.

5) Let us take two straight lines in E_2W_2,

$$a : 1.40 \times_2 x -_2 y = 0$$

and

$$b : x -_2 y = 0,$$

which have a unique common point $O(0,0)$,

$$a \cap b = O.$$

Denote by *h* a half-ray of the straight line *a* emanating from a point *O*,

$$h = \big[(x,y) \in a, x > 0\big],$$

and by *k* a half-ray of the straight line *b* emanating from the same point *O*,

$$k = \big[(x,y) \in b, x > 0\big].$$

Let us take two points $A(0.61, 0.85), A'(0.63, 0.87) \in h$ and two points $B(1,1), B'(1,1) \in k$, that is, $B = B'$. We have

$$\mathbf{OA} = (0.61, 0.85),$$
$$\mathbf{OA'} = (0.63, 0.87),$$
$$\mathbf{OB} = (1,1),$$
$$\mathbf{OB'} = (1,1),$$

and we get

$$(\mathbf{OA},\mathbf{OA}) = 0.61 \times_2 0.61 +_2 0.85 \times_2 0.85 = 1 > 0,$$
$$(\mathbf{OA}',\mathbf{OA}') = 0.63 \times_2 0.63 +_2 0.87 \times_2 0.87 = 1 > 0,$$
$$(\mathbf{OB},\mathbf{OB}) = 1 \times_2 1 +_2 1 \times_2 1 = 2 > 0,$$
$$(\mathbf{OB}',\mathbf{OB}') = 1 \times_2 1 +_2 1 \times_2 1 = 2 > 0.$$

So we have

$$OA \equiv OA',$$
$$OB \equiv OB'.$$

For the pairs of vectors

$$\mathbf{OA},\mathbf{OB}$$

and

$$\mathbf{OA}',\mathbf{OB}',$$

we have the scalar products

$$(\mathbf{OA},\mathbf{OB}) = 0.61 \times_2 1 +_2 0.85 \times_2 1 = 1.46,$$
$$(\mathbf{OA}',\mathbf{OB}') = 0.63 \times_2 1 +_2 0.87 \times_2 1 = 1.50,$$

and their inequality

$$(\mathbf{OA},\mathbf{OB}) \neq (\mathbf{OA}',\mathbf{OB}').$$

So

$$\angle(Ah,Bk) \not\equiv \angle(A'h,B'k).$$

This means that in this case the answer to the question is negative.

So we have proved the following:

Theorem 7.14.
In Mathematics with Observers geometry in the plane E_2W_n, there are an angle $\angle(h,k)$ and a straight line a' with a half-ray h' emanating from a point O' of this line such that there is no half-ray k' such that

$$\angle(h,k) \equiv \angle(h',k').$$

Theorem 7.15.
In Mathematics with Observers geometry in the plane E_2W_n, there are an angle $\angle(h,k)$ and a straight line a' with a half-ray h' emanating from a point O' of this line such that there is a half-ray k' such that

$$\angle(h,k) \equiv \angle(h',k').$$

Theorem 7.16.
In Mathematics with Observers geometry in the plane E_2W_n, there is an angle $\angle(h,k)$ such that

$$\angle(h,k) \equiv \angle(k,h).$$

Theorem 7.17.

In Mathematics with Observers geometry in the plane E_2W_n, there is an angle $\angle(h,k)$ such that

$$\angle(h,k) \not\equiv \angle(k,h).$$

7.7 Fifth property of congruence

Classical geometry states:

> If an angle $\angle(h,k)$ is congruent to an angle $\angle(h',k')$ and to an angle $\angle(h'',k'')$, then the angle $\angle(h',k')$ is congruent to the angle $\angle(h'',k'')$, that is, if
> $$\angle(h,k) \equiv \angle\left(h',k'\right)$$
> and
> $$\angle(h,k) \equiv \angle\left(h'',k''\right),$$
> then
> $$\angle\left(h',k'\right) \equiv \angle\left(h'',k''\right).$$

Question: Is this statement correct in Mathematics with Observers geometry?

Let us consider E_2W_n. Let $a, b \in E_2W_n$ be the straight lines

$$a : a_1 \times_n x +_n a_2 \times_n y +_n a_3 = 0$$

and

$$b : b_1 \times_n x +_n b_2 \times_n y +_n b_3 = 0$$

having a common point $O(x_0, y_0)$.

Let *h*, *k* are two distinct half-rays

$$h \subset a$$

and

$$k \subset b$$

emanating from the point *O*. So we get $\angle(h,k)$.

Also, let $a', b' \in E_2W_n$ be the straight lines

$$a' : a_1' \times_n x +_n a_2' \times_n y +_n a_3' = 0$$

and

$$b' : b_1' \times_n x +_n b_2' \times_n y +_n b_3' = 0$$

having a common point $O'(x_0', y_0')$.

Let h', k' be any two distinct half-rays

$$h' \subset a'$$

and

$$k' \subset b'$$

emanating from the point O'. So we get $\angle(h', k')$.

Because

$$\angle(Ah, Bk) \equiv \angle(A'h', B'k')$$

for some points

$$\begin{aligned} A(x_1, y_1) &\in h, \\ B(x_2, y_2) &\in k, \\ A'(x_1', y_1') &\in h', \\ B'(x_2', y_2') &\in k', \end{aligned}$$

we have the corresponding vectors

$$\begin{aligned} \mathbf{OA} &= (x_1 -_n x_0, y_1 -_n y_0), \\ \mathbf{OB} &= (x_2 -_n x_0, y_2 -_n y_0), \\ \mathbf{O'A'} &= (x_1' -_n x_0', y_1' -_n y_0'), \\ \mathbf{O'B'} &= (x_2' -_n x_0', y_2' -_n y_0') \end{aligned}$$

such that

$$\begin{aligned} (\mathbf{OA}, \mathbf{OA}) &> 0, \\ (\mathbf{OB}, \mathbf{OB}) &> 0, \\ (\mathbf{O'A'}, \mathbf{O'A'}) &> 0, \\ (\mathbf{O'B'}, \mathbf{O'B'}) &> 0, \\ OA &\equiv O'A', \\ OB &\equiv O'B', \\ (\mathbf{OA}, \mathbf{OB}) &= (x_1 -_n x_0) \times_n (x_2 -_n x_0) +_n (y_1 -_n y_0) \times_n (y_2 -_n y_0), \\ (\mathbf{O'A'}, \mathbf{O'B'}) &= (x_1' -_n x_0') \times_n (x_2' -_n x_0') +_n (y_1' -_n y_0') \times_n (y_2' -_n y_0'), \\ (\mathbf{OA}, \mathbf{OB}) &= (\mathbf{O'A'}, \mathbf{O'B'}). \end{aligned}$$

Again, let us consider $\angle(h, k)$. We have two straight lines $a, b \in E_2 W_n$,

$$a : a_1 \times_n x +_n a_2 \times_n y +_n a_3 = 0$$

and

$$b : b_1 \times_n x +_n b_2 \times_n y +_n b_3 = 0,$$

having a common point $O(x_0, y_0)$ and two distinct half-rays

$$h \subset a$$

and

$$k \subset b$$

emanating from the point O. So we get $\angle(h, k)$.

Let us consider $\angle(h'', k'')$. We have two straight lines $a'', b'' \in E_2 W_n$,

$$a'' : a''_1 \times_n x +_n a''_2 \times_n y +_n a''_3 = 0$$

and

$$b'' : b''_1 \times_n x +_n b''_2 \times_n y +_n b''_3 = 0,$$

having a common point $O''(x''_0, y''_0)$ and two distinct half-rays

$$h'' \subset a''$$

and

$$k'' \subset b''$$

emanating from the point O''. So we get $\angle(h'', k'')$.

Because

$$\angle(Ch, Dk) \equiv \angle\left(C'h'', D'k''\right)$$

for some points

$$C(x_3, y_3) \in h,$$
$$D(x_4, y_4) \in k,$$
$$C'\left(x''_1, y''_1\right) \in h'',$$
$$D'\left(x''_2, y''_2\right) \in k'',$$

we have the corresponding vectors

$$\mathbf{OC} = (x_3 -_n x_0, y_3 -_n y_0),$$
$$\mathbf{OD} = (x_4 -_n x_0, y_4 -_n y_0),$$
$$\mathbf{O''C'} = \left(x''_1 -_n x''_0, y''_1 -_n y''_0\right),$$
$$\mathbf{O''D'} = \left(x''_2 -_n x''_0, y''_2 -_n y''_0\right)$$

such that

$$
\begin{aligned}
(\mathbf{OC},\mathbf{OC})&> 0,\\
(\mathbf{OD},\mathbf{OD})&> 0,\\
(\mathbf{O''C'},\mathbf{O''C'})&> 0,\\
(\mathbf{O''D'},\mathbf{O''D'})&> 0,\\
OC&\equiv O''C',\\
OD&\equiv O''D',\\
(\mathbf{OC},\mathbf{OD})&= (x_3 -_n x_0)\times_n (x_4 -_n x_0) +_n (y_3 -_n y_0)\times_n (y_4 -_n y_0),\\
(\mathbf{O''C'},\mathbf{O''D'})&= (x''_1 -_n x''_0)\times_n (x''_2 -_n x''_0) +_n (y''_1 -_n y''_0)\times_n (y''_2 -_n y''_0),\\
(\mathbf{OC},\mathbf{OD})&= (\mathbf{O''C'},\mathbf{O''D'}).
\end{aligned}
$$

The question stated above means: is the statement

$$\angle(A'h', B'k') \equiv \angle(C'h'', D'k'')$$

correct?

The answer does not follow automatically because two sets of points

$$A \in h,\quad B \in k,\quad A' \in h',\quad B' \in k'$$

and

$$C \in h,\quad D \in k,\quad C' \in h'',\quad D' \in k''$$

are different.

1) Let two straight lines $a, b \in E_2W_n$,

$$a : y = 0$$

and

$$b : x = 0,$$

have a common point $O(0,0)$ and two distinct half-rays

$$
\begin{aligned}
&h \subset a,\\
&h : \big[(x,y) \in E_2W_n : x > 0\big],
\end{aligned}
$$

and

$$
\begin{aligned}
&k \subset b,\\
&k : \big[(x,y) \in E_2W_n : y > 0\big],
\end{aligned}
$$

emanating from the point O. So we get $\angle(h,k)$.

We also have two straight lines $a', b' \in E_2W_n$,

$$a' : y = 0$$

and

$$b' : x = 0,$$

having a common point $O(0,0)$, that is,

$$a = a'$$

and

$$b = b'.$$

Let h', k' be any two distinct half-rays

$$h' \subset a',$$
$$h' : \big[(x,y) \in E_2W_n : x > 0\big],$$

that is,

$$h' = h,$$

and

$$k' \subset b',$$
$$k' : \big[(x,y) \in E_2W_n : y < 0\big],$$

emanating from the point O. So we get $\angle(h', k')$.

Also, we have two straight lines $a'', b'' \in E_2W_n$,

$$a'' : y = 0$$

and

$$b'' : x = 0,$$

having a common point $O(0,0)$, that is,

$$a = a''$$

and

$$b = b'',$$

and two distinct half-rays

$$h'' \subset a'',$$
$$h'' : \big[(x,y) \in E_2W_n : x < 0\big],$$

and

$$k'' \subset b'',$$
$$k'' : \big[(x,y) \in E_2W_n : y > 0\big],$$

that is,

$$k'' = k,$$

emanating from the point O. So we get $\angle(h'', k'')$.

Let us take the points

$$A(1,0) \in h,$$
$$B(0,1) \in k,$$
$$A'(1,0) \in h',$$
$$B'(0,-1) \in k',$$

and the corresponding vectors

$$\mathbf{OA} = (1,0),$$
$$\mathbf{OB} = (0,1),$$
$$\mathbf{OA}' = (1,0),$$
$$\mathbf{OB}' = (0,-1).$$

We get

$$(\mathbf{OA}, \mathbf{OA}) = 1 > 0,$$
$$(\mathbf{OB}, \mathbf{OB}) = 1 > 0,$$
$$\left(\mathbf{OA}', \mathbf{OA}'\right) = 1 > 0,$$
$$\left(\mathbf{OB}', \mathbf{OB}'\right) = 1 > 0,$$

and

$$OA \equiv OA',$$
$$OB \equiv OB'.$$

We have the scalar products

$$(\mathbf{OA}, \mathbf{OB}) = 1 \times_n 0 +_n 0 \times_n 1 = 0,$$
$$\left(\mathbf{OA}', \mathbf{OB}'\right) = 1 \times_n 0 +_n 0 \times_n (-1) = 0,$$

and their equality

$$(\mathbf{OA}, \mathbf{OB}) = \left(\mathbf{OA}', \mathbf{OB}'\right) = 0,$$

that is,

$$\angle(Ah, Bk) \equiv \angle\left(A'h', B'k'\right).$$

Let us take the points

$$A(1,0)\in h,$$
$$B(0,1)\in k,$$
$$A''(-1,0)\in h'',$$
$$B''(0,1)\in k'',$$

and the corresponding vectors

$$\mathbf{OA}= (1,0),$$
$$\mathbf{OB}= (0,1),$$
$$\mathbf{OA''}= (-1,0),$$
$$\mathbf{OB''}= (0,1).$$

We get

$$(\mathbf{OA},\mathbf{OA})= 1 > 0,$$
$$(\mathbf{OB},\mathbf{OB})= 1 > 0,$$
$$\left(\mathbf{OA''},\mathbf{OA''}\right)= 1 > 0,$$
$$\left(\mathbf{OB''},\mathbf{OB''}\right)= 1 > 0,$$

and

$$OA\equiv OA'',$$
$$OB\equiv OB''.$$

We have the scalar products

$$(\mathbf{OA},\mathbf{OB})= 1 \times_n 0 +_n 0 \times_n 1 = 0,$$
$$\left(\mathbf{OA''},\mathbf{OB''}\right)= (-1) \times_n 0 +_n 0 \times_n (1) = 0,$$

and their equality

$$(\mathbf{OA},\mathbf{OB}) = \left(\mathbf{OA''},\mathbf{OB''}\right) = 0,$$

that is,

$$\angle(Ah,Bk) \equiv \angle\left(A''h'',B''k''\right).$$

Let us take the points

$$A'(1,0)\in h',$$
$$B'(0,-1)\in k',$$
$$A''(-1,0)\in h'',$$
$$B''(0,1)\in k'',$$

and the corresponding vectors

$$\mathbf{OA}'= (1,0),$$
$$\mathbf{OB}'= (0,-1),$$
$$\mathbf{OA}''= (-1,0),$$
$$\mathbf{OB}''= (0,1).$$

We get

$$\left(\mathbf{OA}',\mathbf{OA}'\right)= 1 > 0,$$
$$\left(\mathbf{OB}',\mathbf{OB}'\right)= 1 > 0,$$
$$\left(\mathbf{OA}'',\mathbf{OA}''\right)= 1 > 0,$$
$$\left(\mathbf{OB}'',\mathbf{OB}''\right)= 1 > 0,$$

and

$$OA'\equiv OA'',$$
$$OB'\equiv OB''.$$

We have the scalar products

$$\left(\mathbf{OA}',\mathbf{OB}'\right)= 1 \times_n 0 +_n 0 \times_n (-1) = 0,$$
$$\left(\mathbf{OA}'',\mathbf{OB}''\right)= (-1) \times_n 0 +_n 0 \times_n (1) = 0,$$

and their equality

$$\left(\mathbf{OA}',\mathbf{OB}'\right) = \left(\mathbf{OA}'',\mathbf{OB}''\right) = 0,$$

that is,

$$\angle\left(A'h',B'k'\right) \equiv \angle\left(A''h'',B''k''\right).$$

This means that in this case the answer to the question is positive.

2) Let us consider E_2W_2, and let two straight lines $a, b \in E_2W_2$,

$$a : x -_2 a_2 y = 0$$

and

$$b : 1.01 \times_2 x -_2 y = 0,$$

have a common point $O(0,0)$.

Let *h*, *k* be any two distinct half-rays,

$$h \subset a,$$
$$h : \left[(x,y) \in a : x > 0\right],$$

and

$$k \subset b,$$
$$k : \left[(x, y) \in b : x > 0\right],$$

emanate from the point O. So we get $\angle(h, k)$.

Also, let $a', b' \in E_2W_2$ be another second pair of straight lines

$$a' : 99.99 \times_2 x -_2 98.37 \times_2 y = 0$$

and

$$b' : 98.37 \times_2 x -_2 99.99 \times_n y = 0$$

having a common point $O(0, 0)$.

Let h', k' are any two distinct half-rays

$$h' \subset a',$$
$$h' : \left[(x, y) \in a' : x > 0\right],$$

and

$$k' \subset b',$$
$$k' : \left[(x, y) \in b' : x > 0\right],$$

emanating from the point O. So we get $\angle(h', k')$.

Let $a'', b'' \in E_2W_2$ be the third pair of straight lines

$$a'' : 99.99 \times_2 x -_2 96.37 \times_2 y = 0$$

and

$$b'' : 96.37 \times_2 x -_2 99.99 \times_2 y = 0$$

having the common point $O(0, 0)$, and let two distinct half-rays

$$h'' \subset a'',$$
$$h'' : \left[(x, y) \in a'' : x > 0\right],$$

and

$$k'' \subset b'',$$
$$k'' : \left[(x, y) \in b'' : x > 0\right],$$

emanate from the point O. So, we get $\angle(h'', k'')$.

Let us take the points

$$A(0.62,0.62) \in h,$$
$$B(0.62,0.62) \in k,$$
$$A'(0.62,0.63) \in h',$$
$$B'(0.63,0.62) \in k',$$

and the corresponding vectors

$$\mathbf{OA} = (0.62,0.62),$$
$$\mathbf{OB} = (0.62,0.62),$$
$$\mathbf{OA}' = (0.62,0.63),$$
$$\mathbf{OB}' = (0.63,0.62).$$

We get

$$(\mathbf{OA},\mathbf{OA}) = 0.72 > 0,$$
$$(\mathbf{OB},\mathbf{OB}) = 0.72 > 0,$$
$$\left(\mathbf{OA}',\mathbf{OA}'\right) = 0.72 > 0,$$
$$\left(\mathbf{OB}',\mathbf{OB}'\right) = 0.72 > 0,$$

and

$$OA \equiv OA',$$
$$OB \equiv OB'.$$

We have the scalar products

$$(\mathbf{OA},\mathbf{OB}) = 0.62 \times_n 0.62 +_n 0.62 \times_n 0.62 = 0.72,$$
$$\left(\mathbf{OA}',\mathbf{OB}'\right) = 0.62 \times_n 0.63 +_n 0.63 \times_n 0.62 = 0.72,$$

and their equality

$$(\mathbf{OA},\mathbf{OB}) = \left(\mathbf{OA}',\mathbf{OB}'\right) = 0.72,$$

that is,

$$\angle(Ah,Bk) \equiv \angle\left(A'h',B'k'\right)$$

Let us take the points

$$A(0.55,0.55) \in h,$$
$$B(0.55,0.55) \in k,$$
$$A''(0.54,0.56) \in h'',$$
$$B''(0.56,0.54) \in k'',$$

and the corresponding vectors

$$\mathbf{OA}= (0.55, 0.55),$$
$$\mathbf{OB}= (0.55, 0.55),$$
$$\mathbf{OA}''= (0.54, 0.56),$$
$$\mathbf{OB}''= (0.56, 0.54).$$

We get

$$(\mathbf{OA}, \mathbf{OA})= 0.50 > 0,$$
$$(\mathbf{OB}, \mathbf{OB})= 0.50 > 0,$$
$$\left(\mathbf{OA}'', \mathbf{OA}''\right)= 0.50 > 0,$$
$$\left(\mathbf{OB}'', \mathbf{OB}''\right)= 0.50 > 0,$$

and

$$OA\equiv OA'',$$
$$OB\equiv OB''.$$

We have the scalar products

$$(\mathbf{OA}, \mathbf{OB})= 0.55 \times_n 0.55 +_n 0.55 \times_n 0.55 = 0.50,$$
$$\left(\mathbf{OA}'', \mathbf{OB}''\right)= 0.54 \times_n 0.\, 0.\, 56 +_n 0.56 \times_n 0.54 = 0.50,$$

and their equality

$$(\mathbf{OA}, \mathbf{OB}) = \left(\mathbf{OA}'', \mathbf{OB}''\right) = 0.50,$$

that is,

$$\angle(Ah, Bk) \equiv \angle\left(A''h'', B''k''\right).$$

Let us now consider $\angle(h', k')$ and $\angle(h'', k'')$. Direct calculations give us the following results:

$$h' = C'(0.62, 0.63),$$

that is, h' contains only one point;

$$k' = D'(0.63, 0.62),$$

that is, k' contains only one point;

$$h'' = \left[E''(0.28, 0.29), F''(0.54, 0.56), G''(0.80, 0.83)\right],$$

that is, h'' contains only three points;

$$k'' = \left[H''(0.29, 0.28), I''(0.56, 0.54), J''(0.83, 0.80)\right],$$

that is, k'' contains only three points.

We have the vectors

$$\mathbf{OC}' = (0.62, 0.63),$$
$$\mathbf{OD}' = (0.63, 0.62),$$
$$\mathbf{OE}'' = (0.28, 0.29),$$
$$\mathbf{OF}'' = (0.54, 0.56),$$
$$\mathbf{OG}'' = (0.80, 0.83),$$
$$\mathbf{OH}'' = (0.29, 0.28),$$
$$\mathbf{OI}'' = (0.56, 0.54),$$
$$\mathbf{OJ}'' = (0.83, 0.80),$$

and we get

$$\left(\mathbf{OC}', \mathbf{OC}'\right) = 0.72 > 0,$$
$$\left(\mathbf{OD}', \mathbf{OD}'\right) = 0.72 > 0,$$
$$\left(\mathbf{OE}'', \mathbf{OE}''\right) = 0.08 > 0,$$
$$\left(\mathbf{OF}'', \mathbf{OF}''\right) = 0.50 > 0,$$
$$\left(\mathbf{OG}'', \mathbf{OG}''\right) = 1.28 > 0,$$
$$\left(\mathbf{OH}'', \mathbf{OH}''\right) = 0.08 > 0,$$
$$\left(\mathbf{OI}'', \mathbf{OI}''\right) = 0.50 > 0,$$
$$\left(\mathbf{OJ}'', \mathbf{OJ}''\right) = 1.28 > 0.$$

So

$$\left(\mathbf{OC}', \mathbf{OC}'\right) \neq \left(\mathbf{OE}'', \mathbf{OE}''\right),$$
$$\left(\mathbf{OC}', \mathbf{OC}'\right) \neq \left(\mathbf{OF}'', \mathbf{OF}''\right),$$
$$\left(\mathbf{OC}', \mathbf{OC}'\right) \neq \left(\mathbf{OG}'', \mathbf{OG}''\right),$$

and

$$\left(\mathbf{OD}', \mathbf{OD}'\right) \neq \left(\mathbf{OH}'', \mathbf{OH}''\right),$$
$$\left(\mathbf{OD}', \mathbf{OD}'\right) \neq \left(\mathbf{OI}'', \mathbf{OI}''\right)$$
$$\left(\mathbf{OD}', \mathbf{OD}'\right) \neq \left(\mathbf{OJ}'', \mathbf{OJ}''\right).$$

This means that

$$\angle(C'h', D'k') \not\equiv \angle(E''h'', H''k''),$$
$$\angle(C'h', D'k') \not\equiv \angle(E''h'', I''k''),$$
$$\angle(C'h', D'k') \not\equiv \angle(E''h'', J''k''),$$
$$\angle(C'h', D'k') \not\equiv \angle(F''h'', H''k''),$$
$$\angle(C'h', D'k') \not\equiv \angle(F''h'', I''k''),$$
$$\angle(C'h', D'k') \not\equiv \angle(F''h'', J''k''),$$
$$\angle(C'h', D'k') \not\equiv \angle(G''h'', H''k''),$$
$$\angle(C'h', D'k') \not\equiv \angle(G''h'', I''k''),$$
$$\angle(C'h', D'k') \not\equiv \angle(G''h'', J''k''),$$

so that in this case the answer to the question is negative.

So we have proved the following:

Theorem 7.18.
In Mathematics with Observers geometry in the plane E_2W_n, there are three distinct angles $\angle(h,k)$, $\angle(h',k')$, $\angle(h'',k'')$ with $\angle(h,k) \equiv \angle(h',k')$ and $\angle(h,k) \equiv \angle(h'',k'')$ such that $\angle(h',k') \equiv \angle(h'',k'')$.

Theorem 7.19.
In Mathematics with Observers geometry in the plane E_2W_n, there are three distinct angles $\angle(h,k)$, $\angle(h',k')$, $\angle(h'',k'')$ with $\angle(h,k) \equiv \angle(h',k')$ and $\angle(h,k) \equiv \angle(h'',k'')$ such that $\angle(h',k') \not\equiv \angle(h'',k'')$.

7.8 Sixth property of congruence

Let (h,k) be two half-rays emanating from a vertex *A* of a triangle ABC and passing, respectively, through *B* and *C*. The angle $\angle(h,k)$ is then said to be the angle included by the sides AB and AC or the one opposite to the side BC in the triangle ABC. It contains all the interior points of the triangle ABC and is denoted by the symbol $\angle BAC$ or $\angle A$.

Classical geometry states:

> If in the two triangles ABC and $A'B'C'$, we have the congruences
> $$AB \equiv A'B', \quad AC \equiv A'C', \quad \angle A \equiv \angle A',$$
> then we also have the congruences
> $$\angle B \equiv \angle B', \quad \angle C \equiv \angle C'.$$

Question: Is this statement correct in Mathematics with Observers geometry?

1) Let us consider E_2W_2, and let $a, b \in E_2W_2$ be two straight lines

$$a : 99.99 \times_2 x -_2 98.37 \times_2 y = 0$$

and

$$b : 98.37 \times_2 x -_2 99.99 \times_n y = 0$$

have a common point $O(0,0)$.

Let *h*, *k* are be two distinct half-rays

$$h \subset a,$$
$$h : \left[(x,y) \in a : x > 0\right],$$

and

$$k \subset b,$$
$$k : \left[(x,y) \in b : x > 0\right],$$

emanating from the point *O*. So we get $\angle(h,k)$.

Let us take the points

$$A(0.62, 0.63) \in h,$$
$$B(0.63, 0.62) \in k,$$

and the corresponding vectors

$$\mathbf{OA} = (0.62, 0.63),$$
$$\mathbf{OB} = (0.63, 0.62).$$

We get

$$(\mathbf{OA}, \mathbf{OA}) = 0.72 > 0,$$
$$(\mathbf{OB}, \mathbf{OB}) = 0.72 > 0.$$

Let us build straight line *c* containing points *A*, *B*:

$$c : c_1 \times_2 x +_2 c_2 \times_2 y +_2 c_3 = 0$$

that is, we have

$$\begin{cases} c_1 \times_2 0.62 +_2 c_2 \times_2 0.63 +_2 c_3 = 0, \\ c_1 \times_2 0.63 +_2 c_2 \times_2 0.62 +_2 c_3 = 0. \end{cases}$$

So

$$\begin{cases} c_1 \times_2 0.62 +_2 c_2 \times_2 0.63 = c_1 \times_2 0.63 +_2 c_2 \times_2 0.62, \\ c_1 \times_2 0.63 +_2 c_2 \times_2 0.62 +_2 c_3 = 0, \end{cases}$$

and

$$\begin{cases} c_1 \times_2 0.62 -_2 c_1 \times_2 0.63 = c_2 \times_2 0.62 -_2 c_2 \times_2 0.63, \\ c_1 \times_2 0.63 +_2 c_2 \times_2 0.62 +_2 c_3 = 0. \end{cases}$$

Let us consider two different solutions of this system:

$$c_1 = 1,$$
$$c_2 = 1,$$
$$c_3 = -1.25,$$

and

$$c_1 = 1.02,$$
$$c_2 = 1.02,$$
$$c_3 = -1.25,$$

that is, the first line $c^{(1)}$ has the equation

$$c^{(1)} : x +_2 y -_2 1.25 = 0,$$

and the second line $c^{(2)}$ has the equation

$$c^{(2)} : 1.02 \times_2 x +_2 1.02 \times_2 y -_2 1.25 = 0.$$

So we get two half-rays

$$l^{(11)} \subset c^{(1)},$$
$$l^{(11)} : \left[(x, y) \in c^{(1)} : x > 0.62\right],$$

and

$$g \subset a,$$
$$g : \left[(x, y) \in a : x < 0.62\right],$$

emanating from the point *A*. So we get $\angle(l^{(11)}, g)$.

Also, we get two half-rays

$$l^{(12)} \subset c^{(1)},$$
$$l^{(12)} : \left[(x, y) \in c^{(1)} : x < 0.63\right],$$

and

$$f \subset b,$$
$$f : \left[(x, y) \in b : x < 0.63\right],$$

emanating from the point *B*. So we get $\angle(l^{(12)}, f)$.

So we get the triangle OAB with vertices *O*, *A*, *B* and two half-rays (h, k) emanating from the vertex *O*, passing, respectively, through *A* and *B*, and forming

$$\angle(h, k) = \angle O.$$

The half-rays $(l^{(11)}, g)$ emanate from the vertex *A*, pass, respectively, through *B* and *O*, and form

$$\angle\big(l^{(11)}, g\big) = \angle A.$$

The half-rays $\big(l^{(12)}, f\big)$ emanate from the vertex *B*, pass, respectively, through *A* and *O*, and form

$$\angle\big(l^{(12)}, f\big) = \angle B.$$

Let us now go to line $c^{(2)}$. In this case, we consider the same points *O*, *A*, *B* but mark them as O', A', B'. We do that because in Mathematics with Observers geometry, two different points of any plane do not define uniquely a straight line containing these points. Also, a triangle has to be considered as the figure formed by the set of three segments of straight lines. In our case, the triangle OAB is the figure formed by the set of three segments of lines *a*, *b*, $c^{(1)}$, and the triangle $O'A'B'$ is the figure formed by the set of three segments of lines *a*, *b*, $c^{(2)}$.

So we get two half-rays

$$l^{(21)} \subset c^{(2)},$$
$$l^{(21)} : \big[(x, y) \in c^{(2)} : x > 0.62\big],$$

and

$$g \subset a,$$
$$g : \big[(x, y) \in a : x < 0.62\big],$$

emanating from the point A'. So we get $\angle(l^{(21)}, g)$.

Also, we get two half-rays

$$l^{(22)} \subset c^{(2)},$$
$$l^{(22)} : \big[(x, y) \in c^{(2)} : x < 0.63\big],$$

and

$$f \subset b,$$
$$f : \big[(x, y) \in b : x < 0.63\big],$$

emanating from the point B'. So we get $\angle(l^{(22)}, f)$.

So we get the triangle $O'A'B'$ with vertices O', A', B' and two half-rays (h, k) emanating from vertex O', passing, respectively, through A' and B', and forming

$$\angle(h, k) = \angle O'.$$

Also, we get two half-rays $\big(l^{(21)}, g\big)$ emanating from the vertex A', passing, respectively, through B' and O', and forming

$$\angle\big(l^{(21)}, g\big) = \angle A',$$

and two half-rays $(l^{(22)}, f)$ emanating from the vertex B', passing, respectively, through A' and O', and forming

$$\angle\big(l^{(22)}, f\big) = \angle B'.$$

So in two triangles OAB and $O'A'B'$, we have the congruences

$$OA \equiv O'A', \quad OB \equiv O'B', \quad \angle AOB \equiv \angle A'O'B'.$$

Let us take three points

$$\begin{aligned} F(-0.62, -0.63) &\in g, \\ K(3.16, -1.91) &\in l^{(11)}, \\ K'(3.14, -1.93) &\in l^{(21)}, \end{aligned}$$

and the corresponding vectors

$$\begin{aligned} \mathbf{AF} &= (-1.24, -1.26), \\ \mathbf{A'F} &= (-1.24, -1.26), \\ \mathbf{AK} &= (2.54, -2.54), \\ \mathbf{A'K'} &= (2.52, -2.56). \end{aligned}$$

We get

$$\begin{aligned} (\mathbf{AF}, \mathbf{AF}) &= 3.08 > 0, \\ \big(\mathbf{A'F}, \mathbf{A'F}\big) &= 3.08 > 0, \\ (\mathbf{AK}, \mathbf{AK}) &= 12.82 > 0, \\ \big(\mathbf{A'K'}, \mathbf{A'K'}\big) &= 12.82 > 0, \end{aligned}$$

and

$$\begin{aligned} (\mathbf{AK}, \mathbf{AF}) &= 0.04, \\ \big(\mathbf{A'K'}, \mathbf{A'F}\big) &= 0.08. \end{aligned}$$

This means that

$$\angle KAF \not\equiv \angle K'A'F,$$

that is, in this case, we get the negative answer to the question.

2) Let us consider the same lines and half-rays as in case 1). Now we take three points

$$\begin{aligned} O(0, 0) &\in g, \\ K(3.16, -1.91) &\in l^{(11)}, \\ K'(3.14, -1.93) &\in l^{(21)}, \end{aligned}$$

and the corresponding vectors

$$\mathbf{A0} = (-0.62, -0.63),$$
$$\mathbf{A'O'} = (-0.62, -0.63),$$
$$\mathbf{AK} = (2.54, -2.54),$$
$$\mathbf{A'K'} = (2.52, -2.56),$$

and we get

$$(\mathbf{AO}, \mathbf{A0}) = 0.72 > 0,$$
$$(\mathbf{A'O'}, \mathbf{A'O'}) = 0.72 > 0,$$
$$(\mathbf{AK}, \mathbf{AK}) = 12.82 > 0,$$
$$(\mathbf{A'K'}, \mathbf{A'K'}) = 12.82 > 0,$$

and

$$(\mathbf{AK}, \mathbf{A0}) = 0.02,$$
$$(\mathbf{A'K'}, \mathbf{A'O'}) = 0.0.$$

This means that

$$\angle OAK \equiv \angle O'A'K',$$

and so the answer to the question in this case is positive.

3) Instead of line $c^{(2)}$, let us now consider line $c^{(3)}$. It contains points A, B satisfying

$$c^{(3)} : c_1 \times_2 x +_2 c_2 \times_2 y +_2 c_3 = 0,$$

that is, as in case 1) of this section,

$$\begin{cases} c_1 \times_2 0.62 +_2 c_2 \times_2 0.63 +_2 c_3 = 0, \\ c_1 \times_2 0.63 +_2 c_2 \times_2 0.62 +_2 c_3 = 0. \end{cases}$$

So

$$\begin{cases} c_1 \times_2 0.62 +_2 c_2 \times_2 0.63 = c_1 \times_2 0.63 +_2 c_2 \times_2 0.62, \\ c_1 \times_2 0.63 +_2 c_2 \times_2 0.62 +_2 c_3 = 0, \end{cases}$$

and

$$\begin{cases} c_1 \times_2 0.62 -_2 c_1 \times_2 0.63 = c_2 \times_2 0.62 -_2 c_2 \times_2 0.63, \\ c_1 \times_2 0.63 +_2 c_2 \times_2 0.62 +_2 c_3 = 0. \end{cases}$$

Now we consider the third solution of this system:

$$c_1 = 1.02,$$
$$c_2 = 1.03,$$
$$c_3 = -1.25,$$

that is, line $c^{(3)}$ has the equation

$$c^{(3)} : 1.02 \times_2 x +_2 1.03 \times_2 y -_2 1.25 = 0.$$

So, as in case 1) of this section, we have the triangle OAB with vertices O, A, B and two half-rays (h, k) emanating from vertex O, passing, respectively, through A and B, and forming

$$\angle(h, k) = \angle O.$$

Also, $(l^{(11)}, g)$ are two half-rays emanating from vertex A, passing, respectively, through B and O, and forming

$$\angle\big(l^{(11)}, g\big) = \angle A,$$

and $(l^{(12)}, f)$ are two half-rays emanating from vertex B, passing, respectively, through A and O, and forming

$$\angle\big(l^{(12)}, f\big) = \angle B.$$

Let us now go to line $c^{(3)}$. In this case, we consider the same points O, A, B but mark them as O'', A'', B''. Now we get two half-rays

$$\begin{aligned} &l^{(31)} \subset c^{(3)}, \\ &l^{(31)} : \big[(x, y) \in c^{(3)} : x > 0.62\big], \end{aligned}$$

and

$$\begin{aligned} &g \subset a, \\ &g : \big[(x, y) \in a : x < 0.62\big], \end{aligned}$$

emanating from the point A''. So we get $\angle(l^{(31)}, g)$.

Also, we get two half-rays

$$\begin{aligned} &l^{(32)} \subset c^{(3)}, \\ &l^{(32)} : \big[(x, y) \in c^{(3)} : x < 0.63\big], \end{aligned}$$

and

$$\begin{aligned} &f \subset b, \\ &f : \big[(x, y) \in b : x < 0.63\big], \end{aligned}$$

emanating from the point B''. So we get $\angle(l^{(32)}, f)$.

So we get the triangle $O''A''B''$ with vertices O'', A'', B'' and two half-rays (h, k) emanating from vertex O'', passing, respectively, through A'' and B'', and forming

$$\angle(h, k) = \angle O''.$$

Also, $(l^{(31)}, g)$ are two half-rays emanating from vertex A'', passing, respectively, through B'' and O'', and forming

$$\angle\left(l^{(31)}, g\right) = \angle A'',$$

and $\left(l^{(32)}, f\right)$ are two half-rays emanating from vertex B'', passing, respectively, through A'' and O'', and forming

$$\angle\left(l^{(32)}, f\right) = \angle B''.$$

So in the triangles OAB and $O''A''B''$, we have the congruences

$$OA \equiv O''A'', \quad OB \equiv O''B'', \quad \angle AOB \equiv \angle A''O''B''.$$

Let us take three points

$$F(-0.62, -0.63) \in g,$$
$$L(7.58, -6.33) \in l^{(11)},$$
$$L''(7.60, -6.31) \in l^{(31)},$$

and the corresponding vectors

$$\mathbf{AF} = (-1.24, -1.26),$$
$$\mathbf{A''F} = (-1.24, -1.26),$$
$$\mathbf{AL} = (6.96, -6.96),$$
$$\mathbf{A''L''} = (6.98, -6.94).$$

We get

$$(\mathbf{AF}, \mathbf{AF}) = 3.08 > 0,$$
$$\left(\mathbf{A''F}, \mathbf{A''F}\right) = 3.08 > 0,$$
$$(\mathbf{AL}, \mathbf{AL}) = 96.66 > 0,$$
$$\left(\mathbf{A''L''}, \mathbf{A''L''}\right) = 96.66 > 0,$$

and

$$(\mathbf{AL}, \mathbf{AF}) = 0.12,$$
$$\left(\mathbf{A''L''}, \mathbf{A''F}\right) = 0.08.$$

This means that these three points do not satisfy the conditions for

$$\angle OAB \equiv \angle O''A''B''.$$

Now we can take the last possible points $O, O'' \in a$, and instead of point *F*, we take again points *L*, L'':

$$O(0, 0) \in g,$$
$$O''(0, 0) \in g,$$
$$L(7.58, -6.33) \in l^{(11)},$$
$$L''(7.60, -6.31) \in l^{(31)},$$

and the corresponding vectors

$$\mathbf{A0} = (-0.62, -0.63),$$
$$\mathbf{A''O''} = (-0.62, -0.63).$$
$$\mathbf{AL} = (6.96, -6.96),$$
$$\mathbf{A''L''} = (6.98, -6.94).$$

We get

$$(\mathbf{AO}, \mathbf{A0}) = 0.72 > 0,$$
$$\left(\mathbf{A''O''}, \mathbf{A'O'}\right) = 0.72 > 0,$$
$$(\mathbf{AL}, \mathbf{AL}) = 96.66 > 0,$$
$$\left(\mathbf{A''L''}, \mathbf{A''L''}\right) = 96.66 > 0,$$

and

$$(\mathbf{AL}, \mathbf{A0}) = 0.06,$$
$$\left(\mathbf{A''L''}, \mathbf{A''O''}\right) = 0.06.$$

This means that there are three points with corresponding congruence

$$\angle OAL \equiv \angle O''A''L''.$$

So the answer to the question in this case is positive.

4) Again let us consider E_2W_2 and two straight lines in E_2W_2,

$$a : x = 0$$

and

$$b : y = 0,$$

having a common point $O(0, 0)$.

Let *h*, *k* be two distinct half-rays

$$h \subset a,$$
$$h : \left[(x, y) \in a : y > 0\right],$$

and

$$k \subset b,$$
$$k : \left[(x, y) \in b : x > 0\right],$$

emanating from the point *O*. So we get $\angle(h, k)$.

Let us take the points

$$A(0, 1) \in h,$$
$$B(1, 0) \in k$$

and the corresponding vectors

$$\mathbf{OA} = (0, 1),$$
$$\mathbf{OB} = (1, 0).$$

We get

$$(\mathbf{OA}, \mathbf{OA}) = 1 > 0,$$
$$(\mathbf{OB}, \mathbf{OB}) = 1 > 0,$$
$$(\mathbf{OA}, \mathbf{OB}) = 0.$$

Let us build straight line *c* containing points *A*, *B* that satisfy

$$c : c_1 \times_2 x +_2 c_2 \times_2 y +_2 c_3 = 0,$$

that is,

$$\begin{cases} c_1 \times_2 0 +_2 c_2 \times_2 1 +_2 c_3 = 0, \\ c_1 \times_2 1 +_2 c_2 \times_2 0 +_2 c_3 = 0. \end{cases}$$

So

$$\begin{cases} c_2 = c_1, \\ c_1 +_2 c_3 = 0. \end{cases}$$

Let us consider a solution of this system,

$$c_1 = 1,$$
$$c_2 = 1,$$
$$c_3 = -1,$$

that is, straight line *c* satisfying the equation

$$c : x +_2 y -_2 1 = 0.$$

So we get two half-rays

$$l^1 \subset c,$$
$$l^1 : \left[(x, y) \in c : x > 0\right],$$

and

$$g \subset a,$$
$$g : \left[(x, y) \in a : y < 1\right],$$

emanating from the point *A*. So we get $\angle(l^1, g)$.

Also, we get two half-rays

$$l^2 \subset c,$$
$$l^2 : \left[(x, y) \in c : y > 0\right],$$

and

$$f \subset b,$$
$$f : \left[(x, y) \in b : x < 1\right],$$

emanating from the point B. So we get $\angle(l^2, f)$.

