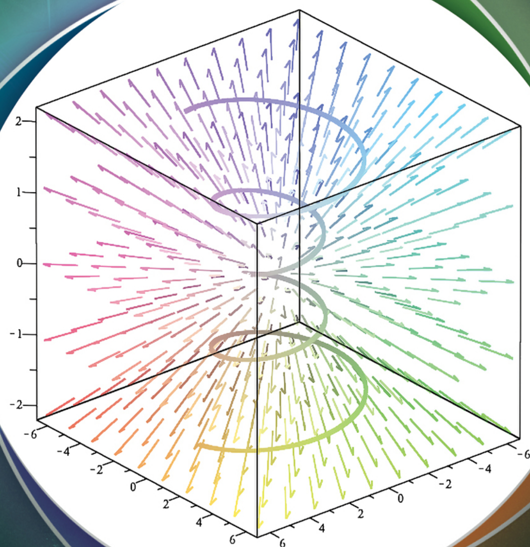
MATHEMATICS FOR ENGINEERS

Volume 3

Systems of Differential Equations, *n*-th Order Differential Equations, Fourier Series and Fourier Transform, Partial Differential Equations, Vector Analysis and Maxwell Equations, Splines

Thomas Westermann



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Preface

This third and final volume in our Mathematics for Engineers series serves as a companion text to the third-semester mathematics preliminaries for students and lecturers in electrical engineering and other engineering disciplines.

Ordinary and partially differential equations play a central role in graduate engineering courses. This topic is the main focus of the third volume: The description of engineering problems leads either to ordinary differential equations as long as there is only one independent variable. When the problem is described by more than one variable, the model equations are partial differential equations, such as the wave equation, the heat transfer equation, the Laplace equation and many more. Differential equations are discussed in detail.

In addition to describing engineering problems using differential equations, we need to analyze signals produced by oscillations and vibrations. This requires frequency analysis of the signals, which leads to the topic of Fourier series for periodic signals and Fourier transform for non-periodic signals. Both techniques are also used in the solution of differential equations.

This volume provides students at universities and colleges with a vivid presentation of these topics as a practical aid to higher mathematics. Mathematical terms are clearly motivated, systematically equated and visualized in many animations. A large number of examples and applications illustrate and deepen the material, and the numerous exercises (with solutions on the book's homepage) make it easier to prepare for exams.

Important formulas and statements are clearly highlighted in order to increase the readability of the books. Many pictures and sketches support the character of modern textbooks. The color-coded layout provides a clear overview of the presentation of the content, e.g. new terms and definitions in light grey, important statements and sentences in grey.

There are additional styles to make the book easier to read:

- The symbol ⚠ **Caution:** draws your attention to passages that are often misinterpreted, overlooked or ignored.
- Tips and rules help you work through the examples and exercises.
- Definitions and important phrases are highlighted in grey boxes.
- Numerous summaries are highlighted in color.
- Important formulas and results are marked.
- Examples and applications are clearly arranged throughout the text.
- Fully worked examples,
- problems with solutions,
- and a lot of illustrations and sketches will help very well for study purposes and for examination preparations.

Alongside the topics covered in the book, additional material is available on the website, as well as MAPLE worksheets that can be downloaded for the current version of MAPLE. The description can be found under the MAPLE tab on the book's website:

https://www.imathonline.de/books/mathe/start.htm

In the book, the following two symbols explicitly refer to additional information information that can be found on the home page:

① Animations, in gif format are available: By clicking on the appropriate location on the web the animations are played through the Internet browser.

② References indicate the MAPLE descriptions. All MAPLE worksheets are available on the website. An overview of all worksheets can be found in *index.mws*.

I would especially like to thank Mayur Shelke for his valuable and intensive help in translating the German book into an adequate English textbook. I would also like to thank the publisher for the careful proofreading and excellent fine-tuning of the English version. Special thanks go to Mrs Rok Ting of World Scientific Publishing for her support and help throughout. She has made it possible to publish these lecture notes in this important and renowned publishing house.

Karlsruhe, April 2025 *Thomas Westermann*

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Chapter 15
Systems of Linear Differential Equations

In many applications, time-dependent quantities $x_1(t)$, $x_2(t)$, ..., $x_n(t)$ are coupled in such a way that the change of a variable $\dot{x}_i(t)$ depends not only on t and $x_i(t)$, but also on the remaining variables and their derivatives. This gives rise to systems of differential equations. In this chapter, first-order systems of linear differential equations with constant coefficients are discussed in detail. The abbreviation **LDEq** is used for these systems.

The homogeneous systems of LDEq are solved using the eigenvectors and eigenvalues of the corresponding system matrix assigned to the problem. If necessary, principal vectors to the eigenvalues are also determined. A fundamental set of the homogeneous solution is obtained using these eigenvectors and principal vectors. The method of variation of constants is used to calculate a solution to the inhomogeneous problem.

5

15 Systems of Linear Differential Equations

In many applications, time-dependent quantities $x_1(t),\ x_2(t), \ldots, x_n(t)$ are coupled in such a way that the change of a variable $\dot{x}_i(t)$ depends not only on t and $x_i(t)$, but also on the remaining variables and their derivatives. This gives rise to systems of differential equations. In this chapter, first-order systems of linear differential equations with constant coefficients are discussed in detail. The abbreviation **LDEq** is used for these systems.

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15.1 Introduction

Examples of LDEq systems can be found in electrical filter circuits consisting of RCL components or in mechanical oscillations in which several spring-mass systems are coupled. For the introduction, a system of coupled double pendulums is considered.

Application Example 15.1 (Coupled Pendulums).

Two pendulums of length l with masses m_1 and m_2 attached to their ends are coupled by a spring with spring constant D (see Fig. 15.1). The two masses are initially deflected by angles φ_1 and φ_2. We consider a frictional force proportional to the velocity acting on the masses, $F_{R_i} = -\gamma\, l\, \dot{\varphi}_i(t)$, with friction coefficient γ. Then the equations of motion for small deflections φ_1 and φ_2 are

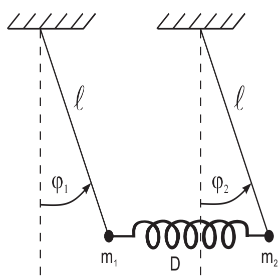


Figure 15.1. Coupled pendulums

$$\ddot{\varphi}_1(t) = -\frac{g}{l}\,\varphi_1(t) - \frac{\gamma}{m_1}\,\dot{\varphi}_1(t) + \frac{D}{m_1}\,(\varphi_2(t) - \varphi_1(t))$$

$$(*)$$

$$\ddot{\varphi}_2(t) = -\frac{g}{l}\,\varphi_2(t) - \frac{\gamma}{m_2}\,\dot{\varphi}_2(t) + \frac{D}{m_2}\,(\varphi_1(t) - \varphi_2(t)),$$

where $\varphi_1(t)$ and $\varphi_2(t)$ are the deflections of the masses m_1 and m_2 at time t. This is a system of **second** order LDEq for the angular deflections $\varphi_1(t)$ and $\varphi_2(t)$.

First, we reduce this system of two second-order differential equations to a system of four first-order differential equations. For this purpose, we include the angular velocities $\dot{\varphi}_1(t)$ and $\dot{\varphi}_2(t)$ belonging to $\varphi_1(t)$ and $\dot{\varphi}_2(t)$ as additional quantities. For a better overview, we introduce a systematic notation

$$\begin{aligned} y_1(t) &= \varphi_1(t) \\ y_2(t) &= \dot{\varphi}_1(t) \\ y_3(t) &= \varphi_2(t) \\ y_4(t) &= \dot{\varphi}_2(t). \end{aligned}$$

We differentiate each of the four unknowns $y_i(t)$ and evaluate them using the DEq $(*)$:

$$\dot{y}_1(t) = \dot{\varphi}_1(t) = y_2(t) \tag{1}$$

$$\begin{aligned} \dot{y}_2(t) = \ddot{\varphi}_1(t) &= -\tfrac{g}{l}\,\varphi_1(t) - \tfrac{\gamma}{m_1}\,\dot{\varphi}_1(t) + \tfrac{D}{m_1}\,(\varphi_2(t) - \varphi_1(t)) \\ &= -\tfrac{g}{l}\,y_1(t) - \tfrac{\gamma}{m_1}\,y_2(t) + \tfrac{D}{m_1}\,(y_3(t) - y_1(t)) \end{aligned} \tag{2}$$

$$\dot{y}_3(t) = \dot{\varphi}_2(t) = y_4(t) \tag{3}$$

$$\begin{aligned} \dot{y}_4(t) = \ddot{\varphi}_2(t) &= -\tfrac{g}{l}\,\varphi_2(t) - \tfrac{\gamma}{m_2}\,\dot{\varphi}_2(t) + \tfrac{D}{m_2}\,(\varphi_1(t) - \varphi_2(t)) \\ &= -\tfrac{g}{l}\,y_3(t) - \tfrac{\gamma}{m_2}\,y_4(t) + \tfrac{D}{m_2}\,(y_1(t) - y_3(t))\,. \end{aligned} \tag{4}$$

We define the vector $\vec{y}(t) := \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{pmatrix}$ with its derivative $\vec{y}'(t) = \begin{pmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \dot{y}_3(t) \\ \dot{y}_4(t) \end{pmatrix}$.

The four first-order differential equations (1) to (4) can then be written in vector notation as the derivative of the vector

$$\vec{y}'(t) = \begin{pmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \dot{y}_3(t) \\ \dot{y}_4(t) \end{pmatrix} = \begin{pmatrix} y_2(t) \\ \left(-\frac{g}{l} - \frac{D}{m_1}\right) y_1(t) - \frac{\gamma}{m_1}\,y_2(t) + \frac{D}{m_1}\,y_3(t) \\ y_4(t) \\ \frac{D}{m_2}\,y_1(t) + \left(-\frac{g}{l} - \frac{D}{m_2}\right) y_3(t) - \frac{\gamma}{m_2}\,y_4(t) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{g}{l} - \frac{D}{m_1} & -\frac{\gamma}{m_1} & \frac{D}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{D}{m_2} & 0 & -\frac{g}{l} - \frac{D}{m_2} & -\frac{\gamma}{m_2} \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{pmatrix}.$$

With the system matrix

$$A := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{g}{l} - \frac{D}{m_1} & -\frac{\gamma}{m_1} & \frac{D}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{D}{m_2} & 0 & -\frac{g}{l} - \frac{D}{m_2} & -\frac{\gamma}{m_2} \end{pmatrix}$$

the problem is abbreviated as:

$$\vec{y}'(t) = A\,\vec{y}(t).$$

The above procedure is applied to any higher order LDEq so that it is transformed into an extended first-order LDEq. Therefore, it is sufficient to consider only first-order systems:

General Problem: Let I be an interval, $\vec{f}(t)\colon I \to \mathbb{R}^n$ a given vector function with continuous components $f_i(t)$ $(i = 1, \ldots, n)$ and A an $(n \times n)$ matrix. We look at the LDEq

$$\vec{y}'(t) = A\,\vec{y}(t) + \vec{f}(t). \qquad (\mathbf{1})$$

If $\vec{f}(t) \neq 0$, then (1) is called an **inhomogeneous** system.

If $\vec{f}(t) = 0$, then (1) is called a **homogeneous** system.

We are looking for a differentiable vector function $\vec{y} : I \to \mathbb{C}^n$ which satisfies LDEq (1).

The vector function $\vec{y}(t)$ consists of n functions

$$\vec{y}(t) = (y_1(t), \ldots, y_n(t))^t,$$

each of these functions can be differentiated and $\vec{y}'(t) = (y_1'(t), \ldots, y_n'(t))^t$. As with first-order linear differential equations, the homogeneous problem is discussed first.

15.2 Homogeneous Systems of LDEq

We consider the homogeneous LDEq

$$\vec{y}'(t) = A\,\vec{y}(t) \tag{2}$$

with an $(n \times n)$ matrix A. Although the solutions are initially unknown, we can make statements about the properties of the solution:

> **Theorem 15.1: Homogeneous LDEq**
>
> The set of all solutions $\mathbb{L}_h$ of a homogeneous LDEq
>
> $$\vec{y}'(t) = A\,\vec{y}(t)$$
>
> with an $(n \times n)$ matrix A is an n-**dimensional vector space.**

This central statement about homogeneous LDEq is illustrated by the example given at the beginning of this chapter. The fact that the solution set is a vector space reflects the superposition principle. In the case of oscillating systems, this means that for two oscillations $\vec{y}_1(t)$ and $\vec{y}_2(t)$, their superposition $\vec{y}_1(t) + \vec{y}_2(t)$ is also a possible oscillation. Furthermore, for every oscillation $\vec{y}(t)$ a multiple $\alpha\,\vec{y}(t)$ is also a valid oscillation. And, the trivial statement that the rest position $\vec{0}$ is a state of the system.

(1) The zero vector $\vec{y}(t) = \vec{0}$ is always a solution:
For $\vec{y}(t) = \vec{0}$ follows $\vec{y}'(t) = \vec{0}' = \vec{0}$. Besides $A\,\vec{0} = \vec{0} \Rightarrow \vec{0}' = A\,\vec{0}$.
$\Rightarrow$ The zero vector $\vec{0}$ is always a solution of the homogeneous LDEq.

(2) $\vec{y}_1(t)$ and $\vec{y}_2(t)$ are solutions, which means that $\vec{y}_1'(t) = A\,\vec{y}_1(t)$ and $\vec{y}_2'(t) = A\,\vec{y}_2(t)$. With these two solutions, the superposition $\vec{y}_1(t) + \vec{y}_2(t)$ is also a solution, because
$(\vec{y}_1(t) + \vec{y}_2(t))' = \vec{y}_1'(t) + \vec{y}_2'(t) = A\,\vec{y}_1(t) + A\,\vec{y}_2(t) = A\,(\vec{y}_1(t) + \vec{y}_2(t))$.

(3) Is $\vec{y}(t)$ a solution, i.e. $\vec{y}'(t) = A\,\vec{y}(t)$, then $\alpha\,\vec{y}(t)$ is also a solution:
$(\alpha\,\vec{y}(t))' = \alpha\,\vec{y}'(t) = \alpha\,A\,\vec{y}(t) = A\,(\alpha\,\vec{y}(t))\,.$ □

We have proved that **for all physical systems that are described by homogeneous LDEq, the superposition law is always valid**! According to the subspace criterion from Volume 1, Section 2.4.2 (1) - (3), we conclude that $\mathbb{L}_h$ is a vector space. Since every finite dimensional vector space has a basis, every solution of the homogeneous LDEq can be expressed as a linear combination of basis functions

$$\vec{y}\,(t) = c_1\,\vec{\varphi}_1\,(t) + c_2\,\vec{\varphi}_2\,(t) + \ldots + c_k\,\vec{\varphi}_k\,(t)\,.$$

The question is how many basic functions there are, or how many free parameters c_i the solution must have.

To clarify this aspect, we return to the pendulum problem from Example 15.1: There are 4 independent ways to excite the system: Deflection φ_1, deflection φ_2, initial velocity $\dot{\varphi}_1$ and initial velocity $\dot{\varphi}_2$. So the solution of the pendulum problem must contain at least 4 free parameters, which can be chosen independently. Therefore, the dimension is $\mathbb{L}_h \geq 4$. Since the pendulum problem requires vector functions $\vec{y}\,(t) = (y_1\,(t)\,, \ldots,\, y_4\,(t))$ with 4 components, the dimension of $\mathbb{L}_h$ is ≤ 4.

$$\Rightarrow \dim\,(\mathbb{L}_h) = 4.$$

The following statement clarifies the conditions that must be met for the LDEq solutions to be linearly independent:

Theorem 15.2: Linearly Independent Functions

Let $\mathbb{L}_h$ be the solution set of the homogeneous LDEq $\vec{y}\,'\,(t) = A\,\vec{y}\,(t)$ with an $(n \times n)$ matrix A. For n solutions $\vec{\varphi}_1\,(t)$, $\vec{\varphi}_2\,(t)$, …, $\vec{\varphi}_n\,(t)$ the following statements are equivalent:

(1) $\vec{\varphi}_1, \ldots, \vec{\varphi}_n$ are **linearly independent functions**.

(2) For any given t the vectors $\vec{\varphi}_1\,(t)$, $\vec{\varphi}_2\,(t)$, …, $\vec{\varphi}_n\,(t)$ are linearly independent.

(3) For a given t_0 the vectors $\vec{\varphi}_1\,(t_0)$, $\vec{\varphi}_2\,(t_0)$, …, $\vec{\varphi}_n\,(t_0)$ are linearly independent.

Consequences of the Theorem:

(1) If $\vec{\varphi}_1, \vec{\varphi}_2, \ldots, \vec{\varphi}_k$ are solutions of the homogeneous LDEq, then every linear combination of

$$\vec{y}\,(t) = c_1\,\vec{\varphi}_1\,(t) + c_2\,\vec{\varphi}_2\,(t) + \ldots + c_k\,\vec{\varphi}_k\,(t) \quad (c_k \in \mathbb{C}\;,\; k \in \mathbb{N})$$

is also a solution.

(2) Since $\mathbb{L}_h$ is an n-dimensional vector space, there exists a basis of n functions, so that the general solution of the homogeneous LDEq can

be represented as a linear combination of these basis functions:

$$\vec{y}(t) = c_1 \vec{\varphi}_1(t) + c_2 \vec{\varphi}_2(t) + \cdots + c_n \vec{\varphi}_n(t).$$

(3) It is not yet clear how to calculate the basis functions, but based on the Theorem 15.2 it is possible to decide for given solutions whether we already have a basis of $\mathbb{L}_h$ or not.

Since the basis functions of the vector space $\mathbb{L}_h$ play a crucial role in the description of all solutions of the homogeneous LDEq, they get their own notation:

Definition: *We call n solutions $(\vec{\varphi}_1(t), \ldots, \vec{\varphi}_n(t))$ of the homogeneous system of LDEq*

$$\vec{y}'(t) = A\,\vec{y}(t) \tag{2}$$

a **fundamental set** *if they form a basis of the vector space $\mathbb{L}_h$ of all homogeneous solutions.*

In the n-dimensional vector space $\mathbb{R}^n$, n vectors are linearly independent if and only if the determinant of these vectors does not disappear.

Theorem 15.3: Fundamental Set

n solutions $(\vec{\varphi}_1, \vec{\varphi}_2, \ldots, \vec{\varphi}_n)$ of (2) are a **fundamental set**

$$\Leftrightarrow \quad \det\left(\vec{\varphi}_1(t_0)\,,\ \vec{\varphi}_2(t_0)\,,\ \ldots, \vec{\varphi}_n(t_0)\right) \neq 0 \quad \text{for an allowed } t_0.$$

Application Example 15.2 (Charge in a Magnetic Field).

The Lorentz equation of motion for a charged particle $q = -e$ in a homogeneous magnetic field $\vec{B} = \begin{pmatrix} 0 \\ 0 \\ B_z \end{pmatrix}$ is

$$m\frac{d}{dt}\vec{v}(t) = q\left(\vec{v} \times \vec{B}\right) = q\begin{vmatrix} \vec{e_x} & v_x & 0 \\ \vec{e_y} & v_y & 0 \\ \vec{e_z} & v_z & B_z \end{vmatrix} = -e\begin{pmatrix} v_y B_z \\ -v_x B_z \\ 0 \end{pmatrix}.$$

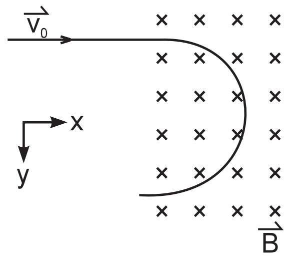


Figure 15.2. Electron in a magnetic field

In components, we get with $\omega = \frac{e}{m} B_z$

$$\begin{aligned} &\text{1st component: } \dot{v}_x(t) = -\tfrac{e}{m} B_z\, v_y(t) = -\omega\, v_y(t), \\ &\text{2nd component: } \dot{v}_y(t) = \tfrac{e}{m} B_z\, v_x(t) \quad = \quad \omega\, v_x(t), \\ &\text{3rd component: } \dot{v}_z(t) = 0. \end{aligned}$$

The third component returns $v_z(t) = const. \Rightarrow v_z(t) = 0$ if there is no initial velocity in z-direction. The first two components $v_x(t), v_y(t)$ form a system of the LDEq:

$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix}' = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} \Rightarrow \vec{v}'(t) = A\,\vec{v}(t) \qquad (*)$$

with the (2×2) matrix $A = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$. We confirm directly that $\vec{v}_1(t) = \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \end{pmatrix}$ and $\vec{v}_2(t) = \begin{pmatrix} -\sin(\omega t) \\ \cos(\omega t) \end{pmatrix}$ are solutions of $(*)$:

$$\begin{aligned} rll\vec{v}_1'(t) &= \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \end{pmatrix}' = \begin{pmatrix} -\omega \sin(\omega t) \\ \omega \cos(\omega t) \end{pmatrix} \\ A\,\vec{v}_1(t) &= \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \end{pmatrix} = \begin{pmatrix} -\omega \sin(\omega t) \\ \omega \cos(\omega t) \end{pmatrix} \end{aligned}$$

$\Rightarrow \vec{v}_1'(t) = A\,\vec{v}_1(t)$. Similarly, this is checked for $\vec{v}_2(t)$. Furthermore, $\vec{v}_1(t)$ and $\vec{v}_2(t)$ are linearly independent: According to Theorem 15.3 it is sufficient to check that $\det(\vec{v}_1(0), \vec{v}_2(0)) \neq 0$:

$$\det(\vec{v}_1(0), \vec{v}_2(0)) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

So $(\vec{v}_1, \vec{v}_2)$ is a fundamental set and every solution to the problem can be written as a linear combination of $\vec{v}_1$ and $\vec{v}_2$

$$\vec{v}(t) = c_1\,\vec{v}_1(t) + c_2\,\vec{v}_2(t)$$

$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = c_1 \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \end{pmatrix} + c_2 \begin{pmatrix} -\sin(\omega t) \\ \cos(\omega t) \end{pmatrix}$$

or in components

$$\begin{aligned} v_x(t) &= c_1 \cos(\omega t) - c_2 \sin(\omega t) \\ v_y(t) &= c_1 \sin(\omega t) + c_2 \cos(\omega t). \end{aligned}$$

The constants c_1 and c_2 are determined by the initial conditions of the problem. In this example $v_x(0) = v_0$ and $v_y(0) = 0$:

$$\begin{aligned} v_x(0) &= c_1 = v_0 \\ v_y(0) &= c_2 = 0 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} v_x(t) &= v_0 \cos(\omega t) \\ v_y(t) &= v_0 \sin(\omega t) \end{aligned} . \qquad \square$$

15.2.1 Solving Homogeneous LDEq with Constant Coefficients

The solution of the homogeneous LDEq is completely reduced to the analysis of the matrix A. This is based on the following statement:

Theorem 15.4: Solution of the Homogeneous System

Let A be an $(n \times n)$ matrix and $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ be a non-zero vector such that there is a $\lambda \in \mathbb{C}$ with $A\,\vec{x} = \lambda\,\vec{x}$. Then the function

$$\vec{\varphi}(t) = \vec{x}\, e^{\lambda t}$$

is a solution of the homogeneous LDEq $\quad \vec{y}'(t) = A\,\vec{y}(t)$.

Proof: Let $\vec{x} \in \mathbb{R}^n$ be a vector and $\lambda \in \mathbb{C}$ be a complex number with $A\,\vec{x} = \lambda\,\vec{x}$. Then the derivative of the vector function $\vec{\varphi}(t) = \vec{x}\, e^{\lambda t}$ is

$$\begin{aligned} \vec{\varphi}'(t) &= \left(\vec{x}\, e^{\lambda t}\right)' = \vec{x}\,\lambda\, e^{\lambda t} = (\lambda\,\vec{x})\; e^{\lambda t} \\ &= (A\,\vec{x})\, e^{\lambda t} = A \left(\vec{x}\, e^{\lambda t}\right) = A\,\vec{\varphi}(t). \end{aligned} \qquad \square$$

But the question is, how do we get vectors $\vec{x}$ with the property $\boxed{A\,\vec{x} = \lambda\,\vec{x}}$? This is exactly the problem of finding eigenvalues and eigenvectors of a given matrix A that is discussed in Volume 2, Section 11.1.

15.2.2 Eigenvalues and Eigenvectors

Definition: *Let A be an $(n \times n)$ matrix. $\vec{x} \neq 0$ is an* **Eigenvector** *of A, if there is a complex number λ with*

$$A\,\vec{x} = \lambda\,\vec{x}.$$

λ is then the **Eigenvalue** *of A to the eigenvector $\vec{x}$.*

The procedure for solving the eigenvalue problem is to first determine all the eigenvalues and then to compute the eigenvectors for each eigenvalue. To do this, we reformulate $A\,\vec{x} = \lambda\,\vec{x}$ into the equivalent equation

$$(A - \lambda\, I_n)\;\vec{x} = \vec{0} \tag{3}$$

where I_n is the identity matrix. To get an eigenvector $\vec{x} \neq \vec{0}$, the determinant of the matrix $A - \lambda\, I_n$ must be zero

$$\det\,(A - \lambda\, I_n) = 0.$$

To summarize Volume 2, Section 11.1 to Section 11.3, we know

Eigenvalues and Eigenvectors of a Matrix

Let A be an $(n \times n)$ matrix.

① λ is an eigenvalue of the matrix $A \;\Leftrightarrow\; \det\,(A - \lambda\, I_n) = 0$.

② $P(\lambda) = \det\,(A - \lambda\, I_n)$ is the **characteristic polynomial**. The zeros of the characteristic polynomial are the eigenvalues of the matrix A.

③ If λ is an eigenvalue of the matrix A, then all eigenvectors to the eigenvalue λ are defined as the solution of the system of linear equations $(A - \lambda\, I_n)\;\vec{x} = \vec{0}$.

④ If λ is an eigenvalue of A, then the set of eigenvectors for the eigenvalue λ forms a vector space.

Example 15.3. Given is the (3×3) matrix $A = \begin{pmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{pmatrix}$. Find all the eigenvalues and the eigenvectors of this matrix.

Step 1: To find the eigenvalues, we set up the matrix $A - \lambda\, I_3$

$$A - \lambda\, I_3 = \begin{pmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5-\lambda & 7 & -5 \\ 0 & 4-\lambda & -1 \\ 2 & 8 & -3-\lambda \end{pmatrix}$$

and calculate its characteristic polynomial

$$\begin{aligned} \det\,(A - \lambda\, I_3) &= \begin{vmatrix} 5-\lambda & 7 & -5 \\ 0 & 4-\lambda & -1 \\ 2 & 8 & -3-\lambda \end{vmatrix} \\ &= (5-\lambda) \begin{vmatrix} 4-\lambda & -1 \\ 8 & -3-\lambda \end{vmatrix} + 2 \begin{vmatrix} 7 & -5 \\ 4-\lambda & -1 \end{vmatrix} \\ &= -\lambda^3 + 6\,\lambda^2 - 11\,\lambda + 6 \\ &= -\,(\lambda - 1)\,(\lambda - 2)\,(\lambda - 3). \end{aligned}$$

From $\det\,(A - \lambda\, I_3) = 0$ we identify the eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$.

Step 2: Once the eigenvalues of a matrix have been determined, the corresponding eigenvectors are calculated for each eigenvalue by solving the system of linear equations $(A - \lambda\, I_n)\, \vec{x} = \vec{0}$:

i) Calculate the eigenvectors for the eigenvalue $\lambda_1 = 1$: We look for vectors $\vec{x} \neq \vec{0}$, so that $(A - \lambda_1\, I_3)\, \vec{x} = \vec{0}$. We solve the system

$$\begin{pmatrix} 5-1 & 7 & -5 \\ 0 & 4-1 & -1 \\ 2 & 8 & -3-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}:$$

$$\hookrightarrow \left(\begin{array}{ccc|c} 4 & 7 & -5 & 0 \\ 0 & 3 & -1 & 0 \\ 2 & 8 & -4 & 0 \end{array}\right) \hookrightarrow \left(\begin{array}{ccc|c} 4 & 7 & -5 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & -9 & 3 & 0 \end{array}\right) \hookrightarrow \left(\begin{array}{ccc|c} 4 & 7 & -5 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Backward substitution gives $x_3 = t$; $3\,x_2 - t = 0 \hookrightarrow x_2 = \frac{1}{3}\,t$; and $4\,x_1 + \frac{7}{3}\,t - 5\,t = 0 \hookrightarrow x_1 = \frac{2}{3}\,t$. We choose $t = 3$ to get $\vec{x}_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ as an eigenvector for the eigenvalue $\lambda_1 = 1$.

ii) Calculate an eigenvector for the eigenvalue $\lambda_2 = 2$: We solve the system of linear equations $(A - \lambda_2 I_3)\,\vec{x} = \vec{0}$:

$$\left(\begin{array}{ccc|c} 5-2 & 7 & -5 & 0 \\ 0 & 4-2 & -1 & 0 \\ 2 & 8 & -3-2 & 0 \end{array}\right) \hookrightarrow \left(\begin{array}{ccc|c} 3 & 7 & -5 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This is $x_3 = t$; $2\,x_2 - t = 0 \hookrightarrow x_2 = \frac{1}{2}\,t$; $3\,x_1 + \frac{7}{2}\,t - 5\,t = 0 \hookrightarrow x_1 = \frac{1}{2}\,t$. An eigenvector for the eigenvalue $\lambda_2 = 2$ is obtained by selecting, for example, $t = 2$: $\vec{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

iii) Calculate an eigenvector for the eigenvalue $\lambda_3 = 3$: By solving the linear system $(A - \lambda_3 I_3)\,\vec{x} = \vec{0}$ we get $\vec{x}_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ as an eigenvector for the eigenvalue $\lambda_3 = 3$. □

15.2.3 Solving Homogeneous LDEq with Eigenvectors

Let us return to our original problem, solving a homogeneous system of LDEq. Using the terms from the previous section, we formulate

> **Fundamental Set**
>
> If the $(n \times n)$ matrix A has a basis of eigenvectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ to the eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, then the vector functions
>
> $$\vec{\varphi}_k\,(t) = \vec{x}_k\, e^{\lambda_k\, t} \qquad (k = 1, \ldots, n)$$
>
> form a fundamental set of the homogeneous LDEq
>
> $$\vec{y}\,'\,(t) = A\,\vec{y}\,(t)\,.$$

The procedure described in Section 15.2.2 for computing the eigenvalues and associated eigenvectors is sufficient to calculate a fundamental set of the LDEq $\vec{y}\,'\,(t) = A\,\vec{y}\,(t)$ if a basis of eigenvectors can be found. The next statement summarizes some important conditions that guarantee the existence of a basis of eigenvectors.

Basis of Eigenvectors

Let A be an $(n \times n)$ matrix.

(1) If the characteristic polynomial $P(\lambda) = \det(A - \lambda I_n)$ has n different zeros, then there is a basis of eigenvectors.

(2) If for each eigenvalue of multiplicity m there are m linear independent eigenvectors, then there exists a basis of eigenvectors.

(3) If A is a real symmetric matrix (i.e. $A = A^t$), then there exists a basis of eigenvectors.

(4) If A is a complex matrix with $A = \overline{A^t}$, then there exists a basis of eigenvectors.

Note: If the dimension of the eigenspace $\mathrm{Eig}(A, \lambda)$ is equal to the multiplicity of the eigenvalue, then there exists a basis of eigenvectors. More generally, it can be shown that this condition is not only necessary but also sufficient: **A basis of eigenvectors exists if and only if for all eigenvalues the multiplicity is equal to the dimension of the eigenspace**. Since such matrices play a special role, they are called *diagonalizable* matrices (see Volume 2, Section 11.3).

Example 15.4 (Eigenvalues and Eigenvectors). Find a fundamental set to

$$\vec{y}'(t) = A\,\vec{y}(t) \qquad \text{with } A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

(i) **Calculate the eigenvalues of A:**

$$P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1-\lambda & 1 \end{vmatrix}$$

$$= -\lambda^2 (\lambda - 3).$$

The eigenvalues are the zeros of the characteristic polynomial:

$P(\lambda) = 0 \quad \hookrightarrow \quad \lambda_1 = 0$ Eigenvalue with multiplicity 2.
$\lambda_2 = 3$ Eigenvalue with multiplicity 1.

(ii) **Calculate the eigenvectors of A:**

Eigenvectors to the eigenvalue $\lambda_1 = 0$:

$$(A - 0 \cdot I_3)\,\vec{x} = 0 \hookrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \hookrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

$\hookrightarrow x_3 = r;\ x_2 = t;\ x_1 = -r - t$. So,

$$\text{Eig}\,(A,\,0) = \left\{ \vec{x} \in \mathbb{R}^3 : \vec{x} = r \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} ;\ r,\, t \in \mathbb{R} \right\}.$$

The dimension of the eigenspace $\text{Eig}(A,\,0)$ is 2 and is equal to the multiplicity of the eigenvalue. For example, two linearly independent eigenvectors are

$$\vec{x}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{x}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Eigenvector to the eigenvalue $\lambda_2 = 3$:

$$(A - 3\,I_3)\,\vec{x} = 0 \hookrightarrow \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \hookrightarrow \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

$\hookrightarrow x_3 = r;\quad x_2 = r;\quad -2\,x_1 + r + r = 0 \hookrightarrow x_1 = r.$

$$\Rightarrow \text{Eig}\,(A,\,3) = \left\{ \vec{x} \in \mathbb{R}^3 :\ \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = r \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} ;\ r \in \mathbb{R} \right\}.$$

The dimension of the eigenspace $\text{Eig}(A,\,3)$ is 1 and is equal to the multiplicity of the eigenvalue. An eigenvector is $\vec{x}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(iii) A **fundamental set** of $\vec{y}\,'(t) = A\,\vec{y}\,(t)$ is

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{0\,t}, \quad \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^{0\,t}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{3\,t}$$

and with constants $c_1,\, c_2,\, c_3$, the general solution is

$$\vec{y}\,(t) = c_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{0\,t} + c_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^{0\,t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{3\,t}. \qquad \square$$

Application Example 15.5 **(Sample Example: Solving LDEq).**

This example summarizes the solution of a first-order LDEq. The system of differential equations is given:

$$\begin{aligned} 4y_2(t) &= y_2'(t) + y_3(t) \\ 5y_1(t) + 7y_2(t) &= y_1'(t) + 5y_3(t) \\ y_3'(t) &= 2y_1(t) + 8y_2(t) - 3y_3(t). \end{aligned} \qquad (*)$$

Find the solutions $y_1(t)$, $y_2(t)$ and $y_3(t)$ for the initial conditions

$$y_1(0) = 3, y_2(0) = 2, y_3(0) = 1.$$

1. **Setting up the LDEq:** All derivatives are placed on the left and the unknown functions on the right.

$$\begin{aligned} y_1'(t) &= 5y_1(t) + 7y_2(t) - 5y_3(t) \\ y_2'(t) &= 4y_2(t) - y_3(t) \\ y_3'(t) &= 2y_1(t) + 8y_2(t) - 3y_3(t) \end{aligned}$$

2. **Setting up the system matrix:** By defining the vector $\vec{y}(t)$ with the unknown functions $y_1(t)$, $y_2(t)$ and $y_3(t)$ as components, we set up the first-order system of linear DEq and identify the system matrix A.

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}' = \begin{pmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} \Rightarrow A = \begin{pmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{pmatrix}.$$

3. **Calculating the eigenvalues and the eigenvectors:** From the Example 15.3 we know that $\vec{x}_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\vec{x}_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ are eigenvectors to the eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$.

4. **Fundamental set:** Knowing the eigenvalues and the corresponding eigenvectors, we specify a fundamental set by

$$\vec{x}_1 e^{\lambda_1 t} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} e^{1t}, \quad \vec{x}_2 e^{\lambda_2 t} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} e^{2t}, \quad \vec{x}_3 e^{\lambda_3 t} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} e^{3t}.$$

5. General solution: The general solution $\vec{y}(t)$ is then a linear combination of the fundamental set

$$\vec{y}(t) = c_1 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} e^{1t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} e^{3t}.$$

The components are the searched functions

$$\begin{aligned} y_1(t) &= \ 2\, c_1 e^{1t} + 1\, c_2 e^{2t} - c_3 e^{3t} \\ y_2(t) &= 1\, c_1 e^{1t} + 1\, c_2 e^{2t} + c_3 e^{3t} \\ y_3(t) &= 3\, c_1 e^{1t} + \ 2\, c_2 e^{2t} + c_3 e^{3t}. \end{aligned}$$

6. Determining the coefficients: The coefficients are determined by the initial conditions:

$$\begin{aligned} y_1(0) &= \ 2\, c_1 + 1\, c_2 - c_3 = 3 \\ y_2(0) &= 1\, c_1 + 1\, c_2 + c_3 = 2 \\ y_3(0) &= 3\, c_1 + \ 2\, c_2 + c_3 = 1. \end{aligned}$$

This is a system of linear equations for the constants c_1, c_2 and c_3 of the form

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 1 & 1 & 1 & 2 \\ 3 & 2 & 1 & 1 \end{array}\right) \hookrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -3 & -1 \\ 0 & 0 & -1 & 4 \end{array}\right),$$

which can be solved, for example, using the Gauss algorithm. This results in the constants $c_1 = -7$, $c_2 = 13$ and $c_3 = -4$. So the solutions of the LDEq $(*)$ are

$$\begin{aligned} y_1(t) &= \ -14\, e^{1t} + 13\, e^{2t} + 4\, e^{3t} \\ y_2(t) &= -7\, e^{1t} + 13\, e^{2t} - 4\, e^{3t} \\ y_3(t) &= -21\, e^{1t} + \ 26\, e^{2t} - 4\, e^{3t}. \end{aligned}$$

□

The procedure described can be applied directly to second-order LDEq of the form

$$\vec{y}''(t) = A\, \vec{y}(t)\,.$$

Vibration problems without friction are described by such LDEq where the first-order derivative does not occur.

Second-order LDEq

Let A be an $(n \times n)$ matrix and $\vec{x}$ an eigenvector to the eigenvalue λ. Then the functions

$$\vec{y}_1(t) = \vec{x}\, e^{+\sqrt{\lambda}\, t} \quad \text{and} \quad \vec{y}_2(t) = \vec{x}\, e^{-\sqrt{\lambda}\, t}$$

are solutions of the second-order LDEq

$$\vec{y}''(t) = A\, \vec{y}(t).$$

Proof: Let $\vec{x}$ be an eigenvector to the eigenvalue λ. Then it holds for $\vec{y}(t) := \vec{x}\, e^{\sqrt{\lambda}\, t}$:

$$\vec{y}''(t) = \left(\vec{x}\, e^{\sqrt{\lambda}\, t}\right)'' = \left(\vec{x}\, \sqrt{\lambda}\, e^{\sqrt{\lambda}\, t}\right)' = \vec{x}\, \sqrt{\lambda}^2\, e^{\sqrt{\lambda}\, t} = \lambda\, \vec{x}\, e^{\sqrt{\lambda}\, t}$$

$$= \vec{x}\, e^{\sqrt{\lambda}\, t} = A\, \vec{y}(t).$$

So $\vec{y}(t)$ is a solution of the second-order LDEq $\vec{y}''(t) = A\, \vec{y}(t)$. Analogously, we show that $\vec{x}\, e^{-\sqrt{\lambda}\, t}$ is also a solution of the LDEq. □

Hint: If A has a basis of eigenvectors $(\vec{x}_1, \ldots, \vec{x}_n)$ to the corresponding eigenvalues $\lambda_i \neq 0 \quad (i = 1, \ldots, n)$, then

$$\vec{x}_1\, e^{\sqrt{\lambda_1}\, t},\ \vec{x}_1\, e^{-\sqrt{\lambda_1}\, t}, \ldots,\ \vec{x}_n\, e^{\sqrt{\lambda_n}\, t},\ \vec{x}_n\, e^{-\sqrt{\lambda_n}\, t}$$

is a fundamental set for $\vec{y}''(t) = A\, \vec{y}(t)$. In this case it is not necessary to go to the first-order system, but the eigenvalues and eigenvectors of the matrix A are used to solve the second-order LDEq!

Application Example 15.6 (Coupled Pendulums).

We return to the introductory Example 15.1 of coupled pendulums. If the frictional forces are neglected, the system of differential equations for the angular deflections $\varphi_1(t)$ and $\varphi_2(t)$ are

$$\ddot{\varphi}_1(t) = -\frac{g}{l}\, \varphi_1(t) + \frac{D}{m}\, (\varphi_2(t) - \varphi_1(t))$$
$$\ddot{\varphi}_2(t) = -\frac{g}{l}\, \varphi_2(t) + \frac{D}{m}\, (\varphi_1(t) - \varphi_2(t)).$$

With $\vec{\varphi}(t) := \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix}$ and $A = \begin{pmatrix} -\frac{g}{l} - \frac{D}{m} & \frac{D}{m} \\ \frac{D}{m} & -\frac{g}{l} - \frac{D}{m} \end{pmatrix}$ we abbreviate

$$\vec{\varphi}''(t) = A\,\vec{\varphi}(t).$$

(1) Calculation of the eigenvalues:

$$P(\lambda) = \det(A - \lambda I_2) = \begin{vmatrix} -\frac{g}{l} - \frac{D}{m} - \lambda & \frac{D}{m} \\ \frac{D}{m} & -\frac{g}{l} - \frac{D}{m} - \lambda \end{vmatrix}$$

$$= \left(-\frac{g}{l} - \frac{D}{m} - \lambda\right)^2 - \left(\frac{D}{m}\right)^2 = 0.$$

The eigenvalues are the zeros of the characteristic polynomial

$$P(\lambda) = 0 \;\Rightarrow\; (\frac{g}{l} + \frac{D}{m}) + \lambda_{1/2} = \pm\frac{D}{m}$$

$$\hookrightarrow \boxed{\lambda_1 = -\frac{g}{l}} \quad \text{and} \quad \boxed{\lambda_2 = -\frac{g}{l} - 2\,\frac{D}{m}}.$$

(2) Calculation of the eigenvectors:

$$\lambda_1 = -\frac{g}{l}:\; (A - \lambda_1 I_2)\,\vec{x} = 0 \hookrightarrow \left(\begin{array}{cc|c} -\frac{D}{m} & \frac{D}{m} & 0 \\ \frac{D}{m} & -\frac{D}{m} & 0 \end{array}\right) \hookrightarrow \left(\begin{array}{cc|c} -\frac{D}{m} & \frac{D}{m} & 0 \\ 0 & 0 & 0 \end{array}\right):$$

Eigenvector to the eigenvalue λ_1 is $\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (*in phase*).

$$\lambda_2 = -\frac{g}{l} - 2\,\frac{D}{m}:\; (A - \lambda_2 I_2)\,\vec{x} = 0 \hookrightarrow \left(\begin{array}{cc|c} \frac{D}{m} & \frac{D}{m} & 0 \\ \frac{D}{m} & \frac{D}{m} & 0 \end{array}\right) \hookrightarrow \left(\begin{array}{cc|c} \frac{D}{m} & \frac{D}{m} & 0 \\ 0 & 0 & 0 \end{array}\right):$$

Eigenvector to the eigenvalue λ_2 is $\vec{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (*out of phase*).

(3) Setting up the fundamental set: With the eigenvectors to the eigenvalues we set up the complex fundamental system

$$\vec{x}_1\, e^{\sqrt{\lambda_1}\,t}, \quad \vec{x}_1\, e^{-\sqrt{\lambda_1}\,t}, \quad \vec{x}_2\, e^{\sqrt{\lambda_2}\,t}, \quad \vec{x}_2\, e^{-\sqrt{\lambda_2}\,t}$$

with

$$\sqrt{\lambda_1} = \sqrt{-\frac{g}{l}} = i\sqrt{\frac{g}{l}} = i\,\omega_1$$

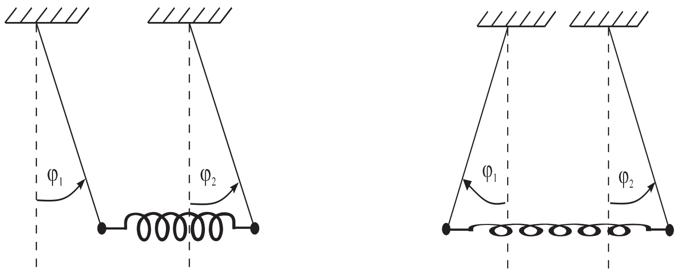
and

$$\sqrt{\lambda_2} = \sqrt{-\frac{g}{l} - 2\,\frac{D}{m}} = i\,\sqrt{\frac{g}{l} + 2\,\frac{D}{m}} = i\,\omega_2.$$

(4) Interpretation:

$\omega_1 = \sqrt{\frac{g}{l}}$ is the natural frequency of the pendulum without spring coupling. This frequency is associated with the eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, which corresponds to an initial deflection of the pendulums (see Fig. 15.2.3, left) in the same direction. The spring is not stretched and both pendulums oscillate at the natural frequency of a single pendulum without coupling.

$\omega_2 = \sqrt{\frac{g}{l} + 2\,\frac{D}{m}}$ is the natural frequency of the spring-pendulum when the two masses are deflected in opposite directions. The corresponding eigenvector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (see Fig. 15.2.3, right). The opposite deflection of the pendulums doubles the expansion of the spring, resulting in a factor $2\,\frac{D}{m}$ at the frequency.



(1) In-phase deflection (2) Out-of-phase deflection

Figure 15.3. Natural oscillations

The oscillations belonging to ω_1 and ω_2 are called *natural oscillations*. When the system is excited with an **eigenvector**, only the corresponding **eigenfrequency** is excited. All other modes are a superposition of these fundamental modes. The general solution with arbitrary complex constants c_1, c_2, c_3, c_4 is given by

$$\vec{\varphi}\,(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\,\omega_1\,t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\,\omega_1\,t} + c_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\,\omega_2\,t} + c_4 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\,\omega_2\,t}.$$

(5) Transition to a real fundamental set: Using Euler's formula (see Volume 1, Section 5.1)

$$e^{i\,\omega\,t} = \cos(\omega\,t) + i\,\sin(\omega\,t)$$

we get for any t

$$\begin{aligned} \cos(\omega\,t) &= \tfrac{1}{2}\,\left(e^{i\,\omega\,t} + e^{-i\,\omega\,t}\right) \\ \sin(\omega\,t) &= \tfrac{1}{2i}\,\left(e^{i\,\omega\,t} - e^{-i\,\omega\,t}\right). \end{aligned}$$

With the solutions $\vec{\psi}_1\,(t) = \vec{x}_1\,e^{i\,\omega_1\,t}$ and $\vec{\psi}_2\,(t) = \vec{x}_1\,e^{-i\,\omega_1\,t}$, the superpositions

$$\frac{1}{2}\,\vec{\psi}_1\,(t) + \frac{1}{2}\,\vec{\psi}_2\,(t) = \vec{x}_1\,\frac{1}{2}\,\left(e^{i\,\omega_1\,t} + e^{-i\,\omega_1\,t}\right) = \vec{x}_1\,\cos\left(\omega_1\,t\right)$$

$$\frac{1}{2i}\,\vec{\psi}_1\,(t) - \frac{1}{2i}\,\vec{\psi}_2\,(t) = \vec{x}_1\,\frac{1}{2i}\,\left(e^{i\,\omega_1\,t} - e^{-i\,\omega_1\,t}\right) = \vec{x}_1\,\sin\left(\omega_1\,t\right)$$

also satisfy the LDEq. Analogously, $\vec{x}_2\,\cos\left(\omega_2\,t\right)$ and $\vec{x}_2\sin\left(\omega_2\,t\right)$ are solutions. In total, we get four real solutions:

$$\vec{x}_1\,\cos\left(\omega_1\,t\right)\,,\vec{x}_1\,\sin\left(\omega_1\,t\right)\,,\vec{x}_2\,\cos\left(\omega_2\,t\right)\,,\vec{x}_2\,\sin\left(\omega_2\,t\right).$$

So the general solution is

$$\begin{aligned} \vec{\varphi}\,(t) = \begin{pmatrix} \varphi_1\,(t) \\ \varphi_2\,(t) \end{pmatrix} &= c_1\,\begin{pmatrix} 1 \\ 1 \end{pmatrix}\,\cos\left(\omega_1\,t\right) + c_2\,\begin{pmatrix} 1 \\ 1 \end{pmatrix}\,\sin\left(\omega_1\,t\right) \\ &\quad + c_3\,\begin{pmatrix} 1 \\ -1 \end{pmatrix}\,\cos\left(\omega_2\,t\right) + c_4\,\begin{pmatrix} 1 \\ -1 \end{pmatrix}\,\sin\left(\omega_2\,t\right) \end{aligned}$$

or in components

$$\begin{aligned} \varphi_1\,(t) &= c_1\,\cos\left(\omega_1\,t\right) + c_2\,\sin\left(\omega_1\,t\right) + c_3\,\cos\left(\omega_2\,t\right) + c_4\,\sin\left(\omega_2\,t\right) \\ \varphi_2\,(t) &= c_1\,\cos\left(\omega_1\,t\right) + c_2\,\sin\left(\omega_1\,t\right) - c_3\,\cos\left(\omega_2\,t\right) - c_4\,\sin\left(\omega_2\,t\right). \end{aligned}$$

The constants c_1, c_2, c_3, c_4 are determined by the initial conditions.

(6) Different initial conditions:
a) With $\varphi_1\,(0) = \varphi_0$, $\varphi_2\,(0) = \varphi_0$, $\dot{\varphi}_1\,(0) = 0$, $\dot{\varphi}_2\,(0) = 0$ the in-phase fundamental oscillation is excited. From the initial conditions we get $c_1 = \varphi_0$, $c_2 = c_3 = c_4 = 0$. The solution is

$$\begin{aligned} \varphi_1\,(t) &= \varphi_0\,\cos\left(\omega_1\,t\right) \\ \varphi_2\,(t) &= \varphi_0\,\cos\left(\omega_1\,t\right). \end{aligned}$$

The pendulums oscillate in-phase at the frequency $\omega_1 = \sqrt{\frac{g}{l}}$.

b) For $\varphi_1(0) = -\varphi_0$, $\varphi_2(0) = \varphi_0$, $\dot{\varphi}_1(0) = 0$, $\dot{\varphi}_2(0) = 0$ the out-of-phase fundamental oscillation is excited. From the initial conditions follows $c_3 = -\varphi_0$ and $c_1 = c_2 = c_4 = 0$

$$\Rightarrow \quad \begin{aligned} \varphi_1(t) &= -\varphi_0 \cos(\omega_2 t) \\ \varphi_2(t) &= \varphi_0 \cos(\omega_2 t). \end{aligned}$$

The pendulums oscillate in opposite phases at the frequency $\omega_2 = \sqrt{\frac{g}{l} + 2\frac{D}{m}}$.

c) We deflect only the first pendulum. Then, the initial conditions are

$$\varphi_1(0) = -\varphi_0,\ \varphi_2(0) = 0,\ \dot{\varphi}_1(0) = 0,\ \dot{\varphi}_2(0) = 0.$$

From these initial conditions we obtain four linear equations for the coefficients to be determined:

$$\begin{aligned} c_1 + c_3 &= -\varphi_0 \\ c_1 - c_3 &= 0 \\ c_2\,\omega_1 + c_4\,\omega_2 &= 0 \\ c_2\,\omega_1 - c_4\,\omega_2 &= 0. \end{aligned}$$

The solution of the system of linear equations is

$$c_1 = \tfrac{1}{2}\varphi_0,\ c_2 = 0,\ c_3 = \tfrac{1}{2}\varphi_0,\ c_4 = 0.$$

Fig. 15.4 shows the solution $\varphi_1(t)$ for the pendulum length $l = 2$, the spring constant $D = 0.2$ and the mass $m = 1$. The initial deflection is $\varphi_0 = -0.1$.

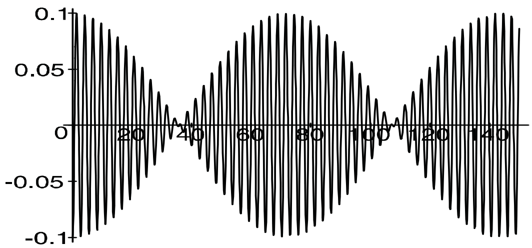


Figure 15.4. Double pendulum without friction: Beats

The resulting oscillation can easily be identified as a beat. □

Visualization: On the homepage there is an animation that visualizes the oscillation of the pendulums. The worksheet is structured in such a way that the initial conditions can be chosen freely. The coefficients in the overall solution are adjusted and the associated animation is displayed.

Note: On the homepage there is an animation, which treats the oscillations of the pendulums with friction. The corresponding worksheet visualizes the oscillations of the pendulums in the form of an animation. The angular deflection for the first pendulum $\varphi_1(t)$ is shown in Fig. 15.5.

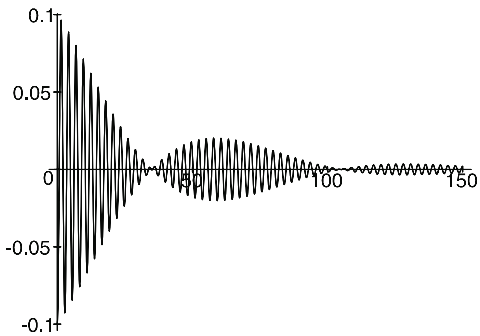


Figure 15.5. Oscillations of a double pendulum with friction

Summary

By computing the eigenvalues and eigenvectors, we can determine a fundamental set of the second-order system

$$\vec{y}''(t) = A\,\vec{y}(t) :$$

If $\vec{x}_k$ is an eigenvector to the eigenvalue λ_k, then two linearly independent solutions are

$$\vec{\varphi}_{k,1}(t) = \vec{x}_k\, e^{\sqrt{\lambda_k}\, t} \quad \text{and} \quad \vec{\varphi}_{k,2}(t) = \vec{x}_k\, e^{-\sqrt{\lambda_k}\, t}.$$

The eigenvalues represent the **eigenfrequencies** of the system and the eigenvectors are the **shapes** associated with the eigenfrequencies. The general vibration is a superposition of these eigenfrequencies.

15.2.4 Main Vectors (Principal Vectors)

As long as the matrix A can be diagonalized, we are able to determine a fundamental set for the first-order linear differential equation system. All we have to do is to calculate the eigenvalues and the corresponding eigenvectors and we will get as many linearly independent solutions as we need for a fundamental set. But what happens if the matrix can't be diagonalized? Consider the next example:

Example 15.7. Find a fundamental set for the LDEq

$$\vec{y}'(t) = A\,\vec{y}\,(t) \qquad \text{with } A = \begin{pmatrix} -3 & 1 \\ -4 & 1 \end{pmatrix}.$$

(i) **Computing the eigenvalues of A:**

$$P(\lambda) = \det\,(A - \lambda\, I_2) = \begin{vmatrix} -3-\lambda & 1 \\ -4 & 1-\lambda \end{vmatrix}$$
$$= (1-\lambda)\cdot(-3-\lambda) + 4 = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2 = 0$$

The eigenvalues are the zeros of the characteristic polynomial. So $\lambda = -1$ is a double root.

(ii) **Computing eigenvectors of the eigenvalue $\lambda = -1$:**

$$(A + 1\cdot I_2)\;\vec{x} = \vec{0}: \left(\begin{array}{cc|c} -2 & 1 & 0 \\ -4 & 2 & 0 \end{array}\right) \hookrightarrow \left(\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

$\hookrightarrow x_2 = r;\; x_1 = \frac{1}{2}r$. Only $\vec{x}_1 = r \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$ are the eigenvectors to the eigenvalue $\lambda = -1$. The dimension of the eigenspace $\mathrm{Eig}(A, -1)$ is 1 but the multiplicity of the eigenvalue is 2. Consequently, we get only one solution to the differential equation system, namely

$$\vec{y}_1(t) = \vec{x}_1 \cdot e^{\lambda\, t} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot e^{-t}.$$

But for a fundamental set we need two independent solutions! □

To find a second solution we choose the approach

$$\vec{y}(t) = (\vec{x}_2 + \vec{x}_1 \cdot t) \cdot e^{\lambda\, t}$$

where $\vec{x}_1$ is the eigenvector to the eigenvalue $\lambda = -1$ and $\vec{x}_2$ is a second unknown vector that we need to determine so that $\vec{y}(t)$ becomes a solution. We insert this approach into the differential equation by differentiating $\vec{y}(t)$

to get the left side of the LDEq and by applying the matrix A to $\vec{y}(t)$ to get the right side.

$$\begin{aligned}\vec{y}\,'(t) &= \vec{x}_1 \cdot e^{\lambda t} + (\vec{x}_2 + \vec{x}_1 \cdot t) \cdot e^{\lambda t} \cdot \lambda \\ &= (\vec{x}_1 + \lambda \vec{x}_2 + \lambda \vec{x}_1 \cdot t) \cdot e^{\lambda t}\end{aligned}$$

$$\begin{aligned}A\,\vec{y}(t) &= A\,(\vec{x}_2 + \vec{x}_1 \cdot t) \cdot e^{\lambda t} \\ &= (A\,\vec{x}_2 + A\,\vec{x}_1 \cdot t) \cdot e^{\lambda t} \\ &= (A\,\vec{x}_2 + \lambda \vec{x}_1 \cdot t) \cdot e^{\lambda t}.\end{aligned}$$

Hence,

$$\vec{x}_1 + \lambda \vec{x}_2 + \lambda \vec{x}_1 \cdot t = A\,\vec{x}_2 + \lambda \vec{x}_1 \cdot t.$$

After a rearrangement we get

$$(A - \lambda\, I_2)\,\vec{x}_2 = \vec{x}_1\ .$$

We have to solve this system of linear equations where the right side is the eigenvector $\vec{x}_1$ to the eigenvalue $\lambda = -1$. Solving the system of linear equations gives us the second vector $\vec{x}_2$, which we will call *main vector* of level 2. If we apply $(A - \lambda\, I_2)$ again on both sides of the equation, we obtain

$$(A - \lambda\, I_2)^2 \vec{x}_2 = (A - \lambda\, I_2)\vec{x}_1 = 0.$$

More generally, we define

Definition: *Let A be an $(n \times n)$ matrix and $\lambda \in \mathbb{C}$ an eigenvalue. A vector $\vec{x}$ is called main vector (principal vector) of the matrix A for the eigenvalue λ if there exists a number $k \in \mathbb{N}$ such that*

$$(A - \lambda I_n)^k\, \vec{x} = \vec{0}.$$

The main vector $\vec{x}$ is called ***main vector of level*** *k if*

$$(A - \lambda I_n)^k\, \vec{x} = \vec{0} \quad but \quad (A - \lambda\, I_n)^{k-1}\, \vec{x} \neq \vec{0}.$$

Here we use the convention that $(A - \lambda\, I_n)^0 = I_n$.

Note: Eigenvectors are also main vectors of the level 1.

Example 15.8 (Main Vectors). Find a fundamental set to the LDEq

$$\vec{y}'(t) = A\,\vec{y}\,(t) \qquad \text{with} \quad A = \begin{pmatrix} -3 & 1 \\ -4 & 1 \end{pmatrix}.$$

From Example 15.7 we know that $\lambda = -1$ is an eigenvalue to the eigenvector $\vec{x}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ where the geometric order of the eigenvector is 2 while the algebraic order of the eigenvalue is 1. So we need a main vector to determine a fundamental set of the problem. Therefore, we solve the linear equation $(A - \lambda I_2)\,\vec{x}_2 = \vec{x}_1$:

$$\left(\begin{array}{cc|c} -2 & 1 & 1 \\ -4 & 2 & 2 \end{array}\right) \hookrightarrow \left(\begin{array}{cc|c} -2 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

$\hookrightarrow x_2 = r;\ x_1 = \frac{1}{2}r - \frac{1}{2}$. Thus, $\vec{x} = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} + r\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$ are level 2 main vectors to the eigenvalue $\lambda = -1$. We set $r = 1$ to get a main vector of level 2: $\vec{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. With the eigenvector $\vec{x}_1$ and the main vector $\vec{x}_2$ we build a fundamental set to the system of LDEq

$$\vec{y}_2(t) = (\vec{x}_2 + \vec{x}_1 \cdot t) \cdot e^{\lambda\, t} = \begin{pmatrix} 0 + t \\ 1 + 2t \end{pmatrix} \cdot e^{-t}$$

along with $\vec{y}_1(t) = \vec{x}_1 \cdot e^{-t} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot e^{-t}$. □

Fundamental Set of Main Vectors

By computing the eigenvalues and eigenvectors and, if necessary, the main vectors, we compute a fundamental set for the homogeneous system of first-order differential equations

$$\vec{y}'(t) = A\,\vec{y}\,(t)\,.$$

If λ is an eigenvalue of order k then we have to compute main vectors with increasing level until we get k main vectors. If $\vec{x}_i$ are main vectors of level i ($i = 1, \ldots, k$) then

$$\vec{y}_1(t) := \vec{x}_1 \cdot e^{\lambda\, t}$$
$$\vec{y}_2(t) := (\vec{x}_2 + \vec{x}_1\, t) \cdot e^{\lambda\, t}$$
$$\ldots$$
$$\vec{y}_k(t) := \left(\vec{x}_k + \ldots + \vec{x}_2\, \tfrac{1}{(k-2)!} t^{k-2} + \vec{x}_1\, \tfrac{1}{(k-1)!} t^{k-1}\right) \cdot e^{\lambda\, t}$$

are linearly independent solutions to the system of LDEq.

Basis of Main Vectors

It is a non-trivial theorem of linear algebra which states: For every $(n \times n)$ matrix A there exists a basis of main vectors.

Example 15.9. Find a fundamental set for the LDEq

$$\vec{y}'(t) = A\,\vec{y}(t) \qquad \text{with } A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(1) **Computing the eigenvalues of A:**

$$P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = -(\lambda-1)^3.$$

So $\lambda = 1$ is an eigenvalue of the algebraic order 3.

(2) **Computing eigenvectors for $\lambda = 1$:**

$$(A - 1 \cdot I_3)\,\vec{x} = 0 \hookrightarrow \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \hookrightarrow \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

So all eigenvectors are $\vec{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and multiples of $\vec{x}_1$. The geometric order of the eigenvector is 1, which means that we need to find main vectors as long as we have enough linearly independent vectors.

(3) **Computing a main vector of level 2:**

$$(A - 1 \cdot I_3)\,\vec{x} = \mathbf{\vec{x}_1} : \left(\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \ldots \hookrightarrow \vec{x} = \begin{pmatrix} \tau \\ 1 \\ 0 \end{pmatrix}.$$

We rewrite this solution $\vec{x} = \tau \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. The first part is the eigenvector $\vec{x}_1$. To get a main vector of level 2, we choose e.g. $\tau = 0$ to obtain $\vec{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

(4) **Computing a main vector of level 3:** To obtain a level 3 main vector, we choose $\vec{x}_2$ as the right side of the system.

$$(A - 1 \cdot I_3)\,\vec{x} = \vec{\mathbf{x}}_\mathbf{2} : \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \hookrightarrow \vec{x} = \begin{pmatrix} \tau \\ 0 \\ 1 \end{pmatrix}.$$

We rewrite this solution $\vec{x} = \tau \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. The first part is again the eigenvector $\vec{x}_1$. So we choose $\tau = 0$ to get a main vector of level 3

$$\vec{x}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Since $\vec{x}_1$, $\vec{x}_2$ and $\vec{x}_3$ are the unit vectors, the eigenvector $\vec{x}_1$ together with its main vectors $\vec{x}_2$ and $\vec{x}_3$ are basis of $\mathbb{R}^3$.

(5) **Fundamental set:** Taking the eigenvector and the main vectors of level 2 and level 3, we obtain a fundamental set for the system of differential equations LDEq

$$\vec{y}_1(t) = \vec{x}_1 \cdot e^{\lambda\, t} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot e^t$$

$$\vec{y}_2(t) = (\vec{x}_2 + \vec{x}_1\, t) \cdot e^{\lambda\, t} = \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} \cdot e^t$$

$$\vec{y}_3(t) = \left(\vec{x}_3 + \vec{x}_2\, t + \vec{x}_1\, \tfrac{1}{2} t^2\right) \cdot e^{\lambda\, t} = \begin{pmatrix} \frac{1}{2} t^2 \\ t \\ 1 \end{pmatrix} \cdot e^t \qquad \square$$

Note: When computing a level 2 main vector, we can alternatively choose $\tau = 1$ and get $\vec{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. With this main vector we compute a level 3 main vector to get $\vec{x}_3 = \begin{pmatrix} \sigma \\ 1 \\ 1 \end{pmatrix}$ for each σ. Again, $\vec{x}_1$, $\vec{x}_2$ and $\vec{x}_3$ are a basis of $\mathbb{R}^3$, since $\det(\vec{x}_1, \vec{x}_2, \vec{x}_3) = 1 \neq 0$.

Example 15.10. Find a fundamental set for the LDEq

$$\vec{y}'(t) = A\,\vec{y}(t) \qquad \text{with} \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{pmatrix}.$$

Similar to Example 15.9, we calculate the characteristic polynomial and find that $\lambda = -1$ is an eigenvalue of order 3. It has an eigenvector $\vec{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ of geometric order 1. So we determine a main vector of level 2, $\vec{x}_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$, and also a main vector of level 3, $\vec{x}_3 = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$.

We will skip the details of the calculation and just refer to the results. The fundamental set is

$$\vec{y}_1(t) = \vec{x}_1 \cdot e^{\lambda t} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot e^{-t}$$

$$\vec{y}_2(t) = (\vec{x}_2 + \vec{x}_1\, t) \cdot e^{\lambda t} = \begin{pmatrix} 2+t \\ -1-t \\ t \end{pmatrix} \cdot e^{-t}$$

$$\vec{y}_3(t) = \left(\vec{x}_3 + \vec{x}_2\, t + \vec{x}_1\, \tfrac{1}{2}t^2\right) \cdot e^{\lambda t} = \begin{pmatrix} 3 + 2t + \frac{1}{2}t^2 \\ -1 - t - \frac{1}{2}t^2 \\ \frac{1}{2}t^2 \end{pmatrix} \cdot e^{-t} \qquad \square$$

Example 15.11. Find a fundamental set for the LDEq

$$\vec{y}'(t) = A\,\vec{y}(t) \qquad \text{with} \quad A = \begin{pmatrix} 8 & 0 & 1 \\ -2 & 9 & 2 \\ -1 & 0 & 10 \end{pmatrix}.$$

Similar to the previous examples, we calculate the characteristic polynomial and find that $\lambda = 9$ is an eigenvalue of algebraic order 3.

(1) **Computing eigenvectors of the eigenvalue $\lambda = 9$:**

$$(A - 9 \cdot I_3)\,\vec{x} = \vec{0}: \quad \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ -2 & 0 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right) \hookrightarrow \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The solution to this system has two parameters and, for example, $\vec{x}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{x}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ are linearly independent eigenvectors. To find a main vector of level 2, we start with $\vec{x}_1$.

(2) **Computing a level 2 main vector for $\vec{x}_1$:** To compute a level 2 main vector, we solve the system

$$(A - 9 \cdot I_3)\, \vec{x} = \mathbf{x_1} : \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ -2 & 0 & 2 & 1 \\ -1 & 0 & 1 & 0 \end{array}\right) \hookrightarrow \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 \end{array}\right).$$

From the second row we can see that the system is not solvable. This means that for the eigenvalue $\vec{x}_1$ there is no main vector of level 2.

(3) **Computing a level 2 main vector for $\vec{x}_2$:** Now we try to solve the system with $\vec{x}_2$

$$(A - 9 \cdot I_3)\, \vec{x} = \mathbf{x_2} : \left(\begin{array}{ccc|c} -1 & 0 & 1 & 1 \\ -2 & 0 & 2 & 0 \\ -1 & 0 & 1 & 1 \end{array}\right) \hookrightarrow \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\mathbf{2} \\ 0 & 0 & 0 & 0 \end{array}\right).$$

From the second row we see again that the system is not solvable. This means that for the eigenvalue $\vec{x}_2$ there is no main vector of level 2.

To find a main vector of level 2, we must therefore use a suitable linear combination of $\vec{x}_1$ and $\vec{x}_2$

$$\alpha\, \vec{x}_1 + \beta\, \vec{x}_2 = \begin{pmatrix} \beta \\ \alpha \\ \beta \end{pmatrix}$$

with which we can compute a main vector:

$$\left(\begin{array}{ccc|c} -1 & 0 & 1 & \beta \\ -2 & 0 & 2 & \alpha \\ -1 & 0 & 1 & \beta \end{array}\right) \hookrightarrow \left(\begin{array}{ccc|c} -1 & 0 & 1 & \beta \\ 0 & 0 & 0 & -\mathbf{2}\beta + \alpha \\ 0 & 0 & 0 & 0 \end{array}\right).$$

If we set $\alpha = 2\beta$, the system is solvable and we will find a main vector of level 2. So instead of $\vec{x}_1$ and $\vec{x}_2$ we take the eigenvector $\vec{x}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ to the eigenvalue $\lambda = 9$ and compute a level 2 main vector for this eigenvector.

(4) **Computing a level 2 main vector for $\vec{x}_3$:**

$$\left(\begin{array}{ccc|c} -1 & 0 & 1 & 1 \\ -2 & 0 & 2 & 2 \\ -1 & 0 & 1 & 1 \end{array}\right) \hookrightarrow \left(\begin{array}{ccc|c} -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The solution to this linear system is

$$\vec{x} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \tau \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \sigma \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

where we identify the two eigenvectors $\vec{x}_1$ and $\vec{x}_2$. By choosing $\sigma = 0$ and $\tau = 0$ we obtain the main vector $\vec{v} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$.

(5) **Fundamental Set:** From the two eigenvectors $\vec{x}_1$ and $\vec{x}_2$ we get two independent solutions

$$\vec{y}_1(t) = \vec{x}_1 \cdot e^{\lambda\, t} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot e^{9t}$$

$$\vec{y}_2(t) = \vec{x}_2 \cdot e^{\lambda\, t} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot e^{9t}$$

For the eigenvector $\vec{x}_3$ we identified a main vector of level 2, $\vec{v}$, so we have an additional solution to the differential system with

$$\vec{y}_3(t) = (\vec{v} + \vec{x}_3\, t) \cdot e^{9t} = \begin{pmatrix} -1+t \\ 2t \\ t \end{pmatrix} \cdot e^{9t}.$$

(6) **Fundamental Matrix:** Especially when we will solve the inhomogeneous problem, it is convenient to summarize the information of the fundamental set in a corresponding matrix. The columns of the fundamental matrix are the vectors of the fundamental set:

$$F(t) := (\vec{y}_1(t), \vec{y}_2(t), \vec{y}_3(t)) = \begin{pmatrix} 0 & e^{9t} & (-1+t)\cdot e^{9t} \\ e^{9t} & 0 & 2t \cdot e^{9t} \\ 0 & e^{9t} & t \cdot e^{9t} \end{pmatrix}. \qquad \square$$

Note: So for any quadratic matrix A we can determine a fundamental set to the homogeneous first-order system $\vec{y}\,'(t) = A\,\vec{y}\,(t)$.

15.3 Inhomogeneous Linear Differential Equation Systems

The solution of an inhomogeneous linear differential equation

$$\vec{y}'(t) = A\,\vec{y}(t) + \vec{f}(t)$$

can be determined elegantly using the Fourier transform (Chapter 18) or the Laplace transform (Volume 2, Chapter 14) if initial conditions are specified. In this section we will calculate a special solution of the inhomogeneous system using the method of **Variation of the Constant**.

Let $(\vec{y}_1(t), \vec{y}_2(t), \ldots, \vec{y}_n(t))$ be a fundamental set of the homogeneous problem, i.e.

$$\vec{y}_i'(t) = A\,\vec{y}_i(t) \qquad (i = 1, \ldots, n) . \tag{$*$}$$

Note that each vector $\vec{y}_i(t)$ has n components $\vec{y}_i(t) = \begin{pmatrix} y_{1i}(t) \\ y_{2i}(t) \\ \cdots \\ y_{ni}(t) \end{pmatrix}$! From the basis functions $\vec{y}_1(t), \ldots, \vec{y}_n(t)$ we form the fundamental matrix, whose columns consist of the basis functions.

Fundamental Matrix

$$F(t) := (\vec{y}_1(t), \ldots, \vec{y}_n(t)) = \begin{pmatrix} y_{11}(t) & y_{12}(t) & \cdots & y_{1n}(t) \\ y_{21}(t) & y_{22}(t) & \cdots & y_{2n}(t) \\ \vdots & \vdots & & \vdots \\ y_{n1}(t) & y_{n2}(t) & \cdots & y_{nn}(t) \end{pmatrix}.$$

With the fundamental matrix we write the general solution of the homogeneous system as

$$\vec{y}(t) = c_1\,\vec{y}_1(t) + c_2\,\vec{y}_2(t) + \ldots + c_n\,\vec{y}_n(t) = F(t) \cdot \begin{pmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{pmatrix} = F(t) \cdot \vec{c}.$$

$F(t)$ is an invertible $(n \times n)$ matrix because the basis functions $\vec{y}_1, \ldots, \vec{y}_n$ are linearly independent and $\det(F(t)) = \det(\vec{y}_1(t), \ldots, \vec{y}_n(t)) \neq 0$. The derivative of a matrix is defined as the derivative of its elements

$$F'(t) := \begin{pmatrix} y'_{11}(t) & \cdots & y'_{1n}(t) \\ y'_{21}(t) & \cdots & y'_{2n}(t) \\ \vdots & & \vdots \\ y'_{n1}(t) & \cdots & y'_{nn}(t) \end{pmatrix} = \left(\vec{y}'_1(t), \ \ldots, \ \vec{y}'_n(t)\right) .$$

Since the vector functions $\vec{y}_i(t)$ are solutions of the homogeneous system $(*)$, $F'(t)$ is

$$\begin{aligned} F'(t) &= \left(\vec{y}'_1(t), \ \vec{y}'_2(t), \ \ldots, \ \vec{y}'_n(t)\right) \\ &= \left(A\,\vec{y}_1(t), \ A\,\vec{y}_2(t), \ \ldots, \ A\,\vec{y}_n(t)\right) \\ &= A \cdot \left(\vec{y}_1(t), \ \vec{y}_2(t), \ \ldots, \ \vec{y}_n(t)\right) \\ &= A \cdot F(t) . \end{aligned}$$

Now we look at the inhomogeneous system

$$\begin{aligned} \vec{y}'(t) &= A\,\vec{y}(t) + \vec{f}(t) \\ \vec{y}(t_0) &= \vec{y}_0 \end{aligned} \qquad (**)$$

with a given continuous function $\vec{f}(t)$ and the initial condition $\vec{y}_0$. We know that

$$\vec{y}(t) = F(t) \cdot \vec{c}$$

is the solution of the homogeneous system with an arbitrary constant vector $\vec{c}$. As an approach to solving the inhomogeneous problem, we vary the constant such that we assume it is also a function of t

$$\vec{y}(t) = F(t) \cdot \vec{c}(t) \qquad \text{(Variation of the Constant),}$$

with an unknown vector function $\vec{c}(t) = \left(c_1(t), c_2(t), \ldots, c_n(t)\right)^t$. We get the derivative of $\vec{y}(t)$ using the product rule

$$\begin{aligned} \vec{y}'(t) &= F'(t)\ \vec{c}(t) + F(t)\ \vec{c}'(t) \\ &= A\,F(t)\ \vec{c}(t) + F(t)\ \vec{c}'(t) \\ &= A\,\vec{y}(t) + F(t)\ \vec{c}'(t) \\ &= A\,\vec{y}(t) + \vec{f}(t). \end{aligned}$$

$\vec{y}(t)$ satisfies the inhomogeneous linear differential equation if and only if

$$F(t)\ \vec{c}'(t) = \vec{f}(t) .$$

Since $F(t)$ is an invertible matrix, we apply its inverse to get

$$\vec{c}'(t) = F^{-1}(t)\ \vec{f}(t)$$

and after integration

$$\vec{c}(t) = \vec{c}_0 + \int_{t_0}^{t} F^{-1}(\xi)\ \vec{f}(\xi)\ d\xi.$$

So the solution to the problem is

$$\vec{y}(t) = F(t) \cdot \vec{c}(t) = F(t) \cdot \left\{ \vec{c}_0 + \int_{t_0}^{t} F^{-1}(\xi)\ \vec{f}(\xi)\ d\xi \right\}$$

$$\vec{y}(t) = \underbrace{F(t)\ \vec{c}_0}_{\text{homogeneous solution}} + \underbrace{F(t) \int_{t_0}^{t} F^{-1}(t)(\xi)\ \vec{f}(\xi)\ d\xi}_{\text{a particular solution}}.$$

The constant vector $\vec{c}_0 = (c_1, \ldots, c_n)^t$ must be chosen so that the vector equation $\vec{y}(t_0) = F(t_0)\ \vec{c}_0 = c_1 \vec{y}_1(t_0) + \ldots + c_n\, \vec{y}_n(t_0)$ is satisfied.

The general solution of the inhomogeneous problem is the sum of the general solution of the homogeneous problem and a special solution of the inhomogeneous differential equation.

Summary: Variation of the Constant

Let $F(t) = (\vec{y}_1(t), \ldots, \vec{y}_n(t))$ be the fundamental matrix of $\vec{y}'(t) = A\, \vec{y}(t)$. Then

$$\vec{y}(t) = F(t)\ \vec{c}_0 + F(t) \int_{t_0}^{t} F^{-1}(\xi)\ \vec{f}(\xi)\ d\xi$$

is the general solution of

$$\vec{y}'(t) = A\, \vec{y}(t) + \vec{f}(t)\ .$$

The constant vector $\vec{c}_0$ must be chosen so that the initial condition $\vec{y}(t_0)$ is satisfied.

Example 15.12 (Inhomogeneous Problem). Find the general solution of the inhomogeneous LDEq

$$\vec{y}'(t) = A\,\vec{y}(t) + \vec{f}(t) \qquad \text{with } A = \begin{pmatrix} -1 & 3 \\ 2 & -2 \end{pmatrix},\ \vec{f}(t) = \begin{pmatrix} 0 \\ e^{-t} \end{pmatrix}.$$

(i) **Computing the fundamental matrix $F(t)$:** To build the fundamental matrix, we need the eigenvalues

$$P(\lambda) = \det\,(A - \lambda\, I_2) = \begin{vmatrix} -1-\lambda & 3 \\ 2 & -2-\lambda \end{vmatrix} = \lambda^2 + 3\lambda - 4 = 0\,.$$

The eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = 1$. By solving the corresponding systems of linear equations

$$(A - \lambda_i \cdot I_2)\,\vec{x} = \vec{0},$$

we get the eigenvectors: $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ to $\lambda_1 = -4$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ to $\lambda_2 = 1$.

So $\vec{y}_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot e^{-4}$ and $\vec{y}_2(t) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot e^{t}$ are a fundamental set and

$$F(t) = \begin{pmatrix} e^{-4t} & 3e^{t} \\ -e^{-4t} & 2e^{t} \end{pmatrix}$$

is the fundamental matrix. The determinant of $F(t)$ is

$$\det(F(t)) = 2\,e^{-3t} + 3\,e^{-3t} = 5\,e^{-3t} \neq 0$$

with its inverse matrix

$$F^{-1}(t) = \begin{pmatrix} \frac{2}{5}e^{4t} & -\frac{3}{5}e^{4t} \\ \frac{1}{5}e^{-t} & \frac{1}{5}e^{-t} \end{pmatrix}.$$

(ii) **Computing $F^{-1}(t) \cdot \vec{f}(t)$ and integrating:** It is

$$F^{-1}(t) \cdot \vec{f}(t) = \begin{pmatrix} \frac{2}{5}e^{4t} & -\frac{3}{5}e^{4t} \\ \frac{1}{5}e^{-t} & \frac{1}{5}e^{-t} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ e^{-t} \end{pmatrix} = \begin{pmatrix} -\frac{3}{5}e^{3t} \\ \frac{1}{5}e^{-2t} \end{pmatrix}$$

and

$$F(t) \cdot \int F^{-1}(t) \cdot \vec{f}(t)\,dt = F(t) \cdot \int \begin{pmatrix} -\frac{3}{5}e^{3t} \\ \frac{1}{5}e^{-2t} \end{pmatrix} dt$$

$$= F(t) \cdot \begin{pmatrix} -\frac{3}{5}e^{3t}\frac{1}{3} \\ \frac{1}{5}e^{-2t}\frac{1}{-2} \end{pmatrix} = \begin{pmatrix} e^{-4t} & 3e^{t} \\ -e^{-4t} & 2e^{t} \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{5}e^{3t} \\ -\frac{1}{10}e^{-2t} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2}e^{-t} \\ 0 \end{pmatrix}.$$

Note: $\vec{f}(t) = \begin{pmatrix} 0 \\ e^{-t} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$ is an exponential function and a particular solution is also of type exponential function. Together with the homogeneous solution we obtain the general solution to the inhomogeneous problem

$$\vec{y}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot e^{-4} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot e^{t} + \begin{pmatrix} -\frac{1}{2}e^{-t} \\ 0 \end{pmatrix}. \qquad \square$$

Example 15.13 (Particular Solution). Find a particular solution to the inhomogeneous LDEq

$$\vec{y}\,'(t) = A\,\vec{y}\,(t) + \vec{f}(t) \quad \text{with} \quad A = \begin{pmatrix} -1 & 3 \\ 2 & -2 \end{pmatrix}, \ \vec{f}(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}.$$

According to Example 15.12 we already know a fundamental set and therefore the fundamental matrix and its inverse

$$F(t) = \begin{pmatrix} e^{-4t} & 3e^{t} \\ -e^{-4t} & 2e^{t} \end{pmatrix} \quad \text{and} \quad F^{-1}(t) = \begin{pmatrix} \frac{2}{5}e^{4t} & -\frac{3}{5}e^{4t} \\ \frac{1}{5}e^{-t} & \frac{1}{5}e^{-t} \end{pmatrix}.$$

We have to calculate $F^{-1}(t) \cdot \vec{f}(t)$ and integrate. It is

$$F^{-1}(t) \cdot \vec{f}(t) = \begin{pmatrix} \frac{2}{5}e^{4t} & -\frac{3}{5}e^{4t} \\ \frac{1}{5}e^{-t} & \frac{1}{5}e^{-t} \end{pmatrix} \cdot \begin{pmatrix} t \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{5}\,t\,e^{4t} \\ \frac{1}{5}\,t\,e^{-t} \end{pmatrix}$$

and a particular solution is

$$\vec{y}_p(t) = F(t) \cdot \int F^{-1}(t) \cdot \vec{f}(t)\,dt = F(t) \cdot \begin{pmatrix} \frac{1}{40}\,(-1+4t)\,e^{4t} \\ \frac{1}{5}\,(-1-t)\,e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-4t} & 3e^{t} \\ -e^{-4t} & 2e^{t} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{40}\,(-1+4t)\,e^{4t} \\ \frac{1}{5}\,(-1-t)\,e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{5}{8} - \frac{1}{2}\,t \\ -\frac{3}{8} - \frac{1}{2}\,t \end{pmatrix}.$$

The particular is also a polynomial. $\square$

Looking at the Examples 15.12 and 15.13 we see that the particular solution is of the same type as the inhomogeneity $\vec{f}(t)$. This leads to a popular way to compute a particular solution by a right-hand-side approach, as we have already considered in the case of first-order differential equations (see Volume 2, Section 13.4).

To clarify the idea, we use the examples already discussed and compute a particular solution by the right-hand-side approach.

