COLLEGE ALGEBRA

9e

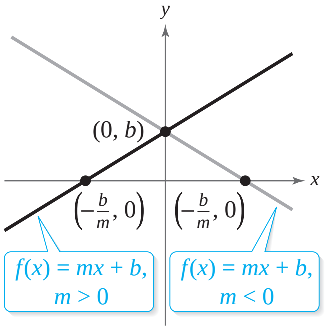
Solutions, Interactivity,
Videos, & Tutorial Help at
LarsonPrecalculus.com

Ron Larson

GRAPHS OF PARENT FUNCTIONS

Linear Function

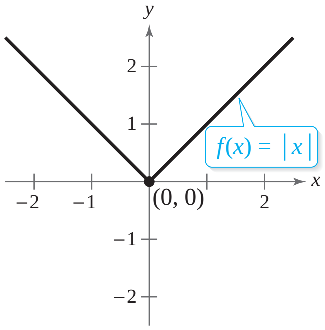
$f(x) = mx + b$



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$
x-intercept: $(-b/m, 0)$
y-intercept: $(0, b)$
Increasing when $m > 0$
Decreasing when $m < 0$

Absolute Value Function

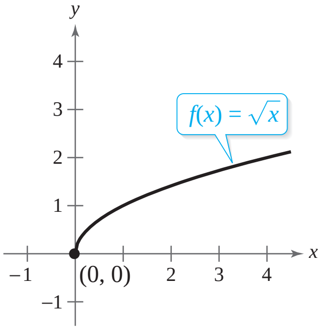
$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$
Intercept: $(0, 0)$
Decreasing on $(-\infty, 0)$
Increasing on $(0, \infty)$
Even function
y-axis symmetry

Square Root Function

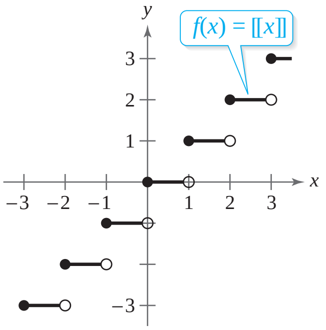
$f(x) = \sqrt{x}$



Domain: $[0, \infty)$
Range: $[0, \infty)$
Intercept: $(0, 0)$
Increasing on $(0, \infty)$

Greatest Integer Function

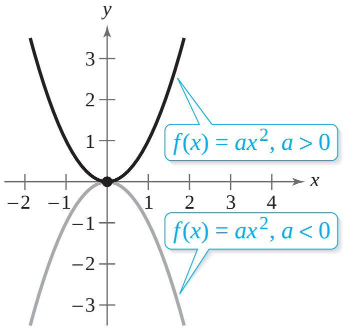
$f(x) = [\![x]\!]$



Domain: $(-\infty, \infty)$
Range: the set of integers
x-intercepts: in the interval $[0, 1)$
y-intercept: $(0, 0)$
Constant between each pair of consecutive integers
Jumps vertically one unit at each integer value

Quadratic (Squaring) Function

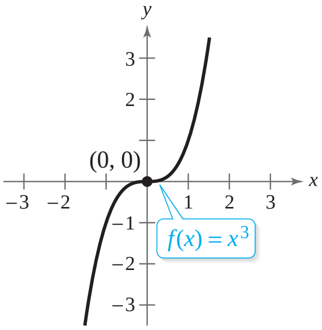
$f(x) = ax^2$



Domain: $(-\infty, \infty)$
Range $(a > 0)$: $[0, \infty)$
Range $(a < 0)$: $(-\infty, 0]$
Intercept: $(0, 0)$
Decreasing on $(-\infty, 0)$ for $a > 0$
Increasing on $(0, \infty)$ for $a > 0$
Increasing on $(-\infty, 0)$ for $a < 0$
Decreasing on $(0, \infty)$ for $a < 0$
Even function
y-axis symmetry
Relative minimum $(a > 0)$, relative maximum $(a < 0)$, or vertex: $(0, 0)$

Cubic Function

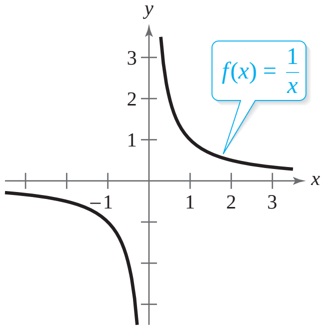
$f(x) = x^3$



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$
Intercept: $(0, 0)$
Increasing on $(-\infty, \infty)$
Odd function
Origin symmetry

Rational (Reciprocal) Function

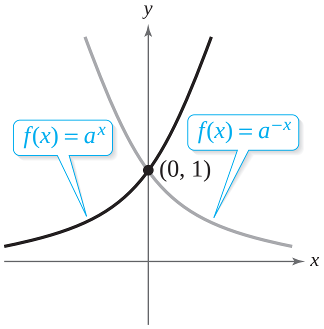
$f(x) = \dfrac{1}{x}$



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$
No intercepts
Decreasing on $(-\infty, 0)$ and $(0, \infty)$
Odd function
Origin symmetry
Vertical asymptote: y-axis
Horizontal asymptote: x-axis

Exponential Function

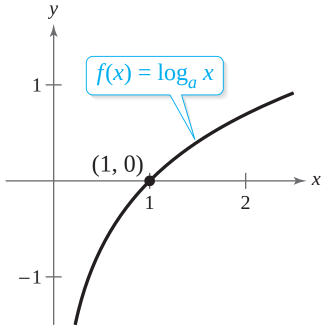
$f(x) = a^x,\ a > 1$



Domain: $(-\infty, \infty)$
Range: $(0, \infty)$
Intercept: $(0, 1)$
Increasing on $(-\infty, \infty)$
for $f(x) = a^x$
Decreasing on $(-\infty, \infty)$
for $f(x) = a^{-x}$
Horizontal asymptote: x-axis
Continuous

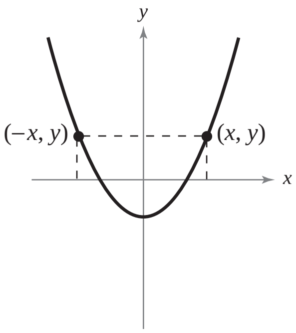
Logarithmic Function

$f(x) = \log_a x,\ a > 1$

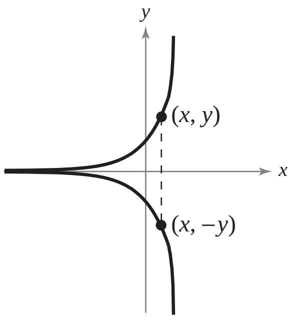


Domain: $(0, \infty)$
Range: $(-\infty, \infty)$
Intercept: $(1, 0)$
Increasing on $(0, \infty)$
Vertical asymptote: y-axis
Continuous
Reflection of graph of $f(x) = a^x$
in the line $y = x$

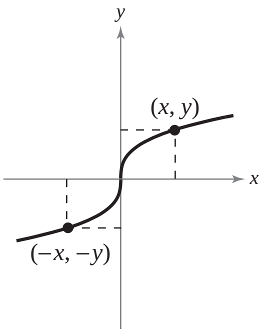
SYMMETRY



***y*-Axis Symmetry**



***x*-Axis Symmetry**



Origin Symmetry

College Algebra

Ninth Edition

College Algebra

Ninth Edition

Ron Larson
The Pennsylvania State University
The Behrend College

With the assistance of David C. Falvo
The Pennsylvania State University
The Behrend College

BROOKS/COLE
CENGAGE Learning™

Australia • Brazil • Japan • Korea • Mexico • Singapore • Spain • United Kingdom • United States

BROOKS/COLE
CENGAGE Learning™

College Algebra
Ninth Edition

Ron Larson

Publisher: Liz Covello
Acquisitions Editor: Gary Whalen
Senior Development Editor: Stacy Green
Assistant Editor: Cynthia Ashton
Editorial Assistant: Samantha Lugtu
Media Editor: Lynh Pham
Senior Content Project Manager: Jessica Rasile
Art Director: Linda May
Rights Acquisition Specialist: Shalice Shah-Caldwell
Manufacturing Planner: Doug Bertke
Senior Marketing Manager: Danae April
Text/Cover Designer: Larson Texts, Inc.
Compositor: Larson Texts, Inc.
Cover Image: diez artwork/Shutterstock.com

Library of Congress Control Number: 2012948316

Student Edition:
ISBN-13: 978-1-133-96302-8
ISBN-10: 1-133-96302-1

Brooks/Cole
20 Channel Center Street
Boston, MA 02210
USA

Cengage Learning is a leading provider of customized learning solutions with office locations around the globe, including Singapore, the United Kingdom, Australia, Mexico, Brazil, and Japan. Locate your local office at: **international.cengage.com/region**

Cengage Learning products are represented in Canada by Nelson Education, Ltd.

For your course and learning solutions, visit **www.cengage.com.**

Purchase any of our products at your local college store or at our preferred online store **www.cengagebrain.com.**

Instructors: Please visit **login.cengage.com** and log in to access instructor-specific resources.

Printed in the United States of America
1 2 3 4 5 6 7 16 15 14 13 12

Contents

3 ▷ Polynomial Functions 241

4 ▷ Rational Functions and Conics 311

5 ▷ Exponential and Logarithmic Functions 361

6 ▷ Systems of Equations and Inequalities 423

7 ▷ Matrices and Determinants 495

8 ▷ Sequences, Series, and Probability 563

Appendices

*Available at the text-specific website *www.cengagebrain.com*

Preface

Welcome to *College Algebra*, Ninth Edition. I am proud to present to you this new edition. As with all editions, I have been able to incorporate many useful comments from you, our user. And while much has changed in this revision, you will still find what you expect—a pedagogically sound, mathematically precise, and comprehensive textbook. Additionally, I am pleased and excited to offer you something brand new—a companion website at **LarsonPrecalculus.com.**

My goal for every edition of this textbook is to provide students with the tools that they need to master algebra. I hope you find that the changes in this edition, together with **LarsonPrecalculus.com,** will help accomplish just that.

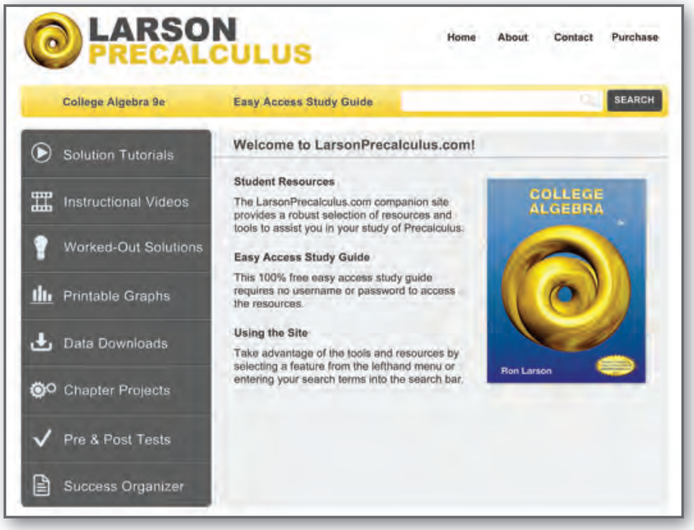
New To This Edition

NEW LarsonPrecalculus.com

This companion website offers multiple tools and resources to supplement your learning. Access to these features is free. View and listen to worked-out solutions of Checkpoint problems in English or Spanish, download data sets, work on chapter projects, watch lesson videos, and much more.

NEW Chapter Opener

Each Chapter Opener highlights real-life applications used in the examples and exercises.



Welcome to LarsonPrecalculus.com!

Student Resources

The LarsonPrecalculus.com companion site provides a robust selection of resources and tools to assist you in your study of Precalculus.

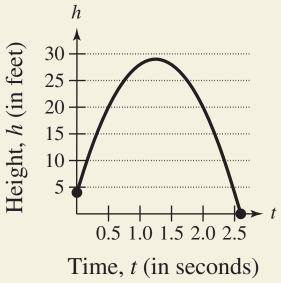
Easy Access Study Guide

This 100% free easy access study guide requires no username or password to access the resources.

Using the Site

Take advantage of the tools and resources by selecting a feature from the lefthand menu or entering your search terms into the search bar.

96. **HOW DO YOU SEE IT?** The graph represents the height h of a projectile after t seconds.



(a) Explain why h is a function of t.

(b) Approximate the height of the projectile after 0.5 second and after 1.25 seconds.

(c) Approximate the domain of h.

(d) Is t a function of h? Explain.

NEW Summarize

The Summarize feature at the end of each section helps you organize the lesson's key concepts into a concise summary, providing you with a valuable study tool.

NEW How Do You See It?

The How Do You See It? feature in each section presents a real-life exercise that you will solve by visual inspection using the concepts learned in the lesson. This exercise is excellent for classroom discussion or test preparation.

NEW Checkpoints

Accompanying every example, the Checkpoint problems encourage immediate practice and check your understanding of the concepts presented in the example. View and listen to worked-out solutions of the Checkpoint problems in English or Spanish at LarsonPrecalculus.com.

NEW Data Spreadsheets

Download these editable spreadsheets from LarsonPrecalculus.com, and use the data to solve exercises.

DATA

Year	Number of Tax Returns Made Through E-File
2003	52.9
2004	61.5
2005	68.5
2006	73.3
2007	80.0
2008	89.9
2009	95.0
2010	98.7

Spreadsheet at LarsonPrecalculus.com

REVISED Exercise Sets

The exercise sets have been carefully and extensively examined to ensure they are rigorous and relevant and to include all topics our users have suggested. The exercises have been **reorganized and titled** so you can better see the connections between examples and exercises. Multi-step, real-life exercises reinforce problem-solving skills and mastery of concepts by giving you the opportunity to apply the concepts in real-life situations.

REVISED Section Objectives

A bulleted list of learning objectives provides you the opportunity to preview what will be presented in the upcoming section.

REVISED Remark

These hints and tips reinforce or expand upon concepts, help you learn how to study mathematics, caution you about common errors, address special cases, or show alternative or additional steps to a solution of an example.

Calc Chat

For the past several years, an independent website—CalcChat.com—has provided free solutions to all odd-numbered problems in the text. Thousands of students have visited the site for practice and help with their homework. For this edition, I used information from CalcChat.com, including which solutions students accessed most often, to help guide the revision of the exercises.

Community 24/7 Ask Questions, Answer Problems, Share your solutions

Solutions 24/7 Step-by-step solutions to odd-numbered exercises

Tutor School Year: Calc Assistant available Sun-Thurs, 4pm-12am, ET
Summer: Calc Assistant available Mon-Fri, 1pm-4pm, ET

Coming to Calc Chat on August 1, 2013

- Calculus 10e
- Algebra & Trigonometry 9e
- College Algebra 9e
- Precalculus 9e
- Precalculus with Limits 3e
- Trigonometry 9e
- Precalculus: A Concise Course 3e

Download the Calc Chat App

Take Calc Chat with you on your iPhone and your Android-powered devices! Learn more about the Calc Chat app.

Visit the Apple App Store to download.

Visit the Android Market to download.

Copyright© 2011 Larson Texts, Inc. All rights reserved.

Trusted Features

Side-By-Side Examples

Throughout the text, we present solutions to many examples from multiple perspectives—algebraically, graphically, and numerically. The side-by-side format of this pedagogical feature helps you to see that a problem can be solved in more than one way and to see that different methods yield the same result. The side-by-side format also addresses many different learning styles.

Algebra Help

Algebra Help directs you to sections of the textbook where you can review algebra skills needed to master the current topic.

Technology

The technology feature gives suggestions for effectively using tools such as calculators, graphing calculators, and spreadsheet programs to help deepen your understanding of concepts, ease lengthy calculations, and provide alternate solution methods for verifying answers obtained by hand.

Historical Notes

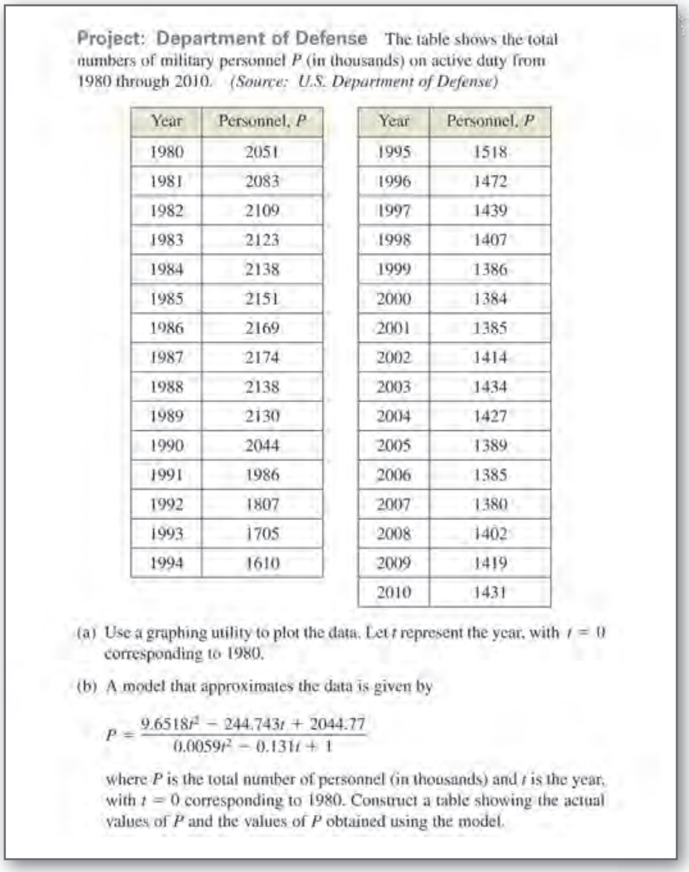
These notes provide helpful information regarding famous mathematicians and their work.

Algebra of Calculus

Throughout the text, special emphasis is given to the algebraic techniques used in calculus. Algebra of Calculus examples and exercises are integrated throughout the text and are identified by the symbol $\int$.

Vocabulary Exercises

The vocabulary exercises appear at the beginning of the exercise set for each section. These problems help you review previously learned vocabulary terms that you will use in solving the section exercises.



Project: Department of Defense The table shows the total numbers of military personnel P (in thousands) on active duty from 1980 through 2010. *(Source: U.S. Department of Defense)*

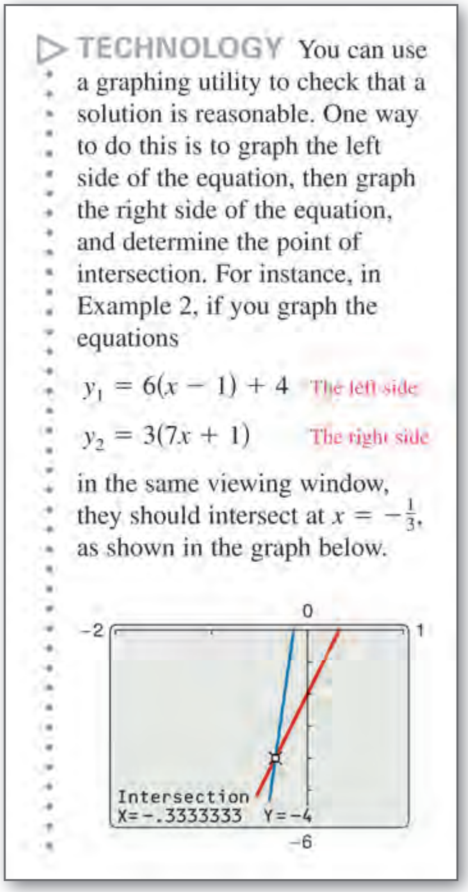
Year	Personnel, P
1980	2051
1981	2083
1982	2109
1983	2123
1984	2138
1985	2151
1986	2169
1987	2174
1988	2138
1989	2130
1990	2044
1991	1986
1992	1807
1993	1705
1994	1610
1995	1518
1996	1472
1997	1439
1998	1407
1999	1386
2000	1384
2001	1385
2002	1414
2003	1434
2004	1427
2005	1389
2006	1385
2007	1380
2008	1402
2009	1419
2010	1431

(a) Use a graphing utility to plot the data. Let t represent the year, with $t = 0$ corresponding to 1980.

(b) A model that approximates the data is given by

$$P = \frac{9.6518t^2 - 244.743t + 2044.77}{0.0059t^2 - 0.131t + 1}$$

where P is the total number of personnel (in thousands) and t is the year, with $t = 0$ corresponding to 1980. Construct a table showing the actual values of P and the values of P obtained using the model.



▷ **TECHNOLOGY** You can use a graphing utility to check that a solution is reasonable. One way to do this is to graph the left side of the equation, then graph the right side of the equation, and determine the point of intersection. For instance, in Example 2, if you graph the equations

$y_1 = 6(x - 1) + 4$ The left side

$y_2 = 3(7x + 1)$ The right side

in the same viewing window, they should intersect at $x = -\frac{1}{3}$, as shown in the graph below.

Project

The projects at the end of selected sections involve in-depth applied exercises in which you will work with large, real-life data sets, often creating or analyzing models. These projects are offered online at LarsonPrecalculus.com.

Chapter Summaries

The Chapter Summary now includes explanations and examples of the objectives taught in each chapter.

ENHANCED WebAssign

Enhanced WebAssign combines exceptional Precalculus content that you know and love with the most powerful online homework solution, WebAssign. Enhanced WebAssign engages you with immediate feedback, rich tutorial content and interactive, fully customizable eBooks (YouBook) helping you to develop a deeper conceptual understanding of the subject matter.

Instructor Resources

Print

Annotated Instructor's Edition
ISBN-13: 978-1-133-96118-5

This AIE is the complete student text plus point-of-use annotations for you, including extra projects, classroom activities, teaching strategies, and additional examples. Answers to even-numbered text exercises, Vocabulary Checks, and Explorations are also provided.

Complete Solutions Manual
ISBN-13: 978-1-133-96134-5

This manual contains solutions to all exercises from the text, including Chapter Review Exercises, and Chapter Tests.

Media

PowerLecture with ExamView™
ISBN-13: 978-1-133-96136-9

The DVD provides you with dynamic media tools for teaching Algebra while using an interactive white board. PowerPoint® lecture slides and art slides of the figures from the text, together with electronic files for the test bank and a link to the Solution Builder, are available. The algorithmic ExamView allows you to create, deliver, and customize tests (both print and online) in minutes with this easy-to-use assessment system. The DVD also provides you with a tutorial on integrating our instructor materials into your interactive whiteboard platform. Enhance how your students interact with you, your lecture, and each other.

Solution Builder
(*www.cengage.com/solutionbuilder*)
This online instructor database offers complete worked-out solutions to all exercises in the text, allowing you to create customized, secure solutions printouts (in PDF format) matched exactly to the problems you assign in class.

ENHANCED WebAssign
www.webassign.net
Printed Access Card: 978-0-538-73810-1
Online Access Code: 978-1-285-18181-3

Exclusively from Cengage Learning, Enhanced WebAssign combines the exceptional mathematics content that you know and love with the most powerful online homework solution, WebAssign. Enhanced WebAssign engages students with immediate feedback, rich tutorial content, and interactive, fully customizable eBooks (YouBook), helping students to develop a deeper conceptual understanding of their subject matter. Online assignments can be built by selecting from thousands of text-specific problems or supplemented with problems from any Cengage Learning textbook.

Student Resources

Print

Student Study and Solutions Manual
ISBN-13: 978-1-133-96294-6

This guide offers step-by-step solutions for all odd-numbered text exercises, Chapter and Cumulative Tests, and Practice Tests with solutions.

Text-Specific DVD
ISBN-13: 978-1-133-96287-8

Keyed to the text by section, these DVDs provide comprehensive coverage of the course—along with additional explanations of concepts, sample problems, and application—to help you review essential topics.

Media

ENHANCED WebAssign

www.webassign.net
Printed Access Card: 978-0-538-73810-1
Online Access Code: 978-1-285-18181-3

Enhanced WebAssign (assigned by the instructor) provides you with instant feedback on homework assignments. This online homework system is easy to use and includes helpful links to textbook sections, video examples, and problem-specific tutorials.

CengageBrain.com

Visit *www.cengagebrain.com* to access additional course materials and companion resources. At the CengageBrain.com home page, search for the ISBN of your title (from the back cover of your book) using the search box at the top of the page. This will take you to the product page where free companion resources can be found.

Acknowledgements

I would like to thank the many people who have helped me prepare the text and the supplements package. Their encouragement, criticisms, and suggestions have been invaluable.

Thank you to all of the instructors who took the time to review the changes in this edition and to provide suggestions for improving it. Without your help, this book would not be possible.

Reviewers

Timothy Andrew Brown, *South Georgia College*
Blair E. Caboot, *Keystone College*
Shannon Cornell, *Amarillo College*
Gayla Dance, *Millsaps College*
Paul Finster, *El Paso Community College*
Paul A. Flasch, *Pima Community College West Campus*
Vadas Gintautas, *Chatham University*
Lorraine A. Hughes, *Mississippi State University*
Shu-Jen Huang, *University of Florida*
Renyetta Johnson, *East Mississippi Community College*
George Keihany, *Fort Valley State University*
Mulatu Lemma, *Savannah State University*
William Mays Jr., *Salem Community College*
Marcella Melby, *University of Minnesota*
Jonathan Prewett, *University of Wyoming*
Denise Reid, *Valdosta State University*
David L. Sonnier, *Lyon College*
David H. Tseng, *Miami Dade College – Kendall Campus*
Kimberly Walters, *Mississippi State University*
Richard Weil, *Brown College*
Solomon Willis, *Cleveland Community College*
Bradley R. Young, *Darton College*

My thanks to Robert Hostetler, The Behrend College, The Pennsylvania State University, and David Heyd, The Behrend College, The Pennsylvania State University, for their significant contributions to previous editions of this text.

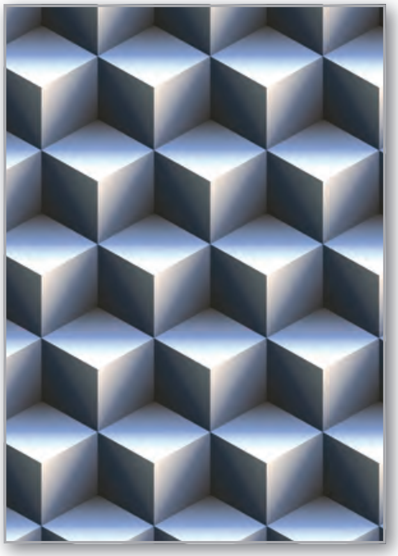
I would also like to thank the staff at Larson Texts, Inc. who assisted with proofreading the manuscript, preparing and proofreading the art package, and checking and typesetting the supplements.

On a personal level, I am grateful to my spouse, Deanna Gilbert Larson, for her love, patience, and support. Also, a special thanks goes to R. Scott O'Neil. If you have suggestions for improving this text, please feel free to write to me. Over the past two decades I have received many useful comments from both instructors and students, and I value these comments very highly.

Ron Larson, Ph.D.
Professor of Mathematics
Penn State University
www.RonLarson.com

P Prerequisites

Autocatalytic Chemical Reaction *(Exercise 92, page 40)*



Computer Graphics *(page 56)*

Steel Beam Loading *(Exercise 93, page 33)*

Gallons of Water on Earth *(page 17)*

Change in Temperature *(page 7)*

Clockwise from top left, mikeledray/Shutterstock.com; Matt Antonino/Shutterstock.com; EpicStockMedia/Shutterstock.com; VladisChern/Shutterstock.com; vyskoczilova/Shutterstock.com

P.1 Review of Real Numbers and Their Properties

Real numbers can represent many real-life quantities. For example, in Exercises 55–58 on page 13, you will use real numbers to represent the federal deficit.

- **Represent and classify real numbers.**
- **Order real numbers and use inequalities.**
- **Find the absolute values of real numbers and find the distance between two real numbers.**
- **Evaluate algebraic expressions.**
- **Use the basic rules and properties of algebra.**

Real Numbers

Real numbers can describe quantities in everyday life such as age, miles per gallon, and population. Symbols such as

$$-5, 9, 0, \tfrac{4}{3}, 0.666\ldots, 28.21, \sqrt{2}, \pi, \text{ and } \sqrt[3]{-32}$$

represent real numbers. Here are some important **subsets** (each member of a subset B is also a member of a set A) of the real numbers. The three dots, called *ellipsis points*, indicate that the pattern continues indefinitely.

$\{1, 2, 3, 4, \ldots\}$ Set of natural numbers

$\{0, 1, 2, 3, 4, \ldots\}$ Set of whole numbers

$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ Set of integers

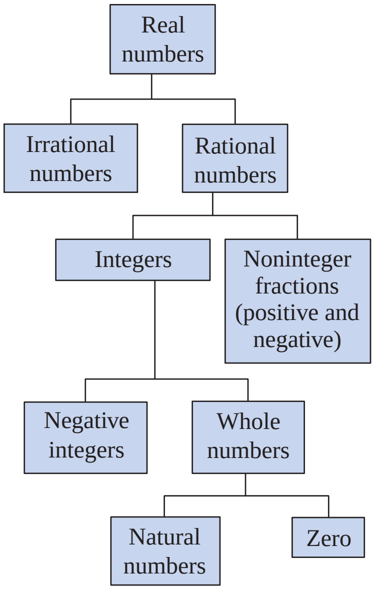
A real number is **rational** when it can be written as the ratio p/q of two integers, where $q \neq 0$. For instance, the numbers

$$\tfrac{1}{3} = 0.3333\ldots = 0.\overline{3}, \quad \tfrac{1}{8} = 0.125, \text{ and } \tfrac{125}{111} = 1.126126\ldots = 1.\overline{126}$$

are rational. The decimal representation of a rational number either repeats $\left(\text{as in } \frac{173}{55} = 3.1\overline{45}\right)$ or terminates $\left(\text{as in } \frac{1}{2} = 0.5\right)$. A real number that cannot be written as the ratio of two integers is called **irrational.** Irrational numbers have infinite nonrepeating decimal representations. For instance, the numbers

$$\sqrt{2} = 1.4142135\ldots \approx 1.41 \quad \text{and} \quad \pi = 3.1415926\ldots \approx 3.14$$

are irrational. (The symbol $\approx$ means "is approximately equal to.") Figure P.1 shows subsets of real numbers and their relationships to each other.



Subsets of real numbers

Figure P.1

EXAMPLE 1 Classifying Real Numbers

Determine which numbers in the set $\left\{-13, -\sqrt{5}, -1, -\frac{1}{3}, 0, \frac{5}{8}, \sqrt{2}, \pi, 7\right\}$ are (a) natural numbers, (b) whole numbers, (c) integers, (d) rational numbers, and (e) irrational numbers.

Solution

a. Natural numbers: $\{7\}$

b. Whole numbers: $\{0, 7\}$

c. Integers: $\{-13, -1, 0, 7\}$

d. Rational numbers: $\left\{-13, -1, -\dfrac{1}{3}, 0, \dfrac{5}{8}, 7\right\}$

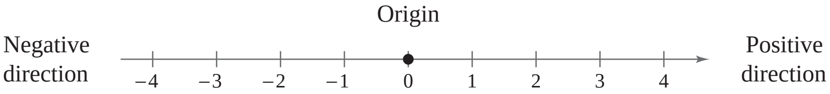
e. Irrational numbers: $\left\{-\sqrt{5}, \sqrt{2}, \pi\right\}$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

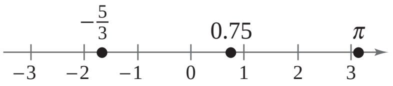
Repeat Example 1 for the set $\left\{-\pi, -\frac{1}{4}, \frac{6}{3}, \frac{1}{2}\sqrt{2}, -7.5, -1, 8, -22\right\}$.

Michael G Smith/Shutterstock.com

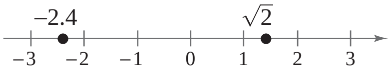
Real numbers are represented graphically on the **real number line.** When you draw a point on the real number line that corresponds to a real number, you are **plotting** the real number. The point 0 on the real number line is the **origin.** Numbers to the right of 0 are positive, and numbers to the left of 0 are negative, as shown below. The term **nonnegative** describes a number that is either positive or zero.



As illustrated below, there is a *one-to-one correspondence* between real numbers and points on the real number line.



Every real number corresponds to exactly one point on the real number line.



Every point on the real number line corresponds to exactly one real number.

EXAMPLE 2 Plotting Points on the Real Number Line

Plot the real numbers on the real number line.

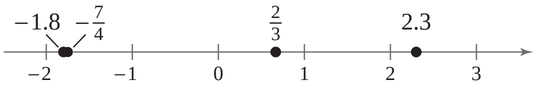
a. $-\frac{7}{4}$

b. 2.3

c. $\frac{2}{3}$

d. -1.8

Solution The following figure shows all four points.



a. The point representing the real number $-\frac{7}{4} = -1.75$ lies between -2 and -1, but closer to -2, on the real number line.

b. The point representing the real number 2.3 lies between 2 and 3, but closer to 2, on the real number line.

c. The point representing the real number $\frac{2}{3} = 0.666 \ldots$ lies between 0 and 1, but closer to 1, on the real number line.

d. The point representing the real number -1.8 lies between -2 and -1, but closer to -2, on the real number line. Note that the point representing -1.8 lies slightly to the left of the point representing $-\frac{7}{4}$.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Plot the real numbers on the real number line.

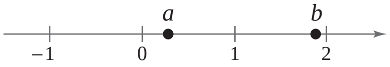
a. $\frac{5}{2}$ **b.** -1.6

c. $-\frac{3}{4}$ **d.** 0.7

Ordering Real Numbers

One important property of real numbers is that they are *ordered*.

> **Definition of Order on the Real Number Line**
>
> If a and b are real numbers, then a is less than b when $b - a$ is positive. The **inequality** $a < b$ denotes the **order** of a and b. This relationship can also be described by saying that b is *greater than* a and writing $b > a$. The inequality $a \le b$ means that a is *less than or equal to* b, and the inequality $b \ge a$ means that b is *greater than or equal to* a. The symbols <, >, ≤, and ≥ are *inequality symbols.*



$a < b$ if and only if a lies to the left of b.

Figure P.2

Geometrically, this definition implies that $a < b$ if and only if a lies to the *left* of b on the real number line, as shown in Figure P.2.

EXAMPLE 3 Ordering Real Numbers

Place the appropriate inequality symbol (< or >) between the pair of real numbers.

a. $-3, 0$ **b.** $-2, -4$ **c.** $\frac{1}{4}, \frac{1}{3}$ **d.** $-\frac{1}{5}, -\frac{1}{2}$

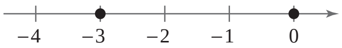


Figure P.3

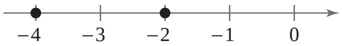


Figure P.4

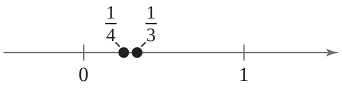


Figure P.5

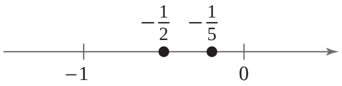


Figure P.6

Solution

a. Because -3 lies to the left of 0 on the real number line, as shown in Figure P.3, you can say that -3 is *less than* 0, and write $-3 < 0$.

b. Because -2 lies to the right of -4 on the real number line, as shown in Figure P.4, you can say that -2 is *greater than* -4, and write $-2 > -4$.

c. Because $\frac{1}{4}$ lies to the left of $\frac{1}{3}$ on the real number line, as shown in Figure P.5, you can say that $\frac{1}{4}$ is *less than* $\frac{1}{3}$, and write $\frac{1}{4} < \frac{1}{3}$.

d. Because $-\frac{1}{5}$ lies to the right of $-\frac{1}{2}$ on the real number line, as shown in Figure P.6, you can say that $-\frac{1}{5}$ is *greater than* $-\frac{1}{2}$, and write $-\frac{1}{5} > -\frac{1}{2}$.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Place the appropriate inequality symbol (< or >) between the pair of real numbers.

a. $1, -5$ **b.** $\frac{3}{2}, 7$ **c.** $-\frac{2}{3}, -\frac{3}{4}$ **d.** $-3.5, 1$

EXAMPLE 4 Interpreting Inequalities

Describe the subset of real numbers that the inequality represents.

a. $x \le 2$ **b.** $-2 \le x < 3$

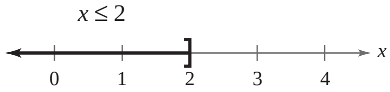


Figure P.7

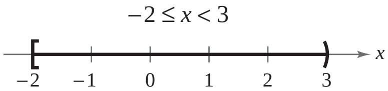


Figure P.8

Solution

a. The inequality $x \le 2$ denotes all real numbers less than or equal to 2, as shown in Figure P.7.

b. The inequality $-2 \le x < 3$ means that $x \ge -2$ *and* $x < 3$. This “double inequality” denotes all real numbers between -2 and 3, including -2 but not including 3, as shown in Figure P.8.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

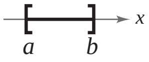
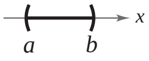
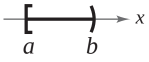
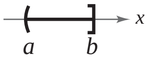
Describe the subset of real numbers that the inequality represents.

a. $x > -3$ **b.** $0 < x \le 4$

Inequalities can describe subsets of real numbers called **intervals.** In the bounded intervals below, the real numbers a and b are the **endpoints** of each interval. The endpoints of a closed interval are included in the interval, whereas the endpoints of an open interval are not included in the interval.

REMARK The reason that the four types of intervals at the right are called *bounded* is that each has a finite length. An interval that does not have a finite length is *unbounded* (see below).

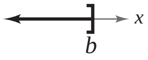
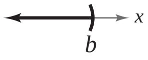
Bounded Intervals on the Real Number Line

Notation	Interval Type	Inequality	Graph
$[a, b]$	Closed	$a \leq x \leq b$	
(a, b)	Open	$a < x < b$	
$[a, b)$		$a \leq x < b$	
$(a, b]$		$a < x \leq b$	

REMARK Whenever you write an interval containing ∞ or $-\infty$, always use a parenthesis and never a bracket next to these symbols. This is because ∞ and $-\infty$ are never an endpoint of an interval and therefore are not included in the interval.

The symbols ∞, **positive infinity,** and $-\infty$, **negative infinity,** do not represent real numbers. They are simply convenient symbols used to describe the unboundedness of an interval such as $(1, \infty)$ or $(-\infty, 3]$.

Unbounded Intervals on the Real Number Line

Notation	Interval Type	Inequality	Graph
$[a, \infty)$		$x \geq a$	
(a, ∞)	Open	$x > a$	
$(-\infty, b]$		$x \leq b$	
$(-\infty, b)$	Open	$x < b$	
$(-\infty, \infty)$	Entire real line	$-\infty < x < \infty$	

EXAMPLE 5 Interpreting Intervals

a. The interval $(-1, 0)$ consists of all real numbers greater than -1 and less than 0.

b. The interval $[2, \infty)$ consists of all real numbers greater than or equal to 2.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Give a verbal description of the interval $[-2, 5)$.

EXAMPLE 6 Using Inequalities to Represent Intervals

a. The inequality $c \leq 2$ can represent the statement "c is at most 2."

b. The inequality $-3 < x \leq 5$ can represent "all x in the interval $(-3, 5]$."

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use inequality notation to represent the statement "x is greater than -2 and at most 4."

Absolute Value and Distance

The **absolute value** of a real number is its *magnitude,* or the distance between the origin and the point representing the real number on the real number line.

Definition of Absolute Value

If a is a real number, then the absolute value of a is

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}.$$

Notice in this definition that the absolute value of a real number is never negative. For instance, if $a = -5$, then $|-5| = -(-5) = 5$. The absolute value of a real number is either positive or zero. Moreover, 0 is the only real number whose absolute value is 0. So, $|0| = 0$.

EXAMPLE 7 Finding Absolute Values

a. $|-15| = 15$ **b.** $\left|\frac{2}{3}\right| = \frac{2}{3}$

c. $|-4.3| = 4.3$ **d.** $-|-6| = -(6) = -6$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Evaluate each expression.

a. $|1|$ **b.** $-\left|\frac{3}{4}\right|$

c. $\frac{2}{|-3|}$ **d.** $-|0.7|$

EXAMPLE 8 Evaluating the Absolute Value of a Number

Evaluate $\frac{|x|}{x}$ for (a) $x > 0$ and (b) $x < 0$.

Solution

a. If $x > 0$, then $|x| = x$ and $\frac{|x|}{x} = \frac{x}{x} = 1$.

b. If $x < 0$, then $|x| = -x$ and $\frac{|x|}{x} = \frac{-x}{x} = -1$.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Evaluate $\frac{|x + 3|}{x + 3}$ for (a) $x > -3$ and (b) $x < -3$. ■

The **Law of Trichotomy** states that for any two real numbers a and b, *precisely* one of three relationships is possible:

$$a = b, \quad a < b, \quad \text{or} \quad a > b. \qquad \text{Law of Trichotomy}$$

EXAMPLE 9 Comparing Real Numbers

Place the appropriate symbol (<, >, or =) between the pair of real numbers.

a. $|-4| \ \square \ |3|$ **b.** $|-10| \ \square \ |10|$ **c.** $-|-7| \ \square \ |-7|$

Solution

a. $|-4| > |3|$ because $|-4| = 4$ and $|3| = 3$, and 4 is greater than 3.

b. $|-10| = |10|$ because $|-10| = 10$ and $|10| = 10$.

c. $-|-7| < |-7|$ because $-|-7| = -7$ and $|-7| = 7$, and -7 is less than 7.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Place the appropriate symbol (<, >, or =) between the pair of real numbers.

a. $|-3| \ \square \ |4|$ **b.** $-|-4| \ \square \ -|4|$ **c.** $|-3| \ \square \ -|-3|$

Properties of Absolute Values

1. $|a| \geq 0$

2. $|-a| = |a|$

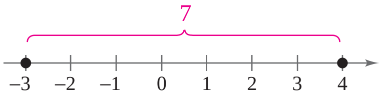
3. $|ab| = |a||b|$

4. $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}, \quad b \neq 0$

Absolute value can be used to define the distance between two points on the real number line. For instance, the distance between -3 and 4 is

$$\begin{aligned} |-3 - 4| &= |-7| \\ &= 7 \end{aligned}$$

as shown in Figure P.9.



The distance between -3 and 4 is 7.
Figure P.9

Distance Between Two Points on the Real Number Line

Let a and b be real numbers. The **distance between a and b** is

$$d(a, b) = |b - a| = |a - b|.$$

One application of finding the distance between two points on the real number line is finding a change in temperature.

EXAMPLE 10 Finding a Distance

Find the distance between -25 and 13.

Solution

The distance between -25 and 13 is

$$|-25 - 13| = |-38| = 38. \qquad \text{Distance between } -25 \text{ and } 13$$

The distance can also be found as follows.

$$|13 - (-25)| = |38| = 38 \qquad \text{Distance between } -25 \text{ and } 13$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

a. Find the distance between 35 and -23.

b. Find the distance between -35 and -23.

c. Find the distance between 35 and 23.

VladisChern/Shutterstock.com

Algebraic Expressions

One characteristic of algebra is the use of letters to represent numbers. The letters are **variables,** and combinations of letters and numbers are **algebraic expressions.** Here are a few examples of algebraic expressions.

$$5x,\qquad 2x - 3,\qquad \frac{4}{x^2 + 2},\qquad 7x + y$$

Definition of an Algebraic Expression

An **algebraic expression** is a collection of letters **(variables)** and real numbers **(constants)** combined using the operations of addition, subtraction, multiplication, division, and exponentiation.

The **terms** of an algebraic expression are those parts that are separated by *addition.* For example, $x^2 - 5x + 8 = x^2 + (-5x) + 8$ has three terms: x^2 and $-5x$ are the **variable terms** and 8 is the **constant term.** The numerical factor of a term is called the **coefficient.** For instance, the coefficient of $-5x$ is -5, and the coefficient of x^2 is 1.

EXAMPLE 11 Identifying Terms and Coefficients

	Algebraic Expression	Terms	Coefficients
a.	$5x - \frac{1}{7}$	$5x, -\frac{1}{7}$	$5, -\frac{1}{7}$
b.	$2x^2 - 6x + 9$	$2x^2, -6x, 9$	$2, -6, 9$
c.	$\frac{3}{x} + \frac{1}{2}x^4 - y$	$\frac{3}{x}, \frac{1}{2}x^4, -y$	$3, \frac{1}{2}, -1$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Identify the terms and coefficients of $-2x + 4$.

To **evaluate** an algebraic expression, substitute numerical values for each of the variables in the expression, as shown in the next example.

EXAMPLE 12 Evaluating Algebraic Expressions

	Expression	Value of Variable	Substitute.	Value of Expression
a.	$-3x + 5$	$x = 3$	$-3(3) + 5$	$-9 + 5 = -4$
b.	$3x^2 + 2x - 1$	$x = -1$	$3(-1)^2 + 2(-1) - 1$	$3 - 2 - 1 = 0$
c.	$\frac{2x}{x + 1}$	$x = -3$	$\frac{2(-3)}{-3 + 1}$	$\frac{-6}{-2} = 3$

Note that you must substitute the value for *each* occurrence of the variable.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Evaluate $4x - 5$ when $x = 0$.

Use the **Substitution Principle** to evaluate algebraic expressions. It states that "If $a = b$, then b can replace a in any expression involving a." In Example 12(a), for instance, 3 is *substituted* for x in the expression $-3x + 5$.

Basic Rules of Algebra

There are four arithmetic operations with real numbers: *addition*, *multiplication*, *subtraction*, and *division*, denoted by the symbols $+$, $\times$ or $\cdot$, $-$, and $\div$ or $/$, respectively. Of these, addition and multiplication are the two primary operations. Subtraction and division are the inverse operations of addition and multiplication, respectively.

Definitions of Subtraction and Division

Subtraction: Add the opposite.

$$a - b = a + (-b)$$

Division: Multiply by the reciprocal.

If $b \neq 0$, then $a/b = a\left(\frac{1}{b}\right) = \frac{a}{b}$.

In these definitions, $-b$ is the **additive inverse** (or opposite) of b, and $1/b$ is the **multiplicative inverse** (or reciprocal) of b. In the fractional form a/b, a is the **numerator** of the fraction and b is the **denominator.**

Because the properties of real numbers below are true for variables and algebraic expressions as well as for real numbers, they are often called the **Basic Rules of Algebra.** Try to formulate a verbal description of each property. For instance, the first property states that *the order in which two real numbers are added does not affect their sum.*

Basic Rules of Algebra

Let a, b, and c be real numbers, variables, or algebraic expressions.

Property		**Example**
Commutative Property of Addition:	$a + b = b + a$	$4x + x^2 = x^2 + 4x$
Commutative Property of Multiplication:	$ab = ba$	$(4 - x)x^2 = x^2(4 - x)$
Associative Property of Addition:	$(a + b) + c = a + (b + c)$	$(x + 5) + x^2 = x + (5 + x^2)$
Associative Property of Multiplication:	$(ab)c = a(bc)$	$(2x \cdot 3y)(8) = (2x)(3y \cdot 8)$
Distributive Properties:	$a(b + c) = ab + ac$	$3x(5 + 2x) = 3x \cdot 5 + 3x \cdot 2x$
	$(a + b)c = ac + bc$	$(y + 8)y = y \cdot y + 8 \cdot y$
Additive Identity Property:	$a + 0 = a$	$5y^2 + 0 = 5y^2$
Multiplicative Identity Property:	$a \cdot 1 = a$	$(4x^2)(1) = 4x^2$
Additive Inverse Property:	$a + (-a) = 0$	$5x^3 + (-5x^3) = 0$
Multiplicative Inverse Property:	$a \cdot \frac{1}{a} = 1, \quad a \neq 0$	$(x^2 + 4)\left(\frac{1}{x^2 + 4}\right) = 1$

Because subtraction is defined as "adding the opposite," the Distributive Properties are also true for subtraction. For instance, the "subtraction form" of $a(b + c) = ab + ac$ is $a(b - c) = ab - ac$. Note that the operations of subtraction and division are neither commutative nor associative. The examples

$$7 - 3 \neq 3 - 7 \quad \text{and} \quad 20 \div 4 \neq 4 \div 20$$

show that subtraction and division are not commutative. Similarly

$$5 - (3 - 2) \neq (5 - 3) - 2 \quad \text{and} \quad 16 \div (4 \div 2) \neq (16 \div 4) \div 2$$

demonstrate that subtraction and division are not associative.

EXAMPLE 13 Identifying Rules of Algebra

Identify the rule of algebra illustrated by the statement.

a. $(5x^3)2 = 2(5x^3)$

b. $(4x + 3) - (4x + 3) = 0$

c. $7x \cdot \frac{1}{7x} = 1, \quad x \neq 0$

d. $(2 + 5x^2) + x^2 = 2 + (5x^2 + x^2)$

Solution

a. This statement illustrates the Commutative Property of Multiplication. In other words, you obtain the same result whether you multiply $5x^3$ by 2, or 2 by $5x^3$.

b. This statement illustrates the Additive Inverse Property. In terms of subtraction, this property states that when any expression is subtracted from itself the result is 0.

c. This statement illustrates the Multiplicative Inverse Property. Note that x must be a nonzero number. The reciprocal of x is undefined when x is 0.

d. This statement illustrates the Associative Property of Addition. In other words, to form the sum $2 + 5x^2 + x^2$, it does not matter whether 2 and $5x^2$, or $5x^2$ and x^2 are added first.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Identify the rule of algebra illustrated by the statement.

a. $x + 9 = 9 + x$ **b.** $5(x^3 \cdot 2) = (5x^3)2$ **c.** $(2 + 5x^2)y^2 = 2 \cdot y^2 + 5x^2 \cdot y^2$

REMARK Notice the difference between the *opposite of a number* and a *negative number.* If a is negative, then its opposite, $-a$, is positive. For instance, if $a = -5$, then

$$-a = -(-5) = 5.$$

Properties of Negation and Equality

Let a, b, and c be real numbers, variables, or algebraic expressions.

Property	**Example**
1. $(-1)a = -a$	$(-1)7 = -7$
2. $-(-a) = a$	$-(-6) = 6$
3. $(-a)b = -(ab) = a(-b)$	$(-5)3 = -(5 \cdot 3) = 5(-3)$
4. $(-a)(-b) = ab$	$(-2)(-x) = 2x$
5. $-(a + b) = (-a) + (-b)$	$-(x + 8) = (-x) + (-8)$ $= -x - 8$
6. If $a = b$, then $a \pm c = b \pm c$.	$\frac{1}{2} + 3 = 0.5 + 3$
7. If $a = b$, then $ac = bc$.	$4^2 \cdot 2 = 16 \cdot 2$
8. If $a \pm c = b \pm c$, then $a = b$.	$1.4 - 1 = \frac{7}{5} - 1 \implies 1.4 = \frac{7}{5}$
9. If $ac = bc$ and $c \neq 0$, then $a = b$.	$3x = 3 \cdot 4 \implies x = 4$

REMARK The "or" in the Zero-Factor Property includes the possibility that either or both factors may be zero. This is an *inclusive or,* and it is generally the way the word "or" is used in mathematics.

Properties of Zero

Let a and b be real numbers, variables, or algebraic expressions.

1. $a + 0 = a$ and $a - 0 = a$

2. $a \cdot 0 = 0$

3. $\frac{0}{a} = 0, \quad a \neq 0$

4. $\frac{a}{0}$ is undefined.

5. Zero-Factor Property: If $ab = 0$, then $a = 0$ or $b = 0$.

Properties and Operations of Fractions

Let a, b, c, and d be real numbers, variables, or algebraic expressions such that $b \neq 0$ and $d \neq 0$.

1. **Equivalent Fractions:** $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$.
2. **Rules of Signs:** $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$ and $\frac{-a}{-b} = \frac{a}{b}$
3. **Generate Equivalent Fractions:** $\frac{a}{b} = \frac{ac}{bc}, \quad c \neq 0$
4. **Add or Subtract with Like Denominators:** $\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$
5. **Add or Subtract with Unlike Denominators:** $\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$
6. **Multiply Fractions:** $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
7. **Divide Fractions:** $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}, \quad c \neq 0$

REMARK In Property 1 of fractions, the phrase "if and only if" implies two statements. One statement is: If $a/b = c/d$, then $ad = bc$. The other statement is: If $ad = bc$, where $b \neq 0$ and $d \neq 0$, then $a/b = c/d$.

EXAMPLE 14 Properties and Operations of Fractions

a. Equivalent fractions: $\frac{x}{5} = \frac{3 \cdot x}{3 \cdot 5} = \frac{3x}{15}$ **b.** Divide fractions: $\frac{7}{x} \div \frac{3}{2} = \frac{7}{x} \cdot \frac{2}{3} = \frac{14}{3x}$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

a. Multiply fractions: $\frac{3}{5} \cdot \frac{x}{6}$ **b.** Add fractions: $\frac{x}{10} + \frac{2x}{5}$

If a, b, and c are integers such that $ab = c$, then a and b are **factors** or **divisors** of c. A **prime number** is an integer that has exactly two positive factors—itself and 1—such as 2, 3, 5, 7, and 11. The numbers 4, 6, 8, 9, and 10 are **composite** because each can be written as the product of two or more prime numbers. The **Fundamental Theorem of Arithmetic** states that every positive integer greater than 1 is a prime number or can be written as the product of prime numbers in precisely one way (disregarding order). For instance, the *prime factorization* of 24 is $24 = 2 \cdot 2 \cdot 2 \cdot 3$.

REMARK The number 1 is neither prime nor composite.

Summarize (Section P.1)

1. Describe how to represent and classify real numbers *(pages 2 and 3)*. For examples of representing and classifying real numbers, see Examples 1 and 2.
2. Describe how to order real numbers and use inequalities *(pages 4 and 5)*. For examples of ordering real numbers and using inequalities, see Examples 3–6.
3. State the absolute value of a real number *(page 6)*. For examples of using absolute value, see Examples 7–10.
4. Explain how to evaluate an algebraic expression *(page 8)*. For examples involving algebraic expressions, see Examples 11 and 12.
5. State the basic rules and properties of algebra *(pages 9–11)*. For examples involving the basic rules and properties of algebra, see Examples 13 and 14.

P.1 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. ________ numbers have infinite nonrepeating decimal representations.

2. The point 0 on the real number line is called the ________.

3. The distance between the origin and a point representing a real number on the real number line is the ________ ________ of the real number.

4. A number that can be written as the product of two or more prime numbers is called a ________ number.

5. The ________ of an algebraic expression are those parts separated by addition.

6. The ________ ________ states that if $ab = 0$, then $a = 0$ or $b = 0$.

Skills and Applications

Classifying Real Numbers

In Exercises 7–10, determine which numbers in the set are (a) natural numbers, (b) whole numbers, (c) integers, (d) rational numbers, and (e) irrational numbers.

7. $\left\{-9, -\frac{7}{2}, 5, \frac{2}{3}, \sqrt{2}, 0, 1, -4, 2, -11\right\}$

8. $\left\{\sqrt{5}, -7, -\frac{7}{3}, 0, 3.12, \frac{5}{4}, -3, 12, 5\right\}$

9. $\{2.01, 0.666\ldots, -13, 0.010110111\ldots, 1, -6\}$

10. $\left\{25, -17, -\frac{12}{5}, \sqrt{9}, 3.12, \frac{1}{2}\pi, 7, -11.1, 13\right\}$

Plotting Points on the Real Number Line

In Exercises 11 and 12, plot the real numbers on the real number line.

11. (a) 3 (b) $\frac{7}{2}$ (c) $-\frac{5}{2}$ (d) -5.2

12. (a) 8.5 (b) $\frac{4}{3}$ (c) -4.75 (d) $-\frac{8}{3}$

Plotting and Ordering Real Numbers

In Exercises 13–16, plot the two real numbers on the real number line. Then place the appropriate inequality symbol (< or >) between them.

13. $-4, -8$ **14.** $1, \frac{16}{3}$ **15.** $\frac{5}{6}, \frac{2}{3}$ **16.** $-\frac{8}{7}, -\frac{3}{7}$

Interpreting an Inequality or an Interval

In Exercises 17–24, (a) give a verbal description of the subset of real numbers represented by the inequality or the interval, (b) sketch the subset on the real number line, and (c) state whether the interval is bounded or unbounded.

17. $x \leq 5$ **18.** $x < 0$

19. $[4, \infty)$ **20.** $(-\infty, 2)$

21. $-2 < x < 2$ **22.** $0 < x \leq 6$

23. $[-5, 2)$ **24.** $(-1, 2]$

Using Inequality and Interval Notation

In Exercises 25–30, use inequality notation and interval notation to describe the set.

25. y is nonnegative. **26.** y is no more than 25.

27. t is at least 10 and at most 22.

28. k is less than 5 but no less than -3.

29. The dog's weight W is more than 65 pounds.

30. The annual rate of inflation r is expected to be at least 2.5% but no more than 5%.

Evaluating an Absolute Value Expression

In Exercises 31–40, evaluate the expression.

31. $|-10|$ **32.** $|0|$

33. $|3 - 8|$ **34.** $|4 - 1|$

35. $|-1| - |-2|$ **36.** $-3 - |-3|$

37. $\dfrac{-5}{|-5|}$ **38.** $-3|-3|$

39. $\dfrac{|x + 2|}{x + 2}, \quad x < -2$ **40.** $\dfrac{|x - 1|}{x - 1}, \quad x > 1$

Comparing Real Numbers

In Exercises 41–44, place the appropriate symbol (<, >, or =) between the two real numbers.

41. $|-4| \ \square\ |4|$ **42.** $-5 \ \square\ -|5|$

43. $-|-6| \ \square\ |-6|$ **44.** $-|-2| \ \square\ -|2|$

Finding a Distance

In Exercises 45–50, find the distance between *a* and *b*.

45. $a = 126, b = 75$ **46.** $a = -126, b = -75$

47. $a = -\frac{5}{2}, b = 0$ **48.** $a = \frac{1}{4}, b = \frac{11}{4}$

49. $a = \frac{16}{5}, b = \frac{112}{75}$ **50.** $a = 9.34, b = -5.65$

Using Absolute Value Notation

In Exercises 51–54, use absolute value notation to describe the situation.

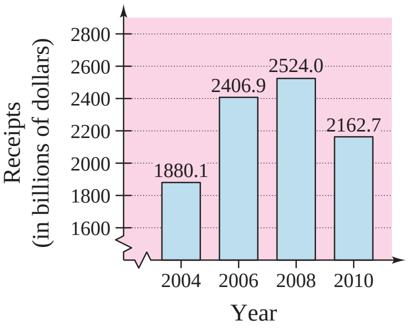
51. The distance between x and 5 is no more than 3.

52. The distance between x and -10 is at least 6.

53. y is at most two units from a.

54. The temperature in Bismarck, North Dakota, was 60°F at noon, then 23°F at midnight. What was the change in temperature over the 12-hour period?

Federal Deficit In Exercises 55–58, use the bar graph, which shows the receipts of the federal government (in billions of dollars) for selected years from 2004 through 2010. In each exercise you are given the expenditures of the federal government. Find the magnitude of the surplus or deficit for the year. *(Source: U.S. Office of Management and Budget)*



	Year	Receipts, R	Expenditures, E	$\lvert R - E\rvert$
55.	2004	▨	\$2292.8 billion	▨
56.	2006	▨	\$2655.1 billion	▨
57.	2008	▨	\$2982.5 billion	▨
58.	2010	▨	\$3456.2 billion	▨

Identifying Terms and Coefficients In Exercises 59–62, identify the terms. Then identify the coefficients of the variable terms of the expression.

59. $7x + 4$

60. $6x^3 - 5x$

61. $4x^3 + 0.5x - 5$

62. $3\sqrt{3}x^2 + 1$

Evaluating an Algebraic Expression In Exercises 63–66, evaluate the expression for each value of x. (If not possible, then state the reason.)

	Expression	Values	
63.	$4x - 6$	(a) $x = -1$	(b) $x = 0$
64.	$9 - 7x$	(a) $x = -3$	(b) $x = 3$
65.	$-x^2 + 5x - 4$	(a) $x = -1$	(b) $x = 1$
66.	$(x + 1)/(x - 1)$	(a) $x = 1$	(b) $x = -1$

Identifying Rules of Algebra In Exercises 67–72, identify the rule(s) of algebra illustrated by the statement.

67. $\dfrac{1}{h+6}(h + 6) = 1, \quad h \neq -6$

68. $(x + 3) - (x + 3) = 0$

69. $2(x + 3) = 2 \cdot x + 2 \cdot 3$

70. $(z - 2) + 0 = z - 2$

71. $x(3y) = (x \cdot 3)y = (3x)y$

72. $\frac{1}{7}(7 \cdot 12) = \left(\frac{1}{7} \cdot 7\right)12 = 1 \cdot 12 = 12$

Operations with Fractions In Exercises 73–76, perform the operation(s). (Write fractional answers in simplest form.)

73. $\frac{5}{8} - \frac{5}{12} + \frac{1}{6}$

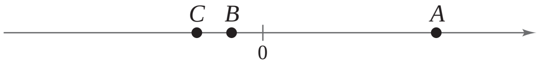
74. $-\left(6 \cdot \frac{4}{8}\right)$

75. $\dfrac{2x}{3} - \dfrac{x}{4}$

76. $\dfrac{5x}{6} \cdot \dfrac{2}{9}$

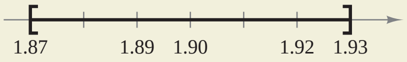
Exploration

77. Determining the Sign of an Expression Use the real numbers A, B, and C shown on the number line to determine the sign of (a) $-A$, (b) $B - A$, (c) $-C$, and (d) $A - C$.

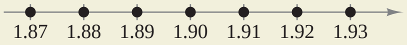


78. HOW DO YOU SEE IT? Match each description with its graph. Which types of real numbers shown in Figure P.1 on page 2 may be included in a range of prices? a range of lengths? Explain.

(i)



(ii)



(a) The price of an item is within \$0.03 of \$1.90.

(b) The distance between the prongs of an electric plug may not differ from 1.9 centimeters by more than 0.03 centimeter.

True or False? In Exercises 79 and 80, determine whether the statement is true or false. Justify your answer.

79. Every nonnegative number is positive.

80. If $a > 0$ and $b < 0$, then $ab > 0$.

81. Conjecture

(a) Use a calculator to complete the table.

n	0.0001	0.01	1	100	10,000
$5/n$					

(b) Use the result from part (a) to make a conjecture about the value of $5/n$ as n (i) approaches 0, and (ii) increases without bound.

Michael G Smith/Shutterstock.com

P.2 Exponents and Radicals

Real numbers and algebraic expressions are often written with exponents and radicals. For instance, in Exercise 79 on page 25, you will use an expression involving rational exponents to find the times required for a funnel to empty for different water heights.

- **Use properties of exponents.**
- **Use scientific notation to represent real numbers.**
- **Use properties of radicals.**
- **Simplify and combine radicals.**
- **Rationalize denominators and numerators.**
- **Use properties of rational exponents.**

Integer Exponents

Repeated *multiplication* can be written in **exponential form.**

Repeated Multiplication	**Exponential Form**
$a \cdot a \cdot a \cdot a \cdot a$	a^5
$(-4)(-4)(-4)$	$(-4)^3$
$(2x)(2x)(2x)(2x)$	$(2x)^4$

Exponential Notation

If a is a real number and n is a positive integer, then

$$a^n = \underbrace{a \cdot a \cdot a \cdot \cdot \cdot a}_{n \text{ factors}}$$

where n is the **exponent** and a is the **base.** You read a^n as "a to the nth **power.**"

An exponent can also be negative. In Property 3 below, be sure you see how to use a negative exponent.

Properties of Exponents

Let a and b be real numbers, variables, or algebraic expressions, and let m and n be integers. (All denominators and bases are nonzero.)

Property	**Example**
1. $a^m a^n = a^{m+n}$	$3^2 \cdot 3^4 = 3^{2+4} = 3^6 = 729$
2. $\dfrac{a^m}{a^n} = a^{m-n}$	$\dfrac{x^7}{x^4} = x^{7-4} = x^3$
3. $a^{-n} = \dfrac{1}{a^n} = \left(\dfrac{1}{a}\right)^n$	$y^{-4} = \dfrac{1}{y^4} = \left(\dfrac{1}{y}\right)^4$
4. $a^0 = 1, \quad a \neq 0$	$(x^2 + 1)^0 = 1$
5. $(ab)^m = a^m b^m$	$(5x)^3 = 5^3x^3 = 125x^3$
6. $(a^m)^n = a^{mn}$	$(y^3)^{-4} = y^{3(-4)} = y^{-12} = \dfrac{1}{y^{12}}$
7. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$	$\left(\dfrac{2}{x}\right)^3 = \dfrac{2^3}{x^3} = \dfrac{8}{x^3}$
8. $\lvert a^2\rvert = \lvert a\rvert^2 = a^2$	$\lvert(-2)^2\rvert = \lvert-2\rvert^2 = (2)^2 = 4$

micropic/iStockphoto.com

TECHNOLOGY You can use a calculator to evaluate exponential expressions. When doing so, it is important to know when to use parentheses because the calculator follows the order of operations. For instance, you would evaluate $(-2)^4$ on a graphing calculator as follows.

$\boxed{(}\ \boxed{(-)}\ 2\ \boxed{)}\ \boxed{\wedge}\ 4\ \boxed{\text{ENTER}}$

The display will be 16. If you omit the parentheses, then the display will be -16.

It is important to recognize the difference between expressions such as $(-2)^4$ and -2^4. In $(-2)^4$, the parentheses indicate that the exponent applies to the negative sign as well as to the 2, but in $-2^4 = -(2^4)$, the exponent applies only to the 2. So, $(-2)^4 = 16$ and $-2^4 = -16$.

The properties of exponents listed on the preceding page apply to *all* integers m and n, not just to positive integers, as shown in the examples below.

EXAMPLE 1 Evaluating Exponential Expressions

a. $(-5)^2 = (-5)(-5) = 25$ Negative sign is part of the base.

b. $-5^2 = -(5)(5) = -25$ Negative sign is *not* part of the base.

c. $2 \cdot 2^4 = 2^{1+4} = 2^5 = 32$ Property 1

d. $\dfrac{4^4}{4^6} = 4^{4-6} = 4^{-2} = \dfrac{1}{4^2} = \dfrac{1}{16}$ Properties 2 and 3

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Evaluate each expression.

a. -3^4 **b.** $(-3)^4$

c. $3^2 \cdot 3$ **d.** $\dfrac{3^5}{3^8}$

EXAMPLE 2 Evaluating Algebraic Expressions

Evaluate each algebraic expression when $x = 3$.

a. $5x^{-2}$ **b.** $\dfrac{1}{3}(-x)^3$

Solution

a. When $x = 3$, the expression $5x^{-2}$ has a value of

$$5x^{-2} = 5(3)^{-2} = \frac{5}{3^2} = \frac{5}{9}.$$

b. When $x = 3$, the expression $\frac{1}{3}(-x)^3$ has a value of

$$\frac{1}{3}(-x)^3 = \frac{1}{3}(-3)^3 = \frac{1}{3}(-27) = -9.$$

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Evaluate each algebraic expression when $x = 4$.

a. $-x^{-2}$ **b.** $\dfrac{1}{4}(-x)^4$

EXAMPLE 3 Using Properties of Exponents

Use the properties of exponents to simplify each expression.

a. $(-3ab^4)(4ab^{-3})$ **b.** $(2xy^2)^3$ **c.** $3a(-4a^2)^0$ **d.** $\left(\frac{5x^3}{y}\right)^2$

Solution

a. $(-3ab^4)(4ab^{-3}) = (-3)(4)(a)(a)(b^4)(b^{-3}) = -12a^2b$

b. $(2xy^2)^3 = 2^3(x)^3(y^2)^3 = 8x^3y^6$

c. $3a(-4a^2)^0 = 3a(1) = 3a, \quad a \neq 0$

d. $\left(\frac{5x^3}{y}\right)^2 = \frac{5^2(x^3)^2}{y^2} = \frac{25x^6}{y^2}$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use the properties of exponents to simplify each expression.

a. $(2x^{-2}y^3)(-x^4y)$ **b.** $(4a^2b^3)^0$ **c.** $(-5z)^3(z^2)$ **d.** $\left(\frac{3x^4}{x^2y^2}\right)^2$

REMARK Rarely in algebra is there only one way to solve a problem. Do not be concerned when the steps you use to solve a problem are not exactly the same as the steps presented in this text. It is important to use steps that you understand and, of course, steps that are justified by the rules of algebra. For instance, you might prefer the following steps for Example 4(d).

$$\left(\frac{3x^2}{y}\right)^{-2} = \left(\frac{y}{3x^2}\right)^2 = \frac{y^2}{9x^4}$$

Note how the first step of this solution uses Property 3. The fractional form of this property is

$$\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m.$$

EXAMPLE 4 Rewriting with Positive Exponents

a. $x^{-1} = \frac{1}{x}$ Property 3

b. $\frac{1}{3x^{-2}} = \frac{1(x^2)}{3}$ The exponent -2 does not apply to 3.

$= \frac{x^2}{3}$ Simplify.

c. $\frac{12a^3b^{-4}}{4a^{-2}b} = \frac{12a^3 \cdot a^2}{4b \cdot b^4}$ Property 3

$= \frac{3a^5}{b^5}$ Property 1

d. $\left(\frac{3x^2}{y}\right)^{-2} = \frac{3^{-2}(x^2)^{-2}}{y^{-2}}$ Properties 5 and 7

$= \frac{3^{-2}x^{-4}}{y^{-2}}$ Property 6

$= \frac{y^2}{3^2x^4}$ Property 3

$= \frac{y^2}{9x^4}$ Simplify.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Rewrite each expression with positive exponents.

a. $2a^{-2}$ **b.** $\frac{3a^{-3}b^4}{15ab^{-1}}$

c. $\left(\frac{x}{10}\right)^{-1}$ **d.** $(-2x^2)^3(4x^3)^{-1}$

HISTORICAL NOTE

The French mathematician Nicolas Chuquet (ca. 1500) wrote *Triparty en la science des nombres,* in which he used a form of exponent notation. He wrote the expressions $6x^3$ and $10x^2$ as $.6.^3$ and $.10.^2$, respectively. Chuquet also represented zero and negative exponents. He wrote x^0 as $.1.^0$ and $3x^{-2}$ as $.3.^{2m}$. Chuquet wrote that $.72.^1$ divided by $.8.^3$ is $.9.^{2m}$. That is, $72x \div 8x^3 = 9x^{-2}$.

There are about 359 billion billion gallons of water on Earth. It is convenient to write such a number in scientific notation.

Scientific Notation

Exponents provide an efficient way of writing and computing with very large (or very small) numbers. For instance, there are about 359 billion billion gallons of water on Earth—that is, 359 followed by 18 zeros.

$$359{,}000{,}000{,}000{,}000{,}000{,}000$$

It is convenient to write such numbers in **scientific notation.** This notation has the form $\pm c \times 10^n$, where $1 \leq c < 10$ and n is an integer. So, the number of gallons of water on Earth, written in scientific notation, is

$$3.59 \times 100{,}000{,}000{,}000{,}000{,}000{,}000 = 3.59 \times 10^{20}.$$

The *positive* exponent 20 indicates that the number is *large* (10 or more) and that the decimal point has been moved 20 places. A *negative* exponent indicates that the number is *small* (less than 1). For instance, the mass (in grams) of one electron is approximately

$$9.0 \times 10^{-28} = 0.\underbrace{0000000000000000000000000009}_{\text{28 decimal places}}.$$

EXAMPLE 5 Scientific Notation

a. $0.0000782 = 7.82 \times 10^{-5}$

b. $836{,}100{,}000 = 8.361 \times 10^{8}$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write 45,850 in scientific notation.

EXAMPLE 6 Decimal Notation

a. $-9.36 \times 10^{-6} = -0.00000936$

b. $1.345 \times 10^{2} = 134.5$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write -2.718×10^{-3} in decimal notation.

TECHNOLOGY Most calculators automatically switch to scientific notation when they are showing large (or small) numbers that exceed the display range.

To enter numbers in scientific notation, your calculator should have an exponential entry key labeled

$\boxed{\text{EE}}$ or $\boxed{\text{EXP}}$.

Consult the user's guide for instructions on keystrokes and how your calculator displays numbers in scientific notation.

EXAMPLE 7 Using Scientific Notation

Evaluate $\dfrac{(2{,}400{,}000{,}000)(0.0000045)}{(0.00003)(1500)}$.

Solution Begin by rewriting each number in scientific notation and simplifying.

$$\begin{aligned}\frac{(2{,}400{,}000{,}000)(0.0000045)}{(0.00003)(1500)} &= \frac{(2.4 \times 10^{9})(4.5 \times 10^{-6})}{(3.0 \times 10^{-5})(1.5 \times 10^{3})}\\ &= \frac{(2.4)(4.5)(10^{3})}{(4.5)(10^{-2})}\\ &= (2.4)(10^{5})\\ &= 240{,}000\end{aligned}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Evaluate (24,000,000,000)(0.00000012)(300,000). ■

EpicStockMedia/Shutterstock.com

Radicals and Their Properties

A **square root** of a number is one of its two equal factors. For example, 5 is a square root of 25 because 5 is one of the two equal factors of 25. In a similar way, a **cube root** of a number is one of its three equal factors, as in $125 = 5^3$.

Definition of *n*th Root of a Number

Let a and b be real numbers and let $n \geq 2$ be a positive integer. If

$$a = b^n$$

then b is an ***n*th root of *a*.** When $n = 2$, the root is a **square root.** When $n = 3$, the root is a **cube root.**

Some numbers have more than one nth root. For example, both 5 and -5 are square roots of 25. The *principal square* root of 25, written as $\sqrt{25}$, is the positive root, 5. The **principal *n*th root** of a number is defined as follows.

Principal *n*th Root of a Number

Let a be a real number that has at least one nth root. The **principal *n*th root of *a*** is the nth root that has the same sign as a. It is denoted by a **radical symbol**

$$\sqrt[n]{a}. \qquad \text{Principal } n\text{th root}$$

The positive integer $n \geq 2$ is the **index** of the radical, and the number a is the **radicand.** When $n = 2$, omit the index and write $\sqrt{a}$ rather than $\sqrt[2]{a}$. (The plural of index is *indices.*)

A common misunderstanding is that the square root sign implies both negative and positive roots. This is not correct. The square root sign implies only a positive root. When a negative root is needed, you must use the negative sign with the square root sign.

Incorrect: $\xcancel{\sqrt{4} = \pm 2}$ *Correct:* $-\sqrt{4} = -2$ *and* $\sqrt{4} = 2$

EXAMPLE 8 Evaluating Expressions Involving Radicals

a. $\sqrt{36} = 6$ because $6^2 = 36$.

b. $-\sqrt{36} = -6$ because $-\left(\sqrt{36}\right) = -\left(\sqrt{6^2}\right) = -(6) = -6$.

c. $\sqrt[3]{\dfrac{125}{64}} = \dfrac{5}{4}$ because $\left(\dfrac{5}{4}\right)^3 = \dfrac{5^3}{4^3} = \dfrac{125}{64}$.

d. $\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$.

e. $\sqrt[4]{-81}$ is not a real number because no real number raised to the fourth power produces -81.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Evaluate each expression (if possible).

a. $-\sqrt{144}$ **b.** $\sqrt{-144}$

c. $\sqrt{\dfrac{25}{64}}$ **d.** $-\sqrt[3]{\dfrac{8}{27}}$

Here are some generalizations about the *n*th roots of real numbers.

Generalizations About *n*th Roots of Real Numbers			
Real Number a	Integer $n > 0$	Root(s) of a	Example
$a > 0$	n is even.	$\sqrt[n]{a},\ -\sqrt[n]{a}$	$\sqrt[4]{81} = 3,\ -\sqrt[4]{81} = -3$
$a > 0$ or $a < 0$	n is odd.	$\sqrt[n]{a}$	$\sqrt[3]{-8} = -2$
$a < 0$	n is even.	No real roots	$\sqrt{-4}$ is not a real number.
$a = 0$	n is even or odd.	$\sqrt[n]{0} = 0$	$\sqrt[5]{0} = 0$

Integers such as 1, 4, 9, 16, 25, and 36 are called **perfect squares** because they have integer square roots. Similarly, integers such as 1, 8, 27, 64, and 125 are called **perfect cubes** because they have integer cube roots.

Properties of Radicals

Let a and b be real numbers, variables, or algebraic expressions such that the indicated roots are real numbers, and let m and n be positive integers.

Property	Example
1. $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$	$\sqrt[3]{8^2} = \left(\sqrt[3]{8}\right)^2 = (2)^2 = 4$
2. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$	$\sqrt{5} \cdot \sqrt{7} = \sqrt{5 \cdot 7} = \sqrt{35}$
3. $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}, \quad b \neq 0$	$\dfrac{\sqrt[4]{27}}{\sqrt[4]{9}} = \sqrt[4]{\dfrac{27}{9}} = \sqrt[4]{3}$
4. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$	$\sqrt[3]{\sqrt{10}} = \sqrt[6]{10}$
5. $\left(\sqrt[n]{a}\right)^n = a$	$\left(\sqrt{3}\right)^2 = 3$
6. For n even, $\sqrt[n]{a^n} = \lvert a \rvert$.	$\sqrt{(-12)^2} = \lvert -12 \rvert = 12$
For n odd, $\sqrt[n]{a^n} = a$.	$\sqrt[3]{(-12)^3} = -12$

A common use of Property 6 is $\sqrt{a^2} = \lvert a \rvert$.

EXAMPLE 9 Using Properties of Radicals

Use the properties of radicals to simplify each expression.

a. $\sqrt{8} \cdot \sqrt{2}$ **b.** $\left(\sqrt[3]{5}\right)^3$

c. $\sqrt[3]{x^3}$ **d.** $\sqrt[6]{y^6}$

Solution

a. $\sqrt{8} \cdot \sqrt{2} = \sqrt{8 \cdot 2} = \sqrt{16} = 4$ **b.** $\left(\sqrt[3]{5}\right)^3 = 5$

c. $\sqrt[3]{x^3} = x$ **d.** $\sqrt[6]{y^6} = \lvert y \rvert$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use the properties of radicals to simplify each expression.

a. $\dfrac{\sqrt{125}}{\sqrt{5}}$ **b.** $\sqrt[3]{125^2}$ **c.** $\sqrt[3]{x^2} \cdot \sqrt[3]{x}$ **d.** $\sqrt{\sqrt{x}}$

Simplifying Radicals

An expression involving radicals is in **simplest form** when the following conditions are satisfied.

1. All possible factors have been removed from the radical.
2. All fractions have radical-free denominators (a process called *rationalizing the denominator* accomplishes this).
3. The index of the radical is reduced.

To simplify a radical, factor the radicand into factors whose exponents are multiples of the index. Write the roots of these factors outside the radical. The "leftover" factors make up the new radicand.

REMARK When you simplify a radical, it is important that both expressions are defined for the same values of the variable. For instance, in Example 10(c), $\sqrt{75x^3}$ and $5x\sqrt{3x}$ are both defined only for nonnegative values of x. Similarly, in Example 10(e), $\sqrt[4]{(5x)^4}$ and $5|x|$ are both defined for all real values of x.

EXAMPLE 10 Simplifying Radicals

a. $\sqrt[3]{24} = \sqrt[3]{\overset{\text{Perfect cube}}{8} \cdot \overset{\text{Leftover factor}}{3}} = \sqrt[3]{2^3 \cdot 3} = 2\sqrt[3]{3}$

b. $\sqrt[4]{48} = \sqrt[4]{\overset{\text{Perfect 4th power}}{16} \cdot \overset{\text{Leftover factor}}{3}} = \sqrt[4]{2^4 \cdot 3} = 2\sqrt[4]{3}$

c. $\sqrt{75x^3} = \sqrt{25x^2 \cdot 3x} = \sqrt{(5x)^2 \cdot 3x} = 5x\sqrt{3x}$

d. $\sqrt[3]{24a^4} = \sqrt[3]{8a^3 \cdot 3a} = \sqrt[3]{(2a)^3 \cdot 3a} = 2a\sqrt[3]{3a}$

e. $\sqrt[4]{(5x)^4} = |5x| = 5|x|$

✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Simplify each radical expression.

a. $\sqrt{32}$ **b.** $\sqrt[3]{250}$ **c.** $\sqrt{24a^5}$ **d.** $\sqrt[3]{-135x^3}$

Radical expressions can be combined (added or subtracted) when they are **like radicals**—that is, when they have the same index and radicand. For instance, $\sqrt{2}$, $3\sqrt{2}$, and $\frac{1}{2}\sqrt{2}$ are like radicals, but $\sqrt{3}$ and $\sqrt{2}$ are unlike radicals. To determine whether two radicals can be combined, you should first simplify each radical.

EXAMPLE 11 Combining Radicals

a.
$$\begin{aligned} 2\sqrt{48} - 3\sqrt{27} &= 2\sqrt{16 \cdot 3} - 3\sqrt{9 \cdot 3} && \text{Find square factors.} \\ &= 8\sqrt{3} - 9\sqrt{3} && \text{Find square roots and multiply by coefficients.} \\ &= (8 - 9)\sqrt{3} && \text{Combine like radicals.} \\ &= -\sqrt{3} && \text{Simplify.} \end{aligned}$$

b.
$$\begin{aligned} \sqrt[3]{16x} - \sqrt[3]{54x^4} &= \sqrt[3]{8 \cdot 2x} - \sqrt[3]{27 \cdot x^3 \cdot 2x} && \text{Find cube factors.} \\ &= 2\sqrt[3]{2x} - 3x\sqrt[3]{2x} && \text{Find cube roots.} \\ &= (2 - 3x)\sqrt[3]{2x} && \text{Combine like radicals.} \end{aligned}$$

✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Simplify each radical expression.

a. $3\sqrt{8} + \sqrt{18}$ **b.** $\sqrt[3]{81x^5} - \sqrt[3]{24x^2}$

Rationalizing Denominators and Numerators

To rationalize a denominator or numerator of the form $a - b\sqrt{m}$ or $a + b\sqrt{m}$, multiply both numerator and denominator by a **conjugate:** $a + b\sqrt{m}$ and $a - b\sqrt{m}$ are conjugates of each other. If $a = 0$, then the rationalizing factor for $\sqrt{m}$ is itself, $\sqrt{m}$. For cube roots, choose a rationalizing factor that generates a perfect cube.

EXAMPLE 12 Rationalizing Single-Term Denominators

Rationalize the denominator of each expression.

a. $\dfrac{5}{2\sqrt{3}}$

b. $\dfrac{2}{\sqrt[3]{5}}$

Solution

a.
$$\begin{aligned}\frac{5}{2\sqrt{3}} &= \frac{5}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} && \sqrt{3} \text{ is rationalizing factor.}\\ &= \frac{5\sqrt{3}}{2(3)} && \text{Multiply.}\\ &= \frac{5\sqrt{3}}{6} && \text{Simplify.}\end{aligned}$$

b.
$$\begin{aligned}\frac{2}{\sqrt[3]{5}} &= \frac{2}{\sqrt[3]{5}} \cdot \frac{\sqrt[3]{5^2}}{\sqrt[3]{5^2}} && \sqrt[3]{5^2} \text{ is rationalizing factor.}\\ &= \frac{2\sqrt[3]{5^2}}{\sqrt[3]{5^3}} && \text{Multiply.}\\ &= \frac{2\sqrt[3]{25}}{5} && \text{Simplify.}\end{aligned}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Rationalize the denominator of each expression.

a. $\dfrac{5}{3\sqrt{2}}$ **b.** $\dfrac{1}{\sqrt[3]{25}}$

EXAMPLE 13 Rationalizing a Denominator with Two Terms

$$\begin{aligned}\frac{2}{3+\sqrt{7}} &= \frac{2}{3+\sqrt{7}} \cdot \frac{3-\sqrt{7}}{3-\sqrt{7}} && \text{Multiply numerator and denominator by conjugate of denominator.}\\ &= \frac{2\left(3-\sqrt{7}\right)}{3(3) + 3\left(-\sqrt{7}\right) + \sqrt{7}(3) - \left(\sqrt{7}\right)\left(\sqrt{7}\right)} && \text{Use Distributive Property.}\\ &= \frac{2\left(3-\sqrt{7}\right)}{(3)^2 - \left(\sqrt{7}\right)^2} && \text{Simplify.}\\ &= \frac{2\left(3-\sqrt{7}\right)}{2} = 3 - \sqrt{7} && \text{Simplify.}\end{aligned}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Rationalize the denominator: $\dfrac{8}{\sqrt{6} - \sqrt{2}}$.

Sometimes it is necessary to rationalize the numerator of an expression. For instance, in Section P.5 you will use the technique shown in the next example to rationalize the numerator of an expression from calculus.

EXAMPLE 14 Rationalizing a Numerator

$$\begin{aligned}\frac{\sqrt{5}-\sqrt{7}}{2} &= \frac{\sqrt{5}-\sqrt{7}}{2}\cdot\frac{\sqrt{5}+\sqrt{7}}{\sqrt{5}+\sqrt{7}} && \text{Multiply numerator and denominator by conjugate of denominator.}\\ &= \frac{(\sqrt{5})^2-(\sqrt{7})^2}{2(\sqrt{5}+\sqrt{7})} && \text{Simplify.}\\ &= \frac{5-7}{2(\sqrt{5}+\sqrt{7})} && \text{Square terms of numerator.}\\ &= \frac{-2}{2(\sqrt{5}+\sqrt{7})} = \frac{-1}{\sqrt{5}+\sqrt{7}} && \text{Simplify.}\end{aligned}$$

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Rationalize the numerator: $\dfrac{2-\sqrt{2}}{3}$.

REMARK Do not confuse the expression $\sqrt{5}+\sqrt{7}$ with the expression $\sqrt{5+7}$. In general, $\sqrt{x+y}$ does not equal $\sqrt{x}+\sqrt{y}$. Similarly, $\sqrt{x^2+y^2}$ does not equal $x+y$.

Rational Exponents

Definition of Rational Exponents

If a is a real number and n is a positive integer such that the principal nth root of a exists, then $a^{1/n}$ is defined as

$a^{1/n} = \sqrt[n]{a}$, where $1/n$ is the **rational exponent** of a.

Moreover, if m is a positive integer that has no common factor with n, then

$$a^{m/n} = (a^{1/n})^m = \left(\sqrt[n]{a}\right)^m \quad\text{and}\quad a^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m}.$$

The numerator of a rational exponent denotes the *power* to which the base is raised, and the denominator denotes the *index* or the *root* to be taken.

Power

Index

$$b^{m/n} = \left(\sqrt[n]{b}\right)^m = \sqrt[n]{b^m}$$

REMARK You must remember that the expression $b^{m/n}$ is not defined unless $\sqrt[n]{b}$ is a real number. This restriction produces some unusual results. For instance, the number $(-8)^{1/3}$ is defined because $\sqrt[3]{-8} = -2$, but the number $(-8)^{2/6}$ is undefined because $\sqrt[6]{-8}$ is not a real number.

When you are working with rational exponents, the properties of integer exponents still apply. For instance, $2^{1/2}2^{1/3} = 2^{(1/2)+(1/3)} = 2^{5/6}$.

EXAMPLE 15 Changing From Radical to Exponential Form

a. $\sqrt{3} = 3^{1/2}$ **b.** $\sqrt{(3xy)^5} = \sqrt[2]{(3xy)^5} = (3xy)^{5/2}$

c. $2x\sqrt[4]{x^3} = (2x)(x^{3/4}) = 2x^{1+(3/4)} = 2x^{7/4}$

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write each expression in exponential form.

a. $\sqrt[3]{27}$ **b.** $\sqrt{x^3y^5z}$ **c.** $3x\sqrt[3]{x^2}$

EXAMPLE 16 Changing From Exponential to Radical Form

a. $(x^2 + y^2)^{3/2} = \left(\sqrt{x^2 + y^2}\right)^3 = \sqrt{(x^2 + y^2)^3}$

b. $2y^{3/4}z^{1/4} = 2(y^3z)^{1/4} = 2\sqrt[4]{y^3z}$

c. $a^{-3/2} = \dfrac{1}{a^{3/2}} = \dfrac{1}{\sqrt{a^3}}$

d. $x^{0.2} = x^{1/5} = \sqrt[5]{x}$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write each expression in radical form.

a. $(x^2 - 7)^{-1/2}$ **b.** $-3b^{1/3}c^{2/3}$

c. $a^{0.75}$ **d.** $(x^2)^{2/5}$

TECHNOLOGY There are four methods of evaluating radicals on most graphing calculators. For square roots, you can use the *square root key* $\boxed{\sqrt{\ }}$. For cube roots, you can use the *cube root key* $\boxed{\sqrt[3]{\ }}$. For other roots, you can first convert the radical to exponential form and then use the *exponential key* $\boxed{\wedge}$, or you can use the *x*th root key $\boxed{\sqrt[x]{\ }}$ (or menu choice). Consult the user's guide for your calculator for specific keystrokes.

Rational exponents are useful for evaluating roots of numbers on a calculator, for reducing the index of a radical, and for simplifying expressions in calculus.

EXAMPLE 17 Simplifying with Rational Exponents

a. $(-32)^{-4/5} = \left(\sqrt[5]{-32}\right)^{-4} = (-2)^{-4} = \dfrac{1}{(-2)^4} = \dfrac{1}{16}$

b. $(-5x^{5/3})(3x^{-3/4}) = -15x^{(5/3)-(3/4)} = -15x^{11/12}, \quad x \neq 0$

c. $\sqrt[9]{a^3} = a^{3/9} = a^{1/3} = \sqrt[3]{a}$ Reduce index.

d. $\sqrt[3]{\sqrt{125}} = \sqrt[6]{125} = \sqrt[6]{(5)^3} = 5^{3/6} = 5^{1/2} = \sqrt{5}$

e. $(2x - 1)^{4/3}(2x - 1)^{-1/3} = (2x - 1)^{(4/3)-(1/3)} = 2x - 1, \quad x \neq \frac{1}{2}$

REMARK The expression in Example 17(e) is not defined when $x = \frac{1}{2}$ because

$$\left(2 \cdot \frac{1}{2} - 1\right)^{-1/3} = (0)^{-1/3}$$

is not a real number.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Simplify each expression.

a. $(-125)^{-2/3}$ **b.** $(4x^2y^{3/2})(-3x^{-1/3}y^{-3/5})$

c. $\sqrt[3]{\sqrt[4]{27}}$ **d.** $(3x + 2)^{5/2}(3x + 2)^{-1/2}$

Summarize (Section P.2)

1. Make a list of the properties of exponents *(page 14)*. For examples that use these properties, see Examples 1–4.
2. State the definition of scientific notation *(page 17)*. For examples of scientific notation, see Examples 5–7.
3. Make a list of the properties of radicals *(page 19)*. For an example that uses these properties, see Example 9.
4. Explain how to simplify a radical expression *(page 20)*. For examples of simplifying radical expressions, see Examples 10 and 11.
5. Explain how to rationalize a denominator or a numerator *(page 21)*. For examples of rationalizing denominators and numerators, see Examples 12–14.
6. State the definition of a rational exponent *(page 22)*. For an example of simplifying expressions with rational exponents, see Example 17.

P.2 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. In the exponential form a^n, n is the ________ and a is the ________.

2. A convenient way of writing very large or very small numbers is called ________ ________.

3. One of the two equal factors of a number is called a __________ __________ of the number.

4. In the radical form $\sqrt[n]{a}$, the positive integer n is the ________ of the radical and the number a is the ________.

5. Radical expressions can be combined (added or subtracted) when they are ________ ________.

6. The expressions $a + b\sqrt{m}$ and $a - b\sqrt{m}$ are ________ of each other.

7. The process used to create a radical-free denominator is known as ________ the denominator.

8. In the expression $b^{m/n}$, m denotes the ________ to which the base is raised and n denotes the ________ or root to be taken.

Skills and Applications

Evaluating Exponential Expressions

In Exercises 9–14, evaluate each expression.

9. (a) $3 \cdot 3^3$ (b) $\dfrac{3^2}{3^4}$

10. (a) $(3^3)^0$ (b) -3^2

11. (a) $(2^3 \cdot 3^2)^2$ (b) $\left(-\frac{3}{5}\right)^3\left(\frac{5}{3}\right)^2$

12. (a) $\dfrac{3}{3^{-4}}$ (b) $48(-4)^{-3}$

13. (a) $\dfrac{4 \cdot 3^{-2}}{2^{-2} \cdot 3^{-1}}$ (b) $(-2)^0$

14. (a) $3^{-1} + 2^{-2}$ (b) $(3^{-2})^2$

Evaluating an Algebraic Expression

In Exercises 15–20, evaluate the expression for the given value of x.

15. $-3x^3, \quad x = 2$

16. $7x^{-2}, \quad x = 4$

17. $6x^0, \quad x = 10$

18. $2x^3, \quad x = -3$

19. $-3x^4, \quad x = -2$

20. $12(-x)^3, \quad x = -\frac{1}{3}$

Using Properties of Exponents

In Exercises 21–26, simplify each expression.

21. (a) $(-5z)^3$ (b) $5x^4(x^2)$

22. (a) $(3x)^2$ (b) $(4x^3)^0, \quad x \neq 0$

23. (a) $6y^2(2y^0)^2$ (b) $(-z)^3(3z^4)$

24. (a) $\dfrac{7x^2}{x^3}$ (b) $\dfrac{12(x + y)^3}{9(x + y)}$

25. (a) $\left(\dfrac{4}{y}\right)^3\left(\dfrac{3}{y}\right)^4$ (b) $\left(\dfrac{b^{-2}}{a^{-2}}\right)\left(\dfrac{b}{a}\right)^2$

26. (a) $[(x^2y^{-2})^{-1}]^{-1}$ (b) $(5x^2z^6)^3(5x^2z^6)^{-3}$

Rewriting with Positive Exponents

In Exercises 27–30, rewrite each expression with positive exponents and simplify.

27. (a) $(x + 5)^0, \quad x \neq -5$ (b) $(2x^2)^{-2}$

28. (a) $(4y^{-2})(8y^4)$ (b) $(z + 2)^{-3}(z + 2)^{-1}$

29. (a) $\left(\dfrac{x^{-3}y^4}{5}\right)^{-3}$ (b) $\left(\dfrac{a^{-2}}{b^{-2}}\right)\left(\dfrac{b}{a}\right)^3$

30. (a) $3^n \cdot 3^{2n}$ (b) $\dfrac{x^2 \cdot x^n}{x^3 \cdot x^n}$

Scientific Notation

In Exercises 31–34, write the number in scientific notation.

31. 10,250.4

32. -0.000125

33. One micron (millionth of a meter): 0.00003937 inch

34. Land area of Earth: 57,300,000 square miles

Decimal Notation

In Exercises 35–38, write the number in decimal notation.

35. 3.14×10^{-4}

36. -1.801×10^5

37. Light year: 9.46×10^{12} kilometers

38. Width of a human hair: 9.0×10^{-5} meter

Using Scientific Notation

In Exercises 39 and 40, evaluate each expression without using a calculator.

39. (a) $(2.0 \times 10^9)(3.4 \times 10^{-4})$
(b) $(1.2 \times 10^7)(5.0 \times 10^{-3})$

40. (a) $\dfrac{6.0 \times 10^8}{3.0 \times 10^{-3}}$ (b) $\dfrac{2.5 \times 10^{-3}}{5.0 \times 10^2}$

Evaluating Expressions Involving Radicals

In Exercises 41 and 42, evaluate each expression without using a calculator.

41. (a) $\sqrt{9}$ (b) $\sqrt[3]{\frac{27}{8}}$

42. (a) $\sqrt[3]{27}$ (b) $\left(\sqrt{36}\right)^3$

Using Properties of Radicals

In Exercises 43 and 44, use the properties of radicals to simplify each expression.

43. (a) $(\sqrt[5]{2})^5$ (b) $\sqrt[5]{32x^5}$

44. (a) $\sqrt{12} \cdot \sqrt{3}$ (b) $\sqrt[4]{(3x^2)^4}$

Simplifying Radical Expressions **In Exercises 45–54, simplify each radical expression.**

45. (a) $\sqrt{20}$ (b) $\sqrt[3]{128}$

46. (a) $\sqrt[3]{\frac{16}{27}}$ (b) $\sqrt{\frac{75}{4}}$

47. (a) $\sqrt{72x^3}$ (b) $\sqrt{54xy^4}$

48. (a) $\sqrt{\dfrac{18^2}{z^3}}$ (b) $\sqrt{\dfrac{32a^4}{b^2}}$

49. (a) $\sqrt[3]{16x^5}$ (b) $\sqrt{75x^2y^{-4}}$

50. (a) $\sqrt[4]{3x^4y^2}$ (b) $\sqrt[5]{160x^8z^4}$

51. (a) $10\sqrt{32} - 6\sqrt{18}$ (b) $\sqrt[3]{16} + 3\sqrt[3]{54}$

52. (a) $5\sqrt{x} - 3\sqrt{x}$ (b) $-2\sqrt{9y} + 10\sqrt{y}$

53. (a) $-3\sqrt{48x^2} + 7\sqrt{75x^2}$

(b) $7\sqrt{80x} - 2\sqrt{125x}$

54. (a) $-\sqrt{x^3 - 7} + 5\sqrt{x^3 - 7}$

(b) $11\sqrt{245x^3} - 9\sqrt{45x^3}$

Rationalizing a Denominator **In Exercises 55–58, rationalize the denominator of the expression. Then simplify your answer.**

55. $\dfrac{1}{\sqrt{3}}$ **56.** $\dfrac{8}{\sqrt[3]{2}}$

57. $\dfrac{5}{\sqrt{14} - 2}$ **58.** $\dfrac{3}{\sqrt{5} + \sqrt{6}}$

Rationalizing a Numerator **In Exercises 59–62, rationalize the numerator of the expression. Then simplify your answer.**

59. $\dfrac{\sqrt{8}}{2}$ **60.** $\dfrac{\sqrt{2}}{3}$

61. $\dfrac{\sqrt{5} + \sqrt{3}}{3}$ **62.** $\dfrac{\sqrt{7} - 3}{4}$

Writing Exponential and Radical Forms **In Exercises 63–68, fill in the missing form of the expression.**

	Radical Form	Rational Exponent Form
63.	$\sqrt[3]{64}$	
64.		$(2ab)^{3/4}$
65.		$3x^{-2/3}$
66.	$x^2\sqrt{x}$	
67.	$x\sqrt{3xy}$	
68.		$a^{0.4}$

Simplifying Expressions **In Exercises 69–78, simplify each expression.**

69. (a) $32^{-3/5}$ (b) $\left(\frac{16}{81}\right)^{-3/4}$

70. (a) $100^{-3/2}$ (b) $\left(\frac{9}{4}\right)^{-1/2}$

71. (a) $(2x^2)^{3/2} \cdot 2^{-1/2} \cdot x^{-4}$ (b) $(x^4y^2)^{1/3}(xy)^{-1/3}$

72. (a) $x^{-3} \cdot x^{1/2} \cdot x^{-3/2} \cdot x$ (b) $5^{-1/2} \cdot 5x^{5/2}(5x)^{-3/2}$

73. (a) $\sqrt[4]{3^2}$ (b) $\sqrt[6]{(x + 1)^4}$

74. (a) $\sqrt[6]{x^3}$ (b) $\sqrt[4]{(3x^2)^4}$

75. (a) $\sqrt{\sqrt{32}}$ (b) $\sqrt{\sqrt[4]{2x}}$

76. (a) $\sqrt{\sqrt{243(x + 1)}}$ (b) $\sqrt{\sqrt[3]{10a^7b}}$

77. (a) $(x - 1)^{1/3}(x - 1)^{2/3}$

(b) $(x - 1)^{1/3}(x - 1)^{-4/3}$

78. (a) $(4x + 3)^{5/2}(4x + 3)^{-5/3}$

(b) $(4x + 3)^{-5/2}(4x + 3)^{1/2}$

79. Mathematical Modeling

A funnel is filled with water to a height of h centimeters. The formula

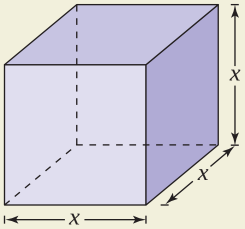
$$t = 0.03[12^{5/2} - (12 - h)^{5/2}], \quad 0 \le h \le 12$$

represents the amount of time t (in seconds) that it will take for the funnel to empty.

(a) Use the *table* feature of a graphing utility to find the times required for the funnel to empty for water heights of $h = 0, h = 1, h = 2, \ldots, h = 12$ centimeters.

(b) What value does t appear to be approaching as the height of the water becomes closer and closer to 12 centimeters?

80. **HOW DO YOU SEE IT?** Package A is a cube with a volume of 500 cubic inches. Package B is a cube with a volume of 250 cubic inches. Is the length x of a side of package A greater than, less than, or equal to twice the length of a side of package B? Explain.



Exploration

True or False? **In Exercises 81–84, determine whether the statement is true or false. Justify your answer.**

81. $\dfrac{x^{k+1}}{x} = x^k$ **82.** $(a^n)^k = a^{n^k}$

83. $(a + b)^2 = a^2 + b^2$ **84.** $\dfrac{a}{\sqrt{b}} = \dfrac{a^2}{(\sqrt{b})^2} = \dfrac{a^2}{b}$

micropic/iStockphoto.com

P.3 Polynomials and Special Products

Polynomials can help you model and solve real-life problems. For instance, in Exercise 93 on page 33, you will use polynomials to model safe uniformly distributed loads on steel beams.

- **Write polynomials in standard form.**
- **Add, subtract, and multiply polynomials.**
- **Use special products to multiply polynomials.**
- **Use polynomials to solve real-life problems.**

Polynomials

One of the most common types of algebraic expressions is the **polynomial.** Some examples are $2x + 5$, $3x^4 - 7x^2 + 2x + 4$, and $5x^2y^2 - xy + 3$. The first two are *polynomials in x* and the third is a *polynomial in x and y*. The terms of a polynomial in x have the form ax^k, where a is the **coefficient** and k is the **degree** of the term. For instance, the polynomial $2x^3 - 5x^2 + 1 = 2x^3 + (-5)x^2 + (0)x + 1$ has coefficients 2, -5, 0, and 1.

Definition of a Polynomial in *x*

Let $a_0, a_1, a_2, . . . , a_n$ be real numbers and let n be a nonnegative integer. A polynomial in x is an expression of the form

$$a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_1x + a_0$$

where $a_n \neq 0$. The polynomial is of **degree** n, a_n is the **leading coefficient,** and a_0 is the **constant term.**

Polynomials with one, two, and three terms are called **monomials, binomials,** and **trinomials,** respectively. A polynomial written with descending powers of x is in **standard form.**

EXAMPLE 1 **Writing Polynomials in Standard Form**

	Polynomial	Standard Form	Degree	Leading Coefficient
a.	$4x^2 - 5x^7 - 2 + 3x$	$-5x^7 + 4x^2 + 3x - 2$	7	-5
b.	$4 - 9x^2$	$-9x^2 + 4$	2	-9
c.	8	$8 \ (8 = 8x^0)$	0	8

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write the polynomial $6 - 7x^3 + 2x$ in standard form. Then identify the degree and leading coefficient of the polynomial. ■

A polynomial that has all zero coefficients is called the **zero polynomial,** denoted by 0. No degree is assigned to the zero polynomial. For polynomials in more than one variable, the degree of a *term* is the sum of the exponents of the variables in the term. The degree of the *polynomial* is the highest degree of its terms. For instance, the degree of the polynomial $-2x^3y^6 + 4xy - x^7y^4$ is 11 because the sum of the exponents in the last term is the greatest. The leading coefficient of the polynomial is the coefficient of the highest-degree term. Expressions are not polynomials when a variable is underneath a radical or when a polynomial expression (with degree greater than 0) is in the denominator of a term. For example, the expressions $x^3 - \sqrt{3x} = x^3 - (3x)^{1/2}$ and $x^2 + (5/x) = x^2 + 5x^{-1}$ are not polynomials.

vyskoczilova/Shutterstock.com

Operations with Polynomials

You can add and subtract polynomials in much the same way you add and subtract real numbers. Add or subtract the *like terms* (terms having the same variables to the same powers) by adding or subtracting their coefficients. For instance, $-3xy^2$ and $5xy^2$ are like terms and their sum is

$$-3xy^2 + 5xy^2 = (-3 + 5)xy^2 = 2xy^2.$$

EXAMPLE 2 Sums and Differences of Polynomials

a. $(5x^3 - 7x^2 - 3) + (x^3 + 2x^2 - x + 8)$

$= (5x^3 + x^3) + (-7x^2 + 2x^2) - x + (-3 + 8)$ Group like terms.

$= 6x^3 - 5x^2 - x + 5$ Combine like terms.

b. $(7x^4 - x^2 - 4x + 2) - (3x^4 - 4x^2 + 3x)$

$= 7x^4 - x^2 - 4x + 2 - 3x^4 + 4x^2 - 3x$ Distributive Property

$= (7x^4 - 3x^4) + (-x^2 + 4x^2) + (-4x - 3x) + 2$ Group like terms.

$= 4x^4 + 3x^2 - 7x + 2$ Combine like terms.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the difference $(2x^3 - x + 3) - (x^2 - 2x - 3)$ and write the resulting polynomial in standard form. ■

REMARK When a negative sign precedes an expression inside parentheses, remember to distribute the negative sign to each term inside the parentheses, as shown.

$$-(3x^4 - 4x^2 + 3x) = -3x^4 + 4x^2 - 3x$$

To find the *product* of two polynomials, use the left and right Distributive Properties. For example, if you treat $5x + 7$ as a single quantity, then you can multiply $3x - 2$ by $5x + 7$ as follows.

$$(3x - 2)(5x + 7) = 3x(5x + 7) - 2(5x + 7)$$

$$= (3x)(5x) + (3x)(7) - (2)(5x) - (2)(7)$$

$$= \underbrace{15x^2}_{\substack{\text{Product of}\\ \text{First terms}}} + \underbrace{21x}_{\substack{\text{Product of}\\ \text{Outer terms}}} - \underbrace{10x}_{\substack{\text{Product of}\\ \text{Inner terms}}} - \underbrace{14}_{\substack{\text{Product of}\\ \text{Last terms}}}$$

$$= 15x^2 + 11x - 14$$

Note in this **FOIL Method** (which can only be used to multiply two binomials) that the outer (O) and inner (I) terms are like terms and can be combined.

EXAMPLE 3 Finding a Product by the FOIL Method

Use the FOIL Method to find the product of $2x - 4$ and $x + 5$.

Solution

$$(2x - 4)(x + 5) = \overset{\text{F}}{2x^2} + \overset{\text{O}}{10x} - \overset{\text{I}}{4x} - \overset{\text{L}}{20}$$

$$= 2x^2 + 6x - 20$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use the FOIL Method to find the product of $3x - 1$ and $x - 5$. ■

When multiplying two polynomials, be sure to multiply each term of one polynomial by *each* term of the other. A vertical arrangement can be helpful.

EXAMPLE 4 A Vertical Arrangement for Multiplication

Multiply $x^2 - 2x + 2$ by $x^2 + 2x + 2$ using a vertical arrangement.

Solution

$$\begin{array}{rrrrrcl}
x^2 & -\,2x & +\;2 & & & & \text{Write in standard form.} \\
\times\; x^2 & +\,2x & +\;2 & & & & \text{Write in standard form.} \\
\hline
x^4 - 2x^3 & +\,2x^2 & & & & ⇦ & x^2(x^2 - 2x + 2) \\
 & 2x^3 - 4x^2 & +\,4x & & & ⇦ & 2x(x^2 - 2x + 2) \\
 & & 2x^2 - 4x + 4 & & & ⇦ & 2(x^2 - 2x + 2) \\
\hline
\multicolumn{5}{l}{x^4 + 0x^3 + 0x^2 + 0x + 4 = x^4 + 4} & & \text{Add and simplify.}
\end{array}$$

So, $(x^2 - 2x + 2)(x^2 + 2x + 2) = x^4 + 4$.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Multiply $x^2 + 2x + 3$ by $x^2 - 2x + 3$ using a vertical arrangement. ■

Special Products

Some binomial products have special forms that occur frequently in algebra. You do not need to memorize these formulas because you can use the Distributive Property to multiply. However, becoming familiar with these formulas will enable you to manipulate the algebra more quickly.

Special Products

Let u and v be real numbers, variables, or algebraic expressions.

Special Product	**Example**
Sum and Difference of Same Terms	
$(u + v)(u - v) = u^2 - v^2$	$(x + 4)(x - 4) = x^2 - 4^2$
	$= x^2 - 16$
Square of a Binomial	
$(u + v)^2 = u^2 + 2uv + v^2$	$(x + 3)^2 = x^2 + 2(x)(3) + 3^2$
	$= x^2 + 6x + 9$
$(u - v)^2 = u^2 - 2uv + v^2$	$(3x - 2)^2 = (3x)^2 - 2(3x)(2) + 2^2$
	$= 9x^2 - 12x + 4$
Cube of a Binomial	
$(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$	$(x + 2)^3 = x^3 + 3x^2(2) + 3x(2^2) + 2^3$
	$= x^3 + 6x^2 + 12x + 8$
$(u - v)^3 = u^3 - 3u^2v + 3uv^2 - v^3$	$(x - 1)^3 = x^3 - 3x^2(1) + 3x(1^2) - 1^3$
	$= x^3 - 3x^2 + 3x - 1$

EXAMPLE 5 Sum and Difference of Same Terms

Find the product of $5x + 9$ and $5x - 9$.

Solution

The product of a sum and a difference of the *same* two terms has no middle term and takes the form $(u + v)(u - v) = u^2 - v^2$.

$$(5x + 9)(5x - 9) = (5x)^2 - 9^2 = 25x^2 - 81$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the product of $3x - 2$ and $3x + 2$.

EXAMPLE 6 Square of a Binomial

Find $(6x - 5)^2$.

Solution

The square of a binomial has the form $(u - v)^2 = u^2 - 2uv + v^2$.

REMARK When squaring a binomial, note that the resulting middle term is always *twice* the product of the two terms.

$$(6x - 5)^2 = (6x)^2 - 2(6x)(5) + 5^2 = 36x^2 - 60x + 25$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find $(x + 10)^2$.

EXAMPLE 7 Cube of a Binomial

Find $(3x + 2)^3$.

Solution

The cube of a binomial has the form $(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$. Note the *decreasing* powers of $u = 3x$ and the *increasing* powers of $v = 2$.

$$(3x + 2)^3 = (3x)^3 + 3(3x)^2(2) + 3(3x)(2^2) + 2^3 = 27x^3 + 54x^2 + 36x + 8$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find $(4x - 1)^3$.

EXAMPLE 8 The Product of Two Trinomials

Find the product of $x + y - 2$ and $x + y + 2$.

Solution

By grouping $x + y$ in parentheses, you can write the product of the trinomials as a special product.

$$\begin{aligned}(x + y - 2)(x + y + 2) &= [(x + y) \overset{\text{Difference}}{\overset{\downarrow}{-}} 2][(x + y) \overset{\text{Sum}}{\overset{\downarrow}{+}} 2]\\ &= (x + y)^2 - 2^2 && \text{Sum and difference of same terms}\\ &= x^2 + 2xy + y^2 - 4\end{aligned}$$

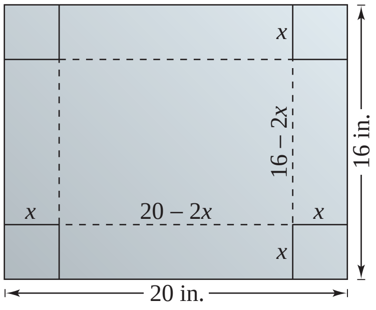
✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

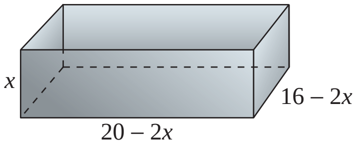
Find the product of $x - 2 + 3y$ and $x - 2 - 3y$.

Application

EXAMPLE 9 Volume of a Box

An open box is made by cutting squares from the corners of a piece of metal that is 16 inches by 20 inches, as shown in the figure. The edge of each cut-out square is x inches. Find the volume of the box in terms of x. Then find the volume of the box when $x = 1$, $x = 2$, and $x = 3$.





Solution

The volume of a rectangular box is equal to the product of its length, width, and height. From the figure, the length is $20 - 2x$, the width is $16 - 2x$, and the height is x. So, the volume of the box is

$$\begin{aligned}\text{Volume} &= (20 - 2x)(16 - 2x)(x)\\ &= (320 - 72x + 4x^2)(x)\\ &= 320x - 72x^2 + 4x^3.\end{aligned}$$

When $x = 1$ inch, the volume of the box is

$$\begin{aligned}\text{Volume} &= 320(1) - 72(1)^2 + 4(1)^3\\ &= 252 \text{ cubic inches.}\end{aligned}$$

When $x = 2$ inches, the volume of the box is

$$\begin{aligned}\text{Volume} &= 320(2) - 72(2)^2 + 4(2)^3\\ &= 384 \text{ cubic inches.}\end{aligned}$$

When $x = 3$ inches, the volume of the box is

$$\begin{aligned}\text{Volume} &= 320(3) - 72(3)^2 + 4(3)^3\\ &= 420 \text{ cubic inches.}\end{aligned}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

In Example 9, find the volume of the box in terms of x when the piece of metal is 10 inches by 12 inches. Then find the volume when $x = 2$ and $x = 3$. ■

Summarize (Section P.3)

1. State the definition of a polynomial in x and explain what is meant by the standard form of a polynomial *(page 26)*. For an example of writing polynomials in standard form, see Example 1.
2. Describe how to add and subtract polynomials *(page 27)*. For an example of adding and subtracting polynomials, see Example 2.
3. Describe the FOIL Method *(page 27)*. For an example of finding a product using the FOIL Method, see Example 3.
4. Explain how to find binomial products that have special forms *(page 28)*. For examples of binomial products that have special forms, see Examples 5–8.

P.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. For the polynomial $a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_1x + a_0$, $a_n \neq 0$, the degree is ________, the leading coefficient is ________, and the constant term is ________.
2. A polynomial with one term is called a ________, while a polynomial with two terms is called a ________ and a polynomial with three terms is called a ________.
3. To add or subtract polynomials, add or subtract the ________ ________ by adding their coefficients.
4. The letters in "FOIL" stand for the following. F ________ O ________ I ________ L ________

Skills and Applications

Polynomials **In Exercises 5–14, (a) write the polynomial in standard form, (b) identify the degree and leading coefficient of the polynomial, and (c) state whether the polynomial is a monomial, a binomial, or a trinomial.**

5. $14x - \frac{1}{2}x^5$
6. $7x$
7. $3 - x^6$
8. $-y + 25y^2 + 1$
9. 3
10. $-8 + t^2$
11. $1 + 6x^4 - 4x^5$
12. $3 + 2x$
13. $4x^3y$
14. $-x^5y + 2x^2y^2 + xy^4$

Identifying Polynomials and Standard Form **In Exercises 15–20, determine whether the expression is a polynomial. If so, write the polynomial in standard form.**

15. $2x - 3x^3 + 8$
16. $5x^4 - 2x^2 + x^{-2}$
17. $\dfrac{3x + 4}{x}$
18. $\dfrac{x^2 + 2x - 3}{2}$
19. $y^2 - y^4 + y^3$
20. $y^4 - \sqrt{y}$

Operations with Polynomials **In Exercises 21–38, perform the operation and write the result in standard form.**

21. $(6x + 5) - (8x + 15)$
22. $(2x^2 + 1) - (x^2 - 2x + 1)$
23. $-(t^3 - 1) + (6t^3 - 5t)$
24. $-(5x^2 - 1) - (-3x^2 + 5)$
25. $(15x^2 - 6) - (-8.3x^3 - 14.7x^2 - 17)$
26. $(15.6w^4 - 14w - 17.4) - (16.9w^4 - 9.2w + 13)$
27. $5z - [3z - (10z + 8)]$
28. $(y^3 + 1) - [(y^2 + 1) + (3y - 7)]$
29. $3x(x^2 - 2x + 1)$
30. $y^2(4y^2 + 2y - 3)$
31. $-5z(3z - 1)$
32. $(-3x)(5x + 2)$
33. $(1 - x^3)(4x)$
34. $-4x(3 - x^3)$
35. $(1.5t^2 + 5)(-3t)$
36. $(2 - 3.5y)(2y^3)$
37. $-2x(0.1x + 17)$
38. $6y(5 - \frac{3}{8}y)$

Operations with Polynomials **In Exercises 39–44, perform the operation.**

39. Add $7x^3 - 2x^2 + 8$ and $-3x^3 - 4$.
40. Add $2x^5 - 3x^3 + 2x + 3$ and $4x^3 + x - 6$.
41. Subtract $x - 3$ from $5x^2 - 3x + 8$.
42. Subtract $-t^4 + 0.5t^2 - 5.6$ from $0.6t^4 - 2t^2$.
43. Multiply $(x + 7)$ and $(2x + 3)$.
44. Multiply $(x^2 + x - 4)$ and $(x^2 - 2x + 1)$.

Multiplying Polynomials **In Exercises 45–76, multiply or find the special product.**

45. $(x + 3)(x + 4)$
46. $(x - 5)(x + 10)$
47. $(3x - 5)(2x + 1)$
48. $(7x - 2)(4x - 3)$
49. $(x^2 - x + 1)(x^2 + x + 1)$
50. $(2x^2 - x + 4)(x^2 + 3x + 2)$
51. $(x + 10)(x - 10)$
52. $(2x + 3)(2x - 3)$
53. $(x + 2y)(x - 2y)$
54. $(4a + 5b)(4a - 5b)$
55. $(2x + 3)^2$
56. $(5 - 8x)^2$
57. $(4x^3 - 3)^2$
58. $(8x + 3)^2$
59. $(x + 1)^3$
60. $(x - 2)^3$
61. $(2x - y)^3$
62. $(3x + 2y)^3$
63. $[(m - 3) + n][(m - 3) - n]$
64. $[(x - 3y) + z][(x - 3y) - z]$
65. $[(x - 3) + y]^2$
66. $[(x + 1) - y]^2$
67. $\left(\frac{1}{5}x - 3\right)\left(\frac{1}{5}x + 3\right)$
68. $\left(3x + \frac{1}{6}\right)\left(3x - \frac{1}{6}\right)$
69. $(1.5x - 4)(1.5x + 4)$
70. $(3a^3 - 4b^2)(3a^3 + 4b^2)$
71. $\left(\frac{1}{4}x - 5\right)^2$
72. $(2.4x + 3)^2$
73. $5x(x + 1) - 3x(x + 1)$
74. $(2x - 1)(x + 3) + 3(x + 3)$
75. $(u + 2)(u - 2)(u^2 + 4)$
76. $(x + y)(x - y)(x^2 + y^2)$

Finding a Product **In Exercises 77–80, find the product. (The expressions are not polynomials, but the formulas can still be used.)**

77. $(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})$

78. $(5 + \sqrt{x})(5 - \sqrt{x})$

79. $(x - \sqrt{5})^2$

80. $(x + \sqrt{3})^2$

81. **Cost, Revenue, and Profit** An electronics manufacturer can produce and sell x MP3 players per week. The total cost C (in dollars) of producing x MP3 players is $C = 73x + 25{,}000$, and the total revenue R (in dollars) is $R = 95x$.

(a) Find the profit P in terms of x.

(b) Find the profit obtained by selling 5000 MP3 players per week.

82. **Compound Interest** After 2 years, an investment of \$500 compounded annually at an interest rate r will yield an amount of $500(1 + r)^2$.

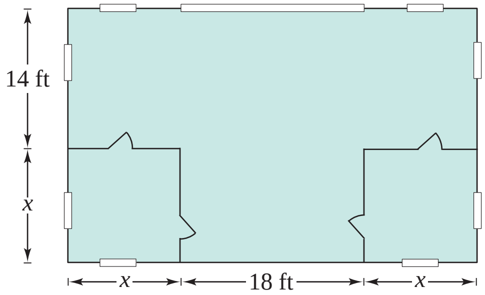
(a) Write this polynomial in standard form.

(b) Use a calculator to evaluate the polynomial for the values of r given in the table.

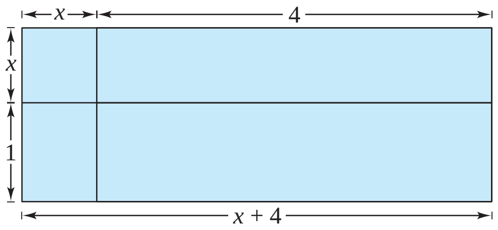
r	$2\frac{1}{2}\%$	3%	4%	$4\frac{1}{2}\%$	5%
$500(1 + r)^2$					

(c) What conclusion can you make from the table?

83. **Geometry** Find a polynomial that represents the total number of square feet for the floor plan shown in the figure.

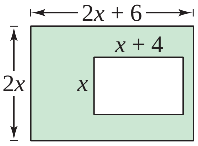


84. **Geometry** Use the area model to write two different expressions for the area. Then equate the two expressions and name the algebraic property illustrated.

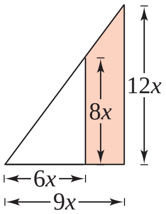


Geometry **In Exercises 85–90, find the area of the shaded region. Write your result as a polynomial in standard form.**

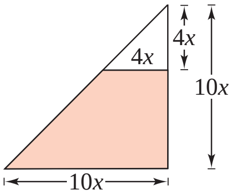
85.



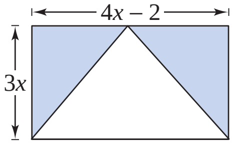
86.



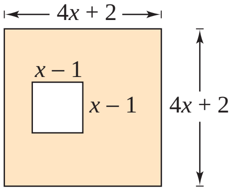
87.



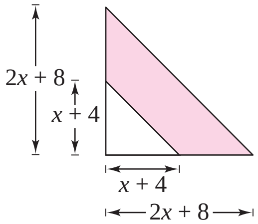
88.



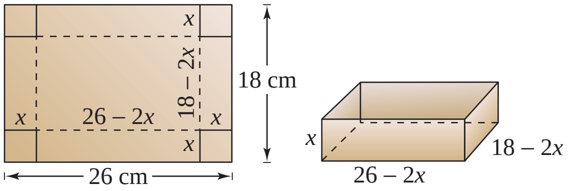
89.



90.



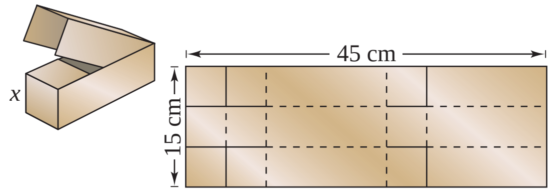
91. **Volume of a Box** A take-out fast-food restaurant is constructing an open box by cutting squares from the corners of a piece of cardboard that is 18 centimeters by 26 centimeters (see figure). The edge of each cut-out square is x centimeters.



(a) Find the volume of the box in terms of x.

(b) Find the volume when $x = 1$, $x = 2$, and $x = 3$.

92. **Volume of a Box** An overnight shipping company is designing a closed box by cutting along the solid lines and folding along the broken lines on the rectangular piece of corrugated cardboard shown in the figure. The length and width of the rectangle are 45 centimeters and 15 centimeters, respectively.



(a) Find the volume of the shipping box in terms of x.

(b) Find the volume when $x = 3$, $x = 5$, and $x = 7$.

93. **Engineering** A one-inch-wide steel beam has a uniformly distributed load. When the span of the beam is x feet and its depth is 6 inches, the safe load S (in pounds) is approximately $S_6 = (0.06x^2 - 2.42x + 38.71)^2$. When the depth is 8 inches, the safe load is approximately $S_8 = (0.08x^2 - 3.30x + 51.93)^2$.

(a) Approximate the difference of the safe loads for these two beams when the span is 12 feet.

(b) How does the difference of the safe loads change as the span increases?

94. **Stopping Distance** The stopping distance of an automobile is the distance traveled during the driver's reaction time plus the distance traveled after the brakes are applied. In an experiment, researchers measured these distances (in feet) when the automobile was traveling at a speed of x miles per hour on dry, level pavement, as shown in the bar graph. The distance traveled during the reaction time R was

$$R = 1.1x$$

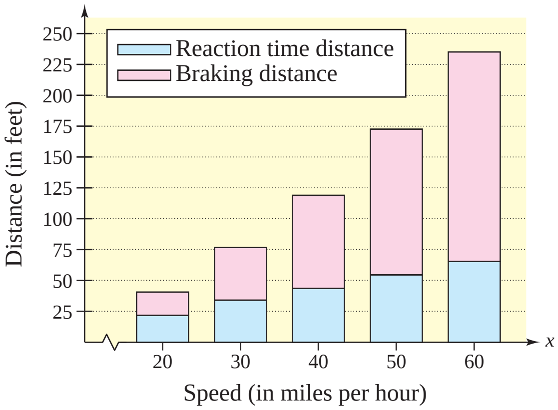
and the braking distance B was

$$B = 0.0475x^2 - 0.001x + 0.23.$$

(a) Determine the polynomial that represents the total stopping distance T.

(b) Use the result of part (a) to estimate the total stopping distance when $x = 30$, $x = 40$, and $x = 55$ miles per hour.

(c) Use the bar graph to make a statement about the total stopping distance required for increasing speeds.



Exploration

True or False? **In Exercises 95 and 96, determine whether the statement is true or false. Justify your answer.**

95. The product of two binomials is always a second-degree polynomial.

96. The sum of two binomials is always a binomial.

97. **Degree of a Product** Find the degree of the product of two polynomials of degrees m and n.

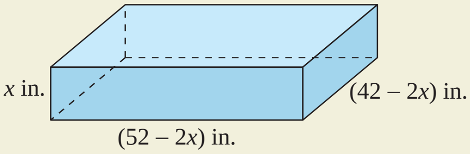
98. **Degree of a Sum** Find the degree of the sum of two polynomials of degrees m and n, where $m < n$.

99. **Writing** A student's homework paper included the following.

$$\xcancel{(x - 3)^2 = x^2 + 9}$$

Write a paragraph fully explaining the error and give the correct method for squaring a binomial.

100. **HOW DO YOU SEE IT?** An open box has a length of $(52 - 2x)$ inches, a width of $(42 - 2x)$ inches, and a height of x inches, as shown.



(a) Describe a way that a box could have been made from a rectangular piece of cardboard. Give the original dimensions of the cardboard.

(b) What degree is the polynomial that represents the volume of the box? Explain.

(c) Describe a procedure for finding the value of x (to the nearest tenth of an inch) that yields the maximum possible volume of the box.

101. **Think About It** Must the sum of two second-degree polynomials be a second-degree polynomial? If not, then give an example.

102. **Think About It** When the polynomial

$$-x^3 + 3x^2 + 2x - 1$$

is subtracted from an unknown polynomial, the difference is $5x^2 + 8$. If it is possible, then find the unknown polynomial.

103. **Logical Reasoning** Verify that $(x + y)^2$ is not equal to $x^2 + y^2$ by letting $x = 3$ and $y = 4$ and evaluating both expressions. Are there any values of x and y for which $(x + y)^2 = x^2 + y^2$? Explain.

vyskoczilova/Shutterstock.com

P.4 Factoring Polynomials

Polynomial factoring can help you solve real-life problems. For instance, in Exercise 92 on page 40, you will use factoring to develop an alternative model for the rate of change of an autocatalytic chemical reaction.

- **Remove common factors from polynomials.**
- **Factor special polynomial forms.**
- **Factor trinomials as the product of two binomials.**
- **Factor polynomials by grouping.**

Polynomials with Common Factors

The process of writing a polynomial as a product is called **factoring.** It is an important tool for solving equations and for simplifying rational expressions.

Unless noted otherwise, when you are asked to factor a polynomial, assume that you are looking for factors that have integer coefficients. If a polynomial does not factor using integer coefficients, then it is **prime** or **irreducible over the integers.** For instance, the polynomial $x^2 - 3$ is irreducible over the integers. Over the *real numbers*, this polynomial factors as

$$x^2 - 3 = (x + \sqrt{3})(x - \sqrt{3}).$$

A polynomial is **completely factored** when each of its factors is prime. For instance,

$$x^3 - x^2 + 4x - 4 = (x - 1)(x^2 + 4) \qquad \text{Completely factored}$$

is completely factored, but

$$x^3 - x^2 - 4x + 4 = (x - 1)(x^2 - 4) \qquad \text{Not completely factored}$$

is not completely factored. Its complete factorization is

$$x^3 - x^2 - 4x + 4 = (x - 1)(x + 2)(x - 2).$$

The simplest type of factoring involves a polynomial that can be written as the product of a monomial and another polynomial. The technique used here is the Distributive Property, $a(b + c) = ab + ac$, in the *reverse* direction.

$$ab + ac = a(b + c) \qquad a \text{ is a common factor.}$$

Removing (factoring out) any common factors is the first step in completely factoring a polynomial.

EXAMPLE 1 Removing Common Factors

Factor each expression.

a. $6x^3 - 4x$ **b.** $-4x^2 + 12x - 16$ **c.** $(x - 2)(2x) + (x - 2)(3)$

Solution

a. $6x^3 - 4x = 2x(3x^2) - 2x(2)$ $\qquad$ $2x$ is a common factor.

$= 2x(3x^2 - 2)$

b. $-4x^2 + 12x - 16 = -4(x^2) + (-4)(-3x) + (-4)4$ $\qquad$ -4 is a common factor.

$= -4(x^2 - 3x + 4)$

c. $(x - 2)(2x) + (x - 2)(3) = (x - 2)(2x + 3)$ $\qquad$ $(x - 2)$ is a common factor.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Factor each expression.

a. $5x^3 - 15x^2$ **b.** $-3 + 6x - 12x^3$ **c.** $(x + 1)(x^2) - (x + 1)(2)$ ■

mikeledray/Shutterstock.com

Factoring Special Polynomial Forms

Some polynomials have special forms that arise from the special product forms on page 28. You should learn to recognize these forms.

Factoring Special Polynomial Forms

Factored Form	**Example**
Difference of Two Squares	
$u^2 - v^2 = (u + v)(u - v)$	$9x^2 - 4 = (3x)^2 - 2^2 = (3x + 2)(3x - 2)$
Perfect Square Trinomial	
$u^2 + 2uv + v^2 = (u + v)^2$	$x^2 + 6x + 9 = x^2 + 2(x)(3) + 3^2 = (x + 3)^2$
$u^2 - 2uv + v^2 = (u - v)^2$	$x^2 - 6x + 9 = x^2 - 2(x)(3) + 3^2 = (x - 3)^2$
Sum or Difference of Two Cubes	
$u^3 + v^3 = (u + v)(u^2 - uv + v^2)$	$x^3 + 8 = x^3 + 2^3 = (x + 2)(x^2 - 2x + 4)$
$u^3 - v^3 = (u - v)(u^2 + uv + v^2)$	$27x^3 - 1 = (3x)^3 - 1^3 = (3x - 1)(9x^2 + 3x + 1)$

The factored form of a difference of two squares is always a set of *conjugate pairs.*

$$u^2 \underset{\substack{\uparrow \\ \text{Difference}}}{-} v^2 = \underbrace{(u \overset{}{+} v)(u \overset{}{-} v)}_{\text{Opposite signs}}$$ Conjugate pairs

To recognize perfect square terms, look for coefficients that are squares of integers and variables raised to *even powers.*

EXAMPLE 2 Removing a Common Factor First

$$\begin{aligned} 3 - 12x^2 &= 3(1 - 4x^2) && \text{3 is a common factor.} \\ &= 3[1^2 - (2x)^2] \\ &= 3(1 + 2x)(1 - 2x) && \text{Difference of two squares} \end{aligned}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Factor $100 - 4y^2$.

REMARK In Example 2, note that the first step in factoring a polynomial is to check for any common factors. Once you have removed any common factors, it is often possible to recognize patterns that were not immediately obvious.

EXAMPLE 3 Factoring the Difference of Two Squares

a. $$\begin{aligned} (x + 2)^2 - y^2 &= [(x + 2) + y][(x + 2) - y] \\ &= (x + 2 + y)(x + 2 - y) \end{aligned}$$

b. $$\begin{aligned} 16x^4 - 81 &= (4x^2)^2 - 9^2 \\ &= (4x^2 + 9)(4x^2 - 9) && \text{Difference of two squares} \\ &= (4x^2 + 9)[(2x)^2 - 3^2] \\ &= (4x^2 + 9)(2x + 3)(2x - 3) && \text{Difference of two squares} \end{aligned}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Factor $(x - 1)^2 - 9y^4$.

A **perfect square trinomial** is the square of a binomial, and it has the form

$$u^2 + 2uv + v^2 = (u + v)^2 \quad \text{or} \quad u^2 - 2uv + v^2 = (u - v)^2.$$

Like signs Like signs

Note that the first and last terms are squares and the middle term is twice the product of u and v.

EXAMPLE 4 Factoring Perfect Square Trinomials

Factor each trinomial.

a. $x^2 - 10x + 25$ **b.** $16x^2 + 24x + 9$

Solution

a. $x^2 - 10x + 25 = x^2 - 2(x)(5) + 5^2 = (x - 5)^2$

b. $16x^2 + 24x + 9 = (4x)^2 + 2(4x)(3) + 3^2 = (4x + 3)^2$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Factor $9x^2 - 30x + 25$.

The next two formulas show the sums and differences of cubes. Pay special attention to the signs of the terms.

Like signs Like signs

$$u^3 + v^3 = (u + v)(u^2 - uv + v^2) \qquad u^3 - v^3 = (u - v)(u^2 + uv + v^2)$$

Unlike signs Unlike signs

EXAMPLE 5 Factoring the Difference of Cubes

$$
\begin{aligned}
x^3 - 27 &= x^3 - 3^3 && \text{Rewrite 27 as } 3^3. \\
&= (x - 3)(x^2 + 3x + 9) && \text{Factor.}
\end{aligned}
$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Factor $64x^3 - 1$.

EXAMPLE 6 Factoring the Sum of Cubes

a.
$$
\begin{aligned}
y^3 + 8 &= y^3 + 2^3 && \text{Rewrite 8 as } 2^3. \\
&= (y + 2)(y^2 - 2y + 4) && \text{Factor.}
\end{aligned}
$$

b.
$$
\begin{aligned}
3x^3 + 192 &= 3(x^3 + 64) && \text{3 is a common factor.} \\
&= 3(x^3 + 4^3) && \text{Rewrite 64 as } 4^3. \\
&= 3(x + 4)(x^2 - 4x + 16) && \text{Factor.}
\end{aligned}
$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Factor each expression.

a. $x^3 + 216$ **b.** $5y^3 + 135$

Trinomials with Binomial Factors

To factor a trinomial of the form $ax^2 + bx + c$, use the following pattern.

$$ax^2 + bx + c = (\overbrace{\boxed{}\,x + \underbrace{\boxed{})(\boxed{}}_{\text{Factors of } c}\,x}^{\text{Factors of } a}\!\!\!\!\!\!\!\! + \boxed{})$$

The goal is to find a combination of factors of a and c such that the outer and inner products add up to the middle term bx. For instance, for the trinomial $6x^2 + 17x + 5$, you can write all possible factorizations and determine which one has outer and inner products that add up to $17x$.

$$(6x + 5)(x + 1),\ (6x + 1)(x + 5),\ (2x + 1)(3x + 5),\ (2x + 5)(3x + 1)$$

You can see that $(2x + 5)(3x + 1)$ is the correct factorization because the outer (O) and inner (I) products add up to $17x$.

$$(2x + 5)(3x + 1) = \overset{\substack{\text{F}\\ \downarrow}}{6x^2} + \overset{\substack{\text{O}\\ \downarrow}}{2x} + \overset{\substack{\text{I}\\ \downarrow}}{15x} + \overset{\substack{\text{L}\\ \downarrow}}{5} = 6x^2 + \overset{\substack{\text{O + I}\\ \downarrow}}{17x} + 5$$

EXAMPLE 7 Factoring a Trinomial: Leading Coefficient Is 1

Factor $x^2 - 7x + 12$.

Solution For this trinomial, you have $a = 1$, $b = -7$, and $c = 12$. Because b is negative and c is positive, both factors of 12 must be negative. So, the possible factorizations of $x^2 - 7x + 12$ are

$$(x - 2)(x - 6),\quad (x - 1)(x - 12),\quad \text{and}\quad (x - 3)(x - 4).$$

Testing the middle term, you will find the correct factorization to be

$$x^2 - 7x + 12 = (x - 3)(x - 4).$$

REMARK Factoring a trinomial can involve trial and error. However, once you have produced the factored form, it is relatively easy to check your answer. For instance, verify the factorization in Example 7 by multiplying $(x - 3)$ by $(x - 4)$ to see that you obtain the original trinomial, $x^2 - 7x + 12$.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Factor $x^2 + x - 6$.

EXAMPLE 8 Factoring a Trinomial: Leading Coefficient Is Not 1

Factor $2x^2 + x - 15$.

Solution For this trinomial, you have $a = 2$ and $c = -15$, which means that the factors of -15 must have unlike signs. The eight possible factorizations are as follows.

$$(2x - 1)(x + 15) \quad (2x + 1)(x - 15) \quad (2x - 3)(x + 5) \quad (2x + 3)(x - 5)$$
$$(2x - 5)(x + 3) \quad (2x + 5)(x - 3) \quad (2x - 15)(x + 1) \quad (2x + 15)(x - 1)$$

Testing the middle term, you will find the correct factorization to be

$$2x^2 + x - 15 = (2x - 5)(x + 3). \qquad \text{O + I} = 6x - 5x = x$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Factor each trinomial.

a. $2x^2 - 5x + 3$ **b.** $12x^2 + 7x + 1$

Factoring by Grouping

Sometimes, polynomials with more than three terms can be **factored by grouping.**

EXAMPLE 9 Factoring by Grouping

$$
\begin{aligned}
x^3 - 2x^2 - 3x + 6 &= (x^3 - 2x^2) - (3x - 6) && \text{Group terms.}\\
&= x^2(x - 2) - 3(x - 2) && \text{Factor each group.}\\
&= (x - 2)(x^2 - 3) && \text{Distributive Property}
\end{aligned}
$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Factor $x^3 + x^2 - 5x - 5$.

REMARK Sometimes, more than one grouping will work. For instance, another way to factor the polynomial in Example 9 is as follows.

$$
\begin{aligned}
&x^3 - 2x^2 - 3x + 6\\
&\quad = (x^3 - 3x) - (2x^2 - 6)\\
&\quad = x(x^2 - 3) - 2(x^2 - 3)\\
&\quad = (x^2 - 3)(x - 2)
\end{aligned}
$$

As you can see, you obtain the same result as in Example 9.

Factoring by grouping can save you some of the trial and error involved in factoring a trinomial. To factor a trinomial of the form $ax^2 + bx + c$ by grouping, rewrite the middle term using the sum of two factors of the product ac that add up to b. Example 10 illustrates this technique.

EXAMPLE 10 Factoring a Trinomial by Grouping

In the trinomial $2x^2 + 5x - 3$, $a = 2$ and $c = -3$, so the product ac is -6. Now, -6 factors as $(6)(-1)$ and $6 - 1 = 5 = b$. So, rewrite the middle term as $5x = 6x - x$. This produces the following.

$$
\begin{aligned}
2x^2 + 5x - 3 &= 2x^2 + 6x - x - 3 && \text{Rewrite middle term.}\\
&= (2x^2 + 6x) - (x + 3) && \text{Group terms.}\\
&= 2x(x + 3) - (x + 3) && \text{Factor groups.}\\
&= (x + 3)(2x - 1) && \text{Distributive Property}
\end{aligned}
$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use factoring by grouping to factor $2x^2 + 5x - 12$.

Guidelines for Factoring Polynomials

1. Factor out any common factors using the Distributive Property.
2. Factor according to one of the special polynomial forms.
3. Factor as $ax^2 + bx + c = (mx + r)(nx + s)$.
4. Factor by grouping.

Summarize (Section P.4)

1. Describe what it means to completely factor a polynomial *(page 34)*. For an example of removing common factors, see Example 1.
2. Make a list of the special polynomial forms of factoring *(page 35)*. For examples of factoring these special forms, see Examples 2–6.
3. Describe how to factor a trinomial of the form $ax^2 + bx + c$ *(page 37)*. For examples of factoring trinomials of this form, see Examples 7 and 8.
4. Explain how to factor a polynomial by grouping *(page 38)*. For examples of factoring by grouping, see Examples 9 and 10.

P.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. The process of writing a polynomial as a product is called ________.
2. A polynomial is ________ ________ when each of its factors is prime.
3. A ________ ________ ________ is the square of a binomial, and it has the form $u^2 + 2uv + v^2$ or $u^2 - 2uv + v^2$.
4. When a polynomial has more than three terms, a method of factoring called ________ ________ ________ may be used.

Skills and Applications

Factoring Out a Common Factor **In Exercises 5–8, factor out the common factor.**

5. $2x^3 - 6x$ 6. $3z^3 - 6z^2 + 9z$

7. $3x(x - 5) + 8(x - 5)$ 8. $(x + 3)^2 - 4(x + 3)$

Greatest Common Factor **In Exercises 9–12, find the greatest common factor such that the remaining factors have only integer coefficients.**

9. $\frac{1}{2}x^3 + 2x^2 - 5x$ 10. $\frac{1}{3}y^4 - 5y^2 + 2y$

11. $\frac{2}{3}x(x - 3) - 4(x - 3)$ 12. $\frac{4}{5}y(y + 1) - 2(y + 1)$

Factoring the Difference of Two Squares **In Exercises 13–22, completely factor the difference of two squares.**

13. $x^2 - 81$ 14. $x^2 - 64$

15. $48y^2 - 27$ 16. $50 - 98z^2$

17. $16x^2 - \frac{1}{9}$ 18. $\frac{4}{25}y^2 - 64$

19. $(x - 1)^2 - 4$ 20. $25 - (z + 5)^2$

21. $81u^4 - 1$ 22. $x^4 - 16y^4$

Factoring a Perfect Square Trinomial **In Exercises 23–28, factor the perfect square trinomial.**

23. $x^2 - 4x + 4$ 24. $4t^2 + 4t + 1$

25. $9u^2 + 24uv + 16v^2$ 26. $36y^2 - 108y + 81$

27. $z^2 + z + \frac{1}{4}$ 28. $9y^2 - \frac{3}{2}y + \frac{1}{16}$

Factoring the Sum or Difference of Cubes **In Exercises 29–36, factor the sum or difference of cubes.**

29. $x^3 - 8$ 30. $27 - x^3$

31. $y^3 + 64$ 32. $x^3 - \frac{8}{27}$

33. $8t^3 - 1$ 34. $27x^3 + 8$

35. $u^3 + 27v^3$ 36. $(x + 2)^3 - y^3$

Factoring a Trinomial **In Exercises 37–46, factor the trinomial.**

37. $x^2 + x - 2$ 38. $x^2 + 5x + 6$

39. $s^2 - 5s + 6$ 40. $t^2 - t - 6$

41. $20 - y - y^2$ 42. $24 + 5z - z^2$

43. $3x^2 - 5x + 2$ 44. $2x^2 - x - 1$

45. $5x^2 + 26x + 5$ 46. $-9z^2 + 3z + 2$

Factoring by Grouping **In Exercises 47–52, factor by grouping.**

47. $x^3 - x^2 + 2x - 2$ 48. $x^3 + 5x^2 - 5x - 25$

49. $2x^3 - x^2 - 6x + 3$ 50. $6 + 2x - 3x^3 - x^4$

51. $x^5 + 2x^3 + x^2 + 2$ 52. $8x^5 - 6x^2 + 12x^3 - 9$

Factoring a Trinomial by Grouping **In Exercises 53–56, factor the trinomial by grouping.**

53. $2x^2 + 9x + 9$ 54. $6x^2 + x - 2$

55. $6x^2 - x - 15$ 56. $12x^2 - 13x + 1$

Factoring Completely **In Exercises 57–84, completely factor the expression.**

57. $6x^2 - 54$ 58. $12x^2 - 48$

59. $x^3 - x^2$ 60. $x^3 - 16x$

61. $x^2 - 2x + 1$ 62. $16 + 6x - x^2$

63. $1 - 4x + 4x^2$ 64. $-9x^2 + 6x - 1$

65. $2x^2 + 4x - 2x^3$ 66. $13x + 6 + 5x^2$

67. $\frac{1}{8}x^2 - \frac{1}{96}x - \frac{1}{16}$ 68. $(t - 1)^2 - 49$

69. $(x^2 + 1)^2 - 4x^2$ 70. $(x^2 + 8)^2 - 36x^2$

71. $5 - x + 5x^2 - x^3$ 72. $3u - 2u^2 + 6 - u^3$

73. $2x^3 + x^2 - 8x - 4$ 74. $\frac{1}{5}x^3 + x^2 - x - 5$

75. $2t^3 - 16$

76. $4x(2x - 1) + (2x - 1)^2$

77. $5(3 - 4x)^2 - 8(3 - 4x)(5x - 1)$

78. $2(x + 1)(x - 3)^2 - 3(x + 1)^2(x - 3)$

79. $7(3x + 2)^2(1 - x)^2 + (3x + 2)(1 - x)^3$

80. $\dfrac{x^2}{2}(x^2 + 1)^4 - (x^2 + 1)^5$

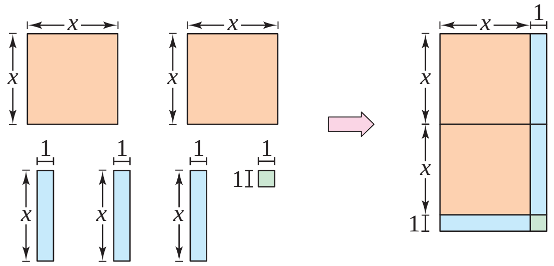
81. $x^4(4)(2x + 1)^3(2x) + (2x + 1)^4(4x^3)$

82. $x^3(3)(x^2 + 1)^2(2x) + (x^2 + 1)^3(3x^2)$

83. $(2x - 5)^4(3)(5x - 4)^2(5) + (5x - 4)^3(4)(2x - 5)^3(2)$

84. $(x^2 - 5)^3(2)(4x + 3)(4) + (4x + 3)^2(3)(x^2 - 5)^2(x^2)$

Geometric Modeling **In Exercises 85–88, draw a "geometric factoring model" to represent the factorization. For instance, a factoring model for $2x^2 + 3x + 1 = (2x + 1)(x + 1)$ is shown below.**



85. $x^2 + 3x + 2 = (x + 2)(x + 1)$

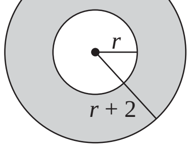
86. $x^2 + 4x + 3 = (x + 3)(x + 1)$

87. $2x^2 + 7x + 3 = (2x + 1)(x + 3)$

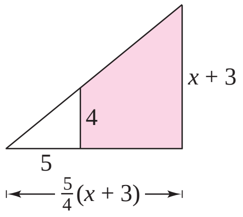
88. $3x^2 + 7x + 2 = (3x + 1)(x + 2)$

Geometry **In Exercises 89 and 90, write an expression in factored form for the area of the shaded portion of the figure.**

89.

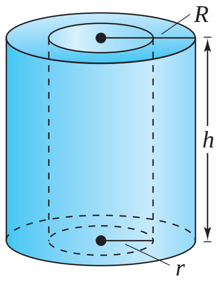


90.



91. Geometry The cylindrical shell shown in the figure has a volume of

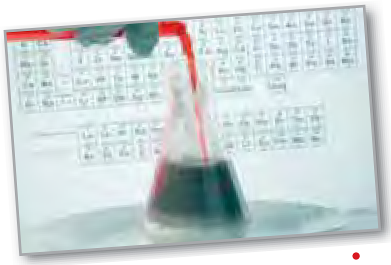
$$V = \pi R^2h - \pi r^2h.$$



(a) Factor the expression for the volume.

(b) From the result of part (a), show that the volume is 2π(average radius)(thickness of the shell)h.

92. Chemistry The rate of change of an autocatalytic chemical reaction is $kQx - kx^2$, where Q is the amount of the original substance, x is the amount of substance formed, and k is a constant of proportionality. Factor the expression.



Factoring a Trinomial **In Exercises 93 and 94, find all values of b for which the trinomial can be factored.**

93. $x^2 + bx - 15$

94. $x^2 + bx + 24$

Factoring a Trinomial **In Exercises 95 and 96, find two integer values of c such that the trinomial can be factored. (There are many correct answers.)**

95. $2x^2 + 5x + c$

96. $3x^2 - x + c$

Exploration

True or False? **In Exercises 97 and 98, determine whether the statement is true or false. Justify your answer.**

97. The difference of two perfect squares can be factored as the product of conjugate pairs.

98. The sum of two perfect squares can be factored as the binomial sum squared.

99. Error Analysis Describe the error.

~~$9x^2 - 9x - 54 = (3x + 6)(3x - 9)$~~

~~$= 3(x + 2)(x - 3)$~~

100. Think About It Is $(3x - 6)(x + 1)$ completely factored? Explain.

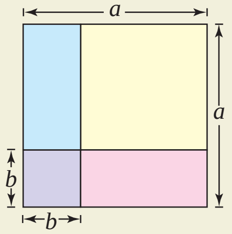
Factoring with Variables in the Exponents **In Exercises 101 and 102, factor the expression as completely as possible.**

101. $x^{2n} - y^{2n}$

102. $x^{3n} + y^{3n}$

103. Think About It Give an example of a polynomial that is prime with respect to the integers.

104. HOW DO YOU SEE IT? The figure shows a large square with an area of a^2 that contains a smaller square with an area of b^2.



(a) Describe the regions that represent $a^2 - b^2$. How can you rearrange these regions to show that $a^2 - b^2 = (a - b)(a + b)$?

(b) How can you use the figure to show that $(a - b)^2 = a^2 - 2ab + b^2$?

(c) Draw another figure to show that $(a + b)^2 = a^2 + 2ab + b^2$. Explain how the figure shows this.

105. Difference of Two Sixth Powers Rewrite $u^6 - v^6$ as the difference of two squares. Then find a formula for completely factoring $u^6 - v^6$. Use your formula to factor $x^6 - 1$ and $x^6 - 64$ completely.

mikeledray/Shutterstock.com

P.5 Rational Expressions

Rational expressions can help you solve real-life problems. For instance, in Exercise 73 on page 49, you will use a rational expression to model the temperature of food in a refrigerator.

- **Find domains of algebraic expressions.**
- **Simplify rational expressions.**
- **Add, subtract, multiply, and divide rational expressions.**
- **Simplify complex fractions and rewrite difference quotients.**

Domain of an Algebraic Expression

The set of real numbers for which an algebraic expression is defined is the **domain** of the expression. Two algebraic expressions are **equivalent** when they have the same domain and yield the same values for all numbers in their domain. For instance,

$$(x + 1) + (x + 2) \quad \text{and} \quad 2x + 3$$

are equivalent because

$$\begin{aligned}(x + 1) + (x + 2) &= x + 1 + x + 2\\ &= x + x + 1 + 2\\ &= 2x + 3.\end{aligned}$$

EXAMPLE 1 Finding the Domain of an Algebraic Expression

a. The domain of the polynomial

$$2x^3 + 3x + 4$$

is the set of all real numbers. In fact, the domain of any polynomial is the set of all real numbers, unless the domain is specifically restricted.

b. The domain of the radical expression

$$\sqrt{x - 2}$$

is the set of real numbers greater than or equal to 2, because the square root of a negative number is not a real number.

c. The domain of the expression

$$\frac{x + 2}{x - 3}$$

is the set of all real numbers except $x = 3$, which would result in division by zero, which is undefined.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the domain of each expression.

a. $4x^3 + 3, \quad x \geq 0$ **b.** $\sqrt{x + 7}$ **c.** $\dfrac{1 - x}{x}$ ■

The quotient of two algebraic expressions is a *fractional expression*. Moreover, the quotient of two *polynomials* such as

$$\frac{1}{x}, \quad \frac{2x - 1}{x + 1}, \quad \text{or} \quad \frac{x^2 - 1}{x^2 + 1}$$

is a **rational expression.**

Filipe B. Varela/Shutterstock.com

Simplifying Rational Expressions

Recall that a fraction is in simplest form when its numerator and denominator have no factors in common aside from ±1. To write a fraction in simplest form, divide out common factors.

$$\frac{a \cdot \cancel{c}}{b \cdot \cancel{c}} = \frac{a}{b}, \quad c \neq 0$$

The key to success in simplifying rational expressions lies in your ability to *factor* polynomials. When simplifying rational expressions, factor each polynomial completely to determine whether the numerator and denominator have factors in common.

EXAMPLE 2 Simplifying a Rational Expression

$$\frac{x^2 + 4x - 12}{3x - 6} = \frac{(x + 6)\cancel{(x - 2)}}{3\cancel{(x - 2)}} \qquad \text{Factor completely.}$$

$$= \frac{x + 6}{3}, \quad x \neq 2 \qquad \text{Divide out common factor.}$$

Note that the original expression is undefined when $x = 2$ (because division by zero is undefined). To make the simplified expression *equivalent* to the original expression, you must list the domain restriction $x \neq 2$ with the simplified expression.

REMARK In Example 2, do not make the mistake of trying to simplify further by dividing out terms.

$$\xcancel{\frac{x + 6}{3} = \frac{x + \cancel{6}}{\cancel{3}} = x + 2}$$

Remember that to simplify fractions, divide out common *factors*, not terms. To learn about other common errors, see Appendix A.

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Write $\dfrac{4x + 12}{x^2 - 3x - 18}$ in simplest form.

Sometimes it may be necessary to change the sign of a factor by factoring out (-1) to simplify a rational expression, as shown in Example 3.

EXAMPLE 3 Simplifying a Rational Expression

$$\frac{12 + x - x^2}{2x^2 - 9x + 4} = \frac{(4 - x)(3 + x)}{(2x - 1)(x - 4)} \qquad \text{Factor completely.}$$

$$= \frac{-\cancel{(x - 4)}(3 + x)}{(2x - 1)\cancel{(x - 4)}} \qquad (4 - x) = -(x - 4)$$

$$= -\frac{3 + x}{2x - 1}, \quad x \neq 4 \qquad \text{Divide out common factor.}$$

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Write $\dfrac{3x^2 - x - 2}{5 - 4x - x^2}$ in simplest form.

In this text, when writing a rational expression, the domain is usually not listed with the expression. It is *implied* that the real numbers that make the denominator zero are excluded from the expression. Also, when performing operations with rational expressions, this text follows the convention of listing *by the simplified expression* all values of x that must be specifically excluded from the domain in order to make the domains of the simplified and original expressions agree. Example 3, for instance, lists the restriction $x \neq 4$ with the simplified expression to make the two domains agree. Note that the value $x = \frac{1}{2}$ is excluded from *both* domains, so it is not necessary to list this value.

Operations with Rational Expressions

To multiply or divide rational expressions, use the properties of fractions discussed in Section P.1. Recall that to divide fractions, you invert the divisor and multiply.

EXAMPLE 4 Multiplying Rational Expressions

$$\frac{2x^2+x-6}{x^2+4x-5}\cdot\frac{x^3-3x^2+2x}{4x^2-6x}=\frac{\cancel{(2x-3)}(x+2)}{(x+5)\cancel{(x-1)}}\cdot\frac{\cancel{x}(x-2)\cancel{(x-1)}}{2\cancel{x}\cancel{(2x-3)}}$$

$$=\frac{(x+2)(x-2)}{2(x+5)},\quad x\neq 0,\ x\neq 1,\ x\neq \tfrac{3}{2}$$

REMARK Note that Example 4 lists the restrictions $x \neq 0$, $x \neq 1$, and $x \neq \frac{3}{2}$ with the simplified expression in order to make the two domains agree. Also note that the value $x = -5$ is excluded from both domains, so it is not necessary to list this value.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Multiply and simplify: $\dfrac{15x^2+5x}{x^3-3x^2-18x}\cdot\dfrac{x^2-2x-15}{3x^2-8x-3}$.