So we get the triangle OAB with vertices O, A, B and two half-rays (h, k) emanating from vertex O, passing, respectively, through A and B, and forming

$$\angle(h, k) = \angle AOB.$$

Also, (l^1, g) are two half-rays emanating from vertex A, passing, respectively, through B and O, and forming

$$\angle\left(l^1, g\right) = \angle OAB,$$

and (l^2, f) are two half-rays emanating from vertex B, passing, respectively, through A and O, and forming

$$\angle\left(l^2, f\right) = \angle OBA.$$

We get the vectors

$$\mathbf{AB} = (-1, 1),$$
$$\mathbf{BA} = (1, -1),$$

and

$$(\mathbf{AO}, \mathbf{AB}) = -1,$$
$$(\mathbf{BO}, \mathbf{BA}) = -1.$$

Let us now we consider two other straight lines $a', b' \in E_2W_2$,

$$a' : x = 1$$

and

$$b' : y = 1,$$

having a common point $O'(1, 1)$. Let h', k' be two distinct half-rays,

$$h' \subset a',$$
$$h' : \left[(x, y) \in a' : y > 1\right],$$

and

$$k' \subset b',$$
$$k' : \left[(x, y) \in b' : x > 1\right],$$

emanating from the point O'. So we get $\angle(h', k')$. Let us take the points

$$A'(1,2)\in h',$$
$$B'(2,1)\in k',$$
and the corresponding vectors

$$\mathbf{O'A'}= (0,1),$$
$$\mathbf{O'B'}= (1,0).$$
We get

$$\big(\mathbf{O'A'},\mathbf{O'A'}\big)= 1 > 0,$$
$$\big(\mathbf{O'B'},\mathbf{O'B'}\big)= 1 > 0,$$
$$\big(\mathbf{O'A'},\mathbf{O'B'}\big)= 0.$$
Let us build straight line c' containing points A', B' satisfying

$$c : c_1' \times_2 x +_2 c_2' \times_2 y +_2 c_3' = 0,$$
that is,

$$\begin{cases} c_1' \times_2 0 +_2 c_2' \times_2 2 +_2 c_3' = 0, \\ c_1' \times_2 2 +_2 c_2' \times_2 0 +_2 c_3' = 0. \end{cases}$$
So

$$\begin{cases} c_2' = c_1', \\ 2 \times_2 c_1' +_2 c_3' = 0. \end{cases}$$
Let us consider a solution of this system,

$$c_1' = 1,$$
$$c_2' = 1,$$
$$c_3' = -2,$$
that is, straight line c' has the equation

$$c' : x +_2 y -_2 2 = 0.$$
So we get two half-rays

$$l'^1 \subset c',$$
$$l'^1 : \big[(x,y) \in c : x > 1\big],$$
and

$$g' \subset a',$$
$$g' : \big[(x,y) \in a' : y < 2\big],$$
emanating from the point A'. So we get $\angle(l'^1, g')$.

Also, we get two half-rays

$$l'^2 \subset c',$$
$$l'^2 : \big[(x, y) \in c' : y > 1\big],$$
and

$$f' \subset b',$$
$$f' : \big[(x, y) \in b' : x < 2\big],$$
emanating from the point B'. So we get $\angle(l'^2, f')$.

So we get the triangle $O'A'B'$ with vertices O', A', B' and two half-rays (h', k') emanating from vertex O', passing, respectively, through A' and B', and forming

$$\angle\big(h', k'\big) = \angle A'O'B'.$$
Also, (l'^1, g') are two half-rays emanating from vertex A', passing, respectively, through B' and O', and forming

$$\angle\big(l'^1, g'\big) = \angle O'A'B',$$
and (l'^2, f') are two half-rays emanating from vertex B', passing, respectively, through A' and O', and forming

$$\angle\big(l'^2, f'\big) = \angle O'B'A'.$$
We get the vectors

$$\mathbf{A'B'} = (-1, 1),$$
$$\mathbf{B'A'} = (1, -1),$$
and

$$\big(\mathbf{A'O'}, \mathbf{A'B'}\big) = -1,$$
$$\big(\mathbf{B'O'}, \mathbf{B'A'}\big) = -1.$$
So in the triangles OAB and $O'A'B'$, we have the congruences

$$OA \equiv O'A', \quad OB \equiv O'B', \quad \angle AOB \equiv \angle A'O'B',$$
and as we proved above,

$$\angle OAB \equiv \angle O'A'B', \quad \angle OBA \equiv \angle O'B'A'.$$
So the answer to the question in this case is positive.

So we have proved the following:

Theorem 7.20.
In Mathematics with Observers geometry in the plane E_2W_n there are two distinct triangles ABC and $A'B'C'$ with

$$AB \equiv A'B', \quad AC \equiv A'C', \quad \angle A \equiv \angle A'$$

such that

$$\angle B \equiv \angle B', \quad \angle C \equiv \angle C'.$$

Theorem 7.21.

In Mathematics with Observers geometry in the plane E_2W_n, there are two distinct triangles ABC and $A'B'C'$ with

$$AB \equiv A'B', \quad AC \equiv A'C', \quad \angle A \equiv \angle A'$$

such that

$$\angle B \not\equiv \angle B', \quad \angle C \not\equiv \angle C'.$$

7.9 Right angles theorem

Classical geometry states:

> “All right angles are congruent to one another.”

Question: Is this statement correct in Mathematics with Observers geometry?

1) Let us consider three straight lines $a, b, c \in E_2W_2$,

$$\begin{aligned} &a : 99.99 \times_2 x -_2 98.37 \times_2 y = 0, \\ &b : 99.99 \times_2 x +_2 98.37 \times_2 y = 0, \\ &c : x +_2 y = 0, \end{aligned}$$

having a common point $O(0, 0)$, and four distinct half-rays *h*, *k*, *l*, *m*:

$$\begin{aligned} &h \subset a, \\ &h : \left[(x, y) \in a : x > 0\right], \\ &k \subset a, \\ &k : \left[(x, y) \in a : x < 0\right], \\ &l \subset b, \\ &l : \left[(x, y) \in b : x > 0\right], \\ &m \subset c, \\ &m : \left[(x, y) \in c : x > 0\right], \end{aligned}$$

emanating from the point *O*. So we get $\angle(l, h)$, $\angle(l, k)$, $\angle(m, h)$, $\angle(m, k)$.

For lines *a*, *b*, all possible positive *x* form the set

$$\Phi = [0.01, 0.02, \ldots, 0.99, 1.00],$$

and we get

$$99.99 \times_2 \Phi = [0.99, 1.98, \ldots, 98.82, 99.99],$$

and all possible positive *y* form the set

$$\Psi = [0.01, 0.02, \ldots, 0.99, 1.00, 1.01],$$

and we get

$$98.37 \times_2 \Psi = [0.98, 1.96, \ldots, 97.29, 98.37, 99.35].$$

Direct calculations show that

$$99.99 \times_2 \Phi \cap 98.37 \times_2 \Psi = 61.92,$$

and we get only one point in the intersection of these two sets, that is,

$$x = 0.62; \quad y = 0.63.$$

That means that

$$\begin{aligned} h &= A(0.62, 0.63), \\ k &= B(-0.62, -0.63), \\ l &= C(0.62, -0.63). \end{aligned}$$

We have the vectors

$$\begin{aligned} \mathbf{OA} &= (0.62, 0.63), \\ \mathbf{OB} &= (-0.62, -0.63), \\ \mathbf{OC} &= (0.62, -0.63), \end{aligned}$$

and we get

$$\begin{aligned} (\mathbf{OA}, \mathbf{OA}) &= 0.72 > 0, \\ (\mathbf{OB}, \mathbf{OB}) &= 0.72 > 0, \\ (\mathbf{OC}, \mathbf{OC}) &= 0.72 > 0, \end{aligned}$$

and

$$\begin{aligned} (\mathbf{OC}, \mathbf{OA}) &= 0, \\ (\mathbf{OC}, \mathbf{OB}) &= 0. \end{aligned}$$

This means that $\angle(Cl, Ah)$, $\angle(Cl, Bk)$ are right angles and

$$\angle(Cl, Ah) \equiv \angle(Cl, Bk).$$

So in this case the answer to the question is positive.

2) Let us now consider straight lines *c*, and take point $D(0.62, 0.62) \in c$. We have the vector

$$\mathbf{OD} = (0.62, 0.62),$$

and we get

$$(\mathbf{OD}, \mathbf{OD}) = 0.72 > 0$$

and

$$(\mathbf{OD}, \mathbf{OA}) = 0,$$
$$(\mathbf{OD}, \mathbf{OB}) = 0.$$

This means that $\angle(Dm, Ah)$, $\angle(Dm, Bk)$ are right angles and

$$\angle(Dm, Ah) \equiv \angle(Dm, Bk).$$

Note that

$$l \not\subset m,$$

because

$$C \notin m.$$

This means that we have two distinct straight lines *b*, *c* perpendicular to line *a* in one point *O*. So in this case the answer to the question is positive.

3) Let us continue to consider straight lines *c*, and take another point $E(1.00, -1.00) \in c$. We have the vector

$$\mathbf{OE} = (1.00, -1.00).$$

We get

$$(\mathbf{OE}, \mathbf{OE}) = 2.00 > 0$$

and

$$(\mathbf{OE}, \mathbf{OA}) = -0.01,$$
$$(\mathbf{OE}, \mathbf{OB}) = 0.01.$$

So $\angle(Em, Ah)$, $\angle(Em, Bk)$ are not right angles, and

$$\angle(Em, Ah) \not\equiv \angle(Em, Bk).$$

This means that in this case the answer to the question is negative, and the rightness of an angle depends not only on lines forming this angle but also on the points in these lines.

So we have proved the following:

Theorem 7.22.
In Mathematics with Observers geometry in the plane E_2W_n, there are two distinct right angles that are congruent to each other.

Theorem 7.23.
In Mathematics with Observers geometry in the plane E_2W_n, there are two distinct right angles that are not congruent to each other.

7.10 Alternate angles theorem

Classical geometry states:

> "If two parallel lines are cut by a third straight line, then the alternate interior angles and also the exterior–interior angles are congruent. Conversely, if the alternate–interior or the exterior–interior angles are congruent, then the given lines are parallel."

Question: Is this statement correct in Mathematics with Observers geometry?

1) Let us consider three straight lines $a, b, c \in E_2W_2$,

$$
\begin{aligned}
&a : 1.26 \times_2 x -_2 1.01 \times_2 y = 0, \\
&b : x -_2 y = 0, \\
&c : x -_2 y +_2 0.27 = 0,
\end{aligned}
$$

which have a common point $O(0,0)$; lines *b*, *c* are parallel in the Euclidean sense.

Let us take three points $A, B, C \in E_2W_2$,

$$
\begin{aligned}
&A(1.12, 1.39) \in a \cap c, \\
&B(0.58, 0.58) \in b, \\
&C(0.61, 0.88) \in c,
\end{aligned}
$$

and four distinct half-rays *h*, *k*, *l*, *m*:

$$
\begin{aligned}
&h \subset a, \\
&h : \big[(x,y) \in a : x > 0\big], \\
&k \subset a, \\
&k : \big[(x,y) \in a : x < 1.12\big], \\
&l \subset b, \\
&l : \big[(x,y) \in b : x > 0\big], \\
&m \subset c, \\
&m : \big[(x,y) \in c : x < 1.12\big].
\end{aligned}
$$

Half-rays *h*, *l* emanate from the point *O*, and half-rays *k*, *m* emanate from the point *A*. So we get the alternate interior angles $\angle(h,l)$, $\angle(k,m)$.

We have the vectors

$$\mathbf{OA}= (1.12, 1.39),$$
$$\mathbf{AO}= (-1.12, -1.39),$$
$$\mathbf{OB}= (0.58, 0.58),$$
$$\mathbf{AC}= (-0.51, -0.51).$$

We get

$$(\mathbf{OA}, \mathbf{OA})= 3.12 > 0,$$
$$(\mathbf{AO}, \mathbf{AO})= 3.12 > 0,$$
$$(\mathbf{OB}, \mathbf{OB})= 0.50 > 0,$$
$$(\mathbf{AC}, \mathbf{AC})= 0.50 > 0,$$

and

$$(\mathbf{OA}, \mathbf{OB})= 1.36,$$
$$(\mathbf{AO}, \mathbf{AC})= 1.22.$$

This means that

$$\mathbf{OA}\equiv \mathbf{AO},$$
$$\mathbf{OB}\equiv \mathbf{AC},$$

but

$$\angle AOB \not\equiv \angle CAO.$$

So in this case the answer to the question is negative.

2) Let us now we have three straight lines $a, b, c \in E_2W_2$:

$$a : x = 0,$$
$$b : y = 0,$$
$$c : y -_2 1 = 0.$$

Lines *a*, *b* have a common point $O(0, 0)$, and lines *b*, *c* are parallel in the Euclidean sense.

Let us take three points $A, B, C \in E_2W_2$:

$$A(0.00, 1.00)\in a \cap c,$$
$$B(1.00, 0.00)\in b,$$
$$C(-1.00, 1.00)\in c,$$

and four distinct half-rays *h*, *k*, *l*, *m*:

$$
\begin{aligned}
&h \subset a,\\
&h : \left[(x,y) \in a : y > 0\right],\\
&k \subset a,\\
&k : \left[(x,y) \in a : y < 1.00\right],\\
&l \subset b,\\
&l : \left[(x,y) \in b : x > 0\right],\\
&m \subset c,\\
&m : \left[(x,y) \in c : x < 0.00\right].
\end{aligned}
$$

Half-rays *h*, *l* emanate from the point *O*, and half-rays *k*, *m* emanate from the point *A*. So we get the alternate interior angles $\angle(h,l)$, $\angle(k,m)$.

We have the vectors

$$
\begin{aligned}
&\mathbf{OA} = (0.00, 1.00),\\
&\mathbf{AO} = (0.00, -1.00),\\
&\mathbf{OB} = (1.00, 0.00)\\
&\mathbf{AC} = (-1.00, 0.00),
\end{aligned}
$$

and we get

$$
\begin{aligned}
&(\mathbf{OA}, \mathbf{OA}) = 1.00 > 0,\\
&(\mathbf{AO}, \mathbf{AO}) = 1.00 > 0,\\
&(\mathbf{OB}, \mathbf{OB}) = 1.00 > 0,\\
&(\mathbf{AC}, \mathbf{AC}) = 1.00 > 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&(\mathbf{OA}, \mathbf{OB}) = 0.00,\\
&(\mathbf{AO}, \mathbf{AC}) = 0.00.
\end{aligned}
$$

This means that

$$
\begin{aligned}
&\mathbf{OA} \equiv \mathbf{AO},\\
&\mathbf{OB} \equiv \mathbf{AC},
\end{aligned}
$$

and

$$\angle AOB \equiv \angle CAO.$$

So in this case the answer to the question is positive.

So we have proved the following:

Theorem 7.24.

In Mathematics with Observers geometry in the plane E_2W_n, there are two parallel lines that are cut by a third straight line such that the alternate interior angles and also the exterior–interior angles are *congruent.*

Theorem 7.25.

In Mathematics with Observers geometry in the plane E_2W_n*, there are two parallel lines that are cut by a third straight line such that the alternate interior angles and also the exterior–interior angles are not congruent.*

Conversely, if the alternate–interior or the exterior–interior angles are congruent, then the given lines may be parallel or nonparallel.

8 Analysis of observability and property of continuity (Archimedes' axiom)

Classical geometry states:

> Let A_1 be a point upon a straight line between arbitrarily chosen points *A* and *B*. Let $A_2, A_3, A_4, \ldots$ be points such that A_1 lies between *A* and A_2, A_2 lies between A_1 and A_3, A_3 lies between A_2 and A_4, etc. Moreover, let the segments
> $$AA_1, A_1A_2, A_2A_3, A_3A_4, \ldots$$
> be all equal. Then, among this series of points, there always exists a point A_n such that *B* lies between *A* and A_n.

Question: Is this statement correct in Mathematics with Observers geometry?

1) Let us take a straight line in E_2W_2,

$$a : 99.99 \times_2 x -_2 98.37 \times_2 y = 0,$$

and let us take the points

$$A(-0.62, -0.63), B(0.62, 0.63) \in a$$

and the point $A_1(0, 0) \in a$ between *A* and *B*.

As we can see from above, line a contains only three points A, B, A_1. So, among any series of points, there is no point A_n such that B lies between A and A_n. This means that in this case the answer to the question is negative.

2) Let us take a straight line in E_2W_2,

$$a : y = 0,$$

the points

$$A(-3, 0), B(3, 0) \in a,$$

and a point $A_1(0, 0) \in a$ between A and B. Let us take the points

$$A_2(1, 0), A_3(2, 0), A_3(3, 0), A_4(4, 0).$$

So A_1 lies between A and A_2, A_2 lies between A_1 and A_3, and A_3 lies between A_2 and A_4. Moreover, the segments

$$AA_1, A_1A_2, A_2A_3, A_3A_4$$

are congruent to each other, and among this series of points, there is a point

$$A_n, \quad n = 4,$$

such that B lies between A and A_n.

This means in this case the answer to the question is positive.

So we have proved the following:

Theorem 8.1.
In Mathematics with Observers geometry in the plane E_2W_n, there are a straight line a, points $A \in a$, $B \in a$, and $A_1, A_2, A_3, \ldots \in a$ such that A_1 lies between A and A_2, A_2 lies between A_1 and A_3, A_3 lies between A_2 and A_4, etc. Moreover, the segments $AA_1, A_1A_2, A_2A_3, A_3A_4, \ldots$ are equal to each other, and among this series of points, there is a point A_n such that B lies between A and A_n.

Theorem 8.2.
In Mathematics with Observers geometry in the plane E_2W_n, there are a straight line a, points $A \in a$, $B \in a$, and $A_1, A_2, A_3, \ldots \in a$ such that A_1 lies between A and A_2, A_2 lies between A_1 and A_3, A_3 lies between A_2 and A_4, etc. Moreover, the segments $AA_1, A_1A_2, A_2A_3, A_3A_4, \ldots$ are equal to each other, and that among this series of points, there is no point A_n such that B lies between A and A_n.

9 Observability and triangle

In classical Euclidean geometry a triangle is a figure formed by set of three distinct points *A*, *B*, *C* not belonging to one straight line in E_2W_n or E_3W_n. These three points are called vertices of the triangle. The segments connecting these three points are call the sides of the triangle. By classical geometry definition two triangles ABC and $A'B'C'$ are said to be congruent if the following congruences are fulfilled:

$$AB\equiv A'B',$$
$$AC\equiv A'C',$$
$$BC\equiv B'C',$$
$$\angle A\equiv \angle A',$$
$$\angle B\equiv \angle B',$$
$$\angle C\equiv \angle C'.$$

We have three classical Euclidean geometry statements.

Statement 1 (First theorem of congruence for triangles).

If for two triangles ABC and $A'B'C'$, the congruences

$$AB\equiv A'B',$$
$$AC\equiv A'C',$$
$$\angle A\equiv \angle A',$$

hold, then the two triangles are congruent.

Statement 2 (Second theorem of congruence for triangles).

If in any two triangles, one side and two adjacent angles are respectively congruent, then the triangles are congruent.

Statement 3 (Third theorem of congruence for triangles).

If two triangles have three sides of one triangle congruent to the corresponding three sides of the other, then the triangles are congruent.

In this section, we consider this definition and these statements from Mathematics with Observers point of view. Note that in Mathematics with Observers geometry, a straight line containing points *A*, *B* or *A*, *C* or *B*, *C* may not exist. Moreover, even if these lines exist, then they are not unique. The segment of an existing line connecting any two points is called a side of the triangle. So a triangle may not have one, two, or three sides, or may have several sides connecting some pairs of vertices. This means that when we deal with triangle ABC, it is necessary to know which sides we consider. Another situation is also possible: in Mathematics with Observers geometry, three sides exist, but vertices do not. In general, in Mathematics with Observers geometry, some sides exist, and some vertices exist too. We consider these situations in the next section.

9.1 Definition of congruence of triangles in Mathematics with Observers geometry: variant of vertices

For E_2W_n, first, let us consider six points

$$A(x_1,y_1), B(x_2,y_2), C(x_3,y_3), A'(x_4,y_4), B'(x_5,y_5), C'(x_6,y_6) \in E_2W_n,$$

where *A*, *B*, *C* are distinct points, A', B', C' are distinct points, and pairs of points

$$(A,B),(A,C),(B,C),\left(A',B'\right),\left(A',C'\right),\left(B',C'\right)$$

may be points of existing straight lines or may be not. For all these points, we have the corresponding vectors

$$\mathbf{a}=(x_1,y_1), \mathbf{b}=(x_2,y_2), \mathbf{c}=(x_3,y_3), \mathbf{a}'=(x_4,y_4), \mathbf{b}'=(x_5,y_5), \mathbf{c}'=(x_6,y_6) \in E_2W_n.$$

Let us consider the vectors in E_2W_n

$$\begin{aligned}
\mathbf{AB}&= \mathbf{b} -_n \mathbf{a} = (x_2 -_n x_1, y_2 -_n y_1),\\
\mathbf{AC}&= \mathbf{c} -_n \mathbf{a} = (x_3 -_n x_1, y_3 -_n y_1),\\
\mathbf{BC}&= \mathbf{c} -_n \mathbf{b} = (x_3 -_n x_2, y_3 -_n y_2),\\
\mathbf{A'B'}&= \mathbf{b}' -_n \mathbf{a}' = (x_5 -_n x_4, y_5 -_n y_4),\\
\mathbf{A'C'}&= \mathbf{c}' -_n \mathbf{a}' = (x_6 -_n x_4, y_6 -_n y_4),\\
\mathbf{B'C'}&= \mathbf{c}' -_n \mathbf{b}' = (x_6 -_n x_5, y_6 -_n y_5),
\end{aligned}$$

and the scalar products

$$
\begin{aligned}
(\mathbf{AB}, \mathbf{AB}) &= (x_2 -_n x_1) \times_n (x_2 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_2 -_n y_1), \\
(\mathbf{AC}, \mathbf{AC}) &= (x_3 -_n x_1) \times_n (x_3 -_n x_1) +_n (y_3 -_n y_1) \times_n (y_3 -_n y_1), \\
(\mathbf{BC}, \mathbf{BC}) &= (x_3 -_n x_2) \times_n (x_3 -_n x_2) +_n (y_3 -_n y_2) \times_n (y_3 -_n y_2), \\
\left(\mathbf{A'B'}, \mathbf{A'B'}\right) &= (x_5 -_n x_4) \times_n (x_5 -_n x_4) +_n (y_5 -_n y_4) \times_n (y_5 -_n y_4), \\
\left(\mathbf{A'C'}, \mathbf{A'C'}\right) &= (x_6 -_n x_4) \times_n (x_6 -_n x_4) +_n (y_6 -_n y_4) \times_n (y_6 -_n y_4), \\
\left(\mathbf{B'C'}, \mathbf{B'C'}\right) &= (x_6 -_n x_5) \times_n (x_6 -_n x_5) +_n (y_6 -_n y_5) \times_n (y_6 -_n y_5), \\
(\mathbf{AB}, \mathbf{AC}) &= (x_2 -_n x_1) \times_n (x_3 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_3 -_n y_1), \\
(\mathbf{AB}, \mathbf{BC}) &= (x_2 -_n x_1) \times_n (x_3 -_n x_2) +_n (y_2 -_n y_1) \times_n (y_3 -_n y_2), \\
(\mathbf{AC}, \mathbf{BC}) &= (x_3 -_n x_1) \times_n (x_3 -_n x_2) +_n (y_3 -_n y_1) \times_n (y_3 -_n y_2), \\
\left(\mathbf{A'B'}, \mathbf{A'C'}\right) &= (x_5 -_n x_4) \times_n (x_6 -_n x_4) +_n (y_5 -_n y_4) \times_n (y_6 -_n y_4), \\
\left(\mathbf{A'B'}, \mathbf{B'C'}\right) &= (x_5 -_n x_4) \times_n (x_6 -_n x_5) +_n (y_5 -_n y_4) \times_n (y_6 -_n y_5), \\
\left(\mathbf{A'C'}, \mathbf{B'C'}\right) &= (x_6 -_n x_4) \times_n (x_6 -_n x_5) +_n (y_6 -_n y_4) \times_n (y_6 -_n y_5)
\end{aligned}
$$

if

$$x_2 -_n x_1 \in W_n,$$
$$y_2 -_n y_1 \in W_n,$$
$$x_3 -_n x_1 \in W_n,$$
$$y_3 -_n y_1 \in W_n,$$
$$x_3 -_n x_2 \in W_n,$$
$$y_3 -_n y_2 \in W_n,$$
$$x_5 -_n x_4 \in W_n,$$
$$y_5 -_n y_4 \in W_n,$$
$$x_6 -_n x_4 \in W_n,$$
$$y_6 -_n y_4 \in W_n,$$
$$x_6 -_n x_5 \in W_n,$$
$$y_6 -_n y_5 \in W_n,$$
$$(x_2 -_n x_1) \times_n (x_2 -_n x_1) \in W_n,$$
$$(y_2 -_n y_1) \times_n (y_2 -_n y_1) \in W_n,$$
$$(x_3 -_n x_1) \times_n (x_3 -_n x_1) \in W_n,$$
$$(y_3 -_n y_1) \times_n (y_3 -_n y_1) \in W_n,$$
$$(x_3 -_n x_2) \times_n (x_3 -_n x_2) \in W_n,$$
$$(y_3 -_n y_2) \times_n (y_3 -_n y_2) \in W_n,$$
$$(x_5 -_n x_4) \times_n (x_5 -_n x_4) \in W_n,$$
$$(y_5 -_n y_4) \times_n (y_5 -_n y_4) \in W_n,$$
$$(x_6 -_n x_4) \times_n (x_6 -_n x_4) \in W_n,$$
$$(y_6 -_n y_4) \times_n (y_6 -_n y_4) \in W_n,$$
$$(x_6 -_n x_5) \times_n (x_6 -_n x_5) \in W_n,$$
$$(y_6 -_n y_5) \times_n (y_6 -_n y_5) \in W_n,$$
$$(x_2 -_n x_1) \times_n (x_3 -_n x_1) \in W_n,$$
$$(y_2 -_n y_1) \times_n (y_3 -_n y_1) \in W_n,$$
$$(x_2 -_n x_1) \times_n (x_3 -_n x_2) \in W_n,$$
$$(y_2 -_n y_1) \times_n (y_3 -_n y_2) \in W_n,$$
$$(x_3 -_n x_1) \times_n (x_3 -_n x_2) \in W_n,$$

$$
\begin{aligned}
&(y_3 -_n y_1) \times_n (y_3 -_n y_2) \in W_n,\\
&(x_5 -_n x_4) \times_n (x_6 -_n x_4) \in W_n,\\
&(y_5 -_n y_4) \times_n (y_6 -_n y_4) \in W_n,\\
&(x_5 -_n x_4) \times_n (x_6 -_n x_5) \in W_n,\\
&(y_5 -_n y_4) \times_n (y_6 -_n y_5),\\
&(x_6 -_n x_4) \times_n (x_6 -_n x_5) \in W_n,\\
&(y_6 -_n y_4) \times_n (y_6 -_n y_5) \in W_n,\\
&(x_2 -_n x_1) \times_n (x_2 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_2 -_n y_1) \in W_n,\\
&(x_3 -_n x_1) \times_n (x_3 -_n x_1) +_n (y_3 -_n y_1) \times_n (y_3 -_n y_1) \in W_n,\\
&(x_3 -_n x_2) \times_n (x_3 -_n x_2) +_n (y_3 -_n y_2) \times_n (y_3 -_n y_2) \in W_n,\\
&(x_5 -_n x_4) \times_n (x_5 -_n x_4) +_n (y_5 -_n y_4) \times_n (y_5 -_n y_4) \in W_n,\\
&(x_6 -_n x_4) \times_n (x_6 -_n x_4) +_n (y_6 -_n y_4) \times_n (y_6 -_n y_4) \in W_n,\\
&(x_6 -_n x_5) \times_n (x_6 -_n x_5) +_n (y_6 -_n y_5) \times_n (y_6 -_n y_5) \in W_n,\\
&(x_2 -_n x_1) \times_n (x_3 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_3 -_n y_1) \in W_n,\\
&(x_2 -_n x_1) \times_n (x_3 -_n x_2) +_n (y_2 -_n y_1) \times_n (y_3 -_n y_2) \in W_n,\\
&(x_3 -_n x_1) \times_n (x_3 -_n x_2) +_n (y_3 -_n y_1) \times_n (y_3 -_n y_2) \in W_n,\\
&(x_5 -_n x_4) \times_n (x_6 -_n x_4) +_n (y_5 -_n y_4) \times_n (y_6 -_n y_4) \in W_n,\\
&(x_5 -_n x_4) \times_n (x_6 -_n x_5) +_n (y_5 -_n y_4) \times_n (y_6 -_n y_5) \in W_n,\\
&(x_6 -_n x_4) \times_n (x_6 -_n x_5) +_n (y_6 -_n y_4) \times_n (y_6 -_n y_5) \in W_n.
\end{aligned}
$$

Now we give the definition of congruence of triangles in Mathematics with Observers geometry. First of all, three points *A*, *B*, *C* form triangle $\triangle ABC$, and three points A', B', C' form a triangle $\triangle A'B'C'$ if

$$\mathbf{AB} \nparallel \mathbf{AC}, \quad \mathbf{A'B'} \nparallel \mathbf{A'C'}.$$

This means that there is no element $\alpha \in W_n$ such that

$$\mathbf{AB} = \alpha \times_n \mathbf{AC}$$

or there is no element $\beta \in W_n$ such that

$$\mathbf{AC} = \beta \times_n \mathbf{AB}.$$

Likewise, this means that there is no element $\gamma \in W_n$ such that

$$\mathbf{A'B'} = \gamma \times_n \mathbf{A'C'},$$

or there is no element $\delta \in W_n$ such that

$$\mathbf{A'C'} = \delta \times_n \mathbf{A'B'}.$$

We call two triangles $\triangle ABC$ and $\triangle A'B'C'$ congruent in Mathematics with Observers if the following conditions are satisfied:

$$
\begin{aligned}
(\mathbf{AB},\mathbf{AB}) &= \left(\mathbf{A'B'},\mathbf{A'B'}\right) > 0,\\
(\mathbf{AC},\mathbf{AC}) &= \left(\mathbf{A'C'},\mathbf{A'C'}\right) > 0,\\
(\mathbf{BC},\mathbf{BC}) &= \left(\mathbf{B'C'},\mathbf{B'C'}\right) > 0,\\
(\mathbf{AB},\mathbf{AC}) &= \left(\mathbf{A'B'},\mathbf{A'C'}\right),\\
(\mathbf{BA},\mathbf{BC}) &= \left(\mathbf{B'A'},\mathbf{B'C'}\right),\\
(\mathbf{CA},\mathbf{CB}) &= \left(\mathbf{C'A'},\mathbf{C'B'}\right).
\end{aligned}
$$

Let us now go to the thee-dimensional case. For E_3W_n, first, let us consider six points

$$A(x_1,y_1,z_1),B(x_2,y_2,z_2),C(x_3,y_3,z_3),A'(x_4,y_4,z_4),B'(x_5,y_5,z_5),C'(x_6,y_6,z_6)\in E_3W_n$$

where A, B, C are distinct points, A', B', C' are distinct points, and the pairs of points

$$(A,B),(A,C),(B,C),\left(A',B'\right),\left(A',C'\right),\left(B',C'\right)$$

may be the points of existing straight lines or may be not. For these points, we have the corresponding vectors

$$
\begin{aligned}
\mathbf{a} &= (x_1,y_1,z_1),\mathbf{b}=(x_2,y_2,z_2),\mathbf{c}=(x_3,y_3,z_3),\\
\mathbf{a}' &= (x_4,y_4,z_4),\mathbf{b}'=(x_5,y_5,z_5),\mathbf{c}'=(x_6,y_6,z_6),\in E_3W_n.
\end{aligned}
$$

Let us consider the vectors in E_3W_n

$$
\begin{aligned}
\mathbf{AB} &= \mathbf{b} -_n \mathbf{a} = (x_2 -_n x_1, y_2 -_n y_1, z_2 -_n z_1)\\
\mathbf{AC} &= \mathbf{c} -_n \mathbf{a} = (x_3 -_n x_1, y_3 -_n y_1, z_3 -_n z_1)\\
\mathbf{BC} &= \mathbf{c} -_n \mathbf{b} = (x_3 -_n x_2, y_3 -_n y_2, z_3 -_n z_2),\\
\mathbf{A'B'} &= \mathbf{b}' -_n \mathbf{a}' = (x_5 -_n x_4, y_5 -_n y_4, z_5 -_n z_4),\\
\mathbf{A'C'} &= \mathbf{c}' -_n \mathbf{a}' = (x_6 -_n x_4, y_6 -_n y_4, z_6 -_n z_4),\\
\mathbf{B'C'} &= \mathbf{c}' -_n \mathbf{b}' = (x_6 -_n x_5, y_6 -_n y_5, z_6 -_n z_5),
\end{aligned}
$$

and the scalar products

$$
\begin{aligned}
(\mathbf{AB},\mathbf{AB}) &= (x_2 -_n x_1) \times_n (x_2 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_2 -_n y_1) +_n (z_2 -_n z_1) \\
(\mathbf{AC},\mathbf{AC}) &= (x_3 -_n x_1) \times_n (x_3 -_n x_1) +_n (y_3 -_n y_1) \times_n (y_3 -_n y_1) +_n (z_3 -_n z_1) \\
(\mathbf{BC},\mathbf{BC}) &= (x_3 -_n x_2) \times_n (x_3 -_n x_2) +_n (y_3 -_n y_2) \times_n (y_3 -_n y_2) +_n (z_3 -_n z_2) \\
(\mathbf{A'B'},\mathbf{A'B'}) &= (x_5 -_n x_4) \times_n (x_5 -_n x_4) +_n (y_5 -_n y_4) \times_n (y_5 -_n y_4) +_n (z_5 -_n z_4) \\
(\mathbf{A'C'},\mathbf{A'C'}) &= (x_6 -_n x_4) \times_n (x_6 -_n x_4) +_n (y_6 -_n y_4) \times_n (y_6 -_n y_4) +_n (z_6 -_n z_4) \\
(\mathbf{B'C'},\mathbf{B'C'}) &= (x_6 -_n x_5) \times_n (x_6 -_n x_5) +_n (y_6 -_n y_5) \times_n (y_6 -_n y_5) +_n (z_6 -_n z_5) \\
(\mathbf{AB},\mathbf{AC}) &= (x_2 -_n x_1) \times_n (x_3 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_3 -_n y_1) +_n (z_2 -_n z_1) \\
(\mathbf{AB},\mathbf{BC}) &= (x_2 -_n x_1) \times_n (x_3 -_n x_2) +_n (y_2 -_n y_1) \times_n (y_3 -_n y_2) +_n (z_2 -_n z_1) \\
(\mathbf{AC},\mathbf{BC}) &= (x_3 -_n x_1) \times_n (x_3 -_n x_2) +_n (y_3 -_n y_1) \times_n (y_3 -_n y_2) +_n (z_3 -_n z_1) \\
(\mathbf{A'B'},\mathbf{A'C'}) &= (x_5 -_n x_4) \times_n (x_6 -_n x_4) +_n (y_5 -_n y_4) \times_n (y_6 -_n y_4) +_n (z_5 -_n z_4) \\
(\mathbf{A'B'},\mathbf{B'C'}) &= (x_5 -_n x_4) \times_n (x_6 -_n x_5) +_n (y_5 -_n y_4) \times_n (y_6 -_n y_5) +_n (z_5 -_n z_4) \\
(\mathbf{A'C'},\mathbf{B'C'}) &= (x_6 -_n x_4) \times_n (x_6 -_n x_5) +_n (y_6 -_n y_4) \times_n (y_6 -_n y_5) +_n (z_6 -_n z_4)
\end{aligned}
$$

if

$x_2 -_n x_1 \in W_n,$
$y_2 -_n y_1 \in W_n,$
$z_2 -_n z_1 \in W_n,$
$x_3 -_n x_1 \in W_n,$
$y_3 -_n y_1 \in W_n,$
$z_3 -_n z_1 \in W_n,$
$x_3 -_n x_2 \in W_n,$
$y_3 -_n y_2 \in W_n,$
$z_3 -_n z_2 \in W_n,$
$x_5 -_n x_4 \in W_n,$
$y_5 -_n y_4 \in W_n,$
$z_5 -_n z_4 \in W_n,$
$x_6 -_n x_4 \in W_n,$
$y_6 -_n y_4 \in W_n,$
$z_6 -_n z_4 \in W_n,$
$x_6 -_n x_5 \in W_n,$
$y_6 -_n y_5 \in W_n,$
$z_6 -_n z_5 \in W_n,$
$(x_2 -_n x_1) \times_n (x_2 -_n x_1) \in W_n,$
$(y_2 -_n y_1) \times_n (y_2 -_n y_1) \in W_n,$
$(z_2 -_n z_1) \times_n (z_2 -_n z_1) \in W_n,$
$(x_3 -_n x_1) \times_n (x_3 -_n x_1) \in W_n,$
$(y_3 -_n y_1) \times_n (y_3 -_n y_1) \in W_n,$

$(z_3 -_n z_1) \times_n (z_3 -_n z_1) \in W_n,$
$(x_3 -_n x_2) \times_n (x_3 -_n x_2) \in W_n,$
$(y_3 -_n y_2) \times_n (y_3 -_n y_2) \in W_n,$
$(z_3 -_n z_2) \times_n (z_3 -_n z_2) \in W_n,$
$(x_5 -_n x_4) \times_n (x_5 -_n x_4) \in W_n,$
$(y_5 -_n y_4) \times_n (y_5 -_n y_4) \in W_n,$
$(z_5 -_n z_4) \times_n (z_5 -_n z_4) \in W_n,$
$(x_6 -_n x_4) \times_n (x_6 -_n x_4) \in W_n,$
$(y_6 -_n y_4) \times_n (y_6 -_n y_4) \in W_n,$
$(z_6 -_n z_4) \times_n (z_6 -_n z_4) \in W_n,$
$(x_6 -_n x_5) \times_n (x_6 -_n x_5) \in W_n,$
$(y_6 -_n y_5) \times_n (y_6 -_n y_5) \in W_n,$
$(z_6 -_n z_5) \times_n (z_6 -_n z_5) \in W_n,$
$(x_2 -_n x_1) \times_n (x_3 -_n x_1) \in W_n,$
$(y_2 -_n y_1) \times_n (y_3 -_n y_1) \in W_n,$
$(z_2 -_n z_1) \times_n (z_3 -_n z_1) \in W_n,$
$(x_2 -_n x_1) \times_n (x_3 -_n x_2) \in W_n,$
$(y_2 -_n y_1) \times_n (y_3 -_n y_2) \in W_n,$
$(z_2 -_n z_1) \times_n (z_3 -_n z_2) \in W_n,$
$(x_3 -_n x_1) \times_n (x_3 -_n x_2) \in W_n,$
$(y_3 -_n y_1) \times_n (y_3 -_n y_2) \in W_n,$
$(z_3 -_n z_1) \times_n (z_3 -_n z_2) \in W_n,$
$(x_5 -_n x_4) \times_n (x_6 -_n x_4) \in W_n,$
$(y_5 -_n y_4) \times_n (y_6 -_n y_4) \in W_n,$
$(z_5 -_n z_4) \times_n (z_6 -_n z_4) \in W_n,$
$(x_5 -_n x_4) \times_n (x_6 -_n x_5) \in W_n,$
$(y_5 -_n y_4) \times_n (y_6 -_n y_5) \in W_n,$
$(z_5 -_n z_4) \times_n (z_6 -_n z_5) \in W_n,$
$(x_6 -_n x_4) \times_n (x_6 -_n x_5) \in W_n,$
$(y_6 -_n y_4) \times_n (y_6 -_n y_5) \in W_n,$
$(z_6 -_n z_4) \times_n (z_6 -_n z_5) \in W_n,$
$(x_2 -_n x_1) \times_n (x_2 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_2 -_n y_1) \in W_n,$
$(x_3 -_n x_1) \times_n (x_3 -_n x_1) +_n (y_3 -_n y_1) \times_n (y_3 -_n y_1) \in W_n,$
$(x_3 -_n x_2) \times_n (x_3 -_n x_2) +_n (y_3 -_n y_2) \times_n (y_3 -_n y_2) \in W_n,$

$$(x_5 -_n x_4) \times_n (x_5 -_n x_4) +_n (y_5 -_n y_4) \times_n (y_5 -_n y_4) \in W_n,$$
$$(x_6 -_n x_4) \times_n (x_6 -_n x_4) +_n (y_6 -_n y_4) \times_n (y_6 -_n y_4) \in W_n,$$
$$(x_6 -_n x_5) \times_n (x_6 -_n x_5) +_n (y_6 -_n y_5) \times_n (y_6 -_n y_5) \in W_n,$$
$$(x_2 -_n x_1) \times_n (x_3 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_3 -_n y_1) \in W_n,$$
$$(x_2 -_n x_1) \times_n (x_3 -_n x_2) +_n (y_2 -_n y_1) \times_n (y_3 -_n y_2) \in W_n,$$
$$(x_3 -_n x_1) \times_n (x_3 -_n x_2) +_n (y_3 -_n y_1) \times_n (y_3 -_n y_2) \in W_n,$$
$$(x_5 -_n x_4) \times_n (x_6 -_n x_4) +_n (y_5 -_n y_4) \times_n (y_6 -_n y_4) \in W_n,$$
$$(x_5 -_n x_4) \times_n (x_6 -_n x_5) +_n (y_5 -_n y_4) \times_n (y_6 -_n y_5) \in W_n,$$
$$(x_6 -_n x_4) \times_n (x_6 -_n x_5) +_n (y_6 -_n y_4) \times_n (y_6 -_n y_5) \in W_n,$$
$$(x_2 -_n x_1) \times_n (x_2 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_2 -_n y_1) +_n (z_2 -_n z_1) \times_n (z_2 -_n z_1) \in W_n,$$
$$(x_3 -_n x_1) \times_n (x_3 -_n x_1) +_n (y_3 -_n y_1) \times_n (y_3 -_n y_1) +_n (z_3 -_n z_1) \times_n (z_3 -_n z_1) \in W_n,$$
$$(x_3 -_n x_2) \times_n (x_3 -_n x_2) +_n (y_3 -_n y_2) \times_n (y_3 -_n y_2) +_n (z_3 -_n z_2) \times_n (z_3 -_n z_2) \in W_n,$$
$$(x_5 -_n x_4) \times_n (x_5 -_n x_4) +_n (y_5 -_n y_4) \times_n (y_5 -_n y_4) +_n (z_5 -_n z_4) \times_n (z_5 -_n z_4) \in W_n,$$
$$(x_6 -_n x_4) \times_n (x_6 -_n x_4) +_n (y_6 -_n y_4) \times_n (y_6 -_n y_4) +_n (z_6 -_n z_4) \times_n (z_6 -_n z_4) \in W_n,$$
$$(x_6 -_n x_5) \times_n (x_6 -_n x_5) +_n (y_6 -_n y_5) \times_n (y_6 -_n y_5) +_n (z_6 -_n z_5) \times_n (z_6 -_n z_5) \in W_n,$$
$$(x_2 -_n x_1) \times_n (x_3 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_3 -_n y_1) +_n (z_2 -_n z_1) \times_n (z_3 -_n z_1) \in W_n,$$
$$(x_2 -_n x_1) \times_n (x_3 -_n x_2) +_n (y_2 -_n y_1) \times_n (y_3 -_n y_2) +_n (z_2 -_n z_1) \times_n (z_3 -_n z_2) \in W_n,$$
$$(x_3 -_n x_1) \times_n (x_3 -_n x_2) +_n (y_3 -_n y_1) \times_n (y_3 -_n y_2) +_n (z_3 -_n z_1) \times_n (z_3 -_n z_2) \in W_n,$$
$$(x_5 -_n x_4) \times_n (x_6 -_n x_4) +_n (y_5 -_n y_4) \times_n (y_6 -_n y_4) +_n (z_5 -_n z_4) \times_n (z_6 -_n z_4) \in W_n$$
$$(x_5 -_n x_4) \times_n (x_6 -_n x_5) +_n (y_5 -_n y_4) \times_n (y_6 -_n y_5) +_n (z_5 -_n z_4) \times_n (z_6 -_n z_5) \in W_n,$$
$$(x_6 -_n x_4) \times_n (x_6 -_n x_5) +_n (y_6 -_n y_4) \times_n (y_6 -_n y_5) +_n (z_6 -_n z_4) \times_n (z_6 -_n z_5) \in W_n.$$

As we did in the two-dimensional case, we give the definition of congruence of triangles in Mathematics with Observers geometry. First, three points *A*, *B*, *C* form a triangle $\triangle ABC$, and three points A', B', C' form a triangle $\triangle A'B'C'$ if

$$\mathbf{AB} \nparallel \mathbf{AC}, \quad \mathbf{A'B'} \nparallel \mathbf{A'C'}.$$

Likewise, this means that there is no element $\gamma \in W_n$ such that

$$\mathbf{A'B'} = \gamma \times_n \mathbf{A'C'},$$

or there is no element $\delta \in W_n$ such that

$$\mathbf{A'C'} = \delta \times_n \mathbf{A'B'}.$$

Also, as in the two-dimensional case, we call two triangles $\triangle ABC$ and $\triangle A'B'C'$ congruent in Mathematics with Observers if the following conditions are satisfied:

$$
\begin{aligned}
(\mathbf{AB},\mathbf{AB})&= (\mathbf{A'B'},\mathbf{A'B'}) > 0,\\
(\mathbf{AC},\mathbf{AC})&= (\mathbf{A'C'},\mathbf{A'C'}) > 0,\\
(\mathbf{BC},\mathbf{BC})&= (\mathbf{B'C'},\mathbf{B'C'}) > 0,\\
(\mathbf{AB},\mathbf{AC})&= (\mathbf{A'B'},\mathbf{A'C'}),\\
(\mathbf{BA},\mathbf{BC})&= (\mathbf{B'A'},\mathbf{B'C'}),\\
(\mathbf{CA},\mathbf{CB})&= (\mathbf{C'A'},\mathbf{C'B'}).
\end{aligned}
$$

Note that for $a, b, c \in W_n$, we assume that

$$
a +_n b +_n c = (a +_n b) +_n c.
$$

9.2 Definition of congruence of triangles in Mathematics with Observers geometry: variant of sides

For E_2W_n or E_3W_n, first, let us consider six points *A*, *B*, *C*, A', B', C', where *A*, *B*, *C* are distinct points, and A', B', C' are distinct points, and the pairs of points (A, B), (A, C), (B, C), (A', B'), (A', C'), (B', C') are points of existing straight lines:

$$
\begin{aligned}
A, B&\in l,\\
A, C&\in m,\\
B, C&\in n\\
A', B'&\in l',\\
A', C'&\in m',\\
B', C'&\in n'.
\end{aligned}
$$

Let us consider vectors in E_2W_n or E_3W_n

$$
\mathbf{AB}, \mathbf{AC}, \mathbf{BC}, \mathbf{A'B'}, \mathbf{A'C'}, \mathbf{B'C'}
$$

and the scalar products

$$
(\mathbf{AB},\mathbf{AB}), (\mathbf{AC},\mathbf{AC}), (\mathbf{BC},\mathbf{BC}), (\mathbf{A'B'},\mathbf{A'B'}), (\mathbf{A'C'},\mathbf{A'C'}), (\mathbf{B'C'},\mathbf{B'C'}).
$$

We get

$$
\begin{aligned}
A&\in l \cap m,\\
B&\in l \cap n,\\
C&\in m \cap n,\\
A'&\in l' \cap m',\\
B'&\in l' \cap n',\\
C'&\in m' \cap n'.
\end{aligned}
$$

Note that, generally, such intersections of lines may have more than one point. Let h, k be any two distinct half-rays

$$A, B \in h \subset l,$$
$$A, C \in k \subset m$$

emanating from the point A, let p, q be any two distinct half-rays

$$B, A \in p \subset l,$$
$$B, C \in q \subset n$$

emanating from the point B, and let r, s be any two distinct half-rays

$$C, B \in r \subset n,$$
$$C, A \in s \subset m$$

emanating from the point C.