Example 15.14 (Right-Hand-Side Approach). Find a particular solution to the inhomogeneous LDEq

$$\vec{y}'(t) = A\,\vec{y}(t) + \vec{f}(t) \quad \text{with} \quad A = \begin{pmatrix} -1 & 3 \\ 2 & -2 \end{pmatrix},\ \vec{f}(t) = \begin{pmatrix} 0 \\ e^{-t} \end{pmatrix}.$$

The right side $\vec{f}(t) = \begin{pmatrix} 0 \\ e^{-t} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$ is an exponential function e^{-t}. Therefore, we use the approach

$$\vec{y}_p(t) = \begin{pmatrix} a \\ b \end{pmatrix} e^{-t} = \begin{pmatrix} a\,e^{-t} \\ b\,e^{-t} \end{pmatrix}.$$

To identify the unknown parameters a and b we evaluate the right and left sides of the system $\vec{y}'(t) = A\,\vec{y}(t) + \vec{f}(t)$:

$$\begin{pmatrix} -a\,e^{-t} \\ -b\,e^{-t} \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 2 & -2 \end{pmatrix} \cdot \begin{pmatrix} a\,e^{-t} \\ b\,e^{-t} \end{pmatrix} + \begin{pmatrix} 0 \\ e^{-t} \end{pmatrix}$$

$$\begin{pmatrix} -a \\ -b \end{pmatrix} e^{-t} = \begin{pmatrix} -a + 3b \\ 2a - 2b + 1 \end{pmatrix} e^{-t}.$$

We compare the two components and get

$$\begin{aligned} -a &= -a + 3b \\ -b &= 2a - 2b + 1 \end{aligned}$$

with the solutions $a = -\frac{1}{2}$ and $b = 0$.

This gives the same particular solution as in Example 15.12 using the fundamental matrix

$$\vec{y}_p(t) = \begin{pmatrix} -\frac{1}{2}e^{-t} \\ 0 \end{pmatrix}.$$

□

Example 15.15 (Right-Hand-Side Approach). Find a particular solution to the inhomogeneous LDEq

$$\vec{y}'(t) = A\,\vec{y}(t) + \vec{f}(t) \quad \text{with} \quad A = \begin{pmatrix} -1 & 3 \\ 2 & -2 \end{pmatrix},\; \vec{f}(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}.$$

The right side $f(\vec{t}) = \begin{pmatrix} t \\ 0 \end{pmatrix}$ is of type polynomial of degree 1, so we use the approach polynomial of degree 1 for each component

$$\vec{y}_p(t) = \begin{pmatrix} a_1 + b_1\,t \\ a_2 + b_2\,t \end{pmatrix}.$$

To identify the unknown parameters a_1, a_2 and b_1, b_2, we evaluate the left and right sides of the system $\vec{y}'(t) = A\,\vec{y}(t) + \vec{f}(t)$:

$$\begin{aligned} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} &= \begin{pmatrix} -1 & 3 \\ 2 & -2 \end{pmatrix} \cdot \begin{pmatrix} a_1 + b_1\,t \\ a_2 + b_2\,t \end{pmatrix} + \begin{pmatrix} t \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -a_1 - b_1\,t + 3a_2 + 3b_2\,t + t \\ 2a_1 + 2b_1\,t - 2a_2 - 2b_2\,t \end{pmatrix} \\ &= \begin{pmatrix} (-a_1 + 3a_2) + (-b_1 + 3b_2 + 1)\,t \\ (2a_1 - 2a_2) + (2b_1 - 2b_2)\,t \end{pmatrix}. \end{aligned}$$

We compare the two polynomials for each component. The first component gives

$$\begin{aligned} t^1 &: \; 0 = -b_1 + 3b_2 + 1 \\ t^0 &: \; b_1 = -a_1 + 3a_2 \end{aligned}$$

and the second component

$$\begin{aligned} t^1 &: \; 0 = 2b_1 - 2b_2 \\ t^0 &: \; b_2 = 2a_1 - 2a_2. \end{aligned}$$

The solutions to these linear equations are $a_1 = -\frac{5}{8}$, $b_1 = -\frac{1}{2}$ and $a_2 = -\frac{3}{8}$, $b_2 = -\frac{1}{2}$.

We obtain the same particular solution as in Example 15.13 using the fundamental matrix

$$\vec{y}_p(t) = \begin{pmatrix} -\frac{5}{8} - \frac{1}{2}\,t \\ -\frac{3}{8} - \frac{1}{2}\,t \end{pmatrix}.$$ □

Example 15.16. Find a particular solution to the inhomogeneous LDEq

$$\vec{y}'(t) = A\,\vec{y}(t) + \vec{f}(t) \quad \text{with} \quad A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix},\ \vec{f}(t) = \begin{pmatrix} -3t\,e^{-t} \\ -(3t+5)\,e^{-t} \end{pmatrix}.$$

The right side $\vec{f}(t)$ is of the type polynomial times exponential e^{-t}. So we check the approach

$$\vec{y}_p(t) = \begin{pmatrix} (a_1 + b_1\,t)\,e^{-t} \\ (a_2 + b_2\,t)\,e^{-t} \end{pmatrix}.$$

To identify the unknown parameters a_1, a_2 and b_1, b_2, we evaluate the right and left sides of the system, $\vec{y}'(t)$ and $A\,\vec{y}(t) + \vec{f}(t)$:

$$\begin{aligned}
\vec{y}_p'(t) &= \begin{pmatrix} b_1\,e^{-t} + (a_1 + b_1\,t)(-1)\,e^{-t} \\ b_2\,e^{-t} + (a_2 + b_2\,t)(-1)\,e^{-t} \end{pmatrix} \\
&= \begin{pmatrix} b_1 - a_1 - b_1\,t \\ b_2 - a_2 - b_2\,t \end{pmatrix} \cdot e^{-t} \\
&= \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} a_1 + b_1\,t \\ a_2 + b_2\,t \end{pmatrix} e^{-t} + \begin{pmatrix} -3t \\ -(3t+5) \end{pmatrix} e^{-t}
\end{aligned}$$

We compare the two components

$$\begin{aligned}
b_1 - a_1 - b_1\,t &= 2a_1 + 2b_1\,t + a_2 + b_2\,t - 3t \\
b_2 - a_2 - b_2\,t &= 3a_1 + 3b_1\,t + 4a_2 + 4b_2\,t - 3t - 5
\end{aligned}$$

with the solution $a_1 = 0$, $b_1 = 1$ and $a_2 = 1$, $b_2 = 0$. A particular solution is

$$\vec{y}_p(t) = \begin{pmatrix} t \\ 1 \end{pmatrix} e^{-t}.$$

□

Note: To obtain the general solution to the LDEq, we have to add the homogeneous solution. The matrix A has the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 5$ with the corresponding eigenvectors $\vec{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\vec{x}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. So the general solution of the inhomogeneous problem is

$$\begin{aligned}
\vec{y}(t) &= \vec{y}_h(t) + \vec{y}_p(t) \\
&= c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{5t} + \begin{pmatrix} t \\ 1 \end{pmatrix} e^{-t}
\end{aligned}$$

Application Example 15.17 (Charged Particle in Electromagnetic Fields). The equation of motion (non-relativistic Lorentz equation) of a charged particle with charge $q = -e$ in electromagnetic fields $\vec{E}$ and $\vec{B}$ is

$$m \frac{d}{dt} \vec{v}(t) = q \left(\vec{E} + \vec{v}(t) \times \vec{B} \right) \quad \text{with} \quad \vec{v}(0) = \vec{v}_0 \ .$$

For $\vec{B} = \begin{pmatrix} 0 \\ 0 \\ B_Z \end{pmatrix}$ and $\vec{E} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$ we get according to Example 15.2

$$\begin{aligned} \text{1st component: } \dot{v}_x(t) &= -\tfrac{e}{m} B_z \, v_y(t) - \tfrac{e}{m} E_x \\ \text{2nd component: } \dot{v}_y(t) &= \tfrac{e}{m} B_z \, v_x(t) - \tfrac{e}{m} E_y \\ \text{3rd component: } \dot{v}_z(t) &= -\tfrac{e}{m} E_z \end{aligned}$$

From the third component we conclude that $v_z(t) = v_{0z} - \frac{e}{m} E_z \cdot t$. From the first two components we get the inhomogeneous linear differential equation

$$\vec{v}'(t) = \begin{pmatrix} v_x'(t) \\ v_y'(t) \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \vec{v}(t) + \begin{pmatrix} -\frac{e}{m} E_x \\ -\frac{e}{m} E_y \end{pmatrix} \quad \text{with} \quad \omega = \frac{e}{m} B \ .$$

Assuming a homogeneous magnetic field, the functions

$$\vec{y}_1(t) = \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \end{pmatrix}, \ \vec{y}_2(t) = \begin{pmatrix} -\sin(\omega t) \\ \cos(\omega t) \end{pmatrix}$$

are a fundamental set according to Example 15.2 and the fundamental matrix is

$$F(t) = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} \quad \text{with} \quad F^{-1}(t) = \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix}.$$

The method Variation of the Constant provides the solution to the inhomogeneous problem:

$$\begin{aligned} \vec{c}(t) &= \vec{v}_0 + \int_0^t F^{-1}(\xi) \cdot \vec{f}(\xi) \, d\xi \\ &= \vec{v}_0 + \int_0^t \begin{pmatrix} \cos(\omega \xi) & \sin(\omega \xi) \\ -\sin(\omega \xi) & \cos(\omega \xi) \end{pmatrix} \cdot \begin{pmatrix} -\frac{e}{m} E_x \\ -\frac{e}{m} E_y \end{pmatrix} d\xi \\ &= \vec{v}_0 - \frac{e}{m} \int_0^t \begin{pmatrix} E_x \cos(\omega \xi) + E_y \sin(\omega \xi) \\ -E_x \sin(\omega \xi) + E_y \cos(\omega \xi) \end{pmatrix} d\xi \end{aligned}$$

$$= \vec{v}_0 - \frac{e}{m}\frac{1}{\omega}\begin{pmatrix} E_x \sin(\omega t) - E_y \cos(\omega t) + E_y \\ E_x \cos(\omega t) - E_x + E_y \sin(\omega t) \end{pmatrix}.$$

The solution of the inhomogeneous problem, which also satisfies the initial condition $\vec{y}(t_0) = \vec{v}_0$, is thus

$$\begin{aligned} \vec{y}(t) &= F(t)\ \vec{c}(t) \\ &= F(t)\ \vec{v}_0 - \frac{e}{m}\frac{1}{\omega} F(t) \begin{pmatrix} E_x \sin(\omega t) - E_y \cos(\omega t) + E_y \\ E_x \cos(\omega t) - E_x + E_y \sin(\omega t) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\omega t)\ v_{0x} - \sin(\omega t)\ v_{0y} \\ \sin(\omega t)\ v_{0x} + \cos(\omega t)\ v_{0y} \end{pmatrix} \\ &\qquad - \frac{e}{m}\frac{1}{\omega}\begin{pmatrix} E_x \sin(\omega t) + E_y\ (\cos(\omega t) - 1) \\ E_x\ (1 - \cos(\omega t)) + E_y \sin(\omega t) \end{pmatrix}. \end{aligned}$$

The components of the velocity are

$$\begin{aligned} v_x(t) &= \cos(\omega t)\ v_{0x} - \sin(\omega t)\ v_{0y} - \frac{1}{B}(E_x \sin(\omega t) + E_y\ (\cos(\omega t) - 1)) \\ v_y(t) &= \sin(\omega t)\ v_{0x} + \cos(\omega t)\ v_{0y} - \frac{1}{B}(E_x\ (1 - \cos(\omega t)) + E_y \sin(\omega t)) \\ v_z(t) &= v_{0z} - \frac{\omega}{B}\ E\ t\,. \end{aligned}$$

For the parameters $\omega = 1,\ B = 0.1,\ E_x = 10,\ E_y = 4,\ E_z = 1$ and the initial velocities $v_{0x} = 1,\ v_{0y} = 0,\ v_{0z} = 0$, the motion is shown in Fig. 15.6.

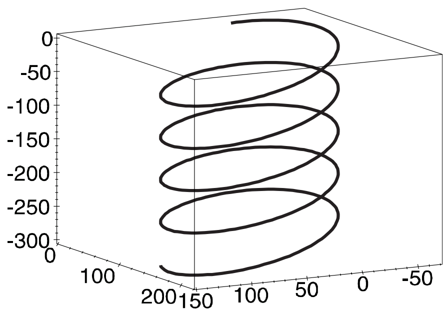


Figure 15.6. Space curve of an electron in an electromagnetic field

□

15.4 Problems on Systems of Linear Differential Equations

15.1 Determine a fundamental set of the first-order LDEq system

a) $\vec{y}'(t) = A\,\vec{y}(t) \quad \text{with } A = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 5 \end{pmatrix}$

b) $\vec{y}'(t) = B\,\vec{y}(t) \quad \text{with } B = \begin{pmatrix} -2 & -9 & 5 \\ -5 & -10 & 7 \\ -9 & -21 & 14 \end{pmatrix}$

Check that the eigenvectors form a basis of $\mathbb{R}^3$.

15.2 Determine a fundamental set of the second-order LDEq system

$\vec{y}''(t) = A\,\vec{y}(t) \quad \text{with } A = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}.$

15.3 a) The equations of motion of a charged particle in a magnetic field are

$$\dot{v}_x = -\frac{e}{m} B_z\, v_y, \qquad \dot{v}_y = \frac{e}{m} B_z\, v_x$$

if $\vec{B} = B_z\,\vec{e}_z$. Determine a real-valued fundamental set.

b) A particular solution is to be found if, in addition to the magnetic field $\vec{B}$, an electric field $\vec{E} = E_0 \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix}$ is effective:

$$\dot{v}_x = -\frac{e}{m} B_z\, v_y \qquad \dot{v}_y = \frac{e}{m} B_z\, v_x + E_0 \cdot t.$$

15.4 Given is the matrix $A = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$.

a) Find all eigenvalues and eigenvectors for the matrix A.
b) Determine a fundamental set of $\vec{y}'(t) = A\,\vec{y}(t)$.
c) Determine a complex fundamental set of $\vec{y}''(t) = A\,\vec{y}(t)$.
d) Determine a real-valued fundamental set of $\vec{y}''(t) = A\,\vec{y}(t)$.
e) Set up the first-order LDEq equivalent to $\vec{y}''(t)$.

15.5 Compute a fundamental set of the first-order system $\vec{y}' = A\,\vec{y}$ in the case of:

a) $A = \begin{pmatrix} 3 & 4 \\ -5 & -5 \end{pmatrix}$ b) $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 3 \end{pmatrix}$

c) $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}$

15.6 Solve the initial value problem:

$$\begin{array}{rcrcrcrl} y_1'(x) & = & 3\,y_1(x) & + & 2\,y_2(x) & - & y_3(x), & y_1(0) = 2 \\ y_2'(x) & = & 2\,y_1(x) & + & 3\,y_2(x) & - & y_3(x), & y_2(0) = 4 \\ y_3'(x) & = & -y_1(x) & - & y_2(x) & + & 4\,y_3(x), & y_3(0) = 0 \end{array}$$

15.7 Solve the second-order differential equation

$$y'' - 5\,y' + 6\,y = 0 \qquad (*)$$

by introducing the new functions $y_1 = y$, $y_2 = y'$ and writing $(*)$ as a system. Solve the first-order system. What is the solution for $y(0) = 1$, $y'(0) = 0$?

15.8 Determine a fundamental set of the first-order LDEq systems

a) $\vec{y}\,'(t) = A\,\vec{y}(t)$ with $A = \begin{pmatrix} 3 & 3 & 1 \\ -1 & 6 & -1 \\ -1 & 1 & 3 \end{pmatrix}$

b) $\vec{y}\,'(t) = B\,\vec{y}(t)$ with $B = \begin{pmatrix} 2 & 1 & -1 \\ 0 & -3 & -1 \\ 0 & 1 & 1 \end{pmatrix}$

c) $\vec{y}\,'(t) = C\,\vec{y}(t)$ with $C = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$

by calculating eigenvectors and main vectors. Check that the eigenvectors with their main vectors form a basis of $\mathbb{R}^3$.

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Chapter 16

Linear Differential Equations of n-th Order

16

Most kinematic problems lead to second-order differential equations according to Newton's law of motion; for solid bodies even to fourth-order or higher differential equations. This chapter deals with the systematic solution of linear differential equations of order n.

The theoretical statements are taken from the chapter on systems of first-order differential equations, since we will reduce a linear differential equation of order n to a system of n first-order differential equations. According to the solution structure, the homogeneous problem is solved first and then the inhomogeneous problem.

6

16 Linear Differential Equations of n-th Order

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The theoretical statements are taken from the chapter on systems of first-order differential equations, since we will reduce a linear differential equation of order n to a system of n first-order differential equations. According to the solution structure, the homogeneous problem is solved first and then the inhomogeneous problem.

16.1 Introduction

Application Example 16.1 (String Pendulum).

A mass m is attached to a string of length l. The angle $\varphi(t)$ varies as a function of time as the mass is deflected by an angle φ_0. The force accelerating the mass m is the component of the gravitational force F_t perpendicular to the string deflection

$$F_t = -F_G \sin\varphi = -m\, g\, \sin\varphi.$$

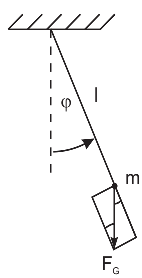

Figure 16.1. Pendulum

Here, we work under the restriction of small angles

$$\sin\varphi \approx \varphi$$

and the frictional force F_R is assumed to be proportional to the velocity

$$F_R = -\gamma\, v = -\gamma\,(l\,\dot{\varphi}).$$

Newton's law of motion, the accelerating force $F_B = m\, a(t) = m\, l\, \ddot{\varphi}(t)$ is equal to the sum of all forces acting on m, giving

$$m\, l\, \ddot{\varphi}(t) = -m\, g\, \varphi(t) - \gamma\, l\, \dot{\varphi}(t)$$

(second-order homogeneous linear DEq). □

Application Example 16.2 (Spring Pendulum).

At the end of a vertical spring with spring constant D there is a mass m. We consider a frictional force proportional to the velocity. The displacement of the mass m at time t is $x(t)$. We look for the *displacement-time-law* $x(t)$ when the mass m is displaced x_0 from the rest position at $t_0 = 0$.

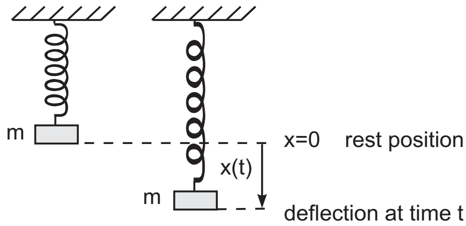

Figure 16.2. Spring pendulum

The forces acting on the mass m are the spring force $F_D = -D\,x(t)$ and the friction force $F_R = -\beta\,\dot{x}(t)$. According to Newton's law of motion, the acceleration force $F_B = m\,\ddot{x}(t)$ is equal to the sum of all the forces acting:

$$m\,\ddot{x}(t) = -\beta\,\dot{x}(t) - D\,x(t) \quad \text{with } x(0) = x_0,\ \dot{x}(0) = 0$$

(second-order homogeneous linear DEq). □

Application Example 16.3 (RLC-Circuit). The RLC circuit is the electromagnetic analogue of the pendulum examples. A circuit is built with an inductance L, a capacitance C and an ohmic resistance R. At $t = 0$ the circuit is closed by applying an external voltage

$$U_B(t) = U_0 \sin(\omega t).$$

The current $I(t)$ is searched for as a function of time.

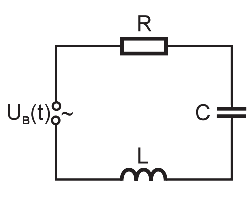

Figure 16.3. RLC circuit

According to the mesh rule, the sum of the voltage drops at R, C and L equals the applied voltage U_B

$$U_R(t) + U_L(t) + U_C(t) = U_B(t).$$

With Ohm's law ($U_R(t) = R \cdot I(t)$), the law of induction ($U_L(t) = L\,\frac{dI(t)}{dt}$) and the voltage across the capacitor $\left(U_C(t) = \frac{1}{C}Q(t) = \frac{1}{C}\int_0^t I(\tau)\,d\tau\right)$, the model equation is

$$R \cdot I(t) + L\,\frac{dI(t)}{dt} + \frac{1}{C}\int_0^t I(\tau)\,d\tau = U_0 \sin(\omega t).$$

After differentiation, we finally get

$$L\,\ddot{I}(t) + R\,\dot{I}(t) + \tfrac{1}{C}\,I(t) = U_0\,\omega\cos(\omega t)$$

(second-order inhomogeneous linear DEq). □

16.1.1 General Problem

Problem: Let $f(x)$ be a continuous function on the interval I and let $a_k \in \mathbb{R} \quad (k = 0, \ldots, n-1)$ be real coefficients. The function $y(x)$, which is assumed to be continuously differentiated n times, must satisfy the linear differential equation

$$y^{(n)}(x) + a_{n-1}\,y^{(n-1)}(x) + \ldots + a_1\,y'(x) + a_0\,y(x) = f(x) \quad (\text{DEq}\,n)$$

for all $x \in I$. (DEq n) is called a differential equation of order n because the n-th derivative of the function $y(x)$ occurs.

If $f(x) = 0$ then (DEq n) is called **homogeneous**.
If $f(x) \neq 0$ then (DEq n) is called **inhomogeneous**.

In the next sections we will discuss the problems: How many solutions does a linear differential equation of order n have? How do we solve the homogeneous problem and how do we solve the inhomogeneous problem? To answer these questions, we first apply the results of Chapter 15 on systems of linear differential equations to linear differential equations of n-th order:

16.1.2 Reduction of an n-th Order DEq to a First-Order System

We start with the n-th order differential equation and construct n first-order differential equations by introducing - generalizing the procedure in Example 15.1 - n functions $y_0(x)$, $y_1(x)$, $\ldots$, $y_{n-1}(x)$ by

$$\left.\begin{array}{rl} y_0(x) & := y(x) \\ y_1(x) & := y'(x) \\ y_2(x) & := y''(x) \\ \vdots & \\ y_{n-1}(x) & := y^{(n-1)}(x) \end{array}\right\} \quad \text{and the vector} \quad \vec{Y}(x) := \begin{pmatrix} y_0(x) \\ y_1(x) \\ \vdots \\ y_{n-2}(x) \\ y_{n-1}(x) \end{pmatrix}.$$

Then, the derivatives of $\vec{Y}(x)$ are

$$\vec{Y}'(x) = \begin{pmatrix} y_0'(x) \\ y_1'(x) \\ \vdots \\ y_{n-2}'(x) \\ y_{n-1}'(x) \end{pmatrix} = \begin{pmatrix} y'(x) \\ y''(x) \\ \vdots \\ y^{(n-1)}(x) \\ y^{(n)}(x) \end{pmatrix}$$

$$= \begin{pmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_{n-1}(x) \\ -a_0\, y(x) - a_1\, y'(x) - \ldots - a_{n-1}\, y^{(n-1)}(x) + f(x) \end{pmatrix}.$$

The identity for the first $(n-1)$ components is according to the definition of the functions $y_i(x)$ and that of the last component is according to the differential equation (DEq n)

$$y^{(n)}(x) = -a_0\, y(x) - a_1\, y'(x) - \ldots - a_{n-1}\, y^{(n-1)}(x) + f(x).$$

We replace the derivatives of $y(x)$ of the last component with the corresponding functions $y_i(x)$:

$$\vec{Y}'(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_{n-1}(x) \\ -a_0\, y_0(x) - a_1\, y_1(x) - \ldots - a_{n-1}\, y_{n-1}(x) + f(x) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix} \vec{Y}(x) + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(x) \end{pmatrix} \qquad \text{(DES 1)}$$

This is an inhomogeneous system of first-order linear differential equations for $\vec{Y}(x)$.

Remarks:

(1) If $y(x)$ is a solution of the inhomogeneous, linear differential equation of n-th order (DEq n), then

$$\vec{Y}(x) = (y(x), y'(x), \dots, y^{(n-1)}(x))^t$$

is a solution of the corresponding inhomogeneous first-order linear differential system (DES 1).

(2) The inverse is also true:
If $\vec{Y}(x) = (y_0(x), y_1(x), \dots, y_{n-1}(x))^t$ is a solution of (DES 1), then the first component of the vector $\vec{Y}(x)$, namely

$$y(x) := y_0(x),$$

is a solution of the n-th order differential equation (DEq n).

Proof: If $\vec{Y}(x)$ is a solution of (DES 1), then its first component $y_0(x)$ is a solution of (DEq n):

$$\begin{aligned} y_0'(x) &= y_1(x) \\ y_0''(x) &= y_1'(x) = y_2(x) \\ &\vdots \\ y_0^{(n-1)}(x) &= y_{n-2}'(x) = y_{n-1}(x) \\ \\ y_0^{(n)}(x) &= y_{n-1}'(x) \\ &= -a_0\, y_0(x) - a_1\, y_1(x) - \ldots - a_{n-1}\, y_{n-1}(x) + f(x) \\ &= -a_0\, y_0(x) - a_1\, y_0'(x) - \ldots - a_{n-1}\, y_0^{(n-1)}(x) + f(x). \end{aligned}$$

So $y_0(x)$ is a solution of the differential equation of order n. The inverse is valid because of the above considerations and the construction of the system (DES 1). □

Example 16.4. Find the system of first-order differential equations belonging to the fourth-order differential equation

$$x''''(t) + 8\,x'''(t) + 22\,x''(t) + 24\,x'(t) + 9\,x(t) = 0.$$

The differential equation is of fourth order, so we need to define four functions

$$\begin{aligned} y_0(t) &= x(t) \\ y_1(t) &= x'(t) \\ y_2(t) &= x''(t) \\ y_3(t) &= x'''(t)\,. \end{aligned}$$

The derivatives of the functions $y_0(t)$, $y_1(t)$, $y_2(t)$ and $y_3(t)$ are

$$\begin{aligned} y_0'(t) &= x'(t) &&= y_1(t) \\ y_1'(t) &= x''(t) &&= y_2(t) \\ y_2'(t) &= x'''(t) &&= y_3(t) \\ y_3'(t) &= x''''(t) &&= -9\,x(t) - 24\,x'(t) - 22\,x''(t) - 8\,x'''(t) \\ & &&= -9\,y_0(t) - 24\,y_1(t) - 22\,y_2(t) - 8\,y_3(t)\,. \end{aligned}$$

For the vector $\vec{Y}(t) := \begin{pmatrix} y_0(t) \\ y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}$ applies

$$\begin{aligned} \vec{Y}'(t) &= \begin{pmatrix} y_0' \\ y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ -9\,y_0 - 24\,y_1 - 22\,y_2 - 8\,y_3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9 & -24 & -22 & -8 \end{pmatrix} \vec{Y}(t)\,. \end{aligned}$$

This is the first-order system of linear differential equations belonging to the fourth-order differential equation. □

Theorem 16.1:

Solving a linear differential equation of order n (DEq n) is equivalent to solving the corresponding first-order system (DES 1).

Remark: According to Theorem 16.1, it is irrelevant whether the differential equation of n-th order is solved or the corresponding system. This theorem has far-reaching consequences for *numerically* solving differential equations of order n: Instead of solving an n-th order differential equation, the first-order system is used and each of these first-order differential equations is solved numerically, e.g. by Euler's method (see Volume 2, Section 13.4.2).

Due to the equivalence established in Theorem 16.1, the theorems on the solution of first-order systems also apply to differential equations of n-th order. According to the Theorems 15.1 and 15.3, the following applies

Theorem 16.2: Solving Linear DEq of n-th Order

(1) Let $\mathbb{L}_h$ be the set of all solutions of the n-th order homogeneous linear differential equation

$$y^{(n)}(x) + a_{n-1}\, y^{(n-1)}(x) + \ldots + a_1\, y'(x) + a_0\, y(x) = 0.$$

$\mathbb{L}_h$ **is an n-dimensional vector space**.

(2) Let $\mathbb{L}_i$ be the set of all solutions of the n-th order **in**homogeneous linear differential equation

$$y^{(n)}(x) + a_{n-1}\, y^{(n-1)}(x) + \ldots + a_1\, y'(x) + a_0\, y(x) = f(x).$$

Then

$$\mathbb{L}_i = y_p(x) + \mathbb{L}_h,$$

where $y_p(x)$ is a *particular* (= *special*) solution of the inhomogeneous differential equation.

(3) n different solutions $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)$ of the homogeneous differential equation are linearly independent if for **one**, and thus for all $x \in I$, the so-called **Wronski determinant** $W(x)$ is non-zero:

$$W(x) = \det \begin{pmatrix} \varphi_1(x) & \varphi_2(x) & \cdots & \varphi_n(x) \\ \varphi_1'(x) & \varphi_2'(x) & \cdots & \varphi_n'(x) \\ \vdots & \vdots & & \vdots \\ \varphi_1^{(n-1)}(x) & \varphi_2^{(n-1)}(x) & \cdots & \varphi_n^{(n-1)}(x) \end{pmatrix} \neq 0.$$

Definition: *A basis* $\varphi_1(x), \ldots, \varphi_n(x)$ *of solutions to the homogeneous differential equation is called* **fundamental set**.

Conclusion: Fundamental Set

$\varphi_1(x), \ldots, \varphi_n(x)$ is a **fundamental set** if and only if the Wronski determinant $W(x_0) \neq 0$ for a $x_0 \in I$.

Example 16.5. For $x > 0$ a second-order homogeneous DEq is given by

$$y''(x) - \frac{1}{2\,x}\,y'(x) + \frac{1}{2\,x^2}\,y(x) = 0.$$

Two solutions are

$$\varphi_1(x) = x \quad \text{and} \qquad \varphi_2(x) = \sqrt{x},$$

which can be confirmed by inserting them into the differential equation. The Wronski determinant of φ_1, φ_2 is

$$W(x) = \det\begin{pmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1'(x) & \varphi_2'(x) \end{pmatrix} = \begin{vmatrix} x & \sqrt{x} \\ 1 & \frac{1}{2\sqrt{x}} \end{vmatrix} = -\frac{1}{2}\sqrt{x}.$$

For $x > 0$ it is $W(x) \neq 0$ and therefore $(\varphi_1(x), \varphi_2(x))$ is a fundamental set. So the general solution of the differential equation is

$$y(x) = c_1\,x + c_2\,\sqrt{x}.$$

The constants c_1 and c_2 are determined by the initial conditions. □

Application Example 16.6 (Electron in a Magnetic Field).

According to Example 15.2, the non-relativistic equations of motion for an electron in a homogeneous magnetic field perpendicular to the direction of motion are

$$\dot{v}_x(t) = -\omega\,v_y(t) \qquad \text{and} \qquad \dot{v}_y(t) = \omega\,v_x(t)$$

with $\omega = \frac{e}{m}\,B \neq 0$. If we differentiate the first equation, $\ddot{v}_x(t) = -\omega\,\dot{v}_y(t)$, and use the second equation, we get a second-order differential equation for the velocity $v_x(t)$:

$$\ddot{v}_x(t) + \omega^2 v_x(t) = 0.$$

This differential equation has two solutions:

$$\varphi_1(t) = \cos(\omega t) \qquad \text{and} \qquad \varphi_2(t) = \sin(\omega t)\ ,$$

which we confirm by inserting them into the differential equation! These two solutions form a fundamental set, since the Wronski determinant is non-zero

$$\begin{aligned} W(t) &= \det\begin{pmatrix} \varphi_1(t) & \varphi_2(t) \\ \varphi_1'(t) & \varphi_2'(t) \end{pmatrix} = \begin{vmatrix} \cos(\omega t) & \sin(\omega t) \\ -\omega\,\sin(\omega t) & \omega\cos(\omega t) \end{vmatrix} \\ &= \omega\,\cos^2(\omega t) + \omega\,\sin^2(\omega t) = \omega \neq 0. \end{aligned}$$

So the general solution for $v_x(t)$ is

$$v_x(t) = c_1\,\cos(\omega t) + c_2\,\sin(\omega t)\,. \qquad \square$$

We will now face the question of how to calculate **all** solutions of the homogeneous problem and a special solution of the inhomogeneous problem: Solving the homogeneous problem is equivalent to finding the zeros of an n-th order polynomial (*characteristic polynomial*) (see Section 16.2). A particular solution of the inhomogeneous differential equation is often obtained by a special approach (see Section 16.3).

16.2 Homogeneous DEq of n-th Order

Example 16.7 (Complex Fundamental Set). Given is the second-order differential equation

$$\ddot{x}(t) + \omega_0^2\,x(t) = 0.$$

Related physical problems include the string pendulum (Example 16.1) and the spring pendulum (Example 16.2) without friction, an LC circuit (Example 16.3) or the equation of motion of an electron in a magnetic field (Example 16.6). We solve the differential equation using the **approach**:

$$x(t) = e^{\lambda\,t}. \qquad (*)$$

Inserting this approach into the differential equation gives

$$\lambda^2 e^{\lambda t} + \omega_0^2 e^{\lambda t} = 0 \hookrightarrow \lambda^2 + \omega_0^2 = 0.$$

The resulting polynomial

$$P(\lambda) = \lambda^2 + \omega_0^2$$

is called the *characteristic polynomial* associated with the differential equation. If λ is a zero of the characteristic polynomial, then $e^{\lambda t}$ is a solution of the differential equation. Here $P(\lambda) = 0$ means $\lambda = \pm\sqrt{-\omega_0^2} = \pm i\,\omega_0$.

$$\Rightarrow \varphi_1(t) = e^{i\,\omega_0\,t} \quad \text{and} \quad \varphi_2(t) = e^{-i\,\omega_0\,t}$$

are solutions of the differential equation. They form a fundamental set, since the Wronski determinant is non-zero:

$$\begin{aligned} W(t) &= \det \begin{pmatrix} \varphi_1(t) & \varphi_2(t) \\ \varphi_1'(t) & \varphi_2'(t) \end{pmatrix} \\ &= \begin{vmatrix} e^{i\,\omega_0\,t} & e^{-i\,\omega_0\,t} \\ i\,\omega_0\,e^{i\,\omega_0\,t} & -i\,\omega\,e^{-i\,\omega_0\,t} \end{vmatrix} = -2\,i\,\omega_0 \neq 0. \end{aligned}$$

Since $\varphi_1(t)$, $\varphi_2(t)$ are complex functions, $(\varphi_1(t), \varphi_2(t))$ is called a *complex fundamental set.* □

Example 16.8 (Real Fundamental Set). To this complex fundamental set, we construct a real-valued by two special linear combinations of $\varphi_1(t)$ and $\varphi_2(t)$.

The superposition principle applies to any linear DEq. With two solutions $\varphi_1(t)$ and $\varphi_2(t)$ every linear combination $c_1\,\varphi_1(t) + c_2\,\varphi_2(t)$ is also a solution of the differential equation. With $\varphi_1(t) = e^{i\,\omega_0\,t}$ and $\varphi_2(t) = e^{-i\,\omega_0\,t}$ we get

$$\begin{aligned} x_1(t) &= \tfrac{1}{2}\,\varphi_1(t) + \tfrac{1}{2}\,\varphi_2(t) = \tfrac{1}{2}\left(e^{i\,\omega_0\,t} + e^{-i\,\omega_0\,t}\right) = \cos(\omega_0 t) \\ x_2(t) &= \tfrac{1}{2i}\,\varphi_1(t) - \tfrac{1}{2i}\,\varphi_2(t) = \tfrac{1}{2i}\left(e^{i\,\omega_0\,t} - e^{-i\,\omega_0\,t}\right) = \sin(\omega_0 t). \end{aligned}$$

These are also solutions of the differential equation. Since the Wronski determinant of these two functions is $W(t) = \omega_0 \neq 0$, $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$ are a *real-valued fundamental set.* So the general solution is

$$x(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t).$$ □

With the knowledge of Examples 16.7 and 16.8 the solution method is transferred to a general, **n-th order homogeneous linear differential equation**

$$y^{(n)}(x) + a_{n-1}\, y^{(n-1)}(x) + \ldots + a_1\, y'(x) + a_0\, y(x) = 0. \qquad (*)$$

With the **approach**

$$y(x) = e^{\lambda x}$$

for the searched function, the k-th derivative of $y(x)$,

$$y^{(k)}(x) = \lambda^k\, e^{\lambda x},$$

is inserted into the differential equation $(*)$, which gives

$$\lambda^n\, e^{\lambda x} + a_{n-1}\, \lambda^{n-1}\, e^{\lambda x} + \ldots + a_1\, \lambda\, e^{\lambda x} + a_0\, e^{\lambda x} = 0$$

$$\Rightarrow \lambda^n + a_{n-1}\, \lambda^{n-1} + \ldots + a_1\, \lambda + a_0 = 0.$$

Definition: (Characteristic Polynomial).

$$P(\lambda) := \lambda^n + a_{n-1}\, \lambda^{n-1} + \ldots + a_1\, \lambda + a_0$$

is called the **Characteristic Polynomial** *related to the differential equation* $(*)$.

If λ_0 is a **zero** of the characteristic polynomial $P(\lambda)$, then

$$y(x) = e^{\lambda_0 x}$$

is a solution of the differential equation. According to the Fundamental Theorem of Algebra (Volume 1, Section 5.2.7) every polynomial of degree n has exactly n complex zeros $\lambda_1, \ldots, \lambda_n$, which can also occur multiple times. If the characteristic polynomial has n *different* zeros, then

$$y_k(x) = e^{\lambda_k x} \qquad k = 1, \ldots, n$$

are n linearly independent functions

Theorem 16.3:
Characteristic Polynomial with n Different Zeros

Given is the *homogeneous linear differential equation of n-th order*

$$y^{(n)}(x) + a_{n-1}\, y^{(n-1)}(x) + \ldots + a_1\, y'(x) + a_0\, y(x) = 0.$$

If the corresponding characteristic polynomial $P(\lambda)$ has n different zeros $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the n different solutions

$$y_k(x) := e^{\lambda_k x} \qquad (k = 1, \ldots, n)$$

are a **fundamental set**.

Proof: From our preliminary considerations it is clear that $e^{\lambda_k x}$ $(k = 1, \ldots, n)$ are solutions of the DEq if the λ_k are zeros of the characteristic polynomial. All that remains is to show the linear independence of the solutions. For this we use the Wronski determinant

$$W(x) = \det \begin{pmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} & \cdots & e^{\lambda_n x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} & \cdots & \lambda_n e^{\lambda_n x} \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} e^{\lambda_1 x} & \lambda_2^{n-1} e^{\lambda_2 x} & \cdots & \lambda_n^{n-1} e^{\lambda_n x} \end{pmatrix}.$$

By complete induction we can show that this so-called Vandermond determinant is non-zero.

$$W(x=0) = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix} = \prod_{i>j} (\lambda_i - \lambda_j) \neq 0.$$

So the solutions form a fundamental set. □

Example 16.9. Find a fundamental set of the differential equation

$$y^{(4)}(x) + 3\, y''(x) - 4\, y(x) = 0.$$

Approach: The function $y(x) = e^{\lambda x}$ is inserted into the differential equation. This produces the characteristic polynomial

$$\lambda^4 e^{\lambda x} + 3\,\lambda^2 e^{\lambda x} - 4\, e^{\lambda x} = 0 \;\Rightarrow\; \boxed{P(\lambda) = \lambda^4 + 3\,\lambda^2 - 4 = 0.}$$

The zeros of the characteristic polynomial are determined with a substitution $z := \lambda^2$. Then $z^2 + 3\,z - 4 = 0 \;\hookrightarrow\; z_1 = 1,\ z_2 = -4$.

$$\Rightarrow \lambda_{1/2} = \pm\sqrt{z_1} = \pm 1 \quad \text{and} \quad \lambda_{3/4} = \pm\sqrt{z_2} = \pm\sqrt{-4} = \pm 2i.$$

Therefore, $P(\lambda)$ has 4 different zeros ± 1, $\pm 2i$ and

$$\Rightarrow e^{1x}\ ,\ e^{-1x}\ ,\ e^{2i\,x}\ ,\ e^{-2i\,x}$$

is a complex fundamental set. With

$$\frac{1}{2}\left(e^{2i\,x} + e^{-2i\,x}\right) = \cos(2x)$$

$$\frac{1}{2i}\left(e^{2i\,x} - e^{-2i\,x}\right) = \sin(2x)$$

a real fundamental set is found:

$$e^{x}, e^{-x}, \cos(2x), \sin(2x)\,. \qquad \square$$

Theorem 16.3 clearly explains how to determine a fundamental set when $P(\lambda)$ has n different zeros. But what happens if the characteristic polynomial has a double or multiple zero? To answer this question, we consider the next example:

Example 16.10. Given is the differential equation

$$\ddot{x}(t) + 2\,\dot{x}(t) + x(t) = 0. \tag{$*$}$$

Approach: If we insert $x(t) = e^{\lambda\,t}$ into the differential equation, we obtain the characteristic polynomial

$$P(\lambda) = \lambda^2 + 2\,\lambda + 1 = (\lambda + 1)^2 = 0.$$

$$P(\lambda) = 0 \quad \hookrightarrow \quad \lambda_{1/2} = -1 \text{ is a zero of multiplicity } 2$$

$$\hookrightarrow \quad x_1(t) = e^{-t} \text{ is a solution of } (*).$$

With this approach we obtain only **one** solution $e^{-1\cdot t}$. But since $(*)$ is a second-order differential equation, $\mathbb{L}_h$ is a 2-dimensional vector space and the fundamental set consists of **two** linearly independent functions! Another solution is given by

$$x_2(t) = t \cdot e^{-t}\,,$$

because $x_2(t)$ and its derivatives

$$\dot{x}_2(t) = e^{-t} - t\,e^{-t}$$
$$\ddot{x}_2(t) = -e^{-t} - e^{-t} + t\,e^{-t}$$

inserted into the differential equation gives

$$\Rightarrow \ddot{x}_2(t) + 2\,\dot{x}_2(t) + x_2(t) = 0.$$

Furthermore, $x_1(t)$ and $x_2(t)$ are linearly independent:

$$W(t) = \det\begin{pmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{pmatrix} = \begin{vmatrix} e^{-t} & t\,e^{-t} \\ -e^{-t} & e^{-t}(1-t) \end{vmatrix}$$
$$= e^{-2t} \neq 0.$$

Therefore, a fundamental set is

$$e^{-t}, \quad t\,e^{-t}.$$

□

If λ_0 is a double zero of the characteristic polynomial $P(\lambda)$, then according to the Example 16.10 $e^{\lambda_0 t}$ and $t\,e^{\lambda_0 t}$ are two linearly independent solutions of the differential equation. If λ_0 is a triple zero, then $e^{\lambda_0 t}$, $t\,e^{\lambda_0 t}$ and $t^2\,e^{\lambda_0 t}$ are three linearly independent solutions and so on.

Generalizing, by applying the method of variation of constants, we obtain the following theorem:

Theorem 16.4: Characteristic Polynomial with Multiple Zeros

Given is the homogeneous linear differential equation of n-th order

$$y^{(n)}(x) + a_{n-1}\,y^{(n-1)}(x) + \ldots + a_1\,y'(x) + a_0\,y(x) = 0.$$

If the corresponding characteristic polynomial $P(\lambda)$ has l different zeros $\lambda_k \in \mathbb{C} \quad (k = 1, \ldots, l)$ with the multiplicity $m_k \quad (k = 1, \ldots, l)$, then

$$e^{\lambda_k x},\ x\,e^{\lambda_k x}, \ldots,\ x^{\mathbf{m_k-1}}\,e^{\lambda_k x}$$

are linearly independent solutions and they form for $k = 1, \ldots, l$ a **fundamental set.**

Example 16.11. Find a real fundamental set and the general solution of the differential equation

$$y^{(4)}(x) + 8\,y''(x) + 16\,y(x) = 0.$$

Approach: Substituting $y(x) = e^{\lambda x}$ into the differential equation gives the characteristic polynomial

$$P(\lambda) = \lambda^4 + 8\,\lambda^2 + 16 = 0.$$

With the substitution $z := \lambda^2$ we obtain $z^2 + 8\,z + 16 = 0 \hookrightarrow z_{1/2} = -4$ is a double zero. And so

$$\lambda_{1/2} = \pm\sqrt{-4} = \pm 2i$$

are double zeros. For each double zero we get two solutions:

$$\begin{array}{lll} \lambda_1 = 2i & \hookrightarrow & \varphi_1(x) = e^{2i\,x}, \quad \varphi_2(x) = x \cdot e^{2i\,x} \\ \lambda_2 = -2i & \hookrightarrow & \varphi_3(x) = e^{-2i\,x}, \quad \varphi_4(x) = x \cdot e^{-2i\,x}. \end{array}$$

This results in a complex fundamental set

$$e^{2i\,x}\;,\quad e^{-2i\,x}\;,\quad x\,e^{2i\,x}\;,\quad x\,e^{-2i\,x}.$$

To obtain a real fundamental set, we choose the linear combinations already introduced:

$$\begin{aligned} \tfrac{1}{2}\left(\varphi_1(x) + \varphi_3(x)\right) &= \tfrac{1}{2}\left(e^{2i\,x} + e^{-2i\,x}\right) &&= \cos(2x) \\ \tfrac{1}{2i}\left(\varphi_1(x) - \varphi_3(x)\right) &= \tfrac{1}{2i}\left(e^{2i\,x} - e^{-2i\,x}\right) &&= \sin(2x) \\ \tfrac{1}{2}\left(\varphi_2(x) + \varphi_4(x)\right) &= x\,\tfrac{1}{2}\left(e^{2i\,x} + e^{-2i\,x}\right) &&= x \cdot \cos(2x) \\ \tfrac{1}{2i}\left(\varphi_2(x) - \varphi_4(x)\right) &= x\,\tfrac{1}{2i}\left(e^{2i\,x} - e^{-2i\,x}\right) &&= x \cdot \sin(2x). \end{aligned}$$

So

$$\cos(2\,x),\ \sin(2\,x),\ x\cos(2\,x),\ x\sin(2\,x)$$

is a real fundamental set. So the general solution of the differential equation is

$$y(x) = c_1\cos(2\,x) + c_2\,\sin(2\,x) + c_3\,x\,\cos(2\,x) + c_4\,x\,\sin(2\,x).$$

Finally, if the initial conditions $y(x_0), y'(x_0), y''(x_0), y'''(x_0)$ are given, then they determine the coefficients c_1, c_2, c_3, c_4. □

Hint: If the differential equation has only real coefficients, then the characteristic polynomial $P(\lambda)$ is also real. Thus, according to the Fundamental Theorem of Algebra, the zeros of $P(\lambda)$ are either real or complex / complex conjugate. In the first case, we get a real solution directly, otherwise, as in Example 16.11, the linear combinations $\frac{1}{2}(e^{\lambda_1 x} + e^{\lambda_2 x})$ and $\frac{1}{2i}(e^{\lambda_1 x} - e^{\lambda_2 x})$ lead to two real solutions.

Example 16.12. Find a real fundamental set for the differential equation

$$y'''(x) - y(x) = 0.$$

We insert the approach $y(x) = e^{\lambda x}$ into the differential equation and obtain the characteristic polynomial

$$P(\lambda) = \lambda^3 - 1 = 0.$$

The zeros of the characteristic polynomial are $\lambda_1 = 1$ and $\lambda_{2/3} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}\, i$, so that

$$e^x, \; e^{\left(-\frac{1}{2}+\frac{1}{2}\sqrt{3}\, i\right) x} \quad \text{and} \quad e^{\left(-\frac{1}{2}-\frac{1}{2}\sqrt{3}\, i\right) x}$$

is a complex fundamental set. With the linear combinations

$$\begin{aligned}\frac{1}{2}\left(e^{\left(-\frac{1}{2}+\frac{1}{2}\sqrt{3}\, i\right) x} + e^{\left(-\frac{1}{2}-\frac{1}{2}\sqrt{3} i\right) x}\right) &= e^{-\frac{1}{2}x}\,\frac{1}{2}\left(e^{\frac{1}{2}\sqrt{3}\, i\, x} + e^{-\frac{1}{2}\sqrt{3}\, i\, x}\right)\\ &= e^{-\frac{1}{2}x}\cos(\tfrac{1}{2}\sqrt{3}\, x)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2i}\left(e^{\left(-\frac{1}{2}+\frac{1}{2}\sqrt{3}\, i\right) x} - e^{\left(-\frac{1}{2}-\frac{1}{2}\sqrt{3}\, i\right) x}\right) &= e^{-\frac{1}{2}x}\,\frac{1}{2i}\left(e^{\frac{1}{2}\sqrt{3}\, i\, x} - e^{-\frac{1}{2}\sqrt{3}\, i\, x}\right)\\ &= e^{-\frac{1}{2}x}\sin(\tfrac{1}{2}\sqrt{3}\, x)\end{aligned}$$

a real fundamental set is achieved

$$e^x, \; e^{-\frac{1}{2}x}\cos(\tfrac{1}{2}\sqrt{3}\, x) \quad \text{and} \quad e^{-\frac{1}{2}x}\sin(\tfrac{1}{2}\sqrt{3}\, x).$$

□

Application Example 16.13 (Free Damped Oscillation).

Returning to the spring pendulum from Example 16.2: For the deflection $x(t)$ of the mass m from rest, we modelled the system with the differential equation

$$m\,\ddot{x}(t) = -\beta\,\dot{x}(t) - D\,x(t) \quad \text{and} \quad x(0) = x_0\,, \quad \dot{x}(0) = 0.$$

With the parameters $\omega_0^2 = \frac{D}{m}$ and $\mu = \frac{1}{2}\frac{\beta}{m}$ we simplify to

$$\ddot{x}(t) + 2\,\mu\,\dot{x}(t) + \omega_0^2\,x(t) = 0\,, \quad x(0) = x_0\,, \quad \dot{x}(0) = 0.$$

The approach

$$x(t) = e^{\lambda t}$$

gives the characteristic polynomial

$$P(\lambda) = \lambda^2 + 2\,\mu\,\lambda + \omega_0^2 = 0$$

with the zeros $\boxed{\lambda_{1/2} = -\mu \pm \sqrt{\mu^2 - \omega_0^2}.}$

The sign of the discriminant

$$\triangle := \mu^2 - \omega_0^2$$

determines the type of vibration.

With weak damping, the mechanical system is capable of real vibrations (*vibration case*). This case occurs when $\mu < \omega_0$. With strong damping $\mu > \omega_0$ the system moves non-periodically (= *aperiodic*) towards the equilibrium position (*creep case*). For $\triangle = 0$, i.e. $\mu = \omega_0$, the aperiodic limit occurs. The following descriptions treat each of these three cases separately:

1. Case: $\triangle < 0$, i.e. $\mu < \omega_0$: damped oscillations.
2. Case: $\triangle = 0$, i.e. $\mu = \omega_0$: aperiodic limit.
3. Case: $\triangle > 0$, i.e. $\mu > \omega_0$: overdamped case (no vibrations).

⮞ 1. Damped Vibration (Weak Damping).

For weak damping ($\mu < \omega_0$), the zeros of the characteristic polynomial are complex conjugate

$$\lambda_{1/2} = -\mu \pm \sqrt{\mu^2 - \omega_0^2} = -\mu \pm i\,\sqrt{\omega_0^2 - \mu^2} = -\mu \pm i\,\omega$$

with $\omega := \sqrt{\omega_0^2 - \mu^2} > 0$. This results in a complex fundamental set

$$\varphi_1(t) = e^{\lambda_1 t} = e^{(-\mu + i\,\omega)\,t} = e^{-\mu\,t}\,e^{i\,\omega\,t}$$

$$\varphi_2(t) = e^{\lambda_2 t} = e^{(-\mu - i\,\omega)\,t} = e^{-\mu\,t}\,e^{-i\,\omega\,t}.$$

With the real fundamental set

$$x_1(t) = \tfrac{1}{2}\,(\varphi_1(t) + \varphi_2(t)) = e^{-\mu\, t}\cos(\omega t)$$

$$x_2(t) = \tfrac{1}{2i}\,(\varphi_1(t) - \varphi_2(t)) = e^{-\mu\, t}\sin(\omega t)$$

the **general solution** is given by

$$x(t) = e^{-\mu\, t}\,(c_1\cos(\omega t) + c_2\sin(\omega t)).$$

The constants c_1, c_2 are determined by the initial conditions $x(0)$ and $\dot{x}(0)$. To use the initial values for $\dot{x}(0)$ we need the derivative of $x(t)$:

$$\begin{aligned}\dot{x}(t) = &-\mu\, e^{-\mu\, t}\,(c_1\cos(\omega t) + c_2\sin(\omega t))\\ &+ e^{-\mu\, t}\,(-c_1\,\omega\sin(\omega t) + c_2\,\omega\cos(\omega t)).\end{aligned}$$

Specifying the initial conditions gives two equations for the constants c_1 and c_2:

$$\begin{aligned}&x(0) = x_0: \quad && x(0) = c_1 = x_0.\\ &\dot{x}(0) = 0: && \dot{x}(0) = c_1\,(-\mu) + c_2\,\omega = 0 \Rightarrow c_2 = x_0\,\left(\tfrac{\mu}{\omega}\right).\end{aligned}$$

$$\Rightarrow x(t) = x_0\, e^{-\mu\, t}\left(\cos(\omega t) + \frac{\beta}{2\,m}\,\frac{1}{\omega}\,\sin(\omega t)\right).$$

Interpretation: The solution consists of a time-dependent decreasing amplitude $x_0\, e^{-\mu\, t}$ and a periodic behavior $(\cos(\omega t) + \frac{\beta}{2\,m}\,\frac{1}{\omega}\sin(\omega t))$. The periodic part of the function can be written in the form $\left(\frac{\omega_0}{\omega}\sin(\omega t + \varphi)\right)$ with $\boxed{\tan\varphi = \frac{\omega}{\mu}}$, so that

$$\boxed{x(t) = x_0\, e^{-\mu\, t}\,\frac{\omega_0}{\omega}\,\sin(\omega t + \varphi).}$$

This is a *damped oscillation*. The spring pendulum oscillates at a **reduced** frequency compared to the undamped oscillation

$$\omega = \sqrt{\omega_0^2 - \mu^2} < \omega_0.$$

Fig. 16.4 shows the typical shape of such a damped oscillation.

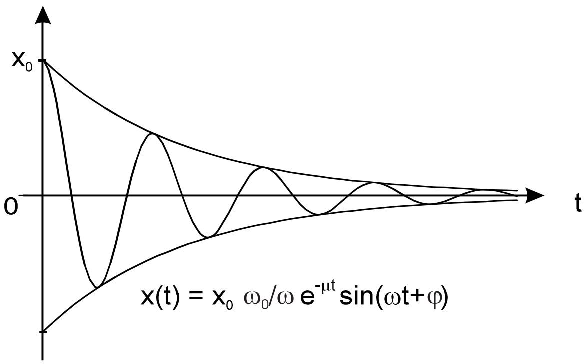

Figure 16.4. Time behavior of a damped oscillation

2. Critically Damped: Aperiodic Limit.

$\triangle = 0$, i.e. $\mu = \omega_0$, describes the aperiodic limit that separates the periodic from the non-periodic behavior. For $\mu = \omega_0$

$$\lambda_{1/2} = -\mu$$

is a **double** zero of the characteristic polynomial $P(\lambda)$ and $\varphi_1(t) = e^{-\mu t}$ and $\varphi_2(t) = t \cdot e^{-\mu t}$ form a real fundamental set. The **general solution** is

$$x(t) = c_1 e^{-\mu t} + c_2 t e^{-\mu t}.$$

The choice of initial conditions determines c_1 and c_2:

$$\begin{aligned} x(0) = x_0&: \quad c_1 = x_0, \\ \dot{x}(0) = 0&: \quad -\mu c_1 + c_2 = 0 \Rightarrow c_2 = \mu x_0. \end{aligned}$$

$$\Rightarrow \boxed{x(t) = x_0 e^{-\mu t}(1 + \mu t).}$$

After its deflection by x_0, the mass point moves non-periodically towards the equilibrium position.

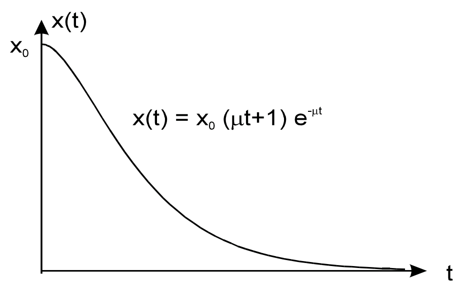

Figure 16.5. Aperiodic limit for $x(0) = x_0$, $x'(0) = 0$

3. Overdamped Mode.

Strong damping occurs when $\mu > \omega_0$. Then the zeros of the characteristic polynomial are two negative real numbers

$$\lambda_{1/2} = -\mu \pm \sqrt{\mu^2 - \omega_0^2} < 0$$

with $k = \sqrt{\mu^2 - \omega_0^2}$. Then, $\varphi_1(t) = e^{(-\mu+k)\,t}$ and $\varphi_2(t) = e^{(-\mu-k)\,t}$ is a real fundamental set. The **general solution** results in

$$x(t) = c_1\, e^{(-\mu+k)\,t} + c_2\, e^{(-\mu-k)\,t}.$$

The mass cannot oscillate due to the strong friction and moves non-periodically towards the equilibrium position. In mechanics this case is called *creep case* or overdamped mode. The exact behavior depends on the initial conditions. For $x(0) = x_0$ and $\dot{x}(0) = 0$ the constants are c_1 and c_2:

$$\begin{aligned} x(0) = x_0: &\qquad x_0 = c_1 + c_2, \\ \dot{x}(0) = 0: &\qquad 0 = (-\mu + k)\; c_1 + (-\mu - k)\; c_2. \end{aligned}$$

The solution of this linear system of equations for c_1 and c_2 is

$$c_1 = x_0\, \frac{k+\mu}{2\,k} \text{ and } c_2 = x_0\, \frac{k-\mu}{2\,k}.$$

With these coefficients we obtain the solution of the overdamped mode

$$\Rightarrow x(t) = \frac{x_0}{2\,k}\, e^{-\mu\,t}\, \left((k+\mu)\; e^{k\,t} + (k-\mu)\; e^{-k\,t}\right).$$

(Exponential decay without vibration).

Animation: On the homepage there is an animation showing the vibration behavior with respect to μ. The animation starts with weak damping and then approaches the aperiodic limit. It can be seen that in the aperiodic limit the system comes to rest most quickly, as is required for shock absorbers or measuring instruments, for example. If the damping is even smaller, the oscillation case is obtained; the period of the oscillation then changes depending on μ. □

16.3 Inhomogeneous DEq of n-th Order

In this section we consider the inhomogeneous problem: Given is a linear DEq of n-th order with inhomogeneity $f(x)$:

$$y^{(n)}(x) + a_{n-1}\, y^{(n-1)}(x) + \ldots + a_1\, y'(x) + a_0\, y(x) = f(x)\,. \qquad (**)$$

The solution to this problem is according to Theorem 16.2

$$y_i(x) = y_h(x) + y_p(x).$$

How to compute $y_h(x)$ is completely discussed in Section 16.2. So the task now is to find a *particular solution* $y_p(x)$.

Using the corresponding first-order system and variation of the constant, we can find a solution formula for any inhomogeneity (see Section 15.3). However, we can shorten the calculation by introducing a special approach: The right-hand side approach.

16.3.1 Inhomogeneity = Exponential Function

First, we consider the case where the inhomogeneity is an exponential function:

$$f(x) = c\, e^{\mu\, x} \qquad \text{with} \quad \mu \in \mathbb{C}.$$

The differential equation is then

$$y^{(n)}(x) + a_{n-1}\, y^{(n-1)}(x) + \ldots + a_1\, y'(x) + a_0\, y(x) = c\, e^{\mu\, x}.$$

A particular solution is of the same type as the inhomogeneity

$$y_p(x) = k\, e^{\mu\, x}$$

with an unknown constant k. We insert $y_p(x)$ with its derivatives into the differential equation,

$$k\, \mu^n\, e^{\mu\, x} + a_{n-1}\, k\, \mu^{n-1}\, e^{\mu\, x} + \ldots + a_1\, k\, \mu\, e^{\mu\, x} + a_0\, k\, e^{\mu\, x} = c\, e^{\mu\, x}$$

$$\Rightarrow k\, \underbrace{\left(\mu^n + a_{n-1}\, \mu^{n-1} + \ldots + a_1\, \mu + a_0\right)}_{P(\mu)} = c \qquad \Rightarrow k\, P(\mu) = c,$$

where the characteristic polynomial $P(\lambda)$ is evaluated at μ. If μ is **not** a zero of the characteristic polynomial, $P(\mu) \neq 0$, then the constant is $k = \frac{c}{P(\mu)}$ and the solution is $y_p(x) = \frac{c}{P(\mu)} \cdot e^{\mu\, x}$:

Theorem 16.5: $f(x)$ = Exponential

Given is an inhomogeneous linear differential equation of n-th order

$$y^{(n)}(x) + a_{n-1}\, y^{(n-1)}(x) + \ldots + a_1\, y'(x) + a_0\, y(x) = c\, e^{\mu\, x}.$$

If μ is **not** a zero of the characteristic polynomial $P(\lambda)$, then

$$y_p(x) = \frac{c}{P(\mu)}\, e^{\mu\, x}$$

is a **particular solution.**

Examples 16.14:

① Find a particular solution $y_p(x)$ of the differential equation

$$y^{(4)}(x) + 2\, y''(x) + y(x) = 25\, e^{2x}.$$

The characteristic polynomial of this differential equation is

$$P(\lambda) = \lambda^4 + 2\,\lambda^2 + 1.$$

$\mu = 2$ is **not** a zero of $P(\lambda)$: $P(2) = 25 \neq 0$. So the approach

$$y_p(x) = k\, e^{2x}$$

gives a particular solution. Inserting $y_p(x)$ into the differential equation determines the constant k:

$$k\, 16\, e^{2x} + k\, 8\, e^{2x} + k\, e^{2x} = s25\, e^{2x}$$

$$\hookrightarrow k = \tfrac{25}{25} = 1 \Rightarrow y_p(x) = e^{2x}.$$

② Find a particular solution $y_p(x)$ of the differential equation

$$y^{(4)}(x) + 2\, y''(x) + y(x) = 25\, e^{i\, 2x}.$$

The characteristic polynomial is

$$P(\lambda) = \lambda^4 + 2\,\lambda^2 + 1.$$

Because $\mu = 2\, i$ and $P(2\, i) = 16\, i^4 + 8\, i^2 + 1 = 9 \neq 0$, μ is **not** a zero of $P(\lambda)$ and therefore a particular solution is

$$y_p(x) = \tfrac{25}{P(2\, i)}\, e^{i\, 2x} = \tfrac{25}{9}\, e^{i\, 2x}.$$ □

Example 16.15. Find a particular solution to the differential equation

$$\boxed{y'''(x) - 2\,y''(x) - 2\,y'(x) + 2\,y(x) = 2\,\sin x.} \qquad (*)$$

(1) **Transferring the DEq to the Complex.** To calculate a particular solution $y_p(x)$ according to Theorem 16.5, the differential equation is extended to the complex:

$$\boxed{\tilde{y}'''(x) - 2\,\tilde{y}''(x) - 2\,\tilde{y}'(x) + 2\,\tilde{y}(x) = 2\,e^{ix}.} \qquad (\tilde{*})$$

If $\tilde{y}_p(x)$ is a solution of the complex differential equation $(\tilde{*})$, then the imaginary part

$$y_p(x) := \text{Im } \tilde{y}_p(x)$$

is a solution of the real DEq $(*)$. To see this, we insert $y_p(x) = \text{Im } \tilde{y}_p(x)$ into the differential equation $(*)$:

$$\begin{aligned}(\text{Im } (\tilde{y}_p))''' & -2\,(\text{Im}\,(\tilde{y}_p))'' - 2\,(\text{Im}\,(\tilde{y}_p))' + 2\,(\text{Im}\,(\tilde{y}_p)) = \\ &= \text{Im } \left(\tilde{y}_p'''\right) - 2 \text{ Im } \left(\tilde{y}_p''\right) - 2 \text{ Im } \left(\tilde{y}_p'\right) + 2 \text{ Im } (\tilde{y}_p) \\ &= \text{Im } \left(\tilde{y}_p''' - 2\,\tilde{y}_p'' - 2\,\tilde{y}_p' + 2\,\tilde{y}_p\right) \overset{(\tilde{*})}{=} \text{Im } \left(2\,e^{ix}\right) = 2\,\sin x.\end{aligned}$$

(2) **Solving the Complex DEq.** To solve the differential equation $(\tilde{*})$, we choose the approach

$$\tilde{y}_p(x) = k\,e^{ix}.$$

Inserting $\tilde{y}_p(x)$ into the differential equation we get

$$k\,i^3\,e^{ix} - k\,2\,i^2\,e^{ix} - k\,2\,i\,e^{ix} + k\,2\,e^{ix} = 2\,e^{ix}$$

$$\hookrightarrow k\,(4 - 3i)\,e^{ix} = 2\,e^{ix} \;\hookrightarrow k = \frac{2}{4 - 3i} \qquad \Rightarrow \tilde{y}_p(x) = \frac{2}{4 - 3i}\,e^{ix}.$$

With this result, we have found the solution of the complex differential equation $(\tilde{*})$ and from this complex solution we have to form the imaginary part. This imaginary part is then the particular solution of $(*)$ that we are looking for.

(3) **Transition to Real Values.** The solution $y_p(x)$ of $(*)$ is

$$y_p(x) = \text{Im } (\tilde{y}_p(x)) = \text{Im } \left(\frac{2}{4 - 3i}\,e^{ix}\right).$$

There are two different ways of calculating the imaginary part. In the first case, both $\frac{2}{4-3i}$ and e^{ix} are decomposed into real and imaginary parts, the product of the two complex quantities is determined in the algebraic normal form and the imaginary part is read from the result. In the second case, $\frac{2}{4-3i}$ is written in exponential form and multiplied by $e^{i\,x}$. The particular solution is again the imaginary part.

> Decomposition of $\frac{2}{4-3i}$ into real and imaginary parts

$$\frac{2}{4-3i} = \frac{2}{4-3i} \cdot \frac{4+3i}{4+3i} = \frac{8}{25} + \frac{6}{25}\,i.$$

$$\begin{aligned} \Rightarrow \tilde{y}_p\,(x) &= \tfrac{2}{4-3i}\, e^{ix} \\ &= \left(\tfrac{8}{25} + \tfrac{6}{25}\, i\right)\,(\cos x + i\,\sin x) \\ &= \left(\tfrac{8}{25}\,\cos x - \tfrac{6}{25}\,\sin x\right) + i\,\left(\tfrac{6}{25}\,\cos x + \tfrac{8}{25}\,\sin x\right). \end{aligned}$$

A particular solution is therefore

$$y_p\,(x) = \mathrm{Im}\,(\tilde{y}\,(x)) = \frac{6}{25}\,\cos x + \frac{8}{25}\,\sin x.$$

> The complex number $c = \frac{2}{4-3i} = \frac{8}{25} + \frac{6}{25}\,i$ is written in exponential form $c = |c|\; e^{i\varphi} = \frac{10}{25}\, e^{i\,36.9^\circ}$, since

$$|c| = \tfrac{1}{25}\,\sqrt{8^2+6^2} = \tfrac{10}{25} \quad \text{and} \quad \tan\varphi = \frac{3}{4} \hookrightarrow \varphi = 36.9^\circ.$$

$$\Rightarrow \tilde{y}_p\,(x) = \tfrac{2}{4-3i}\, e^{ix} = \tfrac{10}{25}\, e^{i\,36.9^\circ} \cdot e^{ix} = \tfrac{10}{25}\, e^{i\,(x+36.9^\circ)}$$

$$= \tfrac{10}{25}\cos(x+36.9^\circ) + i\,\tfrac{10}{25}\sin(x+36.9^\circ).$$

An alternative representation of the particular solution is therefore

$$y_p\,(x) = \mathrm{Im}\,(\tilde{y}_p\,(x)) = \frac{10}{25}\,\sin\,(x+36.9^\circ)\,.$$ □

16.3.2 Inhomogeneity = Polynomial × Exponential Function

The approach for a particular solution from Theorem 16.5 leads to a solution if μ is **not** a zero of the characteristic polynomial $P\,(\lambda)$. But what if μ is a zero? We will consider the more general case of an inhomogeneity $f\,(x) = h\,(x)\; e^{\mu\,x}$ with a polynomial $h\,(x)$:

Theorem 16.6: $f(x)$ = Polynomial $\times$ Exponential

Given is the *inhomogeneous* linear differential equation

$$y^{(n)}(x) + a_{n-1}\, y^{(n-1)}(x) + \ldots + a_1\, y'(x) + a_0\, y(x) = h(x) \cdot e^{\mu\, x}.$$

(i) Let μ be a zero of the characteristic polynomial of order k $(k \geq 0)$

(ii) and let $h(x)$ be a polynomial of degree m.

Then the approach

$$y_p(x) = g(x) \cdot x^k \cdot e^{\mu\, x}$$

provides a particular solution where $g(x)$ is also a **polynomial of degree m**.

Remarks:

(1) Theorem 16.5 is a special case of Theorem 16.6: If $f(x) = c\, e^{\mu\, x}$ and μ is *not* a zero of the characteristic polynomial, then $k = 0$ (i.e. the term $x^0 = 1$ does not appear) and $m = 0$ (c is a polynomial of degree 0). Therefore, the approach function $y_p(x) = k\, e^{\mu\, x}$ returns a particular solution.

(2) If the inhomogeneity is the sum of several functions, then we determine an individual approach function for each inhomogeneity, compute its constants, and finally add all the individual results to obtain the overall $y_p(x)$.

Examples 16.16 (Sample Examples):

① Find a particular solution to the differential equation

$$2\, y''(x) + y'(x) = x\, e^{-x} :$$

The associated characteristic polynomial is

$$P(\lambda) = 2\, \lambda^2 + \lambda$$

and the inhomogeneity is

$$f(x) = x\, e^{-x} \hookrightarrow \mu = -1.$$

$\mu = -1$ is not a zero of $P(\lambda)$, since $P(-1) = 1 \neq 0 \hookrightarrow k = 0$. $h(x) = x$ is a polynomial of degree 1 $\hookrightarrow m = 1$. $\Rightarrow$ The approach function for a particular solution is therefore a polynomial of degree 1 times e^{-x}:

$$y_p(x) = (a_0 + a_1 x)\, e^{-x}.$$

This approach for $y_p(x)$ together with its derivatives

$$\begin{aligned} y_p'(x) &= a_1 e^{-x} - (a_0 + a_1 x)\, e^{-x} \\ y_p''(x) &= -2\, a_1 e^{-x} + (a_0 + a_1 x)\, e^{-x} \end{aligned}$$

is substituted into the differential equation to find a_0 and a_1. Thus,

$$2\, y_p'' + y_p'(x) = [(a_0 - 3\, a_1) + a_1 x]\, e^{-x} = x\, e^{-x}$$

$$\Rightarrow (a_0 - 3\, a_1) + a_1 x = x\,.$$

To determine the coefficients, a coefficient comparison is performed according to descending powers of x

$$\left.\begin{aligned} x^1 &: \quad a_1 = 1 \\ x^0 &: \quad a_0 - 3\, a_1 = 0 \Rightarrow a_0 = 3 \end{aligned}\right\} \Rightarrow \quad y_p(x) = (3 + x)\, e^{-x}.$$

② Find a particular solution to the differential equation

$$2\, y''(x) + y'(x) = x\,:$$

The corresponding characteristic polynomial is

$$P(\lambda) = 2\, \lambda^2 + \lambda$$

and the inhomogeneity is

$$f(x) = x\, e^{0\, x} \hookrightarrow \mu = 0.$$

$\mu = 0$ is a *simple* zero of $P(\lambda) \hookrightarrow k = 1$; $h(x) = x$ is a polynomial of degree 1 $\hookrightarrow m = 1$. So a suitable function for a particular solution is

$$y_p(x) = (a_0 + a_1 x)\, x^1\, e^{0\, x} = a_0\, x + a_1\, x^2.$$

To find the coefficients a_0 and a_1, it is necessary to replace $y_p(x)$ together with its derivatives

$$\begin{aligned} y_p'(x) &= a_0 + 2\, a_1\, x \\ y_p''(x) &= 2\, a_1 \end{aligned}$$

into the differential equation. This gives

$$2\,y_p''(x) + y_p'(x) = 4\,a_1 + a_0 + 2\,a_1\,x = x$$

and with coefficient comparison:

$$\begin{array}{lll} x^1: & 2\,a_1 = 1 & \Rightarrow a_1 = \frac{1}{2} \\ x^0: & 4\,a_1 + a_0 = 0 & \Rightarrow a_0 = -2. \end{array}$$

A particular solution is therefore $y_p(x) = -2\,x + \frac{1}{2}\,x^2$. □

Examples 16.17:

① $y''(x) + y(x) = e^{ix}$:

The characteristic polynomial is $P(\lambda) = \lambda^2 + 1$. The inhomogeneity is $e^{ix} \hookrightarrow \mu = i$. $\mu = i$ is a simple zero $\hookrightarrow k = 1$. Here, $m = 0$.

Approach:
$$\begin{array}{rcl} y_p(x) & = & a_0\,x\,e^{ix} \\ y_p'(x) & = & a_0\,e^{ix} + i\,a_0\,x\,e^{ix} \\ y_p''(x) & = & 2\,i\,a_0\,e^{ix} - a_0\,x\,e^{ix} \end{array}$$

Substitute into the differential equation:

$$\begin{aligned} y_p''(x) + y_p(x) &= 2\,a_0\,i\,e^{ix} - a_0\,x\,e^{ix} + a_0\,x\,e^{ix} \\ &= 2\,a_0\,i\,e^{ix} \\ &= e^{ix} \end{aligned}$$

So $2\,a_0\,i = 1$ or $a_0 = \frac{1}{2i} = -\frac{1}{2}\,i$ and a particular solution is

$$y_p(x) = \frac{1}{2i}\,x\,e^{ix} = -\frac{1}{2}\,i\,x\,e^{ix}.$$

② $y''(x) + y(x) = \cos x$: $\qquad (*)$

As in Example 16.15, the differential equation is extended to the complex

$$\tilde{y}''(x) + \tilde{y}(x) = e^{ix} \qquad (\tilde{*})$$

$\tilde{y}_p(x) = -\frac{1}{2}\,i\,x\,e^{ix}$ is according to ① a particular solution of $(\tilde{*})$. A particular solution of $(*)$ is therefore given by the real part of $\tilde{y}_p(x)$

$$y_p(x) = \text{Re}\,(\tilde{y}_p(x))\,.$$

With

$$-\tfrac{1}{2}\,i\,x\,e^{ix} = +\tfrac{1}{2}\,e^{i\,\frac{3}{2}\,\pi}\,x\,e^{ix} = \tfrac{1}{2}\,x\;e^{i\,\left(x+\frac{3}{2}\,\pi\right)}$$

$$\Rightarrow\; y_p(x) = \text{Re}\left(\tfrac{1}{2}\,x\,e^{i\,\left(x+\frac{3}{2}\,\pi\right)}\right) = \tfrac{1}{2}\,x\,\cos\left(x+\tfrac{3}{2}\,\pi\right) = \tfrac{1}{2}\,x\,\sin x.$$

③ $y''(x) + y(x) = \sin x$:

Following the procedure in ② we get a particular solution

$$y_p(x) = \operatorname{Im}(\tilde{y}_p(x)) = \operatorname{Im}\left(\tfrac{1}{2}\, x\, e^{i\left(x+\frac{3}{2}\pi\right)}\right) = \tfrac{1}{2}\, x \sin\left(x + \tfrac{3}{2}\pi\right).$$
$$= -\tfrac{1}{2}\, x \cos x.$$
□

16.3.3 Special Cases of Theorem 16.6:

To find a particular solution of the inhomogeneous differential equation

$$y^{(n)}(x) + a_{n-1}\, y^{(n-1)}(x) + \ldots + a_1\, y'(x) + a_0\, y(x) = f(x),$$

in some special cases a real approach can be taken directly. Table 16.2 shows the corresponding approach function for common inhomogeneities.

Table 16.2: Approach functions for particular solutions.

Inhomogeneity	Zero of $P(\lambda)$	Approach
$f(x) = \sum_{i=0}^{n} a_i\, x^i$	0 no zero	$y_p(x) = \sum_{i=0}^{n} A_i\, x^i$
	0 is zero of order k	$y_p(x) = x^k \sum_{i=0}^{n} A_i\, x^i$
$f(x) = a\, e^{\mu x}$	μ no zero	$y_p(x) = A\, e^{\mu x}$
	μ is zero of order k	$y_p(x) = A\, x^k\, e^{\mu x}$
$f(x) = a \sin(\beta x)$	$i\beta$ no zero	$y_p(x) = A \sin(\beta x) + B \cos(\beta x)$ $= C \sin(\beta x + \varphi)$
$f(x) = a \cos(\beta x)$	$i\beta$ is zero of order k	$y_p(x) = x^k (A \sin(\beta x) + B \cos(\beta x))$ $= C\, x^k \sin(\beta x + \varphi)$

Example 16.18. Given is the differential equation

$$y''(x) + 2\,y'(x) + y(x) = f(x)$$

with different inhomogeneities f. According to Table 16.2, we choose an approach function for a particular solution and calculate the free parameters.

The characteristic polynomial for the differential equation is

$$P(\lambda) = \lambda^2 + 2\,\lambda + 1 = (\lambda + 1)^2\,.$$

Hence, $\lambda = -1$ is a *double* zero.

$f(x)$	Approach Function	Parameters
$x^2 - 2\,x + 1$	$y_p(x) = a_0 + a_1\,x + a_2\,x^2$ $(\mu = 0$ no zero of $P(\lambda))$	$a_0 = 11,\ a_1 = -6,\ a_2 = 1$
$2\,e^x$	$y_p(x) = A\,e^x$ $(\mu = 1$ no zero of $P(\lambda))$	$A = \frac{1}{2}$
$\cos x$	$\tilde{y}_p(x) = A\,e^{ix} \to$ in complex DE $y_p(x) = \text{Re}\left(A\,e^{ix}\right)$	$A = -\frac{1}{2}\,i$ $y_p(x) = \frac{1}{2}\cos\left(x + \frac{3\pi}{2}\right)$
$\cos x$	$y_p(x) = A\sin x + B\cos x$ $(\mu = i$ no zero of $P(\lambda))$	$A = \frac{1}{2},\ B = 0$
$\sin x$	$y_p(x) = A\sin x + B\cos x$ $(\mu = i$ no zero of $P(\lambda))$	$A = 0,\ B = -\frac{1}{2}$
e^{-x}	$y_p(x) = a_2\,x^2\,e^{-x}$ $(\mu = -1$ is double zero$)$	$a_2 = \frac{1}{2}$
$-x^2\,e^x$	$\left(a_0 + a_1\,x + a_2\,x^2\right)\,e^x$	$a_0 = -\frac{3}{8},\ a_1 = \frac{1}{2},\ a_2 = -\frac{1}{4}$
$x\,e^{-x}$	$(a_0 + a_1\,x)\,x^2\,e^{-x}$	$a_0 = 0,\ a_1 = \frac{1}{6}$

Summary: n-th Order Linear Differential Equations

The general solution of the n-th order inhomogeneous DEq

$$y^{(n)}(x) + a_{n-1}\, y^{(n-1)}(x) + \ldots + a_1\, y'(x) + a_0\, y(x) = f(x) \quad (*)$$

with constant coefficients consists of the **general homogeneous solution** $y_h(x)$ and **a particular solution** $y_p(x)$ of the inhomogeneous DEq:

$$y(x) = y_h(x) + y_p(x)\,.$$

1. Find the general homogeneous solution to

$$y^{(n)}(x) + a_{n-1}\, y^{(n-1)}(x) + \ldots + a_1\, y'(x) + a_0\, y(x) = 0 :$$

(1) Insert $y(x) = e^{\lambda x}$ into differential equation.

(2) Characteristic polynomial

$$P(\lambda) = \lambda^n + a_{n-1}\,\lambda^{n-1} + \ldots + a_1\,\lambda + a_0.$$

(3) Zeros of the characteristic polynomial $\lambda_1, \ldots, \lambda_m$
λ_i single zero $\rightarrow \varphi_i(x) = e^{\lambda_i x}$.
λ_i k-th order zero $\rightarrow \varphi_i(x) = e^{\lambda_i x},\, x\, e^{\lambda_i x}, \ldots,\, x^{k-1}\, e^{\lambda_i x}$.

(4) $\varphi_1(x),\, \varphi_2(x), \ldots, \varphi_n(x)$ is a fundamental set.

(5) The general homogeneous solution is

$$y_h(x) = c_1\,\varphi_1(x) + c_2\,\varphi_2(x) + \ldots + c_n\,\varphi_n(x)\,.$$

2. Determine a particular solution to the inhomogeneous DEq:
According to Table 16.2, special approaches are chosen for a particular solution or according to the Theorem 16.6.

If the inhomogeneity consists of several functions $f_1(x), \ldots, f_l(x)$, a particular approach $y_{p_1}(x), \ldots, y_{p_l}(x)$ is chosen for each term

$$y_{p_i}^{(n)}(x) + a_{n-1}\, y_{p_i}^{(n-1)}(x) + \ldots + a_0\, y_{p_i}(x) = f_i(x)\,.$$

A particular solution for the complete inhomogeneity $f(x) = f_1(x) + \ldots + f_l(x)$ is then

$$y_p(x) = y_{p_1}(x) + \ldots + y_{p_l}(x)\,.$$

Continued

3. The general solution of the differential equation $(*)$ is

$$y(x) = c_1\,\varphi_1(x) + \ldots + c_n\,\varphi_n(x) + y_p(x)\,.$$

4. The coefficients $c_1, \ldots, c_n$ are determined by the initial conditions $y(0)\,,\, y'(0)\,, \ldots,\, y^{(n-1)}(0)$.

Example 16.19 (Sample Example).
Find the solution to

$$y''(x) - 6\,y'(x) + 9\,y(x) = 4\,e^{2x} + 9\,x - 15$$

with the initial conditions $y(0) = y_0$, $y'(0) = 0$.