EXAMPLE 5 Dividing Rational Expressions

$$\frac{x^3-8}{x^2-4}\div\frac{x^2+2x+4}{x^3+8}=\frac{x^3-8}{x^2-4}\cdot\frac{x^3+8}{x^2+2x+4}\qquad\text{Invert and multiply.}$$

$$=\frac{\cancel{(x-2)}\cancel{(x^2+2x+4)}}{\cancel{(x+2)}\cancel{(x-2)}}\cdot\frac{\cancel{(x+2)}(x^2-2x+4)}{\cancel{(x^2+2x+4)}}$$

$$=x^2-2x+4,\quad x\neq\pm2\qquad\text{Divide out common factors.}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Divide and simplify: $\dfrac{x^3-1}{x^2-1}\div\dfrac{x^2+x+1}{x^2+2x+1}$. ■

To add or subtract rational expressions, use the LCD (least common denominator) method or the *basic definition*

$$\frac{a}{b}\pm\frac{c}{d}=\frac{ad\pm bc}{bd},\quad b\neq 0,\ d\neq 0.\qquad\text{Basic definition}$$

This definition provides an efficient way of adding or subtracting *two* fractions that have no common factors in their denominators.

EXAMPLE 6 Subtracting Rational Expressions

$$\frac{x}{x-3}-\frac{2}{3x+4}=\frac{x(3x+4)-2(x-3)}{(x-3)(3x+4)}\qquad\text{Basic definition}$$

$$=\frac{3x^2+4x-2x+6}{(x-3)(3x+4)}\qquad\text{Distributive Property}$$

$$=\frac{3x^2+2x+6}{(x-3)(3x+4)}\qquad\text{Combine like terms.}$$

REMARK When subtracting rational expressions, remember to distribute the negative sign to all the terms in the quantity that is being subtracted.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Subtract: $\dfrac{x}{2x-1}-\dfrac{1}{x+2}$. ■

For three or more fractions, or for fractions with a repeated factor in the denominators, the LCD method works well. Recall that the least common denominator of several fractions consists of the product of all prime factors in the denominators, with each factor given the highest power of its occurrence in any denominator. Here is a numerical example.

$$\frac{1}{6}+\frac{3}{4}-\frac{2}{3}=\frac{1\cdot 2}{6\cdot 2}+\frac{3\cdot 3}{4\cdot 3}-\frac{2\cdot 4}{3\cdot 4}$$

The LCD is 12.

$$=\frac{2}{12}+\frac{9}{12}-\frac{8}{12}$$

$$=\frac{3}{12}$$

$$=\frac{1}{4}$$

Sometimes, the numerator of the answer has a factor in common with the denominator. In such cases, simplify the answer. For instance, in the example above, $\frac{3}{12}$ simplifies to $\frac{1}{4}$.

EXAMPLE 7 Combining Rational Expressions: The LCD Method

Perform the operations and simplify.

$$\frac{3}{x-1}-\frac{2}{x}+\frac{x+3}{x^2-1}$$

Solution Using the factored denominators

$(x-1)$, x, and $(x+1)(x-1)$

you can see that the LCD is $x(x+1)(x-1)$.

$$\frac{3}{x-1}-\frac{2}{x}+\frac{x+3}{(x+1)(x-1)}$$

$$=\frac{3(x)(x+1)}{x(x+1)(x-1)}-\frac{2(x+1)(x-1)}{x(x+1)(x-1)}+\frac{(x+3)(x)}{x(x+1)(x-1)}$$

$$=\frac{3(x)(x+1)-2(x+1)(x-1)+(x+3)(x)}{x(x+1)(x-1)}$$

$$=\frac{3x^2+3x-2x^2+2+x^2+3x}{x(x+1)(x-1)}$$

Distributive Property

$$=\frac{3x^2-2x^2+x^2+3x+3x+2}{x(x+1)(x-1)}$$

Group like terms.

$$=\frac{2x^2+6x+2}{x(x+1)(x-1)}$$

Combine like terms.

$$=\frac{2(x^2+3x+1)}{x(x+1)(x-1)}$$

Factor.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Perform the operations and simplify.

$$\frac{4}{x}-\frac{x+5}{x^2-4}+\frac{4}{x+2}$$

■

Complex Fractions and the Difference Quotient

Fractional expressions with separate fractions in the numerator, denominator, or both are called **complex fractions.** Here are two examples.

$$\frac{\left(\frac{1}{x}\right)}{x^2+1} \quad \text{and} \quad \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x^2+1}\right)}$$

To simplify a complex fraction, combine the fractions in the numerator into a single fraction and then combine the fractions in the denominator into a single fraction. Then invert the denominator and multiply.

EXAMPLE 8 **Simplifying a Complex Fraction**

$$\frac{\left(\frac{2}{x}-3\right)}{\left(1-\frac{1}{x-1}\right)} = \frac{\left[\frac{2-3(x)}{x}\right]}{\left[\frac{1(x-1)-1}{x-1}\right]} \qquad \text{Combine fractions.}$$

$$= \frac{\left(\frac{2-3x}{x}\right)}{\left(\frac{x-2}{x-1}\right)} \qquad \text{Simplify.}$$

$$= \frac{2-3x}{x} \cdot \frac{x-1}{x-2} \qquad \text{Invert and multiply.}$$

$$= \frac{(2-3x)(x-1)}{x(x-2)}, \quad x \neq 1$$

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Simplify the complex fraction $\frac{\left(\frac{1}{x+2}+1\right)}{\left(\frac{x}{3}-1\right)}$. ■

Another way to simplify a complex fraction is to multiply its numerator and denominator by the LCD of all fractions in its numerator and denominator. This method is applied to the fraction in Example 8 as follows.

$$\frac{\left(\frac{2}{x}-3\right)}{\left(1-\frac{1}{x-1}\right)} = \frac{\left(\frac{2}{x}-3\right)}{\left(1-\frac{1}{x-1}\right)} \cdot \frac{x(x-1)}{x(x-1)} \qquad \text{LCD is } x(x-1).$$

$$= \frac{\left(\frac{2-3x}{\cancel{x}}\right) \cdot \cancel{x}(x-1)}{\left(\frac{x-2}{\cancel{x-1}}\right) \cdot x\cancel{(x-1)}}$$

$$= \frac{(2-3x)(x-1)}{x(x-2)}, \quad x \neq 1$$

The next three examples illustrate some methods for simplifying rational expressions involving negative exponents and radicals. These types of expressions occur frequently in calculus.

To simplify an expression with negative exponents, one method is to begin by factoring out the common factor with the *lesser* exponent. Remember that when factoring, you *subtract* exponents. For instance, in $3x^{-5/2} + 2x^{-3/2}$, the lesser exponent is $-\frac{5}{2}$ and the common factor is $x^{-5/2}$.

$$\begin{aligned} 3x^{-5/2} + 2x^{-3/2} &= x^{-5/2}\left[3(1) + 2x^{-3/2-(-5/2)}\right] \\ &= x^{-5/2}(3 + 2x^{1}) \\ &= \frac{3 + 2x}{x^{5/2}} \end{aligned}$$

EXAMPLE 9 Simplifying an Expression

Simplify the following expression containing negative exponents.

$$x(1 - 2x)^{-3/2} + (1 - 2x)^{-1/2}$$

Solution Begin by factoring out the common factor with the *lesser exponent.*

$$\begin{aligned} x(1 - 2x)^{-3/2} + (1 - 2x)^{-1/2} &= (1 - 2x)^{-3/2}\left[x + (1 - 2x)^{(-1/2)-(-3/2)}\right] \\ &= (1 - 2x)^{-3/2}\left[x + (1 - 2x)^{1}\right] \\ &= \frac{1 - x}{(1 - 2x)^{3/2}} \end{aligned}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Simplify $(x - 1)^{-1/3} - x(x - 1)^{-4/3}$.

The next example shows a second method for simplifying an expression with negative exponents.

EXAMPLE 10 Simplifying an Expression

$$\begin{aligned} &\frac{(4 - x^2)^{1/2} + x^2(4 - x^2)^{-1/2}}{4 - x^2} \\ &\qquad = \frac{(4 - x^2)^{1/2} + x^2(4 - x^2)^{-1/2}}{4 - x^2} \cdot \frac{(4 - x^2)^{1/2}}{(4 - x^2)^{1/2}} \\ &\qquad = \frac{(4 - x^2)^{1} + x^2(4 - x^2)^{0}}{(4 - x^2)^{3/2}} \\ &\qquad = \frac{4 - x^2 + x^2}{(4 - x^2)^{3/2}} \\ &\qquad = \frac{4}{(4 - x^2)^{3/2}} \end{aligned}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Simplify

$$\frac{x^2(x^2 - 2)^{-1/2} + (x^2 - 2)^{1/2}}{x^2 - 2}.$$

EXAMPLE 11 Rewriting a Difference Quotient

The following expression from calculus is an example of a *difference quotient*.

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

Rewrite this expression by rationalizing its numerator.

Solution

$$\frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \frac{1}{\sqrt{x+h}+\sqrt{x}}, \quad h \neq 0$$

ALGEBRA HELP You can review the techniques for rationalizing a numerator in Section P.2.

Notice that the original expression is undefined when $h = 0$. So, you must exclude $h = 0$ from the domain of the simplified expression so that the expressions are equivalent.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Rewrite the difference quotient

$$\frac{\sqrt{9+h}-3}{h}$$

by rationalizing its numerator.

Difference quotients, such as that in Example 11, occur frequently in calculus. Often, they need to be rewritten in an equivalent form that can be evaluated when $h = 0$.

Summarize (Section P.5)

1. State the definition of the domain of an algebraic expression *(page 41)*. For an example of finding the domains of algebraic expressions, see Example 1.
2. State the definition of a rational expression and describe how to simplify a rational expression *(pages 41 and 42)*. For examples of simplifying rational expressions, see Examples 2 and 3.
3. Describe how to multiply, divide, add, and subtract rational expressions *(page 43)*. For examples of operations with rational expressions, see Examples 4–7.
4. State the definition of a complex fraction *(page 45)*. For an example of simplifying a complex fraction, see Example 8.
5. Describe how to rewrite a difference quotient and why you would want to do so *(page 47)*. For an example of rewriting a difference quotient, see Example 11.

P.5 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. The set of real numbers for which an algebraic expression is defined is the ________ of the expression.
2. The quotient of two algebraic expressions is a fractional expression, and the quotient of two polynomials is a ________ ________.
3. Fractional expressions with separate fractions in the numerator, denominator, or both are called ________ fractions.
4. Two algebraic expressions that have the same domain and yield the same values for all numbers in their domains are called ________.

Skills and Applications

Finding the Domain of an Algebraic Expression

In Exercises 5–16, find the domain of the expression.

5. $3x^2 - 4x + 7$

6. $6x^2 - 9, \quad x > 0$

7. $\dfrac{1}{3 - x}$

8. $\dfrac{x + 6}{3x + 2}$

9. $\dfrac{x^2 - 1}{x^2 - 2x + 1}$

10. $\dfrac{x^2 - 5x + 6}{x^2 - 4}$

11. $\dfrac{x^2 - 2x - 3}{x^2 - 6x + 9}$

12. $\dfrac{x^2 - x - 12}{x^2 - 8x + 16}$

13. $\sqrt{4 - x}$

14. $\sqrt{2x - 5}$

15. $\dfrac{1}{\sqrt{x - 3}}$

16. $\dfrac{1}{\sqrt{x + 2}}$

Simplifying a Rational Expression

In Exercises 17–30, write the rational expression in simplest form.

17. $\dfrac{15x^2}{10x}$

18. $\dfrac{18y^2}{60y^5}$

19. $\dfrac{3xy}{xy + x}$

20. $\dfrac{4y - 8y^2}{10y - 5}$

21. $\dfrac{x - 5}{10 - 2x}$

22. $\dfrac{12 - 4x}{x - 3}$

23. $\dfrac{y^2 - 16}{y + 4}$

24. $\dfrac{x^2 - 25}{5 - x}$

25. $\dfrac{x^3 + 5x^2 + 6x}{x^2 - 4}$

26. $\dfrac{x^2 + 8x - 20}{x^2 + 11x + 10}$

27. $\dfrac{2 - x + 2x^2 - x^3}{x^2 - 4}$

28. $\dfrac{x^2 - 9}{x^3 + x^2 - 9x - 9}$

29. $\dfrac{z^3 - 8}{z^2 + 2z + 4}$

30. $\dfrac{y^3 - 2y^2 - 3y}{y^3 + 1}$

31. Error Analysis Describe the error.

$$\xcancel{\frac{5x^3}{2x^3 + 4} = \frac{5x^3}{2x^3 + 4} = \frac{5}{2 + 4} = \frac{5}{6}}$$

32. Evaluating a Rational Expression Complete the table. What can you conclude?

x	0	1	2	3	4	5	6
$\dfrac{x - 3}{x^2 - x - 6}$							
$\dfrac{1}{x + 2}$							

Multiplying or Dividing Rational Expressions

In Exercises 33–38, perform the multiplication or division and simplify.

33. $\dfrac{5}{x - 1} \cdot \dfrac{x - 1}{25(x - 2)}$

34. $\dfrac{r}{r - 1} \div \dfrac{r^2}{r^2 - 1}$

35. $\dfrac{4y - 16}{5y + 15} \div \dfrac{4 - y}{2y + 6}$

36. $\dfrac{t^2 - t - 6}{t^2 + 6t + 9} \cdot \dfrac{t + 3}{t^2 - 4}$

37. $\dfrac{x^2 + xy - 2y^2}{x^3 + x^2y} \cdot \dfrac{x}{x^2 + 3xy + 2y^2}$

38. $\dfrac{x^2 - 14x + 49}{x^2 - 49} \div \dfrac{3x - 21}{x + 7}$

Adding or Subtracting Rational Expressions

In Exercises 39–46, perform the addition or subtraction and simplify.

39. $6 - \dfrac{5}{x + 3}$

40. $\dfrac{2x - 1}{x + 3} + \dfrac{1 - x}{x + 3}$

41. $\dfrac{3}{x - 2} + \dfrac{5}{2 - x}$

42. $\dfrac{2x}{x - 5} - \dfrac{5}{5 - x}$

43. $\dfrac{4}{2x + 1} - \dfrac{x}{x + 2}$

44. $\dfrac{1}{x^2 - x - 2} - \dfrac{x}{x^2 - 5x + 6}$

45. $-\dfrac{1}{x} + \dfrac{2}{x^2 + 1} + \dfrac{1}{x^3 + x}$

46. $\dfrac{2}{x + 1} + \dfrac{2}{x - 1} + \dfrac{1}{x^2 - 1}$

Error Analysis **In Exercises 47 and 48, describe the error.**

47. $$\begin{aligned}\frac{x+4}{x+2} - \frac{3x-8}{x+2} &= \frac{x+4-3x-8}{x+2}\\ &= \frac{-2x-4}{x+2}\\ &= \frac{-2(x+2)}{x+2} = -2\end{aligned}$$

48. $$\begin{aligned}\frac{6-x}{x(x+2)} + \frac{x+2}{x^2} + \frac{8}{x^2(x+2)} &= \frac{x(6-x)+(x+2)^2+8}{x^2(x+2)}\\ &= \frac{6x-x^2+x^2+4+8}{x^2(x+2)}\\ &= \frac{6(x+2)}{x^2(x+2)} = \frac{6}{x^2}\end{aligned}$$

Simplifying a Complex Fraction **In Exercises 49–54, simplify the complex fraction.**

49. $\dfrac{\left(\dfrac{x}{2} - 1\right)}{(x-2)}$

50. $\dfrac{(x-4)}{\left(\dfrac{x}{4} - \dfrac{4}{x}\right)}$

51. $\dfrac{\left[\dfrac{x^2}{(x+1)^2}\right]}{\left[\dfrac{x}{(x+1)^3}\right]}$

52. $\dfrac{\left(\dfrac{x^2-1}{x}\right)}{\left[\dfrac{(x-1)^2}{x}\right]}$

53. $\dfrac{\left(\sqrt{x} - \dfrac{1}{2\sqrt{x}}\right)}{\sqrt{x}}$

54. $\dfrac{\left(\dfrac{t^2}{\sqrt{t^2+1}} - \sqrt{t^2+1}\right)}{t^2}$

Factoring an Expression **In Exercises 55–60, factor the expression by removing the common factor with the lesser exponent.**

55. $x^5 - 2x^{-2}$

56. $5x^5 + x^{-3}$

57. $x^2(x^2+1)^{-5} - (x^2+1)^{-4}$

58. $2x(x-5)^{-3} - 4x^2(x-5)^{-4}$

59. $2x^2(x-1)^{1/2} - 5(x-1)^{-1/2}$

60. $4x^3(2x-1)^{3/2} - 2x(2x-1)^{-1/2}$

Simplifying an Expression **In Exercises 61 and 62, simplify the expression.**

61. $\dfrac{3x^{1/3} - x^{-2/3}}{3x^{-2/3}}$

62. $\dfrac{-x^3(1-x^2)^{-1/2} - 2x(1-x^2)^{1/2}}{x^4}$

Mateusz Wolski/Shutterstock.com

Simplifying a Difference Quotient **In Exercises 63–66, simplify the difference quotient.**

63. $\dfrac{\left(\dfrac{1}{x+h} - \dfrac{1}{x}\right)}{h}$

64. $\dfrac{\left[\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}\right]}{h}$

65. $\dfrac{\left(\dfrac{1}{x+h-4} - \dfrac{1}{x-4}\right)}{h}$

66. $\dfrac{\left(\dfrac{x+h}{x+h+1} - \dfrac{x}{x+1}\right)}{h}$

Rewriting a Difference Quotient **In Exercises 67–72, rewrite the difference quotient by rationalizing the numerator.**

67. $\dfrac{\sqrt{x+2} - \sqrt{x}}{2}$

68. $\dfrac{\sqrt{z-3} - \sqrt{z}}{3}$

69. $\dfrac{\sqrt{t+3} - \sqrt{3}}{t}$

70. $\dfrac{\sqrt{x+5} - \sqrt{5}}{x}$

71. $\dfrac{\sqrt{x+h+1} - \sqrt{x+1}}{h}$

72. $\dfrac{\sqrt{x+h-2} - \sqrt{x-2}}{h}$

73. Refrigeration

After placing food (at room temperature) in a refrigerator, the time required for the food to cool depends on the amount of food, the air circulation in the refrigerator, the original temperature of the food, and the temperature of the refrigerator. The model that gives the temperature of food that has an original temperature of 75°F and is placed in a 40°F refrigerator is

$$T = 10\left(\frac{4t^2 + 16t + 75}{t^2 + 4t + 10}\right)$$

where T is the temperature (in degrees Fahrenheit) and t is the time (in hours).

(a) Complete the table.

t	0	2	4	6	8	10	12
T							

t	14	16	18	20	22
T					

(b) What value of T does the mathematical model appear to be approaching?

74. **Rate** A digital copier copies in color at a rate of 50 pages per minute.

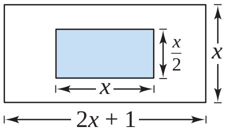
(a) Find the time required to copy one page.

(b) Find the time required to copy x pages.

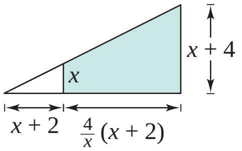
(c) Find the time required to copy 120 pages.

Probability **In Exercises 75 and 76, consider an experiment in which a marble is tossed into a box whose base is shown in the figure. The probability that the marble will come to rest in the shaded portion of the box is equal to the ratio of the shaded area to the total area of the figure. Find the probability.**

75.



76.



77. **Interactive Money Management** The table shows the numbers of U.S. households (in millions) banking online and paying bills online from 2005 through 2010. *(Source: Fiserv, Inc.)*

Year	Banking	Paying Bills
2005	46.7	17.9
2006	58.6	26.3
2007	62.8	28.2
2008	67.0	30.0
2009	69.7	32.6
2010	72.5	36.4

Spreadsheet at LarsonPrecalculus.com

Mathematical models for the data are

$$\text{Number banking online} = \frac{-33.74t + 121.8}{-0.40t + 1.0}$$

and

$$\text{Number paying bills online} = \frac{0.307t^2 - 6.54t + 24.6}{0.015t^2 - 0.28t + 1.0}$$

where t represents the year, with $t = 5$ corresponding to 2005.

(a) Using the models, create a table showing the numbers of households banking online and the numbers of households paying bills online for the given years.

(b) Compare the values given by the models with the actual data.

(c) Determine a model for the ratio of the number of households paying bills online to the number of households banking online.

(d) Use the model from part (c) to find the ratios for the given years. Interpret your results.

78. **Finance** The formula that approximates the annual interest rate r of a monthly installment loan is

$$r = \frac{24(NM - P)}{N} \div \left(P + \frac{NM}{12}\right)$$

where N is the total number of payments, M is the monthly payment, and P is the amount financed.

(a) Approximate the annual interest rate for a five-year car loan of \$28,000 that has monthly payments of \$525.

(b) Simplify the expression for the annual interest rate r, and then rework part (a).

79. **Electrical Engineering** The total resistance R_T (in ohms) of two resistors connected in parallel is given by

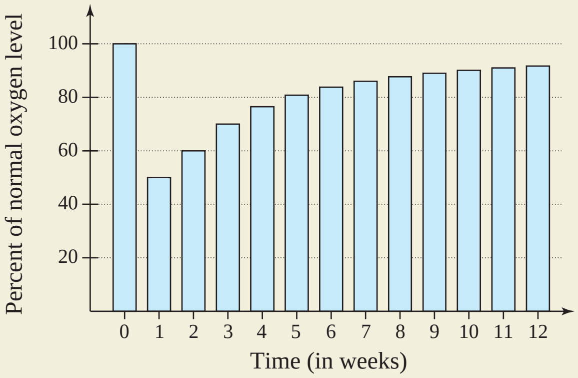
$$R_T = \frac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2}}$$

where R_1 and R_2 are the resistance values of the first and second resistors, respectively. Simplify the expression for the total resistance R_T.

80. **HOW DO YOU SEE IT?** The mathematical model

$$P = 100\left(\frac{t^2 - t + 1}{t^2 + 1}\right), \quad t \geq 0$$

gives the percent P of the normal level of oxygen in a pond, where t is the time (in weeks) after organic waste is dumped into the pond. The bar graph shows the situation. What conclusions can you draw from the bar graph?



Exploration

True or False? **In Exercises 81 and 82, determine whether the statement is true or false. Justify your answer.**

81. $\dfrac{x^{2n} - 1^{2n}}{x^n - 1^n} = x^n + 1^n$

82. $\dfrac{x^2 - 3x + 2}{x - 1} = x - 2$, for all values of x

P.6 The Rectangular Coordinate System and Graphs

The Cartesian plane can help you visualize relationships between two variables. For instance, in Exercise 37 on page 58, given how far north and west one city is from another, plotting points to represent the cities can help you visualize these distances and determine the flying distance between the cities.

- **Plot points in the Cartesian plane.**
- **Use the Distance Formula to find the distance between two points.**
- **Use the Midpoint Formula to find the midpoint of a line segment.**
- **Use a coordinate plane to model and solve real-life problems.**

The Cartesian Plane

Just as you can represent real numbers by points on a real number line, you can represent ordered pairs of real numbers by points in a plane called the **rectangular coordinate system,** or the **Cartesian plane,** named after the French mathematician René Descartes (1596–1650).

Two real number lines intersecting at right angles form the Cartesian plane, as shown in Figure P.10. The horizontal real number line is usually called the ***x*-axis,** and the vertical real number line is usually called the ***y*-axis.** The point of intersection of these two axes is the **origin,** and the two axes divide the plane into four parts called **quadrants.**

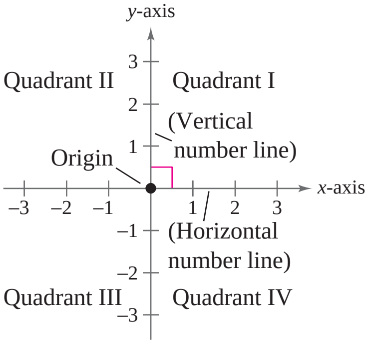


Figure P.10

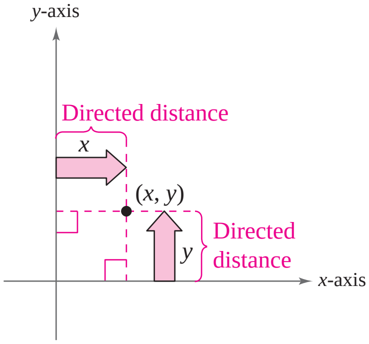
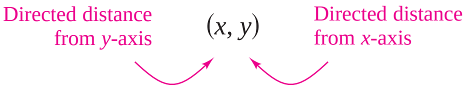


Figure P.11

Each point in the plane corresponds to an **ordered pair** (x, y) of real numbers x and y, called **coordinates** of the point. The ***x*-coordinate** represents the directed distance from the y-axis to the point, and the ***y*-coordinate** represents the directed distance from the x-axis to the point, as shown in Figure P.11.



The notation (x, y) denotes both a point in the plane and an open interval on the real number line. The context will tell you which meaning is intended.

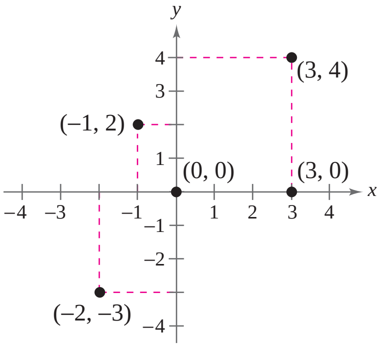


Figure P.12

EXAMPLE 1 Plotting Points in the Cartesian Plane

Plot the points $(-1, 2)$, $(3, 4)$, $(0, 0)$, $(3, 0)$, and $(-2, -3)$.

Solution To plot the point $(-1, 2)$, imagine a vertical line through -1 on the x-axis and a horizontal line through 2 on the y-axis. The intersection of these two lines is the point $(-1, 2)$. Plot the other four points in a similar way, as shown in Figure P.12.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Plot the points $(-3, 2)$, $(4, -2)$, $(3, 1)$, $(0, -2)$, and $(-1, -2)$. ■

Fernando Jose Vasconcelos Soares/Shutterstock.com

The beauty of a rectangular coordinate system is that it allows you to *see* relationships between two variables. It would be difficult to overestimate the importance of Descartes's introduction of coordinates in the plane. Today, his ideas are in common use in virtually every scientific and business-related field.

EXAMPLE 2 Sketching a Scatter Plot

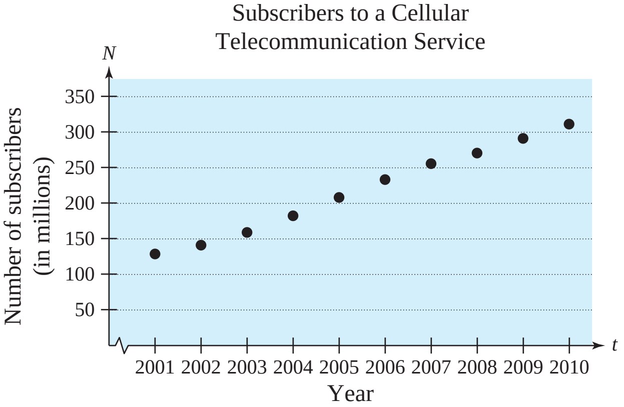
The table shows the numbers N (in millions) of subscribers to a cellular telecommunication service in the United States from 2001 through 2010, where t represents the year. Sketch a scatter plot of the data. *(Source: CTIA-The Wireless Association)*

DATA

Year, t	Subscribers, N
2001	128.4
2002	140.8
2003	158.7
2004	182.1
2005	207.9
2006	233.0
2007	255.4
2008	270.3
2009	290.9
2010	311.0

Spreadsheet at LarsonPrecalculus.com

Solution To sketch a *scatter plot* of the data shown in the table, represent each pair of values by an ordered pair (t, N) and plot the resulting points, as shown below. For instance, the ordered pair (2001, 128.4) represents the first pair of values. Note that the break in the t-axis indicates omission of the years before 2001.



✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The table shows the numbers N (in thousands) of cellular telecommunication service employees in the United States from 2001 through 2010, where t represents the year. Sketch a scatter plot of the data. *(Source: CTIA-The Wireless Association)*

DATA

t	N
2001	203.6
2002	192.4
2003	205.6
2004	226.0
2005	233.1
2006	253.8
2007	266.8
2008	268.5
2009	249.2
2010	250.4

Spreadsheet at LarsonPrecalculus.com

In Example 2, you could have let $t = 1$ represent the year 2001. In that case, there would not have been a break in the horizontal axis, and the labels 1 through 10 (instead of 2001 through 2010) would have been on the tick marks.

▷ **TECHNOLOGY** The scatter plot in Example 2 is only one way to represent the data graphically. You could also represent the data using a bar graph or a line graph. Try using a graphing utility to represent the data given in Example 2 graphically.

The Distance Formula

Recall from the Pythagorean Theorem that, for a right triangle with hypotenuse of length c and sides of lengths a and b, you have

$$a^2 + b^2 = c^2 \qquad \text{Pythagorean Theorem}$$

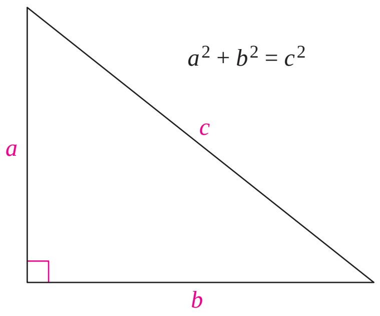


Figure P.13

as shown in Figure P.13. (The converse is also true. That is, if $a^2 + b^2 = c^2$, then the triangle is a right triangle.)

Suppose you want to determine the distance d between two points (x_1, y_1) and (x_2, y_2) in the plane. These two points can form a right triangle, as shown in Figure P.14. The length of the vertical side of the triangle is $|y_2 - y_1|$ and the length of the horizontal side is $|x_2 - x_1|$.

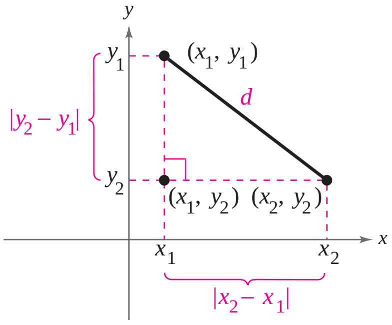


Figure P.14

By the Pythagorean Theorem,

$$\begin{aligned} d^2 &= |x_2 - x_1|^2 + |y_2 - y_1|^2 \\ d &= \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \end{aligned}$$

This result is the **Distance Formula.**

The Distance Formula

The distance d between the points (x_1, y_1) and (x_2, y_2) in the plane is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

EXAMPLE 3 Finding a Distance

Find the distance between the points $(-2, 1)$ and $(3, 4)$.

Algebraic Solution

Let

$$(x_1, y_1) = (-2, 1) \quad \text{and} \quad (x_2, y_2) = (3, 4).$$

Then apply the Distance Formula.

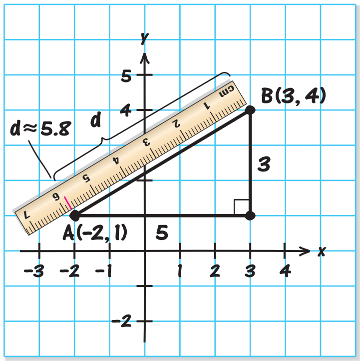
$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} && \text{Distance Formula} \\ &= \sqrt{[3 - (-2)]^2 + (4 - 1)^2} && \text{Substitute for } x_1, y_1, x_2, \text{ and } y_2. \\ &= \sqrt{(5)^2 + (3)^2} && \text{Simplify.} \\ &= \sqrt{34} && \text{Simplify.} \\ &\approx 5.83 && \text{Use a calculator.} \end{aligned}$$

So, the distance between the points is about 5.83 units. Use the Pythagorean Theorem to check that the distance is correct.

$$\begin{aligned} d^2 &\overset{?}{=} 5^2 + 3^2 && \text{Pythagorean Theorem} \\ \left(\sqrt{34}\right)^2 &\overset{?}{=} 5^2 + 3^2 && \text{Substitute for } d. \\ 34 &= 34 && \text{Distance checks. } \checkmark \end{aligned}$$

Graphical Solution

Use centimeter graph paper to plot the points $A(-2, 1)$ and $B(3, 4)$. Carefully sketch the line segment from A to B. Then use a centimeter ruler to measure the length of the segment.



The line segment measures about 5.8 centimeters. So, the distance between the points is about 5.8 units.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the distance between the points $(3, 1)$ and $(-3, 0)$.

EXAMPLE 4 Verifying a Right Triangle

Show that the points

$(2, 1)$, $(4, 0)$, and $(5, 7)$

are vertices of a right triangle.

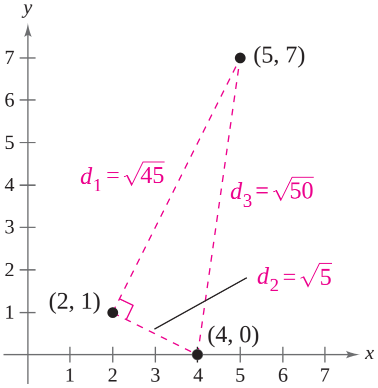


Figure P.15

Solution The three points are plotted in Figure P.15. Using the Distance Formula, the lengths of the three sides are as follows.

$$d_1 = \sqrt{(5-2)^2 + (7-1)^2} = \sqrt{9+36} = \sqrt{45}$$

$$d_2 = \sqrt{(4-2)^2 + (0-1)^2} = \sqrt{4+1} = \sqrt{5}$$

$$d_3 = \sqrt{(5-4)^2 + (7-0)^2} = \sqrt{1+49} = \sqrt{50}$$

Because $(d_1)^2 + (d_2)^2 = 45 + 5 = 50 = (d_3)^2$, you can conclude by the Pythagorean Theorem that the triangle must be a right triangle.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Show that the points $(2, -1)$, $(5, 5)$, and $(6, -3)$ are vertices of a right triangle. ■

The Midpoint Formula

To find the **midpoint** of the line segment that joins two points in a coordinate plane, you can find the average values of the respective coordinates of the two endpoints using the **Midpoint Formula.**

The Midpoint Formula

The midpoint of the line segment joining the points (x_1, y_1) and (x_2, y_2) is given by the Midpoint Formula

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

For a proof of the Midpoint Formula, see Proofs in Mathematics on page 66.

EXAMPLE 5 Finding a Line Segment's Midpoint

Find the midpoint of the line segment joining the points

$(-5, -3)$ and $(9, 3)$.

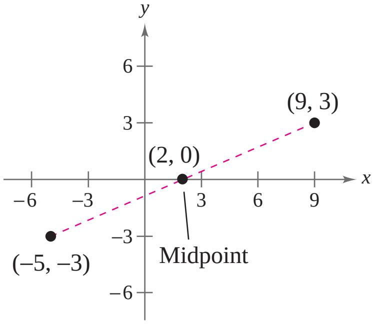


Figure P.16

Solution Let $(x_1, y_1) = (-5, -3)$ and $(x_2, y_2) = (9, 3)$.

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \qquad \text{Midpoint Formula}$$

$$= \left(\frac{-5 + 9}{2}, \frac{-3 + 3}{2}\right) \qquad \text{Substitute for } x_1, y_1, x_2, \text{ and } y_2.$$

$$= (2, 0) \qquad \text{Simplify.}$$

The midpoint of the line segment is $(2, 0)$, as shown in Figure P.16.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the midpoint of the line segment joining the points $(-2, 8)$ and $(4, -10)$. ■

Applications

EXAMPLE 6 Finding the Length of a Pass

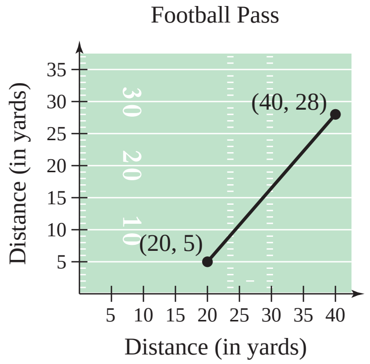


Figure P.17

A football quarterback throws a pass from the 28-yard line, 40 yards from the sideline. A wide receiver catches the pass on the 5-yard line, 20 yards from the same sideline, as shown in Figure P.17. How long is the pass?

Solution You can find the length of the pass by finding the distance between the points (40, 28) and (20, 5).

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} && \text{Distance Formula} \\ &= \sqrt{(40 - 20)^2 + (28 - 5)^2} && \text{Substitute for } x_1, y_1, x_2, \text{ and } y_2. \\ &= \sqrt{20^2 + 23^2} && \text{Simplify.} \\ &= \sqrt{400 + 529} && \text{Simplify.} \\ &= \sqrt{929} && \text{Simplify.} \\ &\approx 30 && \text{Use a calculator.} \end{aligned}$$

So, the pass is about 30 yards long.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

A football quarterback throws a pass from the 10-yard line, 10 yards from the sideline. A wide receiver catches the pass on the 32-yard line, 25 yards from the same sideline. How long is the pass?

In Example 6, the scale along the goal line does not normally appear on a football field. However, when you use coordinate geometry to solve real-life problems, you are free to place the coordinate system in any way that is convenient for the solution of the problem.

EXAMPLE 7 Estimating Annual Sales

Starbucks Corporation had annual sales of approximately \$9.8 billion in 2009 and \$11.7 billion in 2011. Without knowing any additional information, what would you estimate the 2010 sales to have been? *(Source: Starbucks Corporation)*

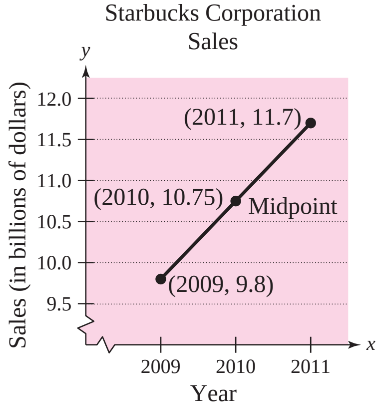


Figure P.18

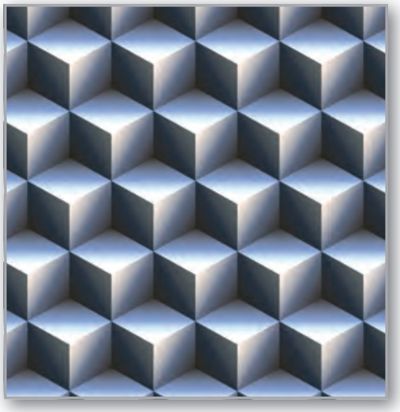
Solution One solution to the problem is to assume that sales followed a linear pattern. With this assumption, you can estimate the 2010 sales by finding the midpoint of the line segment connecting the points (2009, 9.8) and (2011, 11.7).

$$\begin{aligned} \text{Midpoint} &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) && \text{Midpoint Formula} \\ &= \left(\frac{2009 + 2011}{2}, \frac{9.8 + 11.7}{2}\right) && \text{Substitute for } x_1, x_2, y_1, \text{ and } y_2. \\ &= (2010, 10.75) && \text{Simplify.} \end{aligned}$$

So, you would estimate the 2010 sales to have been about \$10.75 billion, as shown in Figure P.18. (The actual 2010 sales were about \$10.71 billion.)

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Yahoo! Inc. had annual revenues of approximately \$7.2 billon in 2008 and \$6.3 billion in 2010. Without knowing any additional information, what would you estimate the 2009 revenue to have been? *(Source: Yahoo! Inc.)*



Much of computer graphics, including this computer-generated tessellation, consists of transformations of points in a coordinate plane. Example 8 illustrates one type of transformation called a translation. Other types include reflections, rotations, and stretches.

EXAMPLE 8 Translating Points in the Plane

The triangle in Figure P.19 has vertices at the points $(-1, 2)$, $(1, -4)$, and $(2, 3)$. Shift the triangle three units to the right and two units up and find the vertices of the shifted triangle, as shown in Figure P.20.

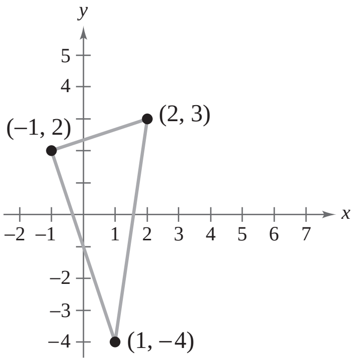


Figure P.19

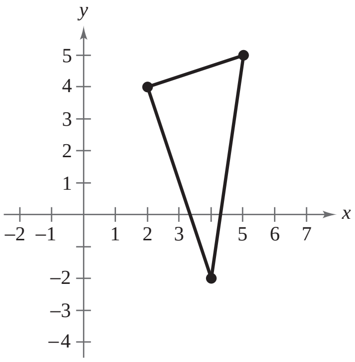


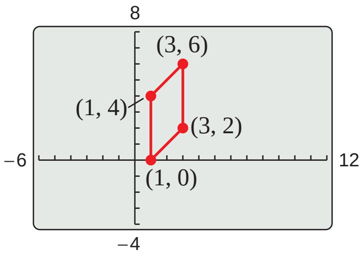
Figure P.20

Solution To shift the vertices three units to the right, add 3 to each of the x-coordinates. To shift the vertices two units up, add 2 to each of the y-coordinates.

Original Point	Translated Point
$(-1, 2)$	$(-1 + 3, 2 + 2) = (2, 4)$
$(1, -4)$	$(1 + 3, -4 + 2) = (4, -2)$
$(2, 3)$	$(2 + 3, 3 + 2) = (5, 5)$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the vertices of the parallelogram shown after translating it two units to the left and four units down.



The figures in Example 8 were not really essential to the solution. Nevertheless, it is strongly recommended that you develop the habit of including sketches with your solutions—even when they are not required.

Summarize (Section P.6)

1. Describe the Cartesian plane *(page 51)*. For an example of plotting points in the Cartesian plane, see Example 1.
2. State the Distance Formula *(page 53)*. For examples of using the Distance Formula to find the distance between two points, see Examples 3 and 4.
3. State the Midpoint Formula *(page 54)*. For an example of using the Midpoint Formula to find the midpoint of a line segment, see Example 5.
4. Describe examples of how to use a coordinate plane to model and solve real-life problems *(pages 55 and 56, Examples 6–8)*.

Matt Antonino/Shutterstock.com

P.6 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. An ordered pair of real numbers can be represented in a plane called the rectangular coordinate system or the _________ plane.
2. The point of intersection of the x- and y-axes is the _________, and the two axes divide the coordinate plane into four parts called _________.
3. The _________ _________ is a result derived from the Pythagorean Theorem.
4. Finding the average values of the representative coordinates of the two endpoints of a line segment in a coordinate plane is also known as using the _________ _________.

Skills and Applications

Plotting Points in the Cartesian Plane **In Exercises 5 and 6, plot the points in the Cartesian plane.**

5. $(-4, 2), (-3, -6), (0, 5), (1, -4), (0, 0), (3, 1)$
6. $\left(1, -\frac{1}{3}\right), (0.5, -1), \left(\frac{3}{7}, 3\right), \left(-\frac{4}{3}, -\frac{3}{7}\right), (-2, 2.5)$

Finding the Coordinates of a Point **In Exercises 7 and 8, find the coordinates of the point.**

7. The point is located three units to the left of the y-axis and four units above the x-axis.
8. The point is on the x-axis and 12 units to the left of the y-axis.

Determining Quadrant(s) for a Point **In Exercises 9–14, determine the quadrant(s) in which (x, y) is located so that the condition(s) is (are) satisfied.**

9. $x > 0$ and $y < 0$
10. $x < 0$ and $y < 0$
11. $x = -4$ and $y > 0$
12. $y < -5$
13. $x < 0$ and $-y > 0$
14. $xy > 0$

Sketching a Scatter Plot **In Exercises 15 and 16, sketch a scatter plot of the data shown in the table.**

15. The table shows the number y of Wal-Mart stores for each year x from 2003 through 2010. *(Source: Wal-Mart Stores, Inc.)*

DATA

Year, x	Number of Stores, y
2003	4906
2004	5289
2005	6141
2006	6779
2007	7262
2008	7720
2009	8416
2010	8970

Spreadsheet at LarsonPrecalculus.com

16. The table shows the lowest temperature on record y (in degrees Fahrenheit) in Duluth, Minnesota, for each month x, where $x = 1$ represents January. *(Source: NOAA)*

DATA

Month, x	Temperature, y
1	−39
2	−39
3	−29
4	−5
5	17
6	27
7	35
8	32
9	22
10	8
11	−23
12	−34

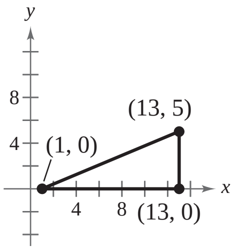
Spreadsheet at LarsonPrecalculus.com

Finding a Distance **In Exercises 17–22, find the distance between the points.**

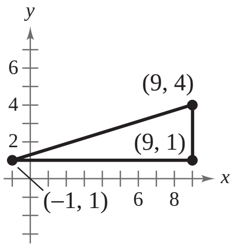
17. $(-2, 6), (3, -6)$
18. $(8, 5), (0, 20)$
19. $(1, 4), (-5, -1)$
20. $(1, 3), (3, -2)$
21. $\left(\frac{1}{2}, \frac{4}{3}\right), (2, -1)$
22. $(9.5, -2.6), (-3.9, 8.2)$

Verifying a Right Triangle **In Exercises 23 and 24, (a) find the length of each side of the right triangle, and (b) show that these lengths satisfy the Pythagorean Theorem.**

23.



24.



Verifying a Polygon **In Exercises 25–28, show that the points form the vertices of the indicated polygon.**

25. Right triangle: $(4, 0), (2, 1), (-1, -5)$

26. Right triangle: $(-1, 3), (3, 5), (5, 1)$

27. Isosceles triangle: $(1, -3), (3, 2), (-2, 4)$

28. Isosceles triangle: $(2, 3), (4, 9), (-2, 7)$

Plotting, Distance, and Midpoint **In Exercises 29–36, (a) plot the points, (b) find the distance between the points, and (c) find the midpoint of the line segment joining the points.**

29. $(6, -3), (6, 5)$

30. $(1, 4), (8, 4)$

31. $(1, 1), (9, 7)$

32. $(1, 12), (6, 0)$

33. $(-1, 2), (5, 4)$

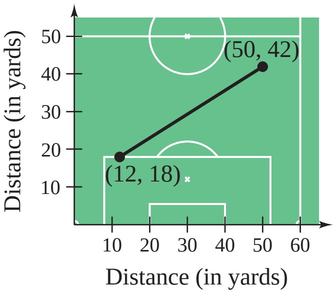
34. $(2, 10), (10, 2)$

35. $(-16.8, 12.3), (5.6, 4.9)$

36. $\left(\frac{1}{2}, 1\right), \left(-\frac{5}{2}, \frac{4}{3}\right)$

37. Flying Distance An airplane flies from Naples, Italy, in a straight line to Rome, Italy, which is 120 kilometers north and 150 kilometers west of Naples. How far does the plane fly?

38. Sports A soccer player passes the ball from a point that is 18 yards from the endline and 12 yards from the sideline. A teammate who is 42 yards from the same endline and 50 yards from the same sideline receives the pass. (See figure.) How long is the pass?

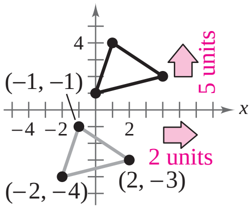


39. Sales The Coca-Cola Company had sales of \$19,564 million in 2002 and \$35,123 million in 2010. Use the Midpoint Formula to estimate the sales in 2006. Assume that the sales followed a linear pattern. *(Source: The Coca-Cola Company)*

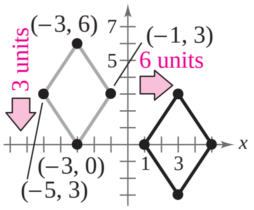
40. Earnings per Share The earnings per share for Big Lots, Inc. were \$1.89 in 2008 and \$2.83 in 2010. Use the Midpoint Formula to estimate the earnings per share in 2009. Assume that the earnings per share followed a linear pattern. *(Source: Big Lots, Inc.)*

Translating Points in the Plane **In Exercises 41–44, find the coordinates of the vertices of the polygon after the indicated translation to a new position in the plane.**

41.



42.



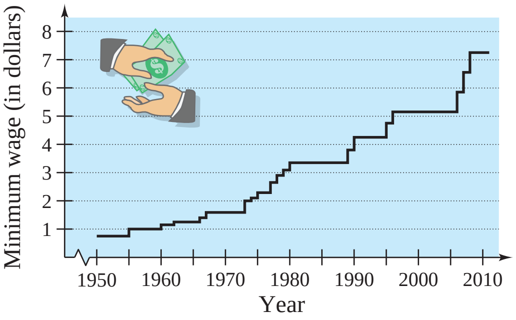
43. Original coordinates of vertices: $(-7, -2)$, $(-2, 2)$, $(-2, -4)$, $(-7, -4)$

Shift: eight units up, four units to the right

44. Original coordinates of vertices: $(5, 8)$, $(3, 6)$, $(7, 6)$

Shift: 6 units down, 10 units to the left

45. Minimum Wage Use the graph below, which shows the minimum wages in the United States (in dollars) from 1950 through 2011. *(Source: U.S. Department of Labor)*



(a) Which decade shows the greatest increase in minimum wage?

(b) Approximate the percent increases in the minimum wage from 1990 to 1995 and from 1995 to 2011.

(c) Use the percent increase from 1995 to 2011 to predict the minimum wage in 2016.

(d) Do you believe that your prediction in part (c) is reasonable? Explain.

46. Data Analysis: Exam Scores The table shows the mathematics entrance test scores x and the final examination scores y in an algebra course for a sample of 10 students.

x	22	29	35	40	44	48	53	58	65	76
y	53	74	57	66	79	90	76	93	83	99

(a) Sketch a scatter plot of the data.

(b) Find the entrance test score of any student with a final exam score in the 80s.

(c) Does a higher entrance test score imply a higher final exam score? Explain.

Fernando Jose Vasconcelos Soares/Shutterstock.com

Exploration

47. **Using the Midpoint Formula** A line segment has (x_1, y_1) as one endpoint and (x_m, y_m) as its midpoint. Find the other endpoint (x_2, y_2) of the line segment in terms of x_1, y_1, x_m, and y_m.

48. **Using the Midpoint Formula** Use the result of Exercise 47 to find the coordinates of the endpoint of a line segment when the coordinates of the other endpoint and midpoint are, respectively,

(a) $(1, -2)$, $(4, -1)$ and (b) $(-5, 11)$, $(2, 4)$.

49. **Using the Midpoint Formula** Use the Midpoint Formula three times to find the three points that divide the line segment joining (x_1, y_1) and (x_2, y_2) into four parts.

50. **Using the Midpoint Formula** Use the result of Exercise 49 to find the points that divide the line segment joining the given points into four equal parts.

(a) $(1, -2)$, $(4, -1)$ (b) $(-2, -3)$, $(0, 0)$

51. **Make a Conjecture** Plot the points $(2, 1)$, $(-3, 5)$, and $(7, -3)$ on a rectangular coordinate system. Then change the signs of the indicated coordinates of each point and plot the three new points on the same rectangular coordinate system. Make a conjecture about the location of a point when each of the following occurs.

(a) The sign of the x-coordinate is changed.

(b) The sign of the y-coordinate is changed.

(c) The signs of both the x- and y-coordinates are changed.

52. **Collinear Points** Three or more points are *collinear* when they all lie on the same line. Use the steps following to determine whether the set of points $\{A(2, 3), B(2, 6), C(6, 3)\}$ and the set of points $\{A(8, 3), B(5, 2), C(2, 1)\}$ are collinear.

(a) For each set of points, use the Distance Formula to find the distances from A to B, from B to C, and from A to C. What relationship exists among these distances for each set of points?

(b) Plot each set of points in the Cartesian plane. Do all the points of either set appear to lie on the same line?

(c) Compare your conclusions from part (a) with the conclusions you made from the graphs in part (b). Make a general statement about how to use the Distance Formula to determine collinearity.

53. **Think About It** When plotting points on the rectangular coordinate system, is it true that the scales on the x- and y-axes must be the same? Explain.

54. **Think About It** What is the y-coordinate of any point on the x-axis? What is the x-coordinate of any point on the y-axis?

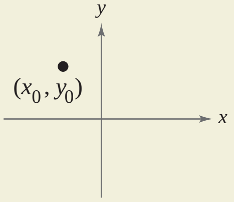
True or False? **In Exercises 55–57, determine whether the statement is true or false. Justify your answer.**

55. In order to divide a line segment into 16 equal parts, you would have to use the Midpoint Formula 16 times.

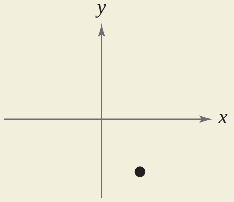
56. The points $(-8, 4)$, $(2, 11)$, and $(-5, 1)$ represent the vertices of an isosceles triangle.

57. If four points represent the vertices of a polygon, and the four sides are equal, then the polygon must be a square.

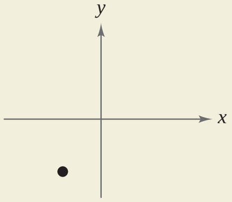
58. **HOW DO YOU SEE IT?** Use the plot of the point (x_0, y_0) in the figure. Match the transformation of the point with the correct plot. Explain your reasoning. [The plots are labeled (i), (ii), (iii), and (iv).]



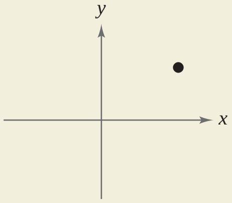
(i)



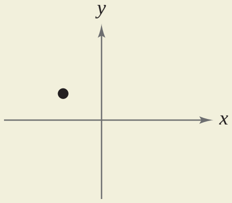
(ii)



(iii)



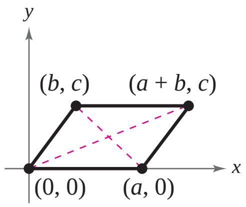
(iv)



(a) (x_0, y_0) (b) $(-2x_0, y_0)$

(c) $(x_0, \frac{1}{2}y_0)$ (d) $(-x_0, -y_0)$

59. **Proof** Prove that the diagonals of the parallelogram in the figure intersect at their midpoints.



Chapter Summary

	What Did You Learn?	Explanation/Examples	Review Exercises
Section P.1	Represent and classify real numbers *(p. 2)*.	**Real numbers:** set of all rational and irrational numbers **Rational numbers:** real numbers that can be written as the ratio of two integers **Irrational numbers:** real numbers that cannot be written as the ratio of two integers Real numbers can be represented on the real number line.	1, 2
	Order real numbers and use inequalities *(p. 4)*.	$a < b$**:** a is less than b. $a > b$**:** a is greater than b. $a \leq b$**:** a is less than or equal to b. $a \geq b$**:** a is greater than or equal to b.	3–6
	Find the absolute values of real numbers and find the distance between two real numbers *(p. 6)*.	**Absolute value of** a**:** $\lvert a\rvert = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$ **Distance between** a **and** b**:** $d(a, b) = \lvert b - a\rvert = \lvert a - b\rvert$	7–12
	Evaluate algebraic expressions *(p. 8)*.	To evaluate an algebraic expression, substitute numerical values for each of the variables in the expression.	13–16
	Use the basic rules and properties of algebra *(p. 9)*.	The basic rules of algebra, the properties of negation and equality, the properties of zero, and the properties and operations of fractions can be used to perform operations.	17–30
Section P.2	Use properties of exponents *(p. 14)*.	**1.** $a^m a^n = a^{m+n}$ **2.** $a^m/a^n = a^{m-n}$ **3.** $a^{-n} = (1/a)^n$ **4.** $a^0 = 1, a \neq 0$ **5.** $(ab)^m = a^m b^m$ **6.** $(a^m)^n = a^{mn}$ **7.** $(a/b)^m = a^m/b^m$ **8.** $\lvert a^2\rvert = a^2$	31–38
	Use scientific notation to represent real numbers *(p. 17)*.	A number written in scientific notation has the form $\pm c \times 10^n$, where $1 \leq c < 10$ and n is an integer.	39–42
	Use properties of radicals *(p. 18)* to simplify and combine radicals *(p. 20)*.	**1.** $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$ **2.** $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$ **3.** $\sqrt[n]{a}/\sqrt[n]{b} = \sqrt[n]{a/b}, b \neq 0$ **4.** $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$ **5.** $\left(\sqrt[n]{a}\right)^n = a$ **6.** n **even:** $\sqrt[n]{a^n} = \lvert a\rvert$, n **odd:** $\sqrt[n]{a^n} = a$ A radical expression is in simplest form when all possible factors have been removed from the radical, all fractions have radical-free denominators, and the index of the radical is reduced. Radical expressions can be combined when they are like radicals.	43–50
	Rationalize denominators and numerators *(p. 21)*.	To rationalize a denominator or numerator of the form $a - b\sqrt{m}$ or $a + b\sqrt{m}$, multiply both numerator and denominator by a conjugate.	51–56
	Use properties of rational exponents *(p. 22)*.	If a is a real number and n is a positive integer such that the principal nth root of a exists, then $a^{1/n}$ is defined as $a^{1/n} = \sqrt[n]{a}$, where $1/n$ is the rational exponent of a.	57–60
Section P.3	Write polynomials in standard form *(p. 26)*, and add, subtract, and multiply polynomials *(p. 27)*.	A polynomial written with descending powers of x is in standard form. To add and subtract polynomials, add or subtract the like terms. To find the product of two binomials, use the FOIL Method.	61–72

	What Did You Learn?	Explanation/Examples	Review Exercises
Section P.3	Use special products to multiply polynomials *(p. 28)*.	**Sum and difference of same terms:** $(u + v)(u - v) = u^2 - v^2$ **Square of a binomial:** $(u + v)^2 = u^2 + 2uv + v^2$ $(u - v)^2 = u^2 - 2uv + v^2$ **Cube of a binomial:** $(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$ $(u - v)^3 = u^3 - 3u^2v + 3uv^2 - v^3$	73–76
	Use polynomials to solve real-life problems *(p. 30)*.	Polynomials can be used to find the volume of a box. (See Example 9.)	77–80
Section P.4	Remove common factors from polynomials *(p. 34)*.	The process of writing a polynomial as a product is called factoring. Removing (factoring out) any common factors is the first step in completely factoring a polynomial.	81, 82
	Factor special polynomial forms *(p. 35)*.	**Difference of two squares:** $u^2 - v^2 = (u + v)(u - v)$ **Perfect square trinomial:** $u^2 + 2uv + v^2 = (u + v)^2$ $u^2 - 2uv + v^2 = (u - v)^2$ **Sum or difference of two cubes:** $u^3 + v^3 = (u + v)(u^2 - uv + v^2)$ $u^3 - v^3 = (u - v)(u^2 + uv + v^2)$	83–86
	Factor trinomials as the product of two binomials *(p. 37)*.	$ax^2 + bx + c = (\boxed{}\,x + \boxed{})(\boxed{}\,x + \boxed{})$ Factors of a Factors of c	87, 88
	Factor polynomials by grouping *(p. 38)*.	Polynomials with more than three terms can sometimes be factored by grouping. (See Examples 9 and 10.)	89, 90
Section P.5	Find domains of algebraic expressions *(p. 41)*.	The set of real numbers for which an algebraic expression is defined is the domain of the expression.	91, 92
	Simplify rational expressions *(p. 42)*.	When simplifying rational expressions, factor each polynomial completely to determine whether the numerator and denominator have factors in common.	93, 94
	Add, subtract, multiply, and divide rational expressions *(p. 43)*.	To add or subtract, use the LCD method or the basic definition $\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$, $b \neq 0$, $d \neq 0$. To multiply or divide, use the properties of fractions.	95–98
	Simplify complex fractions and rewrite difference quotients *(p. 45)*.	To simplify a complex fraction, combine the fractions in the numerator into a single fraction and then combine the fractions in the denominator into a single fraction. Then invert the denominator and multiply.	99–102
Section P.6	Plot points in the Cartesian plane *(p. 51)*.	For an ordered pair (x, y), the x-coordinate is the directed distance from the y-axis to the point, and the y-coordinate is the directed distance from the x-axis to the point.	103–106
	Use the Distance Formula *(p. 53)* and the Midpoint Formula *(p. 54)*.	**Distance Formula:** $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ **Midpoint Formula:** Midpoint $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$	107–110
	Use a coordinate plane to model and solve real-life problems *(p. 55)*.	The coordinate plane can be used to find the length of a football pass. (See Example 6.)	111–114

Review Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

P.1 Classifying Real Numbers

In Exercises 1 and 2, determine which numbers in the set are (a) natural numbers, (b) whole numbers, (c) integers, (d) rational numbers, and (e) irrational numbers.

1. $\left\{11, -14, -\frac{8}{9}, \frac{5}{2}, \sqrt{6}, 0.4\right\}$

2. $\left\{\sqrt{15}, -22, -\frac{10}{3}, 0, 5.2, \frac{3}{7}\right\}$

Plotting and Ordering Rational Numbers

In Exercises 3 and 4, plot the two rational numbers on the real number line. Then place the appropriate inequality symbol (< or >) between them.

3. (a) $\frac{5}{4}$ (b) $\frac{7}{8}$

4. (a) $\frac{9}{25}$ (b) $\frac{5}{7}$

Interpreting an Inequality

In Exercises 5 and 6, give a verbal description of the subset of real numbers represented by the inequality, and sketch the subset on the real number line.

5. $x \leq 7$

6. $x > 1$

Finding a Distance

In Exercises 7 and 8, find the distance between *a* and *b*.

7. $a = -74, b = 48$

8. $a = -112, b = -6$

Using Absolute Value Notation

In Exercises 9–12, use absolute value notation to describe the situation.

9. The distance between x and 7 is at least 4.

10. The distance between x and 25 is no more than 10.

11. The distance between y and -30 is less than 5.

12. The distance between z and -16 is greater than 8.

Evaluating an Algebraic Expression

In Exercises 13–16, evaluate the expression for each value of *x*. (If not possible, then state the reason.)

	Expression	**Values**	
13.	$12x - 7$	(a) $x = 0$	(b) $x = -1$
14.	$x^2 - 6x + 5$	(a) $x = -2$	(b) $x = 2$
15.	$-x^2 + x - 1$	(a) $x = 1$	(b) $x = -1$
16.	$\dfrac{x}{x-3}$	(a) $x = -3$	(b) $x = 3$

Identifying Rules of Algebra

In Exercises 17–22, identify the rule of algebra illustrated by the statement.

17. $2x + (3x - 10) = (2x + 3x) - 10$

18. $4(t + 2) = 4 \cdot t + 4 \cdot 2$

19. $0 + (a - 5) = a - 5$

20. $\dfrac{2}{y+4} \cdot \dfrac{y+4}{2} = 1, \quad y \neq -4$

21. $(t^2 + 1) + 3 = 3 + (t^2 + 1)$

22. $1 \cdot (3x + 4) = 3x + 4$

Performing Operations

In Exercises 23–30, perform the operation(s). (Write fractional answers in simplest form.)

23. $|-3| + 4(-2) - 6$

24. $\dfrac{|-10|}{-10}$

25. $\frac{5}{18} \div \frac{10}{3}$

26. $(16 - 8) \div 4$

27. $6[4 - 2(6 + 8)]$

28. $-4[16 - 3(7 - 10)]$

29. $\dfrac{x}{5} + \dfrac{7x}{12}$

30. $\dfrac{9}{x} \div \dfrac{1}{6}$

P.2 Using Properties of Exponents

In Exercises 31–34, simplify each expression.

31. (a) $3x^2(4x^3)^3$ (b) $\dfrac{5y^6}{10y}$

32. (a) $(3a)^2(6a^3)$ (b) $\dfrac{36x^5}{9x^{10}}$

33. (a) $(-2z)^3$ (b) $\dfrac{(8y)^0}{y^2}$

34. (a) $[(x + 2)^2]^3$ (b) $\dfrac{40(b - 3)^5}{75(b - 3)^2}$

Rewriting with Positive Exponents

In Exercises 35–38, rewrite each expression with positive exponents and simplify.

35. (a) $\dfrac{a^2}{b^{-2}}$ (b) $(a^2b^4)(3ab^{-2})$

36. (a) $\dfrac{6^2u^3v^{-3}}{12u^{-2}v}$ (b) $\dfrac{3^{-4}m^{-1}n^{-3}}{9^{-2}mn^{-3}}$

37. (a) $\dfrac{(5a)^{-2}}{(5a)^2}$ (b) $\dfrac{4(x^{-1})^{-3}}{4^{-2}(x^{-1})^{-1}}$

38. (a) $(x + y^{-1})^{-1}$ (b) $\left(\dfrac{x^{-3}}{y}\right)\left(\dfrac{x}{y}\right)^{-1}$

Scientific Notation

In Exercises 39 and 40, write the number in scientific notation.

39. *Sales for Nautilus, Inc. in 2010:* \$168,500,000 *(Source: Nautilus, Inc.)*

40. *Number of meters in 1 foot:* 0.3048

Decimal Notation

In Exercises 41 and 42, write the number in decimal notation.

41. *Distance between the sun and Jupiter:* 4.84×10^8 miles

42. *Ratio of day to year:* 2.74×10^{-3}

Simplifying Radical Expressions **In Exercises 43–48, simplify each radical expression.**

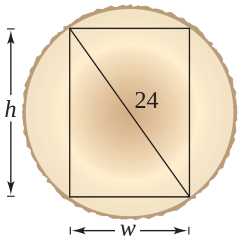
43. (a) $\sqrt[3]{27^2}$ (b) $\sqrt{49^3}$
44. (a) $\sqrt[3]{\frac{64}{125}}$ (b) $\sqrt{\frac{81}{100}}$
45. (a) $\left(\sqrt[3]{216}\right)^3$ (b) $\sqrt[4]{32^4}$
46. (a) $\sqrt[3]{\dfrac{2x^3}{27}}$ (b) $\sqrt[5]{64x^6}$
47. (a) $\sqrt{50} - \sqrt{18}$ (b) $2\sqrt{32} + 3\sqrt{72}$
48. (a) $\sqrt{8x^3} + \sqrt{2x}$ (b) $\sqrt{18x^5} - \sqrt{8x^3}$

49. **Writing** Explain why $\sqrt{5u} + \sqrt{3u} \neq 2\sqrt{2u}$.

50. **Engineering** The rectangular cross section of a wooden beam cut from a log of diameter 24 inches (see figure) will have a maximum strength when its width w and height h are

$$w = 8\sqrt{3} \quad \text{and} \quad h = \sqrt{24^2 - \left(8\sqrt{3}\right)^2}.$$

Find the area of the rectangular cross section and write the answer in simplest form.



Rationalizing a Denominator **In Exercises 51–54, rationalize the denominator of the expression. Then simplify your answer.**

51. $\dfrac{3}{4\sqrt{3}}$
52. $\dfrac{12}{\sqrt[3]{4}}$
53. $\dfrac{1}{2 - \sqrt{3}}$
54. $\dfrac{1}{\sqrt{5} + 1}$

Rationalizing a Numerator **In Exercises 55 and 56, rationalize the numerator of the expression. Then simplify your answer.**

55. $\dfrac{\sqrt{7} + 1}{2}$
56. $\dfrac{\sqrt{2} - \sqrt{11}}{3}$

Simplifying an Expression **In Exercises 57–60, simplify the expression.**

57. $16^{3/2}$
58. $64^{-2/3}$
59. $(3x^{2/5})(2x^{1/2})$
60. $(x - 1)^{1/3}(x - 1)^{-1/4}$

P.3 Polynomials **In Exercises 61–64, write the polynomial in standard form. Identify the degree and leading coefficient.**

61. $3 - 11x^2$
62. $3x^3 - 5x^5 + x - 4$
63. $-4 - 12x^2$
64. $12x - 7x^2 + 6$

Operations with Polynomials **In Exercises 65–68, perform the operation and write the result in standard form.**

65. $-(3x^2 + 2x) + (1 - 5x)$
66. $8y - [2y^2 - (3y - 8)]$
67. $2x(x^2 - 5x + 6)$
68. $(3x^3 - 1.5x^2 + 4)(-3x)$

Operations with Polynomials **In Exercises 69 and 70, perform the operation.**

69. Add $2x^3 - 5x^2 + 10x - 7$ and $4x^2 - 7x - 2$.
70. Subtract $9x^4 - 11x^2 + 16$ from $6x^4 - 20x^2 - x + 3$.

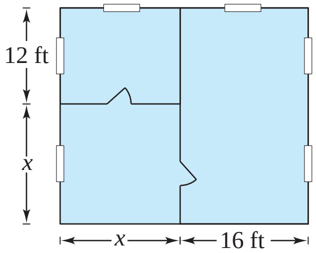
Multiplying Polynomials **In Exercises 71–76, find the product.**

71. $(3x - 6)(5x + 1)$
72. $(x + 2)(x^2 - 2)$
73. $(2x - 3)^2$
74. $(6x + 5)(6x - 5)$
75. $\left(3\sqrt{5} + 2x\right)\left(3\sqrt{5} - 2x\right)$
76. $(x - 4)^3$

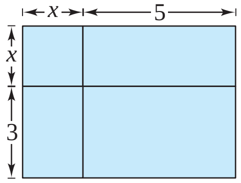
77. **Compound Interest** After 2 years, an investment of \$2500 compounded annually at an interest rate r will yield an amount of $2500(1 + r)^2$. Write this polynomial in standard form.

78. **Surface Area** The surface area S of a right circular cylinder is $S = 2\pi r^2 + 2\pi rh$.
    (a) Draw a right circular cylinder of radius r and height h. Use your drawing to explain how to obtain the surface area formula.
    (b) Find the surface area when the radius is 6 inches and the height is 8 inches.

79. **Geometry** Find a polynomial that represents the total number of square feet for the floor plan shown in the figure.



80. **Geometry** Use the area model to write two different expressions for the area. Then equate the two expressions and name the algebraic property illustrated.



P.4 Factoring Completely **In Exercises 81–90, completely factor the expression.**

81. $x^3 - x$

82. $x(x-3) + 4(x-3)$

83. $25x^2 - 49$

84. $x^2 - 12x + 36$

85. $x^3 - 64$

86. $8x^3 + 27$

87. $2x^2 + 21x + 10$

88. $3x^2 + 14x + 8$

89. $x^3 - x^2 + 2x - 2$

90. $x^3 - 4x^2 + 2x - 8$

P.5 Finding the Domain of an Algebraic Expression **In Exercises 91 and 92, find the domain of the expression.**

91. $\dfrac{1}{x+6}$

92. $\sqrt{x+4}$

Simplifying a Rational Expression **In Exercises 93 and 94, write the rational expression in simplest form.**

93. $\dfrac{x^2 - 64}{5(3x+24)}$

94. $\dfrac{x^3 + 27}{x^2 + x - 6}$

Operations with Rational Expressions **In Exercises 95–98, perform the indicated operation and simplify.**

95. $\dfrac{x^2 - 4}{x^4 - 2x^2 - 8} \cdot \dfrac{x^2 + 2}{x^2}$

96. $\dfrac{4x - 6}{(x-1)^2} \div \dfrac{2x^2 - 3x}{x^2 + 2x - 3}$

97. $\dfrac{1}{x-1} + \dfrac{1-x}{x^2 + x + 1}$

98. $\dfrac{3x}{x+2} - \dfrac{4x^2 - 5}{2x^2 + 3x - 2}$

Simplifying a Complex Fraction **In Exercises 99 and 100, simplify the complex fraction.**

99. $\dfrac{\left[\dfrac{3a}{(a^2/x) - 1}\right]}{\left(\dfrac{a}{x} - 1\right)}$

100. $\dfrac{\left(\dfrac{1}{2x-3} - \dfrac{1}{2x+3}\right)}{\left(\dfrac{1}{2x} - \dfrac{1}{2x+3}\right)}$

Simplifying a Difference Quotient **In Exercises 101 and 102, simplify the difference quotient.**

101. $\dfrac{\left[\dfrac{1}{2(x+h)} - \dfrac{1}{2x}\right]}{h}$

102. $\dfrac{\dfrac{1}{x+h-3} - \dfrac{1}{x-3}}{h}$

P.6 Plotting Points in the Cartesian Plane **In Exercises 103 and 104, plot the points in the Cartesian plane.**

103. $(5, 5), (-2, 0), (-3, 6), (-1, -7)$

104. $(0, 6), (8, 1), (5, -4), (-3, -3)$

Determining Quadrant(s) for a Point **In Exercises 105 and 106, determine the quadrant(s) in which (x, y) is located so that the condition(s) is (are) satisfied.**

105. $x > 0$ and $y = -2$

106. $xy = 4$

Plotting, Distance, and Midpoint **In Exercises 107–110, (a) plot the points, (b) find the distance between the points, and (c) find the midpoint of the line segment joining the points.**

107. $(-3, 8), (1, 5)$

108. $(-2, 6), (4, -3)$

109. $(5.6, 0), (0, 8.2)$

110. $(1.8, 7.4), (-0.6, -14.5)$

Translating Points in the Plane **In Exercises 111 and 112, find the coordinates of the vertices of the polygon after the indicated translation to a new position in the plane.**

111. Original coordinates of vertices:

$(4, 8), (6, 8), (4, 3), (6, 3)$

Shift: eight units down, four units to the left

112. Original coordinates of vertices:

$(0, 1), (3, 3), (0, 5), (-3, 3)$

Shift: three units up, two units to the left

113. **Sales** Barnes & Noble had annual sales of \$5.1 billion in 2008 and \$7.0 billion in 2010. Use the Midpoint Formula to estimate the sales in 2009. Assume that the annual sales followed a linear pattern. *(Source: Barnes & Noble, Inc.)*

114. **Meteorology** The apparent temperature is a measure of relative discomfort to a person from heat and high humidity. The table shows the actual temperatures x (in degrees Fahrenheit) versus the apparent temperatures y (in degrees Fahrenheit) for a relative humidity of 75%.

x	70	75	80	85	90	95	100
y	70	77	85	95	109	130	150

(a) Sketch a scatter plot of the data shown in the table.

(b) Find the change in the apparent temperature when the actual temperature changes from 70°F to 100°F.

Exploration

True or False? **In Exercises 115 and 116, determine whether the statement is true or false. Justify your answer.**

115. A binomial sum squared is equal to the sum of the terms squared.

116. $x^n - y^n$ factors as conjugates for all values of n.

Chapter Test

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

1. Place the appropriate inequality symbol ($<$ or $>$) between the real numbers $-\frac{10}{3}$ and $-|-4|$.

2. Find the distance between the real numbers -5.4 and $3\frac{3}{4}$.

3. Identify the rule of algebra illustrated by $(5 - x) + 0 = 5 - x$.

In Exercises 4 and 5, evaluate each expression without using a calculator.

4. (a) $27\left(-\frac{2}{3}\right)$ (b) $\frac{5}{18} \div \frac{5}{8}$

(c) $\left(-\frac{3}{5}\right)^3$ (d) $\left(\frac{3^2}{2}\right)^{-3}$

5. (a) $\sqrt{5} \cdot \sqrt{125}$ (b) $\frac{\sqrt{27}}{\sqrt{2}}$

(c) $\frac{5.4 \times 10^8}{3 \times 10^3}$ (d) $(3 \times 10^4)^3$

In Exercises 6 and 7, simplify each expression.

6. (a) $3z^2(2z^3)^2$ (b) $(u - 2)^{-4}(u - 2)^{-3}$ (c) $\left(\frac{x^{-2}y^2}{3}\right)^{-1}$

7. (a) $9z\sqrt{8z} - 3\sqrt{2z^3}$ (b) $(4x^{3/5})(x^{1/3})$ (c) $\sqrt[3]{\frac{16}{v^5}}$

8. Write the polynomial $3 - 2x^5 + 3x^3 - x^4$ in standard form. Identify the degree and leading coefficient.

In Exercises 9–12, perform the operation and simplify.

9. $(x^2 + 3) - [3x + (8 - x^2)]$

10. $(x + \sqrt{5})(x - \sqrt{5})$

11. $\frac{5x}{x - 4} + \frac{20}{4 - x}$

12. $\frac{\left(\frac{2}{x} - \frac{2}{x + 1}\right)}{\left(\frac{4}{x^2 - 1}\right)}$

13. Completely factor (a) $2x^4 - 3x^3 - 2x^2$ and (b) $x^3 + 2x^2 - 4x - 8$.

14. Rationalize each denominator and simplify. (a) $\frac{16}{\sqrt[3]{16}}$ (b) $\frac{4}{1 - \sqrt{2}}$

15. Find the domain of $\frac{6 - x}{1 - x}$.

16. Multiply and simplify: $\frac{y^2 + 8y + 16}{2y - 4} \cdot \frac{8y - 16}{(y + 4)^3}$.

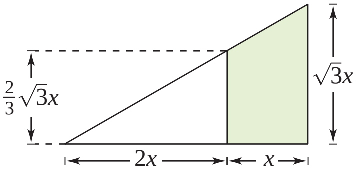


Figure for 19

17. A T-shirt company can produce and sell x T-shirts per day. The total cost C (in dollars) for producing x T-shirts is $C = 1480 + 6x$, and the total revenue R (in dollars) is $R = 15x$. Find the profit obtained by selling 225 T-shirts per day.

18. Plot the points $(-2, 5)$ and $(6, 0)$. Then find the distance between the points and the midpoint of the line segment joining the points.

19. Write an expression for the area of the shaded region in the figure at the left, and simplify the result.

Proofs in Mathematics

What does the word *proof* mean to you? In mathematics, the word *proof* means a valid argument. When you are proving a statement or theorem, you must use facts, definitions, and accepted properties in a logical order. You can also use previously proved theorems in your proof. For instance, the proof of the Midpoint Formula below uses the Distance Formula. There are several different proof methods, which you will see in later chapters.

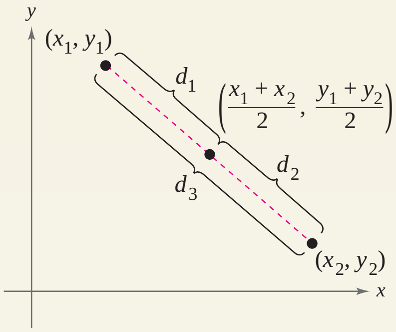
The Midpoint Formula ***(p.54)***

The midpoint of the line segment joining the points (x_1, y_1) and (x_2, y_2) is given by the Midpoint Formula

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Proof

Using the figure, you must show that $d_1 = d_2$ and $d_1 + d_2 = d_3$.



By the Distance Formula, you obtain

$$d_1 = \sqrt{\left(\frac{x_1 + x_2}{2} - x_1\right)^2 + \left(\frac{y_1 + y_2}{2} - y_1\right)^2}$$

$$= \frac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d_2 = \sqrt{\left(x_2 - \frac{x_1 + x_2}{2}\right)^2 + \left(y_2 - \frac{y_1 + y_2}{2}\right)^2}$$

$$= \frac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d_3 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

So, it follows that $d_1 = d_2$ and $d_1 + d_2 = d_3$. ■

THE CARTESIAN PLANE

The Cartesian plane was named after the French mathematician René Descartes (1596–1650). While Descartes was lying in bed, he noticed a fly buzzing around on the square ceiling tiles. He discovered that he could describe the position of the fly by the ceiling tile upon which the fly landed. This led to the development of the Cartesian plane. Descartes felt that using a coordinate plane could facilitate descriptions of the positions of objects.

P.S. Problem Solving

1. **Volume and Density** The NCAA states that the men's and women's shots for track and field competition must comply with the following specifications. *(Source: NCAA)*

	Men's	Women's
Weight (minimum)	7.26 kg	4.0 kg
Diameter (minimum)	110 mm	95 mm
Diameter (maximum)	130 mm	110 mm

(a) Find the maximum and minimum volumes of both the men's and women's shots.

(b) The *density* of an object is an indication of how heavy the object is. To find the density of an object, divide its mass (weight) by its volume. Find the maximum and minimum densities of both the men's and women's shots.

(c) A shot is usually made out of iron. When a ball of cork has the same volume as an iron shot, do you think they would have the same density? Explain your reasoning.

2. **Proof** Find an example for which

$$|a - b| > |a| - |b|$$

and an example for which

$$|a - b| = |a| - |b|.$$

Then prove that

$$|a - b| \geq |a| - |b|$$

for all a, b.

3. **Weight of a Golf Ball** A major feature of Epcot Center at Disney World is called Spaceship Earth. The building is spherically shaped and weighs 1.6×10^7 pounds, which is equal in weight to 1.58×10^8 golf balls. Use these values to find the approximate weight (in pounds) of one golf ball. Then convert the weight to ounces. *(Source: Disney.com)*

4. **Heartbeats** The life expectancies at birth in 2008 for men and women were 75.5 years and 80.5 years, respectively. Assuming an average healthy heart rate of 70 beats per minute, find the numbers of beats in a lifetime for a man and for a woman. *(Source: National Center for Health Statistics)*

5. **Significant Digits** The accuracy of an approximation to a number is related to how many significant digits there are in the approximation. Write a definition of significant digits and illustrate the concept with examples.

6. **Population** The table shows the census population y (in millions) of the United States for each census year x from 1960 through 2010. *(Source: U.S. Census Bureau)*

DATA

Spreadsheet at LarsonPrecalculus.com

Year, x	Population, y
1960	179.32
1970	203.30
1980	226.54
1990	248.72
2000	281.42
2010	308.75

(a) Sketch a scatter plot of the data. Describe any trends in the data.

(b) Find the increase in population from each census year to the next.

(c) Over which decade did the population increase the most? the least?

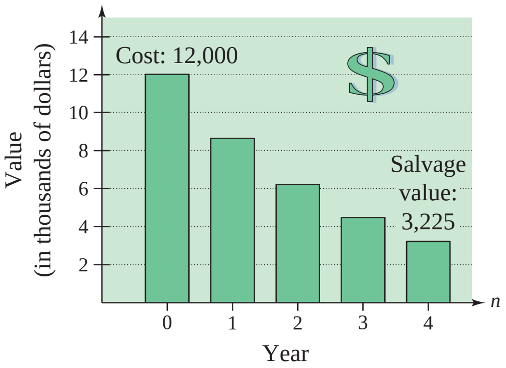
(d) Find the percent increase in population from each census year to the next.

(e) Over which decade was the percent increase the greatest? the least?

7. **Declining Balances Method** Find the annual depreciation rate r from the bar graph below. To find r by the declining balances method, use the formula

$$r = 1 - \left(\frac{S}{C}\right)^{1/n}$$

where n is the useful life of the item (in years), S is the salvage value (in dollars), and C is the original cost (in dollars).

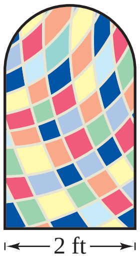


8. Planetary Distance and Period Johannes Kepler (1571–1630), a well-known German astronomer, discovered a relationship between the average distance of a planet from the sun and the time (or period) it takes the planet to orbit the sun. People then knew that planets that are closer to the sun take less time to complete an orbit than planets that are farther from the sun. Kepler discovered that the distance and period are related by an exact mathematical formula.

The table shows the average distances x (in astronomical units) and periods y (in years) for the five planets that are closest to the sun. By completing the table, can you rediscover Kepler's relationship? Write a paragraph that summarizes your conclusions.

Planet	Mercury	Venus	Earth	Mars	Jupiter
x	0.387	0.723	1.000	1.524	5.203
$\sqrt{x}$					
y	0.241	0.615	1.000	1.881	11.860
$\sqrt[3]{y}$					

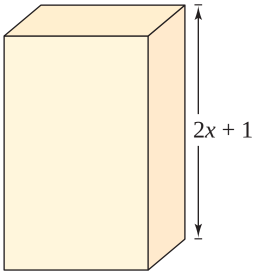
9. Stained Glass Window A stained glass window is in the shape of a rectangle with a semicircular arch (see figure). The width of the window is 2 feet and the perimeter is approximately 13.14 feet. Find the smallest amount of glass required to construct the window.



10. Surface Area of a Box A mathematical model for the volume V (in cubic inches) of the box shown below is

$$V = 2x^3 + x^2 - 8x - 4$$

where x is in inches. Find an expression for the surface area of the box. Then find the surface area when $x = 6$ inches.



11. Nonequivalent and Equivalent Equations Verify that $y_1 \neq y_2$ by letting $x = 0$ and evaluating y_1 and y_2.

$$y_1 = 2x\sqrt{1 - x^2} - \frac{x^3}{\sqrt{1 - x^2}}$$

$$y_2 = \frac{2 - 3x^2}{\sqrt{1 - x^2}}$$

Change y_2 so that $y_1 = y_2$.

12. Proof Prove that

$$\left(\frac{2x_1 + x_2}{3}, \frac{2y_1 + y_2}{3}\right)$$

is one of the points of trisection of the line segment joining (x_1, y_1) and (x_2, y_2). Find the midpoint of the line segment joining

$$\left(\frac{2x_1 + x_2}{3}, \frac{2y_1 + y_2}{3}\right)$$

and (x_2, y_2) to find the second point of trisection.

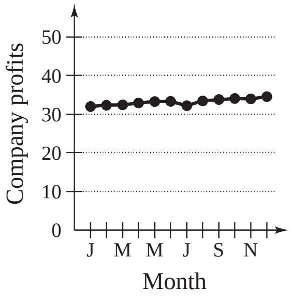
13. Points of Trisection Use the results of Exercise 12 to find the points of trisection of the line segment joining each pair of points.

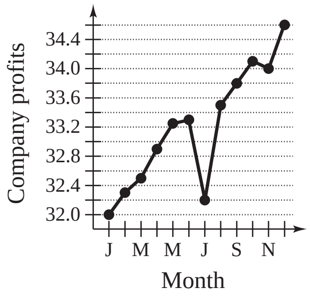
(a) $(1, -2), (4, 1)$ (b) $(-2, -3), (0, 0)$

14. Misleading Graphs Although graphs can help visualize relationships between two variables, they can also be misleading. The graphs shown below represent the same data points.

(a) Which of the two graphs is misleading, and why? Discuss other ways in which graphs can be misleading.

(b) Why would it be beneficial for someone to use a misleading graph?



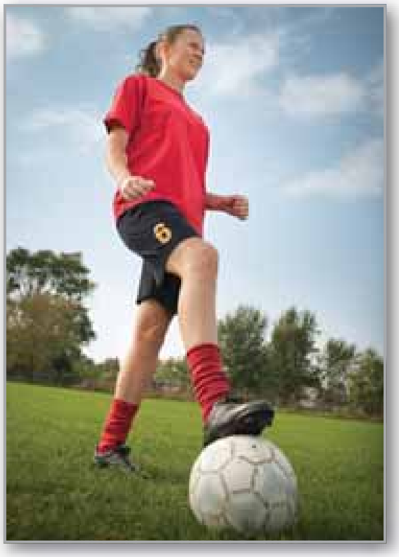


1 Equations, Inequalities, and Mathematical Modeling

Compound Interest *(Example 9, page 127)*

Blood Oxygen Level *(Exercise 125, page 112)*

Forestry *(Exercise 57, page 99)*



Athletic Participation *(Example 6, page 86)*

Population Statistics *(Exercise 79, page 80)*

Clockwise from top left, Apple's Eyes Studio/Shutterstock.com; Sergey Mironov/Shutterstock.com; Fotokostic/Shutterstock.com; John Griffin/The Image Works; Val Thoermer/Shutterstock.com

1.1 Graphs of Equations

The graph of an equation can help you see relationships between real-life quantities. For example, in Exercise 79 on page 80, you will use a graph to predict the life expectancy of a child born in 2015.

ALGEBRA HELP When evaluating an expression or an equation, remember to follow the Basic Rules of Algebra. To review these rules, see Section P.1.

- **Sketch graphs of equations.**
- **Identify *x*- and *y*-intercepts of graphs of equations.**
- **Use symmetry to sketch graphs of equations.**
- **Write equations of and sketch graphs of circles.**
- **Use graphs of equations in solving real-life problems.**

The Graph of an Equation

In Section P.6, you used a coordinate system to graphically represent the relationship between two quantities. There, the graphical picture consisted of a collection of points in a coordinate plane.

Frequently, a relationship between two quantities is expressed as an **equation in two variables.** For instance, $y = 7 - 3x$ is an equation in x and y. An ordered pair (a, b) is a **solution** or **solution point** of an equation in x and y when the substitutions $x = a$ and $y = b$ result in a true statement. For instance, $(1, 4)$ is a solution of $y = 7 - 3x$ because $4 = 7 - 3(1)$ is a true statement.

In this section, you will review some basic procedures for sketching the graph of an equation in two variables. The **graph of an equation** is the set of all points that are solutions of the equation.

EXAMPLE 1 Determining Solution Points

Determine whether (a) $(2, 13)$ and (b) $(-1, -3)$ lie on the graph of $y = 10x - 7$.

Solution

a. $y = 10x - 7$ Write original equation.

$13 \overset{?}{=} 10(2) - 7$ Substitute 2 for x and 13 for y.

$13 = 13$ $(2, 13)$ is a solution. ✓

The point $(2, 13)$ *does* lie on the graph of $y = 10x - 7$ because it is a solution point of the equation.

b. $y = 10x - 7$ Write original equation.

$-3 \overset{?}{=} 10(-1) - 7$ Substitute -1 for x and -3 for y.

$-3 \neq -17$ $(-1, -3)$ is not a solution.

The point $(-1, -3)$ *does not* lie on the graph of $y = 10x - 7$ because it is *not* a solution point of the equation.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Determine whether (a) $(3, -5)$ and (b) $(-2, 26)$ lie on the graph of $y = 14 - 6x$. ■

The basic technique used for sketching the graph of an equation is the **point-plotting method.**

The Point-Plotting Method of Graphing

1. When possible, isolate one of the variables.
2. Construct a table of values showing several solution points.
3. Plot these points in a rectangular coordinate system.
4. Connect the points with a smooth curve or line.

John Griffin/The Image Works

It is important to use negative values, zero, and positive values for x when constructing a table.

EXAMPLE 2 Sketching the Graph of an Equation

Sketch the graph of

$$y = -3x + 7.$$

Solution

Because the equation is already solved for y, construct a table of values that consists of several solution points of the equation. For instance, when $x = -1$,

$$y = -3(-1) + 7$$
$$= 10$$

which implies that

$$(-1, 10)$$

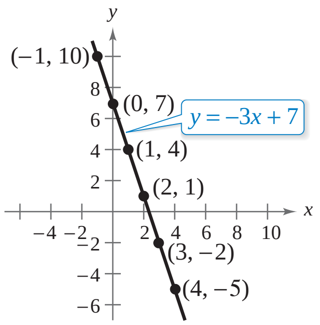
is a solution point of the equation.

x	$y = -3x + 7$	(x, y)
-1	10	$(-1, 10)$
0	7	$(0, 7)$
1	4	$(1, 4)$
2	1	$(2, 1)$
3	-2	$(3, -2)$
4	-5	$(4, -5)$

From the table, it follows that

$$(-1, 10), (0, 7), (1, 4), (2, 1), (3, -2), \quad \text{and} \quad (4, -5)$$

are solution points of the equation. After plotting these points and connecting them, you can see that they appear to lie on a line, as shown below.



✓ Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Sketch the graph of each equation.

a. $y = -3x + 2$

b. $y = 2x + 1$

EXAMPLE 3 Sketching the Graph of an Equation

Sketch the graph of

$$y = x^2 - 2.$$

Solution

Because the equation is already solved for y, begin by constructing a table of values.

x	-2	-1	0	1	2	3
$y = x^2 - 2$	2	-1	-2	-1	2	7
(x, y)	$(-2, 2)$	$(-1, -1)$	$(0, -2)$	$(1, -1)$	$(2, 2)$	$(3, 7)$

REMARK One of your goals in this course is to learn to classify the basic shape of a graph from its equation. For instance, you will learn that the *linear equation* in Example 2 has the form

$$y = mx + b$$

and its graph is a line. Similarly, the *quadratic equation* in Example 3 has the form

$$y = ax^2 + bx + c$$

and its graph is a parabola.

Next, plot the points given in the table, as shown in Figure 1.1. Finally, connect the points with a smooth curve, as shown in Figure 1.2.

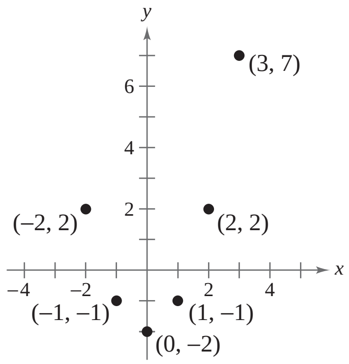


Figure 1.1

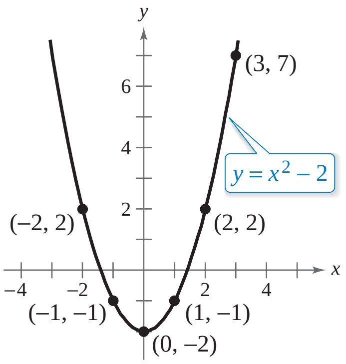


Figure 1.2

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Sketch the graph of each equation.

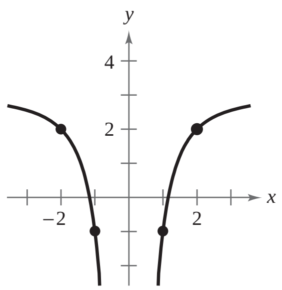
a. $y = x^2 + 3$

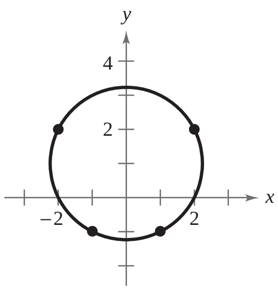
b. $y = 1 - x^2$

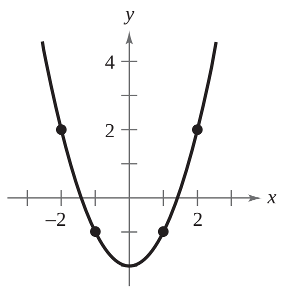
The point-plotting method demonstrated in Examples 2 and 3 is easy to use, but it has some shortcomings. With too few solution points, you can misrepresent the graph of an equation. For instance, when you only plot the four points

$$(-2, 2), (-1, -1), (1, -1), \quad \text{and} \quad (2, 2)$$

in Figure 1.1, any one of the three graphs below is reasonable.





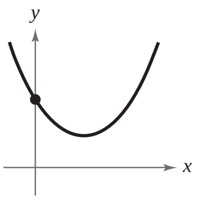


TECHNOLOGY To graph an equation involving x and y on a graphing utility, use the following procedure.

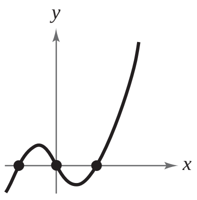
1. Rewrite the equation so that y is isolated on the left side.
2. Enter the equation in the graphing utility.
3. Determine a *viewing window* that shows all important features of the graph.
4. Graph the equation.

Intercepts of a Graph

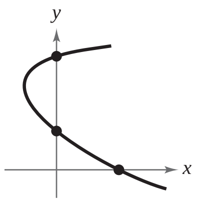
It is often easy to determine the solution points that have zero as either the x-coordinate or the y-coordinate. These points are called **intercepts** because they are the points at which the graph intersects or touches the x- or y-axis. It is possible for a graph to have no intercepts, one intercept, or several intercepts, as shown below.



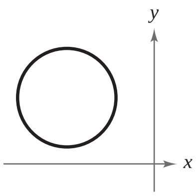
No x-intercepts
One y-intercept



Three x-intercepts
One y-intercept



One x-intercept
Two y-intercepts



No intercepts

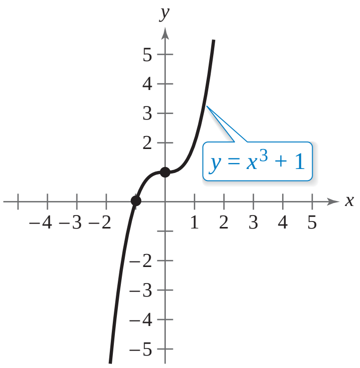
Note that an x-intercept can be written as the ordered pair $(a, 0)$ and a y-intercept can be written as the ordered pair $(0, b)$. Some texts denote the x-intercept as the x-coordinate of the point $(a, 0)$ [and the y-intercept as the y-coordinate of the point $(0, b)$] rather than the point itself. Unless it is necessary to make a distinction, the term *intercept* will refer to either the point or the coordinate.

EXAMPLE 4 Identifying x- and y-Intercepts

Identify the x- and y-intercepts of the graph of

$$y = x^3 + 1$$

shown at the right.



Solution

From the figure, you can see that the graph of the equation

$$y = x^3 + 1$$

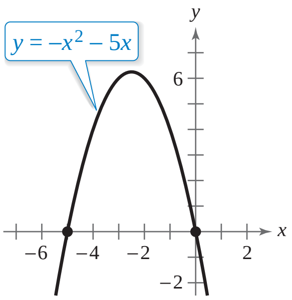
has an x-intercept (where y is zero) at $(-1, 0)$ and a y-intercept (where x is zero) at $(0, 1)$.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Identify the x- and y-intercepts of the graph of

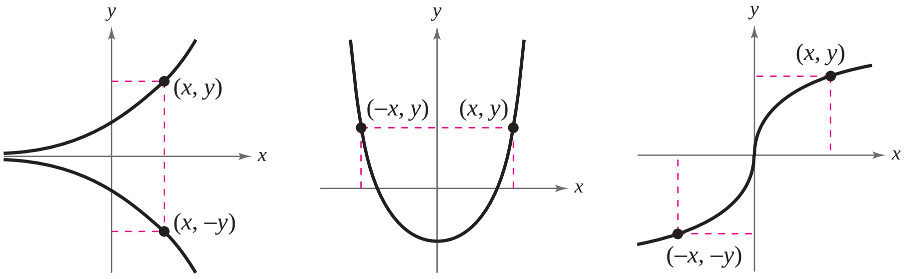
$$y = -x^2 - 5x$$

shown at the right.



Symmetry

Graphs of equations can have **symmetry** with respect to one of the coordinate axes or with respect to the origin. Symmetry with respect to the x-axis means that when the Cartesian plane is folded along the x-axis, the portion of the graph above the x-axis coincides with the portion below the x-axis. Symmetry with respect to the y-axis or the origin can be described in a similar manner, as shown below.



x-Axis symmetry y-Axis symmetry Origin symmetry

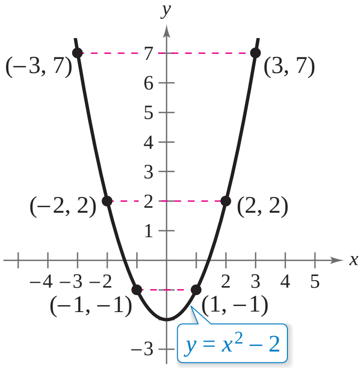
Knowing the symmetry of a graph *before* attempting to sketch it is helpful, because then you need only half as many solution points to sketch the graph. There are three basic types of symmetry, described as follows.

Graphical Tests for Symmetry

1. A graph is **symmetric with respect to the x-axis** if, whenever (x, y) is on the graph, $(x, -y)$ is also on the graph.
2. A graph is **symmetric with respect to the y-axis** if, whenever (x, y) is on the graph, $(-x, y)$ is also on the graph.
3. A graph is **symmetric with respect to the origin** if, whenever (x, y) is on the graph, $(-x, -y)$ is also on the graph.

You can conclude that the graph of $y = x^2 - 2$ is symmetric with respect to the y-axis because the point $(-x, y)$ is also on the graph of $y = x^2 - 2$. (See the table below and Figure 1.3.)

x	-3	-2	-1	1	2	3
y	7	2	-1	-1	2	7
(x, y)	$(-3, 7)$	$(-2, 2)$	$(-1, -1)$	$(1, -1)$	$(2, 2)$	$(3, 7)$



y-Axis symmetry

Figure 1.3

Algebraic Tests for Symmetry

1. The graph of an equation is symmetric with respect to the x-axis when replacing y with $-y$ yields an equivalent equation.
2. The graph of an equation is symmetric with respect to the y-axis when replacing x with $-x$ yields an equivalent equation.
3. The graph of an equation is symmetric with respect to the origin when replacing x with $-x$ and y with $-y$ yields an equivalent equation.

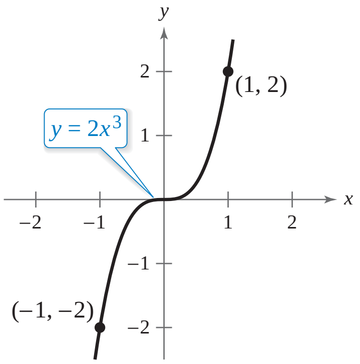


Figure 1.4

EXAMPLE 5 Testing for Symmetry

Test $y = 2x^3$ for symmetry with respect to both axes and the origin.

Solution

x-Axis: $y = 2x^3$ Write original equation.

$-y = 2x^3$ Replace y with $-y$. Result is *not* an equivalent equation.

y-Axis: $y = 2x^3$ Write original equation.

$y = 2(-x)^3$ Replace x with $-x$.

$y = -2x^3$ Simplify. Result is *not* an equivalent equation.

Origin: $y = 2x^3$ Write original equation.

$-y = 2(-x)^3$ Replace y with $-y$ and x with $-x$.

$-y = -2x^3$ Simplify.

$y = 2x^3$ Equivalent equation

Of the three tests for symmetry, the only one that is satisfied is the test for origin symmetry (see Figure 1.4).

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Test $y^2 = 6 - x$ for symmetry with respect to both axes and the origin.

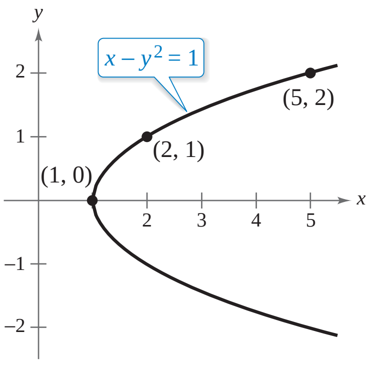


Figure 1.5

EXAMPLE 6 Using Symmetry as a Sketching Aid

Use symmetry to sketch the graph of $x - y^2 = 1$.

Solution Of the three tests for symmetry, the only one that is satisfied is the test for x-axis symmetry because $x - (-y)^2 = 1$ is equivalent to $x - y^2 = 1$. So, the graph is symmetric with respect to the x-axis. Using symmetry, you only need to find the solution points above the x-axis and then reflect them to obtain the graph, as shown in Figure 1.5.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use symmetry to sketch the graph of $y = x^2 - 4$.

ALGEBRA HELP In Example 7, $|x - 1|$ is an absolute value expression. You can review the techniques for evaluating an absolute value expression in Section P.1.

EXAMPLE 7 Sketching the Graph of an Equation

Sketch the graph of $y = |x - 1|$.

Solution This equation fails all three tests for symmetry, and consequently its graph is not symmetric with respect to either axis or to the origin. The absolute value bars indicate that y is always nonnegative. Construct a table of values. Then plot and connect the points, as shown in Figure 1.6. From the table, you can see that $x = 0$ when $y = 1$. So, the y-intercept is $(0, 1)$. Similarly, $y = 0$ when $x = 1$. So, the x-intercept is $(1, 0)$.

x	-2	-1	0	1	2	3	4
$y = \|x - 1\|$	3	2	1	0	1	2	3
(x, y)	$(-2, 3)$	$(-1, 2)$	$(0, 1)$	$(1, 0)$	$(2, 1)$	$(3, 2)$	$(4, 3)$

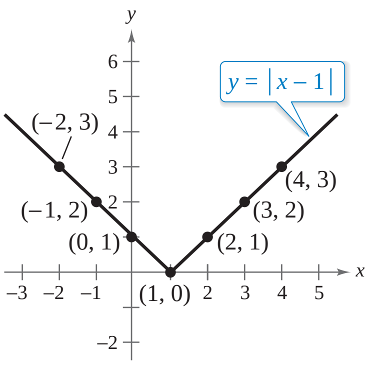


Figure 1.6

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Sketch the graph of $y = |x - 2|$.

Circles

Throughout this course, you will learn to recognize several types of graphs from their equations. For instance, you will learn to recognize that the graph of a second-degree equation of the form

$$y = ax^2 + bx + c$$

is a parabola (see Example 3). The graph of a **circle** is also easy to recognize.

Consider the circle shown in Figure 1.7. A point (x, y) lies on the circle if and only if its distance from the center (h, k) is r. By the Distance Formula,

$$\sqrt{(x - h)^2 + (y - k)^2} = r.$$

By squaring each side of this equation, you obtain the **standard form of the equation of a circle.** For example, a circle with its center at $(h, k) = (1, 3)$ and radius $r = 4$ is given by

$$\sqrt{(x - 1)^2 + (y - 3)^2} = 4 \qquad \text{Substitute for } h, k, \text{ and } r.$$

$$(x - 1)^2 + (y - 3)^2 = 16. \qquad \text{Square each side.}$$

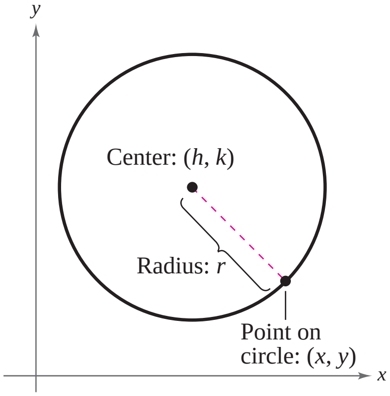


Figure 1.7

Standard Form of the Equation of a Circle

A point (x, y) lies on the circle of **radius** r and **center** (h, k) if and only if

$$(x - h)^2 + (y - k)^2 = r^2.$$

REMARK Be careful when you are finding h and k from the standard form of the equation of a circle. For instance, to find h and k from the equation of the circle in Example 8, rewrite the quantities $(x + 1)^2$ and $(y - 2)^2$ using subtraction.

$$(x + 1)^2 = [x - (-1)]^2,$$

$$(y - 2)^2 = [y - (2)]^2$$

So, $h = -1$ and $k = 2$.

From this result, you can see that the standard form of the equation of a circle *with its center at the origin*, $(h, k) = (0, 0)$, is simply

$$x^2 + y^2 = r^2. \qquad \text{Circle with center at origin}$$

EXAMPLE 8 Writing the Equation of a Circle

The point $(3, 4)$ lies on a circle whose center is at $(-1, 2)$, as shown in Figure 1.8. Write the standard form of the equation of this circle.

Solution

The radius of the circle is the distance between $(-1, 2)$ and $(3, 4)$.

$$r = \sqrt{(x - h)^2 + (y - k)^2} \qquad \text{Distance Formula}$$

$$= \sqrt{[3 - (-1)]^2 + (4 - 2)^2} \qquad \text{Substitute for } x, y, h, \text{ and } k.$$

$$= \sqrt{4^2 + 2^2} \qquad \text{Simplify.}$$

$$= \sqrt{16 + 4} \qquad \text{Simplify.}$$

$$= \sqrt{20} \qquad \text{Radius}$$

Using $(h, k) = (-1, 2)$ and $r = \sqrt{20}$, the equation of the circle is

$$(x - h)^2 + (y - k)^2 = r^2 \qquad \text{Equation of circle}$$

$$[x - (-1)]^2 + (y - 2)^2 = (\sqrt{20})^2 \qquad \text{Substitute for } h, k, \text{ and } r.$$

$$(x + 1)^2 + (y - 2)^2 = 20. \qquad \text{Standard form}$$

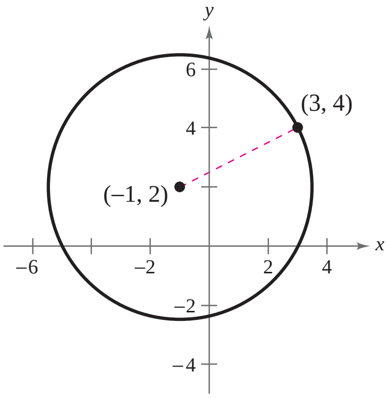


Figure 1.8

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The point $(1, -2)$ lies on a circle whose center is at $(-3, -5)$. Write the standard form of the equation of this circle.

Application

In this course, you will learn that there are many ways to approach a problem. Three common approaches are illustrated in Example 9.

A Numerical Approach: Construct and use a table.

A Graphical Approach: Draw and use a graph.

An Algebraic Approach: Use the rules of algebra.

REMARK You should develop the habit of using at least two approaches to solve every problem. This helps build your intuition and helps you check that your answers are reasonable.

EXAMPLE 9 Recommended Weight

The median recommended weights y (in pounds) for men of medium frame who are 25 to 59 years old can be approximated by the mathematical model

$$y = 0.073x^2 - 6.99x + 289.0, \quad 62 \le x \le 76$$

where x is a man's height (in inches). *(Source: Metropolitan Life Insurance Company)*

a. Construct a table of values that shows the median recommended weights for men with heights of 62, 64, 66, 68, 70, 72, 74, and 76 inches.

b. Use the table of values to sketch a graph of the model. Then use the graph to estimate *graphically* the median recommended weight for a man whose height is 71 inches.

c. Use the model to confirm *algebraically* the estimate you found in part (b).

Solution

a. You can use a calculator to construct the table, as shown on the left.

DATA

Height, x	Weight, y
62	136.2
64	140.6
66	145.6
68	151.2
70	157.4
72	164.2
74	171.5
76	179.4

Spreadsheet at LarsonPrecalculus.com

b. The table of values can be used to sketch the graph of the equation, as shown in Figure 1.9. From the graph, you can estimate that a height of 71 inches corresponds to a weight of about 161 pounds.

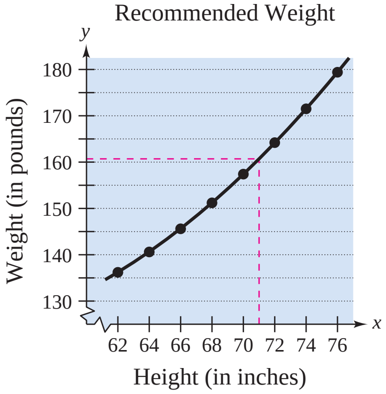


Figure 1.9

c. To confirm algebraically the estimate found in part (b), you can substitute 71 for x in the model.

$$y = 0.073(71)^2 - 6.99(71) + 289.0 \approx 160.70$$

So, the graphical estimate of 161 pounds is fairly good.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use Figure 1.9 to estimate *graphically* the median recommended weight for a man whose height is 75 inches. Then confirm the estimate *algebraically*.

Summarize (Section 1.1)

1. Describe how to sketch the graph of an equation *(page 70)*. For examples of graphing equations, see Examples 1–3.
2. Describe how to identify the x- and y-intercepts of a graph *(page 73)*. For an example of identifying x- and y-intercepts, see Example 4.
3. Describe how to use symmetry to graph an equation *(page 74)*. For an example of using symmetry to graph an equation, see Example 6.
4. State the standard form of the equation of a circle *(page 76)*. For an example of writing the standard form of the equation of a circle, see Example 8.
5. Describe how to use the graph of an equation to solve a real-life problem *(page 77, Example 9)*.

1.1 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. An ordered pair (a, b) is a ________ of an equation in x and y when the substitutions $x = a$ and $y = b$ result in a true statement.
2. The set of all solution points of an equation is the ________ of the equation.
3. The points at which a graph intersects or touches an axis are called the ________ of the graph.
4. A graph is symmetric with respect to the ________ if, whenever (x, y) is on the graph, $(-x, y)$ is also on the graph.
5. The equation $(x - h)^2 + (y - k)^2 = r^2$ is the standard form of the equation of a ________ with center ________ and radius ________.
6. When you construct and use a table to solve a problem, you are using a ________ approach.

Skills and Applications

Determining Solution Points

In Exercises 7–14, determine whether each point lies on the graph of the equation.

	Equation	Points	
7.	$y = \sqrt{x + 4}$	(a) $(0, 2)$	(b) $(5, 3)$
8.	$y = \sqrt{5 - x}$	(a) $(1, 2)$	(b) $(5, 0)$
9.	$y = x^2 - 3x + 2$	(a) $(2, 0)$	(b) $(-2, 8)$
10.	$y = 4 - \lvert x - 2 \rvert$	(a) $(1, 5)$	(b) $(6, 0)$
11.	$y = \lvert x - 1 \rvert + 2$	(a) $(2, 3)$	(b) $(-1, 0)$
12.	$2x - y - 3 = 0$	(a) $(1, 2)$	(b) $(1, -1)$
13.	$x^2 + y^2 = 20$	(a) $(3, -2)$	(b) $(-4, 2)$
14.	$y = \frac{1}{3}x^3 - 2x^2$	(a) $\left(2, -\frac{16}{3}\right)$	(b) $(-3, 9)$

Sketching the Graph of an Equation

In Exercises 15–18, complete the table. Use the resulting solution points to sketch the graph of the equation.

15. $y = -2x + 5$

x	-1	0	1	2	$\frac{5}{2}$
y					
(x, y)					

16. $y = \frac{3}{4}x - 1$

x	-2	0	1	$\frac{4}{3}$	2
y					
(x, y)					

17. $y = x^2 - 3x$

x	-1	0	1	2	3
y					
(x, y)					

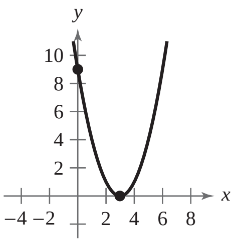
18. $y = 5 - x^2$

x	-2	-1	0	1	2
y					
(x, y)					

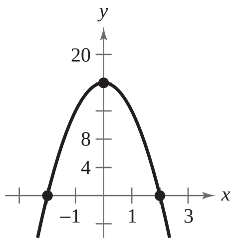
Identifying *x*- and *y*-Intercepts

In Exercises 19–24, identify the *x*- and *y*-intercepts of the graph.

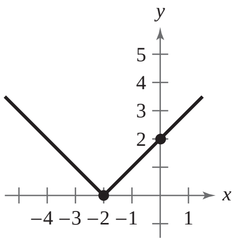
19. $y = (x - 3)^2$



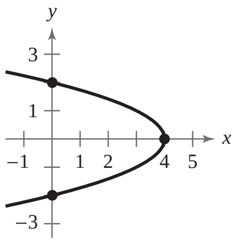
20. $y = 16 - 4x^2$



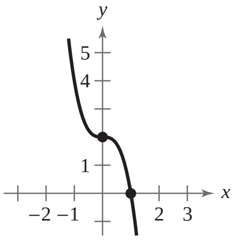
21. $y = \lvert x + 2 \rvert$



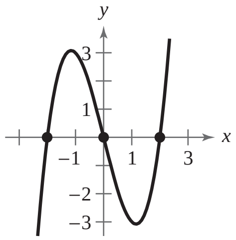
22. $y^2 = 4 - x$



23. $y = 2 - 2x^3$



24. $y = x^3 - 4x$



Testing for Symmetry **In Exercises 25–32, use the algebraic tests to check for symmetry with respect to both axes and the origin.**

25. $x^2 - y = 0$

26. $x - y^2 = 0$

27. $y = x^3$

28. $y = x^4 - x^2 + 3$

29. $y = \dfrac{x}{x^2 + 1}$

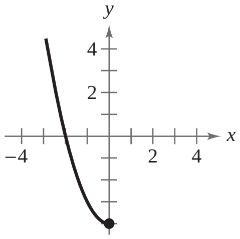
30. $y = \dfrac{1}{x^2 + 1}$

31. $xy^2 + 10 = 0$

32. $xy = 4$

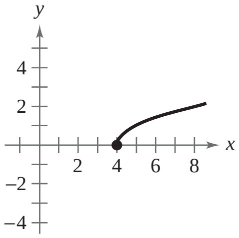
Using Symmetry as a Sketching Aid **In Exercises 33–36, assume that the graph has the indicated type of symmetry. Sketch the complete graph of the equation. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

33.



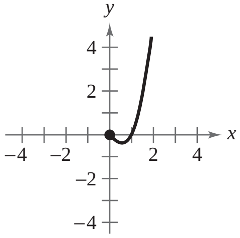
y-Axis symmetry

34.



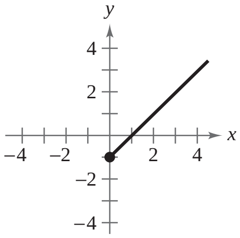
x-Axis symmetry

35.



Origin symmetry

36.



y-Axis symmetry

Sketching the Graph of an Equation **In Exercises 37–48, identify any intercepts and test for symmetry. Then sketch the graph of the equation.**

37. $y = -3x + 1$

38. $y = 2x - 3$

39. $y = x^2 - 2x$

40. $y = -x^2 - 2x$

41. $y = x^3 + 3$

42. $y = x^3 - 1$

43. $y = \sqrt{x - 3}$

44. $y = \sqrt{1 - x}$

45. $y = |x - 6|$

46. $y = 1 - |x|$

47. $x = y^2 - 1$

48. $x = y^2 - 5$

Graphical Analysis **In Exercises 49–60, use a graphing utility to graph the equation. Use a standard setting. Approximate any intercepts.**

49. $y = 5 - \frac{1}{2}x$

50. $y = \frac{2}{3}x - 1$

51. $y = x^2 - 4x + 3$

52. $y = x^2 + x - 2$

53. $y = \dfrac{2x}{x - 1}$

54. $y = \dfrac{4}{x^2 + 1}$

55. $y = \sqrt[3]{x} + 2$

56. $y = \sqrt[3]{x + 1}$

57. $y = x\sqrt{x + 6}$

58. $y = (6 - x)\sqrt{x}$

59. $y = |x + 3|$

60. $y = 2 - |x|$

Writing the Equation of a Circle **In Exercises 61–68, write the standard form of the equation of the circle with the given characteristics.**

61. Center: $(0, 0)$; Radius: 4

62. Center: $(0, 0)$; Radius: 5

63. Center: $(2, -1)$; Radius: 4

64. Center: $(-7, -4)$; Radius: 7

65. Center: $(-1, 2)$; Solution point: $(0, 0)$

66. Center: $(3, -2)$; Solution point: $(-1, 1)$

67. Endpoints of a diameter: $(0, 0)$, $(6, 8)$

68. Endpoints of a diameter: $(-4, -1)$, $(4, 1)$

Sketching the Graph of a Circle **In Exercises 69–74, find the center and radius of the circle. Then sketch the graph of the circle.**

69. $x^2 + y^2 = 25$

70. $x^2 + y^2 = 36$

71. $(x - 1)^2 + (y + 3)^2 = 9$

72. $x^2 + (y - 1)^2 = 1$

73. $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{9}{4}$

74. $(x - 2)^2 + (y + 3)^2 = \frac{16}{9}$

75. Depreciation A hospital purchases a new magnetic resonance imaging (MRI) machine for \$500,000. The depreciated value y (reduced value) after t years is given by $y = 500{,}000 - 40{,}000t$, $0 \le t \le 8$. Sketch the graph of the equation.

76. Consumerism You purchase an all-terrain vehicle (ATV) for \$8000. The depreciated value y after t years is given by $y = 8000 - 900t$, $0 \le t \le 6$. Sketch the graph of the equation.

77. Geometry A regulation NFL playing field (including the end zones) of length x and width y has a perimeter of $346\frac{2}{3}$ or $\frac{1040}{3}$ yards.

(a) Draw a rectangle that gives a visual representation of the problem. Use the specified variables to label the sides of the rectangle.

(b) Show that the width of the rectangle is $y = \frac{520}{3} - x$ and its area is $A = x\left(\frac{520}{3} - x\right)$.

(c) Use a graphing utility to graph the area equation. Be sure to adjust your window settings.

(d) From the graph in part (c), estimate the dimensions of the rectangle that yield a maximum area.

(e) Use your school's library, the Internet, or some other reference source to find the actual dimensions and area of a regulation NFL playing field and compare your findings with the results of part (d).

The symbol indicates an exercise or a part of an exercise in which you are instructed to use a graphing utility.

78. Geometry A soccer playing field of length x and width y has a perimeter of 360 meters.

(a) Draw a rectangle that gives a visual representation of the problem. Use the specified variables to label the sides of the rectangle.

(b) Show that the width of the rectangle is $y = 180 - x$ and its area is $A = x(180 - x)$.

(c) Use a graphing utility to graph the area equation. Be sure to adjust your window settings.

(d) From the graph in part (c), estimate the dimensions of the rectangle that yield a maximum area.

(e) Use your school's library, the Internet, or some other reference source to find the actual dimensions and area of a regulation Major League Soccer field and compare your findings with the results of part (d).

79. Population Statistics

The table shows the life expectancies of a child (at birth) in the United States for selected years from 1930 through 2000. *(Source: U.S. National Center for Health Statistics)*

DATA

Spreadsheet at LarsonPrecalculus.com

Year	Life Expectancy, y
1930	59.7
1940	62.9
1950	68.2
1960	69.7
1970	70.8
1980	73.7
1990	75.4
2000	76.8

A model for the life expectancy during this period is

$$y = -0.002t^2 + 0.50t + 46.6, \quad 30 \le t \le 100$$

where y represents the life expectancy and t is the time in years, with $t = 30$ corresponding to 1930.

(a) Use a graphing utility to graph the data from the table and the model in the same viewing window. How well does the model fit the data? Explain.

(b) Determine the life expectancy in 1990 both graphically and algebraically.

79. Population Statistics (continued)

(c) Use the graph to determine the year when life expectancy was approximately 76.0. Verify your answer algebraically.

(d) One projection for the life expectancy of a child born in 2015 is 78.9. How does this compare with the projection given by the model?

(e) Do you think this model can be used to predict the life expectancy of a child 50 years from now? Explain.

80. Electronics The resistance y (in ohms) of 1000 feet of solid copper wire at 68 degrees Fahrenheit is

$$y = \frac{10{,}370}{x^2}$$

where x is the diameter of the wire in mils (0.001 inch).

(a) Complete the table.

x	5	10	20	30	40	50
y						

x	60	70	80	90	100
y					

(b) Use the table of values in part (a) to sketch a graph of the model. Then use your graph to estimate the resistance when $x = 85.5$.

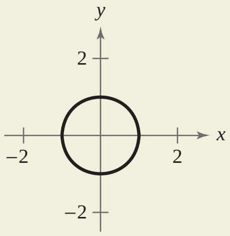
(c) Use the model to confirm algebraically the estimate you found in part (b).

(d) What can you conclude in general about the relationship between the diameter of the copper wire and the resistance?

Exploration

81. Think About It Find a and b if the graph of $y = ax^2 + bx^3$ is symmetric with respect to (a) the y-axis and (b) the origin. (There are many correct answers.)

82. HOW DO YOU SEE IT? The graph of the circle with equation $x^2 + y^2 = 1$ is shown below. Describe the types of symmetry that you observe.



John Griffin/The Image Works

1.2 Linear Equations in One Variable

You can use linear equations in many real-life applications. For example, you can use linear equations in forensics to determine height from femur length. See Exercises 67 and 68 on page 88.

- **Identify different types of equations.**
- **Solve linear equations in one variable.**
- **Solve rational equations that lead to linear equations.**
- **Find *x*- and *y*-intercepts of graphs of equations algebraically.**
- **Use linear equations to model and solve real-life problems.**

Equations and Solutions of Equations

An **equation** in x is a statement that two algebraic expressions are equal. For example,

$$3x - 5 = 7, \quad x^2 - x - 6 = 0, \quad \text{and} \quad \sqrt{2x} = 4$$

are equations. To **solve** an equation in x means to find all values of x for which the equation is true. Such values are **solutions.** For instance, $x = 4$ is a solution of the equation

$$3x - 5 = 7$$

because

$$3(4) - 5 = 7$$

is a true statement.

The solutions of an equation depend on the kinds of numbers being considered. For instance, in the set of rational numbers, $x^2 = 10$ has no solution because there is no rational number whose square is 10. However, in the set of real numbers, the equation has the two solutions

$$x = \sqrt{10} \quad \text{and} \quad x = -\sqrt{10}.$$

An equation that is true for *every* real number in the domain of the variable is called an **identity.** For example,

$$x^2 - 9 = (x + 3)(x - 3) \qquad \text{Identity}$$

is an identity because it is a true statement for any real value of x. The equation

$$\frac{x}{3x^2} = \frac{1}{3x} \qquad \text{Identity}$$

where $x \neq 0$, is an identity because it is true for any nonzero real value of x.

An equation that is true for just *some* (but not all) of the real numbers in the domain of the variable is called a **conditional equation.** For example, the equation

$$x^2 - 9 = 0 \qquad \text{Conditional equation}$$

is conditional because $x = 3$ and $x = -3$ are the only values in the domain that satisfy the equation. The equation

$$2x + 4 = 6$$

is conditional because $x = 1$ is the only value in the domain that satisfies the equation.

A **contradiction** is an equation that is false for *every* real number in the domain of the variable. For example, the equation

$$2x - 4 = 2x + 1 \qquad \text{Contradiction}$$

is a contradiction because there are no real values of x for which the equation is true.

Andrew Douglas/Masterfile

Linear Equations in One Variable

Definition of Linear Equation in One Variable

A **linear equation in one variable** x is an equation that can be written in the standard form

$$ax + b = 0$$

where a and b are real numbers with $a \neq 0$.

A linear equation has exactly one solution. To see this, consider the following steps. (Remember that $a \neq 0$.)

$$ax + b = 0 \qquad \text{Write original equation.}$$

$$ax = -b \qquad \text{Subtract } b \text{ from each side.}$$

$$x = -\frac{b}{a} \qquad \text{Divide each side by } a.$$

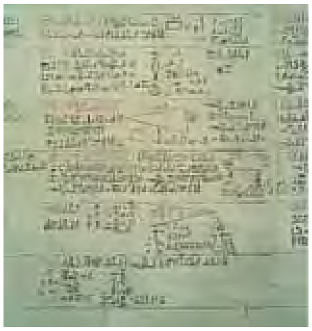
So, the equation

$$ax + b = 0$$

has exactly one solution,

$$x = -\frac{b}{a}.$$

To solve a conditional equation in x, isolate x on one side of the equation by a sequence of **equivalent equations,** each having the same solution(s) as the original equation. The operations that yield equivalent equations come from the properties of equality reviewed in Section P.1.



HISTORICAL NOTE

This ancient Egyptian papyrus, discovered in 1858, contains one of the earliest examples of mathematical writing in existence. The papyrus itself dates back to around 1650 B.C., but it is actually a copy of writings from two centuries earlier. The algebraic equations on the papyrus were written in words. Diophantus, a Greek who lived around A.D. 250, is often called the Father of Algebra. He was the first to use abbreviated word forms in equations.

Generating Equivalent Equations

An equation can be transformed into an *equivalent equation* by one or more of the following steps.

	Given Equation	Equivalent Equation
1. Remove symbols of grouping, combine like terms, or simplify fractions on one or both sides of the equation.	$2x - x = 4$	$x = 4$
2. Add (or subtract) the same quantity to (from) *each* side of the equation.	$x + 1 = 6$	$x = 5$
3. Multiply (or divide) *each* side of the equation by the same *nonzero* quantity.	$2x = 6$	$x = 3$
4. Interchange the two sides of the equation.	$2 = x$	$x = 2$

The steps for solving a linear equation in one variable x written in standard form are shown in the following example.

British Museum Algebra and Trigonometry

EXAMPLE 1 Solving a Linear Equation

a. $3x - 6 = 0$ Original equation

$3x = 6$ Add 6 to each side.

$x = 2$ Divide each side by 3.

b. $5x + 4 = 3x - 8$ Original equation

$2x + 4 = -8$ Subtract $3x$ from each side.

$2x = -12$ Subtract 4 from each side.

$x = -6$ Divide each side by 2.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve each equation.

a. $7 - 2x = 15$

b. $7x - 9 = 5x + 7$

After solving an equation, you should check each solution in the original equation. For instance, you can check the solution of Example 1(a) as follows.

$3x - 6 = 0$ Write original equation.

$3(2) - 6 \overset{?}{=} 0$ Substitute 2 for x.

$0 = 0$ Solution checks. ✓

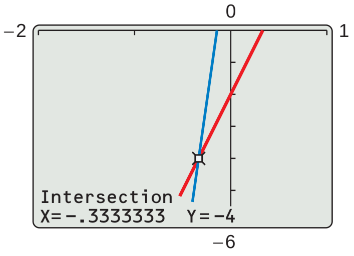
Try checking the solution of Example 1(b).

▷ **TECHNOLOGY** You can use a graphing utility to check that a solution is reasonable. One way to do this is to graph the left side of the equation, then graph the right side of the equation, and determine the point of intersection. For instance, in Example 2, if you graph the equations

$y_1 = 6(x - 1) + 4$ The left side

$y_2 = 3(7x + 1)$ The right side

in the same viewing window, they should intersect at $x = -\frac{1}{3}$, as shown in the graph below.



EXAMPLE 2 Solving a Linear Equation

Solve $6(x - 1) + 4 = 3(7x + 1)$.

Solution

$6(x - 1) + 4 = 3(7x + 1)$ Write original equation.

$6x - 6 + 4 = 21x + 3$ Distributive Property

$6x - 2 = 21x + 3$ Simplify.

$-15x = 5$ Simplify.

$x = -\dfrac{1}{3}$ Divide each side by -15.

Check

$6(x - 1) + 4 = 3(7x + 1)$ Write original equation.

$6\left(-\frac{1}{3} - 1\right) + 4 \overset{?}{=} 3\left[7\left(-\frac{1}{3}\right) + 1\right]$ Substitute $-\frac{1}{3}$ for x.

$6\left(-\frac{4}{3}\right) + 4 \overset{?}{=} 3\left[-\frac{7}{3} + 1\right]$ Simplify.

$-8 + 4 \overset{?}{=} -7 + 3$ Simplify.

$-4 = -4$ Solution checks. ✓

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $4(x + 2) - 12 = 5(x - 6)$.

Rational Equations That Lead to Linear Equations

A **rational equation** is an equation that involves one or more fractional expressions. To solve a rational equation, find the least common denominator (LCD) of all terms and multiply every term by the LCD. This process will clear the original equation of fractions and produce a simpler equation to work with.

REMARK An equation with a *single fraction* on each side can be cleared of denominators by **cross multiplying**. To do this, multiply the left numerator by the right denominator and the right numerator by the left denominator as follows.

$$\begin{aligned} \frac{a}{b} &= \frac{c}{d} && \text{Original equation} \\ ad &= cb && \text{Cross multiply.} \end{aligned}$$

EXAMPLE 3 Solving a Rational Equation

Solve $\dfrac{x}{3} + \dfrac{3x}{4} = 2$.

Solution

$$\begin{aligned} \frac{x}{3} + \frac{3x}{4} &= 2 && \text{Original equation} \\ (12)\frac{x}{3} + (12)\frac{3x}{4} &= (12)2 && \text{Multiply each term by the LCD.} \\ 4x + 9x &= 24 && \text{Simplify.} \\ 13x &= 24 && \text{Combine like terms.} \\ x &= \frac{24}{13} && \text{Divide each side by 13.} \end{aligned}$$

The solution is $x = \frac{24}{13}$. Check this in the original equation.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $\dfrac{4x}{9} - \dfrac{1}{3} = x + \dfrac{5}{3}$.

When multiplying or dividing an equation by a *variable expression*, it is possible to introduce an **extraneous solution** that does not satisfy the original equation.

EXAMPLE 4 An Equation with an Extraneous Solution

Solve $\dfrac{1}{x-2} = \dfrac{3}{x+2} - \dfrac{6x}{x^2-4}$.

Solution The LCD is $x^2 - 4 = (x+2)(x-2)$. Multiply each term by this LCD.

$$\begin{aligned} \frac{1}{x-2}(x+2)(x-2) &= \frac{3}{x+2}(x+2)(x-2) - \frac{6x}{x^2-4}(x+2)(x-2) \\ x + 2 &= 3(x-2) - 6x, \quad x \neq \pm 2 \\ x + 2 &= 3x - 6 - 6x \\ x + 2 &= -3x - 6 \\ 4x &= -8 \quad \Rightarrow \quad x = -2 && \text{Extraneous solution} \end{aligned}$$

In the original equation, $x = -2$ yields a denominator of zero. So, $x = -2$ is an extraneous solution, and the original equation has *no solution.*

REMARK Recall that the least common denominator of two or more fractions consists of the product of all prime factors in the denominators, with each factor given the highest power of its occurrence in any denominator. For instance, in Example 4, by factoring each denominator you can determine that the LCD is $(x+2)(x-2)$.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $\dfrac{3x}{x-4} = 5 + \dfrac{12}{x-4}$.

Finding Intercepts Algebraically

In Section 1.1, you learned to find x- and y-intercepts using a graphical approach. Because all the points on the x-axis have a y-coordinate equal to zero, and all the points on the y-axis have an x-coordinate equal to zero, you can use an algebraic approach to find x- and y-intercepts, as follows.

Finding Intercepts Algebraically

1. To find x-intercepts, set y equal to zero and solve the equation for x.
2. To find y-intercepts, set x equal to zero and solve the equation for y.

EXAMPLE 5 Finding Intercepts Algebraically

Find the x- and y-intercepts of the graph of each equation algebraically.

a. $y = 4x + 1$ **b.** $3x + 2y = 6$

Solution

a. To find the x-intercept, set y equal to zero and solve for x, as follows.

$$\begin{aligned} y &= 4x + 1 && \text{Write original equation.}\\ 0 &= 4x + 1 && \text{Substitute 0 for } y.\\ -1 &= 4x && \text{Subtract 1 from each side.}\\ -\tfrac{1}{4} &= x && \text{Divide each side by 4.} \end{aligned}$$

So, the x-intercept is $\left(-\frac{1}{4}, 0\right)$.

To find the y-intercept, set x equal to zero and solve for y, as follows.

$$\begin{aligned} y &= 4x + 1 && \text{Write original equation.}\\ y &= 4(0) + 1 && \text{Substitute 0 for } x.\\ y &= 1 && \text{Simplify.} \end{aligned}$$

So, the y-intercept is $(0, 1)$.

b. To find the x-intercept, set y equal to zero and solve for x, as follows.

$$\begin{aligned} 3x + 2y &= 6 && \text{Write original equation.}\\ 3x + 2(0) &= 6 && \text{Substitute 0 for } y.\\ 3x &= 6 && \text{Simplify.}\\ x &= 2 && \text{Divide each side by 3.} \end{aligned}$$

So, the x-intercept is $(2, 0)$.

To find the y-intercept, set x equal to zero and solve for y, as follows.

$$\begin{aligned} 3x + 2y &= 6 && \text{Write original equation.}\\ 3(0) + 2y &= 6 && \text{Substitute 0 for } x.\\ 2y &= 6 && \text{Simplify.}\\ y &= 3 && \text{Divide each side by 2.} \end{aligned}$$

So, the y-intercept is $(0, 3)$.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the x- and y-intercepts of the graph of each equation algebraically.

a. $y = -3x - 2$ **b.** $5x + 3y = 15$

Application

EXAMPLE 6 Female Participants in Athletic Programs

The numbers y (in millions) of female participants in high school athletic programs in the United States from 2000 through 2010 can be approximated by the linear model

$$y = 0.045t + 2.70, \quad 0 \leq t \leq 10$$

where $t = 0$ represents 2000. (a) Find algebraically and interpret the y-intercept of the graph of the linear model shown in Figure 1.10. (b) Use the linear model to predict the year in which there will be 3.42 million female participants. *(Source: National Federation of State High School Associations)*

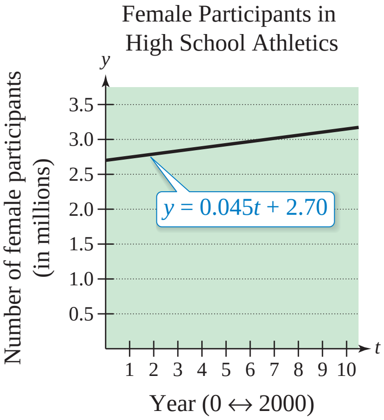


Figure 1.10

Solution

a. To find the y-intercept, let $t = 0$ and solve for y, as follows.

$$\begin{aligned} y &= 0.045t + 2.70 && \text{Write original equation.} \\ &= 0.045(0) + 2.70 && \text{Substitute 0 for } t. \\ &= 2.70 && \text{Simplify.} \end{aligned}$$

So, the y-intercept is $(0, 2.70)$. This means that there were about 2.70 million female participants in 2000.

b. Let $y = 3.42$ and solve for t, as follows.

$$\begin{aligned} y &= 0.045t + 2.70 && \text{Write original equation.} \\ 3.42 &= 0.045t + 2.70 && \text{Substitute 3.42 for } y. \\ 0.72 &= 0.045t && \text{Subtract 2.70 from each side.} \\ 16 &= t && \text{Divide each side by 0.045.} \end{aligned}$$

Because $t = 0$ represents 2000, $t = 16$ must represent 2016. So, from this model, there will be 3.42 million female participants in 2016.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The numbers y (in millions) of male participants in high school athletic programs in the United States from 2000 through 2010 can be approximated by the linear model

$$y = 0.064t + 3.83, \quad 0 \leq t \leq 10$$

where $t = 0$ represents 2000. (a) Find algebraically and interpret the y-intercept of the graph of the linear model. (b) Use the linear model to predict the year in which there will be 4.79 million male participants. *(Source: National Federation of State High School Associations)*

Summarize (Section 1.2)

1. State the definition of an identity, a conditional equation, and a contradiction *(page 81)*.
2. State the definition of a linear equation in one variable and list the four steps that can be used to form an equivalent equation *(page 82)*. For examples of solving linear equations, see Examples 1 and 2.
3. Describe how to solve a rational equation *(page 84)*. For examples of solving rational equations, see Examples 3 and 4.
4. Describe how to find intercepts algebraically *(page 85)*. For an example of finding intercepts algebraically, see Example 5.
5. Describe a real-life application involving a linear equation *(page 86, Example 6)*.

Fotokostic/Shutterstock.com

1.2 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. An ________ is a statement that equates two algebraic expressions.
2. There are three types of equations: ________ , ________ equations, and ________ .
3. A linear equation in one variable x is an equation that can be written in the standard form ________.
4. An ________ equation has the same solution(s) as the original equation.
5. A ________ equation is an equation that involves one or more fractional expressions.
6. An ________ solution is a solution that does not satisfy the original equation.

Skills and Applications

Classifying an Equation **In Exercises 7–14, determine whether the equation is an identity, a conditional equation, or a contradiction.**

7. $2(x - 1) = 2x - 2$ **8.** $3(x + 2) = 5x + 4$

9. $-6(x - 3) + 5 = -2x + 10$

10. $3(x + 2) - 5 = 3x + 1$

11. $4(x + 1) - 2x = 2(x + 2)$

12. $-7(x - 3) + 4x = 3(7 - x)$

13. $2(x - 1) = 2x - 1$ **14.** $\frac{1}{4}(x - 4) = \frac{1}{4}x - 4$

Solving a Linear Equation **In Exercises 15–26, solve the equation and check your solution. (If not possible, explain why.)**

15. $x + 11 = 15$ **16.** $7 - x = 19$

17. $7 - 2x = 25$ **18.** $7x + 2 = 23$

19. $3x - 5 = 2x + 7$ **20.** $5x + 3 = 6 - 2x$

21. $4y + 2 - 5y = 7 - 6y$ **22.** $5y + 1 = 8y - 5 + 6y$

23. $x - 3(2x + 3) = 8 - 5x$

24. $9x - 10 = 5x + 2(2x - 5)$

25. $0.25x + 0.75(10 - x) = 3$

26. $0.60x + 0.40(100 - x) = 50$

Solving a Rational Equation **In Exercises 27–44, solve the equation and check your solution. (If not possible, explain why.)**

27. $\frac{3x}{8} - \frac{4x}{3} = 4$ **28.** $\frac{2x}{5} + 5x = \frac{4}{3}$

29. $\frac{5x}{4} + \frac{1}{2} = x - \frac{1}{2}$ **30.** $\frac{x}{5} - \frac{x}{2} = 3 + \frac{3x}{10}$

31. $\frac{5x - 4}{5x + 4} = \frac{2}{3}$ **32.** $\frac{10x + 3}{5x + 6} = \frac{1}{2}$

33. $10 - \frac{13}{x} = 4 + \frac{5}{x}$ **34.** $\frac{15}{x} - 4 = \frac{6}{x} + 3$

35. $3 = 2 + \frac{2}{z + 2}$ **36.** $\frac{1}{x} + \frac{2}{x - 5} = 0$

37. $\frac{x}{x + 4} + \frac{4}{x + 4} + 2 = 0$

38. $\frac{7}{2x + 1} - \frac{8x}{2x - 1} = -4$

39. $\frac{2}{(x - 4)(x - 2)} = \frac{1}{x - 4} + \frac{2}{x - 2}$

40. $\frac{4}{x - 1} + \frac{6}{3x + 1} = \frac{15}{3x + 1}$

41. $\frac{1}{x - 3} + \frac{1}{x + 3} = \frac{10}{x^2 - 9}$

42. $\frac{1}{x - 2} + \frac{3}{x + 3} = \frac{4}{x^2 + x - 6}$

43. $\frac{3}{x^2 - 3x} + \frac{4}{x} = \frac{1}{x - 3}$

44. $\frac{6}{x} - \frac{2}{x + 3} = \frac{3(x + 5)}{x^2 + 3x}$

Finding Intercepts Algebraically **In Exercises 45–54, find the x- and y-intercepts of the graph of the equation algebraically.**

45. $y = 12 - 5x$ **46.** $y = 16 - 3x$

47. $y = -3(2x + 1)$ **48.** $y = 5 - (6 - x)$

49. $2x + 3y = 10$ **50.** $4x - 5y = 12$

51. $4y - 0.75x + 1.2 = 0$ **52.** $3y + 2.5x - 3.4 = 0$

53. $\frac{2x}{5} + 8 - 3y = 0$ **54.** $\frac{8x}{3} + 5 - 2y = 0$

Graphical Analysis **In Exercises 55–60, use a graphing utility to graph the equation and approximate any x-intercepts. Set $y = 0$ and solve the resulting equation. Compare the results with the graph's x-intercept(s).**

55. $y = 2(x - 1) - 4$ **56.** $y = \frac{4}{3}x + 2$

57. $y = 20 - (3x - 10)$ **58.** $y = 10 + 2(x - 2)$

59. $y = -38 + 5(9 - x)$ **60.** $y = 6x - 6\left(\frac{16}{11} + x\right)$

Solving an Equation **In Exercises 61–64, solve the equation. (Round your solution to three decimal places.)**

61. $0.275x + 0.725(500 - x) = 300$

62. $2.763 - 4.5(2.1x - 5.1432) = 6.32x + 5$

63. $\dfrac{2}{7.398} - \dfrac{4.405}{x} = \dfrac{1}{x}$ **64.** $\dfrac{3}{6.350} - \dfrac{6}{x} = 18$

65. Geometry The surface area S of the circular cylinder shown in the figure is

$$S = 2\pi(25) + 2\pi(5h).$$

Find the height h of the cylinder when the surface area is 471 square feet. Use 3.14 for π.

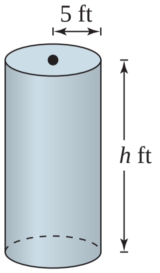


Figure for 65

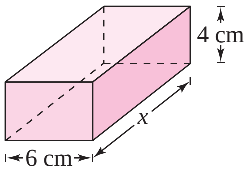


Figure for 66

66. Geometry The surface area S of the rectangular solid in the figure is $S = 2(24) + 2(4x) + 2(6x)$. Find the length x of the solid when the surface area is 248 square centimeters.

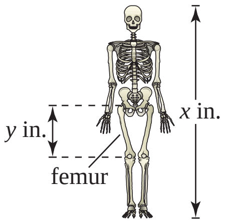
Forensics

In Exercises 67 and 68, use the following information. The relationship between the length of an adult's femur (thigh bone) and the height of the adult can be approximated by the linear equations

$\mathbf{y = 0.432x - 10.44}$ Female

$\mathbf{y = 0.449x - 12.15}$ Male

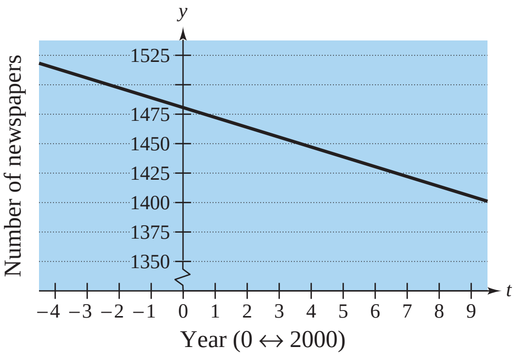
where y is the length of the femur in inches and x is the height of the adult in inches (see figure).



67. A crime scene investigator discovers a femur belonging to an adult human female. The bone is 18 inches long. Estimate the height of the female.

68. Officials search a forest for a missing man who is 6 feet 2 inches tall. They find an adult male femur that is 21 inches long. Is it possible that the femur belongs to the missing man?

69. Newspapers The numbers of daily newspapers y in the United States from 1996 through 2009 (see figure) can be approximated by the model $y = -8.37t + 1480.6$, $-4 \le t \le 9$, where t represents the year, with $t = 0$ corresponding to 2000. *(Source: Editor & Publisher Co.)*

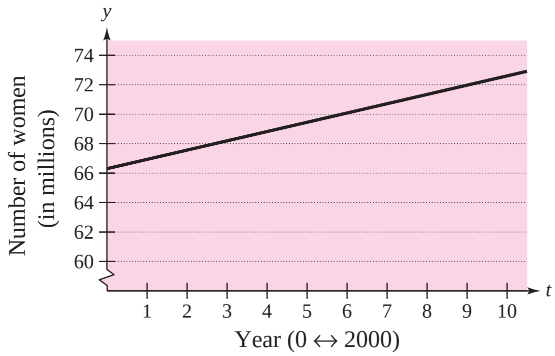


(a) Graphically estimate the y-intercept of the graph.

(b) Find algebraically and interpret the y-intercept of the graph.

(c) Use the model to predict the year in which the number of daily newspapers will be 1355. Does your answer seem reasonable? Explain.

70. Labor Statistics The numbers of women y (in millions) in the civilian work force in the United States from 2000 through 2010 (see figure) can be approximated by the model $y = 0.63t + 66.3$, $0 \le t \le 10$, where t represents the year, with $t = 0$ corresponding to 2000. *(Source: U.S. Bureau of Labor Statistics)*



(a) Find algebraically and interpret the y-intercept of the graph.

(b) According to the model, during which year did the number reach 72 million?

(c) Explain how you can solve part (b) graphically and algebraically.

71. Operating Cost A delivery company has a fleet of vans. The annual operating cost C (in dollars) per van is $C = 0.37m + 2600$, where m is the number of miles traveled by a van in a year. What number of miles will yield an annual operating cost of \$10,000?

18percentgrey/Shutterstock.com

72. Flood Control A river has risen 8 feet above its flood stage. The water begins to recede at a rate of 3 inches per hour. Write a mathematical model that shows the number of feet above flood stage after t hours. If the water continually recedes at this rate, when will the river be 1 foot above its flood stage?

Exploration

True or False? **In Exercises 73–77, determine whether the statement is true or false. Justify your answer.**

73. The equation $x(3 - x) = 10$ is a linear equation.

74. The equation $x^2 + 9x - 5 = 4 - x^3$ has no real solution.

75. The equation $2(x - 3) + 1 = 2x - 5$ has no solution.

76. The equation $3(x - 1) - 2 = 3x - 6$ is an identity and therefore has all real number solutions.

77. The equation $2 - \frac{1}{x - 2} = \frac{3}{x - 2}$ has no solution because $x = 2$ is an extraneous solution.

78. Think About It What is meant by *equivalent* equations? Give an example of two equivalent equations.

79. Think About It

(a) Complete the table.

x	-1	0	1	2	3	4
$3.2x - 5.8$						

(b) Use the table in part (a) to determine the interval in which the solution of the equation $3.2x - 5.8 = 0$ is located. Explain your reasoning.

(c) Complete the table.

x	1.5	1.6	1.7	1.8	1.9	2.0
$3.2x - 5.8$						

(d) Use the table in part (c) to determine the interval in which the solution of the equation $3.2x - 5.8 = 0$ is located. Explain how this process can be used to approximate the solution to any desired degree of accuracy.

80. Approximating a Solution Use the procedure in Exercise 79 to approximate the solution of the equation $0.3(x - 1.5) - 2 = 0$, accurate to two decimal places.

81. Graphical Reasoning

(a) Use a graphing utility to graph the equation $y = 3x - 6$.

(b) Use the result of part (a) to estimate the x-intercept of the graph.

(c) Explain how the x-intercept is related to the solution of the equation $3x - 6 = 0$, as shown in Example 1(a).

82. Finding Intercepts Consider the linear equation

$$y = ax + b$$

where a and b are real numbers.

(a) What is the x-intercept of the graph of the equation when $a \neq 0$?

(b) What is the y-intercept of the graph of the equation?

(c) Use your results from parts (a) and (b) to find the x- and y-intercepts of the graph of $y = 5x + 10$.

83. Finding Intercepts Consider the linear equation

$$ax + by = c$$

where a, b, and c are real numbers.

(a) What is the x-intercept of the graph of the equation when $a \neq 0$?

(b) What is the y-intercept of the graph of the equation when $b \neq 0$?

(c) Use your results from parts (a) and (b) to find the x- and y-intercepts of the graph of $2x + 7y = 11$.

84. HOW DO YOU SEE IT? Use the following information about a possible tax credit for a family consisting of two adults and two children (see figure).

Earned income:

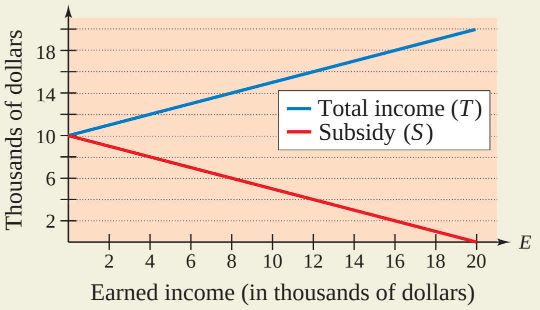
$$E$$

Subsidy (a grant of money):

$$S = 10{,}000 - \tfrac{1}{2}E, \quad 0 \leq E \leq 20{,}000$$

Total income:

$$T = E + S$$



(a) Graphically estimate the intercepts of the red line. Then interpret the meaning of each intercept.

(b) Explain how you can solve part (a) algebraically.

(c) Graphically estimate the earned income for which the total income is $14,000.

(d) Explain how you can solve part (c) algebraically.

1.3 Modeling with Linear Equations

You can use linear equations to determine how many gallons of gasoline to add to a gasoline-oil mixture to bring the mixture to the desired concentration for a chainsaw engine. See Exercise 57 on page 99.

- **Write and use mathematical models to solve real-life problems.**
- **Solve mixture problems.**
- **Use common formulas to solve real-life problems.**

Using Mathematical Models

In this section, you will use algebra to solve problems that occur in real-life situations. The process of translating phrases or sentences into algebraic expressions or equations is called **mathematical modeling.**

A good approach to mathematical modeling is to use two stages. Begin by using the verbal description of the problem to form a *verbal model.* Then, after assigning labels to the quantities in the verbal model, form a *mathematical model* or *algebraic equation.*

Verbal description → Verbal model → Algebraic equation

When you are constructing a verbal model, it is helpful to look for a *hidden equality*—a statement that two algebraic expressions are equal.

EXAMPLE 1 Using a Verbal Model

You accept a job with an annual income of \$32,300. This includes your salary and a year-end bonus of \$500. You are paid twice a month. What is your gross pay (pay before taxes) for each paycheck?

Solution Because there are 12 months in a year and you will be paid twice a month, it follows that you will receive 24 paychecks during the year.

Verbal Model: $\boxed{\text{Income for year}} = \boxed{\text{24 paychecks}} \cdot \boxed{\text{Amount of each paycheck}} + \boxed{\text{Bonus}}$

Labels: Income for year $= 32{,}300$ (dollars)

Amount of each paycheck $= x$ (dollars)

Bonus $= 500$ (dollars)

Equation: $32{,}300 = 24x + 500$

The algebraic equation for this problem is a linear equation in one variable x, which you can solve as follows.

$$32{,}300 = 24x + 500 \qquad \text{Write original equation.}$$

$$31{,}800 = 24x \qquad \text{Subtract 500 from each side.}$$

$$1325 = x \qquad \text{Divide each side by 24.}$$

So, your gross pay for each paycheck is \$1325.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

You accept a job with an annual income of \$58,400. This includes your salary and a \$1200 year-end bonus. You are paid weekly. What is your salary per pay period?

Val Thoermer/Shutterstock.com

A fundamental step in writing a mathematical model to represent a real-life problem is translating key words and phrases into algebraic expressions and equations. The following list gives several examples.

Translating Key Words and Phrases

Key Words and Phrases	Verbal Description	Algebraic Expression or Equation
Equality:		
Equals, equal to, is, are, was, will be, represents	• The sale price S is \$10 less than the list price L.	$S = L - 10$
Addition:		
Sum, plus, increased by, more than, total of	• The sum of 5 and x • Seven more than y	$5 + x$ or $x + 5$ $7 + y$ or $y + 7$
Subtraction:		
Difference, minus, less than, decreased by, subtracted from, reduced by	• The difference of 4 and b • Three less than z	$4 - b$ $z - 3$
Multiplication:		
Product, multiplied by, twice, times, percent of	• Two times x • Three percent of t	$2x$ $0.03t$
Division:		
Quotient, divided by, ratio, per	• The ratio of x to 8	$\frac{x}{8}$

EXAMPLE 2

Finding the Percent of a Raise

You accept a job that pays \$10 per hour. You are told that after a two-month probationary period, your hourly wage will be increased to \$11 per hour. What percent raise will you receive after the two-month period?

Solution

Verbal Model: $\boxed{\text{Raise}} = \boxed{\text{Percent}} \cdot \boxed{\text{Old wage}}$

Labels: Old wage $= 10$ (dollars per hour)

Raise $= 11 - 10 = 1$ (dollars per hour)

Percent $= r$ (in decimal form)

Equation: $1 = r \cdot 10$

$\frac{1}{10} = r$ Divide each side by 10.

$0.1 = r$ Rewrite the fraction as a decimal.

You will receive a raise of 0.1 or 10%.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

You buy stock at \$15 per share. You sell the stock at \$18 per share. What is the percent increase of the stock's value?

EXAMPLE 3 Finding the Percent of Monthly Expenses

Your family has an annual income of $57,000 and the following monthly expenses: mortgage ($1100), car payment ($375), food ($295), utilities ($240), and credit cards ($220). The total value of the monthly expenses represents what percent of your family's annual income?

Solution The total amount of your family's monthly expenses is $2230. The total monthly expenses for 1 year are $26,760.

Verbal Model: Monthly expenses = Percent · Income

Labels:
Income = 57,000 (dollars)
Monthly expenses = 26,760 (dollars)
Percent = r (in decimal form)

Equation:

$$26{,}760 = r \cdot 57{,}000$$

$$\frac{26{,}760}{57{,}000} = r$$ Divide each side by 57,000.

$$0.469 \approx r$$ Use a calculator.

Your family's monthly expenses are approximately 0.469 or 46.9% of your family's annual income.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Your family has annual loan payments equal to 28% of its annual income. During the year, the loan payments total $17,920. What is your family's annual income?

EXAMPLE 4 Finding the Dimensions of a Room

A rectangular kitchen is twice as long as it is wide, and its perimeter is 84 feet. Find the dimensions of the kitchen.