Let h', k' be any two distinct half-rays

$$A', B' \in h' \subset l',$$
$$A', C' \in k' \subset m'$$

emanating from the point A', let p', q' be any two distinct half-rays

$$B', A' \in p' \subset l',$$
$$B', C' \in q' \subset n'$$

emanating from the point B', and let r', s' be any two distinct half-rays

$$C', B' \in r' \subset n',$$
$$C', A' \in s' \subset m'$$

emanating from the point C'.

So we get six angles

$$\angle(h,k), \quad \angle(p,q), \quad \angle(r,s), \quad \angle\left(h',k'\right), \quad \angle\left(p',q'\right), \quad \angle\left(r',s'\right).$$

Now we are ready to formulate the second definition of congruence of triangles, from the point of view of sides. We write

$$\triangle ABC \equiv \triangle A'B'C'$$

if the following conditions are satisfied.

(1) The congruence of sides:

$$(\mathbf{AB},\mathbf{AB}) = \left(\mathbf{A'B'},\mathbf{A'B'}\right) > 0,$$
$$(\mathbf{AC},\mathbf{AC}) = \left(\mathbf{A'C'},\mathbf{A'C'}\right) > 0,$$
$$(\mathbf{BC},\mathbf{BC}) = \left(\mathbf{B'C'},\mathbf{B'C'}\right) > 0;$$

(2) There are four points

$$D \in h, \quad E \in k, \quad D' \in h', \quad E' \in k'$$

such that

$$(\mathbf{AD},\mathbf{AD}) = \left(\mathbf{A'D'},\mathbf{A'D'}\right) > 0,$$
$$(\mathbf{AE},\mathbf{AE}) = \left(\mathbf{A'E'},\mathbf{A'E'}\right) > 0,$$
$$(\mathbf{AD},\mathbf{AE}) = \left(\mathbf{A'D'},\mathbf{A'E'}\right);$$

(3) There are four points

$$F \in p, \quad G \in q, \quad F' \in p', \quad G' \in q'$$

such that

$$(\mathbf{BF},\mathbf{BF}) = \left(\mathbf{B'F'},\mathbf{B'F'}\right) > 0,$$
$$(\mathbf{BG},\mathbf{BG}) = \left(\mathbf{B'G'},\mathbf{B'G'}\right) > 0,$$
$$(\mathbf{BF},\mathbf{BG}) = \left(\mathbf{B'F'},\mathbf{B'G'}\right).$$

(4) There are four points

$$H \in r, \quad I \in s, \quad H' \in r', \quad s' \in q'$$

such that

$$(\mathbf{CH},\mathbf{CH}) = \left(\mathbf{C'H'},\mathbf{C'H'}\right) > 0,$$
$$(\mathbf{CI},\mathbf{CI}) = \left(\mathbf{C'I'},\mathbf{C'I'}\right) > 0,$$
$$(\mathbf{CH},\mathbf{CI}) = \left(\mathbf{C'H'},\mathbf{C'I'}\right).$$

We assume that all elements participating in the previous equalities belong to W_n.

Note that in the situation where three points *A*, *B*, *C* lie by pairs on straight lines and the other three points A', B', C' lie by pairs on straight lines, both variants (vertices and sides) work. If

$$\triangle ABC \equiv \triangle A'B'C'$$

by “vertex variant”, then the same holds by “side variant”. However, if

$$\triangle ABC \equiv \triangle A'B'C'$$

by “side variant”, then the same does not necessarily hold by “vertex variant”.

9.3 Triangles formed by two perpendiculars to one line

Let us consider E_2W_2. Suppose we have two straight lines in E_2W_2,

$$a : 99.99 \times_2 x -_2 98.37 \times_2 y = 0$$

and

$$b : 98.37 \times_2 x -_2 99.99 \times_n y = 0,$$

having a common point $O(0, 0)$.

Let *h*, *k* be two distinct half-rays

$$h \subset a,$$
$$h : \left[(x, y) \in a : x > 0\right],$$

and

$$k \subset b,$$
$$k : \left[(x, y) \in b : x > 0\right],$$

emanating from the point *O*. So we get $\angle(h, k)$.

Let us take the points

$$A(0.62, 0.63) \in h,$$
$$B(0.63, 0.62) \in k$$

and the corresponding vectors

$$\mathbf{OA} = (0.62, 0.63),$$
$$\mathbf{OB} = (0.63, 0.62).$$

We get

$$(\mathbf{OA}, \mathbf{OA}) = 0.72 > 0,$$
$$(\mathbf{OB}, \mathbf{OB}) = 0.72 > 0.$$

Let us build a straight line *c* containing points *A*, *B*:

$$c : c_1 \times_2 x +_2 c_2 \times_2 y +_2 c_3 = 0,$$

that is,

$$\begin{cases} c_1 \times_2 0.62 +_2 c_2 \times_2 0.63 +_2 c_3 = 0, \\ c_1 \times_2 0.63 +_2 c_2 \times_2 0.62 +_2 c_3 = 0. \end{cases}$$

So

$$\begin{cases} c_1 \times_2 0.62 +_2 c_2 \times_2 0.63 = c_1 \times_2 0.63 +_2 c_2 \times_2 0.62, \\ c_1 \times_2 0.63 +_2 c_2 \times_2 0.62 +_2 c_3 = 0, \end{cases}$$

and

$$\begin{cases} c_1 \times_2 0.62 -_2 c_1 \times_2 0.63 = c_2 \times_2 0.62 -_2 c_2 \times_2 0.63, \\ c_1 \times_2 0.63 +_2 c_2 \times_2 0.62 +_2 c_3 = 0. \end{cases}$$

Let us consider a solution of this system:

$$c_1 = 1,$$
$$c_2 = 1,$$
$$c_3 = -1.25,$$

that is, line *c* has the equation

$$c : x +_2 y -_2 1.25 = 0.$$

So we get two half-rays

$$l^{(1)} \subset c,$$
$$l^{(1)} : \left[(x, y) \in c : x > 0.62\right],$$

and

$$g \subset a,$$
$$g : \left[(x, y) \in a : x < 0.62\right],$$

emanating from the point *A*. So we get $\angle(l^{(1)}, g)$.

Also, we get two half-rays

$$l^{(2)} \subset c,$$
$$l^{(2)} : \left[(x, y) \in c : x < 0.63\right],$$

and

$$f \subset b,$$
$$f : \left[(x, y) \in b : x < 0.63\right],$$

emanating from the point *B*. So we get $\angle(l^{(2)}, f)$.

So we get a triangle OAB with vertices *O*, *A*, *B* and two half-rays (h, k) emanating from vertex *O*, passing, respectively, through *A* and *B*, and forming

$$\angle(h, k) = \angle O.$$

Also, $(l^{(1)}, g)$ are two half-rays emanating from vertex *A*, passing, respectively, through *B* and *O*, and forming

$$\angle\left(l^{(1)}, g\right) = \angle A,$$

and $\left(l^{(2)}, f\right)$ are two half-rays emanating from vertex *B*, passing, respectively, through *A* and *O*, and forming

$$\angle\left(l^{(2)}, f\right) = \angle B.$$

Let us consider angles $\angle A$, $\angle B$ and take points $K, L \in c$:

$$K(3, -1.75) \in l^{(1)},$$
$$L(-1.75, 3) \in l^{(2)},$$

and the corresponding vectors

$$\mathbf{AK} = (2.38, -2.38),$$
$$\mathbf{BL} = (-2.38, 2.38).$$

Also, we have

$$\mathbf{AO} = (-0.62, -0.63),$$
$$\mathbf{BO} = (-0.63, -0.62).$$

We get

$$(\mathbf{AK}, \mathbf{AK}) = 11.22 > 0,$$
$$(\mathbf{BL}, \mathbf{BL}) = 11.22 > 0,$$
$$(\mathbf{AO}, \mathbf{AO}) = 0.72 > 0,$$
$$(\mathbf{BO}, \mathbf{BO}) = 0.72 > 0,$$

and

$$(\mathbf{AK}, \mathbf{AO}) = 0,$$
$$(\mathbf{BL}, \mathbf{BO}) = 0.$$

This means that

$$\angle OAK \equiv \angle OBL,$$

and both these angles are right angles, that is, the half-rays *f*, *g* form right angles with corresponding half-rays of line *c* and intersect in point *O*.

So we have proved the following:

Theorem 9.1.
In Mathematics with Observers geometry in the plane E_2W_n, there is a triangle ABC such that $\angle B \in ABC$ and $\angle C \in ABC$ are right angles.

Classical Euclidean geometry states:

> "The sum of the angles of a triangle equals two right angles."

Question: Is this statement correct in Mathematics with Observers geometry?

The answer to this question in this case is negative because we have proved the following:

Theorem 9.2.
In Mathematics with Observers geometry in the plane E_2W_n, there is a triangle ABC such that the sum of the angles of this triangle is greater than two right angles.

9.4 Statement →1. First theorem of congruence for triangles

Let us reformulate the first statement of congruence for triangles in Mathematics with Observers geometry.

For E_2W_n, let us consider six points

$$A(x_1,y_1),B(x_2,y_2),C(x_3,y_3),A'(x_4,y_4),B'(x_5,y_5),C'(x_6,y_6)\in E_2W_n$$

where *A*, *B*, *C* are distinct points, A', B', C' are distinct points, and the pairs of points

$$(A,B),(A,C),(B,C),\left(A',B'\right),\left(A',C'\right),\left(B',C'\right)$$

may or may not be the points of existing straight lines. For all these points, we have the corresponding vectors

$$\mathbf{a}=(x_1,y_1),\mathbf{b}=(x_2,y_2),\mathbf{c}=(x_3,y_3),\mathbf{a}'=(x_4,y_4),\mathbf{b}'=(x_5,y_5),\mathbf{c}'=(x_6,y_6)\in E_2W_n.$$

Let us consider the vectors in E_2W_n

$$\begin{aligned}
\mathbf{AB}&=\mathbf{b}-_n\mathbf{a}=(x_2-_n x_1,y_2-_n y_1),\\
\mathbf{AC}&=\mathbf{c}-_n\mathbf{a}=(x_3-_n x_1,y_3-_n y_1),\\
\mathbf{BC}&=\mathbf{c}-_n\mathbf{b}=(x_3-_n x_2,y_3-_n y_2),\\
\mathbf{A'B'}&=\mathbf{b}'-_n\mathbf{a}'=(x_5-_n x_4,y_5-_n y_4),\\
\mathbf{A'C'}&=\mathbf{c}'-_n\mathbf{a}'=(x_6-_n x_4,y_6-_n y_4),\\
\mathbf{B'C'}&=\mathbf{c}'-_n\mathbf{b}'=(x_6-_n x_5,y_6-_n y_5),
\end{aligned}$$

and the scalar products

$$
\begin{aligned}
(\mathbf{AB},\mathbf{AB})&=(x_2-_n x_1)\times_n(x_2-_n x_1)+_n(y_2-_n y_1)\times_n(y_2-_n y_1),\\
(\mathbf{AC},\mathbf{AC})&=(x_3-_n x_1)\times_n(x_3-_n x_1)+_n(y_3-_n y_1)\times_n(y_3-_n y_1),\\
(\mathbf{BC},\mathbf{BC})&=(x_3-_n x_2)\times_n(x_3-_n x_2)+_n(y_3-_n y_2)\times_n(y_3-_n y_2),\\
(\mathbf{A'B'},\mathbf{A'B'})&=(x_5-_n x_4)\times_n(x_5-_n x_4)+_n(y_5-_n y_4)\times_n(y_5-_n y_4),\\
(\mathbf{A'C'},\mathbf{A'C'})&=(x_6-_n x_4)\times_n(x_6-_n x_4)+_n(y_6-_n y_4)\times_n(y_6-_n y_4),\\
(\mathbf{B'C'},\mathbf{B'C'})&=(x_6-_n x_5)\times_n(x_6-_n x_5)+_n(y_6-_n y_5)\times_n(y_6-_n y_5),\\
(\mathbf{AB},\mathbf{AC})&=(x_2-_n x_1)\times_n(x_3-_n x_1)+_n(y_2-_n y_1)\times_n(y_3-_n y_1),\\
(\mathbf{AB},\mathbf{BC})&=(x_2-_n x_1)\times_n(x_3-_n x_2)+_n(y_2-_n y_1)\times_n(y_3-_n y_2),\\
(\mathbf{AC},\mathbf{BC})&=(x_3-_n x_1)\times_n(x_3-_n x_2)+_n(y_3-_n y_1)\times_n(y_3-_n y_2),\\
(\mathbf{A'B'},\mathbf{A'C'})&=(x_5-_n x_4)\times_n(x_6-_n x_4)+_n(y_5-_n y_4)\times_n(y_6-_n y_4),\\
(\mathbf{A'B'},\mathbf{B'C'})&=(x_5-_n x_4)\times_n(x_6-_n x_5)+_n(y_5-_n y_4)\times_n(y_6-_n y_5),\\
(\mathbf{A'C'},\mathbf{B'C'})&=(x_6-_n x_4)\times_n(x_6-_n x_5)+_n(y_6-_n y_4)\times_n(y_6-_n y_5).
\end{aligned}
$$

So we get a question (analogue of first statement): If

$$
\begin{aligned}
(\mathbf{AB},\mathbf{AB})&=\left(\mathbf{A'B'},\mathbf{A'B'}\right)>0,\\
(\mathbf{AC},\mathbf{AC})&=\left(\mathbf{A'C'},\mathbf{A'C'}\right)>0,\\
(\mathbf{AB},\mathbf{AC})&=\left(\mathbf{A'B'},\mathbf{A'C'}\right),
\end{aligned}
$$

then do we have

$$
\begin{aligned}
(\mathbf{BC},\mathbf{BC})&=\left(\mathbf{B'C'},\mathbf{B'C'}\right)>0,\\
(\mathbf{BA},\mathbf{BC})&=\left(\mathbf{B'A'},\mathbf{B'C'}\right),\\
(\mathbf{CA},\mathbf{CB})&=\left(\mathbf{C'A'},\mathbf{C'B'}\right)?
\end{aligned}
$$

Let us consider several cases.

1) Let $n=2$, and let

$$A(0,0),B(0.98,-0.03),C(-0.02,0.97),A'(0,0),B'(-0.04,0.99),C'(0.96,-0.01).$$

Then

$$
\begin{aligned}
&\mathbf{a}=(0,0),\mathbf{b}=(1,-0.03),\mathbf{c}=(-0.02,1),\mathbf{a'}=(0,0),\mathbf{b'}=(-0.04,1),\mathbf{c'}=(1,-0.01)\\
&\mathbf{AB}=\mathbf{b}-_n\mathbf{a}=(1,-0.03),\\
&\mathbf{AC}=\mathbf{c}-_n\mathbf{a}=(-0.02,1),\\
&\mathbf{BC}=\mathbf{c}-_n\mathbf{b}=(-1.02,1.03),\\
&\mathbf{A'B'}=\mathbf{b'}-_n\mathbf{a'}=(-0.04,1),\\
&\mathbf{A'C'}=\mathbf{c'}-_n\mathbf{a'}=(1,-0.01),\\
&\mathbf{B'C'}=\mathbf{c'}-_n\mathbf{b'}=(1.04,-1.01).
\end{aligned}
$$

We get

$$\begin{aligned}(\mathbf{AB},\mathbf{AB})&= \left(\mathbf{A'B'},\mathbf{A'B'}\right) = 1 > 0,\\ (\mathbf{AC},\mathbf{AC})&= \left(\mathbf{A'C'},\mathbf{A'C'}\right) = 1 > 0,\\ (\mathbf{AB},\mathbf{AC})&= \left(\mathbf{A'B'},\mathbf{A'C'}\right) = -0.05\end{aligned}$$

and

$$\begin{aligned}(\mathbf{BC},\mathbf{BC})&= \left(\mathbf{B'C'},\mathbf{B'C'}\right) = 2.10 > 0,\\ (\mathbf{BA},\mathbf{BC})&= \left(\mathbf{B'A'},\mathbf{B'C'}\right) = 1.05,\\ (\mathbf{CA},\mathbf{CB})&= \left(\mathbf{C'A'},\mathbf{C'B'}\right) = 1.05.\end{aligned}$$

So we get

$$\triangle ABC \equiv \triangle A'B'C'$$

by both “vertex variant” and “side variant”. This means that the answer to the question in this case is positive.

2) Let $n = 2$, and let

$$A(0,0), P(0,1.36), Q(1.50,0), A'(0,2.14), P'(-1.00,1.23), Q'(1.00,1.02).$$

Then

$$\begin{aligned}\mathbf{a}&= (0,0), \mathbf{p} = (0,1.36), \mathbf{q} = (1.50,0), \mathbf{a}' = (0,2.14), \mathbf{p}' = (-1.00,1.23), \mathbf{q}' = (1.00,1.0\\ \mathbf{AP}&= \mathbf{p} -_n \mathbf{a} = (0,1.36),\\ \mathbf{AQ}&= \mathbf{q} -_n \mathbf{a} = (1.50,0),\\ \mathbf{PQ}&= \mathbf{q} -_n \mathbf{p} = (1.50,-1.36),\\ \mathbf{A'P'}&= \mathbf{p}' -_n \mathbf{a}' = (-1.00,-0.91),\\ \mathbf{A'Q'}&= \mathbf{q}' -_n \mathbf{a}' = (1.00,-1.12),\\ \mathbf{P'Q'}&= \mathbf{q}' -_n \mathbf{p}' = (2.00,-0.21).\end{aligned}$$

We get

$$\begin{aligned}(\mathbf{AP},\mathbf{AP})&= \left(\mathbf{A'P'},\mathbf{A'P'}\right) = 1.81 > 0,\\ (\mathbf{AQ},\mathbf{AQ})&= \left(\mathbf{A'Q'},\mathbf{A'Q'}\right) = 2.25 > 0,\\ (\mathbf{AP},\mathbf{AQ})&= \left(\mathbf{A'P'},\mathbf{A'Q'}\right) = 0,\end{aligned}$$

that is,

$$\begin{aligned}\mathbf{AP}&\equiv \mathbf{A'P'},\\ \mathbf{AQ}&\equiv \mathbf{A'Q'},\\ \angle(PAQ)&\equiv \angle\left(P'A'Q'\right),\end{aligned}$$

and we have

$$(\mathbf{PQ}, \mathbf{PQ}) = 4.06,$$
$$(\mathbf{P'Q'}, \mathbf{P'Q'}) = 4.04,$$

that is,

$$(\mathbf{PQ}, \mathbf{PQ}) \neq (\mathbf{P'Q'}, \mathbf{P'Q'}).$$

So we get

$$\triangle APQ \not\equiv \triangle A'P'Q'$$

by both "vertex variant" and "side variant". This means that the answer to the question in this case is negative.

So we have proved th following:

Theorem 9.3.
In Mathematics with Observers geometry in the plane E_2W_n, there are two distinct triangles ABC and $A'B'C'$ with congruences $AB \equiv A'B'$, $AC \equiv A'C'$, $\angle A \equiv \angle A'$ such that these triangles are congruent, that is, $AB \equiv A'B'$, $AC \equiv A'C'$, $BC \equiv B'C'$, $\angle A \equiv \angle A'$, $\angle B \equiv \angle B'$, $\angle C \equiv \angle C'$.

Theorem 9.4.
In Mathematics with Observers geometry in the plane E_2W_n, there are two distinct triangles ABC and $A'B'C'$ with congruences $AB \equiv A'B'$, $AC \equiv A'C'$, $\angle A \equiv \angle A'$ such that these triangles are not congruent.

9.5 Statement →2. Second theorem of congruence for triangles

Let us reformulate the second statement of congruence for triangles in Mathematics with Observers geometry.

For E_2W_n, let us consider six points

$$A(x_1, y_1), B(x_2, y_2), C(x_3, y_3), A'(x_4, y_4), B'(x_5, y_5), C'(x_6, y_6) \in E_2W_n,$$

where *A*, *B*, *C* are distinct points, A', B', C' are distinct points, and pairs of points

$$(A, B), (A, C), (B, C), (A', B'), (A', C'), (B', C')$$

may or may not be the points of existing straight lines. For these points, we have the corresponding vectors

$$\mathbf{a} = (x_1, y_1), \mathbf{b} = (x_2, y_2), \mathbf{c} = (x_3, y_3), \mathbf{a}' = (x_4, y_4), \mathbf{b}' = (x_5, y_5), \mathbf{c}' = (x_6, y_6) \in E_2W_n.$$

Let us consider the vectors in E_2W_n

$$\begin{aligned}
\mathbf{AB} &= \mathbf{b} -_n \mathbf{a} = (x_2 -_n x_1, y_2 -_n y_1),\\
\mathbf{AC} &= \mathbf{c} -_n \mathbf{a} = (x_3 -_n x_1, y_3 -_n y_1),\\
\mathbf{BC} &= \mathbf{c} -_n \mathbf{b} = (x_3 -_n x_2, y_3 -_n y_2),\\
\mathbf{A'B'} &= \mathbf{b'} -_n \mathbf{a'} = (x_5 -_n x_4, y_5 -_n y_4),\\
\mathbf{A'C'} &= \mathbf{c'} -_n \mathbf{a'} = (x_6 -_n x_4, y_6 -_n y_4),\\
\mathbf{B'C'} &= \mathbf{c'} -_n \mathbf{b'} = (x_6 -_n x_5, y_6 -_n y_5),
\end{aligned}$$

and the scalar products

$$\begin{aligned}
(\mathbf{AB},\mathbf{AB}) &= (x_2 -_n x_1) \times_n (x_2 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_2 -_n y_1),\\
(\mathbf{AC},\mathbf{AC}) &= (x_3 -_n x_1) \times_n (x_3 -_n x_1) +_n (y_3 -_n y_1) \times_n (y_3 -_n y_1),\\
(\mathbf{BC},\mathbf{BC}) &= (x_3 -_n x_2) \times_n (x_3 -_n x_2) +_n (y_3 -_n y_2) \times_n (y_3 -_n y_2),\\
\left(\mathbf{A'B'},\mathbf{A'B'}\right) &= (x_5 -_n x_4) \times_n (x_5 -_n x_4) +_n (y_5 -_n y_4) \times_n (y_5 -_n y_4),\\
\left(\mathbf{A'C'},\mathbf{A'C'}\right) &= (x_6 -_n x_4) \times_n (x_6 -_n x_4) +_n (y_6 -_n y_4) \times_n (y_6 -_n y_4),\\
\left(\mathbf{B'C'},\mathbf{B'C'}\right) &= (x_6 -_n x_5) \times_n (x_6 -_n x_5) +_n (y_6 -_n y_5) \times_n (y_6 -_n y_5),\\
(\mathbf{AB},\mathbf{AC}) &= (x_2 -_n x_1) \times_n (x_3 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_3 -_n y_1),\\
(\mathbf{AB},\mathbf{BC}) &= (x_2 -_n x_1) \times_n (x_3 -_n x_2) +_n (y_2 -_n y_1) \times_n (y_3 -_n y_2),\\
(\mathbf{AC},\mathbf{BC}) &= (x_3 -_n x_1) \times_n (x_3 -_n x_2) +_n (y_3 -_n y_1) \times_n (y_3 -_n y_2),\\
\left(\mathbf{A'B'},\mathbf{A'C'}\right) &= (x_5 -_n x_4) \times_n (x_6 -_n x_4) +_n (y_5 -_n y_4) \times_n (y_6 -_n y_4),\\
\left(\mathbf{A'B'},\mathbf{B'C'}\right) &= (x_5 -_n x_4) \times_n (x_6 -_n x_5) +_n (y_5 -_n y_4) \times_n (y_6 -_n y_5),\\
\left(\mathbf{A'C'},\mathbf{B'C'}\right) &= (x_6 -_n x_4) \times_n (x_6 -_n x_5) +_n (y_6 -_n y_4) \times_n (y_6 -_n y_5).
\end{aligned}$$

So we get a question (analogue of the second statement): If

$$\begin{aligned}
(\mathbf{AB},\mathbf{AB}) &= \left(\mathbf{A'B'},\mathbf{A'B'}\right) > 0,\\
(\mathbf{AB},\mathbf{AC}) &= \left(\mathbf{A'B'},\mathbf{A'C'}\right),\\
(\mathbf{BA},\mathbf{BC}) &= \left(\mathbf{B'A'},\mathbf{B'C'}\right),
\end{aligned}$$

then do we have

$$\begin{aligned}
(\mathbf{AC},\mathbf{AC}) &= \left(\mathbf{A'C'},\mathbf{A'C'}\right) > 0,\\
(\mathbf{BC},\mathbf{BC}) &= \left(\mathbf{B'C'},\mathbf{B'C'}\right) > 0,\\
(\mathbf{CA},\mathbf{CB}) &= \left(\mathbf{C'A'},\mathbf{C'B'}\right)?
\end{aligned}$$

Let us consider several cases.

1) Let $n = 2$, and let

$$A(0,0), B(0.98,-0.03), C(-0.02,0.97), A'(0,0), B'(-0.04,0.99), C'(0.96,-0.01).$$

Then

$$\begin{aligned}
\mathbf{a} &= (0,0), \mathbf{b} = (1,-0.03), \mathbf{c} = (-0.02,1), \mathbf{a}' = (0,0), \mathbf{b}' = (-0.04,1), \mathbf{c}' = (1,-0.01) \\
\mathbf{AB} &= \mathbf{b} -_n \mathbf{a} = (1,-0.03), \\
\mathbf{AC} &= \mathbf{c} -_n \mathbf{a} = (-0.02,1), \\
\mathbf{BC} &= \mathbf{c} -_n \mathbf{b} = (-1.02,1.03), \\
\mathbf{A}'\mathbf{B}' &= \mathbf{b}' -_n \mathbf{a}' = (-0.04,1), \\
\mathbf{A}'\mathbf{C}' &= \mathbf{c}' -_n \mathbf{a}' = (1,-0.01), \\
\mathbf{B}'\mathbf{C}' &= \mathbf{c}' -_n \mathbf{b}' = (1.04,-1.01).
\end{aligned}$$

We get

$$\begin{aligned}
(\mathbf{AB},\mathbf{AB}) &= \left(\mathbf{A}'\mathbf{B}',\mathbf{A}'\mathbf{B}'\right) = 1 > 0, \\
(\mathbf{AB},\mathbf{AC}) &= \left(\mathbf{A}'\mathbf{B}',\mathbf{A}'\mathbf{C}'\right) = -0.05, \\
(\mathbf{BA},\mathbf{BC}) &= \left(\mathbf{B}'\mathbf{A}',\mathbf{B}'\mathbf{C}'\right) = 1.05,
\end{aligned}$$

and

$$\begin{aligned}
(\mathbf{AC},\mathbf{AC}) &= \left(\mathbf{A}'\mathbf{C}',\mathbf{A}'\mathbf{C}'\right) = 1 > 0, \\
(\mathbf{BC},\mathbf{BC}) &= \left(\mathbf{B}'\mathbf{C}',\mathbf{B}'\mathbf{C}'\right) = 2.10 > 0, \\
(\mathbf{CA},\mathbf{CB}) &= \left(\mathbf{C}'\mathbf{A}',\mathbf{C}'\mathbf{B}'\right) = 1.05.
\end{aligned}$$

This means that the answer to the question in this case is positive.

2) Let $n = 2$, and let

$$A(0,0), B(2.38,0), C(1.40,2.59), A'(3.17,1.66), B'(5.55,1.66), C'(4.59,4.27).$$

Then

$$\begin{aligned}
\mathbf{a} &= (0,0), \mathbf{b} = (2.38,0), \mathbf{c} = (1.40,2.59), \\
\mathbf{a}' &== (3.17,1.66), \mathbf{b}' = (5.55,1.66), \mathbf{c}' = (4.59,4.27) \in E_2W_2, \\
\mathbf{AB} &= \mathbf{b} -_n \mathbf{a} = (2.38,0), \\
\mathbf{AC} &= \mathbf{c} -_n \mathbf{a} = (1.40,2.59), \\
\mathbf{BC} &= \mathbf{c} -_n \mathbf{b} = (-0.98,2.59), \\
\mathbf{A}'\mathbf{B}' &= \mathbf{b}' -_n \mathbf{a}' = (2.38,0), \\
\mathbf{A}'\mathbf{C}' &= \mathbf{c}' -_n \mathbf{a}' = (1.42,2.61), \\
\mathbf{B}'\mathbf{C}' &= \mathbf{c}' -_n \mathbf{b}' = (-0.96,2.61).
\end{aligned}$$

We get

$$\begin{aligned}
(\mathbf{AB},\mathbf{AB}) &= \left(\mathbf{A}'\mathbf{B}',\mathbf{A}'\mathbf{B}'\right) = 5.61 > 0, \\
(\mathbf{AC},\mathbf{AC}) &= 8.57, \\
\left(\mathbf{A}'\mathbf{C}',\mathbf{A}'\mathbf{C}'\right) &= 8.80,
\end{aligned}$$

that is,

$$(\mathbf{AC},\mathbf{AC}) \neq \left(\mathbf{A'C'},\mathbf{A'C'}\right),$$

and

$$(\mathbf{AB},\mathbf{AC}) = 3.30,$$
$$\left(\mathbf{A'B'},\mathbf{A'C'}\right) = 3.34,$$

that is,

$$(\mathbf{AB},\mathbf{AC}) \neq \left(\mathbf{A'B'},\mathbf{A'C'}\right),$$

and

$$(\mathbf{BC},\mathbf{BC}) = 7.42,$$
$$\left(\mathbf{B'C'},\mathbf{B'C'}\right) = 7.61,$$

that is,

$$(\mathbf{BC},\mathbf{BC}) \neq \left(\mathbf{B'C'},\mathbf{B'C'}\right),$$

and

$$(\mathbf{AB},\mathbf{BC}) = -2.23,$$
$$\left(\mathbf{A'B'},\mathbf{B'C'}\right) = -2.19,$$

that is,

$$(\mathbf{AB},\mathbf{BC}) \neq \left(\mathbf{A'B'},\mathbf{B'C'}\right).$$

So we get

$$\triangle ABC \not\equiv \triangle A'B'C'$$

by “vertex variant”.

Let us now consider the “side variant”. First of all, we have to clarify the situation here:

- are *A*, *B* points of an existing straight line?
- are *A*, *C* points of an existing straight line?
- are *B*, *C* points of an existing straight line?
- are A', B' points of an existing straight line?
- are A', C' points of an existing straight line?
- are B', C' points of an existing straight line?

We get

- *A*, *B* are points of the existing straight line in E_2W_2 with equation
$$l : y = 0;$$
- *A*, *C* are points of the existing straight line in E_2W_2 with equation

$$m : y = 1.87 \times_2 x;$$

- B, C are points of the existing straight line in E_2W_2 with equation
$$p : y = -2.69 \times_2 x +_2 6.32;$$
- A', B' are points of the existing straight line in E_2W_2 with equation
$$l' : y = 1.66;$$
- A', C' are points of the existing straight line in E_2W_2 with equation
$$m' : y = 1.87 \times_2 x -_2 4.20;$$
- B', C' are points of the existing straight line in E_2W_2 with equation
$$p' : y = -2.69 \times_2 x +_2 16.51.$$

We have

$$A \in l \cap m,$$
$$A' \in l' \cap m'.$$

Let h, k be two distinct half-rays

$$h \subset l,$$
$$h : (x, y) \in l, \quad x \geq 0,$$

and

$$k \subset m,$$
$$k : (x, y) \in m, \quad x \geq 0,$$

emanating from the point A. The system formed by these two half-rays h, k is an angle and is represented by the symbol $\angle(h, k)$ or $\angle(k, h)$.

Also, let h', k' are two distinct half-rays

$$h' \subset l',$$
$$h' : (x, y) \in l', \quad x \geq 3.17,$$

and

$$k' \subset m',$$
$$k' : (x, y) \in m', \quad x \geq 3.17,$$

emanating from the point A'. The system formed by these two half-rays h', k' is an angle and is represented by the symbol $\angle(h', k')$ or $\angle(k', h')$.

We also have

$$B \in l \cap p,$$
$$B' \in l' \cap p'.$$

Let q, r be two distinct half-rays

$$q \subset l,$$
$$q : (x, y) \in l, \quad x \leq 2.38,$$

and

$$r \subset p,$$
$$r : (x, y) \in p, \quad x \leq 2.38,$$

emanating from the point B. The system formed by these two half-rays q, r is an angle and is represented by the symbol $\angle(q, r)$ or $\angle(r, q)$.

Also, let q', r' be two distinct half-rays

$$q' \subset l',$$
$$q' : (x, y) \in l', \quad x \leq 5.55,$$

and

$$r' \subset p',$$
$$r' : (x, y) \in p', \quad x \leq 5.55,$$

emanating from the point B'. The system formed by these two half-rays q', r' is an angle and is represented by the symbol $\angle(q', r')$ or $\angle(r', q')$.

Let us take the points

$$P(1, 1.87) \in k,$$
$$P'(4.17, 3.53) \in k'.$$

Then

$$\mathbf{AP} = \mathbf{A'P'} = (1, 1.87),$$

and we get

$$(\mathbf{AP}, \mathbf{AP}) = \left(\mathbf{A'P'}, \mathbf{A'P'}\right) = 4.38 > 0,$$
$$(\mathbf{AB}, \mathbf{AP}) = \left(\mathbf{A'B'}, \mathbf{A'P'}\right),$$

that is,

$$\angle(Bh, Pk) \equiv \angle\left(B'h', P'k'\right).$$

In this case, we say that

$$\angle BAC = \angle(Bh, Pk)$$

is congruent to

$$\angle B'A'C' = \angle\left(B'h', P'k'\right)$$

and write

$$\angle BAC \equiv \angle B'A'C'.$$

This means that in Mathematics with Observers geometry, we do not define the congruence of the angles $\angle(h,k)$ and $\angle(h',k')$. We can define it only in the case where for any points P, Q, P', Q' satisfying the above conditions, we have

$$\angle(Ph,Qk) \equiv \angle\left(P'h',Q'k'\right).$$

Then

$$\angle(h,k) \equiv \angle\left(h',k'\right).$$

Now let us consider the angles $\angle ABC$ and $\angle A'B'C'$. Let us take the points

$$Q(1.38, 2.69) \in r,$$
$$Q'(4.55, 4.35) \in r'.$$

Then

$$\mathbf{QB} = \mathbf{Q'B'} = (1, -2.69),$$

and we get

$$(\mathbf{QB}, \mathbf{QB}) = \left(\mathbf{Q'B'}, \mathbf{Q'B'}\right) = 8.12 > 0,$$
$$(\mathbf{AB}, \mathbf{QB}) = \left(\mathbf{A'B'}, \mathbf{Q'B'}\right),$$

that is,

$$\angle(Aq,Qr) \equiv \angle\left(A'q',Q'r'\right).$$

In this case, we say that

$$\angle ABC = \angle(Aq,Qr)$$

is congruent to

$$\angle A'B'C' = \angle\left(A'q',Q'r'\right)$$

and write

$$\angle ABC \equiv \angle A'B'C'.$$

So in this case, in “side variant”, we have

$$\mathbf{AB} = \mathbf{A'B'},$$

that is,

$$AB \equiv A'B',$$
$$\angle BAC \equiv \angle B'A'C',$$
$$\angle ABC \equiv \angle A'B'C',$$

but

$$AC \not\equiv A'C',$$
$$BC \not\equiv B'C'.$$

This means that in this case,

$$\triangle ABC \not\equiv \triangle A'B'C'$$

by “side variant”, that is, the answer to the question in this case is negative.

So, we proved

Theorem 9.5.
In Mathematics with Observers geometry in the plane E_2W_n, there are two distinct triangles ABC and $A'B'C'$ with congruences $BC \equiv B'C'$, $\angle B \equiv \angle B'$, $\angle C \equiv \angle C'$ such that these triangles are congruent, that is, $AB \equiv A'B'$, $AC \equiv A'C'$, $BC \equiv B'C'$, $\angle A \equiv \angle A'$, $\angle B \equiv \angle B'$, $\angle C \equiv \angle C'$.

Theorem 9.6.
In Mathematics with Observers geometry in the plane E_2W_n, there are two distinct triangles ABC and $A'B'C'$ with congruences $BC \equiv B'C'$, $\angle B \equiv \angle B'$, $\angle C \equiv \angle C'$ such that these triangles are not congruent.

9.6 Statement →3. Third theorem of congruence for triangles

Let us reformulate the third statement of congruence for triangles in Mathematics with Observers geometry.