1. Solve the homogeneous differential equation

$$y''(x) - 6\,y'(x) + 9\,y(x) = 0\,:$$

The characteristic polynomial is

$$P(\lambda) = \lambda^2 - 6\,\lambda + 9.$$

The zeros of $P(\lambda)$ are $\lambda_1 = \lambda_2 = 3$ (double). So the general solution of the homogeneous differential equation is

$$y_h(x) = c_1\,e^{3x} + c_2\,x\,e^{3x}.$$

2. Calculate a particular solution:
The inhomogeneity $f(x) = 4\,e^{2x} + 9\,x - 15$ is the sum of two types of functions. Therefore, a particular solution is found in two steps:

(i) $$y''(x) - 6\,y'(x) + 9\,y(x) = 4\,e^{2x}. \qquad (1)$$

$\mu = 2$ is not a zero of $P(\lambda)$; therefore the approach

$$y_{p_1}(x) = A\,e^{2x}.$$

gives a particular solution. Substituting this into the differential equation (1) we get:

$$A\,(4 - 6\cdot 2 + 9)\,e^{2x} = 4\,e^{2x} \;\Rightarrow\; A = 4 \;\Rightarrow\; y_{p_1}(x) = 4\,e^{2x}.$$

(ii) $$y''(x) - 6\,y'(x) + 9\,y(x) = 9x - 15. \qquad (2)$$

$\mu = 0$ is not a zero of $P(\lambda)$; therefore,

$$y_{p_2}(x) = a_0 + a_1\,x$$

is inserted into the differential equation (2):

$$\begin{aligned} y''_{p_2}(x) - 6\,y'_{p_2}(x) + 9\,y_{p_2}(x) &= -6\,a_1 + 9\,(a_0 + a_1\,x) \\ &= (-6\,a_1 + 9\,a_0) + 9\,a_1\,x \\ &= 9\,x - 15. \end{aligned}$$

Comparing the coefficients gives

x^1: $\quad 9\,a_1 = 9 \quad \Rightarrow \quad a_1 = 1,$

x^0: $\quad -6\,a_1 + 9\,a_0 = -15 \quad \Rightarrow \quad a_0 = -1.$

$$\Rightarrow y_{p_2}(x) = -1 + x.$$

(iii) The particular solution for $4\,e^{2x} + 9\,x - 15$ is the sum of y_{p_1} and y_{p_2}:

$$y_p(x) = 4\,e^{2x} + x - 1.$$

3. The general solution of the differential equation is therefore

$$y(x) = c_1\,e^{3x} + c_2\,x\,e^{3x} + 4\,e^{2x} + x - 1.$$

4. Find the constants c_1, c_2 with the initial conditions:

$$\begin{aligned} y(0) &= c_1 + 4 - 1 = y_0 && \Rightarrow c_1 = y_0 - 3 \\ y'(0) &= 3\,c_1 + c_2 + 9 = 0 && \Rightarrow c_2 = -9 - 3\,c_1 = -3\,y_0. \end{aligned}$$

$$\Rightarrow y(x) = (y_0 - 3)\,e^{3x} - 3\,y_0\,x\,e^{3x} + 4\,e^{2x} + x - 1.$$ □

Application Example 16.20 **(Forced Damped Oscillation).**

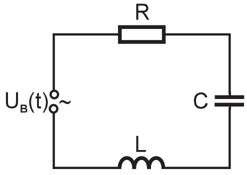

Figure 16.6. Circuit

An electrical resonant circuit consists of an ohmic resistor R, a capacitor of capacity C and a coil of inductance L (see Fig. 16.6). At time $t = 0$, the circuit is closed and an external AC voltage

$$U_B(t) = U_0\,\sin(\omega t)$$

is applied.

According to the mesh rule

$$L \frac{dI(t)}{dt} + R\,I(t) + \frac{1}{C}\int_0^t I(\tau)\,d\tau = U_0 \sin(\omega t).$$

Transferring this to the complex formulation, we get

$$L\,\dot{I}(t) + R\,I(t) + \frac{1}{C}\int_0^t I(\tau)\,d\tau = U_0\,e^{i\,\omega\,t}$$

and with differentiation

$$\ddot{I}(t) + \frac{R}{L}\,\dot{I}(t) + \frac{1}{LC}\,I(t) = \frac{U_0}{L}\,i\,\omega\,e^{i\,\omega\,t}.$$

We introduce the parameters $\beta = \frac{R}{2\,L}$ (damping) and $\omega_0^2 = \frac{1}{LC}$ (undamped eigenfrequency) and obtain

$$\ddot{I}(t) + 2\,\beta\,\dot{I}(t) + \omega_0^2\,I(t) = \frac{U_0}{L}\,i\,\omega\,e^{i\,\omega\,t}.$$

The general, homogeneous solution to this DEq is discussed in Example 16.13, so only one particular solution needs to be found to solve the inhomogeneous differential equation. The characteristic polynomial for the differential equation is

$$P(\lambda) = \lambda^2 + 2\,\beta\,\lambda + \omega_0^2.$$

$\mu = i\,\omega$ is not a zero of the characteristic polynomial. So a particular solution is given by the approach

$$\tilde{I}_p(t) = A\,e^{i\,\omega\,t}.$$

Substituting this into the differential equation, we get

$$(i\,\omega)^2\,A\,e^{i\,\omega\,t} + 2\,\beta\,(i\,\omega)\,A\,e^{i\,\omega\,t} + \omega_0^2\,A\,e^{i\,\omega\,t} = \frac{U_0}{L}\,i\,\omega\,e^{i\,\omega\,t}$$

$$\Rightarrow A = i\,\frac{U_0\,\omega}{L}\,\frac{1}{\omega_0^2 - \omega^2 + 2i\,\beta\,\omega}$$

and

$$\tilde{I}_p(t) = i\,\frac{U_0\,\omega}{L}\,\frac{1}{\omega_0^2 - \omega^2 + 2i\,\beta\,\omega}\,e^{i\,\omega\,t}.$$

Transition to a real particular solution: $I_p(t) = \operatorname{Im}(\tilde{I}_p(t))$.
To calculate the particular solution, we represent the amplitude A in exponential form

$$\begin{aligned} A &= i\,\frac{U_0\,\omega}{L}\,\frac{1}{\omega_0^2-\omega^2+2i\,\beta\,\omega}\cdot\frac{\omega_0^2-\omega^2-i\,2\,\beta\,\omega}{\omega_0^2-\omega^2-i\,2\,\beta\,\omega} \\ &= \frac{U_0\,\omega}{L}\,\frac{1}{\left(\omega_0^2-\omega^2\right)^2+\left(2\,\beta\,\omega\right)^2}\,\left[2\,\beta\,\omega+i\,\left(\omega_0^2-\omega^2\right)\right] \overset{!}{=} |A|\;e^{i\varphi} \end{aligned}$$

with
$$|A| = \frac{U_0\,\omega}{L}\,\frac{1}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+\left(2\,\beta\,\omega\right)^2}} \quad\text{and}\quad \tan\varphi = \frac{\operatorname{Im}\,A}{\operatorname{Re}\,A} = \frac{\omega_0^2-\omega^2}{2\,\beta\,\omega}.$$

$$\Rightarrow \tilde{I}_p(t) = \underbrace{\frac{U_0\,\omega}{L}\,\frac{1}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+\left(2\,\beta\,\omega\right)^2}}\,e^{i\varphi}}_{\text{complex amplitude}}\;e^{i\,\omega\,t}$$

$$\Rightarrow I_p(t) = \operatorname{Im}\left(\tilde{I}_p(t)\right) = \frac{U_0\,\omega}{L}\,\frac{1}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+\left(2\,\beta\,\omega\right)^2}}\,\sin\left(\omega t+\varphi\right).$$

Interpretation: Replacing $\omega_0^2 = \frac{1}{LC}$ and $2\,\beta = \frac{R}{L}$, we obtain the amplitude

$$\frac{U_0\,\omega}{L}\,\frac{1}{\sqrt{\left(2\,\beta\,\omega\right)^2+\left(\omega_0^2-\omega^2\right)^2}} = \frac{U_0}{\sqrt{R^2+\left(\frac{1}{\omega C}-\omega L\right)^2}} = I_0$$

and the phase of the current

$$\tan\varphi(\omega) = \frac{\omega L-\frac{1}{\omega C}}{R}.$$

$$\Rightarrow \boxed{I_p(t) = I_0\sin\left(\omega t+\varphi\right) = \frac{U_0}{\sqrt{R^2+\left(\frac{1}{\omega C}-\omega L\right)^2}}\sin\left(\omega t+\varphi\right).} \qquad (*)$$

Equation $(*)$ is Ohm's law for an alternating current with maximum values U_0 and I_0. The real resistance is

$$Z = \sqrt{R^2+\left(\frac{1}{\omega C}-\omega L\right)^2} = Z(\omega).$$

The current $I_p(t)$ is phase-shifted with respect to the applied voltage by $\varphi(\omega)$. □

16.4 Non-Constant Coefficients

In this section we discuss n-th order linear differential equations, which do not necessarily have constant coefficients. Since the solution structure does not change compared to constant coefficients, we first solve the homogeneous problem $y_h(x)$ and then proceed to find a particular solution $y_p(x)$ to the inhomogeneous problem. Then the inhomogeneous solution $y_i(x)$

$$y_i(x) = y_h(x) + y_p(x)$$

is the sum of the general homogeneous solution and a particular solution.

16.4.1 Homogeneous Problem: d'Alembert's Reduction Method

The basic idea of solving the homogeneous problem is that if we know a part of the solution, we can reduce the order of the DEq, leading to a lower order DEq. This is called the d'Alembert's reduction method. We illustrate this principle with a simple second order DEq

$$y''(x) - y(x) = 0.$$

We directly verify that $y_1(x) = c \cdot e^x$ is a solution to the DEq, for any constant c. The idea of d'Alembert is then to vary the constant $c(x)$ and determine a reduced order DEq to find $c(x)$. We have already used such an approach to find an inhomogeneous solution (see Volume 2, Section 13.3). In that context we called the method *Variation of the Constant*. So we take the approach with its derivatives

$$\begin{aligned} y_2(x) &= c(x) \cdot e^x \qquad (1) \\ y_2'(x) &= c'(x) \cdot e^x + c(x) \cdot e^x \\ y_2''(x) &= c''(x) \cdot e^x + 2\,c'(x) \cdot e^x + c(x) \cdot e^x. \end{aligned}$$

We evaluate the DEq and get

$$y_2''(x) - y_2(x) = c''(x) \cdot e^x + 2\,c'(x) \cdot e^x = 0.$$

Now we reduce the order by setting $w(x) = c'(x)$ and obtain

$$w'(x) \cdot e^x + 2\,w(x) \cdot e^x = 0 \qquad \text{or} \qquad w'(x) + 2\,w(x) = 0.$$

This is a first-order DEq for the function $w(x)$, which is solved using the methods introduced in the Chapter First-Order Differential Equations (see

Volume 2, Section 13.2). The solution for $w(x)$ is

$$w(x) = c_0\, e^{-2x}.$$

Since $w(x) = c'(x)$, we integrate $c(x) = c_1 + \int w(x)\, dx$ and obtain

$$c(x) = c_1 - c_0 \tfrac{1}{2} e^{-2x}.$$

According to our setting (1) we have the solution

$$y_2(x) = c(x) \cdot e^x = (c_1 - c_0 \tfrac{1}{2} e^{-2x}) \cdot e^x = c_1 \cdot e^x - \tfrac{1}{2} c_0 \cdot e^{-x}.$$

From this result we can identify not only a second solution, but also a fundamental set:

$$e^x \quad \text{and} \quad e^{-x}.$$

Note: Important for the d'Alembert reduction method is that we know a solution to the DEq. Usually we try an approach with basic functions. Common approach functions that may give solutions to the DEq are listed in Table 16.1.

Table 16.1: Approach Functions

Power Function	$y(x) = x^{\alpha}$	(1)
Polynomial	$y(x) = a + b\,x + c\,x^2$	(2)
Exponential Function	$y(x) = e^{\alpha\, x}$	(3)

Example 16.21 (Approach function). Given is the second-order DEq with non-constant coefficients

$$(1 - x^2)\, y''(x) + 2x\, y'(x) - 2\, y(x) = 0 \quad \text{for} \quad x > 1.$$

We are looking for a function (power, polynomial, exponential) that is a solution to the DEq.

We start with (1) of the Table 16.1 and evaluate the right side of the DEq by replacing $y(x) = x^{\alpha}$, $y'(x) = \alpha\, x^{\alpha-1}$, $y''(x) = \alpha(\alpha-1)\, x^{\alpha-2}$ in the DEq

$$(1 - x^2)\, \alpha(\alpha - 1)\, x^{\alpha-2} + 2x\, \alpha\, x^{\alpha-1} - 2\, x^{\alpha} = 0.$$

We reorder in terms of the powers of x

$$\alpha(\alpha - 1)\, x^{\alpha-2} - (\alpha - 1)(\alpha - 2)\, x^{\alpha} = 0.$$

This identity is only true if both coefficients are zero at the same time, which is only true for $\alpha = 1$. So $y(x) = x^1$ is a solution.

The calculation for approach (2) is more or less the same. We start with $y(x) = a + b\,x + c\,x^2$ and its derivatives $y'(x) = b + 2\,c\,x$, $y''(x) = 2\,c$ and evaluate the right side of the DEq

$$(1 - x^2)\,2\,c + 2x\,(b + 2\,c\,x) - 2\,(a + b\,x + c\,x^2) = 0,$$

which we simplify to

$$2\,c - 2\,a = 0.$$

We get a solution to the DEq whenever $a = c$ for any b.
If we choose $a = 0$, $c = 0$ and $b = 1$, we obtain $y(x) = x$.
If we choose $a = 1$, $c = 1$ and $b = 0$, we obtain $y(x) = 1 + x^2$.

The approach (3) does not give a solution. Because with $y(x) = e^{\alpha\,x}$ and its derivatives $y'(x) = \alpha\,e^{\alpha\,x}$, $y(x) = \alpha^2\,e^{\alpha\,x}$ we evaluate the DEq and get

$$(1 - x^2)\,(\alpha^2\,e^{\alpha\,x}) + 2x\,(\alpha\,e^{\alpha\,x}) - 2\,(e^{\alpha\,x}) = 0$$

and after simplification

$$(1 - x^2)\,\alpha^2 + 2x\alpha - 2 = 0.$$

But we will not find an α such that the complete right side is zero independently of x. So $y(x) = e^{\alpha\,x}$ is not a solution to the DEq. □

Example 16.22 (Reduction Method). Given is the second-order DEq with non-constant coefficients

$$(1 - x^2)\,y''(x) + 2x\,y'(x) - 2\,y(x) = 0 \quad \text{for} \quad x > 1.$$

According to Example 16.21 the function $y(x) = x$ is a solution to the DEq. We use d'Alembert's reduction method to find a second linearly independent solution.

With $y(x) = x$ also $y(x) = c \cdot x$ is a solution. We will now vary c as a function of x. So we use the approach with its derivatives

$$\begin{aligned} y_2(x) &= c(x) \cdot x \\ y_2'(x) &= c'(x) \cdot x + c(x) \cdot 1 \\ y_2''(x) &= c''(x) \cdot x + 2\,c'(x) \cdot 1. \end{aligned}$$

When we evaluate the DEq, we get

$$\begin{aligned} & (1-x^2)\,y_2''(x) + 2x\,y_2'(x) - 2\,y_2(x) = 0 \\ \Leftrightarrow\ & (1-x^2)\,(c''(x)x + 2\,c'(x)) + 2x\,(c'(x)x + c(x)) - 2\,(c(x)x) = 0 \\ \Leftrightarrow\ & (1-x^2)\,x\,c''(x) + \big(2(1-x^2) + 2x^2\big)\,c'(x) = 0 \\ \Leftrightarrow\ & (1-x^2)\,x\,c''(x) + 2\,c'(x) = 0. \end{aligned}$$

Now we reduce the order by setting $w(x) = c'(x)$ and obtain

$$(1-x^2)x\,w'(x) + 2\,w(x) = 0 \qquad \text{or} \qquad w'(x) = -\frac{2}{(1-x^2)x}\cdot w(x).$$

This is a first-order DEq for the function $w(x)$, which we solve with the method *Separation of Variables* (see Volume 2, Section 13.2)

$$\frac{dw}{w} = -\frac{2}{(1-x^2)x}\,dx.$$

To integrate, we decompose the right side into partial fractions

$$-\frac{2}{(1-x^2)\,x} = \frac{-2}{x} + \frac{1}{1-x} + \frac{1}{1+x}.$$

So we get

$$\int \frac{dw}{w} = \int \left(\frac{-2}{x} + \frac{1}{1-x} + \frac{1}{1+x}\right) dx$$

$$\ln|w| = -2\ln|x| + \ln|1-x| + \ln|1+x| + c = \ln\left(\frac{1-x^2}{x^2}\right) + c.$$

We apply the exponential function to both sides of the equation

$$w(x) = \frac{1-x^2}{x^2}\cdot e^c = \left(\frac{1}{x^2} - 1\right)\cdot c_0$$

with $c_0 = e^c$. Since $w(x) = c'(x)$, we integrate $c(x) = c_1 + \int w(x)\,dx$ and obtain

$$c(x) = c_1 + c_0\left(-x^{-1} - x\right).$$

According to our setting, the solution is

$$y_2(x) = c(x)\cdot x = \left(c_1 + c_0\left(-x^{-1} - x\right)\right)\cdot x = c_1\cdot x - c_0\cdot(1+x^2).$$

From this result we identify a second solution and a fundamental set:

$$x \ \text{ and } \ (1+x^2). \qquad \square$$

Example 16.23 (Sample Example). We look for a fundamental set of the third-order DEq with non-constant coefficients

$$x^3\,y'''(x)+(5x^3-3x^2)\,y''(x)+(6x^3-10x^2+6x)\,y'(x)+(10x-6x^2-6)\,y(x)=0.$$

Guess of a solution: According to the Table 16.1, we look for an approach function (power function, polynomial, exponential) that might give a solution to the DEq.

The power function $y(x)=x$ is a solution to the DEq. Because if we insert $y(x)=x$ with its derivatives $y'(x)=1$ and $y''(x)=0$, $y'''(x)=0$ into the DEq, we get

$$(6x^3-10x^2+6x)+(10x-6x^2-6)\,x=6x^3-10x^2+10x^2-6x^3-6x=0.$$

Reduction of DEq: With $y(x)=x$ also $y(x)=c\cdot x$ is a solution. We will now vary c as a function of x. So we use the approach with its derivatives

$$\begin{aligned} y_2(x) &= c(x)\cdot x \\ y_2'(x) &= c'(x)\cdot x + c(x)\cdot 1 \\ y_2''(x) &= c''(x)\cdot x + 2\,c'(x)\cdot 1 \\ y_2'''(x) &= c'''(x)\cdot x + 3\,c''(x). \end{aligned}$$

We evaluate the DEq and after simplifying the right side of the DEq we get

$$c'''(x)+5\,c''(x)+6\,c'(x)=0.$$

Now we reduce the order by setting $w(x)=c'(x)$ and get

$$w''(x)+5\,w'(x)+6\,w(x)=0.$$

Solving the reduced DEq: The reduced DEq is a second-order homogeneous differential equation which we solve using the characteristic polynomial

$$p(\lambda)=\lambda^2+5\,\lambda+6=(\lambda+3)(\lambda+2)=0.$$

The roots of the characteristic polynomial are $\lambda_1=-3$ and $\lambda_2=-2$. So the general solution of the reduced DEq is

$$w(x)=c_1\,e^{-3\,x}+c_2\,e^{-2\,x}.$$

Solution of the given DEq: Since $w(x) = c'(x)$, we integrate $c(x) = c_1 + \int w(x)\,dx$ and obtain

$$c(x) = c_1 \tfrac{-1}{3} e^{-3\,x} + c_2 \tfrac{-1}{2} e^{-2\,x} + c_3.$$

According to our setting, the solution to the DEq is

$$y_2(x) = c(x) \cdot x = -c_1 \tfrac{1}{3}\, x\, e^{-3\,x} - c_2 \tfrac{1}{2}\, x\, e^{-2\,x} + c_3\, x\,.$$

From this result we identify a fundamental set:

$$x\, e^{-3\,x}, \quad x\, e^{-2\,x}, \quad x\,.$$

□

Example 16.24 (Sample Example). We look for a fundamental set of the second-order DEq with non-constant coefficients

$$x\, y''(x) - (1 + x)\, y'(x) + y(x) = 0 \qquad \text{for} \quad x > 0.$$

Guess of a solution: First, we look for an approach function (see Table 16.1) that might give a solution to the DEq. We check whether $y(x) = e^{\alpha\,x}$ will be a solution to the DEq.

So we insert $y(x) = e^{\alpha\,x}$ with its derivatives $y'(x) = \alpha\, e^{\alpha\,x}$ and $y''(x) = \alpha^2\, e^{\alpha\,x}$ into the DEq and obtain

$$\begin{aligned} x\, y''(x) - (1 + x)\, y'(x) + y(x) &= 0 \\ x\, \alpha^2\, e^{\alpha\,x} - (1 + x)\, \alpha\, e^{\alpha\,x} + e^{\alpha\,x} &= 0 \\ x\, \alpha^2 - (1 + x)\, \alpha + 1 &= 0 \\ x\, \alpha(\alpha - 1) - (\alpha - 1) &= 0 \end{aligned}$$

This identity is only true if both coefficients are zero at the same time, which is only true for $\alpha = 1$. So $y(x) = e^x$ is a solution.

Reduction of DEq: With $y(x) = e^x$ also $y(x) = c \cdot e^x$ is a solution. We will now vary c as a function of x. So we use the approach with its derivatives

$$\begin{aligned} y_2(x) &= c(x) \cdot e^x \\ y_2'(x) &= c'(x) \cdot e^x + c(x) \cdot e^x \\ y_2''(x) &= c''(x) \cdot e^x + 2\, c'(x) \cdot e^x + c(x) \cdot e^x. \end{aligned}$$

We evaluate the DEq and get after simplifying the right side of the DEq

$$x\,c''(x) + (x-1)\,c'(x) = 0.$$

Now we reduce the order by setting $w(x) = c'(x)$:

$$x\,w'(x) + (x-1)\,w(x) = 0.$$

Solving the reduced DEq: The reduced DEq is a first-order homogeneous differential equation solved by Separation of Variables (see Volume 2, Section 13.2). Therefore, we isolate $w'(x)$

$$w'(x) = \frac{1-x}{x} \cdot w(x) = (\frac{1}{x} - 1) \cdot w(x)$$

and separate the variables

$$\frac{dw}{w} = (\frac{1}{x} - 1)\,dx.$$

With integration we get

$$\ln|w| = \ln|x| - x + c$$

and finally using the exponential function ($c_0 = e^c$)

$$w(x) = e^{\ln|x| - x + c} = x\,e^{-x}\,e^{c} = c_0\,x\,e^{-x}.$$

Solution of the given DEq: Since $w(x) = c'(x)$, we integrate

$$c(x) = c_1 + \int w(x)\,dx$$

by parts and obtain

$$c(x) = c_1 + c_0 \int x\,e^{-x}\,dx = c_1 - c_0\,x\,e^{-x} - c_0\,e^{-x}.$$

According to our setting, the solution to the DEq is

$$y_2(x) = c(x) \cdot e^{x} = c_1\,e^{x} - c_0\,(x+1).$$

From this result we identify a fundamental set:

$$x+1\,,e^{x}.$$

□

16.4.2 Inhomogeneous Problem: Variation of Constants

For the discussion of the inhomogeneous problem we start with a second-order DEq. We first transform the second-order DEq into a first-order system and then apply the method Variation of the Constant as described in detail in Section 15.3 using the fundamental matrix.

Suppose we have a second-order DEq of the form

$$y''(x) + a_1(x)\, y'(x) + a_0(x)\, y(x) = g(x). \tag{DEq}$$

We introduce the new functions

$$\begin{aligned} y_0(x) &:= y(x) & y_0' &= y_1(x) \\ y_1(x) &:= y'(x) & y_1' &= -a_0(x)\, y(x) - a_1(x)\, y'(x) + g(x) \\ & & &= -a_0(x)\, y_0(x) - a_1(x)\, y_1'(x) + g(x) \end{aligned}$$

so that we obtain the equivalent first-order system

$$\begin{pmatrix} y_0(x) \\ y_1(x) \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -a_0(x) & -a_1(x) \end{pmatrix} \begin{pmatrix} y_0(x) \\ y_1(x) \end{pmatrix} + \begin{pmatrix} 0 \\ g(x) \end{pmatrix} \tag{SYS}$$

or

$$\vec{y}\,'(x) = A\, \vec{y}(x) + \vec{f}(x)$$

$$\text{with} \quad A = \begin{pmatrix} 0 & 1 \\ -a_0(x) & -a_1(x) \end{pmatrix} \quad \text{and} \quad \vec{f}(x) = \begin{pmatrix} 0 \\ g(x) \end{pmatrix}.$$

The relation between the second-order equation (DEq) and the first-order system (SYS) is: $y_1(x)$, $y_2(x)$ is a fundamental set of (DEq) if and only if $\vec{y}_1(x) = \begin{pmatrix} y_1(x) \\ y_1'(x) \end{pmatrix}$, $\vec{y}_2(x) = \begin{pmatrix} y_2(x) \\ y_2'(x) \end{pmatrix}$ is a fundamental set of the system (SYS). With the fundamental matrix (see Section 15.3)

$$F(x) = (\vec{y}_1(x), \vec{y}_2(x)) = \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix}$$

we write the general solution to the homogeneous DEq as

$$\vec{y}_h(x) = c_1 \cdot \vec{y}_1(x) + c_2 \cdot \vec{y}_2(x) = \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = F(x) \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

To get a particular solution to the inhomogeneous problem, we vary the coefficients

$$\vec{y}_p(x) = c_1(x) \cdot \vec{y}_1(x) + c_2(x) \cdot \vec{y}_2(x)$$

$$= \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} \cdot \begin{pmatrix} c_1(x) \\ c_2(x) \end{pmatrix} = F(x) \begin{pmatrix} c_1(x) \\ c_2(x) \end{pmatrix}.$$

According to the calculation on page 34 we identified a condition for $c_1(x)$ and $c_2(x)$ to get a solution of the inhomogeneous problem:

$$\begin{pmatrix} c_1'(x) \\ c_2'(x) \end{pmatrix} = F^{-1}(x) \begin{pmatrix} 0 \\ g(x) \end{pmatrix},$$

where $F^{-1}(x)$ is the inverse of the fundamental matrix

$$F^{-1}(x) = \frac{1}{y_1(x)y_2'(x) - y_1'(x)y_2(x)} \cdot \begin{pmatrix} y_2\prime(x) & -y_2(x) \\ -y_1'(x) & y_1(x) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ g(x) \end{pmatrix}.$$

In components, we get

$$c_1'(x) = \frac{-y_2(x)g(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)}$$

$$c_2'(x) = \frac{-y_1(x)g(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)}.$$

After integration we find $c_1(x) = \int c_1'(x)\, dx$ and $c_2(x) = \int c_2'(x)\, dx$ and a particular solution to (SYS)

$$\vec{y}_p(x) = F(x) \cdot \begin{pmatrix} c_1(x) \\ c_2(x) \end{pmatrix} = \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} \cdot \begin{pmatrix} c_1(x) \\ c_2(x) \end{pmatrix}$$

Taking the first component, we get a particular solution for the second-order inhomogeneous DEq: $y_p(x) = c_1(x) \cdot y_1(x) + c_2(x) \cdot y_2(x)$.

Generalization: Inhomogeneous n-th Order DEq

To calculate a particular solution of the n-th order DEq

$$y^{(n)}(x) + a_{n-1}(x)\, y^{(n-1)}(x) + \ldots + a_1(x)\, y'(x) + a_0(x)\, y(x) = g(x)$$

we first determine a fundamental set $y_1(x), y_2(x), \ldots, y_n(x)$. With the fundamental set we build the fundamental matrix

$$F(x) = \begin{pmatrix} y_1(x) & y_2(x) & \ldots & y_n(x) \\ y_1'(x) & y_2'(x) & \ldots & y_n'(x) \\ \vdots & \vdots & \ldots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \ldots & y_n^{(n-1)}(x) \end{pmatrix}.$$

The solution to the inhomogeneous DEq is

$$y(x) = c_1(x)\, y_1(x) + c_2(x)\, y_2(x) + \ldots + c_n(x)\, y_n(x) \qquad (*)$$

where the coefficients are defined by

$$\begin{pmatrix} c_1'(x) \\ c_2'(x) \\ \vdots \\ c_n'(x) \end{pmatrix} = F^{-1}(x) \begin{pmatrix} 0 \\ 0 \\ \vdots \\ g(x) \end{pmatrix}.$$

Integrating each component of the vector $\vec{c}\,'(x)$ gives the individual coefficients of $(*)$.

Special Case: Second-Order Inhomogeneous DEq

To calculate a particular solution to the DEq

$$y''(x) + a_1(x)\, y'(x) + a_0(x)\, y(x) = g(x)$$

we first determine a fundamental set $y_1(x), y_2(x)$. We obtain a solution to the inhomogeneous DEq with

$$y_p(x) = c_1(x) \cdot y_1(x) + c_2(x) \cdot y_2(x)$$

where

$$c_1(x) = c_1 + \int \frac{-y_2(x) g(x)}{y_1(x) y_2'(x) - y_1'(x) y_2(x)}\, dx$$

$$c_2(x) = c_2 + \int \frac{y_1(x) g(x)}{y_1(x) y_2'(x) - y_1'(x) y_2(x)}\, dx.$$

Example 16.25 (Sample Example). We look for a solution to the second-order inhomogeneous DEq with non-constant coefficients

$$x\, y''(x) - (1+x)\, y'(x) + y(x) = 3\, x^2 \qquad \text{for} \quad x > 0.$$

⚠ **Caution:** Before we continue, we first rewrite our problem in the standard form, so that the coefficient of the highest derivative is 1:

$$y''(x) - \frac{(1+x)}{x}\, y'(x) + \frac{1}{x} y(x) = 3\, x \qquad \text{for} \quad x > 0.$$

This step is not relevant for the homogeneous problem but for the inhomogeneous problem to identify the correct right-hand side $g(x) =$

$3\,x$. To determine a particular solution, we need a fundamental set of the homogeneous DEq. According to Example 16.24

$$y_1(x) = e^x \qquad \text{and} \qquad y_2(x) = 1 + x$$

is a fundamental set, so

$$F(x) = \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} = \begin{pmatrix} e^x & 1+x \\ e^x & 1 \end{pmatrix}$$

is the fundamental matrix with its inverse

$$\begin{aligned} F^{-1}(x) &= \frac{1}{y_1(x)y_2'(x) - y_1'(x)y_2(x)} \begin{pmatrix} y_2'(x) & -y_2(x) \\ -y_1'(x) & y_1(x) \end{pmatrix} \\ &= \frac{1}{e^x - (1+x)e^x} \begin{pmatrix} 1 & -(1+x) \\ -e^x & e^x \end{pmatrix}. \end{aligned}$$

We compute $F^{-1}(x) \cdot \vec{f}(x)$, which is

$$\begin{aligned} \begin{pmatrix} c_1'(c) \\ c_2'(x) \end{pmatrix} &= \frac{1}{-x\,e^x} \begin{pmatrix} 1 & -(1+x) \\ -e^x & e^x \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 3\,x \end{pmatrix} \\ &= \frac{1}{-x\,e^x} \begin{pmatrix} -(1+x)\,3x \\ 3x\,e^x \end{pmatrix} = \begin{pmatrix} (1+x)\,3\,e^{-x} \\ -3 \end{pmatrix}. \end{aligned}$$

So

$$c_1'(x) = (3+3x)\,e^{-x}$$

$$c_2'(x) = -3$$

and with integration by parts we get

$$c_1(x) = \int (3+3x)\,e^{-x}\,dx = c_1 + (-6-3x)\,e^{-x}$$

and

$$c_2(x) = c_2 - 3\,x.$$

Finally we calculate

$$\begin{aligned} y(x) &= c_1(x) \cdot y_1(x) + c_2(x) \cdot y_2(x) \\ &= (c_1 + (-6-3x)\,e^{-x}) \cdot e^x + (c_2 - 3\,x) \cdot (1+x) \\ &= c_1\,e^x + c_2\,(1+x) + (-6(1+x) - 3\,x^2) \\ &= c_1\,e^x + \tilde{c}_2\,(1+x) - 3\,x^2 \end{aligned}$$

□

16.5 Problems on n-th Order Differential Equations

16.1 Check that $\varphi_1 = 1 - \cos(2\,x)$ and $\varphi_2 = 1 - \cos^2(x)$ are solutions of

$$y'' - (\tan x + \cot x)\; y' + 4\,y = 0.$$

Do they form a fundamental set?

16.2 Show that the two functions $\sinh(k\,x)$ and $\cosh(k\,x)$ form a real-valued fundamental set of the differential equation $y''(x) - k^2\,y(x) = 0$.

16.3 Solve the following 2nd order homogeneous linear differential equations
a) $\ddot{u}(t) + 13\,\dot{u}(t) + 40\,u(t) = 0$ b) $\ddot{v}(t) - 12\,\dot{v}(t) + 36\,v(t) = 0$
c) $y''(x) + 6\,y'(x) + 34\,y(x) = 0$ d) $z''(x) + 16\,z(x) = 0$

16.4 Determine a real fundamental set for
a) $y^{(4)}(x) - 10\,y''(x) + 9\,y(x) = 0$
b) $u^{(3)}(t) - 2\,\ddot{u}(t) + \dot{u}(t) = 0$
c) $y^{(6)}(x) - y(x) = 0$

16.5 Given is the inhomogeneous, linear 2nd order differential equation

$$y''(x) - 3\,y'(x) + 2\,y(x) = s(x)$$

with inhomogeneity $s(x)$. Find particular solutions for
a) $s(x) = 6$ b) $s(x) = x$ c) $s(x) = e^{2\,x}$ d) $s(x) = \cos x$
e) $s(x) = 4\,x + 10\,\cos x$ f) $s(x) = x\,e^{2\,x}$ g) $s(x) = \cos x\,e^{x}$

16.6 Solve the vibration problems
a) $\ddot{x}(t) + 16\,x(t) = 0\,,\quad x(0) = 3\,,\quad \dot{x}(0) = 4$
b) $\ddot{x}(t) + 2\,\dot{x}(x) + 2\,x(t) = 0\,,\quad x(0) = 2\,,\quad \dot{x}(0) = 0$
c) $\ddot{x}(t) + 13\,\dot{x}(t) + 40\,x(t) = 0\,,\quad x(0) = 3\,,\quad \dot{x}(0) = 0$

16.7 Determine all real solutions of the following DEq
a) $y^{(4)}(x) - 10\,y''(x) + 9\,y(x) = \sin(x)$
b) $y'''(x) - 7\,y'(x) - 6\,y(x) = 12\,e^{x}$
c) $y'''(x) - 2\,y''(x) + y'(x) - 2\,y(x) = \cos(x)$
d) $y'''(x) - 6\,y''(x) + 12\,y'(x) - 8\,y(x) = 6\,e^{2\,x}$

16.8 Solve the differential equations from problem 16.6 numerically using the Euler method. Vary the step size and compare the numerical result with the exact solution.

16.9 Transform the third-order DEq

$$y'''(x) - 2\,y''(x) - 4\,y'(x) + 8 = 0$$

into a first-order system.

16.10 Transform the coupled second-order DEq

$$\begin{aligned} x''(t) - y'(t) + x(t) - y(t) &= 0 \\ y''(t) + x'(t) - 3y(t) + x(t) &= 0 \end{aligned}$$

into a first-order system.

16.11 Determine a fundamental set of

$$(x^3 - 2x^2)\, y''(x) - 2x\, y'(x) + 2\, y(x) = 0$$

using the d'Alembert reduction method. Check that x is a solution.

16.12 Solve the second-order DEq with non-constant coefficients

$$x^2\, y''(x) - 2x\, y'(x) + 2\, y(x) = x^3\, \sin(x)\, :$$

a) Find a solution to the homogeneous DEq in the form $y(x) = x^\alpha$.
b) Use the d'Alembert reduction method to find a second solution.
c) Show that $-x\, \sin(x)$ is a particular solution.
d) What is the general solution to the inhomogeneous DEq.

16.13 Given is the DEq

$$(x+1)\, y''(x) + x\, y'(x) - y(x) = (x+1)^2$$

a) Find a solution to the homogeneous DEq in form $y(x) = e^{\alpha x}$.
b) Use the d'Alembert reduction to find a fundamental set.
c) Calculate a particular solution and the general solution.

16.14 The solution of a third-order DEq is given by

$$y(x) = c_1\, e^{-x} + c_2 + c_3\, x + x^2.$$

What is the associated DEq?

16.15 Is

$$y_1(x) = x^2,\; y_2(x) = e^x,\; y_3(x) = e^{-x}$$

a fundamental set to a DEq with constant coefficients? Can it be a fundamental set to a DEq with non-constant coefficients? Find a DEq if it is possible.

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17

Chapter 17
Fourier Series

To analyze *periodic* signals, we need to represent the signal in the form of a Fourier series

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\omega_n t) + \sum_{n=1}^{\infty} b_n \sin(\omega_n t).$$

Such a decomposition of the signal into its harmonic components reveals which frequencies with which amplitudes are contained in the signal.

After an introduction, the formulas for the Fourier series and the Fourier coefficients of 2π-periodic functions are presented in Section 17.2 and applied to examples in Section 17.3. The formulas are transferred to p-periodic functions in Section 17.4 and the complex formulation is discussed in Section 17.5. This complex formulation prepares the transition to the Fourier transform which analyzes *non-periodic* signals.

7

17 Fourier Series

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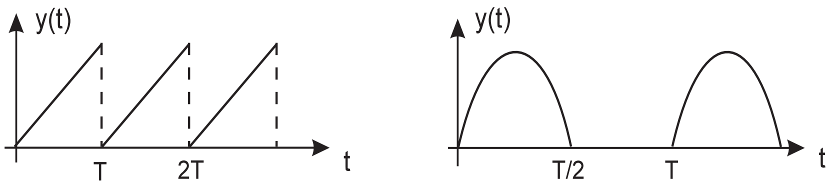
17.1 Introduction

In physics, periodic processes such as the oscillation of a spring pendulum or alternating voltages can be described by the general sinusoidal function

$$y(t) = A \sin(\omega t + \varphi) .$$

This behavior is called a *harmonic oscillation* with frequency ω and amplitude A. Harmonic oscillations also occur in the description of vibrating strings, membranes, pendulums, electromagnetic oscillations, sound and wave propagation, etc.

However, processes that are periodic but no longer sinusoidal are also common. Examples are sawtooth oscillations (sawtooth voltage, chalk squeak) or the sinusoidal pulse of a rectifier, see Fig. 17.1.



Sawtooth oscillation

Sine pulse of a rectifier

Figure 17.1. Periodic but non-harmonic oscillations

For example, when considering the sawtooth oscillation that causes chalk to squeak, the dominant frequencies and associated amplitudes are of interest. If the three notes c^1, g^1, e^2 are played simultaneously on a piano, and the strength of the strokes is chosen so that the pressure generated at the ear is equal to 1.273, 0.424 and 0.255 in normalized units, then the total pressure $p\,(t)$ at the ear is given by their superposition

$$p\,(t) = 1.273\,\sin\,(2\,\pi\,\nu_1\,t) + 0.424\,\sin\,(2\,\pi\,\nu_3\,t) + 0.255\,\sin\,(2\,\pi\,\nu_5\,t)$$

with $\nu_1 = 128\,Hz$ (this is c^1), $\nu_3 = 3\,\nu_1 = 384\,Hz$ (this is g^1) and $\nu_5 = 5\,\nu_1 = 640\,Hz$ (this is e^2), see Fig. 17.2.

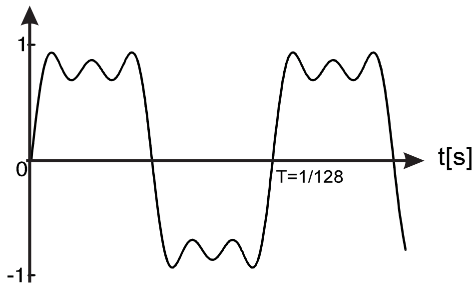


Figure 17.2. Acoustic pressure at the ear

Of general interest in signal analysis is the decomposition of a periodic time signal into its fundamental and harmonics with their associated amplitudes. It turns out that almost any **periodic** function $y\,(t)$ can be represented as a superposition of an infinite number of harmonic oscillations.

The mathematical relationship between a **periodic** signal and its decomposition into fundamental and harmonic oscillations with associated amplitudes is described by the **Fourier series**

$$y\,(t) = a_0 + \sum_{n=1}^{\infty} a_n\,\cos\,(n\,\omega_0 t) + \sum_{n=1}^{\infty} b_n\,\sin\,(n\,\omega_0 t)\ ,$$

where T is the period of the function $y\,(t)$ and $\omega_0 = \frac{2\,\pi}{T}$.

The decomposition of a periodic function into a Fourier series is called **Fourier Analysis**. ω_0 is the fundamental and $n\,\omega_0$ are the harmonics. The coefficients $a_0,\,a_1,\,a_2,\ldots\,;\,b_1,\,b_2,\ldots$ are the Fourier coefficients and represent the amplitudes of the individual frequency components.

17.2 Computation of the Fourier Coefficients

To compute the amplitudes of the Fourier decomposition, we start with a 2π-*periodic* function f (see Fig. 17.3)

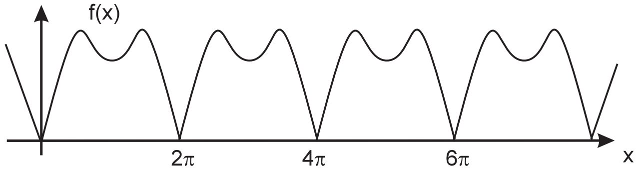


Figure 17.3. 2π-periodic function

and select the approach:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\,x) + \sum_{n=1}^{\infty} b_n \sin(n\,x). \qquad (*)$$

For the formal computation of the coefficients a_0; a_1, $a_2, \ldots$; b_1, $b_2, \ldots$ we need the definite integrals compiled in Table 17.1 where we integrate the sine and cosine functions over a period.

Table 17.1: Summary of elementary sine and cosine integrals

(1)	$\int_0^{2\pi} \sin(n\,x)\,dx = 0$	for $n = 1, 2, 3, \ldots$	
(2)	$\int_0^{2\pi} \cos(n\,x)\,dx = 0$	for $n = 1, 2, 3, \ldots$	
(3)	$\int_0^{2\pi} \cos(n\,x)\,\cos(m\,x)\,dx = \begin{cases} 0 & \text{for } m \neq n \\ \pi & \text{for } m = n \end{cases}$		$= \pi\,\delta(n-m)$
(4)	$\int_0^{2\pi} \sin(n\,x)\,\sin(m\,x)\,dx = \begin{cases} 0 & \text{for } m \neq n \\ \pi & \text{for } m = n \end{cases}$		$= \pi\,\delta(n-m)$
(5)	$\int_0^{2\pi} \sin(n\,x)\,\cos(m\,x)\,dx = 0$	for $n, m = 1, 2, 3, \ldots$	

In Table 17.1 we introduce the *Kronecker symbol* $\delta(k)$, which is defined for all integers $k \in \mathbb{Z}$ by

$$\delta(k) := \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

Note: The integrals (1) and (2) are calculated directly. With the formula $\cos\alpha \, \cos\beta = \frac{1}{2}\,(\cos(\alpha-\beta) + \cos(\alpha+\beta))$ we conclude (3) for $m \neq n$

$$\int_0^{2\pi} \cos(n\,x)\,\cos(m\,x)\,dx = \frac{1}{2}\,(\int_0^{2\pi} \cos((n-m)\,x)\,dx + \int_0^{2\pi} \cos((n+m)\,x)\,dx) = 0.$$

For $n = m$ it is

$$\int_0^{2\pi} \cos^2(n\,x)\,dx = \pi.$$

Formula (4) is calculated analogously to (3) using the relationship

$$\sin\alpha\,\sin\beta = \frac{1}{2}\,(\cos(\alpha-\beta) - \cos(\alpha+\beta)).$$

The formula used to calculate (5) is

$$\sin\alpha\,\cos\beta = \frac{1}{2}\,(\sin(\alpha-\beta) + \sin(\alpha+\beta)).$$

□

Calculating a_0:

We integrate the expression (∗) over the period $[0, 2\pi]$:

$$\int_0^{2\pi} f(x)\,dx = \underbrace{\int_0^{2\pi} a_0\,dx}_{a_0\cdot 2\pi} + \sum_{n=1}^{\infty} a_n \underbrace{\int_0^{2\pi} \cos(n\,x)\,dx}_{=0} + \sum_{n=1}^{\infty} b_n \underbrace{\int_0^{2\pi} \sin(n\,x)\,dx}_{=0}$$

According to Table 17.1, the integrals $\int_0^{2\pi} \cos(n\,x)\,dx$ and $\int_0^{2\pi} \sin(n\,x)\,dx$ are zero. In the above representation, only the first integral $\int_0^{2\pi} a_0\,dx$ appears with a non-zero value $a_0 \cdot 2\pi$, resulting in

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx.$$

Calculating a_n:

We first multiply the expression (∗) by $\cos(m\,x)$ for any integer $m > 0$ and then integrate over the period $[0,\, 2\pi]$:

$$\begin{aligned}\int_0^{2\pi} f(x)\cos(m\,x)\,dx = \quad & a_0 \int_0^{2\pi} \cos(m\,x)\,dx \\ & + \sum_{n=1}^{\infty} a_n \int_0^{2\pi} \cos(n\,x)\cos(m\,x)\,dx \\ & + \sum_{n=1}^{\infty} b_n \int_0^{2\pi} \sin(n\,x)\cos(m\,x)\,dx.\end{aligned}$$

According to Table 17.1 (5), all the terms of the second sum disappear. From the first sum over a_n, only the term where the index n is equal to m is non-zero. Since also $\int_0^{2\pi} \cos(m\,x)\,dx = 0$, we finally get

$$\int_0^{2\pi} f(x)\cos(m\,x)\,dx = \sum_{n=1}^{\infty} a_n\,\pi\,\delta(n-m) = \pi \cdot a_m.$$

$$\Rightarrow \qquad a_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(m\,x)\,dx \qquad m = 1,\, 2,\, 3, \ldots$$

Calculating b_n:

Analogous to the calculation of the coefficients a_n, we first multiply (∗) by $\sin(m\,x)$ with $m > 0$ and then integrate over the period $[0,\, 2\pi]$:

$$\begin{aligned}\int_0^{2\pi} f(x)\sin(m\,x)\,dx = \quad & a_0 \int_0^{2\pi} \sin(m\,x)\,dx \\ & + \sum_{n=1}^{\infty} a_n \int_0^{2\pi} \cos(n\,x)\sin(m\,x)\,dx \\ & + \sum_{n=1}^{\infty} b_n \int_0^{2\pi} \sin(n\,x)\sin(m\,x)\,dx.\end{aligned}$$

All integrals containing $\cos(n\,x)$ disappear according to Table 17.1, also $\int_0^{2\pi} \sin(m\,x)\;dx$. The integrals $\int_0^{2\pi} \sin(n\,x)\;\sin(m\,x)\;dx = \pi\,\delta\,(n-m)$ are all zero except the one where $n = m$. So

$$\int_0^{2\pi} f(x)\;\sin(m\,x)\;dx = \sum_{n=1}^{\infty} b_n\,\pi\,\delta\,(n-m) = \pi \cdot b_m .$$

$$\Rightarrow \qquad \boxed{b_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(m\,x)\,dx} \qquad m = 1,\,2,\,3,\ldots$$

17.3 Fourier Series for 2π-periodic Functions

Following these preliminary considerations, we obtain expressions for the Fourier coefficients of a 2π-periodic function f. The Fourier series converges for most functions and is identical to $f(x)$. However, there are continuous 2π-periodic functions whose Fourier series diverges at an infinite number of points. To ensure that the Fourier series of a 2π-periodic function f converges everywhere and that the limit is $f(x)$, the function f must satisfy certain conditions.

In the following, two conditions are given without proof which together guarantee the convergence and the coincidence of the Fourier series with the function. Both conditions are easy to check and are almost always satisfied in applications.

Condition 1: The period interval $[0,\,2\pi]$ can be divided by a finite number of points $0 = x_1 < x_2 < \ldots < x_N = 2\pi$ so that in any open subinterval $(x_k,\,x_{k+1})\,, 1 \leq k \leq N-1$, the function f is differentiable and f' is bounded. Such functions are called **piecewise continuous differentiable functions**.

Condition 2: At the intermediate points x_k the left and right limits exist

$$f_l(x_k) = \lim_{\varepsilon \to 0} f(x_k - \varepsilon) \qquad \text{and} \qquad f_r(x_k) = \lim_{\varepsilon \to 0} f(x_k + \varepsilon)$$

and the function value is

$$f(x_k) = \tfrac{1}{2}\,\left(f_l(x_k) + f_r(x_k)\right)\,.$$

This property is called the **mean value property**.

Fig. 17.4 shows the graph of a function that satisfies conditions (1) **and** (2).

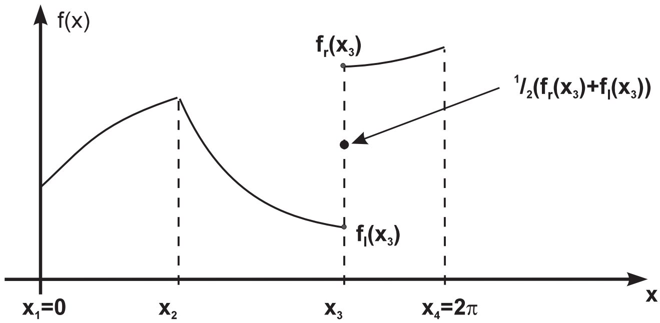


Figure 17.4. Piecewise continuously differentiable function

Piecewise continuously differentiable functions can have many jumps. At continuity points the mean value property is always satisfied, at discontinuity points the function value is the mean of the left limit and the right limit.

Fourier Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function, piecewise continuously differentiable and satisfying the mean value property for all $x \in \mathbb{R}$. Then the **Fourier series**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\,x) + \sum_{n=1}^{\infty} b_n \sin(n\,x)$$

converges for all $x \in \mathbb{R}$ and it is identical to the function f. The **Fourier coefficients** are

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(n\,x)\, dx \qquad n = 1, 2, 3, \ldots$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(n\,x)\, dx \qquad n = 1, 2, 3, \ldots$$

Remark: For a 2π-periodic function $f(x)$, always

$$\int_0^{2\pi} f(x)\,dx = \int_\alpha^{\alpha+2\pi} f(x)\,dx \quad \text{for any} \quad \alpha \in \mathbb{R}.$$

This formula says that any period interval of length 2π can be chosen to calculate the Fourier coefficients. This feature simplifies the calculation of Fourier coefficients for symmetric functions:

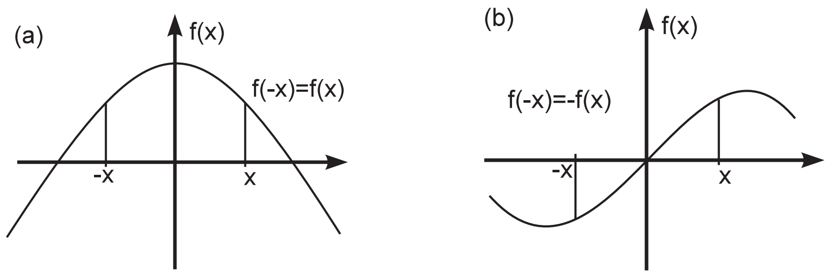


Figure 17.5. Even (a) and odd (b) functions

Symmetry considerations:

(1) For an **even** 2π-periodic function f (i.e. f is axially symmetric with respect to the y-axis, i.e. $f(-x) = f(x)$ for all x), all Fourier coefficients b_k are zero.

$$b_k = 0 \quad \text{for all} \quad k \in \mathbb{N}.$$

Since $f(x)$ is even, $f(x) \cdot \cos(n\,x)$ is an even function and $f(x) \cdot \sin(n\,x)$ is an odd function. We choose the integration interval $[-\pi, \pi]$ so that for the odd function $f(x) \cdot \sin(n\,x)$ (see Fig. 17.5 (b)) the result is

$$b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,\sin(n\,x)\,dx = 0 \qquad n \in \mathbb{N}.$$

For the coefficients a_n we get

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x)\,dx, \quad a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(n\,x)\,dx, \qquad n \in \mathbb{N}.$$

(2) For an **odd** 2π-periodic function f (i.e. f is point symmetric about the origin, i.e. $f(-x) = -f(x)$ for all x), all Fourier coefficients a_k are zero.

$$a_k = 0 \quad \text{for all} \quad k \in \mathbb{N}_0.$$

Since $f(x)$ is odd, $f(x) \cdot \cos(n\,x)$ is an odd function, while $f(x) \cdot \sin(n\,x)$ is even as the product of two odd functions. We choose the integration interval $[-\pi, \pi]$. Taking into account the symmetry, we get (see Fig. 17.5 (a))

$$a_0 = 0 \quad \text{and} \quad a_n = 0, \qquad n \in \mathbb{N}.$$

For the coefficients b_n it is

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(n\,x)\,dx, \qquad n \in \mathbb{N}.$$

□

Hint: Identifying symmetries can greatly reduce the amount of work required to find the Fourier coefficients!

Example 17.1 (With MAPLE-Worksheet). Given is the 2π-periodic function shown in Fig. 17.6. This function is described in the period interval $[0, 2\pi]$ by the expression

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ 0 & x = 0, \pi, 2\pi \\ -1 & \pi < x < 2\pi \end{cases}.$$

The Fourier coefficients and series of the function are searched for.

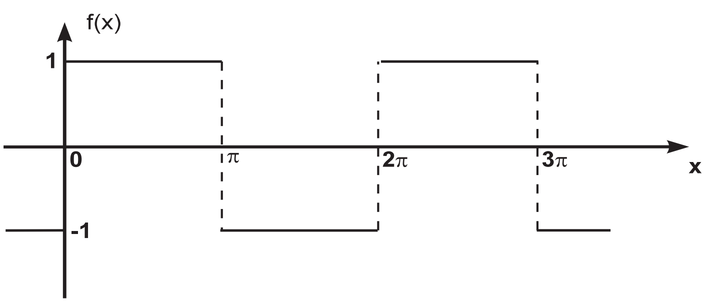


Figure 17.6. 2π-periodic rectangle function

f is piecewise continuously differentiable and fulfills the mean value property at all points. Due to the point symmetry with respect to the origin we have

$$a_n = 0 \qquad \text{for} \;\; n = 0,\, 1,\, 2, \ldots \quad .$$

This leaves only the Fourier coefficients b_n to be calculated. Due to the symmetry consideration (2), the specific integral only needs to be calculated in the range $0, ..., \pi$ when calculating the coefficients b_n:

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(n\,x)\; dx = 2 \cdot \frac{1}{\pi} \int_0^{\pi} f(x) \sin(n\,x)\; dx \\ &= \frac{2}{\pi} \int_0^{\pi} \sin(n\,x)\; dx = \frac{2}{\pi} \left[-\frac{1}{n} \cos(n\,x) \right]_0^{\pi} \\ &= \frac{2}{\pi\,n} \left\{ -\cos(n\,\pi) + \cos(0) \right\} = \frac{2}{\pi\,n} \left\{ -(-1)^n + 1 \right\}, \end{aligned}$$

since $\cos(n\,\pi) = (-1)^n$ and $\cos(0) = 1$. Note that $(-1)^n = +1$ for even n and $(-1)^n = -1$ for odd n, we end up with

$$\Rightarrow \quad b_n = \begin{cases} 0 & \text{for } n = 0,\, 2,\, 4, \ldots \\ \dfrac{4}{\pi\,n} & \text{for } n = 1,\, 3,\, 5, \ldots \quad . \end{cases}$$

The Fourier series of the function f is

$$\begin{aligned} f(x) &= \frac{4}{\pi} \left(\sin(x) + \frac{1}{3} \sin(3\,x) + \frac{1}{5} \sin(5\,x) + \frac{1}{7} \sin(7\,x) + \ldots \right) \\ &= \sum_{\substack{n=1 \\ n\,\text{odd}}}^{\infty} \frac{4}{n\,\pi} \sin(n\,x) = \sum_{n=0}^{\infty} \frac{4}{(2\,n+1)\;\pi} \sin((2\,n+1)\;x)\,. \end{aligned}$$

In Fig. 17.7 (a) the partial sums of this series are shown for $n = 3,\, 5,\, 7$ and in Fig. 17.7 (b) for $n = 40$.

Discussion: We can see that many terms of the partial sum are needed to approximate the function f reasonably well. However, even for large N there are still oscillations before the jump. **The coefficients of the Fourier series b_n are proportional to $\frac{1}{n}$.** □

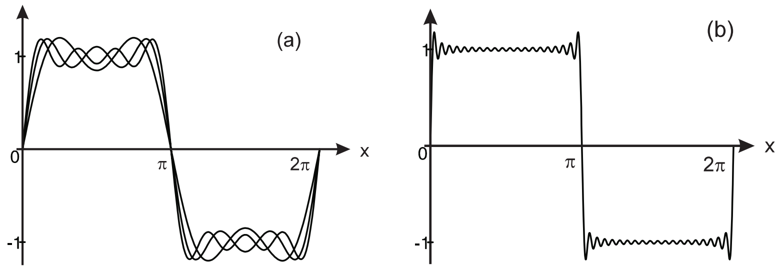


Figure 17.7. Partial sums of the Fourier series (a) for $n = 3, 5, 7$ and (b) for $n = 40$

Animation: On the homepage there is an animation that visualizes the point-by-point convergence of the Fourier series to the function. The Fourier coefficients calculated from the example are used to display the Fourier series.

Example 17.2 (With MAPLE-Worksheet). Find the Fourier series of the triangle function in Fig. 17.8, which is described in the interval $[0,\ 2\pi]$ by

$$f(x) = \begin{cases} x & \text{for } 0 \le x < \pi \\ 0 & \text{for } x = \pi,\ 2\pi \\ x - 2\pi & \text{for } \pi < x < 2\pi\ . \end{cases}$$

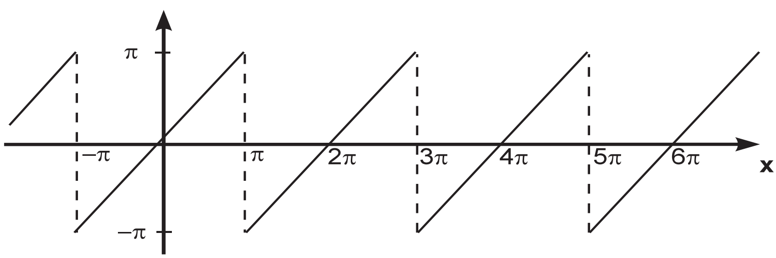


Figure 17.8. Periodic triangle function

f is piecewise continuously differentiable and fulfills the mean value property at all points. Because of the point symmetry with respect to the origin

$$a_n = 0 \qquad \text{for } n = 0,\ 1,\ 2, \ldots\quad .$$

According to the point symmetry, the Fourier coefficients b_n are

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x)\, \sin(n\,x)\; dx$$

$$= 2 \cdot \frac{1}{\pi} \int_0^{\pi} f(x) \sin(n\,x)\, dx = \frac{2}{\pi} \int_0^{\pi} x \cdot \sin(n\,x)\, dx.$$

Partial integration gives

$$b_n = \frac{2}{\pi} \left\{ \left[x \frac{-\cos(n\,x)}{n} \right]_0^{\pi} - \int_0^{\pi} \frac{-\cos(n\,x)}{n}\, dx \right\}$$

$$= \frac{2}{\pi\,n} \left[-\pi \cos(n\,\pi) - 0 \right] = -\frac{2}{n} (-1)^n = \frac{2}{n} (-1)^{n+1},$$

because $\cos(n\,\pi) = (-1)^n$. So the Fourier series of f is

$$f(x) = 2 \left(\sin(x) - \frac{1}{2} \sin(2\,x) + \frac{1}{3} \sin(3\,x) - \frac{1}{4} \sin(4\,x) \pm \ldots \right)$$

$$= 2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \sin(n\,x).$$

As in Example 17.1, the Fourier coefficients also behave as $\sim \frac{1}{n}$. □

Example 17.3 (With MAPLE-Worksheet). Given is the function

$$f(x) = \frac{1}{\pi} (x - \pi)^2$$

in the interval $0 \le x \le 2\pi$. The function is 2π-periodically extended to $\mathbb{R}$.

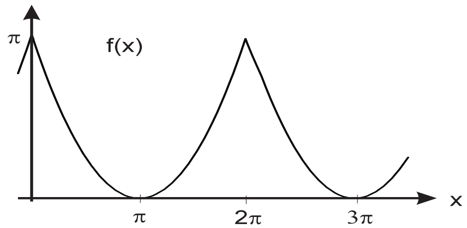


Figure 17.9.

f is piecewise continuously differentiable and fulfills the mean value property. Due to the axis symmetry with respect to the y-axis, the coefficients b_n are zero

$$b_n = 0 \qquad \text{for} \quad n = 1, 2, 3, \ldots .$$

The coefficient a_0 is

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\pi} (x - \pi)^2\, dx = \frac{1}{2\pi^2} \left[\frac{1}{3} (x - \pi)^3 \right]_0^{2\pi} = \frac{1}{3} \pi.$$

The coefficients a_n ($n \in \mathbb{N}$) are calculated taking into account the axis symmetry

$$a_n = \frac{2}{\pi} \int_0^{\pi} \frac{1}{\pi} (x-\pi)^2 \cos(n\,x) dx.$$

Performing a partial integration twice gives the result

$$\begin{aligned} a_n &= \frac{2}{\pi^2} \{ \left[(x-\pi)^2 \frac{1}{n} \sin(n\,x) \right]_0^{\pi} - 2 \int_0^{\pi} (x-\pi) \frac{1}{n} \sin(n\,x)\, dx \} \\ &= -\frac{4}{\pi^2\, n} \int_0^{\pi} (x-\pi) \sin(n\,x) dx \\ &= -\frac{4}{\pi^2\, n} \{ \left[(x-\pi) \frac{-1}{n} \cos(n\,x) \right]_0^{\pi} - \underbrace{\int_0^{\pi} \frac{-1}{n} \cos(n\,x)\, dx \}}_{=0} \\ &= \frac{4}{\pi n^2}. \end{aligned}$$

So

$$f(x) = \frac{\pi}{3} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(n\,x)$$

is the Fourier series. The graph of the function f together with the first terms up to $n = 5$ are shown in Fig. 17.10.

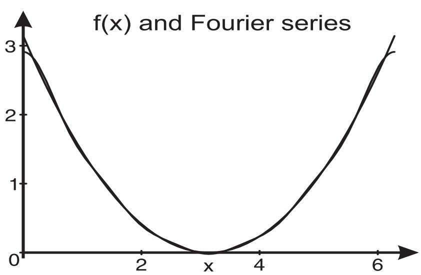


Figure 17.10. The function $f(x) = \frac{1}{\pi}(x-\pi)^2$ and partial sum up to $n = 5$

Discussion: Unlike Examples 17.1 and 17.2, four summation terms are sufficient to approximate the function reasonably well by the Fourier series. No oscillations build up in the period interval. **The coefficients of the Fourier series are proportional to $\frac{1}{n^2}$.** □

Note: Using Fourier series, it is sometimes possible to calculate the value of the series discussed in Volume 2, Section 9.2 on Number Series by inserting special values into the Fourier series. Using the Fourier series of Example 17.3 gives the following two results:

$$x = 0: \; f(0) = \pi = \frac{\pi}{3} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

$$x = \pi: \; f(\pi) = 0 = \frac{\pi}{3} + \frac{4}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2} \quad \Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2} = -\frac{\pi^2}{12}.$$

17.4 Fourier Series for p-periodic Functions

So far we have only considered 2π-periodic functions. To obtain the Fourier coefficients of a general *p-periodic* function f, we compress or stretch the function f so that the modified function F is 2π-periodic.

$$F(x) := f\left(\frac{p}{2\pi} x\right).$$

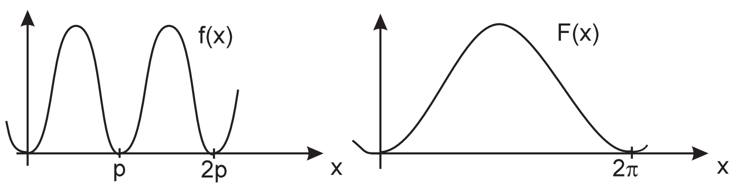


Figure 17.11. p-periodic function f and corresponding 2π-periodic function F

F is 2π-periodic:

$$F(x + 2\pi) = f\left(\frac{p}{2\pi} (x + 2\pi)\right) = f\left(\frac{p}{2\pi} x + p\right) = f\left(\frac{p}{2\pi} x\right) = F(x).$$

For the 2π-periodic function $F(x)$, the Fourier series is set up as follows

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n x) + \sum_{n=1}^{\infty} b_n \sin(n x)$$

with the Fourier coefficients

$$a_0 = \tfrac{1}{2\pi} \int_0^{2\pi} F(x)\, dx; \; a_n = \tfrac{1}{\pi} \int_0^{2\pi} F(x) \cos(n x)\, dx;$$

$$b_n = \tfrac{1}{\pi} \int_0^{2\pi} F(x) \sin(n x)\, dx.$$

A back substitution

$$f(x) = F\left(\frac{2\pi}{p} x\right)$$

constructs the Fourier series of f with the Fourier series of F.

$$f(x) = F(\frac{2\pi}{p}x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\frac{2\pi}{p}x) + \sum_{n=1}^{\infty} b_n \sin(n\frac{2\pi}{p}x).$$

To find the Fourier coefficients, we replace $y = \frac{p}{2\pi}x$ and get:

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} F(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} f(\tfrac{p}{2\pi}x)\,dx = \frac{1}{p}\int_0^{p} f(y)\,dy,$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} F(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{2\pi} f(\tfrac{p}{2\pi}x)\cos(nx)\,dx \\ &= \frac{2}{p}\int_0^{p} f(y)\cos(n\tfrac{2\pi}{p}y)\,dy, \end{aligned}$$

and similarly b_n. In summary:

Fourier Series for p-Periodic Functions

Let $f : \mathbb{R} \to \mathbb{R}$ be a p-periodic function, that is piecewise continuously differentiable and satisfies the mean value property for all $x \in \mathbb{R}$. Then the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{2\pi}{p}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(n\frac{2\pi}{p}x\right)$$

converges for all $x \in \mathbb{R}$ and is identical to the function f. The coefficients are

$$a_0 = \frac{1}{p}\int_0^{p} f(x)\,dx$$

$$a_n = \frac{2}{p}\int_0^{p} f(x)\cos\left(n\frac{2\pi}{p}x\right)dx, \quad n = 1, 2, 3, \ldots$$

$$b_n = \frac{2}{p}\int_0^{p} f(x)\sin\left(n\frac{2\pi}{p}x\right)dx, \quad n = 1, 2, 3, \ldots \quad .$$

Note: The formulas for 2π-periodic functions are the special case $p = 2\pi$. The symmetry considerations (1) and (2) also apply to p-periodic functions.

Application: Fourier Decomposition of T-Periodic Signals

Let $f(t)$ be a periodic oscillation with period T (= oscillation period). Then the Fourier decomposition applies at any time

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\,\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\,\omega_0 t) \qquad (*)$$

with the fundamental frequency $\omega_0 = \frac{2\pi}{T}$.

> Developing the time signal $f(t)$ into an infinite number of sine and cosine functions means decomposing the signal into its harmonic components.

> A periodic signal has a discrete spectrum: It consists of the fundamental at frequency ω_0 and the harmonics at frequencies $n\,\omega_0$. The Fourier coefficients are the amplitudes of these harmonics and thus the contribution of the harmonics to the signal.

However, for a frequency of $n\,\omega_0$, two coefficients are obtained, namely a_n and b_n, since the summands in the Fourier series $(*)$ represent the superposition of two sine and cosine harmonics at the same frequency. If we are looking for **the** amplitude at which the frequency $n\,\omega_0$ occurs in the signal, it is necessary to write this superposition as a cosine or sine signal. For example, we take

$$a_n \cos(n\,\omega_0 t) + b_n \sin(n\,\omega_0 t) = A_n \cos(n\,\omega_0 t - \varphi_n)$$

where A_n is then **the** amplitude and φ_n **the** phase. To find expressions for A_n and φ_n, we use the addition theorem for cosine

$$\begin{aligned} A_n \cos(n\,\omega_0 t - \varphi_n) &= A_n \cos(n\,\omega_0 t) \cos\varphi_n + A_n \sin(n\,\omega_0 t) \sin\varphi_n \\ &= a_n \cos(n\,\omega_0 t) + b_n \sin(n\,\omega_0 t) \end{aligned}$$

and compare the coefficients of $\cos(n\,\omega_0\, t)$ and $\sin(n\,\omega_0\, t)$:

$$a_n = A_n \cos\varphi_n \quad \text{and} \quad b_n = A_n \sin\varphi_n .$$

If we square and add the two equations, we obtain

$$a_n^2 + b_n^2 = A_n^2 \cos^2\varphi_n + A_n^2 \sin^2\varphi_n = A_n^2 \quad \Rightarrow \quad A_n = \sqrt{a_n^2 + b_n^2}\,.$$

Dividing the second by the first equation gives

$$\frac{b_n}{a_n} = \tan \varphi_n \,.$$

Hence, we obtain the Fourier series with the total amplitudes A_n and the phases φ_n.

Summary: T-Periodic Signal

The Fourier series of a T-periodic signal can be written as

$$f(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\,\omega_0 t - \varphi_n)$$

with the amplitude A_n and the phase φ_n specified by

$$A_n = \sqrt{a_n^2 + b_n^2} \qquad \text{and} \qquad \tan(\varphi_n) = \frac{b_n}{a_n}.$$

Alternatively, we can use the complex formulation (see Section 17.5).

A_n is the **total amplitude** at which the frequency $\omega_n = n\,\omega_0$ occurs in the signal. φ_n is the corresponding **phase**. The value a_0 is the coefficient $a_0 = \frac{1}{T}\int_0^T f(t)\,dt$. It corresponds to the **mean value** of the function during one oscillation period. It is called the *direct current component*. For the graphical representation of the coefficients, we choose the following graphs:

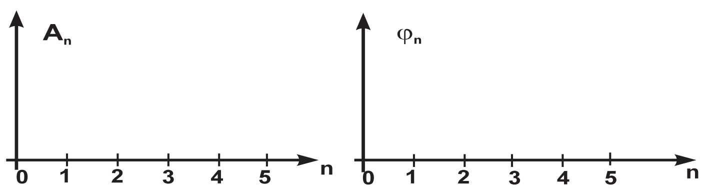


Figure 17.12. Amplitude spectrum A_n and phase spectrum φ_n

In these plots, the amplitudes A_n and the phases φ_n are plotted against the discrete frequencies (= *discrete spectrum* of f). The amplitude plot is called **Amplitude Spectrum** (left) and the phase plot is called **Phase Spectrum** (right).

Example 17.4 (Amplitude Spectrum, with MAPLE-Worksheet). The amplitude spectrum $A_n = \sqrt{a_n^2 + b_n^2}$ for Examples 17.1, 17.2 and 17.3 is shown in Fig. 17.13.

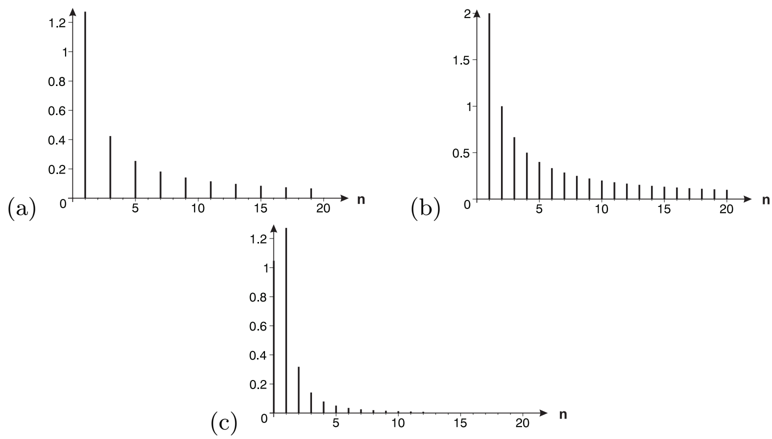


Figure 17.13. Amplitude spectrum for (a) Example 17.1, (b) 17.2 and (c) 17.3

From these plots we can see both the frequencies $n\,\omega_0$ and the corresponding amplitudes A_n. The amplitude spectrum of Examples 17.1 and 17.2 decreases slowly, while in Example 17.3 the coefficients go to zero very quickly. This reflects the fact that in the first two cases the convergence is $\sim \frac{1}{n}$, but in the third case it is $\sim \frac{1}{n^2}$. □

Convergence Considerations: Taking the Fourier series with a finite number of terms gives an approximate function for f in the form of a finite trigonometric sum. As with power series, the more terms considered, the better the approximation. For the examples discussed, this is:

$$a_n \to 0 \quad \text{and} \quad b_n \to 0 \qquad \text{for} \;\; n \to \infty\,.$$

On closer inspection, we find that the approximations for continuous functions converge faster, requiring fewer terms in the Fourier sum to approximate the function sufficiently well. The examples show that

$$a_n \sim \frac{1}{n^2} \quad \text{and} \quad b_n \sim \frac{1}{n^2}\,,$$

if f has no discontinuities. The Fourier series of a p-periodic function converges faster the smoother the function f is. More precisely, it is

Remark: If f is a p-periodic, $(m+1)$-times continuously differentiable function, then the Fourier coefficients of f behave as

$$|a_n| \leq \frac{c}{n^{m+2}} \quad \text{and} \quad |b_n| \leq \frac{c}{n^{m+2}}\,.$$

Examples for $m = 0$ (continuous functions) are 17.3, 17.5 and examples for $m = -1$ (functions with jump point) are 17.1 and 17.2.

Application Example 17.5 **(One-Way Rectifier, with MAPLE-Worksheet).**

Fig. 17.14 shows the sinusoidal signal of a *one-way rectifier* with period T:

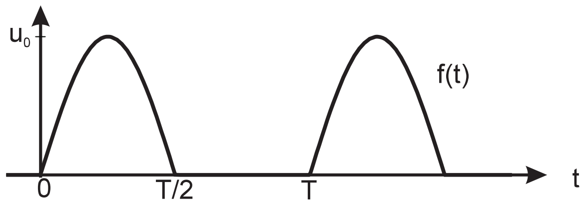


Figure 17.14. Sine wave of a half-wave rectifier

The signal is defined in the period interval $[0, T]$ by

$$f(t) = \begin{cases} u_0 \sin(\omega_0 t) & 0 \le t \le \frac{T}{2} \\ 0 & \frac{T}{2} < t \le T \end{cases}$$

with $\omega_0 = \frac{2\pi}{T}$. Since the signal is zero in the range $\frac{T}{2} \le t \le T$, the integration limits of the integrals are reduced to $t \in [0, \frac{T}{2}]$.

Calculation of the coefficient a_0:

$$\begin{aligned} a_0 &= \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{T}\int_0^{T/2} u_0\ \sin(\omega_0\ t)\,dt \\ &= -\frac{u_0}{T}\left[\frac{1}{\omega_0}\cos(\omega_0\ t)\right]_0^{T/2} = \frac{u_0}{\pi}. \end{aligned}$$

Calculation of the coefficients a_n:

$$a_n = \frac{2}{T}\int_0^T f(t)\ \cos(n\omega_0 t)\,dt = \frac{2u_0}{T}\int_0^{T/2} \sin(\omega_0\ t)\ \cos(n\omega_0\ t)\,dt.$$

Using the trigonometric formula

$$\sin(\alpha)\ \cos(\beta) = \frac{1}{2}\ (\sin(\alpha-\beta) + \sin(\alpha+\beta))$$

we get for $n \neq 1$

$$\begin{aligned} a_n &= \frac{u_0}{T}\left(\int_0^{T/2} \sin((1-n)\omega_0\ t)\,dt + \int_0^{T/2} \sin((1+n)\omega_0\ t)\,dt\right) \\ &= \frac{u_0}{T}\left(\left[-\frac{1}{(1-n)\omega_0}\ \cos((1-n)\omega_0\ t)\right]_0^{T/2}\right. \\ &\qquad\qquad \left. + \left[-\frac{1}{(1+n)\omega_0}\ \cos((1+n)\omega_0\ t)\right]_0^{T/2}\right) \end{aligned}$$

$$= -\frac{u_0}{\pi}\frac{1}{n^2-1}((-1)^n+1).$$

The result for a_n shows that the integral expression is calculated for $n \neq 1$. The coefficient a_1 must be evaluated separately:

$$\begin{aligned} a_1 &= \frac{u_0}{T}\left(\int_0^{T/2} 0\,dt + \int_0^{T/2} \sin(2\omega_0\ t)\,dt\right) \\ &= \frac{u_0}{T}\left(\left[-\frac{1}{2\omega_0}\ \cos(2\omega_0\ t)\right]_0^{T/2}\right) = 0. \end{aligned}$$

So for the coefficients a_n we get

$$a_n = \begin{cases} 0 & \text{for odd } n \\ -\dfrac{2\,u_0}{\pi\ (n^2-1)} & \text{for even } n,\ n>0. \end{cases}$$

The Fourier coefficients b_n are calculated in the same way

$$b_n = \frac{2}{T}\int_0^T f(t)\ \sin(n\omega_0 t)\,dt = \frac{2u_0}{T}\int_0^{T/2} \sin(\omega_0\ t)\ \sin(n\omega_0 t)\,dt$$

with the trigonometric formula

$$\sin(\alpha)\ \sin(\beta) = \frac{1}{2}\ \left(\cos\left(\alpha-\beta\right) - \cos\left(\alpha+\beta\right)\right).$$

For $n \neq 1$ this is

$$b_n = -\frac{\sin(n\,\pi)\,u0}{\pi\,(1+n)\ (-1+n)} = 0.$$

Similarly, the coefficient b_1 is calculated separately:

$$b_1 = \frac{1}{2}\,u_0.$$

The Fourier series of the sine half-wave is therefore

$$\begin{aligned} f(t) &= \frac{u_0}{\pi} + \frac{u_0}{2}\ \sin\left(\omega_0 t\right) - \frac{2}{\pi}\ u_0\ \Big(\tfrac{1}{2^2-1}\ \cos\left(2\,\omega_0 t\right) + \tfrac{1}{4^2-1}\ \cos\left(4\,\omega_0 t\right) \\ &\qquad + \tfrac{1}{6^2-1}\ \cos\left(6\,\omega_0 t\right) + \ldots\Big) \\ &= \frac{u_0}{\pi} + \frac{u_0}{2}\ \sin(\omega_0 t) - \frac{2}{\pi}\ u_0 \sum_{\substack{n=2 \\ n\ \text{even}}}^{\infty} \frac{1}{n^2-1}\ \cos(n\,\omega_0 t). \end{aligned}$$

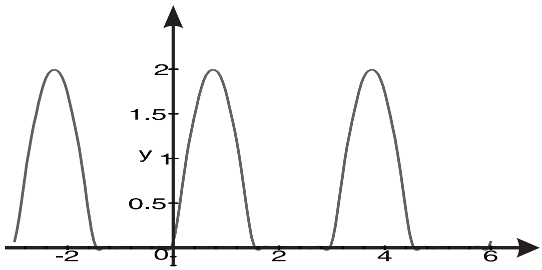


Figure 17.15. Partial sum for $n = 8$

The graphical representation of the Fourier series is shown in Fig. 17.15 for $n = 8$. The Fourier series shows a very good fit to the function with only a few terms in the sum. □

Application Example 17.6 **(Sawtooth waveform, with MAPLE).**

Find the amplitude spectrum of a *sawtooth oscillation* with period T (see Fig. 17.16).

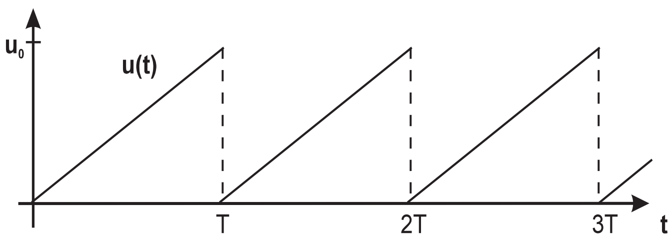


Figure 17.16. Time slope of a sawtooth waveform

The sawtooth is described in $t \in [0,\, T)$ by $u(t) = \frac{u_0}{T}\, t$. We define $\omega_0 = \frac{2\pi}{T}$. As in Example 17.5, the Fourier coefficients are determined.

For a_0 we get directly

$$a_0 = \frac{1}{T} \int_0^T u(t)\, dt = \frac{u_0}{T^2} \int_0^T t\, dt = \frac{1}{2}\, u_0.$$

Using partial integration, the coefficients a_n and b_n are calculated as follows

$$a_n = \frac{2}{T} \int_0^T u(t)\ \cos(n\omega_0 t)\, dt = \frac{2u_0}{T^2} \int_0^T t\ \cos(n\omega_0 t)\, dt = 0,$$

$$b_n = \frac{2}{T} \int_0^T u(t)\ \sin(n\omega_0 t)\, dt = \frac{2u_0}{T^2} \int_0^T t\ \sin(n\omega_0 t)\, dt = -\frac{u_0}{n\,\pi}.$$

The Fourier series of the sawtooth has the form

$$u(t) = \frac{u_0}{2} - \frac{u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(n\,\omega_0 t)$$

$$= \frac{u_0}{2} - \frac{u_0}{\pi} \left(\sin(\omega_0 t) + \tfrac{1}{2} \sin(2\,\omega_0 t) + \tfrac{1}{3} \sin(3\,\omega_0 t) + \ldots\right).$$

Discussion: The spectrum of a sawtooth is characterized by:

(1) The DC component $\frac{u_0}{2}$.

(2) The fundamental oscillation at frequency ω_0 and the amplitude $\frac{u_0}{\pi}$.

(3) The sinusoidal harmonics at the frequencies $2\,\omega_0,\ 3\,\omega_0,\ 4\,\omega_0, \ldots$ and the amplitudes $\frac{u_0}{2\pi}, \frac{u_0}{3\pi}, \frac{u_0}{4\pi}, \ldots$.

Fig. 17.17 shows the partial sum of the Fourier series for $n = 30$

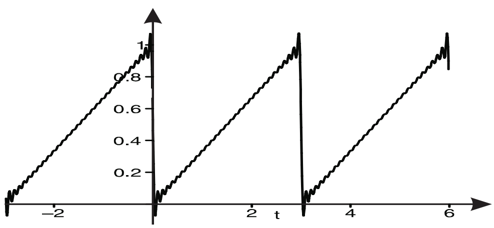


Figure 17.17. Partial sum of the Fourier series for $n = 30$

and Fig. 17.18 shows the amplitude spectrum.