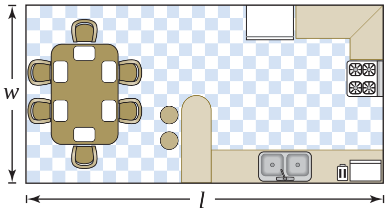


Figure 1.11

Solution For this problem, it helps to draw a diagram, as shown in Figure 1.11.

Verbal Model: 2 · Length + 2 · Width = Perimeter

Labels:
Perimeter = 84 (feet)
Width = w (feet)
Length = $l = 2w$ (feet)

Equation:

$$2(2w) + 2w = 84$$

$$6w = 84$$ Combine like terms.

$$w = 14$$ Divide each side by 6.

Because the length is twice the width, you have

$$\begin{aligned} l &= 2w \\ &= 2(14) \\ &= 28. \end{aligned}$$

So, the dimensions of the kitchen are 14 feet by 28 feet.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

A rectangular family room is 3 times as long as it is wide, and its perimeter is 112 feet. Find the dimensions of the family room.

REMARK Writing the unit for each label in a real-life problem helps you determine the unit for the answer. This is called *unit analysis*. When the same unit of measure occurs in the numerator and denominator of an expression, you can divide out the unit. For instance, unit analysis verifies that the unit for time in the formula below is hours.

$$\begin{aligned}\text{Time} &= \frac{\text{distance}}{\text{rate}}\\ &= \frac{\text{miles}}{\dfrac{\text{miles}}{\text{hour}}}\\ &= \cancel{\text{miles}} \cdot \frac{\text{hours}}{\cancel{\text{miles}}}\\ &= \text{hours}\end{aligned}$$

EXAMPLE 5 Estimating Travel Time

A plane is flying nonstop from Portland, Oregon, to Atlanta, Georgia, a distance of about 2170 miles. After 3 hours in the air, the plane flies over Topeka, Kansas (a distance of about 1440 miles from Portland). Assuming the plane flies at a constant speed, how long will the entire trip take?

Solution

Verbal Model: Distance = Rate · Time

Labels: Distance = 2170 (miles)

Time = t (hours)

Rate = $\dfrac{\text{distance to Topeka}}{\text{time to Topeka}} = \dfrac{1440}{3}$ (miles per hour)

Equation: $2170 = \dfrac{1440}{3}t$

$$2170 = 480t$$

$$\frac{2170}{480} = t$$

$$4.52 \approx t$$

The entire trip will take about 4.52 hours, or about 4 hours and 31 minutes.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

A boat travels at full speed to an island 14 miles away. It takes 0.5 hour to travel the first 5 miles. How long does the entire trip take?

EXAMPLE 6 Determining the Height of a Building

You measure the shadow cast by the Aon Center in Chicago, Illinois, and find it to be 142 feet long, as shown in Figure 1.12. Then you measure the shadow cast by a four-foot post and find it to be 6 inches long. Determine the building's height.

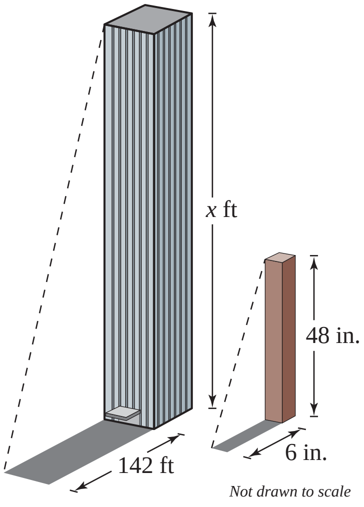


Figure 1.12

Solution To solve this problem, use the result from geometry that the ratios of corresponding sides of similar triangles are equal.

Verbal Model: $\dfrac{\text{Height of building}}{\text{Length of building's shadow}} = \dfrac{\text{Height of post}}{\text{Length of post's shadow}}$

Labels: Height of building = x (feet)

Length of building's shadow = 142 (feet)

Height of post = 4 feet = 48 inches (inches)

Length of post's shadow = 6 (inches)

Equation: $\dfrac{x}{142} = \dfrac{48}{6} \Rightarrow x = 1136$

So, the Aon Center is 1136 feet high.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

You measure the shadow cast by a building and find it to be 55 feet long. Then you measure the shadow cast by a four-foot post and find it to be 1.8 feet long. Determine the building's height.

Mixture Problems

Problems that involve two or more rates are called **mixture problems.**

EXAMPLE 7 A Simple Interest Problem

You invested a total of \$10,000 at $4\frac{1}{2}\%$ and $5\frac{1}{2}\%$ simple interest. During one year, the two accounts earned \$508.75. How much did you invest in each account?

Solution Let x represent the amount invested at $4\frac{1}{2}\%$. Then the amount invested at $5\frac{1}{2}\%$ is $10{,}000 - x$.

Verbal Model: Interest from $4\frac{1}{2}\%$ + Interest from $5\frac{1}{2}\%$ = Total interest

Labels: Interest from $4\frac{1}{2}\% = Prt = (x)(0.045)(1)$ (dollars)
Interest from $5\frac{1}{2}\% = Prt = (10{,}000 - x)(0.055)(1)$ (dollars)
Total interest $= 508.75$ (dollars)

Equation: $0.045x + 0.055(10{,}000 - x) = 508.75$

$$-0.01x = -41.25 \quad \Longrightarrow \quad x = 4125$$

So, \$4125 was invested at $4\frac{1}{2}\%$ and $10{,}000 - x = \$5875$ was invested at $5\frac{1}{2}\%$.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

You invested a total of \$5000 at $2\frac{1}{2}\%$ and $3\frac{1}{2}\%$ simple interest. During one year, the two accounts earned \$151.25. How much did you invest in each account?

REMARK Example 7 uses the simple interest formula $I = Prt$, where I is the interest, P is the principal (original deposit), r is the annual interest rate (in decimal form), and t is the time in years. Notice that in this example the amount invested, \$10,000, is separated into two parts, x and $\$10{,}000 - x$.

EXAMPLE 8 An Inventory Problem

A store has \$30,000 of inventory in single-disc DVD players and multi-disc DVD players. The profit on a single-disc player is 22% and the profit on a multi-disc player is 40%. The profit on the entire stock is 35%. How much was invested in each type?

Solution Let x represent the amount invested in single-disc players. Then the amount invested in multi-disc players is $30{,}000 - x$.

Verbal Model: Profit from single-disc players + Profit from multi-disc players = Total profit

Labels: Inventory of single-disc players $= x$ (dollars)
Inventory of multi-disc players $= 30{,}000 - x$ (dollars)
Profit from single-disc players $= 0.22x$ (dollars)
Profit from multi-disc players $= 0.40(30{,}000 - x)$ (dollars)
Total profit $= 0.35(30{,}000) = 10{,}500$ (dollars)

Equation: $0.22x + 0.40(30{,}000 - x) = 10{,}500$

$$-0.18x = -1500 \quad \Longrightarrow \quad x \approx 8333.33$$

So, \$8333.33 was invested in single-disc DVD players and $30{,}000 - x = \$21{,}666.67$ was invested in multi-disc DVD players.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

In Example 8, the profit on a single-disc player is 24% and the profit on a multi-disc player is 42%. The profit on the entire stock is 36%. How much was invested in each type?

Common Formulas

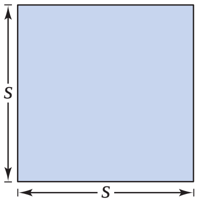
A **literal equation** is an equation that contains more than one variable. Many common types of geometric, scientific, and investment problems use ready-made literal equations called **formulas.** Knowing these formulas will help you translate and solve a wide variety of real-life applications.

Common Formulas for Area A, Perimeter P, Circumference C, and Volume V

Square

$A = s^2$

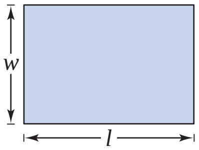
$P = 4s$



Rectangle

$A = lw$

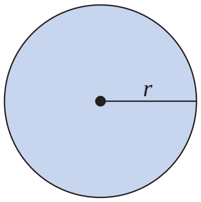
$P = 2l + 2w$



Circle

$A = \pi r^2$

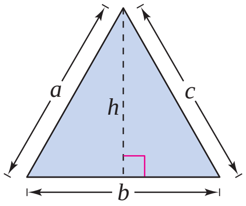
$C = 2\pi r$



Triangle

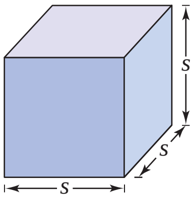
$A = \frac{1}{2}bh$

$P = a + b + c$



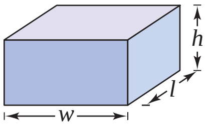
Cube

$V = s^3$



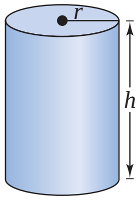
Rectangular Solid

$V = lwh$



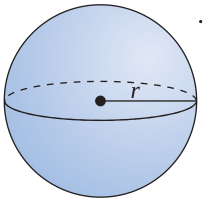
Circular Cylinder

$V = \pi r^2 h$



Sphere

$V = \frac{4}{3}\pi r^3$



Miscellaneous Common Formulas

Temperature:

$F = \frac{9}{5}C + 32$ F = degrees Fahrenheit, C = degrees Celsius

$C = \frac{5}{9}(F - 32)$

Simple Interest:

$I = Prt$ I = interest, P = principal (original deposit),
r = annual interest rate (in decimal form), t = time in years

Compound Interest:

$A = P\left(1 + \frac{r}{n}\right)^{nt}$ n = compoundings (number of times interest is calculated) per year, t = time in years,
A = balance, P = principal (original deposit), r = annual interest rate (in decimal form)

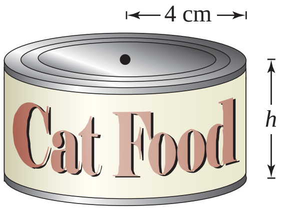
Distance:

$d = rt$ d = distance traveled, r = rate, t = time

When working with applied problems, you often need to rewrite a literal equation in terms of another variable. You can use the methods for solving linear equations to solve some literal equations for a specified variable. For instance, the formula for the perimeter of a rectangle, $P = 2l + 2w$, can be rewritten or solved for w as $w = \frac{1}{2}(P - 2l)$.

EXAMPLE 9 Using a Formula

The cylindrical can below has a volume of 200 cubic centimeters (cm^3). Find the height of the can.



Solution

The formula for the *volume of a cylinder* is $V = \pi r^2 h$. To find the height of the can, solve for h.

$$h = \frac{V}{\pi r^2}$$

Then, using $V = 200$ and $r = 4$, find the height.

$$h = \frac{200}{\pi(4)^2} \qquad \text{Substitute 200 for } V \text{ and 4 for } r.$$

$$= \frac{200}{16\pi} \qquad \text{Simplify denominator.}$$

$$\approx 3.98 \qquad \text{Use a calculator.}$$

So, the height of the can is about 3.98 centimeters. You can use unit analysis to check that your answer is reasonable.

$$\frac{200\text{ cm}^3}{16\pi\text{ cm}^2} = \frac{200\,\cancel{\text{cm}}\cdot\cancel{\text{cm}}\cdot\text{cm}}{16\pi\,\cancel{\text{cm}}\cdot\cancel{\text{cm}}} = \frac{200}{16\pi}\text{ cm} \approx 3.98\text{ cm}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

A cylindrical container has a volume of 84 cubic inches and a radius of 3 inches. Find the height of the container.

Summarize (Section 1.3)

1. Describe the procedure of mathematical modeling *(pages 90 and 91)*. For examples of writing and using mathematical models, see Examples 1–6.
2. Describe a mixture problem *(page 94)*. For examples of mixture problems, see Examples 7 and 8.
3. State the definition of a formula *(page 95)*. For an example that uses a volume formula, see Example 9.

1.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. The process of translating phrases or sentences into algebraic expressions or equations is called ________ ________.

2. A good approach to mathematical modeling is a two-stage approach, using a verbal description to form a ________ ________, and then, after assigning labels to the quantities, forming an ________ ________.

Skills and Applications

Writing a Verbal Description **In Exercises 3–12, write a verbal description of the algebraic expression without using the variable.**

3. $x + 4$

4. $t - 10$

5. $\frac{u}{5}$

6. $\frac{2}{3}x$

7. $\frac{y - 4}{5}$

8. $\frac{z + 10}{7}$

9. $-3(b + 2)$

10. $12x(x - 5)$

11. $\frac{4(p - 1)}{p}$

12. $\frac{(q + 4)(3 - q)}{2q}$

Writing an Algebraic Expression **In Exercises 13–24, write an algebraic expression for the verbal description.**

13. The sum of two consecutive natural numbers

14. The product of two consecutive natural numbers

15. The product of two consecutive odd integers, the first of which is $2n - 1$

16. The sum of the squares of two consecutive even integers, the first of which is $2n$

17. The distance traveled in t hours by a car traveling at 55 miles per hour

18. The travel time for a plane traveling at a rate of r kilometers per hour for 900 kilometers

19. The amount of acid in x liters of a 20% acid solution

20. The sale price of an item that is discounted 33% of its list price L

21. The perimeter of a rectangle with a width x and a length that is twice the width

22. The area of a triangle with base 16 inches and height h inches

23. The total cost of producing x units for which the fixed costs are \$2500 and the cost per unit is \$40

24. The total revenue obtained by selling x units at \$12.99 per unit

Translating a Statement **In Exercises 25–28, translate the statement into an algebraic expression or equation.**

25. Thirty percent of the list price L

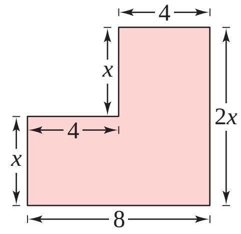
26. The amount of water in q quarts of a liquid that is 28% water

27. The percent of 672 that is represented by the number N

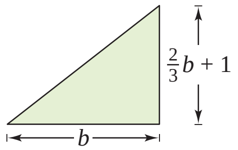
28. The percent change in sales from one month to the next if the monthly sales are S_1 and S_2, respectively

Writing an Expression **In Exercises 29 and 30, write an expression for the area of the region in the figure.**

29.



30.



Number Problems **In Exercises 31–36, write a mathematical model for the problem and solve.**

31. The sum of two consecutive natural numbers is 525. Find the numbers.

32. The sum of three consecutive natural numbers is 804. Find the numbers.

33. One positive number is 5 times another number. The difference between the two numbers is 148. Find the numbers.

34. One positive number is $\frac{1}{5}$ of another number. The difference between the two numbers is 76. Find the numbers.

35. Find two consecutive integers whose product is 5 less than the square of the smaller number.

36. Find two consecutive natural numbers such that the difference of their reciprocals is $\frac{1}{4}$ the reciprocal of the smaller number.

37. Finance A salesperson's weekly paycheck is 15% less than a second salesperson's paycheck. The two paychecks total \$1125. Find the amount of each paycheck.

38. Discount The price of a swimming pool has been discounted by 16.5%. The sale price is \$1210.75. Find the original list price of the pool.

39. Finance A family has annual loan payments equal to 32% of their annual income. During the year, the loan payments total \$15,680. What is the family's annual income?

40. Finance A family has a monthly mortgage payment of \$760, which is 16% of its monthly income. What is the family's monthly income?

41. Dimensions of a Room A rectangular room is 1.5 times as long as it is wide, and its perimeter is 25 meters.

(a) Draw a diagram that gives a visual representation of the problem. Let l represent the length and let w represent the width.

(b) Write l in terms of w and write an equation for the perimeter in terms of w.

(c) Find the dimensions of the room.

42. Dimensions of a Picture Frame A rectangular picture frame has a perimeter of 3 meters. The height of the frame is $\frac{2}{3}$ times its width.

(a) Draw a diagram that gives a visual representation of the problem. Let w represent the width and let h represent the height.

(b) Write h in terms of w and write an equation for the perimeter in terms of w.

(c) Find the dimensions of the picture frame.

43. Course Grade To get an A in a course, you must have an average of at least 90% on four tests worth 100 points each. Your scores on the first three tests were 87, 92, and 84. What must you score on the fourth test to get an A in the course?

44. Course Grade You are taking a course that has four tests. The first three tests are worth 100 points each and the fourth test is worth 200 points. To get an A in the course, you must have an average of at least 90% on the four tests. Your scores on the first three tests were 87, 92, and 84. What must you score on the fourth test to get an A in the course?

45. Travel Time You are driving on a Canadian freeway to a town that is 500 kilometers from your home. After 30 minutes, you pass a freeway exit that you know is 50 kilometers from your home. Assuming that you continue at the same constant speed, how long will it take for the entire trip?

46. Average Speed A truck driver traveled at an average speed of 55 miles per hour on a 200-mile trip to pick up a load of freight. On the return trip (with the truck fully loaded), the average speed was 40 miles per hour. What was the average speed for the round trip?

47. Physics Light travels at the speed of approximately 3.0×10^8 meters per second. Find the time in minutes required for light to travel from the sun to Earth (an approximate distance of 1.5×10^{11} meters).

48. Radio Waves Radio waves travel at the same speed as light, approximately 3.0×10^8 meters per second. Find the time required for a radio wave to travel from Mission Control in Houston to NASA astronauts on the surface of the moon 3.84×10^8 meters away.

49. Height of a Building You measure the shadow cast by One Liberty Place in Philadelphia, Pennsylvania, and find it to be 105 feet long. Then you measure the shadow cast by a three-foot post and find it to be 4 inches long. Determine the building's height.

50. Height of a Tree You measure a tree's shadow and find that it is 8 meters long. Then you measure the shadow of a two-meter lamppost and find that it is 75 centimeters long. How tall is the tree?

51. Flagpole Height A person who is 6 feet tall walks away from a flagpole toward the tip of the shadow of the flagpole. When the person is 30 feet from the flagpole, the tips of the person's shadow and the shadow cast by the flagpole coincide at a point 5 feet in front of the person.

(a) Draw a diagram that gives a visual representation of the problem. Let h represent the height of the flagpole.

(b) Find the height of the flagpole.

52. Shadow Length A person who is 6 feet tall walks away from a 50-foot tower toward the tip of the tower's shadow. At a distance of 32 feet from the tower, the person's shadow begins to emerge beyond the tower's shadow. How much farther must the person walk to be completely out of the tower's shadow?

53. Investment You invested a total of \$12,000 at $4\frac{1}{2}\%$ and 5% simple interest. During one year, the two accounts earned \$580. How much did you invest in each account?

54. Investment You invested a total of \$25,000 at 3% and $4\frac{1}{2}\%$ simple interest. During one year, the two accounts earned \$900. How much did you invest in each account?

55. Inventory A nursery has \$40,000 of inventory in dogwood trees and red maple trees. The profit on a dogwood tree is 25% and the profit on a red maple tree is 17%. The profit for the entire stock is 20%. How much was invested in each type of tree?

56. Inventory An automobile dealer has \$600,000 of inventory in minivans and alternative-fueled vehicles. The profit on a minivan is 24% and the profit on an alternative-fueled vehicle is 28%. The profit for the entire stock is 25%. How much was invested in each type of vehicle?

57. Mixture Problem

A forester is making a gasoline-oil mixture for a chainsaw engine. The forester has 2 gallons of a mixture that is 32 parts gasoline and 1 part oil. How many gallons of gasoline should the forester add to bring the mixture to 50 parts gasoline and 1 part oil?

58. Mixture Problem A grocer mixes peanuts that cost \$1.49 per pound and walnuts that cost \$2.69 per pound to make 100 pounds of a mixture that costs \$2.21 per pound. How much of each kind of nut did the grocer put into the mixture?

59. Area of a Triangle

Solve for h: $A = \frac{1}{2}bh$

60. Volume of a Rectangular Prism

Solve for l: $V = lwh$

61. Markup

Solve for C: $S = C + RC$

62. Discount

Solve for L: $S = L - RL$

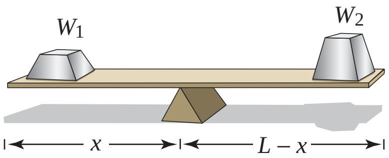
63. Investment at Simple Interest

Solve for r: $A = P + Prt$

64. Area of a Trapezoid

Solve for b: $A = \frac{1}{2}(a + b)h$

Physics **In Exercises 65 and 66, you have a uniform beam of length L with a fulcrum x feet from one end (see figure). Objects with weights W_1 and W_2 are placed at opposite ends of the beam. The beam will balance when $W_1x = W_2(L - x)$. Find x such that the beam will balance.**



65. Two children weighing 50 pounds and 75 pounds are playing on a seesaw that is 10 feet long.

66. A person weighing 200 pounds is attempting to move a 550-pound rock with a bar that is 5 feet long.

67. Volume of a Billiard Ball A billiard ball has a volume of 5.96 cubic inches. Find the radius of a billiard ball.

68. Length of a Tank The diameter of a cylindrical propane gas tank is 4 feet. The total volume of the tank is 603.2 cubic feet. Find the length of the tank.

69. Temperature The average daily temperature in San Diego, California, is 64.4°F. What is this temperature in degrees Celsius?

70. Temperature The average daily temperature in Duluth, Minnesota, is 39.1°F. What is this temperature in degrees Celsius?

71. Temperature The highest temperature ever recorded in Phoenix, Arizona, was 50°C. What is this temperature in degrees Fahrenheit? *(Source: NOAA)*

72. Temperature The lowest temperature ever recorded in Louisville, Kentucky, was −30°C. What is this temperature in degrees Fahrenheit? *(Source: NOAA)*

Exploration

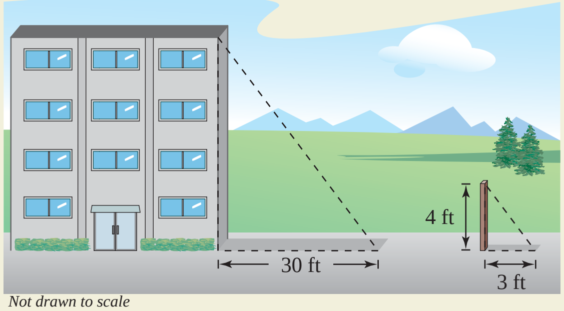
True or False? **In Exercises 73–75, determine whether the statement is true or false. Justify your answer.**

73. "8 less than z cubed divided by the difference of z squared and 9" can be written as $(z^3 - 8)/(z - 9)^2$.

74. The area of a circle with a radius of 2 inches is less than the area of a square with a side length of 4 inches.

75. The volume of a cube with a side length of 9.5 inches is greater than the volume of a sphere with a radius of 5.9 inches.

76. HOW DO YOU SEE IT? To determine a building's height, you measure the shadows cast by the building and a four-foot post (see figure).



(a) Write a verbal model for the situation.

(b) Translate your verbal model into an algebraic equation.

Val Thoermer/Shutterstock.com

1.4 Quadratic Equations and Applications

Quadratic equations can help you model and solve real-life problems. For instance, in Exercise 125 on page 112, you will use a quadratic equation to model a patient's blood oxygen level.

- **Solve quadratic equations by factoring.**
- **Solve quadratic equations by extracting square roots.**
- **Solve quadratic equations by completing the square.**
- **Use the Quadratic Formula to solve quadratic equations.**
- **Use quadratic equations to model and solve real-life problems.**

Solving Quadratic Equations by Factoring

A **quadratic equation** in x is an equation that can be written in the general form

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers with $a \neq 0$. A quadratic equation in x is also called a **second-degree polynomial equation** in x.

In this section, you will study four methods for solving quadratic equations: *factoring*, *extracting square roots*, *completing the square*, and the *Quadratic Formula*. The first method is based on the Zero-Factor Property from Section P.1.

If $ab = 0$, then $a = 0$ or $b = 0$. Zero-Factor Property

To use this property, rewrite the left side of the general form of a quadratic equation as the product of two linear factors. Then find the solutions of the quadratic equation by setting each linear factor equal to zero.

EXAMPLE 1 Solving a Quadratic Equation by Factoring

a. $2x^2 + 9x + 7 = 3$ Original equation

$2x^2 + 9x + 4 = 0$ Write in general form.

$(2x + 1)(x + 4) = 0$ Factor.

$2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$ Set 1st factor equal to 0.

$x + 4 = 0 \Rightarrow x = -4$ Set 2nd factor equal to 0.

The solutions are $x = -\frac{1}{2}$ and $x = -4$. Check these in the original equation.

b. $6x^2 - 3x = 0$ Original equation

$3x(2x - 1) = 0$ Factor.

$3x = 0 \Rightarrow x = 0$ Set 1st factor equal to 0.

$2x - 1 = 0 \Rightarrow x = \frac{1}{2}$ Set 2nd factor equal to 0.

The solutions are $x = 0$ and $x = \frac{1}{2}$. Check these in the original equation.

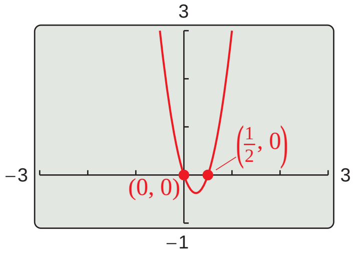
✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $2x^2 - 3x + 1 = 6$ by factoring.

The Zero-Factor Property applies *only* to equations written in general form (in which the right side of the equation is zero). So, all terms must be collected on one side *before* factoring. For instance, in the equation $(x - 5)(x + 2) = 8$, it is *incorrect* to set each factor equal to 8. To solve this equation, first multiply the binomials on the left side of the equation. Then subtract 8 from each side. After simplifying the equation, factor the left side and use the Zero-Factor Property to find the solutions. Try to solve this equation correctly.

Apple's Eyes Studio/Shutterstock.com

TECHNOLOGY You can use a graphing utility to check graphically the real solutions of a quadratic equation. Begin by writing the equation in general form. Then set y equal to the left side and graph the resulting equation. The x-intercepts of the equation represent the real solutions of the original equation. You can use the *zero* or *root* feature of the graphing utility to approximate the x-intercepts of the graph. For instance, to check the solutions of $6x^2 - 3x = 0$, graph $y = 6x^2 - 3x$, and use the *zero* or *root* feature to find the x-intercepts to be $(0, 0)$ and $\left(\frac{1}{2}, 0\right)$, as shown below. These x-intercepts represent the solutions $x = 0$ and $x = \frac{1}{2}$, as found in Example 1(b).



Extracting Square Roots

Consider a quadratic equation of the form $u^2 = d$, where $d > 0$ and u is an algebraic expression. By factoring, you can see that this equation has two solutions.

$$\begin{aligned} u^2 &= d && \text{Write original equation.} \\ u^2 - d &= 0 && \text{Write in general form.} \\ \left(u + \sqrt{d}\right)\left(u - \sqrt{d}\right) &= 0 && \text{Factor.} \\ u + \sqrt{d} &= 0 \;\Rightarrow\; u = -\sqrt{d} && \text{Set 1st factor equal to 0.} \\ u - \sqrt{d} &= 0 \;\Rightarrow\; u = \sqrt{d} && \text{Set 2nd factor equal to 0.} \end{aligned}$$

Because the two solutions differ only in sign, you can write the solutions together, using a "plus or minus sign," as

$$u = \pm\sqrt{d}.$$

This form of the solution is read as "u is equal to plus or minus the square root of d." Solving an equation of the form $u^2 = d$ without going through the steps of factoring is called **extracting square roots.**

Extracting Square Roots

The equation $u^2 = d$, where $d > 0$, has exactly two solutions:

$$u = \sqrt{d} \quad \text{and} \quad u = -\sqrt{d}.$$

These solutions can also be written as

$$u = \pm\sqrt{d}.$$

EXAMPLE 2 Extracting Square Roots

Solve each equation by extracting square roots.

a. $4x^2 = 12$

b. $(x - 3)^2 = 7$

Solution

a.
$$\begin{aligned} 4x^2 &= 12 && \text{Write original equation.} \\ x^2 &= 3 && \text{Divide each side by 4.} \\ x &= \pm\sqrt{3} && \text{Extract square roots.} \end{aligned}$$

The solutions are $x = \sqrt{3}$ and $x = -\sqrt{3}$. Check these in the original equation.

b.
$$\begin{aligned} (x - 3)^2 &= 7 && \text{Write original equation.} \\ x - 3 &= \pm\sqrt{7} && \text{Extract square roots.} \\ x &= 3 \pm \sqrt{7} && \text{Add 3 to each side.} \end{aligned}$$

The solutions are $x = 3 \pm \sqrt{7}$. Check these in the original equation.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve each equation by extracting square roots.

a. $3x^2 = 36$

b. $(x - 1)^2 = 10$

Completing the Square

Suppose that the equation $(x-3)^2 = 7$ from Example 2(b) had been given in the general form $x^2 - 6x + 2 = 0$. Because this equation is equivalent to the original, it has the same two solutions, $x = 3 \pm \sqrt{7}$. However, the left side of the equation is not factorable, and you cannot find its solutions unless you rewrite the equation so it can be solved by extracting square roots. You can do this using a method called **completing the square.**

Completing the Square

To **complete the square** for the expression $x^2 + bx$, add $(b/2)^2$, which is the square of half the coefficient of x. Consequently,

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2.$$

When solving quadratic equations by completing the square, you must add $(b/2)^2$ to *each side* in order to maintain equality.

EXAMPLE 3 Completing the Square: Leading Coefficient Is 1

Solve

$$x^2 + 2x - 6 = 0$$

by completing the square.

Solution

$$x^2 + 2x - 6 = 0 \qquad \text{Write original equation.}$$

$$x^2 + 2x = 6 \qquad \text{Add 6 to each side.}$$

$$x^2 + 2x + \underbrace{1^2}_{(\text{half of } 2)^2} = 6 + 1^2 \qquad \text{Add } 1^2 \text{ to each side.}$$

$$(x+1)^2 = 7 \qquad \text{Simplify.}$$

$$x + 1 = \pm\sqrt{7} \qquad \text{Extract square roots.}$$

$$x = -1 \pm \sqrt{7} \qquad \text{Subtract 1 from each side.}$$

The solutions are $x = -1 \pm \sqrt{7}$. Check these in the original equation as follows.

Check

$$x^2 + 2x - 6 = 0 \qquad \text{Write original equation.}$$

$$\left(-1 + \sqrt{7}\right)^2 + 2\left(-1 + \sqrt{7}\right) - 6 \stackrel{?}{=} 0 \qquad \text{Substitute } -1 + \sqrt{7} \text{ for } x.$$

$$8 - 2\sqrt{7} - 2 + 2\sqrt{7} - 6 \stackrel{?}{=} 0 \qquad \text{Multiply.}$$

$$8 - 2 - 6 = 0 \qquad \text{Solution checks. } \checkmark$$

Check the second solution in the original equation.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve

$$x^2 - 4x - 1 = 0$$

by completing the square.

When the leading coefficient is *not* 1, you must divide each side of the equation by the leading coefficient *before* completing the square.

EXAMPLE 4 Completing the Square: Leading Coefficient Is Not 1

Solve $2x^2 + 8x + 3 = 0$ by completing the square.

Solution

$$2x^2 + 8x + 3 = 0 \qquad \text{Write original equation.}$$

$$2x^2 + 8x = -3 \qquad \text{Subtract 3 from each side.}$$

$$x^2 + 4x = -\frac{3}{2} \qquad \text{Divide each side by 2.}$$

$$x^2 + \underbrace{4x + 2^2}_{(\text{half of } 4)^2} = -\frac{3}{2} + 2^2 \qquad \text{Add } 2^2 \text{ to each side.}$$

$$(x + 2)^2 = \frac{5}{2} \qquad \text{Simplify.}$$

$$x + 2 = \pm\sqrt{\frac{5}{2}} \qquad \text{Extract square roots.}$$

$$x + 2 = \pm\frac{\sqrt{10}}{2} \qquad \text{Rationalize denominator.}$$

$$x = -2 \pm \frac{\sqrt{10}}{2} \qquad \text{Subtract 2 from each side.}$$

▷ **ALGEBRA HELP** You can review rationalizing denominators in Section P.2.

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Solve $2x^2 - 4x + 1 = 0$ by completing the square.

EXAMPLE 5 Completing the Square: Leading Coefficient Is Not 1

Solve $3x^2 - 4x - 5 = 0$ by completing the square.

Solution

$$3x^2 - 4x - 5 = 0 \qquad \text{Write original equation.}$$

$$3x^2 - 4x = 5 \qquad \text{Add 5 to each side.}$$

$$x^2 - \frac{4}{3}x = \frac{5}{3} \qquad \text{Divide each side by 3.}$$

$$x^2 - \frac{4}{3}x + \left(-\frac{2}{3}\right)^2 = \frac{5}{3} + \left(-\frac{2}{3}\right)^2 \qquad \text{Add } \left(-\tfrac{2}{3}\right)^2 \text{ to each side.}$$

$$\left(x - \frac{2}{3}\right)^2 = \frac{19}{9} \qquad \text{Simplify.}$$

$$x - \frac{2}{3} = \pm\frac{\sqrt{19}}{3} \qquad \text{Extract square roots.}$$

$$x = \frac{2}{3} \pm \frac{\sqrt{19}}{3} \qquad \text{Add } \tfrac{2}{3} \text{ to each side.}$$

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Solve $3x^2 - 10x - 2 = 0$ by completing the square. ■

The Quadratic Formula

Often in mathematics you are taught the long way of solving a problem first. Then, the longer method is used to develop shorter techniques. The long way stresses understanding and the short way stresses efficiency.

For instance, you can think of completing the square as a "long way" of solving a quadratic equation. When you use completing the square to solve quadratic equations, you must complete the square for *each* equation separately. In the following derivation, you complete the square *once* for the general form of a quadratic equation

$$ax^2 + bx + c = 0$$

to obtain the **Quadratic Formula**—a shortcut for solving quadratic equations.

$$\begin{aligned}
ax^2 + bx + c &= 0 && \text{Write in general form, } a \neq 0.\\
ax^2 + bx &= -c && \text{Subtract } c \text{ from each side.}\\
x^2 + \frac{b}{a}x &= -\frac{c}{a} && \text{Divide each side by } a.\\
x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 &= -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 && \text{Complete the square.}
\end{aligned}$$

$$\left(\text{half of } \frac{b}{a}\right)^2$$

$$\begin{aligned}
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} && \text{Simplify.}\\
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} && \text{Extract square roots.}\\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2|a|} && \text{Solutions}
\end{aligned}$$

Note that because $\pm 2|a|$ represents the same numbers as $\pm 2a$, the formula simplifies to

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The Quadratic Formula

The solutions of a quadratic equation in the general form

$$ax^2 + bx + c = 0, \quad a \neq 0$$

are given by the **Quadratic Formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The Quadratic Formula is one of the most important formulas in algebra. You can solve every quadratic equation by completing the square or using the Quadratic Formula. You should learn the verbal statement of the Quadratic Formula:

> "Negative b, plus or minus the square root of b squared minus $4ac$, all divided by $2a$."

In the Quadratic Formula, the quantity under the radical sign, $b^2 - 4ac$, is called the **discriminant** of the quadratic expression $ax^2 + bx + c$. It can be used to determine the number of real solutions of a quadratic equation.

Solutions of a Quadratic Equation

The solutions of a quadratic equation

$$ax^2 + bx + c = 0, a \neq 0$$

can be classified as follows. If the discriminant $b^2 - 4ac$ is

1. *positive*, then the quadratic equation has *two* distinct real solutions and its graph has *two* x-intercepts.
2. *zero*, then the quadratic equation has *one* repeated real solution and its graph has *one* x-intercept.
3. *negative*, then the quadratic equation has *no* real solutions and its graph has *no* x-intercepts.

If the discriminant of a quadratic equation is negative, as in case 3 above, then its square root is imaginary (not a real number) and the Quadratic Formula yields two complex solutions. You will study complex solutions in Section 1.5.

When using the Quadratic Formula, remember that *before* the formula can be applied, you must first write the quadratic equation in general form.

EXAMPLE 6 Using the Quadratic Formula

Use the Quadratic Formula to solve

$$x^2 + 3x = 9.$$

Solution The general form of the equation is $x^2 + 3x - 9 = 0$. The discriminant is $b^2 - 4ac = 9 + 36 = 45$, which is positive. So, the equation has two real solutions. You can solve the equation as follows.

$$x^2 + 3x - 9 = 0 \qquad \text{Write in general form.}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Quadratic Formula}$$

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(-9)}}{2(1)} \qquad \text{Substitute } a = 1, b = 3, \text{ and } c = -9.$$

$$x = \frac{-3 \pm \sqrt{45}}{2} \qquad \text{Simplify.}$$

$$x = \frac{-3 \pm 3\sqrt{5}}{2} \qquad \text{Simplify.}$$

The two solutions are

$$x = \frac{-3 + 3\sqrt{5}}{2}$$

and

$$x = \frac{-3 - 3\sqrt{5}}{2}.$$

Check these in the original equation.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use the Quadratic Formula to solve

$$3x^2 + 2x - 10 = 0.$$

Applications

Quadratic equations often occur in problems dealing with area. Here is a simple example.

> A square room has an area of 144 square feet. Find the dimensions of the room.

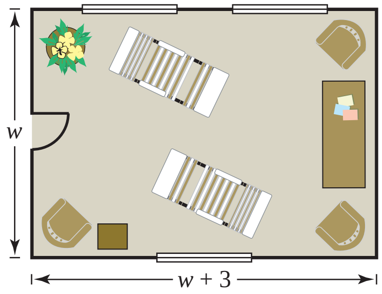
To solve this problem, let x represent the length of each side of the room. Then use the formula for the area of a square to write and solve the equation

$$x^2 = 144$$

to conclude that each side of the room is 12 feet long. Note that although the equation $x^2 = 144$ has two solutions, $x = -12$ and $x = 12$, the negative solution does not make sense in the context of the problem, so you choose the positive solution.

EXAMPLE 7 Finding the Dimensions of a Room

A rectangular sunroom is 3 feet longer than it is wide (see figure) and has an area of 154 square feet. Find the dimensions of the room.



Solution

Verbal Model: $\boxed{\text{Width of room}} \cdot \boxed{\text{Length of room}} = \boxed{\text{Area of room}}$

Labels: Width of room $= w$ (feet)
Length of room $= w + 3$ (feet)
Area of room $= 154$ (square feet)

Equation:

$$\begin{aligned} w(w + 3) &= 154 \\ w^2 + 3w - 154 &= 0 \\ (w - 11)(w + 14) &= 0 \\ w - 11 = 0 &\Rightarrow w = 11 \\ w + 14 = 0 &\Rightarrow w = -14 \end{aligned}$$

Choosing the positive value, you can conclude that the width is 11 feet and the length is

$$w + 3 = 14 \text{ feet.}$$

Check

The length of 14 feet is 3 more than the width of 11 feet. ✓
The area of the sunroom is $11(14) = 154$ square feet. ✓

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

A rectangular kitchen is 6 feet longer than it is wide and has an area of 112 square feet. Find the dimensions of the kitchen.

Another common application of quadratic equations involves an object that is falling (or vertically projected into the air). The general equation that gives the height of such an object is called a **position equation,** and on Earth's surface it has the form

$$s = -16t^2 + v_0 t + s_0.$$

In this equation, s represents the height of the object (in feet), v_0 represents the initial velocity of the object (in feet per second), s_0 represents the initial height of the object (in feet), and t represents the time (in seconds).

REMARK The position equation described here ignores air resistance. It should only be used to model falling objects that have little air resistance and that fall short distances.

EXAMPLE 8 Falling Object

A construction worker accidentally drops a wrench from a height of 235 feet (see Figure 1.13) and yells "Look out below!" Could a person at ground level hear this warning in time to get out of the way? (*Note:* The speed of sound is about 1100 feet per second.)

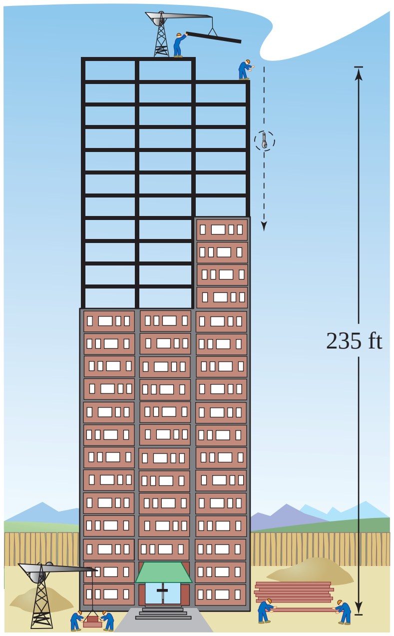


Figure 1.13

Solution

Because sound travels at about 1100 feet per second and the distance to the ground is 235 feet, it follows that a person at ground level hears the warning within 1 second of the time the wrench is dropped. To set up a mathematical model for the height of the wrench, use the position equation

$$s = -16t^2 + v_0 t + s_0.$$

Because the object is dropped rather than thrown, the initial velocity is $v_0 = 0$ feet per second. Moreover, because the initial height is $s_0 = 235$ feet, you have the following model.

$$s = -16t^2 + (0)t + 235 = -16t^2 + 235$$

After 1 second, the wrench's height is

$$-16(1)^2 + 235 = 219 \text{ feet.}$$

After 2 seconds, the wrench's height is

$$-16(2)^2 + 235 = 171 \text{ feet.}$$

To find the number of seconds it takes the wrench to hit the ground, let the height s be zero and solve the equation for t.

$$s = -16t^2 + 235 \qquad \text{Write position equation.}$$

$$0 = -16t^2 + 235 \qquad \text{Substitute 0 for height.}$$

$$16t^2 = 235 \qquad \text{Add } 16t^2 \text{ to each side.}$$

$$t^2 = \frac{235}{16} \qquad \text{Divide each side by 16.}$$

$$t = \frac{\sqrt{235}}{4} \qquad \text{Extract positive square root.}$$

$$t \approx 3.83 \qquad \text{Use a calculator.}$$

The wrench will take about 3.83 seconds to hit the ground. So, a the person who hears the warning within 1 second after the wrench is dropped still has almost 3 seconds to get out of the way.

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

You drop a rock from a height of 196 feet. How long does it take the rock to hit the ground?

A third application of quadratic equations is modeling change over time.

EXAMPLE 9 Quadratic Modeling: Internet Users

From 2000 through 2010, the numbers of Internet users I (in millions) in the United States can be approximated by the quadratic equation

$$I = -0.898t^2 + 19.59t + 126.5, \quad 0 \le t \le 10$$

where t represents the year, with $t = 0$ corresponding to 2000. According to the model, in which year did the number of Internet users reach 200 million? *(Source: International Telecommunication Union/The Nielsen Company)*

Algebraic Solution

To find the year in which the number of Internet users reached 200 million, you can solve the equation

$$-0.898t^2 + 19.59t + 126.5 = 200.$$

To begin, write the equation in general form.

$$-0.898t^2 + 19.59t - 73.5 = 0$$

Then apply the Quadratic Formula.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-19.59 \pm \sqrt{19.59^2 - 4(-0.898)(-73.5)}}{2(-0.898)}$$

$$= \frac{-19.59 \pm \sqrt{119.7561}}{-1.796}$$

$$\approx 4.8 \text{ or } 17.0$$

Choose $t \approx 4.8$ because it is in the domain of I. Because $t = 0$ corresponds to 2000, it follows that $t \approx 4.8$ must correspond to 2004. So, the number of Internet users reached 200 million during the year 2004.

Numerical Solution

You can use a table to determine the year in which the number of Internet users reached 200 million.

Year	t	I
2000	0	126.5
2001	1	145.2
2002	2	162.1
2003	3	177.2
2004	4	190.5
2005	5	202.0
2006	6	211.7
2007	7	219.6
2008	8	225.7
2009	9	230.1
2010	10	232.6

From the table, you can see that sometime during the year 2004 the number of Internet users reached 200 million.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

According to the model in Example 9, in which year did the number of Internet users reach 220 million?

TECHNOLOGY You can also use a graphical approach to solve Example 9. Use a graphing utility to graph

$$y_1 = -0.898t^2 + 19.59t + 126.5$$

and

$$y_2 = 200$$

in the same viewing window. Then use the *intersect* feature to find that the graphs intersect when $t \approx 4.8$ and when $t \approx 17.0$. Choose $t \approx 4.8$ because it is in the domain of I.

A fourth application of quadratic equations involves the **Pythagorean Theorem.** The Pythagorean Theorem states that

$$a^2 + b^2 = c^2 \qquad \text{Pythagorean Theorem}$$

where a and b are the legs of a right triangle and c is the hypotenuse.

EXAMPLE 10 An Application Involving the Pythagorean Theorem

An L-shaped sidewalk from the athletic center to the library on a college campus is shown in Figure 1.14. The length of one sidewalk forming the L is twice as long as the other. The length of the diagonal sidewalk that cuts across the grounds between the two buildings is 32 feet. How many feet does a person save by walking on the diagonal sidewalk?

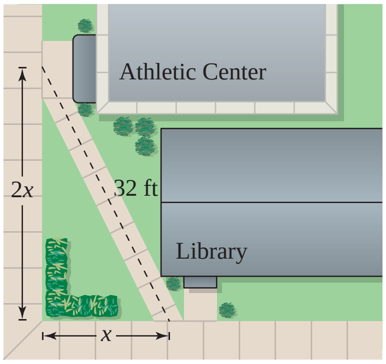


Figure 1.14

Solution Use the Pythagorean Theorem.

$$a^2 + b^2 = c^2 \qquad \text{Pythagorean Theorem}$$

$$x^2 + (2x)^2 = 32^2 \qquad \text{Substitute for } a, b, \text{ and } c.$$

$$5x^2 = 1024 \qquad \text{Simplify.}$$

$$x^2 = 204.8 \qquad \text{Divide each side by 5.}$$

$$x = \sqrt{204.8} \qquad \text{Extract positive square root.}$$

The total distance covered by walking on the L-shaped sidewalk is

$$\begin{aligned} x + 2x &= 3x \\ &= 3\sqrt{204.8} \\ &\approx 42.9 \text{ feet.} \end{aligned}$$

Walking on the diagonal sidewalk saves a person about $42.9 - 32 = 10.9$ feet.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

In Example 10, how many feet does a person save by walking on the diagonal sidewalk when the length of one sidewalk forming the L is three times as long as the other?

Summarize (Section 1.4)

1. Describe how to solve a quadratic equation by factoring *(page 100)*. For an example of solving quadratic equations by factoring, see Example 1.
2. Describe how to solve a quadratic equation by extracting square roots *(page 101)*. For an example of solving quadratic equations by extracting square roots, see Example 2.
3. Describe how to solve a quadratic equation by completing the square *(page 102)*. For examples of solving quadratic equations by completing the square, see Examples 3–5.
4. Describe how to solve a quadratic equation using the Quadratic Formula *(pages 104 and 105)*. For an example of solving a quadratic equation using the Quadratic Formula, see Example 6.
5. Describe a real-life application of quadratic equations *(pages 106–109, Examples 7–10)*.

1.4 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. A ________ ________ in x is an equation that can be written in the general form $ax^2 + bx + c = 0$, where a, b, and c are real numbers with $a \neq 0$.
2. A quadratic equation in x is also called a ________ ________ equation in x.
3. Four methods that can be used to solve a quadratic equation are ________, extracting ________ ________ , ________ the ________, and the ________ ________.
4. The part of the Quadratic Formula, $b^2 - 4ac$, known as the ________, determines the type of solutions of a quadratic equation.
5. The general equation that gives the height of an object that is falling is called a ________ ________.
6. An important theorem that is sometimes used in applications that require solving quadratic equations is the ________ ________.

Skills and Applications

Writing the General Form **In Exercises 7–12, write the quadratic equation in general form.**

7. $2x^2 = 3 - 5x$
8. $x^2 = 16x$
9. $(x - 3)^2 = 3$
10. $13 - 3(x + 7)^2 = 0$
11. $\frac{1}{5}(3x^2 - 10) = 12x$
12. $x(x + 2) = 5x^2 + 1$

Solving a Quadratic Equation by Factoring **In Exercises 13–24, solve the quadratic equation by factoring.**

13. $6x^2 + 3x = 0$
14. $9x^2 - 1 = 0$
15. $x^2 - 2x - 8 = 0$
16. $x^2 - 10x + 9 = 0$
17. $x^2 + 10x + 25 = 0$
18. $4x^2 + 12x + 9 = 0$
19. $3 + 5x - 2x^2 = 0$
20. $2x^2 = 19x + 33$
21. $x^2 + 4x = 12$
22. $-x^2 + 8x = 12$
23. $\frac{3}{4}x^2 + 8x + 20 = 0$
24. $\frac{1}{8}x^2 - x - 16 = 0$

Extracting Square Roots **In Exercises 25–38, solve the equation by extracting square roots. When a solution is irrational, list both the exact solution *and* its approximation rounded to two decimal places.**

25. $x^2 = 49$
26. $x^2 = 144$
27. $x^2 = 11$
28. $x^2 = 32$
29. $3x^2 = 81$
30. $9x^2 = 36$
31. $(x - 12)^2 = 16$
32. $(x - 5)^2 = 25$
33. $(x + 2)^2 = 14$
34. $(x + 9)^2 = 24$
35. $(2x - 1)^2 = 18$
36. $(4x + 7)^2 = 44$
37. $(x - 7)^2 = (x + 3)^2$
38. $(x + 5)^2 = (x + 4)^2$

Completing the Square **In Exercises 39–48, solve the quadratic equation by completing the square.**

39. $x^2 + 4x - 32 = 0$
40. $x^2 - 2x - 3 = 0$
41. $x^2 + 6x + 2 = 0$
42. $x^2 + 8x + 14 = 0$
43. $9x^2 - 18x = -3$
44. $4x^2 - 4x = 1$
45. $7 + 2x - x^2 = 0$
46. $-x^2 + x - 1 = 0$
47. $2x^2 + 5x - 8 = 0$
48. $3x^2 - 4x - 7 = 0$

Rewriting an Expression **In Exercises 49–56, rewrite the quadratic portion of the algebraic expression as the sum or difference of two squares by completing the square.**

49. $\dfrac{1}{x^2 + 2x + 5}$
50. $\dfrac{1}{x^2 + 6x + 10}$
51. $\dfrac{4}{x^2 + 10x + 74}$
52. $\dfrac{5}{x^2 - 18x + 162}$
53. $\dfrac{1}{\sqrt{3 + 2x - x^2}}$
54. $\dfrac{1}{\sqrt{9 + 8x - x^2}}$
55. $\dfrac{1}{\sqrt{12 + 4x - x^2}}$
56. $\dfrac{1}{\sqrt{16 - 6x - x^2}}$

Graphical Analysis **In Exercises 57–64, (a) use a graphing utility to graph the equation, (b) use the graph to approximate any x-intercepts of the graph, (c) set $y = 0$ and solve the resulting equation, and (d) compare the result of part (c) with the x-intercepts of the graph.**

57. $y = (x + 3)^2 - 4$
58. $y = (x - 4)^2 - 1$
59. $y = 1 - (x - 2)^2$
60. $y = 9 - (x - 8)^2$
61. $y = -4x^2 + 4x + 3$
62. $y = 4x^2 - 1$
63. $y = x^2 + 3x - 4$
64. $y = x^2 - 5x - 24$

Using the Discriminant **In Exercises 65–72, use the discriminant to determine the number of real solutions of the quadratic equation.**

65. $2x^2 - 5x + 5 = 0$
66. $-5x^2 - 4x + 1 = 0$
67. $2x^2 - x - 1 = 0$
68. $x^2 - 4x + 4 = 0$
69. $\frac{1}{3}x^2 - 5x + 25 = 0$
70. $\frac{4}{7}x^2 - 8x + 28 = 0$
71. $0.2x^2 + 1.2x - 8 = 0$
72. $9 + 2.4x - 8.3x^2 = 0$

Using the Quadratic Formula **In Exercises 73–96, use the Quadratic Formula to solve the equation.**

73. $2x^2 + x - 1 = 0$

74. $2x^2 - x - 1 = 0$

75. $16x^2 + 8x - 3 = 0$

76. $25x^2 - 20x + 3 = 0$

77. $2 + 2x - x^2 = 0$

78. $x^2 - 10x + 22 = 0$

79. $x^2 + 12x + 16 = 0$

80. $4x = 8 - x^2$

81. $x^2 + 8x - 4 = 0$

82. $2x^2 - 3x - 4 = 0$

83. $12x - 9x^2 = -3$

84. $9x^2 - 37 = 6x$

85. $9x^2 + 30x + 25 = 0$

86. $36x^2 + 24x - 7 = 0$

87. $4x^2 + 4x = 7$

88. $16x^2 - 40x + 5 = 0$

89. $28x - 49x^2 = 4$

90. $3x + x^2 - 1 = 0$

91. $8t = 5 + 2t^2$

92. $25h^2 + 80h + 61 = 0$

93. $(y - 5)^2 = 2y$

94. $(z + 6)^2 = -2z$

95. $\frac{1}{2}x^2 + \frac{3}{8}x = 2$

96. $\left(\frac{5}{7}x - 14\right)^2 = 8x$

Using the Quadratic Formula **In Exercises 97–104, use the Quadratic Formula to solve the equation. (Round your answer to three decimal places.)**

97. $5.1x^2 - 1.7x - 3.2 = 0$

98. $2x^2 - 2.50x - 0.42 = 0$

99. $-0.067x^2 - 0.852x + 1.277 = 0$

100. $-0.005x^2 + 0.101x - 0.193 = 0$

101. $422x^2 - 506x - 347 = 0$

102. $1100x^2 + 326x - 715 = 0$

103. $12.67x^2 + 31.55x + 8.09 = 0$

104. $-3.22x^2 - 0.08x + 28.651 = 0$

Choosing a Method **In Exercises 105–112, solve the equation using any convenient method.**

105. $x^2 - 2x - 1 = 0$

106. $11x^2 + 33x = 0$

107. $(x + 3)^2 = 81$

108. $x^2 - 14x + 49 = 0$

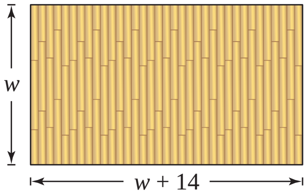
109. $x^2 - x - \frac{11}{4} = 0$

110. $x^2 + 3x - \frac{3}{4} = 0$

111. $(x + 1)^2 = x^2$

112. $3x + 4 = 2x^2 - 7$

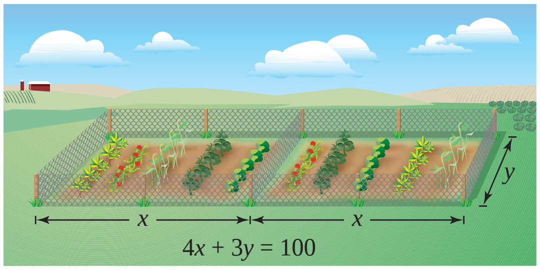
113. Floor Space The floor of a one-story building is 14 feet longer than it is wide (see figure). The building has 1632 square feet of floor space.



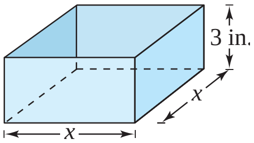
(a) Write a quadratic equation for the area of the floor in terms of w.

(b) Find the length and width of the floor.

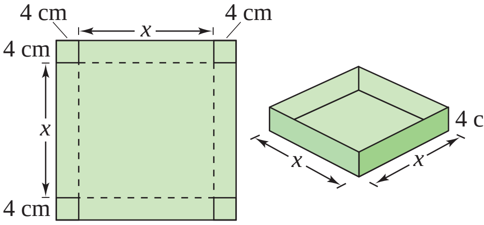
114. Dimensions of a Garden A gardener has 100 meters of fencing to enclose two adjacent rectangular gardens (see figure). The gardener wants the enclosed area to be 350 square meters. What dimensions should the gardener use to obtain this area?



115. Geometry You construct an open box with a square base (see figure) from 108 square inches of material. The height of the box is 3 inches. What are the dimensions of the box? (*Hint:* The surface area is $S = x^2 + 4xh$.)



116. Geometry You construct an open box from a square piece of material by cutting four-centimeter squares from the corners and turning up the sides (see figure). The volume of the box is 576 cubic centimeters. Find the dimensions of the square piece of material that you used to construct the box.



117. Landscaping Two landscapers must mow a rectangular lawn that measures 125 feet by 150 feet. Each wants to mow no more than half of the lawn. The first starts by mowing around the outside of the lawn. The mower has a 24-inch cut. How wide a strip must the first landscaper mow on each of the four sides in order to mow no more than half of the lawn? Approximate the required number of trips around the lawn the first landscaper must take.

118. Seating A rectangular classroom seats 72 students. When the seats are rearranged with three more seats in each row, the classroom has two fewer rows. Find the original number of seats in each row.

Using the Position Equation **In Exercises 119–122, use the position equation given in Example 8 as the model for the problem.**

119. An aircraft flying at 550 feet over level terrain drops a supply package.

(a) How long will it take until the supply package strikes the ground?

(b) The aircraft is flying at 138 miles per hour. How far will the supply package travel horizontally during its descent?

120. You drop a coin from the top of the Eiffel Tower in Paris. The building has a height of 984 feet.

(a) Use the position equation to write a mathematical model for the height of the coin.

(b) Find the height of the coin after 4 seconds.

(c) How long will it take before the coin strikes the ground?

121. Some Major League Baseball pitchers can throw a fastball at speeds of up to and over 100 miles per hour. Assume a Major League Baseball pitcher throws a baseball straight up into the air at 100 miles per hour from a height of 6 feet 3 inches.

(a) Use the position equation to write a mathematical model for the height of the baseball.

(b) Find the height of the baseball after 3 seconds, 4 seconds, and 5 seconds. What must have occurred sometime in the interval $3 \le t \le 5$? Explain.

(c) How many seconds is the baseball in the air?

122. You drop an object from the top of the CN Tower in Toronto, Ontario. The tower has a height of 1815 feet.

(a) Use the position equation to write a mathematical model for the height of the object.

(b) Complete the table.

Time, t	0	2	4	6	8	10	12
Height, s							

(c) From the table in part (b), determine the time interval during which the object reaches the ground. Numerically approximate the time it takes the object to reach the ground.

(d) Find the time it takes the object to reach the ground algebraically. How close was your numerical approximation?

(e) Use a graphing utility with the appropriate viewing window to verify your answer(s) to parts (c) and (d).

Apple's Eyes Studio/Shutterstock.com

123. **Public Debt** The total public debt D (in trillions of dollars) in the United States at the beginning of each year from 2005 through 2011 can be approximated by the model

$$D = 0.157t^2 - 1.46t + 11.1, \quad 5 \le t \le 11$$

where t represents the year, with $t = 0$ corresponding to 2000. *(Source: U.S. Department of the Treasury)*

(a) Use the model to complete the table to determine when the total public debt reached \$10 trillion.

t	5	6	7	8	9	10	11
D							

(b) Verify your result from part (a) algebraically and graphically.

(c) Use the model to predict the total public debt in 2020. Is this prediction reasonable? Explain.

124. **Data Analysis: Movie Tickets** The average ticket prices P at movie theaters from 2001 through 2010 can be approximated by the model

$$P = 0.0133t^2 + 0.096t + 5.57, \quad 1 \le t \le 10$$

where t represents the year, with $t = 1$ corresponding to 2001. *(Source: National Association of Theatre Owners)*

(a) Use a graphing utility to graph the model. Then use the graph to determine the year in which the average ticket price reached \$6.50.

(b) Verify your result from part (a) algebraically.

(c) Use the model to predict the average ticket price at movie theaters in 2020. Is this prediction reasonable? How does this value compare with the ticket price where you live?

125. **Blood Oxygen Level**

Doctors treated a patient at an emergency room from 2:00 P.M. to 7:00 P.M. The patient's blood oxygen level L (in percent) during this time period can be modeled by

$$L = -0.270t^2 + 3.59t + 83.1, \quad 2 \le t \le 7$$

where t represents the time of day, with $t = 2$ corresponding to 2:00 P.M. Use the model to estimate the time (rounded to the nearest hour) when the patient's blood oxygen level was 93%.

126. Biology The metabolic rate of an ectothermic organism increases with increasing temperature within a certain range. Experimental data for the oxygen consumption C (in microliters per gram per hour) of a beetle at certain temperatures can be approximated by the model

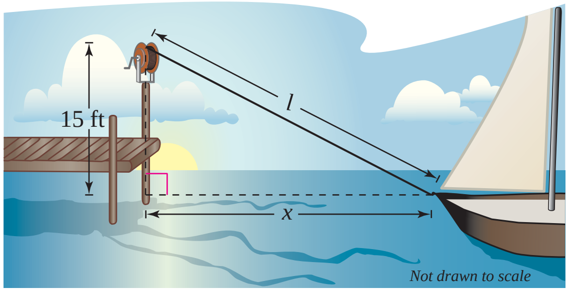
$$C = 0.45x^2 - 1.65x + 50.75, \quad 10 \le x \le 25$$

where x is the air temperature in degrees Celsius.

(a) The oxygen consumption is 150 microliters per gram per hour. What is the air temperature?

(b) When the air temperature increases from 10°C to 20°C, the oxygen consumption increases by approximately what factor?

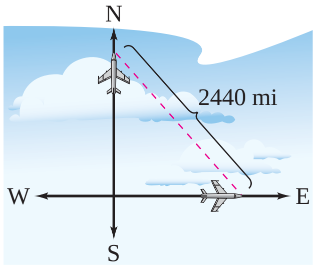
127. Boating A winch is used to tow a boat to a dock. The rope is attached to the boat at a point 15 feet below the level of the winch (see figure).



(a) Use the Pythagorean Theorem to write an equation giving the relationship between l and x.

(b) Find the distance from the boat to the dock when there is 75 feet of rope out.

128. Flying Speed Two planes leave simultaneously from Chicago's O'Hare Airport, one flying due north and the other due east (see figure). The northbound plane is flying 50 miles per hour faster than the eastbound plane. After 3 hours, the planes are 2440 miles apart. Find the speed of each plane.



Exploration

True or False? **In Exercises 129 and 130, determine whether the statement is true or false. Justify your answer.**

129. The quadratic equation $-3x^2 - x = 10$ has two real solutions.

130. If $(2x - 3)(x + 5) = 8$, then either $2x - 3 = 8$ or $x + 5 = 8$.

Think About It **In Exercises 131–136, write a quadratic equation that has the given solutions. (There are many correct answers.)**

131. 3 and -5

132. -6 and -9

133. 8 and 14

134. $\frac{1}{6}$ and $-\frac{2}{5}$

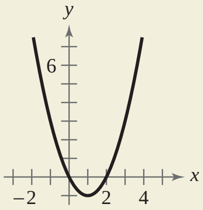
135. $1 + \sqrt{2}$ and $1 - \sqrt{2}$

136. $-3 + \sqrt{5}$ and $-3 - \sqrt{5}$

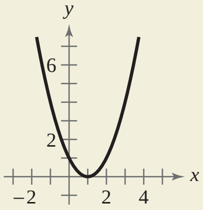
137. Think About It Is it possible for a quadratic equation to have only one x-intercept? Explain.

138. HOW DO YOU SEE IT? From each graph, can you tell whether the discriminant is *positive*, *zero*, or *negative*? Explain your reasoning.

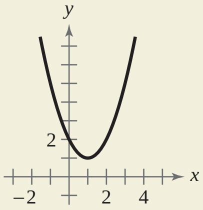
(a) $x^2 - 2x = 0$



(b) $x^2 - 2x + 1 = 0$



(c) $x^2 - 2x + 2 = 0$



Project: Population To work an extended application analyzing the population of the United States, visit this text's website at *LarsonPrecalculus.com*. *(Source: U.S. Census Bureau)*

1.5 Complex Numbers

You can use complex numbers to model and solve real-life problems in electronics. For instance, in Exercise 91 on page 120, you will use complex numbers to find the impedance of an electrical circuit.

- **Use the imaginary unit *i* to write complex numbers.**
- **Add, subtract, and multiply complex numbers.**
- **Use complex conjugates to write the quotient of two complex numbers in standard form.**
- **Find complex solutions of quadratic equations.**

The Imaginary Unit *i*

In Section 1.4, you learned that some quadratic equations have no real solutions. For instance, the quadratic equation

$$x^2 + 1 = 0$$

has no real solution because there is no real number x that can be squared to produce -1. To overcome this deficiency, mathematicians created an expanded system of numbers using the **imaginary unit *i*,** defined as

$$i = \sqrt{-1} \qquad \text{Imaginary unit}$$

where $i^2 = -1$. By adding real numbers to real multiples of this imaginary unit, the set of **complex numbers** is obtained. Each complex number can be written in the **standard form *a* + *bi*.** For instance, the standard form of the complex number $-5 + \sqrt{-9}$ is $-5 + 3i$ because

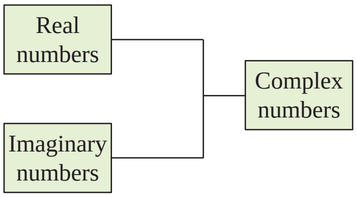
$$-5 + \sqrt{-9} = -5 + \sqrt{3^2(-1)} = -5 + 3\sqrt{-1} = -5 + 3i.$$

Definition of a Complex Number

Let a and b be real numbers. The number $a + bi$ is called a **complex number,** and it is said to be written in **standard form.** The real number a is called the **real part** and the real number b is called the **imaginary part** of the complex number.

When $b = 0$, the number $a + bi$ is a real number. When $b \neq 0$, the number $a + bi$ is called an **imaginary number.** A number of the form bi, where $b \neq 0$, is called a **pure imaginary number.**

The set of real numbers is a subset of the set of complex numbers, as shown below. This is true because every real number a can be written as a complex number using $b = 0$. That is, for every real number a, you can write $a = a + 0i$.



Equality of Complex Numbers

Two complex numbers $a + bi$ and $c + di$, written in standard form, are equal to each other

$$a + bi = c + di \qquad \text{Equality of two complex numbers}$$

if and only if $a = c$ and $b = d$.

Sarunyu_foto/Shutterstock.com

Operations with Complex Numbers

To add (or subtract) two complex numbers, you add (or subtract) the real and imaginary parts of the numbers separately.

Addition and Subtraction of Complex Numbers

For two complex numbers $a + bi$ and $c + di$ written in standard form, the sum and difference are defined as follows.

Sum: $(a + bi) + (c + di) = (a + c) + (b + d)i$

Difference: $(a + bi) - (c + di) = (a - c) + (b - d)i$

The **additive identity** in the complex number system is zero (the same as in the real number system). Furthermore, the **additive inverse** of the complex number $a + bi$ is

$-(a + bi) = -a - bi.$ Additive inverse

So, you have

$(a + bi) + (-a - bi) = 0 + 0i = 0.$

EXAMPLE 1 Adding and Subtracting Complex Numbers

a. $(4 + 7i) + (1 - 6i) = 4 + 7i + 1 - 6i$ Remove parentheses.

$= (4 + 1) + (7 - 6)i$ Group like terms.

$= 5 + i$ Write in standard form.

b. $(1 + 2i) + (3 - 2i) = 1 + 2i + 3 - 2i$ Remove parentheses.

$= (1 + 3) + (2 - 2)i$ Group like terms.

$= 4 + 0i$ Simplify.

$= 4$ Write in standard form.

REMARK Note that the sum of two complex numbers can be a real number.

c. $3i - (-2 + 3i) - (2 + 5i) = 3i + 2 - 3i - 2 - 5i$

$= (2 - 2) + (3 - 3 - 5)i$

$= 0 - 5i$

$= -5i$

d. $(3 + 2i) + (4 - i) - (7 + i) = 3 + 2i + 4 - i - 7 - i$

$= (3 + 4 - 7) + (2 - 1 - 1)i$

$= 0 + 0i$

$= 0$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Perform each operation and write the result in standard form.

a. $(7 + 3i) + (5 - 4i)$

b. $(3 + 4i) - (5 - 3i)$

c. $2i + (-3 - 4i) - (-3 - 3i)$

d. $(5 - 3i) + (3 + 5i) - (8 + 2i)$

Many of the properties of real numbers are valid for complex numbers as well. Here are some examples.

Associative Properties of Addition and Multiplication
Commutative Properties of Addition and Multiplication
Distributive Property of Multiplication Over Addition

Notice below how these properties are used when two complex numbers are multiplied.

$$
\begin{aligned}
(a+bi)(c+di) &= a(c+di) + bi(c+di) && \text{Distributive Property}\\
&= ac + (ad)i + (bc)i + (bd)i^2 && \text{Distributive Property}\\
&= ac + (ad)i + (bc)i + (bd)(-1) && i^2 = -1\\
&= ac - bd + (ad)i + (bc)i && \text{Commutative Property}\\
&= (ac - bd) + (ad + bc)i && \text{Associative Property}
\end{aligned}
$$

Rather than trying to memorize this multiplication rule, you should simply remember how to use the Distributive Property to multiply two complex numbers.

EXAMPLE 2 Multiplying Complex Numbers

a. $$
\begin{aligned}
4(-2+3i) &= 4(-2) + 4(3i) && \text{Distributive Property}\\
&= -8 + 12i && \text{Simplify.}
\end{aligned}
$$

b. $$
\begin{aligned}
(2-i)(4+3i) &= 2(4+3i) - i(4+3i) && \text{Distributive Property}\\
&= 8 + 6i - 4i - 3i^2 && \text{Distributive Property}\\
&= 8 + 6i - 4i - 3(-1) && i^2 = -1\\
&= (8+3) + (6-4)i && \text{Group like terms.}\\
&= 11 + 2i && \text{Write in standard form.}
\end{aligned}
$$

c. $$
\begin{aligned}
(3+2i)(3-2i) &= 3(3-2i) + 2i(3-2i) && \text{Distributive Property}\\
&= 9 - 6i + 6i - 4i^2 && \text{Distributive Property}\\
&= 9 - 6i + 6i - 4(-1) && i^2 = -1\\
&= 9 + 4 && \text{Simplify.}\\
&= 13 && \text{Write in standard form.}
\end{aligned}
$$

d. $$
\begin{aligned}
(3+2i)^2 &= (3+2i)(3+2i) && \text{Square of a binomial}\\
&= 3(3+2i) + 2i(3+2i) && \text{Distributive Property}\\
&= 9 + 6i + 6i + 4i^2 && \text{Distributive Property}\\
&= 9 + 6i + 6i + 4(-1) && i^2 = -1\\
&= 9 + 12i - 4 && \text{Simplify.}\\
&= 5 + 12i && \text{Write in standard form.}
\end{aligned}
$$

REMARK The procedure described above is similar to multiplying two binomials and combining like terms, as in the FOIL Method shown in Section P.3. For instance, you can use the FOIL Method to multiply the two complex numbers from Example 2(b).

$$(2-i)(4+3i) = \overset{F}{8} + \overset{O}{6i} - \overset{I}{4i} - \overset{L}{3i^2}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Perform each operation and write the result in standard form.

a. $(2-4i)(3+3i)$

b. $(4+5i)(4-5i)$

c. $(4+2i)^2$

Complex Conjugates

Notice in Example 2(c) that the product of two complex numbers can be a real number. This occurs with pairs of complex numbers of the form $a + bi$ and $a - bi$, called **complex conjugates.**

$$\begin{aligned}(a + bi)(a - bi) &= a^2 - abi + abi - b^2i^2\\ &= a^2 - b^2(-1)\\ &= a^2 + b^2\end{aligned}$$

ALGEBRA HELP You can compare complex conjugates with the method for rationalizing denominators in Section P.2.

EXAMPLE 3 Multiplying Conjugates

Multiply each complex number by its complex conjugate.

a. $1 + i$ **b.** $4 - 3i$

Solution

a. The complex conjugate of $1 + i$ is $1 - i$.

$$(1 + i)(1 - i) = 1^2 - i^2 = 1 - (-1) = 2$$

b. The complex conjugate of $4 - 3i$ is $4 + 3i$.

$$(4 - 3i)(4 + 3i) = 4^2 - (3i)^2 = 16 - 9i^2 = 16 - 9(-1) = 25$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Multiply each complex number by its complex conjugate.

a. $3 + 6i$

b. $2 - 5i$

To write the quotient of $a + bi$ and $c + di$ in standard form, where c and d are not both zero, multiply the numerator and denominator by the complex conjugate of the *denominator* to obtain

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di}\left(\frac{c - di}{c - di}\right) = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}.$$

REMARK Note that when you multiply the numerator and denominator of a quotient of complex numbers by

$$\frac{c - di}{c - di}$$

you are actually multiplying the quotient by a form of 1. You are not changing the original expression, you are only creating an expression that is equivalent to the original expression.

EXAMPLE 4 Quotient of Complex Numbers in Standard Form

$$\begin{aligned}\frac{2 + 3i}{4 - 2i} &= \frac{2 + 3i}{4 - 2i}\left(\frac{4 + 2i}{4 + 2i}\right) && \text{Multiply numerator and denominator by complex conjugate of denominator.}\\ &= \frac{8 + 4i + 12i + 6i^2}{16 - 4i^2} && \text{Expand.}\\ &= \frac{8 - 6 + 16i}{16 + 4} && i^2 = -1\\ &= \frac{2 + 16i}{20} && \text{Simplify.}\\ &= \frac{1}{10} + \frac{4}{5}i && \text{Write in standard form.}\end{aligned}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write $\dfrac{2 + i}{2 - i}$ in standard form.

Complex Solutions of Quadratic Equations

You can write a number such as $\sqrt{-3}$ in standard form by factoring out $i = \sqrt{-1}$.

$$\sqrt{-3} = \sqrt{3(-1)} = \sqrt{3}\sqrt{-1} = \sqrt{3}i$$

The number $\sqrt{3}i$ is called the *principal square root* of -3.

Principal Square Root of a Negative Number

When a is a positive real number, the **principal square root** of $-a$ is defined as

$$\sqrt{-a} = \sqrt{a}i.$$

REMARK The definition of principal square root uses the rule

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

for $a > 0$ and $b < 0$. This rule is not valid if *both* a and b are negative. For example,

$$\begin{aligned}\sqrt{-5}\sqrt{-5} &= \sqrt{5(-1)}\sqrt{5(-1)}\\ &= \sqrt{5}i\sqrt{5}i\\ &= \sqrt{25}i^2\\ &= 5i^2\\ &= -5\end{aligned}$$

whereas

$$\sqrt{(-5)(-5)} = \sqrt{25} = 5.$$

To avoid problems with square roots of negative numbers, be sure to convert complex numbers to standard form *before* multiplying.

EXAMPLE 5 Writing Complex Numbers in Standard Form

a. $\sqrt{-3}\sqrt{-12} = \sqrt{3}i\sqrt{12}i = \sqrt{36}i^2 = 6(-1) = -6$

b. $\sqrt{-48} - \sqrt{-27} = \sqrt{48}i - \sqrt{27}i = 4\sqrt{3}i - 3\sqrt{3}i = \sqrt{3}i$

c. $\left(-1 + \sqrt{-3}\right)^2 = \left(-1 + \sqrt{3}i\right)^2 = (-1)^2 - 2\sqrt{3}i + \left(\sqrt{3}\right)^2(i^2)$

$$\begin{aligned}&= 1 - 2\sqrt{3}i + 3(-1)\\ &= -2 - 2\sqrt{3}i\end{aligned}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write $\sqrt{-14}\sqrt{-2}$ in standard form.

EXAMPLE 6 Complex Solutions of a Quadratic Equation

Solve $3x^2 - 2x + 5 = 0$.

Solution

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(5)}}{2(3)} \qquad \text{Quadratic Formula}$$

$$= \frac{2 \pm \sqrt{-56}}{6} \qquad \text{Simplify.}$$

$$= \frac{2 \pm 2\sqrt{14}i}{6} \qquad \text{Write } \sqrt{-56} \text{ in standard form.}$$

$$= \frac{1}{3} \pm \frac{\sqrt{14}}{3}i \qquad \text{Write in standard form.}$$

ALGEBRA HELP You can review the techniques for using the Quadratic Formula in Section 1.4.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $8x^2 + 14x + 9 = 0$.

Summarize (Section 1.5)

1. Describe how to write complex numbers using the imaginary unit i *(page 114)*.
2. Describe how to add, subtract, and multiply complex numbers *(pages 115 and 116, Examples 1 and 2)*.
3. Describe how to use complex conjugates to write the quotient of two complex numbers in standard form *(page 117, Example 4)*.
4. Describe how to find complex solutions of a quadratic equation *(page 118, Example 6)*.

1.5 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. A ________ number has the form $a + bi$, where $a \neq 0$, $b = 0$.

2. An ________ number has the form $a + bi$, where $a \neq 0$, $b \neq 0$.

3. A ________ ________ number has the form $a + bi$, where $a = 0$, $b \neq 0$.

4. The imaginary unit i is defined as $i =$ ________, where $i^2 =$ ________.

5. When a is a positive real number, the ________ ________ root of $-a$ is defined as $\sqrt{-a} = \sqrt{a}\,i$.

6. The numbers $a + bi$ and $a - bi$ are called ________ ________, and their product is a real number $a^2 + b^2$.

Skills and Applications

Equality of Complex Numbers

In Exercises 7–10, find real numbers *a* and *b* such that the equation is true.

7. $a + bi = -12 + 7i$

8. $a + bi = 13 + 4i$

9. $(a - 1) + (b + 3)i = 5 + 8i$

10. $(a + 6) + 2bi = 6 - 5i$

Writing a Complex Number in Standard Form

In Exercises 11–22, write the complex number in standard form.

11. $8 + \sqrt{-25}$

12. $5 + \sqrt{-36}$

13. $2 - \sqrt{-27}$

14. $1 + \sqrt{-8}$

15. $\sqrt{-80}$

16. $\sqrt{-4}$

17. 14

18. 75

19. $-10i + i^2$

20. $-4i^2 + 2i$

21. $\sqrt{-0.09}$

22. $\sqrt{-0.0049}$

Performing Operations with Complex Numbers

In Exercises 23–42, perform the operation and write the result in standard form.

23. $(7 + i) + (3 - 4i)$

24. $(13 - 2i) + (-5 + 6i)$

25. $(9 - i) - (8 - i)$

26. $(3 + 2i) - (6 + 13i)$

27. $\left(-2 + \sqrt{-8}\right) + \left(5 - \sqrt{-50}\right)$

28. $\left(8 + \sqrt{-18}\right) - \left(4 + 3\sqrt{2}\,i\right)$

29. $13i - (14 - 7i)$

30. $25 + (-10 + 11i) + 15i$

31. $-\left(\frac{3}{2} + \frac{5}{2}i\right) + \left(\frac{5}{3} + \frac{11}{3}i\right)$

32. $(1.6 + 3.2i) + (-5.8 + 4.3i)$

33. $(1 + i)(3 - 2i)$

34. $(7 - 2i)(3 - 5i)$

35. $12i(1 - 9i)$

36. $-8i(9 + 4i)$

37. $\left(\sqrt{14} + \sqrt{10}\,i\right)\left(\sqrt{14} - \sqrt{10}\,i\right)$

38. $\left(\sqrt{3} + \sqrt{15}\,i\right)\left(\sqrt{3} - \sqrt{15}\,i\right)$

39. $(6 + 7i)^2$

40. $(5 - 4i)^2$

41. $(2 + 3i)^2 + (2 - 3i)^2$

42. $(1 - 2i)^2 - (1 + 2i)^2$

Multiplying Conjugates

In Exercises 43–50, write the complex conjugate of the complex number. Then multiply the number by its complex conjugate.

43. $9 + 2i$

44. $8 - 10i$

45. $-1 - \sqrt{5}\,i$

46. $-3 + \sqrt{2}\,i$

47. $\sqrt{-20}$

48. $\sqrt{-15}$

49. $\sqrt{6}$

50. $1 + \sqrt{8}$

Quotient of Complex Numbers in Standard Form

In Exercises 51–60, write the quotient in standard form.

51. $\dfrac{3}{i}$

52. $-\dfrac{14}{2i}$

53. $\dfrac{2}{4 - 5i}$

54. $\dfrac{13}{1 - i}$

55. $\dfrac{5 + i}{5 - i}$

56. $\dfrac{6 - 7i}{1 - 2i}$

57. $\dfrac{9 - 4i}{i}$

58. $\dfrac{8 + 16i}{2i}$

59. $\dfrac{3i}{(4 - 5i)^2}$

60. $\dfrac{5i}{(2 + 3i)^2}$

Performing Operations with Complex Numbers

In Exercises 61–64, perform the operation and write the result in standard form.

61. $\dfrac{2}{1 + i} - \dfrac{3}{1 - i}$

62. $\dfrac{2i}{2 + i} + \dfrac{5}{2 - i}$

63. $\dfrac{i}{3 - 2i} + \dfrac{2i}{3 + 8i}$

64. $\dfrac{1 + i}{i} - \dfrac{3}{4 - i}$

Writing a Complex Number in Standard Form

In Exercises 65–70, write the complex number in standard form.

65. $\sqrt{-6} \cdot \sqrt{-2}$

66. $\sqrt{-5} \cdot \sqrt{-10}$

67. $\left(\sqrt{-15}\right)^2$

68. $\left(\sqrt{-75}\right)^2$

69. $\left(3 + \sqrt{-5}\right)\left(7 - \sqrt{-10}\right)$

70. $\left(2 - \sqrt{-6}\right)^2$

Complex Solutions of a Quadratic Equation **In Exercises 71–80, use the Quadratic Formula to solve the quadratic equation.**

71. $x^2 - 2x + 2 = 0$

72. $x^2 + 6x + 10 = 0$

73. $4x^2 + 16x + 17 = 0$

74. $9x^2 - 6x + 37 = 0$

75. $4x^2 + 16x + 15 = 0$

76. $16t^2 - 4t + 3 = 0$

77. $\frac{3}{2}x^2 - 6x + 9 = 0$

78. $\frac{7}{8}x^2 - \frac{3}{4}x + \frac{5}{16} = 0$

79. $1.4x^2 - 2x - 10 = 0$

80. $4.5x^2 - 3x + 12 = 0$

Simplifying a Complex Number **In Exercises 81–90, simplify the complex number and write it in standard form.**

81. $-6i^3 + i^2$

82. $4i^2 - 2i^3$

83. $-14i^5$

84. $(-i)^3$

85. $\left(\sqrt{-72}\right)^3$

86. $\left(\sqrt{-2}\right)^6$

87. $\dfrac{1}{i^3}$

88. $\dfrac{1}{(2i)^3}$

89. $(3i)^4$

90. $(-i)^6$

91. Impedance

The opposition to current in an electrical circuit is called its impedance. The impedance z in a parallel circuit with two pathways satisfies the equation

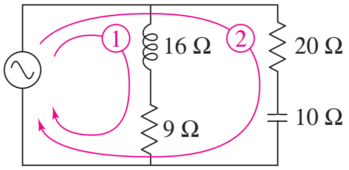
$$\frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2}$$

where z_1 is the impedance (in ohms) of pathway 1 and z_2 is the impedance of pathway 2.

(a) The impedance of each pathway in a parallel circuit is found by adding the impedances of all components in the pathway. Use the table to find z_1 and z_2.

(b) Find the impedance z.

	Resistor	Inductor	Capacitor
Symbol	$a\ \Omega$	$b\ \Omega$	$c\ \Omega$
Impedance	a	bi	$-ci$



92. Cube of a Complex Number Cube each complex number.

(a) $-1 + \sqrt{3}i$ (b) $-1 - \sqrt{3}i$

Exploration

True or False? **In Exercises 93–96, determine whether the statement is true or false. Justify your answer.**

93. There is no complex number that is equal to its complex conjugate.

94. $-i\sqrt{6}$ is a solution of $x^4 - x^2 + 14 = 56$.

95. $i^{44} + i^{150} - i^{74} - i^{109} + i^{61} = -1$

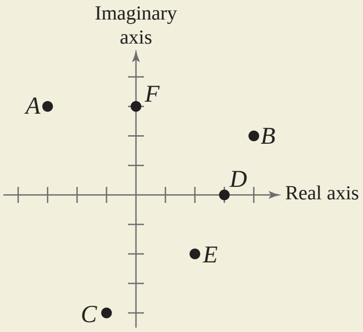
96. The sum of two complex numbers is always a real number.

97. Pattern Recognition Complete the following.

$i^1 = i$	$i^2 = -1$	$i^3 = -i$	$i^4 = 1$
$i^5 =$ ▨	$i^6 =$ ▨	$i^7 =$ ▨	$i^8 =$ ▨
$i^9 =$ ▨	$i^{10} =$ ▨	$i^{11} =$ ▨	$i^{12} =$ ▨

What pattern do you see? Write a brief description of how you would find i raised to any positive integer power.

98. **HOW DO YOU SEE IT?** The coordinate system shown below is called the *complex plane*. In the complex plane, the point that corresponds to the complex number $a + bi$ is (a, b).



Match each complex number with its corresponding point.

(i) 3 (ii) $3i$ (iii) $4 + 2i$

(iv) $2 - 2i$ (v) $-3 + 3i$ (vi) $-1 - 4i$

99. Error Analysis Describe the error.

$$\xcancel{\sqrt{-6}\sqrt{-6} = \sqrt{(-6)(-6)} = \sqrt{36} = 6}$$

100. Proof Prove that the complex conjugate of the product of two complex numbers $a_1 + b_1i$ and $a_2 + b_2i$ is the product of their complex conjugates.

101. Proof Prove that the complex conjugate of the sum of two complex numbers $a_1 + b_1i$ and $a_2 + b_2i$ is the sum of their complex conjugates.

Sarunyu_foto/Shutterstock.com

1.6 Other Types of Equations

You can use polynomial equations, radical equations, rational equations, and absolute value equations to model and solve real-life problems. For instance, in Exercise 103 on page 129, you will use a radical equation to model the relationship between the pressure and temperature of saturated steam.

- **Solve polynomial equations of degree three or greater.**
- **Solve radical equations.**
- **Solve rational equations and absolute value equations.**
- **Use nonlinear and nonquadratic models to solve real-life problems.**

Polynomial Equations

In this section, you will extend the techniques for solving equations to nonlinear and nonquadratic equations. At this point in the text, you have only four basic methods for solving nonlinear equations—*factoring*, *extracting square roots*, *completing the square*, and the *Quadratic Formula*. So, the main goal of this section is to learn to *rewrite* nonlinear equations in a form to which you can apply one of these methods.

Example 1 shows how to use factoring to solve a **polynomial equation,** which is an equation that can be written in the general form

$$a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0 = 0.$$

EXAMPLE 1 Solving a Polynomial Equation by Factoring

Solve $3x^4 = 48x^2$.

Solution First write the polynomial equation in general form with zero on one side. Then factor the other side, set each factor equal to zero, and solve.

$$3x^4 = 48x^2 \qquad \text{Write original equation.}$$

$$3x^4 - 48x^2 = 0 \qquad \text{Write in general form.}$$

$$3x^2(x^2 - 16) = 0 \qquad \text{Factor out common factor.}$$

$$3x^2(x + 4)(x - 4) = 0 \qquad \text{Write in factored form.}$$

$$3x^2 = 0 \Rightarrow x = 0 \qquad \text{Set 1st factor equal to 0.}$$

$$x + 4 = 0 \Rightarrow x = -4 \qquad \text{Set 2nd factor equal to 0.}$$

$$x - 4 = 0 \Rightarrow x = 4 \qquad \text{Set 3rd factor equal to 0.}$$

You can check these solutions by substituting in the original equation, as follows.

Check

$$3(0)^4 = 48(0)^2 \qquad 0 \text{ checks.} ✓$$

$$3(-4)^4 = 48(-4)^2 \qquad -4 \text{ checks.} ✓$$

$$3(4)^4 = 48(4)^2 \qquad 4 \text{ checks.} ✓$$

So, you can conclude that the solutions are $x = 0$, $x = -4$, and $x = 4$.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $9x^4 - 12x^2 = 0$.

A common mistake that is made in solving an equation like that in Example 1 is to divide each side of the equation by the variable factor x^2. This loses the solution $x = 0$. When solving an equation, always write the equation in general form, then factor the equation and set each factor equal to zero. Do not divide each side of an equation by a variable factor in an attempt to simplify the equation.

xtrekx/Shutterstock.com

ALGEBRA HELP For a review of factoring special polynomial forms, see Section P.4.

EXAMPLE 2 Solving a Polynomial Equation by Factoring

Solve $x^3 - 3x^2 + 3x - 9 = 0$.

Solution

$$x^3 - 3x^2 + 3x - 9 = 0 \qquad \text{Write original equation.}$$

$$x^2(x - 3) + 3(x - 3) = 0 \qquad \text{Factor by grouping.}$$

$$(x - 3)(x^2 + 3) = 0 \qquad \text{Distributive Property}$$

$$x - 3 = 0 \Rightarrow x = 3 \qquad \text{Set 1st factor equal to 0.}$$

$$x^2 + 3 = 0 \Rightarrow x = \pm\sqrt{3}i \qquad \text{Set 2nd factor equal to 0.}$$

The solutions are $x = 3$, $x = \sqrt{3}i$, and $x = -\sqrt{3}i$. Check these in the original equation.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve each equation.

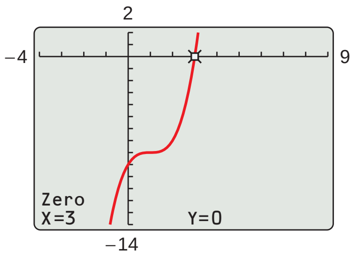
a. $x^3 - 5x^2 - 2x + 10 = 0$

b. $6x^3 - 27x^2 - 54x = 0$

TECHNOLOGY You can use a graphing utility to check graphically the solutions of the equation in Example 2. To do this, graph the equation

$$y = x^3 - 3x^2 + 3x - 9.$$

Then use the *zero* or *root* feature to approximate any x-intercepts. As shown below, the x-intercept of the graph occurs at $(3, 0)$, confirming the *real* solution $x = 3$ found in Example 2.



Try using a graphing utility to check the solutions found in Example 3.

Occasionally, mathematical models involve equations that are of **quadratic type.** In general, an equation is of quadratic type when it can be written in the form

$$au^2 + bu + c = 0$$

where $a \neq 0$ and u is an algebraic expression.

EXAMPLE 3 Solving an Equation of Quadratic Type

Solve $x^4 - 3x^2 + 2 = 0$.

Solution

This equation is of quadratic type with $u = x^2$.

$$x^4 - 3x^2 + 2 = 0 \qquad \text{Write original equation.}$$

$$(x^2)^2 - 3(x^2) + 2 = 0 \qquad \text{Quadratic form}$$

$$u^2 - 3u + 2 = 0 \qquad u = x^2$$

$$(u - 1)(u - 2) = 0 \qquad \text{Factor.}$$

$$u - 1 = 0 \Rightarrow u = 1 \qquad \text{Set 1st factor equal to 0.}$$

$$u - 2 = 0 \Rightarrow u = 2 \qquad \text{Set 2nd factor equal to 0.}$$

Recall that $u = x^2$. Replace u with x^2 and solve for x in each equation.

$$u = 1 \qquad u = 2$$

$$x^2 = 1 \qquad x^2 = 2$$

$$x = \pm 1 \qquad x = \pm\sqrt{2}$$

The solutions are $x = -1$, $x = 1$, $x = \sqrt{2}$, and $x = -\sqrt{2}$. Check these in the original equation.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve each equation.

a. $x^4 - 7x^2 + 12 = 0$ **b.** $9x^4 - 37x^2 + 4 = 0$

Radical Equations

A **radical equation** is an equation that involves one or more radical expressions.

REMARK When squaring each side of an equation or raising each side of an equation to a rational power, it is possible to introduce extraneous solutions. In such cases, checking your solutions is crucial.

EXAMPLE 4 Solving Radical Equations

a.

$$\sqrt{2x+7}-x=2 \qquad \text{Original equation}$$

$$\sqrt{2x+7}=x+2 \qquad \text{Isolate radical.}$$

$$2x+7=x^2+4x+4 \qquad \text{Square each side.}$$

$$0=x^2+2x-3 \qquad \text{Write in general form.}$$

$$0=(x+3)(x-1) \qquad \text{Factor.}$$

$$x+3=0 \Rightarrow x=-3 \qquad \text{Set 1st factor equal to 0.}$$

$$x-1=0 \Rightarrow x=1 \qquad \text{Set 2nd factor equal to 0.}$$

By checking these values, you can determine that the only solution is $x = 1$.

b.

$$\sqrt{2x-5}-\sqrt{x-3}=1 \qquad \text{Original equation}$$

$$\sqrt{2x-5}=\sqrt{x-3}+1 \qquad \text{Isolate } \sqrt{2x-5}.$$

$$2x-5=x-3+2\sqrt{x-3}+1 \qquad \text{Square each side.}$$

$$2x-5=x-2+2\sqrt{x-3} \qquad \text{Combine like terms.}$$

$$x-3=2\sqrt{x-3} \qquad \text{Isolate } 2\sqrt{x-3}.$$

$$x^2-6x+9=4(x-3) \qquad \text{Square each side.}$$

$$x^2-10x+21=0 \qquad \text{Write in general form.}$$

$$(x-3)(x-7)=0 \qquad \text{Factor.}$$

$$x-3=0 \Rightarrow x=3 \qquad \text{Set 1st factor equal to 0.}$$

$$x-7=0 \Rightarrow x=7 \qquad \text{Set 2nd factor equal to 0.}$$

The solutions are $x = 3$ and $x = 7$. Check these in the original equation.

REMARK When an equation contains two radicals, it may not be possible to isolate both. In such cases, you may have to raise each side of the equation to a power at *two* different stages in the solution, as shown in Example 4(b).

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $-\sqrt{40-9x}+2=x$.

EXAMPLE 5 Solving an Equation Involving a Rational Exponent

Solve $(x-4)^{2/3}=25$.

Solution

$$(x-4)^{2/3}=25 \qquad \text{Write original equation.}$$

$$\sqrt[3]{(x-4)^2}=25 \qquad \text{Rewrite in radical form.}$$

$$(x-4)^2=15{,}625 \qquad \text{Cube each side.}$$

$$x-4=\pm 125 \qquad \text{Extract square roots.}$$

$$x=129, \quad x=-121 \qquad \text{Add 4 to each side.}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $(x-5)^{2/3}=16$.

Rational Equations and Absolute Value Equations

In Section 1.2, you learned how to solve rational equations. Recall that the first step is to multiply each term of the equation by the least common denominator (LCD).

EXAMPLE 6 Solving a Rational Equation

Solve $\frac{2}{x} = \frac{3}{x-2} - 1$.

Solution For this equation, the LCD of the three terms is $x(x-2)$, so begin by multiplying each term of the equation by this expression.

REMARK Notice that the values $x = 0$ and $x = 2$ are excluded because they result in division by zero in the original equation.

$$\begin{aligned}
\frac{2}{x} &= \frac{3}{x-2} - 1 && \text{Write original equation.}\\
x(x-2)\frac{2}{x} &= x(x-2)\frac{3}{x-2} - x(x-2)(1) && \text{Multiply each term by the LCD.}\\
2(x-2) &= 3x - x(x-2), \quad x \neq 0, 2 && \text{Simplify.}\\
2x - 4 &= -x^2 + 5x && \text{Simplify.}\\
x^2 - 3x - 4 &= 0 && \text{Write in general form.}\\
(x-4)(x+1) &= 0 && \text{Factor.}\\
x - 4 = 0 \quad &\Rightarrow \quad x = 4 && \text{Set 1st factor equal to 0.}\\
x + 1 = 0 \quad &\Rightarrow \quad x = -1 && \text{Set 2nd factor equal to 0.}
\end{aligned}$$

Both $x = 4$ and $x = -1$ are possible solutions. Because multiplying each side of an equation by a variable expression can introduce extraneous solutions, it is important to check your solutions.

Check $x = 4$

$$\begin{aligned}
\frac{2}{x} &= \frac{3}{x-2} - 1\\
\frac{2}{4} &\stackrel{?}{=} \frac{3}{4-2} - 1\\
\frac{1}{2} &\stackrel{?}{=} \frac{3}{2} - 1\\
\frac{1}{2} &= \frac{1}{2} \checkmark
\end{aligned}$$

Check $x = -1$

$$\begin{aligned}
\frac{2}{x} &= \frac{3}{x-2} - 1\\
\frac{2}{-1} &\stackrel{?}{=} \frac{3}{-1-2} - 1\\
-2 &\stackrel{?}{=} -1 - 1\\
-2 &= -2 \checkmark
\end{aligned}$$

So, the solutions are $x = 4$ and $x = -1$.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $\frac{4}{x} + \frac{2}{x+3} = -3$.

An **absolute value equation** is an equation that involves one or more absolute value expressions. To solve an absolute value equation, remember that the expression inside the absolute value bars can be positive or negative. This results in *two* separate equations, each of which must be solved. For instance, the equation

$$|x - 2| = 3$$

results in the two equations

$$x - 2 = 3$$

and

$$-(x - 2) = 3$$

which implies that the equation has two solutions: $x = 5$ and $x = -1$.

▷ **ALGEBRA HELP** You can review the definition of absolute value in Section P.1.

EXAMPLE 7 Solving an Absolute Value Equation

Solve $|x^2 - 3x| = -4x + 6$.

Solution Because the variable expression inside the absolute value bars can be positive or negative, you must solve the following two equations.

First Equation

$$x^2 - 3x = -4x + 6 \qquad \text{Use positive expression.}$$

$$x^2 + x - 6 = 0 \qquad \text{Write in general form.}$$

$$(x + 3)(x - 2) = 0 \qquad \text{Factor.}$$

$$x + 3 = 0 \quad \Rightarrow \quad x = -3 \qquad \text{Set 1st factor equal to 0.}$$

$$x - 2 = 0 \quad \Rightarrow \quad x = 2 \qquad \text{Set 2nd factor equal to 0.}$$

Second Equation

$$-(x^2 - 3x) = -4x + 6 \qquad \text{Use negative expression.}$$

$$x^2 - 7x + 6 = 0 \qquad \text{Write in general form.}$$

$$(x - 1)(x - 6) = 0 \qquad \text{Factor.}$$

$$x - 1 = 0 \quad \Rightarrow \quad x = 1 \qquad \text{Set 1st factor equal to 0.}$$

$$x - 6 = 0 \quad \Rightarrow \quad x = 6 \qquad \text{Set 2nd factor equal to 0.}$$

Check

$$|(-3)^2 - 3(-3)| \stackrel{?}{=} -4(-3) + 6 \qquad \text{Substitute } -3 \text{ for } x.$$

$$18 = 18 \qquad -3 \text{ checks. } \checkmark$$

$$|(2)^2 - 3(2)| \stackrel{?}{=} -4(2) + 6 \qquad \text{Substitute 2 for } x.$$

$$2 \neq -2 \qquad \text{2 does not check.}$$

$$|(1)^2 - 3(1)| \stackrel{?}{=} -4(1) + 6 \qquad \text{Substitute 1 for } x.$$

$$2 = 2 \qquad \text{1 checks. } \checkmark$$

$$|(6)^2 - 3(6)| \stackrel{?}{=} -4(6) + 6 \qquad \text{Substitute 6 for } x.$$

$$18 \neq -18 \qquad \text{6 does not check.}$$

The solutions are $x = -3$ and $x = 1$.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $|x^2 + 4x| = 5x + 12$.

Applications

It would be virtually impossible to categorize all of the many different types of applications that involve nonlinear and nonquadratic models. However, you will see a variety of applications in the examples and exercises.

EXAMPLE 8 Reduced Rates

A ski club charters a bus for a ski trip at a cost of \$480. To lower the bus fare per skier, the club invites nonmembers to go along. After 5 nonmembers join the trip, the fare per skier decreases by \$4.80. How many club members are going on the trip?

Solution

Verbal Model: Cost per skier · Number of skiers = Cost of trip

Labels:

Cost of trip $= 480$ (dollars)

Number of ski club members $= x$ (people)

Number of skiers $= x + 5$ (people)

Original cost per member $= \dfrac{480}{x}$ (dollars per person)

Cost per skier $= \dfrac{480}{x} - 4.80$ (dollars per person)

Equation:

$$\left(\frac{480}{x} - 4.80\right)(x + 5) = 480$$

$$\left(\frac{480 - 4.8x}{x}\right)(x + 5) = 480 \qquad \text{Rewrite first factor.}$$

$$(480 - 4.8x)(x + 5) = 480x, \quad x \neq 0 \qquad \text{Multiply each side by } x.$$

$$480x + 2400 - 4.8x^2 - 24x = 480x \qquad \text{Multiply.}$$

$$-4.8x^2 - 24x + 2400 = 0 \qquad \text{Subtract } 480x \text{ from each side.}$$

$$x^2 + 5x - 500 = 0 \qquad \text{Divide each side by } -4.8.$$

$$(x + 25)(x - 20) = 0 \qquad \text{Factor.}$$

$$x + 25 = 0 \implies x = -25 \qquad \text{Set 1st factor equal to 0.}$$

$$x - 20 = 0 \implies x = 20 \qquad \text{Set 2nd factor equal to 0.}$$

Only the positive value of x makes sense in the context of the problem, so you can conclude that 20 ski club members are going on the trip. Check this in the original statement of the problem, as follows.

Check

$$\left(\frac{480}{20} - 4.80\right)(20 + 5) \stackrel{?}{=} 480 \qquad \text{Substitute 20 for } x.$$

$$(24 - 4.80)25 \stackrel{?}{=} 480 \qquad \text{Simplify.}$$

$$480 = 480 \qquad \text{20 checks. ✓}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

A high school charters a bus for \$560 to take a group of students to an observatory. When 8 more students join the trip, the cost per student decreases by \$3.50. How many students were in the original group?

EXAMPLE 9 Compound Interest

When you were born, your grandparents deposited \$5000 in a savings account earning interest compounded quarterly. On your 25th birthday, the balance of the account is \$25,062.59. What is the annual interest rate of the account?

Solution

Formula: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

REMARK Recall that the formula for interest that is compounded n times per year is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

In this formula, A is the balance in the account, P is the principal (or original deposit), r is the annual interest rate (in decimal form), n is the number of compoundings per year, and t is the time in years.

Labels:

Balance $= A = 25{,}062.59$	(dollars)
Principal $= P = 5000$	(dollars)
Time $= t = 25$	(years)
Compoundings per year $= n = 4$	(compoundings per year)
Annual interest rate $= r$	(percent in decimal form)

Equation:

$$25{,}062.59 = 5000\left(1 + \frac{r}{4}\right)^{4(25)} \quad \text{Substitute.}$$

$$\frac{25{,}062.59}{5000} = \left(1 + \frac{r}{4}\right)^{100} \quad \text{Divide each side by 5000.}$$

$$5.0125 \approx \left(1 + \frac{r}{4}\right)^{100} \quad \text{Use a calculator.}$$

$$(5.0125)^{1/100} \approx 1 + \frac{r}{4} \quad \text{Raise each side to reciprocal power.}$$

$$1.01625 \approx 1 + \frac{r}{4} \quad \text{Use a calculator.}$$

$$0.01625 \approx \frac{r}{4} \quad \text{Subtract 1 from each side.}$$

$$0.065 \approx r \quad \text{Multiply each side by 4.}$$

The annual interest rate is about 0.065, or 6.5%. Check this in the original statement of the problem.

With compound interest, it is beneficial to begin saving for retirement as soon as you can. Investing \$10,000 in an account earning 5.5% interest compounded quarterly yields \$123,389.89 after 46 years, but only \$60,656.10 after 33 years.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

A deposit of \$2500 in a mutual fund reaches a balance of \$3544.06 after 5 years. What annual interest rate on a certificate of deposit compounded monthly would yield an equivalent return?

Summarize (Section 1.6)

1. Describe how to solve a polynomial equation by factoring *(page 121)*. For examples of solving polynomial equations by factoring, see Examples 1–3.
2. Describe how to solve a radical equation *(page 123)*. For an example of solving equations involving radicals, see Example 4.
3. Describe how to solve an absolute value equation *(page 125)*. For an example of solving an absolute value equation, see Example 7.
4. Describe a real-life example that uses a nonlinear and nonquadratic model *(pages 126 and 127, Examples 8 and 9)*.

Sergey Mironov/Shutterstock.com

1.6 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. The general form of a ________ equation in x is $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 = 0$.
2. An equation is of ________ ________ when it can be written in the form $au^2 + bu + c = 0$.
3. A ________ equation is an equation that involves one or more radical expressions.
4. An ________ ________ equation is an equation that involves one or more absolute value expressions.

Skills and Applications

Solving a Polynomial Equation

In Exercises 5–16, solve the equation. Check your solutions.

5. $6x^4 - 14x^2 = 0$
6. $36x^3 - 100x = 0$
7. $x^4 - 81 = 0$
8. $x^6 - 64 = 0$
9. $x^3 + 512 = 0$
10. $27x^3 - 343 = 0$
11. $5x^3 + 30x^2 + 45x = 0$
12. $9x^4 - 24x^3 + 16x^2 = 0$
13. $x^3 - 3x^2 - x = -3$
14. $x^3 + 2x^2 + 3x = -6$
15. $x^4 - x^3 + x = 1$
16. $x^4 + 2x^3 - 8x = 16$

Solving an Equation of Quadratic Type

In Exercises 17–30, solve the equation. Check your solutions.

17. $x^4 - 4x^2 + 3 = 0$
18. $x^4 + 5x^2 - 36 = 0$
19. $4x^4 - 65x^2 + 16 = 0$
20. $36t^4 + 29t^2 - 7 = 0$
21. $x^6 + 7x^3 - 8 = 0$
22. $x^6 + 3x^3 + 2 = 0$
23. $\frac{1}{x^2} + \frac{8}{x} + 15 = 0$
24. $1 + \frac{3}{x} = \frac{2}{x^2}$
25. $2\left(\frac{x}{x+2}\right)^2 - 3\left(\frac{x}{x+2}\right) - 2 = 0$
26. $6\left(\frac{x}{x+1}\right)^2 + 5\left(\frac{x}{x+1}\right) - 6 = 0$
27. $2x + 9\sqrt{x} = 5$
28. $6x - 7\sqrt{x} - 3 = 0$
29. $3x^{1/3} + 2x^{2/3} = 5$
30. $9t^{2/3} + 24t^{1/3} = -16$

Solving a Radical Equation

In Exercises 31–44, solve the equation. Check your solutions.

31. $\sqrt{3x} - 12 = 0$
32. $7\sqrt{x} - 4 = 0$
33. $\sqrt{x - 10} - 4 = 0$
34. $\sqrt{5 - x} - 3 = 0$
35. $\sqrt[3]{2x + 5} + 3 = 0$
36. $\sqrt[3]{3x + 1} - 5 = 0$
37. $-\sqrt{26 - 11x} + 4 = x$
38. $x + \sqrt{31 - 9x} = 5$
39. $\sqrt{x} - \sqrt{x - 5} = 1$
40. $\sqrt{x} + \sqrt{x - 20} = 10$
41. $\sqrt{x + 5} + \sqrt{x - 5} = 10$
42. $2\sqrt{x + 1} - \sqrt{2x + 3} = 1$
43. $\sqrt{4\sqrt{4x + 9}} = \sqrt{8x + 2}$
44. $\sqrt{16 + 9\sqrt{x}} = 4 + \sqrt{x}$

Solving an Equation Involving a Rational Exponent

In Exercises 45–50, solve the equation. Check your solutions.

45. $(x - 5)^{3/2} = 8$
46. $(x + 2)^{2/3} = 9$
47. $(x^2 - 5)^{3/2} = 27$
48. $(x^2 - x - 22)^{3/2} = 27$
49. $3x(x - 1)^{1/2} + 2(x - 1)^{3/2} = 0$
50. $4x^2(x - 1)^{1/3} + 6x(x - 1)^{4/3} = 0$

Solving a Rational Equation

In Exercises 51–58, solve the equation. Check your solutions.

51. $x = \frac{3}{x} + \frac{1}{2}$
52. $\frac{4}{x} - \frac{5}{3} = \frac{x}{6}$
53. $\frac{1}{x} - \frac{1}{x + 1} = 3$
54. $\frac{4}{x + 1} - \frac{3}{x + 2} = 1$
55. $\frac{30 - x}{x} = x$
56. $4x + 1 = \frac{3}{x}$
57. $\frac{x}{x^2 - 4} + \frac{1}{x + 2} = 3$
58. $\frac{x + 1}{3} - \frac{x + 1}{x + 2} = 0$

Solving an Absolute Value Equation

In Exercises 59–64, solve the equation. Check your solutions.

59. $|2x - 5| = 11$
60. $|3x + 2| = 7$
61. $|x| = x^2 + x - 24$
62. $|x^2 + 6x| = 3x + 18$
63. $|x + 1| = x^2 - 5$
64. $|x - 15| = x^2 - 15x$

Graphical Analysis

In Exercises 65–76, (a) use a graphing utility to graph the equation, (b) use the graph to approximate any x-intercepts of the graph, (c) set $y = 0$ and solve the resulting equation, and (d) compare the result of part (c) with the x-intercepts of the graph.

65. $y = x^3 - 2x^2 - 3x$
66. $y = 2x^4 - 15x^3 + 18x^2$
67. $y = x^4 - 10x^2 + 9$
68. $y = x^4 - 29x^2 + 100$
69. $y = \sqrt{11x - 30} - x$
70. $y = 2x - \sqrt{15 - 4x}$
71. $y = \sqrt{7x + 36} - \sqrt{5x + 16} - 2$
72. $y = 3\sqrt{x} - \frac{4}{\sqrt{x}} - 4$
73. $y = \frac{1}{x} - \frac{4}{x - 1} - 1$
74. $y = x + \frac{9}{x + 1} - 5$
75. $y = |x + 1| - 2$
76. $y = |x - 2| - 3$

Solving an Equation **In Exercises 77–84, find the real solutions of the equation. (Round your answers to three decimal places.)**

77. $3.2x^4 - 1.5x^2 = 2.1$ **78.** $0.1x^4 - 2.4x^2 = 3.6$

79. $7.08x^6 + 4.15x^3 = 9.6$ **80.** $5.25x^6 - 0.2x^3 = 1.55$

81. $1.8x - 6\sqrt{x} = 5.6$

82. $2.4x - 12.4\sqrt{x} = -0.28$

83. $4x^{2/3} + 8x^{1/3} = -3.6$ **84.** $8.4x^{2/3} - 1.2x^{1/3} = 24$

Think About It **In Exercises 85–94, write an equation that has the given solutions. (There are many correct answers.)**

85. $-4, 7$ **86.** $0, 2, 9$

87. $-\frac{7}{3}, \frac{6}{7}$ **88.** $-\frac{1}{8}, -\frac{4}{5}$

89. $\sqrt{3}, -\sqrt{3}, 4$ **90.** $2\sqrt{7}, -\sqrt{7}$

91. $i, -i$ **92.** $2i, -2i$

93. $-1, 1, i, -i$ **94.** $4i, -4i, 6, -6$

95. **Chartering a Bus** A college charters a bus for \$1700 to take a group of students to see a Broadway production. When 6 more students join the trip, the cost per student decreases by \$7.50. How many students were in the original group?

96. **Renting an Apartment** Three students plan to share equally in the rent for an apartment. When they add a fourth student, the cost per student decreases by \$75 per month. How much would the monthly rent have been per student with only three students?

97. **Average Airspeed** An airline runs a commuter flight between Portland, Oregon, and Seattle, Washington, which are 145 miles apart. An increase of 40 miles per hour in the average speed of the plane would decrease the travel time by 12 minutes. What average airspeed is required to obtain this decrease in travel time?

98. **Average Speed** A family drives 1080 miles to a vacation lodge. On the return trip, it takes the family $2\frac{1}{2}$ hours longer, traveling at an average speed that is 6 miles per hour slower, to drive the same distance. Determine the average speed on the way to the lodge.

99. **Compound Interest** A deposit of \$2500 in a savings account reaches a balance of \$3052.49 after 5 years. The interest on the account is compounded monthly. What is the annual interest rate of the account?

100. **Compound Interest** A deposit of \$3000 in a savings account reaches a balance of \$4042.05 after 6 years. The interest on the account is compounded quarterly. What is the annual interest rate of the account?

xtrekx/Shutterstock.com

101. **Airline Passengers** An airline offers daily flights between Chicago and Denver. The total monthly cost C (in millions of dollars) of these flights can be modeled by

$$C = \sqrt{0.2x + 1}$$

where x is the number of passengers (in thousands). The total cost of the flights for June is 2.5 million dollars. How many passengers flew in June?

102. **Number of Doctors** The numbers of medical doctors D (in thousands) in the United States from 2000 through 2009 can be modeled by

$$D = \sqrt{664{,}267.0 + 31{,}003.65t}, \quad 0 \le t \le 9$$

where t represents the year, with $t = 0$ corresponding to 2000. *(Source: American Medical Association)*

(a) In which year did the number of medical doctors reach 925,000?

(b) Use the model to predict when the number of medical doctors will reach 1,100,000. Is this prediction reasonable? Explain.

103. **Saturated Steam**

The temperature T (in degrees Fahrenheit) of saturated steam increases as pressure increases. This relationship can be approximated by the model

$$T = 75.82 - 2.11x + 43.51\sqrt{x}$$

$$5 \le x \le 40$$

where x is the absolute pressure (in pounds per square inch).

(a) The temperature of saturated steam at sea level is 212°F. Find the absolute pressure at this temperature.

(b) Use a graphing utility to verify your solution for part (a).

104. **Voting-Age Population** The total voting-age populations P (in millions) in the United States from 1990 through 2010 can be modeled by

$$P = \frac{181.34 + 0.788t}{1 - 0.007t}, \quad 0 \le t \le 20$$

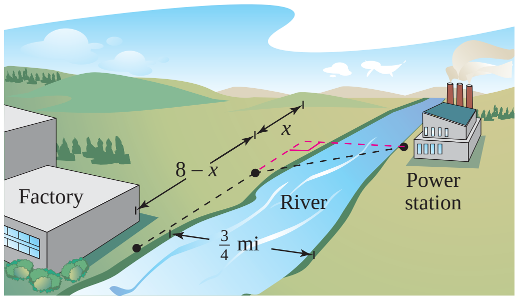
where t represents the year, with $t = 0$ corresponding to 1990. *(Source: U.S. Census Bureau)*

(a) In which year did the total voting-age population reach 210 million?

(b) Use the model to predict when the total voting-age population will reach 280 million. Is this prediction reasonable? Explain.

105. Demand The demand for a video game can be modeled by $p = 40 - \sqrt{0.01x + 1}$ where x is the number of units demanded per day and p is the price per unit. Find the demand when the price is \$37.55.

106. Power Line A power station is on one side of a river that is $\frac{3}{4}$ mile wide, and a factory is 8 miles downstream on the other side of the river. It costs \$24 per foot to run power lines over land and \$30 per foot to run them under water. The project's cost is \$1,098,662.40. Find the length x labeled in the figure.



107. Tiling a Floor Working alone, you can tile a floor in t hours. When you work with a friend, the time y (in hours) it takes to tile the floor satisfies the equation

$$\frac{1}{t} + \frac{1}{t+3} = \frac{1}{y}.$$

Find the time it takes you to tile the floor working alone when you and your friend can tile the floor in 2 hours working together.

108. Painting a Fence Working alone, you can paint a fence in t hours. When you work with a friend, the time y (in hours) it takes to paint the fence satisfies the equation

$$\frac{1}{t} + \frac{1}{t+2} = \frac{1}{y}.$$

Find the time it takes you to paint the fence working alone when you and your friend can paint the fence in 3 hours working together.

109. A Person's Tangential Speed in a Rotor

Solve for g: $v = \sqrt{\dfrac{gR}{\mu s}}$

110. Inductance

Solve for Q: $i = \pm\sqrt{\dfrac{1}{LC}}\sqrt{Q^2 - q}$

Exploration

True or False? **In Exercises 111 and 112, determine whether the statement is true or false. Justify your answer.**

111. An equation can never have more than one extraneous solution.

112. The equation $\sqrt{x + 10} - \sqrt{x - 10} = 0$ has no solution.

Distance **In Exercises 113 and 114, find x such that the distance between the given points is 13. Explain your results.**

113. $(1, 2), (x, -10)$ **114.** $(-8, 0), (x, 5)$

Distance **In Exercises 115 and 116, find y such that the distance between the given points is 17. Explain your results.**

115. $(0, 0), (8, y)$ **116.** $(-8, 4), (7, y)$

Solving an Absolute Value Equation **In Exercises 117 and 118, consider an equation of the form**

$$x + |x - a| = b$$

where a and b are constants.

117. Find a and b when the solution of the equation is $x = 9$. (There are many correct answers.)

118. Write a short paragraph listing the steps required to solve this absolute value equation and explain why it is important to check your solutions.

Solving a Radical Equation **In Exercises 119 and 120, consider an equation of the form**

$$x + \sqrt{x - a} = b$$

where a and b are constants.

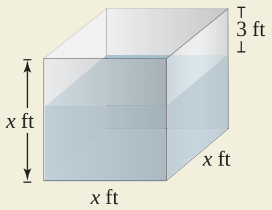
119. Find a and b when the solution of the equation is $x = 20$. (There are many correct answers.)

120. Write a short paragraph listing the steps required to solve this radical equation and explain why it is important to check your solutions.

121. Error Analysis Find the error(s) in the solution.

$$\sqrt{3x} = \sqrt{7x + 4}$$
$$3x^2 = 7x + 4$$
$$x = \frac{-7 \pm \sqrt{7^2 - 4(3)(4)}}{2(3)}$$
$$x = -1 \text{ and } x = -\frac{4}{3}$$

122. HOW DO YOU SEE IT? The figure shows a glass cube partially filled with water.



(a) What does the expression $x^2(x - 3)$ represent?

(b) Given $x^2(x - 3) = 320$, explain how you can find the capacity of the cube.

1.7 Linear Inequalities in One Variable

You can use inequalities to model and solve real-life problems. For instance, in Exercise 114 on page 139, you will use an absolute value inequality to analyze the cover layer thickness of a Blu-ray Disc™.

- **Represent solutions of linear inequalities in one variable.**
- **Use properties of inequalities to create equivalent inequalities.**
- **Solve linear inequalities in one variable.**
- **Solve absolute value inequalities.**
- **Use inequalities to model and solve real-life problems.**

Introduction

Simple inequalities were discussed in Section P.1. There, you used the inequality symbols $<$, $\leq$, $>$, and $\geq$ to compare two numbers and to denote subsets of real numbers. For instance, the simple inequality

$$x \geq 3$$

denotes all real numbers x that are greater than or equal to 3.

Now, you will expand your work with inequalities to include more involved statements such as

$$5x - 7 < 3x + 9 \quad \text{and} \quad -3 \leq 6x - 1 < 3.$$

As with an equation, you **solve an inequality** in the variable x by finding all values of x for which the inequality is true. Such values are **solutions** and are said to **satisfy** the inequality. The set of all real numbers that are solutions of an inequality is the **solution set** of the inequality. For instance, the solution set of

$$x + 1 < 4$$

is all real numbers that are less than 3.

The set of all points on the real number line that represents the solution set is the **graph of the inequality.** Graphs of many types of inequalities consist of intervals on the real number line. See Section P.1 to review the nine basic types of intervals on the real number line. Note that each type of interval can be classified as *bounded* or *unbounded.*

EXAMPLE 1 Intervals and Inequalities

Write an inequality to represent each interval. Then state whether the interval is bounded or unbounded.

a. $(-3, 5]$ **b.** $(-3, \infty)$

c. $[0, 2]$ **d.** $(-\infty, \infty)$

Solution

a. $(-3, 5]$ corresponds to $-3 < x \leq 5$. Bounded

b. $(-3, \infty)$ corresponds to $x > -3$. Unbounded

c. $[0, 2]$ corresponds to $0 \leq x \leq 2$. Bounded

d. $(-\infty, \infty)$ corresponds to $-\infty < x < \infty$. Unbounded

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write an inequality to represent each interval. Then state whether the interval is bounded or unbounded.

a. $[-1, 3]$ **b.** $(-1, 6)$

c. $(-\infty, 4)$ **d.** $[0, \infty)$

Jaroslav74/Shutterstock.com

Properties of Inequalities

The procedures for solving linear inequalities in one variable are much like those for solving linear equations. To isolate the variable, you can make use of the **properties of inequalities.** These properties are similar to the properties of equality, but there are two important exceptions. When each side of an inequality is multiplied or divided by a negative number, the direction of the inequality symbol must be reversed. Here is an example.

$$-2 < 5$$ Original inequality

$$(-3)(-2) > (-3)(5)$$ Multiply each side by -3 and reverse inequality symbol.

$$6 > -15$$ Simplify.

Notice that when you do not reverse the inequality symbol in the example above, you obtain the false statement

$$6 < -15.$$ False statement

Two inequalities that have the same solution set are **equivalent.** For instance, the inequalities

$$x + 2 < 5$$

and

$$x < 3$$

are equivalent. To obtain the second inequality from the first, you can subtract 2 from each side of the inequality. The following list describes the operations that can be used to create equivalent inequalities.

Properties of Inequalities

Let a, b, c, and d be real numbers.

1. Transitive Property

$$a < b \text{ and } b < c \Rightarrow a < c$$

2. Addition of Inequalities

$$a < b \text{ and } c < d \Rightarrow a + c < b + d$$

3. Addition of a Constant

$$a < b \Rightarrow a + c < b + c$$

4. Multiplication by a Constant

$$\text{For } c > 0,\ a < b \Rightarrow ac < bc$$

$$\text{For } c < 0,\ a < b \Rightarrow ac > bc$$ Reverse the inequality symbol.

Each of the properties above is true when the symbol $<$ is replaced by $\le$ and the symbol $>$ is replaced by $\ge$. For instance, another form of the multiplication property is shown below.

$$\text{For } c > 0,\ a \le b \Rightarrow ac \le bc$$

$$\text{For } c < 0,\ a \le b \Rightarrow ac \ge bc$$

On your own, try to verify each of the properties of inequalities by using several examples with real numbers.

Solving a Linear Inequality in One Variable

The simplest type of inequality is a **linear inequality** in one variable. For instance,

$$2x + 3 > 4$$

is a linear inequality in x.

EXAMPLE 2 Solving a Linear Inequality

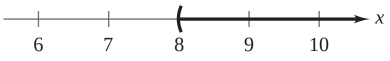
Solve $5x - 7 > 3x + 9$. Then graph the solution set.

REMARK Checking the solution set of an inequality is not as simple as checking the solutions of an equation. You can, however, get an indication of the validity of a solution set by substituting a few convenient values of x. For instance, in Example 2, try substituting $x = 5$ and $x = 10$ into the original inequality.

Solution

$$\begin{aligned} 5x - 7 &> 3x + 9 && \text{Write original inequality.} \\ 2x - 7 &> 9 && \text{Subtract } 3x \text{ from each side.} \\ 2x &> 16 && \text{Add 7 to each side.} \\ x &> 8 && \text{Divide each side by 2.} \end{aligned}$$

The solution set is all real numbers that are greater than 8, which is denoted by $(8, \infty)$. The graph of this solution set is shown below. Note that a parenthesis at 8 on the real number line indicates that 8 *is not* part of the solution set.



Solution interval: $(8, \infty)$

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $7x - 3 \le 2x + 7$. Then graph the solution set.

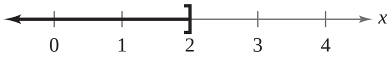
EXAMPLE 3 Solving a Linear Inequality

Solve $1 - \frac{3}{2}x \ge x - 4$.

Algebraic Solution

$$\begin{aligned} 1 - \frac{3x}{2} &\ge x - 4 && \text{Write original inequality.} \\ 2 - 3x &\ge 2x - 8 && \text{Multiply each side by 2.} \\ 2 - 5x &\ge -8 && \text{Subtract } 2x \text{ from each side.} \\ -5x &\ge -10 && \text{Subtract 2 from each side.} \\ x &\le 2 && \text{Divide each side by } -5 \text{ and reverse the inequality symbol.} \end{aligned}$$

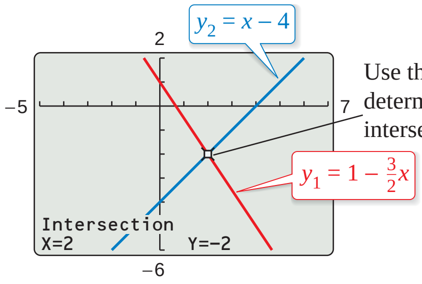
The solution set is all real numbers that are less than or equal to 2, which is denoted by $(-\infty, 2]$. The graph of this solution set is shown below. Note that a bracket at 2 on the real number line indicates that 2 *is* part of the solution set.



Solution interval: $(-\infty, 2]$

Graphical Solution

Use a graphing utility to graph $y_1 = 1 - \frac{3}{2}x$ and $y_2 = x - 4$ in the same viewing window, as shown below.



Use the *intersect* feature to determine that the graphs intersect at $(2, -2)$.

The graph of y_1 lies above the graph of y_2 to the left of their point of intersection, which implies that $y_1 \ge y_2$ for all $x \le 2$.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $2 - \frac{5}{3}x > x - 6$ (a) algebraically and (b) graphically.

Sometimes it is possible to write two inequalities as a **double inequality.** For instance, you can write the two inequalities

$$-4 \le 5x - 2$$

and

$$5x - 2 < 7$$

more simply as

$$-4 \le 5x - 2 < 7. \qquad \text{Double inequality}$$

This form allows you to solve the two inequalities together, as demonstrated in Example 4.

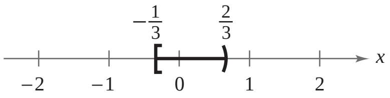
EXAMPLE 4 Solving a Double Inequality

Solve $-3 \le 6x - 1 < 3$. Then graph the solution set.

Solution To solve a double inequality, you can isolate x as the middle term.

$$\begin{aligned}
-3 \le 6x - 1 < 3 &\qquad \text{Write original inequality.}\\
-3 + 1 \le 6x - 1 + 1 < 3 + 1 &\qquad \text{Add 1 to each part.}\\
-2 \le 6x < 4 &\qquad \text{Simplify.}\\
\frac{-2}{6} \le \frac{6x}{6} < \frac{4}{6} &\qquad \text{Divide each part by 6.}\\
-\frac{1}{3} \le x < \frac{2}{3} &\qquad \text{Simplify.}
\end{aligned}$$

The solution set is all real numbers that are greater than or equal to $-\frac{1}{3}$ and less than $\frac{2}{3}$, which is denoted by $\left[-\frac{1}{3}, \frac{2}{3}\right)$. The graph of this solution set is shown below.



Solution interval: $\left[-\frac{1}{3}, \frac{2}{3}\right)$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $1 < 2x + 7 < 11$. Then graph the solution set.

You can solve the double inequality in Example 4 in two parts, as follows.

$$\begin{aligned}
-3 \le 6x - 1 &\quad \text{and} \quad & 6x - 1 < 3\\
-2 \le 6x & & 6x < 4\\
-\frac{1}{3} \le x & & x < \frac{2}{3}
\end{aligned}$$

The solution set consists of all real numbers that satisfy *both* inequalities. In other words, the solution set is the set of all values of x for which

$$-\frac{1}{3} \le x < \frac{2}{3}.$$

When combining two inequalities to form a double inequality, be sure that the inequalities satisfy the Transitive Property. For instance, it is *incorrect* to combine the inequalities $3 < x$ and $x \le -1$ as $3 < x \le -1$. This "inequality" is wrong because 3 is not less than -1.

Absolute Value Inequalities

Solving an Absolute Value Inequality

Let u be an algebraic expression and let a be a real number such that $a > 0$.

1. $|u| < a$ if and only if $-a < u < a$.
2. $|u| \le a$ if and only if $-a \le u \le a$.
3. $|u| > a$ if and only if $u < -a$ or $u > a$.
4. $|u| \ge a$ if and only if $u \le -a$ or $u \ge a$.

EXAMPLE 5 Solving an Absolute Value Inequality

Solve each inequality. Then graph the solution set.

a. $|x - 5| < 2$ **b.** $|x + 3| \ge 7$

Solution

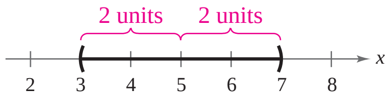
a. $|x - 5| < 2$ Write original inequality.

$-2 < x - 5 < 2$ Write equivalent inequalities.

$-2 + 5 < x - 5 + 5 < 2 + 5$ Add 5 to each part.

$3 < x < 7$ Simplify.

The solution set is all real numbers that are greater than 3 and less than 7, which is denoted by $(3, 7)$. The graph of this solution set is shown below. Note that the graph of the inequality can be described as all real numbers less than two units from 5.



$|x - 5| < 2$: Solutions lie inside $(3, 7)$.

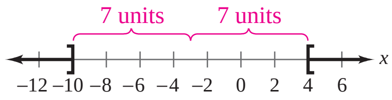
b. $|x + 3| \ge 7$ Write original inequality.

$x + 3 \le -7$ or $x + 3 \ge 7$ Write equivalent inequalities.

$x + 3 - 3 \le -7 - 3$ $x + 3 - 3 \ge 7 - 3$ Subtract 3 from each side.

$x \le -10$ $x \ge 4$ Simplify.

The solution set is all real numbers that are less than or equal to -10 *or* greater than or equal to 4, which is denoted by $(-\infty, -10] \cup [4, \infty)$. The symbol $\cup$ is the *union* symbol, which denotes the combining of two sets. The graph of this solution set is shown below. Note that the graph of the inequality can be described as all real numbers at least seven units from -3.



$|x + 3| \ge 7$: Solutions lie outside $(-10, 4)$.

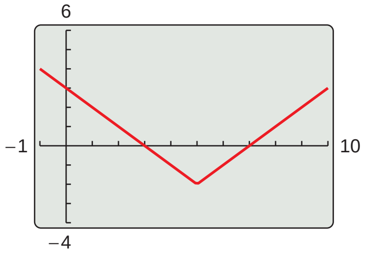
Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $|x - 20| \le 4$. Then graph the solution set.

TECHNOLOGY A graphing utility can be used to identify the solution set of an inequality. For instance, to find the solution set of $|x - 5| < 2$ (see Example 5a), rewrite the inequality as $|x - 5| - 2 < 0$, enter

$$Y1 = \text{abs}\,(X - 5) - 2$$

and press the *graph* key. The graph should look like the one shown below.



Notice that the graph is below the x-axis on the interval $(3, 7)$.

Applications

EXAMPLE 6 Comparative Shopping

Consider the two cell phone plans shown in Figure 1.15. How many *additional* minutes must you use in one month for plan B to cost more than plan A?

Cell Phone Plans
Plan A:
\$49.99 per month for 500 minutes plus \$0.40 for each additional minute
Plan B:
\$45.99 per month for 500 minutes plus \$0.45 for each additional minute

Figure 1.15

Solution Let m represent your additional minutes in one month. Write and solve an inequality.

$$0.45m + 45.99 > 0.40m + 49.99$$

$$0.05m > 4$$

$$m > 80$$

Plan B costs more when you use more than 80 additional minutes in one month.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Rework Example 6 when plan A costs \$54.99 per month for 500 minutes plus \$0.35 for each additional minute.

EXAMPLE 7 Accuracy of a Measurement

You buy chocolates that cost \$9.89 per pound. The scale used to weigh your bag is accurate to within $\frac{1}{32}$ pound. According to the scale, your bag weighs $\frac{1}{2}$ pound and costs \$4.95. How much might you have been undercharged or overcharged?

Solution Let x represent the actual weight of your bag. The difference of the actual weight and the weight shown on the scale is at most $\frac{1}{32}$ pound. That is, $\left|x - \frac{1}{2}\right| \le \frac{1}{32}$. You can solve this inequality as follows.

$$-\frac{1}{32} \le x - \frac{1}{2} \le \frac{1}{32}$$

$$\frac{15}{32} \le x \le \frac{17}{32}$$

The least your bag can weigh is $\frac{15}{32}$ pound, which would have cost \$4.64. The most the bag can weigh is $\frac{17}{32}$ pound, which would have cost \$5.25. So, you might have been overcharged by as much as \$0.31 or undercharged by as much as \$0.30.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Rework Example 7 when the scale is accurate to within $\frac{1}{64}$ pound.

Summarize (Section 1.7)

1. Describe how to use inequalities to represent intervals *(page 131, Example 1).*
2. State the properties of inequalities *(page 132).*
3. Describe how to solve a linear inequality *(pages 133 and 134, Examples 2–4).*
4. Describe how to solve an absolute value inequality *(page 135, Example 5).*
5. Describe a real-life example that uses an inequality *(page 136, Examples 6 and 7).*

1.7 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. The set of all real numbers that are solutions of an inequality is the ________ ________ of the inequality.
2. The set of all points on the real number line that represents the solution set of an inequality is the ________ of the inequality.
3. It is sometimes possible to write two inequalities as one inequality, called a ________ inequality.
4. The symbol ∪ is the ________ symbol, which denotes the combining of two sets.

Skills and Applications

Intervals and Inequalities **In Exercises 5–12, write an inequality that represents the interval. Then state whether the interval is bounded or unbounded.**

5. $[0, 9]$
6. $(-7, 4)$
7. $[-1, 5]$
8. $(2, 10]$
9. $(11, \infty)$
10. $[-5, \infty)$
11. $(-\infty, -2)$
12. $(-\infty, 7]$

Solving a Linear Inequality **In Exercises 13–42, solve the inequality. Then graph the solution set.**

13. $4x < 12$
14. $10x < -40$
15. $-2x > -3$
16. $-6x > 15$
17. $x - 5 \ge 7$
18. $x + 7 \le 12$
19. $2x + 7 < 3 + 4x$
20. $3x + 1 \ge 2 + x$
21. $2x - 1 \ge 1 - 5x$
22. $6x - 4 \le 2 + 8x$
23. $4 - 2x < 3(3 - x)$
24. $4(x + 1) < 2x + 3$
25. $\frac{3}{4}x - 6 \le x - 7$
26. $3 + \frac{2}{7}x > x - 2$
27. $\frac{1}{2}(8x + 1) \ge 3x + \frac{5}{2}$
28. $9x - 1 < \frac{3}{4}(16x - 2)$
29. $3.6x + 11 \ge -3.4$
30. $15.6 - 1.3x < -5.2$
31. $1 < 2x + 3 < 9$
32. $-9 \le -2x - 7 < 5$
33. $0 < 3(x + 7) \le 20$
34. $-1 \le -(x - 4) < 7$
35. $-4 < \frac{2x - 3}{3} < 4$
36. $0 \le \frac{x + 3}{2} < 5$
37. $-1 < \frac{-x - 2}{3} \le 1$
38. $-1 \le \frac{-3x + 5}{7} \le 2$
39. $\frac{3}{4} > x + 1 > \frac{1}{4}$
40. $-1 < 2 - \frac{x}{3} < 1$
41. $3.2 \le 0.4x - 1 \le 4.4$
42. $1.6 < 0.3x + 1 < 2.8$

Solving an Absolute Value Inequality **In Exercises 43–58, solve the inequality. Then graph the solution set. (Some inequalities have no solution.)**

43. $|x| < 5$
44. $|x| \ge 8$
45. $\left|\frac{x}{2}\right| > 1$
46. $\left|\frac{x}{5}\right| > 3$
47. $|x - 5| < -1$
48. $|x - 7| < -5$
49. $|x - 20| \le 6$
50. $|x - 8| \ge 0$
51. $|3 - 4x| \ge 9$
52. $|1 - 2x| < 5$
53. $\left|\frac{x - 3}{2}\right| \ge 4$
54. $\left|1 - \frac{2x}{3}\right| < 1$
55. $|9 - 2x| - 2 < -1$
56. $|x + 14| + 3 > 17$
57. $2|x + 10| \ge 9$
58. $3|4 - 5x| \le 9$

Graphical Analysis **In Exercises 59–68, use a graphing utility to graph the inequality and identify the solution set.**

59. $6x > 12$
60. $3x - 1 \le 5$
61. $5 - 2x \ge 1$
62. $20 < 6x - 1$
63. $4(x - 3) \le 8 - x$
64. $3(x + 1) < x + 7$
65. $|x - 8| \le 14$
66. $|2x + 9| > 13$
67. $2|x + 7| \ge 13$
68. $\frac{1}{2}|x + 1| \le 3$

Graphical Analysis **In Exercises 69–74, use a graphing utility to graph the equation. Use the graph to approximate the values of x that satisfy each inequality.**

	Equation	Inequalities	
69.	$y = 2x - 3$	(a) $y \ge 1$	(b) $y \le 0$
70.	$y = \frac{2}{3}x + 1$	(a) $y \le 5$	(b) $y \ge 0$
71.	$y = -\frac{1}{2}x + 2$	(a) $0 \le y \le 3$	(b) $y \ge 0$
72.	$y = -3x + 8$	(a) $-1 \le y \le 3$	(b) $y \le 0$
73.	$y = \lvert x - 3\rvert$	(a) $y \le 2$	(b) $y \ge 4$
74.	$y = \left\lvert\frac{1}{2}x + 1\right\rvert$	(a) $y \le 4$	(b) $y \ge 1$

Finding an Interval **In Exercises 75–80, find the interval on the real number line for which the radicand is nonnegative.**

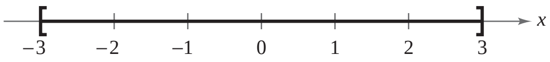
75. $\sqrt{x - 5}$
76. $\sqrt{x - 10}$
77. $\sqrt{x + 3}$
78. $\sqrt{3 - x}$
79. $\sqrt[4]{7 - 2x}$
80. $\sqrt[4]{6x + 15}$

81. **Think About It** The graph of $|x - 5| < 3$ can be described as all real numbers less than three units from 5. Give a similar description of $|x - 10| < 8$.

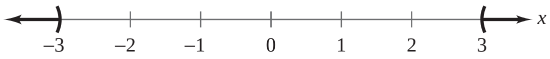
82. **Think About It** The graph of $|x - 2| > 5$ can be described as all real numbers more than five units from 2. Give a similar description of $|x - 8| > 4$.

Using Absolute Value **In Exercises 83–90, use absolute value notation to define the interval (or pair of intervals) on the real number line.**

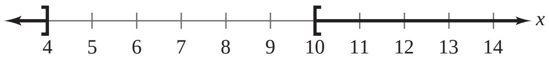
83.



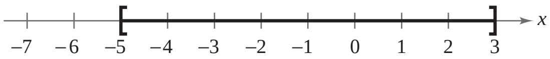
84.



85.



86.



87. All real numbers at least 10 units from 12

88. All real numbers at least five units from 8

89. All real numbers more than four units from -3

90. All real numbers no more than seven units from -6

Writing an Inequality **In Exercises 91–94, write an inequality to describe the situation.**

91. A company expects its earnings per share E for the next quarter to be no less than \$4.10 and no more than \$4.25.

92. The estimated daily oil production p at a refinery is greater than 2 million barrels but less than 2.4 million barrels.

93. The return r on an investment is expected to be no more than 8%.

94. The net income I of a company is expected to be no less than \$239 million.

Physiology **The maximum heart rate r (in beats per minute) of a person in normal health is related to the person's age A (in years) by the equation**

$$r = 220 - A.$$

In Exercises 95 and 96, determine the interval in which the person's heart rate is from 50% to 85% of the maximum heart rate. ***(Source: American Heart Association)***

95. a 20-year-old

96. a 40-year-old

97. **Job Offers** You are considering two job offers. The first job pays \$13.50 per hour. The second job pays \$9.00 per hour plus \$0.75 per unit produced per hour. How many units must you produce per hour for the second job to pay more per hour than the first job?

98. **Job Offers** You are considering two job offers. The first job pays \$3000 per month. The second job pays \$1000 per month plus a commission of 4% of your gross sales. How much must you earn in gross sales for the second job to pay more per month than the first job?

99. **Investment** For what annual interest rates will an investment of \$1000 grow to more than \$1062.50 in 2 years? $[A = P(1 + rt)]$

100. **Investment** For what annual interest rates will an investment of \$750 grow to more than \$825 in 2 years? $[A = P(1 + rt)]$

101. **Cost, Revenue, and Profit** The revenue from selling x units of a product is $R = 115.95x$. The cost of producing x units is $C = 95x + 750$. To obtain a profit, the revenue must be greater than the cost. For what values of x will this product return a profit?

102. **Cost, Revenue, and Profit** The revenue from selling x units of a product is $R = 24.55x$. The cost of producing x units is $C = 15.4x + 150{,}000$. To obtain a profit, the revenue must be greater than the cost. For what values of x will this product return a profit?

103. **Daily Sales** A doughnut shop sells a dozen doughnuts for \$7.95. Beyond the fixed costs (rent, utilities, and insurance) of \$165 per day, it costs \$1.45 for enough materials (flour, sugar, and so on) and labor to produce a dozen doughnuts. The daily profit from doughnut sales varies between \$400 and \$1200. Between what levels (in dozens of doughnuts) do the daily sales vary?

104. **Weight Loss Program** A person enrolls in a diet and exercise program that guarantees a loss of at least $1\frac{1}{2}$ pounds per week. The person's weight at the beginning of the program is 164 pounds. Find the maximum number of weeks before the person attains a goal weight of 128 pounds.

105. **Data Analysis: IQ Scores and GPA** The admissions office of a college wants to determine whether there is a relationship between IQ scores x and grade-point averages y after the first year of school. An equation that models the data the admissions office obtained is $y = 0.067x - 5.638$.

(a) Use a graphing utility to graph the model.

(b) Use the graph to estimate the values of x that predict a grade-point average of at least 3.0.

(c) Verify your estimate from part (b) algebraically.

(d) List other factors that may influence GPA.

106. **Data Analysis: Weightlifting** You want to determine whether there is a relationship between an athlete's weight x (in pounds) and the athlete's maximum bench-press weight y (in pounds). An equation that models the data you obtained is $y = 1.3x - 36$.

(a) Use a graphing utility to graph the model.

(b) Use the graph to estimate the values of x that predict a maximum bench-press weight of at least 200 pounds.

(c) Verify your estimate from part (b) algebraically.

(d) List other factors that might influence an individual's maximum bench-press weight.

107. **Teachers' Salaries** The average salaries S (in thousands of dollars) for public elementary school teachers in the United States from 2001 through 2011 can be modeled by

$$S = 1.36t + 41.1, \quad 1 \le t \le 11$$

where t represents the year, with $t = 1$ corresponding to 2001. *(Source: National Education Association)*

(a) According to the model, when was the average salary at least \$45,000, but no more than \$50,000?

(b) Use the model to predict when the average salary will exceed \$62,000.

108. **Milk Production** Milk production M (in billions of pounds) in the United States from 2002 through 2010 can be modeled by

$$M = 3.24t + 161.5, \quad 2 \le t \le 10$$

where t represents the year, with $t = 2$ corresponding to 2002. *(Source: U.S. Department of Agriculture)*

(a) According to the model, when was the annual milk production greater than 178 billion pounds, but no more than 190 billion pounds?

(b) Use the model to predict when milk production will exceed 225 billion pounds.

109. **Geometry** The side of a square is measured as 10.4 inches with a possible error of $\frac{1}{16}$ inch. Using these measurements, determine the interval containing the possible areas of the square.

110. **Geometry** The side of a square is measured as 24.2 centimeters with a possible error of 0.25 centimeter. Using these measurements, determine the interval containing the possible areas of the square.

111. **Accuracy of Measurement** You stop at a self-service gas station to buy 15 gallons of 87-octane gasoline at \$3.61 per gallon. The gas pump is accurate to within $\frac{1}{10}$ gallon. How much might you be undercharged or overcharged?

112. **Accuracy of Measurement** You buy six T-bone steaks that cost \$8.99 per pound. The weight that is listed on the package is 5.72 pounds. The scale that weighed the package is accurate to within $\frac{1}{32}$ pound. How much might you be undercharged or overcharged?

113. **Time Study** A time study was conducted to determine the length of time required to perform a particular task in a manufacturing process. The times required by approximately two-thirds of the workers in the study satisfied the inequality

$$|t - 15.6| \le 1.9$$

where t is time in minutes. Determine the interval in which these times lie.

Jaroslav74/Shutterstock.com

114. **Error Tolerance**

The protective cover layer of a Blu-ray Disc™ is 100 micrometers thick with an error tolerance of 3 micrometers. Write an absolute value inequality for the possible thicknesses of the cover layer. Then graph the solution set. *(Source: Blu-ray Disc™ Association)*

Exploration

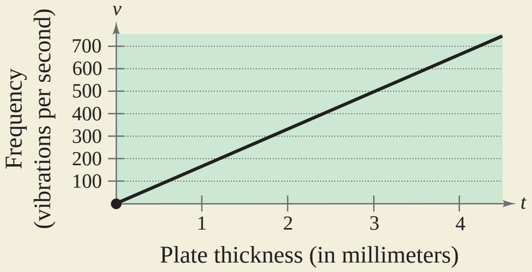
True or False? **In Exercises 115–117, determine whether the statement is true or false. Justify your answer.**

115. If a, b, and c are real numbers, and $a < b$, then $a + c < b + c$.

116. If a, b, and c are real numbers, and $a \le b$, then $ac \le bc$.

117. If $-10 \le x \le 8$, then $-10 \ge -x$ and $-x \ge -8$.

118. **HOW DO YOU SEE IT?** Michael Kasha of Florida State University used physics and mathematics to design a new classical guitar. The model he used for the frequency of the vibrations on a circular plate was $v = (2.6t/d^2)\sqrt{E/\rho}$, where v is the frequency (in vibrations per second), t is the plate thickness (in millimeters), d is the diameter of the plate, E is the elasticity of the plate material, and ρ is the density of the plate material. For fixed values of d, E, and ρ, the graph of the equation is a line (see figure).



(a) Estimate the frequency when the plate thickness is 2 millimeters.

(b) Approximate the interval for the frequency when the plate thickness is greater than or equal to 0 millimeters and less than 3 millimeters.

1.8 Other Types of Inequalities

You can use inequalities to model and solve real-life problems. For instance, in Exercises 73 and 74 on page 148, you will use a polynomial inequality to model the height of a projectile.

- **Solve polynomial inequalities.**
- **Solve rational inequalities.**
- **Use inequalities to model and solve real-life problems.**

Polynomial Inequalities

To solve a polynomial inequality such as $x^2 - 2x - 3 < 0$, you can use the fact that a polynomial can change signs only at its zeros (the x-values that make the polynomial equal to zero). Between two consecutive zeros, a polynomial must be entirely positive or entirely negative. This means that when the real zeros of a polynomial are put in order, they divide the real number line into intervals in which the polynomial has no sign changes. These zeros are the **key numbers** of the inequality, and the resulting intervals are the **test intervals** for the inequality. For instance, the polynomial above factors as

$$x^2 - 2x - 3 = (x + 1)(x - 3)$$

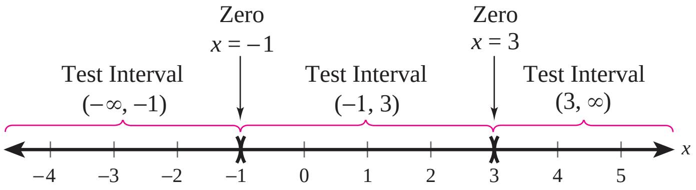
and has two zeros,

$$x = -1 \quad \text{and} \quad x = 3.$$

These zeros divide the real number line into three test intervals:

$$(-\infty, -1), \quad (-1, 3), \quad \text{and} \quad (3, \infty). \qquad \text{(See figure.)}$$

So, to solve the inequality $x^2 - 2x - 3 < 0$, you need only test one value from each of these test intervals. When a value from a test interval satisfies the original inequality, you can conclude that the interval is a solution of the inequality.



Three test intervals for $x^2 - 2x - 3$

You can use the same basic approach to determine the test intervals for any polynomial.

Finding Test Intervals for a Polynomial

To determine the intervals on which the values of a polynomial are entirely negative or entirely positive, use the following steps.

1. Find all real zeros of the polynomial, and arrange the zeros in increasing order (from least to greatest). These zeros are the key numbers of the polynomial.
2. Use the key numbers of the polynomial to determine its test intervals.
3. Choose one representative x-value in each test interval and evaluate the polynomial at that value. When the value of the polynomial is negative, the polynomial has negative values for every x-value in the interval. When the value of the polynomial is positive, the polynomial has positive values for every x-value in the interval.

Vakhrushev Pavel/Shutterstock.com

EXAMPLE 1 Solving a Polynomial Inequality

Solve $x^2 - x - 6 < 0$. Then graph the solution set.

ALGEBRA HELP You can review the techniques for factoring polynomials in Section P.4.

Solution By factoring the polynomial as

$$x^2 - x - 6 = (x + 2)(x - 3)$$

you can see that the key numbers are $x = -2$ and $x = 3$. So, the polynomial's test intervals are

$$(-\infty, -2), \quad (-2, 3), \quad \text{and} \quad (3, \infty). \qquad \text{Test intervals}$$

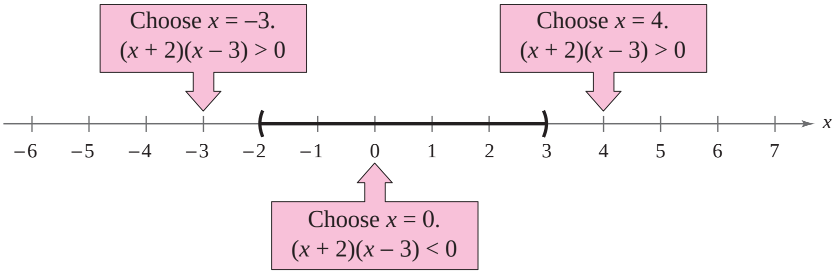
In each test interval, choose a representative x-value and evaluate the polynomial.

Test Interval	x-Value	Polynomial Value	Conclusion
$(-\infty, -2)$	$x = -3$	$(-3)^2 - (-3) - 6 = 6$	Positive
$(-2, 3)$	$x = 0$	$(0)^2 - (0) - 6 = -6$	Negative
$(3, \infty)$	$x = 4$	$(4)^2 - (4) - 6 = 6$	Positive

From this, you can conclude that the inequality is satisfied for all x-values in $(-2, 3)$. This implies that the solution of the inequality

$$x^2 - x - 6 < 0$$

is the interval $(-2, 3)$, as shown below. Note that the original inequality contains a "less than" symbol. This means that the solution set does not contain the endpoints of the test interval $(-2, 3)$.



Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Solve $x^2 - x - 20 < 0$. Then graph the solution set.

As with linear inequalities, you can check the reasonableness of a solution by substituting x-values into the original inequality. For instance, to check the solution found in Example 1, try substituting several x-values from the interval $(-2, 3)$ into the inequality

$$x^2 - x - 6 < 0.$$

Regardless of which x-values you choose, the inequality should be satisfied.

You can also use a graph to check the result of Example 1. Sketch the graph of

$$y = x^2 - x - 6$$

as shown in Figure 1.16. Notice that the graph is below the x-axis on the interval $(-2, 3)$.

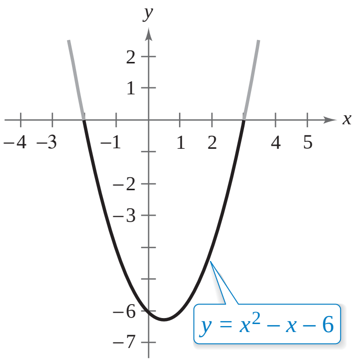


Figure 1.16

In Example 1, the polynomial inequality was given in general form (with the polynomial on one side and zero on the other). Whenever this is not the case, you should begin the solution process by writing the inequality in general form.

EXAMPLE 2 Solving a Polynomial Inequality

Solve $2x^3 - 3x^2 - 32x > -48$. Then graph the solution set.

Solution

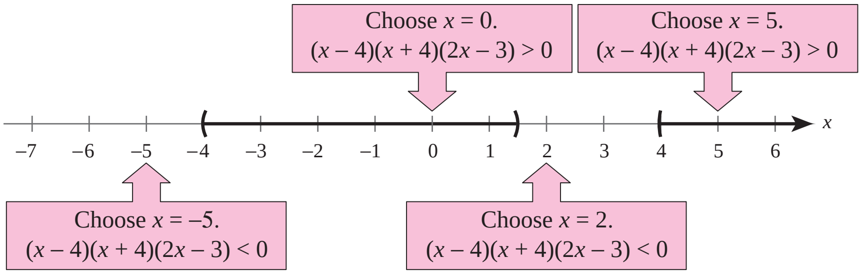
$$2x^3 - 3x^2 - 32x + 48 > 0 \qquad \text{Write in general form.}$$

$$(x - 4)(x + 4)(2x - 3) > 0 \qquad \text{Factor.}$$

The key numbers are $x = -4$, $x = \frac{3}{2}$, and $x = 4$, and the test intervals are $(-\infty, -4)$, $\left(-4, \frac{3}{2}\right)$, $\left(\frac{3}{2}, 4\right)$, and $(4, \infty)$.

Test Interval	x-Value	Polynomial Value	Conclusion
$(-\infty, -4)$	$x = -5$	$2(-5)^3 - 3(-5)^2 - 32(-5) + 48 = -117$	Negative
$\left(-4, \frac{3}{2}\right)$	$x = 0$	$2(0)^3 - 3(0)^2 - 32(0) + 48 = 48$	Positive
$\left(\frac{3}{2}, 4\right)$	$x = 2$	$2(2)^3 - 3(2)^2 - 32(2) + 48 = -12$	Negative
$(4, \infty)$	$x = 5$	$2(5)^3 - 3(5)^2 - 32(5) + 48 = 63$	Positive

From this, you can conclude that the inequality is satisfied on the open intervals $\left(-4, \frac{3}{2}\right)$ and $(4, \infty)$. So, the solution set is $\left(-4, \frac{3}{2}\right) \cup (4, \infty)$, as shown below.



✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $3x^3 - x^2 - 12x > -4$. Then graph the solution set.

EXAMPLE 3 Solving a Polynomial Inequality

Solve $4x^2 - 5x > 6$.

Algebraic Solution

$$4x^2 - 5x - 6 > 0 \qquad \text{Write in general form.}$$

$$(x - 2)(4x + 3) > 0 \qquad \text{Factor.}$$

Key Numbers: $x = -\frac{3}{4}, \quad x = 2$

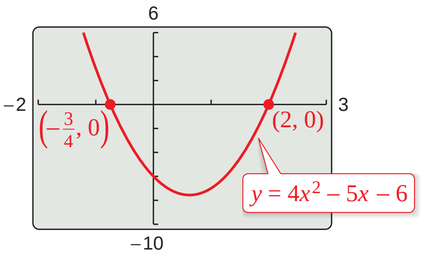
Test Intervals: $\left(-\infty, -\frac{3}{4}\right), \left(-\frac{3}{4}, 2\right), (2, \infty)$

Test: Is $(x - 2)(4x + 3) > 0$?

After testing these intervals, you can see that the polynomial $4x^2 - 5x - 6$ is positive on the open intervals $\left(-\infty, -\frac{3}{4}\right)$ and $(2, \infty)$. So, the solution set of the inequality is $\left(-\infty, -\frac{3}{4}\right) \cup (2, \infty)$.

Graphical Solution

First write the polynomial inequality $4x^2 - 5x > 6$ as $4x^2 - 5x - 6 > 0$. Then use a graphing utility to graph $y = 4x^2 - 5x - 6$. In the figure below, you can see that the graph is *above* the x-axis when x is less than $-\frac{3}{4}$ *or* when x is greater than 2. So, the solution set is $\left(-\infty, -\frac{3}{4}\right) \cup (2, \infty)$.



✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $2x^2 + 3x < 5$ (a) algebraically and (b) graphically.

You may find it easier to determine the sign of a polynomial from its *factored* form. For instance, in Example 3, when you substitute the test value $x = 1$ into the factored form

$$(x - 2)(4x + 3)$$

you can see that the sign pattern of the factors is

$$(\ -\ )(\ +\ )$$

which yields a negative result. Try using the factored forms of the polynomials to determine the signs of the polynomials in the test intervals of the other examples in this section.

When solving a polynomial inequality, be sure to account for the particular type of inequality symbol given in the inequality. For instance, in Example 3, note that the original inequality contained a "greater than" symbol and the solution consisted of two open intervals. If the original inequality had been

$$4x^2 - 5x \geq 6$$

then the solution set would have been $\left(-\infty, -\frac{3}{4}\right] \cup [2, \infty)$.