For E_2W_n, first, let us consider six points

$$A(x_1, y_1), B(x_2, y_2), C(x_3, y_3), A'(x_4, y_4), B'(x_5, y_5), C'(x_6, y_6) \in E_2W_n$$

where *A*, *B*, *C* are distinct points, A', B', C' are distinct points, and pairs of points

$$(A, B), (A, C), (B, C), (A', B'), (A', C'), (B', C')$$

may or may not be points of existing straight lines. For these points, we have the corresponding vectors

$$\mathbf{a} = (x_1, y_1), \mathbf{b} = (x_2, y_2), \mathbf{c} = (x_3, y_3), \mathbf{a}' = (x_4, y_4), \mathbf{b}' = (x_5, y_5), \mathbf{c}' = (x_6, y_6) \in E_2W_n.$$

Let us consider the vectors in E_2W_n

$$\mathbf{AB} = \mathbf{b} -_n \mathbf{a} = (x_2 -_n x_1, y_2 -_n y_1),$$
$$\mathbf{AC} = \mathbf{c} -_n \mathbf{a} = (x_3 -_n x_1, y_3 -_n y_1),$$
$$\mathbf{BC} = \mathbf{c} -_n \mathbf{b} = (x_3 -_n x_2, y_3 -_n y_2),$$
$$\mathbf{A'B'} = \mathbf{b}' -_n \mathbf{a}' = (x_5 -_n x_4, y_5 -_n y_4),$$
$$\mathbf{A'C'} = \mathbf{c}' -_n \mathbf{a}' = (x_6 -_n x_4, y_6 -_n y_4),$$
$$\mathbf{B'C'} = \mathbf{c}' -_n \mathbf{b}' = (x_6 -_n x_5, y_6 -_n y_5),$$

and the scalar products

$$
\begin{aligned}
(\mathbf{AB},\mathbf{AB}) &= (x_2 -_n x_1) \times_n (x_2 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_2 -_n y_1),\\
(\mathbf{AC},\mathbf{AC}) &= (x_3 -_n x_1) \times_n (x_3 -_n x_1) +_n (y_3 -_n y_1) \times_n (y_3 -_n y_1),\\
(\mathbf{BC},\mathbf{BC}) &= (x_3 -_n x_2) \times_n (x_3 -_n x_2) +_n (y_3 -_n y_2) \times_n (y_3 -_n y_2),\\
\left(\mathbf{A'B'},\mathbf{A'B'}\right) &= (x_5 -_n x_4) \times_n (x_5 -_n x_4) +_n (y_5 -_n y_4) \times_n (y_5 -_n y_4),\\
\left(\mathbf{A'C'},\mathbf{A'C'}\right) &= (x_6 -_n x_4) \times_n (x_6 -_n x_4) +_n (y_6 -_n y_4) \times_n (y_6 -_n y_4),\\
\left(\mathbf{B'C'},\mathbf{B'C'}\right) &= (x_6 -_n x_5) \times_n (x_6 -_n x_5) +_n (y_6 -_n y_5) \times_n (y_6 -_n y_5),\\
(\mathbf{AB},\mathbf{AC}) &= (x_2 -_n x_1) \times_n (x_3 -_n x_1) +_n (y_2 -_n y_1) \times_n (y_3 -_n y_1),\\
(\mathbf{AB},\mathbf{BC}) &= (x_2 -_n x_1) \times_n (x_3 -_n x_2) +_n (y_2 -_n y_1) \times_n (y_3 -_n y_2),\\
(\mathbf{AC},\mathbf{BC}) &= (x_3 -_n x_1) \times_n (x_3 -_n x_2) +_n (y_3 -_n y_1) \times_n (y_3 -_n y_2),\\
\left(\mathbf{A'B'},\mathbf{A'C'}\right) &= (x_5 -_n x_4) \times_n (x_6 -_n x_4) +_n (y_5 -_n y_4) \times_n (y_6 -_n y_4),\\
\left(\mathbf{A'B'},\mathbf{B'C'}\right) &= (x_5 -_n x_4) \times_n (x_6 -_n x_5) +_n (y_5 -_n y_4) \times_n (y_6 -_n y_5),\\
\left(\mathbf{A'C'},\mathbf{B'C'}\right) &= (x_6 -_n x_4) \times_n (x_6 -_n x_5) +_n (y_6 -_n y_4) \times_n (y_6 -_n y_5).
\end{aligned}
$$

So we get the following question (analogue of third statement): If

$$
\begin{aligned}
(\mathbf{AB},\mathbf{AB}) &= \left(\mathbf{A'B'},\mathbf{A'B'}\right) > 0,\\
(\mathbf{AC},\mathbf{AC}) &= \left(\mathbf{A'C'},\mathbf{A'C'}\right) > 0,\\
(\mathbf{BC},\mathbf{BC}) &= \left(\mathbf{B'C'},\mathbf{B'C'}\right) > 0,
\end{aligned}
$$

then is it correct that

$$
\begin{aligned}
(\mathbf{AB},\mathbf{AC}) &= \left(\mathbf{A'B'},\mathbf{A'C'}\right),\\
(\mathbf{BA},\mathbf{BC}) &= \left(\mathbf{B'A'},\mathbf{B'C'}\right),\\
(\mathbf{CA},\mathbf{CB}) &= \left(\mathbf{C'A'},\mathbf{C'B'}\right)?
\end{aligned}
$$

Let us consider several cases.

1) Let $n = 2$, and let

$$A(0,0), B(0.98,-0.03), C(-0.02,0.97), A'(0,0), B'(-0.04,0.99), C'(0.96,-0.01).$$

Then

$$
\begin{aligned}
&\mathbf{a} = (0,0), \mathbf{b} = (1,-0.03), \mathbf{c} = (-0.02,1), \mathbf{a}' = (0,0), \mathbf{b}' = (-0.04,1), \mathbf{c}' = (1,-0.01)\\
&\mathbf{AB} = \mathbf{b} -_n \mathbf{a} = (1,-0.03),\\
&\mathbf{AC} = \mathbf{c} -_n \mathbf{a} = (-0.02,1),\\
&\mathbf{BC} = \mathbf{c} -_n \mathbf{b} = (-1.02,1.03),\\
&\mathbf{A'B'} = \mathbf{b}' -_n \mathbf{a}' = (-0.04,1),\\
&\mathbf{A'C'} = \mathbf{c}' -_n \mathbf{a}' = (1,-0.01),\\
&\mathbf{B'C'} = \mathbf{c}' -_n \mathbf{b}' = (1.04,-1.01),
\end{aligned}
$$

and we get

$$(\mathbf{AB},\mathbf{AB}) = \left(\mathbf{A'B'},\mathbf{A'B'}\right) = 1 > 0,$$
$$(\mathbf{AC},\mathbf{AC}) = \left(\mathbf{A'C'},\mathbf{A'C'}\right) = 1 > 0,$$
$$(\mathbf{BC},\mathbf{BC}) = \left(\mathbf{B'C'},\mathbf{B'C'}\right) = 2.10 > 0,$$

and

$$(\mathbf{AB},\mathbf{AC}) = \left(\mathbf{A'B'},\mathbf{A'C'}\right) = -0.05,$$
$$(\mathbf{BA},\mathbf{BC}) = \left(\mathbf{B'A'},\mathbf{B'C'}\right) = 1.05,$$
$$(\mathbf{CA},\mathbf{CB}) = \left(\mathbf{C'A'},\mathbf{C'B'}\right) = 1.05.$$

This means that the answer to the question in this case is positive.

2) Let $n = 2$, and let

$$A(0,0), B(0,1), C(0.82, 0.53), A'(0,0), B'(0,1), C'(0.85, 0.56).$$

Then

$$\mathbf{a} = (0,0), \mathbf{b} = (1,0), \mathbf{c} = (0.82, 0.53), \mathbf{a}' = (0,0), \mathbf{b}' = (1,0), \mathbf{c}' = (0.85, 0.56) \in E_2W_n;$$
$$\mathbf{AB} = \mathbf{b} -_n \mathbf{a} = (1,0),$$
$$\mathbf{AC} = \mathbf{c} -_n \mathbf{a} = (0.82, 0.53),$$
$$\mathbf{BC} = \mathbf{c} -_n \mathbf{b} = (-0.18, 0.53),$$
$$\mathbf{A'B'} = \mathbf{b}' -_n \mathbf{a}' = (1,0),$$
$$\mathbf{A'C'} = \mathbf{c}' -_n \mathbf{a}' = (0.85, 0.56),$$
$$\mathbf{B'C'} = \mathbf{c}' -_n \mathbf{b}' = (-0.15, 0.56),$$

and we get

$$(\mathbf{AB},\mathbf{AB}) = \left(\mathbf{A'B'},\mathbf{A'B'}\right) = 1 > 0,$$
$$(\mathbf{AC},\mathbf{AC}) = \left(\mathbf{A'C'},\mathbf{A'C'}\right) = 0.89 > 0,$$
$$(\mathbf{BC},\mathbf{BC}) = \left(\mathbf{B'C'},\mathbf{B'C'}\right) = 0.26 > 0,$$

and

$$(\mathbf{AB},\mathbf{AC}) = 0.82,$$
$$\left(\mathbf{A'B'},\mathbf{A'C'}\right) = 0.86$$

that is,

$$(\mathbf{AB},\mathbf{AC}) \neq \left(\mathbf{A'B'},\mathbf{A'C'}\right),$$
$$(\mathbf{BA},\mathbf{BC}) = 0.18,$$
$$\left(\mathbf{B'A'},\mathbf{B'C'}\right) = 0.15,$$

that is,

$$(\mathbf{BA},\mathbf{BC}) \neq \left(\mathbf{B}'\mathbf{A}',\mathbf{B}'\mathbf{C}'\right),$$
$$(\mathbf{CA},\mathbf{CB}) = \left(\mathbf{C}'\mathbf{A}',\mathbf{C}'\mathbf{B}'\right) = 0.33.$$

This means that the answer to this question in this case is negative by "vertex variant".

So we have proved the following:

Theorem 9.7.
In Mathematics with Observers geometry in the plane E_2W_n, there are two distinct triangles ABC and $A'B'C'$ with congruences $AB \equiv A'B'$, $AC \equiv A'C'$, $BC \equiv B'C'$ such that these triangles are congruent, that is, $AB \equiv A'B'$, $AC \equiv A'C'$, $BC \equiv B'C'$, $\angle A \equiv \angle A'$, $\angle B \equiv \angle B'$, $\angle C \equiv \angle C'$.

Theorem 9.8.
In Mathematics with Observers geometry in the plane E_2W_n, there are two distinct triangles ABC and $A'B'C'$ with congruences $AB \equiv A'B'$, $AC \equiv A'C'$, $BC \equiv B'C'$ such that these triangles are not congruent.

9.7 Isosceles triangles

A triangle ABC we call an isosceles triangle if two of its sides are congruent, for example, if

$$AB \equiv BC.$$

In this case, the point *B* is called a vertex of the triangle, the side AC is called a base of the triangle, and the sides AB and BC are called lateral sides of the triangle.

In this section, we consider the question: In an isosceles triangle ABC with base AC, is it correct that

$$\angle BAC \equiv \angle BCA?$$

1) First we consider a triangle given by its vertices, not by its sides. Let us take three points as the vertices of a triangle OAB,

$$O(0,0), A(0.66,0.84), B(1,0) \in E_2W_2,$$

and the corresponding vectors

$$\mathbf{OA} = (0.66, 0.84),$$
$$\mathbf{AO} = (-0.66, -0.84),$$
$$\mathbf{OB} = (1, 0),$$
$$\mathbf{BO} = (-1, 0)$$
$$\mathbf{AB} = (0.34, -0.16),$$
$$\mathbf{BA} = (-0.34, 0.16).$$

We get

$$(\mathbf{OA}, \mathbf{OA}) = 1 > 0,$$
$$(\mathbf{AO}, \mathbf{AO}) = 1 > 0,$$
$$(\mathbf{OB}, \mathbf{OB}) = 1 > 0,$$
$$(\mathbf{BO}, \mathbf{BO}) = 1 > 0,$$
$$(\mathbf{AB}, \mathbf{AB}) = 0.10 > 0,$$
$$(\mathbf{BA}, \mathbf{BA}) = 0.10 > 0.$$

We have

$$AO \equiv BO,$$
$$AB \equiv BA,$$

and

$$(\mathbf{AO}, \mathbf{AB}) = -0.10,$$
$$(\mathbf{BO}, \mathbf{BA}) = 0.34,$$

that is,

$$\angle OAB \not\equiv \angle OBA.$$

This means that the answer to the question in this case is negative.

2) Let us take two straight lines in E_2W_2,

$$a : 1.14 \times_2 x -_2 y = 0$$

and

$$b : 1.28 \times_2 x -_2 y = 0,$$

and points $O, A \in a, O, B \in b$:

$$O(0, 0), A(1.71, 1.92), B(1.60, 2.00).$$

Let us take these three points as the vertices of the triangle OAB and, as we did in part 1) of this section, forget straight lines a and b.

We consider the corresponding vectors

$$\mathbf{OA} = (1.71, 1.92),$$
$$\mathbf{AO} = (-1.71, -1.92),$$
$$\mathbf{OB} = (1.60, 2.00),$$
$$\mathbf{BO} = (-1.60, -2.00),$$
$$\mathbf{AB} = (-0.11, 0.08),$$
$$\mathbf{BA} = (0.11, -0.08),$$

and we get

$$(\mathbf{OA},\mathbf{OA})= 1.56 > 0,$$
$$(\mathbf{AO},\mathbf{AO})= 1.56 > 0,$$
$$(\mathbf{OB},\mathbf{OB})= 1.56 > 0,$$
$$(\mathbf{BO},\mathbf{BO})= 1.56 > 0,$$
$$(\mathbf{AB},\mathbf{AB})= 0.01 > 0,$$
$$(\mathbf{BA},\mathbf{BA})= 0.01 > 0.$$

We have

$$AO\equiv BO,$$
$$AB\equiv BA,$$

and

$$(\mathbf{AO},\mathbf{AB})= 0.10,$$
$$(\mathbf{BO},\mathbf{BA})= -0.01,$$

that is,

$$\angle OAB \not\equiv \angle OBA.$$

This means that the answer to this question in this case is negative.

3) Let us consider straight line a containing points O, A,

$$a : a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 = 0,$$

that is,

$$\begin{cases} a_1 \times_2 0 +_2 a_2 \times_2 0 +_2 a_3 = 0, \\ a_1 \times_2 0.66 +_2 a_2 \times_2 0.84 +_2 a_3 = 0. \end{cases}$$

One possible solution is

$$a_3 = 0,$$
$$a_1 = 1.34,$$
$$a_2 = -1,$$

that is, straight line a has the equation

$$a : 1.34 \times_2 x -_2 y = 0$$

Let straight line b containing points O, B have the equation

$$b : y = 0.$$

Let us build strait line c containing points A, B,

$$c : c_1 \times_2 x +_2 c_2 \times_2 y +_2 c_3 = 0,$$

that is,

$$\begin{cases} c_1 \times_2 0.66 +_2 c_2 \times_2 0.84 +_2 c_3 = 0, \\ c_1 \times_2 1 +_2 c_2 \times_2 0 +_2 c_3 = 0. \end{cases}$$

One possible solution is

$$\begin{aligned} c_1 &= 2.40, \\ c_2 &= 1, \\ c_3 &= -2.40, \end{aligned}$$

that is, straight line *c* has the equation

$$c : 2.40 \times_2 x +_2 y -_2 2.40 = 0.$$

This means that the triangle OAB has vertices and sides.

Let us now take the points

$$O'(0,0) \in a \cap b, \quad A'(1.23, 1.63) \in a, \quad B'(2.03, 0) \in b$$

and consider these three points as vertices of the triangle $O'A'B'$. Let us take the corresponding vectors

$$\begin{aligned} \mathbf{O'A'} &= (1.23, 1.63), \\ \mathbf{A'O'} &= (-1.23, -1.63), \\ \mathbf{O'B'} &= (2.03, 0), \\ \mathbf{B'O'} &= (-2.03, 0), \\ \mathbf{A'B'} &= (0.80, -1.63), \\ \mathbf{B'A'} &= (-0.80, 1.63). \end{aligned}$$

We get

$$\begin{aligned} \left(\mathbf{O'A'}, \mathbf{O'A'}\right) &= 4.12 > 0, \\ \left(\mathbf{A'O'}, \mathbf{A'O'}\right) &= 4.12 > 0, \\ \left(\mathbf{O'B'}, \mathbf{O'B'}\right) &= 4.12 > 0, \\ \left(\mathbf{B'O'}, \mathbf{B'O'}\right) &= 4.12 > 0, \\ \left(\mathbf{A'B'}, \mathbf{A'B'}\right) &= 3.26 > 0, \\ \left(\mathbf{B'A'}, \mathbf{B'A'}\right) &= 3.26 > 0. \end{aligned}$$

We have

$$\begin{aligned} A'O' &\equiv B'O', \\ A'B' &\equiv B'A', \end{aligned}$$

and

$$\begin{aligned} \left(\mathbf{A'O'}, \mathbf{A'B'}\right) &= -1.31, \\ \left(\mathbf{B'O'}, \mathbf{B'A'}\right) &= -1.60, \end{aligned}$$

that is,

$$\angle O'A'B' \not\equiv \angle O'B'A'.$$

Lets now take other points

$$O''(0,0) \in a \cap b, \quad A''(2.25, 2.99) \in a, \quad B''(3.72, 0) \in b$$

and consider these three points as vertices of the triangle $O''A''B''$. Let us take the corresponding vectors

$$\begin{aligned}
\mathbf{O''A''} &= (2.25, 2.99),\\
\mathbf{A''O''} &= (-2.25, -2.99),\\
\mathbf{O''B''} &= (3.72, 0),\\
\mathbf{B''O''} &= (-3.72, 0),\\
\mathbf{A''B''} &= (1.47, -2.99),\\
\mathbf{B''A''} &= (-1.47, 2.99).
\end{aligned}$$

We get

$$\begin{aligned}
\left(\mathbf{O''A''}, \mathbf{O''A''}\right) &= 13.81 > 0,\\
\left(\mathbf{A''O''}, \mathbf{A''O''}\right) &= 13.81 > 0,\\
\left(\mathbf{O''B''}, \mathbf{O''B''}\right) &= 13.81 > 0,\\
\left(\mathbf{B''O''}, \mathbf{B''O''}\right) &= 13.81 > 0,\\
\left(\mathbf{A''B''}, \mathbf{A''B''}\right) &= 8.33 > 0,\\
\left(\mathbf{B''A''}, \mathbf{B''A''}\right) &= 8.33 > 0.
\end{aligned}$$

We have

$$\begin{aligned}
A''O'' &\equiv B''O'',\\
A''B'' &\equiv B''A'',
\end{aligned}$$

and

$$\begin{aligned}
\left(\mathbf{A''O''}, \mathbf{A''B''}\right) &= 5.50,\\
\left(\mathbf{B''O''}, \mathbf{B''A''}\right) &= 5.41
\end{aligned}$$

that is,

$$\angle O''A''B'' \not\equiv \angle O''B''A''.$$

This means that the answer to this question in this case is negative.

4) Again, let us consider E_2W_2 and two straight lines in E_2W_2,

$$a : x = 0$$

and

$$b : y = 0,$$

having a common point $O(0, 0)$.

Let h, k be two distinct half-rays

$$h \subset a,$$
$$h : \left[(x, y) \in a : y > 0\right],$$

and

$$k \subset b,$$
$$k : \left[(x, y) \in b : x > 0\right],$$

emanating from the point O. So we get $\angle(h, k)$.

Let us take the points

$$A(0, 1) \in h,$$
$$B(1, 0) \in k,$$

and the corresponding vectors

$$\mathbf{OA} = (0, 1),$$
$$\mathbf{OB} = (1, 0),$$
$$\mathbf{AB} = \mathbf{OB} -_2 \mathbf{OA} = (1, -1).$$

We get

$$(\mathbf{OA}, \mathbf{OA}) = 1 > 0,$$
$$(\mathbf{OB}, \mathbf{OB}) = 1 > 0,$$
$$(\mathbf{AB}, \mathbf{AB}) = 2,$$
$$(\mathbf{OA}, \mathbf{OB}) = 0.$$

Let us build strait line c containing points A, B:

$$c : c_1 \times_2 x +_2 c_2 \times_2 y +_2 c_3 = 0,$$

that is,

$$\begin{cases} c_1 \times_2 0 +_2 c_2 \times_2 1 +_2 c_3 = 0, \\ c_1 \times_2 1 +_2 c_2 \times_2 0 +_2 c_3 = 0, \end{cases}$$

and so

$$\begin{cases} c_2 = c_1, \\ c_1 +_2 c_3 = 0. \end{cases}$$

Let us consider a solution of this system,

$$c_1 = 1,$$
$$c_2 = 1,$$
$$c_3 = -1,$$

that is, straight line c has the equation

$$c : x +_2 y -_2 1 = 0.$$

So we get two half-rays

$$l^1 \subset c,$$
$$l^1 : \big[(x, y) \in c : x > 0\big],$$

and

$$g \subset a,$$
$$g : \big[(x, y) \in a : y < 1\big],$$

emanating from the point A. So we get $\angle(l^1, g)$.

Also, we get two half-rays

$$l^2 \subset c,$$
$$l^2 : \big[(x, y) \in c : y > 0\big],$$

and

$$f \subset b,$$
$$f : \big[(x, y) \in b : x < 1\big],$$

emanating from the point B. So we get $\angle(l^2, f)$.

So we get a triangle OAB with vertices O, A, B and two half-rays (h, k) emanating from vertex O, passing, respectively, through A and B, and forming

$$\angle(h, k) = \angle AOB.$$

Also, (l^1, g) are two half-rays emanating from vertex A, passing, respectively, through B and O, and forming

$$\angle\big(l^1, g\big) = \angle OAB,$$

and (l^2, f) are two half-rays emanating from vertex B, passing, respectively, through A and O, and forming

$$\angle\big(l^2, f\big) = \angle OBA,$$

and we get the vectors

$$\mathbf{AB}=(1,-1),$$
$$\mathbf{BA}=(-1,1),$$
$$\mathbf{AO}=(-1,0),$$
$$\mathbf{BO}=(0,-1),$$

and

$$(\mathbf{AO},\mathbf{AB})=-1,$$
$$(\mathbf{BO},\mathbf{BA})=-1.$$

This means that the triangle AOB is an isosceles triangle with lateral sides

$$AO \equiv OB$$

and base AB, and we get

$$\angle OAB \equiv \angle OBA,$$

that is, in this case the answer to this question is positive.

So we have proved the following:

Theorem 9.9.
In Mathematics with Observers geometry in the plane E_2W_n, there is an isosceles triangle ABC with $AB \equiv BC$ such that $\angle BAC \equiv \angle BCA$.

Theorem 9.10.
In Mathematics with Observers geometry in the plane E_2W_n, there is an isosceles triangle ABC with $AB \equiv BC$ such that $\angle BAC \not\equiv \angle BCA$.

9.8 Similar triangles

1) Let us take two straight lines in E_2W_2,

$$a: x -_2 y = 0$$

and

$$b: y = 0,$$

and the points

$O, A_1, A_2, A_3 \in a; \quad O, B_1, B_2, B_3 \in b:$
$O(0,0), A_1(1.00, 1.00), A_2(2.00, 2.00), A_3(3.00, 3.00), B_1(1.00, 0.00), B_2(2.00, 0.00), B_3(3.00, 0.00).$

Also, let us consider three straight lines $c, d, e \in E_2W_2$,

$$c : x = 1,$$
$$d : x = 2,$$
$$e : x = 3.$$

These three vertical lines are parallel in the Euclidean sense. We have

$$A_1, B_1 \in c; \quad A_2, B_2 \in d; \quad A_3, B_3 \in e.$$

Now let us consider the triangles OA_1B_1, OA_2B_2, OA_3B_3. We consider the corresponding vectors

$$\mathbf{OA}_1 = (1.00, 1.00),$$
$$\mathbf{OA}_2 = (2.00, 2.00),$$
$$\mathbf{OA}_3 = (3.00, 3.00),$$
$$\mathbf{OB}_1 = (1.00, 0.00),$$
$$\mathbf{OB}_2 = (2.00, 0.00),$$
$$\mathbf{OB}_3 = (3.00, 0.00)$$
$$\mathbf{A}_1\mathbf{B}_1 = (0.00, 1.00),$$
$$\mathbf{A}_2\mathbf{B}_2 = (0.00, 2.00),$$
$$\mathbf{A}_3\mathbf{B}_3 = (0.00, 3.00).$$

We have

$$(\mathbf{OA}_1, \mathbf{OA}_1) = 2,$$
$$(\mathbf{OA}_2, \mathbf{OA}_2) = 8,$$
$$(\mathbf{OA}_3, \mathbf{OA}_3) = 18,$$
$$(\mathbf{OB}_1, \mathbf{OB}_1) = 1,$$
$$(\mathbf{OB}_2, \mathbf{OB}_2) = 4,$$
$$(\mathbf{OB}_3, \mathbf{OB}_3) = 9,$$
$$(\mathbf{A}_1\mathbf{B}_1, \mathbf{A}_1\mathbf{B}_1) = 1,$$
$$(\mathbf{A}_2\mathbf{B}_2, \mathbf{A}_2\mathbf{B}_2) = 4$$
$$(\mathbf{A}_3\mathbf{B}_3, \mathbf{A}_3\mathbf{B}_3) = 9.$$

We get in Mathematics with Observers

$$
\begin{aligned}
|\mathbf{OA}_1| &= \sqrt{2} = 1.42, \\
|\mathbf{OA}_2| &= \sqrt{8} = 2.84, \\
|\mathbf{OA}_3| &= \sqrt{18} = NA, \\
|\mathbf{OB}_1| &= \sqrt{1} = 1.00, \\
|\mathbf{OB}_2| &= \sqrt{4} = 2.00, \\
|\mathbf{OB}_3| &= \sqrt{9} = 3.00, \\
|\mathbf{A}_1\mathbf{B}_1| &= \sqrt{1} = 1.00, \\
|\mathbf{A}_2\mathbf{B}_2| &= \sqrt{4} = 2.00, \\
|\mathbf{A}_3\mathbf{B}_3| &= \sqrt{9} = 3.00.
\end{aligned}
$$

Note that

$$4.24 \times_2 4.24 = 17.96; \quad 4.25 \times_2 4.25 = 18.04,$$

and so $\sqrt{18}$ does not exist, that is, NA.

In classical geometry, triangles OA_1B_1, OA_2B_2, OA_3B_3 are similar, and we have the following equalities:

$$
\begin{aligned}
\frac{|\mathbf{OA}_3|}{|\mathbf{OB}_3|} &= \frac{|\mathbf{OA}_2|}{|\mathbf{OB}_2|} = \frac{|\mathbf{OA}_1|}{|\mathbf{OB}_1|}, \\
\frac{|\mathbf{OA}_3|}{|\mathbf{OA}_2|} &= \frac{|\mathbf{OB}_3|}{|\mathbf{OB}_2|} = \frac{|\mathbf{A}_3\mathbf{B}_3|}{|\mathbf{A}_2\mathbf{B}_2|}, \\
\frac{|\mathbf{OA}_3|}{|\mathbf{OA}_1|} &= \frac{|\mathbf{OB}_3|}{|\mathbf{OB}_1|} = \frac{|\mathbf{A}_3\mathbf{B}_3|}{|\mathbf{A}_1\mathbf{B}_1|},
\end{aligned}
$$

and

$$\frac{|\mathbf{OA}_2|}{|\mathbf{OA}_1|} = \frac{|\mathbf{OB}_2|}{|\mathbf{OB}_1|} = \frac{|\mathbf{A}_2\mathbf{B}_2|}{|\mathbf{A}_1\mathbf{B}_1|}.$$

Let us first consider two triangles OA_1B_1 and OA_2B_2 and check these equalities. We get in Mathematics with Observers

$$\frac{|\mathbf{OA}_2|}{|\mathbf{OB}_2|} = \frac{2.84}{2.00} = 1.42,$$
$$\frac{|\mathbf{OA}_1|}{|\mathbf{OB}_1|} = \frac{1.42}{1} = 1.42,$$
$$\frac{|\mathbf{OA}_2|}{|\mathbf{OA}_1|} = \frac{2.84}{1.42} = 2,$$
$$\frac{|\mathbf{OB}_2|}{|\mathbf{OB}_1|} = \frac{2}{1} = 2,$$
$$\frac{|\mathbf{A}_2\mathbf{B}_2|}{|\mathbf{A}_1\mathbf{B}_1|} = \frac{2}{1} = 2,$$

that is, in this case the classical geometry equalities are satisfied in Mathematics with Observers geometry. Note that if we take the inverse classical geometry equalities

$$\frac{|\mathbf{OB}_2|}{|\mathbf{OA}_2|} = \frac{|\mathbf{OB}_1|}{|\mathbf{OA}_1|}$$

and

$$\frac{|\mathbf{OA}_1|}{|\mathbf{OA}_2|} = \frac{|\mathbf{OB}_1|}{|\mathbf{OB}_2|} = \frac{|\mathbf{A}_1\mathbf{B}_1|}{|\mathbf{A}_2\mathbf{B}_2|},$$

we get in Mathematics with Observers

$$\frac{|\mathbf{OB}_2|}{|\mathbf{OA}_2|} = \frac{2.00}{2.84} = 0.72,$$
$$\frac{|\mathbf{OB}_1|}{|\mathbf{OA}_1|} = \frac{1.00}{1.42} = 0.72,$$
$$\frac{|\mathbf{OA}_1|}{|\mathbf{OA}_2|} = \frac{1.42}{2.84} = 0.51,$$
$$\frac{|\mathbf{OB}_1|}{|\mathbf{OB}_2|} = \frac{1}{2} = 0.50,$$
$$\frac{|\mathbf{A}_1\mathbf{B}_1|}{|\mathbf{A}_2\mathbf{B}_2|} = \frac{1}{2} = 0.50,$$

that is, in this case the classical geometry equalities are not satisfied in Mathematics with Observers geometry.

Let us continue to consider the same two triangles OA_1B_1 and OA_2B_2, and instead of the previous equalities, consider the following:

$$(\mathbf{OA}_2, \mathbf{OA}_2) \times_2 (\mathbf{OB}_1, \mathbf{OB}_1) = (\mathbf{OA}_1, \mathbf{OA}_1) \times_2 (\mathbf{OB}_2, \mathbf{OB}_2),$$
$$(\mathbf{OA}_2, \mathbf{OA}_2) \times_2 (\mathbf{A}_1\mathbf{B}_1, \mathbf{A}_1\mathbf{B}_1) = (\mathbf{OA}_1, \mathbf{OA}_1) \times_2 (\mathbf{A}_2\mathbf{B}_2, \mathbf{A}_2\mathbf{B}_2).$$

Let us check these equalities:

$$(\mathbf{OA}_2,\mathbf{OA}_2)\times_2(\mathbf{OB}_1,\mathbf{OB}_1)=8\times_2 1=8,$$
$$(\mathbf{OA}_1,\mathbf{OA}_1)\times_2(\mathbf{OB}_2,\mathbf{OB}_2)=2\times_2 4=8,$$
$$(\mathbf{OA}_2,\mathbf{OA}_2)\times_2(\mathbf{A}_1\mathbf{B}_1,\mathbf{A}_1\mathbf{B}_1)=8\times_2 1=8,$$
$$(\mathbf{OA}_1,\mathbf{OA}_1)\times_2(\mathbf{A}_2\mathbf{B}_2,\mathbf{A}_2\mathbf{B}_2)=2\times_2 4=8,$$

that is, in this case the classical geometry equalities are satisfied in Mathematics with Observers geometry.

Let us consider now the triangles OA_1B_1 and OA_3B_3. Because the length of $\mathbf{OA}_3$ does not exist,

$$|\mathbf{OA}_3|=\sqrt{18}=NA,$$

we can consider the following equalities:

$$(\mathbf{OA}_3,\mathbf{OA}_3)\times_2(\mathbf{OB}_1,\mathbf{OB}_1)=(\mathbf{OA}_1,\mathbf{OA}_1)\times_2(\mathbf{OB}_3,\mathbf{OB}_3),$$
$$(\mathbf{OA}_3,\mathbf{OA}_3)\times_2(\mathbf{A}_1\mathbf{B}_1,\mathbf{A}_1\mathbf{B}_1)=(\mathbf{OA}_1,\mathbf{OA}_1)\times_2(\mathbf{A}_3\mathbf{B}_3,\mathbf{A}_3\mathbf{B}_3).$$

Let us check them:

$$(\mathbf{OA}_3,\mathbf{OA}_3)\times_2(\mathbf{OB}_1,\mathbf{OB}_1)=18\times_2 1=18,$$
$$(\mathbf{OA}_1,\mathbf{OA}_1)\times_2(\mathbf{OB}_3,\mathbf{OB}_3)=2\times_2 9=18,$$
$$(\mathbf{OA}_3,\mathbf{OA}_3)\times_2(\mathbf{A}_1\mathbf{B}_1,\mathbf{A}_1\mathbf{B}_1)=18\times_2 1=18,$$
$$(\mathbf{OA}_1,\mathbf{OA}_1)\times_2(\mathbf{A}_3\mathbf{B}_3,\mathbf{A}_3\mathbf{B}_3)=2\times_2 9=18,$$

that is, in this case the classical geometry equalities are satisfied in Mathematics with Observers geometry.

Finally, let us consider the triangles OA_2B_2 and OA_3B_3. We consider the following equalities:

$$(\mathbf{OA}_3,\mathbf{OA}_3)\times_2(\mathbf{OB}_2,\mathbf{OB}_2)=(\mathbf{OA}_2,\mathbf{OA}_2)\times_2(\mathbf{OB}_3,\mathbf{OB}_3),$$
$$(\mathbf{OA}_3,\mathbf{OA}_3)\times_2(\mathbf{A}_2\mathbf{B}_2,\mathbf{A}_2\mathbf{B}_)=(\mathbf{OA}_2,\mathbf{OA}_2)\times_2(\mathbf{A}_3\mathbf{B}_3,\mathbf{A}_3\mathbf{B}_3).$$

Let us check them:

$$(\mathbf{OA}_3,\mathbf{OA}_3)\times_2(\mathbf{OB}_2,\mathbf{OB}_2)=18\times_2 4=72,$$
$$(\mathbf{OA}_2,\mathbf{OA}_2)\times_2(\mathbf{OB}_3,\mathbf{OB}_3)=8\times_2 9=72,$$
$$(\mathbf{OA}_3,\mathbf{OA}_3)\times_2(\mathbf{A}_2\mathbf{B}_2,\mathbf{A}_2\mathbf{B}_2)=18\times_2 4=72,$$
$$(\mathbf{OA}_2,\mathbf{OA}_2)\times_2(\mathbf{A}_3\mathbf{B}_3,\mathbf{A}_3\mathbf{B}_3)=8\times_2 9=72,$$

that is, in this case the classical geometry equalities are satisfied in Mathematics with Observers geometry.

2) Let us take again two straight lines in E_2W_2,

$$a: x-_2 y=0$$

and

$$b : y = 0,$$

and the points

$$O, A_1, A_2, A_3 \in a; \quad O, B_1, B_2, B_3 \in b :$$
$$O(0,0), A_1(1.83, 1.83), A_2(2.79, 2.79), A_3(3.14, 3.14), B_1(1.83, 0.00), B_2(2.79, 0.00),$$
$$B_3(3.14, 0.00).$$

Let us also consider three straight lines in E_2W_2,

$$c : x = 1.83,$$
$$d : x = 2.79,$$

and

$$e : x = 3.14.$$

These three vertical lines are parallel in the Euclidean sense, and we have

$$A_1, B_1 \in c; \quad A_2, B_2 \in d; \quad A_3, B_3 \in e.$$

Now let us consider the triangles OA_1B_1, OA_2B_2, and OA_3B_3. We consider the corresponding vectors

$$\mathbf{OA}_1 = (1.83, 1.83),$$
$$\mathbf{OA}_2 = (2.79, 2.79),$$
$$\mathbf{OA}_3 = (3.14, 3.14),$$
$$\mathbf{OB}_1 = (1.83, 0.00),$$
$$\mathbf{OB}_2 = (2.79, 0.00),$$
$$\mathbf{OB}_3 = (3.14, 0.00),$$
$$\mathbf{A}_1\mathbf{B}_1 = (0.00, 1.83),$$
$$\mathbf{A}_2\mathbf{B}_2 = (0.00, 2.79),$$
$$\mathbf{A}_3\mathbf{B}_3 = (0.00, 3.14),$$

and we have

$$(\mathbf{OA}_1, \mathbf{OA}_1) = 6.60,$$
$$(\mathbf{OA}_2, \mathbf{OA}_2) = 15.30,$$
$$(\mathbf{OA}_3, \mathbf{OA}_3) = 19.70,$$
$$(\mathbf{OB}_1, \mathbf{OB}_1) = 3.30,$$
$$(\mathbf{OB}_2, \mathbf{OB}_2) = 7.65,$$
$$(\mathbf{OB}_3, \mathbf{OB}_3) = 9.85,$$
$$(\mathbf{A}_1\mathbf{B}_1, \mathbf{A}_1\mathbf{B}_1) = 3.30,$$
$$(\mathbf{A}_2\mathbf{B}_2, \mathbf{A}_2\mathbf{B}_2) = 7.65,$$
$$(\mathbf{A}_3\mathbf{B}_3, \mathbf{A}_3\mathbf{B}_3) = 9.85.$$

We get

$$|\mathbf{OA}_1|=\sqrt{6.60}=NA,$$
$$|\mathbf{OA}_2|=\sqrt{15.30}=NA,$$
$$|\mathbf{OA}_3|=\sqrt{19.70}=NA,$$
$$|\mathbf{OB}_1|=\sqrt{3.30}=1.83,$$
$$|\mathbf{OB}_2|=\sqrt{7.65}=2.79,$$
$$|\mathbf{OB}_3|=\sqrt{9.85}=3.14,$$
$$|\mathbf{A}_1\mathbf{B}_1|=\sqrt{3.30}=1.83,$$
$$|\mathbf{A}_2\mathbf{B}_2|=\sqrt{7.65}=2.79,$$
$$|\mathbf{A}_3\mathbf{B}_3|=\sqrt{9.85}=3.14.$$

Note that

$$2.58 \times_2 2.58 = 6.57; \quad 2.59 \times_2 2.59 = 6.61,$$
$$3.91 \times_2 3.91 = 15.27; \quad 3.92 \times_2 3.92 = 15.33,$$
$$4.44 \times_2 4.44 = 19.68; \quad 4.45 \times_2 4.45 = 19.76.$$

So $\sqrt{6.60}$, $\sqrt{15.30}$, and $\sqrt{19.70}$ do not exist, that is, NA.

In classical geometry the triangles OA_1B_1, OA_2B_2, OA_3B_3 are similar. Let us first consider the triangles OA_1B_1 and OA_3B_3. Because the lengths of $\mathbf{OA}_1$, $\mathbf{OA}_2$ and $\mathbf{OA}_3$ do not exist, we can consider the following equalities:

$$(\mathbf{OA}_3,\mathbf{OA}_3)\times_2(\mathbf{OB}_1,\mathbf{OB}_1)=(\mathbf{OA}_1,\mathbf{OA}_1)\times_2(\mathbf{OB}_3,\mathbf{OB}_3),$$
$$(\mathbf{OA}_3,\mathbf{OA}_3)\times_2(\mathbf{A}_1\mathbf{B}_1,\mathbf{A}_1\mathbf{B}_1)=(\mathbf{OA}_1,\mathbf{OA}_1)\times_2(\mathbf{A}_3\mathbf{B}_3,\mathbf{A}_3\mathbf{B}_3).$$

Let us check them:

$$(\mathbf{OA}_3,\mathbf{OA}_3)\times_2(\mathbf{OB}_1,\mathbf{OB}_1)=19.70\times_2 3.30=65.01,$$
$$(\mathbf{OA}_1,\mathbf{OA}_1)\times_2(\mathbf{OB}_3,\mathbf{OB}_3)=6.60\times_2 9.85=64.98,$$
$$(\mathbf{OA}_3,\mathbf{OA}_3)\times_2(\mathbf{A}_1\mathbf{B}_1,\mathbf{A}_1\mathbf{B}_1)=19.70\times_2 3.30=65.01,$$
$$(\mathbf{OA}_1,\mathbf{OA}_1)\times_2(\mathbf{A}_3\mathbf{B}_3,\mathbf{A}_3\mathbf{B}_3)=6.60\times_2 9.85=64.98,$$

that is, in this case the classical geometry equalities are not satisfied in Mathematics with Observers geometry.

Let us now consider the triangles OA_1B_1 and OA_2B_2. We consider the following equalities:

$$(\mathbf{OA}_2,\mathbf{OA}_2)\times_2(\mathbf{OB}_1,\mathbf{OB}_1)=(\mathbf{OA}_1,\mathbf{OA}_1)\times_2(\mathbf{OB}_2,\mathbf{OB}_2),$$
$$(\mathbf{OA}_2,\mathbf{OA}_2)\times_2(\mathbf{A}_1\mathbf{B}_1,\mathbf{A}_1\mathbf{B}_1)=(\mathbf{OA}_1,\mathbf{OA}_1)\times_2(\mathbf{A}_2\mathbf{B}_2,\mathbf{A}_2\mathbf{B}_2).$$

Let us check them:

$$(\mathbf{OA}_2,\mathbf{OA}_2)\times_2(\mathbf{OB}_1,\mathbf{OB}_1)=15.30\times_2 3.30=50.49,$$
$$(\mathbf{OA}_1,\mathbf{OA}_1)\times_2(\mathbf{OB}_2,\mathbf{OB}_2)=6.60\times_2 7.65=50.46,$$
$$(\mathbf{OA}_2,\mathbf{OA}_2)\times_2(\mathbf{A}_1\mathbf{B}_1,\mathbf{A}_1\mathbf{B}_1)=15.30\times_2 3.30=50.49,$$
$$(\mathbf{OA}_1,\mathbf{OA}_1)\times_2(\mathbf{A}_2\mathbf{B}_2,\mathbf{A}_2\mathbf{B}_2)=6.60\times_2 7.65=50.46,$$

that is, in this case the classical geometry equalities are not satisfied in Mathematics with Observers geometry.