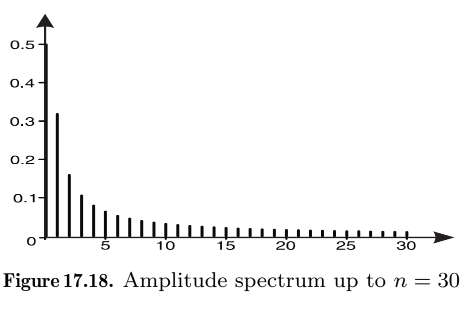


Figure 17.18. Amplitude spectrum up to $n = 30$

Discussion: The Fourier coefficients of the sawtooth are $\sim \frac{1}{n}$. The amplitude spectrum shows how slowly the amplitudes decrease. High frequencies still have relatively large amplitudes. The discontinuity in the time domain can be seen in the graph of the Fourier series. Many harmonics are required to represent the signal. □

17.5 Fourier Series in the Complex Domain

The Fourier representation takes a simple form with complex notation. Using Euler's formulas (see Volume 1, Chapter 5),

$$\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \quad \text{and} \quad \sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right),$$

the real Fourier series of a p-periodic function f is written as follows:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \frac{1}{2}\left(e^{i\,n\frac{2\pi}{p}x} + e^{-i\,n\frac{2\pi}{p}x}\right) + \sum_{n=1}^{\infty} b_n \frac{1}{2i}\left(e^{i\,n\frac{2\pi}{p}x} - e^{-i\,n\frac{2\pi}{p}x}\right).$$

After rearranging the expressions with respect to $e^{i\,n\frac{2\pi}{p}x}$ and a second sum with respect to $e^{-i\,n\frac{2\pi}{p}x}$, we get

$$f(x) = a_0 + \sum_{n=1}^{\infty} \frac{1}{2}(a_n - i\,b_n)\,e^{i\,n\frac{2\pi}{p}x} + \sum_{n=1}^{\infty} \frac{1}{2}(a_n + i\,b_n)\,e^{-i\,n\frac{2\pi}{p}x}.$$

If we examine the three sums of the Fourier series, we find that the last series contains factors $e^{i\,n\frac{2\pi}{p}x}$ for $n = -\infty, \ldots, 1$, the intermediate series $e^{i\,n\frac{2\pi}{p}x}$ for $n = 1, \ldots, \infty$ and the first term $e^{i\,n\frac{2\pi}{p}x}$ for $n = 0$. We define

$$c_0 := a_0;$$

$$\begin{aligned} c_n &:= \frac{1}{2}(a_n - i\,b_n) \\ &= \frac{1}{2}\left(\frac{2}{p}\int_0^p f(x)\cos\left(n\,\tfrac{2\pi}{p}x\right)dx - i\,\frac{2}{p}\int_0^p f(x)\sin\left(n\,\tfrac{2\pi}{p}x\right)dx\right) \\ &= \frac{1}{p}\int_0^p f(x)\left(\cos\left(n\,\tfrac{2\pi}{p}x\right) - i\,\sin\left(n\,\tfrac{2\pi}{p}x\right)\right)dx \\ &= \frac{1}{p}\int_0^p f(x)\,e^{-i\,n\frac{2\pi}{p}x}\,dx, \qquad n \geq 1; \end{aligned}$$

and

$$\begin{aligned} c_{-n} &:= \frac{1}{2}(a_n + i\,b_n) \\ &= \frac{1}{2}\left(\frac{2}{p}\int_0^p f(x)\cos\left(n\,\tfrac{2\pi}{p}x\right)dx + i\,\frac{2}{p}\int_0^p f(x)\sin\left(n\,\tfrac{2\pi}{p}x\right)dx\right) \\ &= \frac{1}{p}\int_0^p f(x)\left(\cos\left(n\,\tfrac{2\pi}{p}x\right) + i\,\sin\left(n\,\tfrac{2\pi}{p}x\right)\right)dx \\ &= \frac{1}{p}\int_0^p f(x)\,e^{i\,n\frac{2\pi}{p}x}\,dx, \qquad n \geq 1. \end{aligned}$$

So the Fourier series of f is represented as **one** series over $e^{i\,n\,\frac{2\pi}{p}\,x}$ for $n = -\infty, \ldots, \infty$:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{i\,n\,\frac{2\pi}{p}\,x}$$

with the coefficients

$$c_n = \frac{1}{p}\int_0^p f(x)\, e^{-i\,n\,\frac{2\pi}{p}\,x}\,dx \qquad \text{for} \quad n \in \mathbb{Z}\,.$$

The following complex formulation of the Fourier Theorem is obtained:

Complex Fourier Series

Let $f : \mathbb{R} \to \mathbb{C}$ be a complex function with real period p. Let f be piecewise continuously differentiable and satisfy the mean value property. Then for all $x \in \mathbb{R}$ the complex Fourier series converges to $f(x)$:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{i\,n\,\frac{2\pi}{p}\,x}.$$

For $n \in \mathbb{Z}$ the complex Fourier coefficients are given by

$$c_n = \frac{1}{p}\int_0^p f(x)\, e^{-i\,n\,\frac{2\pi}{p}\,x}\,dx.$$

Note: This complex formulation is certainly also valid for real-valued functions $f : \mathbb{R} \to \mathbb{R}$ which are piecewise continuously differentiable and satisfy the mean value property.

Remarks: Properties of the Complex Formulation

(1) There is only one series for the Fourier series and the coefficients c_n are determined by a uniform formula.

(2) The sum of the complex Fourier series covers $n = -\infty \ldots \infty$. So it appears that there are negative frequencies. But this is only due to the complex formulation: $e^{i\,\omega_n\,t}$ and $e^{-i\,\omega_n\,t}$ are needed to define the real oscillation $\sin(\omega_n t)$ and $\cos(\omega_n t)$ with the real frequency $\omega_n > 0$, since $\cos(\omega_n\ t) = \frac{1}{2}(e^{i\,\omega_n\,t} + e^{-i\,\omega_n\,t})$ and $\sin(\omega_n\ t) = \frac{1}{2\,i}(e^{i\,\omega_n\,t} - e^{-i\,\omega_n\,t})$.

Advantages of the Complex Approach

The Fourier coefficients c_n of a real signal $f(x)$ have the following properties

(3) $c_n^* = c_{-n}$: The coefficients on negative n are the complex conjugate of the corresponding coefficients on positive n.

(4) Consequently, the absolute values of c_n and c_{-n} are equal, namely

$$|c_n| = \left|\tfrac{1}{2}\,(a_n - i\,b_n)\right| = \tfrac{1}{2}\sqrt{a_n^2 + b_n^2} = \tfrac{1}{2}\,A_n.$$

The magnitude of the complex Fourier coefficients is A_n up to the factor $\frac{1}{2}$. So the **magnitude** $|c_n|$ **represents the amplitude spectrum** half from $-\infty$ to -1 and half from 1 to ∞.

(5) The phase of the complex Fourier coefficients is given by

$$\tan\varphi_n = \frac{\text{Im } c_n}{\text{Re } c_n} = \frac{-\frac{1}{2}\,b_n}{\frac{1}{2}\,a_n} = -\frac{b_n}{a_n}.$$

Up to the sign, this is the **phase spectrum**.

(6) $c_0 = a_0$ represents the **direct current component** of the signal.

The advantage of the complex formulation is that we have a single formula for the coefficients and that these coefficients include both the amplitude spectrum (up to the factor $\frac{1}{2}$) **and** the phase spectrum (up to the sign).

Example 17.7 (With MAPLE-Worksheet). Find the complex Fourier series of the T-periodic function f given in Fig. 17.19.

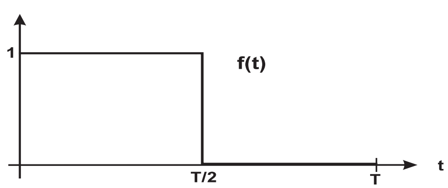


Figure 17.19. Square wave signal

From the magnitude of the complex coefficients we determine the amplitude spectrum of the signal.

With $\omega_0 = \frac{2\pi}{T}$, the complex Fourier coefficients for $n \neq 0$ are

$$\begin{aligned} c_n &= \frac{1}{T} \int_0^T f(t)\, e^{-i\,n\,\omega_0 t}\, dt = \frac{1}{T} \int_0^{\frac{T}{2}} e^{-i\,n\,\omega_0 t}\, dt \\ &= \frac{1}{T} \frac{1}{-i\,n\,\omega_0} \left[e^{-i\,n\,\omega_0 \frac{T}{2}} - 1 \right]. \end{aligned}$$

With $\omega_0 \cdot T = 2\pi$ and $\omega_0 \frac{T}{2} = \pi$ we get $e^{-i\,n\,\omega_0 \frac{T}{2}} = e^{-i\,n\,\pi} = (-1)^n$

$$\Rightarrow \quad c_n = \frac{1}{-i\,n\,2\pi} \left[(-1)^n - 1 \right] = \begin{cases} \frac{1}{i\,n\,\pi} & \text{odd } n \\ 0 & \text{even } n \neq 0 \end{cases}$$

Since the DC component of the signal is $\frac{1}{2}$, we can directly infer $c_0 = \frac{1}{2}$.

$$\Rightarrow \quad f(t) = \frac{1}{2} + \sum_{\substack{n=-\infty \\ odd\ \ n}}^{\infty} \frac{1}{i\,n\,\pi}\, e^{i\,n\,\omega_0 t}.$$

The amplitude spectrum of this function is given by the magnitude of the coefficients.

$$a_0 = |c_0| = \frac{1}{2}$$

$$A_n = 2\,|c_n| = \begin{cases} \frac{2}{n\,\pi} & \text{for odd } \ n \\ 0 & \text{for even } \ n \neq 0\,. \end{cases}$$

□

Calculating the Real Coefficients from the Complex

Although the calculation has been done in the complex, the real Fourier coefficients a_n and b_n are sometimes of interest. From the complex Fourier coefficients c_n, $n \in \mathbb{Z}$, the real Fourier coefficients a_0, a_n and b_n can be recovered so that the real integral formulas do not have to be recalculated. By definition of c_n we have

$$\begin{aligned} c_0 &= a_0 && (1) \\ c_n &= \tfrac{1}{2}\,(a_n - i\,b_n) && (2) \\ c_{-n} &= \tfrac{1}{2}\,(a_n + i\,b_n) && (3) \end{aligned}$$

Directly from (1):	$a_0 = c_0$	
Add (2) and (3):	$a_n = c_n + c_{-n}$	$n = 1, 2, 3, \ldots$
Subtract (3) from (2):	$b_n = i\,(c_n - c_{-n})$	$n = 1, 2, 3, \ldots$

Summary: Fourier Series

Given is a T-periodic signal $f(t)$ that is piecewise continuously differentiable and satisfies the mean value property.

(1) For all $t \in \mathbb{R}$, the real Fourier series is

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(n \frac{2\pi}{T} t\right) + \sum_{n=1}^{\infty} b_n \sin\left(n \frac{2\pi}{T} t\right)$$

$$\text{with} \quad a_0 = \frac{1}{T} \int_0^T f(t)\, dt,$$

$$a_n = \frac{2}{T} \int_0^T f(t) \cos\left(n \frac{2\pi}{T} t\right) dt, \qquad n = 1, 2, 3, \ldots,$$

$$b_n = \frac{2}{T} \int_0^T f(t) \sin\left(n \frac{2\pi}{T} t\right) dt, \qquad n = 1, 2, 3, \ldots.$$

(2) For all $t \in \mathbb{R}$

$$f(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos\left(n \frac{2\pi}{T} t - \varphi_n\right)$$

with $A_n = \sqrt{a_n^2 + b_n^2}$ and $\tan \varphi_n = \dfrac{b_n}{a_n}$ for $n = 1, 2, 3, \ldots$.

(3) For all $t \in \mathbb{R}$, the complex Fourier series is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{i\, n \frac{2\pi}{T} t}$$

$$\text{with} \quad c_n = \frac{1}{T} \int_0^T f(t)\, e^{-i\, n \frac{2\pi}{T} t}\, dt \qquad (n \in \mathbb{Z}).$$

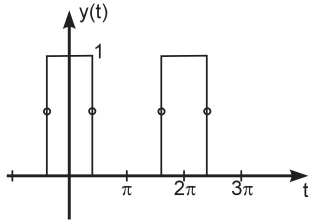
(4) The real coefficients are calculated from the complex coefficients by

$$a_0 = c_0,$$
$$a_n = c_n + c_{-n} \qquad n \in \mathbb{N},$$
$$b_n = i\,(c_n - c_{-n}) \qquad n \in \mathbb{N}.$$

17.6 Compilation of Fourier Series

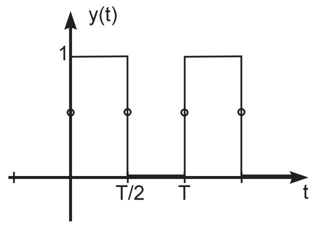
(1) Square wave signals

$$y(t) = \begin{cases} 1 & \text{for } |t| < \delta \\ \frac{1}{2} & \text{for } |t| = \delta \\ 0 & \text{for } \delta < |t| \leq \pi \end{cases}$$



$$y(t) = \frac{\delta}{\pi} + \frac{2}{\pi}\left(\frac{1}{1}\sin(\delta)\cos(t) + \frac{1}{2}\sin(2\delta)\cos(2t) + \ldots\right)$$

$$y(t) = \begin{cases} 1 & \text{for } 0 < t < \frac{T}{2} \\ \frac{1}{2} & \text{for } t = 0,\ \frac{T}{2} \\ 0 & \text{for } \frac{T}{2} < t < T \end{cases}$$

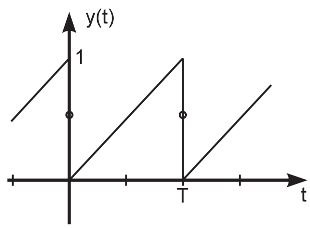


$$y(t) = \frac{1}{2} + \frac{2}{\pi}\left(\sin(\omega_0 t) + \frac{1}{3}\sin(3\,\omega_0 t) + \frac{1}{5}\sin(5\,\omega_0 t) + \ldots\right)$$

---

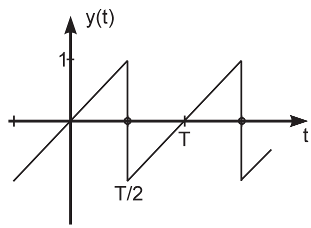
(2) Sawtooth oscillations

$$y(t) = \begin{cases} \frac{1}{T}\,t & \text{for } 0 \leq t < T \\ \frac{1}{2} & \text{for } t = T \end{cases}$$



$$y(t) = \frac{1}{2} - \frac{1}{\pi}\left(\sin(\omega_0 t) + \frac{1}{2}\sin(2\,\omega_0 t) + \frac{1}{3}\sin(3\,\omega_0 t) + \ldots\right)$$

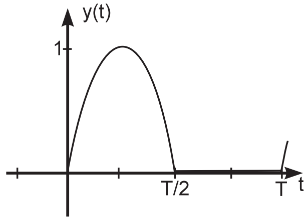
$$y(t) = \begin{cases} t & \text{for } |t| < T \\ 0 & \text{for } t = \frac{T}{2} \end{cases}$$



$$y(t) = 2\left(\sin(\omega_0 t) - \frac{1}{2}\sin(2\,\omega_0 t) + \frac{1}{3}\sin(3\,\omega_0 t) \mp \ldots\right)$$

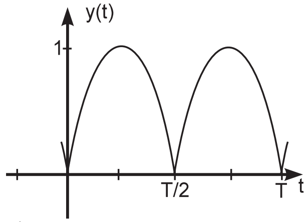
---

(3) Sine impulses



$$y(t) = \begin{cases} \sin(\omega_0 t) & \text{for } 0 \le t \le \frac{T}{2} \\ 0 & \text{for } \frac{T}{2} \le t \le T \end{cases}$$

$$y(t) = \frac{1}{\pi} + \frac{1}{2}\sin(\omega_0 t) - \frac{2}{\pi}\left(\frac{1}{1\cdot 3}\cos(2\,\omega_0 t) + \frac{1}{3\cdot 5}\cos(4\,\omega_0 t) + \ldots\right)$$

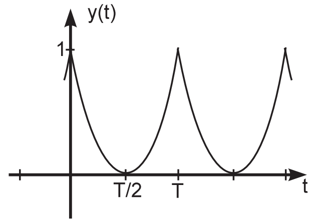


$$y(t) = |\sin(\omega_0 t)| \quad \text{for } 0 \le t \le T$$

$$y(t) = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{1}{1\cdot 3}\cos(2\,\omega_0 t) + \frac{1}{3\cdot 5}\cos(4\,\omega_0 t) + \frac{1}{5\cdot 7}\cos(6\,\omega_0 t) + \ldots\right)$$

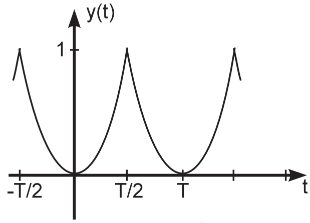
---

(4) Parabolic functions



$$y(t) = \frac{4}{T^2}\left(t - \frac{T}{2}\right)^2 \quad \text{for } 0 \le t \le T$$

$$y(t) = \frac{1}{3} + \frac{4}{\pi^2}\left(\frac{1}{1^2}\cos(\omega_0 t) + \frac{1}{2^2}\cos(2\,\omega_0 t) + \frac{1}{3^2}\cos(3\,\omega_0 t) + \ldots\right)$$



$$y(t) = \left(\frac{t}{T}\right)^2 \quad \text{for } |t| \le \frac{T}{2}$$

$$y(t) = \frac{1}{3} - \frac{4}{\pi^2}\left(\frac{1}{1^2}\cos(\omega_0 t) - \frac{1}{2^2}\cos(2\,\omega_0 t) + \frac{1}{3^2}\cos(3\,\omega_0 t) \mp \ldots\right)$$

17.7 Problems on Fourier Series

17.1 Give an expression for the 2π-periodic functions in the period interval sketched below, look for symmetries and expand them into a Fourier series:

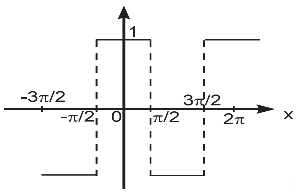


Fig. a

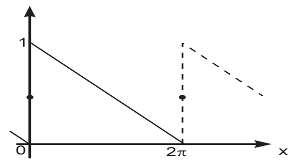


Fig. b

17.2 Sketch the following T-periodic functions and determine the Fourier series and the associated amplitude spectrum:

a) $f(t) = \begin{cases} e^{t} & \text{for } \frac{-T}{2} \le t \le 0 \\ e^{-t} & \text{for } 0 \le t \le \frac{T}{2} \end{cases}$ b) $f(t) = \begin{cases} \frac{2\,h\,t}{T} & \text{for } 0 \le t \le \frac{T}{2} \\ h & \text{for } \frac{T}{2} \le t \le T \end{cases}$

17.3 Calculate the complex Fourier expansion of
a) Problem 17.1a) b) Problem 17.1b)

17.4 Develop $f(t) = \sin^3 t$, $|t| \le \pi$ into a Fourier series (think first!!!).

17.5 a) Expand $f(t) = t^2$, $0 \le t \le T$, into a Fourier series.
b) What is the result for $T = 2\pi$?
c) What do we get with b) for $\sum_{n=1}^{\infty} \frac{1}{n^2}$ at $t = 2\pi$?

17.6 Fig. (c) shows the slope of a sawtooth voltage with period T. Perform a Fourier analysis on this pulse.

$$u(t) = \frac{u_0}{T}\, t \quad (0 < t < T)\,.$$

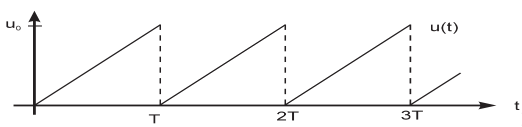


Fig. c

17.7 Calculate the Fourier series taking symmetries into account:

a) $f(x) = \begin{cases} 8 & \text{for } 0 < x < 2 \\ -8 & \text{for } 2 < x < 4 \end{cases}$ with period 4,

b) $f(x) = \begin{cases} -x & \text{for } -4 \le x \le 0 \\ x & \text{for } 0 \le x \le 4 \end{cases}$ with period 8.

17.8 Decompose the trapezoidal impulse into its harmonic components. What is the value of the Fourier series at $t = T$?

$$f(t) = \begin{cases} \frac{2}{T}\,t & 0 \leq t \leq T/2 \\ 1 & T/2 \leq t \leq T \end{cases}$$

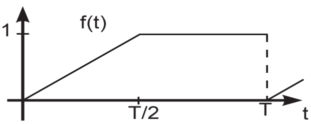


Fig. d

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18

Chapter 18
Fourier Transform

This chapter introduces the Fourier transform to study the frequency behavior of *non-periodic* signals $f(t)$. As with the Fourier series, this analysis is called *frequency analysis* of the time signal $f(t)$. Section 18.1 introduces the formula for the Fourier transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\,\omega\, t}\, dt.$$

Section 18.2 presents important properties of the Fourier transform and discusses their significance. To characterize linear systems, a signal is needed that contains all frequencies with the same amplitude. This leads to the concept of the delta function, which is introduced and whose properties are discussed in Section 18.3.

8

18 Fourier Transform

This chapter introduces the Fourier transform to study the frequency behavior of *non-periodic* signals $f(t)$. As with the Fourier series, this analysis is called *frequency analysis* of the time signal $f(t)$. Section 18.1 introduces the formula for the Fourier transform

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18.1 Fourier Transform and Examples

By specifying the Fourier series, it is possible to analyze periodic processes regardless of the length of the period T. A T-periodic function f can be represented as a superposition of infinitely many harmonic oscillations with a fundamental frequency $\omega_0 = \frac{2\pi}{T}$, harmonics $\omega_n = n\,\frac{2\pi}{T}$ and their associated amplitudes. Periodic functions therefore have a discrete line spectrum. This line spectrum provides a clear assignment between the time domain of the function and its frequency domain.

In the following, a method (*Fourier transform*) is developed which also provides all frequencies with associated amplitudes for non-periodic signals.

Transition from Fourier Series to Fourier Transform

To determine the frequencies in a time signal f, the function f is interpreted as a periodic function with period $T \to \infty$. However, to obtain a representation of the spectrum, we assume a $2p$-periodic function $f(t)$. According to Section 17.5, the corresponding complex Fourier series is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{i\,n\,\frac{2\pi}{2p}\,t}$$

with the complex Fourier coefficients

$$c_n = \frac{1}{2p} \int_0^{2p} f(t)\, e^{-i\,n\,\frac{2\pi}{2p}\,t}\, dt = \frac{1}{2p} \int_{-p}^{p} f(t)\, e^{-i\,n\,\frac{\pi}{p}\,t}\, dt.$$

We insert the coefficients c_n into the Fourier series and replace the frequencies $\omega_n = n\,\frac{\pi}{p}$ and the frequency spacing $\Delta\omega = \omega_n - \omega_{n-1} = \frac{\pi}{p}$. So we get

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} \left[\frac{1}{2p} \int_{-p}^{p} f(t)\, e^{-i\,\omega_n\,t}\, dt \right] e^{i\,\omega_n\,t} \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[\int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} f(t)\, e^{-i\,\omega_n\,t}\, dt \right] e^{i\,\omega_n\,t}\, \Delta\omega. \end{aligned}$$

Now we consider an arbitrary, not necessarily periodic time function $f(t)$ as a periodic function with $p \to \infty$. For $p \to \infty$, the frequency difference $\Delta\omega$ approaches zero and the sum becomes the integral. The frequency spectra move closer together and in the limit we obtain a continuous function in ω:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{i\,\omega\,t}\, d\omega \qquad (FI)$$

with

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\,\omega\,t}\, dt. \qquad (FT)$$

The representation of $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{i\,\omega\,t}\, d\omega$ is called **Fourier integral** and $F(\omega)$ the **Fourier transform** or **Spectrum** of the time function $f(t)$.

Animation: This animation shows how the T-periodic extension of the rectangle converges for $T \to \infty$. We see that the distance between the spectral lines approaches zero ($\Delta\omega \to 0$). So the discrete spectrum is replaced by a continuous frequency function, also called the spectrum. □

The Fourier integral is an improper integral. It can be shown to exist under certain conditions (which are almost always satisfied in practice):

Theorem 18.1: Fourier Transform

Let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise continuously differentiable function and $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$, then for every $\omega \in \mathbb{R}$ the improper integral exists

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\,\omega\,t}\, dt \qquad (\textbf{Fourier Transform of } f).$$

The Fourier transform maps each **time** function $f(t)$ to a **frequency** function $F(\omega) : \mathbb{R} \to \mathbb{C}$. We say that the Fourier transform maps the time domain to the **Spectral Domain (Frequency Domain)** by assigning to each time function $f(t)$ its spectral function $F(\omega)$. To indicate exactly which time function $F(\omega)$ belongs to, the following notation is used

$$\mathcal{F}(f)(\omega) = F(\omega)\ .$$

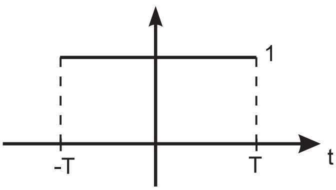
This notation expresses the transform character of the Fourier transform: The function f is assigned a new function $\mathcal{F}(f)$. Sometimes, a correspondence notation is used, as in the case of the Laplace transform:

$$f(t) \ \circ\!\!-\!\!\bullet\ F(\omega)\ .$$

Remark: A function $f : [a, b] \to \mathbb{R}$ is said to be piecewise continuously differentiable if the interval can be divided into a finite number of subintervals I_k, so that f is continuously differentiable within the intervals I_k and there exist both right and left limits at the boundaries.

Example 18.1. For the **rectangular impulse** shown below, the Fourier transform $F(\omega)$ is to be calculated.

$$f(t) := \left\{ \begin{array}{ll} 1 & \text{for } |t| < T \\ 0 & \text{for } |t| > T \end{array} \right\} =: rect\left(\tfrac{t}{T}\right)$$



$$\mathcal{F}(f)(\omega) = F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\,\omega\,t}\, dt$$
$$= \int_{-T}^{T} 1\, e^{-i\,\omega\,t}\, dt = \left.\frac{e^{-i\,\omega\,t}}{-\,i\,\omega}\right|_{-T}^{T}$$

$$= \frac{1}{-i\,\omega}\,\left(e^{-i\,T\,\omega} - e^{i\,T\,\omega}\right) = \frac{2}{\omega}\,\frac{1}{2i}\,\left(e^{i\,T\,\omega} - e^{-i\,T\,\omega}\right)$$

$$= \frac{2}{\omega}\,\sin(\omega\,T) = 2\,T\,\frac{\sin\,(\omega\,T)}{\omega\,T}\,.$$

The value of the transform does not depend on the choice of the function value of $f(t)$ at the points $t = -T$ and $t = -T$! Introducing the si-function si $(x) := \frac{\sin x}{x}$, also known as the sinc-function, the result is

$$\mathcal{F}\left(rect\left(\tfrac{t}{T}\right)\right)(\omega) = \frac{2}{\omega}\,\sin(\omega\,T) = 2\,T\,\mathrm{si}\,(\omega\,T).$$

The graph of the Fourier transform is shown in Fig. 18.1.

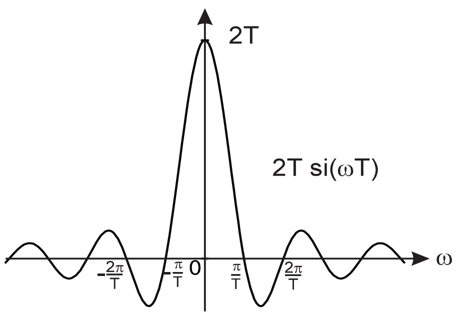


Figure 18.1. Fourier transform of the rectangle function

The spectrum is a sine function in ω whose amplitude decreases with $1/\omega$. The zeros of $F(\omega)$ are the same as those of the sine function

$$\omega_n T = n\pi \hookrightarrow \omega_n = n\,\frac{\pi}{T}$$

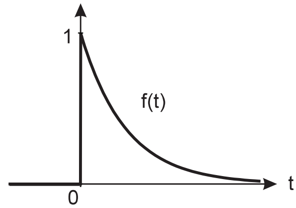
except for $\omega = 0$: To calculate the value of the function at $\omega = 0$, we apply the rule of l'Hospital

$$F(\omega = 0) = \lim_{\omega \to 0} \frac{2\,\sin(\omega\,T)}{\omega} = \lim_{\omega \to 0} \frac{2\,\cos(\omega\,T)\,T}{1} = 2T.$$

Observation: The wider the rectangle $f(t)$ or the larger T, the narrower the maximum of $F(\omega)$ at $\omega = 0$, since the first zero of the spectrum is at $\omega_1 = \frac{\pi}{T}$. On the other hand: The narrower the rectangle $f(t)$, i.e. the smaller T, the wider the maximum. □

Example 18.2. For the **exponential function** shown in the next figure, the corresponding spectral function is calculated:

$$f(t) = e^{-\alpha t} \cdot S(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-\alpha t} & \text{for } t > 0,\ \alpha > 0. \end{cases}$$



Here $S(t)$ is the step function (Heaviside function) $S(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0. \end{cases}$

The spectrum of the function f is its Fourier transform $F(\omega)$:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = \int_0^{\infty} e^{-\alpha t}\, e^{-i\omega t}\, dt = \int_0^{\infty} e^{-(\alpha + i\omega)t}\, dt$$

$$= \left. \frac{e^{-(\alpha+i\omega)t}}{-(\alpha + i\omega)} \right|_{t=0}^{t=\infty} = \lim_{t\to\infty} \frac{e^{-(\alpha+i\omega)t}}{-(\alpha+i\omega)} + \frac{1}{\alpha + i\omega}$$

$$= 0 + \frac{1}{\alpha + i\omega} = \frac{\alpha}{a^2 + \omega^2} - i\frac{\omega}{\alpha^2 + \omega^2}.$$

$$\Rightarrow \quad F(\omega) = \mathcal{F}\left(e^{-\alpha t} \cdot S(t)\right)(\omega) = \frac{1}{\alpha + i\omega}.$$

Note: For most applications it does not matter what value $S(t)$ has at the position $t = 0$. Commonly used values are $S(0) = 0$, $S(0) = 1$, but also $S(0) = \frac{1}{2}$. In the latter case, $S(t) = \frac{1}{2} + \frac{1}{2}\, sign(t)$, where $sign(t)$ is the sign function. □

Representing the Fourier Transform

Example 18.2 shows that the Fourier transform $F(\omega)$ of a time function $f(t)$ is generally complex. The graph of complex-valued functions cannot be drawn directly, but either the real and imaginary parts must be plotted separately, or the complex function

$$F(\omega) = |F(\omega)|\, e^{i\varphi(\omega)}$$

must be split into **magnitude** and **phase**, where

$$|F(\omega)| = \sqrt{F(\omega) \cdot F^*(\omega)} \qquad \text{(Magnitude)}$$

$$\tan\varphi(\omega) = \frac{\operatorname{Im} F(\omega)}{\operatorname{Re} F(\omega)} \qquad \text{(Phase).}$$

Both magnitude and phase are real functions of ω. By analogy with the naming of the Fourier series, we also speak of the *Amplitude Spectrum* and the *Phase Spectrum*, respectively.

Example 18.3 (Amplitude and Phase Spectrum). In Example 18.2 the Fourier transform is

$$F(\omega) = \frac{1}{\alpha + i\,\omega}.$$

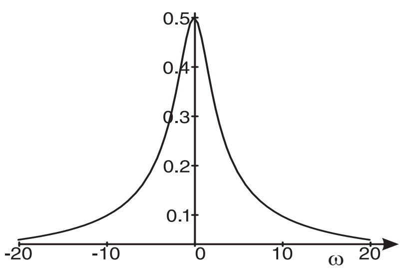
We look for the amplitude and the phase spectrum.

To compute the amplitude and the phase spectrum, we split $F(\omega)$ into the magnitude

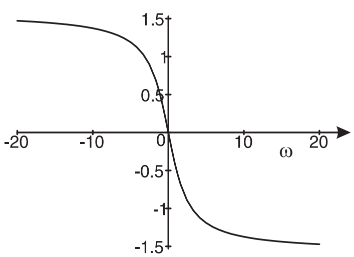
$$|F(\omega)| = \sqrt{F(\omega) \cdot F^*(\omega)} = \sqrt{\frac{1}{\alpha + i\,\omega} \cdot \frac{1}{\alpha - i\,\omega}} = \frac{1}{\sqrt{\alpha^2 + \omega^2}}$$

and the phase

$$\tan\varphi(\omega) = \frac{\operatorname{Im} F(\omega)}{\operatorname{Re} F(\omega)} = -\frac{\omega}{\alpha}.$$



Magnitude of $F(\omega)$



Phase of $F(\omega)$

Figure 18.2. Magnitude and phase of the Fourier transform

The Fig. 18.2 shows both the amplitude spectrum (left) and the phase spectrum (right) for $\alpha = 1$. □

18.2 Inverse Fourier Transform

A time function is completely characterized by its spectrum, as shown by the *inverse Fourier transform.*

Theorem 18.2: Inverse Fourier Transform

If $F(\omega)$ is the Fourier transform of a function $f(t)$, then the function $f(t)$ is given by

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{i\,\omega\,t}\, d\omega \qquad (\textbf{Inverse Fourier Transform}),$$

if $f(t)$ satisfies the mean value property.

The mean value property of a function f means that the function value at any point t is given by the mean value of the left and right side limits: $f(t) = \lim_{\varepsilon \to 0} \frac{1}{2}\left(f(t+\varepsilon) + f(t-\varepsilon)\right)$ for all $t \in \mathbb{D}_f$ (see Section 17.3). For continuous functions the mean value property is always satisfied; for discontinuous functions, the mean value must be ensured at each step.

To be consistent with the mean value property, the step function $S(t)$ should be defined by $S(0) = \frac{1}{2}$ at the position $t_0 = 0$ for Fourier applications!

Theorem 18.2 implies that the Fourier transform of a function f contains the same information as f itself, because the time signal $f(t)$ can be completely reconstructed from $F(\omega)$ using the inverse Fourier transform. In particular, the function f can be characterized solely by its associated spectrum!

Remarks:

(1) The Fourier transform is also defined for complex functions $f(t) = f_1(t) + i\, f_2(t)$. $F(\omega)$ is generally a complex function (see Example 18.2).

(2) If f is a real function (a real signal), then the Fourier transform can be calculated using Euler's identity $e^{-i\,\omega\,t} = \cos(\omega\,t) - i\,\sin(\omega\,t)$:

$$\begin{aligned}\mathcal{F}(f)(\omega) &= \int_{-\infty}^{\infty} f(t)\, e^{-i\,\omega\,t}\, dt = \int_{-\infty}^{\infty} f(t)\, \left(\cos(\omega\,t) - i\,\sin(\omega\,t)\right)\, dt \\ &= \int_{-\infty}^{\infty} f(t)\, \cos(\omega\,t)\, dt - i \int_{-\infty}^{\infty} f(t)\, \sin(\omega\,t)\, dt. \qquad (*)\end{aligned}$$

We call these transformations the *cosine* and *sine transform.*

(3) **Odd and even functions**

(i) For an **even** real function f, that is $f(-t) = f(t)$, it follows

$$F(\omega) = 2 \int_0^\infty f(t) \cos(\omega t)\, dt.$$

Proof: With $f(t)$ also $f(t) \cdot \cos(\omega t)$ is an even function and

$$\int_{-\infty}^\infty f(t) \cos(\omega t)\, dt = 2 \int_0^\infty f(t) \cos(\omega t)\, dt\,,$$

because the integrand is integrated symmetrically about the origin. On the other hand, $f(t) \cdot \sin(\omega t)$ is an odd function. When an odd function is integrated symmetrical to the origin, the integral is zero:

$$\int_{-\infty}^\infty f(t) \sin(\omega t)\, dt = 0.$$

Inserting both integrals into the formula $(*)$ proves the statement.□

(ii) For an **odd** real function f, this is $f(-t) = -f(t)$, we simplify similar to (i)

$$F(\omega) = -i\, 2 \int_0^\infty f(t) \sin(\omega t)\, dt.$$

Example 18.4. Given is the even real function

$$f(t) = e^{-\alpha\,|t|} \qquad \text{with} \;\; \alpha > 0.$$

The Fourier transform of f is calculated according to Note 3 (i)

$$F(\omega) = 2 \int_0^\infty f(t) \cos(\omega t)\, dt = 2 \int_0^\infty e^{-\alpha\, t} \cos(\omega t)\, dt.$$

The absolute values in the argument of the exponential function can be omitted as only positive t are used for integration. To calculate the integral, we replace $\cos(\omega t)$ with $\mathrm{Re}\left(e^{i\,\omega\, t}\right)$ and continue with

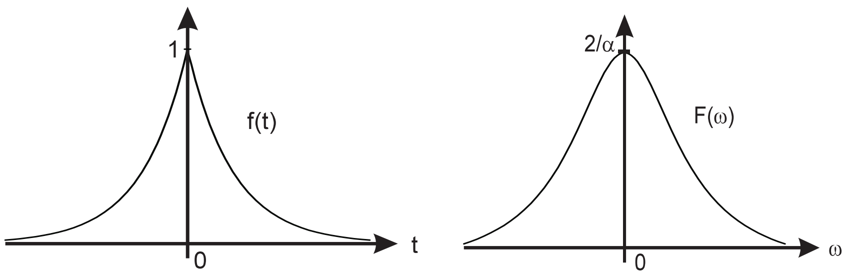
$$F(\omega) = 2 \int_0^\infty e^{-\alpha\, t}\; \mathrm{Re}(e^{i\,\omega\, t})\, dt = 2\, \mathrm{Re} \int_0^\infty e^{(-\alpha + i\,\omega)\, t}\, dt$$

$$= 2\ \mathrm{Re}\ \frac{e^{(-\alpha + i\,\omega)\,t}}{-\alpha + i\,\omega}\Bigg|_{t=0}^{t=\infty} = 2\ \mathrm{Re}\ \frac{-1}{-\alpha + i\,\omega}$$

$$= 2\ \mathrm{Re}\left\{\frac{\alpha}{\alpha^2+\omega^2} + i\,\frac{\omega}{\alpha^2+\omega^2}\right\} = \frac{2\,\alpha}{\alpha^2+\omega^2}\,.$$

The same result can be obtained by partial integration.

$$\Rightarrow \quad F(\omega) = \mathcal{F}\left(e^{-\alpha|\,t|}\right)(\omega) = \frac{2\,\alpha}{\alpha^2+\omega^2}. \qquad \square$$



Time function $f(t)$ Spectral function $F(\omega)$

Figure 18.3. Time function and corresponding spectral function from Example 18.4.

Example 18.5. Given is the odd real function

$$f(t) = e^{-\alpha\,|t|}\ sign\,(t)$$

with $\alpha > 0$ and the sign function $sign\,(t) = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t = 0 \\ -1 & \text{for } t < 0 \end{cases}$.

The Fourier transform according to Note 3 (ii) is given by

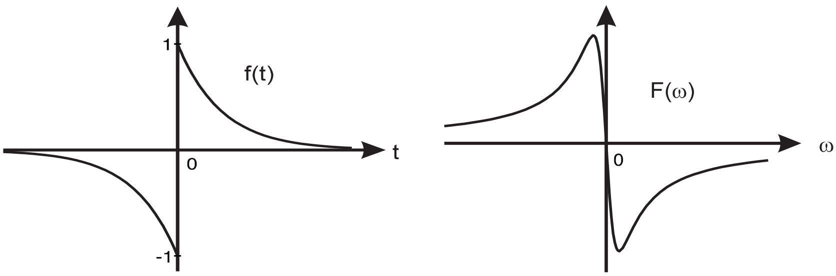
$$F(\omega) = -i\,2\int_0^\infty f(t)\,\sin(\omega\,t)\ dt = -i\,2\int_0^\infty e^{-\alpha\,t}\,\sin(\omega\,t)\ dt.$$

To calculate the integral, we replace $\sin(\omega\,t) = \mathrm{Im}\left(e^{i\,\omega\,t}\right)$ and consider Example 18.4

$$F(\omega) = -i\,2\ \mathrm{Im}\int_0^\infty e^{-\alpha\,t}\,e^{i\,\omega\,t}\,dt$$

$$= -i\,2\ \mathrm{Im}\left\{\frac{\alpha}{\alpha^2+\omega^2} + i\,\frac{\omega}{\alpha^2+\omega^2}\right\}$$

$$\Rightarrow \quad F(\omega) = -i\,\frac{2\,\omega}{\alpha^2+\omega^2}\,. \qquad \square$$



Time function $f(t)$ Fourier transform $F(\omega)$

Figure 18.4. Time function and corresponding Fourier transform from Example 18.5.

Example 18.6. Given is the odd real function $f(t) = \frac{1}{t}$. We look for its Fourier transform.

Since $f(t)$ is an odd function, we compute the sine transform of $\frac{1}{t}$. Although the function f is not defined at $t = 0$ (singularity), $\frac{\sin(\omega\, t)}{t}$ is continuous and bounded for all $t \in \mathbb{R}$. We calculate the Fourier transform of $\frac{1}{t}$ according to Note 3 (ii). With the substitution

$$x = \omega\, t \quad \hookrightarrow \quad dx = \omega\, dt$$

we get for positive values of ω

$$F(\omega) = -i\,2\int_0^\infty \frac{\sin(\omega\, t)}{t}\,dt = -i\,2\int_0^\infty \frac{\sin x}{x}\,dx \qquad (\omega > 0).$$

The value of the definite integral

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

is calculated in the Example 18.8 ③. Using this result, for positive ω we get the value $-i\,\pi$. Similarly, for negative ω we get $i\,\pi$. Both results together give the Fourier transform of the function

$$F(\omega) = \begin{cases} -i\,\pi & \text{for } \omega > 0 \\ i\,\pi & \text{for } \omega < 0 \end{cases} = -i\,\pi\, sign(\omega)\;,$$

where the *sign* function appears again. □

18.3 Properties of the Fourier Transform

This section introduces important properties and illustrates them with examples. Subsequently, we will always assume that the time functions satisfy the requirements of the Fourier transform, so the Fourier transform is defined. $F(\omega) = \mathcal{F}(f)(\omega)$ is always the Fourier transform of f; $F_1(\omega) = \mathcal{F}(f_1)(\omega)$ and $F_2(\omega) = \mathcal{F}(f_2)(\omega)$ are the Fourier transforms of f_1 and f_2, respectively.

18.3.1 Linearity

We compute the spectrum of a superposition $k_1 f_1(t) + k_2 f_2(t)$:

$$\begin{aligned}\mathcal{F}(k_1 f_1 + k_2 f_2)(\omega) &= \int_{-\infty}^{\infty} (k_1 f_1(t) + k_2 f_2(t))\, e^{-i\omega t}\, dt \\ &= k_1 \int_{-\infty}^{\infty} f_1(t)\, e^{-i\omega t}\, dt + k_2 \int_{-\infty}^{\infty} f_2(t)\, e^{-i\omega t}\, dt \\ &= k_1 F_1(\omega) + k_2 F_2(\omega).\end{aligned}$$

(F_1) **Linearity:** $\mathcal{F}(k_1 f_1 + k_2 f_2)(\omega) = k_1 F_1(\omega) + k_2 F_2(\omega)$

Important: Linearity means that the spectrum of a superposition of two time functions consists of the corresponding superposition of the spectra.

Example 18.7. Find the spectrum of the function $4\, rect\left(\frac{t}{T}\right) + 3\, e^{-\alpha|t|}$:

Using Examples 18.1 and 18.4 and applying the property (F_1), we get

$$\begin{aligned}\mathcal{F}\left(4\, rect\left(\tfrac{t}{T}\right) + 3\, e^{-\alpha|t|}\right)(\omega) &= 4\,\mathcal{F}\left(rect\left(\tfrac{t}{T}\right)\right)(\omega) + 3\,\mathcal{F}\left(e^{-\alpha|t|}\right)(\omega) \\ &= 8\,\frac{\sin(\omega T)}{\omega} + 6\,\frac{\alpha}{\alpha^2 + \omega^2}\,. \end{aligned}$$ □

18.3.2 Symmetry Property

(F_2) **Symmetry:** $\mathcal{F}(\mathcal{F}(f))(t) = 2\pi\, f(-t)$

The symmetry property states that the Fourier transform applied twice to a function f will give the function f as the result, but with the factor 2π

and a negative argument. Using the Fourier integral (FI) we conclude that

$$\begin{aligned}\mathcal{F}(F)(t) &= \int_{-\infty}^{\infty} F(\omega)\, e^{-i\,\omega\,t}\, d\omega \\ &= 2\pi\,\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\,\omega\,(-t)}\, d\omega = 2\pi\, f(-t)\,.\end{aligned}$$

The symmetry property is used to calculate the Fourier transform of functions which are themselves Fourier transforms:

Examples 18.8.

① With $\mathcal{F}\left(rect\left(\frac{t}{a}\right)\right)(\omega) = 2\,\frac{\sin(\omega\,a)}{\omega}$ we get for the rectangle function $rect$:

$$\begin{aligned}\mathcal{F}\left(2\,\frac{\sin(\omega\,a)}{\omega}\right)(t) &= \mathcal{F}\left(\mathcal{F}\left(rect\left(\frac{t}{a}\right)\right)\right)(t) \\ &= 2\pi\, rect\left(-\frac{t}{a}\right) = 2\pi\, rect\left(\frac{t}{a}\right).\end{aligned}$$

So, after swapping the variables ω and t, the result is

$$\boxed{\mathcal{F}\left(\frac{\sin(a\ t)}{t}\right)(\omega) = \pi\, rect\left(\frac{\omega}{a}\right).}$$

② From Example 18.4 we conclude with $\alpha = 1$

$$\mathcal{F}\left(e^{-|t|}\right)(\omega) = \frac{2}{1+\omega^2}$$

and with (F_2)

$$\mathcal{F}\left(\frac{2}{1+\omega^2}\right)(t) = \mathcal{F}\left(\mathcal{F}\left(e^{-|t|}\right)\right)(t) = 2\pi\, e^{-|-t|}.$$

After swapping the variables we get

$$\boxed{\mathcal{F}\left(\frac{1}{1+t^2}\right)(\omega) = \pi\, e^{-|\omega|}.}$$

③ We calculate the definite integral $\displaystyle\int_0^{\infty}\frac{\sin x}{x}\, dx = \frac{\pi}{2}$ using the Fourier transform. We start with the result of Example 18.8 ①

$$\mathcal{F}\left(\frac{\sin t}{t}\right)(\omega) = \pi\, rect(\omega)\,.$$

According to the Note 3(i), for the even function $\frac{\sin t}{t}$ we get

$$\mathcal{F}\left(\frac{\sin t}{t}\right)(\omega) = 2\int_0^\infty \frac{\sin t}{t}\,\cos(\omega\,t)\;dt.$$

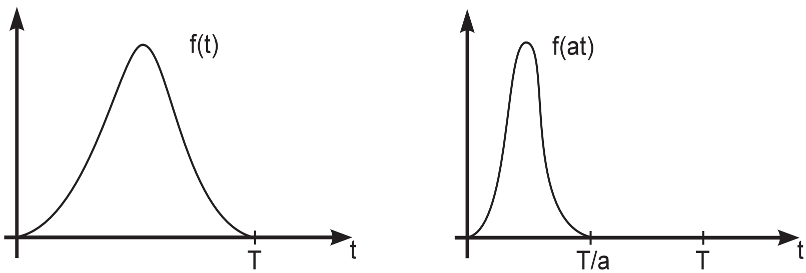
So we evaluate this expression at $\omega = 0$:

$$2\int_0^\infty \frac{\sin t}{t}\,dt = \pi\,rect\,(0) = \pi$$

which is the value of the definite integral. □

18.3.3 Scaling Property

We examine the Fourier transform (the spectrum) of a function $f\,(a\,t)$ which is obtained from $f\,(t)$ by **compression** $(a > 1)$ or **expansion** $(0 < a < 1)$.



Function $f(t)$ Compressed function $f(a\,t)$

Figure 18.5. About the scaling property

For $a > 0$ the Fourier transform of f is calculated with the substitution $\xi = a\,t$

$$\mathcal{F}\left(f\,(a\,t)\right)(\omega) = \int_{-\infty}^{\infty} f\,(a\,t)\;e^{-i\,\omega\,t}\,dt = \int_{-\infty}^{\infty} f\,(\xi)\;e^{-i\,\omega\,\frac{\xi}{a}}\,\frac{d\xi}{a} = \frac{1}{a}\,F\left(\frac{\omega}{a}\right).$$

The argumentation for $a < 0$ is analogous, so the following applies

(F_3) **Scaling:** $\mathcal{F}\left(f\,(a\,t)\right)(\omega) = \frac{1}{|a|}\,F\left(\frac{\omega}{a}\right), \qquad a \in \mathbb{R}_{\neq 0}$

The scaling property indicates that in the compressed signal $f\,(a\,t)$, where $a > 1$, the frequencies $F\left(\frac{\omega}{a}\right)$ occur.

18.3.4 Displacement Properties

Time Shift

The shift theorem is a statement about the Fourier transform of a time-shifted function $f(t-t_0)$.

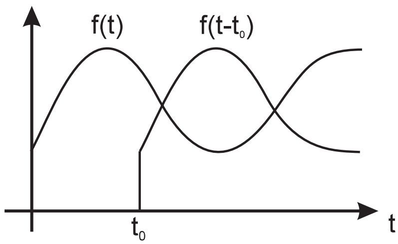


Figure 18.6. Function $f(t)$ and function $f(t-t_0)$ shifted by t_0

With the substitution $\xi = t - t_0$ we compute:

$$\mathcal{F}(f(t-t_0))(\omega) = \int_{-\infty}^{\infty} f(t-t_0)\, e^{-i\,\omega\, t}\, dt = \int_{-\infty}^{\infty} f(\xi)\, e^{-i\,\omega\,(\xi+t_0)}\, d\xi.$$

The exponential term is split into two factors. We exclude $e^{-i\,\omega\, t_0}$ from the integral because it is independent of the integration variable

$$\mathcal{F}(f(t-t_0))(\omega) = e^{-i\,\omega\, t_0} \int_{-\infty}^{\infty} f(\xi)\, e^{-i\,\omega\,\xi}\, d\xi = e^{-i\,\omega\, t_0}\, F(\omega).$$

Consequently, the Fourier transform of a time-shifted function is:

(F_4) **Time Shift:** $\mathcal{F}(f(t-t_0))(\omega) = e^{-i\,\omega\, t_0}\, F(\omega)$

Using $F(\omega) = |F(\omega)|\, e^{i\,\varphi(\omega)}$, the spectrum of the time-shifted function $f(t-t_0)$ is

$$\begin{aligned}\mathcal{F}(f(t-t_0))(\omega) &= e^{-i\,\omega\, t_0}\, |F(\omega)|\, e^{i\,\varphi(\omega)} \\ &= |F(\omega)|\, e^{-i\,(\varphi(\omega)\,-\omega\, t_0)}.\end{aligned}$$

The spectrum of $f(t-t_0)$ has the same amplitude as the spectrum of $f(t)$ only the phase is shifted by $\omega\, t_0$. This means that the same frequencies occur with the same amplitude, but out of phase.

Example 18.9. Find the spectrum of the square wave signal $f(t) = rect\left(\frac{t}{T}\right)$ shifted by $t_0 = T$:

$$\mathcal{F}(f(t-T)) = e^{-i\,\omega\,T}\,\mathcal{F}\left(rect\left(\frac{t}{T}\right)\right)(\omega) = e^{-i\,\omega\,T}\,2\,\frac{\sin(\omega\,T)}{\omega}. \qquad \square$$

Frequency Shift

This property makes a statement about the spectrum of a function $f(t)$ multiplied by $e^{i\,\omega_0\,t}$:

$$\begin{aligned}\mathcal{F}\left(e^{i\,\omega_0\,t}\,f(t)\right)(\omega) &= \int_{-\infty}^{\infty} e^{i\,\omega_0\,t}\,f(t)\,e^{-i\,\omega\,t}\,dt\\ &= \int_{-\infty}^{\infty} f(t)\,e^{-i\,(\omega-\omega_0)\,t}\,dt = F(\omega-\omega_0).\end{aligned}$$

So the spectrum of a function multiplied by $e^{i\,\omega_0\,t}$ is:

(F_5) **Frequency Shift:** $\mathcal{F}\left(e^{i\,\omega_0\,t}\,f(t)\right)(\omega) = F(\omega-\omega_0)$

This frequency shift property means that the spectrum of the function multiplied by $e^{i\,\omega_0\,t}$ is the same as the spectrum of the original function shifted by ω_0. As an application of frequency shifting, the modulation property is discussed:

18.3.5 Modulation Property

We want to find the spectrum of the *amplitude modulated* signal

$$f(t)\,\cos(\omega_0\,t).$$

We start with Euler's formula

$$\cos(\omega_0\,t) = \frac{1}{2}\left(e^{i\,\omega_0\,t} + e^{-i\,\omega_0\,t}\right)$$

and use the linearity property (F_1) and the frequency shift property (F_5)

$$\begin{aligned}\mathcal{F}(\cos(\omega_0\,t)\,f(t))(\omega) &= \mathcal{F}\left(\frac{1}{2}\,e^{i\,\omega_0\,t}\,f(t) + \frac{1}{2}\,e^{-i\,\omega_0\,t}\,f(t)\right)(\omega)\\ &= \frac{1}{2}\,\mathcal{F}\left(e^{i\,\omega_0\,t}\,f(t)\right)(\omega) + \frac{1}{2}\,\mathcal{F}\left(e^{-i\,\omega_0\,t}\,f(t)\right)(\omega)\\ &= \frac{1}{2}\left(F(\omega-\omega_0) + F(\omega+\omega_0)\right).\end{aligned}$$

(F_6) **Modulation:**

$$\mathcal{F}\left(f\left(t\right)\,\cos\left(\omega_0\,t\right)\right)\left(\omega\right)=\frac{1}{2}\,\left(F\left(\omega+\omega_0\right)+F\left(\omega-\omega_0\right)\right)$$

The spectrum of the $\cos\left(\omega_0\,t\right)$**-modulated signal is the spectrum of the original function shifted by** $\pm\omega_0$. The spectrum is shifted exactly by the modulation frequency ω_0!

Examples 18.10. Find the spectrum of a rectangular pulse modulated with $\cos\left(\omega_0\,t\right)$.

According to Example 18.1 the spectrum of the rectangle is

$$\mathcal{F}\left(rect\left(\frac{t}{T}\right)\right)\left(\omega\right)=2\,\frac{\sin\left(\omega\,T\right)}{\omega}.$$

So, using the modulation property

$$\mathcal{F}\left(\cos\left(\omega_0\,t\right)\cdot rect\left(\frac{t}{T}\right)\right)\left(\omega\right)=\frac{\sin(T\,\left(\omega-\omega_0\right))}{\omega-\omega_0}+\frac{\sin(T\,\left(\omega+\omega_0\right))}{\omega+\omega_0}.$$

The amplitude modulation of the square wave corresponds to a shift of the spectrum to the right and to the left by the modulation frequency ω_0.

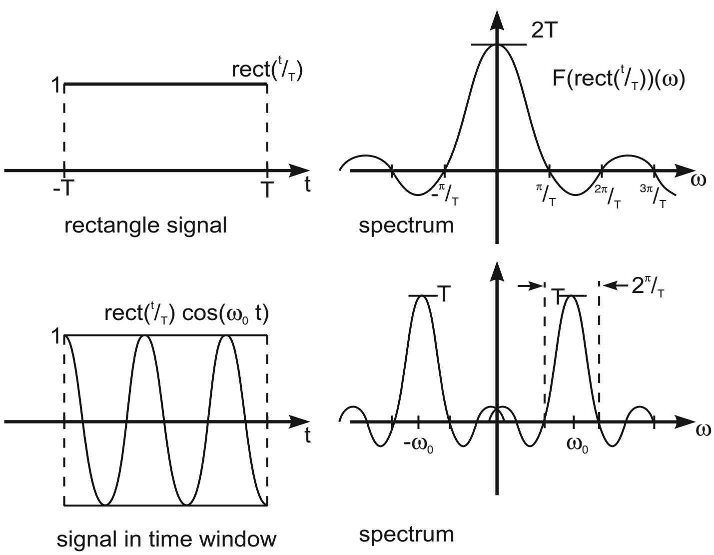


Figure 18.7. Spectrum of the amplitude modulated rectangular signal

Comments/Interpretation:

(1) When messages are transmitted, the signal $f(t)$ is often amplitude modulated, i.e. transmitted with a carrier frequency ω_0: $f(t)\cos(\omega_0 t)$. A receiver can determine the carrier frequency by transmitting an amplitude modulated square wave as a test signal. The spectrum of this signal is then the spectrum of the square wave shifted by the value of the carrier frequency ω_0.

(2) When we analyze real signals, we usually do not get a line spectrum, but a broadening of the lines. Even if we are looking at sine or cosine signals. This effect can be explained by our example:

Since it is not possible to measure the whole range $-\infty < t < \infty$ for all times, but only for a finite time interval, we analyze the function $rect\left(\frac{t}{T}\right)\cdot\cos(\omega_0\, t)$ instead of $\cos(\omega_0\, t)$. According to the Example 18.10 we then get the spectrum $2\,\frac{\sin(\omega\, T)}{\omega}$ shifted by ω_0. This spectrum has its maximum at ω_0, but with a finite width $\frac{2\pi}{T}$. Only in the case of $T \to \infty$ does the spectral width approach 0 and we get a line at ω_0. □

18.3.6 Fourier Transform of the Derivative

For the application of the Fourier transform to differential equations the Fourier transform of the derivative $\mathcal{F}(f')$ is required. As with the Laplace transform there is a relationship between $\mathcal{F}(f')$ and $\mathcal{F}(f)$. Using partial integration, we get for the derivative $\mathcal{F}(f')$

$$\begin{aligned}\mathcal{F}(f')(\omega) &= \int_{-\infty}^{\infty} f'(t)\; e^{-i\,\omega\,t}\,dt \\ &= f(t)\; e^{-i\,\omega\,t}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t)\;(-i\,\omega)\; e^{-i\,\omega\,t}\,dt.\end{aligned}$$

The precondition for using the Fourier transform is the asymptotic behavior of the function f, $\lim_{t\to\pm\infty} f(t) = 0$. So $f(t)\; e^{-i\,\omega\,t}\Big|_{-\infty}^{\infty} = 0$.

$$\Rightarrow \quad \mathcal{F}(f')(\omega) = i\,\omega \int_{-\infty}^{\infty} f(t)\; e^{-i\,\omega\,t}\,dt = i\,\omega\,\mathcal{F}(f)(\omega)\,.$$

Fourier Transform of the Derivative

(F_7) **Derivative:** $\mathcal{F}(f')(\omega) = (i\,\omega)\; F(\omega)$

Important: The spectrum of the derivative f' is equal to the spectrum of the function f multiplied by $i\,\omega$.

Repeated application of the derivative theorem leads inductively to the Fourier transform of the n-th derivative:

Fourier Transform of the n-th Derivative

(F_8) **n-th Derivative:** $\mathcal{F}\left(f^{(n)}\right)(\omega) = (i\,\omega)^n\ F(\omega)$

Application Example 18.11 . Given is the linear differential equation

$$y'(t) + \alpha\, y(t) = f(t)\ S(t)$$

with the constant coefficient α and a continuous function f. $S(t)$ is the step function.

We apply the Fourier transform to this differential equation and take into account the linearity property (F_1). Then on the left side we have

$$\mathcal{F}(y'(t) + \alpha\, y(t)) = \mathcal{F}(y'(t)) + \alpha\,\mathcal{F}(y(t))\,.$$

We replace $\mathcal{F}(y'(t)) = i\,\omega\,\mathcal{F}(y(t))$ and get

$$i\,\omega\,\mathcal{F}(y(t)) + \alpha\,\mathcal{F}(y(t)) = \mathcal{F}(f(t)\ S(t))$$

$$\Rightarrow\quad \mathcal{F}(y(t)) = \frac{1}{\alpha + i\,\omega}\,\mathcal{F}(f(t)\ S(t))\,.$$

This is the Fourier transform of the solution $y(t)$ as the product of $\mathcal{F}(f(t)\ S(t)) \cdot \frac{1}{\alpha + i\,\omega}$.

According to Example 18.2

$$\frac{1}{\alpha + i\,\omega} = \mathcal{F}\left(e^{-\alpha\, t}\, S(t)\right)(\omega)\,.$$

$$\Rightarrow\ \mathcal{F}(y(t)) = \mathcal{F}(f(t)\ S(t)) \cdot \mathcal{F}\left(e^{-\alpha\, t}\, S(t)\right).$$

So the problem arises: Which time function belongs to a product of frequency functions? The answer is given by the next theorem. □

18.3.7 Convolution Theorem

Given is the spectrum of the function f as the product of two frequency functions

$$\mathcal{F}(f) = \mathcal{F}(f_1) \cdot \mathcal{F}(f_2) .$$

The corresponding time function $f(t)$ is then an integral combination of the time functions $f_1(t)$ and $f_2(t)$:

$$f(t) = \int_{-\infty}^{\infty} f_1(\tau)\; f_2(t-\tau)\; d\tau \;,$$

the so-called **Convolution Integral**. The abbreviated notation for the convolution integral is

$$f(t) = (f_1 * f_2)(t) .$$

Convolution Theorem

The Fourier transform of the convolution

$$(f_1 * f_2)(t) := \int_{-\infty}^{\infty} f_1(\tau)\; f_2(t-\tau)\; d\tau$$

is the product of the transform of f_1 and f_2:

(F_9) **Convolution Theorem:** $\mathcal{F}(f_1 * f_2) = \mathcal{F}(f_1) \cdot \mathcal{F}(f_2)$

Proof:

$$\begin{aligned}\mathcal{F}(f_1 * f_2) &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f_1(\tau)\; f_2(t-\tau)\; d\tau \right) e^{-i\,\omega\,t}\, dt \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f_1(\tau)\; f_2(t-\tau)\; e^{-i\,\omega\,t}\, d\tau \right) dt.\end{aligned}$$

After swapping the integration sequence and the subsequent substitution $\xi(t) = t - \tau \quad (\hookrightarrow d\xi = dt)$ we obtain

$$\begin{aligned}\mathcal{F}(f_1 * f_2) &= \int_{-\infty}^{\infty} f_1(\tau) \left(\int_{-\infty}^{\infty} f_2(t-\tau)\; e^{-i\,\omega\,t}\, dt \right) d\tau \\ &= \int_{-\infty}^{\infty} f_1(\tau) \left(\int_{-\infty}^{\infty} f_2(\xi)\; e^{-i\,\omega\,(\xi+\tau)}\, d\xi \right) d\tau \\ &= \int_{-\infty}^{\infty} f_1(\tau)\; e^{-i\,\omega\,\tau}\, d\tau \cdot \int_{-\infty}^{\infty} f_2(\xi)\; e^{-i\,\omega\,\xi}\, d\xi \\ &= \mathcal{F}(f_1) \cdot \mathcal{F}(f_2) .\end{aligned}$$

□

Example 18.12. To complete Example 18.11, we look for the time function $y(t)$, which belongs to the spectrum of

$$\mathcal{F}(f(t)\,S(t)) \cdot \mathcal{F}\left(e^{-\alpha\,t}\,S(t)\right).$$

According to the convolution theorem, the time function $y(t)$ is the convolution of the two functions

$$f_1(t) = f(t)\,S(t) \quad \text{and} \quad f_2(t) = e^{-\alpha\,t}\,S(t):$$

$$\begin{aligned} y(t) &= (f_1 * f_2)(t) = \int_{-\infty}^{\infty} f_1(\tau)\,f_2(t-\tau)\,d\tau \\ &= \int_{-\infty}^{\infty} f(\tau)\,S(\tau)\,e^{-\alpha\,(t-\tau)}\,S(t-\tau)\,d\tau. \end{aligned}$$

Since $S(\tau) = 0$ for $\tau < 0$, the integration is to be calculated only from the lower integration limit $\tau = 0$. For $\tau > 0$, $S(\tau) = 1$, so

$$y(t) = \int_0^{\infty} f(\tau)\,e^{-\alpha\,t}\,e^{\alpha\,\tau}\,S(t-\tau)\,d\tau.$$

We split the integral into two parts

$$\begin{aligned} y(t) = &\int_0^{t} f(\tau)\,e^{-\alpha\,t}\,e^{\alpha\,\tau}\,S(t-\tau)\,d\tau \\ &+ \int_t^{\infty} f(\tau)\,e^{-\alpha\,t}\,e^{\alpha\,\tau}\,S(t-\tau)\,d\tau\,. \end{aligned}$$

The second integral disappears because $\tau > t$ and $S(t-\tau) = 0$ for $\tau > t$. In the first integral $0 < \tau < t$ and for this range $S(t-\tau) = 1$:

$$\Rightarrow \quad \boxed{y(t) = e^{-\alpha\,t}\int_0^{t} e^{\alpha\,\tau}\,f(\tau)\,d\tau.}$$

□

Conclusion: $y(t)$ is, according to Example 18.11, the solution of the inhomogeneous first-order differential equation

$$y'(t) + \alpha\,y(t) = f(t) \qquad \text{with} \quad y(0) = 0.$$

The above formula is also obtained by varying the constants for a linear differential equation with constant coefficients and vanishing initial conditions (see Volume 2, Section 13.3).

Remarks:

(1) The convolution integral of two functions f_1 and f_2 is commutative:

$$\boxed{f_1 * f_2 = f_2 * f_1.}$$

With the substitution $\xi(\tau) = (t-\tau) \quad (\hookrightarrow d\xi = -d\tau)$ we obtain

$$\begin{aligned}(f_1 * f_2)(t) &= \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} f_1(t-\xi)\, f_2(\xi)\, d\xi = (f_2 * f_1)(t).\end{aligned}$$

(2) For integrals of the form

$$\int_{-\infty}^{\infty} g(t-\tau) \quad S(\tau)\, d\tau$$
$$= \underbrace{\int_{-\infty}^{0} g(t-\tau) \underbrace{S(\tau)}_{=0} d\tau}_{=0} + \int_{0}^{\infty} g(t-\tau) \underbrace{S(\tau)}_{=1} d\tau$$

the discontinuous expression $S(0)$ occurs at the upper (lower) limit of the 1st (2nd) partial integral at $\tau = 0$. This function value has **no** effect on the result of the integration, since the area under a function does not change when the function is changed at a finite number of points.

Example 18.13 (Geometric Interpretation). To get a more visual interpretation of the convolution integral, we choose $f_1(t) = S(t)$ and $f_2(t) = S(t)$ and compute

$$(f_1 * f_2)(t) = \int_{-\infty}^{\infty} S(\tau)\, S(t-\tau)\, d\tau.$$

To determine the integral, we consider the sequence of images in Fig. 18.8 (a) to Fig. 18.8 (d).

(a) In (a) the function $S(\tau)$ is drawn. The step function is zero for $\tau < 0$ and one for $\tau > 0$.

(b) The function $S(-\tau)$ is derived from $S(\tau)$ by mirroring (= **convolution**) on the y-axis, see Fig. (b): $S(-\tau) = 0$ for $\tau > 0$ and $S(-\tau) = 1$ for $\tau < 0$.

(c) $S(t-\tau)$ is created from $S(-\tau)$ by shifting the graph of $S(-\tau)$ to the right by t, see Fig. (c).

(d) Then, the product of $S(\tau)$ and $S(t-\tau)$ is shown in (d): $S(t-\tau) \cdot S(\tau) = 0$ for $\tau < 0$ and for $\tau > t$. For the integral $\int_{-\infty}^{\infty} S(\tau)\ S(t-\tau)\ d\tau$ with the integration variable τ only the range between $0 \leq \tau \leq t$ remains non-zero and has the integral value t.

$$\Rightarrow \quad (f_1 * f_2)(t) = \int_{-\infty}^{\infty} S(\tau)\ S(t-\tau)\ d\tau = t\, S(t).$$

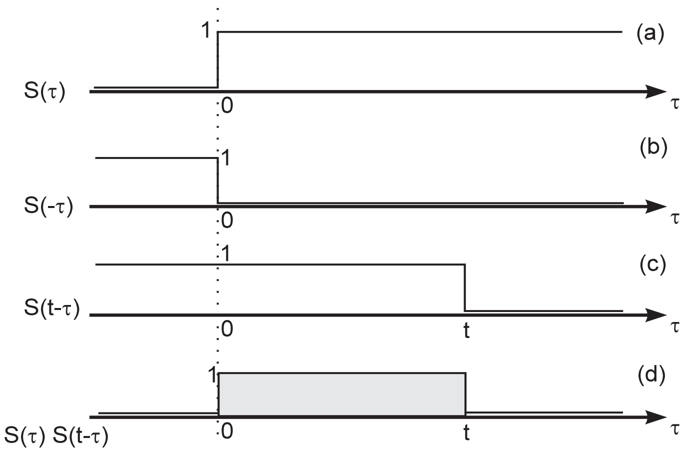


Figure 18.8. The geometric interpretation of the convolution integral

Example 18.14 (Convolution Integral). Find the convolution integral $f * h$, where f and h are the functions shown in Fig. 18.9.

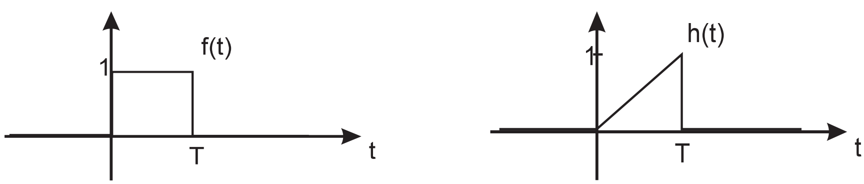


Figure 18.9. Functions f and h

We graphically determine the convolution

$$(f * h)(t) = \int_{-\infty}^{\infty} f(\tau)\ h(t-\tau)\ d\tau.$$

First, as in the Example 18.13, we mirror the function $h(\tau)$ to get $h(-\tau)$. This function is then shifted along the τ-axis by the value T and then multiplied by the rectangle function. These four steps are shown schematically in Fig. 18.10.

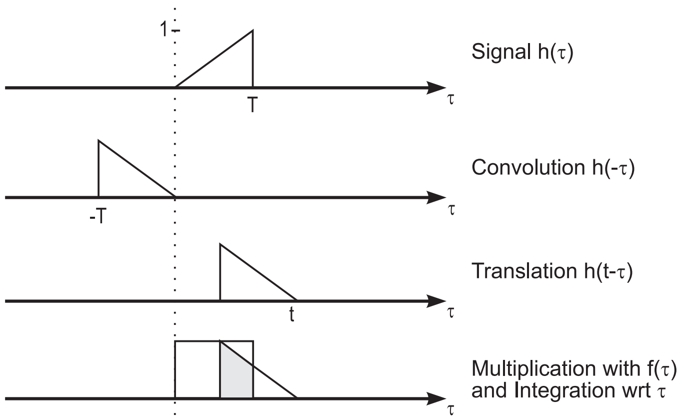


Figure 18.10. Convolution of the rectangle function with the triangle function

As we can see in Fig. 18.11, there are four cases in the computation of the convolution integral:

(1) $t \leq 0$: The function $h(t-\tau)$ does not overlap with the function $f(\tau)$.

(2) $0 \leq t \leq T$: The function $h(t-\tau)$ dips with its peak into the graph of the function $f(\tau)$.

(3) $T \leq t \leq 2T$: The function $h(t-\tau)$ exits the graph of the function $f(\tau)$.

(4) $2T \leq t$: The function $h(t-\tau)$ does not overlap with the function $f(\tau)$.

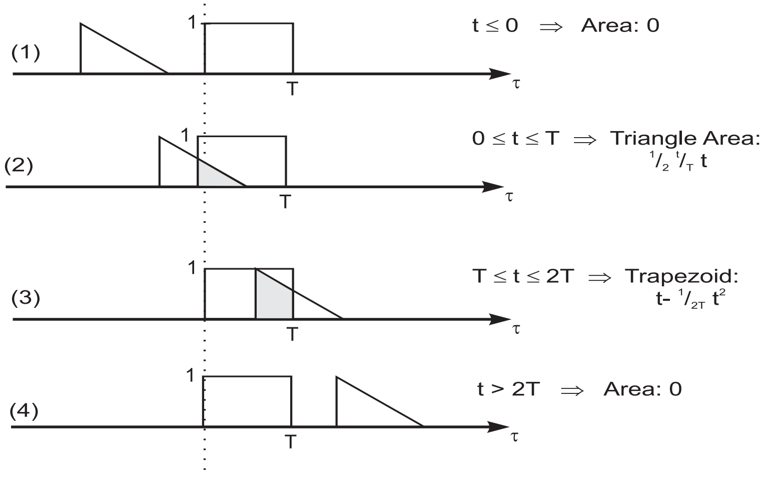


Figure 18.11. Four cases for determining the convolution integral

The result of the convolution can be written as

$$\Rightarrow (f * h)(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ \frac{1}{2T} t^2 & \text{for } 0 \leq t \leq T \\ t - \frac{1}{2T} t^2 & \text{for } T \leq t \leq 2T \\ 0 & \text{for } t > 2T \end{cases}$$

and graphically:

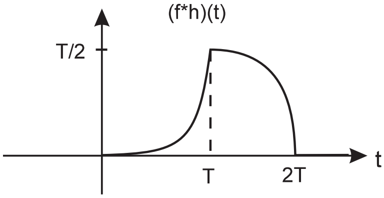


Figure 18.12. Convolution integral $(f * h)(t)$ □

Application Example 18.15 (Solving Differential Equations).

The Fourier transform is not only used to solve first-order differential equations, but also higher order linear differential equations: For example, a second-order differential equation is given by

$$y''(t) - y(t) = f(t)\, S(t).$$

Step 1: We apply the Fourier transform

$$\mathcal{F}(y''(t))(\omega) = (i\,\omega)^2\, \mathcal{F}(y(t))(\omega)$$

and use the linearity property

$$(i\,\omega)^2\, \mathcal{F}(y(t)) - \mathcal{F}(y(t)) = \mathcal{F}(f(t)\, S(t)).$$

Step 2: The algebraic equation for the Fourier transform $\mathcal{F}(y)$ is solved:

$$\begin{aligned} \hookrightarrow \quad \mathcal{F}(y(t)) &= \frac{1}{-1-\omega^2}\, \mathcal{F}(f(t)\, S(t)) \\ &= -\frac{1}{2}\, \frac{2}{1+\omega^2}\, \mathcal{F}(f(t)\, S(t)) \\ &= -\frac{1}{2}\, \mathcal{F}\left(e^{-1\cdot|t|}\right) \cdot \mathcal{F}(f(t)\, S(t)), \end{aligned}$$

because according to Example 18.4: $\mathcal{F}\left(e^{-\alpha|t|}\right) = \dfrac{2\,\alpha}{\alpha^2+\omega^2}$.

Step 3: Inverse transform: Using the convolution theorem, we obtain the solution of the second-order inhomogeneous differential equation

$$\begin{aligned} y(t) &= -\frac{1}{2}\left(e^{-|t|}\right) * (f(t)\,S(t)) \\ &= -\frac{1}{2}\int_{-\infty}^{\infty} e^{-|\tau|}\, f(t-\tau)\,S(t-\tau)\,d\tau \\ &= -\frac{1}{2}\int_{-\infty}^{t} e^{-|\tau|}\, f(t-\tau)\,d\tau. \end{aligned}$$

Then, of course, the integral on the right has to be calculated for a given $f(t)$ to get an expression for $y(t)$. □

Summary: Properties of the Fourier Transform

$F(\omega)$ denotes the Fourier transform of f, $F_1(\omega)$ and $F_2(\omega)$ the transforms of f_1 and f_2, respectively.

(F_1) **Linearity:** $\mathcal{F}(k_1 f_1 + k_2 f_2)(\omega) = k_1 F_1(\omega) + k_2 F_2(\omega)$

(F_2) **Symmetry:** $\mathcal{F}(\mathcal{F}(f))(t) = 2\pi f(-t)$

(F_3) **Scaling:** $\mathcal{F}(f(a\,t))(\omega) = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$ $\quad a \in \mathbb{R}_{\neq 0}$

(F_4) **Time Shift:** $\mathcal{F}(f(t-t_0))(\omega) = e^{-i\,t_0\,\omega} F(\omega)$

(F_5) **Frequency Shift:** $\mathcal{F}\left(e^{i\,\omega_0\,t} f(t)\right)(\omega) = F(\omega - \omega_0)$

(F_6) **Modulation:** $\mathcal{F}(f(t)\cos(\omega_0\,t))(\omega) = \frac{1}{2}\left(F(\omega+\omega_0) + F(\omega-\omega_0)\right)$

(F_7) **Derivative:** $\mathcal{F}(f')(\omega) = i\,\omega\,F(\omega)$

(F_8) ***n*-th Derivative:** $\mathcal{F}\left(f^{(n)}\right)(\omega) = (i\,\omega)^n\, F(\omega)$

($F9$) **Convolution:** $\mathcal{F}(f_1 * f_2)(\omega) = F_1(\omega)\cdot F_2(\omega)$, $\quad (f_1 * f_2)(t) = \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau$

18.4 Fourier Transform of the Delta Function

When analyzing systems with respect to their frequency response, we need a function $\delta(t)$ that contains all frequencies at the same amplitude. With this function as the input signal, the system will be excited equally at all frequencies. With any other input signal, we would excite the system differently at different frequencies. When the output signal is frequency analyzed, this information characterizes the frequency response of the system. So we need a function that contains all frequencies with the same amplitude:

$$\mathcal{F}(\delta)(\omega) \equiv 1.$$

This requirement leads to the *Dirac* or *delta function*.

18.4.1 Delta Function

In systems theory and many other areas of engineering and physics, the impulse function $\delta(t)$ plays an important role. This function is often called the delta function (because of its notation) or the Dirac function, after its inventor. In physics, this function is said to have the following properties:

Properties of the Delta Function:

(1) $\delta(t) = 0 \qquad \text{for } t \neq 0$

(2) $\delta(0) = \infty$

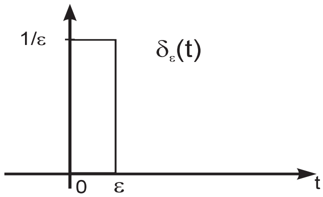
(3) $\displaystyle\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

(4) $\displaystyle\int_{-\infty}^{\infty} \delta(t)\, f(t)\, dt = f(0) \quad \text{for any continuous function } f : \mathbb{R} \to \mathbb{R}.$

Of course, this is not a function in the usual sense, and the theory of *distributions* (= generalized functions) cannot be discussed here. There is only this much: The last equation is the essential one, (3) is the special case $f = 1$, equations (1) and (2) have little meaning!

We will try to explain this phenomenon with the following consideration. A freely moving body is subjected to an impact $F(t) = \frac{1}{\varepsilon}$ with constant force in the finite time ε. Let us define the family of functions $\delta_\varepsilon(t)$

$$\delta_\varepsilon(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{\varepsilon} & \text{for } 0 < t < \varepsilon \\ 0 & \text{for } t > \varepsilon \end{cases}$$



With this notation, the pulse function $F(t)$ can be written as

$$F(t) = m\,v_0\,\delta_\varepsilon(t)\,.$$

The total transmitted impulse is

$$\Delta p = \int_{-\infty}^{\infty} F(t)\;dt = m\,v_0 \int_{-\infty}^{\infty} \delta_\varepsilon(t)\;dt = m\,v_0 \int_0^\varepsilon \frac{1}{\varepsilon}\,dt = m\,v_0.$$

The result is independent of the duration ε! For $\varepsilon \to 0$ the same impulse is transmitted as for a finite ε. The limit of the function family $\delta_\varepsilon(t)$ for $\varepsilon \to 0$ gives the **delta function** (**Dirac function** sometimes just called **impulse function**).

$$\delta(t) \;:= \lim_{\varepsilon\to 0} \delta_\varepsilon(t).$$

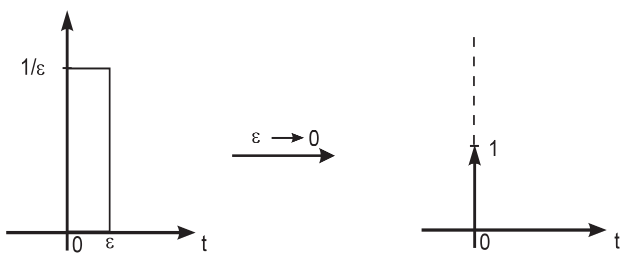


Figure 18.13. From rectangular impulse to delta function

The function defined in this way has properties (1) - (4): Properties (1) to (3) are obviously fulfilled and property (4) can be checked as follows:

$$\int_{-\infty}^{\infty} \delta_\varepsilon(t)\;f(t)\;dt = \int_0^\varepsilon \frac{1}{\varepsilon}\,f(t)\;dt = f(\xi)\int_0^\varepsilon \frac{1}{\varepsilon}\,dt = f(\xi)$$

due to the mean value theorem of integral calculus, where f is evaluated at a suitable, but unknown intermediate position $\xi \in [0, \varepsilon]$. In the limit $\varepsilon \to 0$ we obviously get $\xi \to 0$, since $0 \le \xi \le \varepsilon$, and secondly $\delta_\varepsilon(t) \to \delta(t)$. This is

$$\int_{-\infty}^{\infty} \delta(t)\ f(t)\ dt = \int_{-\infty}^{\infty} \lim_{\varepsilon \to 0} \delta_{\varepsilon}(t)\ f(t)\ dt$$
$$= \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \delta_{\varepsilon}(t)\ f(t)\ dt = f(0).$$

It is important to note that the delta function is a *functional* defined by the integral property

$$\int_{-\infty}^{\infty} f(t)\ \delta(t)\ dt = f(0)$$

which is valid for any continuous function $f(t)$. This is the *universal* property of the delta function. More generally:

Property of the Delta Function

$$\int_{-\infty}^{\infty} f(\tau)\ \delta(t-\tau)\ d\tau = f(t).$$

Because

$$\int_{-\infty}^{\infty} f(\tau)\ \delta(t-\tau)\ d\tau = \int_{-\infty}^{\infty} f(t+\xi)\ \delta(\xi)\ d\xi = f(t+\xi)|_{\xi=0} = f(t).$$

This relationship is called the **hide property** of the delta function, because a single value of the function f is "suppressed" by t.

Remark: The hide property can also be used to show that the delta function is independent of the selected function family $\delta_{\varepsilon}(t)$: Because if $\delta_1(t)$ is the limit value of one function family $\delta_{1\varepsilon}(t)$ and $\delta_2(t)$ is the limit value of another function family $\delta_{2\varepsilon}(t)$, then due to the hide property of $\delta_1(t)$ we get

$$\delta_2(t) = \int_{-\infty}^{\infty} \delta_1(t-\tau)\ \delta_2(\tau)\ d\tau = \int_{-\infty}^{\infty} \delta_1(\xi)\ \delta_2(t-\xi)\ d\xi = \delta_1(t).$$

The last equality holds because of the hide property of $\delta_2(t)$. Hence, $\delta_2(t) = \delta_1(t)$ for all $t \in \mathbb{R}$. □

18.4.2 Fourier Transform of the Delta Function

The basic property of the delta function is that for any continuous function $\varphi(t)$

$$\int_{-\infty}^{\infty} \delta(t)\, \varphi(t)\, dt = \varphi(0)\,.$$

We apply this rule specifically to $\varphi(t) = e^{-i\,\omega\,t}$

$$\int_{-\infty}^{\infty} \delta(t)\, e^{-i\,\omega\,t}\, dt = \left. e^{-i\,\omega\,t}\right|_{t=0} = 1.$$

The left side of the equation is the Fourier transform of the function $\delta(t)$; so the Fourier transform of the delta function is the constant function

$$\mathcal{F}(\delta)(\omega) = 1.$$

The delta function is therefore exactly the function needed for system analysis, containing all frequencies with the same amplitude 1.

Remarks:

(1) If we substitute the result of this transformation into the inverse formula of the Fourier transform (FI), we obtain

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(\delta)(\omega)\, e^{i\,\omega\,t}\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\,\omega\,t}\, d\omega. \qquad (*)$$

We compute the improper integral $(*)$

$$\begin{aligned} \delta(t) &= \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{-1/\varepsilon}^{1/\varepsilon} e^{i\,\omega\,t}\, d\omega = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \frac{1}{i\,t} \left(e^{i\,\omega\,t}\right)\Big|_{\omega=-1/\varepsilon}^{\omega=1/\varepsilon} \\ &= \lim_{\varepsilon \to 0} \frac{1}{2\pi} \frac{1}{i\,t} \left(e^{i\,\frac{1}{\varepsilon}\,t} - e^{-i\,\frac{1}{\varepsilon}\,t}\right) = \lim_{\varepsilon \to 0} \frac{\sin\left(\frac{1}{\varepsilon}\,t\right)}{\pi\,t} \end{aligned}.$$

So the delta function can be identified as the limit $\varepsilon \to 0$ of the family of functions $\frac{\sin\left(\frac{1}{\varepsilon}\,t\right)}{\pi\,t}$ already discussed graphically in Example 18.1.

(2) If the Fourier transform is applied again to the equation $(*)$, the symmetry property (F_2) gives the Fourier transform of the constant function:

$$\mathcal{F}(1)(\omega) = 2\pi\,\delta(\omega).$$

Examples 18.16.