Each of the polynomial inequalities in Examples 1, 2, and 3 has a solution set that consists of a single interval or the union of two intervals. When solving the exercises for this section, watch for unusual solution sets, as illustrated in Example 4.

EXAMPLE 4 Unusual Solution Sets

a. The solution set of

$$x^2 + 2x + 4 > 0$$

consists of the entire set of real numbers, $(-\infty, \infty)$. In other words, the value of the quadratic $x^2 + 2x + 4$ is positive for every real value of x.

b. The solution set of

$$x^2 + 2x + 1 \leq 0$$

consists of the single real number $\{-1\}$, because the quadratic $x^2 + 2x + 1$ has only one key number, $x = -1$, and it is the only value that satisfies the inequality.

c. The solution set of

$$x^2 + 3x + 5 < 0$$

is empty. In other words, the quadratic $x^2 + 3x + 5$ is not less than zero for any value of x.

d. The solution set of

$$x^2 - 4x + 4 > 0$$

consists of all real numbers except $x = 2$. In interval notation, this solution set can be written as $(-\infty, 2) \cup (2, \infty)$.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

What is unusual about the solution set of each inequality?

a. $x^2 + 6x + 9 < 0$

b. $x^2 + 4x + 4 \leq 0$

c. $x^2 - 6x + 9 > 0$

d. $x^2 - 2x + 1 \geq 0$

Rational Inequalities

The concepts of key numbers and test intervals can be extended to rational inequalities. Use the fact that the value of a rational expression can change sign only at its *zeros* (the x-values for which its numerator is zero) and its *undefined values* (the x-values for which its denominator is zero). These two types of numbers make up the *key numbers* of a rational inequality. When solving a rational inequality, begin by writing the inequality in general form with the rational expression on the left and zero on the right.

EXAMPLE 5 Solving a Rational Inequality

Solve $\dfrac{2x - 7}{x - 5} \le 3$. Then graph the solution set.

REMARK In Example 5, write 3 as $\frac{3}{1}$. You should be able to see that the LCD (least common denominator) is $(x - 5)(1) = x - 5$. So, you can rewrite the general form as

$$\frac{2x - 7}{x - 5} - \frac{3(x - 5)}{x - 5} \le 0$$

which simplifies as shown.

Solution

$$\frac{2x - 7}{x - 5} \le 3 \qquad \text{Write original inequality.}$$

$$\frac{2x - 7}{x - 5} - 3 \le 0 \qquad \text{Write in general form.}$$

$$\frac{2x - 7 - 3x + 15}{x - 5} \le 0 \qquad \text{Find the LCD and subtract fractions.}$$

$$\frac{-x + 8}{x - 5} \le 0 \qquad \text{Simplify.}$$

Key Numbers: $x = 5,\ x = 8$ Zeros and undefined values of rational expression

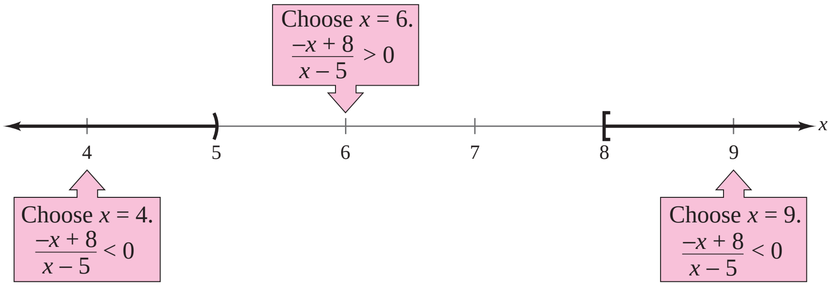
Test Intervals: $(-\infty, 5), (5, 8), (8, \infty)$

Test: Is $\dfrac{-x + 8}{x - 5} \le 0$?

After testing these intervals, as shown below, you can see that the inequality is satisfied on the open intervals $(-\infty, 5)$ and $(8, \infty)$. Moreover, because

$$\frac{-x + 8}{x - 5} = 0$$

when $x = 8$, you can conclude that the solution set consists of all real numbers in the intervals $(-\infty, 5) \cup [8, \infty)$. (Be sure to use a bracket to indicate that x can equal 8.)



✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Solve each inequality. Then graph the solution set.

a. $\dfrac{x - 2}{x - 3} \ge -3$

b. $\dfrac{4x - 1}{x - 6} > 3$

Applications

One common application of inequalities comes from business and involves profit, revenue, and cost. The formula that relates these three quantities is

$$\boxed{\text{Profit}} = \boxed{\text{Revenue}} - \boxed{\text{Cost}}$$

$$P = R - C.$$

EXAMPLE 6 Increasing the Profit for a Product

The marketing department of a calculator manufacturer has determined that the demand for a new model of calculator is

$p = 100 - 0.00001x, \quad 0 \le x \le 10{,}000{,}000$ Demand equation

where p is the price per calculator (in dollars) and x represents the number of calculators sold. (According to this model, no one would be willing to pay \$100 for the calculator. At the other extreme, the company could not *give* away more than 10 million calculators.) The revenue for selling x calculators is

$R = xp = x(100 - 0.00001x)$ Revenue equation

as shown in Figure 1.17. The total cost of producing x calculators is \$10 per calculator plus a one-time development cost of \$2,500,000. So, the total cost is

$C = 10x + 2{,}500{,}000.$ Cost equation

What prices can the company charge per calculator to obtain a profit of at least \$190,000,000?

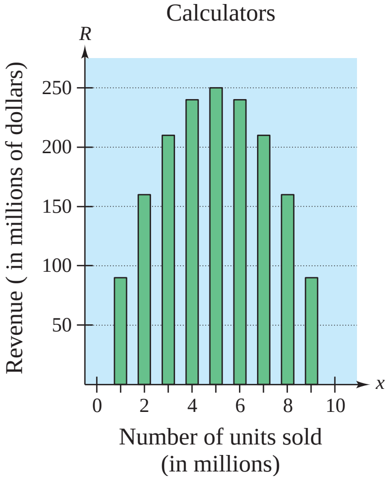


Figure 1.17

Solution

Verbal Model: $\boxed{\text{Profit}} = \boxed{\text{Revenue}} - \boxed{\text{Cost}}$

Equation: $P = R - C$

$$P = 100x - 0.00001x^2 - (10x + 2{,}500{,}000)$$

$$P = -0.00001x^2 + 90x - 2{,}500{,}000$$

To answer the question, solve the inequality

$$P \ge 190{,}000{,}000$$

$$-0.00001x^2 + 90x - 2{,}500{,}000 \ge 190{,}000{,}000.$$

When you write the inequality in general form, find the key numbers and the test intervals, and then test a value in each test interval, you can find the solution to be

$$3{,}500{,}000 \le x \le 5{,}500{,}000$$

as shown in Figure 1.18. Substituting the x-values in the original demand equation shows that prices of

$$\$45.00 \le p \le \$65.00$$

will yield a profit of at least \$190,000,000.

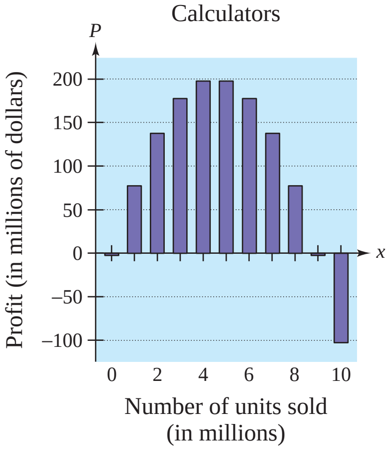


Figure 1.18

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The revenue and cost equations for a product are

$$R = x(60 - 0.0001x) \quad \text{and} \quad C = 12x + 1{,}800{,}000$$

where R and C are measured in dollars and x represents the number of units sold. How many units must be sold to obtain a profit of at least \$3,600,000?

Another common application of inequalities is finding the domain of an expression that involves a square root, as shown in Example 7.

EXAMPLE 7 Finding the Domain of an Expression

Find the domain of $\sqrt{64 - 4x^2}$.

Algebraic Solution

Recall that the domain of an expression is the set of all x-values for which the expression is defined. Because $\sqrt{64 - 4x^2}$ is defined (has real values) only if $64 - 4x^2$ is nonnegative, the domain is given by $64 - 4x^2 \geq 0$.

$$64 - 4x^2 \geq 0 \qquad \text{Write in general form.}$$

$$16 - x^2 \geq 0 \qquad \text{Divide each side by 4.}$$

$$(4 - x)(4 + x) \geq 0 \qquad \text{Write in factored form.}$$

So, the inequality has two key numbers: $x = -4$ and $x = 4$. You can use these two numbers to test the inequality, as follows.

Key numbers: $x = -4, x = 4$

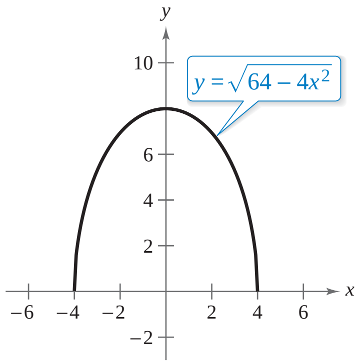
Test intervals: $(-\infty, -4), (-4, 4), (4, \infty)$

Test: Is $(4 - x)(4 + x) \geq 0$?

A test shows that the inequality is satisfied in the *closed interval* $[-4, 4]$. So, the domain of the expression $\sqrt{64 - 4x^2}$ is the interval $[-4, 4]$.

Graphical Solution

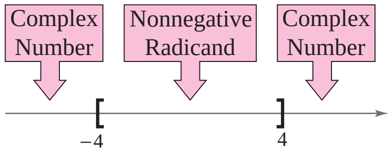
Begin by sketching the graph of the equation $y = \sqrt{64 - 4x^2}$, as shown below. From the graph, you can determine that the x-values extend from -4 to 4 (including -4 and 4). So, the domain of the expression $\sqrt{64 - 4x^2}$ is the interval $[-4, 4]$.



✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the domain of $\sqrt{x^2 - 7x + 10}$.

To analyze a test interval, choose a representative x-value in the interval and evaluate the expression at that value. For instance, in Example 7, when you substitute any number from the interval $[-4, 4]$ into the expression $\sqrt{64 - 4x^2}$, you obtain a nonnegative number under the radical symbol that simplifies to a real number. When you substitute any number from the intervals $(-\infty, -4)$ and $(4, \infty)$, you obtain a complex number. It might be helpful to draw a visual representation of the intervals, as shown below.



Summarize (Section 1.8)

1. Describe how to solve a polynomial inequality *(pages 140–143)*. For examples of solving polynomial inequalities, see Examples 1–4.
2. Describe how to solve a rational inequality *(page 144)*. For an example of solving a rational inequality, see Example 5.
3. Give an example of an application involving a polynomial inequality *(pages 145 and 146, Examples 6 and 7)*.

1.8 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. Between two consecutive zeros, a polynomial must be entirely ________ or entirely ________.
2. To solve a polynomial inequality, find the ________ numbers of the polynomial, and use these numbers to create ________ ________ for the inequality.
3. The key numbers of a rational expression are its ________ and its ________ ________.
4. The formula that relates cost, revenue, and profit is ________.

Skills and Applications

Checking Solutions **In Exercises 5–8, determine whether each value of x is a solution of the inequality.**

Inequality	Values	
5. $x^2 - 3 < 0$	(a) $x = 3$	(b) $x = 0$
	(c) $x = \frac{3}{2}$	(d) $x = -5$
6. $x^2 - x - 12 \geq 0$	(a) $x = 5$	(b) $x = 0$
	(c) $x = -4$	(d) $x = -3$
7. $\dfrac{x+2}{x-4} \geq 3$	(a) $x = 5$	(b) $x = 4$
	(c) $x = -\frac{9}{2}$	(d) $x = \frac{9}{2}$
8. $\dfrac{3x^2}{x^2+4} < 1$	(a) $x = -2$	(b) $x = -1$
	(c) $x = 0$	(d) $x = 3$

Finding Key Numbers **In Exercises 9–12, find the key numbers of the expression.**

9. $3x^2 - x - 2$ **10.** $9x^3 - 25x^2$

11. $\dfrac{1}{x-5} + 1$ **12.** $\dfrac{x}{x+2} - \dfrac{2}{x-1}$

Solving a Polynomial Inequality **In Exercises 13–34, solve the inequality. Then graph the solution set.**

13. $x^2 < 9$ **14.** $x^2 \leq 16$

15. $(x+2)^2 \leq 25$ **16.** $(x-3)^2 \geq 1$

17. $x^2 + 4x + 4 \geq 9$ **18.** $x^2 - 6x + 9 < 16$

19. $x^2 + x < 6$ **20.** $x^2 + 2x > 3$

21. $x^2 + 2x - 3 < 0$ **22.** $x^2 > 2x + 8$

23. $3x^2 - 11x > 20$ **24.** $-2x^2 + 6x \leq -15$

25. $x^2 - 3x - 18 > 0$ **26.** $x^3 + 2x^2 - 4x \leq 8$

27. $x^3 - 3x^2 - x > -3$ **28.** $2x^3 + 13x^2 - 8x \geq 52$

29. $4x^3 - 6x^2 < 0$ **30.** $4x^3 - 12x^2 > 0$

31. $x^3 - 4x \geq 0$ **32.** $2x^3 - x^4 \leq 0$

33. $(x-1)^2(x+2)^3 \geq 0$ **34.** $x^4(x-3) \leq 0$

Unusual Solution Sets **In Exercises 35–38, explain what is unusual about the solution set of the inequality.**

35. $4x^2 - 4x + 1 \leq 0$ **36.** $x^2 + 3x + 8 > 0$

37. $x^2 - 6x + 12 \leq 0$ **38.** $x^2 - 8x + 16 > 0$

Solving a Rational Inequality **In Exercises 39–52, solve the inequality. Then graph the solution set.**

39. $\dfrac{4x-1}{x} > 0$ **40.** $\dfrac{x^2-1}{x} < 0$

41. $\dfrac{3x-5}{x-5} \geq 0$ **42.** $\dfrac{5+7x}{1+2x} \leq 4$

43. $\dfrac{x+6}{x+1} - 2 < 0$ **44.** $\dfrac{x+12}{x+2} - 3 \geq 0$

45. $\dfrac{2}{x+5} > \dfrac{1}{x-3}$ **46.** $\dfrac{5}{x-6} > \dfrac{3}{x+2}$

47. $\dfrac{1}{x-3} \leq \dfrac{9}{4x+3}$ **48.** $\dfrac{1}{x} \geq \dfrac{1}{x+3}$

49. $\dfrac{x^2+2x}{x^2-9} \leq 0$ **50.** $\dfrac{x^2+x-6}{x} \geq 0$

51. $\dfrac{3}{x-1} + \dfrac{2x}{x+1} > -1$ **52.** $\dfrac{3x}{x-1} \leq \dfrac{x}{x+4} + 3$

Graphical Analysis **In Exercises 53–60, use a graphing utility to graph the equation. Use the graph to approximate the values of x that satisfy each inequality.**

Equation	Inequalities	
53. $y = -x^2 + 2x + 3$	(a) $y \leq 0$	(b) $y \geq 3$
54. $y = \frac{1}{2}x^2 - 2x + 1$	(a) $y \leq 0$	(b) $y \geq 7$
55. $y = \frac{1}{8}x^3 - \frac{1}{2}x$	(a) $y \geq 0$	(b) $y \leq 6$
56. $y = x^3 - x^2 - 16x + 16$	(a) $y \leq 0$	(b) $y \geq 36$
57. $y = \dfrac{3x}{x-2}$	(a) $y \leq 0$	(b) $y \geq 6$
58. $y = \dfrac{2(x-2)}{x+1}$	(a) $y \leq 0$	(b) $y \geq 8$
59. $y = \dfrac{2x^2}{x^2+4}$	(a) $y \geq 1$	(b) $y \leq 2$
60. $y = \dfrac{5x}{x^2+4}$	(a) $y \geq 1$	(b) $y \leq 0$

Finding the Domain of an Expression In Exercises 61–66, find the domain of the expression. Use a graphing utility to verify your result.

61. $\sqrt{4 - x^2}$

62. $\sqrt{x^2 - 4}$

63. $\sqrt{x^2 - 9x + 20}$

64. $\sqrt{81 - 4x^2}$

65. $\sqrt{\dfrac{x}{x^2 - 2x - 35}}$

66. $\sqrt{\dfrac{x}{x^2 - 9}}$

Solving an Inequality In Exercises 67–72, solve the inequality. (Round your answers to two decimal places.)

67. $0.4x^2 + 5.26 < 10.2$

68. $-1.3x^2 + 3.78 > 2.12$

69. $-0.5x^2 + 12.5x + 1.6 > 0$

70. $1.2x^2 + 4.8x + 3.1 < 5.3$

71. $\dfrac{1}{2.3x - 5.2} > 3.4$

72. $\dfrac{2}{3.1x - 3.7} > 5.8$

Height of a Projectile

In Exercises 73 and 74, use the position equation

$$s = -16t^2 + v_0 t + s_0$$

where s represents the height of an object (in feet), v_0 represents the initial velocity of the object (in feet per second), s_0 represents the initial height of the object (in feet), and t represents the time (in seconds).

73. A projectile is fired straight upward from ground level ($s_0 = 0$) with an initial velocity of 160 feet per second.

(a) At what instant will it be back at ground level?

(b) When will the height exceed 384 feet?

74. A projectile is fired straight upward from ground level ($s_0 = 0$) with an initial velocity of 128 feet per second.

(a) At what instant will it be back at ground level?

(b) When will the height be less than 128 feet?

75. **Geometry** A rectangular playing field with a perimeter of 100 meters is to have an area of at least 500 square meters. Within what bounds must the length of the rectangle lie?

76. **Geometry** A rectangular parking lot with a perimeter of 440 feet is to have an area of at least 8000 square feet. Within what bounds must the length of the rectangle lie?

77. **Cost, Revenue, and Profit** The revenue and cost equations for a product are $R = x(75 - 0.0005x)$ and $C = 30x + 250{,}000$, where R and C are measured in dollars and x represents the number of units sold. How many units must be sold to obtain a profit of at least \$750,000? What is the price per unit?

78. **Cost, Revenue, and Profit** The revenue and cost equations for a product are $R = x(50 - 0.0002x)$ and $C = 12x + 150{,}000$, where R and C are measured in dollars and x represents the number of units sold. How many units must be sold to obtain a profit of at least \$1,650,000? What is the price per unit?

79. **School Enrollment** The numbers N (in millions) of students enrolled in schools in the United States from 2000 through 2009 are shown in the table. *(Source: U.S. Census Bureau)*

DATA

Year	Number, N
2000	72.2
2001	73.1
2002	74.0
2003	74.9
2004	75.5
2005	75.8
2006	75.2
2007	76.0
2008	76.3
2009	77.3

Spreadsheet at LarsonPrecalculus.com

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 0$ corresponding to 2000.

(b) Use the *regression* feature of the graphing utility to find a quartic model for the data. (A quartic model has the form $at^4 + bt^3 + ct^2 + dt + e$, where a, b, c, d, and e are constant and t is variable.)

(c) Graph the model and the scatter plot in the same viewing window. How well does the model fit the data?

(d) According to the model, when did the number of students enrolled in schools exceed 74 million?

(e) Is the model valid for long-term predictions of student enrollment? Explain.

80. **Safe Load** The maximum safe load uniformly distributed over a one-foot section of a two-inch-wide wooden beam can be approximated by the model

$$\text{Load} = 168.5d^2 - 472.1$$

where d is the depth of the beam.

(a) Evaluate the model for $d = 4$, $d = 6$, $d = 8$, $d = 10$, and $d = 12$. Use the results to create a bar graph.

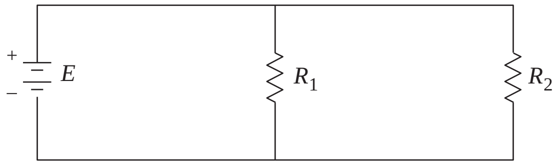
(b) Determine the minimum depth of the beam that will safely support a load of 2000 pounds.

Vakhrushev Pavel/Shutterstock.com

81. Resistors When two resistors of resistances R_1 and R_2 are connected in parallel (see figure), the total resistance R satisfies the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Find R_1 for a parallel circuit in which $R_2 = 2$ ohms and R must be at least 1 ohm.



82. Teachers' Salaries The mean salaries S (in thousands of dollars) of public school classroom teachers in the United States from 2000 through 2011 are shown in the table.

DATA

Year	Salary, S
2000	42.2
2001	43.7
2002	43.8
2003	45.0
2004	45.6
2005	45.9
2006	48.2
2007	49.3
2008	51.3
2009	52.9
2010	54.4
2011	54.2

Spreadsheet at LarsonPrecalculus.com

A model that approximates these data is given by

$$S = \frac{42.16 - 0.236t}{1 - 0.026t}, \quad 0 \leq t \leq 11$$

where t represents the year, with $t = 0$ corresponding to 2000. *(Source: Educational Research Service, Arlington, VA)*

(a) Use a graphing utility to create a scatter plot of the data. Then graph the model in the same viewing window.

(b) How well does the model fit the data? Explain.

(c) Use the model to predict when the salary for classroom teachers will exceed $60,000.

(d) Is the model valid for long-term predictions of classroom teacher salaries? Explain.

Exploration

True or False? **In Exercises 83 and 84, determine whether the statement is true or false. Justify your answer.**

83. The zeros of the polynomial $x^3 - 2x^2 - 11x + 12 \geq 0$ divide the real number line into four test intervals.

84. The solution set of the inequality $\frac{3}{2}x^2 + 3x + 6 \geq 0$ is the entire set of real numbers.

Conjecture **In Exercises 85–88, (a) find the interval(s) for b such that the equation has at least one real solution and (b) write a conjecture about the interval(s) based on the values of the coefficients.**

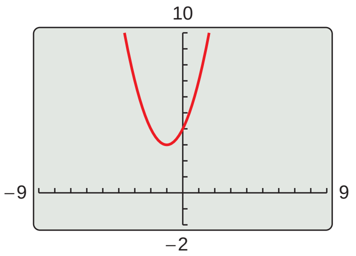
85. $x^2 + bx + 4 = 0$

86. $x^2 + bx - 4 = 0$

87. $3x^2 + bx + 10 = 0$

88. $2x^2 + bx + 5 = 0$

89. Graphical Analysis You can use a graphing utility to verify the results in Example 4. For instance, the graph of $y = x^2 + 2x + 4$ is shown below. Notice that the y-values are greater than 0 for all values of x, as stated in Example 4(a). Use the graphing utility to graph $y = x^2 + 2x + 1$, $y = x^2 + 3x + 5$, and $y = x^2 - 4x + 4$. Explain how you can use the graphs to verify the results of parts (b), (c), and (d) of Example 4.



90. HOW DO YOU SEE IT? Consider the polynomial

$(x - a)(x - b)$

and the real number line shown below.

(a) Identify the points on the line at which the polynomial is zero.

(b) In each of the three subintervals of the line, write the sign of each factor and the sign of the product.

(c) At what x-values does the polynomial change signs?

Chapter Summary

	What Did You Learn?	Explanation/Examples	Review Exercises
Section 1.1	Sketch graphs of equations *(p. 70)*, and identify x- and y-intercepts of graphs of equations *(p. 73)*.	To graph an equation, construct a table of values, plot the points, and connect the points with a smooth curve or line. Intercepts are points where a graph intersects or touches the x- or y-axis.	1–4
	Use symmetry to sketch graphs of equations *(p. 74)*.	Graphs can have symmetry with respect to one of the coordinate axes or with respect to the origin.	5–12
	Write equations of and sketch graphs of circles *(p. 76)*.	A point (x, y) lies on the circle of radius r and center (h, k) if and only if $(x - h)^2 + (y - k)^2 = r^2$.	13–18
	Use graphs of equations in solving real-life problems *(p. 77)*.	You can use the graph of an equation to estimate the recommended weight for a man. (See Example 9.)	19, 20
Section 1.2	Identify different types of equations *(p. 81)*.	**Identity:** true for *every* real number in the domain **Conditional equation:** true for just *some* (but not all) of the real numbers in the domain **Contradiction:** false for *every* real number in the domain	21–24
	Solve linear equations in one variable *(p. 82)*, and solve rational equations that lead to linear equations *(p. 84)*.	**Linear equation in one variable:** an equation that can be written in the standard form $ax + b = 0$, where a and b are real numbers with $a \neq 0$ To solve a rational equation, multiply every term by the LCD.	25–30
	Find x- and y-intercepts algebraically *(p. 85)*.	To find x-intercepts, set y equal to zero and solve for x. To find y-intercepts, set x equal to zero and solve for y.	31–36
	Use linear equations to model and solve real-life problems *(p. 86)*.	You can use a linear equation to model the number of female participants in athletic programs. (See Example 6.)	37, 38
Section 1.3	Use mathematical models to solve real-life problems *(p. 90)*.	You can use mathematical models to find the percent of a raise, and a building's height. (See Examples 2 and 6.)	39–42
	Solve mixture problems *(p. 94)*.	Mixture problems include simple interest problems and inventory problems. (See Examples 7 and 8.)	43, 44
	Use common formulas to solve real-life problems *(p. 95)*.	A literal equation contains more than one variable. A formula is an example of a literal equation. (See Example 9.)	45, 46
Section 1.4	Solve quadratic equations by factoring *(p. 100)*.	The method of factoring is based on the Zero-Factor Property, which states if $ab = 0$, then $a = 0$ or $b = 0$.	47, 48
	Solve quadratic equations by extracting square roots *(p. 101)*.	The equation $u^2 = d$, where $d > 0$, has exactly two solutions: $u = \sqrt{d}$ and $u = -\sqrt{d}$.	49–52
	Solve quadratic equations by completing the square *(p. 102)* and using the Quadratic Formula *(p. 104)*.	To complete the square for $x^2 + bx$, add $(b/2)^2$. **Quadratic Formula:** $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	53–56
	Use quadratic equations to model and solve real-life problems *(p. 106)*.	You can use a quadratic equation to model the numbers of Internet users in the United States from 2000 through 2010. (See Example 9.)	57, 58

	What Did You Learn?	Explanation/Examples	Review Exercises
Section 1.5	Use the imaginary unit i to write complex numbers *(p. 114)*, and add, subtract, and multiply complex numbers *(p. 115)*.	When a and b are real numbers, $a + bi$ is a complex number. **Sum:** $(a + bi) + (c + di) = (a + c) + (b + d)i$ **Difference:** $(a + bi) - (c + di) = (a - c) + (b - d)i$ You can use the Distributive Property to multiply.	59–66
	Use complex conjugates to write the quotient of two complex numbers in standard form *(p. 117)*.	To write $(a + bi)/(c + di)$ in standard form, where c and d are not both zero, multiply the numerator and denominator by the complex conjugate of the denominator, $c - di$.	67–70
	Find complex solutions of quadratic equations *(p. 118)*.	When a is a positive real number, the principal square root of $-a$ is defined as $\sqrt{-a} = \sqrt{a}i$.	71–74
Section 1.6	Solve polynomial equations of degree three or greater *(p. 121)*.	You can solve some polynomial equations of degree three or greater by factoring.	75–78
	Solve radical equations *(p. 123)*.	Solving radical equations usually involves squaring or cubing each side of the equation.	79–82
	Solve rational equations and absolute value equations *(p. 124)*.	To solve a rational equation, multiply each side of the equation by the LCD of all terms in the equation. To solve an absolute value equation, remember that the expression inside the absolute value bars can be positive or negative.	83–88
	Use nonlinear and nonquadratic models to solve real-life problems *(p. 126)*.	You can use nonlinear and nonquadratic models to find the number of ski club members going on a ski trip, and the annual interest rate for an account. (See Examples 8 and 9.)	89, 90
Section 1.7	Represent solutions of linear inequalities in one variable *(p. 131)*.	**Bounded** **Unbounded** $[-1, 2) \rightarrow -1 \le x < 2$ $(3, \infty) \rightarrow x > 3$ $[-4, 5] \rightarrow -4 \le x \le 5$ $(-\infty, \infty) \rightarrow -\infty < x < \infty$	91–94
	Use properties of inequalities to create equivalent inequalities *(p. 132)* and solve linear inequalities in one variable *(p. 133)*.	Solving linear inequalities is similar to solving linear equations. Use the properties of inequalities to isolate the variable. Remember to reverse the inequality symbol when you multiply or divide by a negative number.	95–98
	Solve absolute value inequalities *(p. 135)*.	Let u be an algebraic expression and let a be a real number such that $a > 0$. **1.** $\lvert u \rvert < a$ if and only if $-a < u < a$. **2.** $\lvert u \rvert \le a$ if and only if $-a \le u \le a$. **3.** $\lvert u \rvert > a$ if and only if $u < -a$ or $u > a$. **4.** $\lvert u \rvert \ge a$ if and only if $u \le -a$ or $u \ge a$.	99, 100
	Use inequalities to model and solve real-life problems *(p. 136)*.	You can use an inequality to determine the accuracy of a measurement. (See Example 7.)	101, 102
Section 1.8	Solve polynomial *(p. 140)* and rational *(p. 144)* inequalities.	Use the concepts of key numbers and test intervals to solve both polynomial and rational inequalities.	103–108
	Use inequalities to model and solve real-life problems *(p. 145)*.	A common application of inequalities involves profit P, revenue R, and cost C. (See Example 6.)	109, 110

Review Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

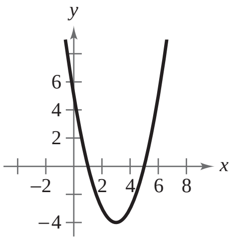
1.1 Sketching the Graph of an Equation **In Exercises 1 and 2, construct a table of values. Use the resulting solution points to sketch the graph of the equation.**

1. $y = -4x + 1$

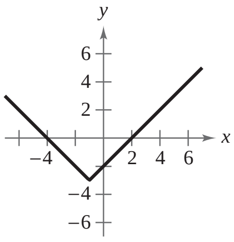
2. $y = x^2 + 2x$

Identifying *x*- and *y*-Intercepts **In Exercises 3 and 4, identify the *x*- and *y*-intercepts of the graph.**

3. $y = (x - 3)^2 - 4$



4. $y = |x + 1| - 3$



Testing for Symmetry **In Exercises 5–12, use the algebraic tests to check for symmetry with respect to both axes and the origin. Then sketch the graph of the equation.**

5. $y = -4x + 2$

6. $y = 5x - 6$

7. $y = 7 - x^2$

8. $y = x^2 + 2$

9. $y = x^3 + 3$

10. $y = -6 - x^3$

11. $y = -|x| - 2$

12. $y = |x| + 9$

Sketching the Graph of a Circle **In Exercises 13–16, find the center and radius of the circle. Then sketch the graph of the circle.**

13. $x^2 + y^2 = 9$

14. $x^2 + y^2 = 4$

15. $(x + 2)^2 + y^2 = 16$

16. $x^2 + (y - 8)^2 = 81$

17. Writing the Equation of a Circle Write the standard form of the equation of the circle for which the endpoints of a diameter are $(0, 0)$ and $(4, -6)$.

18. Writing the Equation of a Circle Write the standard form of the equation of the circle for which the endpoints of a diameter are $(-2, -3)$ and $(4, -10)$.

19. Sales The sales S (in billions of dollars) for Aéropostale for the years 2001 through 2010 can be approximated by the model

$$S = 0.004t^2 + 0.19t + 0.1, \quad 1 \le t \le 10$$

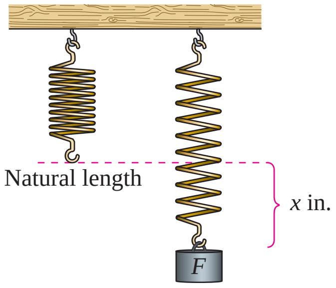
where t represents the year, with $t = 1$ corresponding to 2001. *(Source: Aéropostale, Inc.)*

(a) Sketch a graph of the model.

(b) Use the graph to estimate the year in which the sales were $1.2 billion.

20. Physics The force F (in pounds) required to stretch a spring x inches from its natural length (see figure) is

$$F = \frac{5}{4}x, \quad 0 \le x \le 20.$$



(a) Use the model to complete the table.

x	0	4	8	12	16	20
Force, F						

(b) Sketch a graph of the model.

(c) Use the graph to estimate the force necessary to stretch the spring 10 inches.

1.2 Classifying an Equation **In Exercises 21–24, determine whether the equation is an identity, a conditional equation, or a contradiction.**

21. $2(x - 2) = 2x - 4$

22. $3(x - 2) + 2x = 2(x + 3)$

23. $2(x + 3) = 2x - 2$

24. $5(x - 1) - 2x = 3x - 5$

Solving a Linear Equation **In Exercises 25–30, solve the equation and check your solution. (If not possible, explain why.)**

25. $8x - 5 = 3x + 20$

26. $7x + 3 = 3x - 17$

27. $2(x + 5) - 7 = 3(x - 2)$

28. $3(x + 3) = 5(1 - x) - 1$

29. $\frac{x}{5} - 3 = \frac{x}{3} + 1$

30. $\frac{4x - 3}{6} + \frac{x}{4} = x - 2$

Finding Intercepts Algebraically **In Exercises 31–36, find the *x*- and *y*-intercepts of the graph of the equation algebraically.**

31. $y = 3x - 1$

32. $y = -5x + 6$

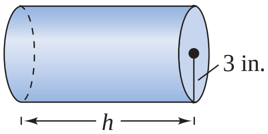
33. $y = 2(x - 4)$

34. $y = 4(7x + 1)$

35. $y = -\frac{1}{2}x + \frac{2}{3}$

36. $y = \frac{3}{4}x - \frac{1}{4}$

37. **Geometry** The surface area S of the cylinder shown in the figure is approximated by the equation $S = 2(3.14)(3)^2 + 2(3.14)(3)h$. The surface area is 244.92 square inches. Find the height h of the cylinder.



38. **Temperature** The Fahrenheit and Celsius temperature scales are related by the equation

$$C = \frac{5}{9}F - \frac{160}{9}.$$

Find the Fahrenheit temperature that corresponds to 100° Celsius.

1.3

39. **Profit** In October, a greeting card company's total profit was 12% more than it was in September. The total profit for the two months was $689,000. Find the profit for each month.

40. **Discount** The price of a digital camera has been discounted by 20%. The sale price is $340. Find the original price.

41. **Business Venture** You are planning to start a small business that will require an investment of $90,000. You have found some people who are willing to share equally in the venture. With 3 more investors, each person's share will decrease by $2500. How many people have you found so far?

42. **Average Speed** You commute 56 miles one way to work. The trip to work takes 10 minutes longer than the trip home. Your average speed on the trip home is 8 miles per hour faster. What is your average speed on the trip home?

43. **Mixture Problem** A car radiator contains 10 liters of a 30% antifreeze solution. How many liters will have to be replaced with pure antifreeze when the resulting solution is to be 50% antifreeze?

44. **Investment** You invested $6000 at $4\frac{1}{2}\%$ and $5\frac{1}{2}\%$ simple interest. During the first year, the two accounts earned $305. How much did you invest in each account?

45. **Volume of a Cone** Solve for h: $V = \frac{1}{3}\pi r^2 h$

46. **Kinetic Energy** Solve for m: $E = \frac{1}{2}mv^2$

1.4 **Choosing a Method** **In Exercises 47–56, solve the equation using any convenient method.**

47. $15 + x - 2x^2 = 0$

48. $2x^2 - x - 28 = 0$

49. $6 = 3x^2$

50. $16x^2 = 25$

51. $(x + 13)^2 = 25$

52. $(x - 5)^2 = 30$

53. $x^2 + 12x = -25$

54. $9x^2 - 12x = 14$

55. $-2x^2 - 5x + 27 = 0$

56. $-20 - 3x + 3x^2 = 0$

57. **Simply Supported Beam** A simply supported 20-foot beam supports a uniformly distributed load of 1000 pounds per foot. The bending moment M (in foot-pounds) x feet from one end of the beam is given by $M = 500x(20 - x)$.

(a) Where is the bending moment zero?

(b) Use a graphing utility to graph the equation.

(c) Use the graph to determine the point on the beam where the bending moment is the greatest.

58. **Sports** You throw a softball straight up into the air at a velocity of 30 feet per second. You release the softball at a height of 5.8 feet and catch it when it falls back to a height of 6.2 feet.

(a) Use the position equation to write a mathematical model for the height of the softball.

(b) What is the height of the softball after 1 second?

(c) How many seconds is the softball in the air?

1.5 **Writing a Complex Number in Standard Form** **In Exercises 59–62, write the complex number in standard form.**

59. $4 + \sqrt{-9}$

60. $3 + \sqrt{-16}$

61. $i^2 + 3i$

62. $-5i + i^2$

Performing Operations with Complex Numbers **In Exercises 63–66, perform the operation and write the result in standard form.**

63. $(7 + 5i) + (-4 + 2i)$

64. $\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) - \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$

65. $6i(5 - 2i)$

66. $(1 + 6i)(5 - 2i)$

Quotient of Complex Numbers in Standard Form **In Exercises 67 and 68, write the quotient in standard form.**

67. $\dfrac{6 - 5i}{i}$

68. $\dfrac{3 + 2i}{5 + i}$

Performing Operations with Complex Numbers **In Exercises 69 and 70, perform the operation and write the result in standard form.**

69. $\dfrac{4}{2 - 3i} + \dfrac{2}{1 + i}$

70. $\dfrac{1}{2 + i} - \dfrac{5}{1 + 4i}$

Complex Solutions of a Quadratic Equation **In Exercises 71–74, use the Quadratic Formula to solve the quadratic equation.**

71. $x^2 - 2x + 10 = 0$

72. $x^2 + 6x + 34 = 0$

73. $4x^2 + 4x + 7 = 0$

74. $6x^2 + 3x + 27 = 0$

1.6 Solving an Equation **In Exercises 75–88, solve the equation. Check your solutions.**

75. $5x^4 - 12x^3 = 0$

76. $4x^3 - 6x^2 = 0$

77. $x^4 - 5x^2 + 6 = 0$

78. $9x^4 + 27x^3 - 4x^2 - 12x = 0$

79. $\sqrt{2x+3} + \sqrt{x-2} = 2$

80. $5\sqrt{x} - \sqrt{x-1} = 6$

81. $(x-1)^{2/3} - 25 = 0$ **82.** $(x+2)^{3/4} = 27$

83. $\dfrac{5}{x} = 1 + \dfrac{3}{x+2}$ **84.** $\dfrac{6}{x} + \dfrac{8}{x+5} = 3$

85. $|x - 5| = 10$ **86.** $|2x + 3| = 7$

87. $|x^2 - 3| = 2x$ **88.** $|x^2 - 6| = x$

89. Demand The demand equation for a hair dryer is $p = 42 - \sqrt{0.001x + 2}$, where x is the number of units demanded per day and p is the price per unit. Find the demand when the price is set at \$29.95.

90. Data Analysis: Newspapers The total numbers N of daily evening newspapers in the United States from 1970 through 2005 can be approximated by the model $N = 1465 - 4.2t^{3/2}$, $0 \le t \le 35$, where t represents the year, with $t = 0$ corresponding to 1970. The actual numbers of newspapers for selected years are shown in the table. *(Source: Editor & Publisher Co.)*

DATA

Spreadsheet at LarsonPrecalculus.com

Year	Newspapers, N
1970	1429
1975	1436
1980	1388
1985	1220
1990	1084
1995	891
2000	727
2005	645

(a) Use a graphing utility to plot the data and graph the model in the same viewing window. How well does the model fit the data?

(b) Use the graph in part (a) to estimate the year in which there were 800 daily evening newspapers.

(c) Use the model to verify algebraically the estimate from part (b).

1.7 Intervals and Inequalities **In Exercises 91–94, write an inequality that represents the interval. Then state whether the interval is bounded or unbounded.**

91. $(-7, 2]$ **92.** $(4, \infty)$

93. $(-\infty, -10]$ **94.** $[-2, 2]$

Solving an Inequality **In Exercises 95–100, solve the inequality. Then graph the solution set.**

95. $3(x+2) + 7 < 2x - 5$

96. $2(x+7) - 4 \ge 5(x-3)$

97. $4(5 - 2x) \le \frac{1}{2}(8 - x)$ **98.** $\frac{1}{2}(3 - x) > \frac{1}{3}(2 - 3x)$

99. $|x - 3| > 4$ **100.** $\left|x - \frac{3}{2}\right| \ge \frac{3}{2}$

101. Geometry The side of a square is measured as 19.3 centimeters with a possible error of 0.5 centimeter. Using these measurements, determine the interval containing the possible areas of the square.

102. Cost, Revenue, and Profit The revenue for selling x units of a product is $R = 125.33x$. The cost of producing x units is $C = 92x + 1200$. To obtain a profit, the revenue must be greater than the cost. For what values of x will this product return a profit?

1.8 Solving an Inequality **In Exercises 103–108, solve the inequality. Then graph the solution set.**

103. $x^2 - 6x - 27 < 0$ **104.** $x^2 - 2x \ge 3$

105. $6x^2 + 5x < 4$ **106.** $2x^2 + x \ge 15$

107. $\dfrac{2}{x+1} \le \dfrac{3}{x-1}$ **108.** $\dfrac{x-5}{3-x} < 0$

109. Investment An investment of P dollars at interest rate r compounded annually increases to an amount $A = P(1 + r)^2$ in 2 years. An investment of \$5000 increases to an amount greater than \$5500 in 2 years. The interest rate must be greater than what percent?

110. Population of a Species A biologist introduces 200 ladybugs into a crop field. The population P of the ladybugs can be approximated by the model $P = [1000(1 + 3t)]/(5 + t)$, where t is the time in days. Find the time required for the population to increase to at least 2000 ladybugs.

Exploration

True or False? **In Exercises 111 and 112, determine whether the statement is true or false. Justify your answer.**

111. $\sqrt{-18}\sqrt{-2} = \sqrt{(-18)(-2)}$

112. The equation $325x^2 - 717x + 398 = 0$ has no solution.

113. Writing Explain why it is essential to check your solutions to radical, absolute value, and rational equations.

114. Error Analysis Describe the error below.

$$|11x + 4| \ge 26$$

~~$11x + 4 \le 26$ or $11x + 4 \ge 26$~~

~~$11x \le 22$ $\quad$ $11x \ge 22$~~

~~$x \le 2$ $\quad$ $x \ge 2$~~

Chapter Test

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

In Exercises 1–6, use the algebraic tests to check for symmetry with respect to both axes and the origin. Then sketch the graph of the equation. Identify any *x*- and *y*-intercepts.

1. $y = 4 - \frac{3}{4}x$

2. $y = 4 - \frac{3}{4}|x|$

3. $y = 4 - (x - 2)^2$

4. $y = x - x^3$

5. $y = \sqrt{5 - x}$

6. $(x - 3)^2 + y^2 = 9$

In Exercises 7–12, solve the equation and check your solution. (If not possible, explain why.)

7. $\frac{2}{3}(x - 1) + \frac{1}{4}x = 10$

8. $(x - 4)(x + 2) = 7$

9. $\dfrac{x - 2}{x + 2} + \dfrac{4}{x + 2} + 4 = 0$

10. $x^4 + x^2 - 6 = 0$

11. $2\sqrt{x} - \sqrt{2x + 1} = 1$

12. $|3x - 1| = 7$

In Exercises 13–16, solve the inequality. Then graph the solution set.

13. $-3 \le 2(x + 4) < 14$

14. $\dfrac{2}{x} > \dfrac{5}{x + 6}$

15. $2x^2 + 5x > 12$

16. $|3x + 5| \ge 10$

17. Perform each operation and write the result in standard form.

(a) $10i - \left(3 + \sqrt{-25}\right)$

(b) $(-1 - 5i)(-1 + 5i)$

18. Write the quotient in standard form:

$$\frac{5}{2 + i}.$$

19. The revenues R (in billions of dollars) for Microsoft from 2001 through 2010 can be approximated by the model

$$R = 4.45t + 19.5, \quad 1 \le t \le 10$$

where t represents the year, with $t = 1$ corresponding to 2001. *(Source: Microsoft Corp.)*

(a) Sketch a graph of the model.

(b) Use the graph in part (a) to predict the sales in 2015.

(c) Use the model to verify algebraically your prediction from part (b).

20. A basketball has a volume of about 455.9 cubic inches. Find the radius of the basketball (accurate to three decimal places).

21. On the first part of a 350-kilometer trip, a salesperson travels 2 hours and 15 minutes at an average speed of 100 kilometers per hour. The salesperson needs to arrive at the destination in another hour and 20 minutes. Find the average speed required for the remainder of the trip.

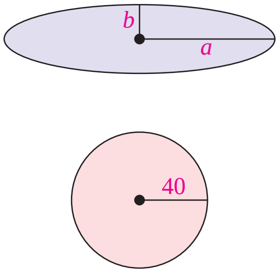


Figure for 22

22. The area of the ellipse in the figure at the left is $A = \pi ab$. If a and b satisfy the constraint $a + b = 100$, find a and b such that the area of the ellipse equals the area of the circle.

Proofs in Mathematics

Conditional Statements

Many theorems are written in the **if-then** form "if p, then q," which is denoted by

$p \to q$ Conditional statement

where p is the **hypothesis** and q is the **conclusion.** Here are some other ways to express the conditional statement $p \to q$.

p implies q. p only if q. p is sufficient for q.

Conditional statements can be either true or false. The conditional statement $p \to q$ is false only when p is true and q is false. To show that a conditional statement is true, you must prove that the conclusion follows for all cases that fulfill the hypothesis. To show that a conditional statement is false, you need to describe only a single **counterexample** that shows that the statement is not always true.

For instance, $x = -4$ is a counterexample that shows that the following statement is false.

If $x^2 = 16$, then $x = 4$.

The hypothesis "$x^2 = 16$" is true because $(-4)^2 = 16$. However, the conclusion "$x = 4$" is false. This implies that the given conditional statement is false.

For the conditional statement $p \to q$, there are three important associated conditional statements.

1. The **converse** of $p \to q$: $q \to p$
2. The **inverse** of $p \to q$: $\sim p \to \sim q$
3. The **contrapositive** of $p \to q$: $\sim q \to \sim p$

The symbol ~ means the **negation** of a statement. For instance, the negation of "The engine is running" is "The engine is not running."

EXAMPLE Writing the Converse, Inverse, and Contrapositive

Write the converse, inverse, and contrapositive of the conditional statement "If I get a B on my test, then I will pass the course."

Solution

a. *Converse:* If I pass the course, then I got a B on my test.

b. *Inverse:* If I do not get a B on my test, then I will not pass the course.

c. *Contrapositive:* If I do not pass the course, then I did not get a B on my test. ■

In the example above, notice that neither the converse nor the inverse is logically equivalent to the original conditional statement. On the other hand, the contrapositive *is* logically equivalent to the original conditional statement. ■

P.S. Problem Solving

1. **Time and Distance** Let x represent the time (in seconds), and let y represent the distance (in feet) between you and a tree. Sketch a possible graph that shows how x and y are related when you are walking toward the tree.

2. **Sum of the First *n* Natural Numbers**

   (a) Find the following sums.

   $1 + 2 + 3 + 4 + 5 =$

   $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 =$

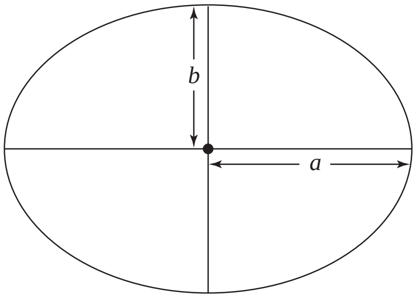
   $1 + 2 + 3 + 4 + 5 + 6$
   $+ 7 + 8 + 9 + 10 =$

   (b) Use the following formula for the sum of the first n natural numbers to verify your answers to part (a).

   $$1 + 2 + 3 + \cdot \cdot \cdot + n = \frac{1}{2}n(n + 1)$$

   (c) Use the formula in part (b) to find n such that the sum of the first n natural numbers is 210.

3. **Area of an Ellipse** The area of an ellipse is given by $A = \pi ab$ (see figure). For a certain ellipse, it is required that $a + b = 20$.



   (a) Show that

   $A = \pi a(20 - a).$

   (b) Complete the table.

a	4	7	10	13	16
A					

   (c) Find two values of a such that $A = 300$.

   (d) Use a graphing utility to graph the area equation.

   (e) Find the a-intercepts of the graph of the area equation. What do these values represent?

   (f) What is the maximum area? What values of a and b yield the maximum area?

4. **Wind Pressure** A building code requires that a building be able to withstand a certain amount of wind pressure. The pressure P (in pounds per square foot) from wind blowing at s miles per hour is given by

   $P = 0.00256s^2.$

   (a) A two-story library is designed. Buildings this tall are often required to withstand wind pressure of 20 pounds per square foot. Under this requirement, how fast can the wind be blowing before it produces excessive stress on the building?

   (b) To be safe, the library is designed so that it can withstand wind pressure of 40 pounds per square foot. Does this mean that the library can survive wind blowing at twice the speed you found in part (a)? Justify your answer.

   (c) Use the pressure formula to explain why even a relatively small increase in the wind speed could have potentially serious effects on a building.

5. **Water Height** For a bathtub with a rectangular base, Toricelli's Law implies that the height h of water in the tub t seconds after it begins draining is given by

   $$h = \left(\sqrt{h_0} - \frac{2\pi d^2\sqrt{3}}{lw}t\right)^2$$

   where l and w are the tub's length and width, d is the diameter of the drain, and h_0 is the water's initial height. (All measurements are in inches.) You completely fill a tub with water. The tub is 60 inches long by 30 inches wide by 25 inches high and has a drain with a two-inch diameter.

   (a) Find the time it takes for the tub to go from being full to half-full.

   (b) Find the time it takes for the tub to go from being half-full to empty.

   (c) Based on your results in parts (a) and (b), what general statement can you make about the speed at which the water drains?

6. **Sum of Squares**

   (a) Consider the sum of squares $x^2 + 9$. If the sum can be factored, then there are integers m and n such that $x^2 + 9 = (x + m)(x + n)$. Write two equations relating the sum and the product of m and n to the coefficients in $x^2 + 9$.

   (b) Show that there are no integers m and n that satisfy both equations you wrote in part (a). What can you conclude?

7. **Pythagorean Triples** A Pythagorean Triple is a group of three integers, such as 3, 4, and 5, that could be the lengths of the sides of a right triangle.

(a) Find two other Pythagorean Triples.

(b) Notice that $3 \cdot 4 \cdot 5 = 60$. Is the product of the three numbers in each Pythagorean Triple evenly divisible by 3? by 4? by 5?

(c) Write a conjecture involving Pythagorean Triples and divisibility by 60.

8. **Sums and Products of Solutions** Determine the solutions x_1 and x_2 of each quadratic equation. Use the values of x_1 and x_2 to fill in the boxes.

Equation	x_1, x_2	$x_1 + x_2$	$x_1 \cdot x_2$
(a) $x^2 - x - 6 = 0$			
(b) $2x^2 + 5x - 3 = 0$			
(c) $4x^2 - 9 = 0$			
(d) $x^2 - 10x + 34 = 0$			

9. **Proof** Given that the solutions of a quadratic equation are $x = \left(-b \pm \sqrt{b^2 - 4ac}\right)/(2a)$, prove that (a) the sum of the solutions is $S = -b/a$ and (b) the product of the solutions is $P = c/a$.

10. **Principal Cube Root**

(a) The principal cube root of 125, $\sqrt[3]{125}$, is 5. Evaluate the expression x^3 for each value of x.

(i) $x = \dfrac{-5 + 5\sqrt{3}i}{2}$

(ii) $x = \dfrac{-5 - 5\sqrt{3}i}{2}$

(b) The principal cube root of 27, $\sqrt[3]{27}$, is 3. Evaluate the expression x^3 for each value of x.

(i) $x = \dfrac{-3 + 3\sqrt{3}i}{2}$

(ii) $x = \dfrac{-3 - 3\sqrt{3}i}{2}$

(c) Use the results of parts (a) and (b) to list possible cube roots of (i) 1, (ii) 8, and (iii) 64. Verify your results algebraically.

11. **Multiplicative Inverse of a Complex Number** The multiplicative inverse of z is a complex number z_m such that $z \cdot z_m = 1$. Find the multiplicative inverse of each complex number.

(a) $z = 1 + i$ (b) $z = 3 - i$ (c) $z = -2 + 8i$

12. **Proof** Prove that the product of a complex number $a + bi$ and its complex conjugate is a real number.

13. **The Mandelbrot Set** A **fractal** is a geometric figure that consists of a pattern that is repeated infinitely on a smaller and smaller scale. The most famous fractal is called the **Mandelbrot Set,** named after the Polish-born mathematician Benoit Mandelbrot. To draw the Mandelbrot Set, consider the following sequence of numbers.

$$c,\ c^2 + c,\ (c^2 + c)^2 + c,\ \left[(c^2 + c)^2 + c\right]^2 + c,\ \dots$$

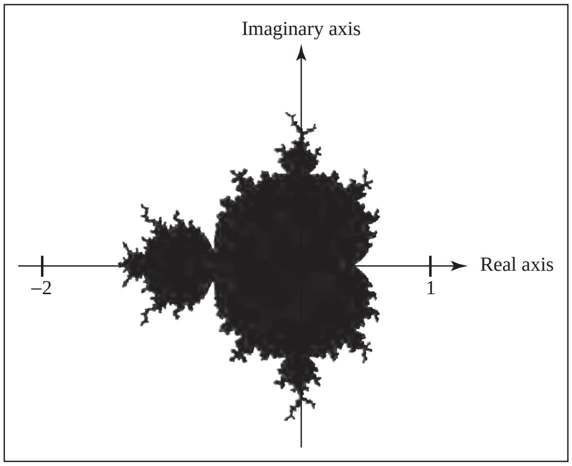
The behavior of this sequence depends on the value of the complex number c. If the sequence is bounded (the absolute value of each number in the sequence,

$$|a + bi| = \sqrt{a^2 + b^2}$$

is less than some fixed number N), then the complex number c is in the Mandelbrot Set, and if the sequence is unbounded (the absolute value of the terms of the sequence become infinitely large), then the complex number c is not in the Mandelbrot Set. Determine whether the complex number c is in the Mandelbrot Set.

(a) $c = i$ (b) $c = 1 + i$ (c) $c = -2$

The figure below shows a black and yellow photo of the Mandelbrot Set.



sciencephotos/Alamy

14. **Finding Values** Use the equation

$$4\sqrt{x} = 2x + k$$

to find three different values of k such that the equation has two solutions, one solution, and no solution. Describe the process you used to find the values.

15. **Using a Graph to Solve an Inequality** Use the graph of

$$y = x^4 - x^3 - 6x^2 + 4x + 8$$

to solve the inequality

$$x^4 - x^3 - 6x^2 + 4x + 8 > 0.$$

2 Functions and Their Graphs

Bacteria *(Example 8, page 218)*

Snowstorm *(Exercise 47, page 204)*

Average Speed *(Example 7, page 192)*

Alternative-Fueled Vehicles *(Example 10, page 180)*

Americans with Disabilities Act *(page 166)*

Clockwise from top left, nulinukas/Shutterstock.com; Fedorov Oleksiy/Shutterstock.com; wellphoto/Shutterstock.com; Jultud/Shutterstock.com; sadwitch/Shutterstock.com

2.1 Linear Equations in Two Variables

Linear equations in two variables can help you model and solve real-life problems. For instance, in Exercise 90 on page 171, you will use a surveyor's measurements to find a linear equation that models a mountain road.

- **Use slope to graph linear equations in two variables.**
- **Find the slope of a line given two points on the line.**
- **Write linear equations in two variables.**
- **Use slope to identify parallel and perpendicular lines.**
- **Use slope and linear equations in two variables to model and solve real-life problems.**

Using Slope

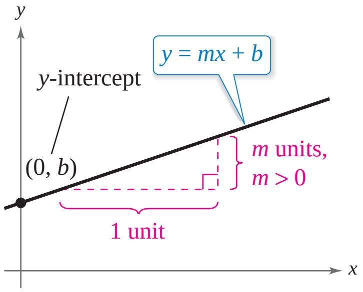
The simplest mathematical model for relating two variables is the **linear equation in two variables** $y = mx + b$. The equation is called *linear* because its graph is a line. (In mathematics, the term *line* means *straight line*.) By letting $x = 0$, you obtain

$$y = m(0) + b = b.$$

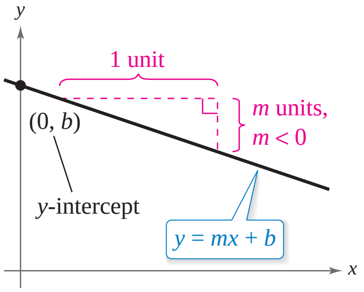
So, the line crosses the y-axis at $y = b$, as shown in the figures below. In other words, the y-intercept is $(0, b)$. The steepness or slope of the line is m.

$$y = \underset{\text{Slope}}{m}x + \underset{y\text{-Intercept}}{b}$$

The **slope** of a nonvertical line is the number of units the line rises (or falls) vertically for each unit of horizontal change from left to right, as shown below.



Positive slope, line rises.



Negative slope, line falls.

A linear equation written in **slope-intercept form** has the form $y = mx + b$.

The Slope-Intercept Form of the Equation of a Line

The graph of the equation

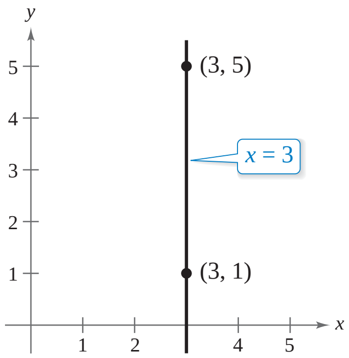
$$y = mx + b$$

is a line whose slope is m and whose y-intercept is $(0, b)$.

Once you have determined the slope and the y-intercept of a line, it is a relatively simple matter to sketch its graph. In the next example, note that none of the lines is vertical. A vertical line has an equation of the form

$$x = a. \qquad \text{Vertical line}$$

The equation of a vertical line cannot be written in the form $y = mx + b$ because the slope of a vertical line is undefined, as indicated in Figure 2.1.



Slope is undefined.
Figure 2.1

Dmitry Kalinovsky/Shutterstock.com

EXAMPLE 1 Graphing a Linear Equation

Sketch the graph of each linear equation.

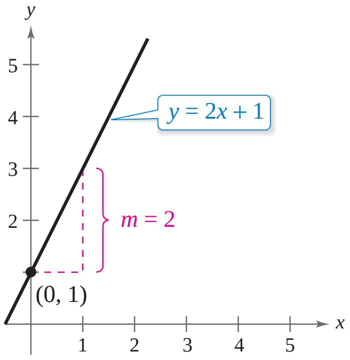
a. $y = 2x + 1$

b. $y = 2$

c. $x + y = 2$

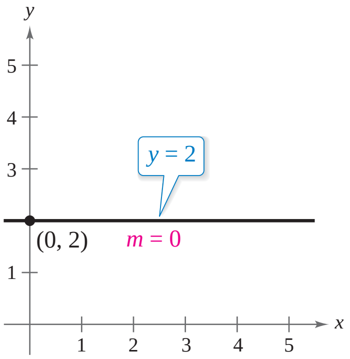
Solution

a. Because $b = 1$, the y-intercept is $(0, 1)$. Moreover, because the slope is $m = 2$, the line *rises* two units for each unit the line moves to the right.



When m is positive, the line rises.

b. By writing this equation in the form $y = (0)x + 2$, you can see that the y-intercept is $(0, 2)$ and the slope is zero. A zero slope implies that the line is horizontal—that is, it does not rise *or* fall.



When m is 0, the line is horizontal.

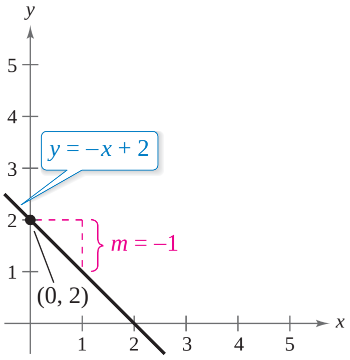
c. By writing this equation in slope-intercept form

$x + y = 2$ Write original equation.

$y = -x + 2$ Subtract x from each side.

$y = (-1)x + 2$ Write in slope-intercept form.

you can see that the y-intercept is $(0, 2)$. Moreover, because the slope is $m = -1$, the line *falls* one unit for each unit the line moves to the right.



When m is negative, the line falls.

✓ Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Sketch the graph of each linear equation.

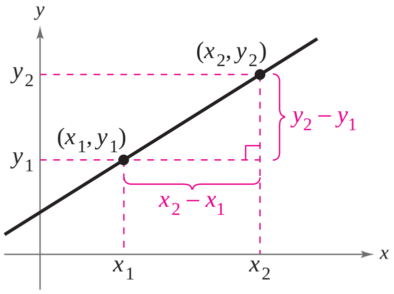
a. $y = -3x + 2$

b. $y = -3$

c. $4x + y = 5$

Finding the Slope of a Line

Given an equation of a line, you can find its slope by writing the equation in slope-intercept form. If you are not given an equation, then you can still find the slope of a line. For instance, suppose you want to find the slope of the line passing through the points (x_1, y_1) and (x_2, y_2), as shown below.



As you move from left to right along this line, a change of $(y_2 - y_1)$ units in the vertical direction corresponds to a change of $(x_2 - x_1)$ units in the horizontal direction.

$$y_2 - y_1 = \text{the change in } y = \text{rise}$$

and

$$x_2 - x_1 = \text{the change in } x = \text{run}$$

The ratio of $(y_2 - y_1)$ to $(x_2 - x_1)$ represents the slope of the line that passes through the points (x_1, y_1) and (x_2, y_2).

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

The Slope of a Line Passing Through Two Points

The **slope** m of the nonvertical line through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

where $x_1 \neq x_2$.

When using the formula for slope, the *order of subtraction* is important. Given two points on a line, you are free to label either one of them as (x_1, y_1) and the other as (x_2, y_2). However, once you have done this, you must form the numerator and denominator using the same order of subtraction.

$$\underbrace{m = \frac{y_2 - y_1}{x_2 - x_1}}_{\text{Correct}} \qquad \underbrace{m = \frac{y_1 - y_2}{x_1 - x_2}}_{\text{Correct}} \qquad \underbrace{\xcancel{m = \frac{y_2 - y_1}{x_1 - x_2}}}_{\text{Incorrect}}$$

For instance, the slope of the line passing through the points $(3, 4)$ and $(5, 7)$ can be calculated as

$$m = \frac{7 - 4}{5 - 3} = \frac{3}{2}$$

or, reversing the subtraction order in both the numerator and denominator, as

$$m = \frac{4 - 7}{3 - 5} = \frac{-3}{-2} = \frac{3}{2}.$$

EXAMPLE 2 Finding the Slope of a Line Through Two Points

Find the slope of the line passing through each pair of points.

a. $(-2, 0)$ and $(3, 1)$ **b.** $(-1, 2)$ and $(2, 2)$

c. $(0, 4)$ and $(1, -1)$ **d.** $(3, 4)$ and $(3, 1)$

Solution

a. Letting $(x_1, y_1) = (-2, 0)$ and $(x_2, y_2) = (3, 1)$, you obtain a slope of

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 0}{3 - (-2)} = \frac{1}{5}. \qquad \text{See Figure 2.2.}$$

b. The slope of the line passing through $(-1, 2)$ and $(2, 2)$ is

$$m = \frac{2 - 2}{2 - (-1)} = \frac{0}{3} = 0. \qquad \text{See Figure 2.3.}$$

c. The slope of the line passing through $(0, 4)$ and $(1, -1)$ is

$$m = \frac{-1 - 4}{1 - 0} = \frac{-5}{1} = -5. \qquad \text{See Figure 2.4.}$$

d. The slope of the line passing through $(3, 4)$ and $(3, 1)$ is

$$m = \frac{1 - 4}{3 - 3} = \cancel{\frac{-3}{0}}. \qquad \text{See Figure 2.5.}$$

Because division by 0 is undefined, the slope is undefined and the line is vertical.

REMARK In Figures 2.2 through 2.5, note the relationships between slope and the orientation of the line.

a. Positive slope: line rises from left to right

b. Zero slope: line is horizontal

c. Negative slope: line falls from left to right

d. Undefined slope: line is vertical

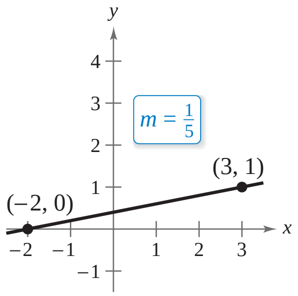


Figure 2.2

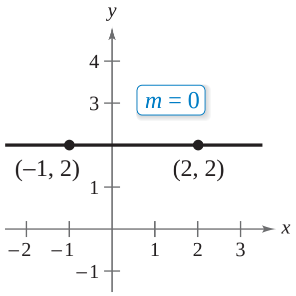


Figure 2.3

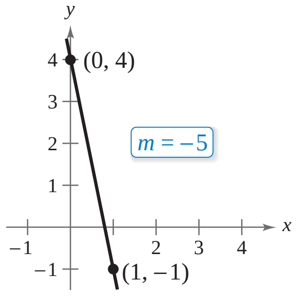


Figure 2.4

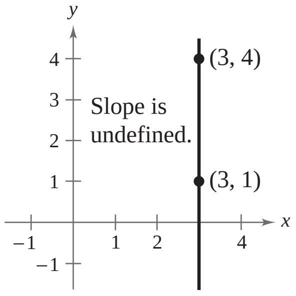


Figure 2.5

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Find the slope of the line passing through each pair of points.

a. $(-5, -6)$ and $(2, 8)$ **b.** $(4, 2)$ and $(2, 5)$

c. $(0, 0)$ and $(0, -6)$ **d.** $(0, -1)$ and $(3, -1)$

Writing Linear Equations in Two Variables

If (x_1, y_1) is a point on a line of slope m and (x, y) is *any other* point on the line, then

$$\frac{y - y_1}{x - x_1} = m.$$

This equation involving the variables x and y, rewritten in the form

$$y - y_1 = m(x - x_1)$$

is the **point-slope form** of the equation of a line.

Point-Slope Form of the Equation of a Line

The equation of the line with slope m passing through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1).$$

The point-slope form is most useful for *finding* the equation of a line. You should remember this form.

EXAMPLE 3 Using the Point-Slope Form

Find the slope-intercept form of the equation of the line that has a slope of 3 and passes through the point $(1, -2)$.

Solution Use the point-slope form with $m = 3$ and $(x_1, y_1) = (1, -2)$.

$y - y_1 = m(x - x_1)$	Point-slope form
$y - (-2) = 3(x - 1)$	Substitute for m, x_1, and y_1.
$y + 2 = 3x - 3$	Simplify.
$y = 3x - 5$	Write in slope-intercept form.

The slope-intercept form of the equation of the line is $y = 3x - 5$. Figure 2.6 shows the graph of this equation.

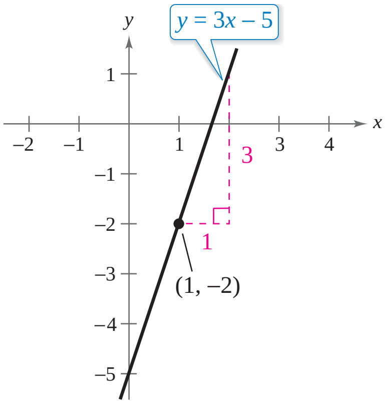


Figure 2.6

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Find the slope-intercept form of the equation of the line that has the given slope and passes through the given point.

a. $m = 2, \quad (3, -7)$

b. $m = -\frac{2}{3}, \quad (1, 1)$

c. $m = 0, \quad (1, 1)$

REMARK When you find an equation of the line that passes through two given points, you only need to substitute the coordinates of one of the points in the point-slope form. It does not matter which point you choose because both points will yield the same result.

The point-slope form can be used to find an equation of the line passing through two points (x_1, y_1) and (x_2, y_2). To do this, first find the slope of the line

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2$$

and then use the point-slope form to obtain the equation

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1). \qquad \text{Two-point form}$$

This is sometimes called the **two-point form** of the equation of a line.

Parallel and Perpendicular Lines

Slope can tell you whether two nonvertical lines in a plane are parallel, perpendicular, or neither.

Parallel and Perpendicular Lines

1. Two distinct nonvertical lines are **parallel** if and only if their slopes are equal. That is,

$$m_1 = m_2.$$

2. Two nonvertical lines are **perpendicular** if and only if their slopes are negative reciprocals of each other. That is,

$$m_1 = \frac{-1}{m_2}.$$

EXAMPLE 4 Finding Parallel and Perpendicular Lines

Find the slope-intercept form of the equations of the lines that pass through the point $(2, -1)$ and are (a) parallel to and (b) perpendicular to the line $2x - 3y = 5$.

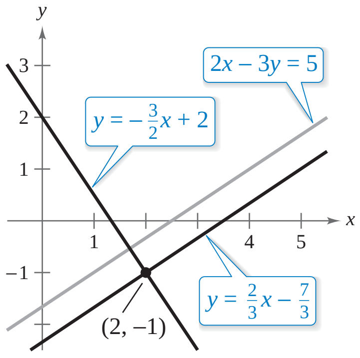


Figure 2.7

Solution By writing the equation of the given line in slope-intercept form

$2x - 3y = 5$ Write original equation.

$-3y = -2x + 5$ Subtract $2x$ from each side.

$y = \frac{2}{3}x - \frac{5}{3}$ Write in slope-intercept form.

you can see that it has a slope of $m = \frac{2}{3}$, as shown in Figure 2.7.

a. Any line parallel to the given line must also have a slope of $\frac{2}{3}$. So, the line through $(2, -1)$ that is parallel to the given line has the following equation.

$y - (-1) = \frac{2}{3}(x - 2)$ Write in point-slope form.

$3(y + 1) = 2(x - 2)$ Multiply each side by 3.

$3y + 3 = 2x - 4$ Distributive Property

$y = \frac{2}{3}x - \frac{7}{3}$ Write in slope-intercept form.

b. Any line perpendicular to the given line must have a slope of $-\frac{3}{2}$ $\left(\text{because } -\frac{3}{2} \text{ is the negative reciprocal of } \frac{2}{3}\right)$. So, the line through $(2, -1)$ that is perpendicular to the given line has the following equation.

$y - (-1) = -\frac{3}{2}(x - 2)$ Write in point-slope form.

$2(y + 1) = -3(x - 2)$ Multiply each side by 2.

$2y + 2 = -3x + 6$ Distributive Property

$y = -\frac{3}{2}x + 2$ Write in slope-intercept form.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the slope-intercept form of the equations of the lines that pass through the point $(-4, 1)$ and are (a) parallel to and (b) perpendicular to the line $5x - 3y = 8$.

Notice in Example 4 how the slope-intercept form is used to obtain information about the graph of a line, whereas the point-slope form is used to write the equation of a line.

▷ **TECHNOLOGY** On a graphing utility, lines will not appear to have the correct slope unless you use a viewing window that has a square setting. For instance, try graphing the lines in Example 4 using the standard setting $-10 \le x \le 10$ and $-10 \le y \le 10$. Then reset the viewing window with the square setting $-9 \le x \le 9$ and $-6 \le y \le 6$. On which setting do the lines $y = \frac{2}{3}x - \frac{5}{3}$ and $y = -\frac{3}{2}x + 2$ appear to be perpendicular?

Applications

In real-life problems, the slope of a line can be interpreted as either a *ratio* or a *rate*. If the x-axis and y-axis have the same unit of measure, then the slope has no units and is a **ratio.** If the x-axis and y-axis have different units of measure, then the slope is a **rate** or **rate of change.**

EXAMPLE 5 Using Slope as a Ratio

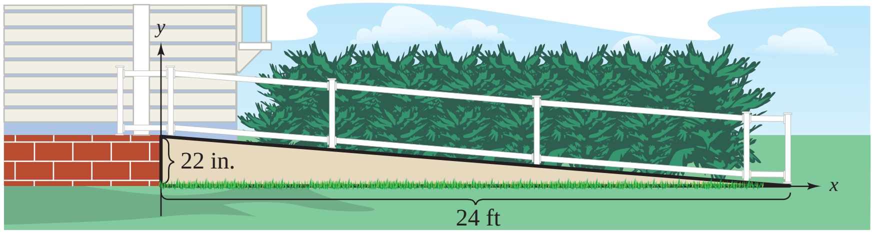
The maximum recommended slope of a wheelchair ramp is $\frac{1}{12}$. A business is installing a wheelchair ramp that rises 22 inches over a horizontal length of 24 feet. Is the ramp steeper than recommended? *(Source: ADA Standards for Accessible Design)*

The Americans with Disabilities Act (ADA) became law on July 26, 1990. It is the most comprehensive formulation of rights for persons with disabilities in U.S. (and world) history.

Solution The horizontal length of the ramp is 24 feet or $12(24) = 288$ inches, as shown below. So, the slope of the ramp is

$$\text{Slope} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{22 \text{ in.}}{288 \text{ in.}} \approx 0.076.$$

Because $\frac{1}{12} \approx 0.083$, the slope of the ramp is not steeper than recommended.



✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The business in Example 5 installs a second ramp that rises 36 inches over a horizontal length of 32 feet. Is the ramp steeper than recommended?

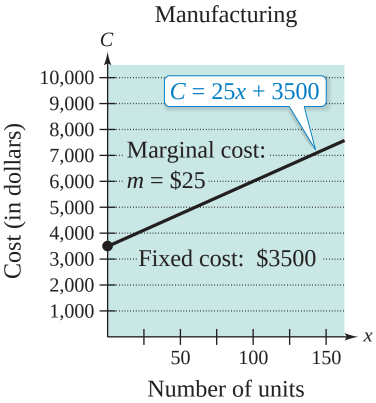
EXAMPLE 6 Using Slope as a Rate of Change

A kitchen appliance manufacturing company determines that the total cost C (in dollars) of producing x units of a blender is

$C = 25x + 3500$. Cost equation

Describe the practical significance of the y-intercept and slope of this line.

Solution The y-intercept $(0, 3500)$ tells you that the cost of producing zero units is \$3500. This is the *fixed cost* of production—it includes costs that must be paid regardless of the number of units produced. The slope of $m = 25$ tells you that the cost of producing each unit is \$25, as shown in Figure 2.8. Economists call the cost per unit the *marginal cost*. If the production increases by one unit, then the "margin," or extra amount of cost, is \$25. So, the cost increases at a rate of \$25 per unit.



Production cost
Figure 2.8

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

An accounting firm determines that the value V (in dollars) of a copier t years after its purchase is

$V = -300t + 1500$.

Describe the practical significance of the y-intercept and slope of this line.

Jultud/Shutterstock.com

Businesses can deduct most of their expenses in the same year they occur. One exception is the cost of property that has a useful life of more than 1 year. Such costs must be *depreciated* (decreased in value) over the useful life of the property. Depreciating the *same amount* each year is called *linear* or *straight-line depreciation*. The *book value* is the difference between the original value and the total amount of depreciation accumulated to date.

EXAMPLE 7 Straight-Line Depreciation

A college purchased exercise equipment worth $12,000 for the new campus fitness center. The equipment has a useful life of 8 years. The salvage value at the end of 8 years is $2000. Write a linear equation that describes the book value of the equipment each year.

Solution Let V represent the value of the equipment at the end of year t. Represent the initial value of the equipment by the data point $(0, 12{,}000)$ and the salvage value of the equipment by the data point $(8, 2000)$. The slope of the line is

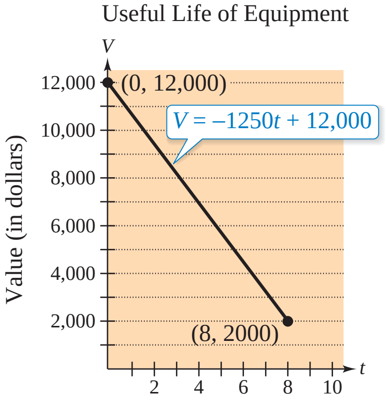
$$m = \frac{2000 - 12{,}000}{8 - 0}$$

$$= -\$1250$$

which represents the annual depreciation in *dollars per year*. Using the point-slope form, you can write the equation of the line as follows.

$$V - 12{,}000 = -1250(t - 0)$$ Write in point-slope form.

$$V = -1250t + 12{,}000$$ Write in slope-intercept form.



Straight-line depreciation

Figure 2.9

The table shows the book value at the end of each year, and Figure 2.9 shows the graph of the equation.

Year, t	Value, V
0	12,000
1	10,750
2	9500
3	8250
4	7000
5	5750
6	4500
7	3250
8	2000

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

A manufacturing firm purchased a machine worth $24,750. The machine has a useful life of 6 years. After 6 years, the machine will have to be discarded and replaced. That is, it will have no salvage value. Write a linear equation that describes the book value of the machine each year.

In many real-life applications, the two data points that determine the line are often given in a disguised form. Note how the data points are described in Example 7.

EXAMPLE 8 Predicting Sales

The sales for Best Buy were approximately \$49.7 billion in 2009 and \$50.3 billion in 2010. Using only this information, write a linear equation that gives the sales in terms of the year. Then predict the sales in 2013. *(Source: Best Buy Company, Inc.)*

Solution Let $t = 9$ represent 2009. Then the two given values are represented by the data points $(9, 49.7)$ and $(10, 50.3)$. The slope of the line through these points is

$$m = \frac{50.3 - 49.7}{10 - 9} = 0.6.$$

You can find the equation that relates the sales y and the year t to be

$$y - 49.7 = 0.6(t - 9) \qquad \text{Write in point-slope form.}$$

$$y = 0.6t + 44.3. \qquad \text{Write in slope-intercept form.}$$

According to this equation, the sales in 2013 will be

$$y = 0.6(13) + 44.3 = 7.8 + 44.3 = \$52.1 \text{ billion. (See Figure 2.10.)}$$

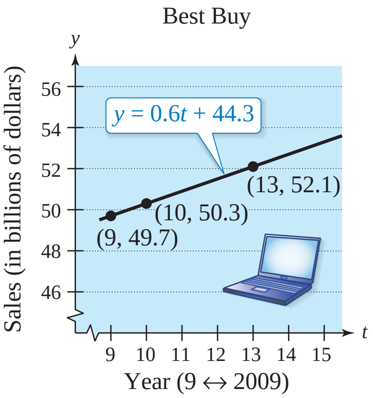


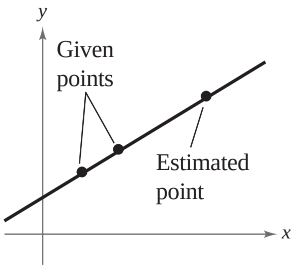
Figure 2.10

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

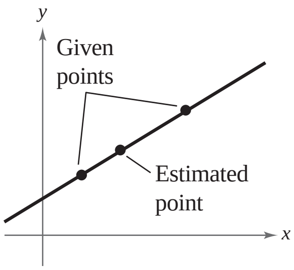
The sales for Nokia were approximately \$58.6 billion in 2009 and \$56.6 billion in 2010. Repeat Example 8 using this information. *(Source: Nokia Corporation)*

The prediction method illustrated in Example 8 is called **linear extrapolation.** Note in Figure 2.11 that an extrapolated point does not lie between the given points. When the estimated point lies between two given points, as shown in Figure 2.12, the procedure is called **linear interpolation.**

Because the slope of a vertical line is not defined, its equation cannot be written in slope-intercept form. However, every line has an equation that can be written in the **general form** $Ax + By + C = 0$, where A and B are not both zero.



Linear extrapolation
Figure 2.11



Linear interpolation
Figure 2.12

Summary of Equations of Lines

1. General form: $Ax + By + C = 0$
2. Vertical line: $x = a$
3. Horizontal line: $y = b$
4. Slope-intercept form: $y = mx + b$
5. Point-slope form: $y - y_1 = m(x - x_1)$
6. Two-point form: $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

Summarize (Section 2.1)

1. Explain how to use slope to graph a linear equation in two variables *(page 160)* and how to find the slope of a line passing through two points *(page 162)*. For examples of using and finding slopes, see Examples 1 and 2.
2. State the point-slope form of the equation of a line *(page 164)*. For an example of using point-slope form, see Example 3.
3. Explain how to use slope to identify parallel and perpendicular lines *(page 165)*. For an example of finding parallel and perpendicular lines, see Example 4.
4. Describe examples of how to use slope and linear equations in two variables to model and solve real-life problems *(pages 166–168, Examples 5–8)*.

2.1 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

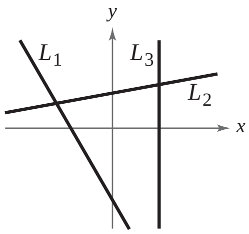
1. The simplest mathematical model for relating two variables is the ________ equation in two variables $y = mx + b$.
2. For a line, the ratio of the change in y to the change in x is called the ________ of the line.
3. The ________-________ form of the equation of a line with slope m passing through (x_1, y_1) is $y - y_1 = m(x - x_1)$.
4. Two lines are ________ if and only if their slopes are equal.
5. Two lines are ________ if and only if their slopes are negative reciprocals of each other.
6. When the x-axis and y-axis have different units of measure, the slope can be interpreted as a ________.
7. The prediction method ________ ________ is the method used to estimate a point on a line when the point does not lie between the given points.
8. Every line has an equation that can be written in ________ form.

Skills and Applications

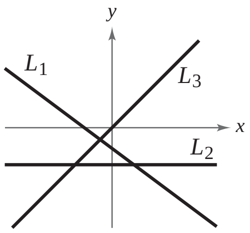
Identifying Lines

In Exercises 9 and 10, identify the line that has each slope.

9. (a) $m = \frac{2}{3}$
(b) m is undefined.
(c) $m = -2$



10. (a) $m = 0$
(b) $m = -\frac{3}{4}$
(c) $m = 1$



Sketching Lines

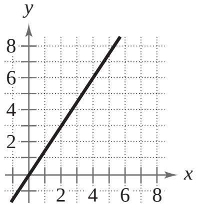
In Exercises 11 and 12, sketch the lines through the point with the indicated slopes on the same set of coordinate axes.

	Point	Slopes	
11.	(2, 3)	(a) 0	(b) 1
		(c) 2	(d) -3
12.	$(-4, 1)$	(a) 3	(b) -3
		(c) $\frac{1}{2}$	(d) Undefined

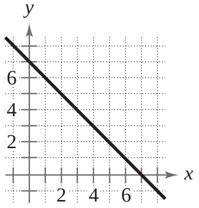
Estimating the Slope of a Line

In Exercises 13 and 14, estimate the slope of the line.

13.



14.



Graphing a Linear Equation

In Exercises 15–24, find the slope and y-intercept (if possible) of the equation of the line. Sketch the line.

15. $y = 5x + 3$
16. $y = -x - 10$
17. $y = -\frac{1}{2}x + 4$
18. $y = \frac{3}{2}x + 6$
19. $y - 3 = 0$
20. $x + 5 = 0$
21. $5x - 2 = 0$
22. $3y + 5 = 0$
23. $7x - 6y = 30$
24. $2x + 3y = 9$

Finding the Slope of a Line Through Two Points

In Exercises 25–34, plot the points and find the slope of the line passing through the pair of points.

25. $(0, 9), (6, 0)$
26. $(12, 0), (0, -8)$
27. $(-3, -2), (1, 6)$
28. $(2, 4), (4, -4)$
29. $(5, -7), (8, -7)$
30. $(-2, 1), (-4, -5)$
31. $(-6, -1), (-6, 4)$
32. $(0, -10), (-4, 0)$
33. $(4.8, 3.1), (-5.2, 1.6)$
34. $\left(\frac{11}{2}, -\frac{4}{3}\right), \left(-\frac{3}{2}, -\frac{1}{3}\right)$

Using a Point and Slope

In Exercises 35–42, use the point on the line and the slope m of the line to find three additional points through which the line passes. (There are many correct answers.)

35. $(2, 1), \quad m = 0$
36. $(3, -2), \quad m = 0$
37. $(-8, 1), \quad m$ is undefined.
38. $(1, 5), \quad m$ is undefined.
39. $(-5, 4), \quad m = 2$
40. $(0, -9), \quad m = -2$
41. $(-1, -6), \quad m = -\frac{1}{2}$
42. $(7, -2), \quad m = \frac{1}{2}$

Finding an Equation of a Line **In Exercises 43–54, find an equation of the line that passes through the given point and has the indicated slope m. Sketch the line.**

43. $(0, -2), \quad m = 3$

44. $(0, 10), \quad m = -1$

45. $(-3, 6), \quad m = -2$

46. $(0, 0), \quad m = 4$

47. $(4, 0), \quad m = -\frac{1}{3}$

48. $(8, 2), \quad m = \frac{1}{4}$

49. $(2, -3), \quad m = -\frac{1}{2}$

50. $(-2, -5), \quad m = \frac{3}{4}$

51. $(6, -1), \quad m$ is undefined.

52. $(-10, 4), \quad m$ is undefined.

53. $\left(4, \frac{5}{2}\right), \quad m = 0$

54. $(-5.1, 1.8), \quad m = 5$

Finding an Equation of a Line **In Exercises 55–64, find an equation of the line passing through the points. Sketch the line.**

55. $(5, -1), (-5, 5)$

56. $(4, 3), (-4, -4)$

57. $(-8, 1), (-8, 7)$

58. $(-1, 4), (6, 4)$

59. $\left(2, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{5}{4}\right)$

60. $(1, 1), \left(6, -\frac{2}{3}\right)$

61. $(1, 0.6), (-2, -0.6)$

62. $(-8, 0.6), (2, -2.4)$

63. $(2, -1), \left(\frac{1}{3}, -1\right)$

64. $\left(\frac{7}{3}, -8\right), \left(\frac{7}{3}, 1\right)$

Parallel and Perpendicular Lines **In Exercises 65–68, determine whether the lines are parallel, perpendicular, or neither.**

65. $L_1\colon\ y = \frac{1}{3}x - 2$
$L_2\colon\ y = \frac{1}{3}x + 3$

66. $L_1\colon\ y = 4x - 1$
$L_2\colon\ y = 4x + 7$

67. $L_1\colon\ y = \frac{1}{2}x - 3$
$L_2\colon\ y = -\frac{1}{2}x + 1$

68. $L_1\colon\ y = -\frac{4}{5}x - 5$
$L_2\colon\ y = \frac{5}{4}x + 1$

Parallel and Perpendicular Lines **In Exercises 69–72, determine whether the lines L_1 and L_2 passing through the pairs of points are parallel, perpendicular, or neither.**

69. $L_1\colon\ (0, -1), (5, 9)$
$L_2\colon\ (0, 3), (4, 1)$

70. $L_1\colon\ (-2, -1), (1, 5)$
$L_2\colon\ (1, 3), (5, -5)$

71. $L_1\colon\ (3, 6), (-6, 0)$
$L_2\colon\ (0, -1), \left(5, \frac{7}{3}\right)$

72. $L_1\colon\ (4, 8), (-4, 2)$
$L_2\colon\ (3, -5), \left(-1, \frac{1}{3}\right)$

Finding Parallel and Perpendicular Lines **In Exercises 73–80, write equations of the lines through the given point (a) parallel to and (b) perpendicular to the given line.**

73. $4x - 2y = 3, \quad (2, 1)$

74. $x + y = 7, \quad (-3, 2)$

75. $3x + 4y = 7, \quad \left(-\frac{2}{3}, \frac{7}{8}\right)$

76. $5x + 3y = 0, \quad \left(\frac{7}{8}, \frac{3}{4}\right)$

77. $y + 3 = 0, \quad (-1, 0)$

78. $x - 4 = 0, \quad (3, -2)$

79. $x - y = 4, \quad (2.5, 6.8)$

80. $6x + 2y = 9, \quad (-3.9, -1.4)$

Intercept Form of the Equation of a Line **In Exercises 81–86, use the *intercept form* to find the equation of the line with the given intercepts. The intercept form of the equation of a line with intercepts $(a, 0)$ and $(0, b)$ is**

$$\frac{x}{a} + \frac{y}{b} = 1, \quad a \neq 0, \quad b \neq 0.$$

81. x-intercept: $(2, 0)$
y-intercept: $(0, 3)$

82. x-intercept: $(-3, 0)$
y-intercept: $(0, 4)$

83. x-intercept: $\left(-\frac{1}{6}, 0\right)$
y-intercept: $\left(0, -\frac{2}{3}\right)$

84. x-intercept: $\left(\frac{2}{3}, 0\right)$
y-intercept: $(0, -2)$

85. Point on line: $(1, 2)$
x-intercept: $(c, 0)$
y-intercept: $(0, c), \quad c \neq 0$

86. Point on line: $(-3, 4)$
x-intercept: $(d, 0)$
y-intercept: $(0, d), \quad d \neq 0$

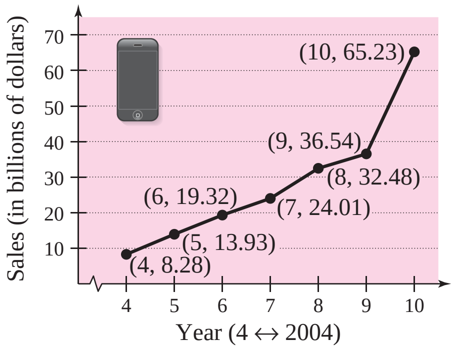
87. **Sales** The following are the slopes of lines representing annual sales y in terms of time x in years. Use the slopes to interpret any change in annual sales for a one-year increase in time.

(a) The line has a slope of $m = 135$.

(b) The line has a slope of $m = 0$.

(c) The line has a slope of $m = -40$.

88. **Sales** The graph shows the sales (in billions of dollars) for Apple Inc. in the years 2004 through 2010. *(Source: Apple Inc.)*



(a) Use the slopes of the line segments to determine the years in which the sales showed the greatest increase and the least increase.

(b) Find the slope of the line segment connecting the points for the years 2004 and 2010.

(c) Interpret the meaning of the slope in part (b) in the context of the problem.

89. Road Grade You are driving on a road that has a 6% uphill grade. This means that the slope of the road is $\frac{6}{100}$. Approximate the amount of vertical change in your position when you drive 200 feet.

90. Road Grade

From the top of a mountain road, a surveyor takes several horizontal measurements x and several vertical measurements y, as shown in the table (x and y are measured in feet).

x	300	600	900	1200
y	-25	-50	-75	-100

x	1500	1800	2100
y	-125	-150	-175

(a) Sketch a scatter plot of the data.

(b) Use a straightedge to sketch the line that you think best fits the data.

(c) Find an equation for the line you sketched in part (b).

(d) Interpret the meaning of the slope of the line in part (c) in the context of the problem.

(e) The surveyor needs to put up a road sign that indicates the steepness of the road. For instance, a surveyor would put up a sign that states "8% grade" on a road with a downhill grade that has a slope of $-\frac{8}{100}$. What should the sign state for the road in this problem?

Rate of Change **In Exercises 91 and 92, you are given the dollar value of a product in 2013 and the rate at which the value of the product is expected to change during the next 5 years. Use this information to write a linear equation that gives the dollar value V of the product in terms of the year t. (Let $t = 13$ represent 2013.)**

	2013 Value	Rate
91.	$2540	$125 decrease per year
92.	$156	$4.50 increase per year

93. Cost The cost C of producing n computer laptop bags is given by

$$C = 1.25n + 15{,}750, \quad 0 < n.$$

Explain what the C-intercept and the slope measure.

94. Monthly Salary A pharmaceutical salesperson receives a monthly salary of $2500 plus a commission of 7% of sales. Write a linear equation for the salesperson's monthly wage W in terms of monthly sales S.

95. Depreciation A sub shop purchases a used pizza oven for $875. After 5 years, the oven will have to be discarded and replaced. Write a linear equation giving the value V of the equipment during the 5 years it will be in use.

96. Depreciation A school district purchases a high-volume printer, copier, and scanner for $24,000. After 10 years, the equipment will have to be replaced. Its value at that time is expected to be $2000. Write a linear equation giving the value V of the equipment during the 10 years it will be in use.

97. Temperature Conversion Write a linear equation that expresses the relationship between the temperature in degrees Celsius C and degrees Fahrenheit F. Use the fact that water freezes at 0°C (32°F) and boils at 100°C (212°F).

98. Brain Weight The average weight of a male child's brain is 970 grams at age 1 and 1270 grams at age 3. *(Source: American Neurological Association)*

(a) Assuming that the relationship between brain weight y and age t is linear, write a linear model for the data.

(b) What is the slope and what does it tell you about brain weight?

(c) Use your model to estimate the average brain weight at age 2.

(d) Use your school's library, the Internet, or some other reference source to find the actual average brain weight at age 2. How close was your estimate?

(e) Do you think your model could be used to determine the average brain weight of an adult? Explain.

99. Cost, Revenue, and Profit A roofing contractor purchases a shingle delivery truck with a shingle elevator for $42,000. The vehicle requires an average expenditure of $9.50 per hour for fuel and maintenance, and the operator is paid $11.50 per hour.

(a) Write a linear equation giving the total cost C of operating this equipment for t hours. (Include the purchase cost of the equipment.)

(b) Assuming that customers are charged $45 per hour of machine use, write an equation for the revenue R derived from t hours of use.

(c) Use the formula for profit $P = R - C$ to write an equation for the profit derived from t hours of use.

(d) Use the result of part (c) to find the break-even point—that is, the number of hours this equipment must be used to yield a profit of 0 dollars.

Dmitry Kalinovsky/Shutterstock.com

100. Geometry The length and width of a rectangular garden are 15 meters and 10 meters, respectively. A walkway of width x surrounds the garden.

(a) Draw a diagram that gives a visual representation of the problem.

(b) Write the equation for the perimeter y of the walkway in terms of x.

(c) Use a graphing utility to graph the equation for the perimeter.

(d) Determine the slope of the graph in part (c). For each additional one-meter increase in the width of the walkway, determine the increase in its perimeter.

Exploration

True or False? **In Exercises 101 and 102, determine whether the statement is true or false. Justify your answer.**

101. A line with a slope of $-\frac{5}{7}$ is steeper than a line with a slope of $-\frac{6}{7}$.

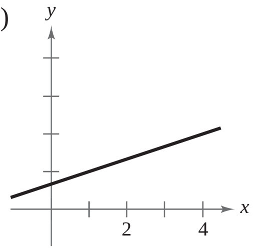
102. The line through $(-8, 2)$ and $(-1, 4)$ and the line through $(0, -4)$ and $(-7, 7)$ are parallel.

103. Right Triangle Explain how you could use slope to show that the points $A(-1, 5)$, $B(3, 7)$, and $C(5, 3)$ are the vertices of a right triangle.

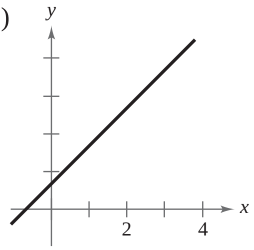
104. Vertical Line Explain why the slope of a vertical line is said to be undefined.

105. Think About It With the information shown in the graphs, is it possible to determine the slope of each line? Is it possible that the lines could have the same slope? Explain.

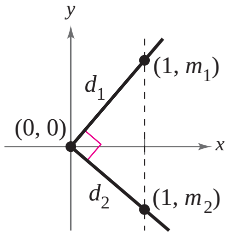
(a)



(b)



106. Perpendicular Segments Find d_1 and d_2 in terms of m_1 and m_2, respectively (see figure). Then use the Pythagorean Theorem to find a relationship between m_1 and m_2.



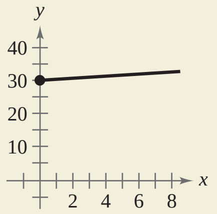
107. Think About It Is it possible for two lines with positive slopes to be perpendicular? Explain.

108. Slope and Steepness The slopes of two lines are -4 and $\frac{5}{2}$. Which is steeper? Explain.

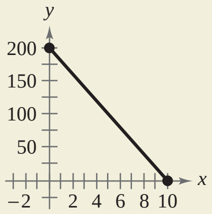
109. Comparing Slopes Use a graphing utility to compare the slopes of the lines $y = mx$, where $m = 0.5$, 1, 2, and 4. Which line rises most quickly? Now, let $m = -0.5$, -1, -2, and -4. Which line falls most quickly? Use a square setting to obtain a true geometric perspective. What can you conclude about the slope and the "rate" at which the line rises or falls?

110. HOW DO YOU SEE IT? Match the description of the situation with its graph. Also determine the slope and y-intercept of each graph and interpret the slope and y-intercept in the context of the situation. [The graphs are labeled (i), (ii), (iii), and (iv).]

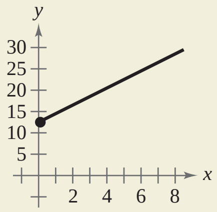
(i)



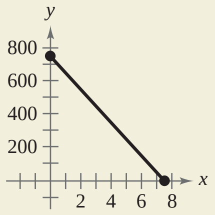
(ii)



(iii)



(iv)



(a) A person is paying \$20 per week to a friend to repay a \$200 loan.

(b) An employee receives \$12.50 per hour plus \$2 for each unit produced per hour.

(c) A sales representative receives \$30 per day for food plus \$0.32 for each mile traveled.

(d) A computer that was purchased for \$750 depreciates \$100 per year.

Finding a Relationship for Equidistance **In Exercises 111–114, find a relationship between x and y such that (x, y) is equidistant (the same distance) from the two points.**

111. $(4, -1), (-2, 3)$

112. $(6, 5), (1, -8)$

113. $\left(3, \frac{5}{2}\right), (-7, 1)$

114. $\left(-\frac{1}{2}, -4\right), \left(\frac{7}{2}, \frac{5}{4}\right)$

Project: Bachelor's Degrees To work an extended application analyzing the numbers of bachelor's degrees earned by women in the United States from 1998 through 2009, visit this text's website at *LarsonPrecalculus.com*. *(Source: National Center for Education Statistics)*

2.2 Functions

Functions can help you model and solve real-life problems. For instance, in Exercise 74 on page 185, you will use a function to model the force of water against the face of a dam.

- **Determine whether relations between two variables are functions, and use function notation.**
- **Find the domains of functions.**
- **Use functions to model and solve real-life problems.**
- **Evaluate difference quotients.**

Introduction to Functions and Function Notation

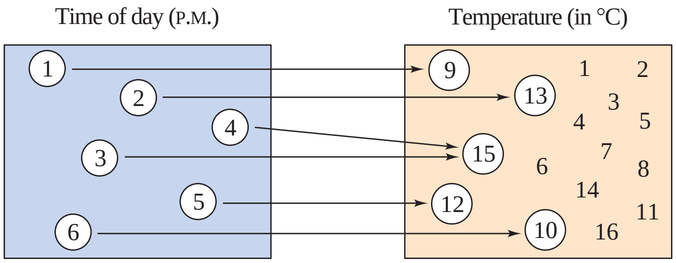
Many everyday phenomena involve two quantities that are related to each other by some rule of correspondence. The mathematical term for such a rule of correspondence is a **relation.** In mathematics, equations and formulas often represent relations. For instance, the simple interest I earned on \$1000 for 1 year is related to the annual interest rate r by the formula $I = 1000r$.

The formula $I = 1000r$ represents a special kind of relation that matches each item from one set with *exactly one* item from a different set. Such a relation is called a **function.**

Definition of Function

A **function** f from a set A to a set B is a relation that assigns to each element x in the set A exactly one element y in the set B. The set A is the **domain** (or set of inputs) of the function f, and the set B contains the **range** (or set of outputs).

To help understand this definition, look at the function below, which relates the time of day to the temperature.



Set A is the domain.
Inputs: 1, 2, 3, 4, 5, 6

Set B contains the range.
Outputs: 9, 10, 12, 13, 15

The following ordered pairs can represent this function. The first coordinate (x-value) is the input and the second coordinate (y-value) is the output.

$$\{(1, 9), (2, 13), (3, 15), (4, 15), (5, 12), (6, 10)\}$$

Characteristics of a Function from Set *A* to Set *B*

1. Each element in A must be matched with an element in B.
2. Some elements in B may not be matched with any element in A.
3. Two or more elements in A may be matched with the same element in B.
4. An element in A (the domain) cannot be matched with two different elements in B.

Lester Lefkowitz/CORBIS

Four common ways to represent functions are as follows.

Four Ways to Represent a Function

1. *Verbally* by a sentence that describes how the input variable is related to the output variable
2. *Numerically* by a table or a list of ordered pairs that matches input values with output values
3. *Graphically* by points on a graph in a coordinate plane in which the horizontal axis represents the input values and the vertical axis represents the output values
4. *Algebraically* by an equation in two variables

To determine whether a relation is a function, you must decide whether each input value is matched with exactly one output value. When any input value is matched with two or more output values, the relation is not a function.

EXAMPLE 1 Testing for Functions

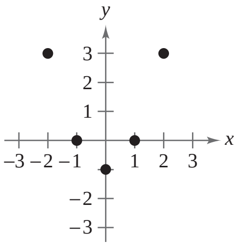
Determine whether the relation represents y as a function of x.

a. The input value x is the number of representatives from a state, and the output value y is the number of senators.

b.

Input, x	Output, y
2	11
2	10
3	8
4	5
5	1

c.



Solution

a. This verbal description *does* describe y as a function of x. Regardless of the value of x, the value of y is always 2. Such functions are called *constant functions*.

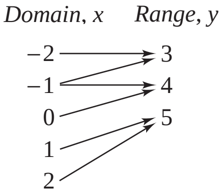
b. This table *does not* describe y as a function of x. The input value 2 is matched with two different y-values.

c. The graph *does* describe y as a function of x. Each input value is matched with exactly one output value.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Determine whether the relation represents y as a function of x.

a.



b.

Input, x	0	1	2	3	4
Output, y	−4	−2	0	2	4

Representing functions by sets of ordered pairs is common in *discrete mathematics*. In algebra, however, it is more common to represent functions by equations or formulas involving two variables. For instance, the equation

$$y = x^2 \qquad y \text{ is a function of } x.$$

represents the variable y as a function of the variable x. In this equation, x is the **independent variable** and y is the **dependent variable.** The domain of the function is the set of all values taken on by the independent variable x, and the range of the function is the set of all values taken on by the dependent variable y.

HISTORICAL NOTE

Many consider Leonhard Euler (1707–1783), a Swiss mathematician, the most prolific and productive mathematician in history. One of his greatest influences on mathematics was his use of symbols, or notation. Euler introduced the function notation $y = f(x)$.

Bettmann/Corbis

EXAMPLE 2 Testing for Functions Represented Algebraically

Which of the equations represent(s) y as a function of x?

a. $x^2 + y = 1$

b. $-x + y^2 = 1$

Solution To determine whether y is a function of x, try to solve for y in terms of x.

a. Solving for y yields

$$x^2 + y = 1 \qquad \text{Write original equation.}$$

$$y = 1 - x^2. \qquad \text{Solve for } y.$$

To each value of x there corresponds exactly one value of y. So, y is a function of x.

b. Solving for y yields

$$-x + y^2 = 1 \qquad \text{Write original equation.}$$

$$y^2 = 1 + x \qquad \text{Add } x \text{ to each side.}$$

$$y = \pm\sqrt{1 + x}. \qquad \text{Solve for } y.$$

The ± indicates that to a given value of x there correspond two values of y. So, y is not a function of x.

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Which of the equations represent(s) y as a function of x?

a. $x^2 + y^2 = 8$ **b.** $y - 4x^2 = 36$

When using an equation to represent a function, it is convenient to name the function for easy reference. For example, you know that the equation $y = 1 - x^2$ describes y as a function of x. Suppose you give this function the name "f." Then you can use the following **function notation.**

Input	**Output**	**Equation**
x	$f(x)$	$f(x) = 1 - x^2$

The symbol $f(x)$ is read as *the value of f at x* or simply *f of x*. The symbol $f(x)$ corresponds to the y-value for a given x. So, you can write $y = f(x)$. Keep in mind that f is the *name* of the function, whereas $f(x)$ is the *value* of the function at x. For instance, the function $f(x) = 3 - 2x$ has *function values* denoted by $f(-1)$, $f(0)$, $f(2)$, and so on. To find these values, substitute the specified input values into the given equation.

For $x = -1$, $\quad f(-1) = 3 - 2(-1) = 3 + 2 = 5.$

For $x = 0$, $\quad f(0) = 3 - 2(0) = 3 - 0 = 3.$

For $x = 2$, $\quad f(2) = 3 - 2(2) = 3 - 4 = -1.$

Although f is often used as a convenient function name and x is often used as the independent variable, you can use other letters. For instance,

$$f(x) = x^2 - 4x + 7, \quad f(t) = t^2 - 4t + 7, \quad \text{and} \quad g(s) = s^2 - 4s + 7$$

all define the same function. In fact, the role of the independent variable is that of a "placeholder." Consequently, the function could be described by

$$f(\square) = (\square)^2 - 4(\square) + 7.$$

EXAMPLE 3 Evaluating a Function

Let $g(x) = -x^2 + 4x + 1$. Find each function value.

a. $g(2)$ **b.** $g(t)$ **c.** $g(x + 2)$

Solution

a. Replacing x with 2 in $g(x) = -x^2 + 4x + 1$ yields the following.

$$\begin{aligned} g(2) &= -(2)^2 + 4(2) + 1 \\ &= -4 + 8 + 1 \\ &= 5 \end{aligned}$$

b. Replacing x with t yields the following.

$$\begin{aligned} g(t) &= -(t)^2 + 4(t) + 1 \\ &= -t^2 + 4t + 1 \end{aligned}$$

c. Replacing x with $x + 2$ yields the following.

$$\begin{aligned} g(x + 2) &= -(x + 2)^2 + 4(x + 2) + 1 \\ &= -(x^2 + 4x + 4) + 4x + 8 + 1 \\ &= -x^2 - 4x - 4 + 4x + 8 + 1 \\ &= -x^2 + 5 \end{aligned}$$

REMARK In Example 3(c), note that $g(x + 2)$ is not equal to $g(x) + g(2)$. In general, $g(u + v) \neq g(u) + g(v)$.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Let $f(x) = 10 - 3x^2$. Find each function value.

a. $f(2)$ **b.** $f(-4)$ **c.** $f(x - 1)$

A function defined by two or more equations over a specified domain is called a **piecewise-defined function.**

EXAMPLE 4 A Piecewise-Defined Function

Evaluate the function when $x = -1$, 0, and 1.

$$f(x) = \begin{cases} x^2 + 1, & x < 0 \\ x - 1, & x \geq 0 \end{cases}$$

Solution Because $x = -1$ is less than 0, use $f(x) = x^2 + 1$ to obtain $f(-1) = (-1)^2 + 1 = 2$. For $x = 0$, use $f(x) = x - 1$ to obtain $f(0) = (0) - 1 = -1$. For $x = 1$, use $f(x) = x - 1$ to obtain $f(1) = (1) - 1 = 0$.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Evaluate the function given in Example 4 when $x = -2$, 2, and 3.

EXAMPLE 5 Finding Values for Which $f(x) = 0$

Find all real values of x such that $f(x) = 0$.

a. $f(x) = -2x + 10$ **b.** $f(x) = x^2 - 5x + 6$

Solution For each function, set $f(x) = 0$ and solve for x.

a. $-2x + 10 = 0$ Set $f(x)$ equal to 0.

$-2x = -10$ Subtract 10 from each side.

$x = 5$ Divide each side by -2.

So, $f(x) = 0$ when $x = 5$.

b. $x^2 - 5x + 6 = 0$ Set $f(x)$ equal to 0.

$(x - 2)(x - 3) = 0$ Factor.

$x - 2 = 0 \Rightarrow x = 2$ Set 1st factor equal to 0.

$x - 3 = 0 \Rightarrow x = 3$ Set 2nd factor equal to 0.

So, $f(x) = 0$ when $x = 2$ or $x = 3$.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find all real values of x such that $f(x) = 0$, where $f(x) = x^2 - 16$.

EXAMPLE 6 Finding Values for Which $f(x) = g(x)$

Find the values of x for which $f(x) = g(x)$.

a. $f(x) = x^2 + 1$ and $g(x) = 3x - x^2$

b. $f(x) = x^2 - 1$ and $g(x) = -x^2 + x + 2$

Solution

a. $x^2 + 1 = 3x - x^2$ Set $f(x)$ equal to $g(x)$.

$2x^2 - 3x + 1 = 0$ Write in general form.

$(2x - 1)(x - 1) = 0$ Factor.

$2x - 1 = 0 \Rightarrow x = \frac{1}{2}$ Set 1st factor equal to 0.

$x - 1 = 0 \Rightarrow x = 1$ Set 2nd factor equal to 0.

So, $f(x) = g(x)$ when $x = \dfrac{1}{2}$ or $x = 1$.

b. $x^2 - 1 = -x^2 + x + 2$ Set $f(x)$ equal to $g(x)$.

$2x^2 - x - 3 = 0$ Write in general form.

$(2x - 3)(x + 1) = 0$ Factor.

$2x - 3 = 0 \Rightarrow x = \frac{3}{2}$ Set 1st factor equal to 0.

$x + 1 = 0 \Rightarrow x = -1$ Set 2nd factor equal to 0.

So, $f(x) = g(x)$ when $x = \dfrac{3}{2}$ or $x = -1$.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the values of x for which $f(x) = g(x)$, where $f(x) = x^2 + 6x - 24$ and $g(x) = 4x - x^2$.

The Domain of a Function

TECHNOLOGY Use a graphing utility to graph the functions $y = \sqrt{4 - x^2}$ and $y = \sqrt{x^2 - 4}$. What is the domain of each function? Do the domains of these two functions overlap? If so, for what values do the domains overlap?

The domain of a function can be described explicitly or it can be *implied* by the expression used to define the function. The **implied domain** is the set of all real numbers for which the expression is defined. For instance, the function

$$f(x) = \frac{1}{x^2 - 4} \qquad \text{Domain excludes } x\text{-values that result in division by zero.}$$

has an implied domain consisting of all real x other than $x = \pm 2$. These two values are excluded from the domain because division by zero is undefined. Another common type of implied domain is that used to avoid even roots of negative numbers. For example, the function

$$f(x) = \sqrt{x} \qquad \text{Domain excludes } x\text{-values that result in even roots of negative numbers.}$$

is defined only for $x \geq 0$. So, its implied domain is the interval $[0, \infty)$. In general, the domain of a function *excludes* values that would cause division by zero *or* that would result in the even root of a negative number.

EXAMPLE 7 Finding the Domain of a Function

Find the domain of each function.

a. $f\colon \{(-3, 0), (-1, 4), (0, 2), (2, 2), (4, -1)\}$

b. $g(x) = \dfrac{1}{x + 5}$

c. Volume of a sphere: $V = \frac{4}{3}\pi r^3$

d. $h(x) = \sqrt{4 - 3x}$

Solution

a. The domain of f consists of all first coordinates in the set of ordered pairs.

$$\text{Domain} = \{-3, -1, 0, 2, 4\}$$

b. Excluding x-values that yield zero in the denominator, the domain of g is the set of all real numbers x except $x = -5$.

c. Because this function represents the volume of a sphere, the values of the radius r must be positive. So, the domain is the set of all real numbers r such that $r > 0$.

d. This function is defined only for x-values for which

$$4 - 3x \geq 0.$$

Using the methods described in Section 1.8, you can conclude that $x \leq \frac{4}{3}$. So, the domain is the interval $\left(-\infty, \frac{4}{3}\right]$.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the domain of each function.

a. $f\colon \{(-2, 2), (-1, 1), (0, 3), (1, 1), (2, 2)\}$

b. $g(x) = \dfrac{1}{3 - x}$

c. Circumference of a circle: $C = 2\pi r$

d. $h(x) = \sqrt{x - 16}$

In Example 7(c), note that the domain of a function may be implied by the physical context. For instance, from the equation

$$V = \tfrac{4}{3}\pi r^3$$

you would have no reason to restrict r to positive values, but the physical context implies that a sphere cannot have a negative or zero radius.

Applications

EXAMPLE 8 The Dimensions of a Container

You work in the marketing department of a soft-drink company and are experimenting with a new can for iced tea that is slightly narrower and taller than a standard can. For your experimental can, the ratio of the height to the radius is 4.

a. Write the volume of the can as a function of the radius r.

b. Write the volume of the can as a function of the height h.

Solution

a. $V(r) = \pi r^2 h = \pi r^2(4r) = 4\pi r^3$

Write V as a function of r.

b. $V(h) = \pi r^2 h = \pi\left(\frac{h}{4}\right)^2 h = \frac{\pi h^3}{16}$

Write V as a function of h.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

For the experimental can described in Example 8, write the *surface area* as a function of (a) the radius r and (b) the height h.

EXAMPLE 9 The Path of a Baseball

A batter hits a baseball at a point 3 feet above ground at a velocity of 100 feet per second and an angle of 45º. The path of the baseball is given by the function

$$f(x) = -0.0032x^2 + x + 3$$

where $f(x)$ is the height of the baseball (in feet) and x is the horizontal distance from home plate (in feet). Will the baseball clear a 10-foot fence located 300 feet from home plate?

Algebraic Solution

When $x = 300$, you can find the height of the baseball as follows.

$$f(x) = -0.0032x^2 + x + 3$$

Write original function.

$$f(300) = -0.0032(300)^2 + 300 + 3$$

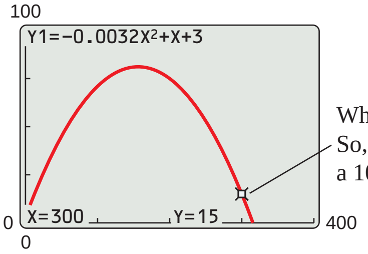
Substitute 300 for x.

$$= 15$$

Simplify.

When $x = 300$, the height of the baseball is 15 feet. So, the baseball will clear a 10-foot fence.

Graphical Solution



When $x = 300$, $y = 15$. So, the ball will clear a 10-foot fence.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

A second baseman throws a baseball toward the first baseman 60 feet away. The path of the baseball is given by

$$f(x) = -0.004x^2 + 0.3x + 6$$

where $f(x)$ is the height of the baseball (in feet) and x is the horizontal distance from the second baseman (in feet). The first baseman can reach 8 feet high. Can the first baseman catch the baseball without jumping?

Alternative fuels for vehicles include electricity, ethanol, hydrogen, compressed natural gas, liquefied natural gas, and liquefied petroleum gas.

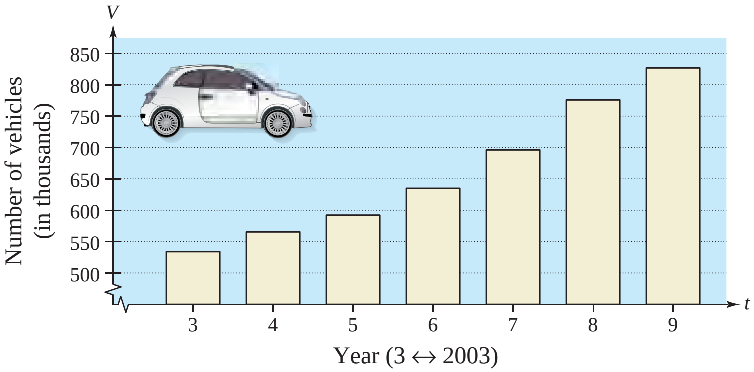
EXAMPLE 10 Alternative-Fueled Vehicles

The number V (in thousands) of alternative-fueled vehicles in the United States increased in a linear pattern from 2003 through 2005, and then increased in a different linear pattern from 2006 through 2009, as shown in the bar graph. These two patterns can be approximated by the function

$$V(t) = \begin{cases} 29.05t + 447.7, & 3 \le t \le 5 \\ 65.50t + 241.9, & 6 \le t \le 9 \end{cases}$$

where t represents the year, with $t = 3$ corresponding to 2003. Use this function to approximate the number of alternative-fueled vehicles for each year from 2003 through 2009. *(Source: U.S. Energy Information Administration)*

Number of Alternative-Fueled Vehicles in the U.S.



Solution From 2003 through 2005, use $V(t) = 29.05t + 447.7$.

$$\underbrace{534.9}_{2003} \quad \underbrace{563.9}_{2004} \quad \underbrace{593.0}_{2005}$$

From 2006 to 2009, use $V(t) = 65.50t + 241.9$.

$$\underbrace{634.9}_{2006} \quad \underbrace{700.4}_{2007} \quad \underbrace{765.9}_{2008} \quad \underbrace{831.4}_{2009}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The number V (in thousands) of 85%-ethanol-fueled vehicles in the United States from 2003 through 2009 can be approximated by the function

$$V(t) = \begin{cases} 33.65t + 77.8, & 3 \le t \le 5 \\ 70.75t - 126.6, & 6 \le t \le 9 \end{cases}$$

where t represents the year, with $t = 3$ corresponding to 2003. Use this function to approximate the number of 85%-ethanol-fueled vehicles for each year from 2003 through 2009. *(Source: U.S. Energy Information Administration)*

Difference Quotients

One of the basic definitions in calculus employs the ratio

$$\frac{f(x + h) - f(x)}{h}, \quad h \neq 0.$$

This ratio is called a **difference quotient,** as illustrated in Example 11.

wellphoto/Shutterstock.com

EXAMPLE 11 Evaluating a Difference Quotient

For $f(x) = x^2 - 4x + 7$, find $\dfrac{f(x+h) - f(x)}{h}$.

Solution

$$\frac{f(x+h) - f(x)}{h} = \frac{[(x+h)^2 - 4(x+h) + 7] - (x^2 - 4x + 7)}{h}$$

$$= \frac{x^2 + 2xh + h^2 - 4x - 4h + 7 - x^2 + 4x - 7}{h}$$

$$= \frac{2xh + h^2 - 4h}{h} = \frac{h(2x + h - 4)}{h} = 2x + h - 4, \quad h \neq 0$$

REMARK You may find it easier to calculate the difference quotient in Example 11 by first finding $f(x + h)$, and then substituting the resulting expression into the difference quotient

$$\frac{f(x+h) - f(x)}{h}.$$

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

For $f(x) = x^2 + 2x - 3$, find $\dfrac{f(x+h) - f(x)}{h}$.

Summary of Function Terminology

Function: A **function** is a relationship between two variables such that to each value of the independent variable there corresponds exactly one value of the dependent variable.

Function Notation: $y = f(x)$

f is the *name* of the function.

y is the **dependent variable.**

x is the **independent variable.**

$f(x)$ is the *value of the function at x.*

Domain: The **domain** of a function is the set of all values (inputs) of the independent variable for which the function is defined. If x is in the domain of f, then f is said to be *defined* at x. If x is not in the domain of f, then f is said to be *undefined* at x.

Range: The **range** of a function is the set of all values (outputs) assumed by the dependent variable (that is, the set of all function values).

Implied Domain: If f is defined by an algebraic expression and the domain is not specified, then the **implied domain** consists of all real numbers for which the expression is defined.

Summarize (Section 2.2)

1. State the definition of a function and describe function notation *(pages 173–176)*. For examples of determining functions and using function notation, see Examples 1–6.
2. State the definition of the implied domain of a function *(page 178)*. For an example of finding the domains of functions, see Example 7.
3. Describe examples of how functions can model real-life problems *(pages 179 and 180, Examples 8–10)*.
4. State the definition of a difference quotient *(page 180)*. For an example of evaluating a difference quotient, see Example 11.

2.2 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

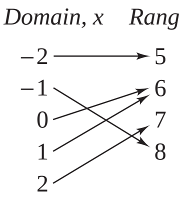
Vocabulary: Fill in the blanks.

1. A relation that assigns to each element x from a set of inputs, or _________, exactly one element y in a set of outputs, or _________, is called a _________.
2. For an equation that represents y as a function of x, the set of all values taken on by the _________ variable x is the domain, and the set of all values taken on by the _________ variable y is the range.
3. If the domain of the function f is not given, then the set of values of the independent variable for which the expression is defined is called the _________ _________.
4. In calculus, one of the basic definitions is that of a _________ _________, given by $\dfrac{f(x+h)-f(x)}{h}$, $h \neq 0$.

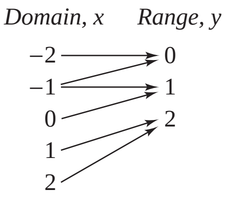
Skills and Applications

Testing for Functions **In Exercises 5–8, determine whether the relation represents y as a function of x.**

5.



6.



7.

Input, x	10	7	4	7	10
Output, y	3	6	9	12	15

8.

Input, x	0	3	9	12	15
Output, y	3	3	3	3	3

Testing for Functions **In Exercises 9 and 10, which sets of ordered pairs represent functions from A to B? Explain.**

9. $A = \{0, 1, 2, 3\}$ and $B = \{-2, -1, 0, 1, 2\}$
   (a) $\{(0, 1), (1, -2), (2, 0), (3, 2)\}$
   (b) $\{(0, -1), (2, 2), (1, -2), (3, 0), (1, 1)\}$
   (c) $\{(0, 0), (1, 0), (2, 0), (3, 0)\}$
   (d) $\{(0, 2), (3, 0), (1, 1)\}$

10. $A = \{a, b, c\}$ and $B = \{0, 1, 2, 3\}$
   (a) $\{(a, 1), (c, 2), (c, 3), (b, 3)\}$
   (b) $\{(a, 1), (b, 2), (c, 3)\}$
   (c) $\{(1, a), (0, a), (2, c), (3, b)\}$
   (d) $\{(c, 0), (b, 0), (a, 3)\}$

Testing for Functions Represented Algebraically **In Exercises 11–20, determine whether the equation represents y as a function of x.**

11. $x^2 + y^2 = 4$
12. $x^2 + y = 4$
13. $2x + 3y = 4$
14. $(x - 2)^2 + y^2 = 4$
15. $y = \sqrt{16 - x^2}$
16. $y = \sqrt{x + 5}$
17. $y = |4 - x|$
18. $|y| = 4 - x$
19. $y = -75$
20. $x - 1 = 0$

Evaluating a Function **In Exercises 21–32, evaluate (if possible) the function at each specified value of the independent variable and simplify.**

21. $f(x) = 2x - 3$
   (a) $f(1)$ (b) $f(-3)$ (c) $f(x - 1)$

22. $V(r) = \frac{4}{3}\pi r^3$
   (a) $V(3)$ (b) $V\left(\frac{3}{2}\right)$ (c) $V(2r)$

23. $g(t) = 4t^2 - 3t + 5$
   (a) $g(2)$ (b) $g(t - 2)$ (c) $g(t) - g(2)$

24. $h(t) = t^2 - 2t$
   (a) $h(2)$ (b) $h(1.5)$ (c) $h(x + 2)$

25. $f(y) = 3 - \sqrt{y}$
   (a) $f(4)$ (b) $f(0.25)$ (c) $f(4x^2)$

26. $f(x) = \sqrt{x + 8} + 2$
   (a) $f(-8)$ (b) $f(1)$ (c) $f(x - 8)$

27. $q(x) = 1/(x^2 - 9)$
   (a) $q(0)$ (b) $q(3)$ (c) $q(y + 3)$

28. $q(t) = (2t^2 + 3)/t^2$
   (a) $q(2)$ (b) $q(0)$ (c) $q(-x)$

29. $f(x) = |x|/x$
   (a) $f(2)$ (b) $f(-2)$ (c) $f(x - 1)$

30. $f(x) = |x| + 4$
   (a) $f(2)$ (b) $f(-2)$ (c) $f(x^2)$

31. $f(x) = \begin{cases} 2x + 1, & x < 0 \\ 2x + 2, & x \geq 0 \end{cases}$
   (a) $f(-1)$ (b) $f(0)$ (c) $f(2)$

32. $f(x) = \begin{cases} 4 - 5x, & x \leq -2 \\ 0, & -2 < x < 2 \\ x^2 + 1, & x \geq 2 \end{cases}$
   (a) $f(-3)$ (b) $f(4)$ (c) $f(-1)$

Evaluating a Function **In Exercises 33–36, complete the table.**

33. $f(x) = x^2 - 3$

x	-2	-1	0	1	2
$f(x)$					

34. $h(t) = \frac{1}{2}|t + 3|$

t	-5	-4	-3	-2	-1
$h(t)$					

35. $f(x) = \begin{cases} -\frac{1}{2}x + 4, & x \le 0 \\ (x - 2)^2, & x > 0 \end{cases}$

x	-2	-1	0	1	2
$f(x)$					

36. $f(x) = \begin{cases} 9 - x^2, & x < 3 \\ x - 3, & x \ge 3 \end{cases}$

x	1	2	3	4	5
$f(x)$					

Finding Values for Which $f(x) = 0$ **In Exercises 37–44, find all real values of x such that $f(x) = 0$.**

37. $f(x) = 15 - 3x$

38. $f(x) = 5x + 1$

39. $f(x) = \dfrac{3x - 4}{5}$

40. $f(x) = \dfrac{12 - x^2}{5}$

41. $f(x) = x^2 - 9$

42. $f(x) = x^2 - 8x + 15$

43. $f(x) = x^3 - x$

44. $f(x) = x^3 - x^2 - 4x + 4$

Finding Values for Which $f(x) = g(x)$ **In Exercises 45–48, find the value(s) of x for which $f(x) = g(x)$.**

45. $f(x) = x^2, \quad g(x) = x + 2$

46. $f(x) = x^2 + 2x + 1, \quad g(x) = 7x - 5$

47. $f(x) = x^4 - 2x^2, \quad g(x) = 2x^2$

48. $f(x) = \sqrt{x} - 4, \quad g(x) = 2 - x$

Finding the Domain of a Function **In Exercises 49–60, find the domain of the function.**

49. $f(x) = 5x^2 + 2x - 1$

50. $g(x) = 1 - 2x^2$

51. $h(t) = \dfrac{4}{t}$

52. $s(y) = \dfrac{3y}{y + 5}$

53. $g(y) = \sqrt{y - 10}$

54. $f(t) = \sqrt[3]{t + 4}$

55. $g(x) = \dfrac{1}{x} - \dfrac{3}{x + 2}$

56. $h(x) = \dfrac{10}{x^2 - 2x}$

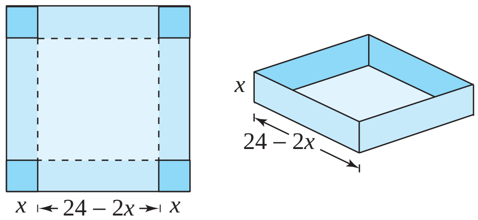
57. $f(s) = \dfrac{\sqrt{s - 1}}{s - 4}$

58. $f(x) = \dfrac{\sqrt{x + 6}}{6 + x}$

59. $f(x) = \dfrac{x - 4}{\sqrt{x}}$

60. $f(x) = \dfrac{x + 2}{\sqrt{x - 10}}$

61. Maximum Volume An open box of maximum volume is to be made from a square piece of material 24 centimeters on a side by cutting equal squares from the corners and turning up the sides (see figure).



(a) The table shows the volumes V (in cubic centimeters) of the box for various heights x (in centimeters). Use the table to estimate themaximum volume.

Height, x	1	2	3	4	5	6
Volume, V	484	800	972	1024	980	864

(b) Plot the points (x, V) from the table in part (a). Does the relation defined by the ordered pairs represent V as a function of x?

(c) Given that V is a function of x, write the function and determine its domain.

62. Maximum Profit The cost per unit in the production of an MP3 player is \$60. The manufacturer charges \$90 per unit for orders of 100 or less. To encourage large orders, the manufacturer reduces the charge by \$0.15 per MP3 player for each unit ordered in excess of 100 (for example, there would be a charge of \$87 per MP3 player for an order size of 120).

(a) The table shows the profits P (in dollars) for various numbers of units ordered, x. Use the table to estimate the maximum profit.

Units, x	130	140	150	160	170
Profit, P	3315	3360	3375	3360	3315

(b) Plot the points (x, P) from the table in part (a). Does the relation defined by the ordered pairs represent P as a function of x?

(c) Given that P is a function of x, write the function and determine its domain. (*Note:* $P = R - C$, where R is revenue and C is cost.)

63. Geometry Write the area A of a square as a function of its perimeter P.

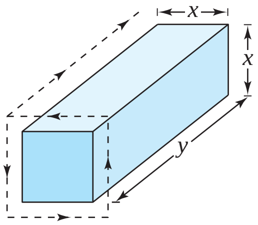
64. Geometry Write the area A of a circle as a function of its circumference C.

65. Path of a Ball The height y (in feet) of a baseball thrown by a child is

$$y = -\frac{1}{10}x^2 + 3x + 6$$

where x is the horizontal distance (in feet) from where the ball was thrown. Will the ball fly over the head of another child 30 feet away trying to catch the ball? (Assume that the child who is trying to catch the ball holds a baseball glove at a height of 5 feet.)

66. Postal Regulations A rectangular package to be sent by the U.S. Postal Service can have a maximum combined length and girth (perimeter of a cross section) of 108 inches (see figure).



(a) Write the volume V of the package as a function of x. What is the domain of the function?

(b) Use a graphing utility to graph the function. Be sure to use an appropriate window setting.

(c) What dimensions will maximize the volume of the package? Explain your answer.

67. Geometry A right triangle is formed in the first quadrant by the x- and y-axes and a line through the point $(2, 1)$ (see figure). Write the area A of the triangle as a function of x, and determine the domain of the function.

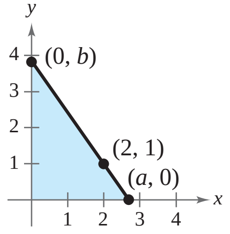


Figure for 67

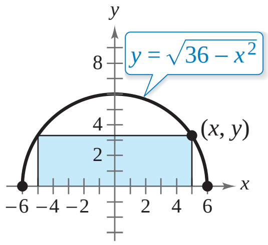


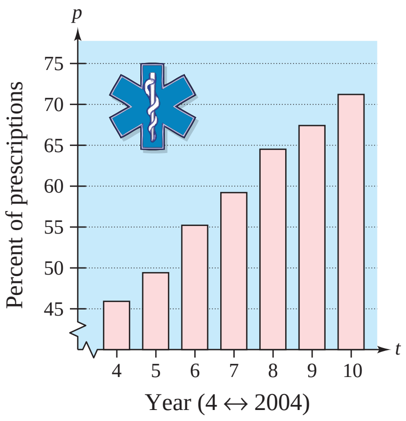
Figure for 68

68. Geometry A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{36 - x^2}$ (see figure). Write the area A of the rectangle as a function of x, and graphically determine the domain of the function.

69. Prescription Drugs The percents p of prescriptions filled with generic drugs in the United States from 2004 through 2010 (see figure) can be approximated by the model

$$p(t) = \begin{cases} 4.57t + 27.3, & 4 \le t \le 7 \\ 3.35t + 37.6, & 8 \le t \le 10 \end{cases}$$

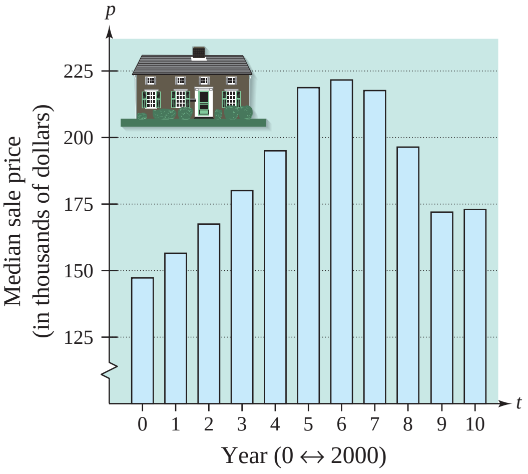
where t represents the year, with $t = 4$ corresponding to 2004. Use this model to find the percent of prescriptions filled with generic drugs in each year from 2004 through 2010. (*Source: National Association of Chain Drug Stores*)



70. Median Sale Price The median sale prices p (in thousands of dollars) of an existing one-family home in the United States from 2000 through 2010 (see figure) can be approximated by the model

$$p(t) = \begin{cases} 0.438t^2 + 10.81t + 145.9, & 0 \le t \le 6 \\ 5.575t^2 - 110.67t + 720.8, & 7 \le t \le 10 \end{cases}$$

where t represents the year, with $t = 0$ corresponding to 2000. Use this model to find the median sale price of an existing one-family home in each year from 2000 through 2010. (*Source: National Association of Realtors*)



71. **Cost, Revenue, and Profit** A company produces a product for which the variable cost is \$12.30 per unit and the fixed costs are \$98,000. The product sells for \$17.98. Let x be the number of units produced and sold.

(a) The total cost for a business is the sum of the variable cost and the fixed costs. Write the total cost C as a function of the number of units produced.

(b) Write the revenue R as a function of the number of units sold.

(c) Write the profit P as a function of the number of units sold. (*Note:* $P = R - C$)

72. **Average Cost** The inventor of a new game believes that the variable cost for producing the game is \$0.95 per unit and the fixed costs are \$6000. The inventor sells each game for \$1.69. Let x be the number of games sold.

(a) The total cost for a business is the sum of the variable cost and the fixed costs. Write the total cost C as a function of the number of games sold.

(b) Write the average cost per unit $\overline{C} = C/x$ as a function of x.

73. **Height of a Balloon** A balloon carrying a transmitter ascends vertically from a point 3000 feet from the receiving station.

(a) Draw a diagram that gives a visual representation of the problem. Let h represent the height of the balloon and let d represent the distance between the balloon and the receiving station.

(b) Write the height of the balloon as a function of d. What is the domain of the function?

74. **Physics**

The function $F(y) = 149.76\sqrt{10}y^{5/2}$ estimates the force F (in tons) of water against the face of a dam, where y is the depth of the water (in feet).

(a) Complete the table. What can you conclude from the table?

y	5	10	20	30	40
$F(y)$					

(b) Use the table to approximate the depth at which the force against the dam is 1,000,000 tons.

(c) Find the depth at which the force against the dam is 1,000,000 tons algebraically.

75. **Transportation** For groups of 80 or more people, a charter bus company determines the rate per person according to the formula

$$\text{Rate} = 8 - 0.05(n - 80), \quad n \geq 80$$

where the rate is given in dollars and n is the number of people.

(a) Write the revenue R for the bus company as a function of n.

(b) Use the function in part (a) to complete the table. What can you conclude?

n	90	100	110	120	130	140	150
$R(n)$							

76. **E-Filing** The table shows the numbers of tax returns (in millions) made through e-file from 2003 through 2010. Let $f(t)$ represent the number of tax returns made through e-file in the year t. (*Source: Internal Revenue Service*)

Year	Number of Tax Returns Made Through E-File
2003	52.9
2004	61.5
2005	68.5
2006	73.3
2007	80.0
2008	89.9
2009	95.0
2010	98.7

Spreadsheet at LarsonPrecalculus.com

(a) Find $\dfrac{f(2010) - f(2003)}{2010 - 2003}$ and interpret the result in the context of the problem.

(b) Make a scatter plot of the data.

(c) Find a linear model for the data algebraically. Let N represent the number of tax returns made through e-file and let $t = 3$ correspond to 2003.

(d) Use the model found in part (c) to complete the table.

t	3	4	5	6	7	8	9	10
N								

(e) Compare your results from part (d) with the actual data.

(f) Use a graphing utility to find a linear model for the data. Let $x = 3$ correspond to 2003. How does the model you found in part (c) compare with the model given by the graphing utility?

Lester Lefkowitz/CORBIS

Evaluating a Difference Quotient **In Exercises 77–84, find the difference quotient and simplify your answer.**

77. $f(x) = x^2 - x + 1, \quad \dfrac{f(2+h) - f(2)}{h}, \quad h \neq 0$

78. $f(x) = 5x - x^2, \quad \dfrac{f(5+h) - f(5)}{h}, \quad h \neq 0$

79. $f(x) = x^3 + 3x, \quad \dfrac{f(x+h) - f(x)}{h}, \quad h \neq 0$

80. $f(x) = 4x^2 - 2x, \quad \dfrac{f(x+h) - f(x)}{h}, \quad h \neq 0$

81. $g(x) = \dfrac{1}{x^2}, \quad \dfrac{g(x) - g(3)}{x - 3}, \quad x \neq 3$

82. $f(t) = \dfrac{1}{t-2}, \quad \dfrac{f(t) - f(1)}{t - 1}, \quad t \neq 1$

83. $f(x) = \sqrt{5x}, \quad \dfrac{f(x) - f(5)}{x - 5}, \quad x \neq 5$

84. $f(x) = x^{2/3} + 1, \quad \dfrac{f(x) - f(8)}{x - 8}, \quad x \neq 8$

Matching and Determining Constants **In Exercises 85–88, match the data with one of the following functions**

$$f(x) = cx,\ g(x) = cx^2,\ h(x) = c\sqrt{|x|},\ \text{and}\ r(x) = \frac{c}{x}$$

and determine the value of the constant *c* that will make the function fit the data in the table.

85.

x	-4	-1	0	1	4
y	-32	-2	0	-2	-32

86.

x	-4	-1	0	1	4
y	-1	$-\frac{1}{4}$	0	$\frac{1}{4}$	1

87.

x	-4	-1	0	1	4
y	-8	-32	Undefined	32	8

88.

x	-4	-1	0	1	4
y	6	3	0	3	6

Exploration

True or False? **In Exercises 89–92, determine whether the statement is true or false. Justify your answer.**

89. Every relation is a function.

90. Every function is a relation.

91. For the function

$$f(x) = x^4 - 1$$

the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

92. The set of ordered pairs $\{(-8, -2), (-6, 0), (-4, 0), (-2, 2), (0, 4), (2, -2)\}$ represents a function.

93. **Think About It** Consider

$$f(x) = \sqrt{x-1} \quad \text{and} \quad g(x) = \frac{1}{\sqrt{x-1}}.$$

Why are the domains of f and g different?

94. **Think About It** Consider

$$f(x) = \sqrt{x-2} \text{ and } g(x) = \sqrt[3]{x-2}.$$

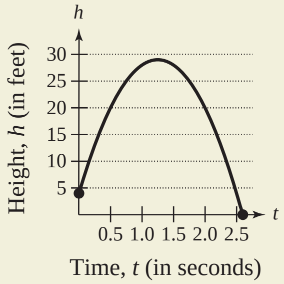
Why are the domains of f and g different?

95. **Think About It** Given

$$f(x) = x^2$$

is f the independent variable? Why or why not?

96. **HOW DO YOU SEE IT?** The graph represents the height h of a projectile after t seconds.



(a) Explain why h is a function of t.

(b) Approximate the height of the projectile after 0.5 second and after 1.25 seconds.

(c) Approximate the domain of h.

(d) Is t a function of h? Explain.

Think About It **In Exercises 97 and 98, determine whether the statements use the word *function* in ways that are mathematically correct. Explain your reasoning.**

97. (a) The sales tax on a purchased item is a function of the selling price.

(b) Your score on the next algebra exam is a function of the number of hours you study the night before the exam.

98. (a) The amount in your savings account is a function of your salary.

(b) The speed at which a free-falling baseball strikes the ground is a function of the height from which it was dropped.

2.3 Analyzing Graphs of Functions

- **Use the Vertical Line Test for functions.**
- **Find the zeros of functions.**
- **Determine intervals on which functions are increasing or decreasing and determine relative maximum and relative minimum values of functions.**
- **Determine the average rate of change of a function.**
- **Identify even and odd functions.**

Graphs of functions can help you visualize relationships between variables in real life. For instance, in Exercise 90 on page 197, you will use the graph of a function to visually represent the temperature of a city over a 24-hour period.

The Graph of a Function

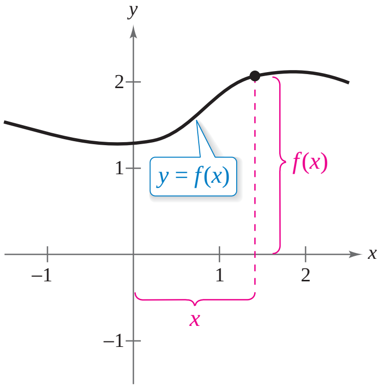
In Section 2.2, you studied functions from an algebraic point of view. In this section, you will study functions from a graphical perspective.

The **graph of a function** f is the collection of ordered pairs $(x, f(x))$ such that x is in the domain of f. As you study this section, remember that

$x =$ the directed distance from the y-axis

$y = f(x) =$ the directed distance from the x-axis

as shown in the figure at the right.



EXAMPLE 1 **Finding the Domain and Range of a Function**

Use the graph of the function f, shown in Figure 2.13, to find (a) the domain of f, (b) the function values $f(-1)$ and $f(2)$, and (c) the range of f.

Solution

a. The closed dot at $(-1, 1)$ indicates that $x = -1$ is in the domain of f, whereas the open dot at $(5, 2)$ indicates that $x = 5$ is not in the domain. So, the domain of f is all x in the interval $[-1, 5)$.

b. Because $(-1, 1)$ is a point on the graph of f, it follows that $f(-1) = 1$. Similarly, because $(2, -3)$ is a point on the graph of f, it follows that $f(2) = -3$.

c. Because the graph does not extend below $f(2) = -3$ or above $f(0) = 3$, the range of f is the interval $[-3, 3]$.

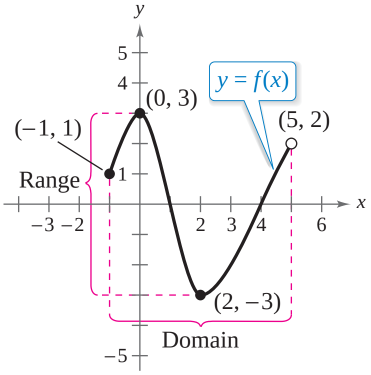
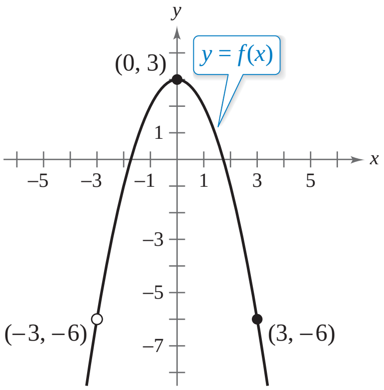


Figure 2.13

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use the graph of the function f to find (a) the domain of f, (b) the function values $f(0)$ and $f(3)$, and (c) the range of f.



REMARK The use of dots (open or closed) at the extreme left and right points of a graph indicates that the graph does not extend beyond these points. If such dots are not on the graph, then assume that the graph extends beyond these points.

gary718/Shutterstock.com

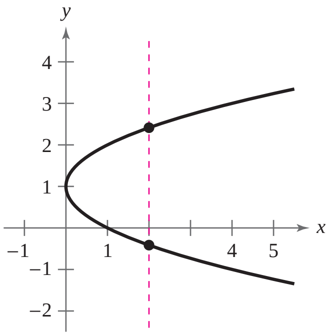
By the definition of a function, at most one y-value corresponds to a given x-value. This means that the graph of a function cannot have two or more different points with the same x-coordinate, and no two points on the graph of a function can be vertically above or below each other. It follows, then, that a vertical line can intersect the graph of a function at most once. This observation provides a convenient visual test called the **Vertical Line Test** for functions.

Vertical Line Test for Functions

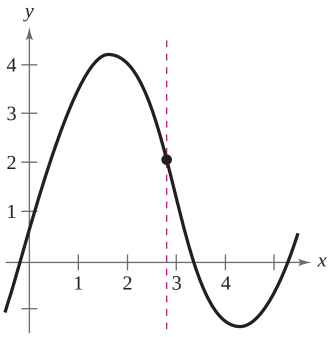
A set of points in a coordinate plane is the graph of y as a function of x if and only if no *vertical* line intersects the graph at more than one point.

EXAMPLE 2 Vertical Line Test for Functions

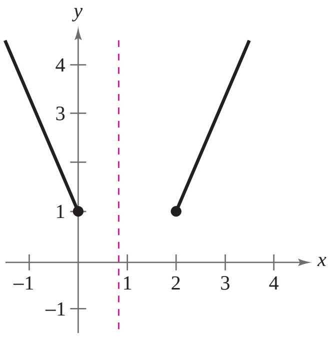
Use the Vertical Line Test to decide whether each of the following graphs represents y as a function of x.



(a)



(b)



(c)

Solution

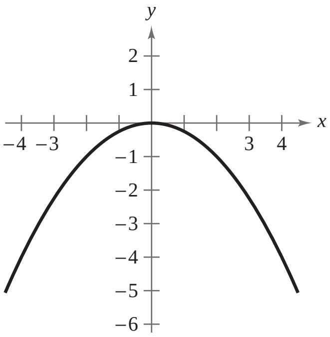
a. This *is not* a graph of y as a function of x, because there are vertical lines that intersect the graph twice. That is, for a particular input x, there is more than one output y.

b. This *is* a graph of y as a function of x, because every vertical line intersects the graph at most once. That is, for a particular input x, there is at most one output y.

c. This *is* a graph of y as a function of x. (Note that when a vertical line does not intersect the graph, it simply means that the function is undefined for that particular value of x.) That is, for a particular input x, there is at most one output y.

✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use the Vertical Line Test to decide whether the graph represents y as a function of x.



TECHNOLOGY Most graphing utilities graph functions of x more easily than other types of equations. For instance, the graph shown in (a) above represents the equation $x - (y - 1)^2 = 0$. To use a graphing utility to duplicate this graph, you must first solve the equation for y to obtain $y = 1 \pm \sqrt{x}$, and then graph the two equations $y_1 = 1 + \sqrt{x}$ and $y_2 = 1 - \sqrt{x}$ in the same viewing window.

Zeros of a Function

If the graph of a function of x has an x-intercept at $(a, 0)$, then a is a **zero** of the function.

Zeros of a Function

The **zeros of a function** f of x are the x-values for which $f(x) = 0$.

EXAMPLE 3 Finding the Zeros of a Function

Find the zeros of each function.

a. $f(x) = 3x^2 + x - 10$

b. $g(x) = \sqrt{10 - x^2}$

c. $h(t) = \dfrac{2t - 3}{t + 5}$

Solution To find the zeros of a function, set the function equal to zero and solve for the independent variable.

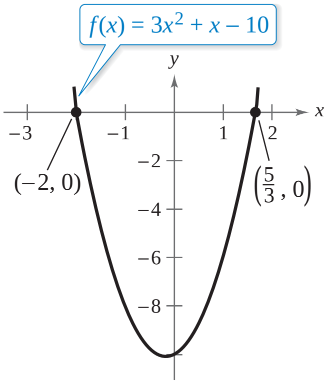
a. $3x^2 + x - 10 = 0$ Set $f(x)$ equal to 0.

$(3x - 5)(x + 2) = 0$ Factor.

$3x - 5 = 0 \Rightarrow x = \frac{5}{3}$ Set 1st factor equal to 0.

$x + 2 = 0 \Rightarrow x = -2$ Set 2nd factor equal to 0.

The zeros of f are $x = \frac{5}{3}$ and $x = -2$. In Figure 2.14, note that the graph of f has $\left(\frac{5}{3}, 0\right)$ and $(-2, 0)$ as its x-intercepts.



Zeros of f: $x = -2, x = \frac{5}{3}$

Figure 2.14

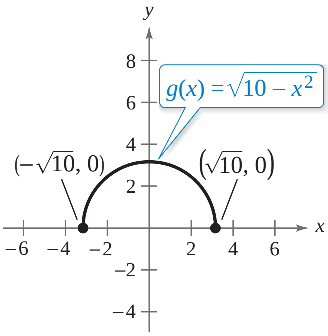
b. $\sqrt{10 - x^2} = 0$ Set $g(x)$ equal to 0.

$10 - x^2 = 0$ Square each side.

$10 = x^2$ Add x^2 to each side.

$\pm\sqrt{10} = x$ Extract square roots.

The zeros of g are $x = -\sqrt{10}$ and $x = \sqrt{10}$. In Figure 2.15, note that the graph of g has $\left(-\sqrt{10}, 0\right)$ and $\left(\sqrt{10}, 0\right)$ as its x-intercepts.



Zeros of g: $x = \pm\sqrt{10}$

Figure 2.15

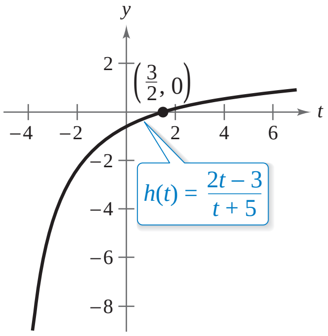
c. $\dfrac{2t - 3}{t + 5} = 0$ Set $h(t)$ equal to 0.

$2t - 3 = 0$ Multiply each side by $t + 5$.

$2t = 3$ Add 3 to each side.

$t = \dfrac{3}{2}$ Divide each side by 2.

The zero of h is $t = \frac{3}{2}$. In Figure 2.16, note that the graph of h has $\left(\frac{3}{2}, 0\right)$ as its t-intercept.



Zero of h: $t = \frac{3}{2}$

Figure 2.16

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the zeros of each function.

a. $f(x) = 2x^2 + 13x - 24$ **b.** $g(t) = \sqrt{t - 25}$ **c.** $h(x) = \dfrac{x^2 - 2}{x - 1}$

Increasing and Decreasing Functions

The more you know about the graph of a function, the more you know about the function itself. Consider the graph shown in Figure 2.17. As you move from *left to right*, this graph falls from $x = -2$ to $x = 0$, is constant from $x = 0$ to $x = 2$, and rises from $x = 2$ to $x = 4$.

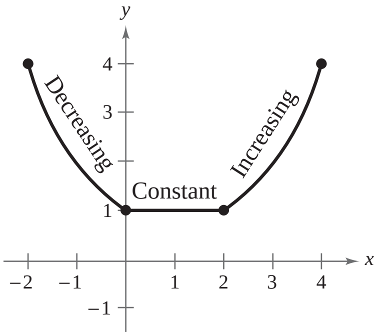


Figure 2.17

Increasing, Decreasing, and Constant Functions

A function f is **increasing** on an interval when, for any x_1 and x_2 in the interval,

$$x_1 < x_2 \quad \text{implies} \quad f(x_1) < f(x_2).$$

A function f is **decreasing** on an interval when, for any x_1 and x_2 in the interval,

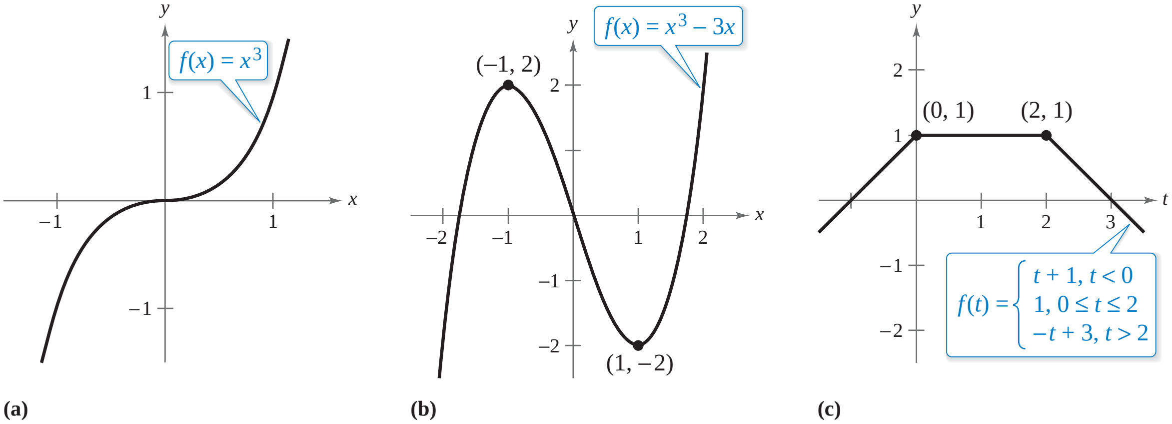
$$x_1 < x_2 \quad \text{implies} \quad f(x_1) > f(x_2).$$

A function f is **constant** on an interval when, for any x_1 and x_2 in the interval,

$$f(x_1) = f(x_2).$$

EXAMPLE 4 Describing Function Behavior

Use the graphs to describe the increasing, decreasing, or constant behavior of each function.



Solution

a. This function is increasing over the entire real line.

b. This function is increasing on the interval $(-\infty, -1)$, decreasing on the interval $(-1, 1)$, and increasing on the interval $(1, \infty)$.

c. This function is increasing on the interval $(-\infty, 0)$, constant on the interval $(0, 2)$, and decreasing on the interval $(2, \infty)$.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Graph the function

$$f(x) = x^3 + 3x^2 - 1.$$

Then use the graph to describe the increasing and decreasing behavior of the function.

To help you decide whether a function is increasing, decreasing, or constant on an interval, you can evaluate the function for several values of x. However, you need calculus to determine, for certain, all intervals on which a function is increasing, decreasing, or constant.

The points at which a function changes its increasing, decreasing, or constant behavior are helpful in determining the **relative minimum** or **relative maximum** values of the function.

REMARK A relative minimum or relative maximum is also referred to as a local minimum or local maximum.

Definitions of Relative Minimum and Relative Maximum

A function value $f(a)$ is called a **relative minimum** of f when there exists an interval (x_1, x_2) that contains a such that

$$x_1 < x < x_2 \quad \text{implies} \quad f(a) \le f(x).$$

A function value $f(a)$ is called a **relative maximum** of f when there exists an interval (x_1, x_2) that contains a such that

$$x_1 < x < x_2 \quad \text{implies} \quad f(a) \ge f(x).$$

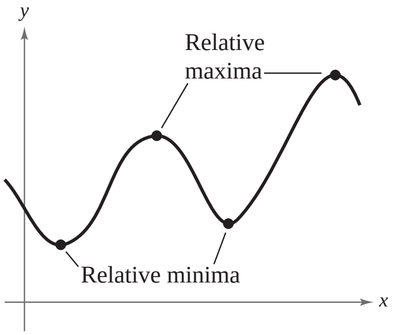


Figure 2.18

Figure 2.18 shows several different examples of relative minima and relative maxima. In Section 3.1, you will study a technique for finding the *exact point* at which a second-degree polynomial function has a relative minimum or relative maximum. For the time being, however, you can use a graphing utility to find reasonable approximations of these points.

EXAMPLE 5 Approximating a Relative Minimum

Use a graphing utility to approximate the relative minimum of the function

$$f(x) = 3x^2 - 4x - 2.$$

Solution The graph of f is shown in Figure 2.19. By using the *zoom* and *trace* features or the *minimum* feature of a graphing utility, you can estimate that the function has a relative minimum at the point

$(0.67, -3.33)$. Relative minimum

Later, in Section 3.1, you will be able to determine that the exact point at which the relative minimum occurs is $\left(\frac{2}{3}, -\frac{10}{3}\right)$.

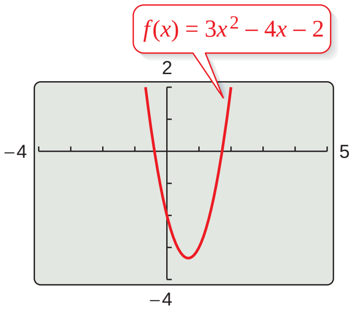


Figure 2.19

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Use a graphing utility to approximate the relative maximum of the function

$$f(x) = -4x^2 - 7x + 3.$$

You can also use the *table* feature of a graphing utility to numerically approximate the relative minimum of the function in Example 5. Using a table that begins at 0.6 and increments the value of x by 0.01, you can approximate that the minimum of $f(x) = 3x^2 - 4x - 2$ occurs at the point $(0.67, -3.33)$.

TECHNOLOGY When you use a graphing utility to estimate the x- and y-values of a relative minimum or relative maximum, the *zoom* feature will often produce graphs that are nearly flat. To overcome this problem, you can manually change the vertical setting of the viewing window. The graph will stretch vertically when the values of Ymin and Ymax are closer together.