Let us finally consider the triangles OA_2B_2 and OA_3B_3. We consider the following equalities:

$$(\mathbf{OA}_3,\mathbf{OA}_3)\times_2(\mathbf{OB}_2,\mathbf{OB}_2)=(\mathbf{OA}_2,\mathbf{OA}_2)\times_2(\mathbf{OB}_3,\mathbf{OB}_3),$$
$$(\mathbf{OA}_3,\mathbf{OA}_3)\times_2(\mathbf{A}_2\mathbf{B}_2,\mathbf{A}_2\mathbf{B}_2)=(\mathbf{OA}_2,\mathbf{OA}_2)\times_2(\mathbf{A}_3\mathbf{B}_3,\mathbf{A}_3\mathbf{B}_3).$$

Let us check them:

$$(\mathbf{OA}_3,\mathbf{OA}_3)\times_2(\mathbf{OB}_2,\mathbf{OB}_2)=19.70\times_2 7.65=NA,$$
$$(\mathbf{OA}_2,\mathbf{OA}_2)\times_2(\mathbf{OB}_3,\mathbf{OB}_3)=15.30\times_2 9.85=NA,$$
$$(\mathbf{OA}_3,\mathbf{OA}_3)\times_2(\mathbf{A}_2\mathbf{B}_2,\mathbf{A}_2\mathbf{B}_2)=19.70\times_2 7.65=NA,$$
$$(\mathbf{OA}_2,\mathbf{OA}_2)\times_2(\mathbf{A}_3\mathbf{B}_3,\mathbf{A}_3\mathbf{B}_3)=15.30\times_2 9.85=NA.$$

Note that

$$19.70\times_2 7.65\notin W_2,$$
$$15.30\times_2 9.85\notin W_2.$$

So

$$(\mathbf{OA}_3,\mathbf{OA}_3)\times_2(\mathbf{OB}_2,\mathbf{OB}_2)=NA,$$
$$(\mathbf{OA}_2,\mathbf{OA}_2)\times_2(\mathbf{OB}_3,\mathbf{OB}_3)=NA,$$
$$(\mathbf{OA}_3,\mathbf{OA}_3)\times_2(\mathbf{A}_2\mathbf{B}_2,\mathbf{A}_2\mathbf{B}_2)=NA,$$
$$(\mathbf{OA}_2,\mathbf{OA}_2)\times_2(\mathbf{A}_3\mathbf{B}_3,\mathbf{A}_3\mathbf{B}_3)=NA,$$

that is, in this case the classical geometry equalities are not satisfied in Mathematics with Observers geometry.

So we have proved the following:

Theorem 9.11.
In Mathematics with Observers geometry in the plane E_2W_n, there are two distinct similar in Euclidean geometry triangles ABC and $A'B'C'$ such that they are similar in Mathematics with Observers geometry.

Theorem 9.12.
In Mathematics with Observers geometry in the plane E_2W_n, there are two distinct similar in Euclidean geometry triangles ABC and $A'B'C'$ such that they are not similar in Mathematics with Observers geometry.

9.9 Pascal's theorem

In classical geometry the following Pascal theorem takes place:

> Given two sets of points A, B, C and A', B', C' situated upon two intersecting straight lines such that none falls at the intersection of these lines. If CB' is parallel to BC' and CA' is parallel to AC', then BA' is parallel to AB'.

Of course, "parallel" here is understood in the Euclidean sense.

Question: Is this theorem correct Mathematics with Observers geometry?

1) Let us take two straight lines in E_2W_2,

$$a : y = 0$$

and

$$a' : x -_2 y = 0,$$

and points $A, B, C \in a$ and $A', B', C' \in a'$:

$$A(8.00, 0.00), B(4.00, 0.00), C(2.00, 0.00), A'(1.00, 1.00), B'(2.00, 2.00), C'(4.00, 4.00).$$

Let us also consider four straight lines $b, c, d, e \in E_2W_2$:

$$\begin{aligned} &b : x +_2 y -_2 2 = 0, \\ &c : x = 2.00, \\ &d : x = 4.00, \\ &e : x +_2 y -_2 8 = 0. \end{aligned}$$

We have

$$A', C \in b; \quad B', C \in c; \quad B, C' \in d; \quad A, C' \in e.$$

Line $A'C$ is parallel to line AC', and line $B'C$ is parallel to line BC' in the Euclidean sense.

Let us consider straight line f containing points A', B and straight line g containing points A, B' if they exist:

$$\begin{aligned} &f : f_1 \times_2 x +_2 f_2 \times_2 y +_2 f_3 = 0, \\ &g : g_1 \times_2 x +_2 g_2 \times_2 y +_2 g_3 = 0. \end{aligned}$$

We have

$$\begin{cases} f_1 \times_2 1 +_2 f_2 \times_2 1 +_2 f_3 = 0, \\ f_1 \times_2 4 +_2 f_2 \times_2 0 +_2 f_3 = 0, \end{cases}$$

that is,

$$\begin{cases} f_1 +_2 f_2 = f_1 \times_2 4, \\ f_1 \times_2 4 +_2 f_3 = 0, \end{cases}$$

that is,

$$\begin{cases} f_2 = f_1 \times_2 3, \\ -f_1 \times_2 4 = f_3. \end{cases}$$

One possible solution of this system is

$$f_1 = 1,$$
$$f_2 = 3,$$
$$f_3 = -4.$$

So one of possible straight lines *f* containing points A', *B* is

$$f : x +_2 3 \times_2 y -_2 4 = 0.$$

Now we go to line *g*. We have

$$\begin{cases} g_1 \times_2 2 +_2 g_2 \times_2 2 +_2 g_3 = 0, \\ g_1 \times_2 8 +_2 g_2 \times_2 0 +_2 g_3 = 0, \end{cases}$$

that is,

$$\begin{cases} g_1 \times_2 2 +_2 g_2 \times_2 2 = g_1 \times_2 8, \\ g_1 \times_2 8 +_2 g_3 = 0, \end{cases}$$

that is,

$$\begin{cases} g_2 \times_2 2 = g_1 \times_2 6, \\ -g_1 \times_2 8 = g_3. \end{cases}$$

One possible solution of this system is

$$g_1 = 1,$$
$$g_2 = 3,$$
$$g_3 = -8.$$

So one of possible straight lines *g* containing points *A*, B' is

$$g : x +_2 3 \times_2 y -_2 8 = 0.$$

We see that lines *f* and *g* are parallel in the Euclidean sense. This means that in this case the answer to the question is positive.

2) Let us take the same two straight lines in E_2W_2,

$$a : y = 0$$

and

$$a' : x -_2 y = 0,$$

and the points $A, B, C \in a$ and $A', B', C' \in a'$:

$$A(9.72, 0.00), B(4.86, 0.00), C(2.14, 0.00), A'(1.07, 1.07), B'(2.14, 2.14), C'(4.86, 4.86).$$

Let us also consider four straight lines $b, c, d, e \in E_2W_2$:

$$b : x +_2 y -_2 2.14 = 0,$$
$$c : x = 2.14,$$
$$d : x = 4.86,$$
$$e : x +_2 y -_2 9.72 = 0.$$

We have

$$A', C \in b; \quad B', C \in c; \quad B, C' \in d; \quad A, C' \in e.$$

Line $A'C$ is parallel to line AC', and line $B'C$ is parallel to line BC' in the Euclidean sense.

Let us consider straight line *f* containing points A', *B* and straight line *g* containing points *A*, B' if they exist:

$$f : f_1 \times_2 x +_2 f_2 \times_2 y +_2 f_3 = 0,$$
$$g : g_1 \times_2 x +_2 g_2 \times_2 y +_2 g_3 = 0$$

We have

$$\begin{cases} f_1 \times_2 1.07 +_2 f_2 \times_2 1.07 +_2 f_3 = 0, \\ f_1 \times_2 4.86 +_2 f_2 \times_2 0 +_2 f_3 = 0, \end{cases}$$

that is,

$$\begin{cases} f_1 \times_2 1.07 +_2 f_2 \times_2 1.07 = f_1 \times_2 4.86, \\ f_1 \times_2 4.86 +_2 f_3 = 0, \end{cases}$$

that is,

$$\begin{cases} f_2 \times_2 1.07 = f_1 \times_2 4.86 -_2 f_1 \times_2 1.07, \\ -f_1 \times_2 4.86 = f_3. \end{cases}$$

One possible solution of this system is

$$f_1 = 1,$$
$$f_2 = 3.58,$$
$$f_3 = -4.86.$$

So one of possible straight lines *f* containing points A', *B* is

$$f : x +_2 3.58 \times_2 y -_2 4.86 = 0.$$

Now we go to line *g*. We have

$$\begin{cases} g_1 \times_2 2.14 +_2 g_2 \times_2 2.14 +_2 g_3 = 0, \\ g_1 \times_2 9.72 +_2 g_2 \times_2 0 +_2 g_3 = 0, \end{cases}$$

that is,

$$\begin{cases} g_1 \times_2 2.14 +_2 g_2 \times_2 2.14 = g_1 \times_2 9.72, \\ g_1 \times_2 9.72 +_2 g_3 = 0, \end{cases}$$

that is,

$$\begin{cases} g_2 \times_2 2.14 = g_1 \times_2 9.72 -_2 g_1 \times_2 2.14, \\ -g_1 \times_2 9.72 = g_3. \end{cases}$$

One possible solution of this system is

$$\begin{aligned} g_1 &= 1.01, \\ g_2 &= 3.59, \\ g_3 &= -9.81. \end{aligned}$$

So one of possible straight lines *g* containing points *A*, B' is

$$g : 1.01 \times_2 x +_2 3.59 \times_2 y -_2 9.81 = 0.$$

Lines *f* and *g* are not parallel in the Euclid sense because straight line g' containing point *A* and parallel to line *f* has the equation

$$g' : x +_2 3.58 \times_2 y -_2 9.72 = 0,$$

and lines *g* and g' are different ($B' \in g$, but $B' \notin g'$).

This means that in this case the answer to the question is negative.

So we have proved the following:

Theorem 9.13.
In Mathematics with Observers geometry in the plane E_2W_n, there are two sets of points A, B, C and A', B', C' situated upon two intersecting straight lines so that none falls at the intersection of these lines, CB' is parallel in the Euclidean sense to BC', CA' is parallel in the Euclidean sense to AC', and BA' is parallel in the Euclidean sense to AB'.

Theorem 9.14.
In Mathematics with Observers geometry in the plane E_2W_n, there are two sets of points A, B, C and A', B', C' situated upon two intersecting straight lines so that none falls at the intersection of these lines, CB' is parallel in the Euclidean sense to BC', CA' is parallel in the Euclidean sense to AC', but BA' is not parallel in the Euclidean sense to AB'.

9.10 Desargues's theorem

In classical geometry the following Desargues theorem takes place:

If two triangles are situated in a plane so that their homologous sides are respectively parallel, then the lines joining the homologous vertices pass through a unique point or are parallel.

Of course, “parallel” here is understood in the Euclidean sense.

Question: Is this theorem correct in Mathematics with Observers geometry?

1) Let us take six points $A, B, C, A', B', C' \in E_2W_2$:

$$A(0,0), B(1,0), C(0,2), A'(2,1), B'(4,1), C'(2,5).$$

Let us consider six straight lines $a, b, c, d, e, f \in E_2W_2$:

$$\begin{aligned}
&a : y = 0,\\
&b : x = 0,\\
&c : 2 \times_2 x +_2 y -_2 2 = 0,\\
&d : y = 1,\\
&e : x = 2,\\
&f : 2 \times_2 x +_2 y -_2 9 = 0.
\end{aligned}$$

We have

$$A, B \in a; \quad A, C \in b; \quad B, C \in c; \quad A', B' \in d; \quad A', C' \in e; \quad B', C' \in f$$

and

$$a \parallel d; \quad b \parallel e; \quad c \parallel f.$$

Let us consider straight lines

$$g \supset [A, A']; \quad h \supset [B, B']; \quad i \supset [C, C']$$

if they exist:

$$\begin{aligned}
&g : g_1 \times_2 x +_2 g_2 \times_2 y +_2 g_3 = 0,\\
&h : h_1 \times_2 x +_2 h_2 \times_2 y +_2 h_3 = 0,\\
&i : i_1 \times_2 x +_2 i_2 \times_2 y +_2 i_3 = 0.
\end{aligned}$$

For line g, we get

$$\begin{cases} g_1 \times_2 0.00 +_2 g_2 \times_2 0.00 +_2 g_3 = 0,\\ g_1 \times_2 2.00 +_2 g_2 \times_2 1.00 +_2 g_3 = 0, \end{cases}$$

that is,

$$\begin{cases} g_3 = 0,\\ g_1 \times_2 2.00 +_2 g_2 = 0. \end{cases}$$

One possible solution of this system is

$$g_1 = -1.00,$$
$$g_2 = 2,$$
$$g_3 = 0,$$

and thus line *g* has the equation

$$g : -x +_2 2 \times_2 y = 0.$$

For line *h*, we get

$$\begin{cases} h_1 \times_2 1.00 +_2 h_2 \times_2 0.00 +_2 h_3 = 0, \\ h_1 \times_2 4.00 +_2 h_2 \times_2 1.00 +_2 h_3 = 0, \end{cases}$$

that is,

$$\begin{cases} h_3 = -h_1, \\ -h_1 \times_2 3.00 = h_2. \end{cases}$$

One possible solution of this system is

$$h_1 = -1.00,$$
$$h_2 = 3,$$
$$h_3 = 1,$$

and thus line *h* has the equation

$$h : -x +_2 3 \times_2 y -_2 1 = 0.$$

For line *i*, we get

$$\begin{cases} i_1 \times_2 0.00 +_2 i_2 \times_2 2.00 +_2 i_3 = 0, \\ i_1 \times_2 2.00 +_2 i_2 \times_2 5.00 +_2 i_3 = 0, \end{cases}$$

that is,

$$\begin{cases} i_3 = -i_2 \times_2 2.00, \\ -i_1 \times_2 2.00 = i_2 \times_2 3.00. \end{cases}$$

One possible solution of this system is

$$i_1 = -3.00,$$
$$i_2 = 2.00,$$
$$i_3 = -4.00,$$

and thus line *i* has the equation

$$i : -3.00 \times_2 x +_2 2.00 \times_2 y -_2 4.00 = 0.$$

Let us now find the intersection of lines *g*, *h*, *i*:

$$-x +_2 2 \times_2 y = 0,$$
$$-x +_2 3 \times_2 y +_2 1 = 0,$$
$$-3.00 \times_2 x +_2 2.00 \times_2 y -_2 4.00 = 0,$$

that is,

$$x = 2 \times_2 y,$$
$$-x +_2 3 \times_2 y +_2 1 = 0,$$
$$-3.00 \times_2 x +_2 2.00 \times_2 y -_2 4.00 = 0,$$

that is,

$$\begin{cases} x = -2.00, \\ y = -1.00, \end{cases}$$

that is, the lines joining the homologous vertices of the triangles ABC and $A'B'C'$ pass through a unique point. This means that in this case the answer to the question is positive.

2) Let us take six points $A, B, C, A', B', C' \in E_2W_2$:

$$A(1.12, 1.12), B(2.12, 1.12), C(1.12, 3.12), A'(2, 1), B'(4, 1), C'(2, 5).$$

Let us consider six straight lines $a, b, c, d, e, f \in E_2W_2$:

$$a : y = 1.12,$$
$$b : x = 1.12,$$
$$c : 2 \times_2 x +_2 y -_2 5.36 = 0,$$
$$d : y = 1,$$
$$e : x = 2,$$
$$f : 2 \times_2 x +_2 y -_2 9 = 0.$$

We have

$$A, B \in a; \quad A, C \in b; \quad B, C \in c; \quad A', B' \in d; \quad A', C' \in e; \quad B', C' \in f$$

and

$$a \parallel d; \quad b \parallel e; \quad c \parallel f.$$

Let us consider straight lines

$$g \supset [A, A']; \quad h \supset [B, B']; \quad i \supset [C, C']$$

if they exist:

$$g : g_1 \times_2 x +_2 g_2 \times_2 y +_2 g_3 = 0,$$
$$h : h_1 \times_2 x +_2 h_2 \times_2 y +_2 h_3 = 0,$$
$$i : i_1 \times_2 x +_2 i_2 \times_2 y +_2 i_3 = 0.$$

For line g, we get

$$\begin{cases} g_1 \times_2 1.12 +_2 g_2 \times_2 1.12 +_2 g_3 = 0, \\ g_1 \times_2 2.00 +_2 g_2 \times_2 1.00 +_2 g_3 = 0, \end{cases}$$

that is,

$$\begin{cases} -g_3 = g_1 \times_2 2.00 +_2 g_2 \times_2 1.00, \\ g_1 \times_2 1.12 -_2 g_1 \times_2 2.00 = g_2 \times_2 1.00 -_2 g_2 \times_2 1.12. \end{cases}$$

One possible solution of this system is

$$\begin{aligned} g_1 &= 1.00, \\ g_2 &= 7.40, \\ g_3 &= -9.40, \end{aligned}$$

and line *g* has the equation

$$g : x +_2 7.40 \times_2 y -_2 9.40 = 0.$$

For line *h*, we get

$$\begin{cases} h_1 \times_2 2.12 +_2 h_2 \times_2 1.12 +_2 h_3 = 0, \\ h_1 \times_2 4.00 +_2 h_2 \times_2 1.00 +_2 h_3 = 0, \end{cases}$$

that is,

$$\begin{cases} -h_3 = h_1 \times_2 4.00 +_2 h_2 \times_2 1.00, \\ h_1 \times_2 2.12 -_2 h_1 \times_2 4.00 = h_2 \times_2 1.00 -_2 h_2 \times_2 1.12. \end{cases}$$

A possible solution of this system is

$$\begin{aligned} h_1 &= 1.00, \\ h_2 &= 15.80, \\ h_3 &= -19.80, \end{aligned}$$

and line *h* has the equation

$$h : x +_2 15.80 \times_2 y -_2 19.80 = 0.$$

For line *i*, we get

$$\begin{cases} i_1 \times_2 1.12 +_2 i_2 \times_2 3.12 +_2 i_3 = 0, \\ i_1 \times_2 2.00 +_2 i_2 \times_2 5.00 +_2 i_3 = 0, \end{cases}$$

that is,

$$\begin{cases} -i_3 = i_1 \times_2 2.00 +_2 i_2 \times_2 5.00, \\ i_1 \times_2 1.12 -_2 i_1 \times_2 2.00 = i_2 \times_2 5.00 -_2 i_2 \times_2 3.12. \end{cases}$$

One possible solution of this system is

$$i_1 = -2.13,$$
$$i_2 = 1.00,$$
$$i_3 = -0.74,$$

and line *i* has the equation

$$i : -2.13 \times_2 x +_2 y -_2 0.74 = 0.$$

Let us now find now the intersection of lines *g*, *h*, *i*:

$$x +_2 7.40 \times_2 y -_2 9.40 = 0,$$
$$x +_2 15.80 \times_2 y -_2 19.80 = 0,$$
$$-2.13 \times_2 x +_2 y -_2 0.74 = 0,$$

that is,

$$x = -7.40 \times_2 y +_2 9.40,$$
$$-7.40 \times_2 y +_2 9.40 +_2 15.80 \times_2 y -_2 19.80 = 0,$$
$$-2.13 \times_2 (-7.40 \times_2 y +_2 9.40) +_2 y -_2 0.74 = 0,$$

that is,

$$x = -7.40 \times_2 y +_2 9.40,$$
$$-7.40 \times_2 y +_2 15.80 \times_2 y -_2 10.40 = 0,$$
$$-2.13 \times_2 (-7.40 \times_2 y +_2 9.40) +_2 y -_2 0.74 = 0,$$

that is,

$$\begin{cases} x = 0.24, \\ y = 1.24, \end{cases}$$

that is, the lines joining the homologous vertices of the triangles ABC and $A'B'C'$ pass through a unique point. This means that in this case the answer to the question is positive.

3) Let us take six points $A, B, C, A', B', C' \in E_2W_2$:

$$A(0.23, 0.98), B(1.11, 2.65), C(1.39, 2.65), A'(2.74, 3.14), B'(8.01, 13.07), C'(8.95, 13.07).$$

Let us consider six straight lines $a, b, c, d, e, f \in E_2W_2$:

$$a : 1.47 \times_2 x -_2 y +_2 0.67 = 0,$$
$$b : 1.87 \times_2 x -_2 y +_2 0.59 = 0,$$
$$c : y -_2 2.65 = 0,$$
$$d : 1.47 \times_2 x -_2 y -_2 0.82 = 0,$$
$$e : 1.87 \times_2 x -_2 y -_2 1.90 = 0,$$
$$f : y -_2 13.07 = 0.$$

We have

$$A, C \in a; \quad A, B \in b; \quad B, C \in c; \quad A', C' \in d; \quad A', B' \in e; \quad B', C' \in f$$

and

$$a \parallel d; \quad b \parallel e; \quad c \parallel f$$

in the Euclidean sense.

Let us consider straight lines

$$g \supset [A, A']; \quad h \supset [B, B']; \quad i \supset [C, C']$$

if they exist:

$$\begin{aligned} g &: g_1 \times_2 x +_2 g_2 \times_2 y +_2 g_3 = 0, \\ h &: h_1 \times_2 x +_2 h_2 \times_2 y +_2 h_3 = 0, \\ i &: i_1 \times_2 x +_2 i_2 \times_2 y +_2 i_3 = 0. \end{aligned}$$

For line *g*, we get

$$\begin{cases} g_1 \times_2 0.23 +_2 g_2 \times_2 0.98 +_2 g_3 = 0, \\ g_1 \times_2 2.74 +_2 g_2 \times_2 3.14 +_2 g_3 = 0, \end{cases}$$

that is,

$$\begin{cases} -g_3 = g_1 \times_2 0.23 +_2 g_2 \times_2 0.98, \\ g_1 \times_2 0.23 -_2 g_1 \times_2 2.74 = g_2 \times_2 3.14 -_2 g_2 \times_2 0.98. \end{cases}$$

One possible solution of this system is

$$\begin{aligned} g_1 &= -0.88, \\ g_2 &= 1.00, \\ g_3 &= -0.82, \end{aligned}$$

and line *g* has the equation

$$g : -0.88 \times_2 x +_2 y -_2 0.82 = 0.$$

For line *h*, we get

$$\begin{cases} h_1 \times_2 1.11 +_2 h_2 \times_2 2.65 +_2 h_3 = 0, \\ h_1 \times_2 8.01 +_2 h_2 \times_2 13.07 +_2 h_3 = 0, \end{cases}$$

that is,

$$\begin{cases} -h_3 = h_1 \times_2 1.11 +_2 h_2 \times_2 2.65, \\ h_1 \times_2 1.11 -_2 h_1 \times_2 8.01 = h_2 \times_2 13.07 -_2 h_2 \times_2 2.65. \end{cases}$$

One possible solution of this system is

$$\begin{aligned} h_1 &= -1.51, \\ h_2 &= 1, \\ h_3 &= -0.98, \end{aligned}$$

and line *h* has the equation

$$h : -1.51 \times_2 x +_2 y -_2 0.98 = 0.$$

For line *i*, we get

$$\begin{cases} i_1 \times_2 1.39 +_2 i_2 \times_2 2.65 +_2 i_3 = 0, \\ i_1 \times_2 8.95 +_2 i_2 \times_2 13.07 +_2 i_3 = 0, \end{cases}$$

that is,

$$\begin{cases} -i_3 = i_1 \times_2 1.39 +_2 i_2 \times_2 2.65, \\ i_1 \times_2 1.39 -_2 i_1 \times_2 8.95 = i_2 \times_2 13.07 -_2 i_2 \times_2 2.65. \end{cases}$$

One possible solution of this system is

$$\begin{aligned} i_1 &= -1.31, \\ i_2 &= 0.95, \\ i_3 &= -0.65, \end{aligned}$$

and line *i* has the equation

$$i : -1.31 \times_2 x +_2 0.95 \times_2 y -_2 0.65 = 0.$$

Let us now find the intersection of lines *g*, *h*, *i*:

$$\begin{aligned} &-0.88 \times_2 x +_2 y -_2 0.82 = 0, \\ &-1.51 \times_2 x +_2 y -_2 0.98 = 0, \\ &-1.31 \times_2 x +_2 0.95 \times_2 y -_2 0.65 = 0, \end{aligned}$$

that is,

$$\begin{aligned} &-0.88 \times_2 x -_2 0.82 = -y, \\ &0.88 \times_2 x +_2 0.82 = 1.51 \times_2 x +_2 0.98, \\ &-1.31 \times_2 x +_2 0.95 \times_2 y -_2 0.65 = 0, \end{aligned}$$

that is

$$\begin{aligned} &x = -0.22, \\ &y = 0.66, \\ &-1.31 \times_2 (-0.22) +_2 0.95 \times_2 0.66 -_2 0.65 = 0.17 \neq 0, \end{aligned}$$

that is, the lines joining the homologous vertexes of the triangles ABC and $A'B'C'$ do not pass through a unique point. This means that in this case the answer to the question is negative.

So we have proved the following:

Theorem 9.15.
In Mathematics with Observers geometry in the plane E_2W_n*, there are two triangles situated in a plane so that their homologous sides are parallel in the Euclidean sense, and the lines joining the homologous vertices pass through a unique point or are parallel in the Euclidean sense.*
Theorem 9.16.
In Mathematics with Observers geometry in the plane E_2W_n*, there are two triangles situated in a plane so that their homologous sides are parallel in the Euclidean sense, and the lines joining the homologous vertices neither pass through a unique point nor are parallel in the Euclidean sense.*

10 Observability and triangles. Special cases

In this chapter, we consider the straight lines with equations

$$y = m \times_n x +_n b$$

or

$$m \times_n x +_n b = 0,$$

where $x, y, m, b, m \times_n x +_n b \in W_n$.

10.1 Angle bisector of triangle theorem

In classical geometry, we have the following statement:

In the angle C of a triangle $\triangle ABC$, the bisector CD divides the side AB proportionally to the corresponding sides:

$$\frac{AC}{AD} = \frac{BC}{BD}.$$

Let us now consider the situation in Mathematics with Observers geometry. First, let us give the definition of the angle bisector line. Let us take a point E on the half-ray AB, a point F on the half-ray AC, and a point G on the half-ray AD and the corresponding three vectors $\mathbf{AE}$, $\mathbf{AF}$, $\mathbf{AG}$. Let

$$(\mathbf{AE}, \mathbf{AE}) > 0,$$
$$(\mathbf{AF}, \mathbf{AF}) > 0,$$
$$(\mathbf{AG}, \mathbf{AG}) > 0,$$
$$(\mathbf{AE}, \mathbf{AE}) = (\mathbf{AF}, \mathbf{AF}).$$

We say that the half-ray AD is an angle A bisector line if

$$(\mathbf{AE}, \mathbf{AG}) = (\mathbf{AG}, \mathbf{AF}).$$

Note that if we take any $\triangle AB'C'$ with points

$$B' \in AB, \quad C' \in AC,$$

then AD is still an angle A bisector.

Let $n = 2$. We would like to check if the following equality is correct:

$$(\mathbf{AB}, \mathbf{AB}) \times_n (\mathbf{DC}, \mathbf{DC}) = (\mathbf{AC}, \mathbf{AC}) \times_n (\mathbf{BD}, \mathbf{BD}).$$

For this, let us consider several cases.

1. Let us consider $\triangle ABC$ with sides

$$\begin{aligned} AB &: y = 2 \times_2 x +_2 2, \\ AC &: y = -2 \times_2 x +_2 2, \\ BC &: y = 0. \end{aligned}$$

Let us determine the coordinates of the vertices of $\triangle ABC$. The coordinates of the point *A* are the solution of the system

$$\begin{cases} y = 2 \times_2 x +_2 2, \\ y = -2 \times_2 x +_2 2, \end{cases}$$

and we get $A(0, 2)$.

The coordinates of the point *B* are the solution of the system

$$\begin{cases} y = 2 \times_2 x +_2 2, \\ y = 0, \end{cases}$$

and we get $B(-1, 0)$.

The coordinates of the point *C* are the solution of the system

$$\begin{cases} y = -2 \times_2 x +_2 2, \\ y = 0, \end{cases}$$

and we get $C(1, 0)$.

The coordinates of the point *D* (base of the angle *A* bisector AD) satisfy

$$\begin{cases} x = 0, \\ y = 0, \end{cases}$$

and we get $D(0, 0)$.

Let us check that the line AD is an angle *A* bisector. Let us consider in this case

$$\begin{aligned} E &= B, \\ F &= C, \\ D &= G, \\ \mathbf{AE} &= (-1, -2), \\ (\mathbf{AE}, \mathbf{AE}) &= -1 \times_2 -1 +_2 -2 \times_2 -2 = 5 > 0, \\ \mathbf{AF} &= (1, -2), \\ (\mathbf{AF}, \mathbf{AF}) &= 1 \times_2 1 +_2 -2 \times_2 -2 = 5 > 0, \\ \mathbf{AG} &= (0, -2), \\ (\mathbf{AG}, \mathbf{AG}) &= 0 \times_2 0 +_2 -2 \times_2 -2 = 4 > 0, \end{aligned}$$

that is,

$$(\mathbf{AE}, \mathbf{AE}) = (\mathbf{AF}, \mathbf{AF}).$$

So we have proved that in this case, the line AD is an angle A bisector. To get the answer to the question above, let us calculate

$$\begin{aligned} \mathbf{BD} &= (1, 0), \\ (\mathbf{BD}, \mathbf{BD}) &= 1 \times_2 1 +_2 0 \times_2 0 = 1 > 0, \\ \mathbf{DC} &= (-1, 0), \\ (\mathbf{DC}, \mathbf{DC}) &= -1 \times_2 -1 +_2 0 \times_2 0 = 1 > 0, \end{aligned}$$

and, finally,

$$\begin{aligned} (\mathbf{AB}, \mathbf{AB}) \times_n (\mathbf{DC}, \mathbf{DC}) &= 5 \times_2 1 = 5, \\ (\mathbf{AC}, \mathbf{AC}) \times_n (\mathbf{BD}, \mathbf{BD}) &= 5 \times_2 1 = 5, \end{aligned}$$

that is,

$$(\mathbf{AB}, \mathbf{AB}) \times_n (\mathbf{DC}, \mathbf{DC}) = (\mathbf{AC}, \mathbf{AC}) \times_n (\mathbf{BD}, \mathbf{BD}).$$

This means that the answer to this question is positive.

2. Let us consider $\triangle ABC'$ with the same points A and B and $C' \in AC$ where C is as in the previous case. So we have

$$A(0, 2), B(-1, 0), C(1, 0),$$

and AD is an angle A bisector. Let

$$BC' : y = 1.01 \times_2 x +_2 1.01.$$

The coordinates of the point C' are the solution of the system

$$\begin{cases} y = -2 \times_2 x +_2 2, \\ y = 1.01 \times_2 x +_2 1.01, \end{cases}$$

and we get $C'(0.33, 1.34)$.

Let D' be the intersection of the angle A bisector AD and line BC'. The coordinates of the point D' are the solution of the system

$$\begin{cases} x = 0, \\ y = 1.01 \times_2 x +_2 1.01, \end{cases}$$

and we get $D'(0, 1.01)$.

We get

$$\begin{aligned}
(\mathbf{AB},\mathbf{AB}) &= 1\times_2 1 +_2 2\times_2 2 = 5 > 0,\\
\left(\mathbf{AC}',\mathbf{AC}'\right) &= 0.33\times_2 0.33 +_2 0.66\times_2 0.66 = 0.45 > 0,\\
\left(\mathbf{BD}',\mathbf{BD}'\right) &= 1\times_2 1 +_2 1.01\times_2 1.01 = 2.02 > 0,\\
\left(\mathbf{D}'\mathbf{C}',\mathbf{D}'\mathbf{C}'\right) &= 0.33\times_2 0.33 +_2 0.33\times_2 0.33 = 0.18 > 0,
\end{aligned}$$

and, finally,

$$\begin{aligned}
(\mathbf{AB},\mathbf{AB})\times_2\left(\mathbf{D}'\mathbf{C}',\mathbf{D}'\mathbf{C}'\right) &= 0.9,\\
\left(\mathbf{AC}',\mathbf{AC}'\right)\times_2\left(\mathbf{BD}',\mathbf{BD}'\right) &= 0.9
\end{aligned}$$

So

$$(\mathbf{AB},\mathbf{AB})\times_2\left(\mathbf{D}'\mathbf{C}',\mathbf{D}'\mathbf{C}'\right) = \left(\mathbf{AC}',\mathbf{AC}'\right)\times_2\left(\mathbf{BD}',\mathbf{BD}'\right).$$

This means that the answer to the question is positive.

3. Let us consider $\triangle ABC$ with sides

$$\begin{aligned}
&AB: y = 4\times_2 x +_2 8,\\
&AC: y = -4\times_2 x +_2 8,\\
&BC: y = 0.
\end{aligned}$$

Let us determine the coordinates of the vertices of $\triangle ABC$. The coordinates of the point *A* are the solution of the system

$$\begin{cases} y = 4\times_2 x +_2 8,\\ y = -4\times_2 x +_2 8, \end{cases}$$

and we get $A(0,8)$.

The coordinates of the point *B* are the solution of the system

$$\begin{cases} y = 4\times_2 x +_2 8,\\ y = 0, \end{cases}$$

and we get $B(-2,0)$.

The coordinates of the point *C* are the solution of the system

$$\begin{cases} y = -4\times_2 x +_2 8,\\ y = 0, \end{cases}$$

and we get $C(2,0)$.

The coordinates of the point *D* (base of the angle *A* bisector AD) are

$$\begin{cases} x = 0,\\ y = 0, \end{cases}$$

and we get $D(0,0)$.

Let us check that line AD is an angle A bisector. Let us consider in this case

$$\begin{aligned} E &= B, \\ F &= C, \\ D &= G, \\ \mathbf{AE} &= (-2,-8), \\ (\mathbf{AE},\mathbf{AE}) &= -2\times_2 -2 +_2 -8\times_2 -8 = 68 > 0, \\ \mathbf{AF} &= (2,-8), \\ (\mathbf{AF},\mathbf{AF}) &= 2\times_2 2 +_2 -8\times_2 -8 = 68 > 0, \\ \mathbf{AG} &= (0,-8), \\ (\mathbf{AG},\mathbf{AG}) &= 0\times_2 0 +_2 -8\times_2 -8 = 64 > 0, \end{aligned}$$

that is,

$$(\mathbf{AE},\mathbf{AE}) = (\mathbf{AF},\mathbf{AF}).$$

So we have proved that in this case line AD is an angle A bisector. To get the answer to the question above, let us calculate

$$\begin{aligned} \mathbf{BD} &= (2,0), \\ (\mathbf{BD},\mathbf{BD}) &= 2\times_2 2 +_2 0\times_2 0 = 4 > 0, \\ \mathbf{DC} &= (-2,0), \\ (\mathbf{DC},\mathbf{DC}) &= -2\times_2 -2 +_2 0\times_2 0 = 4 > 0, \end{aligned}$$

and, finally,

$$\begin{aligned} (\mathbf{AB},\mathbf{AB}) \times_n (\mathbf{DC},\mathbf{DC}) &= 68\times_2 4 \notin W_2, \\ (\mathbf{AC},\mathbf{AC}) \times_n (\mathbf{BD},\mathbf{BD}) &= 68\times_2 4 \notin W_2. \end{aligned}$$

Let

$$BC' : y = 2.02\times_2 x +_2 4.04.$$

The coordinates of the point C' are the solution of the system

$$\begin{cases} y = -4\times_2 x +_2 8, \\ y = 2.02\times_2 x +_2 4.04, \end{cases}$$

and we get $C'(0.66, 5.36)$.

Let D' be the intersection of the angle A bisector AD and line BC'. The coordinates of the point D' are the solution of the system

$$\begin{cases} x = 0, \\ y = 2.02\times_2 x +_2 4.04, \end{cases}$$

and we get $D'(0, 4.04)$.

We get

$$\begin{aligned}(\mathbf{AB}, \mathbf{AB}) &= 68 > 0,\\ (\mathbf{AC'}, \mathbf{AC'}) &= 7.28 > 0,\\ (\mathbf{BD'}, \mathbf{BD'}) &= 20.32 > 0,\\ (\mathbf{D'C'}, \mathbf{D'C'}) &= 2.09 > 0,\end{aligned}$$

and, finally,

$$(\mathbf{AB}, \mathbf{AB}) \times_2 (\mathbf{D'C'}, \mathbf{D'C'}) = 68 \times_2 2.09 \notin W_2.$$

This means that the answer to the question in this case is negative because the element of this equality that we checked does not exist in W_2.

4. Let us consider $\triangle ABC$ with sides

$$\begin{aligned}&AB : y = 3 \times_2 x +_2 6,\\ &AC : y = -3 \times_2 x +_2 6,\\ &BC : y = 0.\end{aligned}$$

Let us determine the coordinates of the vertices of $\triangle ABC$. The coordinates of the point *A* are the solution of the system

$$\begin{cases} y = 3 \times_2 x +_2 6,\\ y = -3 \times_2 x +_2 6,\end{cases}$$

and we get $A(0, 6)$.

The coordinates of the point *B* are the solution of the system

$$\begin{cases} y = 3 \times_2 x +_2 6,\\ y = 0,\end{cases}$$

and we get $B(-2, 0)$.

The coordinates of the point *C* are the solution of the system

$$\begin{cases} y = -3 \times_2 x +_2 6,\\ y = 0,\end{cases}$$

and we get $C(2, 0)$.

The coordinates of the point *D* (base of the angle *A* bisector AD) are

$$\begin{cases} x = 0,\\ y = 0,\end{cases}$$

and we get $D(0,0)$.

Let us check that line AD is an angle A bisector. Let u consider in this case

$$\begin{aligned} E &= B, \\ F &= C, \\ D &= G, \\ \mathbf{AE} &= (-2,-6), \\ (\mathbf{AE},\mathbf{AE}) &= -2 \times_2 -2 +_2 -6 \times_2 -6 = 40 > 0, \\ \mathbf{AF} &= (2,-6), \\ (\mathbf{AF},\mathbf{AF}) &= 2 \times_2 2 +_2 -6 \times_2 -6 = 40 > 0, \\ \mathbf{AG} &= (0,-6), \\ (\mathbf{AG},\mathbf{AG}) &= 0 \times_2 0 +_2 -6 \times_2 -6 = 36 > 0, \end{aligned}$$

that is,

$$(\mathbf{AE},\mathbf{AE}) = (\mathbf{AF},\mathbf{AF}).$$

So we have proved that in this case, line AD is an angle A bisector. To get the answer to the question above, let us calculate

$$\begin{aligned} \mathbf{BD} &= (2,0), \\ (\mathbf{BD},\mathbf{BD}) &= 2 \times_2 2 +_2 0 \times_2 0 = 4 > 0, \\ \mathbf{DC} &= (-2,0), \\ (\mathbf{DC},\mathbf{DC}) &= -2 \times_2 -2 +_2 0 \times_2 0 = 4 > 0, \end{aligned}$$

and, finally,

$$\begin{aligned} (\mathbf{AB},\mathbf{AB}) \times_n (\mathbf{DC},\mathbf{DC}) &= 40 \times_2 4 \notin W_2, \\ (\mathbf{AC},\mathbf{AC}) \times_n (\mathbf{BD},\mathbf{BD}) &= 40 \times_2 4 \notin W_2, \end{aligned}$$

that is,

$$(\mathbf{AB},\mathbf{AB}) \times_n (\mathbf{DC},\mathbf{DC}) = (\mathbf{AC},\mathbf{AC}) \times_n (\mathbf{BD},\mathbf{BD}).$$

Let

$$BC' : y = 1.51 \times_2 x +_2 3.02.$$

The coordinates of the point C' are the solution of the system

$$\begin{cases} y = -3 \times_2 x +_2 6, \\ y = 1.51 \times_2 x +_2 3.02, \end{cases}$$

and we get $C'(0.67, 3.99)$.

Let D' be the intersection of the angle A bisector AD and line BC'. The coordinates of the point D' are the solution of the system

$$\begin{cases} x = 0, \\ y = 1.51 \times_2 x +_2 3.02, \end{cases}$$

and we get $D'(0, 3.02)$.

We get

$$\begin{aligned}(\mathbf{AB}, \mathbf{AB}) &= 40 > 0, \\ (\mathbf{AC'}, \mathbf{AC'}) &= 4.4 > 0, \\ (\mathbf{BD'}, \mathbf{BD'}) &= 13.12 > 0, \\ (\mathbf{D'C'}, \mathbf{D'C'}) &= 1.17 > 0,\end{aligned}$$

and, finally,

$$\begin{aligned}(\mathbf{AB}, \mathbf{AB}) \times_2 (\mathbf{D'C'}, \mathbf{D'C'}) &= 40 \times_2 1.17 = 46.8, \\ (\mathbf{AC'}, \mathbf{AC'}) \times_2 (\mathbf{BD'}, \mathbf{BD'}) &= 4.4 \times_2 13.12 = 57.72.\end{aligned}$$

So

$$46.8 \neq 57.72.$$

This means that the answer to this question in this case is negative.

So we have proved the following:

Theorem 10.1.

In Mathematics with Observers geometry on the plane, there are triangles where the classical angle bisector theorem adopted for Observers' case is correct, and there are triangles where this statement is wrong.

10.2 Middle of segment, median, gravitation center of triangle

Before developing the main theorem of this topic, we need to consider several situations.

1. Let $n = 2$, and let us consider $\triangle(OAB)$ with

$$O(0, 0); \quad A(1, 6); \quad B(1.5, 4.5).$$

Let *L*, *M*, and *N* be the midpoints of OA, AB, and OB, respectively:

$$L(0.5, 3); \quad M(1.25, 5.25); \quad N(0.75, 2.25).$$

Let us find the equation of median BL:

$$y = k \times_2 x +_2 b,$$
$$\begin{cases} 4.5 = 1.5 \times_2 k +_2 b, \\ 3 = 0.5 \times_2 k +_2 b. \end{cases}$$

So

$$\begin{cases} k = 1.5, \\ b = 2.25, \end{cases}$$

and thus the equation of median BL is

$$y = 1.5 \times_2 x +_2 2.25.$$

Let us find the equation of median OM:

$$y = k \times_2 x +_2 b,$$
$$\begin{cases} 0 = 0 \times_2 k +_2 b, \\ 5.25 = 1.25 \times_2 k +_2 b. \end{cases}$$

So

$$\begin{cases} k = 4.21, \\ b = 0, \end{cases}$$

and, finally, the equation of median OM is

$$y = 4.21 \times_2 x.$$

Let us find the equation of the median AN:

$$y = k \times_2 x +_2 b,$$
$$\begin{cases} 6 = 1 \times_2 k +_2 b, \\ 2.25 = 0.75 \times_2 k +_2 b. \end{cases}$$

So

$$\begin{cases} k = 15, \\ b = -9, \end{cases}$$

and, finally, the equation of median AN is

$$y = 15 \times_2 x -_2 9.$$

The centroid of $\triangle(OAB)$ is the point of intersection of all three medians BL, OM, AN, that is, a solution of the following system of equations:

$$y = 1.5 \times_2 x +_2 2.25,$$
$$y = 4.21 \times_2 x,$$
$$y = 15 \times_2 x -_2 9.$$

First, let us find the solution of the system

$$\begin{cases} y = 1.5 \times_2 x +_2 2.25, \\ y = 15 \times_2 x -_2 9, \end{cases}$$
$$0 = 15 \times_2 x -_2 1.5 \times_2 x -_2 11.25,$$

or

$$11.25 = 15 \times_2 x -_2 1.5 \times_2 x.$$

When $x = 0.83$, we get

$$15 \times_2 0.83 -_2 1.5 \times_2 0.83 = 11.22 \neq 11.25.$$

When $x = 0.84$, we get

$$15 \times_2 0.84 -_2 1.5 \times_2 0.84 = 11.36 \neq 11.25.$$

This means that this system has no solution, that is, the medians BL and AN do not intersect.