① We calculate the Fourier transform of the step function

$$S(t) = \frac{1}{2} + \frac{1}{2}\, sign\,(t)\,.$$

According to Example 18.6 the Fourier transform of $f(t) = \frac{1}{t}$ is

$$\mathcal{F}\left(\frac{1}{t}\right)(\omega) = -i\,\pi\, sign\,(\omega)\,.$$

We apply the symmetry property (F_2) on this identity to get the transform of $sign(t)$

$$\begin{aligned}\mathcal{F}\,(sign\,(t))\,(\omega) &= \frac{1}{-i\,\pi}\mathcal{F}(\mathcal{F}(\frac{1}{t}))\,(\omega)\\ &= \frac{1}{-i\,\pi}2\pi\left(\frac{1}{-\omega}\right) = \frac{2}{i\,\omega}.\end{aligned}$$

Using the linearity (F_1) of the Fourier transform we obtain

$$\mathcal{F}\left(\frac{1}{2} + \frac{1}{2}\, sign\,(t)\right)(\omega) = \frac{1}{2}\,\mathcal{F}\,(1)\,(\omega) + \frac{1}{2}\,\mathcal{F}\,(sign\,(t))\,(\omega)$$

$$\Rightarrow \quad \mathcal{F}\,(S\,(t))\,(\omega) = \pi\,\delta\,(\omega) + \frac{1}{i\,\omega}.$$

② With the shift properties (F_4) and (F_5) we calculate the Fourier transforms of the shifted delta function and $e^{i\,\omega_0\,t}$

$$\mathcal{F}\,(\delta\,(t - t_0))\,(\omega) = e^{-i\,\omega\,t_0},$$

$$\mathcal{F}\left(e^{i\,\omega_0\,t}\right)(\omega) = 2\pi\,\delta\,(\omega - \omega_0)\,.$$

③ Applying the modulation property (F_6) for the function $f(t) = 1$, we conclude

$$\mathcal{F}\,(\cos\,(\omega_0\,t))\,(\omega) = \pi\,\delta\,(\omega - \omega_0) + \pi\,\delta\,(\omega + \omega_0).$$

This result means that $\cos(\omega_0\,t)$ contains only one frequency ω_0. The Fourier transform gives only a single line as the spectrum of $\cos(\omega_0\,t)$. However, if the spectrum of $\cos(\omega_0\,t)$ is measured in a finite time interval, this line will be expanded according to Example 18.10!

④ Using the hide property of the delta function we can evaluate

$$\begin{aligned}\delta(t-t_0) * f(t) &= \int_{-\infty}^{\infty} \delta(\tau - t_0)\, f(t-\tau)\, d\tau \\ &= f(t-\tau)\big|_{\tau=t_0} = f(t-t_0)\,.\end{aligned}$$ □

Example 18.17 (Fourier Transform of Periodic Functions). Let f be a T-periodic function. According to the complex formulation of Fourier series for periodic functions with $\omega_0 = \frac{2\pi}{T}$, we have the identity

$$f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{i\,n\,\omega_0\,t}.$$

Because of the linearity of the Fourier transform, we compute according to Example 18.16 ②

$$\mathcal{F}(f(t))(\omega) = \sum_{n=-\infty}^{\infty} c_n\, \mathcal{F}\left(e^{i\,n\,\omega_0\,t}\right)(\omega) = 2\pi \sum_{n=-\infty}^{\infty} c_n\, \delta(\omega - n\,\omega_0)\,.$$

The Fourier transform of a periodic function is given by the spectrum of the Fourier series. Therefore, the **spectrum** of any signal $f(t)$ is the **Fourier transform** $F(\omega)$ of the signal. □

18.4.3 Representation of the Fourier Transform of $\delta(t)$

Visualization: The transition of the spectrum of the function family $\delta_\varepsilon(t)$ for $\varepsilon \to 0$ to the spectrum of the delta function will be visualized. According to Example 18.1 we have the relationship between a rectangular function with length T and height $\frac{1}{T}$ and its spectrum

$$F_T(\omega) = \mathcal{F}\left(\tfrac{1}{T}\, rect(\tfrac{2\,t}{T})\right)(\omega) \quad = \frac{\sin(\omega\, T/2)}{\omega\, T},$$

which is also shown in Fig. 18.14.

For $T \to 0$ the time function $\delta_T(t) = \frac{1}{T}\, rect(\frac{2\,t}{T})$ converges to the delta function. The spectrum $F_T(\omega)$ is now discussed as a function of the parameter T: The maximum amplitude is 1, independent of T. The first two zeros of the spectrum are $\omega = \pm\frac{2\,\pi}{T}$; dependent on T.

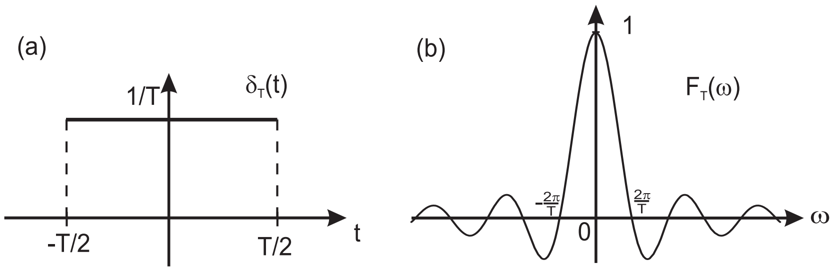


Figure 18.14. a) Time function $\delta_T(t)$, b) Spectrum $F_T(\omega)$

For $T \to 0$ these zeros go towards $\pm\infty$ which means that $F_T\,(\omega)$ approaches the constant function 1:

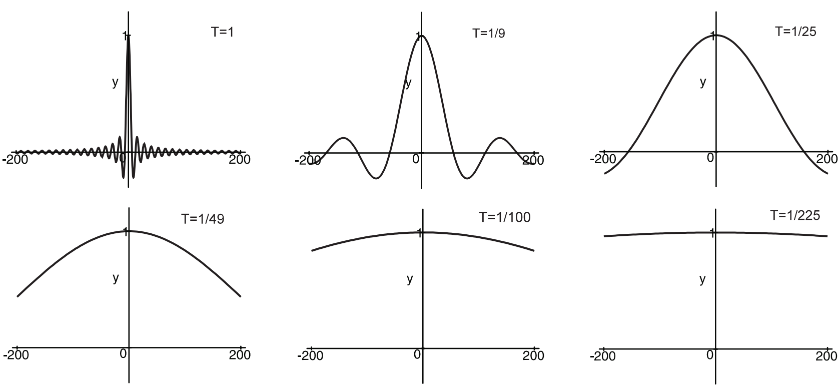
$$\delta_T\,(t) \quad \stackrel{T\to 0}{\longrightarrow} \quad \delta\,(t),$$

$$F_T\,(\omega) \quad \stackrel{T\to 0}{\longrightarrow} \quad 1.$$

The transition $F_T\,(\omega) \to 1$ for $T \to 0$ can be illustrated quite well. We use the spectrum

$$F_T := 2\,\frac{\sin\left(\frac{1}{2}\,\omega\,T\right)}{\omega\,T}$$

and vary T.



Animation: The images show that the maximum amplitude always remains at 1, but the first zero (and all the other zeros) move towards $\pm\infty$, so the limit function is the constant function $F\,(\omega) = 1$. □

18.4.4 Correspondences of the Fourier Transform

Function $f(t)$	Fourier Transform $\mathbf{F}(\omega) = \mathcal{F}(\mathbf{f})(\omega)$
$\delta(t)$	1
1	$2\pi\,\delta(\omega)$
$\cos(\omega_0\,t)$	$\pi\,\delta(\omega-\omega_0)+\pi\,\delta(\omega+\omega_0)$
$\sin(\omega_0\,t)$	$\frac{\pi}{i}\,\delta(\omega-\omega_0)-\frac{\pi}{i}\,\delta(\omega+\omega_0)$
$sign(t)$	$\frac{2}{i\,\omega}$
$S(t)$	$\pi\,\delta(\omega)+\frac{1}{i\,\omega}$
$S(t)\ \cos(\omega_0\,t)$	$\frac{\pi}{2}\,\delta(\omega-\omega_0)+\frac{\pi}{2}\,\delta(\omega+\omega_0)+\frac{i\,\omega}{\omega_0^2-\omega^2}$
$S(t)\ \sin(\omega_0\,t)$	$\frac{\pi}{2\,i}\,\delta(\omega-\omega_0)-\frac{\pi}{2\,i}\,\delta(\omega+\omega_0)+\frac{\omega_0}{\omega_0^2-\omega^2}$
$S(t)\ e^{-a\,t}$	$\frac{1}{a+i\,\omega}\quad(a>0\ \text{ or }\ \operatorname{Re} a>0)$
$S(t)\ t^n\,\frac{e^{-a\,t}}{n!}$	$\frac{1}{(a+i\,\omega)^{n+1}}\quad(a>0\ \text{ or }\ \operatorname{Re} a>0)$
$S(t)\ e^{-a\,t}\,\cos(\omega_0\,t)$	$\frac{i\,\omega+a}{(i\,\omega+a)^2+\omega_0^2}\quad(a>0\ \text{ or }\ \operatorname{Re} a>0)$
$S(t)\ e^{-a\,t}\,\sin(\omega_0\,t)$	$\frac{\omega_0}{(i\,\omega+a)^2+\omega_0^2}\quad(a>0\ \text{ or }\ \operatorname{Re} a>0)$
$e^{-a\,\|t\|},\quad a>0$	$\frac{2\,a}{a^2+\omega^2}$
$e^{-a\,\|t\|}\,\cos(\omega_0\,t),\quad a>0$	$\frac{2\,a\,\left(\omega^2+\omega_0^2+a^2\right)}{(\omega^2-\omega_0^2)^2+a^2\,(2\,\omega^2+2\,\omega_0^2+a^2)}$
$rect\left(\frac{t}{T}\right)=\begin{cases}1 & \text{for } \|t\|<T\\ 0 & \text{for } \|t\|>T\end{cases}$	$\frac{2\,\sin(\omega\,T)}{\omega}$
$\Delta\left(\frac{t}{T}\right)=\begin{cases}1-\frac{\|t\|}{T} & \text{for } \|t\|<T\\ 0 & \text{for } \|t\|>T\end{cases}$	$\frac{4\,\sin^2\left(\frac{\omega\,T}{2}\right)}{T\,\omega^2}$

18.5 Problems on the Fourier Transform

18.1 a) Determine the Fourier transform of

$$f_1(t) = \begin{cases} A & \text{for} \quad -\frac{T}{2} < t < \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$

b) Consider the function $F(f_1)(\omega)$ for $A = \frac{1}{T}$

c) What is the Fourier transform for

$$f_2(t) = \begin{cases} A & \text{for} \quad t_0 < t < t_0 + T \\ 0 & \text{otherwise} \end{cases} ?$$

18.2 Determine the spectrum of the triangular signal

$$f(t) = \begin{cases} A \cdot \left(1 - \left|\frac{t}{T}\right|\right) & |t| \leq T \\ 0 & |t| > T \end{cases}$$

18.3 a) Calculate the Fourier transform of $e^{-\alpha\,|t|}\,sign(t)$ (see Fig. (a)).

b) Calculate the Fourier transform of $\cos^2$-pulse (see Fig. (b)).

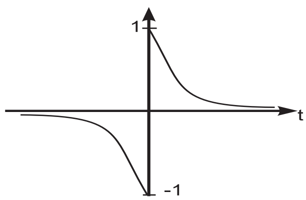


Fig. (a)

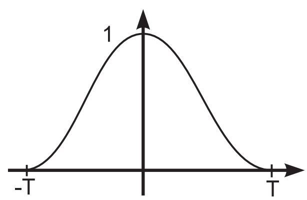


Fig. (b)

18.4 Take the Fourier transforms of 18.1 - 18.3 and plot the result. Can you decide which functions have a jump, which have a non-zero mean and which have?

18.5 Calculate the Fourier transform of the function $f(t)$ shown in Fig. (c).

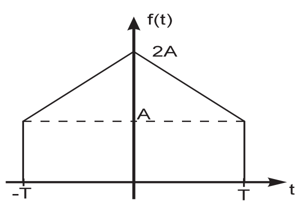


Fig. (c)

18.6 Show that the Fourier transform is a linear transform, i.e. that the superposition law holds:

$$\mathcal{F}(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 \mathcal{F}(f_1) + \alpha_2 \mathcal{F}(f_2).$$

18.7 a) Prove the scaling property $\mathcal{F}\left(f\left(a\, t\right)\right)\left(\omega\right) = \frac{1}{|a|}\, \mathcal{F}\left(f\left(t\right)\right)\left(\frac{\omega}{a}\right)$.
b) Prove the displacement law $\mathcal{F}\left(f\left(t - t_0\right)\right)\left(\omega\right) = e^{-i\omega_0 t}\, \mathcal{F}\left(f\left(t\right)\right)\left(\omega\right)$.

18.8 Demonstrate by mathematical induction that

$$\mathcal{F}\left(\left(-i\, t\right)^n\ f\right) = \frac{d^n}{d\omega^n}\, \mathcal{F}\left(f\right)\left(\omega\right) = F^{(n)}\left(\omega\right)$$

where $F\left(\omega\right) = \mathcal{F}\left(f\right)\left(\omega\right)$.

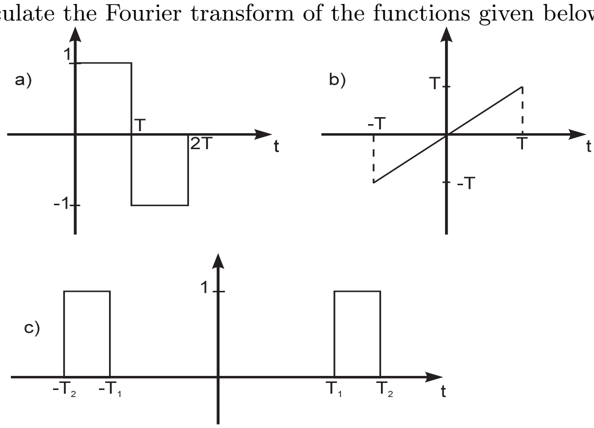
18.9 Use the properties of the Fourier transform to find the transform of
a) $\delta\left(t\right)$ b) $\delta\left(t - t_0\right)$ c) $\frac{i}{2}\ \left(\delta\left(t + t_0\right) - \delta\left(t - t_0\right)\right)$ d) $\sin\left(\omega_0 t\right)$

18.10 Show that the following identities are valid
a) $\mathcal{F}\left(e^{i\, a\, t}\right)\left(\omega\right) = 2\pi\, \delta\left(\omega - a\right)$
b) $\delta\left(t - t_0\right) * f\left(t\right) = f\left(t - t_0\right)$

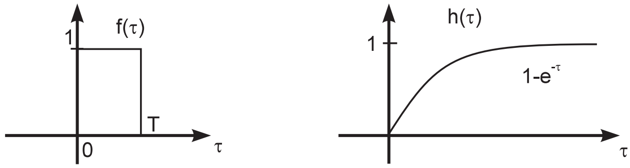
18.11 What is the convolution of the rectangular pulse with itself, where

$$rect\left(\frac{2\, t}{T}\right) = \begin{cases} 1 & |t| \le \frac{T}{2} \\ 0 & |t| > \frac{T}{2} \end{cases}$$

18.12 Calculate the Fourier transform of the functions given below



18.13 Use a sketch to find the convolution $f * h$ of the functions for a) $t \le 0$
b) $0 \le t \le T$ c) $T \le t$.



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Chapter 19
Partial Differential Equations

19

Many important problems in applied mathematics and physics lead to *partial differential equations* (PDE): to equations that establish relationships between one or more functions of several variables and their **partial** derivatives. There are three classical second-order PDE, which we see in many applications and which dominate the theory of PDE:

(1) $\frac{\partial^2}{\partial t^2}\, u\,(x,t) = c^2\, \frac{\partial^2}{\partial x^2}\, u\,(x,t)$ is the **Wave Equation**. $u\,(x,\, t)$ describes, for example, the vibration of a stretched string at the location x at time t. This PDE also occurs when studying acoustic, electromagnetic or water waves.

(2) $\frac{\partial}{\partial t}\, u\,(x,t) = \alpha^2\, \frac{\partial^2}{\partial x^2}\, u\,(x,t)$ is the **Heat Equation**. $u\,(x,\, t)$ represents, for example, the temperature distribution in a bar at the location x at the time t. This PDE is used to describe heat conduction and other diffusion processes.

(3) $\frac{\partial^2}{\partial x^2}\, u\,(x,y) + \frac{\partial^2}{\partial y^2}\, u\,(x,y) = 0$ is the **Laplace Equation**. $u\,(x,\, y)$ describes, for example, the electrostatic potential in a plane problem. This PDE also occurs in other steady-state problems such as a steady-state heat flow, the deflection of a membrane, electric and magnetic potentials.

9

19 Partial Differential Equations

19.1 Introduction

Many important problems in applied mathematics and physics lead to *partial differential equations* (PDE): to equations that establish relationships between one or more functions of several variables and their **partial** derivatives. Two arbitrary examples of PDE are

$$\frac{\partial^3}{\partial x^3}\, u\,(x,\, t) + \left(\frac{\partial}{\partial t}\, u\,(x,\, t)\right)^2 = \frac{\partial^2}{\partial x^2}\, u\,(x,\, t) \ \text{ for } u\,(x,\, t)\,,$$

$$\frac{\partial}{\partial x}\, u\,(x,\, y) = \frac{\partial}{\partial y}\, v\,(x,\, y)\,,\, \frac{\partial}{\partial y}\, u\,(x,\, y) = -\frac{\partial}{\partial x}\, v\,(x,\, y) \ \text{ for } u(x,\, y), v(x,\, y).$$

The order of a PDE is the highest partial derivative occurring in the equation.

There are three classical second-order PDE, which we see in many applications and that dominate the theory of PDE:

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In addition to the PDE, the *initial* and/or *boundary conditions* must be specified to completely describe the behavior of the underlying physical problem. In the following sections we will not cover a systematic approach to PDE, but we will solve special linear PDE, such as those mentioned above.

Notation: The partial derivatives are abbreviated by

$$u_x\,(x,\,t) = \frac{\partial}{\partial x}\,u\,(x,t) \quad \text{or} \quad u_{xx}\,(x,\,t) = \frac{\partial^2}{\partial x^2}\,u\,(x,t)$$

The same applies to the other variables. With this convention the three classical PDE are

$$(1)\;\; u_{tt} = c^2\,u_{xx} \qquad (2)\;\; u_t = \alpha^2\,u_{xx} \qquad (3)\;\; u_{xx} + u_{yy} = 0.$$

Classification of second-order linear PDE. Second-order linear PDE have the general form

$$A\,u_{xx} + 2\,B\,u_{xy} + C\,u_{yy} + D\,u_x + E\,u_y + F = 0 \qquad (*)$$

with functions A, B, C, D, E and F which can depend on x and y. The discriminant of the PDE $(*)$ is the function

$$d := A\,C - B^2.$$

The PDE $(*)$ is called

$$\begin{array}{lll} \textit{parabolic} & \text{if} & d = 0, \\ \textit{hyperbolic} & \text{if} & d < 0, \\ \textit{elliptic} & \text{if} & d > 0. \end{array}$$

These names come from analytic geometry, where

$$a\,x^2 + 2\,b\,x\,y + c\,y^2 + d\,x + e\,y + f = 0$$

represents a parabola, hyperbola or ellipse, depending on whether

$$a\,c - b^2 = 0, < 0 \;\; \text{or} \;\; > 0.$$

Examples 19.1.

① The wave equation $u_{tt} = c^2\,u_{xx}$ is hyperbolic:

$$A = c^2,\; B = 0,\; C = -1 \;\Rightarrow\; d = -c^2 < 0.$$

② The heat equation $u_t = \alpha^2\,u_{xx}$ is parabolic:

$$A = \alpha^2,\; B = C = 0 \;\Rightarrow\; d = 0.$$

③ The Laplace equation $u_{xx} + u_{yyy} = 0$ is elliptic:

$$A = C = 1,\; B = 0 \;\Rightarrow\; d = 1 > 0.$$

□

19.2 The Wave Equation

As a model for the wave equation, we consider an elastic string of length L fixed at the ends and deflected in the vertical plane. We look for the **vertical deflection** depending on the position x at time t: $u\,(x,\,t)$.

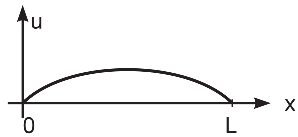


Figure 19.1. Elastic string

19.2.1 Deriving the Wave Equation

For the vertical deflection $u\,(x,\,t)$ of a vibrating string, the PDE is derived under the following conditions

(V_1) The string tension $F_0 = |\vec{F}\,(x)\,|$ is constant along the string.
(V_2) Only small deflections are taken into account for u.

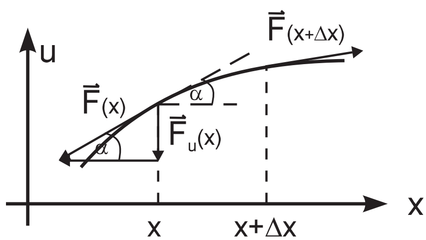


Figure 19.2. Shear force F_u

For the shear force (force in the direction of u) at point x, we compute for small deflections

$$F_u\,(x) = -F_0\,\sin\alpha \approx -F_0\,\alpha \approx -F_0 \cdot \tan\alpha = -F_0\,\left(\frac{\partial u}{\partial x}\right)_x.$$

The same applies to the lateral force F_u at position $x + \triangle x$

$$F_u\,(x + \triangle x) \approx F_0\,\left(\frac{\partial u}{\partial x}\right)_{x+\triangle x} \approx F_0\,\left[\left(\frac{\partial u}{\partial x}\right)_x + \triangle x\,\left(\frac{\partial^2 u}{\partial x^2}\right)_x\right],$$

if $\left(\frac{\partial u}{\partial x}\right)_{x+\triangle x}$ is linearized according to the formula $f\,(x + \triangle x) \approx f\,(x) + f'\,(x) \cdot \triangle x$. On the string element between x and $x + \triangle x$, the total force is

$$\overrightarrow{\triangle F} = \vec{F}\,(x) + \vec{F}\,(x + \triangle x)$$

and its component in u-direction is

$$\triangle F_u = F_u\,(x + \triangle x) - F_u\,(x) \approx F_0\;\triangle\,x\,\frac{\partial^2}{\partial x^2}\,u.$$

This transverse force accelerates the mass element $\triangle m = \rho \cdot \triangle x \cdot A$, where ρ is the density and A is the cross-section of the string. According to Newton's law of motion (the accelerating force $m \cdot a$ is equal to the sum of all the forces acting), we have

$$\triangle m \frac{\partial^2}{\partial t^2} u = \triangle F_u = F_0 \, \triangle \, x \frac{\partial^2}{\partial x^2} u$$

$$\Rightarrow \quad \frac{\partial^2}{\partial t^2} u\,(x,\, t) = \frac{F_0}{\rho A} \frac{\partial^2}{\partial x^2} u\,(x,\, t). \qquad \textbf{(Wave equation)}$$

F_0 is the tension of the string, ρ is the density and A is the cross-sectional area of the string. In the following subsections, this wave equation is solved for different physical conditions:

19.2.2 Infinitely Extended String (Initial Value Problem)

Let f be a 2-times continuously differentiable function of one variable. Then the solution of the wave equation is obtained by the **approach**

$$u\,(x,\, t) = f\,(x + c\,t)\ .$$

Using the chain rule we compute

$$u_{xx}\,(x,\, t) = f''\,(x + c\,t) \qquad \text{and} \qquad u_{tt}\,(x,\, t) = c^2\, f''\,(x + c\,t)\,.$$

Inserting the two second-order derivatives into the PDE gives

$$c^2\, f''\,(x + c\,t) = \frac{F_0}{\rho\, A}\, f''\,(x + c\,t) \;\Rightarrow\; c^2 = \frac{F_0}{\rho\, A} \;\Rightarrow\; c = \pm\sqrt{\frac{F_0}{\rho\, A}}.$$

- $f\,(x + c\,t)$ describes a wave moving in the negative x-direction with velocity c;
- $f\,(x - c\,t)$ describes a wave moving in the positive x-direction with velocity c.

So the solution is

$$u\,(x,\, t) = f_1\,(x + c\,t) + f_2\,(x - c\,t)$$

with any 2-times continuously differentiable functions f_1 and f_2.

Considering Initial Conditions

To specify the solution in more detail, we need to consider an initial deflection $u(x,\, t=0) = u_0(x)$ and an initial velocity $u_t(x,\, t=0) = v_0(x)$. We insert the initial conditions into our solution

$$u(x,\, t=0) = u_0(x) = f_1(x) + f_2(x) \qquad (1)$$

$$u_t(x,\, t=0) = v_0(x) = c\,(f_1'(x) - f_2'(x))\,. \qquad (2)$$

We integrate (2)

$$f_1(x) - f_2(x) = \frac{1}{c}\int_{x_0}^{x} v_0(\xi)\; d\xi + K \qquad (2')$$

and add and subtract equation (1) from (2'). Hence,

$$f_1(x) = \frac{1}{2}\,u_0(x) + \frac{1}{2\,c}\int_{x_0}^{x} v_0(\xi)\; d\xi + \frac{K}{2}$$

$$f_2(x) = \frac{1}{2}\,u_0(x) - \frac{1}{2\,c}\int_{x_0}^{x} v_0(\xi)\; d\xi - \frac{K}{2}.$$

Finally, we obtain the result including the initial deflection $u_0(x)$ and the initial velocity $v_0(x)$.

Solution of the Wave Equation

The solution of the wave equation for the initial displacement $u_0(x)$ and the initial velocity $v_0(x)$ is given by

$$u(x,\, t) = \frac{1}{2}\,[u_0(x + c\,t) + u_0(x - c\,t)] + \frac{1}{2\,c}\int_{x-ct}^{x+ct} v_0(\xi)\; d\xi$$

(**d'Alembert's formula**).

In the case of no initial velocity, $v_0(x) = 0$, the solution simplifies to

$$u(x,\, t) = \tfrac{1}{2}\,u_0(x + c\,t) + \tfrac{1}{2}\,u_0(x - c\,t)\,.$$

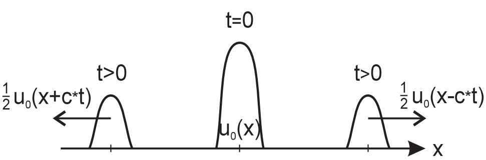


Figure 19.3. Waves moving to the right and to the left

Interpretation: The first term describes the propagation of the initial deflection to the left and the second to the right, each with half the amplitude.

Application: Whip Cracking

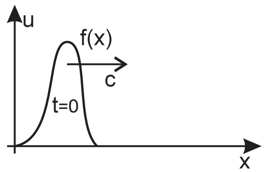


Figure 19.4.
Deflection of the whip

When a whip is cracked, a one-sided "infinite" extended string is led at the other end. If the whip is deflected by raising and lowering the left string end, the string has an initial deflection of the shape $f(x)$ at the beginning (see Fig. 19.4). Then

$$u(x, t) = f(x - c\,t)$$

is the solution of the wave equation with the initial conditions

$$\begin{aligned} u(x, t = 0) &= f(x) \quad (\text{and } f(x) = 0 \text{ for } x \leq 0), \\ u(x = 0, t) &= 0 \qquad \text{for all } t. \end{aligned}$$

The solution is a wave running to the right. The velocity is $c = \sqrt{\frac{F}{\rho\,A}}$. As the cross-section A decreases, the velocity c increases. When the cross-section A is small enough, the velocity c becomes greater than the velocity of sound v_{sound} and the whip cracks.

Visualization: For the graphical representation we have chosen a Gaussian pulse as the initial deflection. We can see how the Gaussian pulse moves to the right and also, that the speed of the pulse increases as the cross-section is reduced.

19.2.3 Tensioned String (Initial Boundary Value Problem)

To determine the deflection of a **guitar** string $u(x,t)$ at time t at location x, we need the PDE

$$u_{tt}(x,t) = c^2\, u_{xx}(x,t),$$

the initial deflection $u_0(x)$, the initial velocity $v_0(x)$

$$\left.\begin{aligned} u(x, t = 0) &= u_0(x) \\ u_t(x, t = 0) &= v_0(x) \end{aligned}\right\} \quad (\textbf{Initial conditions})$$

and two boundary conditions. At $x = 0$ and $x = L$ the deflection of the string is always zero:

$$\left.\begin{array}{ll} u\,(x=0,\,t)=0 & \text{for all} \quad t \\ u\,(x=L,\,t)=0 & \text{for all} \quad t \end{array}\right\} \quad (\textbf{Boundary values}).$$

Since both initial conditions and boundary values are given, this problem is called an **initial boundary value problem**.

If we observe the motion of a stretched string, each point on the string will vibrate. The amplitude of this oscillation depends on the x-position of the point on the string. If the oscillation is denoted by $T(t)$ and the amplitude by $X(x)$, a solution $u(x,t)$ is sought which can be written as the product of a position function $X(x)$ and a time function $T(t)$. The time function $T(t)$ then has a position dependent amplitude $X(x)$. This **product approach**

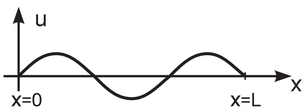


Figure 19.5. Tensioned string

$$u\,(x,\,t) = X\,(x) \cdot T\,(t)$$

is used to separate the variables. Here

$X\,(x)$	is a purely location-dependent function,
$T(t)$	is a purely time-dependent function.

Substituting this product into the PDE gives

$$X\,(x) \cdot T''\,(t) = c^2\; X''\,(x) \cdot T\,(t)$$

and after the separation

$$\frac{T''\,(t)}{T\,(t)} = c^2\,\frac{X''\,(x)}{X\,(x)}.$$

Since the left side of the equation is **not** dependent on x, it is constant with respect to x. Since the right side of the equation is **not** dependent on t, it is constant with respect to t. So the constant does not depend on either t or x:

$$\frac{T''\,(t)}{T\,(t)} = c^2\,\frac{X''\,(x)}{X\,(x)} = const = -\omega^2.$$

The case of a positive would constant would lead to a non-physical solution and is therefore not pursued further. This product approach reduces the **partial** differential equation to two **ordinary** second-order differential equations:

(1) **Time Dependency**:

$$T''(t) + \omega^2 T(t) = 0 \quad \Rightarrow \quad T(t) = A\cos(\omega t) + B\sin(\omega t).$$

(2) **Position Dependency**:

$$X''(x) + \frac{\omega^2}{c^2} X(x) = 0 \quad \Rightarrow \quad X(x) = D\cos(\frac{\omega}{c}x) + E\sin(\frac{\omega}{c}x).$$

Note that the two ordinary differential equations are each oscillation equations. Therefore, the general solution can be given directly as in Example 16.7. The solution of the PDE is

$$u(x,t) = [D\cos(\tfrac{\omega}{c}x) + E\sin(\tfrac{\omega}{c}x)]\,[A\cos(\omega t) + B\sin(\omega t)]. \qquad (*)$$

Considering Boundary Conditions

Not all functions satisfying the representation $(*)$ are solutions of the given problem, because for the **guitar** string the boundary conditions

$$u(x=0,t) = u(x=L,t) = 0$$

must be satisfied for all times t. The solution must therefore take into account that there is no deflection at the edges.

$$x=0: \quad u(0,t) = 0 \text{ for all } t \quad \Rightarrow \quad D\underbrace{\cos(\tfrac{\omega}{c}\cdot 0)}_{=1} + E\underbrace{\sin(\tfrac{\omega}{c}\cdot 0)}_{=0} \overset{!}{=} 0$$

$$\Rightarrow \quad D = 0.$$

$$x=L: \quad u(x=L,t) = 0 \text{ for all } t \quad \Rightarrow \quad E\cdot\sin\left(\tfrac{\omega}{c}L\right) = 0.$$

To get not only the zero solution, it must be $E \neq 0$ and $\sin\left(\tfrac{\omega}{c}L\right) = 0$. The sine term becomes zero if its argument is a multiple of π:

$$\Rightarrow \quad \frac{\omega}{c}\cdot L = n\pi.$$

So only certain discrete frequencies are allowed:

$$\omega_n = n\cdot\frac{\pi}{L}\cdot c, \qquad n \in \mathbb{N}.$$

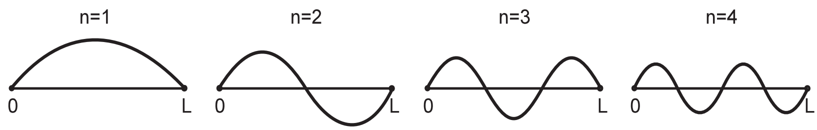


Figure 19.6. The first 4 modes of a stretched string

For each $n \in \mathbb{N}$

$$u_n(x, t) = \sin\left(n \frac{\pi}{L} x\right) \left[a_n \cos\left(n \frac{\pi}{L} c t\right) + b_n \sin\left(n \frac{\pi}{L} c t\right)\right]$$

is a solution of the PDE and satisfies the boundary conditions. Only frequencies ω_n that give rise to standing waves are allowed, so the functions are $2L$-periodic. The general solution for a vibrating string is obtained by superposing all the standing waves $u_n(x, t)$:

$$\begin{aligned} u(x, t) &= \sum_{n=1}^{\infty} u_n(x, t) \\ &= \sum_{n=1}^{\infty} \sin\left(n \frac{\pi}{L} x\right) \left[a_n \cos\left(n \frac{\pi}{L} c t\right) + b_n \sin\left(n \frac{\pi}{L} c t\right)\right]. \end{aligned}$$

In this representation the coefficients $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are still unknown. They result from the initial conditions:

Considering Initial Conditions

The **initial conditions** are

$$u(x, t = 0) = u_0(x) = \sum_{n=1}^{\infty} a_n \sin\left(n \frac{\pi}{L} x\right) = \sum_{n=1}^{\infty} a_n \sin\left(n \frac{2\pi}{2L} x\right).$$

This is the Fourier series of the $2L$-periodic function $\tilde{u}_0(x)$, which is obtained from $u_0(x)$ by mirroring $u_0(x)$ at the origin and then continuing $2L$-periodically to all $x \in \mathbb{R}$.

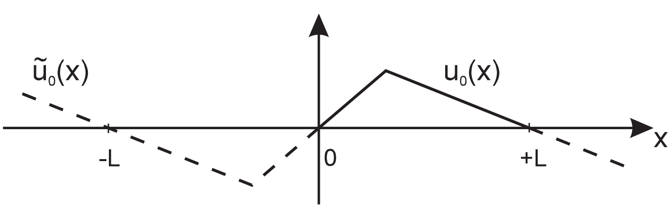


Figure 19.7. Initial deflection $u_0(x)$

According to Chapter 17 (Fourier series), the Fourier coefficients of a $2L$-periodic, point-symmetric function are

$$a_n = \frac{2}{2\,L} \int_0^{2L} \tilde{u}_0\,(x)\,\sin\left(n\,\frac{2\pi}{2L}\,x\right)\,dx$$
$$= 2\frac{1}{L} \int_0^{L} u_0\,(x)\,\sin\left(n\,\frac{\pi}{L}\,x\right)\,dx.$$

The same arguments apply to the **initial velocity**

$$u_t\,(x,\,t=0) = v_0\,(x) = \sum_{n=1}^{\infty} \left(b_n\, \cdot n\frac{\pi}{L}\,c\right)\,\sin\left(n\,\frac{\pi}{L}\,x\right).$$

This is the Fourier series of the $2L$-periodic function $\tilde{v}_0\,(x)$, which is obtained from $v_0\,(x)$ by mirroring $v_0\,(x)$ at the origin and then continuing $2L$-periodically to all $\mathbb{R}$. Consequently,

$$\left(b_n\, \cdot n\frac{\pi}{L}\,c\right) = \frac{2}{2L} \int_0^{2L} \tilde{v}_0\,(x)\,\sin\left(n\,\frac{2\pi}{2L}\,x\right)\,dx$$
$$= 2\,\frac{1}{L} \int_0^{L} v_0\,(x)\,\sin\left(n\,\frac{\pi}{L}\,x\right)\,dx.$$

Summary: Wave Equation with Boundary Conditions

The solution of the wave equation

$$u_{tt}\,(x,\,t) = c^2\,u_{xx}\,(x,\,t)$$

with the boundary values $u\,(x=0,\,t) = u\,(x=L,\,t) = 0$ for all t and the initial values $u\,(x,\,t=0) = u_0\,(x)$ and $u_t\,(x,\,t=0) = v_0\,(x)$ for $0 \le x \le L$ is given by

$$u\,(x,\,t) = \sum_{n=1}^{\infty} \sin\left(n\,\frac{\pi}{L}\,x\right)\,\left[a_n\,\cos\left(n\,\frac{\pi}{L}\,c\,t\right) + b_n\,\sin\left(n\,\frac{\pi}{L}\,c\,t\right)\right].$$

The coefficients a_n and b_n are the Fourier coefficients of $u_0\,(x)$ and $v_0\,(x)$:

$$a_n = \frac{2}{L} \int_0^{L} u_0\,(x) \cdot \sin\left(n\,\frac{\pi}{L}\,x\right)\,dx \qquad n = 1,\,2,\,3,\ldots$$

$$b_n = \frac{2}{n\,\pi\,c} \int_0^{L} v_0\,(x)\,\sin\left(n\,\frac{\pi}{L}\,x\right)\,dx \qquad n = 1,\,2,\,3,\ldots$$

Physical Interpretation: We write the solution as

$$u(x,\,t) = \sum_{n=1}^{\infty} A_n \sin\left(n\,\frac{\pi}{L}\,x\right)\,\sin\left(n\,\frac{\pi}{L}\,c\,t + \varphi_n\right),$$

which represents the superposition of harmonic oscillations with

Amplitudes	$A_n \sin\left(\frac{n\,\pi}{L}\,x\right)$	(dependent on x)
Phases	φ_n	(independent on x)
Frequencies	$\omega_n = n\,\frac{\pi}{L}\,c.$	

The solution is called a **standing wave**. The string performs harmonic oscillations with phases φ_n and position-dependent amplitudes $A_n\ \sin\left(n\,\frac{\pi}{L}\,x\right)$. The string produces a **tone**, whose volume depends on the maximum amplitudes $A_n = \sqrt{a_n^2 + b_n^2}$. For $n = 1$ we get the basic tone, for $n = 2,\ 3,\ 4, \ldots$ we obtain the overtones.

Example 19.2 (With Maple-Worksheet). Given is an initial deflection (see Fig. 19.8), corresponding to plucking a string.

$$u_0(x) = \begin{cases} \dfrac{u_0}{l_1}\,x : & 0 \le x \le l_1 \\[2ex] \dfrac{u_0}{L - l_1}(\,L - x) : & l_1 \le x \le L \end{cases}$$

The initial velocity v_0 is zero.

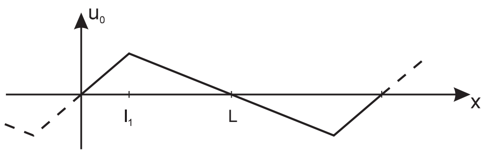


Figure 19.8. Initial deflection $u_0(x)$

Partial integration calculates the Fourier coefficients a_n

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L u_0(x)\cdot \sin\left(n\,\tfrac{\pi}{L}\,x\right)\,dx \\ &= \frac{2}{L}\left(\int_0^{l_1} \frac{u_0}{l_1}\,x\ \cdot \sin(n\tfrac{\pi}{L}\,x)\,dx + \int_{l_1}^{L} \frac{u_0}{L - l_1}\,(L - x)\ \cdot \sin(n\tfrac{\pi}{L}\,x)\,dx\right) \\ &= \frac{2}{L}\left(-\frac{\left(-\sin\left(\frac{n\,\pi\,l_1}{L}\right)L + n\,\pi\,\cos\left(\frac{n\,\pi\,l_1}{L}\right)\,l_1\right)\,L\,u_0}{n^2\,\pi^2\,l1} - \frac{u_0\,L^2\,\sin(n\,\pi)}{(L - l_1)\,n^2\,\pi^2}\right. \\ &\quad \left. - \frac{u_0\,L\,\left(-\sin\left(\frac{n\,\pi\,l_1}{L}\right)\,L - L\,\cos\left(\frac{n\,\pi\,l_1}{L}\right)\,n\,\pi + n\,\pi\,\cos\left(\frac{n\,\pi\,l_1}{L}\right)\,l_1\right)}{(L - l_1)\,n^2\,\pi^2}\right). \end{aligned}$$

With $\sin(n\,\pi) = 0$ the coefficients are simplified to

$$a_n = 2\,\frac{u_0\,L^2\,\sin\left(\frac{n\,\pi\,l_1}{L}\right)}{n^2\,\pi^2\,l_1\,(L - l_1)}.$$

For the initial velocity $v_0\,(x) = 0$ the coefficients $b_n = 0$ do not appear, so the solution is given by

$$u\,(x,\,t) = \frac{2\,u_0\,L^2}{l_1\,(L - l_1)\,\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\,\sin\left(n\,\frac{\pi}{L}\,l_1\right)\,\sin\left(n\,\frac{\pi}{L}\,x\right)\,\cos\left(n\,\frac{\pi}{L}\,c\,t\right).$$

Animation: The corresponding animation shows the time behavior of the standing wave. The tip of the string collapses; the right flank remains stationary at first. The tip moves until it reaches the maximum negative deflection. It is then reflected and returns to its initial deflection. □

About the Timbre

The **timbre** of a sound is obtained by superimposing all the harmonics on the fundamental. The different timbres are due to the fact that the harmonics contribute to the sound with different amplitudes.

Plucking the guitar produces a different sound depending on whether the string is plucked in the middle (1) or near the edge (2).

(1) When plucked in the middle ($l_1 = \frac{L}{2}$ in Example 19.2), the amplitudes are $A_n = |a_n|$

$$(A_n) = \frac{8\,u_0}{\pi^2}\left(1,\,0,\,\frac{1}{9},\,0,\,\frac{1}{25},\,0,\,\frac{1}{49},\,0,\ldots\right).$$

The overtones are very weak because the amplitudes decrease with $\frac{1}{n^2}$. The sound is clear.

(2) When plucked at the right edge ($l_1 = L$) (the slope of the string at the right edge is neglected), the amplitudes are obtained by applying the rule of l'Hospital $A_n = \frac{2\,u_0}{\pi\,n}$ or

$$(A_n) = \frac{2\,u_0}{\pi}\left(1,\,\frac{1}{2},\,\frac{1}{3},\,\frac{1}{4},\ldots\right).$$

The overtones are strongly represented, as the amplitudes decrease with $\frac{1}{n}$. The sound is hard and indistinct.

19.3 Heat Equation

To illustrate the heat equation, we consider a metal bar of length L with a rectangular cross-section of width b and height h. $T(x, t)$ denotes the temperature in the bar at position x at time t. T_u is the ambient temperature. The one-dimensional problem of heat transfer in the x-direction is considered.

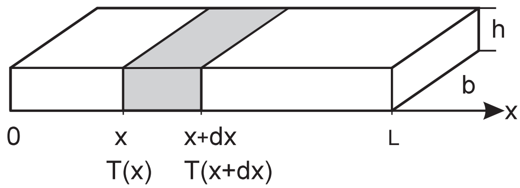


Figure 19.9. Heat transfer in x-direction

19.3.1 Deriving the Heat Equation

To introduce the heat equation, we summarize the physical laws describing heat transfer in solid materials:

(1) The amount of heat δQ flowing in x-direction through a surface $A = b \cdot h$ in the time $\triangle t$ is given by

$$\delta Q = -\lambda\, A\, \triangle\, t\, \frac{\partial T}{\partial x}.$$

The amount of heat is proportional to the temperature gradient $\frac{\partial T}{\partial x}$. Since heat flows from hot to cold, $\delta Q \sim -\frac{\partial T}{\partial x}$ applies. λ is the material specific thermal conductivity.

(2) The amount of heat δQ stored in a volume of mass m and specific heat c is given by

$$\delta Q = c\, m\, (T_2 - T_1) = c\, m\, \triangle\, T,$$

where $\triangle T = (T_2 - T_1)$ is the temperature difference at the ends of the volume.

(3) The amount of heat that the body can give off to its surroundings is proportional to the surface M and the temperature difference between the body and the surroundings

$$\delta Q = -M\, \alpha\, (T - T_u)\,.$$

α is the heat transfer coefficient.

The energy balance is based on the energy consumption shown in Fig. 19.9 $dV = b \cdot h \cdot dx$:

The change in energy per unit time $\triangle t$ in the mass element dm is

= Heat flux into the mass element through the surface A at position x

+ Heat flux out of the mass element through the surface A at position $x + dx$

+ Heat dissipation to the environment.

Using the physical laws (1) to (3) we can write the energy balance in the form of an equation

$$\frac{\delta Q}{\partial t} = c\, dm\, \frac{\partial T}{\partial t} = -\lambda\, A \left(\frac{\partial T}{\partial x}\right)_x + \lambda\, A \left(\frac{\partial T}{\partial x}\right)_{x+dx}$$
$$-2\ (b+h)\ dx\, \alpha\ (T - T_u)\,.$$

If the expression $\left(\frac{\partial T}{\partial x}\right)_{x+dx} \approx \left(\frac{\partial T}{\partial x}\right)_x + dx \left(\frac{\partial^2 T}{\partial x^2}\right)_x$ is linearized and this linearization is inserted into the above equation, it follows with $dm = \rho \cdot dV = \rho \cdot b \cdot h \cdot dx$

$$c\, \rho\, b\, h\, dx\, \frac{\partial T}{\partial t} = -\lambda\, A\, \frac{\partial T}{\partial x} + \lambda\, A \left[\frac{\partial T}{\partial x} + dx\, \frac{\partial^2 T}{\partial x^2}\right] - 2\ (b+h)\ dx\, \alpha\ (T - T_u)$$

$$\Rightarrow \quad \frac{\partial}{\partial t} T(x,t) = \frac{\lambda}{c\, \rho}\, \frac{\partial^2}{\partial x^2} T(x,t) - \alpha\, \frac{2\ (b+h)}{c\, \rho\, b\, h}\ (T(x,t) - T_u). \qquad (*)$$

This is the **heat equation with heat dissipation to the environment.**

We will solve the heat equation for two special cases: In Section 19.3.2 the heat equation is treated without dissipation to the environment ($\alpha = 0$) and in Section 19.3.3 the steady-state heat profile caused by dissipation to the environment is determined $\left(\frac{\partial T}{\partial t} = 0\right)$.

19.3.2 Solution for $\alpha = 0$ and Thermal Insulation at the Ends

With thermal insulation to the environment ($\alpha = 0$) we get the pure **heat equation** in the form

Heat Equation

$$\frac{\partial}{\partial t}T(x,t) = \frac{\lambda}{c\,\rho}\,\frac{\partial^2}{\partial x^2}T(x,t).$$

To determine the temperature in the beam at any time t, both the initial temperature distribution $T\,(x,\,t=0) = T_0\,(x)$ and the boundary conditions at the ends of the beam are required. If the ends are also insulated, there is no heat transfer through the edges at any time, i.e. $T_x\,(x=0,\,t)=0$ and $T_x\,(x=L,\,t)=0$. Note that the heat flow is proportional to the gradient $\frac{\partial T}{\partial x}$! If the ends were kept at a constant temperature, we would have to consider $T\,(x=0,\,t)=T_l$ or $T\,(x=L,\,t)=T_r$.

We have a bar of length L which is insulated over its entire surface, including the ends at $x=0$ and $x=L$. We assume an initial temperature distribution $T_0\,(x)$. The temperature distribution at later times is sought:

$u_t(x,t)$	$=$	$\kappa\,u_{xx}(x,t)$	with $\kappa = \frac{\lambda}{c\,\rho}$	$(*)$
$u\,(x,\,0)$	$=$	$T_0\,(x)$	Initial temperature distribution	
$u_x\,(0,\,t)$	$=$	$u_x\,(L,\,t)=0$	Boundary values.	

⚠ Not to be confused with the time function $T\,(t)$, we write the temperature distribution we are looking for $u\,(x,\,t)$. As with the wave equation, it is helpful to choose a separation approach to solve this problem

$$u\,(x,\,t) = X\,(x)\cdot T\,(t)\ ,$$

where

$X\,(x)$	is a purely position-dependent function,
$T(t)$	is a pure time-dependent function.

Inserting this product into the differential equation $(*)$ yields

$$X\,(x)\cdot T'\,(t) = \kappa\ X''\,(x)\cdot T\,(t)$$

$$\Rightarrow\ \frac{T'\,(t)}{\kappa\,T\,(t)} = \frac{X''\,(x)}{X\,(x)} = const = -k^2.$$

Since the left term does not depend on x, it is constant with respect to x. Since the middle term does not depend on t, it is constant with respect to t.

So the constant does not depend on either x or t: $const = \pm k^2$. A positive constant would lead to an exponential increase in temperature at $x = 0$ and a decrease at $x = L$, which is physically impossible. It contradicts the second main theorem of thermodynamics which states that entropy always increases. Therefore $-k^2$ is assumed to be the constant.

From this product approach we get two ordinary differential equations:

(1) **Time Dependency:** $T'(t) + \kappa\, k^2\, T(t) = 0$. This is a homogeneous first-order linear differential equation with the solution

$$T(t) = D\, e^{-\kappa\, k^2\, t}$$

(see Volume 2, Section 13.2 on First-Order Differential Equations).

(2) **Position Dependency:** $X''(x) + k^2\, X(x) = 0$. This is the oscillation equation with the general solution

$$X(x) = A\, \cos(k\, x) + B\, \sin(k\, x)$$

(see Example 16.7).

So the **solution** of the PDE can be written as

$$u(x,\, t) = e^{-\kappa\, k^2\, t}\, (a\, \cos(k\, x) + b\, \sin(k\, x)).$$

Considering Boundary Conditions

In the case of insulation, there is no heat transport at the ends $x = 0$ and $x = L$: Hence, $u_x(0,\, t) = u_x(L,\, t) = 0$ for all t. To take account of these boundary conditions, we determine the partial derivative of the solution with respect to x

$$u_x(x,\, t) = e^{-\kappa\, k^2\, t}\, (-a\, k\, \sin(k\, x) + b\, k\, \cos(k\, x))$$

and evaluate this derivative at $x = 0$ and at $x = L$:

$x = 0$: $\quad u_x(0,\, t) = 0$ for all $t \Rightarrow b = 0$.

$x = L$: $\quad u_x(L,\, t) = 0$ for all $t \Rightarrow a\, \sin(k\, L) = 0$.

If $a = 0$, then $u(x, t)$ would be zero. So to get a non-zero solution we use:

$$a \neq 0 \quad \text{and} \quad \sin(k\,L) = 0 \;\Rightarrow\; k \cdot L = n\,\pi \qquad n = 0, 1, 2, 3, \ldots .$$

Only discrete wavelengths (*eigenvalues*) $k_n = n\,\frac{\pi}{L}$ are possible. These are only the wavelengths that have bulges at the ends of the bar, because the corresponding functions are $\cos\left(n\,\frac{\pi}{L}\,x\right)$. For each $n \in \mathbb{N}_0$ there is a solution

$$u_n(x, t) = a_n\, e^{-\kappa\,\left(n\,\frac{\pi}{L}\right)^2 t}\, \cos\left(n\,\frac{\pi}{L}\,x\right).$$

However, due to the law of superposition, every linear combination is again a solution of the heat equation

$$\Rightarrow \quad u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = \sum_{n=0}^{\infty} a_n\, e^{-\kappa\,\left(n\,\frac{\pi}{L}\right)^2 t}\, \cos\left(n\,\frac{\pi}{L}\,x\right).$$

The coefficients a_n depend only on the given initial temperature distribution $u(x, 0) = T_0(x)$.

Considering the Initial Condition

$$u(x, t = 0) = T_0(x) = \sum_{n=0}^{\infty} a_n\, \cos\left(n\,\frac{\pi}{L}\,x\right) = \sum_{n=0}^{\infty} a_n\, \cos\left(n\,\frac{2\pi}{2L}\,x\right).$$

This representation is a $2L$-periodic Fourier series. Since it contains only cosine terms, it must be axis-symmetric with respect to the y-axis. So we mirror $T_0(x)$ around the y-axis and then extend it $2\,L$-periodically to $\mathbb{R}$.

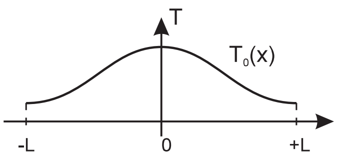


Figure 19.10. Initial temp. $T_0(x)$

The coefficients a_n are the Fourier coefficients of the initial temperature distribution $T_0(x)$ which can be computed using only the information of $T_0(x)$ between 0 and L:

$$a_n = 2\,\frac{1}{L} \int_0^L T_0(x)\, \cos\left(n\,\frac{\pi}{L}\,x\right)\, dx$$

$$a_0 = \frac{1}{L} \int_0^L T_0(x)\, dx.$$

Summary: Solution of the Heat Transfer Equation

The solution of the heat equation

$$u_t(x, t) = \kappa\, u_{xx}(x, t)$$

with $\kappa = \frac{\lambda}{c\,\rho}$, the boundary values $u_x(0, t) = u_x(L, t) = 0$ (thermal insulation) and the initial temperature distribution $T_0(x)$ is given by

$$u(x, t) = \sum_{n=0}^{\infty} a_n\, e^{-\kappa\left(n\,\frac{\pi}{L}\right)^2 t} \cos\left(n\,\frac{\pi}{L}\,x\right),$$

where a_n are the Fourier coefficients of the even, $2\,L$-periodic extension of $T_0(x)$:

$$a_n = \frac{2}{L}\int_0^L T_0(x)\cos\left(n\,\frac{\pi}{L}\,x\right)dx \qquad n = 1, 2, 3, \ldots$$

$$a_0 = \frac{1}{L}\int_0^L T_0(x)\,dx.$$

Example 19.3. We examine the temperature $u(x, t)$ in a thin copper wire $(0 \le x \le L = 1)$ with material constant $\kappa = 1.14$. At the initial time $t = 0$ the wire has the temperature profile

$$T_0(x) = 10 + 5\cos(2\pi\,x) + 4\cos(4\pi\,x) + \cos(6\pi\,x)\,.$$

The ends of the bar are thermally insulated. We are looking for the temperature behavior over time for $t > 0$.

Solution: The temperature $u(x, t)$ satisfies the boundary value problem

$$u_t = \kappa\,u_{xx} \quad \text{with} \quad \begin{cases} u(x, t=0) = T_0(x) \\ u_x(0, t) \quad = u_x(L, t) = 0. \end{cases}$$

The Fourier coefficients of $T_0(x)$ are $a_0 = 10$, $a_2 = 5$, $a_4 = 4$ and $a_6 = 1$; otherwise they are zero. So the temperature for times $t > 0$ is

$$u(x, t) = 10 + 5\,e^{-1.14\cdot 4\pi^2\,t}\cos(2\pi\,x) + 4\,e^{-1.14\cdot 16\pi^2\,t}\cos(4\pi\,x)$$
$$+1\,e^{-1.14\cdot 36\pi^2\,t}\cos(6\pi\,x)\,. \qquad \square$$

Interpreting the Solution

At time $t = 0$ the temperature distribution is given by

$$u\,(x,\,0) = \sum_{n=0}^{\infty} a_n\,\cos(n\,\frac{\pi}{L}\,x) = a_0 + \sum_{n=1}^{\infty} a_n\,\cos(n\,\frac{\pi}{L}\,x) = T_0\,(x)\,.$$

For times $t > 0$ the terms $n > 0$ include the damping factor $e^{-\kappa\,\left(n\,\frac{\pi}{L}\right)^2\,t}$. Therefore, we split the sum into

$$u\,(x,\,t) = a_0 + \sum_{n=1}^{\infty} a_n\,e^{-\kappa\,\left(n\,\frac{\pi}{L}\right)^2\,t}\,\cos(n\,\tfrac{\pi}{L}\,x).$$

For large times this amplitude $e^{-\kappa\,\left(n\,\frac{\pi}{L}\right)^2\,t}$ approaches zero. Then, the temperature distribution approaches the value

$$u\,(x,\,t) \overset{t\to\infty}{\longrightarrow} a_0 = \frac{1}{L}\int_0^L T_0\,(x)\;dx\,.$$

This is the integral mean of the initial temperature distribution. The initial temperature distribution melts and a constant homogeneous mean temperature a_0 is finally reached.

Animation: The animation shows the time course of the melting for the parameters $L = 1$, $\kappa = 0.15$ and the initial temperature distribution $T_0\,(x) = 10 + 5\,\sum_{n=1}^{5} \frac{1}{n^2}\,\cos(n\,\frac{\pi}{L}\,x)$. The process is quite fast at first, but slows down over time. □

19.3.3 Solution for $\alpha = 0$ and Fixed Temperatures at the Ends

The solution to the heat equation for insulation ($\alpha = 0$) is given if the ends of the beam are kept at constant temperatures, T_l at the left end and T_r at the right end. However, we can transform this problem into a zero boundary temperature problem as the next note shows.

Note: In the case of non-zero temperatures at the ends, the problem transforms into a vanishing boundary temperature problem.

If the temperatures at the ends are non-zero,

$$T\,(x = 0,\,t) = T_l \text{ and } T\,(x = L,\,t) = T_r,$$

then a solution $u(x,t)$ of the heat equation is to be found

$$u_t(x,t) = \kappa\,u_{xx}(x,t)$$

with	$u\,(x,\,0) = T_0\,(x)$		initial temperature
and	$u\,(0,\,t) = T_l,$	$u\,(L,\,t) = T_r$	boundary values.

Using

$$\hat{u}(x,t) = u(x,t) - (1 - \frac{x}{L})\,T_l - \frac{x}{L}\,T_r\,,$$

the original problem transforms into a heat problem for $\hat{u}(x,t)$ with vanishing boundary conditions, because for $\hat{u}$ the boundary conditions are

$$\hat{u}(x=0,t) = u(x=0,t) - T_l = T_l - T_l = 0$$
$$\hat{u}(x=L,t) = u(x=L,t) - T_r = T_r - T_r = 0$$

but with modified initial temperature

$$\begin{aligned}\hat{u}(x,t=0) &= u(x,t=0) - (1-\tfrac{x}{L})\,T_l - \tfrac{x}{L}\,T_r \\ &= T_0(x) - (1-\tfrac{x}{L})\,T_l - \tfrac{x}{L}\,T_r.\end{aligned}$$

The heat equation also applies to $\hat{u}(x,t)$. □

According to the note, we consider the case where the ends of the beam have zero temperatures $T\,(x=0,\,t) = 0$ and $T\,(x=L,\,t) = 0$. We look for the solution to the heat transfer equation

$$u_t(x,t) = \kappa\,u_{xx}(x,t)$$

with	$u\,(x,\,0) = T_0\,(x)$		initial temperature
and	$u\,(0,\,t) = 0,$	$u\,(L,\,t) = 0$	boundary values.

Following the same procedure as in Section 19.3.2, the solution of the heat equation is

$$u\,(x,\,t) = e^{-\kappa\,k^2\,t}\;(a\,\cos\,(k\,x) + b\,\sin\,(k\,x))\,.$$

Considering the boundary conditions: We conclude from $u\,(x=0,\,t) = 0$ that $a = 0$ and from $u\,(x=L,\,t) = 0$ that $\sin\,(k\,L) = 0$. The latter equation states that only discrete wavelengths $k_n = n\,\frac{\pi}{L}$ are possible. These wavelengths produce knots at the ends of the beam because the associated solutions are $\sin(n\,\frac{\pi}{L}\,x)$. For each $n \in \mathbb{N}$ we get a solution of the form

$$u_n\,(x,\,t) = b_n\,e^{-\kappa\,\left(n\,\frac{\pi}{L}\right)^2\,t}\,\sin(n\,\tfrac{\pi}{L}\,x)$$

and by superposition

$$u(x,\,t) = \sum_{n=1}^{\infty} b_n\, e^{-\kappa\left(n\,\frac{\pi}{L}\right)^2 t}\, \sin(n\,\frac{\pi}{L}\,x). \qquad (*)$$

Considering the initial condition: The coefficients b_n

$$u(x,\,t=0) = \sum_{n=1}^{\infty} b_n\, \sin(n\,\tfrac{\pi}{L}\,x) = T_0(x)\ .$$

are implicitly defined. The b_n are the coefficients of the Fourier series of the initial temperature distribution: We mirror $T_0(x)$ at the origin to $[-L,0]$ and then extend it $2L$-periodically to $\mathbb{R}$:

$$b_n = \frac{1}{L}\int_0^L T_0(x)\ \sin(n\,\tfrac{\pi}{L}\,x)\,dx \qquad n = 1,\,2,\,3,\ldots\ .$$

Interpretation: Starting from $T_0(x)$, the initial temperature distribution melts again. But since **each** term contains the damping factor $e^{-\kappa\left(n\,\frac{\pi}{L}\right)^2 t}$, the final state is

$$u(x,\,t) \to 0 \quad \text{for large}\ \ t,$$

and the body reaches the constant final temperature $0°$. □

Example 19.4. We look for the temperature $u(x,\,t)$ in a thin copper beam $(0 \leq x \leq L = 1)$ with material constant $\kappa = 1.14$. At $t = 0$ the initial temperature profile is

$$T_0(x) = 2\,\sin(3\pi\,x) + 5\,\sin(8\pi\,x)$$

and the ends of the beam are packed in ice $(0°)$. For $t > 0$ we determine the temperature inside the rod.

Solution: The temperature distribution satisfies the initial boundary value problem

$$u_t = \kappa\,u_{xx} \quad \text{with} \quad \begin{cases} u(x,\,0) = & T_0(x) \\ u(0,\,t) = & u(L,\,t) = 0. \end{cases}$$

According to $(*)$ the solution is

$$u(x,\,t) = 2\,e^{-1.14\cdot 9\pi^2\,t}\ \sin(3\pi\,x) + 5\,e^{-1.14\cdot 64\pi^2\,t}\ \sin(8\pi\,x)\,. \qquad \square$$

19.3.4 Solution of the Stationary Case

A rod of length L emits heat through its surface to the environment. The stationary temperature profile is sought. A temperature profile is called stationary if the temperature does not change with time. Then $\frac{\partial T}{\partial t} = 0$ and the temperature depends only on x. For the stationary temperature distribution $T(x)$ in the bar, our model equation reduces to

$$\frac{d^2}{dx^2} T(x) - 2 \left(\frac{1}{h} + \frac{1}{b} \right) \frac{\alpha}{\lambda} \, (T(x) - T_u) = 0.$$

This is an ordinary differential equation for $T(x)$. Possible boundary conditions are:

A) The left end of the rod is heated with a constant power and the right end is thermally insulated. The power P supplied to the system is defined as the energy supplied per unit time, which gives

$$P = -\lambda\, A \left(\frac{dT}{dx} \right)_{x=0} \quad \text{or} \quad \left(\frac{dT}{dx} \right)_{x=0} = -\frac{P}{b \cdot h \cdot \lambda}.$$

The thermal insulation at $x = L$ means $\left(\frac{dT}{dx}\right)_{x=L} = 0$.

B) The two ends of the rod are kept at a constant temperature. At the left end the temperature is $T(x=0) = T_l$ and at the right end the constant temperature is $T(x=L) = T_r$.

C) Combinations of (A) and (B).

Example 19.5 (With Maple**-Worksheet).** Given is the ordinary inhomogeneous second-order linear differential equation

$$T''(x) - \kappa \, (T(x) - T_u) = 0 \qquad (*)$$

with boundary conditions

$$\left(\frac{dT}{dx} \right)_{x=0} = -\frac{P}{b \cdot h \cdot \lambda} \quad \text{and} \quad \left(\frac{dT}{dx} \right)_{x=L} = 0.$$

(1) The homogeneous differential equation

$$T''(x) - \kappa\, T(x) = 0$$

has the characteristic polynomial

$$P(\lambda) = \lambda^2 - \kappa.$$

$P(\lambda) = 0$ gives $\lambda_{1/2} = \pm\sqrt{\kappa}$. $\Rightarrow e^{\sqrt{\kappa}\,x}, e^{-\sqrt{\kappa}\,x}$ is a real fundamental set. However, to take better account of the boundary values, we choose

$$\left.\begin{aligned}\cosh(\sqrt{\kappa}\,x) &= \tfrac{1}{2}\left(e^{\sqrt{\kappa}\,x} + e^{-\sqrt{\kappa}\,x}\right)\\ \sinh(\sqrt{\kappa}\,x) &= \tfrac{1}{2}\left(e^{\sqrt{\kappa}\,x} - e^{-\sqrt{\kappa}\,x}\right)\end{aligned}\right\}$$

as the fundamental set. Then, the general homogeneous solution is

$$T_h(x) = A\cosh(\sqrt{\kappa}\,x) + B\sinh(\sqrt{\kappa}\,x)\,.$$

(2) A particular solution is given by $T_p(x) = T_u$

(3) and the general solution of $(*)$ is

$$T(x) = T_u + A\cosh(\sqrt{\kappa}\,x) + B\sinh(\sqrt{\kappa}\,x).$$

(4) To take into account the boundary conditions, we compute

$$T'(x) = \sqrt{\kappa}\,A\sinh(\sqrt{\kappa}\,x) + \sqrt{\kappa}\,B\cosh(\sqrt{\kappa}\,x).$$

and apply

$$x = 0: \quad T'(0) = -\frac{P}{b\,h\,\lambda} \quad \Rightarrow \quad \sqrt{\kappa}\,B = -\frac{P}{b\,h\,\lambda}$$

$$x = L: \quad T'(L) = 0 \;\Rightarrow\; \sqrt{\kappa}\,\left(A\sinh(\sqrt{\kappa}\,L) + B\cosh(\sqrt{\kappa}\,L)\right) = 0.$$

This linear system of equations has the solution

$$A = \frac{P}{b\,h\,\lambda}\,\frac{1}{\sqrt{\kappa}}\,\frac{\cosh(\sqrt{\kappa}\,L)}{\sinh(\sqrt{\kappa}\,L)} \quad \text{and} \quad B = -\frac{P}{b\,h\,\lambda}\,\frac{1}{\sqrt{\kappa}}.$$

Fig. 19.11 shows the steady-state temperature profile for $T_u = 20°$; $b = 0.1$; $h = 0.01$; $L = 1$; $\alpha = 0.1$; $\lambda = 1000$; $P = 0.01$ (so $\kappa = 0.022$).

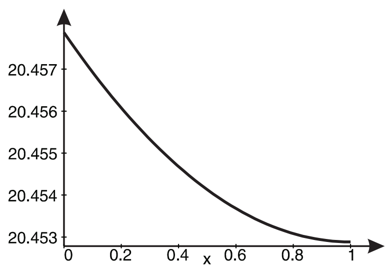


Figure 19.11. Stationary temperature profile in the rod

The curve shows the heating from the left edge, since $\frac{dT}{dx} \neq 0$, while the insulation is on the right edge: $\frac{dT}{dx} = 0$. □

19.4 The Laplace Equation

The Laplace equation is one of the best known partial differential equations. It occurs in many steady-state problems, such as a steady-state heat flow, the deflection of a membrane and electrostatic potentials. The latter is the reason why the Laplace equation is often called the potential equation.

19.4.1 Derivations of the Laplace Equation

① **Electrostatic Potential for Plane Problems.** The basic equations of electrostatics for plane problems are

$$\vec{E}(x,\, y) = -\text{grad}\ \Phi(x,\, y) = -\begin{pmatrix} \frac{\partial}{\partial x}\,\Phi\,(x,\, y) \\ \frac{\partial}{\partial y}\,\Phi\,(x,\, y) \end{pmatrix} \tag{1}$$

$$\text{div}\,\vec{E}(x,\, y) = \frac{1}{\varepsilon}\,\rho(x,y). \tag{2}$$

Where $\Phi\,(x,\, y)$ is the electrostatic potential, $\vec{E}\,(x,\, y) = \begin{pmatrix} E_1\,(x,\, y) \\ E_2\,(x,\, y) \end{pmatrix}$ is the electric field, and $\rho\,(x,y)$ is the charge density at the point (x,y). ε is the dielectric constant. Substituting equation (1) into (2) using the divergence rule gives

$$\begin{aligned} \text{div}\,\vec{E}(x,\, y) &= \tfrac{\partial}{\partial x}\,E_1\,(x,\, y) + \tfrac{\partial}{\partial y}\,E_2\,(x,\, y) \\ &= \tfrac{\partial}{\partial x}\,\left(-\tfrac{\partial}{\partial x}\,\Phi\,(x,\, y)\right) + \tfrac{\partial}{\partial y}\,\left(-\tfrac{\partial}{\partial y}\,\Phi\,(x,\, y)\right) = \frac{1}{\varepsilon}\,\rho(x,y) \end{aligned}$$

$$\Rightarrow \quad \frac{\partial^2}{\partial x^2}\,\Phi\,(x,\, y) + \frac{\partial^2}{\partial y^2}\,\Phi\,(x,\, y) = -\frac{1}{\varepsilon}\,\rho(x,y). \qquad \textbf{(Poisson's Equation)}$$

For a charge-free space, $\rho\,(x,\, y) = 0$, Poisson's equation reduces to

$$\frac{\partial^2}{\partial x^2}\,\Phi\,(x,\, y) + \frac{\partial^2}{\partial y^2}\,\Phi\,(x,\, y) = 0. \qquad \textbf{(Laplace's Equation)}$$

To abbreviate the left side of the two equations, we introduce

$$\boldsymbol{\Delta}\Phi(x,\, y) := \frac{\partial^2}{\partial x^2}\,\Phi\,(x,\, y) + \frac{\partial^2}{\partial y^2}\,\Phi\,(x,\, y)$$

and call $\boldsymbol{\Delta}$ the *Laplace operator*. □

② **2-dimensional Heat Transfer.** We consider heat transfer in both the x- and y-directions. Assuming an isotropic thermal conductivity λ, the energy balance applied to a volume element $dV = h \cdot dx \cdot dy$ is as follows:

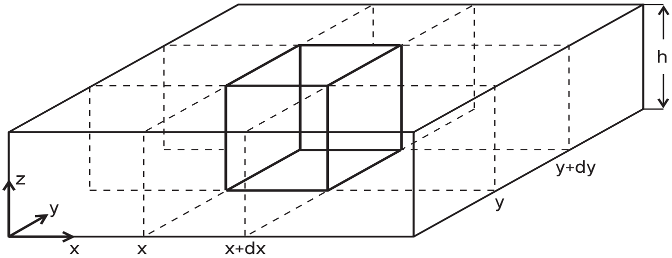


Figure 19.12. Heat transfer in x- and y-direction

Energy change per unit time $\triangle t$ in a mass element dm

$=$ Heat flux through surface $h \cdot dy$ at x

$+$ Heat flux through surface $h \cdot dy$ at $x + dx$

$+$ Heat flux through surface $h \cdot dx$ at y

$+$ Heat flux through surface $h \cdot dx$ at $y + dy$.

Expressed in formulas, the above equation is consistent with the basic laws of thermodynamics (see Section 19.3.1) means

$$\frac{\delta Q}{\partial t} = c\, dm \, \frac{\partial T}{\partial t} = \lambda \; (h\, dy) \; \frac{\partial^2 T}{\partial x^2}\, dx + \lambda \; (h\, dx) \; \frac{\partial^2 T}{\partial y^2}\, dy.$$

With $dm = \rho\, dV = \rho\, h\, dx\, dy$ finally follows

$$\frac{\partial}{\partial t} T = \frac{\lambda}{c\,\rho} \left(\frac{\partial^2}{\partial x^2}\, T + \frac{\partial^2}{\partial y^2}\, T \right).$$

For a stationary temperature profile it is $\frac{\partial T}{\partial t} = 0$, so the temperature depends only on x and y and is given by

$$\boldsymbol{\Delta} T(x, y) = \frac{\partial^2}{\partial x^2} T(x, y) + \frac{\partial^2}{\partial y^2} T(x, y) = 0.$$

The 2-dimensional temperature profile $T\,(x,\, y)$ of a body with surface insulation is given by the Laplace equation. □

③ **Deflection of a membrane.** The vibrations of a membrane are modelled in the same way as those of a vibrating string. In equilibrium and neglecting gravity, the membrane is clamped horizontally at $z = 0$.

The oscillation in z-direction is then described by a function $z\,(x,\, y,\, t)$. Using the same assumptions as in the derivation of the one-dimensional wave equation (see Section 19.2.1), the force acting on an area element $dx\,dy$ of the membrane is calculated.

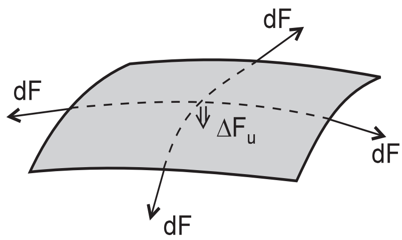


Figure 19.13. Deflection of a membrane

Assuming a constant stress $\gamma\ \left[\frac{N}{m}\right]$, the shear force $\triangle F_u$ is

$$\triangle F_u = (\gamma\, dy)\left(\frac{\partial^2 z}{\partial x^2}\right) dx + (\gamma\, dx)\left(\frac{\partial^2 z}{\partial y^2}\right) dy.$$

This transverse force accelerates the mass element $dm = \rho\, dV = \rho\, h\, dx\, dy$. According to Newton's law of motion, the deflection $z\,(x,\, y,\, t)$ is written as

$$\frac{\partial^2}{\partial t^2}\, z(x,y,t) = \frac{\gamma}{\rho \cdot h}\left(\frac{\partial^2}{\partial x^2}\, z(x,y,t) + \frac{\partial^2}{\partial y^2}\, z(x,y,t)\right)$$

with the stress $\gamma\ \left[\frac{N}{m}\right]$, the density $\rho\ \left[\frac{kg}{m^3}\right]$ and the thickness $h\ [m]$ of the membrane. This is the **two-dimensional wave equation**.

For the steady-state of the membrane, $\frac{\partial}{\partial t}\, z\,(x,\, y,\, t) = 0$ holds. Then, z is only a function of x and y and the steady-state is described by the Laplace equation

$$\boldsymbol{\Delta} z\,(x,\, y) = \frac{\partial^2}{\partial x^2}\, z\,(x,\, y) + \frac{\partial^2}{\partial y^2}\, z\,(x,\, y) = 0.$$ □

19.4.2 Solution of the Laplace Equation (Dirichlet Problem)

We consider a membrane stretched by a rectangular wire. One of the rectangular sides is bent in the z-direction. We look for the shape of the membrane $u(x, y)$ inside the rectangle (see Fig. 19.14).

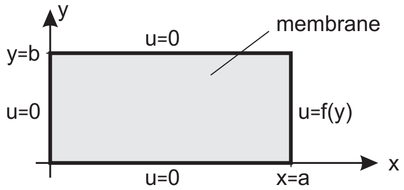


Figure 19.14. Clamped membrane

$u(x, y)$ denotes the deflection of the membrane at location (x, y) in z-direction. This deflection is given by

$$\frac{\partial^2}{\partial x^2} u(x, y) + \frac{\partial^2}{\partial y^2} u(x, y) = 0. \qquad \textbf{(Laplace equation)}$$

Since the PDE is independent of time t, no initial conditions are required for the complete solution of the differential equation, only boundary conditions. We discuss the case where the right side of the rectangle $(x = a)$ is bent in z-direction according to a given function $z = f(y)$:

$$u(x, 0) = 0; \quad u(x, b) = 0;$$
$$u(0, y) = 0; \quad u(a, y) = f(y)$$

(Dirichlet boundary conditions).

The **separation approach**

$$u(x, y) = X(x) \cdot Y(y)$$

is inserted into the PDE

$$X''(x) \cdot Y(y) + X(x) \cdot Y''(y) = 0$$

$$\Rightarrow \frac{Y''(y)}{Y(y)} = -\frac{X''(x)}{X(x)} = const = -k^2 .$$

A positive constant would lead to $k = 0$ in the further analysis and thus only to the zero solution $u(x, y) = 0$. Therefore, we introduce a negative constant $-k^2$. From this product approach we obtain two second-order ordinary differential equations:

(1) y-dependency: $Y''(y) + k^2\, Y(y) = 0$

$$\Rightarrow Y(y) = A\,\cos(k\,y) + B\,\sin(k\,y)\,.$$

(2) x-dependency: $X''(x) - k^2\, X(x) = 0$.

The characteristic polynomial of this differential equation is $P(\lambda) = \lambda^2 - k^2$, so we compute its zeros $P(\lambda) \overset{!}{=} 0$:

$$\lambda_{1/2} = \pm k \quad \Rightarrow \quad e^{k\,x},\, e^{-k\,x}$$

form a real fundamental set. However, to take better account of the boundary conditions, we choose the linear combination

$$\begin{aligned} \cosh(k\,x) &= \tfrac{1}{2}\left(e^{k\,x} + e^{-k\,x}\right) \\ \sinh(k\,x) &= \tfrac{1}{2}\left(e^{k\,x} - e^{-k\,x}\right) \end{aligned}$$

as the fundamental set.

$$\Rightarrow X(x) = C\,\cosh(k\,x) + D\,\sinh(k\,x)\,.$$

The **solution** of the PDE can be written as

$$u(x,\, y) = [A\,\cos(k\,y) + B\,\sin(k\,y)]\,[C\,\cosh(k\,x) + D\,\sinh(k\,x)].$$

Considering Boundary Conditions (Dirichlet Problem):

In the Dirichlet problem, the deflection of the membrane is given on all four sides of the rectangle. In this case the following applies

$u(x,\, 0) = 0$ for all $0 \le x \le a$ $\quad \Rightarrow A\,\underbrace{\cos(0)}_{=1} + B\,\underbrace{\sin(0)}_{=0} = 0 \quad \Rightarrow A = 0.$

$u(0,\, y) = 0$ for all $0 \le y \le b$ $\quad \Rightarrow C\,\underbrace{\cosh(0)}_{=1} + D\,\underbrace{\sinh(0)}_{=0} = 0 \quad \Rightarrow C = 0.$

$u(x,\, b) = 0$ for all $0 \le x \le a$ $\quad \Rightarrow B\,\sin(k\,b) = 0.$

To get not only the zero solution, we have to choose $B \neq 0$ and $\sin(k\,b) = 0$. $\Rightarrow \boxed{k \cdot b = n\,\pi}$. For each $n \in \mathbb{N}$ we obtain a valid constant $k_n = n\,\frac{\pi}{b}$ with the solution

$$u_n(x,\, y) = c_n\,\sinh(n\,\frac{\pi}{b}\,x)\,\sin(n\,\frac{\pi}{b}\,y).$$

All these functions have a shape with knots at $y = 0$ and $y = b$. According to the law of superposition for linear differential equations, the solution is given by

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} c_n \sinh(n \frac{\pi}{b} x) \sin(n \frac{\pi}{b} y).$$

The coefficients c_n are determined by the fourth boundary condition:
$u(a, y) = f(y)$ for all $0 \leq y \leq b$

$$\Rightarrow \sum_{n=1}^{\infty} \left[c_n \sinh(n \frac{\pi}{b} a)\right] \cdot \sin(n \frac{\pi}{b} y) = f(y). \qquad (*)$$

If we extend the function $f(y)$ oddly on the interval $[-b, 0]$ and $2b$-periodically on $\mathbb{R}$, $(*)$ is the Fourier series of the function $f(y)$ with the Fourier coefficients

$$c_n \sinh(n \frac{\pi}{b} a) = \frac{2}{b} \int_0^b f(y) \sin(n \frac{\pi}{b} y)\, dy \qquad n = 1, 2, 3, \ldots.$$

Solution of the Laplace Equation (Dirichlet Problem)

The solution of the Laplace equation

$$u_{xx}(x, y) + u_{yy}(x, y) = 0$$

with the Dirichlet boundary conditions

$$u(x = 0, y) = u(x, y = 0) = u(x, y = b) = 0,$$
$$u(x = a, y) = f(y),$$

is given by

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sinh(n \frac{\pi}{b} x) \sin(n \frac{\pi}{b} y)$$

with the coefficients

$$c_n = \frac{2}{b \sinh(n \frac{\pi}{b} a)} \int_0^b f(y) \sin(n \frac{\pi}{b} y)\, dy \qquad n = 1, 2, 3, \ldots$$

Remark: The separation approach leads to the solution of the Dirichlet problem only if u is zero on three sides of the rectangle. However, any Dirichlet problem for a rectangle can be split into four problems, with three Dirichlet values disappearing on each side. The complete solution is obtained by superimposing the four partial solutions.

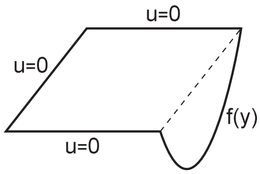


Fig. 19.15. Bending at $x = a$

Example 19.6 (Bending at $x = a$). A membrane is clamped by a rectangular wire (see Fig. 19.15). The membrane is bent in z-direction at $y = b$, where the bending is given by

$$f(y) = y\,(y - b)\,.$$

We look for the deflection in z-direction inside the rectangle.

Step 1: We extend the function $f(y)$ as a point symmetric $2b$-periodic function. b_n are the Fourier coefficients of f. A double integration by parts gives

$$\begin{aligned} b_n &= \frac{2}{b}\int_0^b f(y)\,\sin(n\,\frac{\pi}{b}\,y)\,dy = \frac{2}{b}\int_0^b y(y-b)\,\sin(n\,\frac{\pi}{b}\,y)\,dy \\ &= \frac{4}{n^3\,\pi^3}\,(\cos(n\,\pi) - 1) = \frac{4}{n^3\,\pi^3}\,((-1)^n - 1) \end{aligned}$$

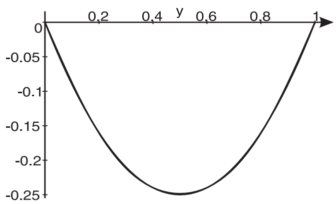


Figure 19.16. Function f and partial sum of the Fourier series with 3 terms

We evaluate the partial sum with only 3 terms $\sum_{n=1}^{3} b_n \sin(n\,\frac{\pi}{b}\,y)$ and compare the sum for $a = 2$ and $b = 1$ with the function $f(y)$. Graphically there is no difference (see Fig. 19.16).

Step 2: We display the solution in a 3-dimensional plot. From the solution formula we evaluate $c_n = b_n \frac{1}{\sinh(n\,\frac{\pi}{b})}$ and take only three terms:

$$u(x, y) = \sum_{n=1}^{3} c_n\,\sinh(n\,\frac{\pi}{b}\,x)\,\sin(n\,\frac{\pi}{b}\,y).$$

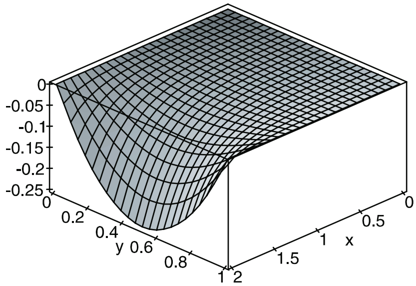


Figure 19.17. Deformation of the membrane

In Fig. 19.17 we see (for $a = 2$ and $b = 1$) that the membrane is clamped on three sides and bends towards the shape of the fourth side. □

19.4.3 Solution of the Laplace Equation (Neumann Problem)

In electrostatic problems, the potential values at all boundaries are usually not known. For example, at open boundaries we only know that the equipotential lines remain perpendicular to the boundary, which mathematically means that the normal derivative of the potential disappears. This leads to the *Neumann* boundary conditions.