Average Rate of Change

In Section 2.1, you learned that the slope of a line can be interpreted as a *rate of change*. For a nonlinear graph whose slope changes at each point, the **average rate of change** between any two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is the slope of the line through the two points (see Figure 2.20). The line through the two points is called the **secant line,** and the slope of this line is denoted as m_{sec}.

$$\text{Average rate of change of } f \text{ from } x_1 \text{ to } x_2 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= \frac{\text{change in } y}{\text{change in } x}$$

$$= m_{sec}$$

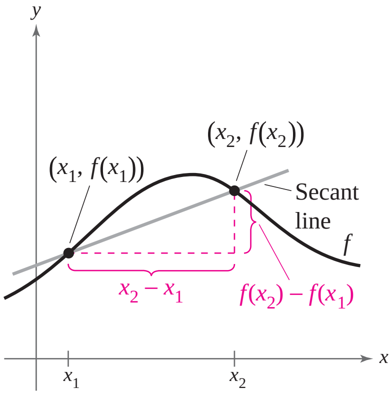


Figure 2.20

EXAMPLE 6 Average Rate of Change of a Function

Find the average rates of change of $f(x) = x^3 - 3x$ (a) from $x_1 = -2$ to $x_2 = -1$ and (b) from $x_1 = 0$ to $x_2 = 1$ (see Figure 2.21).

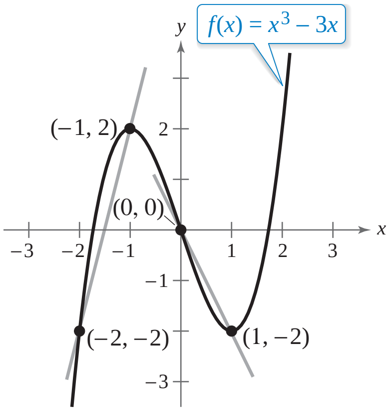


Figure 2.21

Solution

a. The average rate of change of f from $x_1 = -2$ to $x_2 = -1$ is

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(-1) - f(-2)}{-1 - (-2)} = \frac{2 - (-2)}{1} = 4.$$

Secant line has positive slope.

b. The average rate of change of f from $x_1 = 0$ to $x_2 = 1$ is

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(1) - f(0)}{1 - 0} = \frac{-2 - 0}{1} = -2.$$

Secant line has negative slope.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the average rates of change of $f(x) = x^2 + 2x$ (a) from $x_1 = -3$ to $x_2 = -2$ and (b) from $x_1 = -2$ to $x_2 = 0$.

EXAMPLE 7 Finding Average Speed

The distance s (in feet) a moving car is from a stoplight is given by the function

$$s(t) = 20t^{3/2}$$

where t is the time (in seconds). Find the average speed of the car (a) from $t_1 = 0$ to $t_2 = 4$ seconds and (b) from $t_1 = 4$ to $t_2 = 9$ seconds.

Average speed is an average rate of change.

Solution

a. The average speed of the car from $t_1 = 0$ to $t_2 = 4$ seconds is

$$\frac{s(t_2) - s(t_1)}{t_2 - t_1} = \frac{s(4) - s(0)}{4 - 0} = \frac{160 - 0}{4} = 40 \text{ feet per second.}$$

b. The average speed of the car from $t_1 = 4$ to $t_2 = 9$ seconds is

$$\frac{s(t_2) - s(t_1)}{t_2 - t_1} = \frac{s(9) - s(4)}{9 - 4} = \frac{540 - 160}{5} = 76 \text{ feet per second.}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

In Example 7, find the average speed of the car (a) from $t_1 = 0$ to $t_2 = 1$ second and (b) from $t_1 = 1$ second to $t_2 = 4$ seconds.

sadwitch/Shutterstock.com

Even and Odd Functions

In Section 1.1, you studied different types of symmetry of a graph. In the terminology of functions, a function is said to be **even** when its graph is symmetric with respect to the y-axis and **odd** when its graph is symmetric with respect to the origin. The symmetry tests in Section 1.1 yield the following tests for even and odd functions.

Tests for Even and Odd Functions

A function $y = f(x)$ is **even** when, for each x in the domain of f, $f(-x) = f(x)$.

A function $y = f(x)$ is **odd** when, for each x in the domain of f, $f(-x) = -f(x)$.

EXAMPLE 8 Even and Odd Functions

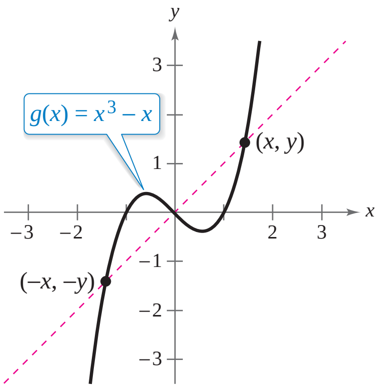
a. The function $g(x) = x^3 - x$ is odd because $g(-x) = -g(x)$, as follows.

$$\begin{aligned} g(-x) &= (-x)^3 - (-x) && \text{Substitute } -x \text{ for } x. \\ &= -x^3 + x && \text{Simplify.} \\ &= -(x^3 - x) && \text{Distributive Property} \\ &= -g(x) && \text{Test for odd function} \end{aligned}$$

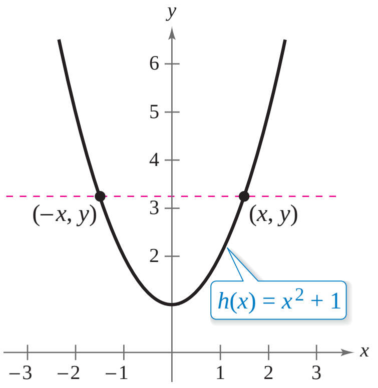
b. The function $h(x) = x^2 + 1$ is even because $h(-x) = h(x)$, as follows.

$$\begin{aligned} h(-x) &= (-x)^2 + 1 && \text{Substitute } -x \text{ for } x. \\ &= x^2 + 1 && \text{Simplify.} \\ &= h(x) && \text{Test for even function} \end{aligned}$$

Figure 2.22 shows the graphs and symmetry of these two functions.



(a) Symmetric to origin: Odd Function



(b) Symmetric to y-axis: Even Function

Figure 2.22

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Determine whether the function is even, odd, or neither. Then describe the symmetry.

a. $f(x) = 5 - 3x$ **b.** $g(x) = x^4 - x^2 - 1$ **c.** $h(x) = 2x^3 + 3x$

Summarize (Section 2.3)

1. State the Vertical Line Test for functions *(page 188)*. For an example of using the Vertical Line Test, see Example 2.
2. Explain how to find the zeros of a function *(page 189)*. For an example of finding the zeros of functions, see Example 3.
3. Explain how to determine intervals on which functions are increasing or decreasing *(page 190)* and how to determine relative maximum and relative minimum values of functions *(page 191)*. For an example of describing function behavior, see Example 4. For an example of approximating a relative minimum, see Example 5.
4. Explain how to determine the average rate of change of a function *(page 192)*. For examples of determining average rates of change, see Examples 6 and 7.
5. State the definitions of an even function and an odd function *(page 193)*. For an example of identifying even and odd functions, see Example 8.

2.3 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. The ________ ________ ________ is used to determine whether the graph of an equation is a function of y in terms of x.

2. The ________ of a function f are the values of x for which $f(x) = 0$.

3. A function f is ________ on an interval when, for any x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

4. A function value $f(a)$ is a relative ________ of f when there exists an interval (x_1, x_2) containing a such that $x_1 < x < x_2$ implies $f(a) \geq f(x)$.

5. The ________ ________ ________ ________ between any two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is the slope of the line through the two points, and this line is called the ________ line.

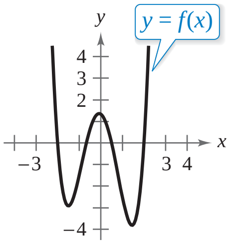
6. A function f is ________ when, for each x in the domain of f, $f(-x) = -f(x)$.

Skills and Applications

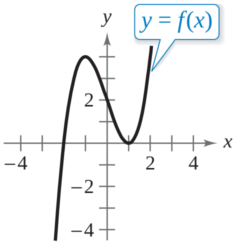
Domain, Range, and Values of a Function

In Exercises 7–10, use the graph of the function to find the domain and range of f and the indicated function values.

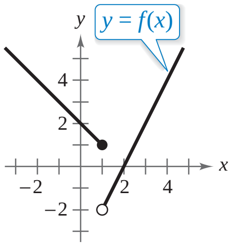
7. (a) $f(-2)$ (b) $f(-1)$
(c) $f\left(\frac{1}{2}\right)$ (d) $f(1)$



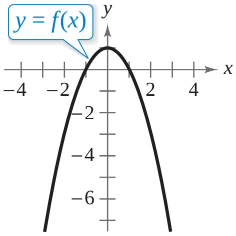
8. (a) $f(-1)$ (b) $f(2)$
(c) $f(0)$ (d) $f(1)$



9. (a) $f(2)$ (b) $f(1)$
(c) $f(3)$ (d) $f(-1)$



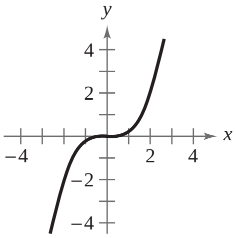
10. (a) $f(-2)$ (b) $f(1)$
(c) $f(0)$ (d) $f(2)$



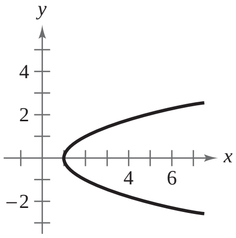
Vertical Line Test for Functions

In Exercises 11–14, use the Vertical Line Test to determine whether y is a function of x. To print an enlarged copy of the graph, go to *MathGraphs.com*.

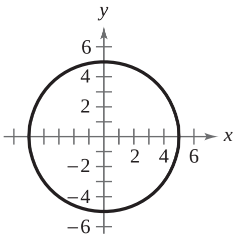
11. $y = \frac{1}{4}x^3$



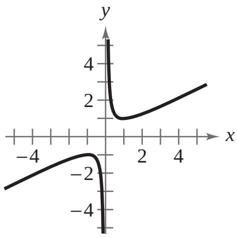
12. $x - y^2 = 1$



13. $x^2 + y^2 = 25$



14. $x^2 = 2xy - 1$



Finding the Zeros of a Function

In Exercises 15–24, find the zeros of the function algebraically.

15. $f(x) = 2x^2 - 7x - 30$

16. $f(x) = 3x^2 + 22x - 16$

17. $f(x) = \dfrac{x}{9x^2 - 4}$

18. $f(x) = \dfrac{x^2 - 9x + 14}{4x}$

19. $f(x) = \frac{1}{2}x^3 - x$

20. $f(x) = 9x^4 - 25x^2$

21. $f(x) = x^3 - 4x^2 - 9x + 36$

22. $f(x) = 4x^3 - 24x^2 - x + 6$

23. $f(x) = \sqrt{2x} - 1$

24. $f(x) = \sqrt{3x + 2}$

Graphing and Finding Zeros

In Exercises 25–30, (a) use a graphing utility to graph the function and find the zeros of the function and (b) verify your results from part (a) algebraically.

25. $f(x) = 3 + \dfrac{5}{x}$

26. $f(x) = x(x - 7)$

27. $f(x) = \sqrt{2x + 11}$

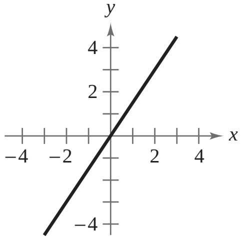
28. $f(x) = \sqrt{3x - 14} - 8$

29. $f(x) = \dfrac{3x - 1}{x - 6}$

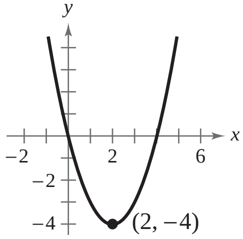
30. $f(x) = \dfrac{2x^2 - 9}{3 - x}$

Describing Function Behavior **In Exercises 31–38, determine the intervals on which the function is increasing, decreasing, or constant.**

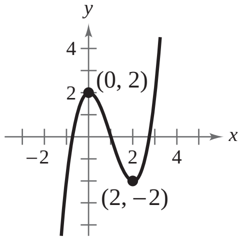
31. $f(x) = \frac{3}{2}x$



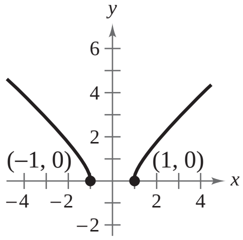
32. $f(x) = x^2 - 4x$



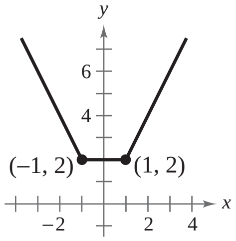
33. $f(x) = x^3 - 3x^2 + 2$



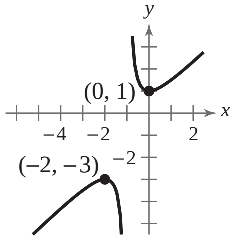
34. $f(x) = \sqrt{x^2 - 1}$



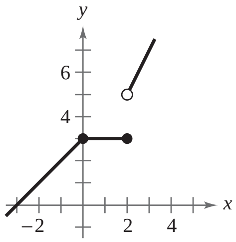
35. $f(x) = |x + 1| + |x - 1|$



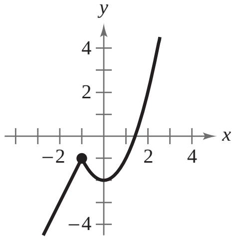
36. $f(x) = \dfrac{x^2 + x + 1}{x + 1}$



37. $f(x) = \begin{cases} x + 3, & x \leq 0 \\ 3, & 0 < x \leq 2 \\ 2x + 1, & x > 2 \end{cases}$



38. $f(x) = \begin{cases} 2x + 1, & x \leq -1 \\ x^2 - 2, & x > -1 \end{cases}$



Describing Function Behavior **In Exercises 39–46, (a) use a graphing utility to graph the function and visually determine the intervals on which the function is increasing, decreasing, or constant, and (b) make a table of values to verify whether the function is increasing, decreasing, or constant on the intervals you identified in part (a).**

39. $f(x) = 3$

40. $g(x) = x$

41. $g(s) = \dfrac{s^2}{4}$

42. $f(x) = 3x^4 - 6x^2$

43. $f(x) = \sqrt{1 - x}$

44. $f(x) = x\sqrt{x + 3}$

45. $f(x) = x^{3/2}$

46. $f(x) = x^{2/3}$

Approximating Relative Minima or Maxima **In Exercises 47–54, use a graphing utility to graph the function and approximate (to two decimal places) any relative minima or maxima.**

47. $f(x) = 3x^2 - 2x - 5$

48. $f(x) = -x^2 + 3x - 2$

49. $f(x) = -2x^2 + 9x$

50. $f(x) = x(x - 2)(x + 3)$

51. $f(x) = x^3 - 3x^2 - x + 1$

52. $h(x) = x^3 - 6x^2 + 15$

53. $h(x) = (x - 1)\sqrt{x}$

54. $g(x) = x\sqrt{4 - x}$

Graphical Analysis **In Exercises 55–60, graph the function and determine the interval(s) for which $f(x) \geq 0$.**

55. $f(x) = 4 - x$

56. $f(x) = 4x + 2$

57. $f(x) = 9 - x^2$

58. $f(x) = x^2 - 4x$

59. $f(x) = \sqrt{x - 1}$

60. $f(x) = -(1 + |x|)$

Average Rate of Change of a Function **In Exercises 61–64, find the average rate of change of the function from x_1 to x_2.**

	Function	**x-Values**
61.	$f(x) = -2x + 15$	$x_1 = 0, x_2 = 3$
62.	$f(x) = x^2 - 2x + 8$	$x_1 = 1, x_2 = 5$
63.	$f(x) = x^3 - 3x^2 - x$	$x_1 = 1, x_2 = 3$
64.	$f(x) = -x^3 + 6x^2 + x$	$x_1 = 1, x_2 = 6$

65. **Research and Development** The amounts y (in millions of dollars) the U.S. Department of Energy spent for research and development from 2005 through 2010 can be approximated by the model

$$y = 56.77t^2 - 366.8t + 8916, \quad 5 \leq t \leq 10$$

where t represents the year, with $t = 5$ corresponding to 2005. *(Source: American Association for the Advancement of Science)*

(a) Use a graphing utility to graph the model.

(b) Find the average rate of change of the model from 2005 to 2010. Interpret your answer in the context of the problem.

66. **Finding Average Speed** Use the information in Example 7 to find the average speed of the car from $t_1 = 0$ to $t_2 = 9$ seconds. Explain why the result is less than the value obtained in part (b) of Example 7.

Physics **In Exercises 67–70, (a) use the position equation $s = -16t^2 + v_0t + s_0$ to write a function that represents the situation, (b) use a graphing utility to graph the function, (c) find the average rate of change of the function from t_1 to t_2, (d) describe the slope of the secant line through t_1 and t_2, (e) find the equation of the secant line through t_1 and t_2, and (f) graph the secant line in the same viewing window as your position function.**

67. An object is thrown upward from a height of 6 feet at a velocity of 64 feet per second.

$t_1 = 0, t_2 = 3$

68. An object is thrown upward from a height of 6.5 feet at a velocity of 72 feet per second.

$t_1 = 0, t_2 = 4$

69. An object is thrown upward from ground level at a velocity of 120 feet per second.

$t_1 = 3, t_2 = 5$

70. An object is dropped from a height of 80 feet.

$t_1 = 1, t_2 = 2$

Even, Odd, or Neither? **In Exercises 71–76, determine whether the function is even, odd, or neither. Then describe the symmetry.**

71. $f(x) = x^6 - 2x^2 + 3$

72. $g(x) = x^3 - 5x$

73. $f(x) = x\sqrt{1 - x^2}$

74. $h(x) = x\sqrt{x + 5}$

75. $f(s) = 4s^{3/2}$

76. $g(s) = 4s^{2/3}$

Even, Odd, or Neither? **In Exercises 77–82, sketch a graph of the function and determine whether it is even, odd, or neither. Verify your answer algebraically.**

77. $f(x) = -9$

78. $f(x) = 5 - 3x$

79. $f(x) = -|x - 5|$

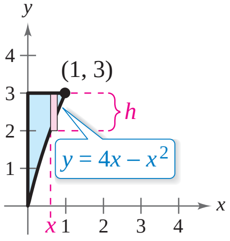
80. $h(x) = x^2 - 4$

81. $f(x) = \sqrt{1 - x}$

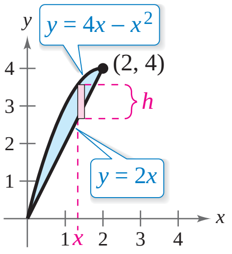
82. $g(t) = \sqrt[3]{t - 1}$

Height of a Rectangle **In Exercises 83 and 84, write the height h of the rectangle as a function of x.**

83.

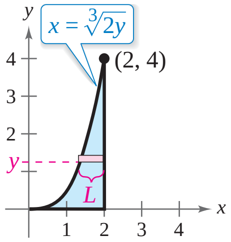


84.

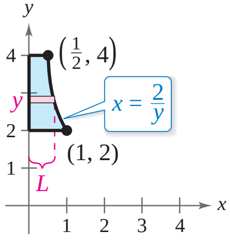


Length of a Rectangle **In Exercises 85 and 86, write the length L of the rectangle as a function of y.**

85.



86.



87. **Lumens** The number of lumens (time rate of flow of light) L from a fluorescent lamp can be approximated by the model

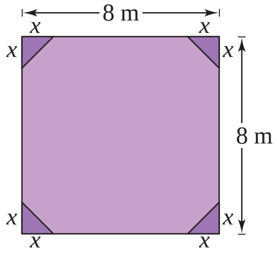
$$L = -0.294x^2 + 97.744x - 664.875, \quad 20 \leq x \leq 90$$

where x is the wattage of the lamp.

(a) Use a graphing utility to graph the function.

(b) Use the graph from part (a) to estimate the wattage necessary to obtain 2000 lumens.

88. **Geometry** Corners of equal size are cut from a square with sides of length 8 meters (see figure).



(a) Write the area A of the resulting figure as a function of x. Determine the domain of the function.

(b) Use a graphing utility to graph the area function over its domain. Use the graph to find the range of the function.

(c) Identify the figure that results when x is the maximum value in the domain of the function. What would be the length of each side of the figure?

89. **Coordinate Axis Scale** Each function described below models the specified data for the years 2003 through 2013, with $t = 3$ corresponding to 2003. Estimate a reasonable scale for the vertical axis (e.g., hundreds, thousands, millions, etc.) of the graph and justify your answer. (There are many correct answers.)

(a) $f(t)$ represents the average salary of college professors.

(b) $f(t)$ represents the U.S. population.

(c) $f(t)$ represents the percent of the civilian work force that is unemployed.

90. Data Analysis: Temperature The table shows the temperatures y (in degrees Fahrenheit) in a city over a 24-hour period. Let x represent the time of day, where $x = 0$ corresponds to 6 A.M.

DATA

Spreadsheet at LarsonPrecalculus.com

Time, x	Temperature, y
0	34
2	50
4	60
6	64
8	63
10	59
12	53
14	46
16	40
18	36
20	34
22	37
24	45

A model that represents these data is given by

$$y = 0.026x^3 - 1.03x^2 + 10.2x + 34, \quad 0 \le x \le 24.$$

(a) Use a graphing utility to create a scatter plot of the data. Then graph the model in the same viewing window.

(b) How well does the model fit the data?

(c) Use the graph to approximate the times when the temperature was increasing and decreasing.

(d) Use the graph to approximate the maximum and minimum temperatures during this 24-hour period.

(e) Could this model predict the temperatures in the city during the next 24-hour period? Why or why not?

91. Writing Use a graphing utility to graph each function. Write a paragraph describing any similarities and differences you observe among the graphs.

(a) $y = x$

(b) $y = x^2$

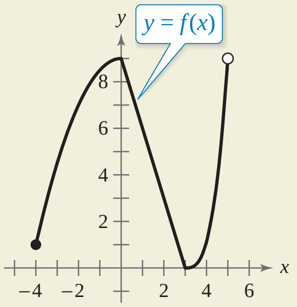
(c) $y = x^3$

(d) $y = x^4$

(e) $y = x^5$

(f) $y = x^6$

92. HOW DO YOU SEE IT? Use the graph of the function to answer (a)–(e).



(a) Find the domain and range of f.

(b) Find the zero(s) of f.

(c) Determine the intervals over which f is increasing, decreasing, or constant.

(d) Approximate any relative minimum or relative maximum values of f.

(e) Is f even, odd, or neither?

Exploration

True or False? **In Exercises 93 and 94, determine whether the statement is true or false. Justify your answer.**

93. A function with a square root cannot have a domain that is the set of real numbers.

94. It is possible for an odd function to have the interval $[0, \infty)$ as its domain.

Think About It **In Exercises 95 and 96, find the coordinates of a second point on the graph of a function f when the given point is on the graph and the function is (a) even and (b) odd.**

95. $\left(-\frac{5}{3}, -7\right)$ **96.** $(2a, 2c)$

97. Graphical Reasoning Graph each of the functions with a graphing utility. Determine whether the function is even, odd, or neither.

$f(x) = x^2 - x^4$ $g(x) = 2x^3 + 1$

$h(x) = x^5 - 2x^3 + x$ $j(x) = 2 - x^6 - x^8$

$k(x) = x^5 - 2x^4 + x - 2$ $p(x) = x^9 + 3x^5 - x^3 + x$

What do you notice about the equations of functions that are odd? What do you notice about the equations of functions that are even? Can you describe a way to identify a function as odd or even by inspecting the equation? Can you describe a way to identify a function as neither odd nor even by inspecting the equation?

98. Even, Odd, or Neither? If f is an even function, determine whether g is even, odd, or neither. Explain.

(a) $g(x) = -f(x)$ (b) $g(x) = f(-x)$

(c) $g(x) = f(x) - 2$ (d) $g(x) = f(x - 2)$

gary718/Shutterstock.com

2.4 A Library of Parent Functions

Piecewise-defined functions can help you model real-life situations. For instance, in Exercise 47 on page 204, you will write a piecewise-defined function to model the depth of snow during a snowstorm.

- **Identify and graph linear and squaring functions.**
- **Identify and graph cubic, square root, and reciprocal functions.**
- **Identify and graph step and other piecewise-defined functions.**
- **Recognize graphs of parent functions.**

Linear and Squaring Functions

One of the goals of this text is to enable you to recognize the basic shapes of the graphs of different types of functions. For instance, you know that the graph of the **linear function** $f(x) = ax + b$ is a line with slope $m = a$ and y-intercept at $(0, b)$. The graph of the linear function has the following characteristics.

- The domain of the function is the set of all real numbers.
- When $m \neq 0$, the range of the function is the set of all real numbers.
- The graph has an x-intercept at $(-b/m, 0)$ and a y-intercept at $(0, b)$.
- The graph is increasing when $m > 0$, decreasing when $m < 0$, and constant when $m = 0$.

EXAMPLE 1 Writing a Linear Function

Write the linear function f for which $f(1) = 3$ and $f(4) = 0$.

Solution To find the equation of the line that passes through $(x_1, y_1) = (1, 3)$ and $(x_2, y_2) = (4, 0)$, first find the slope of the line.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 3}{4 - 1} = \frac{-3}{3} = -1$$

Next, use the point-slope form of the equation of a line.

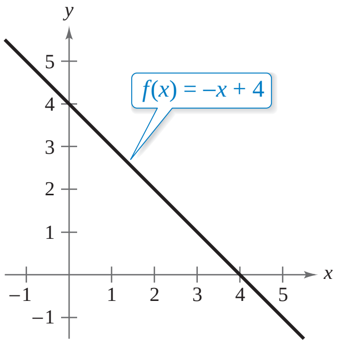
$$y - y_1 = m(x - x_1) \qquad \text{Point-slope form}$$

$$y - 3 = -1(x - 1) \qquad \text{Substitute for } x_1, y_1, \text{ and } m.$$

$$y = -x + 4 \qquad \text{Simplify.}$$

$$f(x) = -x + 4 \qquad \text{Function notation}$$

The figure below shows the graph of this function.



Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write the linear function f for which $f(-2) = 6$ and $f(4) = -9$.

nulinukas/Shutterstock.com

There are two special types of linear functions, the **constant function** and the **identity function.** A constant function has the form

$$f(x) = c$$

and has the domain of all real numbers with a range consisting of a single real number c. The graph of a constant function is a horizontal line, as shown in Figure 2.23. The identity function has the form

$$f(x) = x.$$

Its domain and range are the set of all real numbers. The identity function has a slope of $m = 1$ and a y-intercept at $(0, 0)$. The graph of the identity function is a line for which each x-coordinate equals the corresponding y-coordinate. The graph is always increasing, as shown in Figure 2.24.

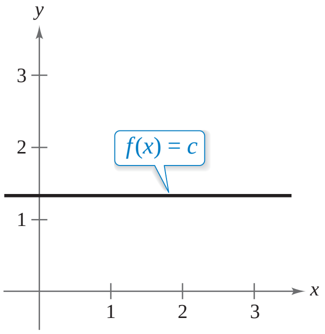


Figure 2.23

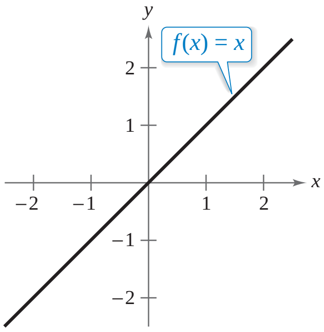


Figure 2.24

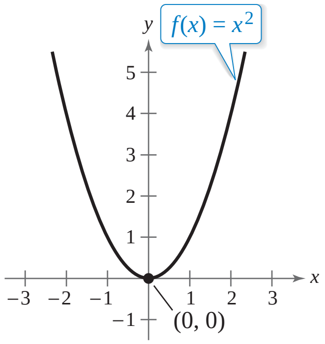
The graph of the **squaring function**

$$f(x) = x^2$$

is a U-shaped curve with the following characteristics.

- The domain of the function is the set of all real numbers.
- The range of the function is the set of all nonnegative real numbers.
- The function is even.
- The graph has an intercept at $(0, 0)$.
- The graph is decreasing on the interval $(-\infty, 0)$ and increasing on the interval $(0, \infty)$.
- The graph is symmetric with respect to the y-axis.
- The graph has a relative minimum at $(0, 0)$.

The figure below shows the graph of the squaring function.



Cubic, Square Root, and Reciprocal Functions

The following summarizes the basic characteristics of the graphs of the **cubic, square root,** and **reciprocal functions.**

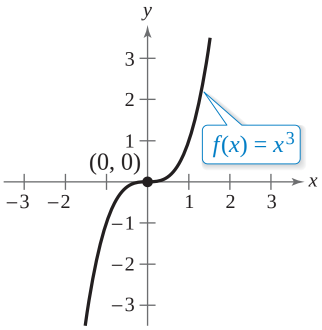
1. The graph of the *cubic* function

$$f(x) = x^3$$

has the following characteristics.

- The domain of the function is the set of all real numbers.
- The range of the function is the set of all real numbers.
- The function is odd.
- The graph has an intercept at $(0, 0)$.
- The graph is increasing on the interval $(-\infty, \infty)$.
- The graph is symmetric with respect to the origin.

The figure shows the graph of the cubic function.



Cubic function

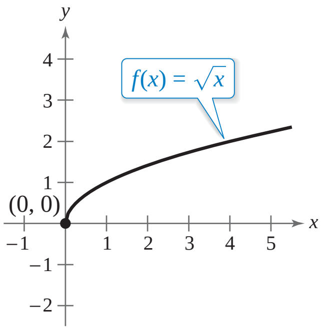
2. The graph of the *square root* function

$$f(x) = \sqrt{x}$$

has the following characteristics.

- The domain of the function is the set of all nonnegative real numbers.
- The range of the function is the set of all nonnegative real numbers.
- The graph has an intercept at $(0, 0)$.
- The graph is increasing on the interval $(0, \infty)$.

The figure shows the graph of the square root function.



Square root function

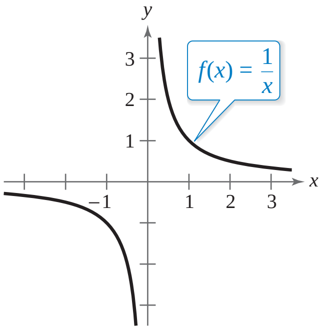
3. The graph of the *reciprocal* function

$$f(x) = \frac{1}{x}$$

has the following characteristics.

- The domain of the function is $(-\infty, 0) \cup (0, \infty)$.
- The range of the function is $(-\infty, 0) \cup (0, \infty)$.
- The function is odd.
- The graph does not have any intercepts.
- The graph is decreasing on the intervals $(-\infty, 0)$ and $(0, \infty)$.
- The graph is symmetric with respect to the origin.

The figure shows the graph of the reciprocal function.



Reciprocal function

Step and Piecewise-Defined Functions

Functions whose graphs resemble sets of stairsteps are known as **step functions.** The most famous of the step functions is the **greatest integer function,** denoted by $[\![x]\!]$ and defined as

$$f(x) = [\![x]\!] = \textit{the greatest integer less than or equal to } x.$$

Some values of the greatest integer function are as follows.

$$[\![-1]\!] = (\text{greatest integer} \le -1) = -1$$

$$[\![-\tfrac{1}{2}]\!] = (\text{greatest integer} \le -\tfrac{1}{2}) = -1$$

$$[\![\tfrac{1}{10}]\!] = (\text{greatest integer} \le \tfrac{1}{10}) = 0$$

$$[\![1.5]\!] = (\text{greatest integer} \le 1.5) = 1$$

$$[\![1.9]\!] = (\text{greatest integer} \le 1.9) = 1$$

The graph of the greatest integer function

$$f(x) = [\![x]\!]$$

has the following characteristics, as shown in Figure 2.25.

- The domain of the function is the set of all real numbers.
- The range of the function is the set of all integers.
- The graph has a y-intercept at $(0, 0)$ and x-intercepts in the interval $[0, 1)$.
- The graph is constant between each pair of consecutive integer values of x.
- The graph jumps vertically one unit at each integer value of x.

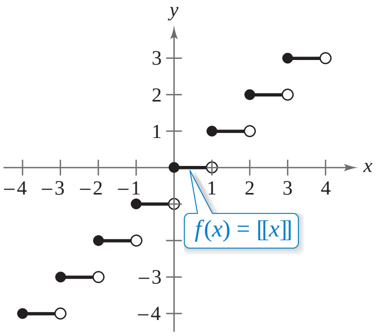


Figure 2.25

TECHNOLOGY When using your graphing utility to graph a step function, you should set your graphing utility to *dot* mode.

EXAMPLE 2 Evaluating a Step Function

Evaluate the function when $x = -1, 2$, and $\frac{3}{2}$.

$$f(x) = [\![x]\!] + 1$$

Solution For $x = -1$, the greatest integer ≤ -1 is -1, so

$$f(-1) = [\![-1]\!] + 1 = -1 + 1 = 0.$$

For $x = 2$, the greatest integer ≤ 2 is 2, so

$$f(2) = [\![2]\!] + 1 = 2 + 1 = 3.$$

For $x = \frac{3}{2}$, the greatest integer $\le \frac{3}{2}$ is 1, so

$$f\left(\tfrac{3}{2}\right) = [\![\tfrac{3}{2}]\!] + 1 = 1 + 1 = 2.$$

Verify your answers by examining the graph of $f(x) = [\![x]\!] + 1$ shown in Figure 2.26.

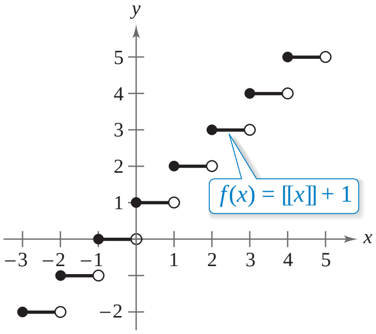


Figure 2.26

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Evaluate the function when $x = -\frac{3}{2}, 1$, and $-\frac{5}{2}$.

$$f(x) = [\![x + 2]\!]$$

Recall from Section 2.2 that a piecewise-defined function is defined by two or more equations over a specified domain. To graph a piecewise-defined function, graph each equation separately over the specified domain, as shown in Example 3.

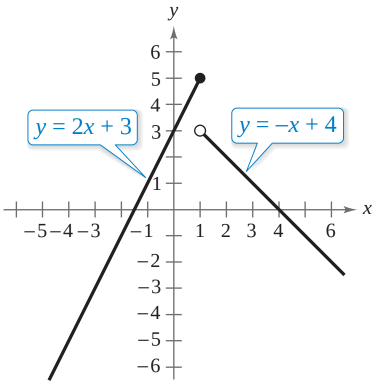


Figure 2.27

EXAMPLE 3 Graphing a Piecewise-Defined Function

Sketch the graph of $f(x) = \begin{cases} 2x + 3, & x \le 1 \\ -x + 4, & x > 1 \end{cases}.$

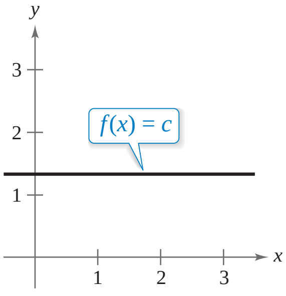
Solution This piecewise-defined function consists of two linear functions. At $x = 1$ and to the left of $x = 1$, the graph is the line $y = 2x + 3$, and to the right of $x = 1$ the graph is the line $y = -x + 4$, as shown in Figure 2.27. Notice that the point $(1, 5)$ is a solid dot and the point $(1, 3)$ is an open dot. This is because $f(1) = 2(1) + 3 = 5$.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

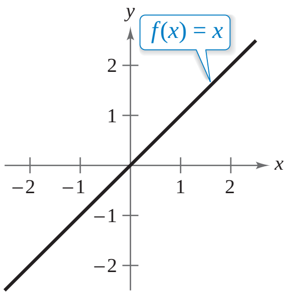
Sketch the graph of $f(x) = \begin{cases} -\frac{1}{2}x - 6, & x \le -4 \\ x + 5, & x > -4 \end{cases}.$

Parent Functions

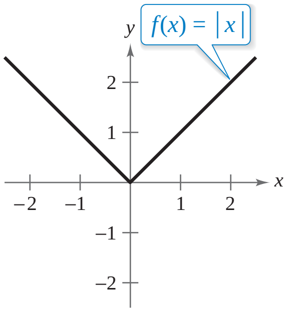
The eight graphs shown below represent the most commonly used functions in algebra. Familiarity with the basic characteristics of these simple graphs will help you analyze the shapes of more complicated graphs—in particular, graphs obtained from these graphs by the rigid and nonrigid transformations studied in the next section.



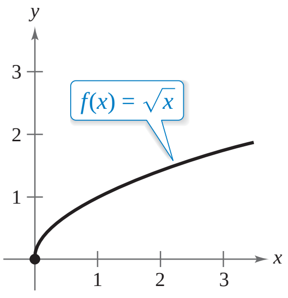
(a) Constant Function



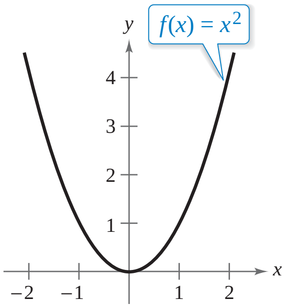
(b) Identity Function



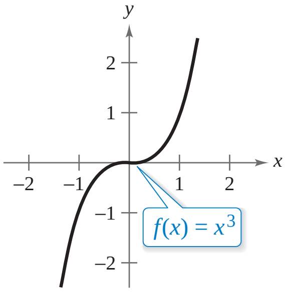
(c) Absolute Value Function



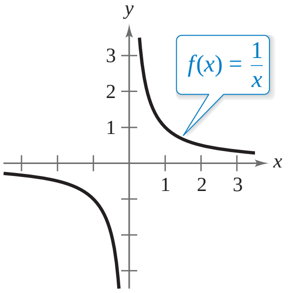
(d) Square Root Function



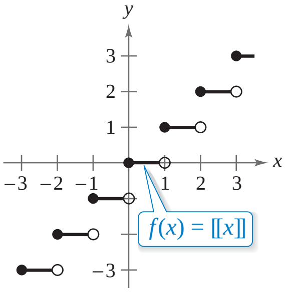
(e) Quadratic Function



(f) Cubic Function



(g) Reciprocal Function



(h) Greatest Integer Function

Summarize (Section 2.4)

1. Explain how to identify and graph linear and squaring functions *(pages 198 and 199)*. For an example involving a linear function, see Example 1.
2. Explain how to identify and graph cubic, square root, and reciprocal functions *(page 200)*.
3. Explain how to identify and graph step and other piecewise-defined functions *(page 201)*. For an example involving a step function, see Example 2. For an example of graphing a piecewise-defined function, see Example 3.
4. State and sketch the graphs of parent functions *(page 202)*.

2.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary

In Exercises 1–9, match each function with its name.

1. $f(x) = [\![x]\!]$

2. $f(x) = x$

3. $f(x) = 1/x$

4. $f(x) = x^2$

5. $f(x) = \sqrt{x}$

6. $f(x) = c$

7. $f(x) = |x|$

8. $f(x) = x^3$

9. $f(x) = ax + b$

(a) squaring function
(b) square root function
(c) cubic function
(d) linear function
(e) constant function
(f) absolute value function
(g) greatest integer function
(h) reciprocal function
(i) identity function

10. Fill in the blank: The constant function and the identity function are two special types of ________ functions.

Skills and Applications

Writing a Linear Function **In Exercises 11–14, (a) write the linear function *f* such that it has the indicated function values and (b) sketch the graph of the function.**

11. $f(1) = 4, \quad f(0) = 6$

12. $f(-3) = -8, \quad f(1) = 2$

13. $f(-5) = -1, \quad f(5) = -1$

14. $f\left(\frac{2}{3}\right) = -\frac{15}{2}, \quad f(-4) = -11$

Graphing a Function **In Exercises 15–26, use a graphing utility to graph the function. Be sure to choose an appropriate viewing window.**

15. $f(x) = 2.5x - 4.25$

16. $f(x) = \frac{5}{6} - \frac{2}{3}x$

17. $g(x) = -2x^2$

18. $f(x) = 3x^2 - 1.75$

19. $f(x) = x^3 - 1$

20. $f(x) = (x - 1)^3 + 2$

21. $f(x) = 4 - 2\sqrt{x}$

22. $h(x) = \sqrt{x + 2} + 3$

23. $f(x) = 4 + (1/x)$

24. $k(x) = 1/(x - 3)$

25. $g(x) = |x| - 5$

26. $f(x) = |x - 1|$

Evaluating a Step Function **In Exercises 27–30, evaluate the function for the indicated values.**

27. $f(x) = [\![x]\!]$

(a) $f(2.1)$ (b) $f(2.9)$ (c) $f(-3.1)$ (d) $f\left(\frac{7}{2}\right)$

28. $h(x) = [\![x + 3]\!]$

(a) $h(-2)$ (b) $h\left(\frac{1}{2}\right)$ (c) $h(4.2)$ (d) $h(-21.6)$

29. $k(x) = [\![\frac{1}{2}x + 6]\!]$

(a) $k(5)$ (b) $k(-6.1)$ (c) $k(0.1)$ (d) $k(15)$

30. $g(x) = -7[\![x + 4]\!] + 6$

(a) $g\left(\frac{1}{8}\right)$ (b) $g(9)$ (c) $g(-4)$ (d) $g\left(\frac{3}{2}\right)$

Graphing a Step Function **In Exercises 31–34, sketch the graph of the function.**

31. $g(x) = -[\![x]\!]$

32. $g(x) = 4[\![x]\!]$

33. $g(x) = [\![x]\!] - 1$

34. $g(x) = [\![x - 3]\!]$

Graphing a Piecewise-Defined Function **In Exercises 35–40, sketch the graph of the function.**

35. $g(x) = \begin{cases} x + 6, & x \le -4 \\ \frac{1}{2}x - 4, & x > -4 \end{cases}$

36. $f(x) = \begin{cases} \sqrt{4 + x}, & x < 0 \\ \sqrt{4 - x}, & x \ge 0 \end{cases}$

37. $f(x) = \begin{cases} 1 - (x - 1)^2, & x \le 2 \\ \sqrt{x - 2}, & x > 2 \end{cases}$

38. $f(x) = \begin{cases} x^2 + 5, & x \le 1 \\ -x^2 + 4x + 3, & x > 1 \end{cases}$

39. $h(x) = \begin{cases} 4 - x^2, & x < -2 \\ 3 + x, & -2 \le x < 0 \\ x^2 + 1, & x \ge 0 \end{cases}$

40. $k(x) = \begin{cases} 2x + 1, & x \le -1 \\ 2x^2 - 1, & -1 < x \le 1 \\ 1 - x^2, & x > 1 \end{cases}$

Graphing a Function **In Exercises 41 and 42, (a) use a graphing utility to graph the function and (b) state the domain and range of the function.**

41. $s(x) = 2\left(\frac{1}{4}x - [\![\frac{1}{4}x]\!]\right)$

42. $k(x) = 4\left(\frac{1}{2}x - [\![\frac{1}{2}x]\!]\right)^2$

43. Wages A mechanic's pay is \$14.00 per hour for regular time and time-and-a-half for overtime. The weekly wage function is

$$W(h) = \begin{cases} 14h, & 0 < h \le 40 \\ 21(h - 40) + 560, & h > 40 \end{cases}$$

where h is the number of hours worked in a week.

(a) Evaluate $W(30)$, $W(40)$, $W(45)$, and $W(50)$.

(b) The company increased the regular work week to 45 hours. What is the new weekly wage function?

44. **Revenue** The table shows the monthly revenue y (in thousands of dollars) of a landscaping business for each month of the year 2013, with $x = 1$ representing January.

Month, x	Revenue, y
1	5.2
2	5.6
3	6.6
4	8.3
5	11.5
6	15.8
7	12.8
8	10.1
9	8.6
10	6.9
11	4.5
12	2.7

Spreadsheet at LarsonPrecalculus.com

A mathematical model that represents these data is

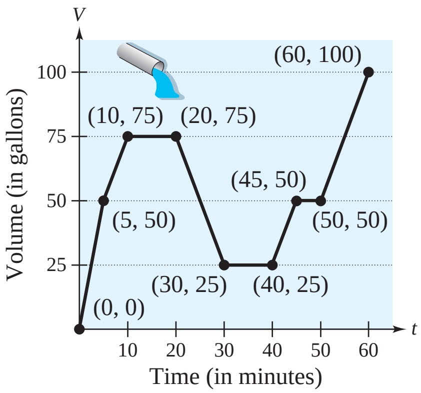
$$f(x) = \begin{cases} -1.97x + 26.3 \\ 0.505x^2 - 1.47x + 6.3. \end{cases}$$

(a) Use a graphing utility to graph the model. What is the domain of each part of the piecewise-defined function? How can you tell? Explain your reasoning.

(b) Find $f(5)$ and $f(11)$, and interpret your results in the context of the problem.

(c) How do the values obtained from the model in part (a) compare with the actual data values?

45. **Fluid Flow** The intake pipe of a 100-gallon tank has a flow rate of 10 gallons per minute, and two drainpipes have flow rates of 5 gallons per minute each. The figure shows the volume V of fluid in the tank as a function of time t. Determine the combination of the input pipe and drain pipes in which the fluid is flowing in specific subintervals of the 1 hour of time shown on the graph. (There are many correct answers.)



nulinukas/Shutterstock.com

46. **Delivery Charges** The cost of sending an overnight package from New York to Atlanta is \$26.10 for a package weighing up to, but not including, 1 pound and \$4.35 for each additional pound or portion of a pound.

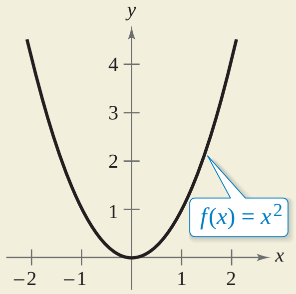
(a) Use the greatest integer function to create a model for the cost C of overnight delivery of a package weighing x pounds, $x > 0$.

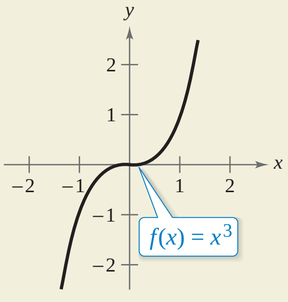
(b) Sketch the graph of the function.

47. **Snowstorm** During a nine-hour snowstorm, it snows at a rate of 1 inch per hour for the first 2 hours, at a rate of 2 inches per hour for the next 6 hours, and at a rate of 0.5 inch per hour for the final hour. Write and graph a piecewise-defined function that gives the depth of the snow during the snowstorm. How many inches of snow accumulated from the storm?

Exploration

48. **HOW DO YOU SEE IT?** For each graph of f shown below, answer (a)–(d).





(a) Find the domain and range of f.

(b) Find the x- and y-intercepts of the graph of f.

(c) Determine the intervals on which f is increasing, decreasing, or constant.

(d) Determine whether f is even, odd, or neither. Then describe the symmetry.

True or False? **In Exercises 49 and 50, determine whether the statement is true or false. Justify your answer.**

49. A piecewise-defined function will always have at least one x-intercept or at least one y-intercept.

50. A linear equation will always have an x-intercept and a y-intercept.

2.5 Transformations of Functions

Transformations of functions can help you model real-life applications. For instance, Exercise 69 on page 212 shows how a transformation of a function can model the number of horsepower required to overcome wind drag on an automobile.

- **Use vertical and horizontal shifts to sketch graphs of functions.**
- **Use reflections to sketch graphs of functions.**
- **Use nonrigid transformations to sketch graphs of functions.**

Shifting Graphs

Many functions have graphs that are transformations of the parent graphs summarized in Section 2.4. For example, you can obtain the graph of

$$h(x) = x^2 + 2$$

by shifting the graph of $f(x) = x^2$ *up* two units, as shown in Figure 2.28. In function notation, h and f are related as follows.

$$h(x) = x^2 + 2 = f(x) + 2 \qquad \text{Upward shift of two units}$$

Similarly, you can obtain the graph of

$$g(x) = (x - 2)^2$$

by shifting the graph of $f(x) = x^2$ to the *right* two units, as shown in Figure 2.29. In this case, the functions g and f have the following relationship.

$$g(x) = (x - 2)^2 = f(x - 2) \qquad \text{Right shift of two units}$$

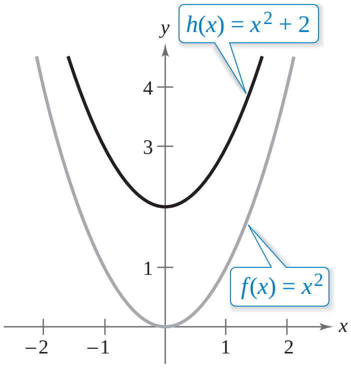


Figure 2.28

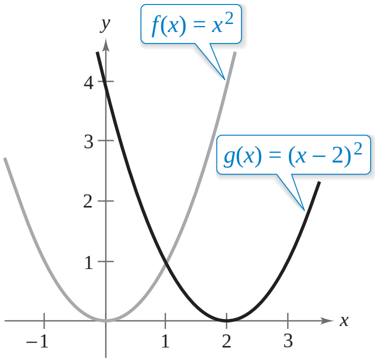


Figure 2.29

The following list summarizes this discussion about horizontal and vertical shifts.

Vertical and Horizontal Shifts

Let c be a positive real number. **Vertical and horizontal shifts** in the graph of $y = f(x)$ are represented as follows.

1. Vertical shift c units *up:* $\quad h(x) = f(x) + c$
2. Vertical shift c units *down:* $\quad h(x) = f(x) - c$
3. Horizontal shift c units to the *right:* $\quad h(x) = f(x - c)$
4. Horizontal shift c units to the *left:* $\quad h(x) = f(x + c)$

REMARK In items 3 and 4, be sure you see that $h(x) = f(x - c)$ corresponds to a *right* shift and $h(x) = f(x + c)$ corresponds to a *left* shift for $c > 0$.

Some graphs can be obtained from combinations of vertical and horizontal shifts, as demonstrated in Example 1(b). Vertical and horizontal shifts generate a *family of functions*, each with the same shape but at a different location in the plane.

Robert Young/Shutterstock.com

EXAMPLE 1 Shifts in the Graph of a Function

Use the graph of $f(x) = x^3$ to sketch the graph of each function.

a. $g(x) = x^3 - 1$

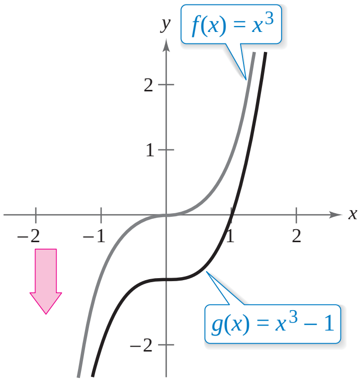
b. $h(x) = (x + 2)^3 + 1$

Solution

a. Relative to the graph of $f(x) = x^3$, the graph of

$$g(x) = x^3 - 1$$

is a downward shift of one unit, as shown below.

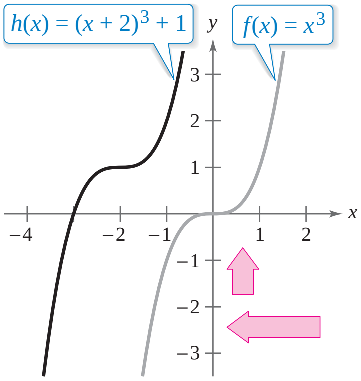


REMARK In Example 1(a), note that $g(x) = f(x) - 1$ and in Example 1(b), $h(x) = f(x + 2) + 1$.

b. Relative to the graph of $f(x) = x^3$, the graph of

$$h(x) = (x + 2)^3 + 1$$

involves a left shift of two units and an upward shift of one unit, as shown below.



Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use the graph of $f(x) = x^3$ to sketch the graph of each function.

a. $h(x) = x^3 + 5$

b. $g(x) = (x - 3)^3 + 2$

In Example 1(b), you obtain the same result when the vertical shift precedes the horizontal shift *or* when the horizontal shift precedes the vertical shift.

Reflecting Graphs

Another common type of transformation is a **reflection.** For instance, if you consider the x-axis to be a mirror, then the graph of $h(x) = -x^2$ is the mirror image (or reflection) of the graph of $f(x) = x^2$, as shown in Figure 2.30.

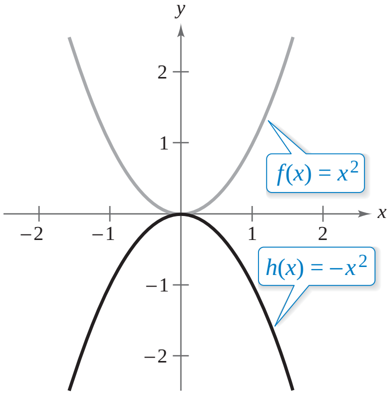


Figure 2.30

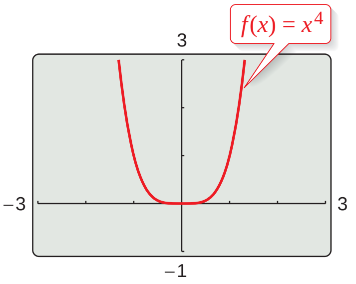


Figure 2.31

Reflections in the Coordinate Axes

Reflections in the coordinate axes of the graph of $y = f(x)$ are represented as follows.

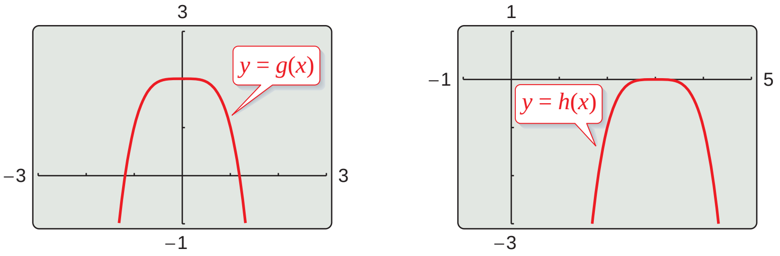
1. Reflection in the x-axis: $h(x) = -f(x)$
2. Reflection in the y-axis: $h(x) = f(-x)$

EXAMPLE 2 Writing Equations from Graphs

The graph of the function

$$f(x) = x^4$$

is shown in Figure 2.31. Each of the graphs below is a transformation of the graph of f. Write an equation for each of these functions.



(a) (b)

Solution

a. The graph of g is a reflection in the x-axis *followed by* an upward shift of two units of the graph of $f(x) = x^4$. So, the equation for g is

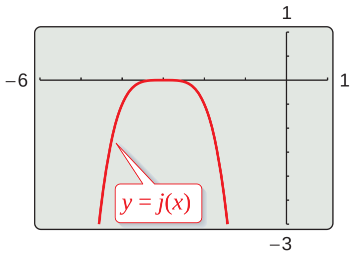
$$g(x) = -x^4 + 2.$$

b. The graph of h is a horizontal shift of three units to the right *followed by* a reflection in the x-axis of the graph of $f(x) = x^4$. So, the equation for h is

$$h(x) = -(x - 3)^4.$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The graph is a transformation of the graph of $f(x) = x^4$. Write an equation for the function.



EXAMPLE 3 Reflections and Shifts

Compare the graph of each function with the graph of $f(x) = \sqrt{x}$.

a. $g(x) = -\sqrt{x}$ **b.** $h(x) = \sqrt{-x}$ **c.** $k(x) = -\sqrt{x+2}$

Algebraic Solution

a. The graph of g is a reflection of the graph of f in the x-axis because

$$\begin{aligned} g(x) &= -\sqrt{x} \\ &= -f(x). \end{aligned}$$

b. The graph of h is a reflection of the graph of f in the y-axis because

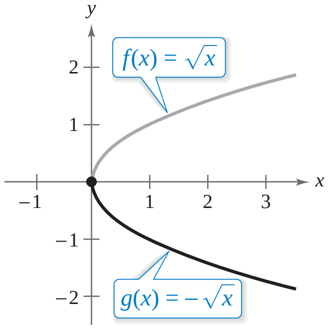
$$\begin{aligned} h(x) &= \sqrt{-x} \\ &= f(-x). \end{aligned}$$

c. The graph of k is a left shift of two units followed by a reflection in the x-axis because

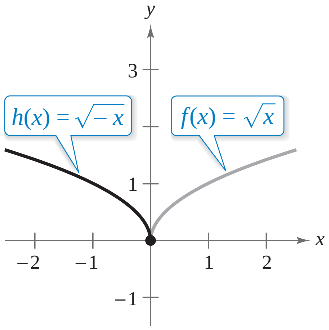
$$\begin{aligned} k(x) &= -\sqrt{x+2} \\ &= -f(x+2). \end{aligned}$$

Graphical Solution

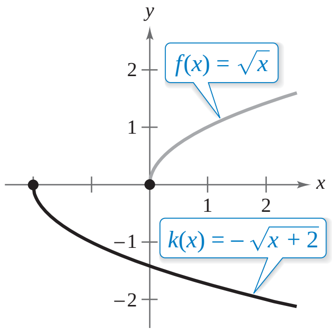
a. Graph f and g on the same set of coordinate axes. From the graph, you can see that the graph of g is a reflection of the graph of f in the x-axis.



b. Graph f and h on the same set of coordinate axes. From the graph, you can see that the graph of h is a reflection of the graph of f in the y-axis.



c. Graph f and k on the same set of coordinate axes. From the graph, you can see that the graph of k is a left shift of two units of the graph of f, followed by a reflection in the x-axis.



✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Compare the graph of each function with the graph of

$$f(x) = \sqrt{x-1}.$$

a. $g(x) = -\sqrt{x-1}$ **b.** $h(x) = \sqrt{-x-1}$

When sketching the graphs of functions involving square roots, remember that you must restrict the domain to exclude negative numbers inside the radical. For instance, here are the domains of the functions in Example 3.

Domain of $g(x) = -\sqrt{x}$: $x \geq 0$

Domain of $h(x) = \sqrt{-x}$: $x \leq 0$

Domain of $k(x) = -\sqrt{x+2}$: $x \geq -2$

Nonrigid Transformations

Horizontal shifts, vertical shifts, and reflections are **rigid transformations** because the basic shape of the graph is unchanged. These transformations change only the *position* of the graph in the coordinate plane. **Nonrigid transformations** are those that cause a *distortion*—a change in the shape of the original graph. For instance, a nonrigid transformation of the graph of $y = f(x)$ is represented by $g(x) = cf(x)$, where the transformation is a **vertical stretch** when $c > 1$ and a **vertical shrink** when $0 < c < 1$. Another nonrigid transformation of the graph of $y = f(x)$ is represented by $h(x) = f(cx)$, where the transformation is a **horizontal shrink** when $c > 1$ and a **horizontal stretch** when $0 < c < 1$.

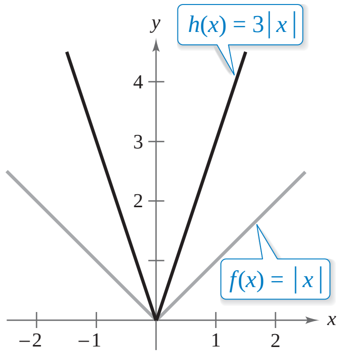


Figure 2.32

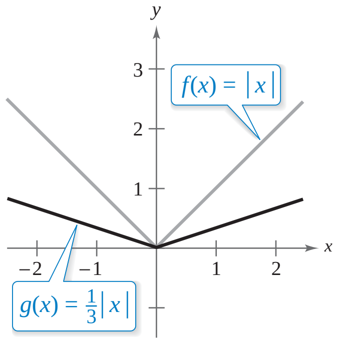


Figure 2.33

EXAMPLE 4 Nonrigid Transformations

Compare the graph of each function with the graph of $f(x) = |x|$.

a. $h(x) = 3|x|$ **b.** $g(x) = \frac{1}{3}|x|$

Solution

a. Relative to the graph of $f(x) = |x|$, the graph of $h(x) = 3|x| = 3f(x)$ is a vertical stretch (each y-value is multiplied by 3) of the graph of f. (See Figure 2.32.)

b. Similarly, the graph of $g(x) = \frac{1}{3}|x| = \frac{1}{3}f(x)$ is a vertical shrink $\left(\text{each } y\text{-value is multiplied by } \frac{1}{3}\right)$ of the graph of f. (See Figure 2.33.)

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Compare the graph of each function with the graph of $f(x) = x^2$.

a. $g(x) = 4x^2$ **b.** $h(x) = \frac{1}{4}x^2$

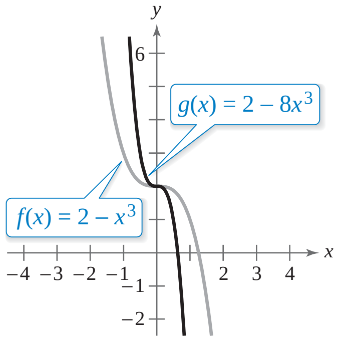


Figure 2.34

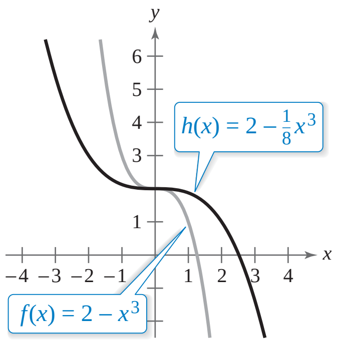


Figure 2.35

EXAMPLE 5 Nonrigid Transformations

Compare the graph of each function with the graph of $f(x) = 2 - x^3$.

a. $g(x) = f(2x)$ **b.** $h(x) = f\left(\frac{1}{2}x\right)$

Solution

a. Relative to the graph of $f(x) = 2 - x^3$, the graph of $g(x) = f(2x) = 2 - (2x)^3 = 2 - 8x^3$ is a horizontal shrink $(c > 1)$ of the graph of f. (See Figure 2.34.)

b. Similarly, the graph of $h(x) = f\left(\frac{1}{2}x\right) = 2 - \left(\frac{1}{2}x\right)^3 = 2 - \frac{1}{8}x^3$ is a horizontal stretch $(0 < c < 1)$ of the graph of f. (See Figure 2.35.)

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Compare the graph of each function with the graph of $f(x) = x^2 + 3$.

a. $g(x) = f(2x)$ **b.** $h(x) = f\left(\frac{1}{2}x\right)$

Summarize (Section 2.5)

1. Describe how to shift the graph of a function vertically and horizontally *(page 205)*. For an example of shifting the graph of a function, see Example 1.
2. Describe how to reflect the graph of a function in the x-axis and in the y-axis *(page 207)*. For examples of reflecting graphs of functions, see Examples 2 and 3.
3. Describe nonrigid transformations of the graph of a function *(page 209)*. For examples of nonrigid transformations, see Examples 4 and 5.

2.5 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary

In Exercises 1–3, fill in the blanks.

1. Horizontal shifts, vertical shifts, and reflections are called ________ transformations.
2. A reflection in the x-axis of $y = f(x)$ is represented by $h(x) =$ ________, while a reflection in the y-axis of $y = f(x)$ is represented by $h(x) =$ ________.
3. A nonrigid transformation of $y = f(x)$ represented by $g(x) = cf(x)$ is a ________ ________ when $c > 1$ and a ________ ________ when $0 < c < 1$.

4. Match the rigid transformation of $y = f(x)$ with the correct representation of the graph of h, where $c > 0$.

(a) $h(x) = f(x) + c$	(i) A horizontal shift of f, c units to the right
(b) $h(x) = f(x) - c$	(ii) A vertical shift of f, c units down
(c) $h(x) = f(x + c)$	(iii) A horizontal shift of f, c units to the left
(d) $h(x) = f(x - c)$	(iv) A vertical shift of f, c units up

Skills and Applications

5. **Shifts in the Graph of a Function** For each function, sketch (on the same set of coordinate axes) a graph of each function for $c = -1, 1$, and 3.
(a) $f(x) = |x| + c$ (b) $f(x) = |x - c|$

6. **Shifts in the Graph of a Function** For each function, sketch (on the same set of coordinate axes) a graph of each function for $c = -3, -1, 1$, and 3.
(a) $f(x) = \sqrt{x} + c$ (b) $f(x) = \sqrt{x - c}$

7. **Shifts in the Graph of a Function** For each function, sketch (on the same set of coordinate axes) a graph of each function for $c = -2, 0$, and 2.
(a) $f(x) = [\![x]\!] + c$ (b) $f(x) = [\![x + c]\!]$

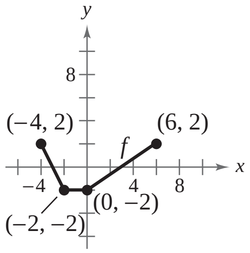
8. **Shifts in the Graph of a Function** For each function, sketch (on the same set of coordinate axes) a graph of each function for $c = -3, -1, 1$, and 3.

(a) $f(x) = \begin{cases} x^2 + c, & x < 0 \\ -x^2 + c, & x \geq 0 \end{cases}$

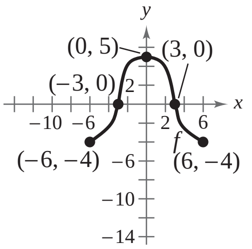
(b) $f(x) = \begin{cases} (x + c)^2, & x < 0 \\ -(x + c)^2, & x \geq 0 \end{cases}$

Sketching Transformations **In Exercises 9 and 10, use the graph of f to sketch each graph. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

9. (a) $y = f(-x)$
(b) $y = f(x) + 4$
(c) $y = 2f(x)$
(d) $y = -f(x - 4)$
(e) $y = f(x) - 3$
(f) $y = -f(x) - 1$
(g) $y = f(2x)$

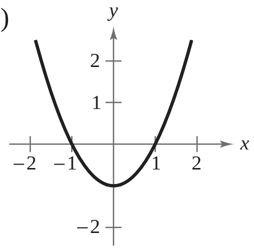


10. (a) $y = f(x - 5)$
(b) $y = -f(x) + 3$
(c) $y = \frac{1}{3}f(x)$
(d) $y = -f(x + 1)$
(e) $y = f(-x)$
(f) $y = f(x) - 10$
(g) $y = f\left(\frac{1}{3}x\right)$

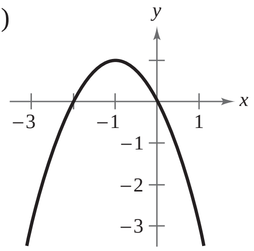


11. **Writing Equations from Graphs** Use the graph of $f(x) = x^2$ to write an equation for each function whose graph is shown.

(a)

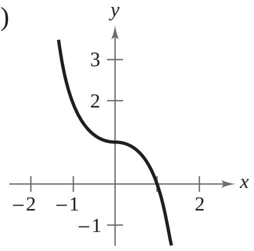


(b)

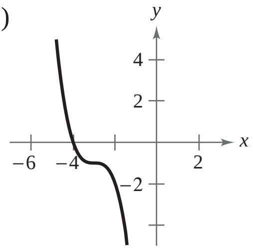


12. **Writing Equations from Graphs** Use the graph of $f(x) = x^3$ to write an equation for each function whose graph is shown.

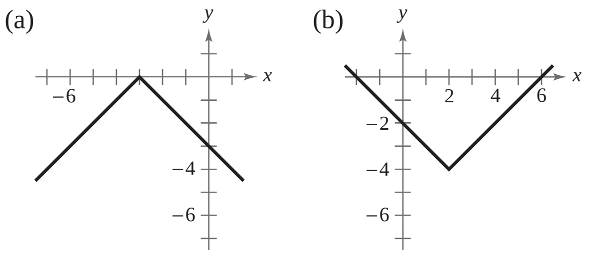
(a)



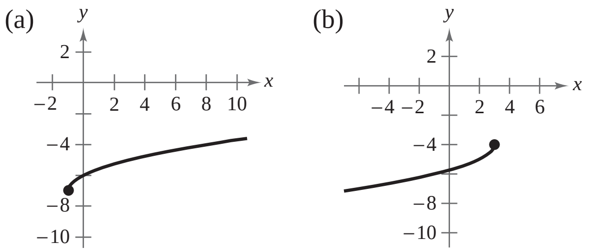
(b)



13. **Writing Equations from Graphs** Use the graph of $f(x) = |x|$ to write an equation for each function whose graph is shown.


(a) (b)

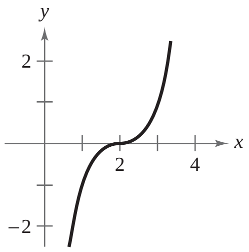
14. **Writing Equations from Graphs** Use the graph of $f(x) = \sqrt{x}$ to write an equation for each function whose graph is shown.


(a) (b)

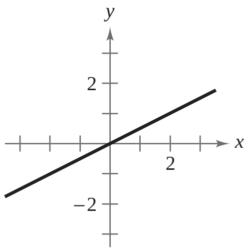
Identifying a Parent Function

In Exercises 15–20, identify the parent function and the transformation shown in the graph. Write an equation for the function shown in the graph.

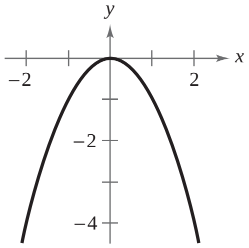
15.



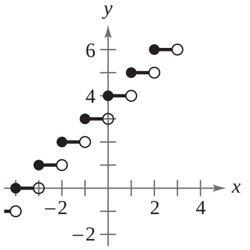
16.



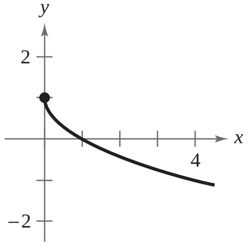
17.



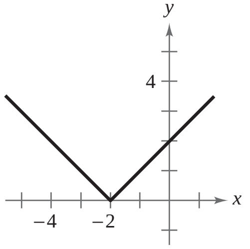
18.



19.



20.



Identifying a Parent Function

In Exercises 21–46, *g* is related to one of the parent functions described in Section 2.4. (a) Identify the parent function *f*. (b) Describe the sequence of transformations from *f* to *g*. (c) Sketch the graph of *g*. (d) Use function notation to write *g* in terms of *f*.

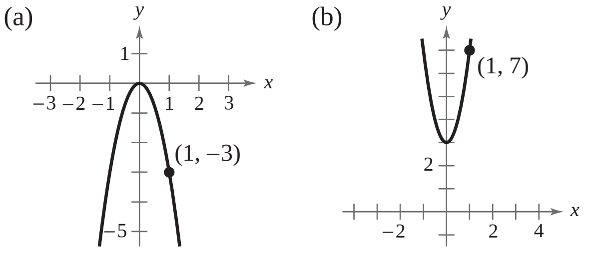
21. $g(x) = 12 - x^2$
22. $g(x) = (x - 8)^2$
23. $g(x) = x^3 + 7$
24. $g(x) = -x^3 - 1$
25. $g(x) = \frac{2}{3}x^2 + 4$
26. $g(x) = 2(x - 7)^2$
27. $g(x) = 2 - (x + 5)^2$
28. $g(x) = -\frac{1}{4}(x + 2)^2 - 2$
29. $g(x) = \sqrt{3x}$
30. $g(x) = \sqrt{\frac{1}{4}x}$
31. $g(x) = (x - 1)^3 + 2$
32. $g(x) = (x + 3)^3 - 10$
33. $g(x) = 3(x - 2)^3$
34. $g(x) = -\frac{1}{2}(x + 1)^3$
35. $g(x) = -|x| - 2$
36. $g(x) = 6 - |x + 5|$
37. $g(x) = -|x + 4| + 8$
38. $g(x) = |-x + 3| + 9$
39. $g(x) = -2|x - 1| - 4$
40. $g(x) = \frac{1}{2}|x - 2| - 3$
41. $g(x) = 3 - [\![x]\!]$
42. $g(x) = 2[\![x + 5]\!]$
43. $g(x) = \sqrt{x - 9}$
44. $g(x) = \sqrt{x + 4} + 8$
45. $g(x) = \sqrt{7 - x} - 2$
46. $g(x) = \sqrt{3x} + 1$

Writing an Equation from a Description

In Exercises 47–54, write an equation for the function described by the given characteristics.

47. The shape of $f(x) = x^2$, but shifted three units to the right and seven units down
48. The shape of $f(x) = x^2$, but shifted two units to the left, nine units up, and then reflected in the *x*-axis
49. The shape of $f(x) = x^3$, but shifted 13 units to the right
50. The shape of $f(x) = x^3$, but shifted six units to the left, six units down, and then reflected in the *y*-axis
51. The shape of $f(x) = |x|$, but shifted 12 units up and then reflected in the *x*-axis
52. The shape of $f(x) = |x|$, but shifted four units to the left and eight units down
53. The shape of $f(x) = \sqrt{x}$, but shifted six units to the left and then reflected in both the *x*-axis and the *y*-axis
54. The shape of $f(x) = \sqrt{x}$, but shifted nine units down and then reflected in both the *x*-axis and the *y*-axis

55. **Writing Equations from Graphs** Use the graph of $f(x) = x^2$ to write an equation for each function whose graph is shown.

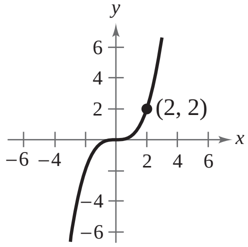

(a) (b)

56. Writing Equations from Graphs Use the graph of

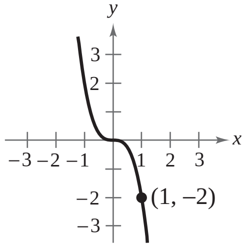
$f(x) = x^3$

to write an equation for each function whose graph is shown.

(a)



(b)

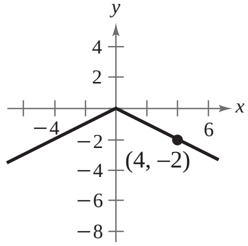


57. Writing Equations from Graphs Use the graph of

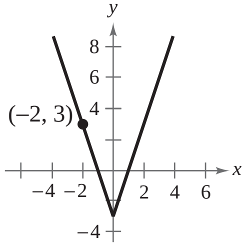
$f(x) = |x|$

to write an equation for each function whose graph is shown.

(a)



(b)

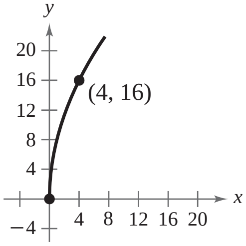


58. Writing Equations from Graphs Use the graph of

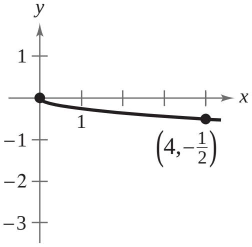
$f(x) = \sqrt{x}$

to write an equation for each function whose graph is shown.

(a)

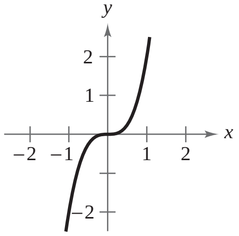


(b)

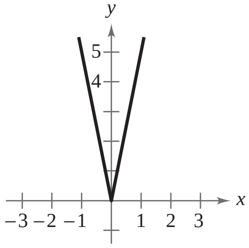


Identifying a Parent Function **In Exercises 59–64, identify the parent function and the transformation shown in the graph. Write an equation for the function shown in the graph. Then use a graphing utility to verify your answer.**

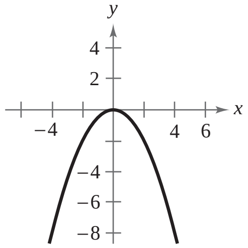
59.



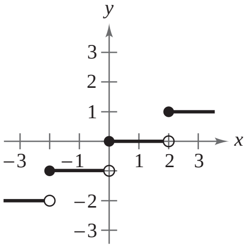
60.



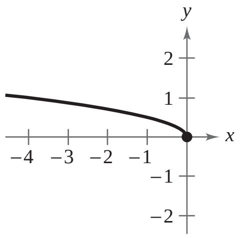
61.



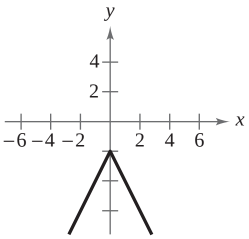
62.



63.

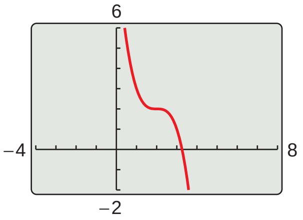


64.

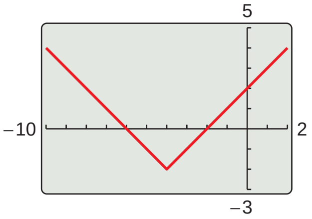


Graphical Analysis **In Exercises 65–68, use the viewing window shown to write a possible equation for the transformation of the parent function.**

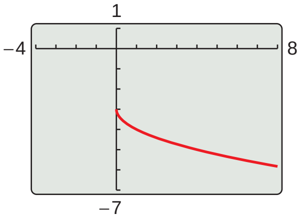
65.



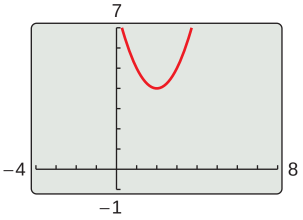
66.



67.



68.



69. Automobile Aerodynamics The number of horsepower H required to overcome wind drag on an automobile is approximated by

$$H(x) = 0.002x^2 + 0.005x - 0.029, \quad 10 \le x \le 100$$

where x is the speed of the car (in miles per hour).

(a) Use a graphing utility to graph the function.

(b) Rewrite the horsepower function so that x represents the speed in kilometers per hour. [Find $H(x/1.6)$.] Identify the type of transformation applied to the graph of the horsepower function.

Robert Young/Shutterstock.com

70. Households The numbers N (in millions) of households in the United States from 2003 through 2010 can be approximated by

$$N = -0.068(x - 13.68)^2 + 119, \quad 3 \le t \le 10$$

where t represents the year, with $t = 3$ corresponding to 2003. *(Source: U.S. Census Bureau)*

(a) Describe the transformation of the parent function $f(x) = x^2$. Then use a graphing utility to graph the function over the specified domain.

(b) Find the average rate of change of the function from 2003 to 2010. Interpret your answer in the context of the problem.

(c) Use the model to predict the number of households in the United States in 2018. Does your answer seem reasonable? Explain.

Exploration

True or False? **In Exercises 71–74, determine whether the statement is true or false. Justify your answer.**

71. The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ in the x-axis.

72. The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the y-axis.

73. The graphs of $f(x) = |x| + 6$ and $f(x) = |-x| + 6$ are identical.

74. If the graph of the parent function $f(x) = x^2$ is shifted six units to the right, three units up, and reflected in the x-axis, then the point $(-2, 19)$ will lie on the graph of the transformation.

75. Finding Points on a Graph The graph of $y = f(x)$ passes through the points $(0, 1)$, $(1, 2)$, and $(2, 3)$. Find the corresponding points on the graph of $y = f(x + 2) - 1$.

76. Think About It You can use either of two methods to graph a function: plotting points or translating a parent function as shown in this section. Which method of graphing do you prefer to use for each function? Explain.

(a) $f(x) = 3x^2 - 4x + 1$

(b) $f(x) = 2(x - 1)^2 - 6$

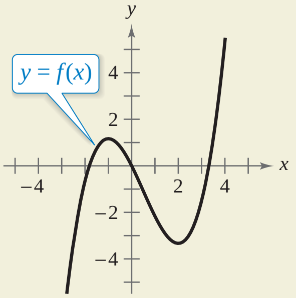
77. Predicting Graphical Relationships Use a graphing utility to graph f, g, and h in the same viewing window. Before looking at the graphs, try to predict how the graphs of g and h relate to the graph of f.

(a) $f(x) = x^2$, $g(x) = (x - 4)^2$,
$h(x) = (x - 4)^2 + 3$

(b) $f(x) = x^2$, $g(x) = (x + 1)^2$,
$h(x) = (x + 1)^2 - 2$

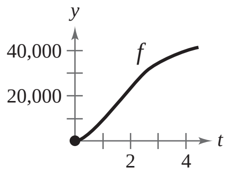
(c) $f(x) = x^2$, $g(x) = (x + 4)^2$,
$h(x) = (x + 4)^2 + 2$

78. HOW DO YOU SEE IT? Use the graph of $y = f(x)$ to find the intervals on which each of the graphs in (a)–(e) is increasing and decreasing. If not possible, then state the reason.

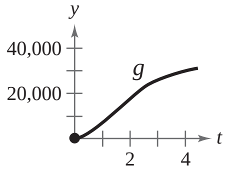


(a) $y = f(-x)$ (b) $y = -f(x)$ (c) $y = \frac{1}{2}f(x)$
(d) $y = -f(x - 1)$ (e) $y = f(x - 2) + 1$

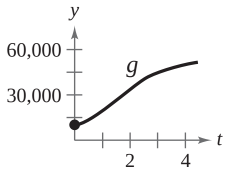
79. Describing Profits Management originally predicted that the profits from the sales of a new product would be approximated by the graph of the function f shown. The actual profits are shown by the function g along with a verbal description. Use the concepts of transformations of graphs to write g in terms of f.



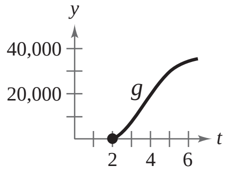
(a) The profits were only three-fourths as large as expected.



(b) The profits were consistently $10,000 greater than predicted.



(c) There was a two-year delay in the introduction of the product. After sales began, profits grew as expected.



80. Reversing the Order of Transformations Reverse the order of transformations in Example 2(a). Do you obtain the same graph? Do the same for Example 2(b). Do you obtain the same graph? Explain.

2.6 Combinations of Functions: Composite Functions

Arithmetic combinations of functions can help you model and solve real-life problems. For instance, in Exercise 57 on page 220, you will use arithmetic combinations of functions to analyze numbers of pets in the United States.

- **Add, subtract, multiply, and divide functions.**
- **Find the composition of one function with another function.**
- **Use combinations and compositions of functions to model and solve real-life problems.**

Arithmetic Combinations of Functions

Just as two real numbers can be combined by the operations of addition, subtraction, multiplication, and division to form other real numbers, two *functions* can be combined to create new functions. For example, the functions $f(x) = 2x - 3$ and $g(x) = x^2 - 1$ can be combined to form the sum, difference, product, and quotient of f and g.

$$f(x) + g(x) = (2x - 3) + (x^2 - 1) = x^2 + 2x - 4 \qquad \text{Sum}$$

$$f(x) - g(x) = (2x - 3) - (x^2 - 1) = -x^2 + 2x - 2 \qquad \text{Difference}$$

$$f(x)g(x) = (2x - 3)(x^2 - 1) = 2x^3 - 3x^2 - 2x + 3 \qquad \text{Product}$$

$$\frac{f(x)}{g(x)} = \frac{2x - 3}{x^2 - 1}, \quad x \neq \pm 1 \qquad \text{Quotient}$$

The domain of an **arithmetic combination** of functions f and g consists of all real numbers that are common to the domains of f and g. In the case of the quotient $f(x)/g(x)$, there is the further restriction that $g(x) \neq 0$.

Sum, Difference, Product, and Quotient of Functions

Let f and g be two functions with overlapping domains. Then, for all x common to both domains, the *sum, difference, product,* and *quotient* of f and g are defined as follows.

1. Sum: $(f + g)(x) = f(x) + g(x)$
2. Difference: $(f - g)(x) = f(x) - g(x)$
3. Product: $(fg)(x) = f(x) \cdot g(x)$
4. Quotient: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$

EXAMPLE 1 Finding the Sum of Two Functions

Given $f(x) = 2x + 1$ and $g(x) = x^2 + 2x - 1$, find $(f + g)(x)$. Then evaluate the sum when $x = 3$.

Solution The sum of f and g is

$$(f + g)(x) = f(x) + g(x) = (2x + 1) + (x^2 + 2x - 1) = x^2 + 4x.$$

When $x = 3$, the value of this sum is

$$(f + g)(3) = 3^2 + 4(3) = 21.$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Given $f(x) = x^2$ and $g(x) = 1 - x$, find $(f + g)(x)$. Then evaluate the sum when $x = 2$.

Michael Pettigrew/Shutterstock.com

EXAMPLE 2 Finding the Difference of Two Functions

Given $f(x) = 2x + 1$ and $g(x) = x^2 + 2x - 1$, find $(f - g)(x)$. Then evaluate the difference when $x = 2$.

Solution The difference of f and g is

$$(f - g)(x) = f(x) - g(x) = (2x + 1) - (x^2 + 2x - 1) = -x^2 + 2.$$

When $x = 2$, the value of this difference is

$$(f - g)(2) = -(2)^2 + 2 = -2.$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Given $f(x) = x^2$ and $g(x) = 1 - x$, find $(f - g)(x)$. Then evaluate the difference when $x = 3$.

EXAMPLE 3 Finding the Product of Two Functions

Given $f(x) = x^2$ and $g(x) = x - 3$, find $(fg)(x)$. Then evaluate the product when $x = 4$.

Solution The product of f and g is

$$(fg)(x) = f(x)g(x) = (x^2)(x - 3) = x^3 - 3x^2.$$

When $x = 4$, the value of this product is

$$(fg)(4) = 4^3 - 3(4)^2 = 16.$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Given $f(x) = x^2$ and $g(x) = 1 - x$, find $(fg)(x)$. Then evaluate the product when $x = 3$.

In Examples 1–3, both f and g have domains that consist of all real numbers. So, the domains of $f + g$, $f - g$, and fg are also the set of all real numbers. Remember to consider any restrictions on the domains of f and g when forming the sum, difference, product, or quotient of f and g.

EXAMPLE 4 Finding the Quotients of Two Functions

Find $(f/g)(x)$ and $(g/f)(x)$ for the functions $f(x) = \sqrt{x}$ and $g(x) = \sqrt{4 - x^2}$. Then find the domains of f/g and g/f.

Solution The quotient of f and g is

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{\sqrt{4 - x^2}}$$

and the quotient of g and f is

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{\sqrt{4 - x^2}}{\sqrt{x}}.$$

The domain of f is $[0, \infty)$ and the domain of g is $[-2, 2]$. The intersection of these domains is $[0, 2]$. So, the domains of f/g and g/f are as follows.

Domain of f/g: $[0, 2)$ Domain of g/f: $(0, 2]$

REMARK Note that the domain of f/g includes $x = 0$, but not $x = 2$, because $x = 2$ yields a zero in the denominator, whereas the domain of g/f includes $x = 2$, but not $x = 0$, because $x = 0$ yields a zero in the denominator.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find $(f/g)(x)$ and $(g/f)(x)$ for the functions $f(x) = \sqrt{x - 3}$ and $g(x) = \sqrt{16 - x^2}$. Then find the domains of f/g and g/f.

Composition of Functions

Another way of combining two functions is to form the **composition** of one with the other. For instance, if $f(x) = x^2$ and $g(x) = x + 1$, then the composition of f with g is

$$\begin{aligned} f(g(x)) &= f(x+1) \\ &= (x+1)^2. \end{aligned}$$

This composition is denoted as $f \circ g$ and reads as "f composed with g."

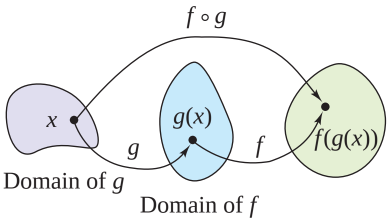


Figure 2.36

Definition of Composition of Two Functions

The **composition** of the function f with the function g is

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. (See Figure 2.36.)

EXAMPLE 5 Composition of Functions

Given $f(x) = x + 2$ and $g(x) = 4 - x^2$, find the following.

a. $(f \circ g)(x)$

b. $(g \circ f)(x)$

c. $(g \circ f)(-2)$

Solution

a. The composition of f with g is as follows.

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g \\ &= f(4 - x^2) && \text{Definition of } g(x) \\ &= (4 - x^2) + 2 && \text{Definition of } f(x) \\ &= -x^2 + 6 && \text{Simplify.} \end{aligned}$$

REMARK The following tables of values help illustrate the composition $(f \circ g)(x)$ in Example 5(a).

x	0	1	2	3
$g(x)$	4	3	0	-5

$g(x)$	4	3	0	-5
$f(g(x))$	6	5	2	-3

x	0	1	2	3
$f(g(x))$	6	5	2	-3

Note that the first two tables can be combined (or "composed") to produce the values in the third table.

b. The composition of g with f is as follows.

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) && \text{Definition of } g \circ f \\ &= g(x + 2) && \text{Definition of } f(x) \\ &= 4 - (x + 2)^2 && \text{Definition of } g(x) \\ &= 4 - (x^2 + 4x + 4) && \text{Expand.} \\ &= -x^2 - 4x && \text{Simplify.} \end{aligned}$$

Note that, in this case, $(f \circ g)(x) \neq (g \circ f)(x)$.

c. Using the result of part (b), write the following.

$$\begin{aligned} (g \circ f)(-2) &= -(-2)^2 - 4(-2) && \text{Substitute.} \\ &= -4 + 8 && \text{Simplify.} \\ &= 4 && \text{Simplify.} \end{aligned}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Given $f(x) = 2x + 5$ and $g(x) = 4x^2 + 1$, find the following.

a. $(f \circ g)(x)$ **b.** $(g \circ f)(x)$ **c.** $(f \circ g)\left(-\frac{1}{2}\right)$

EXAMPLE 6 Finding the Domain of a Composite Function

Find the domain of $(f \circ g)(x)$ for the functions

$$f(x) = x^2 - 9 \quad \text{and} \quad g(x) = \sqrt{9 - x^2}.$$

Algebraic Solution

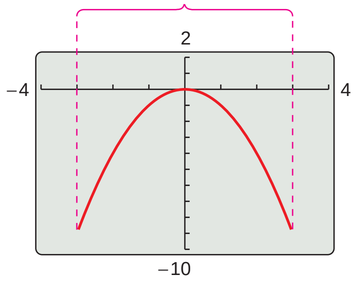
The composition of the functions is as follows.

$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f\left(\sqrt{9 - x^2}\right)\\ &= \left(\sqrt{9 - x^2}\right)^2 - 9\\ &= 9 - x^2 - 9\\ &= -x^2\end{aligned}$$

From this, it might appear that the domain of the composition is the set of all real numbers. This, however, is not true. Because the domain of f is the set of all real numbers and the domain of g is $[-3, 3]$, the domain of $f \circ g$ is $[-3, 3]$.

Graphical Solution

The x-coordinates of the points on the graph extend from -3 to 3. So, the domain of $f \circ g$ is $[-3, 3]$.



✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the domain of $(f \circ g)(x)$ for the functions $f(x) = \sqrt{x}$ and $g(x) = x^2 + 4$.

In Examples 5 and 6, you formed the composition of two given functions. In calculus, it is also important to be able to identify two functions that make up a given composite function. For instance, the function $h(x) = (3x - 5)^3$ is the composition of $f(x) = x^3$ and $g(x) = 3x - 5$. That is,

$$h(x) = (3x - 5)^3 = [g(x)]^3 = f(g(x)).$$

Basically, to "decompose" a composite function, look for an "inner" function and an "outer" function. In the function h above, $g(x) = 3x - 5$ is the inner function and $f(x) = x^3$ is the outer function.

EXAMPLE 7 Decomposing a Composite Function

Write the function $h(x) = \dfrac{1}{(x - 2)^2}$ as a composition of two functions.

Solution One way to write h as a composition of two functions is to take the inner function to be $g(x) = x - 2$ and the outer function to be

$$f(x) = \frac{1}{x^2} = x^{-2}.$$

Then write

$$\begin{aligned}h(x) &= \frac{1}{(x - 2)^2}\\ &= (x - 2)^{-2}\\ &= f(x - 2)\\ &= f(g(x)).\end{aligned}$$

✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write the function $h(x) = \dfrac{\sqrt[3]{8 - x}}{5}$ as a composition of two functions.

Application

EXAMPLE 8 Bacteria Count

The number N of bacteria in a refrigerated food is given by

$$N(T) = 20T^2 - 80T + 500, \quad 2 \le T \le 14$$

where T is the temperature of the food in degrees Celsius. When the food is removed from refrigeration, the temperature of the food is given by

$$T(t) = 4t + 2, \quad 0 \le t \le 3$$

where t is the time in hours.

a. Find the composition $(N \circ T)(t)$ and interpret its meaning in context.

b. Find the time when the bacteria count reaches 2000.

Refrigerated foods can have two types of bacteria: pathogenic bacteria, which can cause foodborne illness, and spoilage bacteria, which give foods an unpleasant look, smell, taste, or texture.

Solution

a.
$$\begin{aligned}(N \circ T)(t) &= N(T(t)) \\ &= 20(4t + 2)^2 - 80(4t + 2) + 500 \\ &= 20(16t^2 + 16t + 4) - 320t - 160 + 500 \\ &= 320t^2 + 320t + 80 - 320t - 160 + 500 \\ &= 320t^2 + 420\end{aligned}$$

The composite function $(N \circ T)(t)$ represents the number of bacteria in the food as a function of the amount of time the food has been out of refrigeration.

b. The bacteria count will reach 2000 when $320t^2 + 420 = 2000$. Solve this equation to find that the count will reach 2000 when $t \approx 2.2$ hours. Note that when you solve this equation, you reject the negative value because it is not in the domain of the composite function.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The number N of bacteria in a refrigerated food is given by

$$N(T) = 8T^2 - 14T + 200, \quad 2 \le T \le 12$$

where T is the temperature of the food in degrees Celsius. When the food is removed from refrigeration, the temperature of the food is given by

$$T(t) = 2t + 2, \quad 0 \le t \le 5$$

where t is the time in hours.

a. Find the composition $(N \circ T)(t)$.

b. Find the time when the bacteria count reaches 1000.

Summarize (Section 2.6)

1. Explain how to add, subtract, multiply, and divide functions *(page 214)*. For examples of finding arithmetic combinations of functions, see Examples 1–4.
2. Explain how to find the composition of one function with another function *(page 216)*. For examples that use compositions of functions, see Examples 5–7.
3. Describe a real-life example that uses a composition of functions *(page 218, Example 8)*.

Fedorov Oleksiy/Shutterstock.com

2.6 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

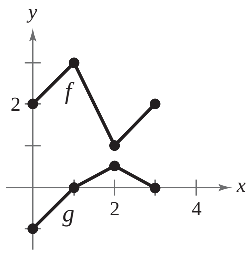
Vocabulary: Fill in the blanks.

1. Two functions f and g can be combined by the arithmetic operations of ________, ________, ________, and ________ to create new functions.
2. The ________ of the function f with g is $(f \circ g)(x) = f(g(x))$.

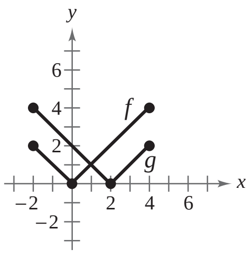
Skills and Applications

Graphing the Sum of Two Functions **In Exercises 3 and 4, use the graphs of f and g to graph $h(x) = (f + g)(x)$. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

3.



4.



Finding Arithmetic Combinations of Functions **In Exercises 5–12, find (a) $(f + g)(x)$, (b) $(f - g)(x)$, (c) $(fg)(x)$, and (d) $(f/g)(x)$. What is the domain of f/g?**

5. $f(x) = x + 2, \quad g(x) = x - 2$
6. $f(x) = 2x - 5, \quad g(x) = 2 - x$
7. $f(x) = x^2, \quad g(x) = 4x - 5$
8. $f(x) = 3x + 1, \quad g(x) = 5x - 4$
9. $f(x) = x^2 + 6, \quad g(x) = \sqrt{1 - x}$
10. $f(x) = \sqrt{x^2 - 4}, \quad g(x) = \dfrac{x^2}{x^2 + 1}$
11. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x^2}$
12. $f(x) = \dfrac{x}{x + 1}, \quad g(x) = x^3$

Evaluating an Arithmetic Combination of Functions **In Exercises 13–24, evaluate the indicated function for $f(x) = x^2 + 1$ and $g(x) = x - 4$.**

13. $(f + g)(2)$
14. $(f - g)(-1)$
15. $(f - g)(0)$
16. $(f + g)(1)$
17. $(f - g)(3t)$
18. $(f + g)(t - 2)$
19. $(fg)(6)$
20. $(fg)(-6)$
21. $(f/g)(5)$
22. $(f/g)(0)$
23. $(f/g)(-1) - g(3)$
24. $(fg)(5) + f(4)$

Graphing Two Functions and Their Sum **In Exercises 25 and 26, graph the functions f, g, and $f + g$ on the same set of coordinate axes.**

25. $f(x) = \frac{1}{2}x, \quad g(x) = x - 1$
26. $f(x) = 4 - x^2, \quad g(x) = x$

Graphical Reasoning **In Exercises 27–30, use a graphing utility to graph f, g, and $f + g$ in the same viewing window. Which function contributes most to the magnitude of the sum when $0 \le x \le 2$? Which function contributes most to the magnitude of the sum when $x > 6$?**

27. $f(x) = 3x, \quad g(x) = -\dfrac{x^3}{10}$
28. $f(x) = \dfrac{x}{2}, \quad g(x) = \sqrt{x}$
29. $f(x) = 3x + 2, \quad g(x) = -\sqrt{x + 5}$
30. $f(x) = x^2 - \frac{1}{2}, \quad g(x) = -3x^2 - 1$

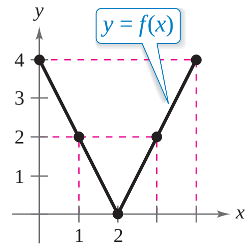
Finding Compositions of Functions **In Exercises 31–34, find (a) $f \circ g$, (b) $g \circ f$, and (c) $g \circ g$.**

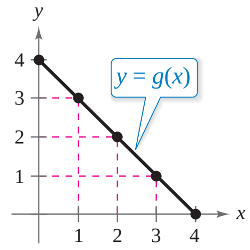
31. $f(x) = x^2, \quad g(x) = x - 1$
32. $f(x) = 3x + 5, \quad g(x) = 5 - x$
33. $f(x) = \sqrt[3]{x - 1}, \quad g(x) = x^3 + 1$
34. $f(x) = x^3, \quad g(x) = \dfrac{1}{x}$

Finding Domains of Functions and Composite Functions **In Exercises 35–42, find (a) $f \circ g$ and (b) $g \circ f$. Find the domain of each function and each composite function.**

35. $f(x) = \sqrt{x + 4}, \quad g(x) = x^2$
36. $f(x) = \sqrt[3]{x - 5}, \quad g(x) = x^3 + 1$
37. $f(x) = x^2 + 1, \quad g(x) = \sqrt{x}$
38. $f(x) = x^{2/3}, \quad g(x) = x^6$
39. $f(x) = |x|, \quad g(x) = x + 6$
40. $f(x) = |x - 4|, \quad g(x) = 3 - x$
41. $f(x) = \dfrac{1}{x}, \quad g(x) = x + 3$
42. $f(x) = \dfrac{3}{x^2 - 1}, \quad g(x) = x + 1$

Evaluating Combinations of Functions **In Exercises 43–46, use the graphs of f and g to evaluate the functions.**





43. (a) $(f + g)(3)$ (b) $(f/g)(2)$

44. (a) $(f - g)(1)$ (b) $(fg)(4)$

45. (a) $(f \circ g)(2)$ (b) $(g \circ f)(2)$

46. (a) $(f \circ g)(1)$ (b) $(g \circ f)(3)$

Decomposing a Composite Function **In Exercises 47–54, find two functions f and g such that $(f \circ g)(x) = h(x)$. (There are many correct answers.)**

47. $h(x) = (2x + 1)^2$ **48.** $h(x) = (1 - x)^3$

49. $h(x) = \sqrt[3]{x^2 - 4}$ **50.** $h(x) = \sqrt{9 - x}$

51. $h(x) = \dfrac{1}{x + 2}$ **52.** $h(x) = \dfrac{4}{(5x + 2)^2}$

53. $h(x) = \dfrac{-x^2 + 3}{4 - x^2}$

54. $h(x) = \dfrac{27x^3 + 6x}{10 - 27x^3}$

55. Stopping Distance The research and development department of an automobile manufacturer has determined that when a driver is required to stop quickly to avoid an accident, the distance (in feet) the car travels during the driver's reaction time is given by $R(x) = \frac{3}{4}x$, where x is the speed of the car in miles per hour. The distance (in feet) traveled while the driver is braking is given by $B(x) = \frac{1}{15}x^2$.

(a) Find the function that represents the total stopping distance T.

(b) Graph the functions R, B, and T on the same set of coordinate axes for $0 \le x \le 60$.

(c) Which function contributes most to the magnitude of the sum at higher speeds? Explain.

56. Vital Statistics Let $b(t)$ be the number of births in the United States in year t, and let $d(t)$ represent the number of deaths in the United States in year t, where $t = 10$ corresponds to 2010.

(a) If $p(t)$ is the population of the United States in year t, then find the function $c(t)$ that represents the percent change in the population of the United States.

(b) Interpret the value of $c(13)$.

Michael Pettigrew/Shutterstock.com

57. Pets

Let $d(t)$ be the number of dogs in the United States in year t, and let $c(t)$ be the number of cats in the United States in year t, where $t = 10$ corresponds to 2010.

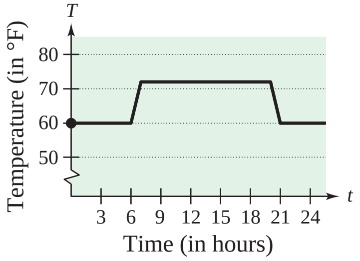
(a) Find the function $p(t)$ that represents the total number of dogs and cats in the United States.

(b) Interpret the value of $p(13)$.

(c) Let $n(t)$ represent the population of the United States in year t, where $t = 10$ corresponds to 2010. Find and interpret

$$h(t) = \frac{p(t)}{n(t)}.$$

58. Graphical Reasoning An electronically controlled thermostat in a home lowers the temperature automatically during the night. The temperature in the house T (in degrees Fahrenheit) is given in terms of t, the time in hours on a 24-hour clock (see figure).



(a) Explain why T is a function of t.

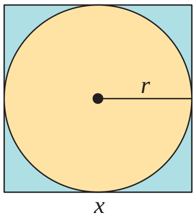
(b) Approximate $T(4)$ and $T(15)$.

(c) The thermostat is reprogrammed to produce a temperature H for which $H(t) = T(t - 1)$. How does this change the temperature?

(d) The thermostat is reprogrammed to produce a temperature H for which $H(t) = T(t) - 1$. How does this change the temperature?

(e) Write a piecewise-defined function that represents the graph.

59. Geometry A square concrete foundation is a base for a cylindrical tank (see figure).



(a) Write the radius r of the tank as a function of the length x of the sides of the square.

(b) Write the area A of the circular base of the tank as a function of the radius r.

(c) Find and interpret $(A \circ r)(x)$.

60. Bacteria Count The number N of bacteria in a refrigerated food is given by

$$N(T) = 10T^2 - 20T + 600, \quad 2 \leq T \leq 20$$

where T is the temperature of the food in degrees Celsius. When the food is removed from refrigeration, the temperature of the food is given by

$$T(t) = 3t + 2, \quad 0 \leq t \leq 6$$

where t is the time in hours.

(a) Find the composition $(N \circ T)(t)$ and interpret its meaning in context.

(b) Find the bacteria count after 0.5 hour.

(c) Find the time when the bacteria count reaches 1500.

61. Salary You are a sales representative for a clothing manufacturer. You are paid an annual salary, plus a bonus of 3% of your sales over \$500,000. Consider the two functions $f(x) = x - 500{,}000$ and $g(x) = 0.03x$. When x is greater than \$500,000, which of the following represents your bonus? Explain your reasoning.

(a) $f(g(x))$

(b) $g(f(x))$

62. Consumer Awareness The suggested retail price of a new hybrid car is p dollars. The dealership advertises a factory rebate of \$2000 and a 10% discount.

(a) Write a function R in terms of p giving the cost of the hybrid car after receiving the rebate from the factory.

(b) Write a function S in terms of p giving the cost of the hybrid car after receiving the dealership discount.

(c) Form the composite functions $(R \circ S)(p)$ and $(S \circ R)(p)$ and interpret each.

(d) Find $(R \circ S)(25{,}795)$ and $(S \circ R)(25{,}795)$. Which yields the lower cost for the hybrid car? Explain.

Exploration

Siblings **In Exercises 63 and 64, three siblings are three different ages. The oldest is twice the age of the middle sibling, and the middle sibling is six years older than one-half the age of the youngest.**

63. (a) Write a composite function that gives the oldest sibling's age in terms of the youngest. Explain how you arrived at your answer.

(b) If the oldest sibling is 16 years old, then find the ages of the other two siblings.

64. (a) Write a composite function that gives the youngest sibling's age in terms of the oldest. Explain how you arrived at your answer.

(b) If the youngest sibling is two years old, then find the ages of the other two siblings.

True or False? **In Exercises 65 and 66, determine whether the statement is true or false. Justify your answer.**

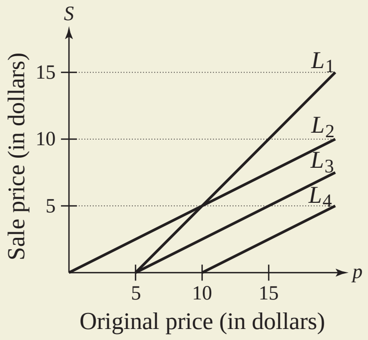
65. If $f(x) = x + 1$ and $g(x) = 6x$, then

$$(f \circ g)(x) = (g \circ f)(x).$$

66. When you are given two functions $f(x)$ and $g(x)$, you can calculate $(f \circ g)(x)$ if and only if the range of g is a subset of the domain of f.

67. Proof Prove that the product of two odd functions is an even function, and that the product of two even functions is an even function.

68. HOW DO YOU SEE IT? The graphs labeled L_1, L_2, L_3, and L_4 represent four different pricing discounts, where p is the original price (in dollars) and S is the sale price (in dollars). Match each function with its graph. Describe the situations in parts (c) and (d).



(a) $f(p)$: A 50% discount is applied.

(b) $g(p)$: A \$5 discount is applied.

(c) $(g \circ f)(p)$

(d) $(f \circ g)(p)$

69. Conjecture Use examples to hypothesize whether the product of an odd function and an even function is even or odd. Then prove your hypothesis.

70. Proof

(a) Given a function f, prove that $g(x)$ is even and $h(x)$ is odd, where $g(x) = \frac{1}{2}[f(x) + f(-x)]$ and

$$h(x) = \tfrac{1}{2}[f(x) - f(-x)].$$

(b) Use the result of part (a) to prove that any function can be written as a sum of even and odd functions. [*Hint:* Add the two equations in part (a).]

(c) Use the result of part (b) to write each function as a sum of even and odd functions.

$$f(x) = x^2 - 2x + 1, \quad k(x) = \frac{1}{x + 1}$$

2.7 Inverse Functions

Inverse functions can help you model and solve real-life problems. For instance, in Exercise 94 on page 230, an inverse function can help you determine the percent load interval for a diesel engine.

- **Find inverse functions informally and verify that two functions are inverse functions of each other.**
- **Use graphs of functions to determine whether functions have inverse functions.**
- **Use the Horizontal Line Test to determine whether functions are one-to-one.**
- **Find inverse functions algebraically.**

Inverse Functions

Recall from Section 2.2 that a set of ordered pairs can represent a function. For instance, the function $f(x) = x + 4$ from the set $A = \{1, 2, 3, 4\}$ to the set $B = \{5, 6, 7, 8\}$ can be written as follows.

$$f(x) = x + 4\text{:}\ \ \{(1, 5), (2, 6), (3, 7), (4, 8)\}$$

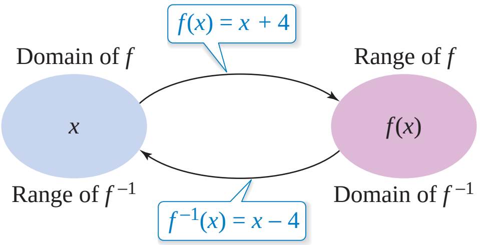
In this case, by interchanging the first and second coordinates of each of these ordered pairs, you can form the **inverse function** of f, which is denoted by f^{-1}. It is a function from the set B to the set A, and can be written as follows.

$$f^{-1}(x) = x - 4\text{:}\ \ \{(5, 1), (6, 2), (7, 3), (8, 4)\}$$

Note that the domain of f is equal to the range of f^{-1}, and vice versa, as shown in the figure below. Also note that the functions f and f^{-1} have the effect of "undoing" each other. In other words, when you form the composition of f with f^{-1} or the composition of f^{-1} with f, you obtain the identity function.

$$f(f^{-1}(x)) = f(x - 4) = (x - 4) + 4 = x$$

$$f^{-1}(f(x)) = f^{-1}(x + 4) = (x + 4) - 4 = x$$



EXAMPLE 1 Finding an Inverse Function Informally

Find the inverse function of $f(x) = 4x$. Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function.

Solution The function f *multiplies* each input by 4. To "undo" this function, you need to *divide* each input by 4. So, the inverse function of $f(x) = 4x$ is

$$f^{-1}(x) = \frac{x}{4}.$$

Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$ as follows.

$$f(f^{-1}(x)) = f\left(\frac{x}{4}\right) = 4\left(\frac{x}{4}\right) = x \qquad f^{-1}(f(x)) = f^{-1}(4x) = \frac{4x}{4} = x$$

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the inverse function of $f(x) = \frac{1}{5}x$. Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function.

Baloncici/Shutterstock.com

Definition of Inverse Function

Let f and g be two functions such that

$$f(g(x)) = x \qquad \text{for every } x \text{ in the domain of } g$$

and

$$g(f(x)) = x \qquad \text{for every } x \text{ in the domain of } f.$$

Under these conditions, the function g is the **inverse function** of the function f. The function g is denoted by f^{-1} (read "f-inverse"). So,

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

The domain of f must be equal to the range of f^{-1}, and the range of f must be equal to the domain of f^{-1}.

Do not be confused by the use of -1 to denote the inverse function f^{-1}. In this text, whenever f^{-1} is written, it *always* refers to the inverse function of the function f and *not* to the reciprocal of $f(x)$.

If the function g is the inverse function of the function f, then it must also be true that the function f is the inverse function of the function g. For this reason, you can say that the functions f and g are *inverse functions of each other.*

EXAMPLE 2 Verifying Inverse Functions

Which of the functions is the inverse function of $f(x) = \dfrac{5}{x-2}$?

$$g(x) = \frac{x-2}{5} \qquad h(x) = \frac{5}{x} + 2$$

Solution By forming the composition of f with g, you have

$$f(g(x)) = f\left(\frac{x-2}{5}\right) = \frac{5}{\left(\frac{x-2}{5}\right) - 2} = \frac{25}{x-12} \neq x.$$

Because this composition is not equal to the identity function x, it follows that g *is not* the inverse function of f. By forming the composition of f with h, you have

$$f(h(x)) = f\left(\frac{5}{x} + 2\right) = \frac{5}{\left(\frac{5}{x} + 2\right) - 2} = \frac{5}{\left(\frac{5}{x}\right)} = x.$$

So, it appears that h *is* the inverse function of f. Confirm this by showing that the composition of h with f is also equal to the identity function, as follows.

$$h(f(x)) = h\left(\frac{5}{x-2}\right) = \frac{5}{\left(\frac{5}{x-2}\right)} + 2 = x - 2 + 2 = x$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Which of the functions is the inverse function of $f(x) = \dfrac{x-4}{7}$?

$$g(x) = 7x + 4 \qquad h(x) = \frac{7}{x-4}$$

The Graph of an Inverse Function

The graphs of a function f and its inverse function f^{-1} are related to each other in the following way. If the point (a, b) lies on the graph of f, then the point (b, a) must lie on the graph of f^{-1}, and vice versa. This means that the graph of f^{-1} is a *reflection* of the graph of f in the line $y = x$, as shown in Figure 2.37.

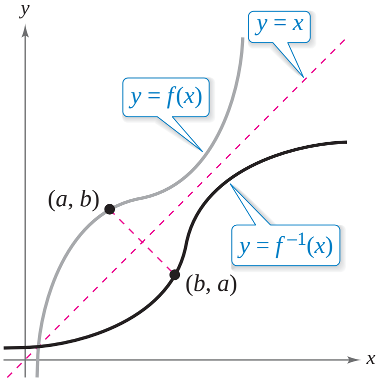


Figure 2.37

EXAMPLE 3 Verifying Inverse Functions Graphically

Sketch the graphs of the inverse functions $f(x) = 2x - 3$ and $f^{-1}(x) = \frac{1}{2}(x + 3)$ on the same rectangular coordinate system and show that the graphs are reflections of each other in the line $y = x$.

Solution The graphs of f and f^{-1} are shown in Figure 2.38. It appears that the graphs are reflections of each other in the line $y = x$. You can further verify this reflective property by testing a few points on each graph. Note in the following list that if the point (a, b) is on the graph of f, then the point (b, a) is on the graph of f^{-1}.

Graph of $f(x) = 2x - 3$	**Graph of $f^{-1}(x) = \frac{1}{2}(x + 3)$**
$(-1, -5)$	$(-5, -1)$
$(0, -3)$	$(-3, 0)$
$(1, -1)$	$(-1, 1)$
$(2, 1)$	$(1, 2)$
$(3, 3)$	$(3, 3)$

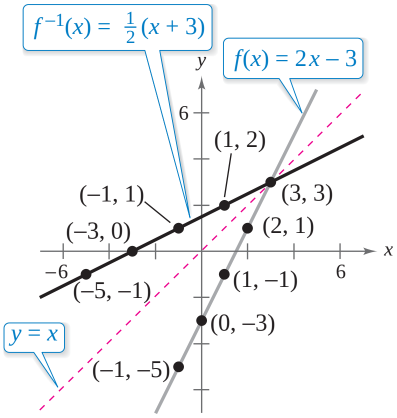


Figure 2.38

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Sketch the graphs of the inverse functions $f(x) = 4x - 1$ and $f^{-1}(x) = \frac{1}{4}(x + 1)$ on the same rectangular coordinate system and show that the graphs are reflections of each other in the line $y = x$.

EXAMPLE 4 Verifying Inverse Functions Graphically

Sketch the graphs of the inverse functions $f(x) = x^2$ $(x \geq 0)$ and $f^{-1}(x) = \sqrt{x}$ on the same rectangular coordinate system and show that the graphs are reflections of each other in the line $y = x$.

Solution The graphs of f and f^{-1} are shown in Figure 2.39. It appears that the graphs are reflections of each other in the line $y = x$. You can further verify this reflective property by testing a few points on each graph. Note in the following list that if the point (a, b) is on the graph of f, then the point (b, a) is on the graph of f^{-1}.

Graph of $f(x) = x^2$, $x \geq 0$	**Graph of $f^{-1}(x) = \sqrt{x}$**
$(0, 0)$	$(0, 0)$
$(1, 1)$	$(1, 1)$
$(2, 4)$	$(4, 2)$
$(3, 9)$	$(9, 3)$

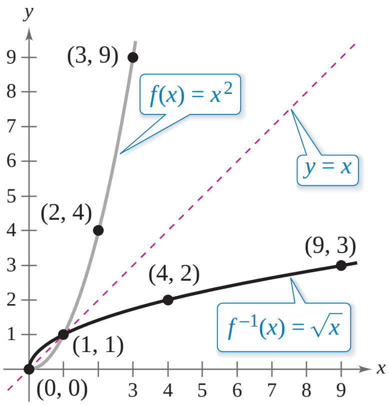


Figure 2.39

Try showing that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Sketch the graphs of the inverse functions $f(x) = x^2 + 1$ $(x \geq 0)$ and $f^{-1}(x) = \sqrt{x - 1}$ on the same rectangular coordinate system and show that the graphs are reflections of each other in the line $y = x$.

One-to-One Functions

The reflective property of the graphs of inverse functions gives you a *geometric* test for determining whether a function has an inverse function. This test is called the **Horizontal Line Test** for inverse functions.

Horizontal Line Test for Inverse Functions

A function f has an inverse function if and only if no *horizontal* line intersects the graph of f at more than one point.

If no horizontal line intersects the graph of f at more than one point, then no y-value matches with more than one x-value. This is the essential characteristic of what are called **one-to-one functions.**

One-to-One Functions

A function f is **one-to-one** when each value of the dependent variable corresponds to exactly one value of the independent variable. A function f has an inverse function if and only if f is one-to-one.

Consider the function $f(x) = x^2$. The table on the left is a table of values for $f(x) = x^2$. The table on the right is the same as the table on the left but with the values in the columns interchanged. The table on the right does not represent a function because the input $x = 4$, for instance, matches with two different outputs: $y = -2$ and $y = 2$. So, $f(x) = x^2$ is not one-to-one and does not have an inverse function.

x	$f(x) = x^2$
-2	4
-1	1
0	0
1	1
2	4
3	9

x	y
4	-2
1	-1
0	0
1	1
4	2
9	3

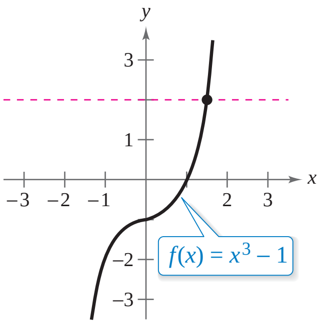


Figure 2.40

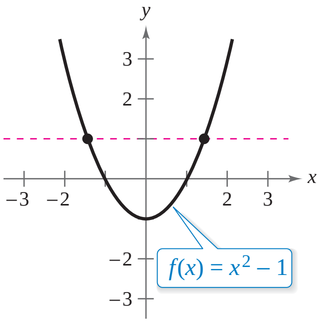


Figure 2.41

EXAMPLE 5 Applying the Horizontal Line Test

a. The graph of the function $f(x) = x^3 - 1$ is shown in Figure 2.40. Because no horizontal line intersects the graph of f at more than one point, f *is* a one-to-one function and *does* have an inverse function.

b. The graph of the function $f(x) = x^2 - 1$ is shown in Figure 2.41. Because it is possible to find a horizontal line that intersects the graph of f at more than one point, f *is not* a one-to-one function and *does not* have an inverse function.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use the graph of f to determine whether the function has an inverse function.

a. $f(x) = \frac{1}{2}(3 - x)$

b. $f(x) = |x|$

Finding Inverse Functions Algebraically

For relatively simple functions (such as the one in Example 1), you can find inverse functions by inspection. For more complicated functions, however, it is best to use the following guidelines. The key step in these guidelines is Step 3—interchanging the roles of x and y. This step corresponds to the fact that inverse functions have ordered pairs with the coordinates reversed.

REMARK Note what happens when you try to find the inverse function of a function that is not one-to-one.

$$\begin{aligned} f(x) &= x^2 + 1 && \text{Original function} \\ y &= x^2 + 1 && \text{Replace } f(x) \text{ with } y. \\ x &= y^2 + 1 && \text{Interchange } x \text{ and } y. \\ x - 1 &= y^2 && \text{Isolate } y\text{-term.} \\ y &= \pm\sqrt{x - 1} && \text{Solve for } y. \end{aligned}$$

You obtain two y-values for each x.

Finding an Inverse Function

1. Use the Horizontal Line Test to decide whether f has an inverse function.
2. In the equation for $f(x)$, replace $f(x)$ with y.
3. Interchange the roles of x and y, and solve for y.
4. Replace y with $f^{-1}(x)$ in the new equation.
5. Verify that f and f^{-1} are inverse functions of each other by showing that the domain of f is equal to the range of f^{-1}, the range of f is equal to the domain of f^{-1}, and $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

EXAMPLE 6 Finding an Inverse Function Algebraically

Find the inverse function of

$$f(x) = \frac{5 - x}{3x + 2}.$$

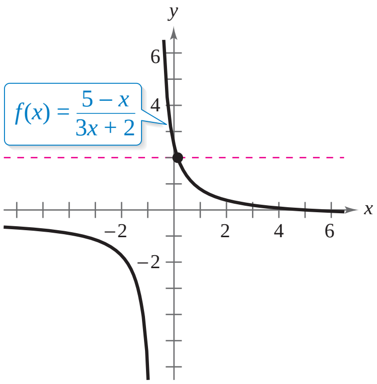


Figure 2.42

Solution The graph of f is shown in Figure 2.42. This graph passes the Horizontal Line Test. So, you know that f is one-to-one and has an inverse function.

$$\begin{aligned} f(x) &= \frac{5 - x}{3x + 2} && \text{Write original function.} \\ y &= \frac{5 - x}{3x + 2} && \text{Replace } f(x) \text{ with } y. \\ x &= \frac{5 - y}{3y + 2} && \text{Interchange } x \text{ and } y. \\ x(3y + 2) &= 5 - y && \text{Multiply each side by } 3y + 2. \\ 3xy + 2x &= 5 - y && \text{Distributive Property} \\ 3xy + y &= 5 - 2x && \text{Collect terms with } y. \\ y(3x + 1) &= 5 - 2x && \text{Factor.} \\ y &= \frac{5 - 2x}{3x + 1} && \text{Solve for } y. \\ f^{-1}(x) &= \frac{5 - 2x}{3x + 1} && \text{Replace } y \text{ with } f^{-1}(x). \end{aligned}$$

Check that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the inverse function of

$$f(x) = \frac{5 - 3x}{x + 2}.$$

EXAMPLE 7 Finding an Inverse Function Algebraically

Find the inverse function of

$$f(x) = \sqrt{2x - 3}.$$

Solution The graph of f is a curve, as shown in the figure below. Because this graph passes the Horizontal Line Test, you know that f is one-to-one and has an inverse function.

$f(x) = \sqrt{2x - 3}$ Write original function.

$y = \sqrt{2x - 3}$ Replace $f(x)$ with y.

$x = \sqrt{2y - 3}$ Interchange x and y.

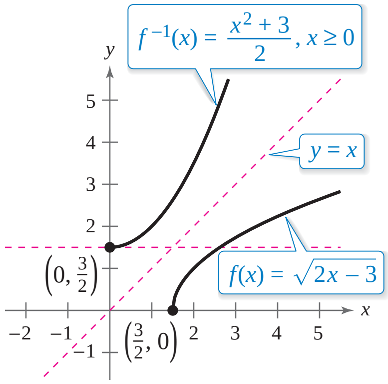
$x^2 = 2y - 3$ Square each side.

$2y = x^2 + 3$ Isolate y-term.

$y = \dfrac{x^2 + 3}{2}$ Solve for y.

$f^{-1}(x) = \dfrac{x^2 + 3}{2}, \quad x \geq 0$ Replace y with $f^{-1}(x)$.

The graph of f^{-1} in the figure is the reflection of the graph of f in the line $y = x$. Note that the range of f is the interval $[0, \infty)$, which implies that the domain of f^{-1} is the interval $[0, \infty)$. Moreover, the domain of f is the interval $\left[\frac{3}{2}, \infty\right)$, which implies that the range of f^{-1} is the interval $\left[\frac{3}{2}, \infty\right)$. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.



Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Find the inverse function of

$$f(x) = \sqrt[3]{10 + x}.$$

Summarize (Section 2.7)

1. State the definition of an inverse function *(page 223)*. For examples of finding inverse functions informally and verifying inverse functions, see Examples 1 and 2.
2. Explain how to use the graph of a function to determine whether the function has an inverse function *(page 224)*. For examples of verifying inverse functions graphically, see Examples 3 and 4.
3. Explain how to use the Horizontal Line Test to determine whether a function is one-to-one *(page 225)*. For an example of applying the Horizontal Line Test, see Example 5.
4. Explain how to find an inverse function algebraically *(page 226)*. For examples of finding inverse functions algebraically, see Examples 6 and 7.

2.7 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: **Fill in the blanks.**

1. If the composite functions $f(g(x))$ and $g(f(x))$ both equal x, then the function g is the ________ function of f.
2. The inverse function of f is denoted by ________.
3. The domain of f is the ________ of f^{-1}, and the ________ of f^{-1} is the range of f.
4. The graphs of f and f^{-1} are reflections of each other in the line ________.
5. A function f is ________ when each value of the dependent variable corresponds to exactly one value of the independent variable.
6. A graphical test for the existence of an inverse function of f is called the _______ Line Test.

Skills and Applications

Finding an Inverse Function Informally **In Exercises 7–12, find the inverse function of f informally. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.**

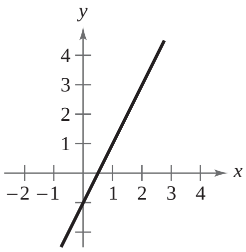
7. $f(x) = 6x$
8. $f(x) = \frac{1}{3}x$
9. $f(x) = 3x + 1$
10. $f(x) = \dfrac{x - 1}{5}$
11. $f(x) = \sqrt[3]{x}$
12. $f(x) = x^5$

Verifying Inverse Functions **In Exercises 13–16, verify that f and g are inverse functions.**

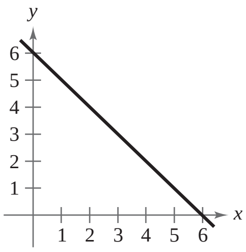
13. $f(x) = -\dfrac{7}{2}x - 3, \quad g(x) = -\dfrac{2x + 6}{7}$
14. $f(x) = \dfrac{x - 9}{4}, \quad g(x) = 4x + 9$
15. $f(x) = x^3 + 5, \quad g(x) = \sqrt[3]{x - 5}$
16. $f(x) = \dfrac{x^3}{2}, \quad g(x) = \sqrt[3]{2x}$

Sketching the Graph of an Inverse Function **In Exercises 17–20, use the graph of the function to sketch the graph of its inverse function $y = f^{-1}(x)$.**

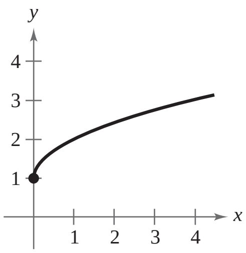
17.



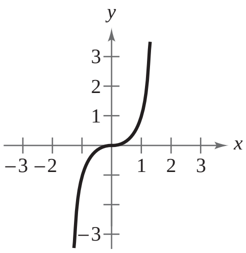
18.



19.



20.



Verifying Inverse Functions **In Exercises 21–32, verify that f and g are inverse functions (a) algebraically and (b) graphically.**

21. $f(x) = 2x, \quad g(x) = \dfrac{x}{2}$
22. $f(x) = x - 5, \quad g(x) = x + 5$
23. $f(x) = 7x + 1, \quad g(x) = \dfrac{x - 1}{7}$
24. $f(x) = 3 - 4x, \quad g(x) = \dfrac{3 - x}{4}$
25. $f(x) = \dfrac{x^3}{8}, \quad g(x) = \sqrt[3]{8x}$
26. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x}$
27. $f(x) = \sqrt{x - 4}, \quad g(x) = x^2 + 4, \quad x \geq 0$
28. $f(x) = 1 - x^3, \quad g(x) = \sqrt[3]{1 - x}$
29. $f(x) = 9 - x^2, \quad x \geq 0, \quad g(x) = \sqrt{9 - x}, \quad x \leq 9$
30. $f(x) = \dfrac{1}{1 + x}, \quad x \geq 0, \quad g(x) = \dfrac{1 - x}{x}, \quad 0 < x \leq 1$
31. $f(x) = \dfrac{x - 1}{x + 5}, \quad g(x) = -\dfrac{5x + 1}{x - 1}$
32. $f(x) = \dfrac{x + 3}{x - 2}, \quad g(x) = \dfrac{2x + 3}{x - 1}$

Using a Table to Determine an Inverse Function **In Exercises 33 and 34, does the function have an inverse function?**

33.

x	-1	0	1	2	3	4
$f(x)$	-2	1	2	1	-2	-6

34.

x	-3	-2	-1	0	2	3
$f(x)$	10	6	4	1	-3	-10

Using a Table to Find an Inverse Function **In Exercises 35 and 36, use the table of values for $y = f(x)$ to complete a table for $y = f^{-1}(x)$.**

35.

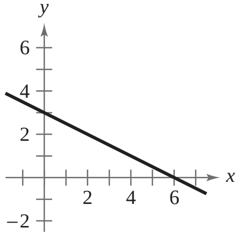
x	-2	-1	0	1	2	3
$f(x)$	-2	0	2	4	6	8

36.

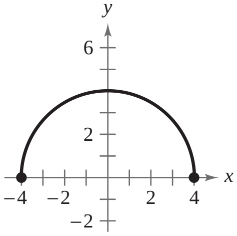
x	-3	-2	-1	0	1	2
$f(x)$	-10	-7	-4	-1	2	5

Applying the Horizontal Line Test **In Exercises 37–40, does the function have an inverse function?**

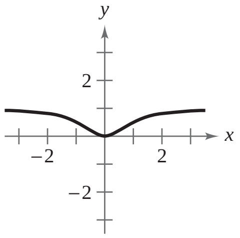
37.



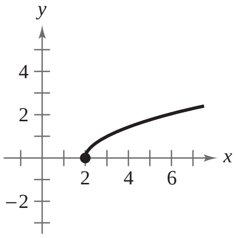
38.



39.



40.



Applying the Horizontal Line Test **In Exercises 41–44, use a graphing utility to graph the function, and use the Horizontal Line Test to determine whether the function has an inverse function.**

41. $g(x) = (x + 5)^3$

42. $f(x) = \frac{1}{8}(x + 2)^2 - 1$

43. $f(x) = -2x\sqrt{16 - x^2}$

44. $h(x) = |x + 4| - |x - 4|$

Finding and Analyzing Inverse Functions **In Exercises 45–56, (a) find the inverse function of f, (b) graph both f and f^{-1} on the same set of coordinate axes, (c) describe the relationship between the graphs of f and f^{-1}, and (d) state the domains and ranges of f and f^{-1}.**

45. $f(x) = 2x - 3$

46. $f(x) = 3x + 1$

47. $f(x) = x^5 - 2$

48. $f(x) = x^3 + 1$

49. $f(x) = \sqrt{4 - x^2}, \quad 0 \le x \le 2$

50. $f(x) = x^2 - 2, \quad x \le 0$

51. $f(x) = \dfrac{4}{x}$

52. $f(x) = -\dfrac{2}{x}$

53. $f(x) = \dfrac{x + 1}{x - 2}$

54. $f(x) = \dfrac{x - 3}{x + 2}$

55. $f(x) = \sqrt[3]{x - 1}$

56. $f(x) = x^{3/5}$

Finding an Inverse Function **In Exercises 57–72, determine whether the function has an inverse function. If it does, then find the inverse function.**

57. $f(x) = x^4$

58. $f(x) = \dfrac{1}{x^2}$

59. $g(x) = \dfrac{x}{8}$

60. $f(x) = 3x + 5$

61. $p(x) = -4$

62. $f(x) = \dfrac{3x + 4}{5}$

63. $f(x) = (x + 3)^2, \quad x \ge -3$

64. $q(x) = (x - 5)^2$

65. $f(x) = \begin{cases} x + 3, & x < 0 \\ 6 - x, & x \ge 0 \end{cases}$

66. $f(x) = \begin{cases} -x, & x \le 0 \\ x^2 - 3x, & x > 0 \end{cases}$

67. $h(x) = -\dfrac{4}{x^2}$

68. $f(x) = |x - 2|, \quad x \le 2$

69. $f(x) = \sqrt{2x + 3}$

70. $f(x) = \sqrt{x - 2}$

71. $f(x) = \dfrac{6x + 4}{4x + 5}$

72. $f(x) = \dfrac{5x - 3}{2x + 5}$

Restricting the Domain **In Exercises 73–82, restrict the domain of the function f so that the function is one-to-one and has an inverse function. Then find the inverse function f^{-1}. State the domains and ranges of f and f^{-1}. Explain your results. (There are many correct answers.)**

73. $f(x) = (x - 2)^2$

74. $f(x) = 1 - x^4$

75. $f(x) = |x + 2|$

76. $f(x) = |x - 5|$

77. $f(x) = (x + 6)^2$

78. $f(x) = (x - 4)^2$

79. $f(x) = -2x^2 + 5$

80. $f(x) = \frac{1}{2}x^2 - 1$

81. $f(x) = |x - 4| + 1$

82. $f(x) = -|x - 1| - 2$

Composition with Inverses **In Exercises 83–88, use the functions $f(x) = \frac{1}{8}x - 3$ and $g(x) = x^3$ to find the indicated value or function.**

83. $(f^{-1} \circ g^{-1})(1)$

84. $(g^{-1} \circ f^{-1})(-3)$

85. $(f^{-1} \circ f^{-1})(6)$

86. $(g^{-1} \circ g^{-1})(-4)$

87. $(f \circ g)^{-1}$

88. $g^{-1} \circ f^{-1}$

Composition with Inverses **In Exercises 89–92, use the functions $f(x) = x + 4$ and $g(x) = 2x - 5$ to find the specified function.**

89. $g^{-1} \circ f^{-1}$

90. $f^{-1} \circ g^{-1}$

91. $(f \circ g)^{-1}$

92. $(g \circ f)^{-1}$

93. Hourly Wage Your wage is \$10.00 per hour plus \$0.75 for each unit produced per hour. So, your hourly wage y in terms of the number of units produced x is $y = 10 + 0.75x$.

(a) Find the inverse function. What does each variable represent in the inverse function?

(b) Determine the number of units produced when your hourly wage is \$24.25.

94. Diesel Mechanics

The function

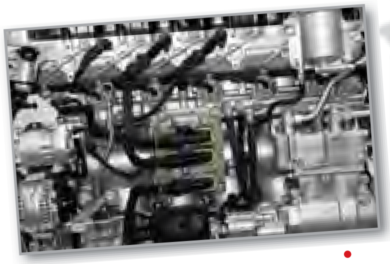
$$y = 0.03x^2 + 245.50, \quad 0 < x < 100$$

approximates the exhaust temperature y in degrees Fahrenheit, where x is the percent load for a diesel engine.

(a) Find the inverse function. What does each variable represent in the inverse function?

(b) Use a graphing utility to graph the inverse function.

(c) The exhaust temperature of the engine must not exceed 500 degrees Fahrenheit. What is the percent load interval?



Exploration

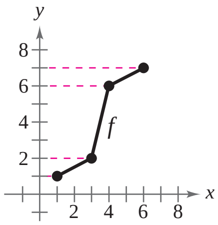
True or False? **In Exercises 95 and 96, determine whether the statement is true or false. Justify your answer.**

95. If f is an even function, then f^{-1} exists.

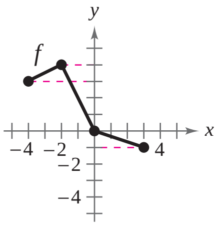
96. If the inverse function of f exists and the graph of f has a y-intercept, then the y-intercept of f is an x-intercept of f^{-1}.

Graphical Analysis **In Exercises 97 and 98, use the graph of the function f to create a table of values for the given points. Then create a second table that can be used to find f^{-1}, and sketch the graph of f^{-1} if possible.**

97.



98.



99. Proof Prove that if f and g are one-to-one functions, then $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

100. Proof Prove that if f is a one-to-one odd function, then f^{-1} is an odd function.

101. Think About It The function $f(x) = k(2 - x - x^3)$ has an inverse function, and $f^{-1}(3) = -2$. Find k.

102. Think About It Consider the functions $f(x) = x + 2$ and $f^{-1}(x) = x - 2$. Evaluate $f(f^{-1}(x))$ and $f^{-1}(f(x))$ for the indicated values of x. What can you conclude about the functions?

x	-10	0	7	45
$f(f^{-1}(x))$				
$f^{-1}(f(x))$				

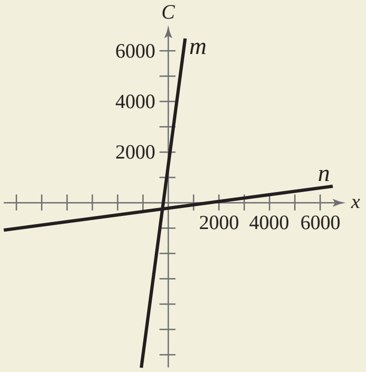
103. Think About It Restrict the domain of $f(x) = x^2 + 1$ to $x \geq 0$. Use a graphing utility to graph the function. Does the restricted function have an inverse function? Explain.

104. HOW DO YOU SEE IT? The cost C for a business to make personalized T-shirts is given by

$$C(x) = 7.50x + 1500$$

where x represents the number of T-shirts.

(a) The graphs of $C(x)$ and $C^{-1}(x)$ are shown below. Match each function with its graph.



(b) Explain what $C(x)$ and $C^{-1}(x)$ represent in the context of the problem.

One-to-One Function Representation **In Exercises 105 and 106, determine whether the situation could be represented by a one-to-one function. If so, then write a statement that best describes the inverse function.**

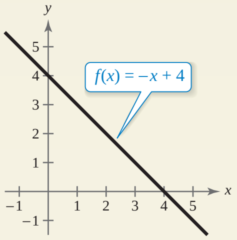
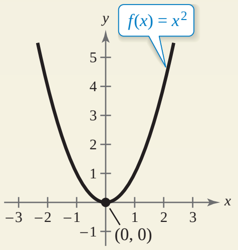
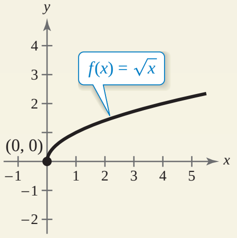
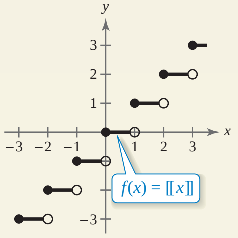
105. The number of miles n a marathon runner has completed in terms of the time t in hours

106. The depth of the tide d at a beach in terms of the time t over a 24-hour period

Baloncici/Shutterstock.com

Chapter Summary

	What Did You Learn?	Explanation/Examples	Review Exercises
Section 2.1	Use slope to graph linear equations in two variables *(p. 160)*.	**The Slope-Intercept Form of the Equation of a Line** The graph of the equation $y = mx + b$ is a line whose slope is m and whose y-intercept is $(0, b)$.	1–4
	Find the slope of a line given two points on the line *(p. 162)*.	The slope m of the nonvertical line through (x_1, y_1) and (x_2, y_2) is $m = (y_2 - y_1)/(x_2 - x_1)$, where $x_1 \neq x_2$.	5, 6
	Write linear equations in two variables *(p. 164)*.	**Point-Slope Form of the Equation of a Line** The equation of the line with slope m passing through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$.	7–10
	Use slope to identify parallel and perpendicular lines *(p. 165)*.	**Parallel lines:** Slopes are equal. **Perpendicular lines:** Slopes are negative reciprocals of each other.	11, 12
	Use slope and linear equations in two variables to model and solve real-life problems *(p. 166)*.	A linear equation in two variables can be used to describe the book value of exercise equipment in a given year. (See Example 7.)	13, 14
Section 2.2	Determine whether relations between two variables are functions, and use function notation *(p. 173)*.	A function f from a set A (domain) to a set B (range) is a relation that assigns to each element x in the set A exactly one element y in the set B. **Equation:** $f(x) = 5 - x^2$ **$f(2)$:** $f(\mathbf{2}) = 5 - \mathbf{2}^2 = 1$	15–20
	Find the domains of functions *(p. 178)*.	**Domain of $f(x) = 5 - x^2$:** All real numbers	21, 22
	Use functions to model and solve real-life problems *(p. 179)*.	A function can be used to model the number of alternative-fueled vehicles in the United States. (See Example 10.)	23, 24
	Evaluate difference quotients *(p. 180)*.	**Difference quotient:** $[f(x + h) - f(x)]/h, \quad h \neq 0$	25, 26
Section 2.3	Use the Vertical Line Test for functions *(p. 188)*.	A set of points in a coordinate plane is the graph of y as a function of x if and only if no *vertical* line intersects the graph at more than one point.	27, 28
	Find the zeros of functions *(p. 189)*.	**Zeros of $f(x)$:** x-values for which $f(x) = 0$	29–32
	Determine intervals on which functions are increasing or decreasing *(p. 190)*, determine relative minimum and relative maximum values of functions *(p. 191)*, and determine the average rates of change of functions *(p. 192)*.	To determine whether a function is increasing, decreasing, or constant on an interval, evaluate the function for several values of x. The points at which the behavior of a function changes can help determine a relative minimum or relative maximum. The average rate of change between any two points is the slope of the line (secant line) through the two points.	33–38
	Identify even and odd functions *(p. 193)*.	**Even:** For each x in the domain of f, $f(-x) = f(x)$. **Odd:** For each x in the domain of f, $f(-x) = -f(x)$.	39–42

	What Did You Learn?	Explanation/Examples	Review Exercises
Section 2.4	Identify and graph linear *(p. 198)*, squaring *(p. 199)*, cubic, square root, reciprocal *(p. 200)*, step, and other piecewise-defined functions *(p. 201)*, and recognize graphs of parent functions *(p. 202)*.	**Linear:** $f(x) = ax + b$  **Squaring:** $f(x) = x^2$  **Square Root:** $f(x) = \sqrt{x}$  **Step:** $f(x) = [\![x]\!]$  Eight of the most commonly used functions in algebra are shown on page 202.	43–50
Section 2.5	Use vertical and horizontal shifts *(p. 205)*, reflections *(p. 207)*, and nonrigid transformations *(p. 209)* to sketch graphs of functions.	**Vertical shifts:** $h(x) = f(x) + c$ or $h(x) = f(x) - c$ **Horizontal shifts:** $h(x) = f(x - c)$ or $h(x) = f(x + c)$ **Reflection in x-axis:** $h(x) = -f(x)$ **Reflection in y-axis:** $h(x) = f(-x)$ **Nonrigid transformations:** $h(x) = cf(x)$ or $h(x) = f(cx)$	51–60
Section 2.6	Add, subtract, multiply, and divide functions *(p. 214)*.	$(f + g)(x) = f(x) + g(x)$ $\quad (f - g)(x) = f(x) - g(x)$ $(fg)(x) = f(x) \cdot g(x)$ $\quad (f/g)(x) = f(x)/g(x), \ g(x) \neq 0$	61, 62
	Find the composition of one function with another function *(p. 216)*.	The composition of the function f with the function g is $(f \circ g)(x) = f(g(x))$.	63, 64
	Use combinations and compositions of functions to model and solve real-life problems *(p. 218)*.	A composite function can be used to represent the number of bacteria in food as a function of the amount of time the food has been out of refrigeration. (See Example 8.)	65, 66
Section 2.7	Find inverse functions informally and verify that two functions are inverse functions of each other *(p. 222)*.	Let f and g be two functions such that $f(g(x)) = x$ for every x in the domain of g and $g(f(x)) = x$ for every x in the domain of f. Under these conditions, the function g is the inverse function of the function f.	67, 68
	Use graphs of functions to determine whether functions have inverse functions *(p. 224)*, use the Horizontal Line Test to determine whether functions are one-to-one *(p. 225)*, and find inverse functions algebraically *(p. 226)*.	If the point (a, b) lies on the graph of f, then the point (b, a) must lie on the graph of f^{-1}, and vice versa. In short, the graph of f^{-1} is a reflection of the graph of f in the line $y = x$. **Horizontal Line Test for Inverse Functions** A function f has an inverse function if and only if no *horizontal* line intersects the graph of f at more than one point. To find an inverse function, replace $f(x)$ with y, interchange the roles of x and y, solve for y, and then replace y with $f^{-1}(x)$.	69–74

Review Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

2.1 Graphing a Linear Equation **In Exercises 1–4, find the slope and *y*-intercept (if possible) of the equation of the line. Sketch the line.**

1. $y = -2x - 7$ **2.** $10x + 2y = 9$

3. $y = 6$ **4.** $x = -3$

Finding the Slope of a Line Through Two Points **In Exercises 5 and 6, plot the points and find the slope of the line passing through the pair of points.**

5. $(6, 4), (-3, -4)$ **6.** $(-3, 2), (8, 2)$

Finding an Equation of a Line **In Exercises 7 and 8, find an equation of the line that passes through the given point and has the indicated slope *m*. Sketch the line.**

7. $(10, -3), \quad m = -\frac{1}{2}$ **8.** $(-8, 5), \quad m = 0$

Finding an Equation of a Line **In Exercises 9 and 10, find an equation of the line passing through the points.**

9. $(-1, 0), (6, 2)$ **10.** $(11, -2), \ (6, -1)$

Finding Parallel and Perpendicular Lines **In Exercises 11 and 12, write the slope-intercept form of the equations of the lines through the given point (a) parallel to and (b) perpendicular to the given line.**

11. $5x - 4y = 8, \ (3, -2)$ **12.** $2x + 3y = 5, \ (-8, 3)$

13. Sales A discount outlet is offering a 20% discount on all items. Write a linear equation giving the sale price S for an item with a list price L.

14. Hourly Wage A microchip manufacturer pays its assembly line workers \$12.25 per hour. In addition, workers receive a piecework rate of \$0.75 per unit produced. Write a linear equation for the hourly wage W in terms of the number of units x produced per hour.

2.2 Testing for Functions Represented Algebraically **In Exercises 15–18, determine whether the equation represents *y* as a function of *x*.**

15. $16x - y^4 = 0$ **16.** $2x - y - 3 = 0$

17. $y = \sqrt{1 - x}$ **18.** $|y| = x + 2$

Evaluating a Function **In Exercises 19 and 20, evaluate the function at each specified value of the independent variable and simplify.**

19. $g(x) = x^{4/3}$

(a) $g(8)$ (b) $g(t + 1)$ (c) $g(-27)$ (d) $g(-x)$

20. $h(x) = \begin{cases} 2x + 1, & x \le -1 \\ x^2 + 2, & x > -1 \end{cases}$

(a) $h(-2)$ (b) $h(-1)$ (c) $h(0)$ (d) $h(2)$

Finding the Domain of a Function **In Exercises 21 and 22, find the domain of the function. Verify your result with a graph.**

21. $f(x) = \sqrt{25 - x^2}$ **22.** $h(x) = \dfrac{x}{x^2 - x - 6}$

Physics **In Exercises 23 and 24, the velocity of a ball projected upward from ground level is given by $v(t) = -32t + 48$, where *t* is the time in seconds and *v* is the velocity in feet per second.**

23. Find the velocity when $t = 1$.

24. Find the time when the ball reaches its maximum height. [*Hint:* Find the time when $v(t) = 0$.]

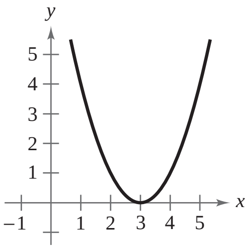
Evaluating a Difference Quotient **In Exercises 25 and 26, find the difference quotient and simplify your answer.**

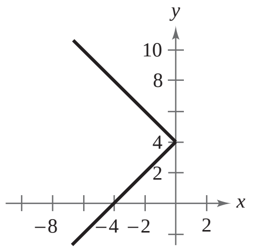
25. $f(x) = 2x^2 + 3x - 1, \quad \dfrac{f(x + h) - f(x)}{h}, \quad h \neq 0$

26. $f(x) = x^3 - 5x^2 + x, \quad \dfrac{f(x + h) - f(x)}{h}, \quad h \neq 0$

2.3 Vertical Line Test for Functions **In Exercises 27 and 28, use the Vertical Line Test to determine whether *y* is a function of *x*. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

27. $y = (x - 3)^2$ **28.** $x = -|4 - y|$





Finding the Zeros of a Function **In Exercises 29–32, find the zeros of the function algebraically.**

29. $f(x) = 5x^2 + 4x - 1$ **30.** $f(x) = \dfrac{8x + 3}{11 - x}$

31. $f(x) = \sqrt{2x + 1}$ **32.** $f(x) = x^3 - x^2$

Describing Function Behavior **In Exercises 33 and 34, use a graphing utility to graph the function and visually determine the intervals on which the function is increasing, decreasing, or constant.**

33. $f(x) = |x| + |x + 1|$

34. $f(x) = (x^2 - 4)^2$

Approximating Relative Minima or Maxima **In Exercises 35 and 36, use a graphing utility to graph the function and approximate (to two decimal places) any relative minima or maxima.**

35. $f(x) = -x^2 + 2x + 1$

36. $f(x) = x^3 - 4x^2 - 1$

Average Rate of Change of a Function **In Exercises 37 and 38, find the average rate of change of the function from x_1 to x_2.**

37. $f(x) = -x^2 + 8x - 4, \quad x_1 = 0, x_2 = 4$

38. $f(x) = 2 - \sqrt{x + 1}, \quad x_1 = 3, x_2 = 7$

Even, Odd, or Neither? **In Exercises 39–42, determine whether the function is even, odd, or neither. Then describe the symmetry.**

39. $f(x) = x^5 + 4x - 7$

40. $f(x) = x^4 - 20x^2$

41. $f(x) = 2x\sqrt{x^2 + 3}$

42. $f(x) = \sqrt[5]{6x^2}$

2.4 Writing a Linear Function **In Exercises 43 and 44, (a) write the linear function f such that it has the indicated function values, and (b) sketch the graph of the function.**

43. $f(2) = -6, \quad f(-1) = 3$

44. $f(0) = -5, \quad f(4) = -8$

Graphing a Function **In Exercises 45–50, sketch the graph of the function.**

45. $f(x) = x^2 + 5$

46. $g(x) = -3x^3$

47. $f(x) = \sqrt{x + 1}$

48. $g(x) = \dfrac{1}{x + 5}$

49. $g(x) = [\![x + 4]\!]$

50. $f(x) = \begin{cases} 5x - 3, & x \geq -1 \\ -4x + 5, & x < -1 \end{cases}$

2.5 Identifying a Parent Function **In Exercises 51–60, h is related to one of the parent functions described in this chapter. (a) Identify the parent function f. (b) Describe the sequence of transformations from f to h. (c) Sketch the graph of h. (d) Use function notation to write h in terms of f.**

51. $h(x) = x^2 - 9$

52. $h(x) = (x - 2)^3 + 2$

53. $h(x) = -\sqrt{x} + 4$

54. $h(x) = |x + 3| - 5$

55. $h(x) = -(x + 2)^2 + 3$

56. $h(x) = \frac{1}{2}(x - 1)^2 - 2$

57. $h(x) = -[\![x]\!] + 6$

58. $h(x) = -\sqrt{x + 1} + 9$

59. $h(x) = 5[\![x - 9]\!]$

60. $h(x) = -\frac{1}{3}x^3$

2.6 Finding Arithmetic Combinations of Functions **In Exercises 61 and 62, find (a) $(f + g)(x)$, (b) $(f - g)(x)$, (c) $(fg)(x)$ and (d) $(f/g)(x)$. What is the domain of f/g?**

61. $f(x) = x^2 + 3, \quad g(x) = 2x - 1$

62. $f(x) = x^2 - 4, \quad g(x) = \sqrt{3 - x}$

Finding Domains of Functions and Composite Functions **In Exercises 63 and 64, find (a) $f \circ g$ and (b) $g \circ f$. Find the domain of each function and each composite function.**

63. $f(x) = \frac{1}{3}x - 3, \quad g(x) = 3x + 1$

64. $f(x) = \sqrt{x + 1}, \quad g(x) = x^2$

Bacteria Count **In Exercises 65 and 66, the number N of bacteria in a refrigerated food is given by $N(T) = 25T^2 - 50T + 300,\ 1 \leq T \leq 19$, where T is the temperature of the food in degrees Celsius. When the food is removed from refrigeration, the temperature of the food is given by $T(t) = 2t + 1,\ 0 \leq t \leq 9$, where t is the time in hours.**

65. Find the composition $(N \circ T)(t)$ and interpret its meaning in context.

66. Find the time when the bacteria count reaches 750.

2.7 Finding an Inverse Function Informally **In Exercises 67 and 68, find the inverse function of f informally. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.**

67. $f(x) = \dfrac{x - 4}{5}$

68. $f(x) = x^3 - 1$

Applying the Horizontal Line Test **In Exercises 69 and 70, use a graphing utility to graph the function, and use the Horizontal Line Test to determine whether the function has an inverse function.**

69. $f(x) = (x - 1)^2$

70. $h(t) = \dfrac{2}{t - 3}$

Finding and Analyzing Inverse Functions **In Exercises 71 and 72, (a) find the inverse function of f, (b) graph both f and f^{-1} on the same set of coordinate axes, (c) describe the relationship between the graphs of f and f^{-1}, and (d) state the domains and ranges of f and f^{-1}.**

71. $f(x) = \frac{1}{2}x - 3$

72. $f(x) = \sqrt{x + 1}$

Restricting the Domain **In Exercises 73 and 74, restrict the domain of the function f to an interval on which the function is increasing, and determine f^{-1} on that interval.**

73. $f(x) = 2(x - 4)^2$

74. $f(x) = |x - 2|$

Exploration

True or False? **In Exercises 75 and 76, determine whether the statement is true or false. Justify your answer.**

75. Relative to the graph of $f(x) = \sqrt{x}$, the function $h(x) = -\sqrt{x + 9} - 13$ is shifted 9 units to the left and 13 units down, then reflected in the x-axis.

76. If f and g are two inverse functions, then the domain of g is equal to the range of f.

Chapter Test

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

In Exercises 1 and 2, find an equation of the line passing through the points. Sketch the line.

1. $(4, -5), (-2, 7)$ **2.** $(3, 0.8), (7, -6)$

3. Write equations of the lines that pass through the point $(0, 4)$ and are (a) parallel to and (b) perpendicular to the line $5x + 2y = 3$.

In Exercises 4 and 5, evaluate the function at each specified value of the independent variable and simplify.

4. $f(x) = |x + 2| - 15$

(a) $f(-8)$ (b) $f(14)$ (c) $f(x - 6)$

5. $f(x) = \dfrac{\sqrt{x + 9}}{x^2 - 81}$

(a) $f(7)$ (b) $f(-5)$ (c) $f(x - 9)$

In Exercises 6 and 7, find the domain of the function.

6. $f(x) = |-x + 6| + 2$ **7.** $f(x) = 10 - \sqrt{3 - x}$

In Exercises 8–10, (a) use a graphing utility to graph the function, (b) approximate the intervals on which the function is increasing, decreasing, or constant, and (c) determine whether the function is even, odd, or neither.

8. $f(x) = 2x^6 + 5x^4 - x^2$ **9.** $f(x) = 4x\sqrt{3 - x}$ **10.** $f(x) = |x + 5|$

11. Use a graphing utility to approximate any relative minima or maxima of $f(x) = -x^3 + 2x - 1$.

12. Find the average rate of change of $f(x) = -2x^2 + 5x - 3$ from $x_1 = 1$ to $x_2 = 3$.

13. Sketch the graph of $f(x) = \begin{cases} 3x + 7, & x \le -3 \\ 4x^2 - 1, & x > -3 \end{cases}.$

In Exercises 14–16, (a) identify the parent function in the transformation, (b) describe the sequence of transformations from *f* to *h*, and (c) sketch the graph of *h*.

14. $h(x) = 3[\![x]\!]$

15. $h(x) = -\sqrt{x + 5} + 8$

16. $h(x) = -2(x - 5)^3 + 3$

In Exercises 17 and 18, find (a) $(f + g)(x)$, (b) $(f - g)(x)$, (c) $(fg)(x)$, (d) $(f/g)(x)$, (e) $(f \circ g)(x)$, and (f) $(g \circ f)(x)$.

17. $f(x) = 3x^2 - 7, \quad g(x) = -x^2 - 4x + 5$ **18.** $f(x) = \dfrac{1}{x}, \quad g(x) = 2\sqrt{x}$

In Exercises 19–21, determine whether the function has an inverse function. If so, find the inverse function.

19. $f(x) = x^3 + 8$ **20.** $f(x) = |x^2 - 3| + 6$ **21.** $f(x) = 3x\sqrt{x}$

22. It costs a company \$58 to produce 6 units of a product and \$78 to produce 10 units. Assuming that the cost function is linear, how much does it cost to produce 25 units?

Cumulative Test for Chapters P–2

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

In Exercises 1 and 2, simplify the expression.

1. $\dfrac{8x^2y^{-3}}{30x^{-1}y^2}$

2. $\sqrt{18x^3y^4}$

In Exercises 3–5, perform the operation(s) and simplify the result.

3. $4x - [2x + 3(2 - x)]$

4. $(x - 2)(x^2 + x - 3)$

5. $\dfrac{2}{s + 3} - \dfrac{1}{s + 1}$

In Exercises 6–8, completely factor the expression.