Let us find the solution of the system

$$\begin{cases} y = 1.5 \times_2 x +_2 2.25, \\ y = 4.21 \times_2 x, \end{cases}$$

$$2.25 = 4.21 \times_2 x -_2 1.5 \times_2 x.$$

When $x = 0.83$, we get

$$4.25 \times_2 0.83 -_2 1.5 \times_2 0.83 = 3.48 -_2 1.23 = 2.25.$$

This means that this system has a solution: the medians BL and OM intersect at the point

$$D(0.83, 3.48).$$

Let us find the solution of the system

$$\begin{cases} y = 15 \times_2 x -_2 9, \\ y = 4.21 \times_2 x, \end{cases}$$

$$15 \times_2 x -_2 4.21 \times_2 x = 9.$$

When $x = 0.83$, we get

$$15 \times_2 0.83 -_2 4.21 \times_2 0.83 = 8.97 < 9.$$

When $x = 0.84$, we get

$$15 \times_2 0.84 -_2 4.21 \times_2 0.84 = 9.08 > 9.$$

This means that this system has no solution, that is, the medians AN and OM do not intersect.

This means that $\triangle(OAB)$ does not have a centroid, so a very important conclusion to classical mechanics is that there are some homogeneous planes of the form of $\triangle(OAB)$ that have no center of gravity.

2. Let $n = 2$, and consider $\triangle(OAB)$ with

$$O(0,0); \quad A(0,4); \quad B(2,0).$$

Let K, C, and L be the midpoints of OA, AB, and OB, respectively. We get

$$K(0,2); \quad C(1,2); \quad L(1,0).$$

Let us find the equation of the median OC:

$$y = k \times_2 x +_2 b,$$
$$\begin{cases} 0 = 0 \times_2 k +_2 b, \\ 2 = 1 \times_2 k +_2 b. \end{cases}$$

So

$$\begin{cases} k = 2, \\ b = 0, \end{cases}$$

and, finally, the equation of median OC is

$$y = 2 \times_2 x.$$

Let us find the equation of median AL:

$$y = k \times_2 x +_2 b,$$
$$\begin{cases} 4 = 0 \times_2 k +_2 b, \\ 0 = 1 \times_2 k +_2 b. \end{cases}$$

So

$$\begin{cases} k = -4, \\ b = 4, \end{cases}$$

and, finally, the equation of median AL is

$$y = -4 \times_2 x +_2 4.$$

Let us find the equation of median BK:

$$y = k \times_2 x +_2 b,$$
$$\begin{cases} 2 = 0 \times_2 k +_2 b, \\ 0 = 2 \times_2 k +_2 b. \end{cases}$$

So

$$\begin{cases} k = -1, \\ b = 2, \end{cases}$$

and, finally, the equation of median AN is

$$y = -x +_2 2.$$

The centroid of $\triangle(OAB)$ is the point of intersection of all three medians OC, AL, BK, that is, is the solution of the system of equations

$$y = 2 \times_2 x,$$
$$y = -4 \times_2 x +_2 4,$$
$$y = -x +_2 2.$$

First, let us find the solution of the system

$$\begin{cases} y = 2 \times_2 x, \\ y = -4 \times_2 x +_2 4, \end{cases}$$
$$2 \times_2 x = -4 \times_2 x +_2 4,$$

or

$$4 = 4 \times_2 x +_2 2 \times_2 x.$$

When $x = 0.66$, we get

$$4 \times_2 0.66 +_2 2 \times_2 0.66 = 3.96 < 4.$$

When $x = 0.67$, we get

$$4 \times_2 0.67 +_2 2 \times_2 0.67 = 4.02 > 4.$$

This means that this system has no solution, that is, the medians OC and AL do not intersect.

Let us find the solution of the system

$$\begin{cases} y = 2 \times_2 x, \\ y = -x +_2 2, \end{cases}$$
$$2 = 2 \times_2 x +_2 x.$$

When $x = 0.66$, we get

$$2 \times_2 0.66 +_2 0.66 = 1.98 < 2.$$

When $x = 0.67$ we get

$$2 \times_2 0.67 +_2 0.67 = 2.01 > 2.$$

This means that this system has no solution, that is, the medians OC and BK do not intersect.

Let us find the solution of the system

$$\begin{cases} y = -4 \times_2 x +_2 4, \\ y = -x +_2 2, \end{cases}$$
$$2 = 4 \times_2 x -_2 x.$$

When $x = 0.66$, we get

$$4 \times_2 0.66 -_2 0.66 = 1.98 < 2.$$

When $x = 0.67$, we get

$$4 \times_2 0.67 -_2 0.67 = 2.01 > 2.$$

This means that this system has no solution, that is, the medians AL and BK do not intersect. So we have showed that each pair of medians does not intersect. This means that $\triangle(OAB)$ does not have a centroid, and again it is a very important conclusion to classical mechanics: we get some homogeneous planes of the form $\triangle(OAB)$ having no center of gravity.

3. Let $n = 3$, and consider $\triangle(OAB)$ with

$$O(0,0); \quad A(2, 1.734); \quad B(4,0).$$

Let *K*, *C*, and *L* be the midpoints of OA, AB, and OB, respectively. We get

$$K(1, 0.867); \quad C(3, 0.867); \quad L(2,0).$$

Let us find the equation of median OC:

$$y = k \times_3 x +_3 b,$$
$$\begin{cases} 0 = 0 \times_3 k +_3 b, \\ 0.867 = 3 \times_3 k +_3 b. \end{cases}$$

So

$$\begin{cases} k = 0.289, \\ b = 0, \end{cases}$$

and, finally, the equation of median OC is

$$y = 0.289 \times_3 x.$$

The equation of median AL is

$$x = 2.$$

Let us find the equation of median BK:

$$y = k \times_3 x +_3 b,$$
$$\begin{cases} 0 = 4 \times_3 k +_3 b, \\ 0.867 = 1 \times_3 k +_3 b. \end{cases}$$

So

$$\begin{cases} k = -0.289, \\ b = 1.156. \end{cases}$$

The centroid of $\triangle(OAB)$ is the point of intersection of all three medians OC, AL, BK, the solution of the system of equations

$$y = 0.289 \times_3 x,$$
$$x = 2,$$
$$y = -0.289 \times_3 x +_3 1.156.$$

First, let us find the solution of the system

$$\begin{cases} y = 0.289 \times_3 x, \\ x = 2. \end{cases}$$

So

$$\begin{cases} x = 2, \\ y = 0.578. \end{cases}$$

Let us find the solution of the system

$$\begin{cases} y = -0.289 \times_3 x +_3 1.156, \\ x = 2. \end{cases}$$

So

$$\begin{cases} x = 2, \\ y = 0.578. \end{cases}$$

This means that the system of equations

$$\begin{aligned} &y = 0.289 \times_3 x, \\ &x = 2, \\ &y = -0.289 \times_3 x +_3 1.156 \end{aligned}$$

has the solution

$$\begin{cases} x = 2, \\ y = 0.578. \end{cases}$$

So we have showed that three medians of $\triangle(OAB)$ intersect, and thus $\triangle(OAB)$ has a centroid. We have proved the following:

Theorem 10.2.
In Mathematics with Observers geometry on the planes, there are triangles having a centroid and triangles having no centroid.
4. Let us continue to consider $\triangle(OAB)$ with vertices

$$O(0,0); \quad A(2,1.734); \quad B(4,0)$$

and the midpoints *K*, *C*, and *L* of OA, AB, and OB, respectively:

$$K(1,0.867); \quad C(3,0.867); \quad L(2,0)$$

from the W_3-observer point of view.

As we showed above, the $\triangle(OAB)$ has the centroid $F(2,0.528)$.

Classical geometry states the following theorem:

"Three medians of any triangle intersect in one point (called the centroid), and this point divides each median in the ratio 1:2."

So in our case, from the classical geometry point of view, we must have

$$|OF| = 2|FC|, \quad |AF| = 2|FL|, \quad |BF| = 2|FK|.$$

Let us check this 1:2 property in our case. Because $|OF|$, $|FC|$, $|AF|$, $|FL|$, $|BF|$, $|FK|$ do not necessarily exist in W_3, we will check this as follows. First, we have to introduce the vectors

$$\begin{aligned} \mathbf{OF} &= (2, 0.528), \\ \mathbf{FC} &= (1, 0.339), \\ \mathbf{AF} &= (0, -1.206), \\ \mathbf{FL} &= (0, -0.528), \\ \mathbf{BF} &= (-2, 0.528), \\ \mathbf{FK} &= (-1, 0.339). \end{aligned}$$

Now we calculate the scalar products of all vectors to themselves:

$$\begin{aligned} (\mathbf{OF}, \mathbf{OF}) &= 2 \times_3 2 +_3 0.528 \times_3 0.528 = 4 +_3 0.27 = 4.27, \\ (\mathbf{FC}, \mathbf{FC}) &= 1 \times_3 1 +_3 0.339 \times_3 0.339 = 1 +_3 0.108 = 1.108, \\ (\mathbf{AF}, \mathbf{AF}) &= 0 \times_3 0 +_3 (-1.206) \times_3 (-1.206) = 0 +_3 1.452 = 1.452, \\ (\mathbf{FL}, \mathbf{FL}) &= 0 \times_3 0 +_3 (-0.528) \times_3 (-0.528) = 0 +_3 0.27 = 0.27, \\ (\mathbf{BF}, \mathbf{BF}) &= (-2) \times_3 (-2) +_3 0.528 \times_3 0.528 = 4 +_3 0.27 = 4.27, \\ (\mathbf{FK}, \mathbf{FK}) &= (-1) \times_3 (-1) +_3 0.339 \times_3 0.339 = 1 +_3 0.108 = 1.108, \end{aligned}$$

and we see that

$$(\mathbf{OF}, \mathbf{OF}) \neq 4 \times_3 (\mathbf{FC}, \mathbf{FC}),$$

because $4.27 \neq 4 \times_3 1.108 = 4.432$,

$$(\mathbf{AF}, \mathbf{AF}) \neq 4 \times_3 (\mathbf{FL}, \mathbf{FL}),$$

because $1.452 \neq 4 \times_3 0.27 = 1.08$, and

$$(\mathbf{BF}, \mathbf{BF}) \neq 4 \times_3 (\mathbf{FK}, \mathbf{FK}),$$

because $4.27 \neq 4 \times_3 1.108 = 4.432$.

5. Let again $n = 3$. Let us consider the $\triangle(OAB)$ with

$$O(0,0); \quad A(2, 3.468); \quad B(4,0).$$

Let *K*, *C*, and *L* be the midpoints of OA, AB, and OB, respectively. We get

$$K(1, 1.734); \quad C(3, 1.734); \quad L(2,0).$$

Let us find the equation of the median OC:

$$y = k \times_3 x +_3 b,$$

$$\begin{cases} 0 = 0 \times_3 k +_3 b, \\ 1.734 = 3 \times_3 k +_3 b, \end{cases}$$

and so

$$\begin{cases} k = 0.578, \\ b = 0. \end{cases}$$

Finally, the equation of median OC is

$$y = 0.578 \times_3 x,$$

and the equation of the median AL is

$$x = 2.$$

Let us find the equation of the median BK:

$$y = k \times_3 x +_3 b,$$

$$\begin{cases} 0 = 4 \times_3 k +_3 b, \\ 1.734 = 1 \times_3 k +_3 b, \end{cases}$$

and so

$$\begin{cases} k = -0.578, \\ b = 2.312. \end{cases}$$

Finally, the equation of the median BK is

$$y = -0.578 \times_3 x +_3 2.312.$$

The centroid of $\triangle(OAB)$ is the point of the intersection of all three medians OC, AL, BK, that is, the solution of the system of equations

$$y = 0.578 \times_3 x,$$
$$x = 2,$$
$$y = -0.578 \times_3 x +_3 2.312.$$

First, let us find the solution of the system

$$\begin{cases} y = 0.578 \times_3 x, \\ x = 2. \end{cases}$$

So

$$\begin{cases} x = 2, \\ y = 1.156. \end{cases}$$

Now let us find the solution of the system

$$\begin{cases} y = -0.578 \times_3 x +_3 2.312, \\ x = 2. \end{cases}$$

So

$$\begin{cases} x = 2, \\ y = 1.156. \end{cases}$$

This means that the system of equations

$$\begin{aligned} &y = 0.578 \times_3 x, \\ &x = 2, \\ &y = -0.578 \times_3 x +_3 2.312 \end{aligned}$$

has the solution

$$\begin{cases} x = 2, \\ y = 1.156. \end{cases}$$

So we have showed that three medians of $\triangle(OAB)$ intersect, which means that $\triangle(OAB)$ has the centroid point $F(2, 1.156)$.

Let us go back to the classical geometry theorem:

> "Three medians of any triangle intersect in one point (called the centroid), and this point divides each median in the ratio 1:2."

So in our case, from the classical geometry point of view, we must have

$$|OF| = 2|FC|, \quad |AF| = 2|FL|, \quad |BF| = 2|FK|.$$

Let us check this 1:2 property in our case. Because $|OF|$, $|FC|$, $|AF|$, $|FL|$, $|BF|$, $|FK|$ do not necessarily exist in W_3, we will check this as follows. First, we introduce the vectors

$$\begin{aligned} \mathbf{OF} &= (2, 1.156), \\ \mathbf{FC} &= (1, 0.578), \\ \mathbf{AF} &= (0, -2.312), \\ \mathbf{FL} &= (0, -1.156), \\ \mathbf{BF} &= (-2, 1.156), \\ \mathbf{FK} &= (-1, 0.578). \end{aligned}$$

Now we calculate the scalar products of all vectors to themselves:

$$
\begin{aligned}
(\mathbf{OF},\mathbf{OF}) &= 2\times_3 2+_3 1.156\times_3 1.156 = 4+_3 1.332 = 5.332,\\
(\mathbf{FC},\mathbf{FC}) &= 1\times_3 1+_3 0.578\times_3 0.578 = 1+_3 0.32 = 1.32,\\
(\mathbf{AF},\mathbf{AF}) &= 0\times_3 0+_3 (-2.312)\times_3 (-2.312) = 0+_3 5.344 = 5.344,\\
(\mathbf{FL},\mathbf{FL}) &= 0\times_3 0+_3 (-1.156)\times_3 (-1.156) = 0+_3 1.332 = 1.332,\\
(\mathbf{BF},\mathbf{BF}) &= (-2)\times_3 (-2)+_3 1.156\times_3 1.156 = 4+_3 1.332 = 5.332,\\
(\mathbf{FK},\mathbf{FK}) &= (-1)\times_3 (-1)+_3 0.578\times_3 0.578 = 1+_3 0.32 = 1.32,
\end{aligned}
$$

and we see that

$$(\mathbf{OF},\mathbf{OF}) \neq 4\times_3 (\mathbf{FC},\mathbf{FC}),$$

because $5.332 \neq 4\times_3 1.32 = 5.28$,

$$(\mathbf{AF},\mathbf{AF}) \neq 4\times_3 (\mathbf{FL},\mathbf{FL}),$$

because $5.344 \neq 4\times_3 1.332 = 5.328$, and

$$(\mathbf{BF},\mathbf{BF}) \neq 4\times_3 (\mathbf{FK},\mathbf{FK}),$$

because $5.332 \neq 4\times_3 1.32 = 5.28$.

6. Let now $n=6$. Let us consider the $\triangle(OAB)$ with

$$O(0,0);\quad A(2,3.464106);\quad B(4,0).$$

Let *K*, *C*, and *L* be the midpoints of OA, AB, and OB, respectively. We get

$$K(1,1.732053);\quad C(3,1.732053);\quad L(2,0).$$

Let us find the equation of the median OC:

$$
\begin{aligned}
&y = k\times_6 x+_6 b,\\
&\begin{cases} 0 = 0\times_6 k+_6 b,\\ 1.732053 = 3\times_6 k+_6 b.\end{cases}
\end{aligned}
$$

So

$$\begin{cases} k = 0.577351,\\ b = 0.\end{cases}$$

Finally, the equation of the median OC is

$$y = 0.577351\times_6 x,$$

and the equation of the median AL is

$$x = 2.$$

Let us find the equation of the median BK:

$$y = k \times_6 x +_6 b,$$
$$\begin{cases} 0 = 4 \times_6 k +_6 b, \\ 1.732053 = 1 \times_6 k +_6 b. \end{cases}$$

So

$$\begin{cases} k = -0.577351, \\ b = 2.309404, \end{cases}$$

and, finally, the equation of the median BK is

$$y = -0.577351 \times_6 x +_6 2.309404.$$

The centroid of $\triangle(OAB)$ is the point of intersection of all three medians OC, AL, BK, that is, a solution of the system of equations

$$\begin{aligned} &y = 0.577351 \times_6 x, \\ &x = 2, \\ &y = -0.577351 \times_6 x +_6 2.309404. \end{aligned}$$

First, let us find the solution of the system

$$\begin{cases} y = 0.577351 \times_6 x, \\ x = 2. \end{cases}$$

So

$$\begin{cases} x = 2, \\ y = 1.154702. \end{cases}$$

Now let us find the solution of the system

$$\begin{cases} y = -0.577351 \times_6 x +_6 2.309404, \\ x = 2. \end{cases}$$

So

$$\begin{cases} x = 2, \\ y = 1.154702. \end{cases}$$

This means that the system of equations

$$\begin{aligned} &y = 0.577351 \times_6 x, \\ &x = 2, \\ &y = -0.577351 \times_6 x +_6 2.309404 \end{aligned}$$

has the solution

$$\begin{cases} x = 2, \\ y = 1.154702. \end{cases}$$

So we have showed that three medians of $\triangle(OAB)$ intersect, and this means that $\triangle(OAB)$ has the centroid point $F(2, 1.154702)$.

Let us go back to the classical geometry theorem:

> “Three medians of any triangle intersect in one point (called the centroid), and this point divides each median in the ratio 1:2.”

In our case, from classical geometry point of view, we must have

$$|OF| = 2|FC|, \quad |AF| = 2|FL|, \quad |BF| = 2|FK|.$$

Let us check this 1:2 property in our case. Because $|OF|$, $|FC|$, $|AF|$, $|FL|$, $|BF|$, $|FK|$ do not necessarily exist in W_6, we will check this as follows. First, we introduce the vectors

$$\begin{aligned}
\mathbf{OF} &= (2, 1.154702),\\
\mathbf{FC} &= (1, 0.577351),\\
\mathbf{AF} &= (0, -2.309404),\\
\mathbf{FL} &= (0, -1.154702),\\
\mathbf{BF} &= (-2, 1.154702),\\
\mathbf{FK} &= (-1, 0.577351).
\end{aligned}$$

Now we calculate the scalar products of all vectors to themselves:

$$\begin{aligned}
(\mathbf{OF}, \mathbf{OF}) &= 2 \times_6 2 +_6 1.154702 \times_6 1.154702 = 4 +_6 1.33333 = 5.33333,\\
(\mathbf{FC}, \mathbf{FC}) &= 1 \times_6 1 +_6 0.577351 \times_6 0.577351 = 1 +_6 0.333321 = 1.333321,\\
(\mathbf{AF}, \mathbf{AF}) &= 0 \times_6 0 +_6 (-2.309404) \times_6 (-2.309404) = 0 +_6 5.333337 = 5.333337,\\
(\mathbf{FL}, \mathbf{FL}) &= 0 \times_6 0 +_6 (-1.154702) \times_6 (-1.154702) = 0 +_6 1.33333 = 1.33333,\\
(\mathbf{BF}, \mathbf{BF}) &= (-2) \times_6 (-2) +_6 1.154702 \times_6 1.154702 = 4 +_6 1.33333 = 5.33333,\\
(\mathbf{FK}, \mathbf{FK}) &= (-1) \times_6 (-1) +_6 0.577351 \times_6 0.577351 = 1 +_6 0.333321 = 1.333321,
\end{aligned}$$

and we see that

$$(\mathbf{OF}, \mathbf{OF}) \neq 4 \times_6 (\mathbf{FC}, \mathbf{FC}),$$

because $5.33333 \neq 4 \times_6 1.333321 = 5.333284$,

$$(\mathbf{AF}, \mathbf{AF}) \neq 4 \times_6 (\mathbf{FL}, \mathbf{FL}),$$

because $5.333337 \neq 4 \times_6 1.33333 = 5.33332$, and

$$(\mathbf{BF}, \mathbf{BF}) \neq 4 \times_6 (\mathbf{FK}, \mathbf{FK}),$$

because $5.33333 \neq 4 \times_6 1.333321 = 5.333284$.

So we have proved the following:

Theorem 10.3.

In Mathematics with Observers geometry on the planes, there are triangles having a centroid that does not divide each median in the ratio 1:2.

7. Let now $n = 6$. Let us consider the $\triangle(OAB)$ with

$$O(0,0); \quad A(2, 3.464102); \quad B(4,0).$$

We have

$$|OA| = |AB| = |OB| = 4,$$

because

$$2 \times_6 2 +_6 3.464102 \times_6 3.464102 = 4 +_6 12 = 16,$$

that is, $\triangle(OAB)$ is an equilateral triangle.

Let *K*, *C*, and *L* be the midpoints of OA, AB, and OB, respectively. We get

$$K(1, 1.732051); \quad C(3, 1.732051); \quad L(2,0)$$

Let us find the equation of the median OC:

$$y = k \times_6 x +_6 b,$$
$$\begin{cases} 0 = 0 \times_6 k +_6 b, \\ 1.732051 = 3 \times_6 k +_6 b. \end{cases}$$

So we get

$$b = 0,$$

but *k* does not exist, because

$$3 \times_6 0.577350 = 1.732050 < 1.732051$$

and

$$3 \times_6 0.577351 = 1.732053 > 1.732051.$$

So the median OC does not exist.

The equation of the median AL is

$$x = 2.$$

Let us find the equation of the median BK:

$$y = k \times_6 x +_6 b,$$
$$\begin{cases} 0 = 4 \times_6 k +_6 b, \\ 1.732051 = 1 \times_6 k +_6 b. \end{cases}$$

So

$$\begin{cases} b = -4 \times_6 k, \\ 1.732051 = 1 \times_6 k -_6 4 \times_6 k, \end{cases}$$

but *k* does not exist, because

$$1 \times_6 (-0.577350) -_6 4 \times_6 (-0.577350) = 1.732050 < 1.732051$$

and

$$1 \times_6 (-0.577351) -_6 4 \times_6 (-0.577351) = 1.732053 > 1.732051.$$

So the median BK does not exist. This means that the centroid of $\triangle(OAB)$ does not exist.

So we have proved the following:

Theorem 10.4.
In Mathematics with Observers geometry on the plane, there is an equilateral triangle having only one median and having no centroid.

10.3 Vertices and sides of triangle

Let us consider a triangle with sides ($n = 2$)

$$\begin{aligned} a &: x = 0, \\ b &: y = 0, \\ c &: y = -x +_2 1. \end{aligned}$$

This triangle has the vertices

$$\begin{aligned} &A(1, 0), \\ &B(0, 1), \\ &C(0, 0). \end{aligned}$$

In this case the triangle has three sides and three vertices. So, in this situation, we can denote the triangle by Δabc or ΔABC

Let us consider the triangle with vertices (again, $n = 2$)

$$\begin{aligned} &A(6.01, 2.01), \\ &B(-4, 4), \\ &C(2, -6). \end{aligned}$$

Let us prove that this triangle has no sides. We will use the formula

$$y = m \times_n x +_n b.$$

The line

$$BC = a$$

has to contain the points *B* and *C*, that is, we have the system of equations

$$\begin{cases} 4 = -4 \times_2 m +_2 b, \\ -6 = 2 \times_2 m +_2 b. \end{cases}$$

We have the new system

$$\begin{cases} b = 4 +_2 4 \times_2 m, \\ -6 = 2 \times_2 m +_2 4 +_2 4 \times_2 m, \end{cases}$$

that is,

$$\begin{cases} b = 4 +_2 4 \times_2 m, \\ -10 = 2 \times_2 m +_2 4 \times_2 m. \end{cases}$$

The second equation of this system has no solution, because if

$$m = 1.66,$$

then

$$2 \times_2 m +_2 4 \times_2 m = 9.96,$$

and if

$$m = 1.6,$$

then

$$2 \times_2 m +_2 4 \times_2 m = 10.02.$$

So

$$a = BC$$

does not exist.

The line

$$AB = c$$

has to contain the points *A* and *B*, that is, we have the system of equations

$$\begin{cases} 2.01 = 6.01 \times_2 m +_2 b, \\ 4 = -4 \times_2 m +_2 b. \end{cases}$$

We have the new system

$$\begin{cases} b = 2.01 -_2 6.01 \times_2 m, \\ 4 = -4 \times_2 m +_2 2.01 -_2 6.01 \times_2 m, \end{cases}$$

that is,

$$\begin{cases} b = 2.01 -_2 6.01 \times_2 m, \\ 1.99 = -4 \times_2 m -_2 6.01 \times_2 m. \end{cases}$$

The second equation of this system has no solution, because if

$$m = -0.19,$$

then

$$-4 \times_2 m -_2 6.01 \times_2 m = 1.9,$$

and if

$$m = -0.2,$$

then

$$-4 \times_2 m -_2 6.01 \times_2 m = 2.$$

So

$$c = AB$$

does not exist.

The line

$$AC = b$$

has to contain the points *A* and *C*, that is, we have the system of equations

$$\begin{cases} 2.01 = 6.01 \times_2 m +_2 b, \\ -6 = 2 \times_2 m +_2 b. \end{cases}$$

We have the new system

$$\begin{cases} b = 2.01 -_2 6.01 \times_2 m, \\ -6 = 2 \times_2 m +_2 2.01 -_2 6.01 \times_2 m, \end{cases}$$

that is,

$$\begin{cases} b = 2.01 -_2 6.01 \times_2 m, \\ -8.01 = 2 \times_2 m -_2 6.01 \times_2 m. \end{cases}$$

The second equation of this system has no solution, because if

$$m = 2,$$

then

$$2 \times_2 m -_2 6.01 \times_2 m = -8.02,$$

and if

$$m = 1.99,$$

then

$$2 \times_2 m -_2 6.01 \times_2 m = -7.97.$$

So

$$b = AC$$

does not exist.

In this case the triangle has no sides and has three vertices. So, in this situation, we can denote the triangle as ΔABC but not as Δabc.

Let us consider three straight lines

$$\begin{aligned} &a : y = 1, \\ &b : y = 3 \times_2 x, \\ &c : y = -6 \times_2 x +_2 11. \end{aligned}$$

Let us try to find a vertex A opposed to line a, which means that

$$A = b \cap c$$

that is, we have the system

$$\begin{cases} y = 3 \times_2 x, \\ y = -6 \times_2 x +_2 11. \end{cases}$$

So we have

$$\begin{cases} y = 3 \times_2 x, \\ 3 \times_2 x = -6 \times_2 x +_2 11, \end{cases}$$

and thus

$$\begin{cases} y = 3 \times_2 x, \\ 3 \times_2 x +_2 6 \times_2 x = 11. \end{cases}$$

The second equation of this system has no solution, because if

$$x = 1.22,$$

then

$$3 \times_2 x +_2 6 \times_2 x = 10.98,$$

and if

$$x = 1.23,$$

then

$$3 \times_2 x +_2 6 \times_2 x = 11.07.$$

So the point *A* does not exist.

Let us try to find a vertex *B* opposed to line *b*, which means that

$$B = a \cap c,$$

that is, we have the system

$$\begin{cases} y = 1, \\ y = -6 \times_2 x +_2 11. \end{cases}$$

So we have

$$\begin{cases} y = 1, \\ 1 = -6 \times_2 x +_2 11, \end{cases}$$

and thus

$$\begin{cases} y = 1, \\ -17 = -6 \times_2 x. \end{cases}$$

The second equation of this system has no solution, because if

$$x = 2.83,$$

then

$$-6 \times_2 x = -16.98,$$

and if

$$x = 2.84,$$

then

$$-6 \times_2 x = -17.04.$$

So the point *B* does not exist.

Let us try to find a vertex *C* opposed to line *c*, which means that

$$C = a \cap b,$$

that is, we have the system

$$\begin{cases} y = 3 \times_2 x, \\ y = 1, \end{cases}$$

or

$$\begin{cases} y = 3 \times_2 x, \\ 1 = 3 \times_2 x. \end{cases}$$

The second equation of this system has no solution, because if

$$x = 0.33,$$

then

$$3 \times_2 x = 0.99,$$

and if

$$x = 0.34,$$

then

$$3 \times_2 x = 1.02.$$

So the point *C* does not exist.

In this case the triangle has three sides and no vertices. So, in this situation, we can denote the triangle as Δabc but not as ΔABC.

Theorem 10.5.
In Mathematics with Observers geometry, there are triangles with three sides and three vertices, with three vertices and no sides, and with three sides and no vertices.

10.4 The center of a circumscribed circle of a triangle

Problem: Find the center of a circumscribed circle around a triangle ABC in E_2W_2.

Points:

$$A(6, 2), B(-4, 4), C(2, -6).$$

Equation of straight line containing segment AB:

$$y = -0.2 \times_2 x +_2 3.2.$$

Equation of straight line containing segment BC: does not exist.

Equation of straight line containing segment AC:

$$y = 2 \times_2 x -_2 10.$$

Center of segment AB: $D(1, 3)$

Despite the nonexistence of the straight line containing segment BC, the center *F* of BC does exist: $F(-1, -1)$. The center of segment AC: $E(4, -2)$. Now we find the perpendicular bisectors of AB and AC. The perpendicular bisector of AB has a slope 5. The perpendicular bisector of AC has a slope –0.5. The equation of perpendicular bisector of AB:

$$y = 5 \times_2 x +_2 b.$$

We have

$$3 = 5 \times_2 1 +_2 b.$$

Then

$$b = -2.$$

So the equation of perpendicular bisector of AB is

$$y = 5 \times_2 x -_2 2,$$

and the equation of perpendicular bisector of AC is

$$y = -0.5 \times_2 x +_2 b.$$

We have

$$-2 = -0.5 \times_2 4 +_2 b.$$

Then

$$b = 0.$$

So the equation of perpendicular bisector of AC is

$$y = -0.5 \times_2 x.$$

Despite the nonexistence of the straight line containing segment BC, vector $\mathbf{BC}$ does exist:

$$\mathbf{BC} = (6, -10),$$

and we can consider straight line *f* containing the point $F(-1, -1)$ and perpendicular to vector $\mathbf{BC}$. We are looking for the equation of line *f* as

$$y = k \times_2 x +_2 b,$$

and we get

$$\begin{cases} -1 = k \times_2 (-1) +_2 b, \\ 6 \times_2 1 -_2 10 \times_2 k = 0, \end{cases}$$

and thus

$$\begin{cases} k = 0.6, \\ b = -0.4, \end{cases}$$

that is, the equation of straight line *f* containing point *F* and perpendicular to vector $\mathbf{BC}$ is

$$y = 0.6 \times_2 x -_2 0.4.$$

The intersection point of the two perpendicular bisectors of AB and AC:

$$\begin{cases} y = 5 \times_2 x -_2 2, \\ y = -0.5 \times_2 x. \end{cases}$$

So the point of intersection is $O(0.37, -0.15)$.

Now we find the squares of distances from this point to the vertices of the triangle:

$$|OA| \times_2 |OA| = (6 -_2 0.37) \times (6 -_2 0.37) +_2 (2 +_2 0.15) \times (2 +_2 0.15) = 36.27,$$
$$|OB| \times_2 |OB| = (-4 -_2 0.37) \times (-4 -_2 0.37) +_2 (4 +_2 0.15) \times (4 +_2 0.15) = 36.26,$$
$$|OC| \times_2 |OC| = (2 -_2 0.37) \times (2 -_2 0.37) +_2 (-6 +_2 0.15) \times (-6 +_2 0.15) = 36.76.$$

So we get that

$$|OA|^2 \neq |OB|^2,$$
$$|OA|^2 \neq |OC|^2,$$
$$|OB|^2 \neq |OC|^2,$$

and thus the point O is not the circumcenter of the triangle ABC.

The intersection point of the perpendicular bisector of AB and line f:

$$\begin{cases} y = 5 \times_2 x -_2 2, \\ y = 0.6 \times_2 x -_2 0.4. \end{cases}$$

So the point of intersection does not exist because we must have

$$5 \times_2 x -_2 2 = 0.6 \times_2 x -_2 0.4,$$

but

$$5 \times_2 x -_2 0.6 \times_2 x = 1.62$$

if $x = 0.36$, and

$$5 \times_2 x -_2 0.6 \times_2 x = 1.57$$

if $x = 0.35$.

The intersection point of the perpendicular bisector of AC and line f:

$$\begin{cases} y = -0.5 \times_2 x, \\ y = 0.6 \times_2 x -_2 0.4. \end{cases}$$

So the point of intersection does not exist because we must have

$$-0.5 \times_2 x = 0.6 \times_2 x -_2 0.4,$$

but

$$-0.5 \times_2 x -_2 0.6 \times_2 x = -0.33$$

if $x = 0.3\star$, and

$$-0.5 \times_2 x -_2 0.6 \times_2 x = -0.44$$

if $x = 0.4\star$, where $\star$=0,1,...,9.

Finally, we can say that the circumcenter of the triangle ABC does not exist.

Now let us try to find the center of circumscribed circle around the triangle ABC in E_2W_2 with

$$A(0,0), B(2,0), C(0,2).$$

Equation of straight line containing segment AB:

$$y = 0.$$

Equation of straight line containing segment BC:

$$y = -x +_2 2.$$

Equation of straight line containing segment AC:

$$x = 0.$$

Center of segment AB:

$$D(1,0).$$

Center of segment AC:

$$E(0,1).$$

Center of segment BC:

$$F(1,1).$$

Now we find equations of the perpendicular bisectors of segments AB, AC, and BC. The equation of perpendicular bisectors of segment AB is

$$x = 1.$$

The equation of perpendicular bisectors of segment AC is

$$y = 1.$$

The equation of perpendicular bisectors of segment BC is

$$y = x$$

The intersection point of these three perpendicular bisectors is

$$O(1,1).$$

Now we find the distance from this point to the vertices of the triangle:

$$|OA| = |OB| = |AB| = \sqrt{2} = 1.42.$$

So the point O is the circumcenter of triangle ABC.

We can formulate a final theorem.

Theorem 10.6.
In Mathematics with Observers geometry for $n = 2$, there are triangles with existing circumcenter, and there are triangles with nonexisting circumcenter.
Note that the theorem is correct for all $n \geq 2$.

10.5 The orthocenter of a triangle

Problem: Find the orthocenter of a triangle ABC in E_2W_2.

Let's try to find the orthocenter of triangle ABC in E_2W_2 with

$$A(0,0), B(2,0), C(0,2).$$

Equation of straight line containing segment AB:

$$y = 0.$$

Equation of straight line containing segment BC:

$$y = -x +_2 2.$$

Equation of straight line containing segment AC:

$$x = 0.$$

Now we find the perpendiculars from vertex *C* to side AB, from vertex *B* to side AC, and from vertex *A* to side BC:

Perpendicular from vertex *C* to side AB has the equation

$$x = 0.$$

Perpendicular from vertex *B* to side AC has the equation

$$y = 0.$$

Perpendicular from vertex *A* to side BC has the equation

$$y = x.$$

The intersection point of the three perpendiculars is

$$A(0,0).$$

So the point *A* is the orthocenter of the triangle ABC.

Now let us try to find the orthocenter of triangle ABC in E_2W_2 with

$$A(0,0), B(0,2), C(0.72, -0.72).$$

Equation of straight line containing segment AB:

$$x = 0.$$

Equation of straight line containing segment BC:

$$y = -3.80 \times_2 x +_2 2.$$

Equation of straight line containing segment AC:

$$y = -x.$$

Now we find the heights from vertex *C* to side AB, from vertex *B* to side AC, and from vertex *A* to side BC.

Height h_C from vertex *C* to side AB has the equation

$$y = -0.72.$$

Height h_B from vertex *B* to side AC has the equation

$$y = x +_2 2.$$

Height h_A from vertex *A* to side BC has the equation

$$y = 0.28 \times_2 x.$$

The intersection points of these heights are

$$\begin{aligned} h_A \cap h_B &= (-2.70, -0.70), \\ h_A \cap h_C &= (-2.80, -0.72). \\ h_B \cap h_C &= (-2.72, -0.72). \end{aligned}$$

So this triangle has no orthocenter.

Let us consider the triangle with vertex points

$$\begin{aligned} &A(6.01, 2.01), \\ &B(-4, 4), \\ &C(2, -6). \end{aligned}$$

We have proved above (see section “Vertices and sides of triangle”) that this triangle has no sides. However, we can consider the vectors $\mathbf{AB}$, $\mathbf{AC}$, $\mathbf{BC}$ and instead of standard heights, consider straight lines *a*, *b*, *c* such that

$$\begin{aligned} A &\in a, \quad a \perp \mathbf{BC}, \\ B &\in b, \quad b \perp \mathbf{AC}, \\ C &\in c, \quad c \perp \mathbf{AB} \end{aligned}$$

We get

$$\mathbf{BC} = (6, -10),$$
$$\mathbf{AC} = (-4.01, -8.01),$$
$$\mathbf{AB} = (-10.01, 1.99).$$

Let straight line *a* have the equation

$$y = k \times_2 x +_2 d.$$

We have

$$\begin{cases} 2.01 = k \times_2 6.01 +_2 d, \\ 6 \times_2 1 -_2 10 \times_2 k = 0, \end{cases}$$

and we get

$$\begin{cases} k = 0.6, \\ d = -1.59, \end{cases}$$

that is, straight line *a* has the equation

$$y = 0.6 \times_2 x -_2 1.59.$$

Let straight line *b* have the equation

$$y = k \times_2 x +_2 d.$$

We have

$$\begin{cases} 4 = k \times_2 (-4) +_2 d, \\ -4.01 \times_2 1 -_2 8.01 \times_2 k = 0, \end{cases}$$

and we get

$$-4.01 \times_2 1 -_2 8.01 \times_2 k = -0.01$$

if $k = -0.50$, and

$$-4.01 \times_2 1 -_2 8.01 \times_2 k = 0.08$$

if $k = -0.51$. This means that line *b* does not exist.

Let straight line *c* have the equation

$$y = k \times_2 x +_2 d.$$

We have

$$\begin{cases} -6 = k \times_2 2 +_2 d, \\ -10.01 \times_2 1 +_2 1.99 \times_2 k = 0, \end{cases}$$

and we get

$$\begin{cases} k = 5.06, \\ d = -16.12, \end{cases}$$

that is, straight line *c* has the equation

$$y = 5.06 \times_2 x -_2 16.12.$$

So this triangle has no orthocenter.

We can formulate the final theorem.

Theorem 10.7.
In Mathematics with Observers geometry for $n = 2$, there are triangles with existing orthocenter, and there are triangles with nonexisting orthocenter.
Note that the theorem is correct for all $n \geq 2$.

10.6 The center of inscribed circle of triangle

In classical geometry the center of inscribed circle of a triangle is the point intersection of the angle bisectors.

Let us first consider several cases.

A. Let $n = 2$. Consider two straight lines

$$a : y = 0$$

and

$$b : x = 0.$$

Then

$$a \cap b = O(0, 0).$$

Let us take the point $P(1, 1)$, and consider the straight line

$$OP : y = x.$$

The equation of circle Ω with center in point *P* with radius 1 is

$$\Omega : (x -_2 1) \times_2 (x -_2 1) +_2 (y -_2 1) \times_2 (y -_2 1) = 1.$$

Let us find the intersection of line OP and circle Ω:

$$OP \cap \Omega : 2 \times_2 \left((x -_2 1) \times_2 (x -_2 1)\right) = 1.$$

Taking $x -_2 1 = 0.7\star$, where $\star$ means any digit from the set 0,1,...,9, we get

$$2 \times_2 0.49 = 0.98 < 1.$$

Taking $x -_2 1 = 0.8\star$, we get

$$2 \times_2 0.64 = 1.28 > 1,$$

that is,

$$OP \cap \Omega = \Lambda,$$

where Λ is the empty set.

This means that there is no triangle with sides a and b and inscribed circle Ω such that the third side of this triangle is perpendicular to line OP and tangent to Ω.