Let us consider the following problem:

$$\frac{\partial^2}{\partial x^2} u(x, y) + \frac{\partial^2}{\partial y^2} u(x, y) = 0 \qquad \text{Laplace equation}$$

$$\left. \begin{array}{ll} u_y(x, 0) = 0 & u_y(x, b) = 0 \\ u_x(0, y) = 0 & u_x(a, y) = f(y) \end{array} \right\} \text{(Neumann boundary conditions).}$$

According to Section 19.4.2, a separation approach provides the solution

$$u(x, y) = [A \cos(k y) + B \sin(k y)]\, [C \cosh(k x) + D \sinh(k x)]\,.$$

To take into account the boundary conditions, the partial derivatives are calculated with respect to x and y

$$u_x(x, y) = [A \cos(k y) + B \sin(k y)]\, [C\, k \sinh(k x) + D\, k \cosh(k x)]$$

$$u_y(x, y) = [-A\, k \sin(k y) + B\, k \cos(k y)]\, [C \cosh(k x) + D \sinh(k x)].$$

We evaluate the solution at the boundaries:

$u_y\,(x,\,0)=0$ for all $0\le x\le a:\quad A\,k\,\underbrace{\sin(0)}_{=0}+B\,k\,\underbrace{\cos(0)}_{=1}=0\;\Rightarrow B=0.$

$u_x\,(0,\,y)=0$ for all $0\le y\le b:\; C\,k\,\underbrace{\sinh(0)}_{=0}+D\,k\,\underbrace{\cosh(0)}_{=1}=0\;\Rightarrow D=0.$

$u_y\,(x,\,b)=0$ for all $0\le x\le a$: $\sin(k\,b)=0\Rightarrow k\,b=n\,\pi$

$$\Rightarrow \boxed{k_n=n\,\tfrac{\pi}{b}} \qquad n=0,\,1,\,2,\,3,\ldots$$

So for each $n=0,\,1,\,2,\ldots$

$$u_n\,(x,\,y)=c_n\,\cosh(n\,\frac{\pi}{b}\,x)\,\cos(n\,\frac{\pi}{b}\,y)$$

is a solution and by superposition we get the general solution

$$u\,(x,\,y)=\sum_{n=0}^{\infty}c_n\,\cosh(n\,\frac{\pi}{b}\,x)\,\cos(n\,\frac{\pi}{b}\,y).$$

The coefficients c_n are again determined by the fourth boundary condition:

$u_x\,(a,\,y)=f\,(y)$ for all $0\le y\le b$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{n\,\pi}{b}\,c_n\,\sinh(n\,\frac{\pi}{b}\,a)\,\cos(n\,\frac{\pi}{b}\,y)=f\,(y)\,. \qquad (*)$$

If the function $f\,(y)$ is extended point symmetrically with respect to the origin on the interval $[-b,\,0]$ and then $2b$-periodically on $\mathbb{R}$, $(*)$ is the Fourier series of $f\,(y)$ except for the constant term c_0.

Therefore, the solution of this boundary value problem is unique up to a constant c_0. So it is necessary to add an additional condition, e.g.

$$c_0=\frac{1}{b}\int_0^b f\,(y)\,dy=0.$$

The remaining c_n are calculated for $n=1,\,2,\,3,\,\ldots$ using the formula

$$c_n=\frac{2}{n\,\pi\,\sinh\left(n\,\frac{\pi}{b}\,a\right)}\int_0^b f\,(y)\,\cos(n\,\frac{\pi}{b}\,y)\,dy.$$

□

19.4.4 The Laplace Equation in Polar Coordinates

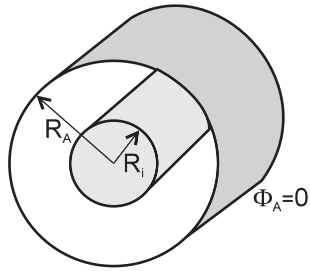


Figure 19.18. Capacitor

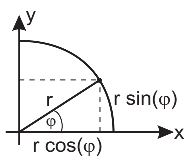
Polar coordinates are usually used to solve the Laplace equation in rotationally symmetric geometries. An example is a cylindrical capacitor where the outer cylindrical part is grounded and the inner part is at potential ϕ_i. Although the equation

$$u_{xx}(x,\, y) + u_{yy}(x,\, y) = 0$$

is valid inside the cylinder, the boundary conditions are not so easy to describe mathematically. Therefore, polar coordinates (r, φ) are introduced, in which the boundary conditions can be easily specified:

$$u(R_i,\, \varphi) = \phi_i \quad \text{and} \quad u(R_A,\, \varphi) = 0 \quad \text{for all} \quad 0 \le \varphi \le 2\pi.$$

The transformation equations are



$$\begin{aligned} x &= r\, \cos\varphi \qquad \Rightarrow \qquad x = x(r,\, \varphi) \\ y &= r\, \sin\varphi \qquad \Rightarrow \qquad y = y(r,\, \varphi) \end{aligned}$$

So the potential $u(x,\, y) = u(x(r,\, \varphi)\,,\, y(r,\, \varphi))$ is a function of r and φ. Using the chain rule, we partially differentiate this function with respect to r and φ.

$$\begin{aligned} \frac{\partial}{\partial r} u(x(r,\, \varphi)\,,\, y(r,\, \varphi)) &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} = u_x \cdot x_r + u_y \cdot y_r \\ &= u_x\, \cos\varphi + u_y\, \sin\varphi \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial \varphi} u(x(r,\, \varphi)\,,\, y(r,\, \varphi)) &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \varphi} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \varphi} = u_x\, x_\varphi + u_y\, y_\varphi \\ &= -u_x\, r\, \sin\varphi + u_y\, r\, \cos\varphi. \end{aligned}$$

This gives the system of linear equations for the two partial derivatives u_x and u_y:

$$\begin{pmatrix} u_r \\ u_\varphi \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -r\, \sin\varphi & r\, \cos\varphi \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix}.$$

After inverting the matrix

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \frac{1}{r} \begin{pmatrix} r\,\cos\varphi & -\sin\varphi \\ r\,\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} u_r \\ u_\varphi \end{pmatrix}$$

we obtain the solution in components by

$$\begin{aligned} u_x &= \cos\varphi\ u_r - \tfrac{1}{r}\ \sin\varphi\ u_\varphi \\ u_y &= \sin\varphi\ u_r + \tfrac{1}{r}\ \cos\varphi\ u_\varphi. \end{aligned}$$

To reformulate the Laplace equation in polar coordinates $(r,\,\varphi)$, we again differentiate u_x partially with respect to x and u_y partially with respect to y:

$$\begin{aligned} u_{xx} &= (u_x)_x = \cos\varphi\ (u_x)_r - \frac{1}{r}\ \sin\varphi\ (u_x)_\varphi \\ &= \cos\varphi\ \left(\cos\varphi\, u_r - \frac{1}{r}\ \sin\varphi\, u_\varphi\right)_r - \frac{1}{r}\ \sin\varphi\ \left(\cos\varphi\, u_r - \frac{1}{r}\ \sin\varphi\, u_\varphi\right)_\varphi \\ &= \cos^2\varphi\, u_{rr} - \frac{\partial}{\partial r}\ \left(\frac{1}{r}\ \cos\varphi\ \sin\varphi\, u_\varphi\right) + \frac{1}{r}\ \sin^2\varphi\, u_r \\ &\qquad -\frac{1}{r}\ \sin\varphi\ \cos\varphi\, u_{r\varphi} + \frac{1}{r^2}\ \sin\varphi\ \cos\varphi\, u_\varphi + \frac{1}{r^2}\ \sin^2\varphi\, u_{\varphi\varphi} \end{aligned}$$

$$\begin{aligned} u_{yy} &= (u_y)_y = \sin\varphi\ (u_y)_r + \frac{1}{r}\ \cos\varphi\ (u_y)_\varphi \\ &= \sin\varphi\ \left(\sin\varphi\, u_r + \frac{1}{r}\ \cos\varphi\, u_\varphi\right)_r + \frac{1}{r}\ \cos\varphi\ \left(\sin\varphi\, u_r + \frac{1}{r}\ \cos\varphi\, u_\varphi\right)_\varphi \\ &= \sin^2\varphi\, u_{rr} + \frac{\partial}{\partial r}\ \left(\frac{1}{r}\ \cos\varphi\ \sin\varphi\, u_\varphi\right) + \frac{1}{r}\ \cos^2\varphi\, u_r \\ &\qquad +\frac{1}{r}\ \cos\varphi\ \sin\varphi\, u_{r\varphi} - \frac{1}{r^2}\ \cos\varphi\ \sin\varphi\, u_\varphi + \frac{1}{r^2}\ \cos^2\varphi\, u_{\varphi\varphi}. \end{aligned}$$

To summarize this result, the Laplace operator is

$$\boldsymbol{\Delta} u\,(r,\,\varphi) = u_{rr}\,(r,\varphi) + \frac{1}{r}\ u_r(r,\varphi) + \frac{1}{r^2}\ u_{\varphi\varphi}\,(r,\varphi)$$

and with the identity $\frac{1}{r}\,\frac{\partial}{\partial r}\,r\,\frac{\partial}{\partial r}\,u\,(r,\,\varphi) = u_{rr} + \frac{1}{r}\,u_r\,(r,\,\varphi)$ finally

$$\boldsymbol{\Delta} u\,(r,\,\varphi) = \frac{1}{r}\,\frac{\partial}{\partial r}\,r\,\frac{\partial}{\partial r}\,u\,(r,\,\varphi) + \frac{1}{r^2}\,\frac{\partial^2}{\partial\varphi^2}\,u\,(r,\,\varphi) = 0.$$

This is the **Laplace equation in polar coordinates** $(r,\,\varphi)$.

Example 19.7. Back to the cylindrical condenser problem from the beginning: We look for a solution $\phi(r, \varphi)$ of

$$\boldsymbol{\Delta}\phi(r, \varphi) = 0$$

with the boundary values $\phi(r = R_i, \varphi) = \phi_i$ and $\phi(r = R_A, \varphi) = 0$.

Due to the symmetry property of the problem, the potential ϕ does not depend on the angle φ, i.e. $\frac{\partial}{\partial\varphi}\phi(r, \varphi) = 0$. So ϕ is only a function of the radius r and the problem is reduced to an ordinary differential equation

$$\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\phi(r) = 0$$

with the boundary values $\phi(R_i) = \phi_i$ and $\phi(R_A) = 0$.

We integrate twice

$$\frac{d}{dr}r\frac{d}{dr}\phi(r) = 0 \Rightarrow r\frac{d}{dr}\phi(r) = \rho$$
$$\Rightarrow \frac{d}{dr}\phi(r) = \frac{\rho}{r} \Rightarrow \phi(r) = \rho\ln(r) + A$$

with the integration constants A and ρ determined by the boundary conditions:

$$\phi(R_i) = \phi_i : \quad \phi(R_i) = \rho\ln(R_i) + A = \phi_i \tag{1}$$

$$\phi(R_A) = 0 : \quad \phi(R_A) = \rho\ln(R_A) + A = 0 \tag{2}$$

Subtracting (2) from (1) gives $\rho = \frac{\phi_i}{\ln\left(\frac{R_i}{R_A}\right)}$ and inserting this into (2) gives $A = -\rho\ln(R_A)$. Finally, we obtain

$$\Rightarrow \quad \phi(r) = \frac{\phi_i}{\ln\left(\frac{R_i}{R_A}\right)}\ln\left(\frac{r}{R_A}\right).$$

This is the potential distribution in a cylindrical capacitor with inner radius R_i at potential ϕ_i and outer radius R_A at potential $\phi_A = 0$. □

19.5 The Two-Dimensional Wave Equation

An example of the *two-dimensional* wave equation is a membrane stretched by a rectangular wire:

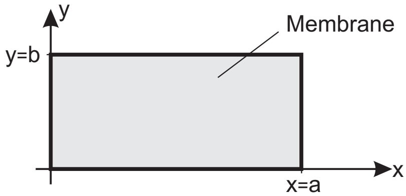


Figure 19.19. Membrane

$u(x,\, y,\, t)$ is the deflection in z-direction. If the edges are horizontal at $z = 0$ and the membrane is deflected (e.g. by a drum beat), the equation for the deflection $u(x,\, y,\, t)$ at the point $(x,\, y)$ at time t is given by the two-dimensional wave equation according to 19.4.1 ③

$$\frac{\partial^2}{\partial t^2}\, u(x,y,t) = c^2 \left(\frac{\partial^2}{\partial x^2}\, u(x,y,t) + \frac{\partial^2}{\partial y^2}\, u(x,y,t) \right)$$

with the initial displacement and velocity

$$\left.\begin{aligned} u(x,\, y,\, t=0) &= u_0(x,\, y) \\ u_t(x,\, y,\, t=0) &= v_0(x,\, y) \end{aligned}\right\} \quad \text{(Initial conditions)}$$

and the boundary conditions

$$\left.\begin{aligned} u(x=0,\, y,\, t) = 0\,;\ \ u(x=a,\, y,\, t) = 0 \quad \text{for all} \quad y,\, t\,. \\ u(x,\, y=0,\, t) = 0\,;\ \ u(x,\, y=b,\, t) = 0 \quad \text{for all} \quad x,\, t\,. \end{aligned}\right\} \quad \text{(BC)}$$

This initial boundary value problem is solved using a product approach

$$u(x,\, y,\, t) = U(x,\, y) \cdot T(t)\ ,$$

where $T(t)$ is a purely time-dependent function and $U(x,\, y)$ is a two-dimensional, location-dependent function. The product approach inserted into the PDE provides

$$T''(t) \cdot U(x,\, y) = c^2\ \left(U_{xx}(x,\, y) + U_{yy}(x,\, y)\right) \cdot T(t)\ .$$

After separating the time from the spacial variables we get

$$\frac{T''(t)}{T(t)} = c^2 \frac{U_{xx}(x, y) + U_{yy}(x, y)}{U(x, y)} = const = -\omega^2 .$$

(1) For the time function the result is a second-order **ordinary** differential equation:

$$T''(t) + \omega^2 T(t) = 0 \Rightarrow T(t) = A \sin(\omega t) + B \cos(\omega t) .$$

(2) For the location-dependent function $U(x, y)$ the result is a second-order **partial** differential equation:

$$U_{xx}(x, y) + U_{yy}(x, y) + \frac{\omega^2}{c^2} U(x, y) = 0.$$

This is the so-called **Helmholtz equation**.

Remark: The Helmholtz equation is also encountered when a separation approach is chosen for the two-dimensional time-dependent heat transfer equation (19.4.1 ②). Only the ordinary differential equation for the time function is then given by $\frac{T'(t)}{T(t)} = const = -\omega^2$.

⊙ The Helmholtz Equation

To solve the Helmholtz equation, again a separation approach is chosen

$$U(x, y) = X(x) \cdot Y(y)$$

which results in

$$X''(x)\, Y(y) + X(x)\, Y''(y) + \frac{\omega^2}{c^2} X(x)\, Y(y) = 0$$

$$\Rightarrow \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -\frac{\omega^2}{c^2} .$$

Since the variable x only appears in the first term and the variable y only in the second term, both terms must be constant in x and y:

$$\frac{X''(x)}{X(x)} = const = -k^2 \quad \text{and} \quad \frac{Y''(y)}{Y(y)} = const = -l^2 .$$

$$\Rightarrow \boxed{k^2 + l^2 = \frac{\omega^2}{c^2}} \quad \text{or} \quad \boxed{\omega = c\sqrt{k^2 + l^2}.}$$

(1) Location dependency on x: $X''(x) + k^2\, X(x) = 0$

$$\Rightarrow X(x) = D\,\sin(k\,x) + E\,\cos(k\,x)\,.$$

(2) Location dependency on y: $Y''(y) + l^2\, Y(y) = 0$

$$\Rightarrow Y(y) = F\,\sin(l\,y) + G\,\cos(l\,y)\,.$$ □

With this solution of the Helmholtz equation and the time-dependent solution $T(t)$, the **solution of the two-dimensional wave equation** is

$$u(x,y,t) = [A\,\sin(\omega t) + B\,\cos(\omega t)] \cdot [D\,\sin(k\,x) + E\,\cos(k\,x)]$$
$$\cdot\,[F\,\sin(l\,y) + G\,\cos(l\,y)]\;.$$

This solution of the two-dimensional wave equation again contains parameters which are determined by the boundary conditions and the initial conditions:

Consideration of the Boundary Conditions:

① $u(x=0,\, y,\, t) = 0$ for all $y,\, t$

$$\Rightarrow X(0) = D\,\sin(0) + E\,\cos(0) \overset{!}{=} 0 \Rightarrow E = 0.$$

② $u(x=a,\, y,\, t) = 0$ for all $y,\, t \quad \Rightarrow X(a) = D\,\sin(k\,a) = 0.$

A non-zero solution for $u(x,\, y,\, t)$ is obtained if

$$\sin(k\,a) = 0 \quad \Rightarrow k\,a = n\,\pi \quad \Rightarrow \boxed{k_n = n\,\frac{\pi}{a}} \qquad n \in \mathbb{N}\,.$$

So in x-direction there are only standing waves with $k_n = n\,\frac{\pi}{a}$ and with the wavelengths $\lambda = \frac{2\pi}{k_n} = \frac{2\,a}{n}$.

③ $u(x,\, y=0,\, t) = 0$ for all $x,\, t$

$$\Rightarrow Y(0) = F\,\sin(0) + G\,\cos(0) = 0 \Rightarrow G = 0.$$

④ $u(x,\, y=b,\, t) = 0$ for all $x,\, t \quad \Rightarrow Y(b) = F\,\sin(l\,b) = 0.$

To get a non-zero solution

$$\sin(l\,b) = 0 \quad \Rightarrow l\,b = m\,\pi \quad \Rightarrow \boxed{l_m = m\,\frac{\pi}{b}} \qquad m \in \mathbb{N}\,.$$

Also in y-direction there are only standing waves with $l_m = m\,\frac{\pi}{b}$ and with wavelengths $\lambda = \frac{2\pi}{l_m} = \frac{2\,b}{m}$.

For each pair $(n,\,m)$ with $n \in \mathbb{N}$ and $m \in \mathbb{N}$ a solution of the two-dimensional wave equation is given by

$$u_{n,\,m}(x,\,y,\,t) = \left[a_{n,\,m}\,\sin(\omega_{n,\,m}\,t) + b_{n,\,m}\,\cos(\omega_{n,\,m}\,t)\right] \cdot \sin(n\,\frac{\pi}{a}\,x)\,\sin(m\,\frac{\pi}{b}\,y)$$

with the frequencies $\omega_{n,\,m} = c\sqrt{\left(n\,\frac{\pi}{a}\right)^2 + \left(m\,\frac{\pi}{b}\right)^2}\,.$

The pair $(n,\,m)$ is called the **oscillation mode**. The solution for the two-dimensional vibrating membrane is a superposition of all the modes:

$$u(x,\,y,\,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{n,\,m}\,\sin(\omega_{n,\,m}\,t)\cdot\sin(n\,\frac{\pi}{a}\,x)\,\sin(m\,\frac{\pi}{b}\,y) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{n,\,m}\,\cos(\omega_{n,\,m}\,t)\cdot\sin(n\,\frac{\pi}{a}\,x)\,\sin(m\,\frac{\pi}{b}\,y)$$

Considering the Initial Conditions:

For the **initial deflection** $u_0(x,\,y)$ we have the relation

$$u(x,\,y,\,t=0) = u_0(x,\,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{n,\,m}\,\sin\left(n\,\frac{\pi}{a}\,x\right)\,\sin\left(m\,\frac{\pi}{b}\,y\right)\,.$$

This is the sine Fourier series of the two-dimensional function u_0. The corresponding formulas for the Fourier coefficients $b_{n,m}$ can be obtained in a similar way to the one-dimensional case (see 17.2); however, this relation will not be discussed in detail here.

Similarly, we have the relation for the **initial velocity**

$$u_t\,(x,\,y,\,t=0) = v_0\,(x,\,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\,(b_{n,\,m}\,\omega_{n,\,m})\,\sin\left(n\,\frac{\pi}{a}\,x\right)\,\sin\left(m\,\frac{\pi}{b}\,y\right).$$

This is the sine Fourier series of the two-dimensional function v_0 with the Fourier coefficients $(b_{n,m}\,\omega_{n,m})$. □

Visualization: The **graphs** show the elementary modes of vibration. For $(n,\,m) = (1,\,1)$; $(n,\,m) = (1,\,2)$; $(n,\,m) = (3,\,1)$; $(n,\,m) = (4,\,4)$ the information is shown in Fig. 19.20. In the $(1,1)$ mode only one half-wave in x-direction and one half-wave in y-direction is established. We do not see any knots, only a bump. In$(1,2)$ mode there is one half-wave in x-direction but two half-waves in y-direction. Finally, in $(4,4)$ we see 4 half-waves in x-direction and 4 half-waves in y-direction.

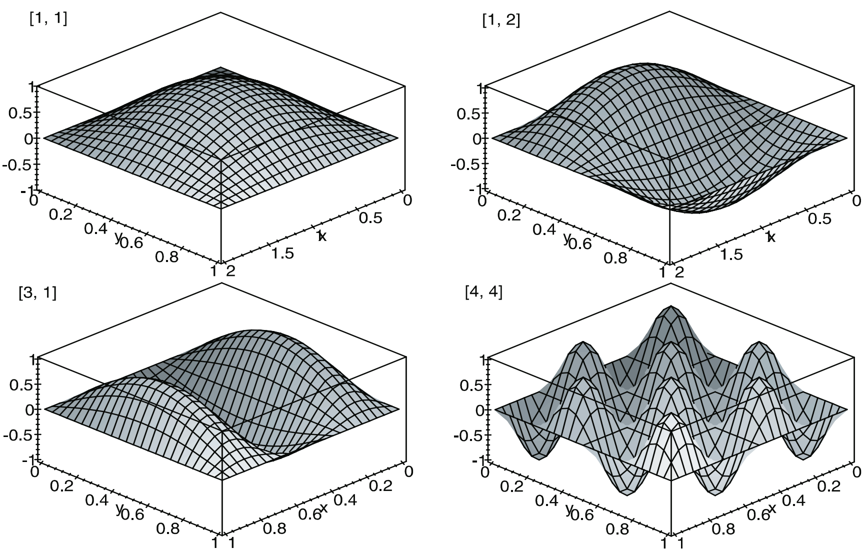


Figure 19.20. Vibration modes of a rectangular diaphragm

Animation: The homepage contains the MAPLE-Animation for the basic vibrations $(n,\,m) = (1,\,1)$. However, in the corresponding Worksheet any other type of oscillation can be selected and animated. □

19.6 The Beam Bending Equation

In this section we model the oscillations of an elastic beam. These oscillations are described by a 4th order partial differential equation.

19.6.1 Deriving the Bending Equation

To model the bending vibration equation, we consider a homogeneous beam (length L, cross-section A, moment of inertia I, modulus of elasticity E) supported on the x-axis. This beam bends under the influence of vertical loads according to the laws of statics:

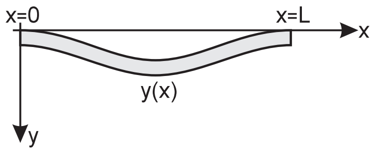


Figure 19.21. Elastic beam

$y(x)$ is the deflection of the beam at point x. For small displacements $y(x)$, the bending moment at point x is given by

$$M(x) = E \cdot I \, \frac{d^2}{dx^2} \, y(x) \; .$$

The associated lateral force $F_q(x)$ at point x is

$$-F_q(x) = \frac{d}{dx} M(x) = E\, I \, \frac{d^3}{dx^3} \, y(x) \; .$$

The transverse force $\triangle F_q$ acting on the mass element is analogous to the transverse force for the vibrating string, using the linearization of $F_q(x + dx) \approx F_q(x) + \frac{d\,F_q(x)}{dx}\, dx$ given by

$$\triangle F_q = F_q(x + dx) - F_q(x) \approx F_q(x) + dx \, \frac{dF_q(x)}{dx} - F_q(x) = dx \, \frac{dF_q(x)}{dx} \; .$$

The dynamic description follows, considering the acceleration force on a mass element $dm = \rho\, A\, dx$

$$F = dm \, \frac{\partial^2}{\partial t^2} y(x,\, t)$$

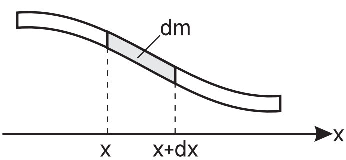


Figure 19.22.

and equals the resulting force $\triangle F_q$:

$$\rho\, A\, dx\, \frac{\partial^2\, y\,(x,\, t)}{\partial t^2} = -E\, I\, \frac{\partial^4\, y\,(x,\, t)}{\partial x^4}\, dx\,.$$

In summary, we get

Beam Bending Equation

$$\frac{\partial^2}{\partial t^2}\, y(x,t) = -\frac{E\, I}{\rho\, A}\, \frac{\partial^4}{\partial x^4}\, y(x,t).$$

This is a 4th order linear PDE with constants ρ (density), A (area), I (moment of inertia) and E (modulus of elasticity).

19.6.2 Solution of the Beam Bending Equation

To determine a solution to the bending equation, a separation approach is used by

$$y\,(x,\, t) = X\,(x) \cdot T\,(t)$$

with

$X\,(x)$	a purely location-dependent function,
$T(t)$	a purely time-dependent function.

Inserting this approach into the PDE gives

$$X\,(x) \cdot T''\,(t) = -\frac{E\, I}{\rho\, A}\, X^{(4)}\,(x) \cdot T\,(t)$$

$$\Rightarrow -\frac{E\, I}{\rho\, A}\, \frac{X^{(4)}\,(x)}{X\,(x)} = \frac{T''\,(t)}{T\,(t)} = const = -\omega^2\,.$$

A positive constant would lead to a non-physical solution. This product approach reduces the PDE to two ordinary differential equations:

(1) Time dependency: $T''\,(t) + \omega^2\, T\,(t) = 0$

$$\Rightarrow\; T\,(t) = A\, \cos\,(\omega t) + B\, \sin\,(\omega t)\,.$$

(2) Location dependency:

$$X^{(4)}\,(x) - \frac{\rho\, A}{E\, I}\, \omega^2\, X\,(x) = 0.$$

This is a 4th order differential equation. With the approach $X(x) = e^{\lambda x}$, the characteristic polynomial is

$$P(\lambda) = \lambda^4 - \frac{\rho A}{E I}\,\omega^2 = 0.$$

The zeros of $P(\lambda)$ are

$$\lambda = \pm\sqrt{\pm\sqrt{\frac{\rho A}{E I}}\,\sqrt{\omega}}\,.$$

With $\kappa = \sqrt[4]{\frac{\rho A}{E I}}\,\sqrt{\omega}$ we get

$$\lambda_1 = \kappa, \quad \lambda_2 = -\kappa, \quad \lambda_3 = i\,\kappa, \quad \lambda_4 = -i\,\kappa$$

and

$$e^{\kappa x}, \quad e^{-\kappa x}, \quad e^{i \kappa x}, \quad e^{-i \kappa x}$$

is a complex fundamental set. With the linear combinations

$$\sinh(\kappa x) = \tfrac{1}{2}\,(e^{\kappa x} - e^{-\kappa x}), \qquad \cosh(\kappa x) = \tfrac{1}{2}\,(e^{\kappa x} + e^{-\kappa x}),$$

$$\sin(\kappa x) = \tfrac{1}{2i}\,\left(e^{i \kappa x} - e^{-i \kappa x}\right), \qquad \cos(\kappa x) = \tfrac{1}{2}\,\left(e^{i \kappa x} + e^{-i \kappa x}\right),$$

we obtain a real fundamental set

$$\cosh(\kappa x)\,, \quad \sinh(\kappa x)\,, \quad \cos(\kappa x)\,, \quad \sin(\kappa x)\,.$$

Then, the solution to the ordinary differential equation is

$$X(x) = A_1 \cosh(\kappa x) + A_2 \sinh(\kappa x) + A_3 \cos(\kappa x) + A_4 \sin(\kappa x).$$

Finally, we get the solution to the beam bending equation

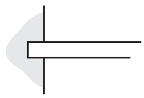
$$u(x,t) = (A_1\,\cosh(\kappa x) + A_2\,\sinh(\kappa x) + A_3\,\cos(\kappa x) + A_4\,\sin(\kappa x)) \cdot (A\,\cos(\omega t) + B\,\sin(\omega t))\,,$$

where the constants A_1, A_2, A_3, A_4, and A, B are determined by the boundary conditions, the initial deflection and the initial velocity.

Considering the Boundary Conditions

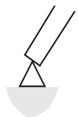
The following boundary conditions for the right boundary (and correspondingly for the left boundary) physically occur.

Fixed:



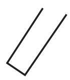
$$X(0) = 0$$
$$X'(0) = 0$$

Flexible:



$$X(0) = 0$$
$$X''(0) = 0$$

Free:



$$X''(0) = 0$$
$$X'''(0) = 0$$

Two cases are discussed in the following subsections: The first is discussed in 19.6.3 flexible/flexible. This is the only case that leads to a closed solution. The second is fixed/fixed in 19.6.4, in this case the corresponding eigenvalue equation can only be solved numerically. To take into account the boundary conditions, the derivatives of $X(x)$ up to order 3 must be calculated

$$X\,(x) = A_1\,\cosh(\kappa\,x) + A_2\,\sinh(\kappa\,x) + A_3\,\cos(\kappa\,x) + A_4\,\sin(\kappa\,x)$$

$$X'\,(x) = A_1\,\kappa\,\sinh(\kappa\,x) + A_2\,\kappa\,\cosh(\kappa\,x) - A_3\,\kappa\,\sin(\kappa\,x) + A_4\,\kappa\,\cos(\kappa\,x)$$

$$X''\,(x) = A_1\,\kappa^2\cosh(\kappa\,x) + A_2\,\kappa^2\sinh(\kappa\,x) - A_3\,\kappa^2\cos(\kappa\,x) - A_4\,\kappa^2\sin(\kappa\,x)$$

$$X'''\,(x) = A_1\,\kappa^3\sinh(\kappa\,x) + A_2\,\kappa^3\cosh(\kappa\,x) + A_3\,\kappa^3\sin(\kappa\,x) - A_4\,\kappa^3\cos(\kappa\,x)$$

and evaluated at the boundaries $x = 0$ and $x = L$

$$\begin{aligned}
X\,(0) &= A_1 + A_3\\
X\,(L) &= A_1\,\cosh(\kappa\,L) + A_2\,\sinh(\kappa\,L) + A_3\,\cos(\kappa\,L) + A_4\,\sin(\kappa\,L)
\end{aligned}$$

$$\begin{aligned}
X'\,(0) &= A_2\,\kappa + A_4\,\kappa\\
X'\,(L) &= A_1\,\kappa\,\sinh(\kappa\,L) + A_2\,\kappa\,\cosh(\kappa\,L) - A_3\,\kappa\,\sin(\kappa\,L) + A_4\,\kappa\,\cos(\kappa\,L)
\end{aligned}$$

$$\begin{aligned}
X''\,(0) &= A_1\,\kappa^2 - A_3\,\kappa^2\\
X''\,(L) &= A_1\,\kappa^2\,\cosh(\kappa\,L) + A_2\,\kappa^2\,\sinh(\kappa\,L) - A_3\,\kappa^2\,\cos(\kappa\,L) - A_4\,\kappa^2\,\sin(\kappa\,L)
\end{aligned}$$

$$\begin{aligned}
X'''\,(0) &= A_2\,\kappa^3 - A_4\,\kappa^3\\
X'''\,(L) &= A_1\,\kappa^3\sinh(\kappa\,L) + A_2\,\kappa^3\cosh(\kappa\,L) + A_3\,\kappa^3\sin(\kappa\,L) - A_4\,\kappa^3\cos(\kappa\,L)
\end{aligned}$$

19.6.3 Boundary Condition: flexible/flexible

$$\begin{array}{lrcl}
X(0)=0: & A_1 + A_3 = 0 & \Rightarrow & A_1 = A_3 = 0 \\
X''(0)=0: & A_1 - A_3 = 0 & & \\
X(L)=0: & A_2 \sinh(\kappa L) + A_4 \sin(\kappa L) = 0 & & \\
X''(L)=0: & A_2 \sinh(\kappa L) - A_4 \sin(\kappa L) = 0 & &
\end{array}$$

The linear system of equations for A_2 and A_4 must not be uniquely solvable, so that there is not only the zero solution $X(x) \equiv 0$. A homogeneous linear system can be solved non-trivially if the determinant of the coefficient matrix is zero

$$-\sinh(\kappa L)\sin(\kappa L) - \sinh(\kappa L)\sin(\kappa L) = 2\sinh(\kappa L)\sin(\kappa L) = 0 .$$

This results in

$$\sin(\kappa L) = 0 \;\Rightarrow\; \kappa L = n\pi \qquad n = 1, 2, 3, \ldots .$$

Only discrete frequencies or wavelengths are possible. The *eigenvalues* are $\boxed{\kappa_n = n\frac{\pi}{L}}$ (or wavelengths $\lambda_n = \frac{2L}{n}$). For each eigenvalue $\kappa_n = n\frac{\pi}{L}$ it is $\sin(n\frac{\pi}{L}x) = 0$ and A_4 is arbitrary. The linear system for A_2 and A_4 reduces to

$$A_2 \sinh(\kappa L) = 0 \Rightarrow A_2 = 0.$$

The form of vibration associated with κ_n is therefore

$$X_n(x) = A \sin\left(n\frac{\pi}{L}x\right).$$

For each $n \in \mathbb{N}$ there is an eigenvalue κ_n with an associated function $X_n(x)$ (= mode of vibration). The solution $y(x,t)$ of the beam bending equation for the boundary condition flexible/flexible is the superposition of all single modes

$$y(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\right) \sin\left(n\frac{\pi}{L}x\right)$$

with the frequencies $\omega_n = \sqrt{\frac{EI}{\rho A}}\, n^2 \frac{\pi^2}{L^2}$.

The coefficients a_n and b_n are defined by the Fourier coefficients of the initial displacement $u_0(x)$ and the initial velocity $v_0(x)$, respectively. Analogous to the formulas for the stretched string, the following applies

$$a_n = 2\,\frac{1}{L}\int_0^L u_0(x)\,\sin(n\,\frac{\pi}{L}\,x)\,dx \qquad n = 1,\,2,\,3,\ldots$$

$$b_n = \frac{1}{\sqrt{\frac{E\,I}{\rho\,A}}\,n^2\,\frac{\pi^2}{L^2}}\,2\,\frac{1}{L}\int_0^L v_0(x)\,\sin(n\,\frac{\pi}{L}\,x)\,dx \qquad n = 1,\,2,\,3,\ldots$$

Physical interpretation: As with the stretched string, the solution is to superimpose harmonic waves:

$$y(x,\,t) = \sum_{n=1}^{\infty} A_n\,\sin\left(n\,\frac{\pi}{L}\,x\right)\cdot\sin(\omega_n\,t + \varphi_n)$$

with the

Amplitudes	$A_n\,\sin\left(\frac{n\,\pi}{L}\,x\right)$	(location dependent on x)
Phases	φ_n	(independent of x)
Frequencies	$\omega_n = \sqrt{\frac{E\,I}{\rho\,A}}\,n^2\,\frac{\pi^2}{L^2}$	(dependend on n^2).

Visualization: We assume an initial displacement $u_0(x) = \frac{1}{2}x\,(x-L)$ with $L = 1$ and an initial velocity $v_0(x) = 0$. So $b_n = 0$ and a_n result from the Fourier analysis of $u_0(x)$: $a_n = 2\,\frac{(-1)^n}{n^3\,\pi^3} - 2\,\frac{1}{n^3\,\pi^3}$. For the material constant $\sqrt{\frac{E\,I}{\rho\,A}} = 0.1$ we compute the solution as a function of x at different times t. Fig. 19.23 shows a sequence for different t-values.

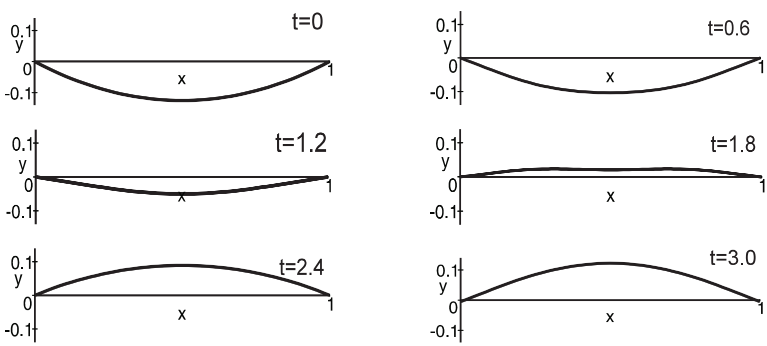


Figure 19.23. Fundamental vibrations of a beam

Interpretation: The animation shows not only an oscillation, as with the stretched string, but the movement is superimposed by a fluttering within the beam.

19.6.4 Boundary Condition: fixed/fixed

$$\begin{array}{lll} X(0)=0: & A_1 + A_3 & = 0 \\ X'(0)=0: & A_2 + A_4 & = 0 \\ X(L)=0: & A_1\,\cosh(\kappa\,L) + A_2\,\sinh(\kappa\,L) + A_3\,\cos(\kappa\,L) + A_4\,\sin(\kappa\,L) & = 0 \\ X'(L)=0: & A_1\,\sinh(\kappa\,L) + A_2\,\cosh(\kappa\,L) - A_3\,\sin(\kappa\,L) + A_4\,\cos(\kappa\,L) & = 0 \end{array}$$

The determinant of the coefficient matrix must disappear, otherwise A_1, A_2, A_3, A_4 are all zero, giving the zero solution $y(x,\,t) \equiv 0$ for all $(x,\,t)$.

$$\det\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \cosh(\kappa\,L) & \sinh(\kappa\,L) & \cos(\kappa\,L) & \sin(\kappa\,L) \\ \sinh(\kappa\,L) & \cosh(\kappa\,L) & -\sin(\kappa\,L) & \cos(\kappa\,L) \end{pmatrix} =$$

$$= 2 - 2\,\cosh(\kappa\,L)\,\cos(\kappa\,L) = 0\,.$$

This gives the eigenvalue equation

$$\boxed{\cosh(\kappa\,L)\cos(\kappa\,L) = 1.}$$

Only discrete $\kappa_n\ \ (n \in \mathbb{N}_0)$ are allowed. The solutions of the eigenvalue equation cannot be given in a closed form and must be calculated approximately. To do this, we set $L = 1$ and solve the non-linear equation

$$\cosh(\kappa)\cdot\cos(\kappa) = 1 \qquad (*)$$

numerically using the Newton algorithm (see Volume 1, Section 7.8).

For large κ the cosine-hyperbolic increases very strongly, so that the eigenvalues (i.e. the solutions of the equation $(*)$) are close to the zeros of the cosine at $n\pi + \frac{\pi}{2}$. To distinguish for larger values of κ, the accuracy of the calculation must be increased. Therefore, we set the calculation accuracy to 20 digits.

To get a better overview of the eigenvalues κ_n, we also calculate $\frac{\kappa_n}{\pi}$ which is shown in the second column of the next table.

Table 19.1: Zeros κ_n and $\frac{\kappa_n}{\pi}$

4.7300407448627040260,	1.5056187311419397690
7.8532046240958375565,	2.4997526700739646572
10.995607838001670907,	3.5000106794359084827
14.137165491257464177,	4.4999995384835765581
17.278759657399481438,	5.5000000199439028337
20.420352245626061091,	6.4999999991381457567
23.561944902040455075,	7.5000000000372440985
26.703537555508186248,	8.4999999999983905363
29.845130209103254267,	9.5000000000000695511
32.986722862692819562,	10.499999999999996994
36.128315516282622650,	11.500000000000000130
39.269908169872415463,	12.499999999999999994
42.411500823462208720,	13.500000000000000000
45.553093477052001958,	14.500000000000000000
48.694686130641795196,	15.500000000000000000

The result shows that from $n = 13$ onward there is no numerical difference between the solution of the equation $(*)$ and the zeros of the cosine. We would have to increase the **precision** again to increase the numerical accuracy.

Instead, we use only the first twelve eigenvalues κ_n. With these values, we obtain for each $n \in \mathbb{N}$ the coefficients $A_1^{(n)}$, $A_2^{(n)}$, $A_3^{(n)}$ and $A_4^{(n)}$ according to the linear system of equations with one free parameter. If we arbitrarily choose $A_1^{(n)}$, the first two equations are

$$A_3^{(n)} = -A_1^{(n)} \quad \text{and} \quad A_4^{(n)} = -A_2^{(n)}.$$

Inserted into the last two equations gives

$$\begin{aligned} A_1^{(n)} \left(\cosh(\kappa_n) - \cos(\kappa_n)\right) + A_2^{(n)} \left(\sinh(\kappa_n) - \sin(\kappa_n)\right) &= 0 \\ A_1^{(n)} \left(\sinh(\kappa_n) + \sin(\kappa_n)\right) + A_2^{(n)} \left(\cosh(\kappa_n) - \cos(\kappa_n)\right) &= 0. \end{aligned}$$

The coefficients $A_2^{(n)}$, $A_3^{(n)}$, $A_4^{(n)}$ depend only on $A_1^{(n)}$ through

$$\begin{aligned} A_2^{(n)} &= A_1^{(n)} \cdot (-1) \frac{\cosh(\kappa_n) - \cos(\kappa_n)}{\sinh(\kappa_n) - \sin(\kappa_n)} \\ A_3^{(n)} &= -A_1^{(n)} \\ A_4^{(n)} &= -A_2^{(n)} = A_1^{(n)} \cdot \frac{\cosh(\kappa_n) - \cos(\kappa_n)}{\sinh(\kappa_n) - \sin(\kappa_n)}. \end{aligned}$$

So for each $n \in \mathbb{N}$ we obtain a natural oscillation $y_n(x,t)$ as the product $T_n(t) \cdot X_n(x)$ where the x-dependency is specified by

$$X_n(x) = A_1^{(n)} \cosh(\kappa_n x) + A_2^{(n)} \sinh(\kappa_n x) + A_3^{(n)} \cos(\kappa_n x) + A_4^{(n)} \sin(\kappa_n x)$$

and the t-dependency by

$$T_n(t) = a_n \cos(\omega_n t) + b_n \sin(\omega_n t) .$$

The complete solution of the PDE $y(x,t)$ is then given by the superposition of all natural oscillations:

$$y(x,\, t) = \sum_{n=1}^{\infty} (a_n \cos(\omega_n t) + b_n \sin(\omega_n t)) \cdot \left(\cosh(\kappa_n x) - \frac{\cosh(\kappa_n) - \cos(\kappa_n)}{\sinh(\kappa_n) - \sin(\kappa_n)} \sinh(\kappa_n x) - \cos(\kappa_n x) + \frac{\cosh(\kappa_n) - \cos(\kappa_n)}{\sinh(\kappa_n) - \sin(\kappa_n)} \sin(\kappa_n x)\right)$$

with

$$\omega_n = \kappa_n^2 \cdot \sqrt{\frac{E\,I}{\rho\,A}} \quad \text{and} \quad \kappa_n \ \text{from Table 19.1.}$$

The coefficients a_n and b_n depend on the initial deflection and the initial velocity.

Visualization: For a given initial deflection without an initial velocity ($\hookrightarrow b_n = 0$) the oscillations are shown. For this visualization we have chosen $\sqrt{\frac{E\,I}{\rho\,A}} = 1$ and an initial deflection $y(x,\, t=0) = y_0(x)$

$$y_0(x) = \sum_{n=1}^{7} \frac{\sin(n\frac{\pi}{2})}{n^2} \left(A_1^{(n)} \cosh(\kappa_n x) + A_2^{(n)} \sinh(\kappa_n x) + A_3^{(n)} \cos(\kappa_n x) + A_4^{(n)} \sin(\kappa_n x)\right),$$

where $A_1^{(n)} = 0.1$ and $A_2^{(n)}$, $A_3^{(n)}$, $A_4^{(n)}$ are calculated according to the formulas above.

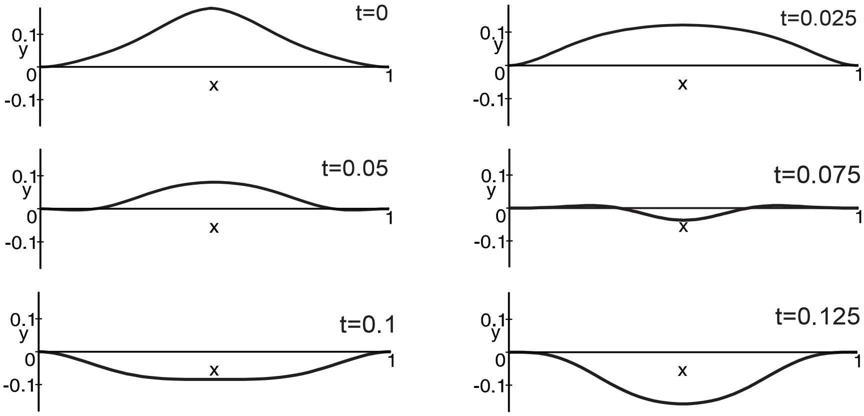


Figure 19.24. Beam bending fixed/fixed

19.7 Problems on Partial Differential Equations

19.1 Check that

$$u(x,\, t) = \cos(\omega t) \cdot \sin(k\, x)$$

is a solution of the wave equation $u_{tt} - c^2\, u_{xx} = 0$ with $k = n\,\frac{\pi}{L}$ and $\omega = c \cdot k\,.$ Also show that this solution fulfills the two initial conditions $u(x = 0,\, t) = u(x = L,\, t) = 0$.

19.2 Show that for $\omega = D \cdot k^2$ the function

$$u(x,\, t) = e^{-\omega t}\, \sin(k\, x)$$

is a solution of the heat equation $u_t - D\, u_{xx} = 0$.

19.3 Show that the functions solve the respective partial DE:
a) $f(x,\, y) = \frac{1}{2}\, \ln\left(x^2 + y^2\right)$ is a solution of $f_{xx} + f_{yy} = 0$
b) $g(x,\, y,\, z) = \left(x^2 + y^2 + z^2\right)^{-1/2}$ is a solution of $g_{xx} + g_{yy} + g_{zz} = 0$.

19.4 a) Show that with any two functions that can be continuously differentiated twice $f_1,\, f_2 : \mathbb{R} \to \mathbb{R}$, the function

$$u(x,\, t) := f_1(x + c\, t) + f_2(x - c\, t)$$

is a solution of the wave equation.
b) Solve the initial value problem

$$u_{tt} - c^2\, u_{xx} = 0, \qquad u(x,\, t = 0) = u_0, \qquad u_t(x,\, t = 0) = v_0$$

for two given functions u_0 and v_0 with the approach

$$u(x,\, t) = f_1(x + c\, t) + f_2(x - c\, t)\,.$$

c) What does this mean for $u_0(x) = \sin(k\, x)\,,\ k = n\,\frac{\pi}{L}$ and $v_0 = 0$?

19.5 Show that $R = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$ fulfills the partial DE $\partial_x^2\, \frac{1}{R} + \partial_y^2\, \frac{1}{R} + \partial_z^2\, \frac{1}{R} = 0$.

19.6 Check that $z(x,\, y) = x\, \varphi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$ fulfills the partial DE

$$x^2\, \frac{\partial^2}{\partial x^2}\, z + 2\, x\, y\, \frac{\partial^2}{\partial x\, \partial y}\, z + y^2\, \frac{\partial^2}{\partial y^2}\, z = 0$$

if φ and ψ are any twice continuously differentiable functions.

19.7 Define k so that $u(x,\, t) = e^{-\kappa\, t}\, \sin\left(\frac{n\,\pi}{L}\, x\right)$ is a solution of the heat equation $u_t = \kappa\, u_{xx}$.

19.8 Determine a general solution for partial DE

$$u_t\,(t,\,x) + u_{xx}\,(t,\,x) = 0\ .$$

19.9 a) Determine the general solution of the partial differential equation

$$u_{xx}\,(x,\,y) - u_{yy}\,(x,\,y) = 0.$$

b) Determine a solution of this partial DE which satisfies the following boundary conditions:

$$\begin{aligned} u\,(x=0,\,y) &= 0 \qquad \text{for all } y \\ u\,(x,\,y=0) &= 0 \qquad \text{for all } x \\ u\,(x,\,y=L) &= 0 \qquad \text{for all } x \end{aligned}$$

19.10 a) Set the parameter k so that

$$u\,(x,\,y,\,t) = \sin\left(n\,\frac{2\pi}{L}\,x\right)\,\sin\left(m\,\frac{2\pi}{L}\,y\right)\,e^{k\,t}$$

is a solution of

$$u_{xx}\,(x,\,y,\,t) + u_{yy}\,(x,\,y,\,t) = u_{tt}\,(x,\,y,\,t) \qquad (*)$$

b) Starting from a) give two real solutions of $(*)$. How to interpret this result?

19.11 Determine the parameter k so that the function

$$u\,(x,\,y,\,t) = \sin\left(n\,\frac{2\pi}{L}\,x\right)\,\sin\left(m\,\frac{2\pi}{L}\,y\right)\,e^{-k\,t}$$

is a solution of the partial DE $\quad u_{xx} + u_{yy} = u_t \quad$.

19.12 a) Show that the function $\quad u\,(x,\,y) = \sin\,(k\,x)\,\left(e^{k\,y} + e^{-k\,y}\right) \quad$ is a solution of the partial DE $\quad u_{xx} + u_{yy} = 0 \quad$.
b) Determine the parameter k in the function

$$u\,(x,\,y) = \sin\,(k\,x)\,\left(e^{k\,y} + e^{-k\,y}\right)$$

so that the function satisfies $u\,(x = L,\,y) = 0$ for all y.

19.13 Find a solution to the partial DE

$$u_t = u_{xx} \qquad 0 \le x \le \pi\,,\ t > 0$$

with $u\,(x,\,0) = 1$ for $0 < x < \pi$ and $u\,(0,\,t) = 1,\ u\,(\pi,\,t) = 0$ for $t > 0$.

19.14 Given is the 3D wave equation $\quad u_{tt} = c^2\,(u_{xx} + u_{yy} + u_{zz})$. Determine solutions of the form

$$u = e^{\alpha\,x + \beta\,y + \gamma\,z - c\,t}.$$

19.15 Given is the heat equation in the plane: Use a separation approach to solve the differential equation

$$u_t = u_{xx} + u_{yy}$$

by a separation $u(x,\, y,\, t) = T(t) \cdot X(x) \cdot Y(y)$ (see Problem 19.14).

19.16 Use a separation approach to solve the following boundary value problems

a) $u_t = u_y;\qquad u(0,\, y) = e^{y} + e^{-2\,y}$

b) $u_t = u_y;\qquad u(t,\, 0) = e^{-3\,t} + e^{2\,t}$

c) $u_t = u_y + u;\qquad u(0,\, y) = 2\,e^{-y} - e^{2\,y}$

d) $u_t = u_y - u;\qquad u(t,\, 0) = e^{-5\,t} + 2\,e^{-7\,t} - 14\,e^{13\,t}$

Chapter 20
Vector Analysis and Integral Theorems

20

Vector analysis plays a fundamental role in the description of physical laws, in mechanics and electrodynamics, where vector fields are considered in $\mathbb{R}^3$

$$\vec{v}(x, y, z) = \begin{pmatrix} v_1(x, y, z) \\ v_2(x, y, z) \\ v_3(x, y, z) \end{pmatrix} : \mathbb{D} \subset \mathbb{R}^3 \to \mathbb{R}^3.$$

A vector field assigns a vector to each point $P(x, y, z)$ in three-dimensional space. The electric field strength $\vec{E}\,(x, y, z)$, the magnetic induction $\vec{B}\,(x, y, z)$ or the velocity profile $\vec{v}\,(x, y, z)$ of a moving medium are examples of vector fields. Physical laws are formulated by either differentiating or integrating these vector fields.

Vector analysis deals with arithmetic operations on vector fields: For differentiation we introduce the *gradient*, the *divergence* and the *curl*. For integration, the *integral theorems of Gauss* and *Stokes* are provided.

0

20 Vector Analysis and Integral Theorems

Vector analysis plays a fundamental role in the description of physical laws, in mechanics and electrodynamics, where **vector fields** are considered in $\mathbb{R}^3$

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A vector field assigns a vector to each point $P(x,y,z)$ in three-dimensional space. The electric field strength $\vec{E}\,(x,y,z)$, the magnetic induction $\vec{B}\,(x,y,z)$ or the velocity profile $\vec{v}\,(x,y,z)$ of a moving medium are examples of vector fields. Physical laws are formulated by either differentiating or integrating these vector fields.

Vector analysis deals with arithmetic operations on vector fields: For differentiation we introduce the *gradient*, the *divergence* and the *curl*. For integration, the *integral theorems of Gauss* and *Stokes* are provided.

A vector field $\vec{v}\colon \mathbb{D} \subset \mathbb{R}^3 \to \mathbb{R}^3$ is said to be *continuous* or *(partially) differentiable* if these properties hold for each component of $\vec{v}$. In the following, vector fields are always continuous and partially differentiable. In addition to vector fields, scalar functions (**scalar fields**)

$$f\,(x,\,y,\,z)\;:\;\;\mathbb{D} \subset \mathbb{R}^3 \to \mathbb{R}$$

are also discussed, because *gradient fields* are identified as the gradient of a scalar field

$$\text{grad } f\,(x,\,y,\,z) = \begin{pmatrix} \frac{\partial}{\partial x} f\,(x,\,y,\,z) \\ \frac{\partial}{\partial y} f\,(x,\,y,\,z) \\ \frac{\partial}{\partial z} f\,(x,\,y,\,z) \end{pmatrix}.$$

Instead of grad (f), the term ∇f is often used with the Nabla operator ∇.

The integral theorems are a generalization of the Fundamental Theorem of Calculus

$$\int_a^b f'\,(x)\;dx = f\,(b) - f\,(a)\,.$$

The definite integral is calculated by evaluating only the antiderivative function at the limits. The Gauss theorem assigns a volume integral to a surface integral and Stokes' theorem assigns a surface integral to a curve integral.

20.1 Line or Curve Integrals

Describing the work in a force field or the voltage in an electric field requires the calculation of an integral along a curve. This leads to the concept of a *line integral*, which is explained with examples from mechanics and electrostatics.

20.1.1 Description of a Curve

The description of a curve $\mathcal{C} \in \mathbb{R}^3$ is given by the parameter representation

$$\mathcal{C} : \vec{r}\,(t) = x\,(t)\ \vec{e}_1 + y\,(t)\ \vec{e}_2 + z\,(t)\ \vec{e}_3 = \begin{pmatrix} x\,(t) \\ y\,(t) \\ z\,(t) \end{pmatrix},$$

where $x\,(t)$, $y\,(t)$, $z\,(t)$ are functions of the variable t. As the parameter t varies, the point P (see Fig. 20.1) moves along the curve $\mathcal{C}$:

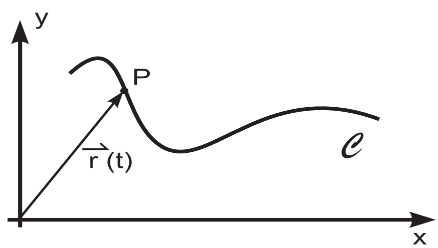


Figure 20.1. Space curve

Example 20.1 (Electron in a Magnetic Field). An electron moves in a homogeneous magnetic field $\vec{B} = B_0\,\vec{e}_z$ on a helical line with radius R. The coordinates of the electron are given at any time by

$$\begin{aligned} x\,(t) &= R\,\cos\,(\omega t) \\ y\,(t) &= R\,\sin\,(\omega t) \\ z\,(t) &= v_z\,t. \end{aligned}$$

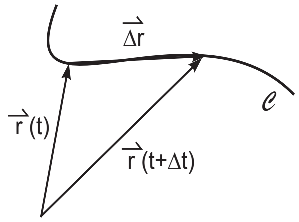
$\omega = \frac{e}{m}\,B_0$ is the angular frequency and v_z is the velocity in z-direction. □

20.1.2 Derivative of a Vector

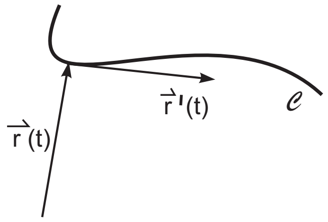
If $\vec{r}(t) = x(t)\,\vec{e}_1 + y(t)\,\vec{e}_2 + z(t)\,\vec{e}_3$ is a parameter representation of a curve $\mathcal{C}$, then the **derivative of the vector** $\vec{r}(t)$ is defined as the limit

$$\vec{r}'(t) = \lim_{\triangle t \to 0} \frac{1}{\triangle t}\left(\vec{r}(t+\triangle t) - \vec{r}(t)\right)$$

of the difference quotient for $\triangle t \to 0$. Geometrically, this corresponds to the transition of the difference vector (= secant vector) into the tangent vector at the point $\vec{r}(t) = (x(t), y(t), z(t))$.



Secant vector $\triangle\vec{r}$



Tangent vector $\vec{r}'(t)$

Due to the vector calculation rules we calculate

$$\begin{aligned}\frac{1}{\triangle t}\triangle\vec{r} &= \frac{1}{\triangle t}\left(\vec{r}(t+\triangle t) - \vec{r}(t)\right)\\ &= \begin{pmatrix} \frac{1}{\triangle t}\left(x(t+\triangle t) - x(t)\right)\\ \frac{1}{\triangle t}\left(y(t+\triangle t) - y(t)\right)\\ \frac{1}{\triangle t}\left(z(t+\triangle t) - z(t)\right)\end{pmatrix} \quad \xrightarrow{\triangle t \to 0} \quad \begin{pmatrix}\dot{x}(t)\\ \dot{y}(t)\\ \dot{z}(t)\end{pmatrix}.\end{aligned}$$

Derivative of a Vector Function

Given is a vector $\vec{r}(t) = \begin{pmatrix} x(t)\\ y(t)\\ z(t)\end{pmatrix}$ depending on t. The derivative of $\vec{r}(t)$ with respect to t is $\vec{r}'(t) = \begin{pmatrix}\dot{x}(t)\\ \dot{y}(t)\\ \dot{z}(t)\end{pmatrix}$.

The derivative of a vector $\vec{r}(t)$ is computed component by component.

Application Example 20.2 **(Space Curve and Velocity).**

If $\vec{r}(t)$ is the time-dependent position vector of the trajectory of a mass, then

$$\begin{array}{ll} \vec{v}(t) = \vec{r}'(t) & \text{is the velocity vector and} \\ \vec{a}(t) = \vec{v}'(t) = \vec{r}''(t) & \text{is the acceleration vector.} \end{array}$$

The velocity vector and the acceleration vector of an electron in a homogeneous magnetic field $B = B_0\,\vec{e}_z$ are given by Example 20.1

$$\vec{v}(t) = \vec{r}'(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{pmatrix} -R\,\omega\,\sin(\omega t) \\ R\,\omega\,\cos(\omega t) \\ v_z \end{pmatrix} = \begin{pmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{pmatrix}$$

$$\vec{a}(t) = \vec{v}'(t) = \begin{pmatrix} \ddot{x}(t) \\ \ddot{y}(t) \\ \ddot{z}(t) \end{pmatrix} = \begin{pmatrix} -R\,\omega^2\,\cos(\omega t) \\ -R\,\omega^2\,\sin(\omega t) \\ 0 \end{pmatrix}.$$

In particular, $\ddot{x}(t) + \omega^2\,x(t) = 0$ and $\ddot{y}(t) + \omega^2\,y(t) = 0$. □

Application Example 20.3 **(Acceleration of a Circular Motion).**

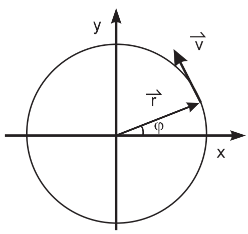


Figure 20.2. Circular motion

For a planar circular motion with a *constant* radius ρ, the coordinates $x(t)$ and $y(t)$ are given in polar coordinates by

$$\begin{array}{l} x(t) = \rho \cdot \cos(\varphi(t)) \\ y(t) = \rho \cdot \sin(\varphi(t)) \end{array} \quad \text{(polar coordinates),}$$

where $\varphi(t)$ is the angle to the positive x-axis.

With this parameter representation, the motion is

$$\begin{aligned} \vec{r}(t) &= x(t)\;\vec{e}_x + y(t)\;\vec{e}_y \\ &= \rho\,\cos(\varphi(t))\,\vec{e}_x + \rho\,\sin(\varphi(t))\,\vec{e}_y = \rho \begin{pmatrix} \cos(\varphi(t)) \\ \sin(\varphi(t)) \end{pmatrix} \end{aligned}$$

with the **velocity vector**

$$\vec{v}(t) = \vec{r}'(t) = \rho \begin{pmatrix} -\sin(\varphi(t))\;\dot{\varphi}(t) \\ \cos(\varphi(t))\,\dot{\varphi}(t) \end{pmatrix} = \rho\,\dot{\varphi}(t) \begin{pmatrix} -\sin(\varphi(t)) \\ \cos(\varphi(t)) \end{pmatrix}.$$

When the radial unit vector $\vec{e}_r$ and the azimuthal unit vector $\vec{e}_\varphi$ are introduced

$$\vec{e}_r = \begin{pmatrix} \cos(\varphi(t)) \\ \sin(\varphi(t)) \end{pmatrix}, \ \vec{e}_\varphi = \begin{pmatrix} -\sin(\varphi(t)) \\ \cos(\varphi(t)) \end{pmatrix},$$

then the velocity vector is

$$\vec{v}(t) = \rho\,\dot{\varphi}(t)\,\vec{e}_\varphi\,.$$

In a circular motion, the velocity $\rho\,\dot{\varphi}(t)$ is perpendicular to the position vector (i.e. tangential to the motion)! The acceleration vector is the derivative of $\vec{v}(t)$ with respect to t:

$$\begin{aligned} \vec{a}(t) = \vec{v}'(t) &= \rho \begin{pmatrix} -\cos(\varphi(t))\,\dot{\varphi}^2(t) - \sin(\varphi(t))\,\ddot{\varphi}(t) \\ -\sin(\varphi(t))\,\dot{\varphi}^2(t) + \cos(\varphi(t))\,\ddot{\varphi}(t) \end{pmatrix} \\ &= -\rho\,\dot{\varphi}^2(t) \begin{pmatrix} \cos(\varphi(t)) \\ \sin(\varphi(t)) \end{pmatrix} + \rho\,\ddot{\varphi}(t) \begin{pmatrix} -\sin(\varphi(t)) \\ \cos(\varphi(t)) \end{pmatrix} \\ &= -\rho\,\dot{\varphi}^2(t)\ \vec{e}_r + \rho\,\ddot{\varphi}(t)\ \vec{e}_\varphi\,. \end{aligned}$$

We calculate the magnitude of $\vec{a}$ using $\vec{e}_r^{\,2} = \vec{e}_\varphi^{\,2} = 1$ and $\vec{e}_r \cdot \vec{e}_\varphi = 0$:

$$\begin{aligned} |\vec{a}| &= \sqrt{\vec{a}\cdot\vec{a}} \\ &= \rho\sqrt{\dot{\varphi}^4(t)\ \vec{e}_r^{\,2} - 2\,\dot{\varphi}^2(t)\,\ddot{\varphi}(t)\ \vec{e}_r\cdot\vec{e}_\varphi + \ddot{\varphi}^2(t)\ \vec{e}_\varphi^{\,2}} \\ &= \rho\sqrt{\dot{\varphi}^4(t) + \ddot{\varphi}^2(t)}\,. \end{aligned}$$

The acceleration force $\vec{F} = m\,\vec{a}(t)$ has a component in the direction $\vec{r}$ (centrifugal force) and another perpendicular to it:

Force in direction $\vec{r}$ (centrifugal force): $\vec{F}_r = -m\,\rho\,\dot{\varphi}^2(t)\ \vec{e}_r$ has the magnitude

$$F_r = \left|\vec{F}_r\right| = m\,\rho\,\dot{\varphi}^2(t).$$

In the case of a circular motion with **constant** angular velocity $\omega = \dot{\varphi}(t) = const$, we get

$$F_r = m\,\rho\,\omega^2.$$

Force in direction of velocity $\vec{e}_\varphi$: $\vec{F}_\varphi = m\,\rho\,\ddot{\varphi}(t)\ \vec{e}_\varphi$. For a circular motion with **constant** angular velocity $\omega = \dot{\varphi}(t) = const$, we get $\ddot{\varphi}(t) = 0 \Rightarrow F_\varphi = 0$. So in this case there is **no** force in the direction of speed. □

20.1.3 Vector Fields (Force Fields)

Definition: *A* **Vector Field** *is a vector function* $\vec{k}: \mathbb{R}^3 \to \mathbb{R}^3$,

$$\vec{k}(x,\, y,\, z) = \begin{pmatrix} k_1(x,\, y,\, z) \\ k_2(x,\, y,\, z) \\ k_3(x,\, y,\, z) \end{pmatrix},$$

with components k_1, k_2, k_3 *depending on the three spacial coordinates* $(x,\, y,\, z)$. $\vec{k}$ *assigns a vector* $\vec{k}(x,\, y,\, z)$ *to any point in space* $(x,\, y,\, z)$.

Example 20.4. According to *Coulomb's Law*, the electric force between two point charges Q and q is inversely proportional to the square of the distance between the charges

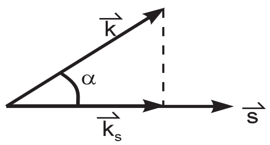
$$\vec{F} = \frac{1}{4\pi\,\varepsilon_0}\,\frac{q\,Q}{r^2}\,\frac{\vec{r}}{|\vec{r}|} = \frac{q\,Q}{4\pi\,\varepsilon_0}\,\frac{1}{r^3}\,\vec{r} = \frac{q\,Q}{4\pi\,\varepsilon_0}\,\frac{1}{\sqrt{x^2+y^2+z^2}^{\,3}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The force vector $\vec{F}$ varies in magnitude and direction. □

20.1.4 Line Integrals (Curve Integrals)

Let $\vec{r}(t) = x(t)\,\vec{e}_1 + y(t)\,\vec{e}_2 + z(t)\,\vec{e}_3$ be a space curve $\mathcal{C}$ and $\vec{k}$ a vector field. $P_A = \vec{r}(t_A)$ is the beginning and $P_E = \vec{r}(t_E)$ the end of the curve. We are looking for the work required to move m along $\mathcal{C}$ from the start to the end points (see Fig. 20.3).

When a mass moves along a direction $\vec{s}$, the work done by a constant force $\vec{k}$ is determined by the scalar product according to Volume I, Chapter 2



$$\boxed{W = \vec{k}\cdot\vec{s}.}$$

To find the work required along a path, we move the mass on the curve $\mathcal{C}$ but split $\mathcal{C}$ into parts

$$P_A = P_0\,,\, P_1\,,\ldots,\, P_N = P_E \quad \text{with} \quad P_i = \vec{r}(t_i) = \vec{r}_i \;\; (i = 0, ..., N)$$

and $t_i = \frac{t_E - t_A}{N}\, i + t_A$. We replace the curve with tracks $\overrightarrow{P_i\, P_{i+1}} = \triangle\vec{r}_i$.

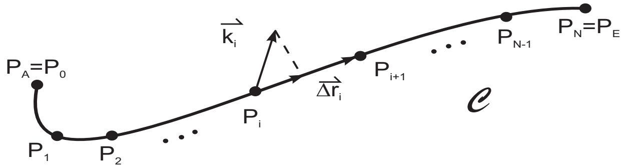


Figure 20.3. Work along the curve $\mathcal{C}$

For each track we compute the scalar product of the local force field $\vec{k}_i = \vec{k}(\vec{r}_i)$ with the direction vector $\triangle\vec{r}_i$: $W_i = \vec{k}(\vec{r}_i) \cdot \triangle\vec{r}_i$. W_i is the work required to move the mass from P_i to P_{i+1}. All contributions are summed to

$$\triangle W = \sum_{i=0}^{N-1} \vec{k}(\vec{r}_i) \cdot \triangle\vec{r}_i = \sum_{i=0}^{N-1} \vec{k}(\vec{r}_i) \frac{1}{\triangle t} (\vec{r}(t_i + \triangle t) - \vec{r}(t_i)) \cdot \triangle t.$$

This sum is an approximation of the total work. The approximation becomes better the finer the subdivision of the curve $\mathcal{C}$ is chosen. The limit $N \to \infty$ (i.e. an arbitrarily fine sub-division of $\mathcal{C}$ with $\triangle\vec{r}_i \to 0$ and $\triangle t \to 0$)

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} \vec{k}(\vec{r}_i) \triangle\vec{r}_i = \lim_{N\to\infty} \sum_{i=0}^{N-1} \vec{k}(\vec{r}_i) \frac{1}{\triangle t} (\vec{r}(t_i + \triangle t) - \vec{r}(t_i)) \cdot \triangle t$$

returns the *curve integral* along $\mathcal{C}$:

Definition: Curve Integral (Line Integral). *Let $\vec{k}(x, y, z)$ be a vector field and $\mathcal{C}$ a curve described by $\vec{r}(t)$ for $t_A \leq t \leq t_E$. Then*

$$\int_{\mathcal{C}} \vec{k} \cdot d\vec{r} = \int_{t_A}^{t_E} \vec{k}(\vec{r}(t)) \cdot \vec{r}'(t) \; dt$$

is the **Line or Curve Integral** *of the vector field $\vec{k}(x, y, z)$ along the curve $\mathcal{C}$. $\vec{r}(t_A)$ is the start and $\vec{r}(t_E)$ marks the end point of the curve.*

Remarks:

(1) The curve integral is independent of the selected sub-division.

(2) The curve integral is therefore independent of the parameterization of the curve $\mathcal{C}$.

(3) The curve integral is obtained by performing the scalar product and evaluating the force field $\vec{k}$ at $\vec{r}(t)$

$$\begin{aligned}\int_{\mathcal{C}} \vec{k} \cdot d\vec{r} = &\int_{t_A}^{t_E} k_1\left(x(t), y(t), z(t)\right)\, \dot{x}(t)\; dt \\ &+ \int_{t_A}^{t_E} k_2\left(x(t), y(t), z(t)\right)\, \dot{y}(t)\; dt \\ &+ \int_{t_A}^{t_E} k_3\left(x(t), y(t), z(t)\right)\, \dot{z}(t)\; dt.\end{aligned}$$

The three integrals depend only on the variable t and are calculated independently using the integration rules for functions with one variable.

(4) The value of the curve integral depends not only on the start and end points of the integration path, but also on the specified path. Exceptions are the so-called *gradient fields*.

(5) For a curve integral along a *closed* curve we use the symbol $\oint_{\mathcal{C}} \vec{k}\, d\vec{r}$.

(6) $\vec{k}(\vec{r}(t)) \cdot \vec{r}'(t)$ is the force acting tangentially on the curve, because $\vec{r}'(t)$ represents the tangent of the curve $\mathcal{C}$ at each point $\vec{r}(t)$.

Procedure for Calculating Curve Integrals

(1) Parameterize the curve $\mathcal{C}$: $\vec{r}(t)$.

(2) Calculate $\vec{r}'(t)$.

(3) Replace the components of $\vec{r}(t)$, namely $x(t)$, $y(t)$, $z(t)$, into the three force components k_1, k_2, k_3, compute the scalar product $\vec{k}(\vec{r}(t)) \cdot \vec{r}'(t)$ and evaluate the integrals with respect to t.

Example 20.5. The force field $\vec{k} = \begin{pmatrix} x\,y^2 \\ x\,y \\ 0 \end{pmatrix}$ is given. Find the curve integral

$$\int_{\mathcal{C}} \vec{k} \cdot d\vec{r} = \int_{\mathcal{C}} \left(x(t)\, y^2(t)\; \dot{x}(t) + x(t)\, y(t)\, \dot{y}(t)\right)\; dt$$

where $\mathcal{C}$ is a path shown in Fig. 20.4. In all cases the start and end points of the curves are $(0, 0)$ and $(1, 1)$.

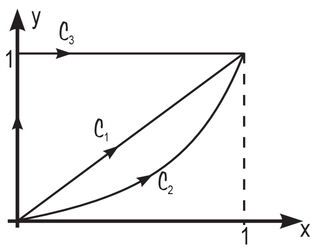


Figure 20.4. Curves from origin $(0, 0)$ to point $(1, 1)$

(1) Integration along path 1: A parameter representation of the path $\mathcal{C}_1$ is

$$\vec{r}(t) = \begin{pmatrix} t \\ t \end{pmatrix} \quad \text{for} \quad 0 \leq t \leq 1 .$$

$$\Rightarrow x(t) = t , \, y(t) = t \quad \Rightarrow \quad \dot{x}(t) = 1 , \, \dot{y}(t) = 1.$$

$$\Rightarrow \int_{\mathcal{C}_1} \vec{k} \, d\vec{r} = \int_0^1 \left(t \cdot t^2 \cdot 1 + t \cdot t \cdot 1\right) \, dt = \int_0^1 \left(t^3 + t^2\right) \, dt = \frac{7}{12}.$$

(2) Integration along path 2: A parameter representation of the path $\mathcal{C}_2$ is

$$\vec{r}(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix} \quad \text{for} \quad 0 \leq t \leq 1 .$$

$$\Rightarrow x(t) = t , \, y(t) = t^2 \quad \Rightarrow \quad \dot{x}(t) = 1 , \, \dot{y}(t) = 2\,t.$$

$$\Rightarrow \int_{\mathcal{C}_2} \vec{k} \, d\vec{r} = \int_0^1 \left(t \cdot t^4 \cdot 1 + t \cdot t^2 \cdot 2\,t\right) \, dt = \int_0^1 \left(t^5 + 2\,t^4\right) \, dt = \frac{17}{30}.$$

(3) Integration along path 3: A parameter representation of the path $\mathcal{C}_3$ is

$$\vec{r}(t) = \begin{cases} \begin{pmatrix} 0 \\ t \end{pmatrix} & \text{for } 0 \leq t \leq \frac{1}{2} \qquad \Rightarrow x(t) = 0, \quad y(t) = t \\ \begin{pmatrix} 2t - 1 \\ 1 \end{pmatrix} & \text{for } \frac{1}{2} \leq t \leq 1 \qquad \Rightarrow x(t) = 2t - 1, \quad y(t) = 1 . \end{cases}$$

$$\Rightarrow \int_{\mathcal{C}_3} \vec{k} \, d\vec{r} = \int_0^{\frac{1}{2}} 0 \, dt + \int_{\frac{1}{2}}^1 (4t - 2) \, dt = \frac{1}{2}.$$

(4) Integration along path 1: We select the path $\mathcal{C}_1$ again, but with a different parameter representation:

$$\vec{r}\,(t) = \begin{pmatrix} t^2 \\ t^2 \end{pmatrix} \quad \text{for} \quad 0 \le t \le 1\,.$$

$$\Rightarrow x\,(t) = t^2\,,\; y\,(t) = t^2 \quad \Rightarrow \quad \dot{x}\,(t) = 2t\,,\; \dot{y}\,(t) = 2t.$$

$$\Rightarrow \int_{\mathcal{C}_1} \vec{k}\, d\vec{r} = \int_0^1 \left(t^6 \cdot 2t + t^4 \cdot 2t\right)\, dt = \int_0^1 \left(2t^7 + 2t^5\right)\, dt = \frac{7}{12}.$$

Example 20.5 shows that the value of the curve integral depends on the selected path (see (1), (2) and (3)), but not on the special parameterization for the same curve (see (1) and (4)). □

Examples 20.6:

① The vector field $\vec{k} = \begin{pmatrix} x\,y \\ y \\ -x \end{pmatrix}$ is integrated along the curve $\mathcal{C}$ with the parameterization $\vec{r}\,(t) = t\,\vec{e}_1 + t^2\,\vec{e}_2 + t^3\,\vec{e}_3$ for $0 \le t \le 1$. The selected representation of $\vec{r}(t)$ gives

$$x\,(t) = t,\; y\,(t) = t^2,\; z\,(t) = t^3.$$

Then

$$\dot{x}(t) = 1, \dot{y}(t) = 2t, \dot{z}(t) = 3t^2$$

and we get

$$\vec{r}\,'\,(t) = \vec{e}_1 + 2\,t\,\vec{e}_2 + 3\,t^2\,\vec{e}_3 = \begin{pmatrix} 1 \\ 2\,t \\ 3\,t^2 \end{pmatrix}, \qquad \vec{k}\,(\vec{r}\,(t)) = \begin{pmatrix} t^3 \\ t^2 \\ -t \end{pmatrix}$$

$$\vec{k}\,(\vec{r}\,(t)) \cdot \vec{r}\,'(t) = \begin{pmatrix} t^3 \\ t^2 \\ -t \end{pmatrix} \begin{pmatrix} 1 \\ 2\,t \\ 3\,t^2 \end{pmatrix} = t^3 + 2\,t^3 - 3\,t^3 = 0.$$

$$\Rightarrow \int_{\mathcal{C}} \vec{k} \cdot d\vec{r} = \int_0^1 0\, dt = 0\,.$$

② Find the curve integral of the vector field $\vec{k} = \begin{pmatrix} x \\ x\,y \end{pmatrix}$ along the parabola $y = x^2$ from the origin to the point $P(1,1)$:

$$\vec{r}\,(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix} \quad \text{for} \quad 0 \le t \le 1 \quad \Rightarrow x\,(t) = t\,,\; y\,(t) = t^2.$$

$$\Rightarrow \vec{r}\,'\,(t) = \begin{pmatrix} 1 \\ 2\,t \end{pmatrix} \quad \text{and} \quad \vec{k}\,(\vec{r}\,(t)) = \begin{pmatrix} x \\ x\,y \end{pmatrix} = \begin{pmatrix} t \\ t^3 \end{pmatrix}$$

$$\Rightarrow \vec{k}\,(\vec{r}\,(t)) \cdot \vec{r}\,'\,(t) = \begin{pmatrix} t \\ t^3 \end{pmatrix} \begin{pmatrix} 1 \\ 2\,t \end{pmatrix} = t + 2\,t^4.$$

$$\Rightarrow \int_0^1 \left(t + 2\,t^4\right)\,dt = \left[\tfrac{1}{2}\,t^2 + \tfrac{2}{5}\,t^5\right]_0^1 = \frac{9}{10}\,.$$

③ Find the curve integral of the vector function $\vec{k} = \begin{pmatrix} x \\ x\,y \end{pmatrix}$ along the curve $\mathcal{C}$ with the parameterization $\vec{r}\,(t) = \begin{pmatrix} t^3 \\ t^4 \end{pmatrix}$ for $0 \le t \le 1$. $\mathcal{C}$ also connects the origin to the point (1, 1):

$$\vec{r}\,(t) = \begin{pmatrix} t^3 \\ t^4 \end{pmatrix} \quad \text{for} \quad 0 \le t \le 1 \quad \Rightarrow x\,(t) = t^3,\; y\,(t) = t^4.$$

$$\Rightarrow \vec{r}\,'\,(t) = \begin{pmatrix} 3\,t^2 \\ 4\,t^3 \end{pmatrix} \quad \text{and} \quad \vec{k}\,(\vec{r}\,(t)) = \begin{pmatrix} x \\ x\,y \end{pmatrix} = \begin{pmatrix} t^3 \\ t^3\,t^4 \end{pmatrix}$$

$$\Rightarrow \vec{k}\,(\vec{r}\,(t)) \cdot \vec{r}\,'\,(t) = \begin{pmatrix} t^3 \\ t^7 \end{pmatrix} \begin{pmatrix} 3\,t^2 \\ 4\,t^3 \end{pmatrix} = 3\,t^5 + 4\,t^{10}.$$

$$\Rightarrow \int_0^1 \left(3\,t^5 + 4\,t^{10}\right)\,dt = \left[\frac{1}{2}\,t^6 + \frac{4}{11}\,t^{11}\right]_0^1 = \frac{19}{22}\,. \qquad \square$$

We observe that the curve integral is *path dependent*. For special vector fields, however, it is path independent, i.e. the value of the curve integral is independent of the chosen path, depending only on the start and end points.

Definition: (Gradient Field).
A vector field $\vec{k}\,(x,\,y,\,z)$ *is called a* **gradient field (potential field)**, *if there exists a continuously differentiable function* $\Phi(x,\,y,\,z)\colon \mathbb{R}^3 \to \mathbb{R}$ *such that*

$$\vec{k}\,(x,\,y,\,z) = \operatorname{grad}\Phi\,(x,\,y,\,z)\,.$$

This means for the components of the vector field $\vec{k}(x,y,z)$

$$k_1\,(x,\,y,\,z) = \frac{\partial}{\partial x}\,\Phi\,(x,\,y,\,z)\,,$$
$$k_2\,(x,\,y,\,z) = \frac{\partial}{\partial y}\,\Phi\,(x,\,y,\,z)\,,$$
$$k_3\,(x,\,y,\,z) = \frac{\partial}{\partial z}\,\Phi\,(x,\,y,\,z)\,.$$

The function $\Phi\,(x,\,y,\,z)$ *is a* **potential function of** $\vec{k}$.

A potential function, in the context of physics, is a scalar function whose gradient equals the given vector field $\vec{k}$. This function describes the potential energy or potential of a force field. The gradient fields are those vector fields for which the curve integrals are always path independent. This important theorem is stated in the next theorem (without proof):

Theorem: Main Condition on Curve Integrals

Let $G \subset \mathbb{R}^3$ be an axis-parallel cube and $\vec{k} : G \to \mathbb{R}^3$ be a vector field with continuous partial derivatives in G. Then the following statements are equivalent:

(1) $\vec{k}$ is a gradient field.

(2) In G the **integrability conditions** are satisfied

$$\frac{\partial k_1}{\partial y} = \frac{\partial k_2}{\partial x}; \qquad \frac{\partial k_2}{\partial z} = \frac{\partial k_3}{\partial y}; \qquad \frac{\partial k_1}{\partial z} = \frac{\partial k_3}{\partial x}.$$

(3) The curve integral $\int_{\mathcal{C}} \vec{k}\,d\vec{r}$ depends **only** on the start and end points of the curves for all curves in G.

(4) The curve integral $\oint_{\mathcal{C}} \vec{k}\,d\vec{r}$ is zero for all **closed** curves $\mathcal{C}$ in G.

Remark: The **integrability condition** for a two-dimensional vector field $\vec{k} = \begin{pmatrix} k_1(x, y) \\ k_2(x, y) \end{pmatrix}$ is:

$$\frac{\partial k_1}{\partial y} = \frac{\partial k_2}{\partial x}.$$

Note:

(1) Potential functions belonging to $\vec{k}$ differ by at most one constant.

(2) In physics, force fields that have an associated potential function are called **conservative** force fields.

(3) In physics, $\vec{k}$ is sometimes defined as the negative gradient $-\operatorname{grad}(\Phi)$. This is included in the above definition of a gradient field.

If $\vec{k}$ is a gradient field, then the line integral $\int_{\mathcal{C}} \vec{k}\, d\vec{r}$ can easily be evaluated using its potential function $\Phi(x, y, z)$:

$$\vec{k}(x, y, z) = \begin{pmatrix} k_1(x, y, z) \\ k_2(x, y, z) \\ k_3(x, y, z) \end{pmatrix} \stackrel{!}{=} \operatorname{grad} \Phi(x, y, z) = \begin{pmatrix} \partial_x \Phi(x, y, z) \\ \partial_y \Phi(x, y, z) \\ \partial_z \Phi(x, y, z) \end{pmatrix}.$$

The total differential of Φ is

$$\begin{aligned} d\Phi &= \frac{\partial \Phi}{\partial x}\, dx + \frac{\partial \Phi}{\partial y}\, dy + \frac{\partial \Phi}{\partial z}\, dz \\ &= k_1\, dx + k_2\, dy + k_3\, dz \\ &= \vec{k}\, d\vec{r}. \end{aligned}$$

Then

$$\int_{\mathcal{C}} \vec{k}\, d\vec{r} = \int_{P_1}^{P_2} d\Phi = \Phi\Big|_{P_2} - \Phi\Big|_{P_1},$$

where P_1 is the start point and P_2 is the end point of the curve $\mathcal{C}$.