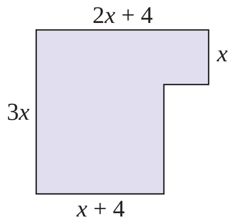
6. $25 - (x - 2)^2$

7. $x - 5x^2 - 6x^3$

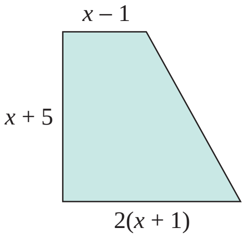
8. $54x^3 + 16$

In Exercises 9 and 10, write an expression for the area of the region as a polynomial in standard form.

9.



10.



In Exercises 11–13, sketch the graph of the equation.

11. $x - 3y + 12 = 0$

12. $y = x^2 - 9$

13. $y = \sqrt{4 - x}$

In Exercises 14–16, solve the equation and check your solution.

14. $3x - 5 = 6x + 8$

15. $-(x + 3) = 14(x - 6)$

16. $\dfrac{1}{x - 2} = \dfrac{10}{4x + 3}$

In Exercises 17–22, solve the equation using any convenient method and check your solutions. State the method you used.

17. $x^2 - 4x + 3 = 0$

18. $-2x^2 + 8x + 12 = 0$

19. $\frac{2}{3}x^2 = 24$

20. $3x^2 + 5x - 6 = 0$

21. $3x^2 + 9x + 1 = 0$

22. $\frac{1}{2}x^2 - 7 = 25$

In Exercises 23–28, solve the equation (if possible).

23. $x^4 + 12x^3 + 4x^2 + 48x = 0$

24. $8x^3 - 48x^2 + 72x = 0$

25. $x^{2/3} + 13 = 17$

26. $\sqrt{x + 10} = x - 2$

27. $|3(x - 4)| = 27$

28. $|x - 12| = -2$

In Exercises 29–32, solve the inequality. Then graph the solution set.

29. $|x + 1| \leq 6$

30. $|5 + 6x| > 3$

31. $5x^2 + 12x + 7 \geq 0$

32. $-x^2 + x + 4 < 0$

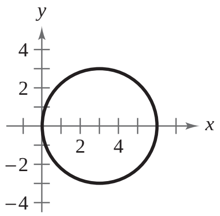


Figure for 34

33. Find the slope-intercept form of the equation of the line passing through $\left(-\frac{1}{2}, 1\right)$ and $(3, 8)$.

34. Explain why the graph at the left does not represent y as a function of x.

35. Evaluate (if possible) the function $f(x) = \dfrac{x}{x - 2}$ at each specified value of the independent variable.

(a) $f(6)$ (b) $f(2)$ (c) $f(s + 2)$

In Exercises 36–38, determine whether the function is even, odd, or neither.

36. $f(x) = 5 + \sqrt{4 - x}$

37. $f(x) = x^5 - x^3 + 2$

38. $f(x) = 2x^4 - 4$

39. Compare the graph of each function with the graph of $y = \sqrt[3]{x}$. (*Note:* It is not necessary to sketch the graphs.)

(a) $r(x) = \frac{1}{2}\sqrt[3]{x}$

(b) $h(x) = \sqrt[3]{x} + 2$

(c) $g(x) = \sqrt[3]{x + 2}$

In Exercises 40 and 41, find (a) $(f + g)(x)$, (b) $(f - g)(x)$, (c) $(fg)(x)$, and (d) $(f/g)(x)$. What is the domain of f/g?

40. $f(x) = x - 4, \quad g(x) = 3x + 1$

41. $f(x) = \sqrt{x - 1}, \quad g(x) = x^2 + 1$

In Exercises 42 and 43, find (a) $f \circ g$ and (b) $g \circ f$. Find the domain of each composite function.

42. $f(x) = 2x^2, \quad g(x) = \sqrt{x + 6}$

43. $f(x) = x - 2, \quad g(x) = |x|$

44. Determine whether $h(x) = 3x - 4$ has an inverse function. If so, find the inverse function.

45. A group of n people decide to buy a $36,000 minibus. Each person will pay an equal share of the cost. When three additional people join the group, the cost per person will decrease by $1000. Find n.

46. For groups of 60 or more, a charter bus company determines the rate per person according to the formula

$$\text{Rate} = \$10.00 - \$0.05(n - 60), \quad n \geq 60.$$

(a) Write the revenue R as a function of n.

(b) Use a graphing utility to graph the revenue function. Move the cursor along the function to estimate the number of passengers that will maximize the revenue.

47. The height of an object thrown vertically upward from a height of 8 feet at a velocity of 36 feet per second can be modeled by $s(t) = -16t^2 + 36t + 8$, where s is the height (in feet) and t is the time (in seconds). Find the average rate of change of the function from $t_1 = 0$ to $t_2 = 2$. Interpret your answer in the context of the problem.

Proofs in Mathematics

Biconditional Statements

Recall from the Proofs in Mathematics in Chapter 1 that a conditional statement is a statement of the form "if *p*, then *q*." A statement of the form "*p* if and only if *q*" is called a **biconditional statement.** A biconditional statement, denoted by

$p \leftrightarrow q$ Biconditional statement

is the conjunction of the conditional statement $p \rightarrow q$ and its converse $q \rightarrow p$.

A biconditional statement can be either true or false. To be true, *both* the conditional statement and its converse must be true.

EXAMPLE 1 Analyzing a Biconditional Statement

Consider the statement $x = 3$ if and only if $x^2 = 9$.

a. Is the statement a biconditional statement?

b. Is the statement true?

Solution

a. The statement is a biconditional statement because it is of the form "*p* if and only if *q*."

b. Rewrite the statement as the following conditional statement and its converse.

Conditional statement: If $x = 3$, then $x^2 = 9$.

Converse: If $x^2 = 9$, then $x = 3$.

The first of these statements is true, but the second is false because x could also equal -3. So, the biconditional statement is false. ■

Knowing how to use biconditional statements is an important tool for reasoning in mathematics.

EXAMPLE 2 Analyzing a Biconditional Statement

Determine whether the biconditional statement is true or false. If it is false, then provide a counterexample.

A number is divisible by 5 if and only if it ends in 0.

Solution Rewrite the biconditional statement as the following conditional statement and its converse.

Conditional statement: If a number is divisible by 5, then it ends in 0.

Converse: If a number ends in 0, then it is divisible by 5.

The conditional statement is false. A counterexample is the number 15, which is divisible by 5 but does not end in 0. ■

P.S. Problem Solving

1. **Monthly Wages** As a salesperson, you receive a monthly salary of \$2000, plus a commission of 7% of sales. You receive an offer for a new job at \$2300 per month, plus a commission of 5% of sales.

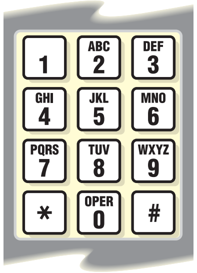
   (a) Write a linear equation for your current monthly wage W_1 in terms of your monthly sales S.

   (b) Write a linear equation for the monthly wage W_2 of your new job offer in terms of the monthly sales S.

   (c) Use a graphing utility to graph both equations in the same viewing window. Find the point of intersection. What does it signify?

   (d) You think you can sell \$20,000 per month. Should you change jobs? Explain.

2. **Telephone Keypad** For the numbers 2 through 9 on a telephone keypad (see figure), create two relations: one mapping numbers onto letters, and the other mapping letters onto numbers. Are both relations functions? Explain.



3. **Sums and Differences of Functions** What can be said about the sum and difference of each of the following?

   (a) Two even functions

   (b) Two odd functions

   (c) An odd function and an even function

4. **Inverse Functions** The two functions

   $f(x) = x \quad \text{and} \quad g(x) = -x$

   are their own inverse functions. Graph each function and explain why this is true. Graph other linear functions that are their own inverse functions. Find a general formula for a family of linear functions that are their own inverse functions.

5. **Proof** Prove that a function of the following form is even.

   $$y = a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + \cdot \cdot \cdot + a_2x^2 + a_0$$

6. **Miniature Golf** A miniature golf professional is trying to make a hole-in-one on the miniature golf green shown. The golf ball is at the point (2.5, 2) and the hole is at the point (9.5, 2). The professional wants to bank the ball off the side wall of the green at the point (x, y). Find the coordinates of the point (x, y). Then write an equation for the path of the ball.

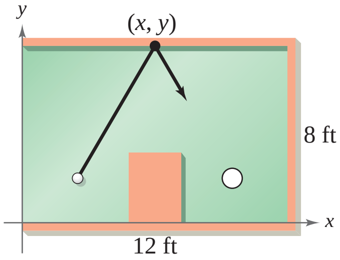


Figure for 6

7. **Titanic** At 2:00 P.M. on April 11, 1912, the *Titanic* left Cobh, Ireland, on her voyage to New York City. At 11:40 P.M. on April 14, the *Titanic* struck an iceberg and sank, having covered only about 2100 miles of the approximately 3400-mile trip.

   (a) What was the total duration of the voyage in hours?

   (b) What was the average speed in miles per hour?

   (c) Write a function relating the distance of the *Titanic* from New York City and the number of hours traveled. Find the domain and range of the function.

   (d) Graph the function from part (c).

8. **Average Rate of Change** Consider the function $f(x) = -x^2 + 4x - 3$. Find the average rate of change of the function from x_1 to x_2.

   (a) $x_1 = 1, x_2 = 2$

   (b) $x_1 = 1, x_2 = 1.5$

   (c) $x_1 = 1, x_2 = 1.25$

   (d) $x_1 = 1, x_2 = 1.125$

   (e) $x_1 = 1, x_2 = 1.0625$

   (f) Does the average rate of change seem to be approaching one value? If so, then state the value.

   (g) Find the equations of the secant lines through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ for parts (a)–(e).

   (h) Find the equation of the line through the point $(1, f(1))$ using your answer from part (f) as the slope of the line.

9. **Inverse of a Composition** Consider the functions $f(x) = 4x$ and $g(x) = x + 6$.

   (a) Find $(f \circ g)(x)$.

   (b) Find $(f \circ g)^{-1}(x)$.

   (c) Find $f^{-1}(x)$ and $g^{-1}(x)$.

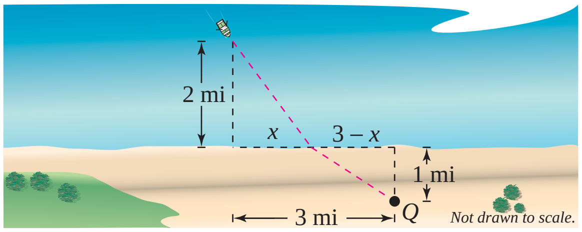
   (d) Find $(g^{-1} \circ f^{-1})(x)$ and compare the result with that of part (b).

   (e) Repeat parts (a) through (d) for $f(x) = x^3 + 1$ and $g(x) = 2x$.

   (f) Write two one-to-one functions f and g, and repeat parts (a) through (d) for these functions.

   (g) Make a conjecture about $(f \circ g)^{-1}(x)$ and $(g^{-1} \circ f^{-1})(x)$.

10. Trip Time You are in a boat 2 miles from the nearest point on the coast. You are to travel to a point Q, 3 miles down the coast and 1 mile inland (see figure). You row at 2 miles per hour and walk at 4 miles per hour.



(a) Write the total time T of the trip as a function of x.

(b) Determine the domain of the function.

(c) Use a graphing utility to graph the function. Be sure to choose an appropriate viewing window.

(d) Find the value of x that minimizes T.

(e) Write a brief paragraph interpreting these values.

11. Heaviside Function The **Heaviside function** $H(x)$ is widely used in engineering applications. (See figure.) To print an enlarged copy of the graph, go to *MathGraphs.com*.

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Sketch the graph of each function by hand.

(a) $H(x) - 2$

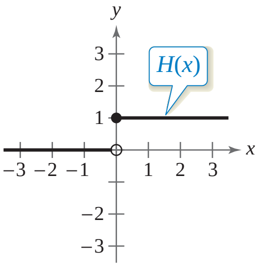
(b) $H(x - 2)$

(c) $-H(x)$

(d) $H(-x)$

(e) $\frac{1}{2}H(x)$

(f) $-H(x - 2) + 2$



12. Repeated Composition Let $f(x) = \dfrac{1}{1 - x}$.

(a) What are the domain and range of f?

(b) Find $f(f(x))$. What is the domain of this function?

(c) Find $f(f(f(x)))$. Is the graph a line? Why or why not?

13. Associative Property with Compositions Show that the Associative Property holds for compositions of functions—that is,

$$(f \circ (g \circ h))(x) = ((f \circ g) \circ h)(x).$$

14. Graphical Analysis Consider the graph of the function f shown in the figure. Use this graph to sketch the graph of each function. To print an enlarged copy of the graph, go to *MathGraphs.com*.

(a) $f(x + 1)$

(b) $f(x) + 1$

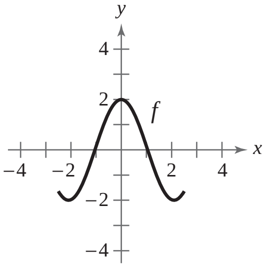
(c) $2f(x)$

(d) $f(-x)$

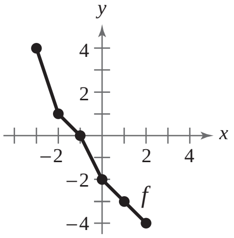
(e) $-f(x)$

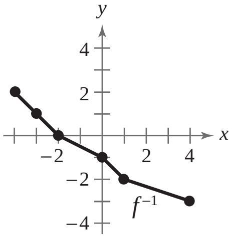
(f) $|f(x)|$

(g) $f(|x|)$



15. Graphical Analysis Use the graphs of f and f^{-1} to complete each table of function values.





(a)

x	-4	-2	0	4
$(f(f^{-1}(x)))$				

(b)

x	-3	-2	0	1
$(f + f^{-1})(x)$				

(c)

x	-3	-2	0	1
$(f \cdot f^{-1})(x)$				

(d)

x	-4	-3	0	4
$\lvert f^{-1}(x) \rvert$				

3 Polynomial Functions

Candle Making *(Example 11, page 284)*

Product Demand
(Example 5, page 293)

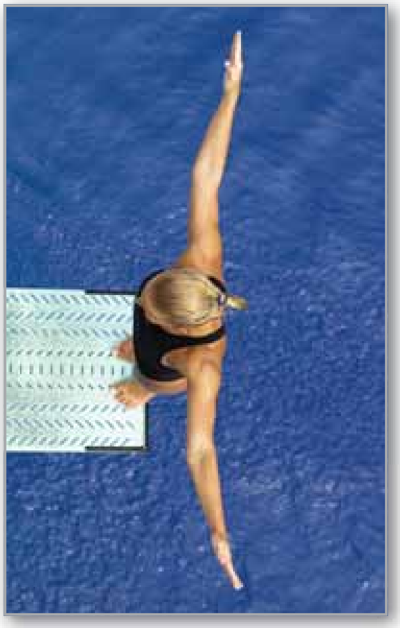
Lyme Disease *(Exercise 86, page 274)*

Tree Growth
(Exercise 104, page 264)

Path of a Diver *(Exercise 75, page 249)*

Clockwise from top left, hfng/Shutterstock.com; Marcin Balcerzak/Shutterstock.com;
S.Borisov/Shutterstock.com; wellphoto/Shutterstock.com; Dariusz Majgier/Shutterstock.com

3.1 Quadratic Functions and Models



You can use quadratic functions to model the path of an object or person. For instance, in Exercise 75 on page 249, you will use a quadratic function to model the path of a diver.

- **Analyze graphs of quadratic functions.**
- **Write quadratic functions in standard form and use the results to sketch graphs of functions.**
- **Find minimum and maximum values of quadratic functions in real-life applications.**

The Graph of a Quadratic Function

In this and the next section, you will study the graphs of polynomial functions. In Section 2.4, you were introduced to the following basic functions.

$f(x) = ax + b$ Linear function

$f(x) = c$ Constant function

$f(x) = x^2$ Squaring function

These functions are examples of **polynomial functions.**

Definition of Polynomial Function

Let n be a nonnegative integer and let $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ be real numbers with $a_n \neq 0$. The function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$$

is called a **polynomial function of x with degree n.**

Polynomial functions are classified by degree. For instance, a constant function $f(x) = c$ with $c \neq 0$ has degree 0, and a linear function $f(x) = ax + b$ with $a \neq 0$ has degree 1. In this section, you will study second-degree polynomial functions, which are called **quadratic functions.**

For instance, each of the following functions is a quadratic function.

$f(x) = x^2 + 6x + 2$

$g(x) = 2(x + 1)^2 - 3$

$h(x) = 9 + \frac{1}{4}x^2$

$k(x) = -3x^2 + 4$

$m(x) = (x - 2)(x + 1)$

Note that the squaring function is a simple quadratic function that has degree 2.

Definition of Quadratic Function

Let a, b, and c be real numbers with $a \neq 0$. The function

$f(x) = ax^2 + bx + c$ Quadratic function

is called a **quadratic function.**

The graph of a quadratic function is a special type of "U"-shaped curve called a **parabola.** Parabolas occur in many real-life applications—especially those involving reflective properties of satellite dishes and flashlight reflectors. You will study these properties in Section 4.3.

wellphoto/Shutterstock.com

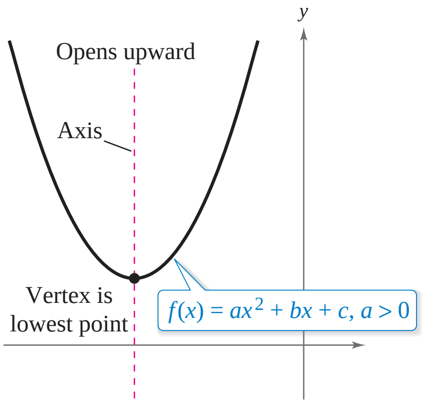
All parabolas are symmetric with respect to a line called the **axis of symmetry,** or simply the **axis** of the parabola. The point where the axis intersects the parabola is the **vertex** of the parabola, as shown below. When the leading coefficient is positive, the graph of

$$f(x) = ax^2 + bx + c$$

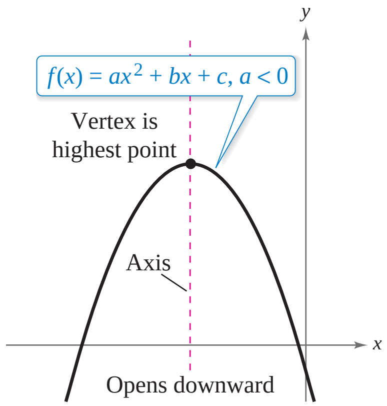
is a parabola that opens upward. When the leading coefficient is negative, the graph of

$$f(x) = ax^2 + bx + c$$

is a parabola that opens downward.



Leading coefficient is positive.

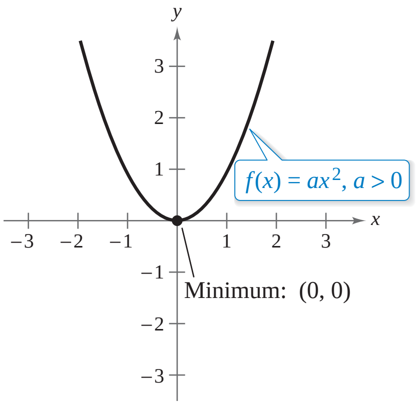


Leading coefficient is negative.

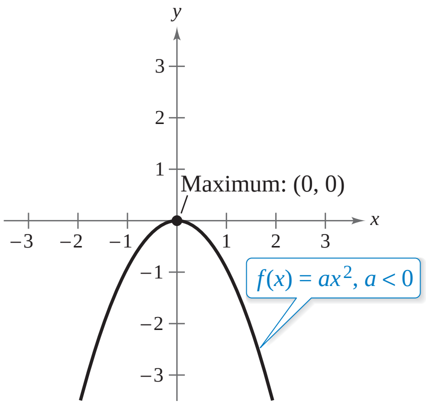
The simplest type of quadratic function is

$$f(x) = ax^2.$$

Its graph is a parabola whose vertex is $(0, 0)$. When $a > 0$, the vertex is the point with the *minimum* y-value on the graph, and when $a < 0$, the vertex is the point with the *maximum* y-value on the graph, as shown below.



Leading coefficient is positive.



Leading coefficient is negative.

When sketching the graph of

$$f(x) = ax^2$$

it is helpful to use the graph of $y = x^2$ as a reference, as discussed in Section 2.5. There you learned that when $a > 1$, the graph of $y = af(x)$ is a vertical stretch of the graph of $y = f(x)$. When $0 < a < 1$, the graph of $y = af(x)$ is a vertical shrink of the graph of $y = f(x)$. This is demonstrated again in Example 1.

EXAMPLE 1 Sketching Graphs of Quadratic Functions

Sketch the graph of each quadratic function and compare it with the graph of $y = x^2$.

a. $f(x) = \frac{1}{3}x^2$ **b.** $g(x) = 2x^2$

ALGEBRA HELP You can review the techniques for shifting, reflecting, and stretching graphs in Section 2.5.

Solution

a. Compared with $y = x^2$, each output of $f(x) = \frac{1}{3}x^2$ "shrinks" by a factor of $\frac{1}{3}$, creating the broader parabola shown in Figure 3.1.

b. Compared with $y = x^2$, each output of $g(x) = 2x^2$ "stretches" by a factor of 2, creating the narrower parabola shown in Figure 3.2.

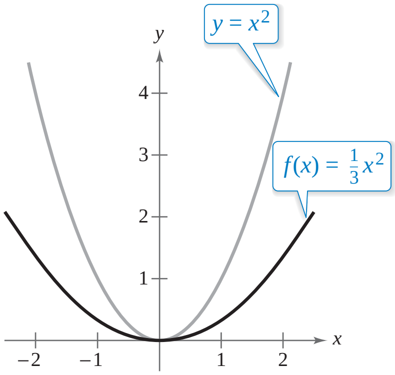


Figure 3.1

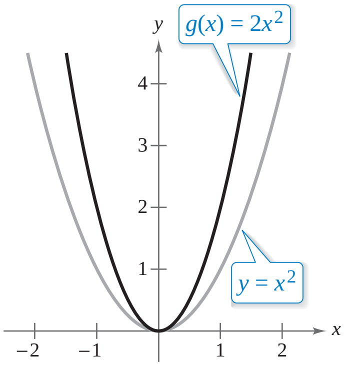


Figure 3.2

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Sketch the graph of each quadratic function and compare it with the graph of $y = x^2$.

a. $f(x) = \frac{1}{4}x^2$ **b.** $g(x) = -\frac{1}{6}x^2$

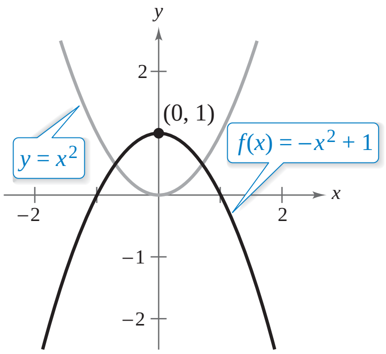
c. $h(x) = \frac{5}{2}x^2$ **d.** $k(x) = -4x^2$

In Example 1, note that the coefficient a determines how wide the parabola $f(x) = ax^2$ opens. When $|a|$ is small, the parabola opens wider than when $|a|$ is large. Recall from Section 2.5 that the graphs of

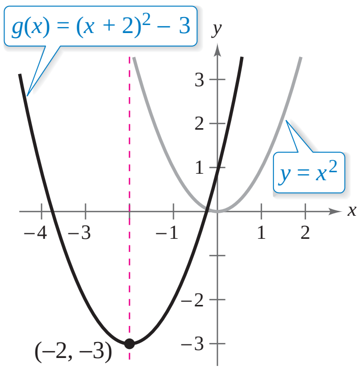
$$y = f(x \pm c), \quad y = f(x) \pm c, \quad y = f(-x), \quad \text{and} \quad y = -f(x)$$

are rigid transformations of the graph of $y = f(x)$. For instance, in the figures below, notice how the graph of $y = x^2$ can be transformed to produce the graphs of

$$f(x) = -x^2 + 1 \quad \text{and} \quad g(x) = (x + 2)^2 - 3.$$



Reflection in *x*-axis followed by an upward shift of one unit



Left shift of two units followed by a downward shift of three units

The Standard Form of a Quadratic Function

The **standard form** of a quadratic function is $f(x) = a(x - h)^2 + k$. This form is especially convenient for sketching a parabola because it identifies the vertex of the parabola as (h, k).

REMARK The standard form of a quadratic function identifies four basic transformations of the graph of $y = x^2$.

a. The factor a produces a vertical stretch or shrink.

b. When $a < 0$, the graph is reflected in the x-axis.

c. The factor $(x - h)^2$ represents a horizontal shift of h units.

d. The term k represents a vertical shift of k units.

Standard Form of a Quadratic Function

The quadratic function

$$f(x) = a(x - h)^2 + k, \quad a \neq 0$$

is in **standard form.** The graph of f is a parabola whose axis is the vertical line $x = h$ and whose vertex is the point (h, k). When $a > 0$, the parabola opens upward, and when $a < 0$, the parabola opens downward.

To graph a parabola, it is helpful to begin by writing the quadratic function in standard form using the process of completing the square, as illustrated in Example 2. In this example, notice that when completing the square, you *add and subtract* the square of half the coefficient of x within the parentheses instead of adding the value to each side of the equation as is done in Section 1.4.

EXAMPLE 2 Using Standard Form to Graph a Parabola

Sketch the graph of

$$f(x) = 2x^2 + 8x + 7.$$

Identify the vertex and the axis of the parabola.

Solution Begin by writing the quadratic function in standard form. Notice that the first step in completing the square is to factor out any coefficient of x^2 that is not 1.

ALGEBRA HELP You can review the techniques for completing the square in Section 1.4.

$$f(x) = 2x^2 + 8x + 7 \qquad \text{Write original function.}$$

$$= 2(x^2 + 4x) + 7 \qquad \text{Factor 2 out of } x\text{-terms.}$$

$$= 2(x^2 + 4x \underbrace{+\ 4}_{(4/2)^2} - 4) + 7 \qquad \text{Add and subtract 4 within parentheses.}$$

After adding and subtracting 4 within the parentheses, you must now regroup the terms to form a perfect square trinomial. The -4 can be removed from inside the parentheses; however, because of the 2 outside of the parentheses, you must multiply -4 by 2, as shown below.

$$f(x) = 2(x^2 + 4x + 4) - 2(4) + 7 \qquad \text{Regroup terms.}$$

$$= 2(x^2 + 4x + 4) - 8 + 7 \qquad \text{Simplify.}$$

$$= 2(x + 2)^2 - 1 \qquad \text{Write in standard form.}$$

From this form, you can see that the graph of f is a parabola that opens upward and has its vertex at $(-2, -1)$. This corresponds to a left shift of two units and a downward shift of one unit relative to the graph of $y = 2x^2$, as shown in Figure 3.3. In the figure, you can see that the axis of the parabola is the vertical line through the vertex, $x = -2$.

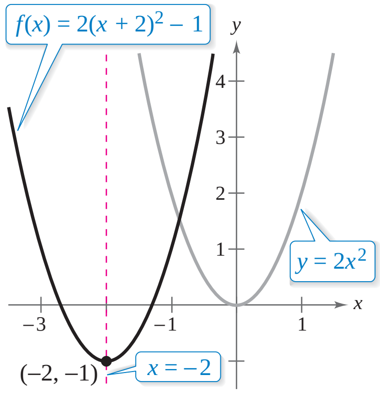


Figure 3.3

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Sketch the graph of

$$f(x) = 3x^2 - 6x + 4.$$

Identify the vertex and the axis of the parabola. ■

ALGEBRA HELP You can review the techniques for using the Quadratic Formula in Section 1.4.

To find the x-intercepts of the graph of $f(x) = ax^2 + bx + c$, you must solve the equation $ax^2 + bx + c = 0$. When $ax^2 + bx + c$ does not factor, you can use the Quadratic Formula to find the x-intercepts. Remember, however, that a parabola may not have x-intercepts.

EXAMPLE 3 Finding the Vertex and x-Intercepts of a Parabola

Sketch the graph of $f(x) = -x^2 + 6x - 8$. Identify the vertex and x-intercepts.

Solution

$$\begin{aligned} f(x) &= -x^2 + 6x - 8 && \text{Write original function.} \\ &= -(x^2 - 6x) - 8 && \text{Factor } -1 \text{ out of } x\text{-terms.} \\ &= -(x^2 - 6x + \underbrace{9}_{(-6/2)^2} - 9) - 8 && \text{Add and subtract 9 within parentheses.} \\ &= -(x^2 - 6x + 9) - (-9) - 8 && \text{Regroup terms.} \\ &= -(x - 3)^2 + 1 && \text{Write in standard form.} \end{aligned}$$

From this form, you can see that f is a parabola that opens downward with vertex $(3, 1)$. The x-intercepts of the graph are determined as follows.

$$\begin{aligned} -(x^2 - 6x + 8) &= 0 && \text{Factor out } -1. \\ -(x - 2)(x - 4) &= 0 && \text{Factor.} \\ x - 2 &= 0 \quad \Rightarrow \quad x = 2 && \text{Set 1st factor equal to 0.} \\ x - 4 &= 0 \quad \Rightarrow \quad x = 4 && \text{Set 2nd factor equal to 0.} \end{aligned}$$

So, the x-intercepts are $(2, 0)$ and $(4, 0)$, as shown in Figure 3.4.

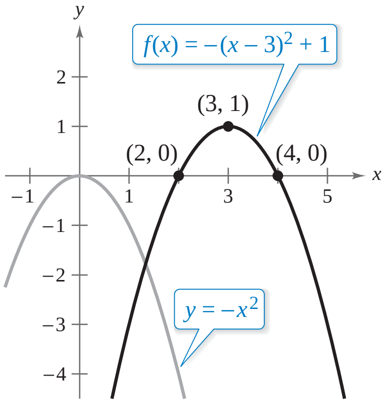


Figure 3.4

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Sketch the graph of $f(x) = x^2 - 4x + 3$. Identify the vertex and x-intercepts.

EXAMPLE 4 Writing the Equation of a Parabola

Write the standard form of the equation of the parabola whose vertex is $(1, 2)$ and that passes through the point $(3, -6)$.

Solution Because the vertex is $(h, k) = (1, 2)$, the equation has the form

$$f(x) = a(x - 1)^2 + 2. \qquad \text{Substitute for } h \text{ and } k \text{ in standard form.}$$

Because the parabola passes through the point $(3, -6)$, it follows that $f(3) = -6$. So,

$$\begin{aligned} f(x) &= a(x - 1)^2 + 2 && \text{Write in standard form.} \\ -6 &= a(3 - 1)^2 + 2 && \text{Substitute 3 for } x \text{ and } -6 \text{ for } f(x). \\ -6 &= 4a + 2 && \text{Simplify.} \\ -8 &= 4a && \text{Subtract 2 from each side.} \\ -2 &= a. && \text{Divide each side by 4.} \end{aligned}$$

The equation in standard form is $f(x) = -2(x - 1)^2 + 2$. The graph of f is shown in Figure 3.5.

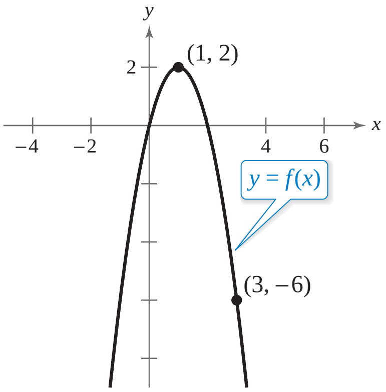


Figure 3.5

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write the standard form of the equation of the parabola whose vertex is $(-4, 11)$ and that passes through the point $(-6, 15)$.

Finding Minimum and Maximum Values

Many applications involve finding the maximum or minimum value of a quadratic function. By completing the square of the quadratic function $f(x) = ax^2 + bx + c$, you can rewrite the function in standard form (see Exercise 93).

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) \qquad \text{Standard form}$$

So, the vertex of the graph of f is $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$, which implies the following.

Minimum and Maximum Values of Quadratic Functions

Consider the function $f(x) = ax^2 + bx + c$ with vertex $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

1. When $a > 0$, f has a *minimum* at $x = -\frac{b}{2a}$. The minimum value is $f\left(-\frac{b}{2a}\right)$.
2. When $a < 0$, f has a *maximum* at $x = -\frac{b}{2a}$. The maximum value is $f\left(-\frac{b}{2a}\right)$.

EXAMPLE 5 The Maximum Height of a Baseball

The path of a baseball after being hit is given by the function $f(x) = -0.0032x^2 + x + 3$, where $f(x)$ is the height of the baseball (in feet) and x is the horizontal distance from home plate (in feet). What is the maximum height of the baseball?

Algebraic Solution

For this quadratic function, you have

$$f(x) = ax^2 + bx + c = -0.0032x^2 + x + 3$$

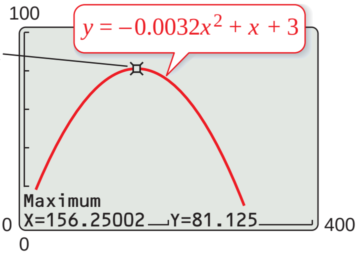
which implies that $a = -0.0032$ and $b = 1$. Because $a < 0$, the function has a maximum at $x = -b/(2a)$. So, the baseball reaches its maximum height when it is

$$x = -\frac{b}{2a} = -\frac{1}{2(-0.0032)} = 156.25 \text{ feet from home plate.}$$

At this distance, the maximum height is

$$f(156.25) = -0.0032(156.25)^2 + 156.25 + 3 = 81.125 \text{ feet.}$$

Graphical Solution



The maximum height is $y = 81.125$ feet at $x = 156.25$ feet.

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Rework Example 5 when the path of the baseball is given by the function

$$f(x) = -0.007x^2 + x + 4.$$

Summarize (Section 3.1)

1. State the definition of a quadratic function and describe its graph *(pages 242–244)*. For an example of sketching the graphs of quadratic functions, see Example 1.
2. State the standard form of a quadratic function *(page 245)*. For examples that use the standard form of a quadratic function, see Examples 2–4.
3. Describe how to find the minimum or maximum value of a quadratic function *(page 247)*. For a real-life example that involves finding the maximum height of a baseball, see Example 5.

3.1 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

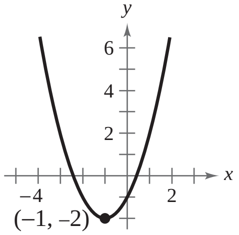
Vocabulary: Fill in the blanks.

1. Linear, constant, and squaring functions are examples of ________ functions.
2. A polynomial function of x with degree n has the form $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_1x + a_0$ $(a_n \neq 0)$, where n is a ________ ________ and $a_n, a_{n-1}, . . . , a_1, a_0$ are ________ numbers.
3. A ________ function is a second-degree polynomial function, and its graph is called a ________.
4. The graph of a quadratic function is symmetric about its ________.
5. When the graph of a quadratic function opens upward, its leading coefficient is ________ and the vertex of the graph is a ________.
6. When the graph of a quadratic function opens downward, its leading coefficient is ________ and the vertex of the graph is a ________.

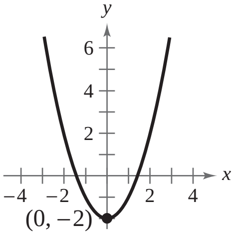
Skills and Applications

Matching **In Exercises 7–12, match the quadratic function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]**

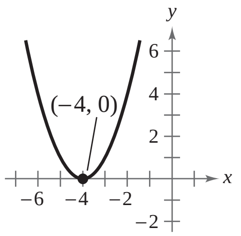
(a)



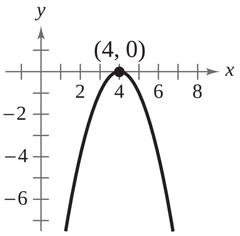
(b)



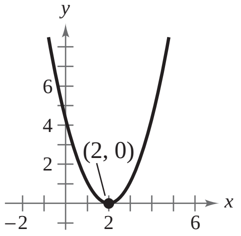
(c)



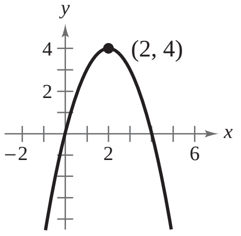
(d)



(e)



(f)



7. $f(x) = (x - 2)^2$
8. $f(x) = (x + 4)^2$
9. $f(x) = x^2 - 2$
10. $f(x) = (x + 1)^2 - 2$
11. $f(x) = 4 - (x - 2)^2$
12. $f(x) = -(x - 4)^2$

Sketching Graphs of Quadratic Functions **In Exercises 13–16, sketch the graph of each quadratic function and compare it with the graph of $y = x^2$.**

13. (a) $f(x) = \frac{1}{2}x^2$ (b) $g(x) = -\frac{1}{8}x^2$
(c) $h(x) = \frac{3}{2}x^2$ (d) $k(x) = -3x^2$
14. (a) $f(x) = x^2 + 1$ (b) $g(x) = x^2 - 1$
(c) $h(x) = x^2 + 3$ (d) $k(x) = x^2 - 3$
15. (a) $f(x) = (x - 1)^2$ (b) $g(x) = (3x)^2 + 1$
(c) $h(x) = \left(\frac{1}{3}x\right)^2 - 3$ (d) $k(x) = (x + 3)^2$
16. (a) $f(x) = -\frac{1}{2}(x - 2)^2 + 1$
(b) $g(x) = \left[\frac{1}{2}(x - 1)\right]^2 - 3$
(c) $h(x) = -\frac{1}{2}(x + 2)^2 - 1$
(d) $k(x) = [2(x + 1)]^2 + 4$

Using Standard Form to Graph a Parabola **In Exercises 17–34, write the quadratic function in standard form and sketch its graph. Identify the vertex, axis of symmetry, and x-intercept(s).**

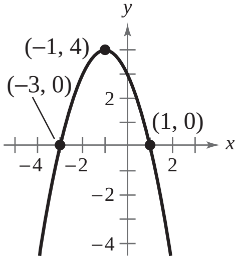
17. $f(x) = x^2 - 6x$
18. $g(x) = x^2 - 8x$
19. $h(x) = x^2 - 8x + 16$
20. $g(x) = x^2 + 2x + 1$
21. $f(x) = x^2 + 8x + 13$
22. $f(x) = x^2 - 12x + 44$
23. $f(x) = x^2 - 14x + 54$
24. $h(x) = x^2 + 16x - 17$
25. $f(x) = x^2 + 34x + 289$
26. $f(x) = x^2 - 30x + 225$
27. $f(x) = x^2 - x + \frac{5}{4}$
28. $f(x) = x^2 + 3x + \frac{1}{4}$
29. $f(x) = -x^2 + 2x + 5$
30. $f(x) = -x^2 - 4x + 1$
31. $h(x) = 4x^2 - 4x + 21$
32. $f(x) = 2x^2 - x + 1$
33. $f(x) = \frac{1}{4}x^2 - 2x - 12$
34. $f(x) = -\frac{1}{3}x^2 + 3x - 6$

Graphical Analysis **In Exercises 35–42, use a graphing utility to graph the quadratic function. Identify the vertex, axis of symmetry, and x-intercept(s). Then check your results algebraically by writing the quadratic function in standard form.**

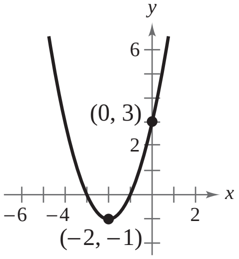
35. $f(x) = -(x^2 + 2x - 3)$
36. $f(x) = -(x^2 + x - 30)$
37. $g(x) = x^2 + 8x + 11$
38. $f(x) = x^2 + 10x + 14$
39. $f(x) = 2x^2 - 16x + 32$
40. $f(x) = -4x^2 + 24x - 41$
41. $g(x) = \frac{1}{2}(x^2 + 4x - 2)$
42. $f(x) = \frac{3}{5}(x^2 + 6x - 5)$

Writing the Equation of a Parabola **In Exercises 43–46, write an equation for the parabola in standard form.**

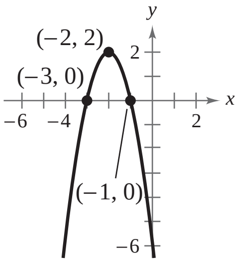
43.



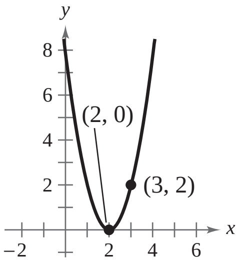
44.



45.



46.



Writing the Equation of a Parabola **In Exercises 47–56, write the standard form of the equation of the parabola that has the indicated vertex and passes through the given point.**

47. Vertex: $(-2, 5)$; point: $(0, 9)$

48. Vertex: $(4, -1)$; point: $(2, 3)$

49. Vertex: $(1, -2)$; point: $(-1, 14)$

50. Vertex: $(2, 3)$; point: $(0, 2)$

51. Vertex: $(5, 12)$; point: $(7, 15)$

52. Vertex: $(-2, -2)$; point: $(-1, 0)$

53. Vertex: $\left(-\frac{1}{4}, \frac{3}{2}\right)$; point: $(-2, 0)$

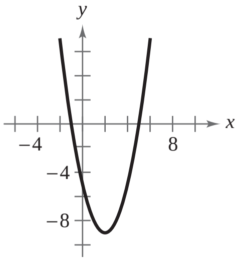
54. Vertex: $\left(\frac{5}{2}, -\frac{3}{4}\right)$; point: $(-2, 4)$

55. Vertex: $\left(-\frac{5}{2}, 0\right)$; point: $\left(-\frac{7}{2}, -\frac{16}{3}\right)$

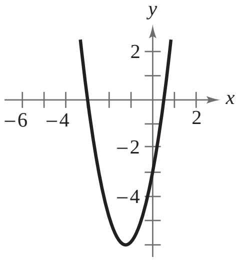
56. Vertex: $(6, 6)$; point: $\left(\frac{61}{10}, \frac{3}{2}\right)$

Graphical Reasoning **In Exercises 57 and 58, determine the x-intercept(s) of the graph visually. Then find the x-intercept(s) algebraically to confirm your results.**

57. $y = x^2 - 4x - 5$



58. $y = 2x^2 + 5x - 3$



wellphoto/Shutterstock.com

Graphical Analysis **In Exercises 59–64, use a graphing utility to graph the quadratic function. Find the x-intercept(s) of the graph and compare them with the solutions of the corresponding quadratic equation when $f(x) = 0$.**

59. $f(x) = x^2 - 4x$

60. $f(x) = -2x^2 + 10x$

61. $f(x) = x^2 - 9x + 18$

62. $f(x) = x^2 - 8x - 20$

63. $f(x) = 2x^2 - 7x - 30$

64. $f(x) = \frac{7}{10}(x^2 + 12x - 45)$

Finding Quadratic Functions **In Exercises 65–70, find two quadratic functions, one that opens upward and one that opens downward, whose graphs have the given x-intercepts. (There are many correct answers.)**

65. $(-1, 0), (3, 0)$

66. $(-5, 0), (5, 0)$

67. $(0, 0), (10, 0)$

68. $(4, 0), (8, 0)$

69. $(-3, 0), \left(-\frac{1}{2}, 0\right)$

70. $\left(-\frac{5}{2}, 0\right), (2, 0)$

Number Problems **In Exercises 71–74, find two positive real numbers whose product is a maximum.**

71. The sum is 110.

72. The sum is S.

73. The sum of the first and twice the second is 24.

74. The sum of the first and three times the second is 42.

75. Path of a Diver The path of a diver is given by the function

$$f(x) = -\frac{4}{9}x^2 + \frac{24}{9}x + 12$$

where $f(x)$ is the height (in feet) and x is the horizontal distance from the end of the diving board (in feet). What is the maximum height of the diver?

76. Height of a Ball The path of a punted football is given by the function

$$f(x) = -\frac{16}{2025}x^2 + \frac{9}{5}x + 1.5$$

where $f(x)$ is the height (in feet) and x is the horizontal distance (in feet) from the point at which the ball is punted.

(a) How high is the ball when it is punted?

(b) What is the maximum height of the punt?

(c) How long is the punt?

77. Minimum Cost A manufacturer of lighting fixtures has daily production costs of $C = 800 - 10x + 0.25x^2$, where C is the total cost (in dollars) and x is the number of units produced. How many fixtures should be produced each day to yield a minimum cost?

78. Maximum Profit The profit P (in hundreds of dollars) that a company makes depends on the amount x (in hundreds of dollars) the company spends on advertising according to the model $P = 230 + 20x - 0.5x^2$. What expenditure for advertising will yield a maximum profit?

79. Maximum Revenue The total revenue R earned (in thousands of dollars) from manufacturing handheld video games is given by

$$R(p) = -25p^2 + 1200p$$

where p is the price per unit (in dollars).

(a) Find the revenues when the prices per unit are \$20, \$25, and \$30.

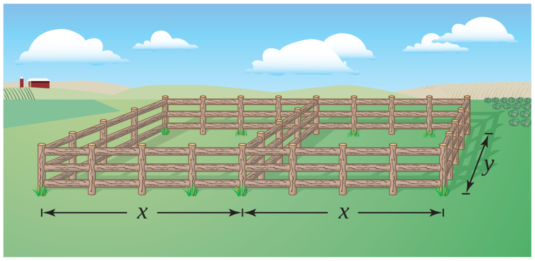
(b) Find the unit price that will yield a maximum revenue. What is the maximum revenue? Explain your results.

80. Maximum Revenue The total revenue R earned per day (in dollars) from a pet-sitting service is given by $R(p) = -12p^2 + 150p$, where p is the price charged per pet (in dollars).

(a) Find the revenues when the prices per pet are \$4, \$6, and \$8.

(b) Find the unit price that will yield a maximum revenue. What is the maximum revenue? Explain your results.

81. Numerical, Graphical, and Analytical Analysis A rancher has 200 feet of fencing to enclose two adjacent rectangular corrals (see figure).



(a) Write the area A of the corrals as a function of x.

(b) Construct a table showing possible values of x and the corresponding areas of the corral. Use the table to estimate the dimensions that will produce the maximum enclosed area.

(c) Use a graphing utility to graph the area function. Use the graph to approximate the dimensions that will produce the maximum enclosed area.

(d) Write the area function in standard form to find analytically the dimensions that will produce the maximum area.

(e) Compare your results from parts (b), (c), and (d).

82. Geometry An indoor physical fitness room consists of a rectangular region with a semicircle on each end. The perimeter of the room is to be a 200-meter single-lane running track.

(a) Draw a diagram that gives a visual representation of the problem. Let x and y represent the length and width of the rectangular region, respectively.

(b) Determine the radius of each semicircular end of the room. Determine the distance, in terms of y, around the inside edge of each semicircular part of the track.

(c) Use the result of part (b) to write an equation, in terms of x and y, for the distance traveled in one lap around the track. Solve for y.

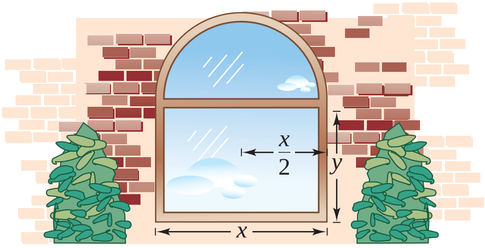
(d) Use the result of part (c) to write the area A of the rectangular region as a function of x. What dimensions will produce a rectangle of maximum area?

83. Maximum Revenue A small theater has a seating capacity of 2000. When the ticket price is \$20, attendance is 1500. For each \$1 decrease in price, attendance increases by 100.

(a) Write the revenue R of the theater as a function of ticket price x.

(b) What ticket price will yield a maximum revenue? What is the maximum revenue?

84. Maximum Area A Norman window is constructed by adjoining a semicircle to the top of an ordinary rectangular window (see figure). The perimeter of the window is 16 feet.



(a) Write the area A of the window as a function of x.

(b) What dimensions will produce a window of maximum area?

85. Graphical Analysis From 1950 through 2005, the per capita consumption C of cigarettes by Americans (age 18 and older) can be modeled by $C = 3565.0 + 60.30t - 1.783t^2$, $0 \le t \le 55$, where t is the year, with $t = 0$ corresponding to 1950. (*Source: Tobacco Outlook Report*)

(a) Use a graphing utility to graph the model.

(b) Use the graph of the model to approximate the maximum average annual consumption. Beginning in 1966, all cigarette packages were required by law to carry a health warning. Do you think the warning had any effect? Explain.

(c) In 2005, the U.S. population (age 18 and over) was 296,329,000. Of those, about 59,858,458 were smokers. What was the average annual cigarette consumption *per smoker* in 2005? What was the average daily cigarette consumption *per smoker*?

86. **Data Analysis: Sales** The sales y (in billions of dollars) for Harley-Davidson from 2000 through 2010 are shown in the table. *(Source: U.S. Harley-Davidson, Inc.)*

Spreadsheet at LarsonPrecalculus.com

Year	Sales, y
2000	2.91
2001	3.36
2002	4.09
2003	4.62
2004	5.02
2005	5.34
2006	5.80
2007	5.73
2008	5.59
2009	4.78
2010	4.86

(a) Use a graphing utility to create a scatter plot of the data. Let x represent the year, with $x = 0$ corresponding to 2000.

(b) Use the *regression* feature of the graphing utility to find a quadratic model for the data.

(c) Use the graphing utility to graph the model in the same viewing window as the scatter plot. How well does the model fit the data?

(d) Use the *trace* feature of the graphing utility to approximate the year in which the sales for Harley-Davidson were the greatest.

(e) Verify your answer to part (d) algebraically.

(f) Use the model to predict the sales for Harley-Davidson in 2013.

Exploration

True or False? **In Exercises 87 and 88, determine whether the statement is true or false. Justify your answer.**

87. The graph of $f(x) = -12x^2 - 1$ has no x-intercepts.

88. The graphs of

$$f(x) = -4x^2 - 10x + 7$$

and

$$g(x) = 12x^2 + 30x + 1$$

have the same axis of symmetry.

Think About It **In Exercises 89–92, find the values of b such that the function has the given maximum or minimum value.**

89. $f(x) = -x^2 + bx - 75$; Maximum value: 25

90. $f(x) = -x^2 + bx - 16$; Maximum value: 48

91. $f(x) = x^2 + bx + 26$; Minimum value: 10

92. $f(x) = x^2 + bx - 25$; Minimum value: -50

93. **Verifying the Vertex** Write the quadratic function

$$f(x) = ax^2 + bx + c$$

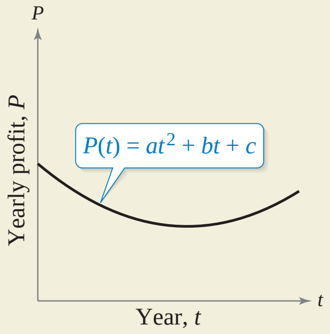
in standard form to verify that the vertex occurs at

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right).$$

94. **HOW DO YOU SEE IT?** The graph shows a quadratic function of the form

$$P(t) = at^2 + bt + c$$

which represents the yearly profits for a company, where $P(t)$ is the profit in year t.



(a) Is the value of a positive, negative, or zero? Explain.

(b) Write an expression in terms of a and b that represents the year t when the company made the least profit.

(c) The company made the same yearly profits in 2004 and 2012. Estimate the year in which the company made the least profit.

(d) Assume that the model is still valid today. Are the yearly profits currently increasing, decreasing, or constant? Explain.

95. **Proof** Assume that the function

$$f(x) = ax^2 + bx + c, \quad a \neq 0$$

has two real zeros. Prove that the x-coordinate of the vertex of the graph is the average of the zeros of f. (*Hint:* Use the Quadratic Formula.)

Project: Height of a Basketball To work an extended application analyzing the height of a basketball after it has been dropped, visit this text's website at *LarsonPrecalculus.com.*

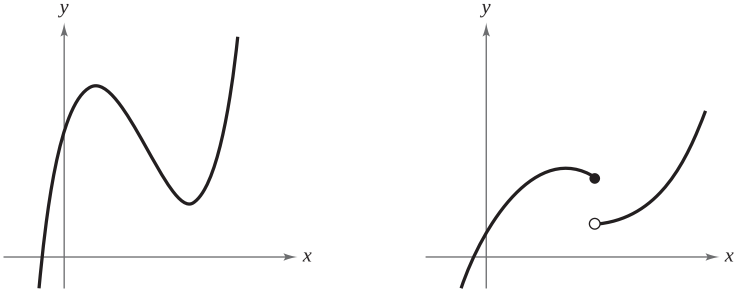
3.2 Polynomial Functions of Higher Degree

You can use polynomial functions to analyze nature. For instance, in Exercise 104 on page 264, you will use a polynomial function to analyze the growth of a red oak tree.

- **Use transformations to sketch graphs of polynomial functions.**
- **Use the Leading Coefficient Test to determine the end behaviors of graphs of polynomial functions.**
- **Find and use the real zeros of polynomial functions as sketching aids.**
- **Use the Intermediate Value Theorem to help locate the real zeros of polynomial functions.**

Graphs of Polynomial Functions

In this section, you will study basic features of the graphs of polynomial functions. The first feature is that the graph of a polynomial function is **continuous.** Essentially, this means that the graph of a polynomial function has no breaks, holes, or gaps, as shown in Figure 3.6(a). The graph shown in Figure 3.6(b) is an example of a piecewise-defined function that is not continuous.

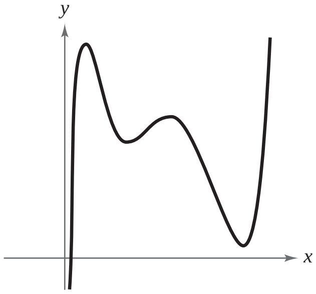


(a) Polynomial functions have continuous graphs.

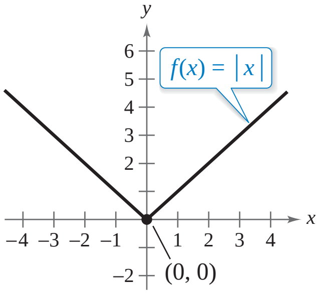
(b) Functions with graphs that are not continuous are not polynomial functions.

Figure 3.6

The second feature is that the graph of a polynomial function has only smooth, rounded turns, as shown in Figure 3.7. A polynomial function cannot have a sharp turn. For instance, the function $f(x) = |x|$, the graph of which has a sharp turn at the point $(0, 0)$, as shown in Figure 3.8, is not a polynomial function.



Polynomial functions have graphs with smooth, rounded turns.
Figure 3.7



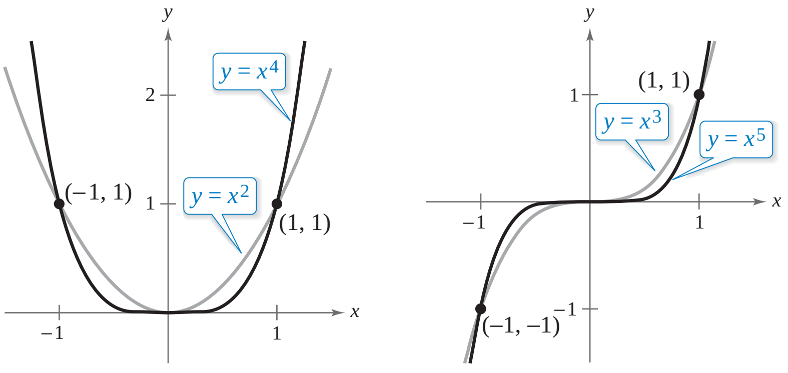
Graphs of polynomial functions cannot have sharp turns.
Figure 3.8

The graphs of polynomial functions of degree greater than 2 are more difficult to analyze than the graphs of polynomial functions of degree 0, 1, or 2. However, using the features presented in this section, coupled with your knowledge of point plotting, intercepts, and symmetry, you should be able to make reasonably accurate sketches *by hand*.

S.Borisov/Shutterstock.com

REMARK For functions given by $f(x) = x^n$, when n is even, the graph of the function is symmetric with respect to the y-axis, and when n is odd, the graph of the function is symmetric with respect to the origin.

The polynomial functions that have the simplest graphs are monomial functions of the form $f(x) = x^n$, where n is an integer greater than zero. From Figure 3.9, you can see that when n is *even*, the graph is similar to the graph of $f(x) = x^2$, and when n is *odd*, the graph is similar to the graph of $f(x) = x^3$. Moreover, the greater the value of n, the flatter the graph near the origin.



(a) When n is even, the graph of $y = x^n$ touches the axis at the x-intercept.

(b) When n is odd, the graph of $y = x^n$ crosses the axis at the x-intercept.

Figure 3.9

EXAMPLE 1 Sketching Transformations of Monomial Functions

Sketch the graph of each function.

a. $f(x) = -x^5$ **b.** $h(x) = (x + 1)^4$

Solution

a. Because the degree of $f(x) = -x^5$ is odd, its graph is similar to the graph of $y = x^3$. In Figure 3.10, note that the negative coefficient has the effect of reflecting the graph in the x-axis.

b. The graph of $h(x) = (x + 1)^4$, as shown in Figure 3.11, is a left shift by one unit of the graph of $y = x^4$.

ALGEBRA HELP You can review the techniques for shifting, reflecting, and stretching graphs in Section 2.5.

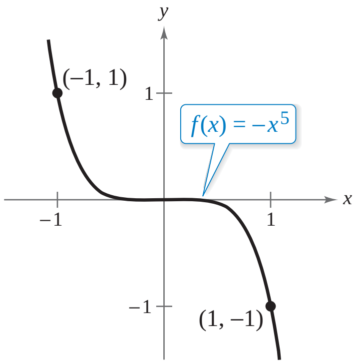


Figure 3.10

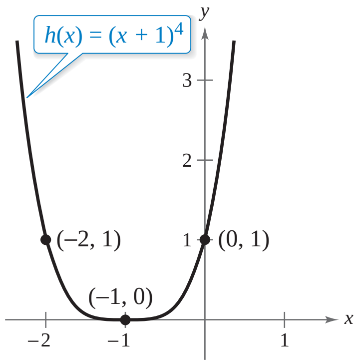


Figure 3.11

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Sketch the graph of each function.

a. $f(x) = (x + 5)^4$

b. $g(x) = x^4 - 7$

c. $h(x) = 7 - x^4$

d. $k(x) = \frac{1}{4}(x - 3)^4$

The Leading Coefficient Test

In Example 1, note that both graphs eventually rise or fall without bound as x moves to the left or to the right. Whether the graph of a polynomial function eventually rises or falls can be determined by the function's degree (even or odd) and by its leading coefficient (positive or negative), as indicated in the **Leading Coefficient Test.**

REMARK The notation "$f(x) \to -\infty$ as $x \to -\infty$" indicates that the graph falls to the left. The notation "$f(x) \to \infty$ as $x \to \infty$" indicates that the graph rises to the right.

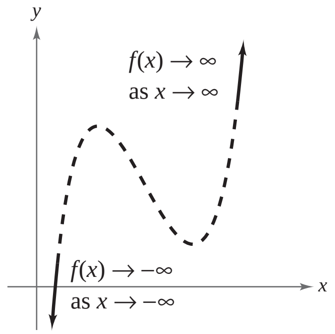
Leading Coefficient Test

As x moves without bound to the left or to the right, the graph of the polynomial function

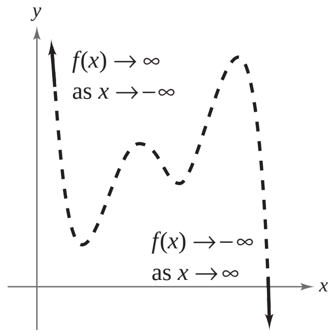
$$f(x) = a_n x^n + \cdot \cdot \cdot + a_1 x + a_0$$

eventually rises or falls in the following manner.

1. When n is *odd:*

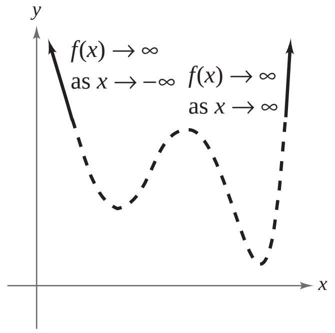


If the leading coefficient is positive $(a_n > 0)$, then the graph falls to the left and rises to the right.

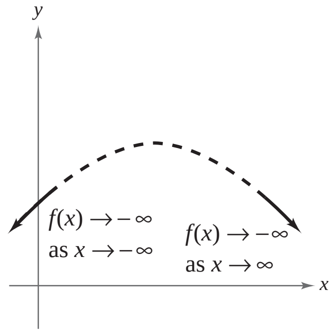


If the leading coefficient is negative $(a_n < 0)$, then the graph rises to the left and falls to the right.

2. When n is *even:*



If the leading coefficient is positive $(a_n > 0)$, then the graph rises to the left and to the right.



If the leading coefficient is negative $(a_n < 0)$, then the graph falls to the left and to the right.

The dashed portions of the graphs indicate that the test determines *only* the right-hand and left-hand behavior of the graph.

As you continue to study polynomial functions and their graphs, you will notice that the degree of a polynomial plays an important role in determining other characteristics of the polynomial function and its graph.

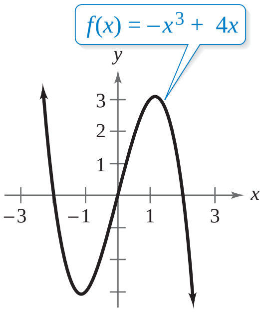
EXAMPLE 2 Applying the Leading Coefficient Test

Describe the right-hand and left-hand behavior of the graph of each function.

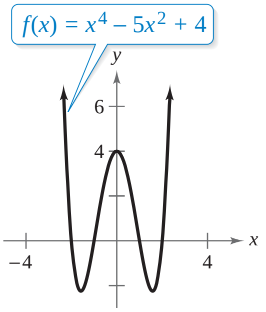
a. $f(x) = -x^3 + 4x$ **b.** $f(x) = x^4 - 5x^2 + 4$ **c.** $f(x) = x^5 - x$

Solution

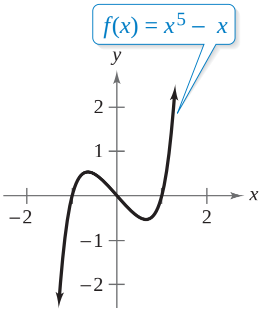
a. Because the degree is odd and the leading coefficient is negative, the graph rises to the left and falls to the right, as shown below.



b. Because the degree is even and the leading coefficient is positive, the graph rises to the left and to the right, as shown below.



c. Because the degree is odd and the leading coefficient is positive, the graph falls to the left and rises to the right, as shown below.



✓ Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Describe the right-hand and left-hand behavior of the graph of each function.

a. $f(x) = \frac{1}{4}x^3 - 2x$ **b.** $f(x) = -3.6x^5 + 5x^3 - 1$

In Example 2, note that the Leading Coefficient Test tells you only whether the graph *eventually* rises or falls to the right or to the left. Other characteristics of the graph, such as intercepts and minimum and maximum points, must be determined by other tests.

Real Zeros of Polynomial Functions

It can be shown that for a polynomial function f of degree n, the following statements are true.

1. The function f has, at most, n real zeros. (You will study this result in detail in the discussion of the Fundamental Theorem of Algebra in Section 3.4.)
2. The graph of f has, at most, $n - 1$ turning points. (Turning points, also called relative minima or relative maxima, are points at which the graph changes from increasing to decreasing or vice versa.)

REMARK Remember that the *zeros* of a function of x are the x-values for which the function is zero.

Finding the zeros of a polynomial function is one of the most important problems in algebra. There is a strong interplay between graphical and algebraic approaches to this problem.

ALGEBRA HELP To do Example 3 algebraically, you need to be able to completely factor polynomials. You can review the techniques for factoring in Section P.4.

Real Zeros of Polynomial Functions

When f is a polynomial function and a is a real number, the following statements are equivalent.

1. $x = a$ is a *zero* of the function f.
2. $x = a$ is a *solution* of the polynomial equation $f(x) = 0$.
3. $(x - a)$ is a *factor* of the polynomial $f(x)$.
4. $(a, 0)$ is an *x-intercept* of the graph of f.

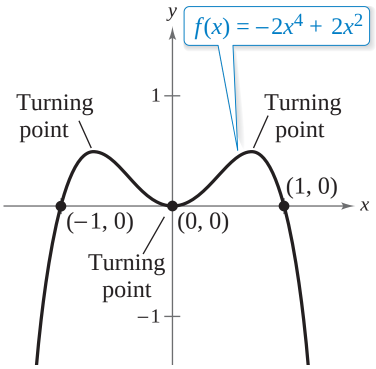


Figure 3.12

EXAMPLE 3 Finding Real Zeros of a Polynomial Function

Find all real zeros of $f(x) = -2x^4 + 2x^2$. Then determine the maximum possible number of turning points of the graph of the function.

Solution To find the real zeros of the function, set $f(x)$ equal to zero and solve for x.

$$-2x^4 + 2x^2 = 0 \qquad \text{Set } f(x) \text{ equal to 0.}$$

$$-2x^2(x^2 - 1) = 0 \qquad \text{Remove common monomial factor.}$$

$$-2x^2(x - 1)(x + 1) = 0 \qquad \text{Factor completely.}$$

So, the real zeros are $x = 0$, $x = 1$, and $x = -1$. Because the function is a fourth-degree polynomial, the graph of f can have at most $4 - 1 = 3$ turning points. In this case, the graph of f has three turning points, as shown in Figure 3.12.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find all real zeros of $f(x) = x^3 - 12x^2 + 36x$. Then determine the maximum possible number of turning points of the graph of the function. ■

In Example 3, note that because the exponent is greater than 1, the factor $-2x^2$ yields the *repeated* zero $x = 0$.

Repeated Zeros

A factor $(x - a)^k$, $k > 1$, yields a **repeated zero** $x = a$ of **multiplicity** k.

1. When k is odd, the graph *crosses* the x-axis at $x = a$.
2. When k is even, the graph *touches* the x-axis (but does not cross the x-axis) at $x = a$.

A polynomial function is in **standard form** when its terms are in descending order of exponents from left to right. Before applying the Leading Coefficient Test to a polynomial function, it is a good idea to check that the polynomial function is in standard form.

EXAMPLE 4 Sketching the Graph of a Polynomial Function

Sketch the graph of

$$f(x) = 3x^4 - 4x^3.$$

Solution

1. *Apply the Leading Coefficient Test.* Because the leading coefficient is positive and the degree is even, you know that the graph eventually rises to the left and to the right (see Figure 3.13).

2. *Find the Real Zeros of the Polynomial.* By factoring

$$f(x) = 3x^4 - 4x^3 = x^3(3x - 4) \qquad \text{Remove common monomial factor.}$$

you can see that the real zeros of f are $x = 0$ and $x = \frac{4}{3}$ (both of odd multiplicity). So, the x-intercepts occur at $(0, 0)$ and $\left(\frac{4}{3}, 0\right)$. Add these points to your graph, as shown in Figure 3.13.

3. *Plot a Few Additional Points.* To sketch the graph by hand, find a few additional points, as shown in the table.

x	-1	0.5	1	1.5
$f(x)$	7	-0.3125	-1	1.6875

Then plot the points (see Figure 3.14).

4. *Draw the Graph.* Draw a continuous curve through the points, as shown in Figure 3.14. Because both zeros are of odd multiplicity, you know that the graph should cross the x-axis at $x = 0$ and $x = \frac{4}{3}$. If you are unsure of the shape of a portion of a graph, then you can plot some additional points.

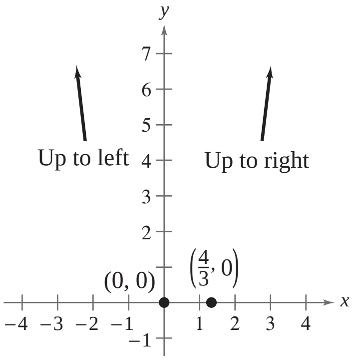


Figure 3.13

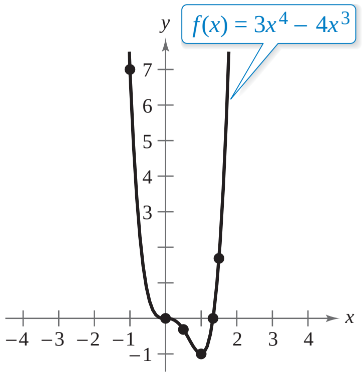


Figure 3.14

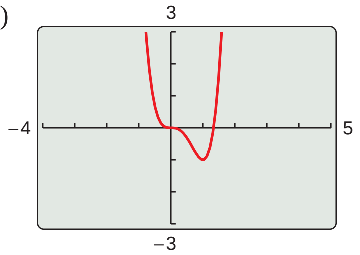
Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Sketch the graph of

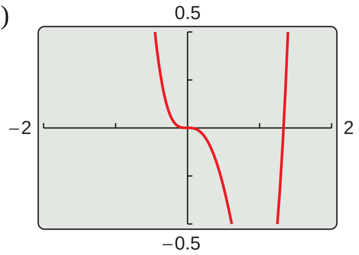
$$f(x) = 2x^3 - 6x^2.$$

TECHNOLOGY Example 4 uses an *algebraic approach* to describe the graph of the function. A graphing utility can complement this approach. Remember to find a viewing window that shows all significant features of the graph. For instance, viewing window (a) illustrates all of the significant features of the function in Example 4 while viewing window (b) does not.

(a)



(b)



EXAMPLE 5 Sketching the Graph of a Polynomial Function

Sketch the graph of $f(x) = -2x^3 + 6x^2 - \frac{9}{2}x$.

Solution

1. *Apply the Leading Coefficient Test.* Because the leading coefficient is negative and the degree is odd, you know that the graph eventually rises to the left and falls to the right (see Figure 3.15).

2. *Find the Real Zeros of the Polynomial.* By factoring

$$\begin{aligned} f(x) &= -2x^3 + 6x^2 - \tfrac{9}{2}x \\ &= -\tfrac{1}{2}x(4x^2 - 12x + 9) \\ &= -\tfrac{1}{2}x(2x - 3)^2 \end{aligned}$$

you can see that the real zeros of f are $x = 0$ (odd multiplicity) and $x = \frac{3}{2}$ (even multiplicity). So, the x-intercepts occur at $(0, 0)$ and $\left(\frac{3}{2}, 0\right)$. Add these points to your graph, as shown in Figure 3.15.

3. *Plot a Few Additional Points.* To sketch the graph by hand, find a few additional points, as shown in the table.

x	$f(x)$
-0.5	4
0.5	-1
1	-0.5
2	-1

Then plot the points (see Figure 3.16).

4. *Draw the Graph.* Draw a continuous curve through the points, as shown in Figure 3.16. As indicated by the multiplicities of the zeros, the graph crosses the x-axis at $(0, 0)$ but does not cross the x-axis at $\left(\frac{3}{2}, 0\right)$.

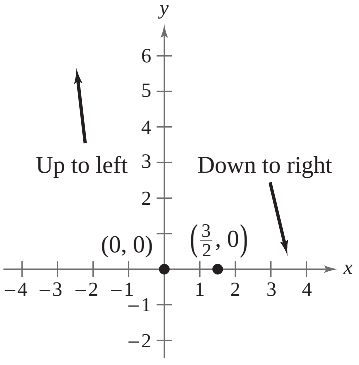


Figure 3.15

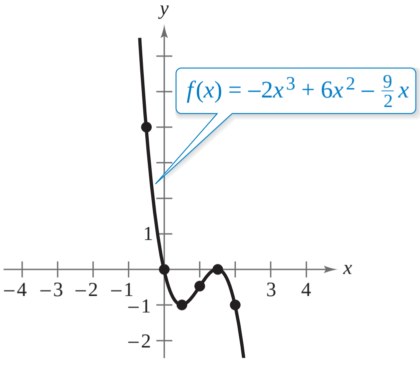


Figure 3.16

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Sketch the graph of

$$f(x) = -\tfrac{1}{4}x^4 + \tfrac{3}{2}x^3 - \tfrac{9}{4}x^2.$$

REMARK Observe in Example 5 that the sign of $f(x)$ is positive to the left of and negative to the right of the zero $x = 0$. Similarly, the sign of $f(x)$ is negative to the left and to the right of the zero $x = \frac{3}{2}$. This suggests that if the zero of a polynomial function is of *odd* multiplicity, then the sign of $f(x)$ changes from one side to the other side of the zero. If the zero is of *even* multiplicity, then the sign of $f(x)$ does not change from one side to the other side of the zero. The following table helps to illustrate this concept.

x	-0.5	0	0.5
$f(x)$	4	0	-1
Sign	$+$		$-$

x	1	$\frac{3}{2}$	2
$f(x)$	-0.5	0	-1
Sign	$-$		$-$

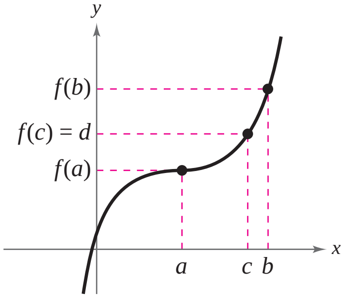
This sign analysis may be helpful in graphing polynomial functions.

The Intermediate Value Theorem

The **Intermediate Value Theorem** implies that if

$$(a, f(a)) \quad \text{and} \quad (b, f(b))$$

are two points on the graph of a polynomial function such that $f(a) \neq f(b)$, then for any number d between $f(a)$ and $f(b)$ there must be a number c between a and b such that $f(c) = d$. (See figure.)



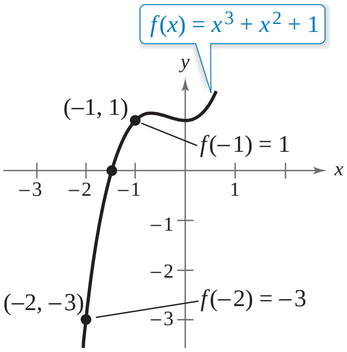
Intermediate Value Theorem

Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a) \neq f(b)$, then, in the interval $[a, b]$, f takes on every value between $f(a)$ and $f(b)$.

The Intermediate Value Theorem helps you locate the real zeros of a polynomial function in the following way. If you can find a value $x = a$ at which a polynomial function is positive, and another value $x = b$ at which it is negative, then you can conclude that the function has at least one real zero between these two values. For example, the function

$$f(x) = x^3 + x^2 + 1$$

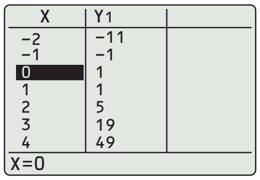
is negative when $x = -2$ and positive when $x = -1$. So, it follows from the Intermediate Value Theorem that f must have a real zero somewhere between -2 and -1, as shown below.



The function f must have a real zero somewhere between -2 and -1.

By continuing this line of reasoning, you can approximate any real zeros of a polynomial function to any desired accuracy. This concept is further demonstrated in Example 6.

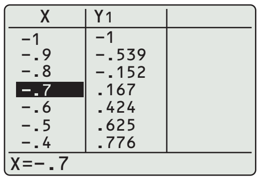
TECHNOLOGY You can use the *table* feature of a graphing utility to approximate the real zero of the polynomial function in Example 6. Construct a table that shows function values for $-2 \le x \le 4$, as shown below.



X	Y1
-2	-11
-1	-1
0	1
1	1
2	5
3	19
4	49

X=0

Scroll through the table looking for consecutive values that differ in sign. From the table, you can see that $f(-1)$ and $f(0)$ differ in sign. So, you can conclude from the Intermediate Value Theorem that the function has a real zero between -1 and 0. You can adjust your table to show function values for $-1 \le x \le 0$ using increments of 0.1, as shown below.



X	Y1
-1	-1
-.9	-.539
-.8	-.152
-.7	.167
-.6	.424
-.5	.625
-.4	.776

X=-.7

By scrolling through the table, you can see that $f(-0.8)$ and $f(-0.7)$ differ in sign. So, the function has a real zero between -0.8 and -0.7. Repeating this process several times, you should obtain $x \approx -0.755$ as the real zero of the function. Use the *zero* or *root* feature of the graphing utility to confirm this result.

EXAMPLE 6 Using the Intermediate Value Theorem

Use the Intermediate Value Theorem to approximate the real zero of

$$f(x) = x^3 - x^2 + 1.$$

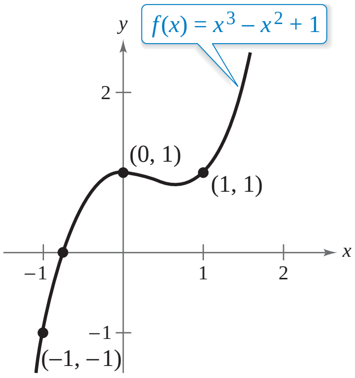
Solution Begin by computing a few function values, as follows.

x	-2	-1	0	1
$f(x)$	-11	-1	1	1

Because $f(-1)$ is negative and $f(0)$ is positive, you can apply the Intermediate Value Theorem to conclude that the function has a real zero between -1 and 0. To pinpoint this zero more closely, divide the interval $[-1, 0]$ into tenths and evaluate the function at each point. When you do this, you will find that

$$f(-0.8) = -0.152 \quad \text{and} \quad f(-0.7) = 0.167.$$

So, f must have a real zero between -0.8 and -0.7, as shown below.



The function f has a real zero between -0.8 and -0.7.

For a more accurate approximation, compute function values between $f(-0.8)$ and $f(-0.7)$ and apply the Intermediate Value Theorem again. By continuing this process, you can approximate this zero to any desired accuracy.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use the Intermediate Value Theorem to approximate the real zero of

$$f(x) = x^3 - 3x^2 - 2.$$

Summarize (Section 3.2)

1. Describe the graphs of monomial functions of the form $f(x) = x^n$ *(page 253)*. For an example of sketching transformations of monomial functions, see Example 1.
2. Describe the Leading Coefficient Test *(page 254)*. For an example of applying the Leading Coefficient Test, see Example 2.
3. Describe the real zeros of polynomial functions *(page 256)*. For examples involving the real zeros of polynomial functions, see Examples 3–5.
4. State the Intermediate Value Theorem *(page 259)*. For an example of using the Intermediate Value Theorem, see Example 6.

3.2 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

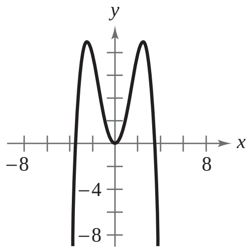
Vocabulary: Fill in the blanks.

1. The graphs of all polynomial functions are _________, which means that the graphs have no breaks, holes, or gaps.
2. The _________ _________ _________ is used to determine the left-hand and right-hand behavior of the graph of a polynomial function.
3. A polynomial function of degree n has at most _________ real zeros and at most _________ turning points.
4. When $x = a$ is a zero of a polynomial function f, the following three statements are true.
   (a) $x = a$ is a _________ of the polynomial equation $f(x) = 0$.
   (b) _________ is a factor of the polynomial $f(x)$.
   (c) $(a, 0)$ is an _________ of the graph of f.
5. When a real zero of a polynomial function is of even multiplicity, the graph of f _________ the x-axis at $x = a$, and when it is of odd multiplicity, the graph of f _________ the x-axis at $x = a$.
6. A factor $(x - a)^k$, $k > 1$, yields a _________ _________ $x = a$ of _________ k.
7. A polynomial function is written in _________ form when its terms are written in descending order of exponents from left to right.
8. The _________ _________ Theorem states that if f is a polynomial function such that $f(a) \neq f(b)$, then, in the interval $[a, b]$, f takes on every value between $f(a)$ and $f(b)$.

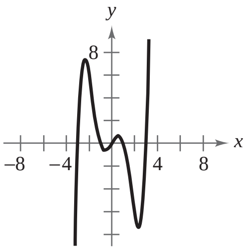
Skills and Applications

Matching **In Exercises 9–14, match the polynomial function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]**

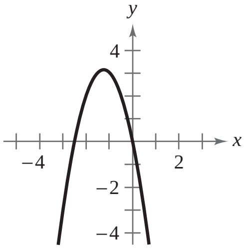
(a)



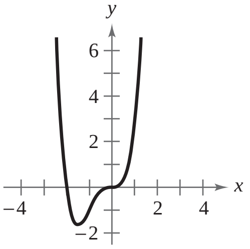
(b)



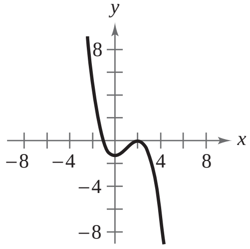
(c)



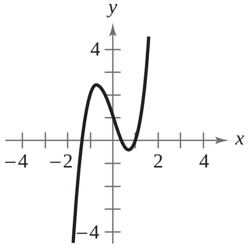
(d)



(e)



(f)



9. $f(x) = -2x^2 - 5x$
10. $f(x) = 2x^3 - 3x + 1$
11. $f(x) = -\frac{1}{4}x^4 + 3x^2$
12. $f(x) = -\frac{1}{3}x^3 + x^2 - \frac{4}{3}$
13. $f(x) = x^4 + 2x^3$
14. $f(x) = \frac{1}{5}x^5 - 2x^3 + \frac{9}{5}x$

Sketching Transformations of Monomial Functions **In Exercises 15–18, sketch the graph of $y = x^n$ and each transformation.**

15. $y = x^3$
   (a) $f(x) = (x - 4)^3$ (b) $f(x) = x^3 - 4$
   (c) $f(x) = -\frac{1}{4}x^3$ (d) $f(x) = (x - 4)^3 - 4$
16. $y = x^5$
   (a) $f(x) = (x + 1)^5$ (b) $f(x) = x^5 + 1$
   (c) $f(x) = 1 - \frac{1}{2}x^5$ (d) $f(x) = -\frac{1}{2}(x + 1)^5$
17. $y = x^4$
   (a) $f(x) = (x + 3)^4$ (b) $f(x) = x^4 - 3$
   (c) $f(x) = 4 - x^4$ (d) $f(x) = \frac{1}{2}(x - 1)^4$
   (e) $f(x) = (2x)^4 + 1$ (f) $f(x) = \left(\frac{1}{2}x\right)^4 - 2$
18. $y = x^6$
   (a) $f(x) = -\frac{1}{8}x^6$ (b) $f(x) = (x + 2)^6 - 4$
   (c) $f(x) = x^6 - 5$ (d) $f(x) = -\frac{1}{4}x^6 + 1$
   (e) $f(x) = \left(\frac{1}{4}x\right)^6 - 2$ (f) $f(x) = (2x)^6 - 1$

Apply the Leading Coefficient Test **In Exercises 19–30, describe the right-hand and left-hand behavior of the graph of the polynomial function.**

19. $f(x) = \frac{1}{5}x^3 + 4x$

20. $f(x) = 2x^2 - 3x + 1$

21. $g(x) = 5 - \frac{7}{2}x - 3x^2$

22. $h(x) = 1 - x^6$

23. $g(x) = -x^3 + 3x^2$

24. $g(x) = -x^4 + 4x - 6$

25. $f(x) = -2.1x^5 + 4x^3 - 2$

26. $f(x) = 4x^5 - 7x + 6.5$

27. $f(x) = 6 - 2x + 4x^2 - 5x^7$

28. $f(x) = (3x^4 - 2x + 5)/4$

29. $h(t) = -\frac{3}{4}(t^2 - 3t + 6)$

30. $f(s) = -\frac{7}{8}(s^3 + 5s^2 - 7s + 1)$

Graphical Analysis **In Exercises 31–34, use a graphing utility to graph the functions *f* and *g* in the same viewing window. Zoom out sufficiently far to show that the right-hand and left-hand behaviors of *f* and *g* appear identical.**

31. $f(x) = 3x^3 - 9x + 1, \quad g(x) = 3x^3$

32. $f(x) = -\frac{1}{3}(x^3 - 3x + 2), \quad g(x) = -\frac{1}{3}x^3$

33. $f(x) = -(x^4 - 4x^3 + 16x), \quad g(x) = -x^4$

34. $f(x) = 3x^4 - 6x^2, \quad g(x) = 3x^4$

Finding Real Zeros of a Polynomial Function **In Exercises 35–50, (a) find all real zeros of the polynomial function, (b) determine the multiplicity of each zero, (c) determine the maximum possible number of turning points of the graph of the function, and (d) use a graphing utility to graph the function and verify your answers.**

35. $f(x) = x^2 - 36$

36. $f(x) = 81 - x^2$

37. $h(t) = t^2 - 6t + 9$

38. $f(x) = x^2 + 10x + 25$

39. $f(x) = \frac{1}{3}x^2 + \frac{1}{3}x - \frac{2}{3}$

40. $f(x) = \frac{1}{2}x^2 + \frac{5}{2}x - \frac{3}{2}$

41. $f(x) = 3x^3 - 12x^2 + 3x$

42. $g(x) = 5x(x^2 - 2x - 1)$

43. $f(t) = t^3 - 8t^2 + 16t$

44. $f(x) = x^4 - x^3 - 30x^2$

45. $g(t) = t^5 - 6t^3 + 9t$

46. $f(x) = x^5 + x^3 - 6x$

47. $f(x) = 3x^4 + 9x^2 + 6$

48. $f(x) = 2x^4 - 2x^2 - 40$

49. $g(x) = x^3 + 3x^2 - 4x - 12$

50. $f(x) = x^3 - 4x^2 - 25x + 100$

Graphical Analysis **In Exercises 51–54, (a) use a graphing utility to graph the function, (b) use the graph to approximate any *x*-intercepts of the graph, (c) set $y = 0$ and solve the resulting equation, and (d) compare the results of part (c) with any *x*-intercepts of the graph.**

51. $y = 4x^3 - 20x^2 + 25x$

52. $y = 4x^3 + 4x^2 - 8x - 8$

53. $y = x^5 - 5x^3 + 4x$

54. $y = \frac{1}{4}x^3(x^2 - 9)$

Finding a Polynomial Function **In Exercises 55–64, find a polynomial function that has the given zeros. (There are many correct answers.)**

55. 0, 8

56. $0, -7$

57. $2, -6$

58. $-4, 5$

59. $0, -4, -5$

60. 0, 1, 10

61. $4, -3, 3, 0$

62. $-2, -1, 0, 1, 2$

63. $1 + \sqrt{3}, 1 - \sqrt{3}$

64. $2, 4 + \sqrt{5}, 4 - \sqrt{5}$

Finding a Polynomial Function **In Exercises 65–74, find a polynomial of degree *n* that has the given zero(s). (There are many correct answers.)**

	Zero(s)	**Degree**
65.	$x = -3$	$n = 2$
66.	$x = -12, -6$	$n = 2$
67.	$x = -5, 0, 1$	$n = 3$
68.	$x = -2, 4, 7$	$n = 3$
69.	$x = 0, \sqrt{3}, -\sqrt{3}$	$n = 3$
70.	$x = 9$	$n = 3$
71.	$x = -5, 1, 2$	$n = 4$
72.	$x = -4, -1, 3, 6$	$n = 4$
73.	$x = 0, -4$	$n = 5$
74.	$x = -1, 4, 7, 8$	$n = 5$

Sketching the Graph of a Polynomial Function **In Exercises 75–88, sketch the graph of the function by (a) applying the Leading Coefficient Test, (b) finding the real zeros of the polynomial, (c) plotting sufficient solution points, and (d) drawing a continuous curve through the points.**

75. $f(x) = x^3 - 25x$

76. $g(x) = x^4 - 9x^2$

77. $f(t) = \frac{1}{4}(t^2 - 2t + 15)$

78. $g(x) = -x^2 + 10x - 16$

79. $f(x) = x^3 - 2x^2$

80. $f(x) = 8 - x^3$

81. $f(x) = 3x^3 - 15x^2 + 18x$

82. $f(x) = -4x^3 + 4x^2 + 15x$

83. $f(x) = -5x^2 - x^3$

84. $f(x) = -48x^2 + 3x^4$

85. $f(x) = x^2(x - 4)$

86. $h(x) = \frac{1}{3}x^3(x - 4)^2$

87. $g(t) = -\frac{1}{4}(t - 2)^2(t + 2)^2$

88. $g(x) = \frac{1}{10}(x + 1)^2(x - 3)^3$

Graphical Analysis **In Exercises 89–92, use a graphing utility to graph the function. Use the *zero* or *root* feature to approximate the real zeros of the function. Then determine the multiplicity of each zero.**

89. $f(x) = x^3 - 16x$

90. $f(x) = \frac{1}{4}x^4 - 2x^2$

91. $g(x) = \frac{1}{5}(x + 1)^2(x - 3)(2x - 9)$

92. $h(x) = \frac{1}{5}(x + 2)^2(3x - 5)^2$

Using the Intermediate Value Theorem In Exercises 93–96, (a) use the Intermediate Value Theorem and the *table* feature of a graphing utility to find intervals one unit in length in which the polynomial function is guaranteed to have a zero. (b) Adjust the table to approximate the zeros of the function. Use the *zero* or *root* feature of the graphing utility to verify your results.

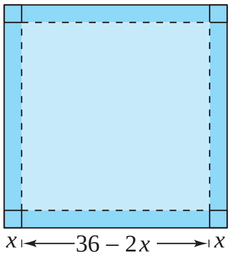
93. $f(x) = x^3 - 3x^2 + 3$

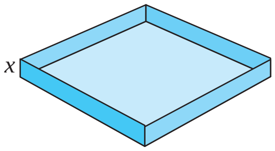
94. $f(x) = 0.11x^3 - 2.07x^2 + 9.81x - 6.88$

95. $g(x) = 3x^4 + 4x^3 - 3$

96. $h(x) = x^4 - 10x^2 + 3$

97. **Numerical and Graphical Analysis** You plan to construct an open box from a square piece of material, 36 inches on a side, by cutting equal squares with sides of length x from the corners and turning up the sides (see figure).





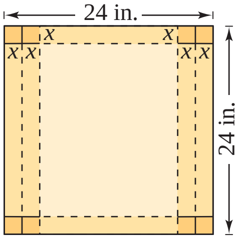
(a) Write a function V that represents the volume of the box.

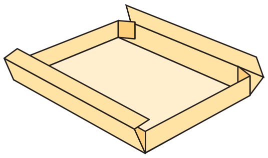
(b) Determine the domain of the function V.

(c) Use a graphing utility to construct a table that shows the box heights x and the corresponding volumes $V(x)$. Use the table to estimate the dimensions that will produce a maximum volume.

(d) Use the graphing utility to graph V and use the graph to estimate the value of x for which $V(x)$ is a maximum. Compare your result with that of part (c).

98. **Maximum Volume** You plan to construct an open box with locking tabs from a square piece of material 24 inches on a side by cutting equal squares from the corners and folding along the dashed lines (see figure).



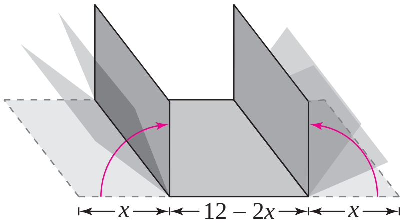


(a) Write a function V that represents the volume of the box.

(b) Determine the domain of the function V.

(c) Sketch a graph of the function and estimate the value of x for which $V(x)$ is a maximum.

99. **Construction** A roofing contractor is fabricating gutters from 12-inch aluminum sheeting. The contractor plans to use an aluminum siding folding press to create the gutter by creasing equal lengths for the sidewalls (see figure).



(a) Let x represent the height of the sidewall of the gutter. Write a function A that represents the cross-sectional area of the gutter.

(b) The length of the aluminum sheeting is 16 feet. Write a function V that represents the volume of one run of gutter in terms of x.

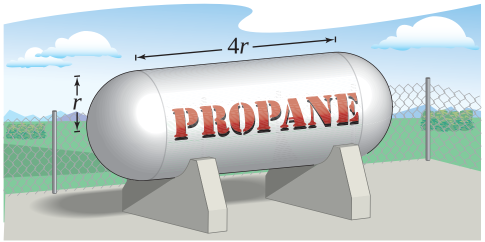
(c) Determine the domain of the function V.

(d) Use a graphing utility to construct a table that shows the sidewall heights x and the corresponding volumes $V(x)$. Use the table to estimate the dimensions that will produce a maximum volume.

(e) Use the graphing utility to graph V. Use the graph to estimate the value of x for which $V(x)$ is a maximum. Compare your result with that of part (d).

(f) Would changing the length of the aluminum sheet affect the value of x for which $V(x)$ is a maximum? Explain.

100. **Construction** An industrial propane tank is formed by adjoining two hemispheres to the ends of a right circular cylinder. The length of the cylindrical portion of the tank is four times the radius of the hemispherical components (see figure).



(a) Write a function V that represents the total volume of the tank in terms of r.

(b) Find the domain of the function V.

(c) Use a graphing utility to graph the function.

(d) Use the graph from part (c) to estimate the radius and length of the cylindrical portion of the tank when the total volume of the tank is 120 cubic feet.

101. Revenue The revenues R (in millions of dollars) for a company from 2003 through 2010 can be modeled by

$$R = 6.212t^3 - 132.87t^2 + 863.2t - 1115,\ 3 \le t \le 10$$

where t represents the year, with $t = 3$ corresponding to 2003.

(a) Use a graphing utility to approximate any relative extrema of the model over its domain.

(b) Use the graphing utility to approximate the intervals on which the revenue for the company is increasing and decreasing over its domain.

(c) Use the results of parts (a) and (b) to describe the company's revenue during this time period.

102. Revenue The revenues R (in millions of dollars) for a company from 2003 through 2010 can be modeled by

$$R = -0.1685t^4 + 4.298t^3 - 39.044t^2 + 149.9t - 185,$$

$$3 \le t \le 10$$

where t represents the year, with $t = 3$ corresponding to 2003.

(a) Use a graphing utility to approximate any relative extrema of the model over its domain.

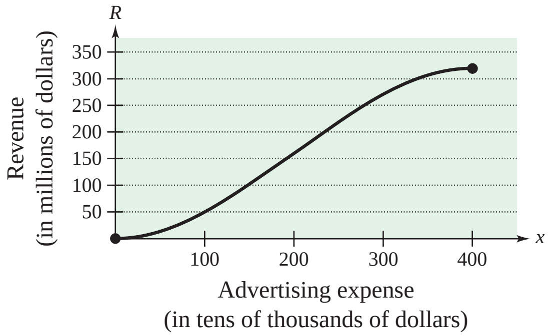
(b) Use the graphing utility to approximate the intervals on which the revenue for the company is increasing and decreasing over its domain.

(c) Use the results of parts (a) and (b) to describe the company's revenue during this time period.

103. Revenue The revenue R (in millions of dollars) for a company is related to its advertising expense by the function

$$R = \frac{1}{100{,}000}(-x^3 + 600x^2), \quad 0 \le x \le 400$$

where x is the amount spent on advertising (in tens of thousands of dollars). Use the graph of this function to estimate the point on the graph at which the function is increasing most rapidly. This point is called the *point of diminishing returns* because any expense above this amount will yield less return per dollar invested in advertising.



Artistas/Shutterstock.com

104. Tree Growth

The growth of a red oak tree is approximated by the function

$$G = -0.003t^3 + 0.137t^2 + 0.458t - 0.839,$$

$$2 \le t \le 34$$

where G is the height of the tree (in feet) and t is its age (in years).

(a) Use a graphing utility to graph the function.

(b) Estimate the age of the tree when it is growing most rapidly. This point is called the *point of diminishing returns* because the increase in size will be less with each additional year.

(c) Using calculus, the point of diminishing returns can be found by finding the vertex of the parabola

$$y = -0.009t^2 + 0.274t + 0.458.$$

Find the vertex of this parabola.

(d) Compare your results from parts (b) and (c).

Exploration

True or False? **In Exercises 105–111, determine whether the statement is true or false. Justify your answer.**

105. A fifth-degree polynomial function can have five turning points in its graph.

106. If f is a polynomial function of x such that $f(2) = -6$ and $f(6) = 6$, then f has at most one real zero between $x = 2$ and $x = 6$.

107. A polynomial function cannot have more real zeros than it has turning points.

108. If the graph of a polynomial function falls to the right, then its leading coefficient is negative.

109. If the graph of a polynomial function rises to the left, then its leading coefficient is positive.

110. The graph of the function

$$f(x) = x^4 - 6x^3 - 3x - 8$$

falls to the left and to the right.

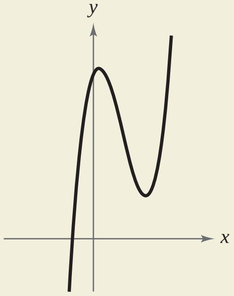
111. The graph of the function

$$f(x) = 2 + x - x^2 + x^3 - x^4 + x^5 + x^6 - x^7$$

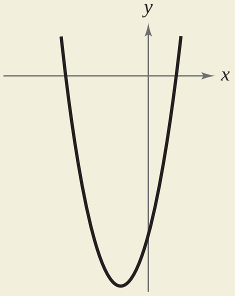
rises to the left and falls to the right.

112. HOW DO YOU SEE IT? For each graph, describe a polynomial function that could represent the graph. (Indicate the degree of the function and the sign of its leading coefficient.)

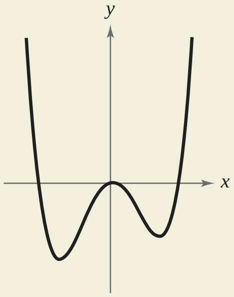
(a)



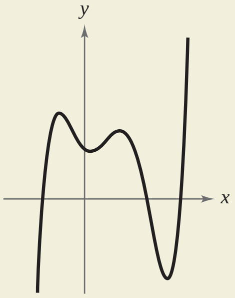
(b)



(c)



(d)



113. Modeling Polynomials Sketch the graph of a polynomial function that is of fourth degree, has a zero of multiplicity 2, and has a negative leading coefficient. Sketch the graph of another polynomial function with the same characteristics except that the leading coefficient is positive.

114. Modeling Polynomials Sketch the graph of a polynomial function that is of fifth degree, has a zero of multiplicity 2, and has a negative leading coefficient. Sketch the graph of another polynomial function with the same characteristics except that the leading coefficient is positive.

115. Graphical Reasoning Sketch a graph of the function $f(x) = x^4$. Explain how the graph of each function g differs (if it does) from the graph of f. Determine whether g is even, odd, or neither.

(a) $g(x) = f(x) + 2$

(b) $g(x) = f(x + 2)$

(c) $g(x) = f(-x)$

(d) $g(x) = -f(x)$

(e) $g(x) = f\left(\frac{1}{2}x\right)$

(f) $g(x) = \frac{1}{2}f(x)$

(g) $g(x) = f\left(x^{3/4}\right)$

(h) $g(x) = (f \circ f)(x)$

116. Think About It For each function, identify the degree of the function and whether the degree of the function is even or odd. Identify the leading coefficient and whether the leading coefficient is positive or negative. Use a graphing utility to graph each function. Describe the relationship between the degree of the function and the sign of the leading coefficient of the function and the right-hand and left-hand behavior of the graph of the function.

(a) $f(x) = x^3 - 2x^2 - x + 1$

(b) $f(x) = 2x^5 + 2x^2 - 5x + 1$

(c) $f(x) = -2x^5 - x^2 + 5x + 3$

(d) $f(x) = -x^3 + 5x - 2$

(e) $f(x) = 2x^2 + 3x - 4$

(f) $f(x) = x^4 - 3x^2 + 2x - 1$

(g) $f(x) = x^2 + 3x + 2$

117. Think About It Sketch the graph of each polynomial function. Then count the number of real zeros of the function and the numbers of relative minima and relative maxima. Compare these numbers with the degree of the polynomial. What do you observe?

(a) $f(x) = -x^3 + 9x$

(b) $f(x) = x^4 - 10x^2 + 9$

(c) $f(x) = x^5 - 16x$

118. Think About It Explore transformations of the form

$$g(x) = a(x - h)^5 + k.$$

(a) Use a graphing utility to graph the functions

$$y_1 = -\tfrac{1}{3}(x - 2)^5 + 1$$

and

$$y_2 = \tfrac{3}{5}(x + 2)^5 - 3.$$

Determine whether the graphs are increasing or decreasing. Explain.

(b) Will the graph of g always be increasing or decreasing? If so, then is this behavior determined by a, h, or k? Explain.

(c) Use the graphing utility to graph the function

$$H(x) = x^5 - 3x^3 + 2x + 1.$$

Use the graph and the result of part (b) to determine whether H can be written in the form

$$H(x) = a(x - h)^5 + k.$$

Explain.

3.3 Polynomial and Synthetic Division

- **Use long division to divide polynomials by other polynomials.**
- **Use synthetic division to divide polynomials by binomials of the form $(x - k)$.**
- **Use the Remainder Theorem and the Factor Theorem.**

Synthetic division can help you evaluate polynomial functions. For instance, in Exercise 86 on page 274, you will use synthetic division to evaluate a model for the number of confirmed cases of Lyme disease in Maine.

Long Division of Polynomials

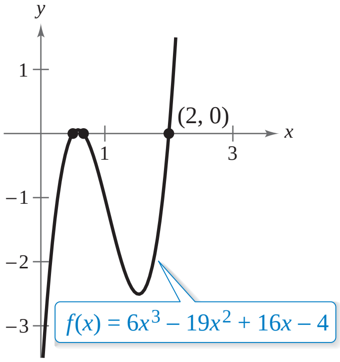
Suppose you are given the graph of

$$f(x) = 6x^3 - 19x^2 + 16x - 4.$$

Notice that a zero of f occurs at $x = 2$, as shown at the right. Because $x = 2$ is a zero of f, you know that $(x - 2)$ is a factor of $f(x)$. This means that there exists a second-degree polynomial $q(x)$ such that

$$f(x) = (x - 2) \cdot q(x).$$

To find $q(x)$, you can use **long division,** as illustrated in Example 1.



EXAMPLE 1 Long Division of Polynomials

Divide the polynomial $6x^3 - 19x^2 + 16x - 4$ by $x - 2$, and use the result to factor the polynomial completely.

Solution

Think $\frac{6x^3}{x} = 6x^2$.

Think $\frac{-7x^2}{x} = -7x$.

Think $\frac{2x}{x} = 2$.

$$\begin{array}{rl}
6x^2 - 7x + 2 & \\
x - 2\,\overline{)\,6x^3 - 19x^2 + 16x - 4} & \\
\underline{6x^3 - 12x^2} & \text{Multiply: } 6x^2(x - 2). \\
-7x^2 + 16x & \text{Subtract.} \\
\underline{-7x^2 + 14x} & \text{Multiply: } -7x(x - 2). \\
2x - 4 & \text{Subtract.} \\
\underline{2x - 4} & \text{Multiply: } 2(x - 2). \\
0 & \text{Subtract.}
\end{array}$$

From this division, you can conclude that

$$6x^3 - 19x^2 + 16x - 4 = (x - 2)(6x^2 - 7x + 2)$$

and by factoring the quadratic $6x^2 - 7x + 2$, you have

$$6x^3 - 19x^2 + 16x - 4 = (x - 2)(2x - 1)(3x - 2).$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Divide the polynomial $3x^2 + 19x + 28$ by $x + 4$, and use the result to factor the polynomial completely.

Dariusz Majgier/Shutterstock.com

In Example 1, $x - 2$ is a factor of the polynomial

$6x^3 - 19x^2 + 16x - 4$

and the long division process produces a remainder of zero. Often, long division will produce a nonzero remainder. For instance, when you divide $x^2 + 3x + 5$ by $x + 1$, you obtain the following.

$$\begin{array}{rrl}
 & x + 2 & \longleftarrow \text{Quotient} \\
\text{Divisor} \longrightarrow \quad x + 1 & \overline{)\,x^2 + 3x + 5} & \longleftarrow \text{Dividend} \\
 & \underline{x^2 + x\phantom{{}+5}} & \\
 & 2x + 5 & \\
 & \underline{2x + 2} & \\
 & 3 & \longleftarrow \text{Remainder}
\end{array}$$

In fractional form, you can write this result as follows.

$$\frac{\overbrace{x^2 + 3x + 5}^{\text{Dividend}}}{\underbrace{x + 1}_{\text{Divisor}}} = \overbrace{x + 2}^{\text{Quotient}} + \frac{\overset{\substack{\text{Remainder}\\ \downarrow}}{3}}{\underbrace{x + 1}_{\text{Divisor}}}$$

This implies that

$$x^2 + 3x + 5 = (x + 1)(x + 2) + 3 \qquad \text{Multiply each side by } (x + 1).$$

which illustrates the following theorem, called the **Division Algorithm.**

The Division Algorithm

If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$\underset{\substack{\uparrow\\ \text{Dividend}}}{f(x)} = \underset{\substack{\uparrow\\ \text{Divisor}}}{d(x)}\underset{\substack{\uparrow\\ \text{Quotient}}}{q(x)} + \underset{\substack{\uparrow\\ \text{Remainder}}}{r(x)}$$

where $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$. If the remainder $r(x)$ is zero, $d(x)$ *divides evenly* into $f(x)$.

The Division Algorithm can also be written as

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

In the Division Algorithm, the rational expression $f(x)/d(x)$ is **improper** because the degree of $f(x)$ is greater than or equal to the degree of $d(x)$. On the other hand, the rational expression $r(x)/d(x)$ is **proper** because the degree of $r(x)$ is less than the degree of $d(x)$.

Before you apply the Division Algorithm, follow these steps.

1. Write the dividend and divisor in descending powers of the variable.

2. Insert placeholders with zero coefficients for missing powers of the variable.

Note how these steps are applied in the next example.

EXAMPLE 2 Long Division of Polynomials

Divide $x^3 - 1$ by $x - 1$.

Solution Because there is no x^2-term or x-term in the dividend $x^3 - 1$, you need to rewrite the dividend as $x^3 + 0x^2 + 0x - 1$ before you apply the Division Algorithm.

$$\begin{array}{rrl}
 & x^2 + x + 1 & \\
x - 1 \big) & \overline{x^3 + 0x^2 + 0x - 1} & \\
 & \underline{x^3 - x^2} & \text{Multiply } x^2 \text{ by } x - 1. \\
 & x^2 + 0x & \text{Subtract and bring down } 0x. \\
 & \underline{x^2 - x} & \text{Multiply } x \text{ by } x - 1. \\
 & x - 1 & \text{Subtract and bring down } -1. \\
 & \underline{x - 1} & \text{Multiply 1 by } x - 1. \\
 & 0 & \text{Subtract.}
\end{array}$$

So, $x - 1$ divides evenly into $x^3 - 1$, and you can write

$$\frac{x^3 - 1}{x - 1} = x^2 + x + 1, \quad x \neq 1.$$

You can check this result by multiplying.

$$\begin{aligned}
(x - 1)(x^2 + x + 1) &= x^3 + x^2 + x - x^2 - x - 1 \\
&= x^3 - 1
\end{aligned}$$

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Divide $x^3 - 2x^2 - 9$ by $x - 3$.

Note that the condition $x \neq 1$ in Example 1 is equivalent to the condition in the Division Algorithm that $d(x) \neq 0$. Statement of this condition will be omitted for the remainder of this section, but you should be aware that the results of polynomial division are only valid when the divisor does not equal zero.

EXAMPLE 3 Long Division of Polynomials

Divide $-5x^2 - 2 + 3x + 2x^4 + 4x^3$ by $2x - 3 + x^2$.

Solution Begin by writing the dividend and divisor in descending powers of x.

$$\begin{array}{rrl}
 & 2x^2 + 1 & \\
x^2 + 2x - 3 \big) & \overline{2x^4 + 4x^3 - 5x^2 + 3x - 2} & \\
 & \underline{2x^4 + 4x^3 - 6x^2} & \text{Multiply } 2x^2 \text{ by } x^2 + 2x - 3. \\
 & x^2 + 3x - 2 & \text{Subtract and bring down } 3x - 2. \\
 & \underline{x^2 + 2x - 3} & \text{Multiply 1 by } x^2 + 2x - 3. \\
 & x + 1 & \text{Subtract.}
\end{array}$$

Note that the first subtraction eliminated two terms from the dividend. When this happens, the quotient skips a term. You can write the result as

$$\frac{2x^4 + 4x^3 - 5x^2 + 3x - 2}{x^2 + 2x - 3} = 2x^2 + 1 + \frac{x + 1}{x^2 + 2x - 3}.$$

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

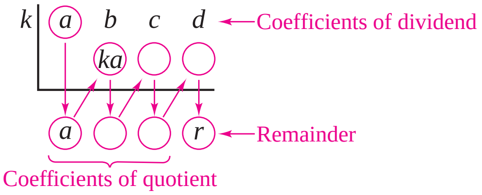
Divide $-x^3 + 9x + 6x^4 - x^2 - 3$ by $1 + 3x$.

Synthetic Division

There is a nice shortcut for long division of polynomials by divisors of the form $x - k$. This shortcut is called **synthetic division.** The pattern for synthetic division of a cubic polynomial is summarized as follows. (The pattern for higher-degree polynomials is similar.)

Synthetic Division (for a Cubic Polynomial)

To divide $ax^3 + bx^2 + cx + d$ by $x - k$, use the following pattern.



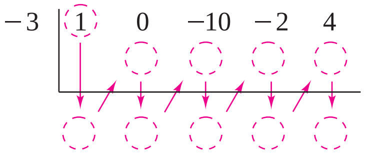
Vertical pattern: Add terms in columns.
Diagonal pattern: Multiply results by k.

This algorithm for synthetic division works only for divisors of the form $x - k$. Remember that $x + k = x - (-k)$.

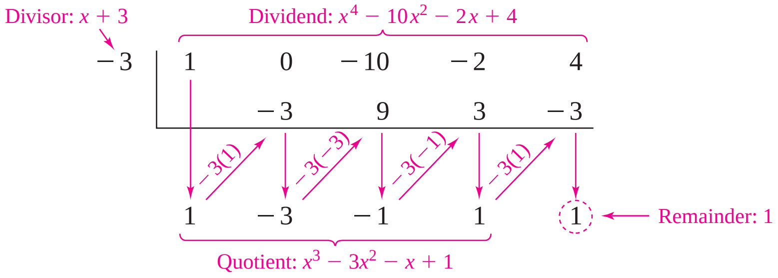
EXAMPLE 4 Using Synthetic Division

Use synthetic division to divide $x^4 - 10x^2 - 2x + 4$ by $x + 3$.

Solution You should set up the array as follows. Note that a zero is included for the missing x^3-term in the dividend.



Then, use the synthetic division pattern by adding terms in columns and multiplying the results by -3.



So, you have

$$\frac{x^4 - 10x^2 - 2x + 4}{x + 3} = x^3 - 3x^2 - x + 1 + \frac{1}{x + 3}.$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use synthetic division to divide $5x^3 + 8x^2 - x + 6$ by $x + 2$.

The Remainder and Factor Theorems

The remainder obtained in the synthetic division process has an important interpretation, as described in the **Remainder Theorem.**

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is

$$r = f(k).$$

For a proof of the Remainder Theorem, see Proofs in Mathematics on page 307.

The Remainder Theorem tells you that synthetic division can be used to evaluate a polynomial function. That is, to evaluate a polynomial function $f(x)$ when $x = k$, divide $f(x)$ by $x - k$. The remainder will be $f(k)$, as illustrated in Example 5.

EXAMPLE 5 Using the Remainder Theorem

Use the Remainder Theorem to evaluate

$$f(x) = 3x^3 + 8x^2 + 5x - 7$$

when $x = -2$.

Solution Using synthetic division, you obtain the following.

$$\begin{array}{r|rrrr} -2 & 3 & 8 & 5 & -7 \\ & & -6 & -4 & -2 \\ \hline & 3 & 2 & 1 & -9 \end{array}$$

Because the remainder is $r = -9$, you can conclude that

$$f(-2) = -9. \qquad r = f(k)$$

This means that $(-2, -9)$ is a point on the graph of f. You can check this by substituting $x = -2$ in the original function.

Check

$$\begin{aligned} f(-2) &= 3(-2)^3 + 8(-2)^2 + 5(-2) - 7 \\ &= 3(-8) + 8(4) - 10 - 7 \\ &= -24 + 32 - 10 - 7 \\ &= -9 \end{aligned}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use the Remainder Theorem to find each function value given

$$f(x) = 4x^3 + 10x^2 - 3x - 8.$$

a. $f(-1)$ **b.** $f(4)$

c. $f\left(\frac{1}{2}\right)$ **d.** $f(-3)$

TECHNOLOGY You can evaluate a function with your graphing utility by entering the function in the equation editor and using the *table* feature in *ask* mode. When the table is set to *ask* mode, you can enter values in the X column and the corresponding function values will be displayed in the function column.

Another important theorem is the **Factor Theorem,** which is stated below.

The Factor Theorem

A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.

For a proof of the Factor Theorem, see Proofs in Mathematics on page 307.

Using the Factor Theorem, you can test whether a polynomial has $(x - k)$ as a factor by evaluating the polynomial at $x = k$. If the result is 0, then $(x - k)$ is a factor.

EXAMPLE 6 Factoring a Polynomial: Repeated Division

Show that $(x - 2)$ and $(x + 3)$ are factors of $f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18$. Then find the remaining factors of $f(x)$.

Algebraic Solution

Using synthetic division with the factor $(x - 2)$, you obtain the following.

$$\begin{array}{r|rrrrr} 2 & 2 & 7 & -4 & -27 & -18 \\ & & 4 & 22 & 36 & 18 \\ \hline & 2 & 11 & 18 & 9 & 0 \longrightarrow \end{array}$$

0 remainder, so $f(2) = 0$ and $(x - 2)$ is a factor.

Take the result of this division and perform synthetic division again using the factor $(x + 3)$.

$$\begin{array}{r|rrrr} -3 & 2 & 11 & 18 & 9 \\ & & -6 & -15 & -9 \\ \hline & \multicolumn{3}{c}{\underbrace{2 \qquad 5 \qquad 3}_{2x^2 + 5x + 3}} & 0 \longrightarrow \end{array}$$

0 remainder, so $f(-3) = 0$ and $(x + 3)$ is a factor.

Because the resulting quadratic expression factors as

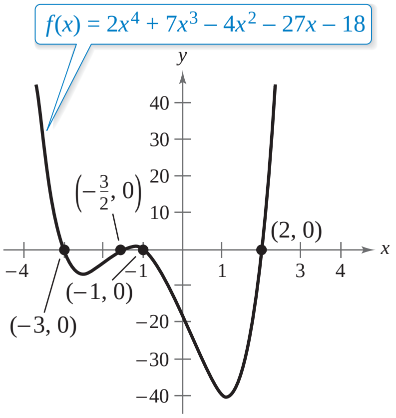
$$2x^2 + 5x + 3 = (2x + 3)(x + 1)$$

the complete factorization of $f(x)$ is

$$f(x) = (x - 2)(x + 3)(2x + 3)(x + 1).$$

Graphical Solution

From the graph of $f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18$, you can see that there are four x-intercepts (see figure below). These occur at $x = -3$, $x = -\frac{3}{2}$, $x = -1$, and $x = 2$. (Check this algebraically.) This implies that $(x + 3)$, $\left(x + \frac{3}{2}\right)$, $(x + 1)$, and $(x - 2)$ are factors of $f(x)$. $\left[\text{Note that } \left(x + \frac{3}{2}\right) \text{ and } (2x + 3) \text{ are equivalent factors because they both yield the same zero, } x = -\frac{3}{2}.\right]$



✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Show that $(x + 3)$ is a factor of $f(x) = x^3 - 19x - 30$. Then find the remaining factors of $f(x)$.

Summarize (Section 3.3)

1. Describe how to divide one polynomial by another polynomial using long division *(pages 266 and 267)*. For examples of long division of polynomials, see Examples 1–3.
2. Describe the algorithm for synthetic division *(page 269)*. For an example of using synthetic division, see Example 4.
3. State the Remainder Theorem and the Factor Theorem *(pages 270 and 271)*. For an example of using the Remainder Theorem, see Example 5. For an example of using the Factor Theorem, see Example 6.

3.3 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary

1. Two forms of the Division Algorithm are shown below. Identify and label each term or function.

$$f(x) = d(x)q(x) + r(x) \qquad \frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

In Exercises 2–6, fill in the blanks.

2. In the Division Algorithm, the rational expression $r(x)/d(x)$ is ________ because the degree of $r(x)$ is less than the degree of $d(x)$.

3. In the Division Algorithm, the rational expression $f(x)/d(x)$ is ________ because the degree of $f(x)$ is greater than or equal to the degree of $d(x)$.

4. An alternative method to long division of polynomials is called ________ ________, in which the divisor must be of the form $x - k$.

5. The ________ Theorem states that a polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.

6. The ________ Theorem states that if a polynomial $f(x)$ is divided by $x - k$, then the remainder is $r = f(k)$.

Skills and Applications

Analytical Analysis

In Exercises 7 and 8, use long division to verify that $y_1 = y_2$.

7. $y_1 = \dfrac{x^2}{x + 2}, \quad y_2 = x - 2 + \dfrac{4}{x + 2}$

8. $y_1 = \dfrac{x^4 - 3x^2 - 1}{x^2 + 5}, \quad y_2 = x^2 - 8 + \dfrac{39}{x^2 + 5}$

Graphical Analysis

In Exercises 9 and 10, (a) use a graphing utility to graph the two equations in the same viewing window, (b) use the graphs to verify that the expressions are equivalent, and (c) use long division to verify the results algebraically.

9. $y_1 = \dfrac{x^2 + 2x - 1}{x + 3}, \quad y_2 = x - 1 + \dfrac{2}{x + 3}$

10. $y_1 = \dfrac{x^4 + x^2 - 1}{x^2 + 1}, \quad y_2 = x^2 - \dfrac{1}{x^2 + 1}$

Long Division of Polynomials

In Exercises 11–26, use long division to divide.

11. $(2x^2 + 10x + 12) \div (x + 3)$

12. $(5x^2 - 17x - 12) \div (x - 4)$

13. $(4x^3 - 7x^2 - 11x + 5) \div (4x + 5)$

14. $(6x^3 - 16x^2 + 17x - 6) \div (3x - 2)$

15. $(x^4 + 5x^3 + 6x^2 - x - 2) \div (x + 2)$

16. $(x^3 + 4x^2 - 3x - 12) \div (x - 3)$

17. $(x^3 - 27) \div (x - 3)$

18. $(x^3 + 125) \div (x + 5)$

19. $(7x + 3) \div (x + 2)$

20. $(8x - 5) \div (2x + 1)$

21. $(x^3 - 9) \div (x^2 + 1)$

22. $(x^5 + 7) \div (x^3 - 1)$

23. $(3x + 2x^3 - 9 - 8x^2) \div (x^2 + 1)$

24. $(5x^3 - 16 - 20x + x^4) \div (x^2 - x - 3)$

25. $\dfrac{x^4}{(x - 1)^3}$

26. $\dfrac{2x^3 - 4x^2 - 15x + 5}{(x - 1)^2}$

Using Synthetic Division

In Exercises 27–46, use synthetic division to divide.

27. $(3x^3 - 17x^2 + 15x - 25) \div (x - 5)$

28. $(5x^3 + 18x^2 + 7x - 6) \div (x + 3)$

29. $(6x^3 + 7x^2 - x + 26) \div (x - 3)$

30. $(2x^3 + 14x^2 - 20x + 7) \div (x + 6)$

31. $(4x^3 - 9x + 8x^2 - 18) \div (x + 2)$

32. $(9x^3 - 16x - 18x^2 + 32) \div (x - 2)$

33. $(-x^3 + 75x - 250) \div (x + 10)$

34. $(3x^3 - 16x^2 - 72) \div (x - 6)$

35. $(5x^3 - 6x^2 + 8) \div (x - 4)$

36. $(5x^3 + 6x + 8) \div (x + 2)$

37. $\dfrac{10x^4 - 50x^3 - 800}{x - 6}$

38. $\dfrac{x^5 - 13x^4 - 120x + 80}{x + 3}$

39. $\dfrac{x^3 + 512}{x + 8}$

40. $\dfrac{x^3 - 729}{x - 9}$

41. $\dfrac{-3x^4}{x - 2}$

42. $\dfrac{-3x^4}{x + 2}$

43. $\dfrac{180x - x^4}{x - 6}$

44. $\dfrac{5 - 3x + 2x^2 - x^3}{x + 1}$

45. $\dfrac{4x^3 + 16x^2 - 23x - 15}{x + \frac{1}{2}}$

46. $\dfrac{3x^3 - 4x^2 + 5}{x - \frac{3}{2}}$

Using the Remainder Theorem **In Exercises 47–54, write the function in the form $f(x) = (x - k)q(x) + r$ for the given value of k, and demonstrate that $f(k) = r$.**

47. $f(x) = x^3 - x^2 - 14x + 11, \quad k = 4$

48. $f(x) = x^3 - 5x^2 - 11x + 8, \quad k = -2$

49. $f(x) = 15x^4 + 10x^3 - 6x^2 + 14, \quad k = -\frac{2}{3}$

50. $f(x) = 10x^3 - 22x^2 - 3x + 4, \quad k = \frac{1}{5}$

51. $f(x) = x^3 + 3x^2 - 2x - 14, \quad k = \sqrt{2}$

52. $f(x) = x^3 + 2x^2 - 5x - 4, \quad k = -\sqrt{5}$

53. $f(x) = -4x^3 + 6x^2 + 12x + 4, \quad k = 1 - \sqrt{3}$

54. $f(x) = -3x^3 + 8x^2 + 10x - 8, \quad k = 2 + \sqrt{2}$

Using the Remainder Theorem **In Exercises 55–58, use the Remainder Theorem and synthetic division to find each function value. Verify your answers using another method.**

55. $f(x) = 2x^3 - 7x + 3$

(a) $f(1)$ (b) $f(-2)$ (c) $f\left(\frac{1}{2}\right)$ (d) $f(2)$

56. $g(x) = 2x^6 + 3x^4 - x^2 + 3$

(a) $g(2)$ (b) $g(1)$ (c) $g(3)$ (d) $g(-1)$

57. $h(x) = x^3 - 5x^2 - 7x + 4$

(a) $h(3)$ (b) $h(2)$ (c) $h(-2)$ (d) $h(-5)$

58. $f(x) = 4x^4 - 16x^3 + 7x^2 + 20$

(a) $f(1)$ (b) $f(-2)$ (c) $f(5)$ (d) $f(-10)$

Using the Factor Theorem **In Exercises 59–66, use synthetic division to show that x is a solution of the third-degree polynomial equation, and use the result to factor the polynomial completely. List all real solutions of the equation.**

59. $x^3 - 7x + 6 = 0, \quad x = 2$

60. $x^3 - 28x - 48 = 0, \quad x = -4$

61. $2x^3 - 15x^2 + 27x - 10 = 0, \quad x = \frac{1}{2}$

62. $48x^3 - 80x^2 + 41x - 6 = 0, \quad x = \frac{2}{3}$

63. $x^3 + 2x^2 - 3x - 6 = 0, \quad x = \sqrt{3}$

64. $x^3 + 2x^2 - 2x - 4 = 0, \quad x = \sqrt{2}$

65. $x^3 - 3x^2 + 2 = 0, \quad x = 1 + \sqrt{3}$

66. $x^3 - x^2 - 13x - 3 = 0, \quad x = 2 - \sqrt{5}$

Factoring a Polynomial **In Exercises 67–74, (a) verify the given factors of $f(x)$, (b) find the remaining factor(s) of $f(x)$, (c) use your results to write the complete factorization of $f(x)$, (d) list all real zeros of f, and (e) confirm your results by using a graphing utility to graph the function.**

	Function	**Factors**
67.	$f(x) = 2x^3 + x^2 - 5x + 2$	$(x + 2), (x - 1)$
68.	$f(x) = 3x^3 + 2x^2 - 19x + 6$	$(x + 3), (x - 2)$
69.	$f(x) = x^4 - 4x^3 - 15x^2 + 58x - 40$	$(x - 5), (x + 4)$
70.	$f(x) = 8x^4 - 14x^3 - 71x^2 - 10x + 24$	$(x + 2), (x - 4)$
71.	$f(x) = 6x^3 + 41x^2 - 9x - 14$	$(2x + 1), (3x - 2)$
72.	$f(x) = 10x^3 - 11x^2 - 72x + 45$	$(2x + 5), (5x - 3)$
73.	$f(x) = 2x^3 - x^2 - 10x + 5$	$(2x - 1), \left(x + \sqrt{5}\right)$
74.	$f(x) = x^3 + 3x^2 - 48x - 144$	$\left(x + 4\sqrt{3}\right), (x + 3)$

Graphical Analysis **In Exercises 75–80, (a) use the *zero* or *root* feature of a graphing utility to approximate the zeros of the function accurate to three decimal places, (b) determine the exact value of one of the zeros, and (c) use synthetic division to verify your result from part (b), and then factor the polynomial completely.**

75. $f(x) = x^3 - 2x^2 - 5x + 10$

76. $g(x) = x^3 - 4x^2 - 2x + 8$

77. $h(t) = t^3 - 2t^2 - 7t + 2$

78. $f(s) = s^3 - 12s^2 + 40s - 24$

79. $h(x) = x^5 - 7x^4 + 10x^3 + 14x^2 - 24x$

80. $g(x) = 6x^4 - 11x^3 - 51x^2 + 99x - 27$

Simplifying Rational Expressions **In Exercises 81–84, simplify the rational expression by using long division or synthetic division.**

81. $\dfrac{4x^3 - 8x^2 + x + 3}{2x - 3}$

82. $\dfrac{x^3 + x^2 - 64x - 64}{x + 8}$

83. $\dfrac{x^4 + 6x^3 + 11x^2 + 6x}{x^2 + 3x + 2}$

84. $\dfrac{x^4 + 9x^3 - 5x^2 - 36x + 4}{x^2 - 4}$

85. Profit A company that produces calculators estimated that the profit P (in dollars) from selling a particular model of calculator was

$$P = -152x^3 + 7545x^2 - 169{,}625, \quad 0 \le x \le 45$$

where x was the advertising expense (in tens of thousands of dollars). For this model of calculator, the advertising expense was \$400,000 ($x = 40$) and the profit was \$2,174,375.

(a) Use a graphing utility to graph the profit function.

(b) Use the graph from part (a) to estimate another amount the company could have spent on advertising that would have produced the same profit.

(c) Use synthetic division to confirm the result of part (b) algebraically.

86. Data Analysis: Lyme Disease

The numbers N of confirmed cases of Lyme disease in Maine from 2003 to 2010 are shown in the table, where t represents the year, with $t = 3$ corresponding to 2003. *(Source: Centers for Disease Control and Prevention)*

Year, t	Number, N
3	175
4	225
5	247
6	338
7	529
8	780
9	791
10	559

Spreadsheet at LarsonPrecalculus.com

(a) Use a graphing utility to create a scatter plot of the data.

(b) Use the *regression* feature of the graphing utility to find a quartic model for the data. Graph the model in the same viewing window as the scatter plot.

(c) Use the model to create a table of estimated values of N. Compare the model with the original data.

(d) Use synthetic division to confirm algebraically your estimated value for the year 2010.

Exploration

True or False? **In Exercises 87–90, determine whether the statement is true or false. Justify your answer.**

87. If $(7x + 4)$ is a factor of some polynomial function $f(x)$, then $\frac{4}{7}$ is a zero of f.

88. $(2x - 1)$ is a factor of the polynomial

$$6x^6 + x^5 - 92x^4 + 45x^3 + 184x^2 + 4x - 48.$$

89. The rational expression

$$\frac{x^3 + 2x^2 - 13x + 10}{x^2 - 4x - 12}$$

is improper.

90. The equation

$$\frac{x^3 - 3x^2 + 4}{x + 1} = x^2 - 4x + 4$$

is true for all values of x.

Think About It **In Exercises 91 and 92, perform the division by assuming that n is a positive integer.**

91. $\dfrac{x^{3n} + 9x^{2n} + 27x^n + 27}{x^n + 3}$

92. $\dfrac{x^{3n} - 3x^{2n} + 5x^n - 6}{x^n - 2}$

93. Writing Briefly explain what it means for a divisor to divide evenly into a dividend.

94. Writing Briefly explain how to check polynomial division, and justify your reasoning. Give an example.

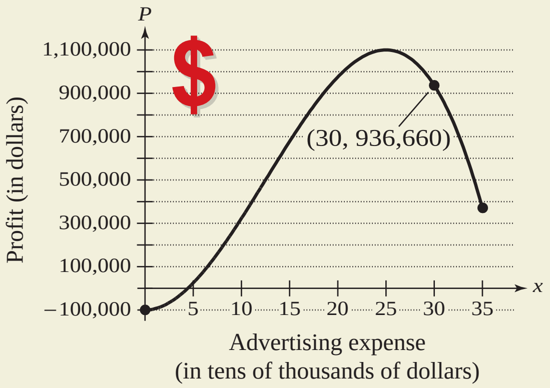
Exploration **In Exercises 95 and 96, find the constant c such that the denominator will divide evenly into the numerator.**

95. $\dfrac{x^3 + 4x^2 - 3x + c}{x - 5}$

96. $\dfrac{x^5 - 2x^2 + x + c}{x + 2}$

97. Think About It Find the value of k such that $x - 4$ is a factor of $x^3 - kx^2 + 2kx - 8$.

98. HOW DO YOU SEE IT? The graph below shows a company's estimated profits for different advertising expenses. The company's actual profit was \$936,660 for an advertising expense of \$300,000.



(a) From the graph, it appears that the company could have obtained the same profit for a lesser advertising expense. Use the graph to estimate this expense.

(b) The company's model is

$$P = -140.75x^3 + 5348.3x^2 - 76,560,$$

$$0 \le x \le 35$$

where P is the profit (in dollars) and x is the advertising expense (in tens of thousands of dollars). Explain how you could verify the lesser expense from part (a) algebraically.

Dariusz Majgier/Shutterstock.com

3.4 Zeros of Polynomial Functions

Finding zeros of polynomial functions is an important part of solving real-life problems. For instance, in Exercise 113 on page 287, the zeros of a polynomial function can help you redesign a storage bin so that it can hold five times as much food.

- **Use the Fundamental Theorem of Algebra to determine the number of zeros of polynomial functions.**
- **Find rational zeros of polynomial functions.**
- **Find conjugate pairs of complex zeros.**
- **Find zeros of polynomials by factoring.**
- **Use Descartes's Rule of Signs and the Upper and Lower Bound Rules to find zeros of polynomials.**
- **Find zeros of polynomials in real-life applications.**

The Fundamental Theorem of Algebra

In the complex number system, every nth-degree polynomial function has *precisely* n zeros. This important result is derived from the **Fundamental Theorem of Algebra,** first proved by the German mathematician Carl Friedrich Gauss (1777–1855).

The Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has at least one zero in the complex number system.

Using the Fundamental Theorem of Algebra and the equivalence of zeros and factors, you obtain the **Linear Factorization Theorem.**

Linear Factorization Theorem

If $f(x)$ is a polynomial of degree n, where $n > 0$, then $f(x)$ has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$

where $c_1, c_2, \ldots, c_n$ are complex numbers.

For a proof of the Linear Factorization Theorem, see Proofs in Mathematics on page 308.

Note that the Fundamental Theorem of Algebra and the Linear Factorization Theorem tell you only that the zeros or factors of a polynomial exist, not how to find them. Such theorems are called **existence theorems.**

REMARK Recall that in order to find the zeros of a function f, set $f(x)$ equal to 0 and solve the resulting equation for x. For instance, the function in Example 1(a) has a zero at $x = 2$ because

$$x - 2 = 0$$
$$x = 2.$$

EXAMPLE 1 **Zeros of Polynomial Functions**

a. The first-degree polynomial function $f(x) = x - 2$ has exactly *one* zero: $x = 2$.

b. Counting multiplicity, the second-degree polynomial function

$$f(x) = x^2 - 6x + 9 = (x - 3)(x - 3)$$

has exactly *two* zeros: $x = 3$ and $x = 3$. (This is called a *repeated zero*.)

c. The third-degree polynomial function

$$f(x) = x^3 + 4x = x(x^2 + 4) = x(x - 2i)(x + 2i)$$

has exactly *three* zeros: $x = 0$, $x = 2i$, and $x = -2i$.

ALGEBRA HELP Examples 1(b) and 1(c) involve factoring polynomials. You can review the techniques for factoring polynomials in Section P.4.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Determine the number of zeros of the polynomial function $f(x) = x^4 - 1$.

k45025/Shutterstock.com

The Rational Zero Test

The **Rational Zero Test** relates the possible rational zeros of a polynomial (having integer coefficients) to the leading coefficient and to the constant term of the polynomial.

The Rational Zero Test

If the polynomial $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdot\ \cdot\ \cdot + a_2x^2 + a_1x + a_0$ has *integer* coefficients, then every rational zero of f has the form

$$\text{Rational zero} = \frac{p}{q}$$

where p and q have no common factors other than 1, and

$p =$ a factor of the constant term a_0

$q =$ a factor of the leading coefficient a_n.

Although they were not contemporaries, Jean Le Rond d'Alembert (1717–1783) worked independently of Carl Gauss in trying to prove the Fundamental Theorem of Algebra. His efforts were such that, in France, the Fundamental Theorem of Algebra is frequently known as the Theorem of d'Alembert.

To use the Rational Zero Test, you should first list all rational numbers whose numerators are factors of the constant term and whose denominators are factors of the leading coefficient.

$$\text{Possible rational zeros} = \frac{\text{factors of constant term}}{\text{factors of leading coefficient}}$$

Having formed this list of *possible rational zeros*, use a trial-and-error method to determine which, if any, are actual zeros of the polynomial. Note that when the leading coefficient is 1, the possible rational zeros are simply the factors of the constant term.

EXAMPLE 2 Rational Zero Test with Leading Coefficient of 1

Find the rational zeros of

$f(x) = x^3 + x + 1.$

Solution Because the leading coefficient is 1, the possible rational zeros are the factors of the constant term.

Possible rational zeros: 1 and -1

By testing these possible zeros, you can see that neither works.

$$f(1) = (1)^3 + 1 + 1$$
$$= 3$$
$$f(-1) = (-1)^3 + (-1) + 1$$
$$= -1$$

So, you can conclude that the given polynomial has *no* rational zeros. Note from the graph of f in Figure 3.17 that f does have one real zero between -1 and 0. However, by the Rational Zero Test, you know that this real zero is *not* a rational number.

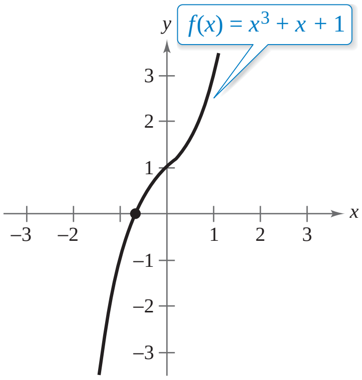


Figure 3.17

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the rational zeros of each function.

a. $f(x) = x^3 - 5x^2 + 2x + 8$

b. $f(x) = x^3 + 2x^2 + 6x - 4$

c. $f(x) = x^3 - 3x^2 + 2x - 6$

Fogg Art Museum/Harvard University

EXAMPLE 3 Rational Zero Test with Leading Coefficient of 1

Find the rational zeros of

$$f(x) = x^4 - x^3 + x^2 - 3x - 6.$$

Solution Because the leading coefficient is 1, the possible rational zeros are the factors of the constant term.

Possible rational zeros: $\pm1, \pm2, \pm3, \pm6$

REMARK When the list of possible rational zeros is small, as in Example 2, it may be quicker to test the zeros by evaluating the function. When the list of possible rational zeros is large, as in Example 3, it may be quicker to use a different approach to test the zeros, such as using synthetic division or sketching a graph.

By applying synthetic division successively, you can determine that $x = -1$ and $x = 2$ are the only two rational zeros.

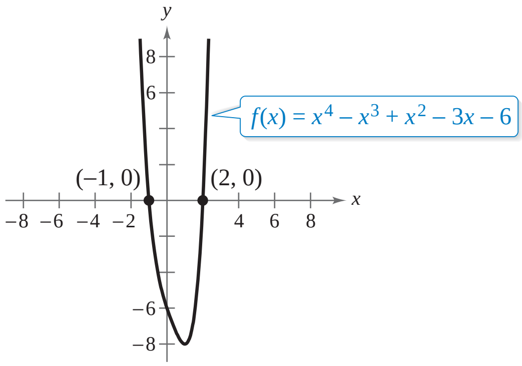
$$\begin{array}{r|rrrrr} -1 & 1 & -1 & 1 & -3 & -6 \\ & & -1 & 2 & -3 & 6 \\ \hline & 1 & -2 & 3 & -6 & 0 \end{array} \longrightarrow \text{0 remainder, so } x = -1 \text{ is a zero.}$$

$$\begin{array}{r|rrrr} 2 & 1 & -2 & 3 & -6 \\ & & 2 & 0 & 6 \\ \hline & 1 & 0 & 3 & 0 \end{array} \longrightarrow \text{0 remainder, so } x = 2 \text{ is a zero.}$$

So, $f(x)$ factors as

$$f(x) = (x + 1)(x - 2)(x^2 + 3).$$

Because the factor $(x^2 + 3)$ produces no real zeros, you can conclude that $x = -1$ and $x = 2$ are the only *real* zeros of f, which is verified in the figure below.



✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Find the rational zeros of

$$f(x) = x^3 - 15x^2 + 75x - 125.$$

When the leading coefficient of a polynomial is not 1, the list of possible rational zeros can increase dramatically. In such cases, the search can be shortened in several ways: (1) a programmable calculator can be used to speed up the calculations; (2) a graph, drawn either by hand or with a graphing utility, can give good estimates of the locations of the zeros; (3) the Intermediate Value Theorem along with a table generated by a graphing utility can give approximations of the zeros; and (4) synthetic division can be used to test the possible rational zeros.

Finding the first zero is often the most difficult part. After that, the search is simplified by working with the lower-degree polynomial obtained in synthetic division, as shown in Example 3.

EXAMPLE 4 Using the Rational Zero Test

Find the rational zeros of $f(x) = 2x^3 + 3x^2 - 8x + 3$.

Solution The leading coefficient is 2 and the constant term is 3.

$$\textit{Possible rational zeros: } \frac{\text{Factors of 3}}{\text{Factors of 2}} = \frac{\pm 1, \pm 3}{\pm 1, \pm 2} = \pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$$

By synthetic division, you can determine that $x = 1$ is a rational zero.

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -8 & 3 \\ & & 2 & 5 & -3 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

So, $f(x)$ factors as

$$\begin{aligned} f(x) &= (x - 1)(2x^2 + 5x - 3) \\ &= (x - 1)(2x - 1)(x + 3) \end{aligned}$$

and you can conclude that the rational zeros of f are $x = 1$, $x = \frac{1}{2}$, and $x = -3$.

REMARK Remember that when you try to find the rational zeros of a polynomial function with many possible rational zeros, as in Example 4, you must use trial and error. There is no quick algebraic method to determine which of the possibilities is an actual zero; however, sketching a graph may be helpful.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the rational zeros of

$$f(x) = 2x^4 - 9x^3 - 18x^2 + 71x - 30.$$

Recall from Section 3.2 that if $x = a$ is a zero of the polynomial function f, then $x = a$ is a solution of the polynomial equation $f(x) = 0$.

EXAMPLE 5 Solving a Polynomial Equation

Find all real solutions of $-10x^3 + 15x^2 + 16x - 12 = 0$.

Solution The leading coefficient is -10 and the constant term is -12.

$$\textit{Possible rational solutions: } \frac{\text{Factors of } -12}{\text{Factors of } -10} = \frac{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12}{\pm 1, \pm 2, \pm 5, \pm 10}$$

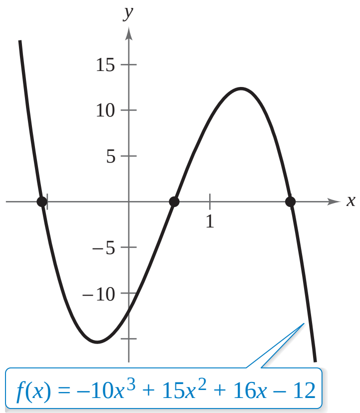


Figure 3.18

With so many possibilities (32, in fact), it is worth your time to stop and sketch a graph. From Figure 3.18, it looks like three reasonable solutions would be $x = -\frac{6}{5}$, $x = \frac{1}{2}$, and $x = 2$. Testing these by synthetic division shows that $x = 2$ is the only rational solution. So, you have

$$(x - 2)(-10x^2 - 5x + 6) = 0.$$

Using the Quadratic Formula for the second factor, you find that the two additional solutions are irrational numbers.

$$x = \frac{-5 - \sqrt{265}}{20} \approx -1.0639$$

and

$$x = \frac{-5 + \sqrt{265}}{20} \approx 0.5639$$

ALGEBRA HELP You can review the techniques for using the Quadratic Formula in Section 1.4.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find all real solutions of

$$x^3 + 4x^2 - 15x - 18 = 0.$$

Conjugate Pairs

In Example 1(c), note that the pair of complex zeros are conjugates. That is, they are of the form

$$a + bi$$

and

$$a - bi.$$

Complex Zeros Occur in Conjugate Pairs

Let f be a polynomial function that has *real coefficients*. If $a + bi$, where $b \neq 0$, is a zero of the function, then the conjugate $a - bi$ is also a zero of the function.

Be sure you see that this result is true only when the polynomial function has *real coefficients*. For instance, the result applies to the function

$$f(x) = x^2 + 1$$

but not to the function

$$g(x) = x - i.$$

EXAMPLE 6 Finding a Polynomial Function with Given Zeros

Find a fourth-degree polynomial function with real coefficients that has

$$-1, \quad -1, \quad \text{and} \quad 3i$$

as zeros.

Solution Because $3i$ is a zero *and* the polynomial is stated to have real coefficients, you know that the conjugate $-3i$ must also be a zero. So, the four zeros are

$$-1, \quad -1, \quad 3i, \quad \text{and} \quad -3i.$$

Then, using the Linear Factorization Theorem, $f(x)$ can be written as

$$f(x) = a(x + 1)(x + 1)(x - 3i)(x + 3i).$$

For simplicity, let $a = 1$. Then multiply the factors with real coefficients to obtain

$$(x + 1)(x + 1) = x^2 + 2x + 1$$

and multiply the complex conjugates to obtain

$$(x - 3i)(x + 3i) = x^2 + 9.$$

So, you obtain the following fourth-degree polynomial function.

$$\begin{aligned} f(x) &= (x^2 + 2x + 1)(x^2 + 9) \\ &= x^4 + 2x^3 + 10x^2 + 18x + 9 \end{aligned}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find a fourth-degree polynomial function with real coefficients that has the given zeros.

a. $2, -2$, and $-7i$

b. $1, 3$, and $4 - i$

c. $-1, 2$, and $3 + i$

Factoring a Polynomial

The Linear Factorization Theorem shows that you can write any nth-degree polynomial as the product of n linear factors.

$$f(x) = a_n(x - c_1)(x - c_2)(x - c_3) \cdot \cdot \cdot (x - c_n)$$

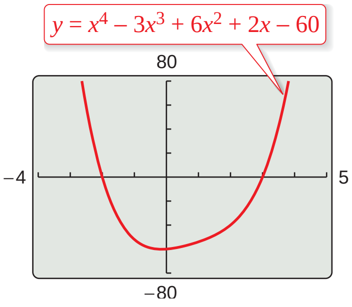
However, this result includes the possibility that some of the values of c_i are complex. The following theorem says that even if you do not want to get involved with "complex factors," you can still write $f(x)$ as the product of linear and/or quadratic factors. For a proof of this theorem, see Proofs in Mathematics on page 308.

Factors of a Polynomial

Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

A quadratic factor with no real zeros is said to be *prime* or **irreducible over the reals.** Note that this is not the same as being *irreducible over the rationals*. For example, the quadratic $x^2 + 1 = (x - i)(x + i)$ is irreducible over the reals (and therefore over the rationals). On the other hand, the quadratic $x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$ is irreducible over the rationals but *reducible* over the reals.

TECHNOLOGY In Example 7, you can use a graphing utility to graph the function, as shown below.



Then you can use the *zero* or *root* feature of the graphing utility to determine that $x = -2$ and $x = 3$ are zeros of the function.

EXAMPLE 7 Finding the Zeros of a Polynomial Function

Find all the zeros of $f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$ given that $1 + 3i$ is a zero of f.

Solution Because complex zeros occur in conjugate pairs, you know that $1 - 3i$ is also a zero of f. This means that both $[x - (1 + 3i)]$ and $[x - (1 - 3i)]$ are factors of $f(x)$. Multiplying these two factors produces

$$\begin{aligned}[x - (1 + 3i)][x - (1 - 3i)] &= [(x - 1) - 3i][(x - 1) + 3i] \\ &= (x - 1)^2 - 9i^2 \\ &= x^2 - 2x + 10.\end{aligned}$$

Using long division, you can divide $x^2 - 2x + 10$ into $f(x)$ to obtain the following.

$$\begin{array}{r} x^2 - x - 6 \\ x^2 - 2x + 10 \overline{\smash{)}\, x^4 - 3x^3 + 6x^2 + 2x - 60} \\ \underline{x^4 - 2x^3 + 10x^2} \\ -x^3 - 4x^2 + 2x \\ \underline{-x^3 + 2x^2 - 10x} \\ -6x^2 + 12x - 60 \\ \underline{-6x^2 + 12x - 60} \\ 0 \end{array}$$

So, you have

$$f(x) = (x^2 - 2x + 10)(x^2 - x - 6) = (x^2 - 2x + 10)(x - 3)(x + 2)$$

and you can conclude that the zeros of f are $x = 1 + 3i$, $x = 1 - 3i$, $x = 3$, and $x = -2$.

ALGEBRA HELP You can review the techniques for polynomial long division in Section 3.3.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find all the zeros of $f(x) = 3x^3 - 2x^2 + 48x - 32$ given that $4i$ is a zero of f. ■

In Example 7, without being told that $1 + 3i$ is a zero of f, you could still find all the zeros of the function by using synthetic division to find the real zeros -2 and 3. Then you could factor the polynomial as

$$(x + 2)(x - 3)(x^2 - 2x + 10).$$

Finally, by using the Quadratic Formula, you could determine that the zeros are

$$x = -2, \quad x = 3, \quad x = 1 + 3i, \quad \text{and} \quad x = 1 - 3i.$$

Example 8 shows how to find all the zeros of a polynomial function, including complex zeros.

TECHNOLOGY In Example 8, the fifth-degree polynomial function has three real zeros. In such cases, you can use the *zoom* and *trace* features or the *zero* or *root* feature of a graphing utility to approximate the real zeros. You can then use these real zeros to determine the complex zeros algebraically.

EXAMPLE 8 Finding the Zeros of a Polynomial Function

Write

$$f(x) = x^5 + x^3 + 2x^2 - 12x + 8$$

as the product of linear factors and list all the zeros of the function.

Solution Because the leading coefficient is 1, the possible rational zeros are the factors of the constant term.

Possible rational zeros: ±1, ±2, ±4, and ±8

Synthetic division produces the following.

$$\begin{array}{r|rrrrrr} 1 & 1 & 0 & 1 & 2 & -12 & 8 \\ & & 1 & 1 & 2 & 4 & -8 \\ \hline & 1 & 1 & 2 & 4 & -8 & 0 \end{array} \longrightarrow \text{1 is a zero.}$$

$$\begin{array}{r|rrrrr} -2 & 1 & 1 & 2 & 4 & -8 \\ & & -2 & 2 & -8 & 8 \\ \hline & 1 & -1 & 4 & -4 & 0 \end{array} \longrightarrow -2 \text{ is a zero.}$$

So, you have

$$\begin{aligned} f(x) &= x^5 + x^3 + 2x^2 - 12x + 8 \\ &= (x - 1)(x + 2)(x^3 - x^2 + 4x - 4). \end{aligned}$$

You can factor $x^3 - x^2 + 4x - 4$ as $(x - 1)(x^2 + 4)$, and by factoring $x^2 + 4$ as

$$\begin{aligned} x^2 - (-4) &= \left(x - \sqrt{-4}\right)\left(x + \sqrt{-4}\right) \\ &= (x - 2i)(x + 2i) \end{aligned}$$

you obtain

$$f(x) = (x - 1)(x - 1)(x + 2)(x - 2i)(x + 2i)$$

which gives the following five zeros of f.

$$x = 1, \quad x = 1, \quad x = -2, \quad x = 2i, \quad \text{and} \quad x = -2i$$

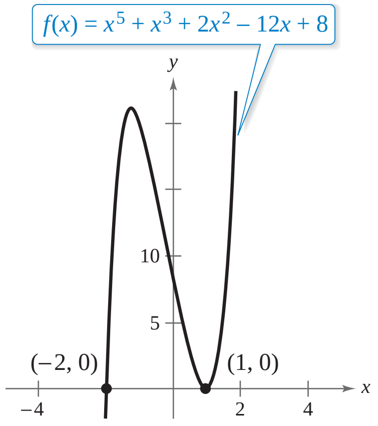


Figure 3.19

From the graph of f shown in Figure 3.19, you can see that the *real* zeros are the only ones that appear as x-intercepts. Note that $x = 1$ is a repeated zero.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write each function as the product of linear factors and list all the zeros of the function.

a. $f(x) = x^4 + 8x^2 - 9$

b. $g(x) = x^3 - 3x^2 + 7x - 5$

c. $h(x) = x^3 - 11x^2 + 41x - 51$

Other Tests for Zeros of Polynomials

You know that an nth-degree polynomial function can have *at most n* real zeros. Of course, many nth-degree polynomials do not have that many real zeros. For instance, $f(x) = x^2 + 1$ has no real zeros, and $f(x) = x^3 + 1$ has only one real zero. The following theorem, called **Descartes's Rule of Signs,** sheds more light on the number of real zeros of a polynomial.

Descartes's Rule of Signs

Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$ be a polynomial with real coefficients and $a_0 \neq 0$.

1. The number of *positive real zeros* of f is either equal to the number of variations in sign of $f(x)$ or less than that number by an even integer.
2. The number of *negative real zeros* of f is either equal to the number of variations in sign of $f(-x)$ or less than that number by an even integer.

A **variation in sign** means that two consecutive coefficients have opposite signs.

When using Descartes's Rule of Signs, a zero of multiplicity k should be counted as k zeros. For instance, the polynomial $x^3 - 3x + 2$ has two variations in sign, and so it has either two positive or no positive real zeros. Because

$$x^3 - 3x + 2 = (x - 1)(x - 1)(x + 2)$$

you can see that the two positive real zeros are $x = 1$ of multiplicity 2.

EXAMPLE 9 Using Descartes's Rule of Signs

Determine the possible numbers of positive and negative real zeros of

$$f(x) = 3x^3 - 5x^2 + 6x - 4.$$

Solution The original polynomial has *three* variations in sign.

$$\begin{array}{cccccccc} & +\ \text{to} & - & & + & \text{to} & - & \\ & \downarrow & \downarrow & & \downarrow & & \downarrow & \\ f(x) = & 3x^3 & - & 5x^2 & + & 6x & - & 4 \\ & & \uparrow & & \uparrow & & & \\ & & - & \text{to} & + & & & \end{array}$$

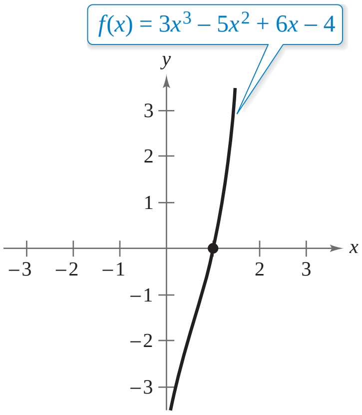


Figure 3.20

The polynomial

$$\begin{aligned} f(-x) &= 3(-x)^3 - 5(-x)^2 + 6(-x) - 4 \\ &= -3x^3 - 5x^2 - 6x - 4 \end{aligned}$$

has no variations in sign. So, from Descartes's Rule of Signs, the polynomial

$$f(x) = 3x^3 - 5x^2 + 6x - 4$$

has either three positive real zeros or one positive real zero, and has no negative real zeros. From the graph in Figure 3.20, you can see that the function has only one real zero, $x = 1$.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Determine the possible numbers of positive and negative real zeros of

$$f(x) = -2x^3 + 5x^2 - x + 8.$$

Another test for zeros of a polynomial function is related to the sign pattern in the last row of the synthetic division array. This test can give you an upper or lower bound for the real zeros of f. A real number c is an **upper bound** for the real zeros of f when no zeros are greater than c. Similarly, c is a **lower bound** when no real zeros of f are less than c.

Upper and Lower Bound Rules

Let $f(x)$ be a polynomial with real coefficients and a positive leading coefficient. Suppose $f(x)$ is divided by $x - c$, using synthetic division.

1. If $c > 0$ and each number in the last row is either positive or zero, then c is an **upper bound** for the real zeros of f.
2. If $c < 0$ and the numbers in the last row are alternately positive and negative (zero entries count as positive or negative), then c is a **lower bound** for the real zeros of f.

EXAMPLE 10 Finding the Zeros of a Polynomial Function

Find all real zeros of

$$f(x) = 6x^3 - 4x^2 + 3x - 2.$$

Solution The possible real zeros are as follows.

$$\frac{\text{Factors of } -2}{\text{Factors of } 6} = \frac{\pm1, \pm2}{\pm1, \pm2, \pm3, \pm6} = \pm1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm\frac{2}{3}, \pm2$$

The original polynomial $f(x)$ has three variations in sign. The polynomial

$$\begin{aligned} f(-x) &= 6(-x)^3 - 4(-x)^2 + 3(-x) - 2 \\ &= -6x^3 - 4x^2 - 3x - 2 \end{aligned}$$

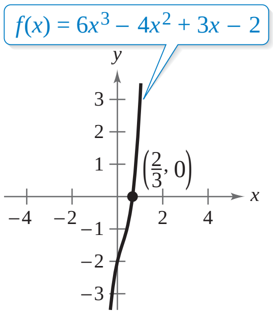
has no variations in sign. As a result of these two findings, you can apply Descartes's Rule of Signs to conclude that there are three positive real zeros or one positive real zero, and no negative real zeros. Trying $x = 1$ produces the following.

$$\begin{array}{r|rrrr} 1 & 6 & -4 & 3 & -2 \\ & & 6 & 2 & 5 \\ \hline & 6 & 2 & 5 & 3 \end{array}$$

So, $x = 1$ is not a zero, but because the last row has all positive entries, you know that $x = 1$ is an upper bound for the real zeros. So, you can restrict the search to zeros between 0 and 1. By trial and error, you can determine that $x = \frac{2}{3}$ is a zero. So,

$$f(x) = \left(x - \frac{2}{3}\right)(6x^2 + 3).$$

Because $6x^2 + 3$ has no real zeros, it follows that $x = \frac{2}{3}$ is the only real zero, as shown at the right.



Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Find all real zeros of

$$f(x) = 8x^3 - 4x^2 + 6x - 3.$$

Application

According to the National Candle Association, 7 out of 10 U.S. households use candles.

EXAMPLE 11 Using a Polynomial Model

You are designing candle making kits. Each kit contains 25 cubic inches of candle wax and a mold for making a pyramid-shaped candle. You want the height of the candle to be 2 inches less than the length of each side of the candle's square base. What should the dimensions of your candle mold be?

Solution The volume of a pyramid is $V = \frac{1}{3}Bh$, where B is the area of the base and h is the height. The area of the base is x^2 and the height is $(x - 2)$. So, the volume of the pyramid is $V = \frac{1}{3}x^2(x - 2)$. Substituting 25 for the volume yields the following.

$$25 = \frac{1}{3}x^2(x - 2) \qquad \text{Substitute 25 for } V.$$

$$75 = x^3 - 2x^2 \qquad \text{Multiply each side by 3, and distribute } x^2.$$

$$0 = x^3 - 2x^2 - 75 \qquad \text{Write in general form.}$$

The possible rational solutions are $x = \pm1, \pm3, \pm5, \pm15, \pm25, \pm75$. Use synthetic division to test some of the possible solutions. Note that in this case it makes sense to test only positive x-values. Using synthetic division, you can determine that $x = 5$ is a solution.

$$\begin{array}{r|rrrr} 5 & 1 & -2 & 0 & -75 \\ & & 5 & 15 & 75 \\ \hline & 1 & 3 & 15 & 0 \end{array}$$

The other two solutions that satisfy $x^2 + 3x + 15 = 0$ are imaginary and can be discarded. You can conclude that the base of the candle mold should be 5 inches by 5 inches and the height should be $5 - 2 = 3$ inches.

✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Rework Example 11 when each kit contains 147 cubic inches of candle wax and you want the height of the pyramid-shaped candle to be 2 inches more than the length of each side of the candle's square base.

Before concluding this section, here is an additional hint that can help you find the real zeros of a polynomial. When the terms of $f(x)$ have a common monomial factor, it should be factored out before applying the tests in this section. For instance, by writing

$$f(x) = x^4 - 5x^3 + 3x^2 + x = x(x^3 - 5x^2 + 3x + 1)$$

you can see that $x = 0$ is a zero of f and that the remaining zeros can be obtained by analyzing the cubic factor.

Summarize (Section 3.4)

1. State the Fundamental Theorem of Algebra and the Linear Factorization Theorem *(page 275, Example 1).*
2. Describe the Rational Zero Test *(page 276, Examples 2–5).*
3. State the definition of conjugates *(page 279, Example 6).*
4. Describe what it means for a quadratic factor to be irreducible over the reals *(page 280, Examples 7 and 8).*
5. State Descartes's Rule of Signs and the Upper and Lower Bound Rules *(pages 282 and 283, Examples 9 and 10).*
6. Describe a real-life situation involving finding the zeros of a polynomial *(page 284, Example 11).*

hfng/Shutterstock.com

3.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. The ________ ________ of ________ states that if $f(x)$ is a polynomial of degree n $(n > 0)$, then f has at least one zero in the complex number system.
2. The ________ ________ ________ states that if $f(x)$ is a polynomial of degree n $(n > 0)$, then $f(x)$ has precisely n linear factors, $f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$, where $c_1, c_2, . . . , c_n$ are complex numbers.
3. The test that gives a list of the possible rational zeros of a polynomial function is the ________ ________ Test.
4. If $a + bi$ is a complex zero of a polynomial with real coefficients, then so is its ________, $a - bi$.
5. Every polynomial of degree $n > 0$ with real coefficients can be written as the product of ________ and ________ factors with real coefficients, where the ________ factors have no real zeros.
6. A quadratic factor that cannot be factored further as a product of linear factors containing real numbers is said to be ________ over the ________.
7. The theorem that can be used to determine the possible numbers of positive real zeros and negative real zeros of a function is called ________ ________ of ________.
8. A real number b is a ________ bound for the real zeros of f when no real zeros are less than b, and is a ________ bound when no real zeros are greater than b.

Skills and Applications

Zeros of Polynomial Functions **In Exercises 9–14, determine the number of zeros of the polynomial function.**

9. $f(x) = x + 3$

10. $f(x) = x^2 + 5x - 6$

11. $g(x) = 1 - x^3$

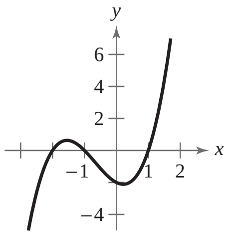
12. $f(x) = x^6 - x^7$

13. $f(x) = (x + 5)^2$

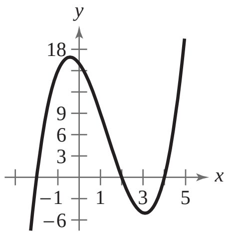
14. $h(t) = (t - 1)^2 - (t + 1)^2$

Using the Rational Zero Test **In Exercises 15–18, use the Rational Zero Test to list the possible rational zeros of f. Verify that the zeros of f shown on the graph are contained in the list.**

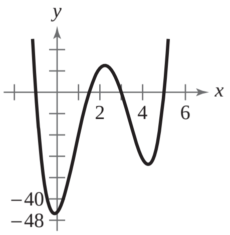
15. $f(x) = x^3 + 2x^2 - x - 2$



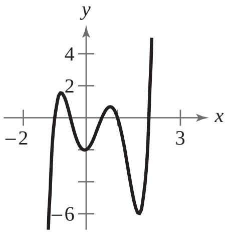
16. $f(x) = x^3 - 4x^2 - 4x + 16$



17. $f(x) = 2x^4 - 17x^3 + 35x^2 + 9x - 45$



18. $f(x) = 4x^5 - 8x^4 - 5x^3 + 10x^2 + x - 2$



Using the Rational Zero Test **In Exercises 19–28, find the rational zeros of the function.**

19. $f(x) = x^3 - 7x - 6$

20. $f(x) = x^3 - 13x + 12$

21. $g(x) = x^3 - 4x^2 - x + 4$

22. $h(x) = x^3 - 9x^2 + 20x - 12$

23. $h(t) = t^3 + 8t^2 + 13t + 6$

24. $p(x) = x^3 - 9x^2 + 27x - 27$

25. $C(x) = 2x^3 + 3x^2 - 1$

26. $f(x) = 3x^3 - 19x^2 + 33x - 9$

27. $f(x) = 9x^4 - 9x^3 - 58x^2 + 4x + 24$

28. $f(x) = 2x^4 - 15x^3 + 23x^2 + 15x - 25$

Solving a Polynomial Equation **In Exercises 29–32, find all real solutions of the polynomial equation.**

29. $z^4 + z^3 + z^2 + 3z - 6 = 0$

30. $x^4 - 13x^2 - 12x = 0$

31. $2y^4 + 3y^3 - 16y^2 + 15y - 4 = 0$

32. $x^5 - x^4 - 3x^3 + 5x^2 - 2x = 0$

Using the Rational Zero Test **In Exercises 33–36, (a) list the possible rational zeros of *f*, (b) sketch the graph of *f* so that some of the possible zeros in part (a) can be disregarded, and then (c) determine all real zeros of *f*.**

33. $f(x) = x^3 + x^2 - 4x - 4$

34. $f(x) = -3x^3 + 20x^2 - 36x + 16$

35. $f(x) = -4x^3 + 15x^2 - 8x - 3$

36. $f(x) = 4x^3 - 12x^2 - x + 15$

Using the Rational Zero Test **In Exercises 37–40, (a) list the possible rational zeros of *f*, (b) use a graphing utility to graph *f* so that some of the possible zeros in part (a) can be disregarded, and then (c) determine all real zeros of *f*.**

37. $f(x) = -2x^4 + 13x^3 - 21x^2 + 2x + 8$

38. $f(x) = 4x^4 - 17x^2 + 4$

39. $f(x) = 32x^3 - 52x^2 + 17x + 3$

40. $f(x) = 4x^3 + 7x^2 - 11x - 18$

Graphical Analysis **In Exercises 41–44, (a) use the *zero* or *root* feature of a graphing utility to approximate the zeros of the function accurate to three decimal places, (b) determine the exact value of one of the zeros (use synthetic division to verify your result), and (c) factor the polynomial completely.**

41. $f(x) = x^4 - 3x^2 + 2$

42. $P(t) = t^4 - 7t^2 + 12$

43. $h(x) = x^5 - 7x^4 + 10x^3 + 14x^2 - 24x$

44. $g(x) = 6x^4 - 11x^3 - 51x^2 + 99x - 27$

Finding a Polynomial Function with Given Zeros **In Exercises 45–50, find a polynomial function with real coefficients that has the given zeros. (There are many correct answers.)**

45. $1, 5i$

46. $4, -3i$

47. $2, 5 + i$

48. $5, 3 - 2i$

49. $\frac{2}{3}, -1, 3 + \sqrt{2}i$

50. $-5, -5, 1 + \sqrt{3}i$

Factoring a Polynomial **In Exercises 51–54, write the polynomial (a) as the product of factors that are irreducible over the *rationals*, (b) as the product of linear and quadratic factors that are irreducible over the *reals*, and (c) in completely factored form.**

51. $f(x) = x^4 + 6x^2 - 27$

52. $f(x) = x^4 - 2x^3 - 3x^2 + 12x - 18$

(*Hint:* One factor is $x^2 - 6$.)

53. $f(x) = x^4 - 4x^3 + 5x^2 - 2x - 6$

(*Hint:* One factor is $x^2 - 2x - 2$.)

54. $f(x) = x^4 - 3x^3 - x^2 - 12x - 20$

(*Hint:* One factor is $x^2 + 4$.)

Finding the Zeros of a Polynomial Function **In Exercises 55–62, use the given zero to find all the zeros of the function.**

	Function	**Zero**
55.	$f(x) = x^3 - x^2 + 4x - 4$	$2i$
56.	$f(x) = 2x^3 + 3x^2 + 18x + 27$	$3i$
57.	$f(x) = 2x^4 - x^3 + 49x^2 - 25x - 25$	$5i$
58.	$g(x) = x^3 - 7x^2 - x + 87$	$5 + 2i$
59.	$g(x) = 4x^3 + 23x^2 + 34x - 10$	$-3 + i$
60.	$h(x) = 3x^3 - 4x^2 + 8x + 8$	$1 - \sqrt{3}i$
61.	$f(x) = x^4 + 3x^3 - 5x^2 - 21x + 22$	$-3 + \sqrt{2}i$
62.	$f(x) = x^3 + 4x^2 + 14x + 20$	$-1 - 3i$

Finding the Zeros of a Polynomial Function **In Exercises 63–80, write the polynomial as the product of linear factors and list all the zeros of the function.**

63. $f(x) = x^2 + 36$

64. $f(x) = x^2 - x + 56$

65. $h(x) = x^2 - 2x + 17$

66. $g(x) = x^2 + 10x + 17$

67. $f(x) = x^4 - 16$

68. $f(y) = y^4 - 256$

69. $f(z) = z^2 - 2z + 2$

70. $h(x) = x^3 - 3x^2 + 4x - 2$

71. $g(x) = x^3 - 3x^2 + x + 5$

72. $f(x) = x^3 - x^2 + x + 39$

73. $h(x) = x^3 - x + 6$

74. $h(x) = x^3 + 9x^2 + 27x + 35$

75. $f(x) = 5x^3 - 9x^2 + 28x + 6$

76. $g(x) = 2x^3 - x^2 + 8x + 21$

77. $g(x) = x^4 - 4x^3 + 8x^2 - 16x + 16$

78. $h(x) = x^4 + 6x^3 + 10x^2 + 6x + 9$

79. $f(x) = x^4 + 10x^2 + 9$

80. $f(x) = x^4 + 29x^2 + 100$

Finding the Zeros of a Polynomial Function **In Exercises 81–86, find all the zeros of the function. When there is an extended list of possible rational zeros, use a graphing utility to graph the function in order to disregard any of the possible rational zeros that are obviously not zeros of the function.**

81. $f(x) = x^3 + 24x^2 + 214x + 740$

82. $f(s) = 2s^3 - 5s^2 + 12s - 5$

83. $f(x) = 16x^3 - 20x^2 - 4x + 15$

84. $f(x) = 9x^3 - 15x^2 + 11x - 5$

85. $f(x) = 2x^4 + 5x^3 + 4x^2 + 5x + 2$

86. $g(x) = x^5 - 8x^4 + 28x^3 - 56x^2 + 64x - 32$

Using Descartes's Rule of Signs **In Exercises 87–94, use Descartes's Rule of Signs to determine the possible numbers of positive and negative real zeros of the function.**

87. $g(x) = 2x^3 - 3x^2 - 3$

88. $h(x) = 4x^2 - 8x + 3$

89. $h(x) = 2x^3 + 3x^2 + 1$

90. $h(x) = 2x^4 - 3x + 2$

91. $g(x) = 5x^5 - 10x$

92. $f(x) = 4x^3 - 3x^2 + 2x - 1$

93. $f(x) = -5x^3 + x^2 - x + 5$

94. $f(x) = 3x^3 + 2x^2 + x + 3$

Verifying Upper and Lower Bounds **In Exercises 95–98, use synthetic division to verify the upper and lower bounds of the real zeros of *f*.**

95. $f(x) = x^3 + 3x^2 - 2x + 1$

(a) Upper: $x = 1$ (b) Lower: $x = -4$

96. $f(x) = x^3 - 4x^2 + 1$

(a) Upper: $x = 4$ (b) Lower: $x = -1$

97. $f(x) = x^4 - 4x^3 + 16x - 16$

(a) Upper: $x = 5$ (b) Lower: $x = -3$

98. $f(x) = 2x^4 - 8x + 3$

(a) Upper: $x = 3$ (b) Lower: $x = -4$

Finding the Zeros of a Polynomial Function **In Exercises 99–102, find all real zeros of the function.**

99. $f(x) = 4x^3 - 3x - 1$

100. $f(z) = 12z^3 - 4z^2 - 27z + 9$

101. $f(y) = 4y^3 + 3y^2 + 8y + 6$

102. $g(x) = 3x^3 - 2x^2 + 15x - 10$

Finding the Rational Zeros of a Polynomial **In Exercises 103–106, find the rational zeros of the polynomial function.**

103. $P(x) = x^4 - \frac{25}{4}x^2 + 9 = \frac{1}{4}(4x^4 - 25x^2 + 36)$

104. $f(x) = x^3 - \frac{3}{2}x^2 - \frac{23}{2}x + 6 = \frac{1}{2}(2x^3 - 3x^2 - 23x + 12)$

105. $f(x) = x^3 - \frac{1}{4}x^2 - x + \frac{1}{4} = \frac{1}{4}(4x^3 - x^2 - 4x + 1)$

106. $f(z) = z^3 + \frac{11}{6}z^2 - \frac{1}{2}z - \frac{1}{3} = \frac{1}{6}(6z^3 + 11z^2 - 3z - 2)$

Rational and Irrational Zeros **In Exercises 107–110, match the cubic function with the numbers of rational and irrational zeros.**

(a) Rational zeros: 0; irrational zeros: 1

(b) Rational zeros: 3; irrational zeros: 0

(c) Rational zeros: 1; irrational zeros: 2

(d) Rational zeros: 1; irrational zeros: 0

107. $f(x) = x^3 - 1$

108. $f(x) = x^3 - 2$

109. $f(x) = x^3 - x$

110. $f(x) = x^3 - 2x$

k45025/Shutterstock.com

111. Geometry You want to make an open box from a rectangular piece of material, 15 centimeters by 9 centimeters, by cutting equal squares from the corners and turning up the sides.

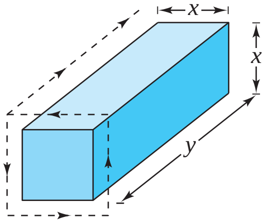
(a) Let x represent the side length of each of the squares removed. Draw a diagram showing the squares removed from the original piece of material and the resulting dimensions of the open box.

(b) Use the diagram to write the volume V of the box as a function of x. Determine the domain of the function.

(c) Sketch the graph of the function and approximate the dimensions of the box that will yield a maximum volume.

(d) Find values of x such that $V = 56$. Which of these values is a physical impossibility in the construction of the box? Explain.

112. Geometry A rectangular package to be sent by a delivery service (see figure) can have a maximum combined length and girth (perimeter of a cross section) of 120 inches.



(a) Use the diagram to write the volume V of the package as a function of x.

(b) Use a graphing utility to graph the function and approximate the dimensions of the package that will yield a maximum volume.

(c) Find values of x such that $V = 13{,}500$. Which of these values is a physical impossibility in the construction of the package? Explain.

113. Geometry

A bulk food storage bin with dimensions 2 feet by 3 feet by 4 feet needs to be increased in size to hold five times as much food as the current bin. (Assume each dimension is increased by the same amount.)

(a) Write a function that represents the volume V of the new bin.

(b) Find the dimensions of the new bin.

114. Cost The ordering and transportation cost C (in thousands of dollars) for machine parts is

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x+30}\right), \quad x \geq 1$$

where x is the order size (in hundreds). In calculus, it can be shown that the cost is a minimum when

$$3x^3 - 40x^2 - 2400x - 36{,}000 = 0.$$

Use a calculator to approximate the optimal order size to the nearest hundred units.

Exploration

True or False? **In Exercises 115 and 116, decide whether the statement is true or false. Justify your answer.**

115. It is possible for a third-degree polynomial function with integer coefficients to have no real zeros.

116. If $x = -i$ is a zero of the function

$$f(x) = x^3 + ix^2 + ix - 1$$

then $x = i$ must also be a zero of f.

Think About It **In Exercises 117–122, determine (if possible) the zeros of the function g when the function f has zeros at $x = r_1$, $x = r_2$, and $x = r_3$.**

117. $g(x) = -f(x)$

118. $g(x) = 3f(x)$

119. $g(x) = f(x - 5)$

120. $g(x) = f(2x)$

121. $g(x) = 3 + f(x)$

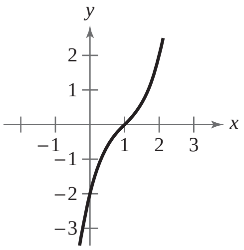
122. $g(x) = f(-x)$

123. Think About It A cubic polynomial function f has real zeros -2, $\frac{1}{2}$, and 3, and its leading coefficient is negative. Write an equation for f and sketch its graph. How many different polynomial functions are possible for f?

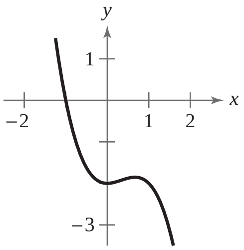
124. Think About It Sketch the graph of a fifth-degree polynomial function whose leading coefficient is positive and that has a zero at $x = 3$ of multiplicity 2.

Writing an Equation **In Exercises 125 and 126, the graph of a cubic polynomial function $y = f(x)$ is shown. One of the zeros is $1 + i$. Write an equation for f.**

125.



126.



127. Think About It Let $y = f(x)$ be a quartic polynomial with leading coefficient $a = 1$ and $f(i) = f(2i) = 0$. Write an equation for f.

128. Think About It Let $y = f(x)$ be a cubic polynomial with leading coefficient $a = -1$ and $f(2) = f(i) = 0$. Write an equation for f.

129. Writing an Equation Write the equation for a quadratic function f (with integer coefficients) that has the given zeros. Assume that b is a positive integer.

(a) $\pm\sqrt{bi}$

(b) $a \pm bi$

130. HOW DO YOU SEE IT? Use the information in the table to answer each question.

Interval	Value of $f(x)$
$(-\infty, -2)$	Positive
$(-2, 1)$	Negative
$(1, 4)$	Negative
$(4, \infty)$	Positive

(a) What are the three real zeros of the polynomial function f?

(b) What can be said about the behavior of the graph of f at $x = 1$?

(c) What is the least possible degree of f? Explain. Can the degree of f ever be odd? Explain.

(d) Is the leading coefficient of f positive or negative? Explain.

(e) Sketch a graph of a function that exhibits the behavior described in the table.

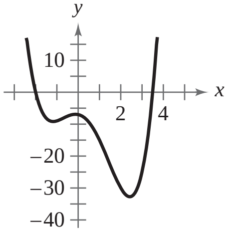
131. Graphical Reasoning The graph of one of the following functions is shown below. Identify the function shown in the graph. Explain why each of the others is not the correct function. Use a graphing utility to verify your result.

(a) $f(x) = x^2(x + 2)(x - 3.5)$

(b) $g(x) = (x + 2)(x - 3.5)$

(c) $h(x) = (x + 2)(x - 3.5)(x^2 + 1)$

(d) $k(x) = (x + 1)(x + 2)(x - 3.5)$



3.5 Mathematical Modeling and Variation

You can use functions as models to represent a wide variety of real-life data sets. For instance, in Exercise 75 on page 299, a variation model can be used to model the water temperatures of the ocean at various depths.

- **Use mathematical models to approximate sets of data points.**
- **Use the *regression* feature of a graphing utility to find the equation of a least squares regression line.**
- **Write mathematical models for direct variation, direct variation as an *n*th power, inverse variation, combined variation, and joint variation.**

Introduction

In this section, you will study the following techniques for fitting models to data: *least squares regression* and *direct and inverse variation*.

EXAMPLE 1 A Mathematical Model

The populations y (in millions) of the United States from 2003 through 2010 are shown in the table. *(Source: U.S. Census Bureau)*

Year, t	2003	2004	2005	2006	2007	2008	2009	2010
Population, y	290.1	292.8	295.5	298.4	301.2	304.1	306.8	308.7

Spreadsheet at LarsonPrecalculus.com

A linear model that approximates the data is

$$y = 2.72t + 282.0, \quad 3 \le t \le 10$$

where t represents the year, with $t = 3$ corresponding to 2003. Plot the actual data *and* the model on the same set of coordinate axes. How closely does the model represent the data?

REMARK Note that the linear model in Example 1 was found using the *regression* feature of a graphing utility and is the line that *best* fits the data. This concept of a "best-fitting" line is discussed on the next page.

Solution The actual data are plotted in Figure 3.21, along with the graph of the linear model. From the graph, it appears that the model is a "good fit" for the actual data. You can see how well the model fits by comparing the actual values of y with the values of y given by the model. The values given by the model are labeled $y*$ in the table below.

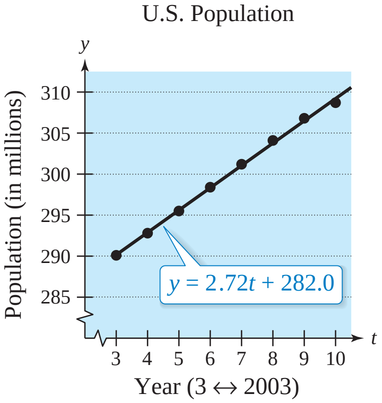


Figure 3.21

t	3	4	5	6	7	8	9	10
y	290.1	292.8	295.5	298.4	301.2	304.1	306.8	308.7
$y*$	290.2	292.9	295.6	298.3	301.0	303.8	306.5	309.2

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The median sales prices y (in thousands of dollars) of new homes sold in a neighborhood from 2003 through 2010 are given by the following ordered pairs. *(Spreadsheet at LarsonPrecalculus.com)*

(2003, 179.4)	(2005, 191.0)	(2007, 202.6)	(2009, 214.9)
(2004, 185.4)	(2006, 196.7)	(2008, 208.7)	(2010, 221.4)

A linear model that approximates the data is

$$y = 5.96t + 161.3, \quad 3 \le t \le 10$$

where t represents the year, with $t = 3$ corresponding to 2003. Plot the actual data *and* the model on the same set of coordinate axes. How closely does the model represent the data?

Khoroshunova Olga/Shutterstock.com

Least Squares Regression and Graphing Utilities

So far in this text, you have worked with many different types of mathematical models that approximate real-life data. In some instances the model was given (as in Example 1), whereas in other instances you were asked to find the model using simple algebraic techniques or a graphing utility.

To find a model that approximates a set of data most accurately, statisticians use a measure called the **sum of square differences,** which is the sum of the squares of the differences between actual data values and model values. The "best-fitting" linear model, called the **least squares regression line,** is the one with the least sum of square differences.

Recall that you can approximate this line visually by plotting the data points and drawing the line that appears to best fit the data—or you can enter the data points into a calculator or computer and use the *linear regression* feature of the calculator or computer.

When you use the *regression* feature of a graphing calculator or computer program, you will notice that the program may also output an "r-value." This r-value is the **correlation coefficient** of the data and gives a measure of how well the model fits the data. The closer the value of $|r|$ is to 1, the better the fit.

EXAMPLE 2 Finding a Least Squares Regression Line

The data in the table show the numbers E (in millions) of Medicare enrollees from 2003 through 2010. Construct a scatter plot that represents the data and find the least squares regression line for the data. *(Source: U.S. Centers for Medicare and Medicaid Services)*

DATA

Spreadsheet at LarsonPrecalculus.com

Year	Medicare Enrollees, E
2003	41.2
2004	41.9
2005	42.6
2006	43.4
2007	44.4
2008	45.5
2009	46.6
2010	47.5

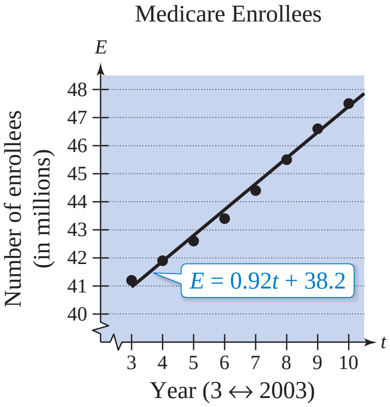


Figure 3.22

Solution Let $t = 3$ represent 2003. The scatter plot for the data is shown in Figure 3.22. Using the *regression* feature of a graphing utility, you can determine that the equation of the least squares regression line is

$$E = 0.92t + 38.2$$

To check this model, compare the actual E-values with the E-values given by the model, which are labeled E^* in the table at the left. The correlation coefficient for this model is $r \approx 0.996$, which implies that the model is a good fit.

t	E	E^*
3	41.2	41.0
4	41.9	41.9
5	42.6	42.8
6	43.4	43.7
7	44.4	44.6
8	45.5	45.6
9	46.6	46.5
10	47.5	47.4

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The total Medicare disbursements (in billions of dollars) are given by the following ordered pairs. *(Spreadsheet at LarsonPrecalculus.com)* Construct a scatter plot that represents the data and find the least squares regression line for the data. *(Source: U.S. Centers for Medicare and Medicaid Services)*

DATA

(2003, 277.8)	(2005, 336.9)	(2007, 434.8)	(2009, 498.2)
(2004, 301.5)	(2006, 380.4)	(2008, 455.1)	(2010, 521.1)

Direct Variation

There are two basic types of linear models. The more general model has a y-intercept that is nonzero.

$$y = mx + b, \quad b \neq 0$$

The simpler model

$$y = kx$$

has a y-intercept that is zero. In the simpler model, y is said to **vary directly** as x, or to be **directly proportional** to x.

> **Direct Variation**
>
> The following statements are equivalent.
>
> **1.** y **varies directly** as x.
>
> **2.** y is **directly proportional** to x.
>
> **3.** $y = kx$ for some nonzero constant k.
>
> k is the **constant of variation** or the **constant of proportionality.**

EXAMPLE 3 Direct Variation

In Pennsylvania, the state income tax is directly proportional to *gross income*. You are working in Pennsylvania and your state income tax deduction is \$46.05 for a gross monthly income of \$1500. Find a mathematical model that gives the Pennsylvania state income tax in terms of gross income.

Solution

Verbal Model: State income tax = k · Gross income

Labels:
State income tax $= y$ (dollars)
Gross income $= x$ (dollars)
Income tax rate $= k$ (percent in decimal form)

Equation: $y = kx$

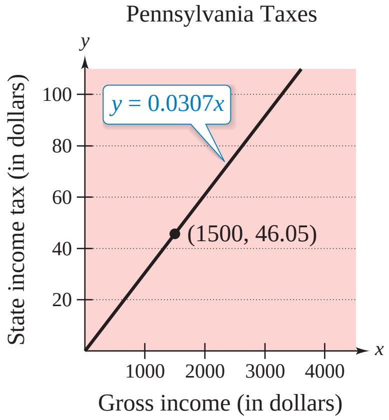


Figure 3.23

To solve for k, substitute the given information into the equation $y = kx$, and then solve for k.

$y = kx$ Write direct variation model.

$46.05 = k(1500)$ Substitute 46.05 for y and 1500 for x.

$0.0307 = k$ Simplify.

So, the equation (or model) for state income tax in Pennsylvania is

$$y = 0.0307x.$$

In other words, Pennsylvania has a state income tax rate of 3.07% of gross income. The graph of this equation is shown in Figure 3.23.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The simple interest on an investment is directly proportional to the amount of the investment. For instance, an investment of \$2500 will earn \$187.50 after 1 year. Find a mathematical model that gives the interest I after 1 year in terms of the amount invested P.

Direct Variation as an *n*th Power

Another type of direct variation relates one variable to a *power* of another variable. For example, in the formula for the area of a circle

$$A = \pi r^2$$

the area A is directly proportional to the square of the radius r. Note that for this formula, π is the constant of proportionality.

REMARK Note that the direct variation model $y = kx$ is a special case of $y = kx^n$ with $n = 1$.

Direct Variation as an *n*th Power

The following statements are equivalent.

1. y **varies directly as the *n*th power** of x.
2. y is **directly proportional to the *n*th power** of x.
3. $y = kx^n$ for some nonzero constant k.

EXAMPLE 4 Direct Variation as an *n*th Power

The distance a ball rolls down an inclined plane is directly proportional to the square of the time it rolls. During the first second, the ball rolls 8 feet. (See Figure 3.24.)

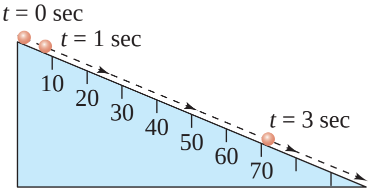


Figure 3.24

a. Write an equation relating the distance traveled to the time.

b. How far will the ball roll during the first 3 seconds?

Solution

a. Letting d be the distance (in feet) the ball rolls and letting t be the time (in seconds), you have

$$d = kt^2.$$

Now, because $d = 8$ when $t = 1$, you can see that $k = 8$, as follows.

$d = kt^2$ Write direct variation model.

$8 = k(1)^2$ Substitute 8 for d and 1 for t.

$8 = k$ Simplify.

So, the equation relating distance to time is

$$d = 8t^2.$$

b. When $t = 3$, the distance traveled is

$d = 8(3)^2$ Substitute 3 for t.

$= 8(9)$ Simplify.

$= 72$ feet. Simplify.

So, the ball will roll 72 feet during the first 3 seconds.

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Neglecting air resistance, the distance s an object falls varies directly as the square of the duration t of the fall. An object falls a distance of 144 feet in 3 seconds. How far will it fall in 6 seconds?

In Examples 3 and 4, the direct variations are such that an *increase* in one variable corresponds to an *increase* in the other variable. This is also true in the model $d = \frac{1}{5}F$, $F > 0$, where an increase in F results in an increase in d. You should not, however, assume that this always occurs with direct variation. For example, in the model $y = -3x$, an increase in x results in a *decrease* in y, and yet y is said to vary directly as x.

Inverse Variation

Inverse Variation

The following statements are equivalent.

1. y **varies inversely** as x.
2. y is **inversely proportional** to x.
3. $y = \frac{k}{x}$ for some nonzero constant k.

If x and y are related by an equation of the form $y = k/x^n$, then y varies inversely as the nth power of x (or y is inversely proportional to the nth power of x).

EXAMPLE 5 Inverse Variation

A company has found that the demand for one of its products varies inversely as the price of the product. When the price is \$6.25, the demand is 400 units. Approximate the demand when the price is \$5.75.

According to the National Association of Manufacturers, "manufacturing supports an estimated 17 million jobs in the U.S.—about one in six private sector jobs."

Solution

Let p be the price and let x be the demand. Because x varies inversely as p, you have

$$x = \frac{k}{p}.$$

Now, because $x = 400$ when $p = 6.25$, you have

$$x = \frac{k}{p} \qquad \text{Write inverse variation model.}$$

$$400 = \frac{k}{6.25} \qquad \text{Substitute 400 for } x \text{ and 6.25 for } p.$$

$$(400)(6.25) = k \qquad \text{Multiply each side by 6.25.}$$

$$2500 = k. \qquad \text{Simplify.}$$

So, the equation relating price and demand is

$$x = \frac{2500}{p}.$$

When $p = 5.75$, the demand is

$$x = \frac{2500}{p} \qquad \text{Write inverse variation model.}$$

$$= \frac{2500}{5.75} \qquad \text{Substitute 5.75 for } p.$$

$$\approx 435 \text{ units.} \qquad \text{Simplify.}$$

So, the demand for the product is 435 units when the price of the product is \$5.75.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The company in Example 5 has found that the demand for another of its products varies inversely as the price of the product. When the price is \$2.75, the demand is 600 units. Approximate the demand when the price is \$3.25.

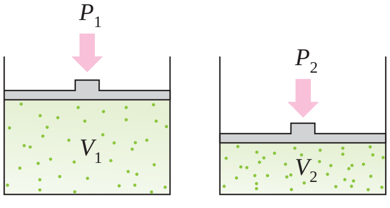
Marcin Balcerzak/Shutterstock.com

Combined Variation

Some applications of variation involve problems with *both* direct and inverse variations in the same model. These types of models are said to have **combined variation.**

EXAMPLE 6 Combined Variation

A gas law states that the volume of an enclosed gas varies directly as the temperature *and* inversely as the pressure, as shown in Figure 3.25. The pressure of a gas is 0.75 kilogram per square centimeter when the temperature is 294 K and the volume is 8000 cubic centimeters.



If $P_2 > P_1$, then $V_2 < V_1$.

If the temperature is held constant and pressure increases, then the volume decreases.

Figure 3.25

a. Write an equation relating pressure, temperature, and volume.

b. Find the pressure when the temperature is 300 K and the volume is 7000 cubic centimeters.

Solution

a. Let V be volume (in cubic centimeters), let P be pressure (in kilograms per square centimeter), and let T be temperature (in Kelvin). Because V varies directly as T and inversely as P, you have

$$V = \frac{kT}{P}.$$

Now, because $P = 0.75$ when $T = 294$ and $V = 8000$, you have

$$V = \frac{kT}{P} \qquad \text{Write combined variation model.}$$

$$8000 = \frac{k(294)}{0.75} \qquad \text{Substitute 8000 for } V\text{, 294 for } T\text{, and 0.75 for } P.$$

$$\frac{6000}{294} = k \qquad \text{Simplify.}$$

$$\frac{1000}{49} = k. \qquad \text{Simplify.}$$

So, the equation relating pressure, temperature, and volume is

$$V = \frac{1000}{49}\left(\frac{T}{P}\right).$$

b. Isolate P on one side of the equation by multiplying each side by P and dividing each side by V to obtain $P = \frac{1000}{49}\left(\frac{T}{V}\right)$. When $T = 300$ and $V = 7000$, the pressure is

$$P = \frac{1000}{49}\left(\frac{T}{V}\right) \qquad \text{Combined variation model solved for } P.$$

$$= \frac{1000}{49}\left(\frac{300}{7000}\right) \qquad \text{Substitute 300 for } T \text{ and 7000 for } V.$$

$$= \frac{300}{343} \qquad \text{Simplify.}$$

$$\approx 0.87 \text{ kilogram per square centimeter.} \qquad \text{Simplify.}$$

So, the pressure is about 0.87 kilogram per square centimeter when the temperature is 300 K and the volume is 7000 cubic centimeters.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The resistance of a copper wire carrying an electrical current is directly proportional to its length and inversely proportional to its cross-sectional area. Given that #28 copper wire (which has a diameter of 0.0126 inch) has a resistance of 66.17 ohms per thousand feet, what length of #28 copper wire will produce a resistance of 33.5 ohms? ■

Joint Variation

Joint Variation

The following statements are equivalent.

1. z **varies jointly** as x and y.
2. z is **jointly proportional** to x and y.
3. $z = kxy$ for some nonzero constant k.

If x, y, and z are related by an equation of the form $z = kx^n y^m$, then z varies jointly as the nth power of x and the mth power of y.

EXAMPLE 7 Joint Variation

The *simple* interest for a certain savings account is jointly proportional to the time and the principal. After one quarter (3 months), the interest on a principal of \$5000 is \$43.75. (a) Write an equation relating the interest, principal, and time. (b) Find the interest after three quarters.

Solution

a. Let I = interest (in dollars), P = principal (in dollars), and t = time (in years). Because I is jointly proportional to P and t, you have

$$I = kPt.$$

For $I = 43.75$, $P = 5000$, and $t = \frac{1}{4}$, you have

$$43.75 = k(5000)\left(\frac{1}{4}\right)$$

which implies that $k = 4(43.75)/5000 = 0.035$. So, the equation relating interest, principal, and time is

$$I = 0.035Pt$$

which is the familiar equation for simple interest where the constant of proportionality, 0.035, represents an annual interest rate of 3.5%.

b. When $P = \$5000$ and $t = \frac{3}{4}$, the interest is

$$I = (0.035)(5000)\left(\frac{3}{4}\right) = \$131.25.$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The kinetic energy E of an object varies jointly with the object's mass m and the square of the object's velocity v. An object with a mass of 50 kilograms traveling at 16 meters per second has a kinetic energy of 6400 joules. What is the kinetic energy of an object with mass of 70 kilograms traveling at 20 meters per second? ■

Summarize (Section 3.5)

1. Give an example of how you can use a mathematical model to approximate a set of data points *(page 289, Example 1)*.
2. Describe how you can use the *regression* feature of a graphing utility to find the equation of a least squares regression line *(page 290, Example 2)*.
3. Describe direct variation, direct variation as an nth power, inverse variation, combined variation, and joint variation *(pages 291–295, Examples 3–7)*.

3.5 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. Two techniques for fitting models to data are called direct and inverse ________ and least squares ________.
2. Statisticians use a measure called the ________ of ________ ________ to find a model that approximates a set of data most accurately.
3. The linear model with the least sum of square differences is called the ________ ________ ________ line.
4. An r-value of a set of data, also called a ________ ________, gives a measure of how well a model fits a set of data.
5. Direct variation models can be described as "y varies directly as x," or "y is ________ ________ to x."
6. In direct variation models of the form $y = kx$, k is called the ________ of ________.
7. The direct variation model $y = kx^n$ can be described as "y varies directly as the nth power of x," or "y is ________ ________ to the nth power of x."
8. The mathematical model $y = \frac{k}{x}$ is an example of ________ variation.
9. Mathematical models that involve both direct and inverse variation are said to have ________ variation.
10. The joint variation model $z = kxy$ can be described as "z varies jointly as x and y," or "z is ________ ________ to x and y."

Skills and Applications

11. Labor Force The total numbers of people 16 years of age and over (in thousands) not in the U.S. civilian labor force from 1998 through 2010 are given by the following ordered pairs. *(Spreadsheet at LarsonPrecalculus.com)*

(1998, 67,547)
(1999, 68,385)
(2000, 69,994)
(2001, 71,359)
(2002, 72,707)
(2003, 74,658)
(2004, 75,956)
(2005, 76,762)
(2006, 77,387)
(2007, 78,743)
(2008, 79,501)
(2009, 81,659)
(2010, 83,941)

A linear model that approximates the data is $y = 1298.8t + 57{,}094$, $8 \leq t \leq 20$, where y represents the total number of people 16 years of age and over (in thousands) not in the U.S. civilian labor force and $t = 8$ represents 1998. Plot the actual data and the model on the same set of coordinate axes. How closely does the model represent the data? *(Source: U.S. Bureau of Labor Statistics)*

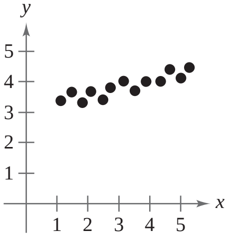
12. Sports The winning times (in minutes) in the women's 400-meter freestyle swimming event in the Olympics from 1948 through 2008 are given by the following ordered pairs. *(Spreadsheet at LarsonPrecalculus.com)*

(1948, 5.30)
(1952, 5.20)
(1956, 4.91)
(1960, 4.84)
(1964, 4.72)
(1968, 4.53)
(1972, 4.32)
(1976, 4.16)
(1980, 4.15)
(1984, 4.12)
(1988, 4.06)
(1992, 4.12)
(1996, 4.12)
(2000, 4.10)
(2004, 4.09)
(2008, 4.05)

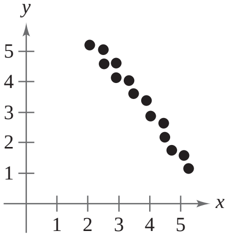
A linear model that approximates the data is $y = -0.020t + 5.00$, $-2 \leq t \leq 58$, where y represents the winning time (in minutes) and $t = -2$ represents 1948. Plot the actual data and the model on the same set of coordinate axes. How closely does the model represent the data? Does it appear that another type of model may be a better fit? Explain. *(Source: International Olympic Committee)*

Sketching a Line In Exercises 13–16, sketch the line that you think best approximates the data in the scatter plot. Then find an equation of the line. To print an enlarged copy of the graph, go to *MathGraphs.com*.

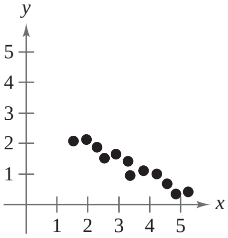
13.



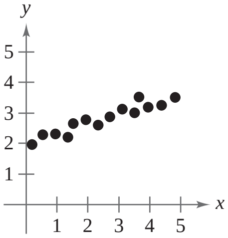
14.



15.



16.



17. Sports The lengths (in feet) of the winning men's discus throws in the Olympics from 1920 through 2008 are given by the following ordered pairs. *(Spreadsheet at LarsonPrecalculus.com) (Source: International Olympic Committee)*

(1920, 146.6)
(1924, 151.3)
(1928, 155.3)
(1932, 162.3)
(1936, 165.6)
(1948, 173.2)
(1952, 180.5)
(1956, 184.9)
(1960, 194.2)
(1964, 200.1)
(1968, 212.5)
(1972, 211.3)
(1976, 221.5)
(1980, 218.7)
(1984, 218.5)
(1988, 225.8)
(1992, 213.7)
(1996, 227.7)
(2000, 227.3)
(2004, 229.3)
(2008, 225.8)

(a) Sketch a scatter plot of the data. Let y represent the length of the winning discus throw (in feet) and let $t = 20$ represent 1920.

(b) Use a straightedge to sketch the best-fitting line through the points and find an equation of the line.

(c) Use the *regression* feature of a graphing utility to find the least squares regression line that fits the data.

(d) Compare the linear model you found in part (b) with the linear model given by the graphing utility in part (c).

18. Data Analysis: Broadway Shows The annual gross ticket sales S (in millions of dollars) for Broadway shows in New York City from 1995 through 2011 are given by the following ordered pairs. *(Spreadsheet at LarsonPrecalculus.com) (Source: The Broadway League)*

(1995, 406)
(1996, 436)
(1997, 499)
(1998, 558)
(1999, 588)
(2000, 603)
(2001, 666)
(2002, 643)
(2003, 721)
(2004, 771)
(2005, 769)
(2006, 862)
(2007, 939)
(2008, 938)
(2009, 943)
(2010, 1020)
(2011, 1080)

(a) Use a graphing utility to create a scatter plot of the data. Let $t = 5$ represent 1995.

(b) Use the *regression* feature of the graphing utility to find the equation of the least squares regression line that fits the data.

(c) Use the graphing utility to graph the scatter plot you created in part (a) and the model you found in part (b) in the same viewing window. How closely does the model represent the data?

(d) Use the model to predict the annual gross ticket sales in 2017.

(e) Interpret the meaning of the slope of the linear model in the context of the problem.

Direct Variation **In Exercises 19–26, assume that y is directly proportional to x. Use the given x-value and y-value to find a linear model that relates y and x.**

19. $x = 2, y = 14$
20. $x = 5, y = 12$
21. $x = 10, y = 2050$
22. $x = 6, y = 580$
23. $x = 5, y = 1$
24. $x = -24, y = 3$
25. $x = 4, y = 8\pi$
26. $x = \pi, y = -1$

Direct Variation as an *n*th Power **In Exercises 27–30, use the given values of k and n to complete the table for the direct variation model $y = kx^n$. Plot the points in a rectangular coordinate system.**

x	2	4	6	8	10
$y = kx^n$					

27. $k = 1, n = 2$
28. $k = 2, n = 2$
29. $k = \frac{1}{2}, n = 3$
30. $k = \frac{1}{4}, n = 3$

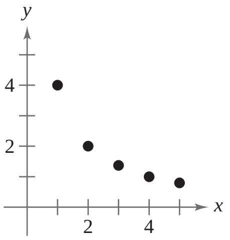
Inverse Variation as an *n*th Power **In Exercises 31–34, use the given values of k and n to complete the table for the inverse variation model $y = k/x^n$. Plot the points in a rectangular coordinate system.**

x	2	4	6	8	10
$y = k/x^n$					

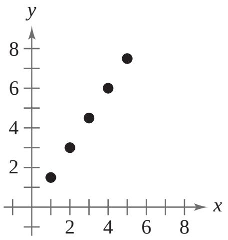
31. $k = 2, n = 1$
32. $k = 5, n = 1$
33. $k = 10, n = 2$
34. $k = 20, n = 2$

Think About It **In Exercises 35 and 36, use the graph to determine whether y varies directly as some power of x or inversely as some power of x. Explain.**

35.



36.



Determining Variation **In Exercises 37–40, determine whether the variation model represented by the ordered pairs (x, y) is of the form $y = kx$ or $y = k/x$, and find k. Then write a model that relates y and x.**

37. $(5, 1), \left(10, \frac{1}{2}\right), \left(15, \frac{1}{3}\right), \left(20, \frac{1}{4}\right), \left(25, \frac{1}{5}\right)$
38. $(5, 2), (10, 4), (15, 6), (20, 8), (25, 10)$
39. $(5, -3.5), (10, -7), (15, -10.5), (20, -14), (25, -17.5)$
40. $(5, 24), (10, 12), (15, 8), (20, 6), \left(25, \frac{24}{5}\right)$

Finding a Mathematical Model **In Exercises 41–50, find a mathematical model for the verbal statement.**

41. A varies directly as the square of r.

42. V varies directly as the cube of e.

43. y varies inversely as the square of x.

44. h varies inversely as the square root of s.

45. F varies directly as g and inversely as r^2.

46. z varies jointly as the square of x and the cube of y.

47. ***Newton's Law of Cooling:*** The rate of change R of the temperature of an object is directly proportional to the difference between the temperature T of the object and the temperature T_e of the environment in which the object is placed.

48. ***Boyle's Law:*** For a constant temperature, the pressure P of a gas is inversely proportional to the volume V of the gas.

49. ***Logistic Growth:*** The rate of growth R of a population is jointly proportional to the size S of the population and the difference between S and the maximum population size L that the environment can support.

50. ***Newton's Law of Universal Gravitation:*** The gravitational attraction F between two objects of masses m_1 and m_2 is jointly proportional to the masses and inversely proportional to the square of the distance r between the objects.

Describing a Formula **In Exercises 51–54, write a sentence using the variation terminology of this section to describe the formula.**

51. ***Surface area of a sphere:***

$S = 4\pi r^2$

52. ***Average speed:***

$r = d/t$

53. ***Area of a triangle:***

$A = \frac{1}{2}bh$

54. ***Volume of a right circular cylinder:***

$V = \pi r^2 h$

Finding a Mathematical Model **In Exercises 55–62, find a mathematical model that represents the statement. (Determine the constant of proportionality.)**

55. A varies directly as r^2. ($A = 9\pi$ when $r = 3$.)

56. y varies inversely as x. ($y = 3$ when $x = 25$.)

57. y is inversely proportional to x. ($y = 7$ when $x = 4$.)

58. z varies jointly as x and y. ($z = 64$ when $x = 4$ and $y = 8$.)

59. F is jointly proportional to r and the third power of s. ($F = 4158$ when $r = 11$ and $s = 3$.)

60. P varies directly as x and inversely as the square of y. $\left(P = \frac{28}{3}\text{ when } x = 42 \text{ and } y = 9.\right)$

61. z varies directly as the square of x and inversely as y. ($z = 6$ when $x = 6$ and $y = 4$.)

62. v varies jointly as p and q and inversely as the square of s. ($v = 1.5$ when $p = 4.1$, $q = 6.3$, and $s = 1.2$.)

63. **Simple Interest** The simple interest on an investment is directly proportional to the amount of the investment. An investment of \$3250 will earn \$113.75 after 1 year. Find a mathematical model that gives the interest I after 1 year in terms of the amount invested P.

64. **Simple Interest** The simple interest on an investment is directly proportional to the amount of the investment. An investment of \$6500 will earn \$211.25 after 1 year. Find a mathematical model that gives the interest I after 1 year in terms of the amount invested P.

65. **Measurement** Use the fact that 13 inches is approximately the same length as 33 centimeters to find a mathematical model that relates centimeters y to inches x. Then use the model to find the numbers of centimeters in 10 inches and 20 inches.

66. **Measurement** Use the fact that 14 gallons is approximately the same amount as 53 liters to find a mathematical model that relates liters y to gallons x. Then use the model to find the numbers of liters in 5 gallons and 25 gallons.

Hooke's Law **In Exercises 67–70, use Hooke's Law for springs, which states that the distance a spring is stretched (or compressed) varies directly as the force on the spring.**

67. A force of 220 newtons stretches a spring 0.12 meter. What force is required to stretch the spring 0.16 meter?

68. A force of 265 newtons stretches a spring 0.15 meter.

(a) What force is required to stretch the spring 0.1 meter?

(b) How far will a force of 90 newtons stretch the spring?

69. The coiled spring of a toy supports the weight of a child. The spring is compressed a distance of 1.9 inches by the weight of a 25-pound child. The toy will not work properly if its spring is compressed more than 3 inches. What is the maximum weight for which the toy will work properly?

70. An overhead garage door has two springs, one on each side of the door. A force of 15 pounds is required to stretch each spring 1 foot. Because of a pulley system, the springs stretch only one-half the distance the door travels. The door moves a total of 8 feet, and the springs are at their natural lengths when the door is open. Find the combined lifting force applied to the door by the springs when the door is closed.

71. **Ecology** The diameter of the largest particle that can be moved by a stream varies approximately directly as the square of the velocity of the stream. A stream with a velocity of $\frac{1}{4}$ mile per hour can move coarse sand particles about 0.02 inch in diameter. Approximate the velocity required to carry particles 0.12 inch in diameter.

72. **Music** The frequency of vibrations of a piano string varies directly as the square root of the tension on the string and inversely as the length of the string. The middle A string has a frequency of 440 vibrations per second. Find the frequency of a string that has 1.25 times as much tension and is 1.2 times as long.

73. **Work** The work W done when lifting an object varies jointly with the object's mass m and the height h that the object is lifted. The work done when a 120-kilogram object is lifted 1.8 meters is 2116.8 joules. How much work is done when lifting a 100-kilogram object 1.5 meters?

74. **Beam Load** The maximum load that can be safely supported by a horizontal beam varies jointly as the width of the beam and the square of its depth and inversely as the length of the beam. Determine the changes in the maximum safe load under the following conditions.

(a) The width and length of the beam are doubled.

(b) The width and depth of the beam are doubled.

75. **Data Analysis: Ocean Temperatures**

An oceanographer took readings of the water temperatures C (in degrees Celsius) at several depths d (in meters). The data collected are shown as ordered pairs (d, C). *(Spreadsheet at LarsonPrecalculus.com)*

(1000, 4.2)	(4000, 1.2)
(2000, 1.9)	(5000, 0.9)
(3000, 1.4)	

(a) Sketch a scatter plot of the data.

(b) Does it appear that the data can be modeled by the inverse variation model $C = k/d$? If so, find k for each pair of coordinates.

(c) Determine the mean value of k from part (b) to find the inverse variation model $C = k/d$.

(d) Use a graphing utility to plot the data points and the inverse model from part (c).

(e) Use the model to approximate the depth at which the water temperature is 3°C.

76. **Data Analysis: Light Intensity** A light probe is located x centimeters from a light source, and the intensity y (in microwatts per square centimeter) of the light is measured. The results are shown as ordered pairs (x, y). *(Spreadsheet at LarsonPrecalculus.com)*

(30, 0.1881)	(34, 0.1543)	(38, 0.1172)
(42, 0.0998)	(46, 0.0775)	(50, 0.0645)

A model for the data is $y = 262.76/x^{2.12}$.

(a) Use a graphing utility to plot the data points and the model in the same viewing window.

(b) Use the model to approximate the light intensity 25 centimeters from the light source.

Exploration

True or False? **In Exercises 77 and 78, decide whether the statement is true or false. Justify your answer.**

77. In the equation for the area of a circle, $A = \pi r^2$, the area A varies jointly with π and the square of the radius r.

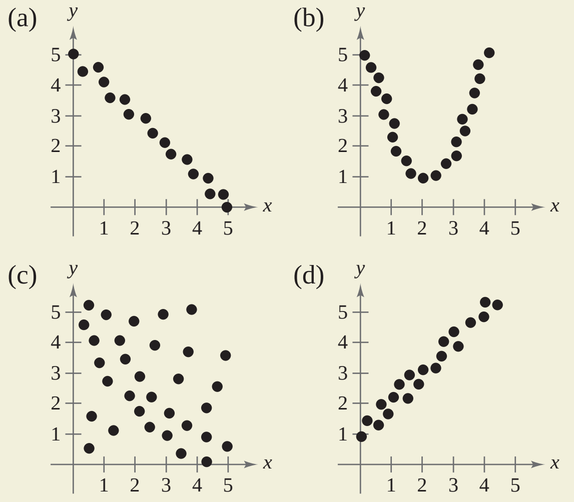
78. If the correlation coefficient for a least squares regression line is close to -1, then the regression line cannot be used to describe the data.

79. **Writing**

(a) Given that y varies directly as the square of x and x is doubled, how will y change? Explain.

(b) Given that y varies inversely as the square of x and x is doubled, how will y change? Explain.

80. **HOW DO YOU SEE IT?** Discuss how well the data shown in each scatter plot can be approximated by a linear model.



Project: Fraud and Identity Theft To work an extended application analyzing the numbers of fraud complaints and identity theft victims in the United States in 2010, visit this text's website at *LarsonPrecalculus.com*. *(Source: U.S. Federal Trade Commission)*

Khoroshunova Olga/Shutterstock.com

Chapter Summary

	What Did You Learn?	Explanation/Examples	Review Exercises
Section 3.1	Analyze graphs of quadratic functions *(p. 242).*	Let a, b, and c be real numbers with $a \neq 0$. The function $f(x) = ax^2 + bx + c$ is called a quadratic function. Its graph is a "U"-shaped curve called a parabola.	1, 2
	Write quadratic functions in standard form and use the results to sketch graphs of functions *(p. 245).*	The quadratic function $f(x) = a(x - h)^2 + k$, $a \neq 0$, is in standard form. The graph of f is a parabola whose axis is the vertical line $x = h$ and whose vertex is (h, k). When $a > 0$, the parabola opens upward. When $a < 0$, the parabola opens downward.	3–20
	Find minimum and maximum values of quadratic functions in real-life applications *(p. 247).*	Consider $f(x) = ax^2 + bx + c$ with vertex $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$. When $a > 0$, f has a *minimum* at $x = -b/(2a)$. When $a < 0$, f has a *maximum* at $x = -b/(2a)$.	21–26
Section 3.2	Use transformations to sketch graphs of polynomial functions *(p. 252).*	The graph of a polynomial function is continuous (no breaks, holes, or gaps) and has only smooth, rounded turns.	27–32
	Use the Leading Coefficient Test to determine the end behaviors of graphs of polynomial functions *(p. 254).*	Consider the graph of $f(x) = a_n x^n + \cdot \cdot \cdot + a_1 x + a_0$. **When *n* is odd:** If $a_n > 0$, then the graph falls to the left and rises to the right. If $a_n < 0$, then the graph rises to the left and falls to the right. **When *n* is even:** If $a_n > 0$, then the graph rises to the left and to the right. If $a_n < 0$, then the graph falls to the left and to the right.	33–36
	Find and use the real zeros of polynomial functions as sketching aids *(p. 256).*	When f is a polynomial function and a is a real number, the following are equivalent: (1) $x = a$ is a *zero* of f, (2) $x = a$ is a *solution* of the equation $f(x) = 0$, (3) $(x - a)$ is a *factor* of $f(x)$, and (4) $(a, 0)$ is an *x-intercept* of the graph of f.	37–46
	Use the Intermediate Value Theorem to help locate the real zeros of polynomial functions *(p. 259).*	Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a) \neq f(b)$, then, in $[a, b]$, f takes on every value between $f(a)$ and $f(b)$.	47–50
Section 3.3	Use long division to divide polynomials by other polynomials *(p. 266).*	$\dfrac{\overset{\text{Dividend}}{x^2 + 3x + 5}}{\underset{\text{Divisor}}{x + 1}} = \overbrace{x + 2}^{\text{Quotient}} + \dfrac{\overset{\text{Remainder}}{3}}{\underset{\text{Divisor}}{x + 1}}$	51–56
	Use synthetic division to divide polynomials by binomials of the form $(x - k)$ *(p. 269).*	Divisor: $x + 3$ Dividend: $\overbrace{x^4 - 10x^2 - 2x + 4}$ $\begin{array}{r\|rrrrr} -3 & 1 & 0 & -10 & -2 & 4 \\ & & -3 & 9 & 3 & -3 \\ \hline & 1 & -3 & -1 & 1 & \boxed{1} \end{array}$ ← Remainder: 1 Quotient: $x^3 - 3x^2 - x + 1$	57–60
	Use the Remainder Theorem and the Factor Theorem *(p. 270).*	**The Remainder Theorem:** If a polynomial $f(x)$ is divided by $x - k$, then the remainder is $r = f(k)$. **The Factor Theorem:** A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.	61–68

	What Did You Learn?	Explanation/Examples	Review Exercises
Section 3.4	Use the Fundamental Theorem of Algebra to determine the number of zeros of polynomial functions *(p. 275).*	**The Fundamental Theorem of Algebra** If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has at least one zero in the complex number system. **Linear Factorization Theorem** If $f(x)$ is a polynomial of degree n, where $n > 0$, then $f(x)$ has precisely n linear factors $f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$ where $c_1, c_2, . . ., c_n$ are complex numbers.	69–74
	Find rational zeros of polynomial functions *(p. 276).*	The Rational Zero Test relates the possible rational zeros of a polynomial to the leading coefficient and to the constant term of the polynomial.	75–82
	Find conjugate pairs of complex zeros *(p. 279).*	**Complex Zeros Occur in Conjugate Pairs** Let f be a polynomial function that has real coefficients. If $a + bi$ $(b \neq 0)$ is a zero of the function, then the conjugate $a - bi$ is also a zero of the function.	83, 84
	Find zeros of polynomials by factoring *(p. 280).*	Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.	85–96
	Use Descartes's Rule of Signs *(p. 282)* and the Upper and Lower Bound Rules *(p. 283)* to find zeros of polynomials.	**Descartes's Rule of Signs** Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$ be a polynomial with real coefficients and $a_0 \neq 0$. **1.** The number of *positive real zeros* of f is either equal to the number of variations in sign of $f(x)$ or less than that number by an even integer. **2.** The number of *negative real zeros* of f is either equal to the number of variations in sign of $f(-x)$ or less than that number by an even integer.	97–100
Section 3.5	Use mathematical models to approximate sets of data points *(p. 289).*	To see how well a model fits a set of data, compare the actual values and model values of y. (See Example 1.)	101
	Use the *regression* feature of a graphing utility to find the equation of a least squares regression line *(p. 290).*	The sum of square differences is the sum of the squares of the differences between actual data values and model values. The least squares regression line is the linear model with the least sum of square differences. The *regression* feature of a graphing utility can be used to find the least squares regression line. The correlation coefficient (r-value) of the data gives a measure of how well the model fits the data. The closer the value of $\lvert r \rvert$ is to 1, the better the fit.	102
	Write mathematical models for direct variation *(p. 291)*, direct variation as an *n*th power *(p. 292)*, inverse variation *(p. 293)*, combined variation *(p. 294)*, and joint variation *(p. 295).*	**Direct variation:** $y = kx$ for some nonzero constant k. **Direct variation as an *n*th power:** $y = kx^n$ for some nonzero constant k. **Inverse variation:** $y = k/x$ for some nonzero constant k. **Combined variation:** Model has *both* direct and inverse variation. **Joint variation:** $z = kxy$ for some nonzero constant k.	103–108

Review Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

3.1 Sketching Graphs of Quadratic Functions

In Exercises 1 and 2, sketch the graph of each quadratic function and compare it with the graph of $y = x^2$.

1. (a) $f(x) = 2x^2$
(b) $g(x) = -2x^2$
(c) $h(x) = x^2 + 2$
(d) $k(x) = (x + 2)^2$

2. (a) $f(x) = x^2 - 4$
(b) $g(x) = 4 - x^2$
(c) $h(x) = (x - 3)^2$
(d) $k(x) = \frac{1}{2}x^2 - 1$

Using Standard Form to Graph a Parabola **In Exercises 3–14, write the quadratic function in standard form and sketch its graph. Identify the vertex, axis of symmetry, and x-intercept(s).**

3. $g(x) = x^2 - 2x$

4. $f(x) = 6x - x^2$

5. $f(x) = x^2 + 8x + 10$

6. $h(x) = 3 + 4x - x^2$

7. $f(t) = -2t^2 + 4t + 1$

8. $f(x) = x^2 - 8x + 12$

9. $h(x) = 4x^2 + 4x + 13$

10. $f(x) = x^2 - 6x + 1$

11. $h(x) = x^2 + 5x - 4$

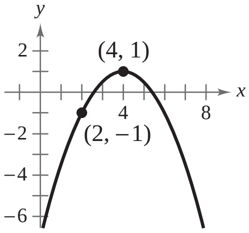
12. $f(x) = 4x^2 + 4x + 5$

13. $f(x) = \frac{1}{3}(x^2 + 5x - 4)$

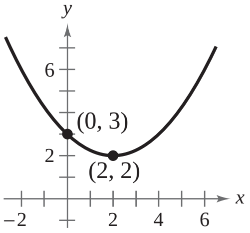
14. $f(x) = \frac{1}{2}(6x^2 - 24x + 22)$

Writing the Equation of a Parabola **In Exercises 15–20, write the standard form of the equation of the parabola that has the indicated vertex and passes through the given point.**

15.



16.



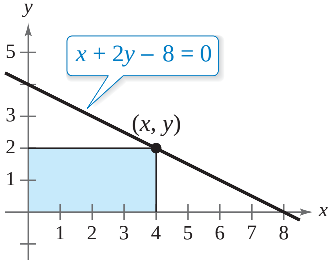
17. Vertex: $(1, -4)$; point: $(2, -3)$

18. Vertex: $(2, 3)$; point: $(-1, 6)$

19. Vertex: $\left(-\frac{3}{2}, 0\right)$; point: $\left(-\frac{9}{2}, -\frac{11}{4}\right)$

20. Vertex: $(3, 3)$; point: $\left(\frac{1}{4}, \frac{4}{5}\right)$

21. Numerical, Graphical, and Analytical Analysis A rectangle is inscribed in the region bounded by the x-axis, the y-axis, and the graph of $x + 2y - 8 = 0$, as shown in the figure.



(a) Write the area A of the rectangle as a function of x.

(b) Determine the domain of the function in the context of the problem.

(c) Construct a table showing possible values of x and the corresponding areas of the rectangle. Use the table to estimate the dimensions that will produce the maximum area.

(d) Use a graphing utility to graph the area function. Use the graph to approximate the dimensions that will produce the maximum area.

(e) Write the area function in standard form to find analytically the dimensions that will produce the maximum area.

22. Geometry The perimeter of a rectangle is 200 meters.

(a) Draw a diagram that gives a visual representation of the problem. Let x and y represent the length and width of the rectangle, respectively.

(b) Write y as a function of x. Use the result to write the area A as a function of x.

(c) Of all possible rectangles with perimeters of 200 meters, find the dimensions of the one with the maximum area.

23. Maximum Revenue The total revenue R earned (in dollars) from producing a gift box of candles is given by

$$R(p) = -10p^2 + 800p$$

where p is the price per box (in dollars).

(a) Find the revenues when the prices per box are \$20, \$25, and \$30.

(b) Find the unit price that will yield a maximum revenue. What is the maximum revenue? Explain your results.

24. **Maximum Profit** A real estate office handles an apartment building that has 50 units. When the rent is \$540 per month, all units are occupied. However, for each \$30 increase in rent, one unit becomes vacant. Each occupied unit requires an average of \$18 per month for service and repairs. What rent should be charged to obtain the maximum profit?

25. **Minimum Cost** A soft-drink manufacturer has daily production costs of

$$C = 70{,}000 - 120x + 0.055x^2$$

where C is the total cost (in dollars) and x is the number of units produced. How many units should be produced each day to yield a minimum cost?

26. **Sociology** The average age of the groom at a first marriage for a given age of the bride can be approximated by the model

$$y = -0.107x^2 + 5.68x - 48.5, \quad 20 \le x \le 25$$

where y is the age of the groom and x is the age of the bride. Sketch a graph of the model. For what age of the bride is the average age of the groom 26? *(Source: U.S. Census Bureau)*

3.2 Sketching a Transformation of a Monomial Function

In Exercises 27–32, sketch the graphs of $y = x^n$ and the transformation.

27. $y = x^3, \quad f(x) = -(x - 2)^3$

28. $y = x^3, \quad f(x) = -4x^3$

29. $y = x^4, \quad f(x) = 6 - x^4$

30. $y = x^4, \quad f(x) = 2(x - 8)^4$

31. $y = x^5, \quad f(x) = (x - 5)^5$

32. $y = x^5, \quad f(x) = \frac{1}{2}x^5 + 3$

Applying the Leading Coefficient Test

In Exercises 33–36, describe the right-hand and left-hand behavior of the graph of the polynomial function.

33. $f(x) = -2x^2 - 5x + 12$

34. $f(x) = \frac{1}{2}x^3 + 2x$

35. $g(x) = \frac{3}{4}(x^4 + 3x^2 + 2)$

36. $h(x) = -x^7 + 8x^2 - 8x$

Finding Real Zeros of a Polynomial Function

In Exercises 37–42, (a) find all real zeros of the polynomial function, (b) determine the multiplicity of each zero, (c) determine the maximum possible number of turning points of the graph of the function, and (d) use a graphing utility to graph the function and verify your answers.

37. $f(x) = 3x^2 + 20x - 32$

38. $f(x) = x(x + 3)^2$

39. $f(t) = t^3 - 3t$

40. $f(x) = x^3 - 8x^2$

41. $f(x) = -18x^3 + 12x^2$

42. $g(x) = x^4 + x^3 - 12x^2$

Sketching the Graph of a Polynomial Function

In Exercises 43–46, sketch the graph of the function by (a) applying the Leading Coefficient Test, (b) finding the real zeros of the polynomial, (c) plotting sufficient solution points, and (d) drawing a continuous curve through the points.

43. $f(x) = -x^3 + x^2 - 2$

44. $g(x) = 2x^3 + 4x^2$

45. $f(x) = x(x^3 + x^2 - 5x + 3)$

46. $h(x) = 3x^2 - x^4$

Using the Intermediate Value Theorem

In Exercises 47–50, (a) use the Intermediate Value Theorem and the *table* feature of a graphing utility to find intervals one unit in length in which the polynomial function is guaranteed to have a zero. (b) Adjust the table to approximate the zeros of the function. Use the *zero* or *root* feature of the graphing utility to verify your results.

47. $f(x) = 3x^3 - x^2 + 3$

48. $f(x) = 0.25x^3 - 3.65x + 6.12$

49. $f(x) = x^4 - 5x - 1$

50. $f(x) = 7x^4 + 3x^3 - 8x^2 + 2$

3.3 Long Division of Polynomials

In Exercises 51–56, use long division to divide.

51. $\dfrac{30x^2 - 3x + 8}{5x - 3}$

52. $\dfrac{4x + 7}{3x - 2}$

53. $\dfrac{5x^3 - 21x^2 - 25x - 4}{x^2 - 5x - 1}$

54. $\dfrac{3x^4}{x^2 - 1}$

55. $\dfrac{x^4 - 3x^3 + 4x^2 - 6x + 3}{x^2 + 2}$

56. $\dfrac{6x^4 + 10x^3 + 13x^2 - 5x + 2}{2x^2 - 1}$

Using Synthetic Division

In Exercises 57–60, use synthetic division to divide.

57. $\dfrac{6x^4 - 4x^3 - 27x^2 + 18x}{x - 2}$

58. $\dfrac{0.1x^3 + 0.3x^2 - 0.5}{x - 5}$

59. $\dfrac{2x^3 - 25x^2 + 66x + 48}{x - 8}$

60. $\dfrac{5x^3 + 33x^2 + 50x - 8}{x + 4}$

Using the Remainder Theorem **In Exercises 61 and 62, use the Remainder Theorem and synthetic division to find each function value.**

61. $f(x) = x^4 + 10x^3 - 24x^2 + 20x + 44$

(a) $f(-3)$ (b) $f(-1)$

62. $g(t) = 2t^5 - 5t^4 - 8t + 20$

(a) $g(-4)$ (b) $g(\sqrt{2})$

Using the Factor Theorem **In Exercises 63 and 64, use synthetic division to determine whether the given values of x are zeros of the function.**

63. $f(x) = 20x^4 + 9x^3 - 14x^2 - 3x$

(a) $x = -1$ (b) $x = \frac{3}{4}$ (c) $x = 0$ (d) $x = 1$

64. $f(x) = 3x^3 - 8x^2 - 20x + 16$

(a) $x = 4$ (b) $x = -4$ (c) $x = \frac{2}{3}$ (d) $x = -1$

Factoring a Polynomial **In Exercises 65–68, (a) verify the given factor(s) of $f(x)$, (b) find the remaining factors of $f(x)$, (c) use your results to write the complete factorization of $f(x)$, (d) list all real zeros of f, and (e) confirm your results by using a graphing utility to graph the function.**

Function	Factor(s)
65. $f(x) = x^3 + 4x^2 - 25x - 28$	$(x - 4)$
66. $f(x) = 2x^3 + 11x^2 - 21x - 90$	$(x + 6)$
67. $f(x) = x^4 - 4x^3 - 7x^2 + 22x + 24$	$(x + 2), (x - 3)$
68. $f(x) = x^4 - 11x^3 + 41x^2 - 61x + 30$	$(x - 2), (x - 5)$

3.4 Zeros of Polynomial Functions **In Exercises 69–74, determine the number of zeros of the polynomial function.**

69. $f(x) = x - 6$

70. $g(x) = x^2 - 2x - 8$

71. $h(t) = t^2 - t^5$

72. $f(x) = x^8 + x^9$

73. $f(x) = (x - 8)^3$

74. $g(t) = (2t - 1)^2 - (t + 1)^2$

Using the Rational Zero Test **In Exercises 75 and 76, use the Rational Zero Test to list the possible rational zeros of f.**

75. $f(x) = -4x^3 + 8x^2 - 3x + 15$

76. $f(x) = 3x^4 + 4x^3 - 5x^2 - 8$

Using the Rational Zero Test **In Exercises 77–82, find the rational zeros of the function.**

77. $f(x) = x^3 + 3x^2 - 28x - 60$

78. $f(x) = 4x^3 - 27x^2 + 11x + 42$

79. $f(x) = x^3 - 10x^2 + 17x - 8$

80. $f(x) = x^3 + 9x^2 + 24x + 20$

81. $f(x) = x^4 + x^3 - 11x^2 + x - 12$

82. $f(x) = 25x^4 + 25x^3 - 154x^2 - 4x + 24$

Finding a Polynomial Function with Given Zeros **In Exercises 83 and 84, find a polynomial function with real coefficients that has the given zeros. (There are many correct answers.)**

83. $\frac{2}{3}, 4, \sqrt{3}i$

84. $2, -3, 1 - 2i$

Finding the Zeros of a Polynomial Function **In Exercises 85–88, use the given zero to find all the zeros of the function.**

Function	Zero
85. $f(x) = x^3 - 4x^2 + x - 4$	i
86. $h(x) = -x^3 + 2x^2 - 16x + 32$	$-4i$
87. $g(x) = 2x^4 - 3x^3 - 13x^2 + 37x - 15$	$2 + i$
88. $f(x) = 4x^4 - 11x^3 + 14x^2 - 6x$	$1 - i$

Finding the Zeros of a Polynomial Function **In Exercises 89–92, write the polynomial as the product of linear factors and list all the zeros of the function.**

89. $f(x) = x^3 + 4x^2 - 5x$

90. $g(x) = x^3 - 7x^2 + 36$

91. $g(x) = x^4 + 4x^3 - 3x^2 + 40x + 208$

92. $f(x) = x^4 + 8x^3 + 8x^2 - 72x - 153$

Finding the Zeros of a Polynomial Function **In Exercises 93–96, find all the zeros of the function. When there is an extended list of possible rational zeros, use a graphing utility to graph the function in order to disregard any of the possible rational zeros that are obviously not zeros of the function.**

93. $f(x) = x^3 - 16x^2 + x - 16$

94. $f(x) = 4x^4 - 12x^3 - 71x^2 - 3x - 18$

95. $g(x) = x^4 - 3x^3 - 14x^2 - 12x - 72$

96. $g(x) = 9x^5 - 27x^4 - 86x^3 + 204x^2 - 40x + 96$

Using Descartes's Rule of Signs **In Exercises 97 and 98, use Descartes's Rule of Signs to determine the possible numbers of positive and negative real zeros of the function.**

97. $g(x) = 5x^3 + 3x^2 - 6x + 9$

98. $h(x) = -2x^5 + 4x^3 - 2x^2 + 5$

Verifying Upper and Lower Bounds **In Exercises 99 and 100, use synthetic division to verify the upper and lower bounds of the real zeros of f.**

99. $f(x) = 4x^3 - 3x^2 + 4x - 3$

(a) Upper: $x = 1$

(b) Lower: $x = -\frac{1}{4}$

100. $f(x) = 2x^3 - 5x^2 - 14x + 8$

(a) Upper: $x = 8$

(b) Lower: $x = -4$

3.5

101. **Compact Discs** The values V (in billions of dollars) of shipments of compact discs in the United States from 2004 through 2010 are shown in the table. *(Source: Recording Industry Association of America)*

Spreadsheet at LarsonPrecalculus.com

Year, t	Value, V
2004	11.45
2005	10.52
2006	9.37
2007	7.45
2008	5.47
2009	4.27
2010	3.36

A linear model that approximates these data is

$$V = -1.453t + 17.58, \quad 4 \le t \le 10$$

where t represents the year, with $t = 4$ corresponding to 2004. Plot the actual data and the model on the same set of coordinate axes. How closely does the model represent the data?

102. **Data Analysis: Radio Stations** The table shows the numbers N of noncommercial FM radio stations in the United States from 2002 through 2009. (*Source: Federal Communications Commission)*

Spreadsheet at LarsonPrecalculus.com

Year, t	Stations, N
2002	2354
2003	2552
2004	2533
2005	2672
2006	2817
2007	2892
2008	3040
2009	3151

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 2$ corresponding to 2002.

(b) Use the *regression* feature of the graphing utility to find the equation of the least squares regression line that fits the data. Then graph the model and the scatter plot you found in part (a) in the same viewing window. How closely does the model represent the data?

(c) Use the model to predict the number of noncommercial FM radio stations in 2020.

(d) Interpret the meaning of the slope of the linear model in the context of the problem.

103. **Measurement** You notice a billboard indicating that it is 2.5 miles or 4 kilometers to the next restaurant of a national fast-food chain. Use this information to find a mathematical model that relates miles to kilometers. Then use the model to find the numbers of kilometers in 2 miles and 10 miles.

104. **Energy** The power P produced by a wind turbine is directly proportional to the cube of the wind speed S. A wind speed of 27 miles per hour produces a power output of 750 kilowatts. Find the output for a wind speed of 40 miles per hour.

105. **Frictional Force** The frictional force F between the tires and the road required to keep a car on a curved section of a highway is directly proportional to the square of the speed s of the car. If the speed of the car is doubled, the force will change by what factor?

106. **Demand** A company has found that the daily demand x for its boxes of chocolates is inversely proportional to the price p. When the price is \$5, the demand is 800 boxes. Approximate the demand when the price is increased to \$6.

107. **Travel Time** The travel time between two cities is inversely proportional to the average speed. A train travels between the cities in 3 hours at an average speed of 65 miles per hour. How long would it take to travel between the cities at an average speed of 80 miles per hour?

108. **Cost** The cost of constructing a wooden box with a square base varies jointly as the height of the box and the square of the width of the box. A box of height 16 inches and of width 6 inches costs \$28.80. How much would a box of height 14 inches and of width 8 inches cost?

Exploration

True or False? **In Exercises 109 and 110, determine whether the statement is true or false. Justify your answer.**

109. A fourth-degree polynomial with real coefficients can have

$$-5, \quad -8i, \quad 4i, \quad \text{and} \quad 5$$

as its zeros.

110. If y is directly proportional to x, then x is directly proportional to y.

111. **Writing** Explain how to determine the maximum or minimum value of a quadratic function.

112. **Writing** Explain the connections between factors of a polynomial, zeros of a polynomial function, and solutions of a polynomial equation.

Chapter Test

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

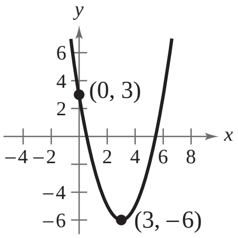


Figure for 3

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

1. Sketch the graph of each quadratic function and compare it with the graph of $y = x^2$.

(a) $g(x) = 2 - x^2$ (b) $g(x) = \left(x - \frac{3}{2}\right)^2$

2. Identify the vertex and intercepts of the graph of $f(x) = x^2 + 4x + 3$.

3. Write the standard form of the equation of the parabola shown at the left.

4. The path of a ball is given by the function $f(x) = -\frac{1}{20}x^2 + 3x + 5$, where $f(x)$ is the height (in feet) of the ball and x is the horizontal distance (in feet) from where the ball was thrown.

(a) What is the maximum height of the ball?

(b) Which number determines the height at which the ball was thrown? Does changing this value change the coordinates of the maximum height of the ball? Explain.

5. Describe the right-hand and left-hand behavior of the graph of the function $h(t) = -\frac{3}{4}t^5 + 2t^2$. Then sketch its graph.

6. Divide using long division.

$$\frac{3x^3 + 4x - 1}{x^2 + 1}$$

7. Divide using synthetic division.

$$\frac{2x^4 - 5x^2 - 3}{x - 2}$$

8. Use synthetic division to show that $x = \sqrt{3}$ is a zero of the function

$f(x) = 2x^3 - 5x^2 - 6x + 15.$

Use the result to factor the polynomial function completely and list all the zeros of the function.

In Exercises 9 and 10, find the rational zeros of the function.

9. $g(t) = 2t^4 - 3t^3 + 16t - 24$

10. $h(x) = 3x^5 + 2x^4 - 3x - 2$

In Exercises 11 and 12, find a polynomial function with real coefficients that has the given zeros. (There are many correct answers.)

11. $0, 3, 2 + i$

12. $1 - \sqrt{3}i, 2, 2$

In Exercises 13 and 14, find all the zeros of the function.

13. $f(x) = 3x^3 + 14x^2 - 7x - 10$

14. $f(x) = x^4 - 9x^2 - 22x - 24$

In Exercises 15–17, find a mathematical model that represents the statement. (Determine the constant of proportionality.)

15. v varies directly as the square root of s. ($v = 24$ when $s = 16$.)

16. A varies jointly as x and y. ($A = 500$ when $x = 15$ and $y = 8$.)

17. b varies inversely as a. ($b = 32$ when $a = 1.5$.)

Year, t	Insured Commercial Banks, y
6	7401
7	7284
8	7087
9	6840
10	6530

Spreadsheet at LarsonPrecalculus.com

18. The table at the left shows the numbers of insured commercial banks y in the United States for the years 2006 through 2010, where $t = 6$ represents 2006. Use the *regression* feature of a graphing utility to find the equation of the least squares regression line that fits the data. How well does the model represent the data? *(Source: Federal Deposit Insurance Corporation)*

Proofs in Mathematics

These two pages contain proofs of four important theorems about polynomial functions. The first two theorems are from Section 3.3, and the second two theorems are from Section 3.4.

The Remainder Theorem ***(p. 270)***

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is

$$r = f(k).$$

Proof

From the Division Algorithm, you have

$$f(x) = (x - k)q(x) + r(x)$$

and because either $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $x - k$, you know that $r(x)$ must be a constant. That is, $r(x) = r$. Now, by evaluating $f(x)$ at $x = k$, you have

$$\begin{aligned} f(k) &= (k - k)q(k) + r \\ &= (0)q(k) + r = r. \end{aligned}$$

■

To be successful in algebra, it is important that you understand the connection among *factors* of a polynomial, *zeros* of a polynomial function, and *solutions* or *roots* of a polynomial equation. The Factor Theorem is the basis for this connection.

The Factor Theorem ***(p. 271)***

A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.

Proof

Using the Division Algorithm with the factor $(x - k)$, you have

$$f(x) = (x - k)q(x) + r(x).$$

By the Remainder Theorem, $r(x) = r = f(k)$, and you have

$$f(x) = (x - k)q(x) + f(k)$$

where $q(x)$ is a polynomial of lesser degree than $f(x)$. If $f(k) = 0$, then

$$f(x) = (x - k)q(x)$$

and you see that $(x - k)$ is a factor of $f(x)$. Conversely, if $(x - k)$ is a factor of $f(x)$, then division of $f(x)$ by $(x - k)$ yields a remainder of 0. So, by the Remainder Theorem, you have $f(k) = 0$. ■

THE FUNDAMENTAL THEOREM OF ALGEBRA

The Linear Factorization Theorem is closely related to the Fundamental Theorem of Algebra. The Fundamental Theorem of Algebra has a long and interesting history. In the early work with polynomial equations, the Fundamental Theorem of Algebra was thought to have been not true, because imaginary solutions were not considered. In fact, in the very early work by mathematicians such as Abu al-Khwarizmi (c. 800 A.D.), negative solutions were also not considered.

Once imaginary numbers were accepted, several mathematicians attempted to give a general proof of the Fundamental Theorem of Algebra. These included Gottfried von Leibniz (1702), Jean d'Alembert (1746), Leonhard Euler (1749), Joseph-Louis Lagrange (1772), and Pierre Simon Laplace (1795). The mathematician usually credited with the first correct proof of the Fundamental Theorem of Algebra is Carl Friedrich Gauss, who published the proof in his doctoral thesis in 1799.

Linear Factorization Theorem *(p. 275)*

If $f(x)$ is a polynomial of degree n, where $n > 0$, then $f(x)$ has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$

where $c_1, c_2, \ldots, c_n$ are complex numbers.

Proof

Using the Fundamental Theorem of Algebra, you know that f must have at least one zero, c_1. Consequently, $(x - c_1)$ is a factor of $f(x)$, and you have

$$f(x) = (x - c_1)f_1(x).$$

If the degree of $f_1(x)$ is greater than zero, then you again apply the Fundamental Theorem of Algebra to conclude that f_1 must have a zero c_2, which implies that

$$f(x) = (x - c_1)(x - c_2)f_2(x).$$

It is clear that the degree of $f_1(x)$ is $n - 1$, that the degree of $f_2(x)$ is $n - 2$, and that you can repeatedly apply the Fundamental Theorem of Algebra n times until you obtain

$$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$

where a_n is the leading coefficient of the polynomial $f(x)$. ■

Factors of a Polynomial *(p. 280)*

Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

Proof

To begin, you use the Linear Factorization Theorem to conclude that $f(x)$ can be *completely* factored in the form

$$f(x) = d(x - c_1)(x - c_2)(x - c_3) \cdot \cdot \cdot (x - c_n).$$

If each c_i is real, then there is nothing more to prove. If any c_i is complex $(c_i = a + bi,$ $b \neq 0)$, then, because the coefficients of $f(x)$ are real, you know that the conjugate $c_j = a - bi$ is also a zero. By multiplying the corresponding factors, you obtain

$$\begin{aligned}(x - c_i)(x - c_j) &= [x - (a + bi)][x - (a - bi)]\\ &= x^2 - 2ax + (a^2 + b^2)\end{aligned}$$

where each coefficient is real. ■

P.S. Problem Solving

1. **Exploring Quadratic Functions**

(a) Find the zeros of each quadratic function $g(x)$.

(i) $g(x) = x^2 - 4x - 12$
(ii) $g(x) = x^2 + 5x$
(iii) $g(x) = x^2 + 3x - 10$
(iv) $g(x) = x^2 - 4x + 4$
(v) $g(x) = x^2 - 2x - 6$
(vi) $g(x) = x^2 + 3x + 4$

(b) For each function in part (a), use a graphing utility to graph $f(x) = (x - 2) \cdot g(x)$. Verify that $(2, 0)$ is an x-intercept of the graph of $f(x)$. Describe any similarities or differences in the behaviors of the six functions at this x-intercept.

(c) For each function in part (b), use the graph of $f(x)$ to approximate the other x-intercepts of the graph.

(d) Describe the connections that you find among the results of parts (a), (b), and (c).

2. **Building a Quonset Hut** Quonset huts were developed during World War II. They were temporary housing structures that could be assembled quickly and easily. A Quonset hut is shaped like a half cylinder. A manufacturer has 600 square feet of material with which to build a Quonset hut.

(a) The formula for the surface area of half a cylinder is $S = \pi r^2 + \pi rl$, where r is the radius and l is the length of the hut. Solve this equation for l when $S = 600$.

(b) The formula for the volume of the hut is $V = \frac{1}{2}\pi r^2 l$. Write the volume V of the Quonset hut as a polynomial function of r.

(c) Use the function you wrote in part (b) to find the maximum volume of a Quonset hut with a surface area of 600 square feet. What are the dimensions of the hut?

3. **Verifying the Remainder Theorem** Show that if $f(x) = ax^3 + bx^2 + cx + d$, then $f(k) = r$, where $r = ak^3 + bk^2 + ck + d$, using long division. In other words, verify the Remainder Theorem for a third-degree polynomial function.

4. **Babylonian Mathematics** In 2000 B.C., the Babylonians solved polynomial equations by referring to tables of values. One such table gave the values of $y^3 + y^2$. To be able to use this table, the Babylonians sometimes had to manipulate the equation, as shown below.

$$ax^3 + bx^2 = c \qquad \text{Original equation}$$

$$\frac{a^3x^3}{b^3} + \frac{a^2x^2}{b^2} = \frac{a^2c}{b^3} \qquad \text{Multiply each side by } \frac{a^2}{b^3}.$$

$$\left(\frac{ax}{b}\right)^3 + \left(\frac{ax}{b}\right)^2 = \frac{a^2c}{b^3} \qquad \text{Rewrite.}$$

Then they would find $(a^2c)/b^3$ in the $y^3 + y^2$ column of the table. Because they knew that the corresponding y-value was equal to $(ax)/b$, they could conclude that $x = (by)/a$.

(a) Calculate

$y^3 + y^2$

for

$y = 1, 2, 3, \ldots, 10.$

Record the values in a table.

(b) Use the table from part (a) and the method above to solve each equation.

(i) $x^3 + x^2 = 252$
(ii) $x^3 + 2x^2 = 288$
(iii) $3x^3 + x^2 = 90$
(iv) $2x^3 + 5x^2 = 2500$
(v) $7x^3 + 6x^2 = 1728$
(vi) $10x^3 + 3x^2 = 297$

(c) Using the methods from this chapter, verify your solution of each equation.

5. **Finding Dimensions** At a glassware factory, molten cobalt glass is poured into molds to make paperweights. Each mold is a rectangular prism whose height is 3 inches greater than the length of each side of the square base. A machine pours 20 cubic inches of liquid glass into each mold. What are the dimensions of the mold?

6. **Sums and Products of Zeros**

(a) Complete the table.

Function	Zeros	Sum of Zeros	Product of Zeros
$f_1(x) = x^2 - 5x + 6$			
$f_2(x) = x^3 - 7x + 6$			
$f_3(x) = x^4 + 2x^3 + x^2 + 8x - 12$			
$f_4(x) = x^5 - 3x^4 - 9x^3 + 25x^2 - 6x$			

(b) Use the table to make a conjecture relating the sum of the zeros of a polynomial function to the coefficients of the polynomial function.

(c) Use the table to make a conjecture relating the product of the zeros of a polynomial function to the coefficients of the polynomial function.

7. True or False? Determine whether the statement is true or false. If false, provide one or more reasons why the statement is false and correct the statement. Let

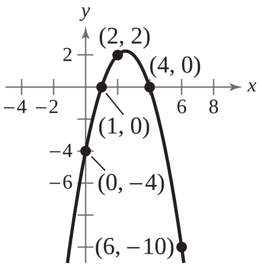
$$f(x) = ax^3 + bx^2 + cx + d, \quad a \neq 0$$

and let $f(2) = -1$. Then

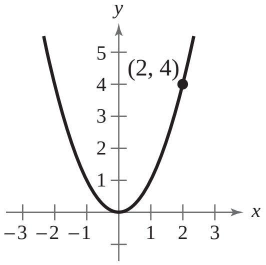
$$\frac{f(x)}{x+1} = q(x) + \frac{2}{x+1}$$

where $q(x)$ is a second-degree polynomial.

8. Finding the Equation of a Parabola The parabola shown in the figure has an equation of the form $y = ax^2 + bx + c$. Find the equation of this parabola using the following methods. (a) Find the equation analytically. (b) Use the *regression* feature of a graphing utility to find the equation.



9. Finding the Slope of a Tangent Line One of the fundamental themes of calculus is to find the slope of the tangent line to a curve at a point. To see how this can be done, consider the point (2, 4) on the graph of the quadratic function $f(x) = x^2$, as shown in the figure.



(a) Find the slope m_1 of the line joining (2, 4) and (3, 9). Is the slope of the tangent line at (2, 4) greater than or less than the slope of the line through (2, 4) and (3, 9)?

(b) Find the slope m_2 of the line joining (2, 4) and (1, 1). Is the slope of the tangent line at (2, 4) greater than or less than the slope of the line through (2, 4) and (1, 1)?

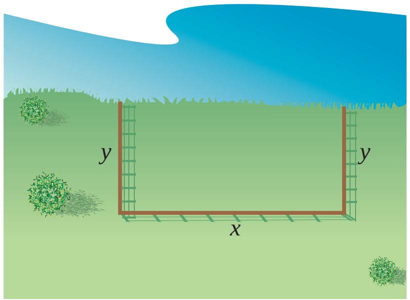
(c) Find the slope m_3 of the line joining (2, 4) and (2.1, 4.41). Is the slope of the tangent line at (2, 4) greater than or less than the slope of the line through (2, 4) and (2.1, 4.41)?

(d) Find the slope m_h of the line joining (2, 4) and $(2 + h, f(2 + h))$ in terms of the nonzero number h.

(e) Evaluate the slope formula from part (d) for $h = -1$, 1, and 0.1. Compare these values with those in parts (a)–(c).

(f) What can you conclude the slope $m_{\tan}$ of the tangent line at (2, 4) to be? Explain your answer.

10. Maximum Area A rancher plans to fence a rectangular pasture adjacent to a river (see figure). The rancher has 100 meters of fencing, and no fencing is needed along the river.



(a) Write the area A of the pasture as a function of x, the length of the side parallel to the river. What is the domain of A?

(b) Graph the function A and estimate the dimensions that yield the maximum area of the pasture.

(c) Find the exact dimensions that yield the maximum area of the pasture by writing the quadratic function in standard form.

11. Maximum and Minimum Area A wire 100 centimeters in length is cut into two pieces. One piece is bent to form a square and the other to form a circle. Let x equal the length of the wire used to form the square.

(a) Write the function that represents the combined area of the two figures.

(b) Determine the domain of the function.

(c) Find the value(s) of x that yield a maximum area and a minimum area. Explain your reasoning.

12. Roots of a Cubic Equation Can a cubic equation with real coefficients have two real zeros and one complex zero? Explain your reasoning.

4

Rational Functions and Conics

Solar System *(Page 332)*

Satellite Orbit
(Exercise 41, page 350)

Suspension Bridge *(Exercise 33, page 340)*

Medicine
(Exercise 84, page 328)

Mining *(Example 4, page 315)*

Clockwise from top left, Orla/Shutterstock.com; Paul Fleet/Shutterstock.com;
chungking/Shutterstock.com; kilukilu/Shutterstock.com; Cosmo Condina/The Image Bank/Getty Images

4.1 Rational Functions and Asymptotes

You can use rational functions to model and solve real-life problems relating to environmental scenarios. For instance, in Exercise 41 on page 318, you will use a rational function to determine the cost of supplying recycling bins to the population of a rural township.

- **Find the domains of rational functions.**
- **Find the vertical and horizontal asymptotes of graphs of rational functions.**
- **Use rational functions to model and solve real-life problems.**

Introduction

A **rational function** is a quotient of polynomial functions. It can be written in the form

$$f(x) = \frac{N(x)}{D(x)}$$

where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial.

In general, the *domain* of a rational function of x includes all real numbers except x-values that make the denominator zero. Much of the discussion of rational functions will focus on their graphical behavior near the x-values excluded from the domain.

EXAMPLE 1 Finding the Domain of a Rational Function

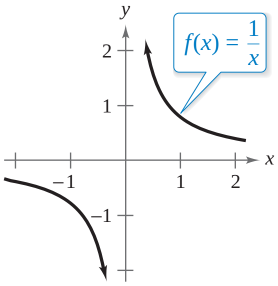
Find the domain of $f(x) = \frac{1}{x}$ and discuss the behavior of f near any excluded x-values.

Solution Because the denominator is zero when $x = 0$, the domain of f is all real numbers except $x = 0$. To determine the behavior of f near this excluded value, evaluate $f(x)$ to the left and right of $x = 0$, as indicated in the following tables.

x	-1	-0.5	-0.1	-0.01	-0.001	$\longrightarrow 0$
$f(x)$	-1	-2	-10	-100	-1000	$\longrightarrow -\infty$

x	$0 \longleftarrow$	0.001	0.01	0.1	0.5	1
$f(x)$	$\infty \longleftarrow$	1000	100	10	2	1

Note that as x approaches 0 *from the left*, $f(x)$ decreases without bound. In contrast, as x approaches 0 *from the right*, $f(x)$ increases without bound. The graph of f is shown below.



Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the domain of $f(x) = \frac{3x}{x-1}$ and discuss the behavior of f near any excluded x-values.

ZQFotography/Shutterstock.com

Vertical and Horizontal Asymptotes

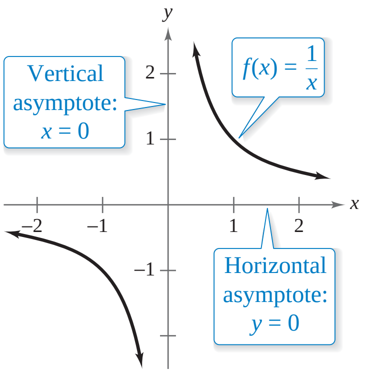


Figure 4.1

In Example 1, the behavior of f near $x = 0$ is denoted as follows.

$$\underbrace{f(x) \longrightarrow -\infty \text{ as } x \longrightarrow 0^-}_{\substack{f(x) \text{ decreases without bound} \\ \text{as } x \text{ approaches 0 from the left.}}} \qquad \underbrace{f(x) \longrightarrow \infty \text{ as } x \longrightarrow 0^+}_{\substack{f(x) \text{ increases without bound} \\ \text{as } x \text{ approaches 0 from the right.}}}$$

The line $x = 0$ is a **vertical asymptote** of the graph of f, as shown in Figure 4.1. From this figure, you can see that the graph of f also has a **horizontal asymptote**—the line $y = 0$. The behavior of f near $y = 0$ is denoted as follows.

$$\underbrace{f(x) \longrightarrow 0 \text{ as } x \longrightarrow -\infty}_{\substack{f(x) \text{ approaches 0 as } x \\ \text{decreases without bound.}}} \qquad \underbrace{f(x) \longrightarrow 0 \text{ as } x \longrightarrow \infty}_{\substack{f(x) \text{ approaches 0 as } x \\ \text{increases without bound.}}}$$

Definitions of Vertical and Horizontal Asymptotes

1. The line $x = a$ is a **vertical asymptote** of the graph of f when

$$f(x) \longrightarrow \infty \quad \text{or} \quad f(x) \longrightarrow -\infty$$

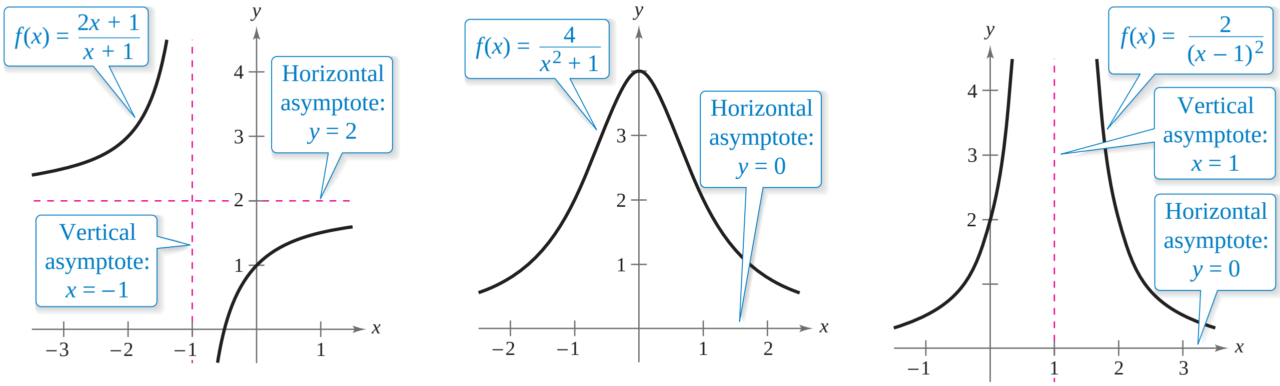
as $x \longrightarrow a$, either from the right or from the left.

2. The line $y = b$ is a **horizontal asymptote** of the graph of f when

$$f(x) \longrightarrow b$$

as $x \longrightarrow \infty$ or $x \longrightarrow -\infty$.

Eventually (as $x \longrightarrow \infty$ or $x \longrightarrow -\infty$), the distance between the horizontal asymptote and the points on the graph must approach zero. Figure 4.2 shows the vertical and horizontal asymptotes of the graphs of three rational functions.



(a) (b) (c)

Figure 4.2

The graphs of

$$f(x) = \frac{1}{x}$$

in Figure 4.1 and

$$f(x) = \frac{2x + 1}{x + 1}$$

in Figure 4.2(a) are **hyperbolas.** You will study hyperbolas in Sections 4.3 and 4.4.

Vertical and Horizontal Asymptotes of a Rational Function

Let f be the rational function

$$f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdot \cdot \cdot + b_1 x + b_0}$$

where $N(x)$ and $D(x)$ have no common factors.

1. The graph of f has *vertical* asymptotes at the zeros of $D(x)$.
2. The graph of f has one or no *horizontal* asymptote determined by comparing the degrees of $N(x)$ and $D(x)$.
   - **a.** When $n < m$, the graph of f has the line $y = 0$ (the x-axis) as a horizontal asymptote.
   - **b.** When $n = m$, the graph of f has the line $y = a_n/b_m$ (ratio of the leading coefficients) as a horizontal asymptote.
   - **c.** When $n > m$, the graph of f has no horizontal asymptote.

EXAMPLE 2 Finding Vertical and Horizontal Asymptotes

Find all vertical and horizontal asymptotes of the graph of each rational function.

a. $f(x) = \dfrac{2x}{3x^2 + 1}$ **b.** $f(x) = \dfrac{2x^2}{x^2 - 1}$

Solution

a. For this rational function, the degree of the numerator is *less than* the degree of the denominator, so the graph has the line $y = 0$ as a horizontal asymptote. To find any vertical asymptotes, set the denominator equal to zero and solve the resulting equation for x. Because the equation

$$3x^2 + 1 = 0$$

has no real solutions, you can conclude that the graph has no vertical asymptote. The graph of the function is shown in Figure 4.3.

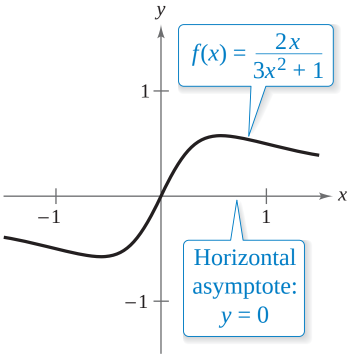


Figure 4.3

b. For this rational function, the degree of the numerator is *equal* to the degree of the denominator. The leading coefficient of the numerator is 2 and the leading coefficient of the denominator is 1, so the graph has the line $y = 2$ as a horizontal asymptote. To find any vertical asymptotes, set the denominator equal to zero and solve the resulting equation for x.

$$x^2 - 1 = 0 \qquad \text{Set denominator equal to zero.}$$

$$(x + 1)(x - 1) = 0 \qquad \text{Factor.}$$

$$x + 1 = 0 \;\Rightarrow\; x = -1 \qquad \text{Set 1st factor equal to 0.}$$

$$x - 1 = 0 \;\Rightarrow\; x = 1 \qquad \text{Set 2nd factor equal to 0.}$$

This equation has two real solutions, $x = -1$ and $x = 1$, so the graph has the lines $x = -1$ and $x = 1$ as vertical asymptotes. The graph of the function is shown in Figure 4.4.

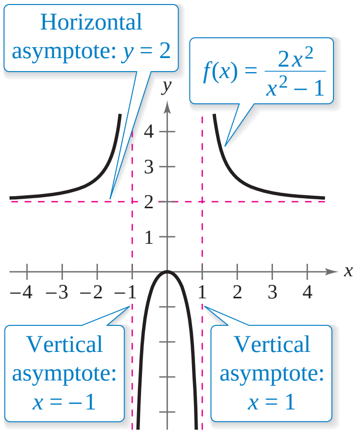


Figure 4.4

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find all vertical and horizontal asymptotes of the graph of $f(x) = \dfrac{5x^2}{x^2 - 1}$.

EXAMPLE 3 Finding Vertical and Horizontal Asymptotes

Find all vertical and horizontal asymptotes of the graph of

$$f(x) = \frac{x^2 + x - 2}{x^2 - x - 6}.$$

Solution For this rational function, the degree of the numerator is *equal to* the degree of the denominator. The leading coefficient of both the numerator and denominator is 1, so the graph has the line $y = 1$ as a horizontal asymptote. To find any vertical asymptotes, first factor the numerator and denominator as follows.

$$f(x) = \frac{x^2 + x - 2}{x^2 - x - 6} = \frac{(x - 1)\cancel{(x + 2)}}{\cancel{(x + 2)}(x - 3)} = \frac{x - 1}{x - 3}, \quad x \neq -2$$

By setting the denominator $x - 3$ (of the simplified function) equal to zero, you can determine that the graph has the line $x = 3$ as a vertical asymptote.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find all vertical and horizontal asymptotes of the graph of

$$f(x) = \frac{3x^2 + 7x - 6}{x^2 + 4x + 3}.$$

Applications

Worldwide, coal is the largest source of energy used to generate electricity.

EXAMPLE 4 Cost-Benefit Model

A utility company burns coal to generate electricity. The cost C (in dollars) of removing $p\%$ of the smokestack pollutants is given by $C = 80{,}000p/(100 - p)$, $0 \leq p < 100$. Sketch the graph of this function. You are a member of a state legislature considering a law that would require utility companies to remove 90% of the pollutants from their smokestack emissions. The current law requires 85% removal. How much additional cost would the utility company incur as a result of the new law?

Solution The graph of this function is shown in Figure 4.5. Note that the graph has a vertical asymptote at $p = 100$. Because the current law requires 85% removal, the current cost to the utility company is

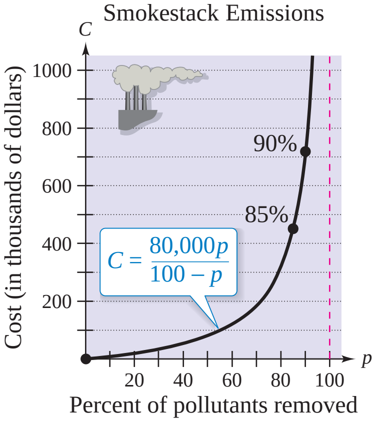


Figure 4.5

$$C = \frac{80{,}000(85)}{100 - 85} \approx \$453{,}333. \qquad \text{Evaluate } C \text{ when } p = 85.$$

The cost to remove 90% of the pollutants would be

$$C = \frac{80{,}000(90)}{100 - 90} = \$720{,}000. \qquad \text{Evaluate } C \text{ when } p = 90.$$

So, the new law would require the utility company to spend an additional

$$720{,}000 - 453{,}333 = \$266{,}667. \qquad \text{Subtract 85\% removal cost from 90\% removal cost.}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The cost C (in millions of dollars) of removing $p\%$ of the industrial and municipal pollutants discharged into a river is given by $C = 255p/(100 - p)$, $0 \leq p < 100$.

a. Find the costs of removing 20%, 45%, and 80% of the pollutants.

b. According to the model, would it be possible to remove 100% of the pollutants? Explain.

kilukilu/Shutterstock.com

EXAMPLE 5 Ultraviolet Radiation

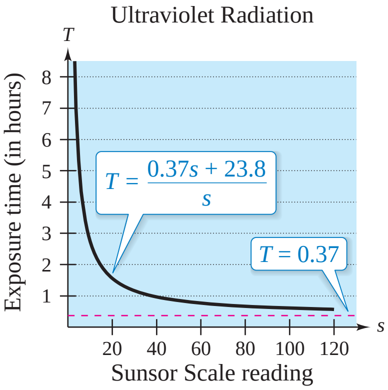


Figure 4.6

For a person with sensitive skin, the amount of time T (in hours) the person can be exposed to the sun with minimal burning can be modeled by

$$T = \frac{0.37s + 23.8}{s}, \quad 0 < s \le 120$$

where s is the Sunsor Scale reading. The Sunsor Scale is based on the level of intensity of UVB rays. *(Source: Sunsor, Inc.)*

a. Find the amounts of time a person with sensitive skin can be exposed to the sun with minimal burning when $s = 10$, $s = 25$, and $s = 100$.

b. Suppose the model is valid for all $s > 0$. What would be the horizontal asymptote of the graph of this function, and what would it represent?

Solution

a. When $s = 10$, $T = \frac{0.37(10) + 23.8}{10}$

$= 2.75$ hours.

When $s = 25$, $T = \frac{0.37(25) + 23.8}{25}$

≈ 1.32 hours.

When $s = 100$, $T = \frac{0.37(100) + 23.8}{100}$

≈ 0.61 hour.

b. As shown in Figure 4.6, the horizontal asymptote is the line $T = 0.37$. This line represents the shortest possible exposure time with minimal burning.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

A business has a cost function of $C = 0.4x + 8000$, where C is measured in dollars and x is the number of units produced. The average cost per unit is given by

$$\overline{C} = \frac{C}{x} = \frac{0.4x + 8000}{x}, \quad x > 0.$$

a. Find the average costs per unit when $x = 1000$, $x = 8000$, $x = 20{,}000$, and $x = 100{,}000$.

b. What is the horizontal asymptote for this function?

Summarize (Section 4.1)

1. State the definition of a rational function and describe the domain of a rational function *(page 312)*. For an example of finding the domain of a rational function, see Example 1.
2. State the definitions of vertical and horizontal asymptotes *(page 313)*. For examples of finding vertical and horizontal asymptotes of the graphs of rational functions, see Examples 2 and 3.
3. Describe real-life problems that can be modeled by rational functions *(pages 315 and 316, Examples 4 and 5)*.

4.1 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. Functions of the form $f(x) = N(x)/D(x)$, where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial, are called ________ ________.
2. When $f(x) \to \pm\infty$ as $x \to a$ from the left or the right, $x = a$ is a ________ ________ of the graph of f.
3. When $f(x) \to b$ as $x \to \pm\infty$, $y = b$ is a ________ ________ of the graph of f.
4. The graph of $f(x) = 1/x$ is called a ________.

Skills and Applications

Finding the Domain of a Rational Function

In Exercises 5–12, find the domain of the function, and discuss the behavior of f near any excluded x-values.

5. $f(x) = \dfrac{1}{x - 1}$ **6.** $f(x) = \dfrac{4}{x + 3}$

7. $f(x) = \dfrac{5x}{x + 2}$ **8.** $f(x) = \dfrac{6x}{x - 3}$

9. $f(x) = \dfrac{3x^2}{x^2 - 1}$ **10.** $f(x) = \dfrac{2x}{x^2 - 4}$

11. $f(x) = \dfrac{x^2 + 3x + 2}{x^2 - 2x + 1}$ **12.** $f(x) = \dfrac{x^2 + x - 20}{x^2 + 8x + 16}$

Finding Vertical and Horizontal Asymptotes

In Exercises 13–28, find all vertical and horizontal asymptotes of the graph of the function.

13. $f(x) = \dfrac{4}{x^2}$ **14.** $f(x) = \dfrac{1}{(x - 2)^3}$

15. $f(x) = \dfrac{5 + x}{5 - x}$ **16.** $f(x) = \dfrac{3 - 7x}{3 + 2x}$

17. $f(x) = \dfrac{x^3}{x^2 - 1}$ **18.** $f(x) = \dfrac{2x^2}{x + 1}$

19. $f(x) = \dfrac{3x^2 + 1}{x^2 + x + 9}$ **20.** $f(x) = \dfrac{3x^2 + x - 5}{x^2 + 1}$

21. $f(x) = \dfrac{x - 4}{x^2 - 16}$ **22.** $f(x) = \dfrac{x + 3}{x^2 - 9}$

23. $f(x) = \dfrac{x^2 - 1}{x^2 - 2x - 3}$

24. $f(x) = \dfrac{x^2 - 4}{x^2 - 3x + 2}$

25. $f(x) = \dfrac{x^2 - 3x - 4}{2x^2 + x - 1}$

26. $f(x) = \dfrac{x^2 + x - 2}{2x^2 + 5x + 2}$

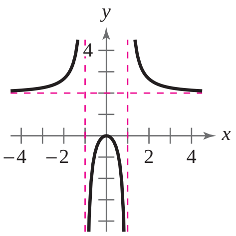
27. $f(x) = \dfrac{6x^2 + 5x - 6}{3x^2 - 8x + 4}$

28. $f(x) = \dfrac{6x^2 - 11x + 3}{6x^2 - 7x - 3}$

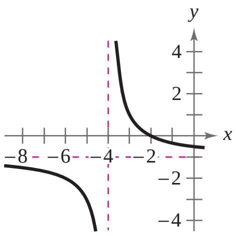
Matching

In Exercises 29–36, match the rational function with its graph. [The graphs are labeled (a)–(h).]

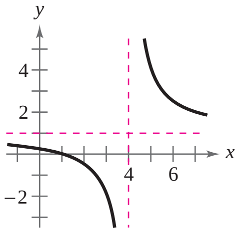
(a)



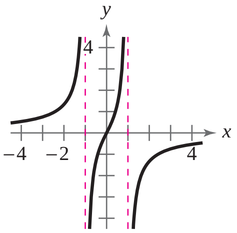
(b)



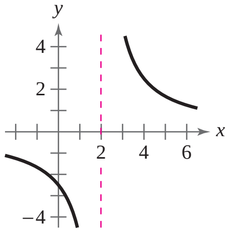
(c)



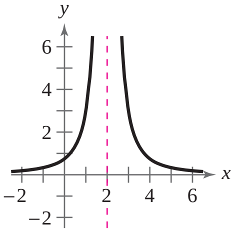
(d)



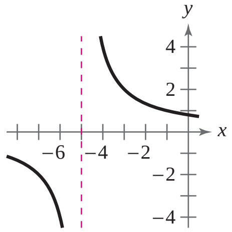
(e)



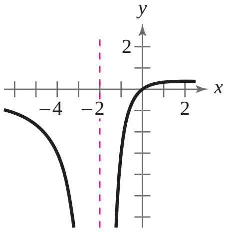
(f)



(g)



(h)



29. $f(x) = \dfrac{4}{x + 5}$ **30.** $f(x) = \dfrac{5}{x - 2}$

31. $f(x) = \dfrac{x - 1}{x - 4}$ **32.** $f(x) = -\dfrac{x + 2}{x + 4}$

33. $f(x) = \dfrac{2x^2}{x^2 - 1}$ **34.** $f(x) = \dfrac{-2x}{x^2 - 1}$

35. $f(x) = \dfrac{3}{(x - 2)^2}$ **36.** $f(x) = \dfrac{3x}{(x + 2)^2}$

Analytical and Numerical Analysis **In Exercises 37–40, (a) determine the domains of f and g, (b) simplify f and find any vertical asymptotes of f, (c) complete the table, and (d) explain how the two functions differ.**

37. $f(x) = \dfrac{x^2 - 4}{x + 2}, \quad g(x) = x - 2$

x	-4	-3	-2.5	-2	-1.5	-1	0
$f(x)$							
$g(x)$							

38. $f(x) = \dfrac{x^2(x + 3)}{x^2 + 3x}, \quad g(x) = x$

x	-3	-2	-1	0	1	2	3
$f(x)$							
$g(x)$							

39. $f(x) = \dfrac{2x - 1}{2x^2 - x}, \quad g(x) = \dfrac{1}{x}$

x	-1	-0.5	0	0.5	2	3	4
$f(x)$							
$g(x)$							

40. $f(x) = \dfrac{2x - 8}{x^2 - 9x + 20}, \quad g(x) = \dfrac{2}{x - 5}$

x	0	1	2	3	4	5	6
$f(x)$							
$g(x)$							

41. Recycling

The cost C (in dollars) of supplying recycling bins to $p\%$ of the population of a rural township is given by

$$C = \frac{25{,}000p}{100 - p}, \quad 0 \le p < 100.$$

(a) Use a graphing utility to graph the cost function.

(b) Find the costs of supplying bins to 15%, 50%, and 90% of the population.

(c) According to this model, would it be possible to supply bins to 100% of the residents? Explain.

42. Pollution The cost C (in millions of dollars) of removing $p\%$ of the industrial and municipal pollutants discharged into a river is given by

$$C = \frac{225p}{100 - p}, \quad 0 \le p < 100.$$

(a) Use a graphing utility to graph the cost function.

(b) Find the costs of removing 10%, 40%, and 75% of the pollutants.

(c) According to this model, would it be possible to remove 100% of the pollutants? Explain.

43. Population Growth The game commission introduces 100 deer into newly acquired state game lands. The population N of the herd is modeled by

$$N = \frac{20(5 + 3t)}{1 + 0.04t}, \quad t \ge 0$$

where t is the time in years.

(a) Use a graphing utility to graph this model.

(b) Find the populations when $t = 5$, $t = 10$, and $t = 25$.

(c) What is the limiting size of the herd as time increases?

44. Food Consumption A biology class performs an experiment comparing the quantity of food consumed by a certain kind of moth with the quantity supplied. The model for the experimental data is

$$y = \frac{1.568x - 0.001}{6.360x + 1}, \quad x > 0$$

where x is the quantity (in milligrams) of food supplied and y is the quantity (in milligrams) of food consumed.

(a) Use a graphing utility to graph this model.

(b) At what level of consumption will the moth become satiated?

45. Human Memory Model Psychologists have developed mathematical models to predict memory performance as a function of the number of trials n of a certain task. Consider the learning curve

$$P = \frac{0.5 + 0.9(n - 1)}{1 + 0.9(n - 1)}, \quad n > 0$$

where P is the fraction of correct responses after n trials.

(a) Complete the table for this model. What does it suggest?

n	1	2	3	4	5	6	7	8	9	10
P										

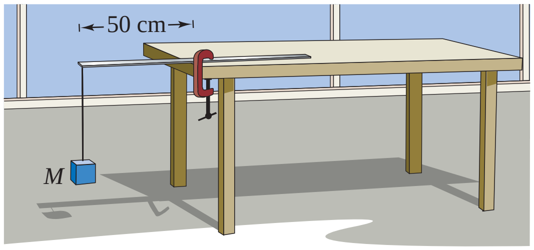
(b) According to this model, what is the limiting percent of correct responses as n increases?

ZQFotography/Shutterstock.com

46. **Data Analysis: Physics Experiment** Consider a physics laboratory experiment designed to determine an unknown mass. A flexible metal meter stick is clamped to a table with 50 centimeters overhanging the edge (see figure). Known masses M ranging from 200 grams to 2000 grams are attached to the end of the meter stick. For each mass, the meter stick is displaced vertically and then allowed to oscillate. The average time t (in seconds) of one oscillation for each mass is recorded in the table.

Spreadsheet at LarsonPrecalculus.com

Mass, M	Time, t
200	0.450
400	0.597
600	0.712
800	0.831
1000	0.906
1200	1.003
1400	1.088
1600	1.168
1800	1.218
2000	1.338



A model for the data that can be used to predict the time of one oscillation is

$$t = \frac{38M + 16{,}965}{10(M + 5000)}.$$

(a) Use this model to create a table showing the predicted time for each of the masses shown in the table.

(b) Compare the predicted times with the experimental times. What can you conclude?

(c) Use the model to approximate the mass of an object for which $t = 1.056$ seconds.

Exploration

True or False? **In Exercises 47 and 48, determine whether the statement is true or false. Justify your answer.**

47. The graph of a polynomial function can have infinitely many vertical asymptotes.

48. $f(x) = x^3 - 2x^2 - 5x + 6$ is a rational function.

Determining Function Behavior **In Exercises 49–52, (a) determine the value that the function f approaches as the magnitude of x increases. Is $f(x)$ greater than or less than this functional value when (b) x is positive and large in magnitude and (c) x is negative and large in magnitude?**

49. $f(x) = 4 - \dfrac{1}{x}$

50. $f(x) = 2 + \dfrac{1}{x - 3}$

51. $f(x) = \dfrac{2x - 1}{x - 3}$

52. $f(x) = \dfrac{2x - 1}{x^2 + 1}$

Think About It **In Exercises 53 and 54, write a rational function f that has the specified characteristics. (There are many correct answers.)**

53. Vertical asymptote: None
Horizontal asymptote: $y = 2$

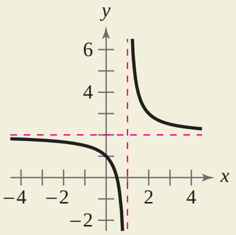
54. Vertical asymptotes: $x = -2, x = 1$
Horizontal asymptote: None

55. **Think About It** Give an example of a rational function whose domain is the set of all real numbers. Give an example of a rational function whose domain is the set of all real numbers except $x = 15$.

56. **HOW DO YOU SEE IT?** The graph of a rational function

$$f(x) = \frac{N(x)}{D(x)}$$

is shown below. Determine which of the statements about the function is false. Justify your answer.



(a) When $x = 1$, $D(x) = 0$.

(b) The degrees of $N(x)$ and $D(x)$ are equal.

(c) The ratio of the leading coefficients of $N(x)$ and $D(x)$ is 1.

4.2 Graphs of Rational Functions

You can use a rational function to model the concentration of a chemical in the bloodstream after injection. For instance, see Exercise 84 on page 328.

- **Sketch the graphs of rational functions.**
- **Sketch the graphs of rational functions that have slant asymptotes.**
- **Use the graphs of rational functions to model and solve real-life problems.**

Sketching the Graph of a Rational Function

To sketch the graph of a rational function, use the following guidelines.

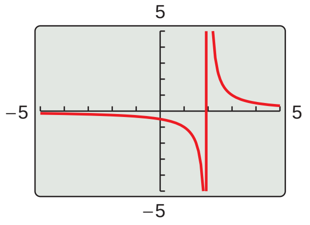
Guidelines for Graphing Rational Functions

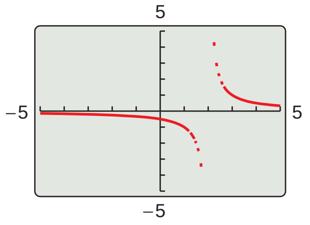
Let $f(x) = N(x)/D(x)$, where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial.

1. Simplify f, if possible.
2. Find and plot the y-intercept (if any) by evaluating $f(0)$.
3. Find the zeros of the numerator (if any) by solving the equation $N(x) = 0$. Then plot the corresponding x-intercepts.
4. Find the zeros of the denominator (if any) by solving the equation $D(x) = 0$. Then sketch the corresponding vertical asymptotes.
5. Find and sketch the horizontal asymptote (if any) by using the rule for finding the horizontal asymptote of a rational function.
6. Plot at least one point *between* and one point *beyond* each x-intercept and vertical asymptote.
7. Use smooth curves to complete the graph between and beyond the vertical asymptotes.

You may also want to test for symmetry when graphing rational functions, especially for simple rational functions. Recall from Section 2.4 that the graph of $f(x) = 1/x$ is symmetric with respect to the origin.

TECHNOLOGY Some graphing utilities have difficulty graphing rational functions that have vertical asymptotes. Often, the utility will connect parts of the graph that are not supposed to be connected. For instance, the screen on the left below shows the graph of $f(x) = 1/(x - 2)$. Notice that the graph should consist of two unconnected portions—one to the left of $x = 2$ and the other to the right of $x = 2$. To eliminate this problem, you can try changing the mode of the graphing utility to *dot* mode. The problem with this is that the graph is then represented as a collection of dots (as shown in the screen on the right) rather than as a smooth curve.





chungking/Shutterstock.com

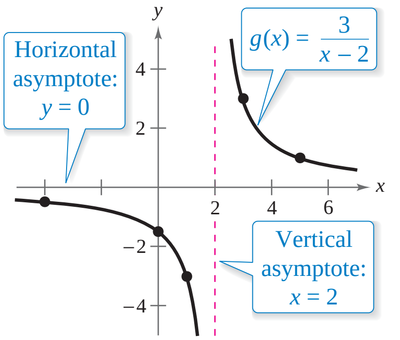


Figure 4.7

EXAMPLE 1 Sketching the Graph of a Rational Function

Sketch the graph of $g(x) = \dfrac{3}{x - 2}$ and state its domain.

Solution

y-intercept: $\left(0, -\frac{3}{2}\right)$, because $g(0) = -\frac{3}{2}$

x-intercept: none, because $3 \neq 0$

Vertical asymptote: $x = 2$, zero of denominator

Horizontal asymptote: $y = 0$, because degree of $N(x) <$ degree of $D(x)$

Additional points:

x	-4	1	2	3	5
$g(x)$	-0.5	-3	Undefined	3	1

REMARK Note in the examples in this section that the vertical asymptotes are included in the table of additional points. This is done to emphasize numerically the behavior of the graph of the function.

By plotting the intercept, asymptotes, and a few additional points, you can obtain the graph shown in Figure 4.7. The domain of g is all real numbers $x = 2$.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Sketch the graph of $f(x) = \dfrac{1}{x + 3}$ and state its domain.

The graph of g in Example 1 is a vertical stretch and a right shift of the graph of $f(x) = 1/x$, because

$$g(x) = \frac{3}{x - 2} = 3\left(\frac{1}{x - 2}\right) = 3f(x - 2).$$

EXAMPLE 2 Sketching the Graph of a Rational Function

Sketch the graph of $f(x) = \dfrac{2x - 1}{x}$ and state its domain.

Solution

y-intercept: none, because $x = 0$ is not in the domain

x-intercept: $\left(\frac{1}{2}, 0\right)$, because $f\left(\frac{1}{2}\right) = 0$

Vertical asymptote: $x = 0$, zero of denominator

Horizontal asymptote: $y = 2$, because degree of $N(x) =$ degree of $D(x)$

Additional points:

x	-4	-1	0	$\frac{1}{4}$	4
$f(x)$	2.25	3	Undefined	-2	1.75

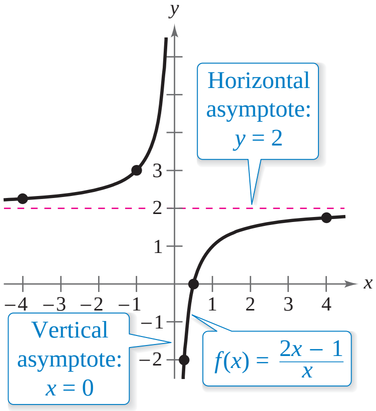


Figure 4.8

By plotting the intercept, asymptotes, and a few additional points, you can obtain the graph shown in Figure 4.8. The domain of f is all real numbers $x = 0$.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Sketch the graph of $C(x) = \dfrac{3 + 2x}{1 + x}$ and state its domain.

EXAMPLE 3 Sketching the Graph of a Rational Function

Sketch the graph of $f(x) = \dfrac{x}{x^2 - x - 2}$ and state its domain.

Solution Factor the denominator to determine more easily the zeros of the denominator.

$$f(x) = \frac{x}{x^2 - x - 2} = \frac{x}{(x + 1)(x - 2)}$$

y-intercept: $(0, 0)$, because $f(0) = 0$

x-intercept: $(0, 0)$, because $f(0) = 0$

Vertical asymptotes: $x = -1$, $x = 2$, zeros of denominator

Horizontal asymptote: $y = 0$, because degree of $N(x) <$ degree of $D(x)$

Additional points:

x	-3	-1	-0.5	1	2	3
$f(x)$	-0.3	Undefined	0.4	-0.5	Undefined	0.75

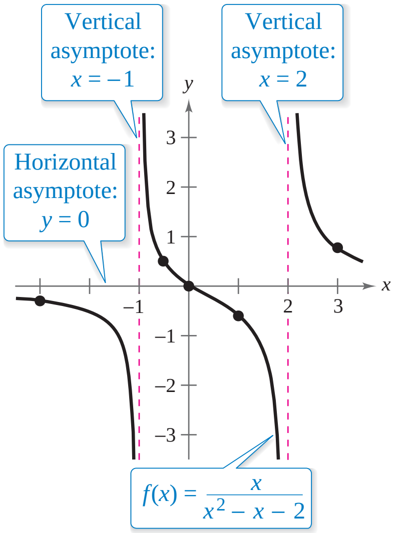


Figure 4.9

The graph is shown in Figure 4.9. The domain of f is all real numbers except $x = -1$ and $x = 2$.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Sketch the graph of $f(x) = \dfrac{3x}{x^2 + x - 2}$ and state its domain.

EXAMPLE 4 Sketching the Graph of a Rational Function

Sketch the graph of $f(x) = \dfrac{x^2 - 9}{x^2 - 2x - 3}$ and state its domain.

Solution By factoring the numerator and denominator, you have

$$f(x) = \frac{x^2 - 9}{x^2 - 2x - 3} = \frac{\cancel{(x - 3)}(x + 3)}{\cancel{(x - 3)}(x + 1)} = \frac{x + 3}{x + 1}, \quad x \neq 3.$$

y-intercept: $(0, 3)$, because $f(0) = 3$

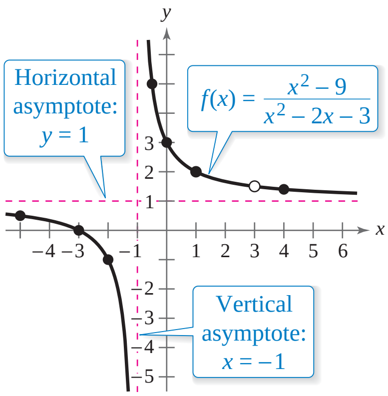
x-intercept: $(-3, 0)$, because $f(-3) = 0$

Vertical asymptote: $x = -1$, zero of (simplified) denominator

Horizontal asymptote: $y = 1$, because degree of $N(x) =$ degree of $D(x)$

Additional points:

x	-5	-2	-1	-0.5	1	3	4
$f(x)$	0.5	-1	Undefined	5	2	Undefined	1.4



Hole at $x = 3$

Figure 4.10

The graph is shown in Figure 4.10. Notice that there is a hole in the graph at $x = 3$ because the function is not defined when $x = 3$. The domain of f is all real numbers except $x = -1$ and $x = 3$.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Sketch the graph of $f(x) = \dfrac{x^2 - 4}{x^2 - x - 6}$ and state its domain.

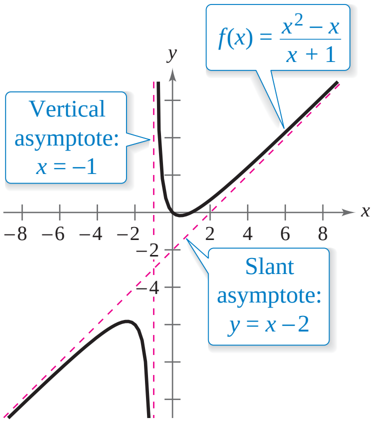


Figure 4.11

Slant Asymptotes

Consider a rational function whose denominator is of degree 1 or greater. If the degree of the numerator is exactly *one more* than the degree of the denominator, then the graph of the function has a **slant** (or **oblique**) **asymptote.** For example, the graph of

$$f(x) = \frac{x^2 - x}{x + 1}$$

has a slant asymptote, as shown in Figure 4.11. To find the equation of a slant asymptote, use long division. For instance, by dividing $x + 1$ into $x^2 - x$, you obtain

$$f(x) = \frac{x^2 - x}{x + 1} = \underbrace{x - 2}_{\substack{\text{Slant asymptote} \\ (y = x - 2)}} + \frac{2}{x + 1}.$$

As x increases or decreases without bound, the remainder term $2/(x + 1)$ approaches 0, so the graph of f approaches the line $y = x - 2$, as shown in Figure 4.11.

EXAMPLE 5 A Rational Function with a Slant Asymptote

Sketch the graph of $f(x) = \frac{x^2 - x - 2}{x - 1}$ and state its domain.

Solution First write $f(x)$ in two different ways. Factoring the numerator

$$f(x) = \frac{x^2 - x - 2}{x - 1}$$
$$= \frac{(x - 2)(x + 1)}{x - 1}$$

allows you to recognize the x-intercepts. Long division

$$f(x) = \frac{x^2 - x - 2}{x - 1}$$
$$= x - \frac{2}{x - 1}$$

allows you to recognize that the line $y = x$ is a slant asymptote of the graph.

y-intercept: (0, 2), because $f(0) = 2$

x-intercepts: $(-1, 0)$ and $(2, 0)$, because $f(-1) = 0$ and $f(2) = 0$

Vertical asymptote: $x = 1$, zero of denominator

Slant asymptote: $y = x$

Additional points:

x	-2	0.5	1	1.5	3
$f(x)$	-1.33	4.5	Undefined	-2.5	2

The graph is shown in Figure 4.12. The domain of f is all real numbers except $x = 1$.

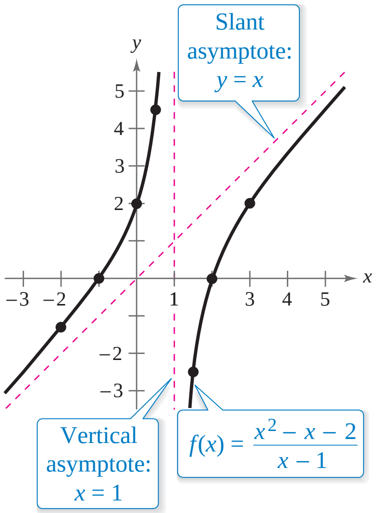


Figure 4.12

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Sketch the graph of $f(x) = \frac{3x^2 + 1}{x}$ and state its domain.

Application

EXAMPLE 6 Finding a Minimum Area

A rectangular page is designed to contain 48 square inches of print. The margins at the top and bottom of the page are each 1 inch deep. The margins on each side are $1\frac{1}{2}$ inches wide. What should the dimensions of the page be so that the least amount of paper is used?

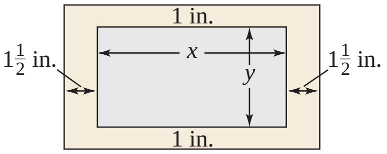


Figure 4.13

Graphical Solution

Let A be the area to be minimized. From Figure 4.13, you can write

$$A = (x + 3)(y + 2).$$

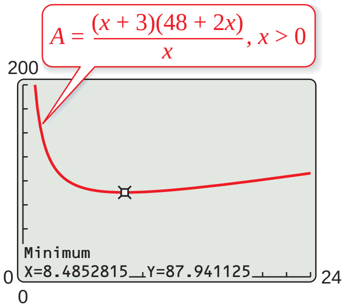
The printed area inside the margins is modeled by $48 = xy$ or $y = 48/x$. To find the minimum area, rewrite the equation for A in terms of just one variable by substituting $48/x$ for y.

$$A = (x + 3)\left(\frac{48}{x} + 2\right)$$

$$= \frac{(x + 3)(48 + 2x)}{x}, \quad x > 0$$

The graph of this rational function is shown below. Because x represents the width of the printed area, you need to consider only the portion of the graph for which x is positive. Using a graphing utility, you can approximate the minimum value of A to occur when $x \approx 8.5$ inches. The corresponding value of y is $48/8.5 \approx 5.6$ inches. So, the dimensions should be

$$x + 3 \approx 11.5 \text{ inches} \quad \text{by} \quad y + 2 \approx 7.6 \text{ inches}.$$



```
Minimum
X=8.4852815  Y=87.941125
```

Numerical Solution

Let A be the area to be minimized. From Figure 4.13, you can write

$$A = (x + 3)(y + 2).$$

The printed area inside the margins is modeled by $48 = xy$ or $y = 48/x$. To find the minimum area, rewrite the equation for A in terms of just one variable by substituting $48/x$ for y.

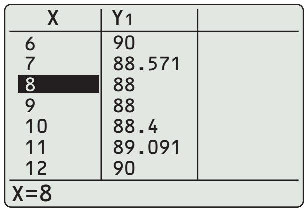
$$A = (x + 3)\left(\frac{48}{x} + 2\right) = \frac{(x + 3)(48 + 2x)}{x}, \quad x > 0$$

Use the *table* feature of a graphing utility to create a table of values for the function

$$y_1 = \frac{(x + 3)(48 + 2x)}{x}$$

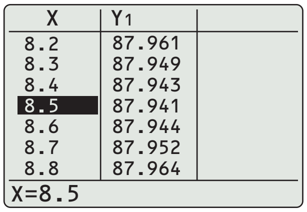
beginning at $x = 1$. From the table, you can see that the minimum value of y_1 occurs when x is somewhere between 8 and 9, as shown in Figure 4.14. To approximate the minimum value of y_1 to one decimal place, change the table so that it starts at $x = 8$ and increases by 0.1. The minimum value of y_1 occurs when $x \approx 8.5$, as shown in Figure 4.15. The corresponding value of y is $48/8.5 \approx 5.6$ inches. So, the dimensions should be

$$x + 3 \approx 11.5 \text{ inches} \quad \text{by} \quad y + 2 \approx 7.6 \text{ inches}.$$



```
  X     Y1
  6     90
  7     88.571
  8     88
  9     88
  10    88.4
  11    89.091
  12    90
X=8
```

Figure 4.14



```
  X     Y1
  8.2   87.961
  8.3   87.949
  8.4   87.943
  8.5   87.941
  8.6   87.944
  8.7   87.952
  8.8   87.964
X=8.5
```

Figure 4.15

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Rework Example 6 when the margins on each side are 2 inches wide and the page is designed to contain 40 square inches of print.

Summarize (Section 4.2)

1. State the guidelines for graphing rational functions *(page 320, Examples 1–4)*.
2. Describe when a rational function has a slant asymptote *(page 323, Example 5)*.
3. Describe a real-life problem that can be modeled by a rational function *(page 324, Example 6)*.

4.2 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

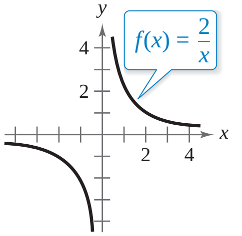
Vocabulary: Fill in the blanks.

1. For the rational function $f(x) = N(x)/D(x)$, if the degree of $N(x)$ is exactly one more than the degree of $D(x)$, then the graph of f has a ________ (or oblique) ________.
2. The graph of $g(x) = 3/(x - 2)$ has a ________ asymptote at $x = 2$.

Skills and Applications

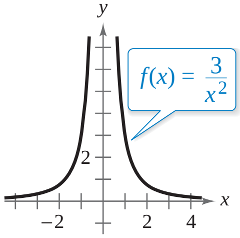
Sketching a Transformation of a Rational Function In Exercises 3–6, use the graph of $f(x) = \dfrac{2}{x}$ to sketch the graph of g.

3. $g(x) = \dfrac{2}{x} + 4$
4. $g(x) = \dfrac{2}{x - 5}$
5. $g(x) = -\dfrac{2}{x}$
6. $g(x) = \dfrac{1}{x + 2}$



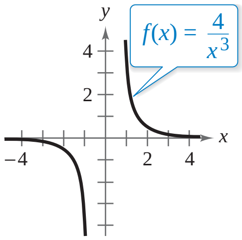
Sketching a Transformation of a Rational Function In Exercises 7–10, use the graph of $f(x) = \dfrac{3}{x^2}$ to sketch the graph of g.

7. $g(x) = \dfrac{3}{x^2} - 1$
8. $g(x) = -\dfrac{3}{x^2}$
9. $g(x) = \dfrac{3}{(x - 1)^2}$
10. $g(x) = \dfrac{1}{x^2}$



Sketching a Transformation of a Rational Function In Exercises 11–14, use the graph of $f(x) = \dfrac{4}{x^3}$ to sketch the graph of g.

11. $g(x) = \dfrac{4}{(x + 2)^3}$
12. $g(x) = \dfrac{4}{x^3} + 2$
13. $g(x) = -\dfrac{4}{x^3}$
14. $g(x) = \dfrac{2}{x^3}$



Sketching the Graph of a Rational Function In Exercises 15–44, (a) state the domain of the function, (b) identify all intercepts, (c) find any vertical or horizontal asymptotes, and (d) plot additional solution points as needed to sketch the graph of the rational function.

15. $f(x) = \dfrac{1}{x + 2}$
16. $f(x) = \dfrac{1}{x - 3}$
17. $h(x) = \dfrac{-1}{x + 4}$
18. $g(x) = \dfrac{1}{6 - x}$
19. $C(x) = \dfrac{7 + 2x}{2 + x}$
20. $P(x) = \dfrac{1 - 3x}{1 - x}$
21. $g(x) = \dfrac{1}{x + 2} + 2$
22. $f(x) = 2 - \dfrac{3}{x^2}$
23. $f(x) = \dfrac{x^2}{x^2 + 9}$
24. $f(t) = \dfrac{1 - 2t}{t}$
25. $h(x) = \dfrac{x^2}{x^2 - 9}$
26. $g(x) = \dfrac{x}{x^2 - 9}$
27. $g(s) = \dfrac{4s}{s^2 + 4}$
28. $f(x) = -\dfrac{1}{(x - 2)^2}$
29. $g(x) = \dfrac{4(x + 1)}{x(x - 4)}$
30. $h(x) = \dfrac{2}{x^2(x - 2)}$
31. $f(x) = \dfrac{2x}{x^2 - 3x - 4}$
32. $f(x) = \dfrac{3x}{x^2 + 2x - 3}$
33. $h(x) = \dfrac{x^2 - 5x + 4}{x^2 - 4}$
34. $g(x) = \dfrac{x^2 - 2x - 8}{x^2 - 9}$
35. $f(x) = \dfrac{6x}{x^2 - 5x - 14}$
36. $f(x) = \dfrac{3(x^2 + 1)}{x^2 + 2x - 15}$
37. $f(x) = \dfrac{2x^2 - 5x - 3}{x^3 - 2x^2 - x + 2}$
38. $f(x) = \dfrac{x^2 - x - 2}{x^3 - 2x^2 - 5x + 6}$
39. $f(x) = \dfrac{x^2 + 3x}{x^2 + x - 6}$
40. $f(x) = \dfrac{5(x + 4)}{x^2 + x - 12}$
41. $f(x) = \dfrac{2x^2 - 5x + 2}{2x^2 - x - 6}$
42. $f(x) = \dfrac{3x^2 - 8x + 4}{2x^2 - 3x - 2}$
43. $f(t) = \dfrac{t^2 - 1}{t - 1}$
44. $f(x) = \dfrac{x^2 - 36}{x + 6}$

Analytical and Graphical Analysis **In Exercises 45 and 46, (a) determine the domains of f and g, (b) use a graphing utility to graph f and g in the same viewing window, and (c) explain why the graphing utility may not show the difference in the domains of f and g.**

45. $f(x) = \dfrac{x^2 - 1}{x + 1}, \quad g(x) = x - 1$

46. $f(x) = \dfrac{x - 2}{x^2 - 2x}, \quad g(x) = \dfrac{1}{x}$

A Rational Function with a Slant Asymptote **In Exercises 47–62, (a) state the domain of the function, (b) identify all intercepts, (c) find any vertical or slant asymptotes, and (d) plot additional solution points as needed to sketch the graph of the rational function.**

47. $h(x) = \dfrac{x^2 - 9}{x}$

48. $g(x) = \dfrac{x^2 + 5}{x}$

49. $f(x) = \dfrac{2x^2 + 1}{x}$

50. $f(x) = \dfrac{1 - x^2}{x}$

51. $g(x) = \dfrac{x^2 + 1}{x}$

52. $h(x) = \dfrac{x^2}{x - 1}$

53. $f(t) = -\dfrac{t^2 + 1}{t + 5}$

54. $f(x) = \dfrac{x^2}{3x + 1}$

55. $f(x) = \dfrac{x^3}{x^2 - 4}$

56. $g(x) = \dfrac{x^3}{2x^2 - 8}$

57. $f(x) = \dfrac{x^3 - 1}{x^2 - x}$

58. $f(x) = \dfrac{x^4 + x}{x^3}$

59. $f(x) = \dfrac{x^2 - x + 1}{x - 1}$

60. $f(x) = \dfrac{2x^2 - 5x + 5}{x - 2}$

61. $f(x) = \dfrac{2x^3 - x^2 - 2x + 1}{x^2 + 3x + 2}$

62. $f(x) = \dfrac{2x^3 + x^2 - 8x - 4}{x^2 - 3x + 2}$

Graphical Analysis **In Exercises 63–66, use a graphing utility to graph the rational function. Give the domain of the function and find any asymptotes. Then zoom out sufficiently far so that the graph appears as a line. Identify the line.**

63. $f(x) = \dfrac{x^2 + 5x + 8}{x + 3}$

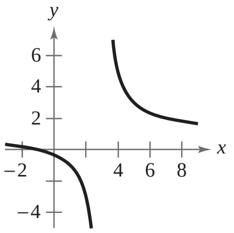
64. $f(x) = \dfrac{2x^2 + x}{x + 1}$

65. $g(x) = \dfrac{1 + 3x^2 - x^3}{x^2}$

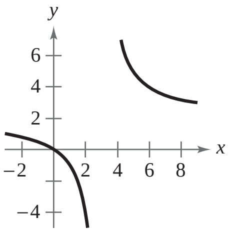
66. $h(x) = \dfrac{12 - 2x - x^2}{2(4 + x)}$

Graphical Reasoning **In Exercises 67–70, (a) use the graph to determine any x-intercepts of the graph of the rational function and (b) set $y = 0$ and solve the resulting equation to confirm your result in part (a).**

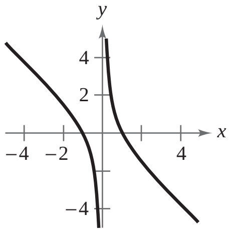
67. $y = \dfrac{x + 1}{x - 3}$



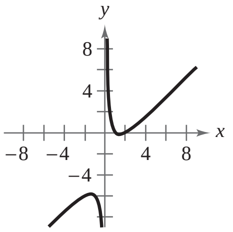
68. $y = \dfrac{2x}{x - 3}$



69. $y = \dfrac{1}{x} - x$



70. $y = x - 3 + \dfrac{2}{x}$



Graphical Reasoning **In Exercises 71–74, (a) use a graphing utility to graph the rational function and determine any x-intercepts of the graph and (b) set $y = 0$ and solve the resulting equation to confirm your result in part (a).**

71. $y = \dfrac{1}{x + 5} + \dfrac{4}{x}$

72. $y = 20\left(\dfrac{2}{x + 1} - \dfrac{3}{x}\right)$

73. $y = x - \dfrac{6}{x - 1}$

74. $y = x - \dfrac{9}{x}$

75. Concentration of a Mixture A 1000-liter tank contains 50 liters of a 25% brine solution. You add x liters of a 75% brine solution to the tank.

(a) Show that the concentration C, the proportion of brine to total solution, in the final mixture is

$$C = \frac{3x + 50}{4(x + 50)}.$$

(b) Determine the domain of the function based on the physical constraints of the problem.

(c) Sketch a graph of the concentration function.

(d) As the tank is filled, what happens to the rate at which the concentration of brine is increasing? What percent does the concentration of brine appear to approach?

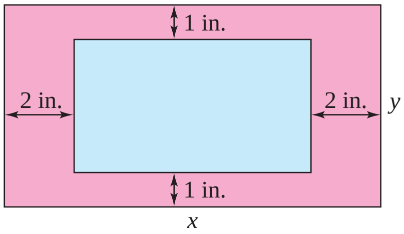
76. Geometry A rectangular region of length x and width y has an area of 500 square meters.

(a) Write the width y as a function of x.

(b) Determine the domain of the function based on the physical constraints of the problem.

(c) Sketch a graph of the function and determine the width of the rectangle when $x = 30$ meters.

77. Page Design A page that is x inches wide and y inches high contains 30 square inches of print. The top and bottom margins are each 1 inch deep and the margins on each side are 2 inches wide (see figure).



(a) Show that the total area A on the page is

$$A = \frac{2x(x + 11)}{x - 4}.$$

(b) Determine the domain of the function based on the physical constraints of the problem.

(c) Use a graphing utility to graph the area function and approximate the page size for which the least amount of paper is used. Verify your answer numerically using the *table* feature of the graphing utility.

78. Page Design A rectangular page is designed to contain 64 square inches of print. The margins at the top and bottom of the page are each 1 inch deep. The margins on each side are $1\frac{1}{2}$ inches wide. What should the dimensions of the page be so that the least amount of paper is used?

Graphical Analysis **In Exercises 79 and 80, use a graphing utility to graph the function and locate any relative maximum or minimum points on the graph.**

79. $f(x) = \dfrac{3(x + 1)}{x^2 + x + 1}$

80. $C(x) = x + \dfrac{32}{x}$

81. Minimum Cost The ordering and transportation cost C (in thousands of dollars) for the components used in manufacturing a product is given by

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right), \quad x \geq 1$$

where x is the order size (in hundreds). Use a graphing utility to graph the cost function. From the graph, estimate the order size that minimizes cost.

82. Minimum Cost The cost C of producing x units of a product is given by

$$C = 0.2x^2 + 10x + 5$$

and the average cost per unit is given by

$$\overline{C} = \frac{C}{x} = \frac{0.2x^2 + 10x + 5}{x}, \quad x > 0.$$

Sketch the graph of the average cost function and estimate the number of units that should be produced to minimize the average cost per unit.

83. Average Speed A driver averaged 50 miles per hour on the round trip between two cities 100 miles apart. The average speeds for going and returning were x and y miles per hour, respectively.

(a) Show that

$$y = \frac{25x}{x - 25}.$$

(b) Determine the vertical and horizontal asymptotes of the graph of the function.

(c) Use a graphing utility to graph the function.

(d) Complete the table.

x	30	35	40	45	50	55	60
y							

(e) Are the results in the table what you expected? Explain.

(f) Is it possible to average 20 miles per hour in one direction and still average 50 miles per hour on the round trip? Explain.

84. Medicine The concentration C of a chemical in the bloodstream t hours after injection into muscle tissue is given by

$$C = \frac{3t^2 + t}{t^3 + 50}, \quad t > 0.$$

(a) Determine the horizontal asymptote of the graph of the function and interpret its meaning in the context of the problem.

(b) Use a graphing utility to graph the function and approximate the time when the bloodstream concentration is greatest.

(c) Use the graphing utility to determine when the concentration is less than 0.345.

Exploration

True or False? **In Exercises 85–88, determine whether the statement is true or false. Justify your answer.**

85. When the graph of a rational function f has a vertical asymptote at $x = 5$, it is possible to sketch the graph without lifting your pencil from the paper.

86. The graph of a rational function can never cross one of its asymptotes.

87. The graph of $f(x) = \dfrac{2x^3}{x + 1}$ has a slant asymptote.

88. Every rational function has a horizontal asymptote.

Think About It **In Exercises 89 and 90, use a graphing utility to graph the function. Explain why there is no vertical asymptote when a superficial examination of the function may indicate that there should be one.**

89. $h(x) = \dfrac{6 - 2x}{3 - x}$

90. $g(x) = \dfrac{x^2 + x - 2}{x - 1}$

91. Writing (a) Given a rational function f, how can you determine whether f has a slant asymptote? (b) You determine that a rational function f has a slant asymptote. How do you find it?

92. Writing and Graphing a Rational Function Write a rational function satisfying the following criteria. Then sketch a graph of your function.

Vertical asymptote: $x = 2$

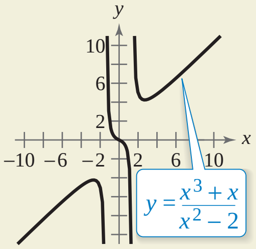
Slant asymptote: $y = x + 1$

Zero of the function: $x = -2$

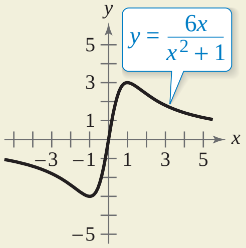
93. Think About It Is it possible for a rational function to have all three types of asymptotes (vertical, horizontal, and slant)? Why or why not?

94. HOW DO YOU SEE IT? For each statement below, determine which graph(s) show that the statement is incorrect.

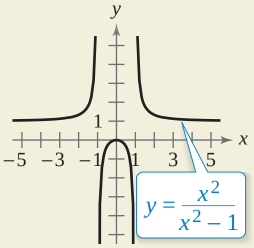
(i)



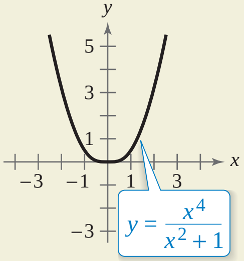
(ii)



(iii)



(iv)



(a) Every rational function has a vertical asymptote.

(b) Every rational function has at least one vertical, horizontal, or slant asymptote.

(c) A rational function can have at most one vertical asymptote.

(d) The graph of a rational function with a slant asymptote cannot cross the slant asymptote.

Project: Department of Defense To work an extended application analyzing the total numbers of military personnel on active duty from 1980 through 2010, visit this text's website at *LarsonPrecalculus.com*. (*Source: U.S. Department of Defense*)

Ensuper/Shutterstock.com

4.3 Conics

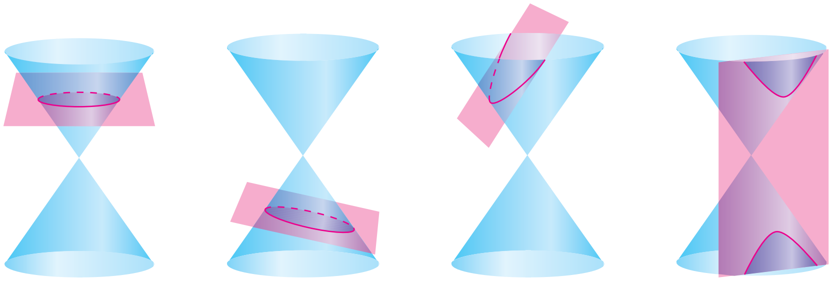
Conics have been used for hundreds of years to model and solve engineering problems. For instance, in Exercise 33 on page 340, a parabola can be used to model the cables of the Golden Gate Bridge.

- **Recognize the four basic conics: circle, ellipse, parabola, and hyperbola.**
- **Recognize, graph, and write equations of parabolas (vertex at origin).**
- **Recognize, graph, and write equations of ellipses (center at origin).**
- **Recognize, graph, and write equations of hyperbolas (center at origin).**

Introduction

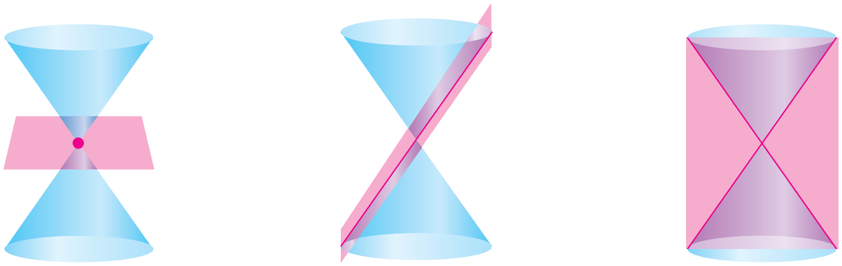
Conic sections were discovered during the classical Greek period, 600 to 300 B.C. This early Greek study was largely concerned with the geometric properties of conics. It was not until the early 17th century that the broad applicability of conics became apparent and played a prominent role in the early development of calculus.

A **conic section** (or simply **conic**) is the intersection of a plane and a double-napped cone. Notice in Figure 4.16 that in the formation of the four basic conics, the intersecting plane does not pass through the vertex of the cone. When the plane does pass through the vertex, the resulting figure is a **degenerate conic,** as shown in Figure 4.17.



Circle Ellipse Parabola Hyperbola

Figure 4.16 Basic Conics



Point Line Two Intersecting Lines

Figure 4.17 Degenerate Conics

There are several ways to approach the study of conics. You could begin by defining conics in terms of the intersections of planes and cones, as the Greeks did, or you could define them algebraically, in terms of the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

However, you will study a third approach, in which each of the conics is defined as a *locus* (collection) of points satisfying a certain geometric property. For example, in Section 1.1 you saw how the definition of a circle as *the collection of all points* (x, y) *that are equidistant from a fixed point* (h, k) led easily to the standard form of the equation of a circle

$$(x - h)^2 + (y - k)^2 = r^2. \qquad \text{Equation of a circle}$$

Recall from Section 1.1 that the center of a circle is at (h, k) and that the radius of the circle is r.

Cosmo Condina/The Image Bank/Getty Images

Parabolas

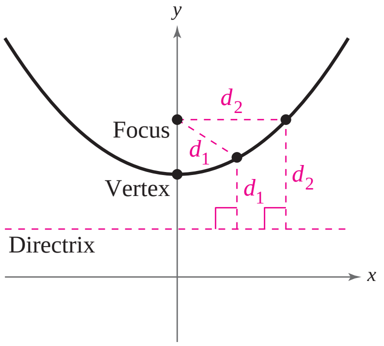
In Section 3.1, you learned that the graph of the quadratic function

$$f(x) = ax^2 + bx + c$$

is a parabola that opens upward or downward. The following definition of a parabola is more general in the sense that it is independent of the orientation of the parabola.

Definition of a Parabola

A **parabola** is the set of all points (x, y) in a plane that are equidistant from a fixed line, the **directrix,** and a fixed point, the **focus,** not on the line. (See figure.) The **vertex** is the midpoint between the focus and the directrix. The **axis** of the parabola is the line passing through the focus and the vertex.



Note in the figure above that a parabola is symmetric with respect to its axis. Using the definition of a parabola, you can derive the following **standard form of the equation of a parabola** whose directrix is parallel to the x-axis or to the y-axis.

Standard Equation of a Parabola (Vertex at Origin)

The **standard form of the equation of a parabola** with vertex at $(0, 0)$ and directrix $y = -p$ is

$$x^2 = 4py, \quad p \neq 0. \qquad \text{Vertical axis}$$

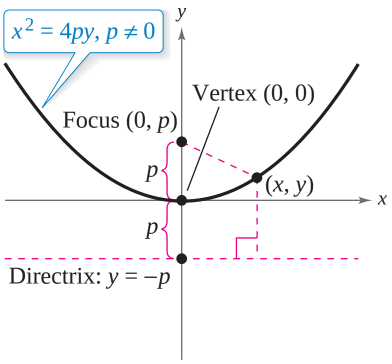
For directrix $x = -p$, the equation is

$$y^2 = 4px, \quad p \neq 0. \qquad \text{Horizontal axis}$$

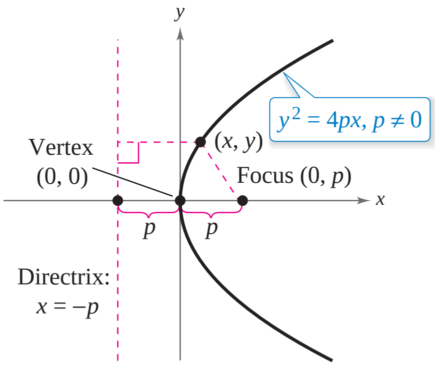
The focus is on the axis p units (directed distance) from the vertex.

For a proof of the standard form of the equation of a parabola, see Proofs in Mathematics on page 358.

Notice that a parabola can have a vertical or a horizontal axis. Examples of each are shown below.



Parabola with vertical axis



Parabola with horizontal axis

EXAMPLE 1 Finding the Focus and Directrix of a Parabola

Find the focus and directrix of the parabola $y = -2x^2$. Then sketch the parabola.