B. Let $n = 2$. Consider two straight lines

$$a : y = 0$$

and

$$b : x = 0.$$

Then

$$a \cap b = O(0, 0).$$

Let us take the point $P(1.29, 1.29)$ and consider the straight line

$$OP : y = x.$$

The equation of circle Ω with center in point P with radius 1.29 is

$$\Omega : (x -_2 1.29) \times_2 (x -_2 1.29) +_2 (y -_2 1.29) \times_2 (y -_2 1.29) = 1.29 \times_2 1.29 = 1.62.$$

Let us find the intersection of line OP and circle Ω:

$$OP \cap \Omega : 2 \times_2 \big((x -_2 1.29) \times_2 (x -_2 1.29)\big) = 1.62.$$

Taking $x -_2 1.29 = 0.9\star$, where $\star$ means any digit from the set 0,1,...,9, we get

$$2 \times_2 (0.9 \star \times_2 0.9\star) = 2 \times_2 0.81 = 1.62,$$

that is,

$$x -_2 1.29 = \pm 0.9\star$$

or

$$x -_2 1.29 = 0.90,$$
$$x -_2 1.29 = 0.91,$$
$$x -_2 1.29 = 0.92,$$
$$x -_2 1.29 = 0.93,$$
$$x -_2 1.29 = 0.94,$$
$$x -_2 1.29 = 0.95,$$
$$x -_2 1.29 = 0.96,$$
$$x -_2 1.29 = 0.97,$$
$$x -_2 1.29 = 0.98,$$
$$x -_2 1.29 = 0.99$$

or

$$x -_2 1.29 = -0.90,$$
$$x -_2 1.29 = -0.91,$$
$$x -_2 1.29 = -0.92,$$
$$x -_2 1.29 = -0.93,$$
$$x -_2 1.29 = -0.94,$$
$$x -_2 1.29 = -0.95,$$
$$x -_2 1.29 = -0.96,$$
$$x -_2 1.29 = -0.97,$$
$$x -_2 1.29 = -0.98,$$
$$x -_2 1.29 = -0.99.$$

This means that in the first case, we have

$$x_1 = 2.19,$$
$$x_2 = 2.20,$$
$$x_3 = 2.21,$$
$$x_4 = 2.22,$$
$$x_5 = 2.23,$$
$$x_6 = 2.24,$$
$$x_7 = 2.25,$$
$$x_8 = 2.26,$$
$$x_9 = 2.27,$$
$$x_{10} = 2.28,$$

and in the second case, we have

$$x_{11}= 0.39,$$
$$x_{12}= 0.38,$$
$$x_{13}= 0.37,$$
$$x_{14}= 0.36,$$
$$x_{15}= 0.35,$$
$$x_{16}= 0.34,$$
$$x_{17}= 0.33,$$
$$x_{18}= 0.32,$$
$$x_{19}= 0.31,$$
$$x_{20}= 0.30.$$

This means that in the first case, we have ten points $L = OP \cap \Omega$:

$$L_1(2.19, 2.19),$$
$$L_2(2.20, 2.20),$$
$$L_3(2.21, 2.21),$$
$$L_4(2.22, 2.22),$$
$$L_5(2.23, 2.23),$$
$$L_6(2.24, 2.24),$$
$$L_7(2.25, 2.25),$$
$$L_8(2.26, 2.26),$$
$$L_9(2.27, 2.27),$$
$$L_{10}(2.28, 2.28).$$

This means that in the second case, we have ten points $M = OP \cap \Omega$

$$M_1(0.39, 0.39),$$
$$M_2(0.38, 0.38),$$
$$M_3(0.37, 0.37),$$
$$M_4(0.36, 0.36),$$
$$M_5(0.35, 0.35),$$
$$M_6(0.34, 0.34),$$
$$M_7(0.33, 0.33),$$
$$M_8(0.32, 0.32),$$
$$M_9(0.31, 0.31),$$
$$M_{10}(0.30, 0.30).$$

Let us consider the straight lines c_i ($i = 1, 2, \ldots, 10$) containing points L_i and perpendicular to line OP:

$$c_i : y = -x +_2 d_i.$$

So

$$c_1 : y = -x +_2 4.38,$$
$$c_2 : y = -x +_2 4.40,$$
$$c_3 : y = -x +_2 4.42,$$
$$c_4 : y = -x +_2 4.44,$$
$$c_5 : y = -x +_2 4.46,$$
$$c_6 : y = -x +_2 4.48,$$
$$c_7 : y = -x +_2 4.50,$$
$$c_8 : y = -x +_2 4.52,$$
$$c_9 : y = -x +_2 4.54,$$
$$c_{10} : y = -x +_2 4.56.$$

Let us consider the straight lines c_i ($i = 11, 12, \ldots, 20$) containing point M_i and perpendicular to line OP:

$$c_i : y = -x +_2 e_i.$$

So

$$c_{11} : y = -x +_2 0.78,$$
$$c_{12} : y = -x +_2 0.76,$$
$$c_{13} : y = -x +_2 0.74,$$
$$c_{14} : y = -x +_2 0.72,$$
$$c_{15} : y = -x +_2 0.70,$$
$$c_{16} : y = -x +_2 0.68,$$
$$c_{17} : y = -x +_2 0.66,$$
$$c_{18} : y = -x +_2 0.64,$$
$$c_{19} : y = -x +_2 0.62,$$
$$c_{20} : y = -x +_2 0.60.$$

Note that all straight lines $c_i, i = 1, 2, \ldots, 20$, are parallel to each other and tangent to one circle Ω.

So we have obtained an intermediate theorem.

Theorem 10.8.
In Mathematics with Observers geometry, there is a circle with more than two parallel lines tangent to this circle.

Let us now consider intersections of lines $c_i, i = 1, 2, \ldots, 10$, with lines *a* and *b*. We get

$$
\begin{gathered}
a \cap c_1 = C_1(4.38, 0), \\
a \cap c_2 = C_2(4.40, 0), \\
a \cap c_3 = C_3(4.42, 0), \\
a \cap c_4 = C_4(4.44, 0), \\
a \cap c_5 = C_5(4.46, 0), \\
a \cap c_6 = C_6(4.48, 0), \\
a \cap c_7 = C_7(4.50, 0), \\
a \cap c_8 = C_8(4.52, 0), \\
a \cap c_9 = C_9(4.54, 0), \\
a \cap c_{10} = C_{10}(4.56, 0), \\
b \cap c_1 = B_1(0, 4.38), \\
b \cap c_2 = B_2(0, 4.40), \\
b \cap c_3 = B_3(0, 4.42), \\
b \cap c_4 = B_4(0, 4.44), \\
b \cap c_5 = B_5(0, 4.46), \\
b \cap c_6 = B_6(0, 4.48), \\
b \cap c_7 = B_7(0, 4.50), \\
b \cap c_8 = B_8(0, 4.52), \\
b \cap c_9 = B_9(0, 4.54), \\
b \cap c_{10} = B_{10}(0, 4.56).
\end{gathered}
$$

1. Let us take three points $O(0, 0)$, $B_1(0, 4.38)$, and $C_1(4.38, 0)$. The straight line $B_1C_1 = c_1$ has the equation

$$B_1C_1 : y = -x +_2 4.38$$

Now taking the points

$$K(0, 1.29) \in b$$

and

$$N(1.29, 0) \in a,$$

we get

$$PK = PN = PL_1 = 1.29.$$

Now we are interested in whether the line B_1P exists or not. Let us find the equation of the straight line B_1P:

$$
\begin{aligned}
&y = k \times_2 x +_2 b, \\
&\begin{cases} 4.38 = 0 \times_2 k +_2 b, \\ 1.29 = 1.29 \times_2 k +_2 b. \end{cases}
\end{aligned}
$$

So

$$\begin{cases} k = -2.43, \\ b = 4.38, \end{cases}$$

and, finally, the equation of the straight line B_1P is

$$y = -2.43 \times_2 x +_2 4.38.$$

Let us consider three vectors

$$B_1K = (0, -3.09),$$
$$B_1L_1 = (2.19, -2.19),$$
$$B_1P = (1.29, -3.09).$$

We get

$$(B_1K, B_1K) = 0 \times_2 0 +_2 (-3.09) \times_2 (-3.09) = 9.54 > 0,$$
$$(B_1L_1, B_1L_1) = 2.19 \times_2 2.19 +_2 (-2.19) \times_2 (-2.19) = 9.54 > 0,$$

that is,

$$(B_1K, B_1K) = (B_1L_1, B_1L_1).$$

Now let us consider

$$(B_1K, B_1P) = 0 \times_2 1.29 +_2 (-3.09) \times_2 (-3.09) = 9.54 > 0,$$
$$(B_1L_1, B_1P) = 2.19 \times_2 1.29 +_2 (-2.19) \times_2 (-3.09) = 2.79 +_2 6.75 = 9.54 > 0,$$

that is,

$$(B_1K, B_1P) = (B_1L_1, B_1P).$$

This means that the straight line B_1P is an angle B_1 bisector and that the point *P* is the intersection of the bisectors of angles *O* and B_1.

So we have obtained an intermediate theorem.

Theorem 10.9.
In Mathematics with Observers geometry on the plane, there are a circle and a point out of this circle such that there are two tangent lines to this circle through this point with equal segments between this point and touch points.

2. Let us take three points $O(0,0)$, $B_2(0,4.40)$, and $C_3(4.40,0)$. The straight line $B_2C_2 = c_2$ has the equation

$$B_2C_2 : y = -x +_2 4.40.$$

We get

$$PK = PN = PL_2 = 1.29.$$

We are interested whether the line B_2P exists or not.

Let us find the equation of the straight line B_2P:

$$y = k \times_2 x +_2 b,$$
$$\begin{cases} 4.40 = 0 \times_2 k +_2 b, \\ 1.29 = 1.29 \times_2 k +_2 b. \end{cases}$$

So

$$\begin{cases} k = -2.45, \\ b = 4.40, \end{cases}$$

and, finally, the equation of the straight line B_2P is

$$y = -2.45 \times_2 x +_2 4.40.$$

Let us now consider three vectors

$$B_2K = (0, -3.11),$$
$$B_2L_2 = (2.20, -2.20),$$
$$B_2P = (1.29, -3.11).$$

We get

$$(B_2K, B_2K) = 0 \times_2 0 +_2 (-3.11) \times_2 (-3.11) = 9.67 > 0,$$
$$(B_2L_2, B_2L_2) = 2.20 \times_2 2.20 +_2 (-2.20) \times_2 (-2.20) = 9.68 > 0,$$

that is,

$$(B_2K, B_2K) \neq (B_2L_2, B_2L_2).$$

We have

$$(B_2K, B_2P) = 0 \times_2 1.29 +_2 (-3.11) \times_2 (-3.11) = 9.67 > 0,$$
$$(B_2L_2, B_2P) = 2.20 \times_2 1.29 +_2 (-2.20) \times_2 (-3.11) = 2.82 +_2 6.84 = 9.66 > 0,$$

that is,

$$(B_2K, B_2P) \neq (B_2L_2, B_2P).$$

This means that with chosen vectors B_2K, B_2L_2, B_2P, the straight line B_2P is not an angle B_2 bisector and the point *P* is not an intersection of the bisectors of angles *O* and B_2.

3. Let us take three points $O(0, 0)$, $B_3(0, 4.42)$, and $C_3(4.42, 0)$. The straight line $B_3C_3 = c_3$ has the equation

$$B_3C_3 : y = -x +_2 4.42.$$

We get

$$PK = PN = PL_3 = 1.29.$$

We are interested in whether the line B_3P exists or not.

Let us find the equation of the straight line B_3P:

$$y = k \times_2 x +_2 b,$$
$$\begin{cases} 4.42 = 0 \times_2 k +_2 b, \\ 1.29 = 1.29 \times_2 k +_2 b. \end{cases}$$

So

$$\begin{cases} k = -2.47, \\ b = 4.42, \end{cases}$$

and, finally, the equation of the straight line B_3P is

$$y = -2.47 \times_2 x +_2 4.42.$$

Let us consider three vectors

$$B_3K = (0, -3.13),$$
$$B_3L_3 = (2.21, -2.21),$$
$$B_3P = (1.29, -3.13).$$

Then

$$(B_3K, B_3K) = 0 \times_2 0 +_2 (-3.13) \times_2 (-3.13) = 9.79 > 0,$$
$$(B_3L_3, B_3L_3) = 2.21 \times_2 2.21 +_2 (-2.21) \times_2 (-2.21) = 9.76 > 0,$$

that is,

$$(B_3K, B_3K) \neq (B_3L_3, B_3L_3).$$

We have

$$(B_3K, B_3P) = 0 \times_2 1.29 +_2 (-3.13) \times_2 (-3.13) = 9.79 > 0,$$
$$(B_3L_3, B_3P) = 2.21 \times_2 1.29 +_2 (-2.21) \times_2 (-3.13) = 2.82 +_2 6.84 = 9.74 > 0,$$

that is,

$$(B_3K, B_3P) \neq (B_3L_3, B_3P).$$

This means that with chosen vectors B_3K, B_3L_3, B_3P, the straight line B_3P is not an angle B_3 bisector and the point *P* is not an intersection of bisectors of angles *O* and B_3.

4. Let us take three points $O(0, 0)$, $B_4(0, 4.44)$, and $C_4(4.44, 0)$. The straight line $B_4C_4 = c_4$ has the equation

$$B_4C_4 : y = -x +_2 4.44.$$

Then

$$PK = PN = PL_4 = 1.29.$$

We are interested in whether the line B_4P exists or not.

Let us find the equation of the straight line B_4P:

$$y = k \times_2 x +_2 b,$$
$$\begin{cases} 4.44 = 0 \times_2 k +_2 b, \\ 1.29 = 1.29 \times_2 k +_2 b. \end{cases}$$

So

$$\begin{cases} k = -2.49, \\ b = 4.44, \end{cases}$$

and, finally, the equation of the straight line B_4P is

$$y = -2.49 \times_2 x +_2 4.44.$$

Let us consider three vectors

$$B_4K = (0, -3.15),$$
$$B_4L_4 = (2.22, -2.22),$$
$$B_4P = (1.29, -3.15).$$

Then

$$(B_4K, B_4K) = 0 \times_2 0 +_2 (-3.15) \times_2 (-3.15) = 9.91 > 0,$$
$$(B_4L_4, B_4L_4) = 2.22 \times_2 2.22 +_2 (-2.22) \times_2 (-2.22) = 9.84 > 0,$$

that is,

$$(B_4K, B_4K) \neq (B_4L_4, B_4L_4),$$

and

$$(B_4K, B_4P) = 0 \times_2 1.29 +_2 (-3.15) \times_2 (-3.15) = 9.91 > 0,$$
$$(B_4L_4, B_4P) = 2.22 \times_2 1.29 +_2 (-2.22) \times_2 (-3.15) = 2.84 +_2 6.98 = 9.82 > 0,$$

that is,

$$(B_4K, B_4P) \neq (B_4L_4, B_4P).$$

This means that with chosen vectors B_4K, B_4L_4, B_4P, the straight line B_4P is not an angle B_4 bisector and the point *P* is not an intersection of bisectors of angles *O* and B_4.

5. Let us take three points $O(0,0)$, $B_5(0, 4.46)$, and $C_5(4.46, 0)$. The straight line $B_5C_5 = c_5$ has the equation

$$B_5C_5 : y = -x +_2 4.46.$$

Then

$$PK = PN = PL_5 = 1.29.$$

We are now interested in whether the line B_5P exists or not.

Let us find the equation of the straight line B_5P:

$$y = k \times_2 x +_2 b,$$
$$\begin{cases} 4.46 = 0 \times_2 k +_2 b, \\ 1.29 = 1.29 \times_2 k +_2 b. \end{cases}$$

Taking $k = -2.49$, we get

$$1.29 + 3.15 = 4.44 < 4.46,$$

whereas taking $k = -2.50$, we get

$$1.29 + 3.18 = 4.47 > 4.46.$$

So the line B_5P does not exist.

6. Let us take three points $O(0, 0)$, $B_6(0, 4.48)$, and $C_6(4.48, 0)$. The straight line $B_6C_6 = c_6$ has the equation

$$B_6C_6 : y = -x +_2 4.48.$$

We get

$$PK = PN = PL_6 = 1.29.$$

We are now interested whether the line B_6P exists or not.

Let us find the equation of the straight line B_6P:

$$y = k \times_2 x +_2 b,$$
$$\begin{cases} 4.48 = 0 \times_2 k +_2 b, \\ 1.29 = 1.29 \times_2 k +_2 b. \end{cases}$$

Taking $k = -2.51$, we get

$$1.29 + 3.19 = 4.48,$$

and, finally, the equation of the straight line B_6P is

$$y = -2.51 \times_2 x +_2 4.48.$$

Let us consider three vectors

$$B_6K = (0, -3.19),$$
$$B_6L_6 = (2.24, -2.24),$$
$$B_6P = (1.29, -3.19).$$

Then

$$(B_6K, B_6K) = 0 \times_2 0 +_2 (-3.19) \times_2 (-3.19) = 10.15 > 0,$$
$$(B_6L_6, B_6L_6) = 2.24 \times_2 2.24 +_2 (-2.24) \times_2 (-2.24) = 10.00 > 0,$$

that is,

$$(B_6K, B_6K) \neq (B_6L_6, B_6L_6),$$

and

$$(B_6K, B_6P) = 0 \times_2 1.29 +_2 (-3.19) \times_2 (-3.19) = 10.15 > 0,$$
$$(B_6L_6, B_6P) = 2.24 \times_2 1.29 +_2 (-2.24) \times_2 (-3.19) = 2.86 +_2 7.12 = 9.98 > 0,$$

that is,

$$(B_6K, B_6P) \neq (B_6L_6, B_6P).$$

This means that with chosen vectors B_6K, B_6L_6, B_6P, the straight line B_6P is not an angle B_6 bisector and the point *P* is not an intersection of bisectors of angles *O* and B_6.

7. Let us take three points $O(0, 0)$, $B_7(0, 4.50)$, and $C_7(4.50, 0)$. The straight line $B_7C_7 = c_7$ has the equation

$$B_7C_7 : y = -x +_2 4.50.$$

Then

$$PK = PN = PL_7 = 1.29.$$

We are now interested in whether the line B_7P exists or not.

Let us find the equation of the straight line B_7P:

$$y = k \times_2 x +_2 b,$$
$$\begin{cases} 4.50 = 0 \times_2 k +_2 b, \\ 1.29 = 1.29 \times_2 k +_2 b. \end{cases}$$

Taking $k = -2.53$, we get

$$1.29 + 3.21 = 4.50,$$

and, finally, the equation of the straight line B_7P is

$$y = -2.53 \times_2 x +_2 4.50.$$

Let us consider three vectors

$$B_7K = (0, -3.21),$$
$$B_7L_7 = (2.25, -2.25),$$
$$B_7P = (1.29, -3.21).$$

We have

$$(B_7K, B_7K) = 0 \times_2 0 +_2 (-3.21) \times_2 (-3.21) = 10.30 > 0,$$
$$(B_7L_7, B_7L_7) = 2.25 \times_2 2.25 +_2 (-2.25) \times_2 (-2.25) = 10.08 > 0,$$

that is,

$$(B_7K, B_7K) \neq (B_7L_7, B_7L_7),$$

and

$$(B_7K, B_7P) = 0 \times_2 1.29 +_2 (-3.21) \times_2 (-3.21) = 10.30 > 0,$$
$$(B_7L_7, B_7P) = 2.25 \times_2 1.29 +_2 (-2.25) \times_2 (-3.21) = 2.87 +_2 7.21 = 10.08 > 0,$$

that is,

$$(B_7K, B_7P) \neq (B_7L_7, B_7P).$$

This means that with chosen vectors B_7K, B_7L_7, B_7P, the straight line B_7P is not an angle B_7 bisector and the point *P* is not an intersection of bisectors of angles *O* and B_7.

8. Let us take three points $O(0,0)$, $B_8(0, 4.52)$, and $C_8(4.52, 0)$. The straight line $B_8C_8 = c_8$ has the equation

$$B_8C_8 : y = -x +_2 4.52.$$

We have

$$PK = PN = PL_8 = 1.29.$$

We are now interested in whether the line B_8P exists or not.

Let us find the equation of the straight line B_8P:

$$y = k \times_2 x +_2 b,$$
$$\begin{cases} 4.52 = 0 \times_2 k +_2 b, \\ 1.29 = 1.29 \times_2 k +_2 b. \end{cases}$$

Taking $k = -2.55$, we get

$$1.29 + 3.23 = 4.52,$$

and, finally, the equation of the straight line B_7P is

$$y = -2.55 \times_2 x +_2 4.52.$$

Let us consider three vectors

$$B_8K = (0, -3.23),$$
$$B_8L_8 = (2.26, -2.26),$$
$$B_8P = (1.29, -3.23).$$

We get

$$(B_8K, B_8K) = 0 \times_2 0 +_2 (-3.23) \times_2 (-3.23) = 10.42 > 0,$$
$$(B_8L_8, B_8L_8) = 2.26 \times_2 2.26 +_2 (-2.26) \times_2 (-2.26) = 10.16 > 0,$$

that is,

$$(B_8K, B_8K) \neq (B_8L_8, B_8L_8),$$

and

$$(B_8K, B_8P) = 0 \times_2 1.29 +_2 (-3.23) \times_2 (-3.23) = 10.42 > 0,$$
$$(B_8L_8, B_8P) = 2.26 \times_2 1.29 +_2 (-2.26) \times_2 (-3.23) = 2.88 +_2 7.28 = 10.16 > 0,$$

that is,

$$(B_8K, B_8P) \neq (B_8L_8, B_8P).$$

This means that with chosen vectors B_8K, B_8L_8, B_8P, the straight line B_8P is not an angle B_8 bisector and the point *P* is not an intersection of bisectors of angles *O* and B_8.

9. Let us take three points $O(0,0)$, $B_9(0, 4.54)$, and $C_9(4.54, 0)$. The straight line $B_9C_9 = c_9$ has the equation

$$B_9C_9 : y = -x +_2 4.54.$$

We have

$$PK = PN = PL_9 = 1.29.$$

We are now interested in whether the line B_9P exists or no.

Let us find the equation of the straight line B_9P:

$$y = k \times_2 x +_2 b,$$
$$\begin{cases} 4.54 = 0 \times_2 k +_2 b, \\ 1.29 = 1.29 \times_2 k +_2 b. \end{cases}$$

Taking $k = -2.57$, we get

$$1.29 + 3.25 = 4.54,$$

and, finally, the equation of the straight line B_9P is

$$y = -2.57 \times_2 x +_2 4.54.$$

Let us consider three vectors

$$B_9K = (0, -3.25),$$
$$B_9L_9 = (2.27, -2.27),$$
$$B_9P = (1.29, -3.25).$$

We have

$$(B_9K, B_9K) = 0 \times_2 0 +_2 (-3.25) \times_2 (-3.25) = 10.54 > 0,$$
$$(B_9L_9, B_9L_9) = 2.27 \times_2 2.27 +_2 (-2.27) \times_2 (-2.27) = 10.24 > 0,$$

that is,

$$(B_9K, B_9K) \neq (B_9L_9, B_9L_9),$$

and

$$(B_9K, B_9P) = 0 \times_2 1.29 +_2 (-3.25) \times_2 (-3.25) = 10.54 > 0,$$
$$(B_9L_9, B_9P) = 2.27 \times_2 1.29 +_2 (-2.27) \times_2 (-3.25) = 2.89 +_2 7.35 = 10.24 > 0,$$

that is,

$$(B_9K, B_9P) \neq (B_9L_9, B_9P).$$

This means that with chosen vectors B_9K, B_9L_9, B_9P, the straight line B_9P is not an angle B_9 bisector and the point P is not an intersection of bisectors of angles O and B_9.

10. Let us take three points $O(0,0)$, $B_{10}(0, 4.56)$, and $C_{10}(4.56, 0)$. The straight line $B_{10}C_{10} = c_{10}$ has the equation

$$B_{10}C_{10} : y = -x +_2 4.56.$$

We have

$$PK = PN = PL_{10} = 1.29.$$

We are interested in whether the line $B_{10}P$ exists or not.

Let us find the equation of the straight line $B_{10}P$:

$$y = k \times_2 x +_2 b,$$
$$\begin{cases} 4.56 = 0 \times_2 k +_2 b, \\ 1.29 = 1.29 \times_2 k +_2 b. \end{cases}$$

Taking $k = -2.59$, we get

$$1.29 + 3.27 = 4.56,$$

and, finally, the equation of the straight line $B_{10}P$ is

$$y = -2.59 \times_2 x +_2 4.56.$$

Let us consider three vectors

$$B_{10}K = (0, -3.27),$$
$$B_{10}L_{10} = (2.28, -2.28),$$
$$B_{10}P = (1.29, -3.27).$$

We get

$$(B_{10}K, B_{10}K) = 0 \times_2 0 +_2 (-3.27) \times_2 (-3.27) = 10.66 > 0,$$
$$(B_{10}L_{10}, B_{10}L_{10}) = 2.28 \times_2 2.28 +_2 (-2.28) \times_2 (-2.28) = 10.32 > 0,$$

that is,

$$(B_{10}K, B_{10}K) \neq (B_{10}L_{10}, B_{10}L_{10}),$$

and

$$(B_{10}K, B_{10}P) = 0 \times_2 1.29 +_2 (-3.27) \times_2 (-3.27) = 10.66 > 0,$$
$$(B_{10}L_{10}, B_{10}P) = 2.28 \times_2 1.29 +_2 (-2.28) \times_2 (-3.27) = 2.90 +_2 7.42 = 10.32 > 0,$$

that is,

$$(B_{10}K, B_{10}P) \neq (B_{10}L_{10}, B_{10}P).$$

This means that with chosen vectors $B_{10}K$, $B_{10}L_{10}$, $B_{10}P$, the straight line $B_{10}P$ is not an angle B_{10} bisector and the point *P* is not an intersection of bisectors of angles *O* and B_{10}.

So we have obtained an intermediate theorem.

Theorem 10.10.
In Mathematics with Observers geometry on the plane, there are a circle and a point out of this circle such that there are two tangent lines to this circle through this point with not equal segments between this point and touch points.

Let us now consider vertices C_i in triangles $OB_iC_i, i = 1, 2, \ldots, 10$.

1. Now we have to check whether the straight line C_1P exists or not.

Let us find the equation of the straight line C_1P:

$$y = k \times_2 x +_2 b,$$
$$\begin{cases} 0 = 4.38 \times_2 k +_2 b, \\ 1.29 = 1.29 \times_2 k +_2 b. \end{cases}$$

So

$$\begin{cases} b = -4.38 \times_2 k, \\ 1.29 = 1.29 \times_2 k -_2 4.38 \times_2 k. \end{cases}$$

For $k = -0.40$, we get

$$1.29 < 1.34,$$

and for $k = -0.39$, we get

$$1.29 > 1.20,$$

that is, the straight line C_1P in the class of straight lines

$$y = k \times_2 x +_2 b$$

does not exist.

2. Now we have to check whether the straight line C_2P exists or not in this class of straight lines.

Let us find the equation of the straight line C_2P:

$$y = k \times_2 x +_2 b,$$
$$\begin{cases} 0 = 4.40 \times_2 k +_2 b, \\ 1.29 = 1.29 \times_2 k +_2 b. \end{cases}$$

So

$$\begin{cases} b = -4.40 \times_2 k, \\ 1.29 = 1.29 \times_2 k -_2 4.40 \times_2 k. \end{cases}$$

For $k = -0.40$, we get

$$1.29 > 1.28,$$

and for $k = -0.41$, we get

$$1.29 < 1.31,$$

that is, the straight line C_2P in the class of straight lines

$$y = k \times_2 x +_2 b$$

does not exist.

3. Now we have to check whether the straight line C_3P exists or not in this class of straight lines.

Let us find the equation of the straight line C_3P:

$$y = k \times_2 x +_2 b,$$
$$\begin{cases} 0 = 4.42 \times_2 k +_2 b, \\ 1.29 = 1.29 \times_2 k +_2 b. \end{cases}$$

So

$$\begin{cases} b = -4.42 \times_2 k, \\ 1.29 = 1.29 \times_2 k -_2 4.42 \times_2 k. \end{cases}$$

For $k = -0.40$, we get

$$1.29 > 1.28,$$

and for $k = -0.41$, we get

$$1.29 < 1.31,$$

that is, the straight line C_3P in the class of straight lines

$$y = k \times_2 x +_2 b$$

does not exist.

4. Now we have to check whether the straight line C_4P exists or not in this class of straight lines.

Let us find the equation of the straight line C_4P:

$$y = k \times_2 x +_2 b,$$
$$\begin{cases} 0 = 4.44 \times_2 k +_2 b, \\ 1.29 = 1.29 \times_2 k +_2 b. \end{cases}$$

So

$$\begin{cases} b = -4.44 \times_2 k, \\ 1.29 = 1.29 \times_2 k -_2 4.44 \times_2 k. \end{cases}$$

For $k = -0.40$, we get

$$1.29 > 1.28,$$

and for $k = -0.41$, we get

$$1.29 < 1.31,$$

that is, the straight line C_4P in the class of straight lines

$$y = k \times_2 x +_2 b$$

does not exist.

5. Now we have to check whether the straight line C_5P exists or not in this class of straight lines.

Let us find the equation of the straight line C_5P:

$$y = k \times_2 x +_2 b,$$
$$\begin{cases} 0 = 4.46 \times_2 k +_2 b, \\ 1.29 = 1.29 \times_2 k +_2 b. \end{cases}$$

So

$$\begin{cases} b = -4.46 \times_2 k, \\ 1.29 = 1.29 \times_2 k -_2 4.46 \times_2 k. \end{cases}$$

For $k = -0.40$, we get

$$1.29 > 1.28,$$

and for $k = -0.41$, we get

$$1.29 < 1.31,$$

that is, the straight line C_5P in the class of straight lines

$$y = k \times_2 x +_2 b$$

does not exist.

6. Now we have to check whether the straight line C_6P exists or not in this class of straight lines.

Let us find the equation of the straight line C_6P:

$$y = k \times_2 x +_2 b,$$
$$\begin{cases} 0 = 4.48 \times_2 k +_2 b, \\ 1.29 = 1.29 \times_2 k +_2 b. \end{cases}$$

So

$$\begin{cases} b = -4.48 \times_2 k, \\ 1.29 = 1.29 \times_2 k -_2 4.48 \times_2 k. \end{cases}$$

For $k = -0.40$, we get

$$1.29 > 1.28,$$

and for $k = -0.41$, we get

$$1.29 < 1.31,$$

that is, the straight line C_6P in the class of straight lines

$$y = k \times_2 x +_2 b$$

does not exist.

7. Now we have to check whether the straight line C_7P exists or not in this class of straight lines.

Let us find the equation of the straight line C_7P:

$$y = k \times_2 x +_2 b,$$
$$\begin{cases} 0 = 4.50 \times_2 k +_2 b, \\ 1.29 = 1.29 \times_2 k +_2 b. \end{cases}$$

So

$$\begin{cases} b = -4.50 \times_2 k, \\ 1.29 = 1.29 \times_2 k -_2 4.50 \times_2 k. \end{cases}$$

For $k = -0.39$, we get

$$1.29 > 1.26,$$

and for $k = -0.40$, we get

$$1.29 < 1.32,$$

that is, the straight line C_7P in the class of straight lines

$$y = k \times_2 x +_2 b$$

does not exist.

8. Now we have to check whether the straight line C_8P exists or not in this class of straight lines.

Let us find the equation of the straight line C_8P:

$$y = k \times_2 x +_2 b,$$
$$\begin{cases} 0 = 4.52 \times_2 k +_2 b, \\ 1.29 = 1.29 \times_2 k +_2 b. \end{cases}$$

So

$$\begin{cases} b = -4.52 \times_2 k, \\ 1.29 = 1.29 \times_2 k -_2 4.52 \times_2 k. \end{cases}$$

For $k = -0.39$, we get

$$1.29 > 1.26,$$

and for $k = -0.40$, we get

$$1.29 < 1.32,$$

that is, the straight line C_8P in the class of straight lines

$$y = k \times_2 x +_2 b$$

does not exist.

9. Now we have to check whether the straight line C_9P exists or not in this class of straight lines.

Let us find the equation of the straight line C_9P:

$$y = k \times_2 x +_2 b,$$
$$\begin{cases} 0 = 4.54 \times_2 k +_2 b, \\ 1.29 = 1.29 \times_2 k +_2 b. \end{cases}$$

So

$$\begin{cases} b = -4.54 \times_2 k, \\ 1.29 = 1.29 \times_2 k -_2 4.54 \times_2 k. \end{cases}$$

For $k = -0.39$, we get

$$1.29 > 1.26,$$

and for $k = -0.40$, we get

$$1.29 < 1.32,$$

that is, the straight line C_9P in the class of straight lines

$$y = k \times_2 x +_2 b$$

does not exist.

10. Finally, we have to check whether the straight line $C_{10}P$ exists or not in this class of straight lines.

Let us find the equation of the straight line $C_{10}P$:

$$y = k \times_2 x +_2 b,$$
$$\begin{cases} 0 = 4.56 \times_2 k +_2 b, \\ 1.29 = 1.29 \times_2 k +_2 b. \end{cases}$$

So

$$\begin{cases} b = -4.56 \times_2 k, \\ 1.29 = 1.29 \times_2 k -_2 4.56 \times_2 k. \end{cases}$$

For $k = -0.39$, we get

$$1.29 > 1.26,$$

and for $k = -0.40$, we get

$$1.29 < 1.32,$$

that is, the straight line $C_{10}P$ in the class of straight lines

$$y = k \times_2 x +_2 b$$

does not exist.

So the lines

$$C_iP, \quad i = 1, 2, \ldots, 10,$$

do not exist in the considered class of lines

$$y = k \times_2 x +_2 b, \quad 0 = k \times_2 x +_2 b.$$

However, let us consider the lines C_iP in the general class of straight lines

$$e \times_2 x +_2 f \times_2 y +_2 g = 0$$

with

$$n = 2, \quad e, f, g, e \times_2 x, f \times_2 y, e \times_2 x +_2 f \times_2 y \in W_2.$$

1. Let us check the existence for line C_1P ($i = 1$). So let us find the equation of the straight line C_1P in the general class of straight lines

$$e \times_2 x +_2 f \times_2 y +_2 g = 0,$$
$$\begin{cases} e \times_2 4.38 +_2 f \times_2 0 +_2 g = 0, \\ e \times_2 1.29 +_2 f \times_2 1.29 +_2 g = 0. \end{cases}$$

So

$$e = 0.50,$$
$$f = 1.22,$$
$$g = -2.15,$$

that is, the straight line C_1P in the general class of straight lines

$$e \times_2 x +_2 f \times_2 y +_2 g = 0$$

exists and has the equation

$$0.50 \times_2 x +_2 1.22 \times_2 y -_2 2.15 = 0.$$

We get

$$PK = PN = PL_1 = 1.29.$$

So we now know that the line C_1P exists. Let us consider three vectors

$$C_1N = (-3.09, 0),$$
$$C_1L_1 = (-2.19, 2.19),$$
$$C_1P = (-3.09, 1.29).$$

We get

$$(C_1N, C_1N) = (-3.09) \times_2 (-3.09) +_2 0 \times_2 0 = 9.54 > 0,$$
$$(C_1L_1, C_1L_1) = (-2.19) \times_2 (-2.19) +_2 2.19 \times_2 2.19 = 9.54 > 0,$$

that is,

$$(C_1N, C_1N) = (C_1L_1, C_1L_1),$$

and

$$(C_1N, C_1P) = (-3.09) \times_2 (-3.09) +_2 0 \times_2 1.29 = 9.54 > 0,$$
$$(C_1L_1, C_1P) = (-2.19) \times_2 (-3.09) +_2 2.19 \times_2 1.29 = 6.75 +_2 2.79 = 9.54 > 0,$$

that is,

$$(C_1N, C_1P) = (C_1L_1, C_1P).$$

This means that with chosen vectors C_1N, C_1L_1, C_1P, the straight line C_1P in the general class of straight lines is an angle C_1 bisector and the point *P* is an intersection of three bisectors of angles *O*, B_1, and C_1.

2. Let us check that for line C_2P ($i = 2$). Let us find the equation of the straight line C_2P in the general class of straight lines

$$e \times_2 x +_2 f \times_2 y +_2 g = 0,$$
$$\begin{cases} e \times_2 4.40 +_2 f \times_2 0 +_2 g = 0, \\ e \times_2 1.29 +_2 f \times_2 1.29 +_2 g = 0. \end{cases}$$

So

$$e = 0.50,$$
$$f = 1.27,$$
$$g = -2.20,$$

that is, the straight line C_2P in the general class of straight lines

$$e \times_2 x +_2 f \times_2 y +_2 g = 0$$

exists and has the equation

$$0.50 \times_2 x +_2 1.27 \times_2 y -_2 2.20 = 0.$$

Let us take three points $O(0,0)$, $B_2(0, 4.40)$, and $C_2(4.40, 0)$. The straight line $B_2C_2 = c_2$ has the equation

$$B_2C_2 : y = -x +_2 4.40.$$

We get

$$PK = PN = PL_2 = 1.29.$$

So we now know that the line C_2P exists. Let us consider three vectors

$$C_2N = (-3.11, 0),$$
$$C_2L_2 = (-2.20, 2.20),$$
$$C_2P = (-3.11, 1.29).$$

We get

$$(C_2N, C_2N) = (-3.11) \times_2 (-3.11) +_2 0 \times_2 0 = 9.67 > 0,$$
$$(C_2L_2, C_2L_2) = (-2.20) \times_2 (-2.20) +_2 2.20 \times_2 2.20 = 9.68 > 0,$$

that is,

$$(C_2N, C_2N) \neq (C_2L_2, C_2L_2),$$

and

$$(C_2N, C_2P) = (-3.11) \times_2 (-3.11) +_2 0 \times_2 1.29 = 9.67 > 0,$$
$$(C_2L_2, C_2P) = (-2.20) \times_2 (-3.11) +_2 2.20 \times_2 1.29 = 6.84 +_2 2.80 = 9.64 > 0,$$

that is,

$$(C_2N, C_2P) \neq (C_2L_2, C_2P).$$

This means that with chosen vectors C_2N, C_2L_2, C_2P, the straight line C_2P in the general class of straight lines is not an angle C_2 bisector and the point *P* is not an intersection of bisectors of angles *O* and C_2.

We can formulate final theorems.

Theorem 10.11.
In Mathematics with Observers geometry, there are triangles with existing inscribed circle with center in the intersection of three angle bisectors.
Theorem 10.12.
In Mathematics with Observers geometry, there are triangles with nonexisting three angle bisectors but with an existing inscribed circle.

10.7 Special equilateral triangle

Let $n = 6$. Let us consider the $\triangle(OAB)$ with

$$O(0,0); \quad A(2, 3.464102); \quad B(4,0).$$

Then

$$|OA| = |AB| = |OB| = 4,$$

because

$$2 \times_6 2 +_6 3.464102 \times_6 3.464102 = 4 +_6 12 = 16,$$

that is, $\triangle(OAB)$ is the equilateral triangle with sides 4.

Let us find the equation of the side OA:

$$y = k \times_6 x +_6 b,$$
$$\begin{cases} 0 = 0 \times_6 k +_6 b, \\ 3.464102 = 2 \times_6 k +_6 b. \end{cases}$$

We get

$$b = 0, \quad k = 1.732051.$$

So the equation of side OA is

$$y = 1.732051 \times_6 x.$$

Let us find the equation of the side OB:

$$y = k \times_6 x +_6 b,$$
$$\begin{cases} 0 = 0 \times_6 k +_6 b, \\ 0 = 4 \times_6 k +_6 b. \end{cases}$$

We get

$$b = 0, \quad k = 0.$$

So the equation of side OB is

$$y = 0.$$

Let us find the equation of the side AB:

$$y = k \times_6 x +_6 b,$$
$$\begin{cases} 3.464102 = 2 \times_6 k +_6 b, \\ 0 = 4 \times_6 k +_6 b, \end{cases}$$
$$\begin{cases} 3.464102 -_6 2 \times_6 k = b, \\ 0 = 4 \times_6 k +_6 3.464102 -_6 2 \times_6 k. \end{cases}$$

We get

$$b = 6.928204, \quad k = -1.732051.$$

So the equation of side AB is

$$y = -1.732051 \times_6 x +_6 6.928204.$$

Let *K*, *C*, and *L* be the midpoints of OA, AB, and OB, respectively. We get

$$K(1, 1.732051); \quad C(3, 1.732051); \quad L(2, 0).$$

Let us now consider the vector

$$\mathbf{OA} = (2, 3.464102)$$

and find the equation of straight line *f* containing the point $K(1, 1.732051)$ and perpendicular to the vector $\mathbf{OA}$:

$$y = k \times_6 x +_6 b.$$

We get

$$\begin{cases} 1.732051 = k \times_6 1 +_6 b, \\ 2 \times_6 1 +_6 3.464102 \times_6 k = 0, \end{cases}$$
$$\begin{cases} 1.732051 -_6 k \times_6 1 = b, \\ 2 \times_6 1 +_6 3.464102 \times_6 k = 0. \end{cases}$$

So line *f* does not exist, because

$$2 \times_6 1 +_6 3.464102 \times_6 (-0.577352) = 0.000001 > 0$$

and

$$2 \times_6 1 +_6 3.464102 \times_6 (-0.577353) = -0.000002 < 0.$$

The equation of straight line *g* containing the point $L(2, 0)$ and perpendicular to the straight line OB is

$$x = 2.$$

Let us now consider the vector

$$\mathbf{AB} = (2, -3.464102)$$

and find the equation of straight line *h* containing the point $C(3, 1.732051)$ and perpendicular to the vector $\mathbf{AB}$:

$$y = k \times_6 x +_6 b.$$

We get

$$\begin{cases} 1.732051 = k \times_6 3 +_6 b, \\ 2 \times_6 1 -_6 3.464102 \times_6 k = 0, \end{cases}$$
$$\begin{cases} 1.732051 -_6 k \times_6 3 = b, \\ 2 \times_6 1 -_6 3.464102 \times_6 k = 0. \end{cases}$$

So line *h* does not exist, because

$$2 \times_6 1 -_6 3.464102 \times_6 0.577352 = 0.000001 > 0$$

and

$$2 \times_6 1 -_6 3.464102 \times_6 0.577353 = -0.000002 < 0.$$

This means that $\triangle(OAB)$ has only one side bisector and only one height and has no center of circumscribed circle and no orthocenter.

Let us find the equation of the median OC:

$$y = k \times_6 x +_6 b,$$
$$\begin{cases} 0 = 0 \times_6 k +_6 b, \\ 1.732051 = 3 \times_6 k +_6 b. \end{cases}$$

So we get

$$b = 0,$$

but *k* does not exist, because

$$3 \times_6 0.577350 = 1.732050 < 1.732051$$

and

$$3 \times_6 0.577351 = 1.732053 > 1.732051.$$

So the median OC does not exist.

The equation of the median AL is

$$x = 2.$$

Let us find the equation of the median BK:

$$y = k \times_6 x +_6 b,$$
$$\begin{cases} 0 = 4 \times_6 k +_6 b, \\ 1.732051 = 1 \times_6 k +_6 b. \end{cases}$$

So

$$\begin{cases} b = -4 \times_6 k, \\ 1.732051 = 1 \times_6 k -_6 4 \times_6 k, \end{cases}$$

but *k* does not exist, because

$$1 \times_6 (-0.577350) -_6 4 \times_6 (-0.577350) = 1.732050 < 1.732051$$

and

$$1 \times_6 (-0.577351) -_6 4 \times_6 (-0.577351) = 1.732053 > 1.732051.$$

So the median BK does not exist. This means that the centroid of $\triangle(OAB)$ does not exist.