Integration of a Gradient Field

If $\vec{k}$ is a gradient field with potential Φ, e.g. $\vec{k}(x, y, z) = \operatorname{grad}(\Phi)$, then the line integral is

$$\int_{\mathcal{C}} \vec{k}\, d\vec{r} = \int_{\mathcal{C}} d\Phi = \Phi\Big|_{\text{End point of } \mathcal{C}} - \Phi\Big|_{\text{Start point of } \mathcal{C}}.$$

If the start point is identical to the end point, then

$$\oint_C \vec{k}\, d\vec{r} = \Phi\Big|_{P_1} - \Phi\Big|_{P_1} = 0\,.$$

The main theorem not only tells us whether a curve integral is path independent, but also how the integrability conditions can be used to check whether a gradient field exists or not. If the potential of a gradient field is available, the addition to the main theorem shows how to calculate the curve integral: Analogous to the main theorem of calculus (see Volume 2; Section 8.2), the curve integral is the difference of the potential function evaluated at the end point and the start point.

Examples 20.7:

① There are many gradient fields in physics. Examples are the electrostatic potential, the Newtonian gravitational field, or the magnetic field produced by an electric current flowing through a wire.

② The vector field $\vec{k}\,(x,\, y) = \begin{pmatrix} 3\,x^2\,y \\ x^3 \end{pmatrix}$ is a gradient field because

$$\left.\begin{aligned} \frac{\partial k_1}{\partial y} &= \frac{\partial}{\partial y}\,(3\,x^2\,y) = 3\,x^2 \\ \frac{\partial k_2}{\partial x} &= \frac{\partial}{\partial x}\,x^3 = 3\,x^2 \end{aligned}\right\} \Rightarrow \frac{\partial k_1}{\partial y} = \frac{\partial k_2}{\partial x}\,.$$

Then the potential field Φ associated with $\vec{k}$ can be calculated: Since $\vec{k}$ has a potential function $\Phi\,(x,\, y)$ with $\vec{k} = \operatorname{grad}\Phi$, we know that

$$\vec{k} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \operatorname{grad}\Phi = \begin{pmatrix} \partial_x\,\Phi \\ \partial_y\,\Phi \end{pmatrix} \Rightarrow \begin{aligned} \partial_x\,\Phi &= 3\,x^2\,y && (1) \\ \partial_y\,\Phi &= x^3\,. && (2) \end{aligned}$$

When the equation (1) is integrated with respect to x, the result is

$$\Phi(x,y) = x^3\,y + K\,(y)$$

with an integration constant that may depend on y. We differentiate (2) with respect to y

$$\frac{\partial}{\partial y}\,\Phi(x,y) = x^3 + K'\,(y) = x^3 = k_2(x,y)$$

$$\Rightarrow K'\,(y) = 0 \Rightarrow K\,(y) = C = const\,.$$

$$\Rightarrow \boxed{\Phi(x,\, y) = x^3\, y + C.}$$

The curve integral of $\vec{k}$ along a curve $\mathcal{C}$ with start point $(x_0,\, y_0)$ and end point $(x_1,\, y_1)$ is given by

$$\int_{\mathcal{C}} \vec{k}\, d\vec{r} = \int_{\mathcal{C}} d\Phi = \Phi\Big|_{(x_0,\, y_0)}^{(x_1,\, y_1)} = x_1^3\, y_1 - x_0^3\, y_0\ .$$

③ The vector field $\vec{k} = \begin{pmatrix} x\, y^2 \\ x\, y \end{pmatrix}$ is **not** a gradient field, because the integrability condition is violated

$$\left.\begin{aligned} \frac{\partial k_1}{\partial y} &= \frac{\partial}{\partial y}\left(x\, y^2\right) = 2\, x\, y \\ \frac{\partial k_2}{\partial x} &= \frac{\partial}{\partial x}\left(x\, y\right) = y \end{aligned}\right\} \Rightarrow \frac{\partial k_1}{\partial y} \neq \frac{\partial k_2}{\partial x}\ .$$

(Compare this statement with the result of Example 20.5!)

④ Check whether the vector field

$$\vec{k}(x,\, y,\, z) = \frac{1}{x^2 + y^2 + z^2} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

is a gradient field with the potential function

$$\Phi(x,\, y,\, z) = \ln\left(x^2 + y^2 + z^2\right) + K\ .$$ □

The integrability conditions are explicitly checked to decide whether a gradient field exists or not. If $\vec{k}$ is a gradient field, then the task is to find its potential.

Example 20.8. Given is the vector field $\vec{v}(x,\, y,\, z) = \begin{pmatrix} 2\, x + y \\ x + 2\, y\, z \\ y^2 + 2\, z \end{pmatrix}$. For this vector field we can explicitly check the integrability conditions. Then, the potential Φ belonging to $\vec{v}$ is searched for

$$\vec{v} = \operatorname{grad} \Phi = \begin{pmatrix} \partial_x \Phi \\ \partial_y \Phi \\ \partial_z \Phi \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 2\, x + y \\ x + 2\, y\, z \\ y^2 + 2\, z \end{pmatrix}\ .$$

We integrate the first component $\partial_x \Phi$ with respect to x.

$$\partial_x \Phi = 2\, x + y \Rightarrow \quad \Phi = x^2 + y\, x + f(y,\, z) \qquad (*)$$

returns an integration constant $f(y, z)$, which may depend on y and z. If the second component of $\vec{v}$ is compared to the partial derivative of Φ with respect to y, the following holds for $f_y(y, z)$:

$$\partial_y \Phi = v_2 = x + 2\,y\,z \quad \text{and} \quad \partial_y \Phi \overset{(*)}{=} x + f_y(y, z)$$

$$\Rightarrow f_y(y, z) = 2\,y\,z\,.$$

Integration with respect to y gives

$$f(y, z) = y^2\,z + g(z)\,,$$

where the function g can still depend on z but no longer on x or y.

$$\Rightarrow \Phi(x, y, z) = x^2 + y\,x + y^2\,z + g(z)\,. \tag{$**$}$$

Comparing the third component v_3 with the partial derivative of Φ with respect to z, we get for $g'(z)$:

$$\partial_z \Phi = v_3 = y^2 + 2\,z \quad \text{and} \quad \partial_z \Phi \overset{(**)}{=} y^2 + g'(z)$$

$$\Rightarrow g'(z) = 2\,z \Rightarrow g(z) = z^2 + K\,.$$

The integration constant K depends neither on x, y nor on z.

$$\Rightarrow \quad \boxed{\Phi(x, y, z) = x^2 + y\,x + y^2\,z + z^2 + K.} \qquad \square$$

Example 20.9. Find the potential Φ associated with the gradient field

$$\vec{k} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} z + y \\ x + z \\ x + y \end{pmatrix} = \operatorname{grad} \Phi = \begin{pmatrix} \partial_x \Phi \\ \partial_y \Phi \\ \partial_z \Phi \end{pmatrix} :$$

We start with the first component and integrate over x. The integration constant may depend on the variables y and z. The result is differentiated with respect to y and gives the second component of $\vec{k}$:

$$\partial_x \Phi = k_1 = z + y \quad \Rightarrow \quad \Phi = z\,x + y\,x + f(y, z)$$

$$\Downarrow \partial y$$

$$\partial_y \Phi = x + f_y(y, z) = k_2 = x + z$$

$$\Rightarrow f_y(y, z) = z \quad \Rightarrow \quad f(y, z) = z \cdot y + g(z).$$

$$\Rightarrow \Phi(x, y, z) = z\,x + y\,x + z\,y + g(z).$$

We take the derivative with respect to z which is the third component of $\vec{k}$. Comparing the left and right sides gives $g(z)$.

$$\partial_z \Phi = x + y + g'(z) = k_3 = x + y \quad \Rightarrow \quad g'(z) = 0 \quad \Rightarrow \quad g(z) = K$$

$$\Rightarrow \quad \boxed{\Phi(x, y, z) = z\,x + yx + z\,y + K.} \qquad \square$$

20.1.5 Application Examples

The curve integral is used to calculate the work required to move a mass m in a force field $\vec{k}(x, y, z)$ along the curve $\mathcal{C}$ from a start point P_A to an end point P_E. According to the definition of the line integral, the *work* is

$$W = \int_{\mathcal{C}} \vec{k}\, d\vec{r} = \int_{t_A}^{t_E} \vec{k}(\vec{r}(t)) \cdot \vec{r}'(t)\; dt,$$

where $\vec{r}(t)$ is a parameter representation of the curve $\mathcal{C}$, $\vec{r}(t_A)$ is the start point and $\vec{r}(t_E)$ is the end point. W is positive if the force field is acting on the mass, otherwise it is negative.

Application Example 20.10 (Radially Symmetric Force Fields).

A force field $\vec{k}$ is called *radially symmetric* if the magnitude of $\vec{k}$ depends only on the distance r and the vector $\vec{k}$ points radially outwards. A radially symmetric field has the form

$$\vec{k}(\vec{r}) = f(r)\, \vec{r} = \begin{pmatrix} f(r)\, x \\ f(r)\, y \\ f(r)\, z \end{pmatrix}$$

where f is a function in a variable and $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$. Physical examples of radially symmetric force fields are

$$\vec{F}(\vec{r}) = f_g\, \frac{m\,M}{r^2}\, \frac{\vec{r}}{r} \qquad \text{(Newtonian gravity)}$$

$$\vec{F}(\vec{r}) = \frac{1}{4\pi\,\varepsilon_0}\, \frac{q\,Q}{r^2}\, \frac{\vec{r}}{r} \qquad \text{(Coulomb force).}$$

All rotationally symmetric force fields are *conservative*: The work integral depends only on the start and end points of the path, not on the path chosen! We show that the first integrability condition is satisfied. To do this, we use the chain rule to form the partial derivatives of $\vec{k}(\vec{r}) = \begin{pmatrix} f(r)\,x \\ f(r)\,y \\ f(r)\,z \end{pmatrix}$:

$$\begin{aligned} \frac{\partial}{\partial y} k_1(\vec{r}) &= \frac{\partial}{\partial y}(f(r)\,x) = x\,f'(r)\,\frac{\partial}{\partial y}\sqrt{x^2+y^2+z^2} \\ &= x\,f'(r)\,\frac{y}{\sqrt{x^2+y^2+z^2}} = f'(r)\,\frac{x\,y}{r}. \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial x} k_2(\vec{r}) &= \frac{\partial}{\partial x}(f(r)\,y) = y\,f'(r)\,\frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} \\ &= y\,f'(r)\,\frac{x}{\sqrt{x^2+y^2+z^2}} = f'(r)\,\frac{x\,y}{r}. \end{aligned}$$

$$\Rightarrow \frac{\partial}{\partial y} k_1(\vec{r}) = \frac{\partial}{\partial x} k_2(\vec{r}).$$

The other two integrability conditions are checked in the same way. □

Application Example 20.11 **(Coulomb Force).**

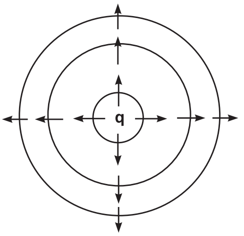


Figure 20.5. Coulomb force

The potential of the Coulomb force

$$\vec{k}(\vec{r}) = \frac{1}{4\pi\,\varepsilon_0}\,\frac{q\,Q}{r^2}\,\frac{\vec{r}}{r}$$

is the electrostatic potential

$$\begin{aligned} \Phi(x,\,y,\,z) &= \frac{1}{4\pi\,\varepsilon_0}\,\frac{q\,Q}{r} \\ &= \frac{1}{4\pi\,\varepsilon_0}\,\frac{q\,Q}{\sqrt{x^2+y^2+z^2}}\,. \end{aligned}$$

We calculate directly that $\vec{k} = -\operatorname{grad}\Phi$. Thus, according to the main theorem on curve integrals

$$\oint_C \vec{k}\,d\vec{r} = 0$$

for all closed curves **not** containing the origin. There $\vec{k}$ becomes singular! To compute the behavior of $\vec{k}$ including its singularity, we look at the curve integral along a circle with radius R

$$\vec{r}(t) = R \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} \quad \text{for} \quad 0 \le t \le 2\pi.$$

Then, the singularity of $\vec{k}$ is in the center of the circle: Because of $x(t) = R\cos t$ ($\hookrightarrow \dot{x}(t) = -R\sin t$), $y(t) = R\sin t$ ($\hookrightarrow \dot{y}(t) = R\cos t$) and $z(t) = 0$, we obtain for the force field

$$\vec{k}(\vec{r}) = \frac{1}{4\pi\,\varepsilon_0}\frac{q\,Q}{r^3}\,\vec{r} = \frac{q\,Q}{4\pi\,\varepsilon_0}\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}\begin{pmatrix} x \\ y \\ z \end{pmatrix} :$$

$$\begin{aligned}
\int_{\mathcal{C}} \vec{k}\,d\vec{r} &= \frac{q\,Q}{4\pi\,\varepsilon_0}\int_0^{2\pi}\left\{\frac{1}{(x^2(t)+y^2(t)+z^2(t))^{\frac{3}{2}}}\,x(t)\cdot\dot{x}(t)\right. \\
&\qquad\left. + \frac{1}{(x^2(t)+y^2(t)+z^2(t))^{\frac{3}{2}}}\,y(t)\cdot\dot{y}(t)\right\} dt \\
&= \frac{q\,Q}{4\pi\,\varepsilon_0}\int_0^{2\pi}\frac{1}{R^{\frac{3}{2}}}\left(x(t)\,\dot{x}(t) + y(t)\,\dot{y}(t)\right) dt \\
&= \frac{q\,Q}{4\pi\,\varepsilon_0}\frac{1}{R^{\frac{3}{2}}}\int_0^{2\pi}\left(-R^2\cos t\,\sin t + R^2\sin t\,\cos t\right) dt = 0.
\end{aligned}$$

□

Application Example 20.12 (Voltage).

For the electrostatic field $\vec{E}(\vec{r}) = \begin{pmatrix} E_1(x,y,z) \\ E_2(x,y,z) \\ E_3(x,y,z) \end{pmatrix}$, the curve integral

$$U = \int_{\mathcal{C}} \vec{E}\,d\vec{r} = \int_{\mathcal{C}} (E_1\,dx + E_2\,dy + E_3\,dz)$$

gives the voltage between the start and end points of the curve $\mathcal{C}$. Since the electric field is a gradient field ($\vec{E} = -\text{grad}(\Phi)$), the curve integral does not depend on the selected path. So the voltage is just the difference between the potential at the start and end points:

$$U = \int_{\mathcal{C}} \vec{E}\,d\vec{r} = -\Phi\Big|_{end} + \Phi\Big|_{start}$$

□

Application Example 20.13 (Magnetic Field of a Conductor).

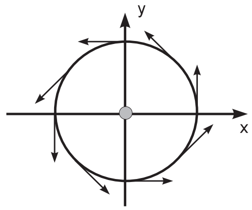


Figure 20.6.
Current carrying conductor

A homogeneous current-carrying wire is along the z-direction. The current generates a magnetic field in the $(x,\, y)$-plane proportional to $\frac{1}{r}$, the direction of the magnetic field is tangential to the circular rings:

$$\vec{B} = \frac{\mu_0\, I}{2\pi} \begin{pmatrix} -y \\ x \end{pmatrix} \frac{1}{x^2 + y^2}\,.$$

The magnetic field becomes singular at the origin $(0, 0)$. We first compute the line integral in the (x, y)-plane in the case where the origin is excluded. In a second calculation we choose a curve that includes the origin.

For $(x,\, y) \neq (0,\, 0)$, the integrability condition in the plane is satisfied:

$$\frac{\partial B_1}{\partial y} = \frac{\partial B_2}{\partial x} = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}\,.$$

Therefore, all closed curve integrals that **do not include the origin** are zero.

However, if the integral along the circle of radius R includes the singularity $(0,\, 0)$, we must explicitly calculate the line integral.

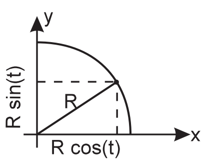


Figure 20.7.

A parameterization of the circle is

$$\vec{r}\,(t) = R \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \quad \text{for} \quad 0 \leq t \leq 2\pi\,.$$

$$\Rightarrow \quad \begin{aligned} x\,(t) &= R\,\cos t \quad (\hookrightarrow \dot{x}\,(t) = -R\,\sin t) \\ y\,(t) &= R\,\sin t \quad (\hookrightarrow \dot{y}\,(t) = R\,\cos t). \end{aligned}$$

So the curve integral along $\mathcal{C}$ containing the origin is

$$\begin{aligned} \oint_{\mathcal{C}} \vec{B}\, d\vec{r} &= \frac{\mu_0\, I}{2\pi} \int_0^{2\pi} \frac{1}{x^2\,(t) + y^2\,(t)} \left(-y\,(t)\ \dot{x}\,(t) + x\,(t)\ \dot{y}\,(t)\right)\, dt \\ &= \frac{\mu_0\, I}{2\pi} \frac{1}{R^2} \int_0^{2\pi} \left(R^2\, \sin^2 t + R^2\, \cos^2 t\right)\, dt = \mu_0\, I. \end{aligned}$$

This curve integral results in the total current I flowing through the area bounded by the curve $\mathcal{C}$. □

20.2 Surface Integrals

Many applications require the area of curved surfaces. These so-called surface integrals are also needed to calculate electric, magnetic or mass flow through a surface. Therefore, the concept of integrals is extended to include integrals of functions over a surface in $\mathbb{R}^3$.

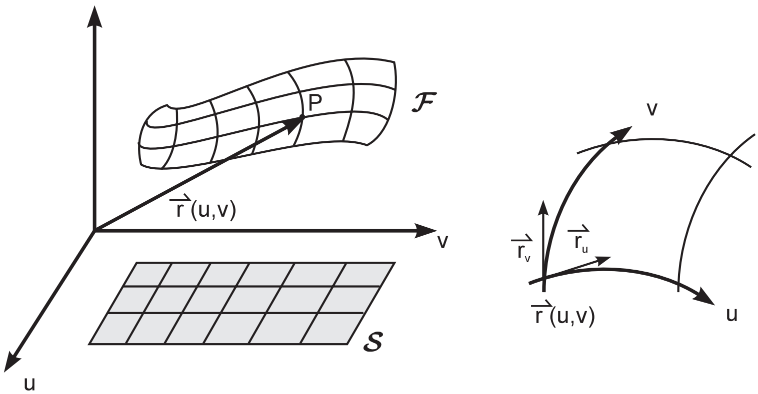


Figure 20.8. Parameterization of a curved surface

Extending the description of a curve $\mathcal{C}$ using the parameter representation

$$\vec{r}\,(t) = x\,(t)\;\vec{e}_1 + y\,(t)\;\vec{e}_2 + z\,(t)\;\vec{e}_3$$

with **one** parameter t, we define curved surfaces using a parameter representation with **two** parameters $(u,\, v)$:

Definition: (Surface). *Let $S \subset \mathbb{R}^2$ be an area in a plane (e.g. a rectangle) with the parameter variables $(u,\, v) \in S$ (e.g. $u_1\,(v) \leq u \leq u_2\,(v)$, $v_1 \leq v \leq v_2$). A surface $F \subset \mathbb{R}^3$ (see Fig. 20.8) is defined by the* **parameter representation**

$$F: \quad \vec{r}\,(u,\, v) = x\,(u,\, v)\;\vec{e}_1 + y\,(u,\, v)\;\vec{e}_2 + z\,(u,\, v)\;\vec{e}_3$$

where x, y, z are functions depending on the two variables $(u,\, v)$. If the $(u,\, v)$-values change, the point P (see Fig. 20.8) moves on the surface F.

We interpret (u, v) as the **coordinates** of point P. We assume that the components of the mapping $\vec{r}: \mathbb{R}^2 \to \mathbb{R}^3$, $(u, v) \mapsto \vec{r}(u, v)$, are continuously partially differentiable functions. The **tangential plane** at the point P is generated by the direction vectors $\frac{\partial \vec{r}}{\partial u}$ and $\frac{\partial \vec{r}}{\partial v}$. For the direction vectors we also write

$$\vec{r}_u = \frac{\partial \vec{r}}{\partial u} = \begin{pmatrix} \partial_u\, x(u,v) \\ \partial_u\, y(u,v) \\ \partial_u\, z(u,v) \end{pmatrix} \quad \text{and} \quad \vec{r}_v = \frac{\partial \vec{r}}{\partial v} = \begin{pmatrix} \partial_v\, x(u,v) \\ \partial_v\, y(u,v) \\ \partial_v\, z(u,v) \end{pmatrix} .$$

Examples 20.14:

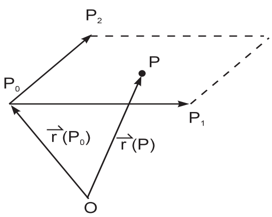


Figure 20.9.

① A parallelogram surface defined by the 3 points $P_0,\ P_1,\ P_2$ has the point-direction representation of a plane

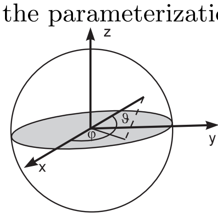
$$\vec{r}\,(P) = \vec{r}\,(P_0) + u\ \overrightarrow{P_0\,P_1} + v\ \overrightarrow{P_0\,P_2}$$

where $0 \le u \le 1$ and $0 \le v \le 1$.

② A spherical surface with radius R at the center 0 and spherical coordinates $u = \varphi,\ v = \vartheta$ has the parameterization

$$\vec{r}\,(u,\ v) = \begin{pmatrix} R\,\cos u\,\cos v \\ R\,\sin u\,\cos v \\ R\,\sin v \end{pmatrix}$$

where $0 \le \varphi \le 2\pi$ and $-\frac{\pi}{2} \le \vartheta \le \frac{\pi}{2}$. □



To determine the area of a curved surface F, we divide the base area S into rectangles with side lengths $(\triangle u,\ \triangle v)$. With this division of S, the surface F is divided into *surface elements* $\triangle F_i$:

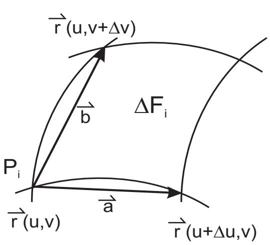


Figure 20.10. Surface element $\triangle F_i$

The content of each surface element $\triangle F_i$ is approximated by the parallelogram area $F_p = \left|\vec{a} \times \vec{b}\right|$, where the parallelogram is spanned by the direction vectors $\vec{a}$ and $\vec{b}$. According to the Taylor theorem in linear approximation $(n = 1)$ we get

$$\vec{a} = \vec{r}\,(u + \triangle u,\, v) - \vec{r}\,(u,\, v) \approx \frac{\partial \vec{r}}{\partial u} \cdot \triangle u$$
$$\vec{b} = \vec{r}\,(u,\, v + \triangle v) - \vec{r}\,(u,\, v) \approx \frac{\partial \vec{r}}{\partial v} \cdot \triangle v.$$

$$\Rightarrow \triangle F_i \approx \left|\vec{a} \times \vec{b}\right| \approx \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| \triangle u \, \triangle v.$$

The vector $\vec{n} := \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$ is perpendicular to the tangent plane. It is called the **normal vector** of the surface F at the point $P\,(u,\, v)$. The sign is reversed if we change the order of the parameters u and v, because $\vec{r}_v \times \vec{r}_u = -\vec{r}_u \times \vec{r}_v$. The sum of all parts $\triangle F_i$ is

$$Z_n = \sum_{i=1}^{n} \left|\frac{\partial \vec{r}}{\partial u}\,(P_i) \times \frac{\partial \vec{r}}{\partial v}\,(P_i)\right| \triangle u \, \triangle v.$$

This subtotal is an approximation of the area of F. This approximation gets better the finer the subdivision of the surface S is. For $\triangle u \to 0$, $\triangle v \to 0$ we define

$$\iint_{(F)} dF = \iint_{(S)} |\vec{r}_u\,(u,\, v) \times \vec{r}_v\,(u,\, v)| \; du\, dv$$

as the **surface integral of the surface** F. This is a double integral using the function $|\vec{r}_u \times \vec{r}_v|$ in the domain S, as described in Volume 2, Section 12.1.

Example 20.15. ① Find the surface of the hemisphere with radius R. Using the parameterization from Example 20.14 ② we get

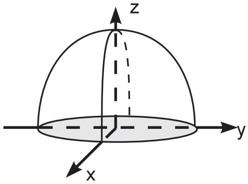


Figure 20.11.

$$\vec{r}\,(u,\, v) = R \begin{pmatrix} \cos u \, \cos v \\ \sin u \, \cos v \\ \sin v \end{pmatrix} :$$

$$\vec{r}_u = R \begin{pmatrix} -\sin u \, \cos v \\ \cos u \, \cos v \\ 0 \end{pmatrix} , \quad \vec{r}_v = R \begin{pmatrix} -\cos u \, \sin v \\ -\sin u \, \sin v \\ \cos v \end{pmatrix} .$$

$$\Rightarrow \vec{r}_u \times \vec{r}_v = R^2 \cos v \, (\cos u \cos v \, \vec{e}_1 + \sin u \cos v \, \vec{e}_2 + \sin v \, \vec{e}_3)$$
$$\Rightarrow |\vec{r}_u \times \vec{r}_v| = R^2 \cos v.$$

We insert this parameterization into the surface integral:

$$\iint_{(F)} dF = \int_{v=0}^{\pi/2} \int_{u=0}^{2\pi} R^2 \cos v \, du \, dv$$
$$= 2\pi R^2 \int_{v=0}^{\pi/2} \cos v \, dv = 2\pi R^2 .$$

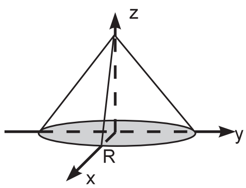


Figure 20.12.

② Find the surface of the circular cone $z = R - \sqrt{x^2 + y^2}$, $0 \le z \le R$. With

$$x = u$$
$$y = v$$
$$z = R - \sqrt{u^2 + v^2}, \qquad u^2 + v^2 \le R^2,$$

we get the parameterization of the surface F.

Therefore,

$$\vec{r}(u, v) = \begin{pmatrix} u \\ v \\ R - \sqrt{u^2 + v^2} \end{pmatrix} \quad \Rightarrow \quad \vec{r}_u = \begin{pmatrix} 1 \\ 0 \\ \frac{-u}{\sqrt{u^2+v^2}} \end{pmatrix}, \; \vec{r}_v = \begin{pmatrix} 0 \\ 1 \\ \frac{-v}{\sqrt{u^2+v^2}} \end{pmatrix}$$

$$\Rightarrow \vec{r}_u \times \vec{r}_v = \begin{pmatrix} \frac{u}{\sqrt{u^2+v^2}} \\ \frac{v}{\sqrt{u^2+v^2}} \\ 1 \end{pmatrix} \quad \Rightarrow \quad |\vec{r}_u \times \vec{r}_v| = \sqrt{2}$$

$$\Rightarrow \iint_{(F)} dF = \iint_{(S)} \sqrt{2} \, du \, dv = \sqrt{2} \, \pi \, R^2.$$ □

20.2.1 Surface Integral of a Vector Field

If $\vec{v}(x, y, z)$ is the *velocity field* of e.g. a flowing liquid, then the scalar product

$$\vec{v} \cdot \vec{n} = \vec{v} \cdot (\vec{r}_u \times \vec{r}_v) \, \triangle u \, \triangle v$$

is the amount of liquid per unit time $\triangle t$ passing through the area $\triangle F$.

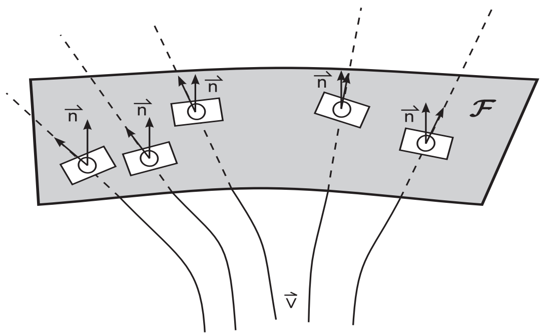


Figure 20.13. Flow of a liquid through a surface F

The component of the liquid flowing parallel to the surface **does not** contribute to the flow through the surface! So the flow of $\vec{v}$ through the surface F is approximately

$$\sum_{k=1}^{N} \vec{v}_k \, \vec{n}_k = \sum_{k=1}^{N} \vec{v}_k \cdot (\vec{r}_u \times \vec{r}_v) \; \triangle u \, \triangle v,$$

where $\triangle F_i$ is summed over all surface elements. In case $N \to \infty$ (i.e. $\triangle u \to 0$ and $\triangle v \to 0$) we obtain the surface integral $\iint\limits_{(F)} \vec{v} \, d\vec{F}$.

Definition: (Surface Integral). *Let $\vec{v}\,(\vec{r})$ be a vector field on the surface F. The surface is represented by the parameterization $\vec{r}\,(u, v)$ with $(u, v) \in S$. Then*

$$\iint\limits_{(F)} \vec{v} \, d\vec{F} = \iint\limits_{(S)} \vec{v}\,(\vec{r}\,(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) \; du \, dv$$

is the **Surface Integral** *(if it exists) of $\vec{v}\,(\vec{r})$ on the surface F.*

Remarks:

(1) If the surface integral exists, it is independent of the particular parameterization of S.

(2) If F is a closed surface (e.g. the surface of a body), then instead of

$$\iint\limits_{(F)} \vec{v} \, d\vec{F} \quad \text{we write} \quad \oiint\limits_{(F)} \vec{v} \, d\vec{F} \, .$$

(3) Often the surface F is not given in a parameter representation, but in an explicit expression $z = f(x, y)$. Then we must first parameterize the surface in order to calculate $\iint\limits_{(F)} \vec{v}\, d\vec{F}$. We will always get such a parameterization with

$$x = u\,,\ y = v\,,\ z = f(u, v) \quad \text{and} \quad \vec{r}(u, v) = \begin{pmatrix} u \\ v \\ f(u, v) \end{pmatrix}.$$

(4) The surface integrals are exactly what we need when we want to calculate, for example, the mass flow of a fluid through a surface or the magnetic flux as the next examples show.

Mass Flow Rate

Let $\vec{v}$ be the velocity field of a flowing medium. Then

$$\iint\limits_{(F)} \vec{v}\, d\vec{A} \text{ is the volume per unit time and}$$

$$\rho \iint\limits_{(F)} \vec{v}\, d\vec{A} \text{ is the mass per unit time}$$

of the fluid flowing through the surface F, if ρ is the homogeneous density of the fluid.

Application Example 20.16 (Mass Flow Rate).

Given is the velocity field of a fluid $\vec{v}(\vec{r}) = \begin{pmatrix} x \\ y \\ \sqrt{x^2 + y^2} \end{pmatrix}$. We calculate the mass flow through a hemispherical surface $x^2 + y^2 + z^2 = R \quad (z > 0)$ in 2 time units ($\rho = 1$). A parameterization of the sphere's surface is given in Example 20.14 ②:

$$\vec{r}(u, v) = R \begin{pmatrix} \cos u \cos v \\ \sin u \cos v \\ \sin v \end{pmatrix} \qquad \begin{matrix} 0 \le u \le 2\pi \\ 0 \le v \le \frac{\pi}{2} \end{matrix}$$

with the normal vector

$$\vec{r}_u \times \vec{r}_v = R^2 \cos v \begin{pmatrix} \cos u \cos v \\ \sin u \cos v \\ \sin v \end{pmatrix}.$$

$$\Rightarrow \vec{v}\,(\vec{r}) \cdot (\vec{r}_u \times \vec{r}_v) = \begin{pmatrix} R\cos u \cos v \\ R\sin u \cos v \\ R\cos v \end{pmatrix} \cdot R^2 \cos v \cdot \begin{pmatrix} \cos u \cos v \\ \sin u \cos v \\ \sin v \end{pmatrix}$$
$$= R^3 \cos v \left(\cos^2 v + \cos v \sin v\right).$$

The volume of fluid per unit time $\iint\limits_{(F)} \vec{v}\, d\vec{F}$ is

$$\iint\limits_{(F)} \vec{v}\, d\vec{F} = \int_{u=0}^{2\pi} \int_{v=0}^{\pi/2} \vec{v}\,(\vec{r}\,(u,\,v)) \cdot (\vec{r}_u \times \vec{r}_v)\; dv\, du$$
$$= R^3 \int_{u=0}^{2\pi} \int_{v=0}^{\pi/2} \cos v \left(\cos^2 v + \cos v \sin v\right) dv\, du$$
$$= R^3 \int_{u=0}^{2\pi} 1\, du = 2\pi\, R^3.$$

The mass M flowing through the surface in two units of time is

$$M = 2 \cdot 1 \cdot 2\pi\, R^3 = 4\pi\, R^3\,.$$

□

Application Example 20.17 (Magnetic Flux).

The magnetic flux Φ of a magnetic field $\vec{B}$ penetrating an area A is

$$\Phi = \iint\limits_{(A)} \vec{B}\, d\vec{A}\,.$$

Given is the magnetic field of a straight, current-carrying conductor

$$\vec{B} = \frac{\mu_0\, I}{2\pi} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \frac{1}{x^2 + y^2}\,.$$

We look for the magnetic flux through the circular surface A.

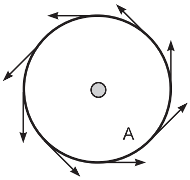


Figure 20.14.

The magnetic field is in the $(x,\,y)$-plane, the surface vector $d\vec{A} = \vec{r}_u \times \vec{r}_v\, du\, dv$ is perpendicular to the surface A, i.e. in z-direction. So $\vec{B} \cdot d\vec{A} = 0$ and the magnetic flux through the surface A is zero. □

20.3 The Divergence

Let $\vec{v}$ be the velocity field of a flowing liquid, given in a volume $G \subset \mathbb{R}^3$. Let O be the surface of the volume. According to Section 20.2, the integral

$$\oiint_{(O)} \vec{v}\, d\vec{O} = \oiint_{(O)} \vec{v}\, \vec{n}\, dA$$

is the net flow of the fluid through the surface O. Instead of $d\vec{O}$ we write $\vec{n}\, dA$, where $\vec{n}$ is the outward normal unit vector and dA is the area element. If the integral is positive, then the domain G contains *sources*. Otherwise, G contains *sinks*. We introduce a method to calculate the sources by an appropriate differentiation of the vector field $\vec{v}$ without evaluating the surface integral.

The surface integral $\oiint_{(O)} \vec{v}\, d\vec{O}$ allows only a global statement in the sense of a total balance over the whole domain G. To obtain a local statement about the properties of the velocity field $\vec{v}$ at point P, we select a volume V with surface O including point P.

Definition (Divergence):
The limit (as the volume around P goes to zero) is called the **local source density** or **divergence** of $\vec{v}$. We write:

$$\operatorname{div}(\vec{v})\Big|_P = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oiint_{(O)} \vec{v}\, \vec{n}\, dA,$$

where O is the surface of the volume V.

To simplify the discussion, we assume the point P to be at origin and select a cuboid centred at P with edge lengths $2\,\Delta x$, $2\,\Delta y$, $2\,\Delta z$ as the volume (see Fig. 20.15).

The total mass flow out of the cuboid is balanced by determining the flow through two opposite surfaces. The flow through the surfaces A_1 and A_2 is given by

$$\Phi_z = \iint_{(A_1)} \vec{v}\, \vec{n}_1\, dA_1 + \iint_{(A_2)} \vec{v}\, \vec{n}_2\, dA_2.$$

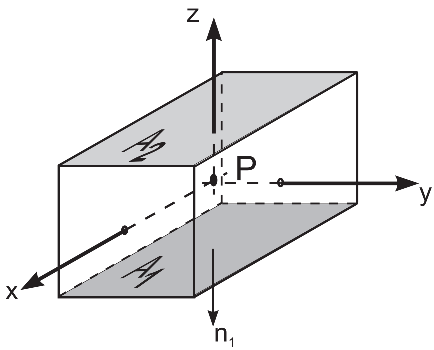


Figure 20.15. Outflow through area A_1 and A_2

The surface A_1 at $z = -\Delta z$ has the normal $\vec{n}_1 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = -\vec{e}_3$ and $\vec{v}\,\vec{n}_1 = -\vec{v}\,\vec{e}_3 = -v_3$. The surface element is $dx\,dy$ (Cartesian coordinates). With $-\Delta x \leq x \leq \Delta x\,,\ -\Delta y \leq y \leq \Delta y$ we calculate

$$\iint\limits_{(A_1)} \vec{v}\,\vec{n}_1\,dA_1 = \int_{y=-\Delta y}^{\Delta y} \int_{x=-\Delta x}^{\Delta x} -\vec{v}\,\vec{e}_3\,dx\,dy$$
$$= -\int_{y=-\Delta y}^{\Delta y} \int_{x=-\Delta x}^{\Delta x} v_3\,(x,\,y,\,-\Delta z)\;dx\,dy.$$

Similarly, for the second surface integral at position $z = \Delta z$, with $\vec{n}_2 = -\vec{n}_1 = \vec{e}_3$, we get

$$\iint\limits_{(A_2)} \vec{v}\,\vec{n}_2\,dA_2 = \int_{y=-\Delta y}^{\Delta y} \int_{x=-\Delta x}^{\Delta x} v_3\,(x,\,y,\,\Delta z)\;dx\,dy.$$

Therefore,

$$\Phi_z = \int_{-\Delta y}^{\Delta y} \int_{-\Delta x}^{\Delta x} \left(v_3\,(x,\,y,\,\Delta z) - v_3\,(x,\,y,\,-\Delta z)\right)\,dx\,dy.$$

We linearize the function v_3 for small Δz with respect to the variable z. Then

$$v_3\,(x,\,y,\,\Delta z) - v_3\,(x,\,y,\,-\Delta z) \approx \frac{\partial}{\partial z}\,v_3\,(x,\,y,\,0)\cdot 2\,\Delta z$$

$$\Rightarrow\ \Phi_z \approx \int_{-\Delta y}^{\Delta y} \int_{-\Delta x}^{\Delta x} \frac{\partial}{\partial z}\,v_3\,(x,\,y,\,0)\cdot 2\,\Delta z\cdot dx\,dy$$

Using the Mean Value Theorem for Integrals (see Volume 2, Section 8.6.5), the function $\frac{\partial}{\partial z} v_3(x, y, 0)$ is excluded from the integral if it is evaluated at a suitable intermediate point (ξ_1, η_1) with $-\Delta x \le \xi_1 \le \Delta x$ and $-\Delta y \le \eta_1 \le \Delta y$:

$$\begin{aligned}\Phi_z &\approx \frac{\partial}{\partial z} v_3(\xi_1, \eta_1, 0) \cdot \int_{-\Delta y}^{\Delta y} \int_{-\Delta x}^{\Delta x} 2\,\Delta z \cdot dx\,dy \\ &\approx \frac{\partial}{\partial z} v_3(\xi_1, \eta_1, 0)\; 2\Delta z\, 2\Delta x\, 2\Delta y.\end{aligned}$$

Similar expressions are obtained for the other two adjacent surfaces for Φ_x and Φ_y. The total flux outside the box is given by the sum of all contributions from Φ_x, Φ_y and Φ_z. So for $\Delta V = 8\,\Delta x\,\Delta y\,\Delta z$ we get

$$\begin{aligned}\operatorname{div}(\vec{v})\Big|_{P_0} &= \lim_{\Delta V \to 0} \frac{1}{\Delta V} (\Phi_x + \Phi_y + \Phi_z) \\ &= \lim_{\Delta V \to 0} \frac{1}{8\,\Delta x\,\Delta y\,\Delta z}\, 8\,\Delta x\,\Delta y\,\Delta z \\ &\qquad \cdot \left[\frac{\partial v_3}{\partial z}(\xi_1, \eta_1, 0) + \frac{\partial v_1}{\partial x}(0, \eta_2, \tau_2) + \frac{\partial v_2}{\partial y}(\xi_3, 0, \tau_3)\right]\end{aligned}$$

where $-\Delta x \le \xi_1, \xi_3 \le \Delta x\,, \ -\Delta y \le \eta_1, \eta_2 \le \Delta y\,, \ -\Delta z \le \tau_2, \tau_3 \le \Delta z$. For $\Delta V \to 0$ ($\Delta x \to 0$, $\Delta y \to 0$, $\Delta z \to 0$) all points in the cuboid tend to P. Especially for the intermediate points we get $\xi_1, \xi_3, \eta_1, \eta_2, \tau_2, \tau_3 \to 0$.

It can be shown that the limit exists at any point P of the volume and is independent of the selected volume (or surface):

Divergence

The scalar function

$$\operatorname{div}(\vec{v}) = \frac{\partial v_1}{\partial x}(x, y, z) + \frac{\partial v_2}{dy}(x, y, z) + \frac{\partial v_3}{\partial z}(x, y, z)$$

is the **divergence** of the vector field $\vec{v}$ at (x, y, z).

The divergence $\operatorname{div}(\vec{v})$ **of a vector field** $\vec{v}$ **specifies the local source density at point** $P(x, y, z)$. If $\operatorname{div}(\vec{v}) = 0$, then the vector field has no local sources. The divergence of a vector field $\vec{v}$ is not a vector field but a *scalar* field.

Examples 20.18:

① The divergence of $\vec{v} = \begin{pmatrix} x^2 - y\,z \\ y^2 - z\,x \\ z^2 - x\,y \end{pmatrix}$ is

$$\begin{aligned} \operatorname{div}(\vec{v}) &= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \\ &= \frac{\partial}{\partial x}\,(x^2 - y\,z) + \frac{\partial}{\partial y}\,(y^2 - z\,x) + \frac{\partial}{\partial z}\,(z^2 - x\,y) \\ &= 2\,x + 2\,y + 2\,z \end{aligned}$$

At point $P\,(1,\,3,\,2)$ the divergence becomes

$$\operatorname{div}(\vec{v})\Big|_P = 2 + 6 + 4 = 12.$$

② A local source density (= charge density) $\rho\,(x, y, z)$ induces an electric field $\vec{E} = \begin{pmatrix} \frac{1}{3}\,x^3 + y \\ (z+1)\;y \\ z^2\;(x-y) + 2\,z \end{pmatrix}$. Applying the divergence to the electric field, we compute ρ:

$$\begin{aligned} \frac{1}{\epsilon_0}\;\rho\,(x,\,y,\,z) &= \operatorname{div}(\vec{E}) \\ &= \frac{\partial}{\partial x}\left(\frac{1}{3}\,x^3 + y\right) + \frac{\partial}{\partial y}\,(\,(z+1)\;y) \\ &\qquad + \frac{\partial}{\partial z}\;(z^2\;(x-y) + 2\,z) \\ &= x^2 + (z+1) + 2\,z\;(x-y) + 2. \end{aligned}$$

The charge density at the origin is $\rho\,(0,\,0,\,0) = 3\,\epsilon_0$. □

20.4 Integral Theorem of Gauss

Let $G \subset \mathbb{R}^3$ be a contiguous domain divided into subdomains G_k with volumes ΔV_k, each containing the points P_k. The surface belonging to the subdomain G_k is A_k. Then the flow through the subdomain G_k is

$$\Delta V_k\;\operatorname{div}(\vec{v})\Big|_{P_k} \approx \mathop{\iint\!\!\!\!\!\!\bigcirc}_{(A_k)} \vec{v}\,\vec{n}\,dA \qquad (k = 1, \ldots, n)\,.$$

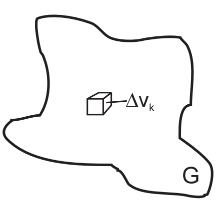


Figure 20.16.

If two domains G_k and G_l are adjacent, the fluxes at the adjacent surfaces cancel each other out because the outwardly directed normals of the common surfaces are opposite. This means that the flux through G_k and G_l can be determined by balancing only the outer surfaces!

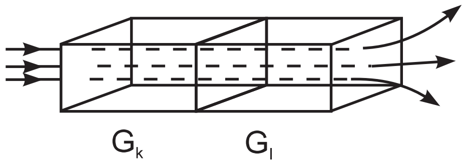


Figure 20.17. Flow through two adjacent areas

Adding all the subdomains we obtain

$$\sum_{k=1}^{n} \operatorname{div}(\vec{v})\Big|_{P_k} \cdot \Delta V_k \approx \underset{(A)}{\oint\!\!\oint} \vec{v}\,\vec{n}\,dA$$

and for $n \to \infty$ (i.e. any fine subdivision $\Delta V_k \to 0$):

$$\underset{(V)}{\iiint} \operatorname{div}(\vec{v})\,dV = \underset{(A)}{\oint\!\!\oint} \vec{v}\,\vec{n}\,dA.$$

This integral relation is called the *Gaussian Integral Theorem*: The flow of a vector field $\vec{v}$ through a closed surface A is equal to the triple integral over the source density div$(\vec{v})$ in the enclosed volume.

Gaussian Integral Theorem: Divergence Theorem

Let $V \subset \mathbb{R}^3$ be a 3-dimensional domain with surface $A = \partial V$. Let $\vec{n}$ be the normal of length 1 pointing outwards on the surface A. Let $\vec{v}\,(x, y, z)$ be a vector field. Then

$$\underset{(V)}{\iiint} \operatorname{div}(\vec{v})\,dV = \underset{(\partial V)}{\oint\!\!\oint} \vec{v}\,\vec{n}\,dA.$$

Example 20.19. The velocity field is given

$$\vec{v} = \begin{pmatrix} x - z \\ x^3 + y\,z \\ -3 + y \end{pmatrix}.$$

We are looking for the flux Φ through the surface of the cone with height $h = 2$ and radius $R = 2$:

The flux Φ through the surface of a volume is

$$\Phi = \oiint_{(\partial V)} \vec{v}\,\vec{n}\,dA = \iiint_{(V)} \operatorname{div}(\vec{v})\,dV$$

and the divergence $\operatorname{div}(\vec{v})$ of the velocity field is

$$\operatorname{div}(\vec{v}) = \frac{\partial}{\partial x}\,(x - z) + \frac{\partial}{\partial y}\,\left(x^3 + y\,z\right) + \frac{\partial}{\partial z}\,(-3 + y) = 1 + z.$$

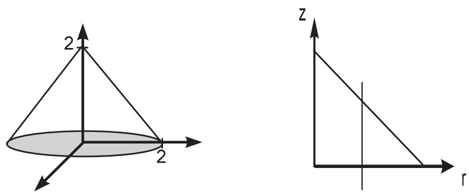


Figure 20.18. Cone and integration along z-axis

For the integration we introduce cylindrical coordinates

$$x = r\,\cos\varphi$$
$$y = r\,\sin\varphi$$
$$z = z$$

and use the parameterization of the cone $(R = 2)$

φ-Integration	:	$0 \le \varphi \le 2\pi$	
z-Integration	:	$0 \le z \le R - r$	for given r,
r-Integration	:	$0 \le r \le R.$	

$$\Phi = \int_{\varphi=0}^{2\pi} \int_{r=0}^{2} \int_{z=0}^{2-r} (1 + z)\;r\,dz\,dr\,d\varphi = 2\pi \int_{r=0}^{2} \left[z + \frac{1}{2}\,z^2\right]_{z=0}^{2-r} r\,dr$$
$$= 2\pi \int_{r=0}^{2} \left(4\,r - 3\,r^2 + \frac{1}{2}\,r^3\right)\,dr = 4\pi.$$

□

Example 20.20. Given is the electric field $\vec{E} = \begin{pmatrix} \frac{1}{3}\,x^3 + y \\ z + 1 \\ x - y \end{pmatrix}$. We calculate the electric flow through the surface of a sphere with radius R.

The electric flow through the surface is calculated using the Gaussian Integral Theorem

$$\Phi = \oiint_{(A)} \vec{E}\,\vec{n}\,dA = \iiint_{(V)} \operatorname{div}(\vec{E})\,dV = \iiint_{(V)} x^2\,dV.$$

To describe the volume V of the triple integral, we introduce spherical coordinates

$$\begin{aligned} x &= r\,\cos\varphi\,\cos\vartheta, \\ y &= r\,\sin\varphi\,\cos\vartheta, \\ z &= r\,\sin\vartheta. \end{aligned}$$

With these coordinates the divergence of the electric field becomes

$$\operatorname{div}(\vec{E}) = x^2 = r^2\,\cos^2\varphi\,\cos^2\vartheta$$

and the volume integral is

$$\begin{aligned} \iiint\limits_{(V)} \operatorname{div}(\vec{E})\,dV &= \int_{r=0}^{R} \int_{\vartheta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\varphi=0}^{2\pi} r^2\,\cos^2\varphi\,\cos^2\vartheta\,r^2\,\cos\vartheta\,d\varphi\,d\vartheta\,dr. \\ &= \int_{r=0}^{R} r^4 \int_{\vartheta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3(\vartheta) \int_{\varphi=0}^{2\pi} \cos^2(\varphi)\,d\varphi\,d\vartheta\,dr. \end{aligned}$$

Using the integrals

$$\int \cos(x)^2\,dx = \frac{1}{2}\cos(x)\sin(x) + \frac{1}{2}x + C$$

$$\int \cos(x)^3\,dx = \frac{1}{3}\cos(x)^2\sin(x) + \frac{2}{3}\sin(x) + C,$$

we finally get the result

$$\iiint\limits_{(V)} \operatorname{div}(\vec{E})\,dV = \frac{4}{15}\,R^5\,\pi. \qquad \square$$

Examples 20.21:

① The total charge Q in a volume V with surface A at a charge density $\rho\,(x,\,y,\,z)$ is given by

$$Q = \iiint\limits_{(V)} \rho\,(x,\,y,\,z)\;dV \approx \oiint\limits_{(A)} \vec{E}\,\vec{n}\,dA.$$

The electric flow through the surface A is proportional to the total charge in the volume V. We introduce the constant ε_0 and obtain

$$Q = \iiint\limits_{(V)} \rho\,(x,\,y,\,z)\;dV = \varepsilon_0\,\oiint\limits_{(A)} \vec{E}\,\vec{n}\,dA. \qquad (1)$$

According to the Gaussian Integral Theorem, the surface integral is

$$Q = \varepsilon_0 \oiint_{(A)} \vec{E}\,\vec{n}\,dA = \varepsilon_0 \iiint_{(V)} \operatorname{div}(\vec{E})\,dV. \qquad (2)$$

With the identity of the integrals (1) and (2) for each volume V we conclude with the Mean Value Theorem of Integral Calculus the identity of the integrands

$$\boxed{\varepsilon_0 \operatorname{div}(\vec{E}) = \rho.}$$

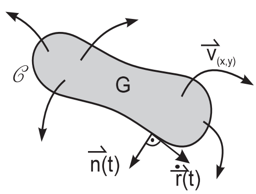
The sources of the electric field are the charge densities.

② Since there are no magnetic charges, the divergence of the magnetic field is zero:

$$\operatorname{div}(\vec{B}) = 0.$$

□

At the end of this section we note that the Gaussian Integral Theorem is also valid for *planes*:



Gauss' Integral Theorem in the Plane

Let $G \subset \mathbb{R}^2$ be a plane with the boundary curve $\mathcal{C}$ and let $\vec{v}\,(x,\,y) = \begin{pmatrix} v_1\,(x,\,y) \\ v_2\,(x,\,y) \end{pmatrix}$ be a two-dimensional vector field. Then

$$\iint_{(G)} \operatorname{div}(\vec{v})\;dx\;dy = \oint_{(\mathcal{C})} \vec{v}(\vec{r}(t))\,\vec{n}(t)\,dt,$$

where $\vec{r}\,(t) = \begin{pmatrix} x\,(t) \\ y\,(t) \end{pmatrix}$ is a parameterization of the curve $\mathcal{C}$ and $\vec{n}\,(t) = \begin{pmatrix} \dot{y}\,(t) \\ -\dot{x}\,(t) \end{pmatrix}$ is the normal vector pointing outwards to $\vec{r}\,'\,(t)$.

20.5 The Curl

To determine the local sources of a vector field, we have introduced the divergence. The calculation of the vortices of a vector field leads to the concept of the *curl*. Roughly speaking, the sources in a volume are determined by calculating the flow through its surface. To describe the *vortices* (*circulation*) in a surface A, however, the vector field is integrated **along** the boundary curve $\mathcal{C}$.

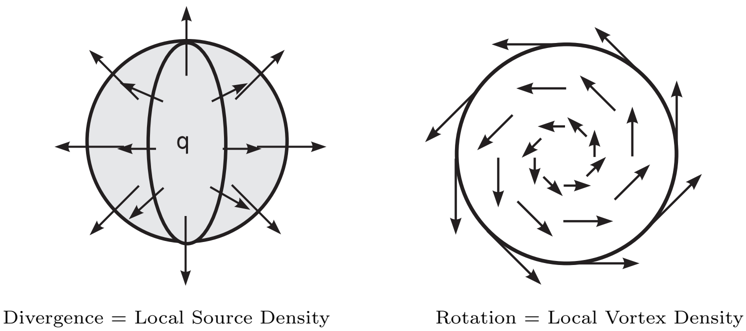


Figure 20.19. Divergence and Curl

Let $\vec{v}$ be the velocity field of a flowing liquid. Let $\mathcal{C}$ be a closed curve around a point P. Then the line integral

$$\oint_{(\mathcal{C})} \vec{v}\, d\vec{r} = \oint_{(\mathcal{C})} \vec{v}\ (\vec{r}\,(t)) \cdot \vec{r}\,'\ (t)\ dt$$

is a measure of the circulation of the liquid **near the point** P. Note that $\vec{r}\,'\ (t)$ is tangent to the curve $\mathcal{C}$ and therefore $\vec{v}\,(\vec{r}\,(t)) \cdot \vec{r}\,'\ (t)$ shows the component of $\vec{v}$ along the curve. To obtain a local statement **at a point** P (see Fig. 20.20), we select a surface A near the point P with the boundary curve $\mathcal{C}$ and the surface normal vector $\vec{n}$. The vector $\vec{n}$ is perpendicular to A. The orientation of $\mathcal{C}$ and $\vec{n}$ is chosen such that they form a right-handed helix.

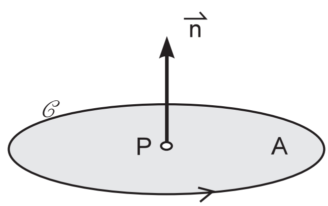


Figure 20.20. Surface A with boundary curve $\mathcal{C}$ and normal $\vec{n}$

Definition (Curl, Rotation):
The **Curl** (**local circulation**) of a vector field $\vec{v}$ at a point P is a vector $\text{rot}(\vec{v})$ whose component in direction $\vec{n}$ is defined by

$$\vec{n} \cdot \text{rot}(\vec{v}) = \lim_{A \to 0} \frac{1}{A} \oint_{(\mathcal{C})} \vec{v} \cdot d\vec{r}.$$

By a suitable choice of the domain A, all components of $\text{rot}(\vec{v})$ are obtained. It can be shown that the curl definition is independent on the specific choice of the surface A, provided that the partial derivatives of the vector field $\vec{v}$ are continuous.

We determine $\text{rot}(\vec{v})$ for the vector field $\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$ in Cartesian coordinates, taking the point P as the origin and choosing a rectangle with center P and edge lengths $2\Delta x$ and $2\Delta y$.

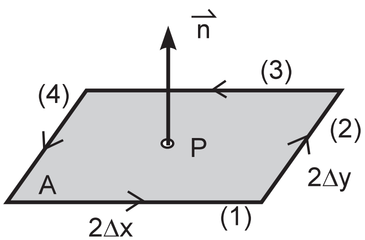


Figure 20.21. Circulation in z-direction

Then $\vec{n} = \vec{e}_z$ and $\vec{n} \cdot \text{rot}(\vec{v}) = (\text{rot}(\vec{v}))_z$ is the z-component of the curl. The line integral over the boundary curve $\mathcal{C}$ is divided into four partial integrals:

$$(\text{rot}\,\vec{v})_z = \lim_{\substack{\Delta y \to 0 \\ \Delta x \to 0}} \frac{1}{4\,\Delta x\,\Delta y} \left\{ \underbrace{\int_{-\Delta x}^{\Delta x} v_x\,(t,\ -\Delta y, 0)\ dt}_{(1)} + \underbrace{\int_{-\Delta y}^{\Delta y} v_y\,(\Delta x, t, 0)\ dt}_{(2)} \right.$$

$$\left. \underbrace{\int_{\Delta x}^{-\Delta x} v_x\,(t,\ \Delta y,\ 0)\ dt}_{(3)} + \underbrace{\int_{\Delta y}^{-\Delta y} v_y\,(\ -\Delta x,\ t,\ 0)\ dt}_{(4)} \right\}.$$

Note that the last two integrals vary from $x = \Delta x$ to $-\Delta x$ and from $y = \Delta y$ to $-\Delta y$. By swapping the integration limits, these integrals get negative signs. By linearizing v_x with respect to the second and v_y with respect to the first variable

$$\begin{aligned} v_x\,(t,\,-\Delta y,\,0) - v_x\,(t,\,\Delta y,\,0) &\approx -\frac{\partial v_x}{\partial y}\,(t,\,0,\,0)\cdot 2\,\Delta y \\ v_y\,(\Delta x,\,t,\,0) - v_y\,(-\Delta x,\,t,\,0) &\approx \frac{\partial v_y}{\partial x}\,(0,\,t,\,0)\cdot 2\,\Delta x \end{aligned}$$

we get for the z-component of the curl

$$(\operatorname{rot}\vec{v})_z \approx \frac{1}{4\,\Delta x\,\Delta y}\left\{\int_{-\Delta x}^{\Delta x} -\frac{\partial v_x}{\partial y}\,(t,0,0)\;2\Delta y\,dt + \int_{-\Delta y}^{\Delta y}\frac{\partial v_y}{\partial x}\,(0,t,0)\;2\Delta x\,dy\right\}.$$

According to the Mean Value Theorem of Integrals (see Volume 2) the term $\frac{\partial}{\partial y}v_x\,(t,\,0,\,0)$ can be excluded, if it is evaluated at a suitable but unknown intermediate point $-\Delta x \le \xi_1 \le \Delta x$. The same applies to $\frac{\partial}{\partial x}v_y\,(0,\,t,\,0)$ for an intermediate point $-\Delta y \le \eta_1 \le \Delta y$. Therefore,

$$(\operatorname{rot}\vec{v})_z \approx \frac{1}{4\,\Delta x\,\Delta y}\left\{-\frac{\partial v_x}{\partial y}\,(\xi_1,0,0)\;2\Delta x\,2\Delta y + \frac{\partial v_y}{\partial x}\,(0,\eta_1,0)\;2\Delta x\,2\Delta y\right\}.$$

For $\Delta x \to 0$ and $\Delta y \to 0$ all points of the surface converge to the origin ($\xi_1 \to 0,\,\eta_1 \to 0$), so that

$$(\text{ rot }\vec{v})_z = \lim_{\Delta x,\Delta y\to 0}\oint_{(C)}\vec{v}\,d\vec{r} = -\left.\frac{\partial v_x}{\partial y}\right|_P + \left.\frac{\partial v_y}{\partial x}\right|_P .$$

Similarly, the first and second components of the curl are obtained by selecting areas in the $(y,\,z)$- and $(x,\,z)$-plane, respectively. In summary

Curl of a Vector Field

Let $\vec{v}(x,y,z) = \begin{pmatrix} v_x(x,y,z) \\ v_y(x,y,z) \\ v_z(x,y,z) \end{pmatrix}$ be a vector field with differentiable components. Then, the **curl** of $\vec{v}(x,y,z)$ at $(x,\,y,\,z)$ is

$$\operatorname{rot}(\vec{v}) = \begin{pmatrix} \dfrac{\partial v_z}{\partial y} - \dfrac{\partial v_y}{\partial z} \\ \dfrac{\partial v_x}{\partial z} - \dfrac{\partial v_z}{\partial x} \\ \dfrac{\partial v_y}{\partial x} - \dfrac{\partial v_x}{\partial y} \end{pmatrix}.$$

The curl $\text{rot}(\vec{v})$ **of the vector field** $\vec{v}$ **indicates the local circulation (vortex density) of** $\vec{v}$ **at point** $P\,(x,\, y,\, z)$. If $\text{rot}(\vec{v}) = \vec{0}$, then the vector field has no local vortices and is called **vortex-free**.

If we replace the integrability condition in the main clause for line integrals (see page 236) by the curl, we get

Statement 20.1:

In a simple contiguous domain, the following three conditions are equivalent:

(1) $\displaystyle\int_{(\mathcal{C})} \vec{v}\, d\vec{r}$ is path independent.

(2) $\displaystyle\oint \vec{v}\, d\vec{r} = 0.$

(3) $\text{rot}\,\vec{v} = \vec{0}.$

Note: When evaluating the curl in Cartesian coordinates, the determinant notation is usually used:

$$\text{rot}(\vec{v}) = \begin{vmatrix} \vec{e}_x & \partial_x & v_x \\ \vec{e}_y & \partial_y & v_y \\ \vec{e}_z & \partial_z & v_z \end{vmatrix}.$$

Example 20.22. We calculate the curl of the vector field $\vec{v} = \begin{pmatrix} x^2\, y \\ -2\, x\, z \\ 2\, y\, z \end{pmatrix}$:

$$\begin{aligned} \text{rot}(\vec{v}) &= \begin{vmatrix} \vec{e}_x & \partial_x & x^2\, y \\ \vec{e}_y & \partial_y & -2\, x\, z \\ \vec{e}_z & \partial_z & 2\, y\, z \end{vmatrix} \\ &= \vec{e}_x\, \left(\partial_y\, 2\, y\, z - \partial_z\, (-2\, x\, z)\right) \\ &\quad -\vec{e}_y\, \left(\partial_x\, 2\, y\, z - \partial_z\, x^2\, y\right) \\ &\quad +\vec{e}_z\, \left(\partial_x\, (-2\, x\, z) - \partial_y\, x^2\, y\right) \\ &= \begin{pmatrix} 2\, z + 2\, x \\ 0 \\ -2\, z - x^2 \end{pmatrix}. \end{aligned}$$

□

Examples 20.23:

① For **radially symmetric force fields** $\vec{k}(\vec{r}) = f(r)\,\vec{r}$ the integrability conditions are always satisfied. Therefore

$$\operatorname{rot}(\vec{k}(\vec{r})) = \vec{0}.$$

② For all vector fields with $\operatorname{rot}(\vec{k}) = \vec{0}$ the integrability conditions are fulfilled. Therefore, there is always a potential function $\Phi(x, y, z)$ with $\vec{k} = \operatorname{grad}\Phi$:

$$\operatorname{rot}(\vec{k}) = \vec{0} \quad \Leftrightarrow \quad \text{There exists a scalar field } \Phi \text{ with } \vec{k} = \operatorname{grad}\Phi.$$

③ Given is the vector field $\vec{k} = \begin{pmatrix} -\frac{y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \\ 0 \end{pmatrix}$. Show that $\operatorname{rot}(\vec{k}) = \vec{0}$. □

Application Example 20.24 (Rotating Bodies).

The velocity of a **rotating rigid body** is $\vec{v} = \vec{\omega} \times \vec{r}$. Here $\vec{\omega}$ is the vector whose direction is parallel to the axis of the curl and whose magnitude indicates the angular velocity.

$$\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \vec{e}_x & \omega_x & x \\ \vec{e}_y & \omega_y & y \\ \vec{e}_z & \omega_z & z \end{vmatrix} = \begin{pmatrix} \omega_y\, z - \omega_z\, y \\ \omega_z\, x - \omega_x\, z \\ \omega_x\, y - \omega_y\, x \end{pmatrix}.$$

Using the determinant notation of the curl

$$\operatorname{rot}(\vec{v}) = \begin{vmatrix} e_x & \partial_x & \omega_y\, z - \omega_z\, y \\ e_y & \partial_y & \omega_z\, x - \omega_x\, z \\ e_z & \partial_z & \omega_x\, y - \omega_y\, x \end{vmatrix}$$

we obtain the x-component

$$(\operatorname{rot}\ \vec{v})_x = \partial_y\ (\omega_x\, y - \omega_y\, x) - \partial_z\ (\omega_z\, x - \omega_x\, z) = 2\,\omega_x,$$

and so on:

$$\operatorname{rot}(\vec{v}) = 2\,\vec{\omega}.$$

If a small sample is placed in the velocity field $\vec{v}$ of a flowing liquid, the angular velocity $\vec{\omega}$ is proportional to the curl $\frac{1}{2}\operatorname{rot}(\vec{v})$. □

20.6 Stokes' Integral Theorem

To explain Stokes' integral theorem on a given plane A with boundary curve $\mathcal{C}$, we divide A into sub-areas $\Delta A_i, \quad i = 1, \ldots, n$. These subdivisions contain points P_i and are bounded by curves $\mathcal{C}_i$.

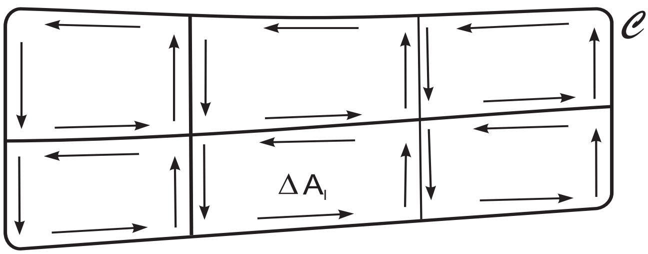


Figure 20.22. Stokes' Integral Theorem

For the circulation of the vector field $\vec{v}$ in the area element ΔA_i we get approximately

$$\mathrm{rot}(\vec{v})\Big|_{P_i} \cdot \Delta\vec{A}_i \approx \oint_{(\mathcal{C}_i)} \vec{v} \cdot d\vec{r}.$$

Looking at the boundaries of two adjacent sub-areas A_k and A_l,

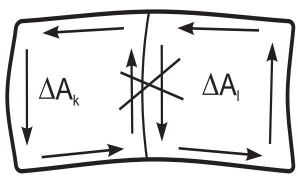


Figure 20.23.

the contributions at the boundaries cancel each other out because they are oriented in opposite directions. Therefore, the circulation in the area A_k and A_l can be determined by balancing the outer boundary curves. By summing all the sub-areas, we obtain the approximation

$$\sum_{k=1}^{n} \mathrm{rot}(\vec{v})\Big|_{P_k} \cdot \Delta\vec{A}_k \approx \oint_{(\mathcal{C})} \vec{v} \cdot d\vec{r}.$$

For $n \to \infty$ (i.e. $\Delta\vec{A}_k \to 0$) the sum $\sum\limits_{k=1}^{n} \operatorname{rot}(\vec{v})\Big|_{P_k} \cdot \Delta\vec{A}_k$ converges to the integral $\iint\limits_{(A)} \operatorname{rot}(\vec{v}) \cdot d\vec{A}$ and thus

$$\iint\limits_{(A)} \operatorname{rot}(\vec{v}) \cdot d\vec{A} = \oint\limits_{(\mathcal{C})} \vec{v} \cdot d\vec{r}.$$

This integral relation is the content of *Stokes' Theorem*.

Stokes' Integral Theorem

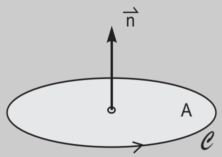


Figure 20.24.

Let A be a surface with boundary curve $\mathcal{C}$ and let $\vec{v}\,(x,\, y,\, z)$ be a vector field defined on A. Then

$$\iint\limits_{(A)} \operatorname{rot}(\vec{v}) \cdot d\vec{A} = \oint\limits_{(\mathcal{C})} \vec{v} \cdot d\vec{r}.$$

The orientation of $\mathcal{C}$ and the surface normal $d\vec{A} = \vec{n}\, dA$ form a right-hand screw; the line element is $d\vec{r} = \vec{r}\,'\,(t)\; dt$ with the tangent vector $\vec{r}\,'\,(t)$ to the curve $\mathcal{C}$.

Example 20.25. In a time-varying magnetic field $\vec{B}$, the law of induction holds

$$U_i = -\iint\limits_{(A)} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}.$$

The voltage between two points in an electric field $\vec{E}$ is given by the line integral along $\mathcal{C}$

$$U = \int_{(\mathcal{C})} \vec{E}\, d\vec{r}.$$

Following Stokes' theorem, we obtain

$$\iint\limits_{(A)} \operatorname{rot}(\vec{E}) \cdot d\vec{A} = -\iint\limits_{(A)} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$$

for any area A. So the identity holds for the integrands:

$$\operatorname{rot}(\vec{E}) = -\frac{\partial}{\partial t}\, \vec{B}.$$

□

20.7 Working with Operators

This section summarizes important operations on scalar and vector fields. A function $\Phi : \mathbb{R}^3 \to \mathbb{R}$ with $\Phi(x, y, z)$ is a **scalar field**. Examples of scalar fields are spatial temperature profiles $T(x, y, z)$ or the charge density $\rho(x, y, z)$. A function $\vec{k} : \mathbb{R}^3 \to \mathbb{R}^3$ with

$$\vec{k}(x, y, z) = \begin{pmatrix} k_1(x, y, z) \\ k_2(x, y, z) \\ k_3(x, y, z) \end{pmatrix}$$

is a **vector field**. Examples of vector fields are the magnetic field $\vec{B}$, the electric field $\vec{E}$ or force fields $\vec{F}$. A vector field $\vec{k}$ is a **potential field (gradient field)**, if a function Φ exists with $\vec{k} = \operatorname{grad}(\Phi)$. In physics, an operator notation has been established for grad, div and rot using the **Nabla** operator:

$$\nabla := \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}.$$

Formally, the Nabla operator is a vector that is always placed to the left of the function to be differentiated. When using the Nabla operator for the multiplications of vectors (scalar multiplication $\alpha \cdot \vec{v}$, scalar product $\vec{v} \cdot \vec{\omega}$, cross product $\vec{v} \times \vec{\omega}$), the following holds

$$\begin{aligned} \operatorname{grad}(\Phi) &= \nabla\Phi \\ \operatorname{div}(\vec{k}) &= \nabla \cdot \vec{k} \\ \operatorname{rot}(\vec{k}) &= \nabla \times \vec{k}. \end{aligned}$$

Summary: Gradient, Divergence, Curl.

Let $\Phi(x, y, z)$ be a scalar and $\vec{k} = \begin{pmatrix} k_1(x,y,z) \\ k_2(x,y,z) \\ k_3(x,y,z) \end{pmatrix}$ a vector field.

(1) $\operatorname{grad}\Phi(x, y, z) = \nabla\Phi(x, y, z) = \begin{pmatrix} \partial_x \Phi(x,y,z) \\ \partial_y \Phi(x,y,z) \\ \partial_z \Phi(x,y,z) \end{pmatrix}$
is the **gradient** of $\Phi(x, y, z)$. The gradient is a vector field.

(2) $\operatorname{div}(\vec{k}) = \nabla \cdot \vec{k} = \partial_x k_1(x, y, z) + \partial_y k_2(x, y, z) + \partial_z k_3(x, y, z)$
is the **divergence** of $\vec{k}$. The divergence is a scalar field.

$$(3)\ \operatorname{rot}(\vec{k}) = \nabla \times \vec{k} = \begin{pmatrix} \partial_y\, k_3(x,y,z) - \partial_z\, k_2(x,y,z) \\ \partial_z\, k_1(x,y,z) - \partial_x\, k_3(x,y,z) \\ \partial_x\, k_2(x,y,z) - \partial_y\, k_1(x,y,z) \end{pmatrix}$$

is the **curl** of the vector field $\vec{k}$. The curl is a vector field.

Example 20.26. Given is the scalar function

$$\Phi(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2+1}}\,.$$

Its gradient is the vector field $\vec{k}$

$$\vec{k}(x,y,z) = \text{ grad } \Phi(x,y,z) = \begin{pmatrix} -\frac{x}{(x^2+y^2+z^2+1)^{\frac{3}{2}}} \\ -\frac{y}{(x^2+y^2+z^2+1)^{\frac{3}{2}}} \\ -\frac{z}{(x^2+y^2+z^2+1)^{\frac{3}{2}}} \end{pmatrix}.$$

The divergence of the vector field $\vec{k}$ leads to a scalar field

$$\operatorname{div}(\vec{k}) = -3\,\frac{1}{(x^2+y^2+z^2+1)^{\frac{5}{2}}}\,.$$

The curl is zero because it is the gradient of a scalar potential. □

The Nabla operator can be used as a normal vector. For example, we can check that $\nabla \cdot \left(\nabla \times \vec{k}\right) = 0$ and $\nabla \times (\nabla\Phi) = \vec{0}$ are valid:

Theorem 20.2:

Let $\vec{k}(x,y,z)$ be a differentiable vector field and $\Phi(x,y,z)$ be doubly differentiable scalar field, then

$$\operatorname{div}(\operatorname{rot}\vec{k}) = 0 \qquad \text{and} \qquad \operatorname{rot}\,(\operatorname{grad}\,\Phi) = \vec{0}.$$

① We check the first identity directly using the Nabla notation:

$$\operatorname{div}(\operatorname{rot}(\vec{k})) = \nabla \cdot (\nabla \times \vec{k}) = \nabla \cdot \begin{pmatrix} \partial_y\, k_3(x,y,z) - \partial_z\, k_2(x,y,z) \\ \partial_z\, k_1(x,y,z) - \partial_x\, k_3(x,y,z) \\ \partial_x\, k_2(x,y,z) - \partial_y\, k_1(x,y,z) \end{pmatrix}$$

$$\begin{aligned} &= \partial_x \left(\partial_y\, k_3(x,y,z) - \partial_z\, k_2(x,y,z)\right) \\ &\quad +\partial_y \left(\partial_z\, k_1(x,y,z) - \partial_x\, k_3(x,y,z)\right) \\ &\quad +\partial_z \left(\partial_x\, k_2(x,y,z) - \partial_y\, k_1(x,y,z)\right) = 0 \end{aligned}$$

② We also check the second identity using the Nabla notation and taking into account that mixed second-order derivatives are equal:

$$\begin{aligned} \operatorname{rot}(\operatorname{grad}(\Phi)) = \nabla \times (\nabla \phi) &= \nabla \times \begin{pmatrix} \partial_x \Phi \\ \partial_y \Phi \\ \partial_z \Phi \end{pmatrix} \\ &= \begin{pmatrix} \partial_y\, \partial_z \Phi - \partial_z\, \partial_y \Phi \\ \partial_z\, \partial_x \Phi - \partial_x\, \partial_z \Phi \\ \partial_x\, \partial_y \Phi - \partial_y\, \partial_x \Phi \end{pmatrix} = \vec{0} \end{aligned}$$ □

The following differentiation rules can also be verified directly, assuming that Φ is always a differentiable scalar field, $\vec{v}$ and $\vec{\omega}$ are differentiable vector fields.

Identities with Differential Operators

a)	$\operatorname{div}(\vec{v} + \vec{\omega})$	$=$	$\operatorname{div}(\vec{v}) + \operatorname{div}(\vec{\omega})$
b)	$\operatorname{rot}(\vec{v} + \vec{\omega})$	$=$	$\operatorname{rot}(\vec{v}) + \operatorname{rot}(\vec{\omega})$
c)	$\operatorname{div}(\Phi\, \vec{v})$	$=$	$\operatorname{grad} \Phi \cdot \vec{v} + \Phi \operatorname{div}(\vec{v})$
d)	$\operatorname{rot}(\Phi\, \vec{v})$	$=$	$\operatorname{grad} \Phi \times \vec{v} + \Phi \operatorname{rot}(\vec{v})$
e)	$\operatorname{div}(\vec{v} \times \vec{\omega})$	$=$	$\vec{\omega} \cdot \operatorname{rot}(\vec{v}) - \vec{v} \cdot \operatorname{rot}(\vec{\omega})$
f)	$\operatorname{rot}(\vec{v} \times \vec{\omega})$	$=$	$\operatorname{div}(\vec{\omega})\, \vec{v} - \operatorname{div}(\vec{v})\, \vec{\omega} + (\vec{\omega} \cdot \nabla)\, \vec{v} - (\vec{v} \cdot \nabla)\, \vec{\omega}$
g)	$\operatorname{grad}(\vec{v} \cdot \vec{\omega})$	$=$	$\vec{\omega} \times \operatorname{rot}(\vec{v}) + \vec{v} \times \operatorname{rot}(\vec{\omega}) + (\vec{\omega} \cdot \nabla)\, \vec{v} + (\vec{v} \cdot \nabla)\, \vec{\omega}$

where

$$\begin{aligned} (\vec{\omega} \cdot \nabla)\, \vec{v} &= (\omega_1\, \partial_x + \omega_2\, \partial_y + \omega_3\, \partial_z) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \\ &= \begin{pmatrix} \omega_1\, \partial_x\, v_1 + \omega_2\, \partial_y\, v_1 + \omega_3\, \partial_z\, v_1 \\ \omega_1\, \partial_x\, v_2 + \omega_2\, \partial_y\, v_2 + \omega_3\, \partial_z\, v_2 \\ \omega_1\, \partial_x\, v_3 + \omega_2\, \partial_y\, v_3 + \omega_3\, \partial_z\, v_3 \end{pmatrix}. \end{aligned}$$

Finally, two important consequences are noted for source-free ($\text{div}(\vec{k}) = 0$) and vortex-free ($\text{rot}(\vec{k}) = \vec{0}$) vector fields.

Theorem 20.3:

(1) The vector field $\vec{k}$ is vortex-free if and only if there exists a scalar field Φ with $\vec{k} = \text{grad}\ \Phi$. Φ is then its scalar potential:

$$\boxed{\text{There exists a } \Phi \text{ with } \quad \vec{k} = \text{grad}\,\Phi \qquad \Leftrightarrow \qquad \text{rot}(\vec{k}) = \vec{0}.}$$

(2) The vector field $\vec{k}$ is source-free if and only if there exists a vector field $\vec{A}$ with $\vec{k} = \text{rot}(\vec{A})$. $\vec{A}$ is then its vector potential:

$$\boxed{\text{There exists an } \vec{A} \text{ with } \quad \vec{k} = \text{rot}(\vec{A}) \qquad \Leftrightarrow \qquad \text{div}(\vec{k}) = 0.}$$

Example 20.27. Given is the scalar potential

$$\Phi(x, y, z) = \arctan\left(\frac{y}{x}\right) \qquad \text{for} \quad x \geq 0.$$

The corresponding vector field $\vec{k}$ is

$$\vec{k} = \ \text{grad}\ \ \Phi = \begin{pmatrix} -\frac{y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \\ 0 \end{pmatrix}.$$

Since $\vec{k}$ is a potential field, $\text{rot}(\vec{k}) = \vec{0}$. The divergence of $\vec{k}$ is

$$\begin{aligned} \text{div}(\vec{k}) &= \partial_x \left(-\frac{y}{x^2+y^2}\right) + \partial_y \left(\frac{x}{x^2+y^2}\right) + \partial_z\ (0) \\ &= \frac{y \cdot 2\,x}{(x^2+y^2)^2} + \frac{-x \cdot 2\,y}{(x^2+y^2)^2} + 0 = 0. \end{aligned}$$

The vector field $\vec{k}$ is free of both divergence and curl. For the scalar potential Φ we get

$$\text{div}\,(\text{grad}\ \Phi) = \nabla \cdot \nabla\phi = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \Phi = \partial_x^2\,\Phi + \partial_y^2\,\Phi + \partial_z^2\,\Phi = 0.$$

We call

$$\mathbf{\Delta}\Phi = \partial_x^2\,\Phi + \partial_y^2\,\Phi + \partial_z^2\,\Phi$$

the **Laplace** operator (see Section 19.4) which plays an important role in electrostatics: All electrostatic problems can be described by

$$\mathbf{\Delta}\Phi = -\frac{\rho}{\varepsilon_0},$$

where $\rho(x, y, z)$ is the charge density and ε_0 is the permittivity (dielectric constant). □

Example 20.28. A scalar function $\Phi\,(x,\,y,\,z)$ is called a **harmonic** function if at each point of the domain $\mathbf{\Delta}\Phi = 0$. The next functions

$$\begin{aligned}
\Phi\,(x,\,y) &= x^2 - y^2\\
\Phi\,(x,\,y) &= \cos x\,\cosh y\\
\Phi\,(x,\,y) &= \ln\left(\sqrt{x^2+y^2}\right)\\
\Phi\,(x,\,y) &= \arctan\left(\frac{y}{x}\right)
\end{aligned}$$

are all harmonic functions which can be checked directly. □

Examples 20.29:

① Given is the force field $\vec{F} = \begin{pmatrix} x\,y \\ x\,z \\ x^2\,y\,z^2 \end{pmatrix}$. Is $\vec{F}$ a gradient field? We compute the curl of $\vec{F}$:

$$\operatorname{rot}(\vec{F}) = \begin{vmatrix} \vec{e}_x & \partial_x & x\,y \\ \vec{e}_y & \partial_y & x\,z \\ \vec{e}_z & \partial_z & x^2\,y\,z^2 \end{vmatrix} = \begin{pmatrix} x^2\,z^2 - x \\ -2\,x\,y\,z^2 \\ z - x \end{pmatrix}.$$

Since $\operatorname{rot}(\vec{F}) \neq \vec{0}$, $\vec{F}$ is **not** a gradient field.

② The force field

$$\vec{F}\,(\vec{r}) = c\,\frac{\vec{r}}{|\vec{r}|^3} = c\,\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

is a radial force field. We can check that

$$\operatorname{rot}(\vec{F}) = \vec{0} \quad \text{and} \quad \operatorname{div}(\vec{F}) = 0.$$

Since $\operatorname{rot}(\vec{F}) = \vec{0}$, we conclude that $\vec{F}$ is a gradient field, i.e. there is a potential Φ with

$$\vec{F} = \text{ grad } \Phi = \begin{pmatrix} \partial_x \Phi \\ \partial_y \Phi \\ \partial_z \Phi \end{pmatrix} \overset{!}{=} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} c\,\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}} \\ c\,\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}} \\ c\,\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \end{pmatrix}.$$

This potential Φ is determined by

$$\Phi(x, y, z) = -\frac{c}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} + const = \frac{c}{|\vec{r}|} + const$$

It is $\operatorname{div}(\vec{F}) = \operatorname{div}(grad\, \Phi) = \Delta\Phi = 0$. So $\Phi(x, y, z)$ is a harmonic function. Physical examples include the **gravitational field** or the **Coulomb field** of an electric charge. □

Example 20.30. Consider a viscous fluid flowing through a pipe of radius r in the y-direction. Its velocity in y-direction is given by

$$\vec{v} = c \cdot \begin{pmatrix} 0 \\ r^2 - x^2 - z^2 \\ 0 \end{pmatrix}$$

We discuss whether the fluid has sources or whether it has a circulation.