Let us now find the bisectors of angles *O*, *A*, *B* if they exist. Let us start from angle AOB. First, we have the vectors

$$\mathbf{OA} = (2, 3.464102)$$

and

$$\mathbf{OB} = (4, 0)$$

Moreover,

$$(\mathbf{OA},\mathbf{OA}) = 16 > 0,$$
$$(\mathbf{OB},\mathbf{OB}) = 16 > 0,$$

and thus

$$(\mathbf{OA},\mathbf{OA}) = (\mathbf{OB},\mathbf{OB}).$$

The equation of the straight line *l*, angle AOB bisector is

$$y = k \times_6 x +_6 b.$$

Since $O \in l$, we get

$$b = 0,$$

that is, the equation of straight line *l* is

$$y = k \times_6 x,$$

and the direction vector **l** of line *l* is

$$\mathbf{l} = (1, k).$$

We must have

$$(\mathbf{OA},\mathbf{l}) = (\mathbf{OB},\mathbf{l}),$$

that is,

$$2 \times_6 1 +_6 3.464102 \times_6 k = 4 \times_6 1 +_6 0 \times_6 k,$$
$$3.464102 \times_6 k = 2.$$

However, *k* does not exist, because

$$3.464102 \times_6 0.577352 = 1.999999 < 2$$

and

$$3.464102 \times_6 0.577353 = 2.000002 > 2.$$

So an angle AOB bisector does not exist.

The equation of straight line *m*, angle OAB bisector is

$$x = 2.$$

Let us now find the angle *B* bisector. We have

$$\mathbf{BA} = (-2, 3.464102),$$
$$\mathbf{BO} = (-4, 0),$$
$$(\mathbf{BA},\mathbf{BA}) = 16 > 0,$$
$$(\mathbf{BO},\mathbf{BO}) = 16 > 0$$
$$(\mathbf{BA},\mathbf{BA}) = (\mathbf{BO},\mathbf{BO}).$$

The equation of straight line *p*, angle OBA bisector is

$$y = k \times_6 x +_6 b.$$

The direction vector **p** of line *p* is

$$\mathbf{p} = (1, k).$$

We must have

$$(\mathbf{BA}, \mathbf{p}) = (\mathbf{BO}, \mathbf{p}),$$

that is,

$$(-2) \times_6 1 +_6 3.464102 \times_6 k = (-4) \times_6 1 +_6 0 \times_6 k,$$
$$3.464102 \times_6 k = -2.$$

However, *k* does not exist, because

$$3.464102 \times_6 (-0.577352) = -1.999999 > -2$$

and

$$3.464102 \times_6 (-0.577353) = -2.000002 < -2.$$

So the angle OBA bisector does not exist.

So we have proved the following:

Theorem 10.13.
The special equilateral triangle has only one side bisector, only one height, only one median, and only one angle bisector, and thus this triangle does not have the center of a circumscribed circle, orthocenter, centroid, and center of an inscribed circle.

10.8 Amazing triangle

Let us consider the triangle OAB with sides

$$OA : y = 6 \times_n x; \quad OB : y = 3 \times_n x; \quad AB : y = -3 \times_n x +_n 9.$$

Then $O(0, 0)$.

A:

$$\begin{cases} y = 6 \times_n x, \\ y = -3 \times_n x +_n 9, \end{cases}$$
$$6 \times_n x = -3 \times_n x +_n 9,$$
$$9 \times_n x = 9,$$
$$x = 1,$$
$$y = 6,$$

that is, $A(1, 6)$.

B:

$$\begin{cases} y = 3 \times_n x, \\ y = -3 \times_n x +_n 9, \end{cases}$$
$$3 \times_n x = -3 \times_n x +_n 9,$$
$$6 \times_n x = 9,$$
$$x = 1.5,$$
$$y = 4.5,$$

that is, $B(1.5, 4.5)$.

Let us consider all these lines and points in E_2W_2, that is, $n = 2$.

Theorem 10.14.
There are no lines perpendicular to sides AO*,* BO*, and* AB*.*

Proof.

Let us find *k* that satisfies the condition

$$6 \times_2 k = -1,$$
$$k = -0.16,$$
$$6 \times_2 -0.16 = -0.96 > -1,$$
$$k = -0.17,$$
$$6 \times_2 -0.17 = -1.02 < -1.$$

So *k* does not exist. This means that there are no lines perpendicular to AO.

Let us find *k* that satisfies the condition

$$3 \times_2 k = -1,$$
$$k = -0.33,$$
$$3 \times_2 -0.33 = -0.99 > -1,$$
$$k = -0.34,$$
$$3 \times_2 -0.34 = -1.02 < -1.$$

So *k* does not exist. This means that there are no lines perpendicular to BO.

Let us find *k* that satisfies the condition

$$-3 \times_2 k = -1,$$
$$k = 0.33,$$
$$6 \times_2 0.33 = 0.96 > -1,$$
$$k = 0.34,$$
$$-3 \times_2 0.34 = -1.02 < -1.$$

So *k* does not exist. This means that there are no lines perpendicular to AB. □

Theorem 10.15.

The triangle OAB has no perpendicular bisector for each side.

Theorem 10.16.

The triangle OAB has no center of a circumscribed circle.

Theorem 10.17.

The triangle OAB has no center of an inscribed circle.

Theorem 10.18.

The triangle OAB has no heights.

Theorem 10.19.

The triangle OAB has no orthocenter.

Theorem 10.20.

The triangle OAB has three medians but has no centroid.

Proof of the last theorem.

Let *L*, *M* and *N* be the midpoints of OA, AB, and OB:

$$L(O.5, 3); \quad M(1.25, 5.25); \quad N(0.75, 2.25).$$

Median BL:

$$y = 1.5 \times_2 x +_2 2.25.$$

Median OM:

$$y = 4.21 \times_2 x.$$

Median AN:

$$y = 15 \times_2 x -_2 9.$$

Let us first find $BL \cap AN$ and consider the system

$$\begin{cases} y = 1.5 \times_2 x +_2 2.25, \\ y = 15 \times_2 x -_2 9. \end{cases}$$

Let us try to solve this system:

$$0 = 15 \times_2 x -_2 1.5 \times_2 x -_2 11.25.$$

If

$$x = 0.84,$$

then

$$12.6 -_2 1.24 -_2 11.25 > 0,$$

and if

$$x = 0.83,$$

then

$$12.45 -_2 1.23 -_2 11.25 < 0.$$

So this system has no solution, and

$$BL \cap AN = \Lambda.$$

Let us find $BL \cap OM$ and consider the system

$$\begin{cases} y = 1.5 \times_2 x +_2 2.25, \\ y = 4.21 \times_2 x. \end{cases}$$

Let us try to solve this system

$$0 = 1.5 \times_2 x -_2 4.21 \times_2 x +_2 2.25.$$

The solution of this system is

$$x = 0.83,$$

and

$$BL \cap OM = (0.83, 3.48).$$

Let us find $AN \cap OM$ and consider the system

$$\begin{cases} y = 15 \times_2 x -_2 9, \\ y = 4.21 \times_2 x. \end{cases}$$

Let us try to solve this system:

$$0 = 15 \times_2 x -_2 4.21 \times_2 x -_2 9.$$

If

$$x = 0.83,$$

then

$$12.45 -_2 3.48 -_2 9 = -0.03 < 0,$$

and if

$$x = 0.84,$$

then

$$12.6 -_2 3.52 -_2 9 = 0.08 > 0.$$

So this system has no solution, and

$$AN \cap OM = \Lambda.$$

Thus the triangle OAB has no centroid. □

10.9 The length of segment

Let us consider the segment

$$AB = [0, 1]$$

with middle point

$$x = 0.50.$$

The same situation takes a place in Mathematics with Observers geometry. So we have $n = 2$. Let us consider the segment

$$AB = [0, 1.01].$$

This segment does not have a middle point because if

$$x = 0.5,$$

then

$$0.5 -_2 0 = 0.5,$$
$$1.01 -_2 0.5 = 0.51,$$

and

$$0.5 < 0.51,$$

and if

$$x = 0.51,$$

then

$$0.51 -_2 0 = 0.51,$$
$$1.01 -_2 0.51 = 0.5,$$

and

$$0.51 > 0.5.$$

Let us consider the segment AB on the plane with

$$A = (0.08, 0.01), \quad B = (0.06, 0.03).$$

Then the length of the segment AB can be calculated by the formula

$$|AB| = \sqrt{(0.06 -_2 0.08) \times_2 (0.06 -_2 0.08) +_2 (0.03 -_2 0.01) \times_2 (0.03 -_2 0.01)} = \sqrt{0 +_2}$$

So the length of the segment AB is

$$|AB| = 0.00, 0.01, \dots, 0.09.$$

Let us consider the segment AB on the plane with $A = (0, 1)$ and $B = (1, 0)$. This segment AB is a part of straight line

$$a : y = -x +_2 1.$$

Since $C(0.01, 0.99) \in a$, we have

$$|AC| = \sqrt{(0 -_2 0.01) \times_2 (0 -_2 0.01) +_2 (1 -_2 0.99) \times_2 (1 -_2 0.99)} = \sqrt{0},$$

that is,

$$|AC| = 0.00, 0.01, \dots, 0.09.$$

We have

$$|BC| = \sqrt{(1 -_2 0.01) \times_2 (1 -_2 0.01) +_2 (0 -_2 0.99) \times_2 (0 -_2 0.99)} = \sqrt{0.81 +_2 0.81} = \sqrt{1}$$

that is,

$$|BC| = 1.29.$$

We have

$$|AB| = \sqrt{(0 -_2 1) \times_2 (0 -_2 1) +_2 (1 -_2 0) \times_2 (1 -_2 0)} = \sqrt{1 +_2 1} = \sqrt{2},$$

that is,

$$|AB| = 1.42.$$

We have the following question: is the equality

$$|AB| = |AC| +_2 |BC|$$

correct? We have

$$1.42 > 1.29 +_2 [0.00, 0.01, \dots, 0.09] = [1.29, 1.3, \dots, 1.38],$$

that is, the length of the segment in this case is greater than the sum of lengths of its parts. Note that

$$AB = AC \cup CB.$$

Let C be the midpoint of segment AB. Then $C(0.5, 0.5)$. Let us find the lengths of AC and CB. We have

$$|AC| = \sqrt{(0 -_2 0.5) \times_2 (0 -_2 0.5) +_2 (1 -_2 0.5) \times_2 (1 -_2 0.5)} = \sqrt{0.25 +_2 0.25} = \sqrt{0.5}.$$

Suppose

$$\sqrt{0.5} = 0.7.$$

However,

$$0.7 \times_2 0.7 = 0.49 \neq 0.5.$$

If

$$\sqrt{0.5} = 0.71,$$

then

$$0.71 \times_2 0.71 = 0.49 \neq 0.5,$$

and so on. If

$$\sqrt{0.5} = 0.79,$$

then

$$0.79 \times_2 0.79 = 0.49 \neq 0.5.$$

If

$$\sqrt{0.5} = 0.8,$$

then

$$0.8 \times_2 0.8 = 0.64 > 0.5.$$

So $|AC|$ does not exist.

We have

$$|CB| = \sqrt{(1 -_2 0.5) \times_2 (1 -_2 0.5) +_2 (0 -_2 0.5) \times_2 (0 -_2 0.5)} = \sqrt{0.25 +_2 0.25} = \sqrt{0.5}.$$

Suppose

$$\sqrt{0.5} = 0.7.$$

However,

$$0.7 \times_2 0.7 = 0.49 \neq 0.5.$$

If

$$\sqrt{0.5} = 0.71,$$

then

$$0.71 \times_2 0.71 = 0.49 \neq 0.5,$$

and so on. If

$$\sqrt{0.5} = 0.79,$$

then

$$0.79 \times_2 0.79 = 0.49 \neq 0.5.$$

If

$$\sqrt{0.5} = 0.8,$$

then

$$0.8 \times_2 0.8 = 0.64 > 0.5.$$

So $|BC|$ does not exist. Let us find the length of AB:

$$|AB| = \sqrt{(0 -_2 1) \times_2 (0 -_2 1) +_2 (1 -_2 0) \times_2 (1 -_2 0)} = \sqrt{1 +_2 1} = \sqrt{2} = 1.42.$$

So $|AB| = 1.42$, and thus

$$AB = AC \cup CB.$$

However, $|AB|$ is defined and equals 1.42, but $|AC|$ and $|BC|$ are not defined.

Let us consider the straight line

$$y = -x +_2 2$$

and take the intersection of this line with positive half-axis *x*, *y*: $A(0, 2)$ and $B(2, 0)$, and then $C(1, 1)$ is the middle point of segment AB. Let us find the lengths of AC and CB:

$$|AC| = \sqrt{(2 -_2 1) \times_2 (2 -_2 1) +_2 (0 -_2 1) \times_2 (0 -_2 1)} = \sqrt{1 +_2 1} = \sqrt{2} = 1.42,$$

$$|CB| = \sqrt{(0 -_2 1) \times_2 (0 -_2 1) +_2 (2 -_2 1) \times_2 (2 -_2 1)} = \sqrt{1 +_2 1} = \sqrt{2} = 1.42.$$

Let us find the length of AB:

$$|AB| = \sqrt{(2 -_2 0) \times_2 (2 -_2 0) +_2 (0 -_2 2) \times_2 (0 -_2 2)} = \sqrt{4 +_2 4} = \sqrt{8} = 2.84.$$

We have

$$2.84 = 1.42 +_2 1.42.$$

So we get the situation where

$$AB = AC \cup CB$$

and

$$|AB| = |AC| +_2 |CB|.$$

Let us consider the straight line

$$y = -x +_2 3$$

and intersection of this line with positive half-axis x, y, that is, the points $A(0,3)$ and $B(3,0)$. Then $C(1.5, 1.5)$ is the middle point of segment $|AB|$. Let us find the lengths of AC and CB. We have

$$|AC| = \sqrt{(3 -_2 1.5) \times_2 (3 -_2 1.5) +_2 (0 -_2 1.5) \times_2 (0 -_2 1.5)} = \sqrt{2.25 +_2 2.25} = \sqrt{4.5}.$$

If

$$\sqrt{4.5} = 2.12,$$

then

$$2.12 \times_2 2.12 = 4.49 < 4.5,$$

and if

$$\sqrt{4.5} = 2.13,$$

then

$$2.13 \times_2 2.13 = 4.53 > 4.5.$$

Thus $|AC|$ does not exist. We have

$$|CB| = \sqrt{(0 -_2 1.5) \times_2 (0 -_2 1.5) +_2 (3 -_2 1.5) \times_2 (3 -_2 1.5)} = \sqrt{2.25 +_2 2.25} = \sqrt{4.5},$$

and so $|CB|$ also does not exist.

Let us find the length of AB:

$$|AB| = \sqrt{(3 -_2 0) \times_2 (3 -_2 0) +_2 (0 -_2 3) \times_2 (0 -_2 3)} = \sqrt{9 +_2 9} = \sqrt{18}.$$

If

$$\sqrt{18} = 4.24,$$

then

$$4.24 \times_2 4.24 = 17.96 < 18,$$

and if

$$\sqrt{18} = 4.25,$$

then

$$4.25 \times_2 4.25 = 18.04 > 18,$$

and so $|AB|$ does not exist. We get the situation where

$$AB = AC \cup CB,$$

but the lengths of AB, AC, and CB do not exist.

Let us consider the straight line

$$y = -x +_2 4$$

and the intersection of this line with positive half-axis *x*, *y*, that is, the points $A(0, 4)$ and $B(4, 0)$. Then $C(2, 2)$ is the middle point of segment AB. Let us find the lengths of AC and CB:

$$|AC| = \sqrt{(4 -_2 2) \times_2 (4 -_2 2) +_2 (0 -_2 2) \times_2 (0 -_2 2)} = \sqrt{4 +_2 4} = \sqrt{8} = 2.84,$$
$$|CB| = \sqrt{(0 -_2 2) \times_2 (0 -_2 2) +_2 (4 -_2 2) \times_2 (4 -_2 2)} = \sqrt{4 +_2 4} = \sqrt{8} = 2.84$$

Let us find the length of AB:

$$|AB| = \sqrt{(4 -_2 0) \times_2 (4 -_2 0) +_2 (0 -_2 4) \times_2 (0 -_2 4)} = \sqrt{16 +_2 16} = \sqrt{32}$$

If

$$\sqrt{32} = 5.66,$$

then

$$5.66 \times_2 5.66 = 31.96 < 32,$$

and if

$$\sqrt{32} = 5.67,$$

then

$$5.67 \times_2 5.67 = 32.06 > 32,$$

and so $|AB|$ does not exist. So we get the situation where

$$AB = AC \cup CB,$$

but the length of AB does not exist, and

$$|AC| + |CB| = 5.68.$$

Let us consider the straight line

$$y = -x +_2 0.09$$

and the intersection of this line with positive half-axis *x*, *y*, that is, the points $A(0, 0.09)$ and $B(0.09, 0)$. The segment AB has 10 points. So the middle point of segment AB does not exist. Let us take the point $C(0.05, 0.04)$ and find the lengths of AC and CB:

$$|AC|= \sqrt{(0.09 -_2 0.04) \times_2 (0.09 -_2 0.04) +_2 (0 -_2 0.05) \times_2 (0 -_2 0.05)} = \sqrt{0 +_2 0} = \sqrt{0.}$$
$$|AC|= 0, 0.01, \ldots, 0.09,$$
$$|CB|= \sqrt{(0 -_2 0.05) \times_2 (0 -_2 0.05) +_2 (0.04 -_2 0.09) \times_2 (0.04 -_2 0.09)} = \sqrt{0 +_2 0} = \sqrt{0.}$$
$$|CB|= 0, 0.01, \ldots, 0.09.$$

Let us find the length of AB:

$$|AB|= \sqrt{(0.09 -_2 0) \times_2 (0.09 -_2 0) +_2 (0 -_2 0.09) \times_2 (0 -_2 0.09)} = \sqrt{0 +_2 0} = \sqrt{0},$$
$$|AB|= 0, 0.01, \ldots, 0.09.$$

So we get the situation where

$$AB = AC \cup CB,$$

and we have three different possibilities:

$$|AB| = |AC| + |CB|,$$

for example,

$$0.04 = 0.03 + 0.01;$$
$$|AB| > |AC| + |CB|,$$

for example,

$$0.04 > 0.02 + 0.01;$$

and

$$|AB| < |AC| + |CB|,$$

for example,

$$0.04 < 0.05 + 0.05.$$

Let us consider the straight line

$$y = -x +_2 2$$

and three points on this line: $A(0, 2)$, $B(2, 0)$, and $C(0.82, 1.18)$. Let us find the lengths of AC and CB. We have

$$|AC| = \sqrt{(2 -_2 1.18) \times_2 (2 -_2 1.18) +_2 (0 -_2 0.82) \times_2 (0 -_2 0.82)} = \sqrt{0.64 +_2 0.64} = \sqrt{(}$$

If

$$\sqrt{(1.28)} = 1.13,$$

then

$$1.13 \times_2 1.13 = 1.27 < 1.28,$$

and if

$$\sqrt{(1.28)} = 1.14,$$

then

$$1.14 \times_2 1.14 = 1.29 > 1.28,$$

and so $|AC|$ does not exist. Also,

$$\begin{aligned}|CB| &= \sqrt{(0 -_2 1.18) \times_2 (0 -_2 1.18) +_2 (2 -_2 0.82) \times_2 (2 -_2 0.82)} \\ &= \sqrt{1.37 +_2 1.37} = \sqrt{2.74} = 1.69.\end{aligned}$$

Let us find the length of AB:

$$|AB| = \sqrt{(2 -_2 0) \times_2 (2 -_2 0) +_2 (0 -_2 2) \times_2 (0 -_2 2)} = \sqrt{4 +_2 4} = \sqrt{8} = 2.84.$$

So we get the situation where

$$AB = AC \cup CB,$$

$|AB|$ and $|CB|$ both exist, but $|AC|$ does not exist.

Let us consider the straight line

$$y = -x +_2 2$$

and its three points $A(0.15, 1.85)$, $B(2, 0)$, and $C(1, 1)$. Let us find the lengths of AC and CB:

$$|AC| = \sqrt{(0.15 -_2 1) \times_2 (0.15 -_2 1) +_2 (1 -_2 1.85) \times_2 (1 -_2 1.85)} = \sqrt{0.64 +_2 0.64} = \sqrt{(}$$

and thus $|AC|$ does not exist,

$$|CB| = \sqrt{(1 -_2 2) \times_2 (1 -_2 2) +_2 (1 -_2 0) \times_2 (1 -_2 0)} = \sqrt{1 +_2 1} = \sqrt{(2)} = 1.42.$$

Let us find the length of AB:

$$|AB| = \sqrt{(0.15 -_2 2) \times_2 (0.15 -_2 2) +_2 (1.85 -_2 0) \times_2 (1.85 -_2 0)} = \sqrt{3.34 +_2 3.34} = \sqrt{(}$$

If

$$\sqrt{6.68} = 2.59,$$

then

$$2.59 \times_2 2.58 = 6.61 < 6.68,$$

and if

$$\sqrt{6.68} = 2.6,$$

then

$$2.6 \times_2 2.6 = 6.76 > 6.68,$$

and so $|AB|$ does not exist. So we get the situation where

$$AB = AC \cup CB,$$

$|CB|$ does exist, but $|AC|$ and $|AB|$ do not.

From the previous examples we have the following:

Theorem 10.21.
Segments on the plane E_2W_n may have unique lengths, several lengths, or no lengths. If

$$AB = AC \cup CB,$$

then we have several possibilities:

1. $|AB|$, $|AC|$, *and* $|CB|$ *do not exist.*
2. $|AB|$ *does not exist, but* $|AC|$ *and* $|CB|$ *do exist.*
3. $|AB|$ *does exist, but* $|AC|$ *and* $|CB|$ *do not exist.*
4. $|AB|$ *and* $|BC|$ *exist, but* $|AC|$ *does not exist.*
5. $|AB|$ *and* $|AC|$ *do not exist, but* $|CB|$ *does exist.*
6. $|AB|$, $|AC|$, *and* $|CB|$ *exist, and*
   - **(6.1)** $|AB| = |AC| + |CB|$,
   - **(6.2)** $|AB| > |AC| + |CB|$, *or*
   - **(6.3)** $|AB| < |AC| + |CB|$.

Notes.

1. In the case where the length of segment is not unique, then we have to consider different ways of comparison and summation.

2. We have illustrated these statements for $n = 2$, that is, for E_2W_2. However, the theorem is correct for all *n*.

10.10 Midsegments of a triangle

Let us consider triangle ABC with $A(x_A, y_A)$, $B(x_B, y_B)$, $C(x_C, y_C)$, and the midpoints $L(x_L, y_L)$, $M(x_M, y_M)$, $N(x_N, y_N)$ of segments AB, AC, and BC, respectively.

Suppose we know *L*, *M*, *N* and we have to find *A*, *B*, *C*. For the *x*-coordinates, we have

$$
\begin{aligned}
&2 \times_n x_L = x_A +_n x_B,\\
&2 \times_n x_M = x_A +_n x_C,\\
&2 \times_n x_N = x_B +_n x_C,\\
&x_B = 2 \times_n x_L -_n x_A,\\
&2 \times_n x_M = x_A +_n x_C,\\
&x_C = 2 \times_n x_N -_n 2 \times_n x_L +_n x_A,\\
&x_B = 2 \times_n x_L -_n x_A,\\
&x_C = 2 \times_n x_M -_n x_A,\\
&x_C = 2 \times_n x_N -_n 2 \times_n x_L +_n x_A,\\
&x_C = 2 \times_n x_N -_n 2 \times_n x_L +_n x_A,\\
&x_C = 2 \times_n x_M -_n x_A,\\
&x_C = 2 \times_n x_N -_n 2 \times_n x_L +_n x_A,\\
&\begin{cases} 2 \times_n x_L +_n 2 \times_n x_M -_n 2 \times_n x_N = 2 \times_n x_A,\\ x_C = 2 \times_n x_M -_n x_A,\end{cases}\\
&x_A = x_L +_n x_M -_n x_N,\\
&x_B = 2 \times_n x_L -_n x_L -_n x_M +_n x_N,\\
&x_C = 2 \times_n x_M -_n x_L -_n x_A,\\
&x_A = x_L +_n x_M -_n x_N,\\
&x_B = x_L +_n x_N -_n x_M,\\
&x_C = x_M +_n x_N -_n x_L.
\end{aligned}
$$

For the *y*-coordinates, we have

$$
\begin{aligned}
y_A &= y_L +_n y_M -_n y_N,\\
y_B &= y_L +_n y_N -_n y_M,\\
y_C &= y_M +_n y_N -_n y_L,
\end{aligned}
$$

that is, the coordinates of the vertices of the triangle ABC are

$$
\begin{aligned}
&A(x_L +_n x_M -_n x_N, y_L +_n y_M -_n y_N),\\
&B(x_L +_n x_N -_n x_M, y_L +_n y_N -_n y_M),\\
&C(x_M +_n x_N -_n x_L, y_M +_n y_N -_n y_L).
\end{aligned}
$$

These formulas coincide with the formulas of classical Euclidean geometry.

Let us consider several examples with $n = 2$.

Example 1.

$L(-1, 0)$, $M(0, 1)$, $N(1, -1)$. In this case, $A(-2, 2)$, $B(0, -2)$, $C(2, 0)$.

Let us check two classic statements:

***a*:** $2 \times_2 |MN| = |AB|$, $2 \times_2 |LN| = |AC|$, $2 \times_2 |LM| = |BC|$;

***b*:** $MN \parallel AB$, $LN \parallel AC$, $LM \parallel BC$.

Let us start from *a*:

$$|MN| = \sqrt{1^2 +_2 2^2} = \sqrt{5} = 2.24,$$
$$|AB| = \sqrt{2^2 +_2 4^2} = \sqrt{20} = 4.48,$$
$$|NL| = \sqrt{2^2 +_2 1^2} = \sqrt{5} = 2.24,$$
$$|AC| = \sqrt{4^2 +_2 2^2} = \sqrt{5} = 4.48,$$
$$|ML| = \sqrt{1^2 +_2 1^2} = \sqrt{5} = 1.42,$$
$$|BC| = \sqrt{2^2 +_2 2^2} = \sqrt{5} = 2.84,$$

that is,

$$2 \times_2 |MN| = |AB|, \quad 2 \times_2 |LN| = |AC|, \quad 2 \times_2 |LM| = |BC|.$$

Let us go to *b*:

$$MN : y = -2 \times_2 x +_2 1,$$
$$AB : y = -2 \times_2 x -_2 2,$$
$$NL : y = -0.5 \times_2 x -_2 0.5,$$
$$AC : y = -0.5 \times_2 x +_2 1,$$
$$ML : y = x +_2 1,$$
$$BC : y = x +_2 1,$$

that is,

$$MN \parallel AB, \quad LN \parallel AC, \quad LM \parallel BC.$$

Example 2.

$L(-1.08, 0.24)$, $M(-0.14, 1.22)$, $N(1.16 - 1.28)$. In this case, $A(-2.38, 2.74)$, $B(0.22, -2.26)$, $C(2.1, -0.30)$.

Let us check two classic statements:

***a*:** $2 \times_2 |MN| = |AB|, 2 \times_2 |LN| = |AC|, 2 \times_2 |LM| = |BC|;$

***b*:** $MN \parallel AB, LN \parallel AC, LM \parallel BC.$

Let us start from *a*:

$$|MN| = \sqrt{1.3^2 +_2 2.5^2} = \sqrt{7.94},$$

that is, $|MN|$ does not exist,

$$|AB| = \sqrt{2.6^2 +_2 5^2} = \sqrt{31.76} = 5.64,$$
$$|NL| = \sqrt{2.24^2 +_2 -1.54^2} = \sqrt{7.33} = 2.71,$$
$$|AC| = \sqrt{-4.48^2 +_2 3.04^2} = \sqrt{29.24},$$

that is, $|AC|$ does not exist,

$$|ML| = \sqrt{0.94^2 +_2 0.98^2} = \sqrt{1.62} = 1.29,$$
$$|BC| = \sqrt{-1.88^2 +_2 -1.96^2} = \sqrt{7.13},$$

that is, $|BC|$ does not exist. So statement *a* is not true in this case.

Let us go to *b*: the lines MN and AB do not exist, that is, statement *b* is not true in this case.

So we have proved the following theorem for $n = 2$, but it is correct for all *n*.

Theorem 10.22.

In Mathematics with Observers geometry, for all $n \geq 2$, the statements

$$2 \times_2 |MN| = |AB|, \quad 2 \times_2 |LN| = |AC|, \quad 2 \times_2 |LM| = |BC|,$$
$$MN \parallel AB, \quad LN \parallel AC, \quad LM \parallel BC$$

are correct for some triangles and incorrect for some other triangles.

11 Observability and planes, lines, and vectors

11.1 Plane and vectors

1) Let us consider plane $\alpha \in E_3W_n$ containing the origin point $O(0,0,0)$:

$$\alpha : a_1 \times_2 x +_2 a_2 \times_2 y +_2 a_3 \times_2 z = 0$$

with

$$(a_1, a_2, a_3) \neq (0,0,0).$$

Let us take any point $A(x,y,z) \in \alpha$ and consider two vectors

$$\mathbf{a} = (a_1, a_2, a_3); \quad \mathbf{OA} = (x, y, z).$$

Since the scalar product

$$(\mathbf{a}, \mathbf{OA}) = 0,$$

we have that

$$\mathbf{a} \perp \mathbf{OA}$$

for all points $A(x,y,z) \in \alpha$.

Now take two points $B(x_1, y_1, z_1), C(x_2, y_2, z_2) \in \alpha$ and the corresponding vectors

$$\mathbf{OB} = (x_1, y_1, z_1); \quad \mathbf{OC} = (x_2, y_2, z_2).$$

Let us assume that

$$\mathbf{OB} \nparallel \mathbf{OC}.$$

We have the scalar products

$$(\mathbf{a}, \mathbf{OB}) = 0,$$
$$(\mathbf{a}, \mathbf{OC}) = 0$$

and the vector product

$$\mathbf{OB} \times \mathbf{OC} = (y_1 \times_n z_2 -_n z_1 \times_n y_2, -x_1 \times_n z_2 +_n z_1 \times_n x_2, x_1 \times_n y_2 -_n y_1 \times_n x_2).$$

Question: Is it true that

$$\mathbf{a} \parallel \mathbf{OB} \times \mathbf{OC}?$$

(Here $\parallel$ is understood in the Euclidean sense.)

2) Let $n = 2$, and we are in E_3W_2. Let us plane α have the equation

$$\alpha : x +_2 y +_2 z = 0.$$

So

$$\mathbf{a} = (1, 1, 1).$$

Let us take two points $B(1, 1, -2), C(-1, -3, 4) \in \alpha$ and the corresponding vectors

$$\mathbf{OB} = (1, 1, -2); \quad \mathbf{OC} = (-1, -3, 4).$$

We get

$$\begin{aligned}\mathbf{OB} \times \mathbf{OC} &= \big(1 \times_2 4 -_2 (-2) \times_2 (-3), -1 \times_2 4 +_2 (-2) \times_2 (-1), 1 \times_2 (-3) -_2 1 \times_2 \\ &= (-2, -2, -2), \\ (\mathbf{OB}, \mathbf{OB} \times \mathbf{OC}) &= 1 \times_2 (-2) +_2 1 \times_2 (-2) +_2 (-2) \times_2 (-2) = 0, \\ (\mathbf{OC}, \mathbf{OB} \times \mathbf{OC}) &= (-1) \times_2 (-2) +_2 (-3) \times_2 (-2) +_2 4 \times_2 (-2) = 0.\end{aligned}$$

In this case, we have the positive answer,

$$\mathbf{a} \parallel \mathbf{OB} \times \mathbf{OC},$$

because

$$-2 \times_2 \mathbf{a} = \mathbf{OB} \times \mathbf{OC}.$$

3) Let $n = 2$, and we are again in E_3W_2. Let plane α have the equation

$$\alpha : 0.81 \times_2 x +_2 0.63 \times_2 y +_2 z = 0,$$

and so

$$\mathbf{a} = (0.81, 0.63, 1).$$

Let us take two points $B(0.73, 0.34, -0.74), C(0.26, 0.91, -0.70) \in \alpha$ and the corresponding vectors

$$\mathbf{OB} = (0.73, 0.34, -0.74); \quad \mathbf{OC} = (0.26, 0.91, -0.70).$$

We get

$$\begin{aligned}&\mathbf{OB} \times \mathbf{OC} \\ &\quad = \big(0.34 \times_2 (-0.70) -_2 (-0.74) \times_2 0.91, \\ &\qquad -0.73 \times_2 (-0.70) +_2 (-0.74) \times_2 0.26, \\ &\qquad 0.73 \times_2 0.91 -_2 0.34 \times_2 0.26\big) \\ &\quad = (0.42, 0.35, 0.57), \\ &(\mathbf{OB}, \mathbf{OB} \times \mathbf{OC}) = 0.73 \times_2 0.42 +_2 0.34 \times_2 0.35 +_2 (-0.74) \times_2 0.57 = 0.02 \neq 0, \\ &(\mathbf{OC}, \mathbf{OB} \times \mathbf{OC}) = 0.26 \times_2 0.42 +_2 0.91 \times_2 0.35 +_2 (-0.70) \times_2 0.57 = 0.\end{aligned}$$

So in this case, we have the negative answer,

$$\mathbf{a} \nparallel \mathbf{OB} \times \mathbf{OC},$$

because

$$
\begin{aligned}
2\times_2 \mathbf{OB}\times\mathbf{OC} &= (0.84, 0.70, 1.14) \neq \mathbf{a},\\
1.99\times_2 \mathbf{OB}\times\mathbf{OC} &= (0.78, 0.62, 1.02) \neq \mathbf{a},
\end{aligned}
$$

and

$$
\begin{aligned}
&0.50\times_2 \mathbf{a} = (0.40, 0.30, 0.50) \neq \mathbf{OB}\times\mathbf{OC},\\
&0.51\times_2 \mathbf{a} = 0.50\times_2 \mathbf{a},\\
&\qquad\qquad \dots\\
&0.59\times_2 \mathbf{a} = 0.50\times_2 \mathbf{a},\\
&0.60\times_2 \mathbf{a} = (0.48, 0.36, 0.60) \neq \mathbf{OB}\times\mathbf{OC},\\
&0.61\times_2 \mathbf{a} = 0.60\times_2 \mathbf{a},\\
&\qquad\qquad \dots\\
&0.69\times_2 \mathbf{a} = 0.60\times_2 \mathbf{a}.
\end{aligned}
$$

So we have proved the following theorem for $n = 2$, but it is correct for all *n*.

Theorem 11.1.
In Mathematics with Observers geometry, for all $n \geq 2$, the statement

$$\mathbf{a} \parallel \mathbf{OB}\times\mathbf{OC}$$

is correct for some situations and incorrect for some other situations.

11.2 Line and vectors

1) Let us consider straight line $a \in E_3W_n$ containing the origin point $O(0,0,0)$ with the system of equations

$$
\begin{cases}
a_1\times_n x +_n a_2\times_n y +_n a_3\times_n z = 0,\\
b_1\times_n x +_n b_2\times_n y +_n b_3\times_n z = 0,
\end{cases}
$$

with planes in E_3W_n,

$$
\begin{aligned}
&\alpha : a_1\times_n x +_n a_2\times_n y +_n a_3\times_n z = 0,\\
&(a_1, a_2, a_3) \neq (0,0,0),\\
&\beta : b_1\times_n x +_n b_2\times_n y +_n b_3\times_n z = 0,\\
&(b_1, b_2, b_3) \neq (0,0,0),
\end{aligned}
$$

such that

$$\alpha\cap\beta \neq \alpha; \quad \alpha\cap\beta \neq \beta.$$

Now let us take any two points $A(x_1, y_1, z_1), B(x_2, y_2, z_2) \in a$; $A \neq B$, and consider the vectors

$$\mathbf{a} = (a_1, a_2, a_3); \quad \mathbf{b} = (b_1, b_2, b_3); \quad \mathbf{OA} = (x_1, y_1, z_1); \quad \mathbf{OB} = (x_2, y_2, z_2).$$

Since the scalar products

$$(\mathbf{a},\mathbf{OA})= 0,$$
$$(\mathbf{b},\mathbf{OA})= 0,$$
$$(\mathbf{a},\mathbf{OB})= 0,$$
$$(\mathbf{b},\mathbf{OB})= 0,$$

we have

$$\mathbf{a}\perp \mathbf{OA},$$
$$\mathbf{b}\perp \mathbf{OA},$$
$$\mathbf{a}\perp \mathbf{OB},$$
$$\mathbf{b}\perp \mathbf{OB}$$

for all points $A, B \in a$.

We also have the vector product

$$\mathbf{a}\times\mathbf{b} = (a_2 \times_n b_3 -_n a_3 \times_n b_2, -a_1 \times_n b_3 +_n a_3 \times_n b_1, a_1 \times_n b_2 -_n b_1 \times_n a_2).$$

Questions: a) $\mathbf{OA} \parallel \mathbf{a}\times\mathbf{b}$?

b) $\mathbf{OB} \parallel \mathbf{OA}$?

2) Let $n = 2$, and we are in E_3W_2. Let plane α have the equation

$$\alpha : x +_2 y +_2 z = 0,$$

and let plane β have the equation

$$\beta : x -_2 y +_2 z = 0.$$

So

$$\mathbf{a}= (1, 1, 1),$$
$$\mathbf{b}= (1, -1, 1).$$

Let us take two points $A(1, 0, -1), B(-2, 0, 2) \in a$ and the corresponding vectors

$$\mathbf{OA} = (1, 0, -1); \quad \mathbf{OB} = (-2, 0, 2).$$

We get

$$\mathbf{a}\times\mathbf{b}= \big(1 \times_2 1 -_2 1 \times_2 (-1), 1 \times_2 1 -_2 1 \times_2 1, 1 \times_2 (-1) -_2 1 \times_2 1\big) = (2, 0, -2),$$
$$2 \times_2 \mathbf{OA}= \mathbf{a}\times\mathbf{b},$$
$$-2 \times_2 \mathbf{OA}= \mathbf{OB},$$

and in this case, we have the positive answers to questions a) and b):

$$\mathbf{OA} \parallel \mathbf{a}\times\mathbf{b},$$

and

$$\mathbf{OB} \parallel \mathbf{OA}.$$

3) Let $n = 2$, and we are again in E_3W_2. Let plane α have the equation

$$\alpha : 0.81 \times_2 x +_2 0.63 \times_2 y +_2 z = 0,$$

and let plane β have the equation

$$\beta : 0.51 \times_2 x +_2 0.08 \times_2 y +_2 0.59 \times_2 z = 0.$$

So

$$\mathbf{a} = (0.81, 0.63, 1),$$
$$\mathbf{b} = (0.51, 0.08, 0.59).$$

Let us take two points $A(0.73, 0.34, -0.74), B(0.70, 0.39, -0.74) \in a$ and the corresponding vectors

$$\mathbf{OA} = (0.73, 0.34, -0.74); \quad \mathbf{OB} = (0.70, 0.39, -0.74).$$

We get

$$\mathbf{a} \times \mathbf{b} = (0.63 \times_2 0.59 -_2 1 \times_2 0.08, 1 \times_2 0.51 -_2 0.81 \times_2 0.59, 0.81 \times_2 0.08 -_2 0.63 \times_2 0.51)$$
$$= (0.22, 0.11, -0.30).$$

We have

$$0.49 \times_2 \mathbf{OA} = (0.28, 0.12, -0.28) \neq \mathbf{a} \times \mathbf{b},$$
$$0.50 \times_2 \mathbf{OA} = (0.35, 0.15, -0.35) \neq \mathbf{a} \times \mathbf{b},$$

and

$$3.39 \times_2 \mathbf{a} \times \mathbf{b} = (0.72, 0.36, -0.99) \neq \mathbf{OA},$$
$$3.40 \times_2 \mathbf{a} \times \mathbf{b} = (0.74, 0.37, -1.02) \neq \mathbf{OA}.$$

This means that

$$\mathbf{OA} \nparallel \mathbf{a} \times \mathbf{b}.$$

We also have

$$0.49 \times_2 \mathbf{OB} = (0.28, 0.12, -0.28) \neq \mathbf{a} \times \mathbf{b},$$
$$0.50 \times_2 \mathbf{OB} = (0.35, 0.15, -0.35) \neq \mathbf{a} \times \mathbf{b},$$

and

$$3.19 \times_2 \mathbf{a} \times \mathbf{b} = (0.68, 0.34, -0.93) \neq \mathbf{OB},$$

$$3.20 \times_2 \mathbf{a} \times \mathbf{b} = (0.70, 0.35, -0.96) \neq \mathbf{OB},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$3.29 \times_2 \mathbf{a} \times \mathbf{b} = (0.70, 0.35, -0.96) \neq \mathbf{OB},$$

$$3.30 \times_2 \mathbf{a} \times \mathbf{b} = (0.72, 0.36, -0.99) \neq \mathbf{OB}.$$

This means that

$$\mathbf{OB} \nparallel \mathbf{a} \times \mathbf{b}.$$

We also get

$$\begin{aligned}
1.00 \times_2 \mathbf{OA} &= (0.73, 0.34, -0.74) \neq \mathbf{OB}, \\
0.99 \times_2 \mathbf{OA} &= (0.63, 0.27, -0.63) \neq \mathbf{OB}, \\
1.10 \times_2 \mathbf{OB} &= (0.77, 0.42, -0.81) \neq \mathbf{OA}, \\
1.09 \times_2 \mathbf{OB} &= (0.70, 0.39, -0.74) \neq \mathbf{OA}.
\end{aligned}$$

This means that

$$\mathbf{OB} \nparallel \mathbf{OA},$$

and in this case, we have the negative answers for questions a) and b).

So we have proved the following theorem for $n = 2$, but it is correct for all n.

Theorem 11.2.
In Mathematics with Observers geometry, for all $n \geq 2$, the statements

$$\mathbf{OA} \parallel \mathbf{a} \times \mathbf{b}$$

and

$$\mathbf{OB} \parallel \mathbf{OA}$$

are correct for some situations and incorrect for some other situations.

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