To determine the sources, we apply the divergence operator to the vector field.

$$\operatorname{div}(\vec{v}) = \frac{\partial}{\partial x}\,(0) + \frac{\partial}{\partial y}\,\left(r^2 - x^2 - z^2\right) + \frac{\partial}{\partial z}\,(0) = 0.$$

This field has no sources since $\operatorname{div}(\vec{v}) = 0$. For the circulation, we compute the rotation:

$$\operatorname{rot}\,(\vec{v}) = \begin{pmatrix} 2\,c\,z \\ 0 \\ -2\,c\,x \end{pmatrix}$$

This field has a circulation perpendicular to its flow direction in the $(x,\, z)$-plane. □

20.8 Maxwell's Equations

Maxwell's equations are one of the highlights of 19th century mathematical physics. They describe all the phenomena of classical electrodynamics. They are based on four laws of physics:

1. Faraday's Law of Induction

Faraday's law of induction (1831) states that the change of magnetic flux in a loop of conductor over time induces a voltage

$$U_i = -\frac{\partial}{\partial t}\,\Phi = -\frac{\partial}{\partial t} \iint\limits_{(A)} \vec{B}\; d\vec{A}.$$

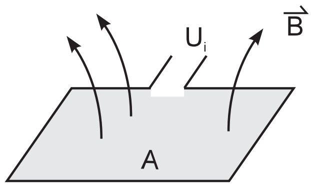


Figure 20.25. Magnetic flux through A

If the area A penetrated by the magnetic field is constant in time, then

$$U_i = -\iint\limits_{(A)} \left(\frac{\partial}{\partial t}\,\vec{B}\right)\, d\vec{A}.$$

The induced voltage is related to an electric field

$$U_i = \oint\limits_{(\mathcal{C})} \vec{E}\, d\vec{r} = \iint\limits_{(A)} \operatorname{rot}(\vec{E}) \cdot d\vec{A}$$

when the Stokes theorem is applied to the area A with boundary curve $\mathcal{C}$.

$$\Rightarrow \iint\limits_{(A)} \operatorname{rot}(\vec{E}) \cdot d\vec{A} = \iint\limits_{(A)} \left(-\frac{\partial}{\partial t}\,\vec{B}\right)\, d\vec{A}.$$

This identity holds for all surfaces A (even for arbitrarily small ones). Using the Mean Value Theorem of Integral Calculus, we conclude that the identity must then already hold for the integrands

$$rot(\vec{E}) = -\frac{\partial}{\partial t}\,\vec{B}.$$

2. The Gaussian Law

The Gaussian law of electrostatics states that the flux of an electric field from a volume V through its surface A is proportional to the total charge Q in the volume: $\oiint\limits_{(A)} \vec{E}\, d\vec{A} \sim Q$. The proportionality constant is $\frac{1}{\varepsilon_0}$ (ε_0:

dielectric constant). If $\rho(x, y, z)$ is the charge density within the volume V, then the total charge is

$$Q = \iiint\limits_{(V)} \rho(x, y, z)\, dV.$$

$$\Rightarrow \frac{1}{\varepsilon_0} Q = \frac{1}{\varepsilon_0} \iiint\limits_{(V)} \rho(x, y, z) = \oiint\limits_{(A)} \vec{E}\, d\vec{A},$$

where A is the surface area enclosing the volume V. According to the Gaussian Integral Theorem

$$\oiint\limits_{(A)} \vec{E}\, d\vec{A} = \iiint\limits_{(V)} \operatorname{div}(\vec{E})\; dV.$$

$$\Rightarrow \iiint\limits_{(V)} \operatorname{div}(\vec{E})\; dV = \frac{1}{\varepsilon_0} \iiint\limits_{(V)} \rho\,(x,\, y,\, z)\; dV.$$

This identity holds for any volume and therefore for the integrands

$$\operatorname{div}(\vec{E}) = \frac{\rho}{\varepsilon_0}\ .$$

The sources of the electric field are the charge densities.

3. Ampere's Law

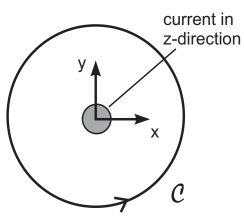


Figure 20.26.
Current carrying conductor

Ampere's law (1825) states that a conductor carrying an electric current induces a magnetic field

$$\oint\limits_{(\mathcal{C})} \vec{B}\, d\vec{r} = \mu_0\, I.$$

If $\vec{j}$ is the current density distribution within an area A defined by its boundary curve $\mathcal{C}$, then the current I is given by

$$I = \iint\limits_{(A)} \vec{j}\, d\vec{A}.$$

Using Stokes' theorem, we get

$$\mu_0 \, I = \mu_0 \iint\limits_{(A)} \vec{j} \; d\vec{A} = \oint\limits_{(\mathcal{C})} \vec{B} \, d\vec{r} = \iint\limits_{(A)} \operatorname{rot}(\vec{B}) \; d\vec{A}$$

$$\Rightarrow \iint\limits_{(A)} \mu_0 \, \vec{j} \, d\vec{A} = \iint\limits_{(A)} \operatorname{rot}(\vec{B}) \; d\vec{A}$$

for all areas A. So we conclude the identity for the integrands

$$\mu_0 \, \vec{j} \; = \; \operatorname{rot}(\vec{B}).$$

4. Source of the Magnetic Field

Since the magnetic field is source-free (there are no magnetic monopoles),

$$\operatorname{div}(\vec{B}) = 0.$$

□

5. The Continuity Equation

Summarizing all four laws gives the four equations

$$\operatorname{div}(\vec{E}) = \frac{\rho}{\varepsilon_0} \tag{1}$$

$$\operatorname{rot}(\vec{B}) = \mu_0 \, \vec{j} \tag{2}$$

$$\operatorname{rot}(\vec{E}) = -\frac{\partial}{\partial t} \, \vec{B} \tag{3}$$

$$\operatorname{div}(\vec{B}) = 0 \tag{4}$$

However, there is a contradiction in these four equations: If the divergence of equation (2) is formed, then

$$\operatorname{div}\left(\mu_0 \, \vec{j}\right) = \operatorname{div}\left(\operatorname{rot}(\vec{B})\right) = 0.$$

The divergence of the current density is zero. This statement contradicts the continuity equation: The time variation of the total charge in a volume, $\frac{\partial}{\partial t} \iiint\limits_{(V)} \rho \, dV$, is equal to the current flowing through its surface

$$- \oiint_{(A)} \vec{j} \, d\vec{A} = - \iiint\limits_{(V)} \operatorname{div}(\vec{j}) \; dV$$

$$\Rightarrow \iiint_{(V)} \frac{\partial}{\partial t}\, \rho \, dV = \iiint_{(V)} -\operatorname{div}(\vec{j}) \, dV.$$

Since this identity holds for all volumes V, it also holds for the integrands

$$\frac{\partial}{\partial t}\, \rho = -\operatorname{div}(\vec{j}) \qquad \textbf{(Continuity Equation).}$$

From the continuity equation we conclude that

$$\operatorname{div}(\vec{j}) = -\frac{\partial}{\partial t}\, \rho \overset{(1)}{=} -\frac{\partial}{\partial t}\, \varepsilon_0 \operatorname{div}(\vec{E}) = \operatorname{div}\left(-\varepsilon_0 \frac{\partial}{\partial t}\, \vec{E}\right)$$

$$\Rightarrow \operatorname{div}\left(\vec{j} + \varepsilon_0 \frac{\partial}{\partial t}\, \vec{E}\right) = 0.$$

$\vec{j} + \varepsilon_0 \frac{\partial}{\partial t} \vec{E}$ is the Maxwell total current and $\varepsilon_0 \frac{\partial}{\partial t} \vec{E}$ is the displacement current. So we replace $\vec{j}$ in equation (2) by

$$\vec{j} + \varepsilon_0 \frac{\partial}{\partial t}\, \vec{E}.$$

Then equations (1)−(4) are free of contradictions and we have the complete set of **Maxwell's equations for the vacuum:**

Maxwell's Equations

Inner Field Equations	Field Generation
$\operatorname{rot}(\vec{E}) = -\frac{\partial}{\partial t}\, \vec{B}$	$\operatorname{div}(\vec{E}) = \frac{\rho}{\varepsilon_0}$
$\operatorname{div}(\vec{B}) = 0$	$\operatorname{rot}(\vec{B}) = \mu_0\, \vec{j} + \varepsilon_0\, \mu_0 \frac{\partial}{\partial t}\, \vec{E}$

Maxwell's equations are four coupled linear *partial* differential equations that describe the relationship between the electromagnetic fields $\vec{E}$ and $\vec{B}$ and their sources, the charge and current densities.

20.9 Problems on Vector Analysis

20.1 Determine the velocity $\vec{v}\,(t)$ and the acceleration $\vec{a}\,(t)$ for the motion of a mass on a

a) circular path $\vec{r}\,(t) = R \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \end{pmatrix}$

b) cycloid path $\vec{r}\,(t) = R \begin{pmatrix} t - \sin t \\ 1 - \cos t \end{pmatrix}$.

20.2 Find the associated potential belonging to the vector fields

a) $\vec{k} = \begin{pmatrix} 2\,x\,y + 4\,x \\ x^2 - 1 \end{pmatrix}$ b) $\vec{k} = \begin{pmatrix} e^y \\ x\,e^y \end{pmatrix}$ c) $\vec{k} = \begin{pmatrix} 3\,x^2\,y + y^3 \\ x^3 + 3\,x\,y^2 \end{pmatrix}$.

20.3 Given is the force field $\vec{F} = \begin{pmatrix} x \\ y \end{pmatrix}$.

a) Show that $\vec{F}$ is conservative.
b) Determine the associated potential.
c) Calculate the work $\int_C \vec{F}\,d\vec{r}$ to move a mass from $P_1\,(1,\,0)$ to $P_2\,(3,\,5)$.

20.4 Check whether the next vector fields are gradient fields and calculate the corresponding potentials if possible

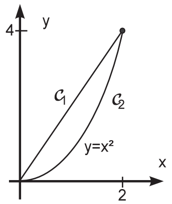
a) $\vec{f}_1 = \begin{pmatrix} y\,z + 1 \\ x\,z + 1 \\ x\,y + 1 \end{pmatrix}$ b) $\vec{f}_2 = \begin{pmatrix} z + y \\ x + z \\ x + y \end{pmatrix}$ c) $\vec{f}_3 = \begin{pmatrix} 2\,x + y \\ x + 2\,y\,z \\ y^2 + 2z \end{pmatrix}$

d) $\vec{f}_4 = \begin{pmatrix} x \\ x\,y \\ x\,y\,z \end{pmatrix}$ e) $\vec{f}_5 = \begin{pmatrix} 1 + y + y\,z \\ x + x\,z \\ x\,y \end{pmatrix}$

20.5 Calculate the line integral

$$\int_C \left(y\,dx + \left(x^2 + x\,y\right)\,dy\right)$$

along the adjacent lines between the points $A\,(0,\,0)$ and $B\,(2,\,4)$.



20.6 Determine the value of the surface integral $\oint\!\!\oint_O \vec{v}\,d\vec{A}$ for

$\vec{v} = \begin{pmatrix} 1 + z^4 \\ 1 + z^4 \\ 1 + x^2\,y^2 \end{pmatrix}$. The parameterization of the surface is given by

$$F: \quad \vec{r}\,(u,\,v) = u\,\vec{e}_x + v\,\vec{e}_y + \frac{1}{4}\,u\,v\,\vec{e}_z = \begin{pmatrix} u \\ v \\ \frac{1}{4}\,u\,v \end{pmatrix}$$

for $-1 \le u \le 1$ and $-1 \le v \le 1$.

20.7 Find the flow of $\vec{v}=\begin{pmatrix} 2\,z \\ x+y \\ 0 \end{pmatrix}$ through the surface of $x^2+y^2+z^2=R^2$.

20.8 Find the divergence of the vector field $\vec{v}=\begin{pmatrix} x^2-y\,z \\ y\,z-y^2 \\ z^2+x\,z \end{pmatrix}$ at the points $(2,\,-1,\,3)$, $(2,\,9,\,4)$ and $(-1,\,1,\,-2)$.

20.9 Specify $f(x,\,y)$ such that $\vec{v}=\begin{pmatrix} x\,y \\ x\,y \\ z\cdot f(x,\,y) \end{pmatrix}$ is source-free.

20.10 Calculate the divergence of the vector fields

a) $\vec{k}_1=\begin{pmatrix} y+z \\ x+2\,x\,y \\ x+2\,z \end{pmatrix}$ b) $\vec{k}_2=\begin{pmatrix} 2\,x^2-y\,z \\ e^z\,y \\ e^z\,x+y \end{pmatrix}$ c) $\vec{k}_3=\dfrac{\vec{r}}{|r|}$

20.11 Determine the curl of $\vec{v}=\begin{pmatrix} x^2\,y \\ -2\,x\,z \\ 2\,y\,z \end{pmatrix}$.

20.12 a) Calculate the curl for the vector fields $\vec{v}$ and $\vec{w}$.
b) Is $\vec{v}$ or $\vec{w}$ vortex-free?
c) Is $\vec{v}$ or $\vec{w}$ source-free?

$$\vec{v}=\begin{pmatrix} y\,z \\ z\,x \\ x\,y \end{pmatrix},\quad \vec{w}=\begin{pmatrix} x+y-z \\ z-x+y \\ y+z-x \end{pmatrix}.$$

20.13 a) Calculate the curl of $\vec{v}=(\vec{a}\cdot\vec{r})\cdot\vec{r}$ with $\vec{a}=\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$.
b) At which points is $\operatorname{rot}(\vec{v})=\vec{0}$?

20.14 Calculate $\operatorname{rot}\left(\operatorname{rot}\begin{pmatrix} z^2 \\ x+y \\ z-x^2-y^2 \end{pmatrix}\right)$.

20.15 Calculate the curl and divergence of the following vector fields

a) $\vec{f}_1=\begin{pmatrix} x\,y \\ x\,z \\ x^2\,y\,z^2 \end{pmatrix}$ b) $\vec{f}_2=\begin{pmatrix} x^2\,y+z \\ y^2\,e^x-z^2 \\ z^2\,x+y^2 \end{pmatrix}$ c) $\vec{f}_3=c\begin{pmatrix} 0 \\ r^2-x^2-y^2 \\ 0 \end{pmatrix}$

20.16 Let $\vec{v}$ be a vector field and Φ a scalar field. Prove that the following identities hold:

$$\operatorname{div}(\operatorname{rot}(\vec{v}))=0 \quad\text{and}\quad \operatorname{rot}(\operatorname{grad}\Phi)=\vec{0}.$$

20.17 Show that $\vec{f} = \begin{pmatrix} 1+y+y\,z \\ x+x\,z \\ x\,y \end{pmatrix}$ is a gradient field.

20.18 Determine the associated scalar and vector potentials from the vector field 20.17.

20.19 Check by differentiating whether

a) $\int \begin{pmatrix} x\cos(y) \\ x\sin(y) \\ x^2+y^2 \end{pmatrix} d\vec{r}$ is path-independent?

b) $\vec{v} = \begin{pmatrix} z\sin^2(y) \\ 2\,x\,z\sin(y)\cos(y) \\ x\sin^2(y) \end{pmatrix}$ is a gradient field?

20.20 Check the Gaussian Integral Theorem of the plane for a circle around the origin of radius 2, where $\vec{v} = \begin{pmatrix} x^2-5\,x\,y+3\,y \\ 6\,x\,y^2-x \end{pmatrix}$.

20.21 Calculate the flow of $\vec{v} = \begin{pmatrix} \frac{x^2-y^2}{z} \\ \frac{x^2+y^2}{z} \\ -(x+y)\ln z \end{pmatrix}$ out of the cube $0 \le x \le 1\,,\ -1 \le y \le 2\,,\ 1 \le z \le 4$.

20.22 Calculate the flow of $\vec{v} = \begin{pmatrix} x^3 \\ z-x^2\,y \\ y-z\,x^2 \end{pmatrix}$ out of the cylinder $0 \le z \le H\,,$ $x^2+y^2 \le R^2$.

20.23 Calculate the flow of $\vec{v} = \begin{pmatrix} 0 \\ y\cos^2(x)+y^3 \\ z\left(\sin^2(x)-3\,y^2\right) \end{pmatrix}$ from the surface of the sphere $x^2+y^2+z^2=4$.

20.24 Verify Stokes' integral theorem for

a) $\vec{v} = \begin{pmatrix} x\,y \\ y\,z \\ x\,z \end{pmatrix}$, $V = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 1\,,\ y \ge 0\,,\ z \ge 0\right\}$.

b) $\vec{v} = \begin{pmatrix} x-z \\ x^3+y\,z \\ -3\,x\,y^2 \end{pmatrix}$, $V = \left\{(x,y,z) \in \mathbb{R}^3 : z = 2-\sqrt{x^2+y^2}\,,\ z \ge 0\right\}$.

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Chapter 21
Splines

To describe technical shapes, series of measurements are often taken. The dependent variable is measured at sample points x_i. A list of pairs corresponding to an unknown function $f(x)$ is then available. Finding an approximate function that allows the calculation of intermediate points is called interpolation.

Apart from conventional interpolation methods such as Lagrange or Newton interpolation from Volume 1, splines are the most commonly used today. These splines can be used to construct the most complicated shapes, such as those used in CAD design, by specifying only a few points.

1

21 Splines

To describe technical shapes, series of measurements are often taken. The dependent variable is measured at sample points x_i. A list of pairs corresponding to an unknown function $f(x)$ is then available. Finding an approximate function that allows the calculation of intermediate points is called interpolation.

Apart from conventional interpolation methods such as Lagrange or Newton interpolation from Volume 1, splines are the most commonly used today. These splines can be used to construct the most complicated shapes, such as those used in CAD design, by specifying only a few points.

When many points are considered, conventional interpolation results in a high degree polynomial: When interpolating $n+1$ points, a polynomial of degree n is required, resulting in a large waviness of the graph. These polynomials fluctuate greatly with small changes in the measured values.

Example 21.1 (With Maple-Worksheet).
Given is the function

$$f(x) = 4\,e^{-x^2}$$

shown in Fig. 21.1. We calculate the 8th degree polynomial that interpolates the 9 measurement points at

$$x = -4,\ -3,\ \ldots,\ 3,\ 4$$

using the methods discussed in Volume 1, Section 4.2.5. The result, together with the 9 pairs and the original function, is shown in Fig. 21.2.

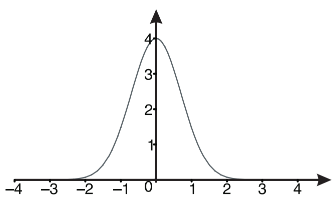

Figure 21.1. Function $f(x) = 4\,e^{-x^2}$

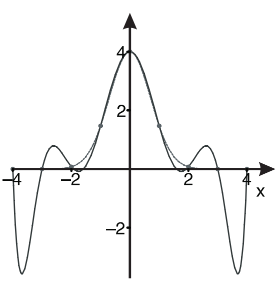

Figure 21.2. Function (dashed) and interpolation polynomial (solid)

It can be seen that the interpolation polynomial fits the given points, but has a large waviness that is not present in the original function. □

To get rid of the waviness with higher-order polynomials, the interval is divided into sub-intervals and a polynomial of small degree is determined for each sub-interval. The simplest way to do this is to connect two consecutive points with a straight line. However, this results in a polygon which is discontinuous in its first derivative: The slope of the computed polygon is not smooth.

It is therefore advisable to combine several polynomials of small degree to form a function that can be differentiated as often as necessary over the entire interval. This method is called *spline interpolation*. Multiple differentiation is necessary to avoid buckling of the overall function at the interfaces.

This chapter describes two commonly used spline methods: For **cubic splines**, a polynomial of degree 3 is chosen between two values and a smooth overall curve is constructed by a suitable transition and appropriate boundary conditions. For the **Bezier splines** a constructive approach to generating a given curve from multiple Bezier segments is described.

21.1 Interpolation with Cubic Splines

With the **cubic splines**, polynomials of degree 3 are selected in each subinterval and the coefficients of the functions are determined such that the spline curve passes smoothly through all points.

In addition, two boundary conditions are selected at the edges. Depending on these conditions, a distinction is made between natural cubic splines, periodic cubic splines, natural parametric splines and periodic parametric cubic splines. The details of the calculation are only presented for the case of natural cubic splines. The procedure is similar for the other cases.

Definition: The *cubic spline function* $\mathcal{S}$ for the values $(x_i,\, y_i)_{i=1,\ldots,n}$ with monotonically arranged nodes x_i at $a = x_1 < x_2 < x_3 \cdots < x_n = b$ is defined by the four properties

(1) The spline function $\mathcal{S}$ can be continuously differentiated twice over the entire interval $[a, b]$.

(2) In each sub-interval $[x_i,\, x_{i+1}]$ for $i = 1, \ldots, n-1$, $\mathcal{S}$ is a polynomial $p_i(x)$ of degree 3.

(3) $\mathcal{S}$ satisfies the interpolation condition: $\mathcal{S}(x_i) = y_i$ for $i = 1, \ldots, n$.

(4) Depending on the spline function (e.g. natural cubic, periodic cubic, etc.) special boundary conditions are defined.

The cubic spline $\mathcal{S}$ results in a cubic polynomial $p_i(x)$ in each sub-interval $[x_i, x_{i+1}]$ for $i = 1, \ldots, n-1$:

Spline function in the sub-interval $[x_i,\, x_{i+1}]$

$$S\Big|_{[x_i,\, x_{i+1}]} : \quad p_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3.$$

For the $n-1$ sub-intervals we obtain $n-1$ polynomials $p_1(x)$, $p_2(x)$, ..., $p_{n-1}(x)$ with their derivatives:

$$\begin{aligned} p_i'(x) &= b_i + 2\,c_i(x - x_i) + 3\,d_i(x - x_i)^2 \\ p_i''(x) &= 2\,c_i + 6\,d_i(x - x_i). \end{aligned}$$

It is necessary to find $4 \cdot (n-1)$ coefficients a_i, b_i, c_i, d_i for $i = 1, \ldots, n-1$. This requires $4n-4$ equations. Using properties (1) and (3) we obtain $4n-6$ conditions:

Conditions for Spline Function

(a) $p_i(x_i) = y_i$ for $i = 1, \ldots, n-1$ and $p_{n-1}(x_n) = y_n$.

(b) $p_i(x_i) = p_{i-1}(x_i)$ for $i = 2, \ldots, n-1$.

(c) $p_i'(x_i) = p_{i-1}'(x_i)$ for $i = 2, \ldots, n-1$.

(d) $p_i''(x_i) = p_{i-1}''(x_i)$ for $i = 2, \ldots, n-1$.

(a) is the interpolation condition, (b) - (d) are the connection conditions for the polynomials. The missing two equations are obtained from property (4) by specifying two boundary conditions (BC).

21.1.1 Natural Cubic Splines

Natural cubic splines define the boundary conditions such that the curvature disappears at the outer boundaries:

$$p_1''(x_1) = 0 \quad\text{and}\quad p_{n-1}''(x_n) = 0.$$

From the definition of the cubic splines and the boundary conditions for the natural cubic splines, all coefficients a_i, b_i, c_i, d_i are determined. To make the formulas clearer, we define for $i = 1, \ldots, n-1$

$$h_i := x_{i+1} - x_i.$$

Conditions (a) to (d) are used to determine the unknown coefficients. Everything can be traced back to the solution of a linear system of equations for $c_1, \ldots, c_{n-1}$. From condition (a) we compute the coefficients a_i directly by

$$\boxed{a_i = y_i} \quad \text{for} \quad i = 1, \ldots, n-1. \tag{1}$$

For formal reasons, we introduce $a_n := y_n$. From condition (d) we see that the coefficients d_i are computed directly using c_i:

$$\boxed{d_i = \frac{1}{3\,h_i}(c_{i+1} - c_i)} \quad \text{for} \quad i = 1, \ldots, n-2. \tag{2}$$

From condition (b) we get b_i, while for d_i the identity (2) is inserted:

$$\boxed{b_i = \frac{1}{h_i}(a_{i+1} - a_i) - \tfrac{h_i}{3}(c_{i+1} + 2\,c_i)} \quad \text{for} \quad i = 1, \ldots, n-2. \tag{3}$$

And from condition (c) follows for $i = 2, \ldots, n-1$:

$$b_i - b_{i-1} = h_{i-1}\,(c_i + c_{i-1}). \tag{4}$$

We insert equation (3) into (4) and obtain for $i = 2, \ldots, n-2$:

$$h_{i-1}\,c_{i-1} + 2\,c_i\,(h_{i-1} + h_i) + h_i\,c_{i+1} = \frac{3\,(a_{i+1} - a_i)}{h_i} - \frac{3\,(a_i - a_{i-1})}{h_{i-1}}. \tag{5}$$

The above equation also holds for $i = n-1$ if we define $c_n := 0$.

The boundary conditions for natural cubic splines are that the curvature disappears at the interval boundaries. This results in two additional equations

$$p_1''(x_1) = c_1 = 0 \quad \text{and} \quad p_{n-1}''(x_n) = 2\,c_{n-1} + 6\,d_{n-1}\,h_{n-1} = 0. \qquad \text{(BC)}$$

We define $c_n = 0$, then equation (2) holds for $i = 1, \dots, n-1$ according to (BC). So the left side of the system of linear equations can be represented by the matrix

$$A = \begin{pmatrix} 2(h_1+h_2) & h_2 & & & \\ h_2 & 2(h_2+h_3) & h_3 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & h_{n-2} \\ & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) \end{pmatrix},$$

Note: The matrix A is a tridiagonal matrix because only the three central diagonals, the major diagonal and the two adjacent minor diagonals, are non-zero. All other coefficients are zero. This system of equations is solved by a modified Gauss method for tridiagonal matrices, the *Thomas algorithm* (see Section 21.3.1).

Summary: Natural Cubic Splines

Given are the values $(x_i, y_i)_{i=1,\dots,n}$ at ordered nodes x_i with $a = x_1 < x_2 < x_3 < \dots < x_n = b$. The natural cubic spline $\mathcal{S}$ consists in each sub-interval of a cubic polynomial

$$p_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

where $i = 1, \dots, n-1$.

For $h_i = x_{i+1} - x_i$ the coefficients are calculated by

$$\begin{aligned} a_i &= y_i \quad \text{for} \quad i = 1, \dots, n, \\ b_i &= \frac{1}{h_i}(a_{i+1} - a_i) - \frac{h_i}{3}(c_{i+1} + 2\,c_i) \quad \text{for} \quad i = 1, \dots, n-1, \\ d_i &= \frac{1}{3\,h_i}(c_{i+1} - c_i) \quad \text{for} \quad i = 1, \dots, n-1, \\ c_1 &= 0, \quad c_n = 0. \end{aligned}$$

The coefficients c_i, $i = 2, \ldots, n-1$, are the solutions of the system of linear equations $A\vec{c} = \vec{r}$ with the matrix

$$A = \begin{pmatrix} 2(h_1+h_2) & h_2 & & & \\ h_2 & 2(h_2+h_3) & h_3 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & h_{n-2} \\ & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) \end{pmatrix},$$

the vector $\vec{c}$ containing the unknown coefficients and the right-hand side $\vec{r}$:

$$\vec{c} = \begin{pmatrix} c_2 \\ c_3 \\ \vdots \\ c_{n-1} \end{pmatrix}, \ \vec{r} = \begin{pmatrix} \frac{3}{h_2}(a_3 - a_2) - \frac{3}{h_1}(a_2 - a_1) \\ \vdots \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \end{pmatrix}.$$

Example 21.2 (With MAPLE-Worksheet). The points $(1,\ 1)$, $(4,\ 3)$, $(5,\ 2)$, $(6,\ 4)$ and $(9,\ 1)$ are given. We search for the natural cubic spline $\mathcal{S}$ through these points.

First, we set up the system of linear equations $A\vec{c} = \vec{r}$. For this, we define

$$h_1 = 3,\ h_2 = 1,\ h_3 = 1,\ h_4 = 3$$

and according to the summary of natural cubic splines we get

$$\begin{pmatrix} 8 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 8 \end{pmatrix} \begin{pmatrix} c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} -5 \\ 9 \\ -9 \end{pmatrix}.$$

The solution is $c_2 = -\frac{59}{60}$, $c_3 = \frac{43}{15}$, $c_4 = -\frac{89}{60}$ and $c_1 = 0$. With $a_1 = y_1 = 1$, $a_2 = y_2 = 3$, $a_3 = y_3 = 2$, $a_4 = y_4 = 4$ we compute

$$b_1 = -\frac{33}{20},\ b_2 = -\frac{13}{10},\ b_3 = \frac{7}{12},\ b_4 = -\frac{59}{30}$$

and

$$d_1 = -\frac{59}{540},\ d_2 = -\frac{77}{60},\ d_3 = -\frac{29}{20},\ d_4 = \frac{89}{540}.$$

The cubic spline $\mathcal{S}$ associated with these coefficients is

$$\mathcal{S} = \begin{cases} -\frac{59}{540}\,x^3 + \frac{59}{180}\,x^2 + \frac{119}{90}\,x - \frac{73}{135} & \text{for } 1 \le x \le 4 \\ \frac{77}{60}\,x^3 - \frac{983}{60}\,x^2 + \frac{409}{6}\,x - \frac{269}{3} & \text{for } 4 \le x \le 5 \\ -\frac{29}{20}\,x^3 + \frac{1477}{60}\,x^2 - \frac{821}{6}\,x + 252 & \text{for } 5 \le x \le 6 \\ \frac{89}{540}\,x^3 - \frac{89}{20}\,x^2 + \frac{1127}{30}\,x - \frac{484}{5} & \text{for } 6 \le x \le 9 \end{cases}.$$

The graph of $\mathcal{S}$ with the given points is shown in Fig. 21.3. □

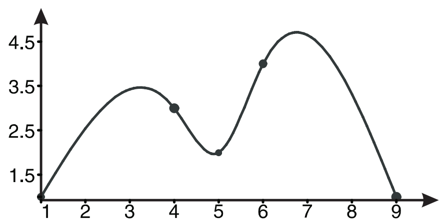

Figure 21.3. Natural cubic spline $\mathcal{S}$ for Example 21.2

Example 21.3 (With MAPLE). Given is the function $f(x) = 4\,e^{-x^2}$ from Example 21.1. We search for the natural cubic spline $\mathcal{S}$ through the points at $x = -4, -3, \ldots 3, 4$ and its graphical representation.

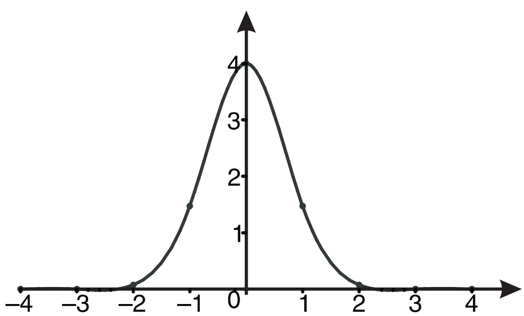

Figure 21.4. Natural cubic spline for Example 21.1

The spline function $\mathcal{S}$ is shown in Fig. 21.4. It can be seen very clearly that the spline function, unlike the interpolation polynomial from Example 21.1, has no waviness and represents the original function quite well. □

21.1.2 Periodic Cubic Splines

Periodic cubic splines are defined for **periodic** boundary conditions:

$$\mathcal{S}(x_1) = \mathcal{S}(x_n),\ S'(x_1) = S'(x_n) \text{ and } S''(x_1) = S''(x_n).$$

It is assumed that the spline $\mathcal{S}$ behaves periodically at the edges, i.e. it has the same function values and the same first and second derivatives. Applying these conditions to the first and last cubic polynomials, we get

$$p_1(x_1) = p_{n-1}(x_n), \quad p_1'(x_1) = p_{n-1}'(x_n) \quad \text{and} \quad p_1''(x_1) = p_{n-1}''(x_n).$$

In the case of periodic cubic splines, we get the equations

$$\begin{aligned} a_1 &= a_{n-1} + b_{n-1}\,h_{n-1} + c_{n-1}\,h_{n-1}^2 + d_{n-1}\,h_{n-1}^3 \\ b_1 &= b_{n-1} + 2\,c_{n-1}\,h_{n-1} + 3\,d_{n-1}\,h_{n-1}^2 \\ 2\,c_1 &= 2c_{n-1} + 6\,d_{n-1}\,h_{n-1} \end{aligned} \tag{BC'}$$

instead of the equations (BC). We isolate b_{n-1} in the first equation and insert the result into the second equation. Then

$$\begin{aligned} b_1 &= \frac{1}{3h_{n-1}}(3\,a_1 - 3\,a_{n-1} + c_{n-1}\,h_{n-1}^2 + 2\,h_{n-1}^2 c_1) \\ b_{n-1} &= \frac{1}{3h_{n-1}}(3\,a_1 - 3\,a_{n-1} - 2\,c_{n-1}\,h_{n-1}^2 - h_{n-1}^2\,c_1) \\ d_{n-1} &= -\frac{1}{3h_{n-1}}(-c_1 + c_{n-1}). \end{aligned}$$

We use the above formulas and modify the system of linear equations for the coefficients c_i as described in the following summary

Periodic Cubic Splines

Given are the values $(x_i,\, y_i)_{i=1,\dots,n}$ with ordered nodes x_i at $a = x_1 < x_2 < x_3 < \cdots < x_n = b$. The periodic cubic spline $\mathcal{S}$ consists in each sub-interval of a cubic polynomial

$$p_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

where $i = 1, \dots, n-1$. With $h_i := x_{i+1} - x_i$ and $h_n := h_1$, the coefficients are given by

$$
\begin{aligned}
a_i &= y_i \quad \text{for} \quad i = 1, \ldots, n-1, \\
a_n &= a_1, \; a_{n+1} := a_2, \\
b_i &= \frac{1}{h_i}(a_{i+1} - a_i) - \frac{h_i}{3}(c_{i+1} + 2\,c_i) \quad \text{for} \quad i = 1, \ldots, n-1, \\
d_i &= \frac{1}{3\,h_i}(c_{i+1} - c_i) \quad \text{for} \quad i = 1, \ldots, n-1, \\
c_1 &= c_n.
\end{aligned}
$$

The coefficients c_i, $i = 2, \ldots, n$ are the solution of the system of linear equations $A\vec{c} = \vec{r}$ with the matrix

$$
A = \begin{pmatrix}
2(h_1 + h_2) & h_2 & & & & h_1 \\
h_2 & 2(h_2 + h_3) & h_3 & & & \\
 & \ddots & \ddots & \ddots & & \\
 & & & \ddots & \ddots & h_{n-1} \\
h_1 & & & & h_{n-1} & 2(h_{n-1} + h_n)
\end{pmatrix},
$$

the vector $\vec{c}$ of the unknown coefficients and the right-hand side $\vec{r}$:

$$
\vec{c} = \begin{pmatrix} c_2 \\ c_3 \\ \vdots \\ c_n \end{pmatrix}, \quad
\vec{r} = \begin{pmatrix}
\frac{3}{h_2}(a_3 - a_2) - \frac{3}{h_1}(a_2 - a_1) \\
\vdots \\
\vdots \\
\frac{3}{h_n}(a_{n+1} - a_n) - \frac{3}{h_{n-1}}(a_n - a_{n-1})
\end{pmatrix}.
$$

Example 21.4 (With Maple). The points (1, 1), (4, 3), (5, 2), (6, 4) and (9, 1) are given from Example 21.2. We look for the periodic cubic spline $\mathcal{S}$ that fits these points.

We will skip the details of the calculation and just present the result of the spline function $\mathcal{S}$

$$
\mathcal{S} = \left\{
\begin{array}{ll}
-\frac{67}{216}\,x^3 + \frac{20}{9}\,x^2 - \frac{283}{72}\,x + \frac{163}{54} & \text{for} \;\; 1 \le x \le 4 \\
\frac{37}{24}\,x^3 - 20\,x^2 + \frac{2039}{24}\,x - \frac{231}{2} & \text{for} \;\; 4 \le x \le 5 \\
-\frac{41}{24}\,x^3 + \frac{115}{4}\,x^2 - \frac{3811}{24}\,x + \frac{1163}{4} & \text{for} \;\; 5 \le x \le 6 \\
\frac{79}{216}\,x^3 - \frac{103}{12}\,x^2 + \frac{1565}{24}\,x - \frac{629}{4} & \text{for} \;\; 6 \le x \le 9
\end{array}
\right| .
$$

Fig. 21.5 displays the given points together with the graph of $\mathcal{S}$.

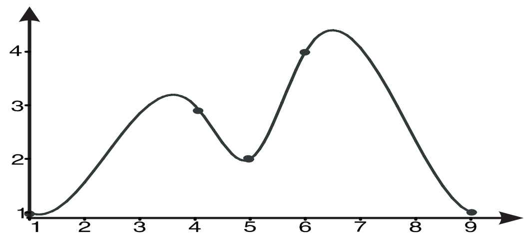

Figure 21.5. Periodic cubic spline for Example 21.4

It can be seen that the qualitative behavior of the periodic cubic spline $\mathcal{S}$ within the interval $[1, 9]$ is the same as for the natural cubic spline. However, due to the modified boundary conditions, we observe deviations near the interval boundaries. □

21.1.3 Natural Parametric Cubic Splines

The natural parametric cubic splines are particularly useful for approximating a curve in space that does not have a monotonous distribution of points, such as spirals or (non-smooth) closed curves.

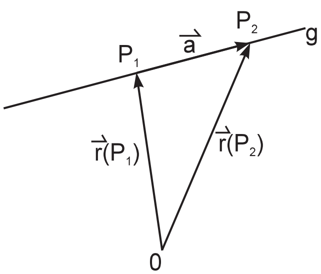

Figure 21.6. Line g through P_1 and P_2

The idea of the procedure is based on the following geometrical consideration: The line through the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is described in parameter form by

$$g: \quad \vec{r}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{r}(P_1) + \lambda\,\overrightarrow{P_1P_2}.$$

For $\lambda = 0$ the line starts at point P_1 and for $\lambda = 1$ it ends at point P_2. Alternatively, we can use the notation

$$g: \quad \vec{r}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{r}(P_1) + (t - t_1)\,\frac{1}{|\overrightarrow{P_1P_2}|}\,\overrightarrow{P_1P_2},$$

where $|\overrightarrow{P_1P_2}|$ is given by

$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

The line g gives a linear connection between points P_1 and P_2. It starts at point P_1 for $t = t_1$ and ends at point P_2 for $t = t_1 + |\overrightarrow{P_1P_2}|$.

If we choose a cubic approach, instead of the linear approach in the parameter t, we define

$$\vec{r}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{r}(P_1) + (t - t_1)\, k_1 \frac{1}{|\overrightarrow{P_1P_2}|} \overrightarrow{P_1P_2} + (t - t_1)^2\, k_2 \frac{1}{|\overrightarrow{P_1P_2}|} \overrightarrow{P_1P_2} + (t - t_1)^3\, k_3 \frac{1}{|\overrightarrow{P_1P_2}|} \overrightarrow{P_1P_2}.$$

With a suitable choice of k_1, k_2, k_3 we get a connection of P_1 with P_2 through a cubic curve.

Motivated by these considerations of a non-linear connection between the points P_1 and P_2, the *parametric* spline $\mathcal{S}$ is composed of two components $\mathcal{S}_x$ and $\mathcal{S}_y$, where the components for $t \in [t_i,\, t_{i+1}]$ are defined by

$$\mathcal{S}_x(t) = p_{ix}(t) = a_{ix} + b_{ix}(t - t_i) + c_{ix}(t - t_i)^2 + d_{ix}(t - t_i)^3$$
$$S_y(t) = p_{iy}(t) = a_{iy} + b_{iy}(t - t_i) + c_{iy}(t - t_i)^2 + d_{iy}(t - t_i)^3.$$

Here, we define

$$t_0 = 0; \quad t_{i+1} = t_i + \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2} \;\text{ for }\; i = 1, \ldots, n - 1.$$

The x-component of the spline $\mathcal{S}_x$ is obtained by using the equation system for natural cubic splines, replacing x_i with t_i and y_i with x_i. The y-component $\mathcal{S}_y$ results from replacing x_i with t_i, leaving y_i. Geometrically, t_n is the length of the polygon through the points $P_1, \ldots, P_n$.

Example 21.5 (With Maple). A spiral is defined by the following 13 points: $(0,0)$, $(3,4)$, $(2,6)$, $(0,4)$, $(4,0)$, $(7,4)$, $(6,6)$, $(4,4)$, $(8,0)$, $(11,4)$, $(10,6)$, $(8,4)$, $(12,0)$. We look for the natural **parametric** cubic spline $\mathcal{S}$ that fits these points.

Although the method of calculating the spline interpolation curve is clear and straightforward, the amount of work involved is too great to describe here. Instead, we have implemented the algorithm in a Maple-procedure.

The procedure **ParaCubicSpline** determines this spline and displays the values along with the graph of $\mathcal{S}$ (see Fig. 21.7). □

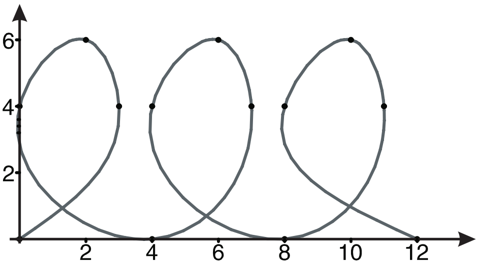

Figure 21.7. Parametric cubic spline for a spiral

21.1.4 Periodic Parametric Cubic Splines

The periodic parametric splines are ideal for smooth closed curves such as contours. The spline $\mathcal{S}$ again has two components, S_x and S_y. The basic system of equations is a *periodic* cubic spline, the exchange of variables is analogous to the natural parametric spline.

Example 21.6 (With MAPLE**).** To show the behavior of a periodic parametric cubic spline, we select the points $\underline{(5,1)}$, $(8,3)$, $(9,6)$, $(7,7)$, $(6,5)$, $8(,4)$, $(10,6)$, $(7,9)$, $(2,4)$, $\underline{(5,1)}$. In this example, the last and first points of the list are the same.

Figure 21.8 shows the **parametric** cubic spline $\mathcal{S}$ as a closed curve for the selected points. The calculation uses the MAPLE-procedure **PeriodicParaSpline** implemented in the corresponding worksheet. This procedure calculates and graphically displays the spline curve $\mathcal{S}$. □

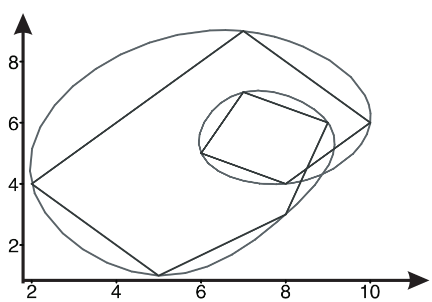

Figure 21.8. Periodic parametric spline

21.2 Bezier Splines

Bezier spline curves and Bezier spline surfaces play an important role in the modelling of free-form geometries in CAD/CAM design. They were discovered independently by the engineers Bezier (Renault) and de Casteljau (Citroen).

Bezier and de Casteljau introduced a basic element - the Bezier segment - which can be used to model virtually any geometric shape. A Bezier curve or Bezier surface is made up of many Bezier segments. A constructive algorithm is presented in Section 21.2.1 and an algebraic method for constructing of Bezier segments is given in Section 21.2.2.

Note: Although the method of calculating the Bezier segments is clear and straightforward, the computational time involved is too great to describe here. Instead, we have implemented the algorithms in MAPLE-procedures.

The great advantage of the Bezier splines is their data reduction, i.e. with a few points (Bezier points) a curve or surface consisting of hundreds or even thousands of points can easily be generated. We will see that the calculation of spline curves involves the solution of large linear systems of equations. The Section 21.3 deals specifically with solving such large systems of linear equations.

21.2.1 Creating a Bezier Segment

The Bezier segment is constructed from four points. The construction of a segment is realized by a constructive algorithm (de Casteljau algorithm), which is illustrated in Fig. 21.9.

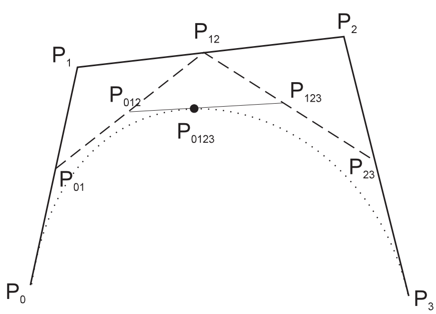

Figure 21.9. Constructing a point on the Bezier segment

To construct a Bezier segment, four Bezier **points** (P_0 to P_3) are specified. P_0 is the start point and P_3 is the end point of the Bezier segment; P_1 and P_2 define the shape of the curve.

In the first step of the procedure, the three lines connecting the points P_0 to P_3 are divided by a fixed ratio t. This creates three new points P_{01}, P_{12}, P_{23}. There are two connecting lines to these three points, again divided by the same ratio. This defines two points P_{012}, P_{123} and only one line. Dividing by t again gives P_{0123} which is a point of the Bezier segment shown in Fig. 21.9.

The ratio t varies from 0 to 1, for $t = 0$ we get the point P_0 and for $t = 1$ the point P_3. The number of divisions of the lines is called an iteration. So 3 iterations are needed to find a point on the segment with 4 starting points. This procedure is repeated for other values of t. The result are points on the Bezier segment. The resolution depends on the number of t-parameters chosen.

Example 21.7 (With MAPLE). The Bezier segment is searched for the points $P_0(1,1)$, $P_1(1.5,5)$, $P_2(2.5,3)$, $P_3(3.5,1)$. The constructive method according to de Casteljau gives a curve as shown in Fig. 21.10. The worksheet used is structured such that Bezier points other than the default ones can be specified.

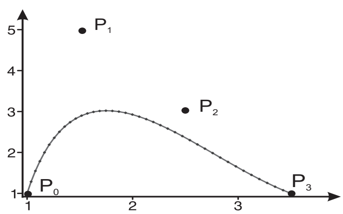

Figure 21.10. Bezier segment to points $P_0, \ldots, P_4$

Visualization. To illustrate the influence of the position of point P_2 on the overall curve, we choose $P_0(1,1)$, $P_1(1.5,5)$, $\underline{P_2(x,3)}$, $P_3(3.5,1)$. P_0, P_1, P_3 are fixed but the x-coordinate of P_2 varies from $x = 0$ to $x = 6$. In Fig. 21.11 all curves have been superimposed to show the influence of the point P_2 on the behavior of the whole curve: If the x-value of P_2 is smaller than P_1, a peaked curve is obtained; for larger x-values the curve becomes more bulbous.

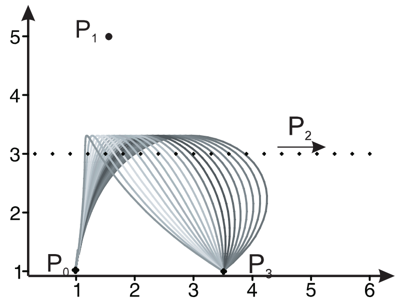

Figure 21.11. Bezier segments depending on P_2

21.2.2 Parameterization of a Bezier Segment

An explicit parametric representation of Bezier segments can be described using the Bernstein polynomials. The Bernstein polynomials of degree n are defined for $i = 0, \dots, n$ by

$$B_i^{(n)}(t) := \binom{n}{i} t^i (1-t)^{n-i}.$$

There are $n+1$ Bernstein polynomials of degree n. $\binom{n}{i}$ are the binomial coefficients: $\binom{n}{i} = \frac{n!}{i!\,(n-i)!}$.

Example 21.8 (Bernstein polynomials, with MAPLE**-Worksheet).** We define all third degree Bernstein polynomials and show their graphs.

The following list contains all third degree Bernstein polynomials:

$$\begin{aligned} B_0^{(3)}(t) &= (1-t)^3 = 1 - 3\,t + 3\,t^2 - t^3 \\ B_1^{(3)}(t) &= 3\,t\,(1-t)^2 = 3\,t - 6\,t^2 + 3\,t^3 \\ B_2^{(3)}(t) &= 3\,t^2\,(1-t) = 3\,t^2 - 3\,t^3 \\ B_3^{(3)}(t) &= t^3. \end{aligned}$$

The Fig. 21.12 shows these four polynomials for $t \in [0,1]$. □

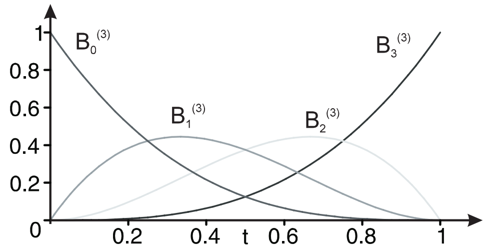

Figure 21.12. Bernstein polynomials of degree three

Using the Bernstein polynomials the Bezier segments can be described by a simple formula. Using the polynomials from Example 21.8 we obtain the Bezier segment with 4 points:

$$\vec{\mathrm{r}}(t) = \vec{r}(P_0)\,(1-t)^3 + \vec{r}(P_1)\,3\,(1-t)^2\,t + \vec{r}(P_2)\,3\,(1-t)\,t^2 + \vec{r}(P_3)\,t^3\,. \qquad (1)$$

The vectors $\vec{r}(P_0)$, $\vec{r}(P_1)$, $\vec{r}(P_2)$ and $\vec{r}(P_3)$ are the position vectors of the Bezier points P_0 to P_3. The result $\vec{r}(t)$ is a parameterization of the segment. By varying t between 0 and 1, a complete Bezier segment of arbitrary precision is obtained.

Summary: Bezier Segment

The Bezier segment defined by the points P_0, P_1, P_2, P_3 is given by the parametric representation

$$\vec{\mathrm{r}}(t) = \sum_{i=0}^{3} \vec{r}(P_i)\ \mathrm{B}_i^{(3)}(t) \qquad \text{for} \quad t \in [0,1], \qquad (2)$$

where $B_i^{(3)}(t) = \begin{pmatrix} 3 \\ i \end{pmatrix} t^i\,(1-t)^{3-i}$ are the Bernstein polynomials of degree 3 $(i = 0, 1, 2, 3)$.

Example 21.9 (With Maple-Worksheet). Find the description of the Bezier segment by the points $P_0(0,0)$, $P_1(1,5)$, $P_2(2,2)$, $P_3(6,0)$ using the Bernstein polynomials and the graphical representation of the segment.

With the position vectors $\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\5\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}, \begin{pmatrix}6\\0\end{pmatrix}$ the parameterization is

$$\vec{r}(t) = \begin{pmatrix} 3\,(1-t)^2\,t + 6\,(1-t)\,t^2 + 6\,t^3 \\ 15\,(1-t)^2\,t + 6\,(1-t)\,t^2 \end{pmatrix} = \begin{pmatrix} 3\,t^3 + 3\,t \\ 9\,t^3 - 24\,t^2 + 15\,t \end{pmatrix}.$$

The Bezier segment is shown in Fig. 21.13. □

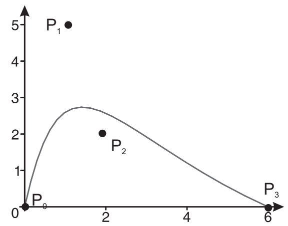

Figure 21.13. Bezier segment generated using Bernstein polynomials

Interpretation: The degree 3 of the Bezier segment in equation (2) is equal to the number of Casteljau iterations. With equation (1) it follows that $\vec{r}(0) = \vec{r}(P_0)$ and $\frac{d}{dt}\,\vec{r}(0) = 3\,\overrightarrow{P_0P_1}$. So for $t = 0$ the resulting point on the Bezier segment corresponds to the first point P_0 and the tangent at P_0 corresponds to the line between P_0 and P_1. Similarly, it can be shown that the segment at P_3 is the tangent to the line defined by P_2 and P_3. Segments of degree 3 are also known as cubic Bezier segments.

Higher Order Bezier Segments

Running the de Casteljau algorithm with $n+1$ points and n iterations instead of 4 points and 3 iterations produces a Bezier segment of degree n. Generalizing to equation (2) we obtain

$$\vec{r}(t) := \sum_{i=0}^{n} \vec{r}(P_i)\,\mathrm{B}_i^{(n)}(t). \tag{3}$$

The Bezier coefficients $\vec{r}(P_i)$ are again the position vectors of the points P_i and $\mathrm{B}_i^{(n)}(t)$ (for $i = 0, \ldots, n$) are the Bernstein polynomials of degree n.

Example 21.10 (With Maple-Worksheet). Find the description of the Bezier segment at points $P_0(1,1)$, $P_1(1.5,5)$, $P_2(2.5,3)$, $P_3(3.5,1)$, $P_4(4.5,6)$ using the 4th degree Bernstein polynomials.

With the position vectors $\vec{r}(P_0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{r}(P_1) = \begin{pmatrix} 1.5 \\ 5 \end{pmatrix}$, $\vec{r}(P_2) = \begin{pmatrix} 2.5 \\ 3 \end{pmatrix}$, $\vec{r}(P_3) = \begin{pmatrix} 3.5 \\ 1 \end{pmatrix}$, $\vec{r}(P_4) = \begin{pmatrix} 4.5 \\ 6 \end{pmatrix}$ and the 4th degree Bernstein polynomials and the 4 points

$$\begin{aligned} \vec{r}(t) = \; & \vec{r}(P_0)\,(1-t)^4 + 4\vec{r}(P_1)\,(1-t)^3\,t \\ & + 6\vec{r}(P_2)\,(1-t)^2\,t^2 + 4\vec{r}(P_3)\,(1-t)\,t^3 + \vec{r}(P_4)\,t^4 \end{aligned}$$

we get the parameterization

$$\begin{aligned} \vec{r}(t) &= \begin{pmatrix} (1-t)^4 + 6\,(1-t)^3\,t + 15\,(1-t)^2\,t^2 + 14\,(1-t)\,t^3 + 4.5\,t^4 \\ (1-t)^4 + 20\,(1-t)^3\,t + 18\,(1-t)^2\,t^2 + 4\,(1-t)\,t^3 + 6\,t^4 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}\,t^4 - 2\,t^3 + 2\,t^2 + 2\,t + 1 \\ t^4 + 24\,t^3 - 36\,t^2 + 16\,t + 1 \end{pmatrix}. \end{aligned}$$

The Bezier segment is shown in Fig. 21.14. □

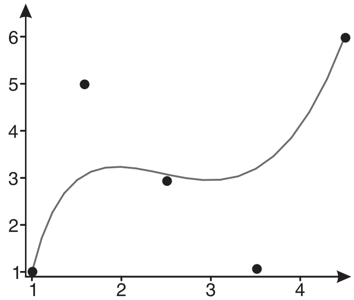

Figure 21.14. 4th order Bezier segment

Visualization with Maple. The Bernstein polynomial approach has been implemented in Maple-worksheets. Depending on the point P_2 the animation gives the same results as the constructive method. However, the parameterized approach offers a significant time advantage over the constructive solution. The worksheet 3BSeg.mws contains several examples of higher order Bezier segments and additional animations.

21.2.3 Bezier Curves

To display complicated curves, one Bezier **segment** is usually not sufficient, even at a higher level. Therefore, several simple Bezier segments of lower degree (e.g. $n = 3$) are combined. The resulting curves are called Bezier **curves**.

When modelling shapes, however, spline curves with smooth transitions play a more important role. The condition for a smooth transition is that a Bezier curve consisting of n-th degree segments is $(n-1)$ times continuously differentiable at the nodes. We will use the notation (C^{n-1}-continuous) for a point where the spline curve is $(n-1)$ times continuously differentiable. This means that a cubic Bezier spline curve ($n = 3$) must be C^2-continuous at its transitions.

The following list gives the geometric conditions for the C^0-, C^1- and C^2-continuity:

C^0 denotes a continuous junction, usually with a kink. The last point of the first segment $P_3^{(k)}$ coincides with the first point of the second segment $P_0^{(k+1)}$:

$$P_3{}^{(k)} = P_0{}^{(k+1)}$$

for $k = 1, 2, \ldots, l-1$, where k is the number of the individual segment and l is the total number of segments.

C^1 denotes a continuously differentiable transition, i.e. the spline curve has the same gradient at the contact point from the right as from the left. More specifically, this means that

(1) the C^0-continuity must be satisfied;

(2) the penultimate point of the first segment, the two connecting points and the second point of the second segment lie on a straight line (collinear) (see Fig. 21.15).

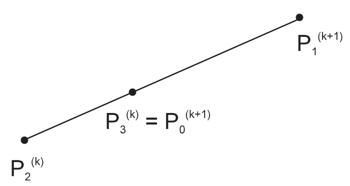

Figure 21.15. Construction of a C^1-continuous Bezier curve

C^2 denotes a continuous differentiable transition. The curvature of the curve is preserved at the transition points. This means that the C^1-continuity is satisfied and there is a point C_k with the following properties (see Fig. 21.16):

(1) C_k is collinear with the points $P_1{}^{(k)}$ and $P_2{}^{(k)}$;

(2) $P_2^{(k)}$ divides the distance $P_1^{(k)}C_k$ by the ratio v_k;

(3) C_k is collinear with the points $P_1{}^{(k+1)}$ and $P_2{}^{(k+1)}$;

(4) $P_1^{(k+1)}$ divides the distance $C_k P_2^{(k+1)}$ by the ratio v_k.

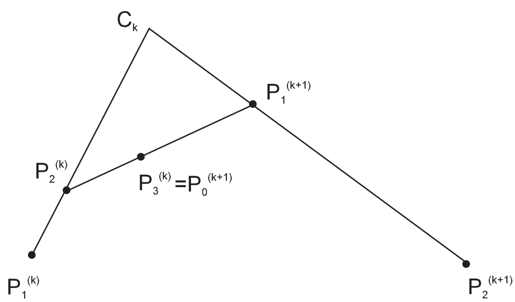

Figure 21.16. Construction of a C^2-continuous Bezier curve

Example 21.11 (C^0-continuous curve, with MAPLE-Worksheet). Find the C^0-continuous Bezier curve to the given Bezier points for two Bezier segments:

$$P_0^{(1)} = (0,0),\ P_1^{(1)} = (1,-7),\ P_2^{(1)} = (6,-2),\ P_3^{(1)} = (10,0);$$
$$P_1^{(2)} = (15,-4),\ P_2^{(2)} = (22,4),\ P_3^{(2)} = (24,0).$$

For the C^0-continuity we define $P_0^{(2)} = P_3^{(1)}$ and determine the first and second segments separately by third-order Bernstein polynomials.

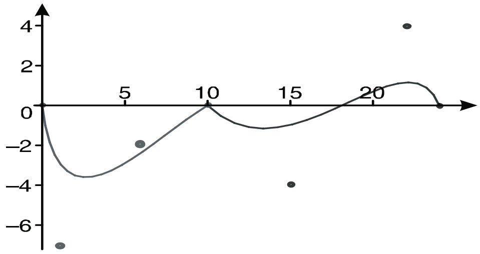

Figure 21.17. C^0-continuous Bezier curve

The Fig. 21.17 shows the Bezier curve together with the Bezier points. The bend at the transition from the first to the second segment can be clearly seen at the point $P_3^{(1)}$. □

Example 21.12 (C^1-continuous curve, with MAPLE-**Worksheet).** Find the C^1-continuous Bezier curve to the given Bezier points for two Bezier segments:

$$P_0^{(1)} = (0,0),\ P_1^{(1)} = (1,-7),\ P_2^{(1)} = (6,-2),\ P_3^{(1)} = (10,0);$$
$$P_2^{(2)} = (14,4),\ P_3^{(2)} = (24,0).$$

To make the transition between the first and second segments continuously differentiable, two conditions must be met:

(1) $P_0^{(2)} = P_3^{(1)}$ and

(2) the points $P_2^{(1)}$, $P_3^{(1)}$ and $P_1^{(2)}$ lie on a straight line. To do this, for example, we set $\lambda = 7$ and define

$$P_1^{(2)} = P_2^{(1)} + \frac{\lambda}{|P_3^{(1)} - P_2^{(1)}|}(P_3^{(1)} - P_2^{(1)}).$$

This definition of the points $P_0^{(2)}$ and $P_1^{(2)}$ determines the first and second segments:

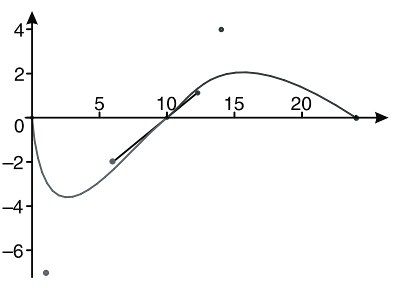

Figure 21.18. C^1-continuous Bezier curve

The Fig. 21.18 shows the connection between points $P_2^{(1)}$ and $P_1^{(2)}$. The difference between the first and second segments is clearly visible. The slope of the curve at point $P_3^{(1)}$ is equal to the slope of this straight line, i.e. the straight line is the tangent to the Bezier curve at the transition point. However, we can also see at point $P_3^{(1)}$ that the curvatures from the left and right are different: The curvature of the left curve is negative (right curvature), while the curvature of the right curve is positive (left curvature). □

To get the same curvature from both sides, it is necessary to switch to the C^2-continuity, as the next example shows:

Example 21.13 (C^2-continuous curve, with MAPLE-**Worksheet).** Find the C^2-continuous Bezier curve to the given Bezier points for two Bezier segments:

$$P_0^{(1)} = (0,0),\ P_1^{(1)} = (1,-7) \ \text{ and } \ P_2^{(2)} = (12,4),\ P_3^{(2)} = (24,0).$$

This example constructs a Bezier curve that is twice continuously differentiable at its interface. According to the C^2-continuity, we need to define a point C and a division ratio v. With these choices we define two additional points for each segment: $P_2^{(1)}$, $P_3^{(1)}$ and $P_0^{(2)}$ and $P_1^{(2)}$ are constructed from them. We choose $C = (20, 20)$, the division ratio $v = 0.7$ and define

$$P_2^{(1)} = P_1^{(1)} + \frac{v}{|C - P_1^{(1)}|}(C - P_1^{(1)}),$$

$$P_1^{(2)} = P_2^{(2)} + \frac{v}{|C - P_2^{(2)}|}(C - P_2^{(2)}),$$

$$P_3^{(1)} = P_2^{(1)} + v(P_1^{(2)} - P_2^{(1)}),$$

$$P_0^{(2)} = P_3^{(1)}.$$

With the calculated Bezier points

$$P_2^{(1)} = (1.40,\ -6.42),\ P_1^{(2)} = (13.75,\ 3.34),$$
$$P_3^{(1)} = (10.04,\ 0.41),\ P_0^{(2)} = (10.04,\ 0.41),$$

we determine the two Bezier segments and draw the Bezier curve:

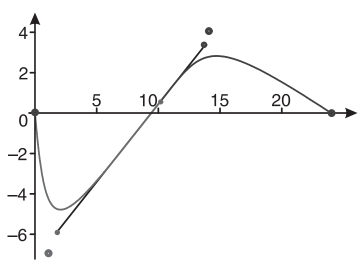

Figure 21.19. C^2-continuous Bezier curve

The Fig. 21.19 shows that the slope of the left curve corresponds qualitatively to the right curve at point $P_3^{(1)}$, as does the curvature. □

21.2.4 Creating Bezier Surfaces

The creation of Bezier surfaces is similar to the parametric creation of a spline curve. The starting point is again a base segment defined by adding two Bernstein polynomials. This results in a vector $\vec{r}$ which depends on two parameters (u, v). Both parameters vary in the interval $[0, 1]$. To define a surface segment $(n+1) \cdot (m+1)$ Bezier points are needed, i.e. for a bicubic surface segment $(n = 3, m = 3)$, we need 16 Bezier points.

Equation (5) gives the general definition of a Bezier surface segment.

$$\vec{\mathrm{r}}(u,\, v) = \sum_{i=0}^{n} \left(\sum_{j=0}^{m} P_{ij}\, \mathrm{B}_i^{(n)}(u)\, \mathrm{B}_j^{(m)}(v) \right) \qquad (5)$$

Visualization with MAPLE**:** The creation of the Bezier surface segments is realized in the worksheet (5B3dSeg.mws).

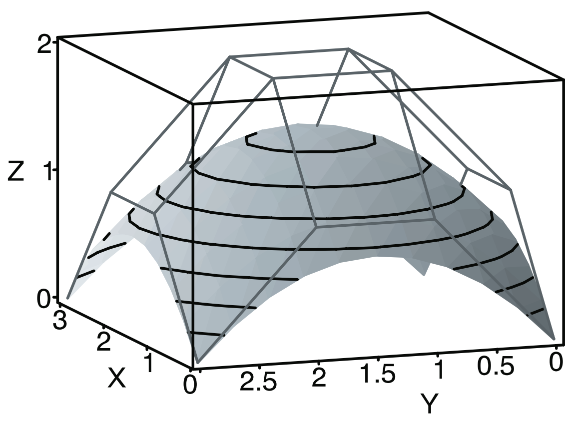

Figure 21.20. Construction of a Bezier surface

This worksheet shows some Bezier surfaces and provides animations to show the influence of the Bezier points on the surface segment. As with Bezier curves, Bezier surfaces are created by connecting individual surface segments. Similar continuity conditions apply to the connections.

21.3 Solving Systems of Linear Equations

Calculating the spline curves leads to large systems of linear equations that must be solved to determine the unknown values. These linear systems are sparsely populated because very few elements of the matrix are non-zero. Therefore, the efficient solution of sparse systems of linear equations is an important task.

In the case of simple matrix structures, such as those arising from the spline problem, special variants of the Gaussian method play an important role. The next two subsections describe the Thomas algorithm for tridiagonal matrices and the Cholesky method for symmetric positive definite matrices.

21.3.1 Thomas Algorithm

We return to the summary of the spline problem. The calculation of the coefficients of the spline function is related to the solution of a system of linear equations with a tridiagonal matrix $\mathbf{T}$. In the description of the algorithm, we start with a system of linear equations

$$\mathbf{T}\,\vec{x} = \vec{d},$$

where the matrix on the left side is a *tridiagonal matrix*. This is why the system is also called a *tridiagonal equation system*. The solution of a system of linear equations is done by the Gaussian algorithm, which leads to a particularly simple method, the *tridiagonal* or *Thomas algorithm*.

To describe the Thomas algorithm, we assume that T is given by

$$T := \left(\begin{array}{cccccc|c} b_1 & c_1 & 0 & \dots & 0 & 0 & d_1 \\ a_2 & b_2 & c_2 & \dots & 0 & 0 & d_2 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a_{n-1} & b_{n-1} & c_{n-1} & d_{n-1} \\ 0 & 0 & \dots & 0 & a_n & b_n & d_n \end{array}\right),$$

where the diagonal elements $b_i \neq 0$ for all $i = 1 \dots n$. Then the Gauss elimination can be formulated explicitly to solve the system of linear equations:

Thomas Algorithm

Given is the tridiagonal matrix T from above. We define

$$\left.\begin{aligned} q &= \tfrac{a_i}{b_{i-1}} \\ b_i' &= b_i - q c_{i-1} \\ d_i' &= d_i - q d_{i-1} \end{aligned}\right\} i = 2, \ldots, n\,.$$

The backward substitutions give the solution

$$x_n = \frac{d_n'}{b_n'}$$
$$x_i = \frac{d_i' - c_i x_{i+1}}{b_i'} \qquad i = n-1, \ldots, 1\,.$$

Example 21.14 (With MAPLE**-Worksheet).** We are looking for the solution to $\mathbf{T}\,\vec{x} = \vec{d}$ with

$$\mathbf{T} = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix} \quad \text{and} \quad \vec{d} = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}.$$

The **Thomas** algorithm is implemented in a MAPLE procedure and is used to solve the systems of linear equations:

$$x_1 = \frac{5}{2},\ x_2 = 4,\ x_3 = \frac{9}{2},\ x_4 = 4,\ x_5 = \frac{5}{2}.$$ □

21.3.2 The Cholesky Method

Like the Gaussian algorithm, the Cholesky method is a direct method for solving systems of linear equations with a symmetric *positive definite* matrix, such as those encountered when solving boundary value problems for partial differential equations.

Before we discuss the decomposition of the matrix $\mathbf{A}$, some criteria for positive definite matrices are given below. An important property is its diagonal dominance. This notion is descriptively clear, but we will give a precise definition.

Definition: Let $\mathbf{A}$ be an $(n \times n)$ matrix.

(1) The matrix $\mathbf{A}$ is called *diagonal dominant* if for all $i = 1, \ldots, n$ it holds: $|a_{ii}| \geq \sum_{k=1,k\neq i} |a_{ik}|$; at least for one i the greater sign must hold.

(2) The matrix $\mathbf{A}$ is called *strictly diagonal dominant* if for all $i = 1, \ldots, n$ it holds: $|a_{ii}| > \sum_{k=1,k\neq i} |a_{ik}|$.

Criteria for Positive Definite Matrices

Let $\mathbf{A}$ be a symmetric $(n \times n)$ matrix.

① $\mathbf{A}$ is positive definite $\Rightarrow$ $a_{ii} > 0$ for all $i = 1 \ldots n$.

② $\mathbf{A}$ is positive definite $\Leftrightarrow$
All principal determinants are positive, i.e.
$\det (a_{ij})_{i=1..k,j=1..k} > 0$ for all $k = 1 \ldots n$.

③ $\mathbf{A}$ is strictly diagonal dominant $\Rightarrow$ $\mathbf{A}$ is positive definite.

④ $\mathbf{A}$ is diagonal dominant and $a_{ii} > 0$, $a_{ij} < 0$ for all $i \neq j$
$\Rightarrow$ $\mathbf{A}$ is positive definite.

⑤ $\mathbf{A}$ is tridiagonal, diagonal dominant and $a_{ii} > 0$,
$a_{ij} \neq 0$ for all $|i - j| = 1$
$\Rightarrow$ $\mathbf{A}$ is positive definite.

Now we look at the Cholesky decomposition. The first step is to decompose the matrix $\mathbf{A}$ into the product of the lower triangular matrix $\mathbf{R}^t$ and the upper triangular matrix $\mathbf{R}$ such that

$$\mathbf{A} = \mathbf{R^t} \cdot \mathbf{R}.$$

Example 21.15. We look for the Cholesky decomposition of the symmetric matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 4 & 25 \end{pmatrix}.$$

We perform the matrix multiplication $\mathbf{A} = \mathbf{R^t} \cdot \mathbf{R}$ with an upper triangular (2×2) matrix $\mathbf{R}$.

$$\begin{pmatrix} 1 & 4 \\ 4 & 25 \end{pmatrix} = \mathbf{R^t} \cdot \mathbf{R} = \begin{pmatrix} r_{11} & 0 \\ r_{12} & r_{22} \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix} = \begin{pmatrix} r_{11}^2 & r_{11}r_{12} \\ (r_{12}r_{11}) & r_{12}^2 + r_{22}^2 \end{pmatrix}.$$

Due to the symmetry of the original matrix A we get only 3 equations. By comparing the coefficients with the original matrix elements we get

$$\begin{aligned} r_{11}^2 &= 1 \quad \curvearrowright \quad r_{11} = 1 \\ r_{11}r_{12} &= 4 \quad \curvearrowright \quad r_{12} = 4 \\ r_{12}^2 + r_{22}^2 &= 25 \quad \curvearrowright \quad r_{22} = 3. \end{aligned}$$

So $\mathbf{R} = \begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix}$ and $\mathbf{A} = \mathbf{R^t} \cdot \mathbf{R} = \begin{pmatrix} 1 & 0 \\ 4 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 4 & 25 \end{pmatrix}$. □

Example 21.16. We are looking for the Cholesky decomposition into a lower and an upper triangular matrix $\mathbf{R}$ for the symmetric matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 8 & 4 \\ 0 & 4 & 20 \end{pmatrix}.$$

We perform a matrix multiplication $\mathbf{A} = \mathbf{R^t} \cdot \mathbf{R}$ with an upper triangular (3×3) matrix $\mathbf{R}$:

$$\begin{aligned} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 8 & 4 \\ 0 & 4 & 20 \end{pmatrix} = \mathbf{R^t} \cdot \mathbf{R} &= \begin{pmatrix} r_{11} & 0 & 0 \\ r_{12} & r_{22} & 0 \\ r_{13} & r_{23} & r_{33} \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix} \\ &= \begin{pmatrix} r_{11}^2 & r_{11}r_{12} & r_{11}r_{13} \\ (r_{12}r_{11}) & r_{12}^2 + r_{22}^2 & r_{12}r_{13} + r_{22}r_{23} \\ (r_{13}r_{11}) & (r_{13}r_{12} + r_{23}r_{22}) & r_{13}^2 + r_{23}^2 + r_{33}^2 \end{pmatrix}. \end{aligned}$$

Due to the symmetry of A and also of the matrix product, we skip the terms in brackets. Comparing the coefficients with the original matrix elements we get

$$\begin{aligned} r_{11}^2 &= 1 \quad \curvearrowright \quad r_{11} = 1 \\ 1r_{12} &= 2 \quad \curvearrowright \quad r_{12} = 2 \\ 1r_{13} &= 0 \quad \curvearrowright \quad r_{13} = 0 \\ 4 + r_{22}^2 &= 8 \quad \curvearrowright \quad r_{22} = 2 \end{aligned}$$

$$2 \cdot 0 + 2r_{23} = 4 \quad \curvearrowright \quad r_{23} = 2$$
$$4 + r_{33}^2 = 20 \quad \curvearrowright \quad r_{33} = 4.$$

So the upper triangular matrix is

$$\mathbf{R} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

and the Cholesky decomposition is

$$\mathbf{A} = \mathbf{R^t} \cdot \mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$ □

Similar to the examples above, we now consider the general case of an $n \times n$ matrix. The algorithm for the Cholesky decomposition is obtained by comparing the coefficients of the symmetric matrix $\mathbf{A}$ with the product $\mathbf{R^t} \cdot \mathbf{R}$.

Cholesky Decomposition

Let $\mathbf{A}$ be a symmetric, positive definite $(n \times n)$ matrix. Then $\mathbf{A} = \mathbf{R^t} \cdot \mathbf{R}$ is decomposed with an upper triangular matrix $\mathbf{R}$. The coefficients of $\mathbf{R}$ are computed using the following algorithm:

(1) For $i = 1 \dots n$ we define

$$r_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} r_{ki}^2} \qquad \text{(Diagonal elements).}$$

(2) For $j = i + 1 \dots n$ we set

$$r_{ij} = \frac{1}{r_{ii}} \left(a_{ij} - \sum_{k=1}^{i-1} r_{kj} \cdot r_{ki} \right) \qquad \text{(Non-diagonal elements).}$$

Let us return to the solution of the system of linear equations $\mathbf{A}\,\vec{x} = \vec{b}$. After decomposing the matrix $\mathbf{A}$ into $\mathbf{R^t} \cdot \mathbf{R}$, we replace the system of linear equations $\mathbf{A}\,\vec{x} = \vec{b}$ by $\mathbf{R^t} \cdot \mathbf{R}\,\vec{x} = \vec{b}$ and introduce the vector $\vec{c}$ by $\mathbf{R^t}\vec{c} = \vec{b}$.

The problem is then solved in two steps:

Cholesky Algorithm

The system of linear equations $\mathbf{A}\,\vec{x} = \vec{b}$ with a symmetric positive definite matrix $\mathbf{A}$ is processed in two steps using the Cholesky decomposition $\mathbf{A} = \mathbf{R^t} \cdot \mathbf{R}$:

(1) **Forward step:** Calculate the solution of the system of linear equations

$$\mathbf{R^t}\,\vec{c} = \vec{b}$$

by forward insertion. The result is the vector $\vec{c}$.

(2) **Reverse step:** Find the solution of the system of linear equations

$$\mathbf{R}\,\vec{x} = \vec{c}$$

by backward substitution.

Example 21.17. We are looking for the solution to the system of linear equations $\mathbf{A}\,\vec{x} = \vec{b}$ with

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 4 & 25 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

According to Example 21.15 the Cholesky decomposition is

$$\mathbf{A} = \mathbf{R^t} \cdot \mathbf{R} = \begin{pmatrix} 1 & 0 \\ 4 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 4 & 25 \end{pmatrix}.$$

Forward step: First, we solve $\mathbf{R^t}\vec{c} = \vec{b}$: $\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 4 & 3 & 1 \end{array}\right).$

The first equation gives $1 \cdot c_1 = 1 \curvearrowright c_1 = 1$ and the second gives $4 \cdot 1 + 3\,c_2 = 1 \curvearrowright c_2 = -1$.

Reverse step: Then, we solve $\mathbf{R}\vec{x} = \vec{c}$: $\left(\begin{array}{cc|c} 1 & 4 & 1 \\ 0 & 3 & -1 \end{array}\right).$

The second equation gives $3\,x_2 = -1 \curvearrowright x_2 = -\frac{1}{3}$ and from the first $1\,x_1 - \frac{4}{3} = 1 \curvearrowright x_1 = \frac{7}{3}$. □

Example 21.18 (With MAPLE-Worksheet). We are looking for the solution to the system of linear equations $\mathbf{A}\,\vec{x} = \vec{b}$ with

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 8 & 4 \\ 0 & 4 & 20 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 1 \\ 6 \\ 0 \end{pmatrix}.$$

According to Example 21.16 the decomposition is

$$\mathbf{A} = \mathbf{R^t} \cdot \mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{pmatrix}.$$

Forward step: First, we solve $\mathbf{R^t}\,\vec{c} = \vec{b}$:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 2 & 2 & 0 & 6 \\ 0 & 2 & 4 & 0 \end{array}\right).$$

From the first equation we get $1 \cdot c_1 = 1 \curvearrowright c_1 = 1$; from the second $2 + 2\,c_2 = 6 \curvearrowright c_2 = 2$ and from the third $4 + 4\,c_3 = 0 \curvearrowright c_3 = -1$.

Reverse step: Then, we solve $\mathbf{R}\,\vec{x} = \vec{c}$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 4 & -1 \end{array}\right).$$

From the third equation we get $4\,x_3 = -1 \curvearrowright x_3 = -\frac{1}{4}$, from the second $2\,x_2 - \frac{1}{2} = 2 \curvearrowright x_2 = \frac{5}{4}$ and from the first $x_1 + \frac{10}{4} = 1 \curvearrowright x_1 = -\frac{3}{2}$. □

21.4 Problems on Splines

21.1 Given are the value pairs (1, 1), (3, 3), (5, 3), (7, 1). Find the *natural* cubic spline function $\mathcal{S}$ through these points.

21.2 Given is the function $f(x) = \cos(x)$. Show that the *natural* cubic spline function $\mathcal{S}$ is replaced by the points

x_k	0	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$
$f(x_k)$	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0

is determined by the polynomials

$$\begin{aligned}
p_0(x) &= 1 - 0.148\,x - 0.394\,x^2\\
p_1(x) &= 0.866 - 0.472\,(x - \frac{1}{6}\,\pi) - 0.619\,(x - \frac{1}{6}\,\pi)^2 + 0.398\,(x - \frac{1}{6}\,\pi)^3\\
p_2(x) &= 0.707 - 0.715\,(x - \frac{1}{4}\,\pi) - 0.307\,(x - \frac{1}{4}\,\pi)^2 + 0.058\,(x - \frac{1}{4}\,\pi)^3\\
p_3(x) &= 0.5 - 0.864\,(x - \frac{1}{3}\,\pi) - 0.262\,(x - \frac{1}{3}\,\pi)^2 + 0.166\,(x - \frac{1}{3}\,\pi)^3.
\end{aligned}$$

21.3 Given is the function $f(x) = \sin(x)$. Find the *periodic* cubic spline function $\mathcal{S}$ for the points $x = 0,\ \frac{1}{4}\,\pi,\ \ldots\ \frac{7}{4}\,\pi,\ \pi$ and their graphical representation.

21.4 Given is the function $f(x) = x^{1/3}$.
a) Determine the interpolation polynomial of degree 5 for the values $x = -5, -3, -1, 1, 3, 5$ using the Newton interpolation method. Draw the output function and the interpolation polynomial in a diagram.
b) Determine the interpolation polynomial of 10th degree for the values $x = -5, -4, -3, -2, -1, 0.1, 1, 2, 3, 4, 5$ using the Newton interpolation method. Draw the output function and the interpolation polynomial in a diagram.
c) Determine the *natural* cubic spline function $\mathcal{S}$ to the values given in part (b).

21.5 The unit circle is given. Select 8 points on the unit circle and define the *parametric* spline function for these points.

21.6 a) Determine the second-order Bernstein polynomials and display the polynomials in the interval $[0, 1]$ graphically.
b) Determine the fourth-order Bernstein polynomials and display the polynomials in the interval $[0, 1]$ graphically.

21.7 Given are the Bezier points

$$P_0(0,0), P_1(1,5), P_2(2,4), P_3(6,-2), P_4(10,0).$$

Find the description of the Bezier segment by the points
a) $P_0(0,0)$, $P_1(1,5)$, $P_2(2,2)$
b) $P_0(0,0)$, $P_1(1,5)$, $P_2(2,2)$, $P_3(6,0)$
c) $P_0(0,0)$, $P_1(1,5)$, $P_2(2,4)$, $P_3(6,-2)$, $P_4(10,0)$
using the Bernstein polynomials and the graphical representation of the respective segment.

21.8 Create a C^2-continuous Bezier curve consisting of the two segments defined by the points

$$P_0^{(1)}(0,0),\ P_1^{(1)}(5,-5),\ P_2^{(1)}(10,-5),\ P_3^{(1)}(15,0)$$
$$\text{and } P_2^{(2)}(20,5),\ P_3^{(2)}(25,0).$$

21.9 Solve the system of linear equations $\mathbf{A}\ \vec{x} = \vec{b}$ using the Thomas Algorithm

$$\mathbf{A} = \begin{pmatrix} -4 & 2 & 0 & 0 & 0 \\ 2 & -4 & 2 & 0 & 0 \\ 0 & 2 & -4 & 2 & 0 \\ 0 & 0 & 2 & -4 & 2 \\ 0 & 0 & 0 & 2 & -4 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

21.10 Given are the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 2 & -3 \\ 2 & 0 & 8 & 2 & 2 \\ 0 & 2 & 2 & 9 & -4 \\ 1 & -3 & 2 & -4 & 12 \end{pmatrix} \quad \text{and} \quad \mathbf{R} = \begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

a) Show that the matrix $\mathbf{A}$ is positive definite by finding all principal determinants.

b) Show that the matrix $\mathbf{R}$ can be Cholesky decomposed.

c) Solve the system of linear equations $\mathbf{A}\ \vec{x} = \vec{b}$ with the vector $\vec{b} = (1,0,1,0,1)^t$ using the Cholesky algorithm.

21.11 Determine the Cholesky decomposition of the following matrices.

$$\mathbf{A_1} = \begin{pmatrix} 9 & -3 \\ -3 & 5 \end{pmatrix}, \quad \mathbf{A_2} = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 5 & -3 \\ 2 & -3 & 14 \end{pmatrix}, \quad \mathbf{A_3} = \begin{pmatrix} 9 & -3 & 0 & -12 \\ -3 & 5 & 8 & 12 \\ 0 & 8 & 20 & 14 \\ -12 & 12 & 14 & 34 \end{pmatrix}.$$

21.12 Solve the system of linear equations $\mathbf{A_i}\ \vec{x} = \vec{b_i}$ using the Cholesky algorithm for the matrices from Problem 21.11 with the inhomogeneities

$$\vec{b_1} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \vec{b_2} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \quad \vec{b_3} = \begin{pmatrix} 0 \\ 4 \\ 0 \\ 4 \end{pmatrix}.$$

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Homepage of the Book: Additional Material

iMath: iMath is an interactive maths application: In this pedagogically appealing app, easy-to-understand exercises from this book are solved in detail. The app can also be used for exam preparation. It can be launched directly from

https://www.imathonline.de/iMatheWeb

YouTube videos: On the Westermann YouTube channel, many of the topics covered in this book are explained in short videos in German. In the form of summaries, the most important aspects are summarized in a short and easily understandable way. The corresponding links to the videos can be found at

https://www.youtube.com/channel/UChzktnND8kk9pmwQmybSx-w

Additional material is available on the homepage of the book. All information, Maple procedures, Maple worksheets and additional chapters can be downloaded free of charge from

https://www.imathonline.de/books/mathe/start.htm

Maple-**Worksheets:** All the Maple worksheets for the problems and examples mentioned in the text. In particular, the worksheets for all visualizations can be found here.

Animations: All animations shown or described in the text are available on the homepage as Animated-Gif, so that they can be started directly in the browser.

https://www.imathonline.de/animations

Solutions to the problems: Complete solutions are given for all problems.