Solution Because the squared term in the equation involves x, you know that the axis is vertical, and the equation is of the form

$$x^2 = 4py. \qquad \text{Standard form, vertical axis}$$

You can write the original equation in this form as follows.

$$-2x^2 = y \qquad \text{Write original equation.}$$

$$x^2 = -\frac{1}{2}y \qquad \text{Divide each side by } -2.$$

$$x^2 = 4\left(-\frac{1}{8}\right)y \qquad \text{Write in standard form.}$$

So, $p = -\frac{1}{8}$. Because p is negative, the parabola opens downward. The focus of the parabola is

$$(0, p) = \left(0, -\frac{1}{8}\right) \qquad \text{Focus}$$

and the directrix of the parabola is

$$y = -p = -\left(-\tfrac{1}{8}\right) = \tfrac{1}{8}. \qquad \text{Directrix}$$

The parabola is shown in Figure 4.18.

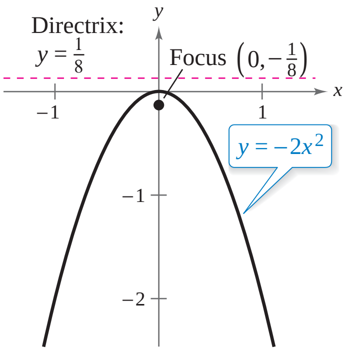


Figure 4.18

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the focus and directrix of the parabola $y = \frac{1}{4}x^2$. Then sketch the parabola.

EXAMPLE 2 Finding the Standard Equation of a Parabola

Find the standard form of the equation of the parabola with vertex at the origin and focus at $(2, 0)$.

Solution The axis of the parabola is horizontal, passing through $(0, 0)$ and $(2, 0)$, as shown in Figure 4.19. So, the standard form is

$$y^2 = 4px. \qquad \text{Standard form, horizontal axis}$$

Because the focus is $p = 2$ units from the vertex, the equation is

$$y^2 = 4(2)x$$

$$y^2 = 8x.$$

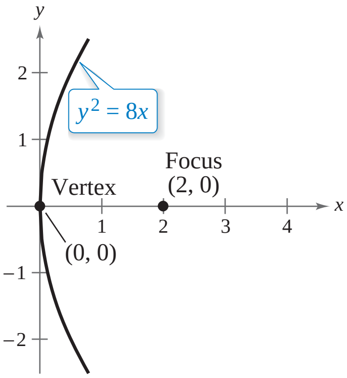


Figure 4.19

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the standard form of the equation of the parabola with vertex at the origin and focus at $\left(0, \frac{3}{8}\right)$.

Parabolas occur in a wide variety of applications. For instance, a parabolic reflector can be formed by revolving a parabola about its axis. The resulting surface has the property that all incoming rays parallel to the axis are reflected through the focus of the parabola. This is the principle behind the construction of the parabolic mirrors used in reflecting telescopes. Conversely, the light rays emanating from the focus of a parabolic reflector used in a flashlight are all parallel to one another, as shown in Figure 4.20.

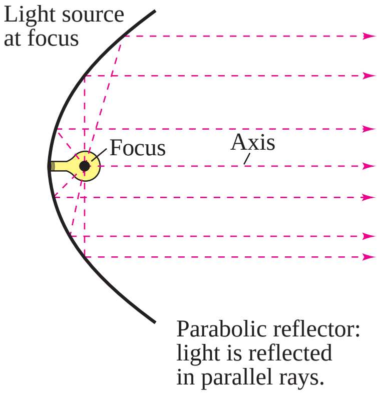


Figure 4.20

Ellipses

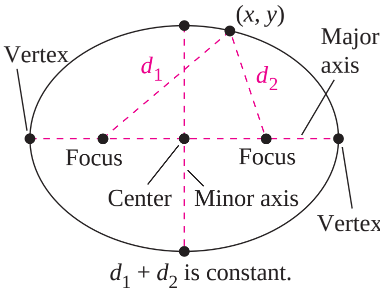


Figure 4.21

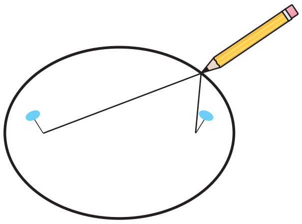


Figure 4.22

Each planet in the solar system moves in an elliptical orbit with the sun at one of the foci.

Definition of an Ellipse

An **ellipse** is the set of all points (x, y) in a plane the sum of whose distances from two distinct fixed points (**foci**) is constant. See Figure 4.21.

The line through the foci intersects the ellipse at two points (**vertices**). The chord joining the vertices is the **major axis,** and its midpoint is the **center** of the ellipse. The chord perpendicular to the major axis at the center is the **minor axis.** (See Figure 4.21.)

You can visualize the definition of an ellipse by imagining two thumbtacks placed at the foci, as shown in Figure 4.22. If the ends of a fixed length of string are fastened to the thumbtacks and the string is *drawn taut* with a pencil, then the path traced by the pencil will be an ellipse.

The standard form of the equation of an ellipse takes one of two forms, depending on whether the major axis is horizontal or vertical.

Standard Equation of an Ellipse (Center at Origin)

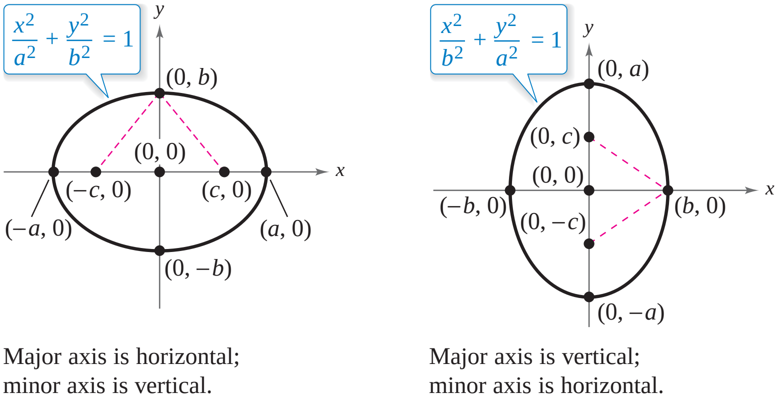
The **standard form of the equation of an ellipse** centered at the origin with major and minor axes of lengths $2a$ and $2b$, respectively, where $0 < b < a$, is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$ Major axis is horizontal.

or

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$ Major axis is vertical.

The vertices and foci lie on the major axis, a and c units, respectively, from the center. Moreover, a, b, and c are related by the equation $c^2 = a^2 - b^2$. See the figures below.



Major axis is horizontal; minor axis is vertical.

Major axis is vertical; minor axis is horizontal.

In the figure on the left above, note that because the sum of the distances from a point on the ellipse to the two foci is constant, it follows that

$$\text{(Sum of distances from } (0, b) \text{ to foci)} = \text{(sum of distances from } (a, 0) \text{ to foci)}$$

$$2\sqrt{b^2 + c^2} = (a + c) + (a - c)$$

$$\sqrt{b^2 + c^2} = a$$

$$c^2 = a^2 - b^2.$$

Orla/Shutterstock.com

EXAMPLE 3 Finding the Standard Equation of an Ellipse

Find the standard form of the equation of the ellipse shown in Figure 4.23.

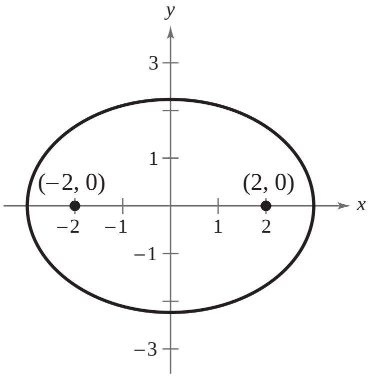


Figure 4.23

Solution From Figure 4.23, the foci occur at $(-2, 0)$ and $(2, 0)$. So, the center of the ellipse is $(0, 0)$, the major axis is horizontal, and the ellipse has an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$ Standard form, horizontal major axis

Also from Figure 4.23, the length of the major axis is $2a = 6$. This implies that $a = 3$. Moreover, the distance from the center to either focus is $c = 2$. Finally,

$$b^2 = a^2 - c^2 = 3^2 - 2^2 = 9 - 4 = 5.$$

Substituting $a^2 = 3^2$ and $b^2 = (\sqrt{5})^2$ yields the following equation in standard form.

$$\frac{x^2}{3^2} + \frac{y^2}{(\sqrt{5})^2} = 1$$ Standard form

This equation simplifies to

$$\frac{x^2}{9} + \frac{y^2}{5} = 1.$$

✓ Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Find the standard form of the equation of the ellipse that has a major axis of length 10 and foci at $(0, -3)$ and $(0, 3)$.

EXAMPLE 4 Sketching an Ellipse

Sketch the ellipse

$$x^2 + 4y^2 = 16$$

and identify the vertices.

Solution

$$x^2 + 4y^2 = 16$$ Write original equation.

$$\frac{x^2}{16} + \frac{4y^2}{16} = \frac{16}{16}$$ Divide each side by 16.

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$ Simplify.

$$\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1$$ Write in standard form.

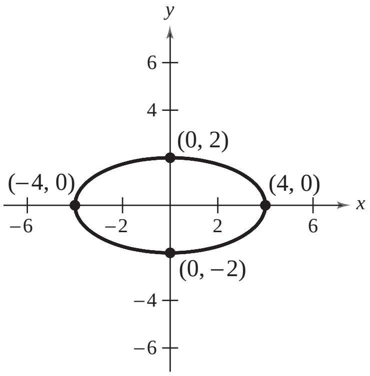


Figure 4.24

Because the denominator of the x^2-term is greater than the denominator of the y^2-term, you can conclude that the major axis is horizontal. Moreover, because $a^2 = 4^2$, the endpoints of the major axis lie four units *right* and *left* of the center $(0, 0)$. So the vertices of the ellipse are $(4, 0)$ and $(-4, 0)$. Similarly, because the denominator of the y^2-term is $b^2 = 2^2$, the endpoints of the minor axis (or *co-vertices*) lie two units *up* and *down* from the center at $(0, 2)$ and $(0, -2)$. The ellipse is shown in Figure 4.24.

✓ Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Sketch the ellipse

$$4x^2 + y^2 = 64$$

and identify the vertices.

EXAMPLE 5 Sketching an Ellipse

Sketch the ellipse

$$4x^2 + y^2 = 36$$

and identify the vertices.

Algebraic Solution

$$4x^2 + y^2 = 36 \qquad \text{Write original equation.}$$

$$\frac{4x^2}{36} + \frac{y^2}{36} = \frac{36}{36} \qquad \text{Divide each side by 36.}$$

$$\frac{x^2}{9} + \frac{y^2}{36} = 1 \qquad \text{Simplify.}$$

$$\frac{x^2}{3^2} + \frac{y^2}{6^2} = 1 \qquad \text{Write in standard form.}$$

Because the denominator of the y^2-term is larger than the denominator of the x^2-term, you can conclude that the major axis is vertical. Moreover, because

$$a^2 = 6^2$$

the endpoints of the major axis lie six units *up* and *down* from the center $(0, 0)$. So, the vertices of the ellipse are

$$(0, 6) \quad \text{and} \quad (0, -6).$$

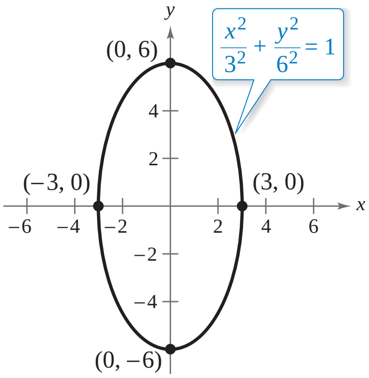
Similarly, because the denominator of the x^2-term is

$$b^2 = 3^2$$

the endpoints of the minor axis (or co-vertices) lie three units to the *right* and *left* of the center at

$$(3, 0) \quad \text{and} \quad (-3, 0).$$

The ellipse is shown below.



Graphical Solution

Solve the equation of the ellipse for y as follows.

$$4x^2 + y^2 = 36$$

$$y^2 = 36 - 4x^2$$

$$y = \pm\sqrt{36 - 4x^2}$$

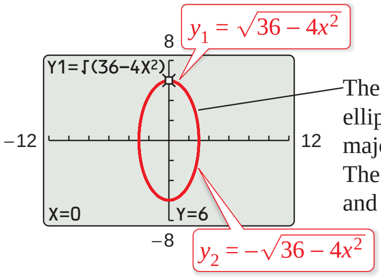
Then, use a graphing utility to graph

$$y_1 = \sqrt{36 - 4x^2}$$

and

$$y_2 = -\sqrt{36 - 4x^2}$$

in the same viewing window, as shown below. Be sure to use a square setting.



The center of the ellipse is (0, 0) and the major axis is vertical. The vertices are (0, 6) and (0, −6).

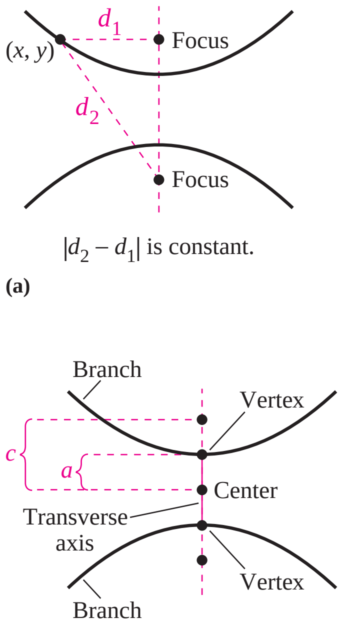
✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Sketch the ellipse $x^2 + 9y^2 = 81$ and identify the vertices.

Note in Example 5 that you can graph conics using a graphing utility by first solving for y. You may have to graph the conic using two separate equations.

Hyperbolas

The definition of a **hyperbola** is similar to that of an ellipse. For an ellipse, the *sum* of the distances between the foci and a point on the ellipse is constant, whereas for a hyperbola, the absolute value of the *difference* of the distances between the foci and a point on the hyperbola is constant.



(a)

(b)

Figure 4.25

Definition of a Hyperbola

A **hyperbola** is the set of all points (x, y) in a plane for which the absolute value of the difference of the distances from two distinct fixed points called **foci** is constant. See Figure 4.25(a).

The graph of a hyperbola has two disconnected parts (**branches**). The line through the two foci intersects the hyperbola at two points (**vertices**). The line segment connecting the vertices is the **transverse axis,** and the midpoint of the transverse axis is the **center** of the hyperbola. See Figure 4.25(b).

Standard Equation of a Hyperbola (Center at Origin)

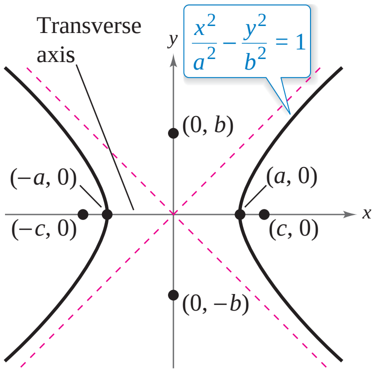
The **standard form of the equation of a hyperbola** with center at the origin is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{Transverse axis is horizontal.}$$

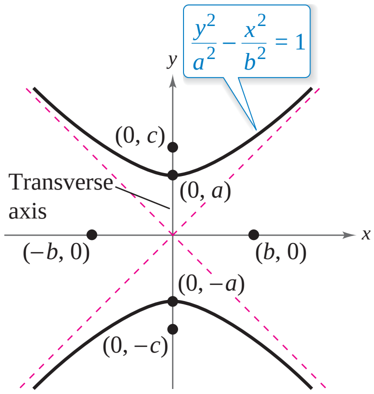
or

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \qquad \text{Transverse axis is vertical.}$$

where $a > 0$ and $b > 0$. The vertices and foci are, respectively, a and c units from the center. Moreover, a, b, and c are related by the equation $b^2 = c^2 - a^2$. See the figures below.



Transverse axis is horizontal



Transverse axis is vertical

Be careful when finding the foci of ellipses and hyperbolas. Notice that the relationships among a, b, and c differ slightly.

Finding the foci of an ellipse:

$$c^2 = a^2 - b^2$$

Finding the foci of a hyperbola:

$$c^2 = a^2 + b^2$$

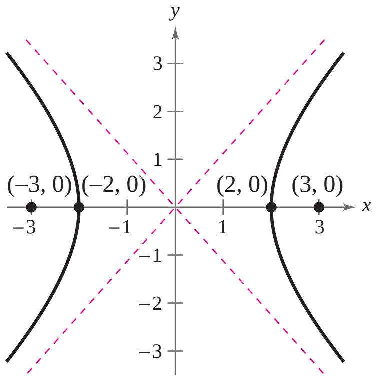


Figure 4.26

EXAMPLE 6 Finding the Standard Equation of a Hyperbola

Find the standard form of the equation of the hyperbola with foci at $(-3, 0)$ and $(3, 0)$ and vertices at $(-2, 0)$ and $(2, 0)$, as shown in Figure 4.26.

Solution From the graph, you can determine that $c = 3$, because the foci are three units from the center. Moreover, $a = 2$ because the vertices are two units from the center. So, it follows that

$$\begin{aligned} b^2 &= c^2 - a^2 \\ &= 3^2 - 2^2 \\ &= 9 - 4 \\ &= 5. \end{aligned}$$

Because the transverse axis is horizontal, the standard form of the equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \qquad \text{Standard form, horizontal transverse axis}$$

Substitute

$$a^2 = 2^2 \quad \text{and} \quad b^2 = \left(\sqrt{5}\right)^2$$

to obtain

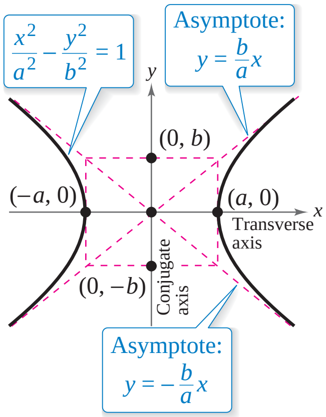
$$\frac{x^2}{2^2} - \frac{y^2}{\left(\sqrt{5}\right)^2} = 1 \qquad \text{Write in standard form.}$$

$$\frac{x^2}{4} - \frac{y^2}{5} = 1. \qquad \text{Simplify.}$$

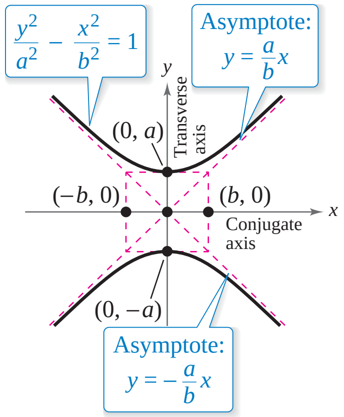
✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the standard form of the equation of the hyperbola with foci at $(0, 6)$ and $(0, -6)$ and vertices at $(0, -3)$ and $(0, 3)$.

An important aid in sketching the graph of a hyperbola is the determination of its *asymptotes*, as shown below. Each hyperbola has two asymptotes that intersect at the center of the hyperbola. Furthermore, the asymptotes pass through the corners of a rectangle of dimensions $2a$ by $2b$. The line segment of length $2b$ joining $(0, b)$ and $(0, -b)$ $[$or $(-b, 0)$ and $(b, 0)]$ is the **conjugate axis** of the hyperbola.



Transverse axis is horizontal; conjugate axis is vertical.



Transverse axis is vertical; conjugate axis is horizontal.

Asymptotes of a Hyperbola (Center at Origin)

The asymptotes of a hyperbola with center at $(0, 0)$ are

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x \qquad \text{Transverse axis is horizontal.}$$

or

$$y = \frac{a}{b}x \quad \text{and} \quad y = -\frac{a}{b}x. \qquad \text{Transverse axis is vertical.}$$

EXAMPLE 7 Sketching a Hyperbola

Sketch the hyperbola

$$4x^2 - y^2 = 16.$$

Algebraic Solution

$$4x^2 - y^2 = 16 \qquad \text{Write original equation.}$$

$$\frac{4x^2}{16} - \frac{y^2}{16} = \frac{16}{16} \qquad \text{Divide each side by 16.}$$

$$\frac{x^2}{4} - \frac{y^2}{16} = 1 \qquad \text{Simplify.}$$

$$\frac{x^2}{2^2} - \frac{y^2}{4^2} = 1 \qquad \text{Write in standard form.}$$

Because the x^2-term is positive, you can conclude that the transverse axis is horizontal and the vertices occur at $(-2, 0)$ and $(2, 0)$. Moreover, the endpoints of the conjugate axis occur at $(0, -4)$ and $(0, 4)$, and you can sketch the rectangle shown in Figure 4.27. Finally, by drawing the asymptotes through the corners of this rectangle, you can complete the sketch shown in Figure 4.28. Note that the asymptotes are $y = 2x$ and $y = -2x$.

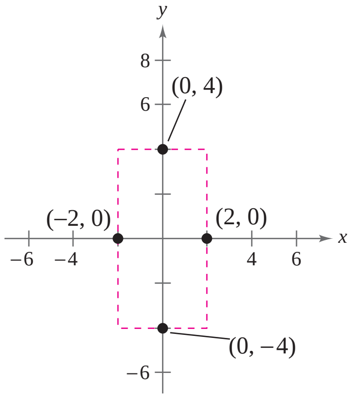


Figure 4.27

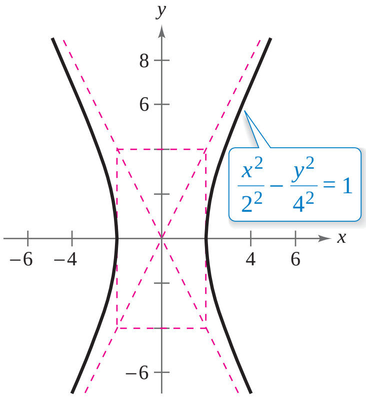


Figure 4.28

Graphical Solution

Solve the equation of the hyperbola for y as follows.

$$4x^2 - y^2 = 16$$

$$4x^2 - 16 = y^2$$

$$\pm\sqrt{4x^2 - 16} = y$$

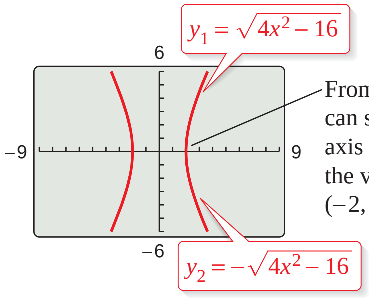
Then use a graphing utility to graph

$$y_1 = \sqrt{4x^2 - 16}$$

and

$$y_2 = -\sqrt{4x^2 - 16}$$

in the same viewing window, as shown below. Be sure to use a square setting.



From the graph, you can see that the transverse axis is horizontal and the vertices are $(-2, 0)$ and $(2, 0)$.

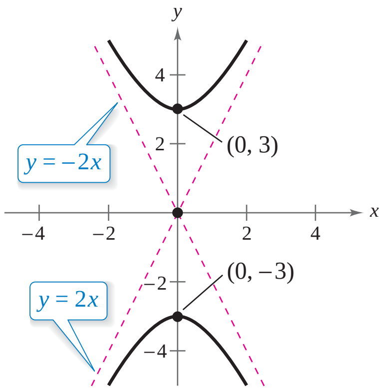
✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Sketch the hyperbola

$$x^2 - \frac{y^2}{4} = 1.$$

EXAMPLE 8 Finding the Standard Equation of a Hyperbola

Find the standard form of the equation of the hyperbola that has vertices at $(0, -3)$ and $(0, 3)$ and asymptotes $y = -2x$ and $y = 2x$, as shown in the figure.



Solution

Because the transverse axis is vertical, the asymptotes are of the forms

$$y = \frac{a}{b}x \quad \text{and} \quad y = -\frac{a}{b}x.$$

Using the fact that $y = 2x$ and $y = -2x$, you can determine that

$$\frac{a}{b} = 2.$$

Because $a = 3$, you can determine that $b = \frac{3}{2}$. Finally, you can conclude that the hyperbola has the following equation.

$$\frac{y^2}{3^2} - \frac{x^2}{\left(\frac{3}{2}\right)^2} = 1 \qquad \text{Write in standard form.}$$

$$\frac{y^2}{9} - \frac{x^2}{\frac{9}{4}} = 1 \qquad \text{Simplify.}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the standard form of the equation of the hyperbola that has vertices at $(-1, 0)$ and $(1, 0)$ and asymptotes $y = -4x$ and $y = 4x$.

Summarize (Section 4.3)

1. List the four basic conic sections *(page 329)*.
2. State the definition of a parabola *(page 330)*. For examples involving parabolas, see Examples 1 and 2.
3. State the definition of an ellipse *(page 332)*. For examples involving ellipses, see Examples 3–5.
4. State the definition of a hyperbola *(page 335)*. For examples involving hyperbolas, see Examples 6–8.

4.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

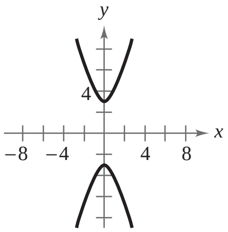
Vocabulary: Fill in the blanks.

1. A ________ is the intersection of a plane and a double-napped cone.
2. The equation $(x - h)^2 + (y - k)^2 = r^2$ is the standard form of the equation of a ________ with center ________ and radius ________.
3. A ________ is the set of all points (x, y) in a plane that are equidistant from a fixed line, called the ________, and a fixed point, called the ________, not on the line.
4. The ________ of a parabola is the midpoint between the focus and the directrix.
5. The line that passes through the focus and the vertex of a parabola is called the ________ of the parabola.
6. An ________ is the set of all points (x, y) in a plane the sum of whose distances from two distinct fixed points, called ________, is constant.
7. The chord joining the vertices of an ellipse is called the ________ ________, and its midpoint is the ________ of the ellipse.
8. The chord perpendicular to the major axis at the center of an ellipse is the ________ ________ of the ellipse.
9. A ________ is the set of all points (x, y) in a plane the difference of whose distances from two distinct fixed points, called ________, is a positive constant.
10. The line segment connecting the vertices of a hyperbola is called the ________ ________, and the midpoint of the line segment is the ________ of the hyperbola.

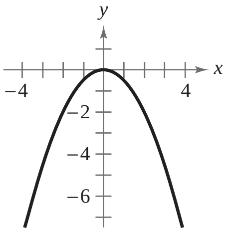
Skills and Applications

Matching **In Exercises 11–16, match the equation with its graph. [The graphs are labeled (a)–(f).]**

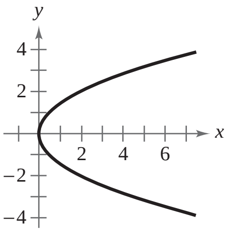
(a)



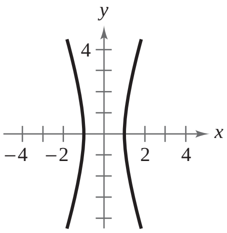
(b)



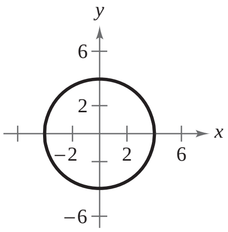
(c)



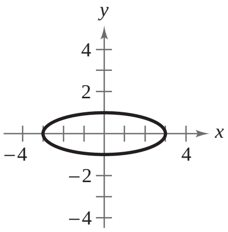
(d)



(e)



(f)



11. $x^2 = -2y$
12. $y^2 = 2x$
13. $x^2 + 9y^2 = 9$
14. $9x^2 - y^2 = 9$
15. $y^2 - 9x^2 = 9$
16. $x^2 + y^2 = 16$

Finding the Focus and Directrix of a Parabola **In Exercises 17–22, find the focus and directrix of the parabola. Then sketch the parabola.**

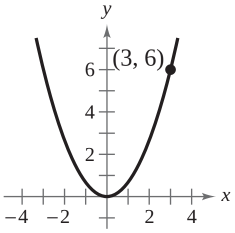
17. $y = \frac{1}{2}x^2$
18. $y = -4x^2$
19. $y^2 = -6x$
20. $y^2 = 3x$
21. $x^2 + 12y = 0$
22. $x + y^2 = 0$

Finding the Standard Equation of a Parabola **In Exercises 23–28, find the standard form of the equation of the parabola with the given characteristic(s) and vertex at the origin.**

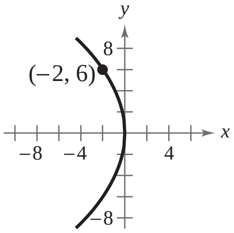
23. Focus: $(-2, 0)$
24. Focus: $\left(0, \frac{1}{2}\right)$
25. Directrix: $y = -2$
26. Directrix: $x = 4$
27. Passes through the point $(4, 6)$; horizontal axis
28. Passes through the point $\left(-2, \frac{1}{4}\right)$; vertical axis

Finding the Standard Equation of a Parabola **In Exercises 29 and 30, find the standard form of the equation of the parabola and determine the coordinates of the focus.**

29.



30.



31. Flashlight The light bulb in a flashlight is at the focus of the parabolic reflector, 1.5 centimeters from the vertex of the reflector (see figure). Write an equation for a cross section of the flashlight's reflector with its focus on the positive x-axis and its vertex at the origin.

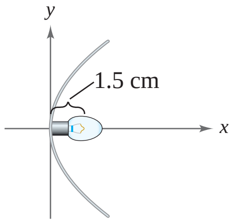


Figure for 31

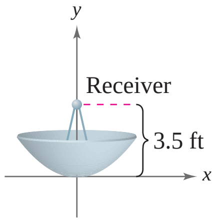


Figure for 32

32. Satellite Antenna Write an equation for a cross section of the parabolic satellite dish antenna shown in the figure.

33. Suspension Bridge

Each cable of the Golden Gate Bridge is suspended (in the shape of a parabola) between two towers that are 1280 meters apart. The top of each tower is 152 meters above the roadway. The cables touch the roadway at the midpoint between the towers.

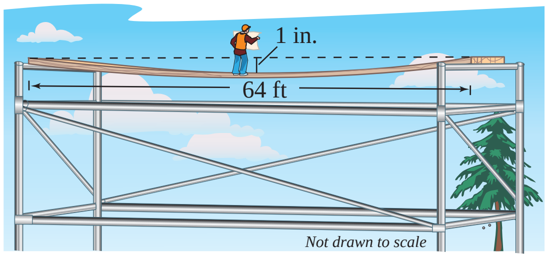
(a) Sketch the bridge in a rectangular coordinate system with the origin at the center of the roadway. Label the coordinates of known points.

(b) Write an equation that models the cables.

(c) Complete the table by finding the height y of the suspension cables over the roadway at a distance of x meters from the center of the bridge.

Distance, x	0	200	400	500	600
Height, y					

34. Beam Deflection A simply supported beam (see figure) is 64 feet long and has a load at the center. The deflection of the beam at its center is 1 inch. The shape of the deflected beam is parabolic.

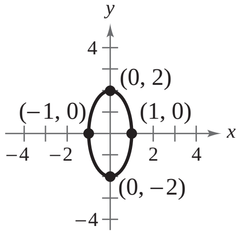


(a) Find an equation of the parabola. (Assume that the origin is at the center of the beam.)

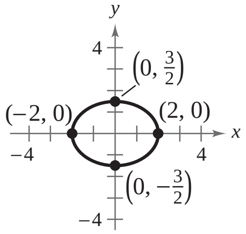
(b) How far from the center of the beam is the deflection $\frac{1}{2}$ inch?

Finding the Standard Equation of an Ellipse **In Exercises 35–44, find the standard form of the equation of the ellipse with the given characteristics and center at the origin.**

35.



36.



37. Vertices: $(\pm 5, 0)$; foci: $(\pm 2, 0)$

38. Vertices: $(0, \pm 8)$; foci: $(0, \pm 4)$

39. Foci: $(\pm 5, 0)$; major axis of length 14

40. Foci: $(\pm 2, 0)$; major axis of length 10

41. Vertices: $(\pm 9, 0)$; minor axis of length 6

42. Vertices: $(0, \pm 10)$; minor axis of length 2

43. Vertices: $(0, \pm 5)$; passes through the point $(4, 2)$

44. Passes through the points $(0, 4)$ and $(2, 0)$

Sketching an Ellipse **In Exercises 45–54, find the vertices of the ellipse. Then sketch the ellipse.**

45. $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$

46. $\dfrac{x^2}{121} + \dfrac{y^2}{144} = 1$

47. $\dfrac{x^2}{25/9} + \dfrac{y^2}{16/9} = 1$

48. $\dfrac{x^2}{4} + \dfrac{y^2}{1/4} = 1$

49. $\dfrac{x^2}{36} + \dfrac{y^2}{7} = 1$

50. $\dfrac{x^2}{28} + \dfrac{y^2}{64} = 1$

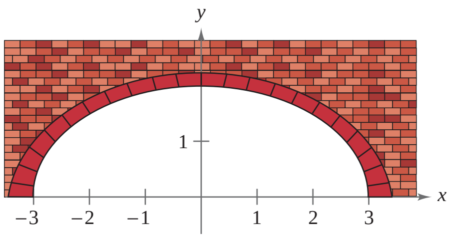
51. $4x^2 + y^2 = 1$

52. $4x^2 + 9y^2 = 36$

53. $\dfrac{1}{16}x^2 + \dfrac{1}{81}y^2 = 1$

54. $\dfrac{1}{100}x^2 + \dfrac{1}{49}y^2 = 1$

55. Architecture A fireplace arch is to be constructed in the shape of a semiellipse. The opening is to have a height of 2 feet at the center and a width of 6 feet along the base (see figure). The contractor draws the outline of the ellipse on the wall by the method shown in Figure 4.22. Give the required positions of the tacks and the length of the string.



Cosmo Condina/The Image Bank/Getty Images

56. Architecture A semielliptical arch over a tunnel for a one-way road through a mountain has a major axis of 50 feet and a height at the center of 10 feet.

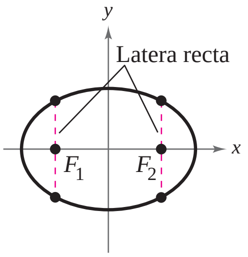
(a) Sketch the arch of the tunnel on a rectangular coordinate system with the center of the road entering the tunnel at the origin. Identify the coordinates of the known points.

(b) Find an equation of the semielliptical arch over the tunnel.

(c) You are driving a moving truck that has a width of 8 feet and a height of 9 feet. Will the moving truck clear the opening of the arch?

57. Architecture Repeat Exercise 56 for a semielliptical arch with a major axis of 40 feet and a height at the center of 15 feet. The dimensions of the truck are 10 feet wide by 14 feet high.

58. Geometry A line segment through a focus of an ellipse with endpoints on the ellipse and perpendicular to the major axis is called a **latus rectum** of the ellipse. An ellipse has two latera recta. Knowing the length of the latera recta is helpful in sketching an ellipse because it yields other points on the curve (see figure). Show that the length of each latus rectum is $2b^2/a$.



Using Latera Recta **In Exercises 59–62, sketch the ellipse, using the latera recta (see Exercise 58).**

59. $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$

60. $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

61. $9x^2 + 4y^2 = 36$

62. $5x^2 + 3y^2 = 15$

Finding the Standard Equation of a Hyperbola **In Exercises 63–70, find the standard form of the equation of the hyperbola with the given characteristics and center at the origin.**

63. Vertices: $(0, \pm 2)$; foci: $(0, \pm 6)$

64. Vertices: $(\pm 4, 0)$; foci: $(\pm 5, 0)$

65. Vertices: $(\pm 1, 0)$; asymptotes: $y = \pm 3x$

66. Vertices: $(0, \pm 3)$; asymptotes: $y = \pm 3x$

67. Foci: $(0, \pm 8)$; asymptotes: $y = \pm 4x$

68. Foci: $(\pm 10, 0)$; asymptotes: $y = \pm \frac{3}{4}x$

69. Vertices: $(0, \pm 3)$; passes through the point $(-2, 5)$

70. Vertices: $(\pm 2, 0)$; passes through the point $(3, \sqrt{3})$

Sketching a Hyperbola **In Exercises 71–80, find the vertices of the hyperbola. Then sketch the hyperbola using the asymptotes as an aid.**

71. $x^2 - y^2 = 1$

72. $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

73. $\dfrac{y^2}{1} - \dfrac{x^2}{4} = 1$

74. $\dfrac{y^2}{9} - \dfrac{x^2}{1} = 1$

75. $\dfrac{y^2}{49} - \dfrac{x^2}{196} = 1$

76. $\dfrac{x^2}{36} - \dfrac{y^2}{4} = 1$

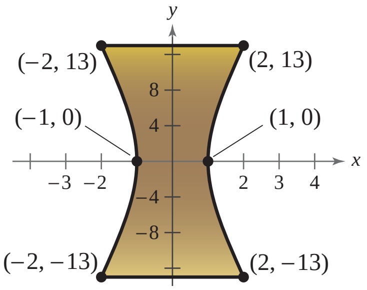
77. $4y^2 - x^2 = 1$

78. $4y^2 - 9x^2 = 36$

79. $\dfrac{1}{36}y^2 - \dfrac{1}{100}x^2 = 1$

80. $\dfrac{1}{144}x^2 - \dfrac{1}{169}y^2 = 1$

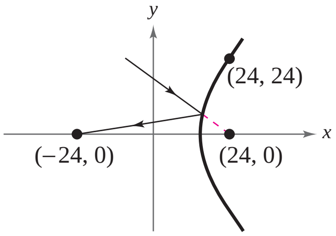
81. Art A sculpture has a hyperbolic cross section (see figure).



(a) Write an equation that models the curved sides of the sculpture.

(b) Each unit on the coordinate plane represents 1 foot. Find the width of the sculpture at a height of 5 feet.

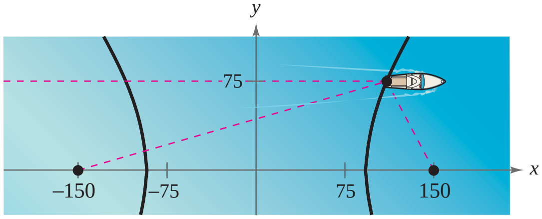
82. Optics A hyperbolic mirror (used in some telescopes) has the property that a light ray directed at the focus will be reflected to the other focus. The focus of a hyperbolic mirror (see figure) has coordinates (24, 0). Find the vertex of the mirror when its mount at the top edge of the mirror has coordinates (24, 24).



83. Aeronautics When an airplane travels faster than the speed of sound, the sound waves form a cone behind the airplane. If the airplane is flying parallel to the ground, then the sound waves intersect the ground in a hyperbola with the airplane directly above its center. A sonic boom is heard along the hyperbola. You hear a sonic boom that is audible along a hyperbola with the equation $(x^2/100) - (y^2/4) = 1$, where x and y are measured in miles. What is the shortest horizontal distance you could be from the airplane?

84. Navigation Long-distance radio navigation for aircraft and ships uses synchronized pulses transmitted by widely separated transmitting stations. These pulses travel at the speed of light (186,000 miles per second). The difference in the times of arrival of these pulses at an aircraft or ship is constant on a hyperbola having the transmitting stations as foci.

Assume that two stations 300 miles apart are positioned on a rectangular coordinate system at points with coordinates $(-150, 0)$ and $(150, 0)$ and that a ship is traveling on a path with coordinates $(x, 75)$, as shown in the figure. Find the x-coordinate of the position of the ship when the time difference between the pulses from the transmitting stations is 1000 micro-seconds (0.001 second).



Exploration

True or False? **In Exercises 85–88, determine whether the statement is true or false. Justify your answer.**

85. The equation $x^2 - y^2 = 144$ represents a circle.

86. The major axis of the ellipse $y^2 + 16x^2 = 64$ is vertical.

87. It is possible for a parabola to intersect its directrix.

88. If the vertex and focus of a parabola are on a horizontal line, then the directrix of the parabola is vertical.

89. Area of an Ellipse Consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a + b = 20.$$

(a) The area of the ellipse is given by $A = \pi ab$. Write the area of the ellipse as a function of a.

(b) Find the equation of an ellipse with an area of 264 square centimeters.

(c) Complete the table using your equation from part (a), and make a conjecture about the shape of the ellipse with maximum area.

a	8	9	10	11	12	13
A						

(d) Use a graphing utility to graph the area function and use the graph to support your conjecture in part (c).

90. HOW DO YOU SEE IT? In parts (a)–(d), describe in words how a plane could intersect the double-napped cone to form the conic section (see figure).

(a) Circle (b) Ellipse

(c) Parabola (d) Hyperbola

91. Think About It How can you tell if an ellipse is a circle from the equation?

92. Think About It Is the graph of $x^2 + 4y^4 = 4$ an ellipse? Explain.

93. Think About It The graph of $x^2 - y^2 = 0$ is a degenerate conic. Sketch this graph and identify the degenerate conic.

94. Think About It Which part of the graph of the ellipse $4x^2 + 9y^2 = 36$ is represented by each equation? (Do not graph.)

(a) $x = -\frac{3}{2}\sqrt{4 - y^2}$ (b) $y = \frac{2}{3}\sqrt{9 - x^2}$

95. Writing At the beginning of this section, you learned that each type of conic section can be formed by the intersection of a plane and a double-napped cone. Write a short paragraph describing examples of physical situations in which hyperbolas are formed.

96. Writing Write a paragraph discussing the changes in the shape and orientation of the graph of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{4^2} = 1$$

as a increases from 1 to 8.

97. Using a Definition Use the definition of an ellipse to derive the standard form of the equation of an ellipse.

98. Using a Definition Use the definition of a hyperbola to derive the standard form of the equation of a hyperbola.

99. Drawing Ellipses An ellipse can be drawn using two thumbtacks placed at the foci of the ellipse, a string of fixed length (greater than the distance between the tacks), and a pencil, as shown in Figure 4.22. Try doing this. Vary the length of the string and the distance between the thumbtacks. Explain how to obtain ellipses that are almost circular. Explain how to obtain ellipses that are long and narrow.

4.4 Translations of Conics

In some real-life applications, it is not convenient to use conics whose centers or vertices are at the origin. For instance, in Exercise 41 on page 350, you can use a parabola to model the path of a satellite as it escapes Earth's gravity.

- **Recognize equations of conics that have been shifted vertically or horizontally in the plane.**
- **Write and graph equations of conics that have been shifted vertically or horizontally in the plane.**

Vertical and Horizontal Shifts of Conics

In Section 4.3, you studied conic sections whose graphs were in *standard position*. In this section, you will study the equations of conic sections that have been shifted vertically or horizontally in the plane.

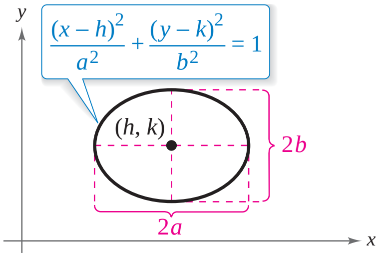
Standard Forms of Equations of Conics

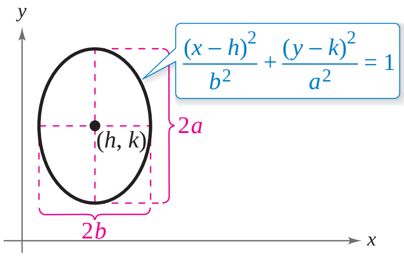
Circle: Center $= (h, k)$; radius $= r$

$$(x - h)^2 + (y - k)^2 = r^2$$

Ellipse: Center $= (h, k)$

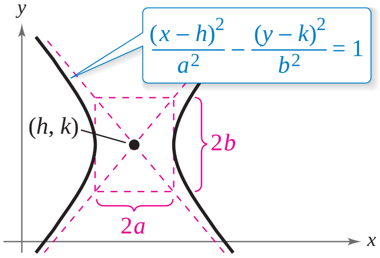
Major axis length $= 2a$; minor axis length $= 2b$

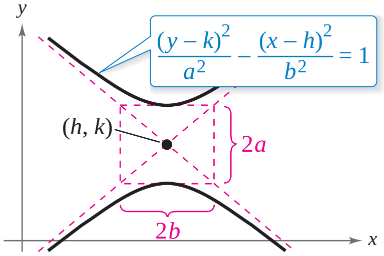




Hyperbola: Center $= (h, k)$

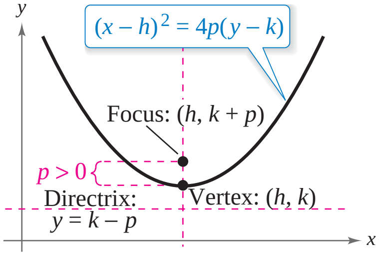
Transverse axis length $= 2a$; conjugate axis length $= 2b$

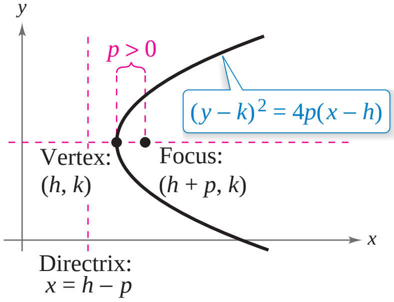




Parabola: Vertex $= (h, k)$

Directed distance from vertex to focus $= p$





REMARK Consider the equation of the ellipse

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1.$$

If you let $a = b$, then the equation can be rewritten as

$$(x - h)^2 + (y - k)^2 = a^2$$

which is the standard form of the equation of a circle with radius $r = a$ (see Section 1.1). Geometrically, when $a = b$ for an ellipse, the major and minor axes are of equal length, and so the graph is a circle [see Example 1(a)].

Paul Fleet/Shutterstock.com

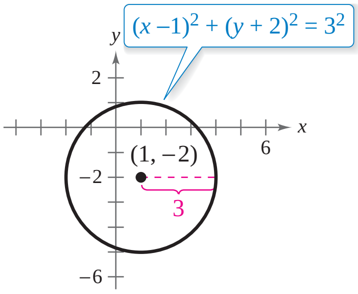


Figure 4.29 Circle

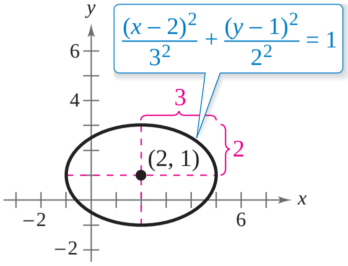


Figure 4.30 Ellipse

EXAMPLE 1 Equations of Conic Sections

Identify each conic. Then describe the translation of the graph of the conic.

a. $(x-1)^2 + (y+2)^2 = 3^2$ **b.** $\dfrac{(x-2)^2}{3^2} + \dfrac{(y-1)^2}{2^2} = 1$

c. $\dfrac{(x-3)^2}{1^2} - \dfrac{(y-2)^2}{3^2} = 1$ **d.** $(x-2)^2 = 4(-1)(y-3)$

Solution

a. The graph of $(x-1)^2 + (y+2)^2 = 3^2$ is a *circle* whose center is the point $(1, -2)$ and whose radius is 3, as shown in Figure 4.29. The graph of the circle has been shifted one unit to the right and two units down from standard position.

b. The graph of

$$\frac{(x-2)^2}{3^2} + \frac{(y-1)^2}{2^2} = 1$$

is an *ellipse* whose center is the point $(2, 1)$. The major axis of the ellipse is horizontal and of length $2(3) = 6$, and the minor axis of the ellipse is vertical and of length $2(2) = 4$, as shown in Figure 4.30. The graph of the ellipse has been shifted two units to the right and one unit up from standard position.

c. The graph of

$$\frac{(x-3)^2}{1^2} - \frac{(y-2)^2}{3^2} = 1$$

is a *hyperbola* whose center is the point $(3, 2)$. The transverse axis is horizontal and of length $2(1) = 2$, and the conjugate axis is vertical and of length $2(3) = 6$, as shown in Figure 4.31. The graph of the hyperbola has been shifted three units to the right and two units up from standard position.

d. The graph of $(x-2)^2 = 4(-1)(y-3)$ is a *parabola* whose vertex is the point $(2, 3)$. The axis of the parabola is vertical. Because $p = -1$, the focus lies one unit below the vertex, and the directrix is $y = k - p = 3 - (-1) = 3 + 1 = 4$ as shown in Figure 4.32. The graph of the parabola has been reflected in the x-axis, shifted two units to the right and three units up from standard position.

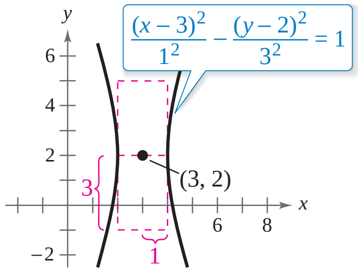


Figure 4.31 Hyperbola

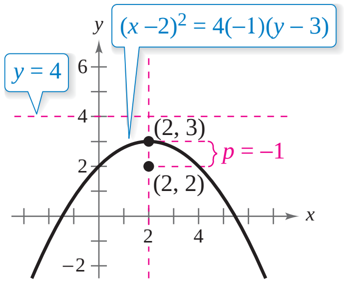


Figure 4.32 Parabola

✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Identify each conic. Then describe the translation of the graph of the conic.

a. $(x+1)^2 + (y-1)^2 = 2^2$ **b.** $\dfrac{(x+1)^2}{9} + \dfrac{(y-2)^2}{25} = 1$

c. $\dfrac{(x-3)^2}{4} - \dfrac{(y-1)^2}{1} = 1$ **d.** $(x+4)^2 - 8(y-3) = 0$

Equations of Conics in Standard Form

EXAMPLE 2 Finding Characteristics of a Parabola

Find the vertex, focus, and directrix of the parabola $x^2 - 2x + 4y - 3 = 0$. Then sketch the parabola.

Solution

Complete the square to write the equation in standard form.

$$\begin{aligned} x^2 - 2x + 4y - 3 &= 0 && \text{Write original equation.} \\ x^2 - 2x &= -4y + 3 && \text{Collect } x\text{-terms on one side of the equation.} \\ x^2 - 2x + 1 &= -4y + 3 + 1 && \text{Complete the square.} \\ (x - 1)^2 &= -4y + 4 && \text{Write in completed square form.} \\ (x - 1)^2 &= 4(-1)(y - 1) && \text{Write in standard form, } (x - h)^2 = 4p(y - k). \end{aligned}$$

From this standard form, it follows that $h = 1$, $k = 1$, and $p = -1$. Because the axis is vertical and p is negative, the parabola opens downward.

The vertex is

$$(h, k) = (1, 1) \qquad \text{Vertex}$$

the focus is

$$(h, k + p) = (1, 0) \qquad \text{Focus}$$

and the directrix is

$$y = k - p = 2. \qquad \text{Directrix}$$

The parabola is shown in Figure 4.33.

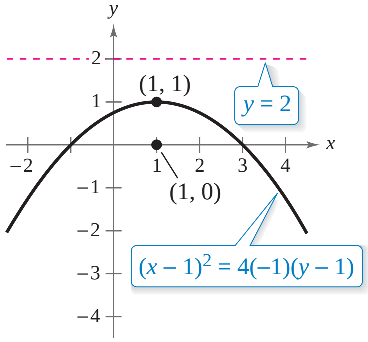


Figure 4.33

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the vertex, focus, and directrix of the parabola $y^2 - 6y + 4x + 17 = 0$. Then sketch the parabola. ■

Note in Example 2 that p is the directed distance from the vertex to the focus. Because the axis of the parabola is vertical and $p = -1$, the focus is one unit below the vertex.

EXAMPLE 3 Finding the Standard Equation of a Parabola

Find the standard form of the equation of the parabola whose vertex is $(2, 1)$ and whose focus is $(4, 1)$, as shown in Figure 4.34.

Solution

Because the vertex and the focus lie on a horizontal line and the focus lies two units to the right of the vertex, it follows that the axis of the parabola is horizontal and $p = 2$. So the standard form of the parabola is

$$\begin{aligned} (y - k)^2 &= 4p(x - h) && \text{Standard form, horizontal axis} \\ (y - 1)^2 &= 4(2)(x - 2) && \text{Write in standard form.} \\ (y - 1)^2 &= 8(x - 2). && \text{Simplify.} \end{aligned}$$

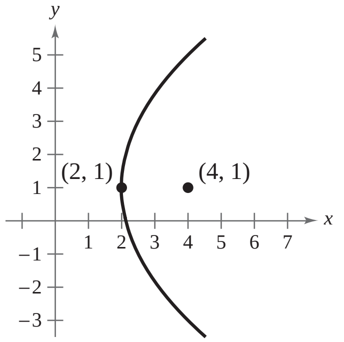


Figure 4.34

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the standard form of the equation of the parabola whose vertex is $(-1, 1)$ and whose directrix is $y = 0$. ■

EXAMPLE 4 Sketching an Ellipse

Sketch the ellipse $x^2 + 4y^2 + 6x - 8y + 9 = 0$.

Solution

Complete the square to write the equation in standard form.

$$\begin{aligned}
x^2 + 4y^2 + 6x - 8y + 9 &= 0 && \text{Write original equation.}\\
(x^2 + 6x + \boxed{}) + (4y^2 - 8y + \boxed{}) &= -9 && \text{Group terms.}\\
(x^2 + 6x + \boxed{}) + 4(y^2 - 2y + \boxed{}) &= -9 && \text{Factor 4 out of } y\text{-terms.}\\
(x^2 + 6x + 9) + 4(y^2 - 2y + 1) &= -9 + 9 + 4(1) && \text{Complete the squares.}\\
(x + 3)^2 + 4(y - 1)^2 &= 4 && \text{Write in completed square form.}\\
\frac{(x + 3)^2}{4} + \frac{(y - 1)^2}{1} &= 1 && \text{Divide each side by 4.}\\
\frac{(x + 3)^2}{2^2} + \frac{(y - 1)^2}{1^2} &= 1 && \frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1
\end{aligned}$$

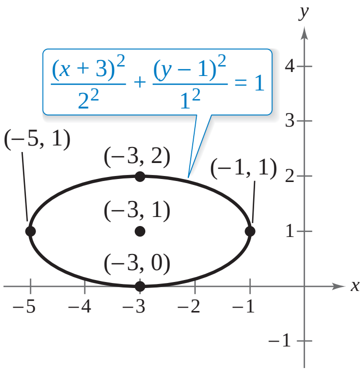


Figure 4.35

From this standard form, it follows that the center is $(h, k) = (-3, 1)$. Because the denominator of the x-term is $a^2 = 2^2$, the endpoints of the major axis lie two units to the right and left of the center. Similarly, because the denominator of the y-term is $b^2 = 1^2$, the endpoints of the minor axis lie one unit up and down from the center. The ellipse is shown in Figure 4.35.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Sketch the ellipse $9x^2 + 4y^2 - 36x + 24y + 36 = 0$.

EXAMPLE 5 Finding the Standard Equation of an Ellipse

Find the standard form of the equation of the ellipse whose vertices are $(2, -2)$ and $(2, 4)$. The length of the minor axis of the ellipse is 4, as shown in Figure 4.36.

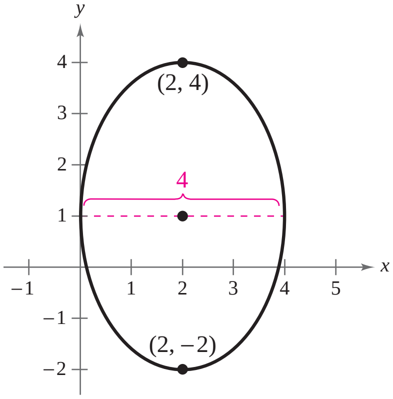


Figure 4.36

Solution

The center of the ellipse lies at the midpoint of its vertices. So, the center is

$$(h, k) = \left(\frac{2 + 2}{2}, \frac{4 + (-2)}{2}\right) = (2, 1). \qquad \text{Center.}$$

Because the vertices lie on a vertical line and are six units apart, it follows that the major axis is vertical and has a length of $2a = 6$. So, $a = 3$. Moreover, because the minor axis has a length of 4, it follows that $2b = 4$, which implies that $b = 2$. So, the standard form of the ellipse is

$$\begin{aligned}
\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} &= 1 && \text{Major axis is vertical.}\\
\frac{(x - 2)^2}{2^2} + \frac{(y - 1)^2}{3^2} &= 1 && \text{Write in standard form.}\\
\frac{(x - 2)^2}{4} + \frac{(y - 1)^2}{9} &= 1. && \text{Simplify.}
\end{aligned}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the standard form of the equation of the ellipse whose vertices are $(3, 0)$ and $(3, 10)$. The length of the minor axis of the ellipse is 6. ■

EXAMPLE 6 Sketching a Hyperbola

Sketch the hyperbola

$$y^2 - 4x^2 + 4y + 24x - 41 = 0.$$

Solution

Complete the square to write the equation in standard form.

$$\begin{aligned}
y^2 - 4x^2 + 4y + 24x - 41 &= 0 && \text{Write original equation.}\\
(y^2 + 4y + \boxed{}) - (4x^2 - 24x + \boxed{}) &= 41 && \text{Group terms.}\\
(y^2 + 4y + \boxed{}) - 4(x^2 - 6x + \boxed{}) &= 41 && \text{Factor 4 out of } x\text{-terms.}\\
(y^2 + 4y + 4) - 4(x^2 - 6x + 9) &= 41 + 4 - 4(9) && \text{Complete the squares.}\\
(y + 2)^2 - 4(x - 3)^2 &= 9 && \text{Write in completed square form.}\\
\frac{(y + 2)^2}{9} - \frac{4(x - 3)^2}{9} &= 1 && \text{Divide each side by 9.}\\
\frac{(y + 2)^2}{9} - \frac{(x - 3)^2}{\frac{9}{4}} &= 1 && \text{Change 4 to } \frac{1}{\frac{1}{4}}.\\
\frac{(y + 2)^2}{3^2} - \frac{(x - 3)^2}{\left(\frac{3}{2}\right)^2} &= 1 && \frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1
\end{aligned}$$

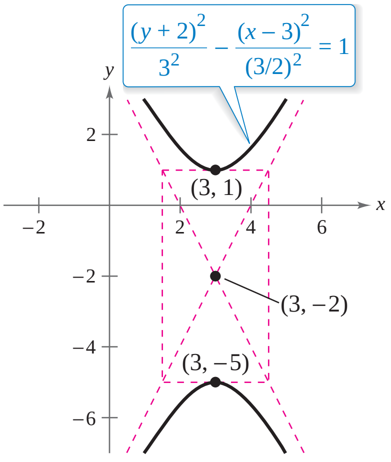


Figure 4.37

From this standard form, it follows that the transverse axis is vertical and the center lies at $(h, k) = (3, -2)$. Because the denominator of the y-term is $a^2 = 3^2$, you know that the vertices occur three units above and below the center.

$(3, 1)$ and $(3, -5)$ Vertices

To sketch the hyperbola, draw a rectangle whose top and bottom pass through the vertices. Because the denominator of the x-term is $b^2 = \left(\frac{3}{2}\right)^2$, locate the sides of the rectangle $\frac{3}{2}$ units to the right and left of the center, as shown in Figure 4.37. Finally, sketch the asymptotes by drawing lines through the opposite corners of the rectangle. Using these asymptotes, you can complete the graph of the hyperbola, as shown in Figure 4.37.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Sketch the hyperbola

$$9x^2 - y^2 - 18x - 6y - 9 = 0.$$

To find the foci in Example 6, first find c. Recall from Section 4.3 that $b^2 = c^2 - a^2$. So,

$$\begin{aligned}
c^2 &= a^2 + b^2\\
&= 9 + \frac{9}{4}\\
&= \frac{45}{4} \quad \Rightarrow \quad c = \frac{3\sqrt{5}}{2}
\end{aligned}$$

Because the transverse axis is vertical, the foci lie c units above and below the center.

$\left(3, -2 + \frac{3}{2}\sqrt{5}\right)$ and $\left(3, -2 - \frac{3}{2}\sqrt{5}\right)$ Foci

EXAMPLE 7 Finding the Standard Equation of a Hyperbola

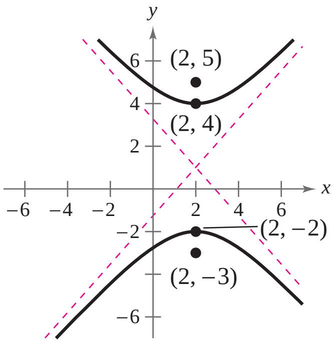


Figure 4.38

Find the standard form of the equation of the hyperbola whose vertices are $(2, -2)$ and $(2, 4)$ and whose foci are $(2, -3)$ and $(2, 5)$, as shown in Figure 4.38.

Solution The center of the hyperbola lies at the midpoint of its vertices. So, the center is $(h, k) = (2, 1)$. Because the vertices lie on a vertical line and are six units apart, it follows that the transverse axis is vertical and has a length of $2a = 6$. So, $a = 3$. Because the foci are four units from the center, it follows that $c = 4$. So,

$$b^2 = c^2 - a^2 = 4^2 - 3^2 = 16 - 9 = 7.$$

Because the transverse axis is vertical, the standard form of the equation is

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1 \qquad \text{Standard form, vertical transverse axis}$$

$$\frac{(y-1)^2}{3^2} - \frac{(x-2)^2}{(\sqrt{7})^2} = 1 \qquad \text{Write in standard form.}$$

$$\frac{(y-1)^2}{9} - \frac{(x-2)^2}{7} = 1. \qquad \text{Simplify.}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the standard form of the equation of the hyperbola whose vertices are $(3, -1)$ and $(5, -1)$ and whose foci are $(1, -1)$ and $(7, -1)$.

An interesting application of conic sections involves the orbits of comets in our solar system. Of the 610 comets identified prior to 1970, 245 have elliptical orbits, 295 have parabolic orbits, and 70 have hyperbolic orbits. For example, Halley's comet has an elliptical orbit, and reappearance of this comet can be predicted every 76 years. The center of the sun is a focus of each of these orbits, and each orbit has a vertex at the point where the comet is closest to the sun, as shown in Figure 4.39.

If p is the distance between the vertex and the focus (in meters), and v is the speed of the comet at the vertex (in meters per second), then the type of orbit is determined as follows.

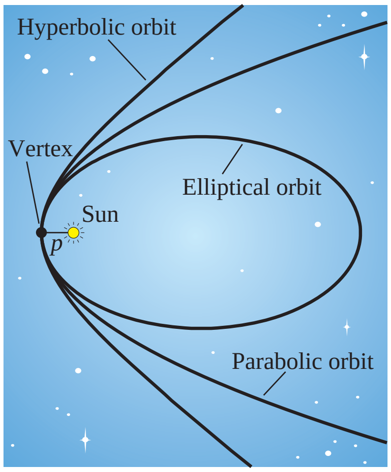


Figure 4.39

1. Ellipse: $v < \sqrt{\dfrac{2GM}{p}}$

2. Parabola: $v = \sqrt{\dfrac{2GM}{p}}$

3. Hyperbola: $v > \sqrt{\dfrac{2GM}{p}}$

In each of these relations, $M = 1.989 \times 10^{30}$ kilograms (the mass of the sun) and $G \approx 6.67 \times 10^{-11}$ cubic meter per kilogram-second squared (the universal gravitational constant).

Summarize (Section 4.4)

1. List the standard forms of the equations of the four basic conic sections *(page 343)*. For an example involving the equations of conic sections, see Example 1.
2. Describe how to write and graph equations of conics that have been shifted vertically or horizontally in the plane *(pages 344–348)*. For examples of writing and graphing equations of conics that have been shifted vertically or horizontally in the plane, see Examples 2–7.

4.4 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: **Match the description of the conic with its standard equation. The equations are labeled (a)–(f).**

(a) $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$ (b) $\dfrac{(x-h)^2}{a^2}-\dfrac{(y-k)^2}{b^2}=1$ (c) $\dfrac{(y-k)^2}{a^2}-\dfrac{(x-h)^2}{b^2}=1$

(d) $\dfrac{(x-h)^2}{b^2}+\dfrac{(y-k)^2}{a^2}=1$ (e) $(x-h)^2=4p(y-k)$ (f) $(y-k)^2=4p(x-h)$

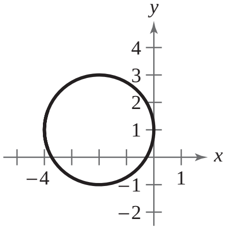
1. Hyperbola with horizontal transverse axis
2. Ellipse with vertical major axis
3. Parabola with vertical axis
4. Hyperbola with vertical transverse axis
5. Ellipse with horizontal major axis
6. Parabola with horizontal axis

Skills and Applications

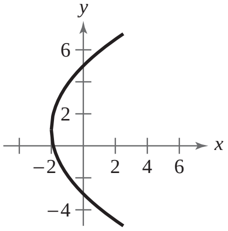
Equations of Conic Sections

In Exercises 7–14, identify the conic. Then describe the translation of the graph of the conic.

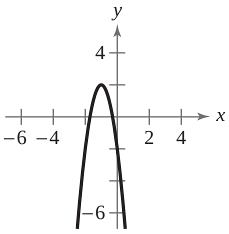
7. $(x+2)^2+(y-1)^2=4$



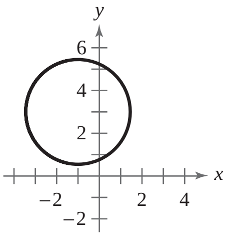
8. $(y-1)^2=4(2)(x+2)$



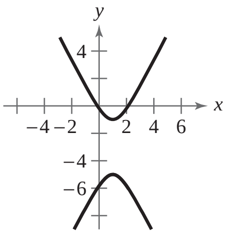
9. $y-2=4(-1)(x+1)^2$



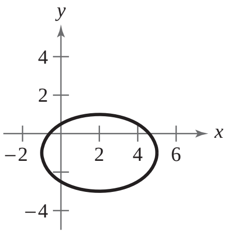
10. $(x+1)^2+(y-3)^2=6$



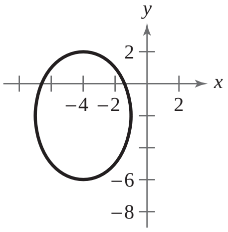
11. $\dfrac{(y+3)^2}{4}-(x-1)^2=1$



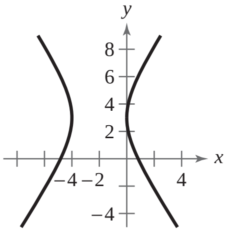
12. $\dfrac{(x-2)^2}{9}+\dfrac{(y+1)^2}{4}=1$



13. $\dfrac{(x+4)^2}{9}+\dfrac{(y+2)^2}{16}=1$



14. $\dfrac{(x+2)^2}{4}-\dfrac{(y-3)^2}{9}=1$



Finding Characteristics of a Circle

In Exercises 15–20, find the center and radius of the circle.

15. $x^2+y^2=49$
16. $x^2+y^2=1$
17. $(x-4)^2+(y-5)^2=36$
18. $(x+8)^2+(y+1)^2=144$
19. $(x-1)^2+y^2=10$
20. $x^2+(y+12)^2=24$

Writing the Equation of a Circle in Standard Form

In Exercises 21–26, write the equation of the circle in standard form, and then find its center and radius.

21. $x^2+y^2-2x+6y+9=0$
22. $x^2+y^2-10x-6y+25=0$
23. $x^2+y^2-8x=0$
24. $2x^2+2y^2-2x-2y-7=0$
25. $4x^2+4y^2+12x-24y+41=0$
26. $9x^2+9y^2+54x-36y+17=0$

Finding Characteristics of a Parabola

In Exercises 27–34, find the vertex, focus, and directrix of the parabola. Then sketch the parabola.

27. $(x-1)^2+8(y+2)=0$
28. $(x+2)+(y-4)^2=0$
29. $\left(y+\frac{1}{2}\right)^2=2(x-5)$
30. $\left(x+\frac{1}{2}\right)^2=4(y-3)$
31. $y=\frac{1}{4}(x^2-2x+5)$
32. $4x-y^2-2y-33=0$
33. $y^2+6y+8x+25=0$
34. $y^2-4y-4x=0$

Finding the Standard Equation of a Parabola

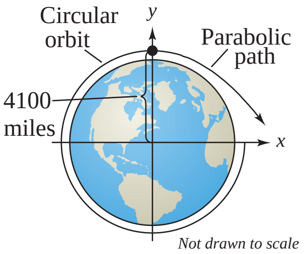
In Exercises 35–40, find the standard form of the equation of the parabola with the given characteristics.

35. Vertex: $(3, 2)$; focus: $(1, 2)$
36. Vertex: $(-1, 2)$; focus: $(-1, 0)$
37. Vertex: $(0, 4)$; directrix: $y=2$
38. Vertex: $(-2, 1)$; directrix: $x=1$
39. Focus: $(4, 4)$; directrix: $x=-4$
40. Focus: $(0, 0)$; directrix: $y=4$

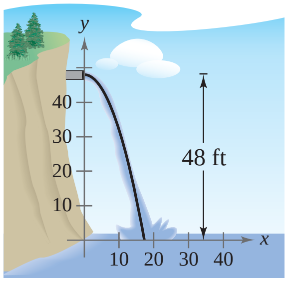
41. Satellite Orbit A satellite in a 100-mile-high circular orbit around Earth has a velocity of approximately 17,500 miles per hour (see figure). When this velocity is multiplied by $\sqrt{2}$, the satellite has the minimum velocity necessary to escape Earth's gravity and follow a parabolic path with the center of Earth as the focus.

(a) Find the escape velocity of the satellite.

(b) Find an equation of its path (assume that the radius of Earth is 4000 miles).



42. Fluid Flow Water is flowing from a horizontal pipe 48 feet above the ground. The falling stream of water has the shape of a parabola whose vertex (0, 48) is at the end of the pipe (see figure). The stream of water strikes the ground at the point $(10\sqrt{3}, 0)$. Find the equation of the path taken by the water.



Projectile Motion **In Exercises 43 and 44, consider the path of an object projected horizontally with a velocity of v feet per second at a height of s feet, where the model for the path is $x^2 = (-v^2/16)(y - s)$. In this model (in which air resistance is disregarded), y is the height (in feet) of the projectile and x is the horizontal distance (in feet) the projectile travels.**

43. A ball is thrown from the top of a 100-foot tower with a velocity of 28 feet per second.

(a) Find the equation of the parabolic path.

(b) How far does the ball travel horizontally before striking the ground?

44. A cargo plane is flying at an altitude of 500 feet and a speed of 135 miles per hour. A supply crate is dropped from the plane. How many *feet* will the crate travel horizontally before it hits the ground?

Sketching an Ellipse **In Exercises 45–52, find the center, foci, and vertices of the ellipse. Then sketch the ellipse.**

45. $\dfrac{(x-1)^2}{9} + \dfrac{(y-5)^2}{25} = 1$

46. $\dfrac{(x-6)^2}{4} + \dfrac{(y+7)^2}{16} = 1$

47. $(x+2)^2 + \dfrac{(y+4)^2}{1/4} = 1$

48. $\dfrac{(x-3)^2}{25/9} + (y-8)^2 = 1$

49. $9x^2 + 25y^2 - 36x - 50y + 52 = 0$

50. $16x^2 + 25y^2 - 32x + 50y + 16 = 0$

51. $9x^2 + 4y^2 + 36x - 16y + 16 = 0$

52. $9x^2 + 4y^2 - 36x + 8y + 31 = 0$

Finding the Standard Equation of an Ellipse **In Exercises 53–60, find the standard form of the equation of the ellipse with the given characteristics.**

53. Vertices: (3, 3), (3, −3); minor axis of length 2

54. Vertices: (−2, 3), (6, 3); minor axis of length 6

55. Foci: (0, 0), (4, 0); major axis of length 8

56. Foci: (0, 0), (0, 8); major axis of length 16

57. Center: (0, 4); $a = 2c$; vertices: (−4, 4), (4, 4)

58. Center: (3, 2); $a = 3c$; foci: (1, 2), (5, 2)

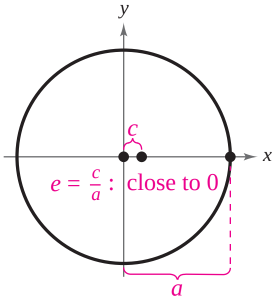
59. Vertices: (−3, 0), (7, 0); foci: (0, 0), (4, 0)

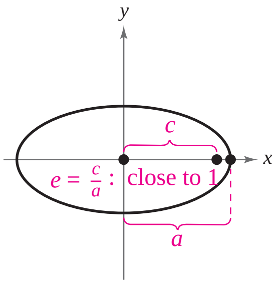
60. Vertices: (2, 0), (2, 4); foci: (2, 1), (2, 3)

Eccentricity of an Ellipse **In Exercises 61 and 62, use the following information. The flatness of an ellipse is measured by its eccentricity e, defined by**

$$e = \frac{c}{a}, \quad \text{where} \quad 0 < e < 1.$$

When an ellipse is nearly circular, e is close to 0. When an ellipse is elongated, e is close to 1 (see figures).



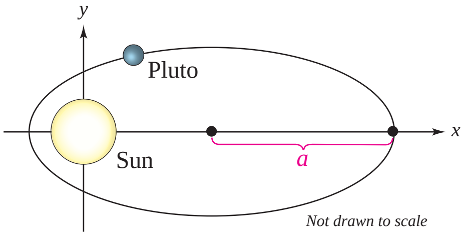


61. Find the standard form of the equation of the ellipse with vertices (±5, 0) and eccentricity $e = \frac{3}{5}$.

62. Find the standard form of the equation of the ellipse with vertices (0, ±8) and eccentricity $e = \frac{1}{2}$.

Paul Fleet/Shutterstock.com

63. Planetary Motion The dwarf planet Pluto moves in an elliptical orbit with the sun at one of the foci, as shown in the figure. The length of half of the major axis, a, is 3.67×10^9 miles, and the eccentricity is 0.249. Find the least distance (*perihelion*) and the greatest distance (*aphelion*) of Pluto from the center of the sun.



64. Australian Football In Australia, football by Australian Rules (or rugby) is played on elliptical fields. The field can be a maximum of 155 meters wide and a maximum of 185 meters long. Let the center of a field of maximum size be represented by the point (0, 77.5). Find the standard form of the equation of the ellipse that represents this field. *(Source: Australian Football League)*

Sketching a Hyperbola

In Exercises 65–72, find the center, foci, and vertices of the hyperbola. Then sketch the hyperbola using the asymptotes as an aid.

65. $\dfrac{(x-2)^2}{16} - \dfrac{(y+1)^2}{9} = 1$

66. $\dfrac{(x-1)^2}{144} - \dfrac{(y-4)^2}{25} = 1$

67. $(y+6)^2 - (x-2)^2 = 1$

68. $\dfrac{(y-1)^2}{1/4} - \dfrac{(x+3)^2}{1/9} = 1$

69. $x^2 - 9y^2 + 2x - 54y - 85 = 0$

70. $16y^2 - x^2 + 2x + 64y + 62 = 0$

71. $9x^2 - y^2 - 36x - 6y + 18 = 0$

72. $x^2 - 9y^2 + 36y - 72 = 0$

Finding the Standard Equation of a Hyperbola

In Exercises 73–80, find the standard form of the equation of the hyperbola with the given characteristics.

73. Vertices: (0, 2), (0, 0); foci: (0, 3), (0, −1)

74. Vertices: (1, 2), (5, 2); foci: (0, 2), (6, 2)

75. Vertices: (2, 0), (6, 0); foci: (0, 0), (8, 0)

76. Vertices: (2, 3), (2, −3); foci: (2, 5), (2, −5)

77. Vertices: (2, 3), (2, −3); passes through the point (0, 5)

78. Vertices: (−2, 1), (2, 1); passes through the point (4, 3)

79. Vertices: (0, 2), (6, 2); asymptotes: $y = \frac{2}{3}x,\ y = 4 - \frac{2}{3}x$

80. Vertices: (3, 0), (3, 4); asymptotes: $y = \frac{2}{3}x,\ y = 4 - \frac{2}{3}x$

Identifying a Conic

In Exercises 81–90, identify the conic by writing its equation in standard form. Then sketch its graph.

81. $x^2 + y^2 - 6x + 4y + 9 = 0$

82. $x^2 + 4y^2 - 6x + 16y + 21 = 0$

83. $y^2 - x^2 + 4y = 0$

84. $y^2 - 4y - 4x = 0$

85. $16y^2 + 128x + 8y - 7 = 0$

86. $4x^2 - y^2 - 4x - 3 = 0$

87. $9x^2 + 16y^2 + 36x + 128y + 148 = 0$

88. $25x^2 - 10x - 200y - 119 = 0$

89. $16x^2 + 16y^2 - 16x + 24y - 3 = 0$

90. $4x^2 + 3y^2 + 8x - 24y + 51 = 0$

Exploration

True or False?

In Exercises 91 and 92, determine whether the statement is true or false. Justify your answer.

91. The conic represented by the equation $3x^2 + 2y^2 - 18x - 16y + 58 = 0$ is an ellipse.

92. The graphs of $x^2 + 10y - 10x + 5 = 0$ and $x^2 + 16y^2 + 10x - 32y - 23 = 0$ do not intersect.

93. Alternate Form of the Equation of an Ellipse Consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

(a) Show that the equation of the ellipse can be written as

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2(1-e^2)} = 1$$

where e is the eccentricity (see Exercises 61 and 62).

(b) Use a graphing utility to graph the ellipse

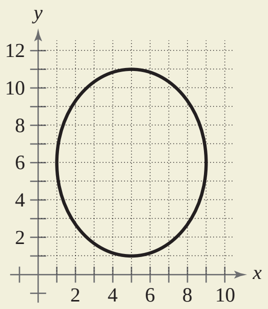
$$\frac{(x-2)^2}{4} + \frac{(y-3)^2}{4(1-e^2)} = 1$$

for $e = 0.95$, 0.75, 0.5, 0.25, and 0. Make a conjecture about the change in the shape of the ellipse as e approaches 0.

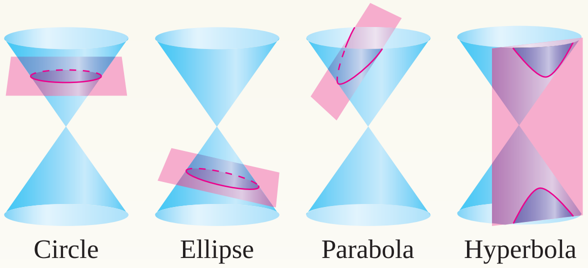
94. HOW DO YOU SEE IT? Consider the ellipse shown.

(a) Find the center, vertices, and foci of the ellipse.

(b) Find the standard form of the equation of the ellipse.



Chapter Summary

	What Did You Learn?	**Explanation/Examples**	**Review Exercises**
Section 4.1	Find the domains of rational functions *(p. 312)*.	A rational function is a quotient of polynomial functions. It can be written in the form $f(x) = N(x)/D(x)$, where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial. In general, the domain of a rational function of x includes all real numbers except x-values that make the denominator zero.	1–4
	Find the vertical and horizontal asymptotes of graphs of rational functions *(p. 313)*.	The line $x = a$ is a vertical asymptote of the graph of f if $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a$, either from the right or from the left. The line $y = b$ is a horizontal asymptote of the graph of f if $f(x) \to b$ as $x \to \infty$ or $x \to -\infty$.	5–10
	Use rational functions to model and solve real-life problems *(p. 315)*.	A rational function can be used to model the cost of removing a given percent of the smokestack pollutants at a utility company that burns coal. (See Example 4.)	11, 12
Section 4.2	Sketch the graphs of rational functions *(p. 320)*.	Let $f(x) = N(x)/D(x)$, where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial. **1.** Simplify f, if possible. **2.** Find and plot the y-intercept (if any) by evaluating $f(0)$. **3.** Find the zeros of the numerator (if any) by solving the equation $N(x) = 0$. Then plot the corresponding x-intercepts. **4.** Find the zeros of the denominator (if any) by solving the equation $D(x) = 0$. Then sketch the corresponding vertical asymptotes. **5.** Find and sketch the horizontal asymptote (if any). **6.** Plot at least one point *between* and one point *beyond* each x-intercept and vertical asymptote. **7.** Use smooth curves to complete the graph between and beyond the vertical asymptotes.	13–24
	Sketch the graphs of rational functions that have slant asymptotes *(p. 323)*.	Consider a rational function whose denominator is of degree 1 or greater. If the degree of the numerator is exactly *one more* than the degree of the denominator, then the graph of the function has a slant asymptote.	25–30
	Use the graphs of rational functions to model and solve real-life problems *(p. 324)*.	You can use the graph of a rational function to model the printed area of a rectangular page that is to be minimized and to find the page dimensions that use the least amount of paper. (See Example 6.)	31–34
Section 4.3	Recognize the four basic conics: circle, ellipse, parabola, and hyperbola *(p. 329)*.	A conic section (or simply conic) is the intersection of a plane and a double-napped cone. 	35–42

	What Did You Learn?	Explanation/Examples	Review Exercises
Section 4.3	Recognize, graph, and write equations of parabolas (vertex at origin) *(p. 330)*.	The standard form of the equation of a parabola with vertex at $(0, 0)$ and directrix $y = -p$ is $x^2 = 4py,\ p \neq 0.$ Vertical axis For directrix $x = -p$, the equation is $y^2 = 4px,\ p \neq 0.$ Horizontal axis The focus is on the axis p units (directed distance) from the vertex.	43–50
	Recognize, graph, and write equations of ellipses (center at origin) *(p. 332)*.	The standard form of the equation of an ellipse centered at the origin with major and minor axes of lengths $2a$ and $2b$, respectively, where $0 < b < a$, is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Major axis is horizontal. or $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$ Major axis is vertical. The vertices and foci lie on the major axis, a and c units, respectively, from the center. Moreover, a, b, and c are related by the equation $c^2 = a^2 - b^2$.	51–58
	Recognize, graph, and write equations of hyperbolas (center at origin) *(p. 335)*.	The standard form of the equation of a hyperbola with center at the origin is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ Transverse axis is horizontal. or $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ Transverse axis is vertical. where $a > 0$ and $b > 0$. The vertices and foci are, respectively, a and c units from the center. Moreover, a, b, and c are related by the equation $b^2 = c^2 - a^2$.	59–62
Section 4.4	Recognize equations of conics that have been shifted vertically or horizontally in the plane *(p. 343)* and write and graph equations of conics that have been shifted vertically or horizontally in the plane *(p. 345)*.	**Circle:** The graph of $(x - 2)^2 + (y + 1)^2 = 5^2$ is a circle whose center is the point $(2, -1)$ and whose radius is 5. The graph has been shifted two units to the right and one unit down from standard position. **Ellipse:** The graph of $\frac{(x - 1)^2}{4^2} + \frac{(y - 2)^2}{3^2} = 1$ is an ellipse whose center is the point $(1, 2)$. The major axis is horizontal and of length 8, and the minor axis is vertical and of length 6. The graph has been shifted one unit to the right and two units up from standard position. **Hyperbola:** The graph of $\frac{(x - 5)^2}{1^2} - \frac{(y - 4)^2}{2^2} = 1$ is a hyperbola whose center is the point $(5, 4)$. The transverse axis is horizontal and of length 2, and the conjugate axis is vertical and of length 4. The graph has been shifted five units to the right and four units up from standard position. **Parabola:** The graph of $(x - 1)^2 = 4(-1)(y - 6)$ is a parabola whose vertex is the point $(1, 6)$. The axis of the parabola is vertical. Because $p = -1$, the focus lies one unit below the vertex, and the directrix is $y = 6 - (-1) = 7$. The graph has been reflected in the x-axis, shifted one unit to the right and six units up from standard position.	63–87

Review Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

4.1 Finding the Domain of a Rational Function

In Exercises 1–4, find the domain of the rational function and discuss the behavior of *f* near any excluded *x*-values.

1. $f(x) = \dfrac{3x}{x + 10}$

2. $f(x) = \dfrac{4x^3}{2 + 5x}$

3. $f(x) = \dfrac{8}{x^2 - 10x + 24}$

4. $f(x) = \dfrac{x^2 + x - 2}{x^2 - 4x + 4}$

Finding Vertical and Horizontal Asymptotes

In Exercises 5–10, find all vertical and horizontal asymptotes of the graph of the function.

5. $f(x) = \dfrac{4}{x + 3}$

6. $f(x) = \dfrac{2x^2 + 5x - 3}{x^2 + 2}$

7. $g(x) = \dfrac{x^2}{x^2 - 4}$

8. $g(x) = \dfrac{1}{(x - 3)^2}$

9. $h(x) = \dfrac{5x + 20}{x^2 - 2x - 24}$

10. $h(x) = \dfrac{x^3 - 4x^2}{x^2 + 3x + 2}$

11. Average Cost A business has a cost function of $C = 0.5x + 500$, where C is measured in dollars and x is the number of units produced. The average cost per unit function is given by

$$\overline{C} = C/x = (0.5x + 500)/x, \quad x > 0.$$

Determine the average cost per unit as x increases without bound. (Find the horizontal asymptote.)

12. Seizure of Illegal Drugs The cost C (in millions of dollars) for the federal government to seize p% of an illegal drug as it enters the country is given by

$$C = \frac{528p}{100 - p}, \quad 0 \le p < 100.$$

(a) Use a graphing utility to graph the cost function.

(b) Find the costs of seizing 25%, 50%, and 75% of the drug.

(c) According to this model, would it be possible to seize 100% of the drug?

4.2 Sketching the Graph of a Rational Function

In Exercises 13–24, (a) state the domain of the function, (b) identify all intercepts, (c) find any vertical or horizontal asymptotes, and (d) plot additional solution points as needed to sketch the graph of the rational function.

13. $f(x) = \dfrac{-3}{2x^2}$

14. $f(x) = \dfrac{4}{x}$

15. $g(x) = \dfrac{2 + x}{1 - x}$

16. $h(x) = \dfrac{x - 4}{x - 7}$

17. $p(x) = \dfrac{5x^2}{4x^2 + 1}$

18. $f(x) = \dfrac{2x}{x^2 + 4}$

19. $f(x) = \dfrac{x}{x^2 + 1}$

20. $h(x) = \dfrac{9}{(x - 3)^2}$

21. $f(x) = \dfrac{-6x^2}{x^2 + 1}$

22. $y = \dfrac{2x^2}{x^2 - 4}$

23. $f(x) = \dfrac{6x^2 - 11x + 3}{3x^2 - x}$

24. $f(x) = \dfrac{6x^2 - 7x + 2}{4x^2 - 1}$

A Rational Function with a Slant Asymptote

In Exercises 25–30, (a) state the domain of the function, (b) identify all intercepts, (c) find any vertical or slant asymptotes, and (d) plot additional solution points as needed to sketch the graph of the rational function.

25. $f(x) = \dfrac{2x^3}{x^2 + 1}$

26. $f(x) = \dfrac{x^2 + 1}{x + 1}$

27. $f(x) = \dfrac{x^2 + 3x - 10}{x + 2}$

28. $f(x) = \dfrac{x^3}{x^2 - 25}$

29. $f(x) = \dfrac{3x^3 - 2x^2 - 3x + 2}{3x^2 - x - 4}$

30. $f(x) = \dfrac{3x^3 - 4x^2 - 12x + 16}{3x^2 + 5x - 2}$

31. Minimum Cost The cost of producing x units of a product is C, and the average cost per unit $\overline{C}$ is given by

$$\overline{C} = \frac{C}{x} = \frac{100{,}000 + 0.9x}{x}, \quad x > 0.$$

(a) Sketch the graph of the average cost function.

(b) Find the average costs of producing $x = 1000$, 10,000, and 100,000 units.

(c) By increasing the level of production, what is the least average cost per unit you can obtain? Explain your reasoning.

32. Page Design A page that is x inches wide and y inches high contains 30 square inches of print. The top and bottom margins are each 2 inches deep and the margins on each side are 2 inches wide.

(a) Show that the total area A of the page is

$$A = \frac{2x(2x + 7)}{x - 4}.$$

(b) Determine the domain of the function based on the physical constraints of the problem.

(c) Use a graphing utility to graph the area function and approximate the page size for which the least amount of paper is used. Verify your answer numerically using the *table* feature of the graphing utility.

33. Photosynthesis The amount y of CO_2 uptake (in milligrams per square decimeter per hour) at optimal temperatures and with the natural supply of CO_2 is approximated by the model

$$y = \frac{18.47x - 2.96}{0.23x + 1}, \quad x > 0$$

where x is the light intensity (in watts per square meter). Use a graphing utility to graph the function and determine the limiting amount of CO_2 uptake.

34. Medicine The concentration C of a chemical in the bloodstream t hours after injection into muscle tissue is given by $C(t) = (2t + 1)/(t^2 + 4)$, $t > 0$.

(a) Determine the horizontal asymptote of the graph of the function and interpret its meaning in the context of the problem.

(b) Use a graphing utility to graph the function and approximate the time when the bloodstream concentration is greatest.

4.3 Identifying a Conic **In Exercises 35–42, identify the conic.**

35. $y^2 = -16x$

36. $16x^2 + y^2 = 16$

37. $\dfrac{x^2}{64} - \dfrac{y^2}{4} = 1$

38. $\dfrac{x^2}{1} + \dfrac{y^2}{36} = 1$

39. $x^2 + 20y = 0$

40. $x^2 + y^2 = 400$

41. $\dfrac{y^2}{49} - \dfrac{x^2}{144} = 1$

42. $\dfrac{x^2}{49} + \dfrac{y^2}{144} = 1$

Finding the Standard Equation of a Parabola **In Exercises 43–48, find the standard form of the equation of the parabola with the given characteristic(s) and vertex at the origin.**

43. Focus: $(-6, 0)$

44. Focus: $(0, 7)$

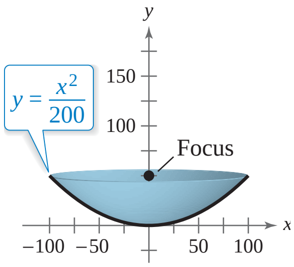
45. Directrix: $y = -3$

46. Directrix: $x = 3$

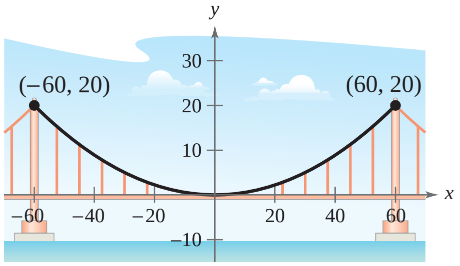
47. Passes through the point $(3, 6)$; horizontal axis

48. Passes through the point $(4, -2)$; vertical axis

49. Satellite Antenna A cross section of a large parabolic antenna (see figure) is modeled by $y = x^2/200$, $-100 \le x \le 100$. The receiving and transmitting equipment is positioned at the focus. Find the coordinates of the focus.



50. Suspension Bridge Each cable of a suspension bridge is suspended (in the shape of a parabola) between two towers (see figure).

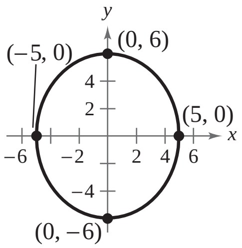


(a) Find the coordinates of the focus.

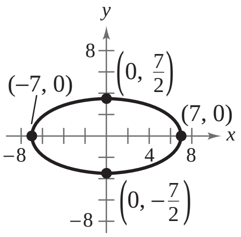
(b) Write an equation that models the cables.

Finding the Standard Equation of an Ellipse **In Exercises 51–56, find the standard form of the equation of the ellipse with the given characteristics and center at the origin.**

51.



52.



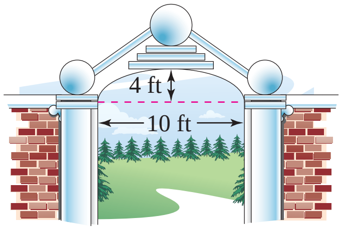
53. Vertices: $(\pm 7, 0)$; foci: $(\pm 6, 0)$

54. Foci: $(\pm 3, 0)$; major axis of length 12

55. Foci: $(\pm 14, 0)$; minor axis of length 10

56. Vertices: $(0, \pm 6)$; passes through the point $(2, 2)$

57. Architecture A semielliptical archway is to be formed over the entrance to an estate (see figure). The arch is to be set on pillars that are 10 feet apart and is to have a height (atop the pillars) of 4 feet. Where should the foci be placed in order to sketch the arch?



58. Wading Pool You are building a wading pool that is in the shape of an ellipse. Your plans give an equation for the elliptical shape of the pool measured in feet as

$$\frac{x^2}{324} + \frac{y^2}{196} = 1.$$

Find the longest distance across the pool, the shortest distance, and the distance between the foci.

Finding the Standard Equation of a Hyperbola

In Exercises 59–62, find the standard form of the equation of the hyperbola with the given characteristics and center at the origin.

59. Vertices: $(0, \pm 1)$; foci: $(0, \pm 5)$

60. Vertices: $(\pm 4, 0)$; foci: $(\pm 6, 0)$

61. Vertices: $(\pm 1, 0)$; asymptotes: $y = \pm 2x$

62. Vertices: $(0, \pm 2)$; asymptotes: $y = \pm \frac{2}{\sqrt{5}}x$

4.4 Finding the Standard Equation of a Parabola

In Exercises 63–66, find the standard form of the equation of the parabola with the given characteristics.

63. Vertex: $(4, 2)$; focus: $(4, 0)$

64. Vertex: $(2, 0)$; focus: $(0, 0)$

65. Vertex: $(-8, 8)$; directrix: $y = 1$

66. Focus: $(0, 5)$; directrix: $x = 6$

Finding the Standard Equation of an Ellipse

In Exercises 67–70, find the standard form of the equation of the ellipse with the given characteristics.

67. Vertices: $(0, 2), (4, 2)$; minor axis of length 2

68. Vertices: $(5, 0), (5, 12)$; minor axis of length 10

69. Vertices: $(2, -2), (2, 8)$; foci: $(2, 0), (2, 6)$

70. Vertices: $(-9, -4), (11, -4)$; foci: $(-5, -4), (7, -4)$

Finding the Standard Equation of a Hyperbola

In Exercises 71–76, find the standard form of the equation of the hyperbola with the given characteristics.

71. Vertices: $(-10, 3), (6, 3)$; foci: $(-12, 3), (8, 3)$

72. Vertices: $(2, 2), (-2, 2)$; foci: $(4, 2), (-4, 2)$

73. Vertices: $(0, 0), (0, -4)$; passes through the point $\left(2, 2\left(\sqrt{5} - 1\right)\right)$

74. Vertices: $(\pm 6, 7)$;
asymptotes: $y = -\frac{1}{2}x + 7, y = \frac{1}{2}x + 7$

75. Foci: $(0, 0), (8, 0)$; asymptotes: $y = \pm 2(x - 4)$

76. Foci: $(3, \pm 2)$; asymptotes: $y = \pm 2(x - 3)$

Identifying a Conic

In Exercises 77–84, identify the conic by writing its equation in standard form. Then sketch its graph and describe the translation.

77. $x^2 - 6x + 2y + 9 = 0$

78. $y^2 - 12y - 8x + 20 = 0$

79. $x^2 + y^2 - 2x - 4y + 5 = 0$

80. $16x^2 + 16y^2 - 16x + 24y - 3 = 0$

81. $x^2 + 9y^2 + 10x - 18y + 25 = 0$

82. $4x^2 + y^2 - 16x + 15 = 0$

83. $9x^2 - y^2 - 72x + 8y + 119 = 0$

84. $x^2 - 9y^2 + 10x + 18y + 7 = 0$

85. **Architecture** A parabolic archway is 12 meters high at the vertex. At a height of 10 meters, the width of the archway is 8 meters (see figure). How wide is the archway at ground level?

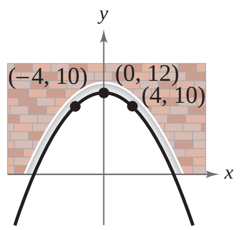


Figure for 85

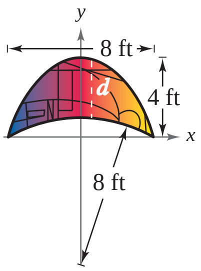


Figure for 86

86. **Architecture** A church window (see figure) is bounded above by a parabola and below by the arc of a circle.

(a) Find equations for the parabola and the circle.

(b) Complete the table by filling in the vertical distance d between the circle and the parabola for each given value of x.

x	0	1	2	3	4
d					

87. **Running Path** Let $(0, 0)$ represent a water fountain located in a city park. Each day you run through the park along a path given by

$$x^2 + y^2 - 200x - 52{,}500 = 0$$

where x and y are measured in meters.

(a) What type of conic is your path? Explain your reasoning.

(b) Write the equation of the path in standard form. Sketch a graph of the equation.

(c) After you run, you walk to the water fountain. If you stop running at $(-100, 150)$, how far must you walk for a drink of water?

Exploration

True or False?

In Exercises 88 and 89, determine whether the statement is true or false. Justify your answer.

88. The domain of a rational function can never be the set of all real numbers.

89. The graph of the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

can be a single point.

Chapter Test

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

In Exercises 1–3, find the domain of the function and identify any asymptotes.

1. $y = \dfrac{3x}{x+1}$ **2.** $f(x) = \dfrac{3-x^2}{3+x^2}$ **3.** $g(x) = \dfrac{x^2-7x+12}{x-3}$

In Exercises 4–9, identify any intercepts and asymptotes of the graph of the function. Then sketch a graph of the function.

4. $h(x) = \dfrac{4}{x^2} - 1$ **5.** $g(x) = \dfrac{x^2+2}{x-1}$

6. $f(x) = \dfrac{x+1}{x^2+x-12}$ **7.** $f(x) = \dfrac{2x^2-5x-12}{x^2-16}$

8. $f(x) = \dfrac{2x^2+9}{5x^2+9}$ **9.** $g(x) = \dfrac{2x^3-7x^2+4x+4}{x^2-x-2}$

10. A rectangular page is designed to contain 36 square inches of print. The margins at the top and bottom of the page are each 2 inches deep. The margins on each side are 1 inch wide. What should the dimensions of the page be so that the least amount of paper is used?

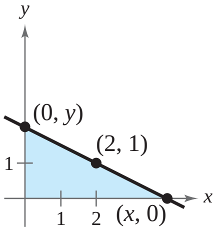


Figure for 11

11. A triangle is formed by the coordinate axes and a line through the point $(2, 1)$, as shown in the figure.

(a) Verify that $y = 1 + \dfrac{2}{x-2}$.

(b) Write the area A of the triangle as a function of x. Determine the domain of the function in the context of the problem.

(c) Sketch a graph of the area function. Estimate the minimum area of the triangle from the graph.

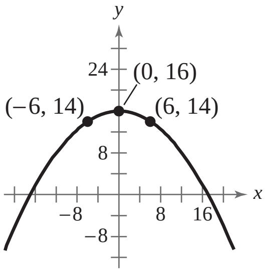


Figure for 20

In Exercises 12–17, sketch the conic and identify the center, vertices, and foci, if applicable.

12. $y^2 - 4x = 0$ **13.** $x^2 + y^2 - 10x + 4y + 4 = 0$

14. $x^2 - 10x - 2y + 19 = 0$

15. $x^2 - \dfrac{y^2}{4} = 1$ **16.** $\dfrac{y^2}{4} - x^2 = 1$

17. $x^2 + 3y^2 - 2x + 36y + 100 = 0$

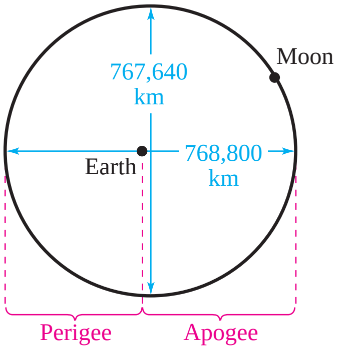


Figure for 21

18. Find the standard form of the equation of the ellipse with vertices $(0, 2)$ and $(8, 2)$ and minor axis of length 4.

19. Find the standard form of the equation of the hyperbola with vertices $(0, \pm 3)$ and asymptotes $y = \pm\frac{3}{2}x$.

20. A parabolic archway is 16 meters high at the vertex. At a height of 14 meters, the width of the archway is 12 meters, as shown in the figure. How wide is the archway at ground level?

21. The moon orbits Earth in an elliptical path with the center of Earth at one focus, as shown in the figure. The major and minor axes of the orbit have lengths of 768,800 kilometers and 767,640 kilometers, respectively. Find the least distance (perigee) and the greatest distance (apogee) from the center of the moon to the center of Earth.

Proofs in Mathematics

You can use the definition of a parabola to derive the standard form of the equation of a parabola whose directrix is parallel to the x-axis or to the y-axis.

PARABOLIC PATTERNS

There are many natural occurrences of parabolas in real life. For instance, the famous astronomer Galileo discovered in the 17th century that an object that is projected upward and obliquely to the pull of gravity travels in a parabolic path. Examples of this are the center of gravity of a jumping dolphin and the path of water molecules of a drinking water fountain.

Standard Equation of a Parabola (Vertex at Origin) *(p. 330)*

The **standard form of the equation of a parabola** with vertex at $(0, 0)$ and directrix $y = -p$ is

$$x^2 = 4py, \quad p \neq 0. \qquad \text{Vertical axis}$$

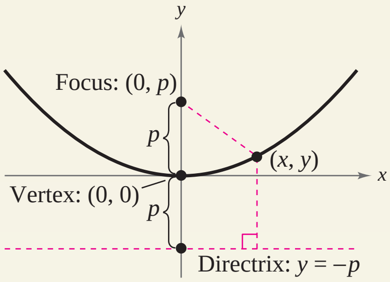
For directrix $x = -p$, the equation is

$$y^2 = 4px, \quad p \neq 0. \qquad \text{Horizontal axis}$$

The focus is on the axis p units (directed distance) from the vertex.

Proof

For the first case, suppose the directrix $(y = -p)$ is parallel to the x-axis. In the figure, you assume that $p > 0$, and because p is the directed distance from the vertex to the focus, the focus must lie above the vertex. Because the point (x, y) is equidistant from $(0, p)$ and $y = -p$, you can apply the Distance Formula to obtain



Parabola with vertical axis

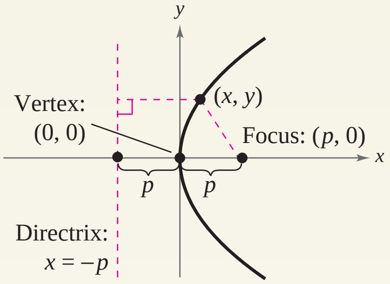
$$\sqrt{(x-0)^2 + (y-p)^2} = y + p \qquad \text{Distance Formula}$$

$$x^2 + (y-p)^2 = (y+p)^2 \qquad \text{Square each side.}$$

$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2 \qquad \text{Expand.}$$

$$x^2 = 4py. \qquad \text{Simplify.}$$

A proof of the second case is similar to the proof of the first case. Suppose the directrix $(x = -p)$ is parallel to the y-axis. In the figure, you assume that $p > 0$, and because p is the directed distance from the vertex to the focus, the focus must lie to the right of the vertex. Because the point (x, y) is equidistant from $(p, 0)$ and $x = -p$, you can apply the Distance Formula to obtain



Parabola with horizontal axis

$$\sqrt{(x-p)^2 + (y-0)^2} = x + p \qquad \text{Distance Formula}$$

$$(x-p)^2 + y^2 = (x+p)^2 \qquad \text{Square each side.}$$

$$x^2 - 2px + p^2 + y^2 = x^2 + 2px + p^2 \qquad \text{Expand.}$$

$$y^2 = 4px. \qquad \text{Simplify.}$$

■

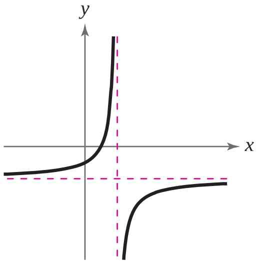
P.S. Problem Solving

1. Matching Match the graph of the rational function

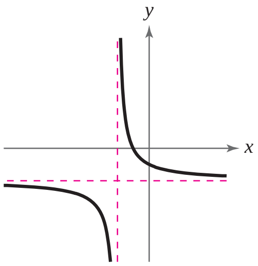
$$f(x) = \frac{ax + b}{cx + d}$$

with the given conditions.

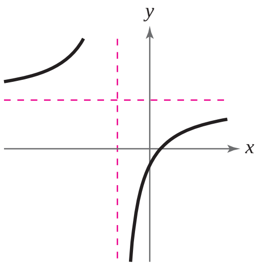
(a)



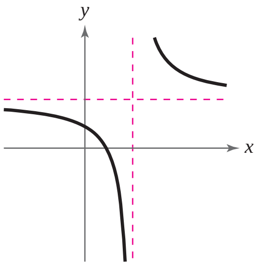
(b)



(c)



(d)



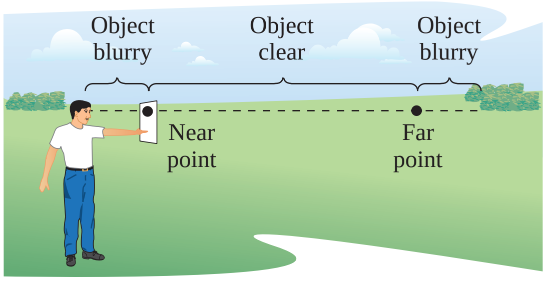
(i) $a > 0$	(ii) $a > 0$	(iii) $a < 0$	(iv) $a > 0$
$b < 0$	$b > 0$	$b > 0$	$b < 0$
$c > 0$	$c < 0$	$c > 0$	$c > 0$
$d < 0$	$d < 0$	$d < 0$	$d > 0$

2. Effects of Values on a Graph Consider the function $f(x) = (ax)/(x - b)^2$.

(a) Determine the effect on the graph of f when $b \neq 0$ and a is varied. Consider cases in which a is positive and a is negative.

(b) Determine the effect on the graph of f when $a \neq 0$ and b is varied.

3. Distinct Vision The endpoints of the interval over which distinct vision is possible are called the *near point* and *far point* of the eye (see figure). With increasing age, these points normally change. The table shows the approximate near points y (in inches) for various ages x (in years).



Age, x	Near Point, y
16	3.0
32	4.7
44	9.8
50	19.7
60	39.4

(a) Use the *regression* feature of a graphing utility to find a quadratic model for the data. Use the graphing utility to plot the data and graph the model in the same viewing window.

(b) Find a rational model for the data. Take the reciprocals of the near points to generate the points $(x, 1/y)$. Use the *regression* feature of the graphing utility to find a linear model for the data. The resulting line has the form

$$\frac{1}{y} = ax + b.$$

Solve for y. Use the graphing utility to plot the data and graph the model in the same viewing window.

(c) Use the *table* feature of the graphing utility to construct a table showing the predicted near point based on each model for each of the ages in the original table. How well do the models fit the original data?

(d) Use both models to estimate the near point for a person who is 25 years old. Which model is a better fit?

(e) Do you think either model can be used to predict the near point for a person who is 70 years old? Explain.

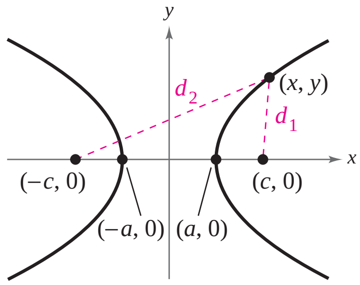
4. Statuary Hall Statuary Hall is an elliptical room in the United States Capitol in Washington, D.C. The room is also called the Whispering Gallery because a person standing at one focus of the room can hear even a whisper spoken by a person standing at the other focus. This occurs because any sound that is emitted from one focus of an ellipse will reflect off the side of the ellipse to the other focus. Statuary Hall is 46 feet wide and 97 feet long.

(a) Find an equation that models the shape of the room.

(b) How far apart are the two foci?

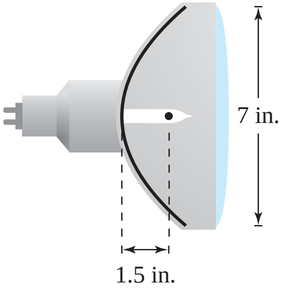
(c) What is the area of the floor of the room? (The area of an ellipse is $A = \pi ab$.)

5. Property of a Hyperbola Use the figure to show that $|d_2 - d_1| = 2a$.



6. Finding the Equation of a Hyperbola Find an equation of a hyperbola such that for any point on the hyperbola, the difference between its distances from the points $(2, 2)$ and $(10, 2)$ is 6.

7. Car Headlight The filament of a light bulb is a thin wire that glows when electricity passes through it. The filament of a car headlight is at the focus of a parabolic reflector, which sends light out in a straight beam. Given that the filament is 1.5 inches from the vertex, find an equation for the cross section of the reflector. The reflector is 7 inches wide. How deep is it?



8. Analyzing Parabolas Consider the parabola $x^2 = 4py$.

(a) Use a graphing utility to graph the parabola for $p = 1$, $p = 2$, $p = 3$, and $p = 4$. Describe the effect on the graph when p increases.

(b) Locate the focus for each parabola in part (a).

(c) For each parabola in part (a), find the length of the chord passing through the focus and parallel to the directrix. How can the length of this chord be determined directly from the standard form of the equation of the parabola?

(d) Explain how the result of part (c) can be used as an aid when sketching parabolas.

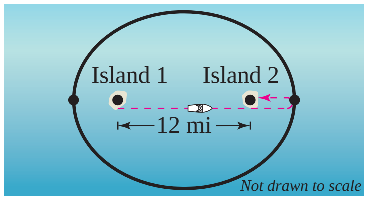
9. Tangent Line Let (x_1, y_1) be the coordinates of a point on the parabola $x^2 = 4py$. The equation of the line that just touches the parabola at the point (x_1, y_1), called a *tangent line*, is given by

$$y - y_1 = \frac{x_1}{2p}(x - x_1).$$

(a) What is the slope of the tangent line?

(b) For each parabola in Exercise 8, find the equations of the tangent lines at the endpoints of the chord. Use a graphing utility to graph the parabola and tangent lines.

10. Tour Boat A tour boat travels between two islands that are 12 miles apart (see figure). For each trip between the islands, there is enough fuel for a 20-mile trip.



(a) Explain why the region in which the boat can travel is bounded by an ellipse.

(b) Let $(0, 0)$ represent the center of the ellipse. Find the coordinates of the center of each island.

(c) The boat travels from Island 1, past Island 2 to one vertex of the ellipse, and then to Island 2 (see figure). How many miles does the boat travel? Use your answer to find the coordinates of the vertex.

(d) Use the results of parts (b) and (c) to write an equation of the ellipse that bounds the region in which the boat can travel.

11. Proof Prove that the graph of the equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

is one of the following (except in degenerate cases).

Conic	**Condition**
(a) Circle	$A = C$
(b) Parabola	$A = 0$ or $C = 0$ (but not both)
(c) Ellipse	$AC > 0$
(d) Hyperbola	$AC < 0$

5 Exponential and Logarithmic Functions

Earthquakes *(Example 6, page 406)*

Trees per Acre *(Exercise 83, page 398)*

Sound Intensity *(Exercises 85–88, page 388)*

Human Memory Model *(Exercise 81, page 382)*

Nuclear Reactor Accident *(Example 9, page 369)*

Clockwise from top left, James Marshall/CORBIS; Darrenp/Shutterstock.com;
Sebastian Kaulitzki/Shutterstock.com; Hellen Sergeyeva/Shutterstock.com; kentoh/Shutterstock.com

5.1 Exponential Functions and Their Graphs

Exponential functions can help you model and solve real-life problems. For instance, Exercise 72 on page 372 uses an exponential function to model the concentration of a drug in the bloodstream.

- **Recognize and evaluate exponential functions with base *a*.**
- **Graph exponential functions and use the One-to-One Property.**
- **Recognize, evaluate, and graph exponential functions with base e.**
- **Use exponential functions to model and solve real-life problems.**

Exponential Functions

So far, this text has dealt mainly with **algebraic functions,** which include polynomial functions and rational functions. In this chapter, you will study two types of nonalgebraic functions—*exponential functions* and *logarithmic functions*. These functions are examples of **transcendental functions.**

Definition of Exponential Function

The **exponential function f with base a** is denoted by

$$f(x) = a^x$$

where $a > 0$, $a \neq 1$, and x is any real number.

The base $a = 1$ is excluded because it yields $f(x) = 1^x = 1$. This is a constant function, not an exponential function.

You have evaluated a^x for integer and rational values of x. For example, you know that $4^3 = 64$ and $4^{1/2} = 2$. However, to evaluate 4^x for any real number x, you need to interpret forms with *irrational* exponents. For the purposes of this text, it is sufficient to think of $a^{\sqrt{2}}$ (where $\sqrt{2} \approx 1.41421356$) as the number that has the successively closer approximations

$$a^{1.4}, a^{1.41}, a^{1.414}, a^{1.4142}, a^{1.41421}, \ldots .$$

EXAMPLE 1 Evaluating Exponential Functions

Use a calculator to evaluate each function at the indicated value of x.

	Function	**Value**
a.	$f(x) = 2^x$	$x = -3.1$
b.	$f(x) = 2^{-x}$	$x = \pi$
c.	$f(x) = 0.6^x$	$x = \frac{3}{2}$

Solution

	Function Value	**Graphing Calculator Keystrokes**	**Display**
a.	$f(-3.1) = 2^{-3.1}$	2 $\boxed{\wedge}$ $\boxed{(-)}$ 3.1 $\boxed{\text{ENTER}}$	0.1166291
b.	$f(\pi) = 2^{-\pi}$	2 $\boxed{\wedge}$ $\boxed{(-)}$ π $\boxed{\text{ENTER}}$	0.1133147
c.	$f\left(\frac{3}{2}\right) = (0.6)^{3/2}$	.6 $\boxed{\wedge}$ $\boxed{(}$ 3 $\boxed{\div}$ 2 $\boxed{)}$ $\boxed{\text{ENTER}}$	0.4647580

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use a calculator to evaluate $f(x) = 8^{-x}$ at $x = \sqrt{2}$.

When evaluating exponential functions with a calculator, remember to enclose fractional exponents in parentheses. Because the calculator follows the order of operations, parentheses are crucial in order to obtain the correct result.

Sura Nualpradid/Shutterstock.com

Graphs of Exponential Functions

The graphs of all exponential functions have similar characteristics, as shown in Examples 2, 3, and 5.

EXAMPLE 2 Graphs of $y = a^x$

ALGEBRA HELP You can review the techniques for sketching the graph of an equation in Section 1.1.

In the same coordinate plane, sketch the graph of each function.

a. $f(x) = 2^x$

b. $g(x) = 4^x$

Solution The following table lists some values for each function, and Figure 5.1 shows the graphs of the two functions. Note that both graphs are increasing. Moreover, the graph of $g(x) = 4^x$ is increasing more rapidly than the graph of $f(x) = 2^x$.

x	-3	-2	-1	0	1	2
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
4^x	$\frac{1}{64}$	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16

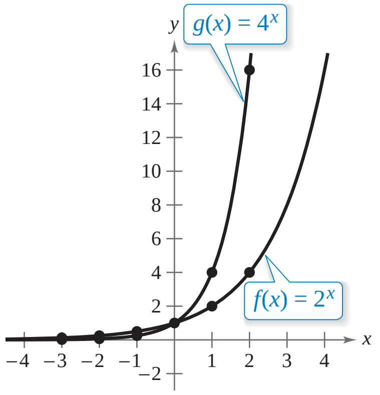


Figure 5.1

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

In the same coordinate plane, sketch the graph of each function.

a. $f(x) = 3^x$ **b.** $g(x) = 9^x$

The table in Example 2 was evaluated by hand. You could, of course, use a graphing utility to construct tables with even more values.

EXAMPLE 3 Graphs of $y = a^{-x}$

In the same coordinate plane, sketch the graph of each function.

a. $F(x) = 2^{-x}$

b. $G(x) = 4^{-x}$

Solution The following table lists some values for each function, and Figure 5.2 shows the graphs of the two functions. Note that both graphs are decreasing. Moreover, the graph of $G(x) = 4^{-x}$ is decreasing more rapidly than the graph of $F(x) = 2^{-x}$.

x	-2	-1	0	1	2	3
2^{-x}	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
4^{-x}	16	4	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$

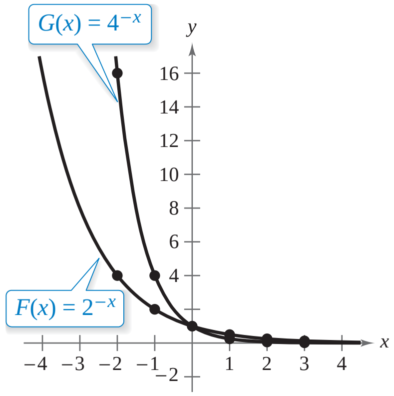


Figure 5.2

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

In the same coordinate plane, sketch the graph of each function.

a. $f(x) = 3^{-x}$ **b.** $g(x) = 9^{-x}$

Note that it is possible to use one of the properties of exponents to rewrite the functions in Example 3 with positive exponents, as follows.

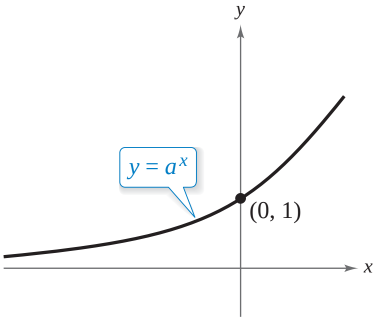
$$F(x) = 2^{-x} = \frac{1}{2^x} = \left(\frac{1}{2}\right)^x \quad \text{and} \quad G(x) = 4^{-x} = \frac{1}{4^x} = \left(\frac{1}{4}\right)^x$$

Comparing the functions in Examples 2 and 3, observe that

$$F(x) = 2^{-x} = f(-x) \quad \text{and} \quad G(x) = 4^{-x} = g(-x).$$

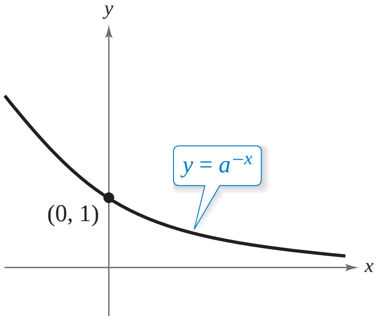
Consequently, the graph of F is a reflection (in the y-axis) of the graph of f. The graphs of G and g have the same relationship. The graphs in Figures 5.1 and 5.2 are typical of the exponential functions $y = a^x$ and $y = a^{-x}$. They have one y-intercept and one horizontal asymptote (the x-axis), and they are continuous. The following summarizes the basic characteristics of these exponential functions.

REMARK Notice that the range of an exponential function is $(0, \infty)$, which means that $a^x > 0$ for all values of x.



Graph of $y = a^x$, $a > 1$

- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- y-intercept: $(0, 1)$
- Increasing
- x-axis is a horizontal asymptote ($a^x \rightarrow 0$ as $x \rightarrow -\infty$).
- Continuous



Graph of $y = a^{-x}$, $a > 1$

- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- y-intercept: $(0, 1)$
- Decreasing
- x-axis is a horizontal asymptote ($a^{-x} \rightarrow 0$ as $x \rightarrow \infty$).
- Continuous

Notice that the graph of an exponential function is always increasing or always decreasing. As a result, the graphs pass the Horizontal Line Test, and therefore the functions are one-to-one functions. You can use the following **One-to-One Property** to solve simple exponential equations.

For $a > 0$ and $a \neq 1$, $a^x = a^y$ if and only if $x = y$. One-to-One Property

EXAMPLE 4 Using the One-to-One Property

a. $9 = 3^{x+1}$ Original equation

$3^2 = 3^{x+1}$ $9 = 3^2$

$2 = x + 1$ One-to-One Property

$1 = x$ Solve for x.

b. $\left(\frac{1}{2}\right)^x = 8$ Original equation

$2^{-x} = 2^3$ $\left(\frac{1}{2}\right)^x = 2^{-x}$, $8 = 2^3$

$x = -3$ One-to-One Property

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use the One-to-One Property to solve the equation for x.

a. $8 = 2^{2x-1}$ **b.** $\left(\frac{1}{3}\right)^{-x} = 27$

In the following example, notice how the graph of $y = a^x$ can be used to sketch the graphs of functions of the form $f(x) = b \pm a^{x+c}$.

ALGEBRA HELP You can review the techniques for transforming the graph of a function in Section 2.5.

EXAMPLE 5 Transformations of Graphs of Exponential Functions

Each of the following graphs is a transformation of the graph of $f(x) = 3^x$.

a. Because $g(x) = 3^{x+1} = f(x + 1)$, the graph of g can be obtained by shifting the graph of f one unit to the *left*, as shown in Figure 5.3.

b. Because $h(x) = 3^x - 2 = f(x) - 2$, the graph of h can be obtained by shifting the graph of f *down* two units, as shown in Figure 5.4.

c. Because $k(x) = -3^x = -f(x)$, the graph of k can be obtained by *reflecting* the graph of f in the x-axis, as shown in Figure 5.5.

d. Because $j(x) = 3^{-x} = f(-x)$, the graph of j can be obtained by *reflecting* the graph of f in the y-axis, as shown in Figure 5.6.

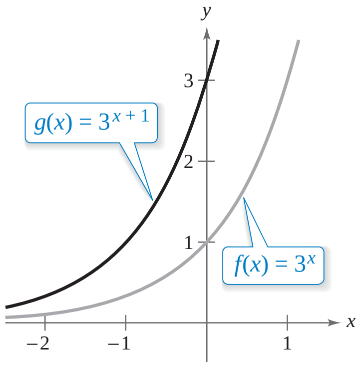


Figure 5.3 Horizontal shift

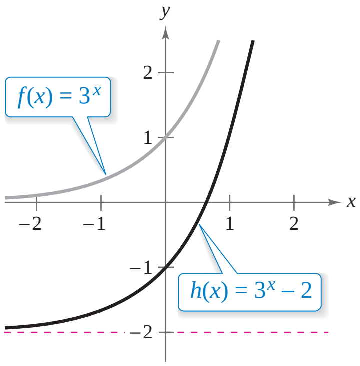


Figure 5.4 Vertical shift

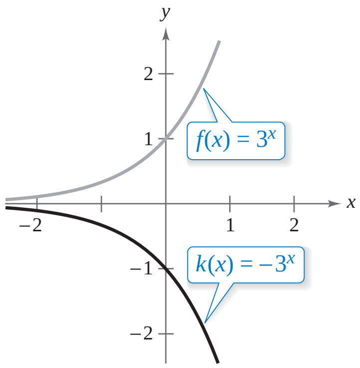


Figure 5.5 Reflection in x-axis

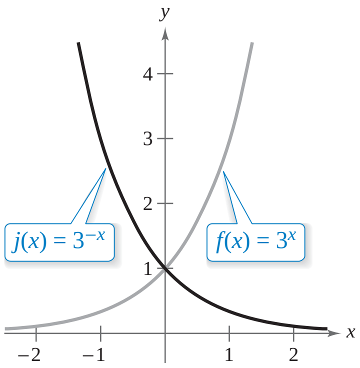


Figure 5.6 Reflection in y-axis

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use the graph of $f(x) = 4^x$ to describe the transformation that yields the graph of each function.

a. $g(x) = 4^{x-2}$ **b.** $h(x) = 4^x + 3$ **c.** $k(x) = 4^{-x} - 3$

Notice that the transformations in Figures 5.3, 5.5, and 5.6 keep the x-axis as a horizontal asymptote, but the transformation in Figure 5.4 yields a new horizontal asymptote of $y = -2$. Also, be sure to note how each transformation affects the y-intercept.

The Natural Base e

In many applications, the most convenient choice for a base is the irrational number

$$e \approx 2.718281828 \ . \ . \ . \ .$$

This number is called the **natural base.** The function $f(x) = e^x$ is called the **natural exponential function.** Figure 5.7 shows its graph. Be sure you see that for the exponential function $f(x) = e^x$, e is the constant 2.718281828 . . . , whereas x is the variable.

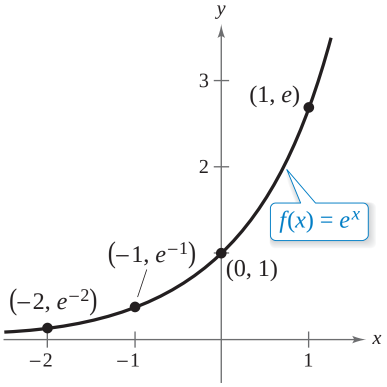


Figure 5.7

EXAMPLE 6 Evaluating the Natural Exponential Function

Use a calculator to evaluate the function $f(x) = e^x$ at each value of x.

a. $x = -2$

b. $x = -1$

c. $x = 0.25$

d. $x = -0.3$

Solution

Function Value	Graphing Calculator Keystrokes	Display
a. $f(-2) = e^{-2}$	$\boxed{e^x}$ $\boxed{(-)}$ 2 $\boxed{\text{ENTER}}$	0.1353353
b. $f(-1) = e^{-1}$	$\boxed{e^x}$ $\boxed{(-)}$ 1 $\boxed{\text{ENTER}}$	0.3678794
c. $f(0.25) = e^{0.25}$	$\boxed{e^x}$ 0.25 $\boxed{\text{ENTER}}$	1.2840254
d. $f(-0.3) = e^{-0.3}$	$\boxed{e^x}$ $\boxed{(-)}$ 0.3 $\boxed{\text{ENTER}}$	0.7408182

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use a calculator to evaluate the function $f(x) = e^x$ at each value of x.

a. $x = 0.3$

b. $x = -1.2$

c. $x = 6.2$

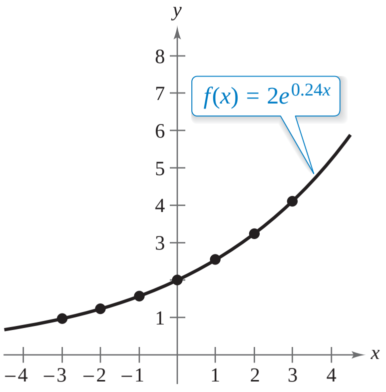


Figure 5.8

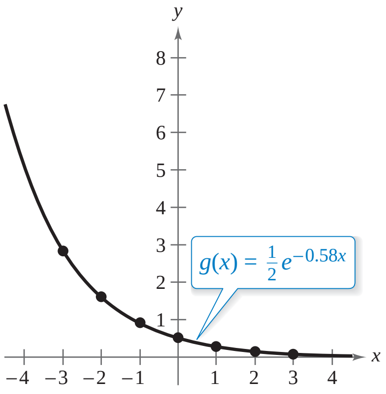


Figure 5.9

EXAMPLE 7 Graphing Natural Exponential Functions

Sketch the graph of each natural exponential function.

a. $f(x) = 2e^{0.24x}$

b. $g(x) = \frac{1}{2}e^{-0.58x}$

Solution To sketch these two graphs, use a graphing utility to construct a table of values, as follows. After constructing the table, plot the points and connect them with smooth curves, as shown in Figures 5.8 and 5.9. Note that the graph in Figure 5.8 is increasing, whereas the graph in Figure 5.9 is decreasing.

x	-3	-2	-1	0	1	2	3
$f(x)$	0.974	1.238	1.573	2.000	2.542	3.232	4.109
$g(x)$	2.849	1.595	0.893	0.500	0.280	0.157	0.088

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Sketch the graph of $f(x) = 5e^{0.17x}$.

Applications

One of the most familiar examples of exponential growth is an investment earning *continuously compounded interest*. Recall that the formula for *interest compounded n times per year* is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

In this formula, A is the balance in the account, P is the principal (or original deposit), r is the annual interest rate (in decimal form), n is the number of compoundings per year, and t is the time in years. Using exponential functions, you can now *develop* this formula and show how it leads to continuous compounding.

Suppose you invest a principal P at an annual interest rate r, compounded once per year. If the interest is added to the principal at the end of the year, then the new balance P_1 is

$$\begin{aligned} P_1 &= P + Pr \\ &= P(1 + r). \end{aligned}$$

This pattern of multiplying the previous principal by $1 + r$ repeats each successive year, as follows.

Year	**Balance After Each Compounding**
0	$P = P$
1	$P_1 = P(1 + r)$
2	$P_2 = P_1(1 + r) = P(1 + r)(1 + r) = P(1 + r)^2$
3	$P_3 = P_2(1 + r) = P(1 + r)^2(1 + r) = P(1 + r)^3$
$\vdots$	$\vdots$
t	$P_t = P(1 + r)^t$

To accommodate more frequent (quarterly, monthly, or daily) compounding of interest, let n be the number of compoundings per year and let t be the number of years. Then the rate per compounding is r/n, and the account balance after t years is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Amount (balance) with n compoundings per year

When you let the number of compoundings n increase without bound, the process approaches what is called **continuous compounding.** In the formula for n compoundings per year, let $m = n/r$. This produces

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Amount with n compoundings per year

$$= P\left(1 + \frac{r}{mr}\right)^{mrt}$$

Substitute mr for n.

$$= P\left(1 + \frac{1}{m}\right)^{mrt}$$

Simplify.

$$= P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}.$$

Property of exponents

m	$\left(1 + \frac{1}{m}\right)^m$
1	2
10	2.59374246
100	2.704813829
1,000	2.716923932
10,000	2.718145927
100,000	2.718268237
1,000,000	2.718280469
10,000,000	2.718281693
$\downarrow$	$\downarrow$
∞	e

As m increases without bound (that is, as $m \to \infty$), the table at the left shows that $[1 + (1/m)]^m \to e$. From this, you can conclude that the formula for continuous compounding is

$$A = Pe^{rt}.$$

Substitute e for $(1 + 1/m)^m$.

REMARK Be sure you see that, when using the formulas for compound interest, you must write the annual interest rate in decimal form. For instance, you must write 6% as 0.06.

Formulas for Compound Interest

After t years, the balance A in an account with principal P and annual interest rate r (in decimal form) is given by the following formulas.

1. For n compoundings per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$
2. For continuous compounding: $A = Pe^{rt}$

EXAMPLE 8 Compound Interest

You invest \$12,000 at an annual rate of 3%. Find the balance after 5 years when the interest is compounded

a. quarterly.
b. monthly.
c. continuously.

Solution

a. For quarterly compounding, you have $n = 4$. So, in 5 years at 3%, the balance is

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \qquad \text{Formula for compound interest}$$

$$= 12{,}000\left(1 + \frac{0.03}{4}\right)^{4(5)} \qquad \text{Substitute for } P, r, n, \text{ and } t.$$

$$\approx \$13{,}934.21. \qquad \text{Use a calculator.}$$

b. For monthly compounding, you have $n = 12$. So, in 5 years at 3%, the balance is

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \qquad \text{Formula for compound interest}$$

$$= 12{,}000\left(1 + \frac{0.03}{12}\right)^{12(5)} \qquad \text{Substitute for } P, r, n, \text{ and } t.$$

$$\approx \$13{,}939.40. \qquad \text{Use a calculator.}$$

c. For continuous compounding, the balance is

$$A = Pe^{rt} \qquad \text{Formula for continuous compounding}$$

$$= 12{,}000e^{0.03(5)} \qquad \text{Substitute for } P, r, \text{ and } t.$$

$$\approx \$13{,}942.01. \qquad \text{Use a calculator.}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

You invest \$6000 at an annual rate of 4%. Find the balance after 7 years when the interest is compounded

a. quarterly.
b. monthly.
c. continuously.

In Example 8, note that continuous compounding yields more than quarterly and monthly compounding. This is typical of the two types of compounding. That is, for a given principal, interest rate, and time, continuous compounding will always yield a larger balance than compounding n times per year.

EXAMPLE 9 Radioactive Decay

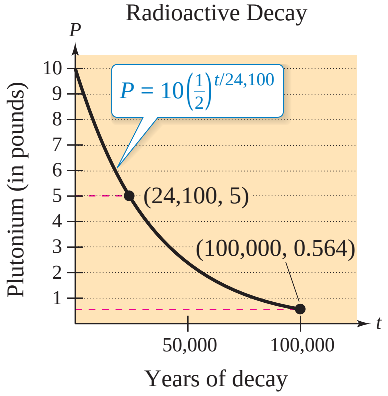
In 1986, a nuclear reactor accident occurred in Chernobyl in what was then the Soviet Union. The explosion spread highly toxic radioactive chemicals, such as plutonium (^{239}Pu), over hundreds of square miles, and the government evacuated the city and the surrounding area. To see why the city is now uninhabited, consider the model

$$P = 10\left(\frac{1}{2}\right)^{t/24{,}100}$$

which represents the amount of plutonium P that remains (from an initial amount of 10 pounds) after t years. Sketch the graph of this function over the interval from $t = 0$ to $t = 100{,}000$, where $t = 0$ represents 1986. How much of the 10 pounds will remain in the year 2017? How much of the 10 pounds will remain after 100,000 years?

The International Atomic Energy Authority ranks nuclear incidents and accidents by severity using a scale from 1 to 7 called the International Nuclear and Radiological Event Scale (INES). A level 7 ranking is the most severe. To date, the Chernobyl accident is the only nuclear accident in history to be given an INES level 7 ranking.

Solution The graph of this function is shown in the figure at the right. Note from this graph that plutonium has a *half-life* of about 24,100 years. That is, after 24,100 years, *half* of the original amount will remain. After another 24,100 years, one-quarter of the original amount will remain, and so on. In the year 2017 ($t = 31$), there will still be



Radioactive Decay

$$\begin{aligned}P &= 10\left(\frac{1}{2}\right)^{31/24{,}100}\\ &\approx 10\left(\frac{1}{2}\right)^{0.0012863}\\ &\approx 9.991 \text{ pounds}\end{aligned}$$

of plutonium remaining. After 100,000 years, there will still be

$$\begin{aligned}P &= 10\left(\frac{1}{2}\right)^{100{,}000/24{,}100}\\ &\approx 0.564 \text{ pound}\end{aligned}$$

of plutonium remaining.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

In Example 9, how much of the 10 pounds will remain in the year 2089? How much of the 10 pounds will remain after 125,000 years?

Summarize (Section 5.1)

1. State the definition of an exponential function f with base a *(page 362)*. For an example of evaluating exponential functions, see Example 1.
2. Describe the basic characteristics of the exponential functions $y = a^x$ and $y = a^{-x}$, $a > 1$ *(page 364)*. For examples of graphing exponential functions, see Examples 2, 3, and 5.
3. State the definitions of the natural base and the natural exponential function *(page 366)*. For examples of evaluating and graphing natural exponential functions, see Examples 6 and 7.
4. Describe examples of how to use exponential functions to model and solve real-life problems *(pages 368 and 369, Examples 8 and 9)*.

Hellen Sergeyeva/Shutterstock.com

5.1 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. Polynomial and rational functions are examples of ________ functions.

2. Exponential and logarithmic functions are examples of nonalgebraic functions, also called ________ functions.

3. You can use the ________ Property to solve simple exponential equations.

4. The exponential function $f(x) = e^x$ is called the ________ ________ function, and the base e is called the ________ base.

5. To find the amount A in an account after t years with principal P and an annual interest rate r compounded n times per year, you can use the formula ________.

6. To find the amount A in an account after t years with principal P and an annual interest rate r compounded continuously, you can use the formula ________.

Skills and Applications

Evaluating an Exponential Function

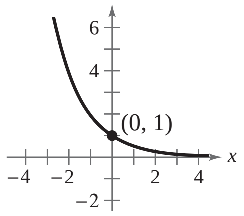
In Exercises 7–12, evaluate the function at the indicated value of x. Round your result to three decimal places.

Function	Value
7. $f(x) = 0.9^x$	$x = 1.4$
8. $f(x) = 2.3^x$	$x = \frac{3}{2}$
9. $f(x) = 5^x$	$x = -\pi$
10. $f(x) = \left(\frac{2}{3}\right)^{5x}$	$x = \frac{3}{10}$
11. $g(x) = 5000(2^x)$	$x = -1.5$
12. $f(x) = 200(1.2)^{12x}$	$x = 24$

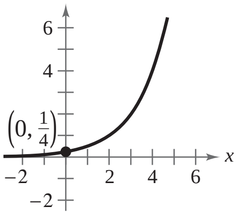
Matching an Exponential Function with Its Graph

In Exercises 13–16, match the exponential function with its graph. [The graphs are labeled (a), (b), (c), and (d).]

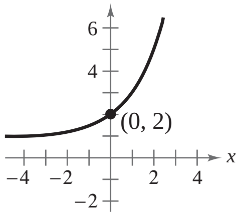
(a)



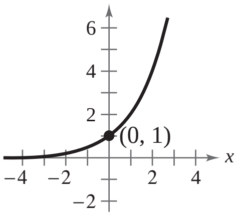
(b)



(c)



(d)



13. $f(x) = 2^x$

14. $f(x) = 2^x + 1$

15. $f(x) = 2^{-x}$

16. $f(x) = 2^{x-2}$

Graphing an Exponential Function

In Exercises 17–22, use a graphing utility to construct a table of values for the function. Then sketch the graph of the function.

17. $f(x) = \left(\frac{1}{2}\right)^x$

18. $f(x) = \left(\frac{1}{2}\right)^{-x}$

19. $f(x) = 6^{-x}$

20. $f(x) = 6^x$

21. $f(x) = 2^{x-1}$

22. $f(x) = 4^{x-3} + 3$

Using the One-to-One Property

In Exercises 23–26, use the One-to-One Property to solve the equation for x.

23. $3^{x+1} = 27$

24. $2^{x-3} = 16$

25. $\left(\frac{1}{2}\right)^x = 32$

26. $5^{x-2} = \frac{1}{125}$

Transforming the Graph of an Exponential Function

In Exercises 27–30, use the graph of f to describe the transformation that yields the graph of g.

27. $f(x) = 3^x, \quad g(x) = 3^x + 1$

28. $f(x) = 10^x, \quad g(x) = 10^{-x+3}$

29. $f(x) = \left(\frac{7}{2}\right)^x, \quad g(x) = -\left(\frac{7}{2}\right)^{-x}$

30. $f(x) = 0.3^x, \quad g(x) = -0.3^x + 5$

Graphing an Exponential Function

In Exercises 31–34, use a graphing utility to graph the exponential function.

31. $y = 2^{-x^2}$

32. $y = 3^{-|x|}$

33. $y = 3^{x-2} + 1$

34. $y = 4^{x+1} - 2$

Evaluating a Natural Exponential Function

In Exercises 35–38, evaluate the function at the indicated value of x. Round your result to three decimal places.

Function	Value
35. $f(x) = e^x$	$x = 3.2$
36. $f(x) = 1.5e^{x/2}$	$x = 240$
37. $f(x) = 5000e^{0.06x}$	$x = 6$
38. $f(x) = 250e^{0.05x}$	$x = 20$

Graphing a Natural Exponential Function **In Exercises 39–44, use a graphing utility to construct a table of values for the function. Then sketch the graph of the function.**

39. $f(x) = e^x$ **40.** $f(x) = e^{-x}$

41. $f(x) = 3e^{x+4}$ **42.** $f(x) = 2e^{-0.5x}$

43. $f(x) = 2e^{x-2} + 4$ **44.** $f(x) = 2 + e^{x-5}$

Graphing a Natural Exponential Function **In Exercises 45–50, use a graphing utility to graph the exponential function.**

45. $y = 1.08e^{-5x}$ **46.** $y = 1.08e^{5x}$

47. $s(t) = 2e^{0.12t}$ **48.** $s(t) = 3e^{-0.2t}$

49. $g(x) = 1 + e^{-x}$ **50.** $h(x) = e^{x-2}$

Using the One-to-One Property **In Exercises 51–54, use the One-to-One Property to solve the equation for *x*.**

51. $e^{3x+2} = e^3$ **52.** $e^{2x-1} = e^4$

53. $e^{x^2-3} = e^{2x}$ **54.** $e^{x^2+6} = e^{5x}$

Compound Interest **In Exercises 55–58, complete the table to determine the balance *A* for *P* dollars invested at rate *r* for *t* years and compounded *n* times per year.**

n	1	2	4	12	365	Continuous
A						

55. $P = \$1500$, $r = 2\%$, $t = 10$ years

56. $P = \$2500$, $r = 3.5\%$, $t = 10$ years

57. $P = \$2500$, $r = 4\%$, $t = 20$ years

58. $P = \$1000$, $r = 6\%$, $t = 40$ years

Compound Interest **In Exercises 59–62, complete the table to determine the balance *A* for $12,000 invested at rate *r* for *t* years, compounded continuously.**

t	10	20	30	40	50
A					

59. $r = 4\%$ **60.** $r = 6\%$

61. $r = 6.5\%$ **62.** $r = 3.5\%$

63. **Trust Fund** On the day of a child's birth, a parent deposits $30,000 in a trust fund that pays 5% interest, compounded continuously. Determine the balance in this account on the child's 25th birthday.

64. **Trust Fund** A philanthropist deposits $5000 in a trust fund that pays 7.5% interest, compounded continuously. The balance will be given to the college from which the philanthropist graduated after the money has earned interest for 50 years. How much will the college receive?

65. **Inflation** Assuming that the annual rate of inflation averages 4% over the next 10 years, the approximate costs C of goods or services during any year in that decade will be modeled by $C(t) = P(1.04)^t$, where t is the time in years and P is the present cost. The price of an oil change for your car is presently $23.95. Estimate the price 10 years from now.

66. **Computer Virus** The number V of computers infected by a virus increases according to the model $V(t) = 100e^{4.6052t}$, where t is the time in hours. Find the number of computers infected after (a) 1 hour, (b) 1.5 hours, and (c) 2 hours.

67. **Population Growth** The projected populations of the United States for the years 2020 through 2050 can be modeled by $P = 290.323e^{0.0083t}$, where P is the population (in millions) and t is the time (in years), with $t = 20$ corresponding to 2020. *(Source: U.S. Census Bureau)*

(a) Use a graphing utility to graph the function for the years 2020 through 2050.

(b) Use the *table* feature of the graphing utility to create a table of values for the same time period as in part (a).

(c) According to the model, during what year will the population of the United States exceed 400 million?

68. **Population** The populations P (in millions) of Italy from 2000 through 2012 can be approximated by the model $P = 57.563e^{0.0052t}$, where t represents the year, with $t = 0$ corresponding to 2000. *(Source: U.S. Census Bureau, International Data Base)*

(a) According to the model, is the population of Italy increasing or decreasing? Explain.

(b) Find the populations of Italy in 2000 and 2012.

(c) Use the model to predict the populations of Italy in 2020 and 2025.

69. **Radioactive Decay** Let Q represent a mass of radioactive plutonium (^{239}Pu) (in grams), whose half-life is 24,100 years. The quantity of plutonium present after t years is $Q = 16\left(\frac{1}{2}\right)^{t/24,100}$.

(a) Determine the initial quantity (when $t = 0$).

(b) Determine the quantity present after 75,000 years.

(c) Use a graphing utility to graph the function over the interval $t = 0$ to $t = 150,000$.

70. **Radioactive Decay** Let Q represent a mass of carbon 14 (^{14}C) (in grams), whose half-life is 5715 years. The quantity of carbon 14 present after t years is $Q = 10\left(\frac{1}{2}\right)^{t/5715}$.

(a) Determine the initial quantity (when $t = 0$).

(b) Determine the quantity present after 2000 years.

(c) Sketch the graph of this function over the interval $t = 0$ to $t = 10,000$.

71. Depreciation After t years, the value of a wheelchair conversion van that originally cost \$49,810 depreciates so that each year it is worth $\frac{7}{8}$ of its value for the previous year.

(a) Find a model for $V(t)$, the value of the van after t years.

(b) Determine the value of the van 4 years after it was purchased.

72. Drug Concentration Immediately following an injection, the concentration of a drug in the bloodstream is 300 milligrams per milliliter. After t hours, the concentration is 75% of the level of the previous hour.

(a) Find a model for $C(t)$, the concentration of the drug after t hours.

(b) Determine the concentration of the drug after 8 hours.

Exploration

True or False? **In Exercises 73 and 74, determine whether the statement is true or false. Justify your answer.**

73. The line $y = -2$ is an asymptote for the graph of $f(x) = 10^x - 2$.

74. $e = \dfrac{271{,}801}{99{,}990}$

Think About It **In Exercises 75–78, use properties of exponents to determine which functions (if any) are the same.**

75. $f(x) = 3^{x-2}$
$g(x) = 3^x - 9$
$h(x) = \frac{1}{9}(3^x)$

76. $f(x) = 4^x + 12$
$g(x) = 2^{2x+6}$
$h(x) = 64(4^x)$

77. $f(x) = 16(4^{-x})$
$g(x) = \left(\frac{1}{4}\right)^{x-2}$
$h(x) = 16(2^{-2x})$

78. $f(x) = e^{-x} + 3$
$g(x) = e^{3-x}$
$h(x) = -e^{x-3}$

79. Solving Inequalities Graph the functions $y = 3^x$ and $y = 4^x$ and use the graphs to solve each inequality.

(a) $4^x < 3^x$ (b) $4^x > 3^x$

80. Graphical Analysis Use a graphing utility to graph each function. Use the graph to find where the function is increasing and decreasing, and approximate any relative maximum or minimum values.

(a) $f(x) = x^2e^{-x}$ (b) $g(x) = x2^{3-x}$

81. Graphical Analysis Use a graphing utility to graph $y_1 = (1 + 1/x)^x$ and $y_2 = e$ in the same viewing window. Using the *trace* feature, explain what happens to the graph of y_1 as x increases.

82. Graphical Analysis Use a graphing utility to graph

$$f(x) = \left(1 + \frac{0.5}{x}\right)^x \quad \text{and} \quad g(x) = e^{0.5}$$

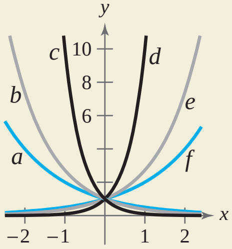
in the same viewing window. What is the relationship between f and g as x increases and decreases without bound?

83. Graphical Analysis Use a graphing utility to graph each pair of functions in the same viewing window. Describe any similarities and differences in the graphs.

(a) $y_1 = 2^x, y_2 = x^2$

(b) $y_1 = 3^x, y_2 = x^3$

84. HOW DO YOU SEE IT? The figure shows the graphs of $y = 2^x$, $y = e^x$, $y = 10^x$, $y = 2^{-x}$, $y = e^{-x}$, and $y = 10^{-x}$. Match each function with its graph. [The graphs are labeled (a) through (f).] Explain your reasoning.



85. Think About It Which functions are exponential?

(a) $3x$

(b) $3x^2$

(c) 3^x

(d) 2^{-x}

86. Compound Interest Use the formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

to calculate the balance of an investment when $P = \$3000$, $r = 6\%$, and $t = 10$ years, and compounding is done (a) by the day, (b) by the hour, (c) by the minute, and (d) by the second. Does increasing the number of compoundings per year result in unlimited growth of the balance? Explain.

Project: Population per Square Mile To work an extended application analyzing the population per square mile of the United States, visit this text's website at *LarsonPrecalculus.com*. *(Source: U.S. Census Bureau)*

Sura Nualpradid/Shutterstock.com

5.2 Logarithmic Functions and Their Graphs

Logarithmic functions can often model scientific observations. For instance, Exercise 81 on page 382 uses a logarithmic function to model human memory.

- **Recognize and evaluate logarithmic functions with base *a*.**
- **Graph logarithmic functions.**
- **Recognize, evaluate, and graph natural logarithmic functions.**
- **Use logarithmic functions to model and solve real-life problems.**

Logarithmic Functions

In Section 2.7, you studied the concept of an inverse function. There, you learned that when a function is one-to-one—that is, when the function has the property that no horizontal line intersects the graph of the function more than once—the function must have an inverse function. By looking back at the graphs of the exponential functions introduced in Section 5.1, you will see that every function of the form $f(x) = a^x$ passes the Horizontal Line Test and therefore must have an inverse function. This inverse function is called the **logarithmic function with base *a*.**

Definition of Logarithmic Function with Base *a*

For $x > 0$, $a > 0$, and $a \neq 1$,

$$y = \log_a x \text{ if and only if } x = a^y.$$

The function

$$f(x) = \log_a x$$ Read as "log base *a* of *x*."

is called the **logarithmic function with base *a*.**

The equations $y = \log_a x$ and $x = a^y$ are equivalent. The first equation is in logarithmic form and the second is in exponential form. For example, $2 = \log_3 9$ is equivalent to $9 = 3^2$, and $5^3 = 125$ is equivalent to $\log_5 125 = 3$.

When evaluating logarithms, remember that *a logarithm is an exponent*. This means that $\log_a x$ is the exponent to which a must be raised to obtain x. For instance, $\log_2 8 = 3$ because 2 raised to the third power is 8.

EXAMPLE 1 Evaluating Logarithms

Evaluate each logarithm at the indicated value of x.

a. $f(x) = \log_2 x, \quad x = 32$ **b.** $f(x) = \log_3 x, \quad x = 1$

c. $f(x) = \log_4 x, \quad x = 2$ **d.** $f(x) = \log_{10} x, \quad x = \frac{1}{100}$

Solution

a. $f(32) = \log_2 32 = 5$ because $2^5 = 32$.

b. $f(1) = \log_3 1 = 0$ because $3^0 = 1$.

c. $f(2) = \log_4 2 = \frac{1}{2}$ because $4^{1/2} = \sqrt{4} = 2$.

d. $f\left(\frac{1}{100}\right) = \log_{10} \frac{1}{100} = -2$ because $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Evaluate each logarithm at the indicated value of x.

a. $f(x) = \log_6 x, \ x = 1$ **b.** $f(x) = \log_5 x, \ x = \frac{1}{125}$ **c.** $f(x) = \log_{10} x, \ x = 10{,}000$

Sebastian Kaulitzki/Shutterstock.com

The logarithmic function with base 10 is called the **common logarithmic function.** It is denoted by $\log_{10}$ or simply by log. On most calculators, this function is denoted by $\boxed{\text{LOG}}$. Example 2 shows how to use a calculator to evaluate common logarithmic functions. You will learn how to use a calculator to calculate logarithms to any base in the next section.

EXAMPLE 2 Evaluating Common Logarithms on a Calculator

Use a calculator to evaluate the function $f(x) = \log x$ at each value of x.

a. $x = 10$ **b.** $x = \frac{1}{3}$ **c.** $x = 2.5$ **d.** $x = -2$

Solution

Function Value	Graphing Calculator Keystrokes	Display
a. $f(10) = \log 10$	$\boxed{\text{LOG}}$ 10 $\boxed{\text{ENTER}}$	1
b. $f(\frac{1}{3}) = \log \frac{1}{3}$	$\boxed{\text{LOG}}$ $\boxed{(}$ 1 $\boxed{\div}$ 3 $\boxed{)}$ $\boxed{\text{ENTER}}$	-0.4771213
c. $f(2.5) = \log 2.5$	$\boxed{\text{LOG}}$ 2.5 $\boxed{\text{ENTER}}$	0.3979400
d. $f(-2) = \log(-2)$	$\boxed{\text{LOG}}$ $\boxed{(-)}$ 2 $\boxed{\text{ENTER}}$	ERROR

Note that the calculator displays an error message (or a complex number) when you try to evaluate $\log(-2)$. The reason for this is that there is no real number power to which 10 can be raised to obtain -2.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use a calculator to evaluate the function $f(x) = \log x$ at each value of x.

a. $x = 275$ **b.** $x = 0.275$ **c.** $x = -\frac{1}{2}$ **d.** $x = \frac{1}{2}$

The following properties follow directly from the definition of the logarithmic function with base a.

Properties of Logarithms

1. $\log_a 1 = 0$ because $a^0 = 1$.

2. $\log_a a = 1$ because $a^1 = a$.

3. $\log_a a^x = x$ and $a^{\log_a x} = x$ Inverse Properties

4. If $\log_a x = \log_a y$, then $x = y$. One-to-One Property

EXAMPLE 3 Using Properties of Logarithms

a. Simplify $\log_4 1$. **b.** Simplify $\log_{\sqrt{7}} \sqrt{7}$. **c.** Simplify $6^{\log_6 20}$.

Solution

a. Using Property 1, $\log_4 1 = 0$.

b. Using Property 2, $\log_{\sqrt{7}} \sqrt{7} = 1$.

c. Using the Inverse Property (Property 3), $6^{\log_6 20} = 20$.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

a. Simplify $\log_9 9$. **b.** Simplify $20^{\log_{20} 3}$. **c.** Simplify $\log_{\sqrt{3}} 1$.

You can use the One-to-One Property (Property 4) to solve simple logarithmic equations, as shown in Example 4.

EXAMPLE 4 Using the One-to-One Property

a. $\log_3 x = \log_3 12$ Original equation

$x = 12$ One-to-One Property

b. $\log(2x + 1) = \log 3x \implies 2x + 1 = 3x \implies 1 = x$

c. $\log_4(x^2 - 6) = \log_4 10 \implies x^2 - 6 = 10 \implies x^2 = 16 \implies x = \pm 4$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $\log_5(x^2 + 3) = \log_5 12$ for x.

Graphs of Logarithmic Functions

To sketch the graph of $y = \log_a x$, use the fact that the graphs of inverse functions are reflections of each other in the line $y = x$.

EXAMPLE 5 Graphs of Exponential and Logarithmic Functions

In the same coordinate plane, sketch the graph of each function.

a. $f(x) = 2^x$ **b.** $g(x) = \log_2 x$

Solution

a. For $f(x) = 2^x$, construct a table of values. By plotting these points and connecting them with a smooth curve, you obtain the graph shown in Figure 5.10.

x	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

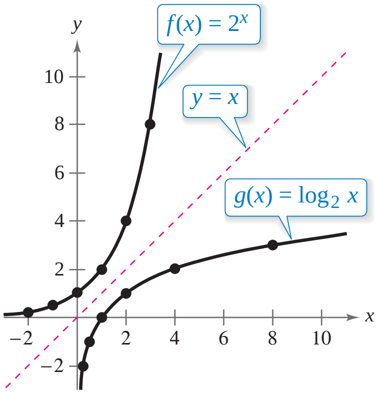


Figure 5.10

b. Because $g(x) = \log_2 x$ is the inverse function of $f(x) = 2^x$, the graph of g is obtained by plotting the points $(f(x), x)$ and connecting them with a smooth curve. The graph of g is a reflection of the graph of f in the line $y = x$, as shown in Figure 5.10.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

In the same coordinate plane, sketch the graphs of (a) $f(x) = 8^x$ and (b) $g(x) = \log_8 x$.

EXAMPLE 6 Sketching the Graph of a Logarithmic Function

Sketch the graph of $f(x) = \log x$. Identify the vertical asymptote.

Solution Begin by constructing a table of values. Note that some of the values can be obtained without a calculator by using the properties of logarithms. Others require a calculator. Next, plot the points and connect them with a smooth curve, as shown in Figure 5.11. The vertical asymptote is $x = 0$ (y-axis).

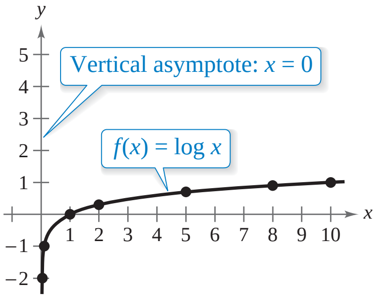


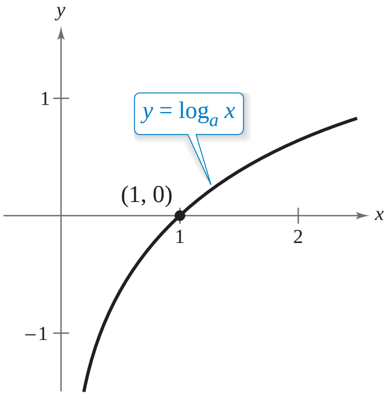
Figure 5.11

	Without calculator				With calculator		
x	$\frac{1}{100}$	$\frac{1}{10}$	1	10	2	5	8
$f(x) = \log x$	-2	-1	0	1	0.301	0.699	0.903

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Sketch the graph of $f(x) = \log_9 x$. Identify the vertical asymptote.

The nature of the graph in Figure 5.11 is typical of functions of the form $f(x) = \log_a x$, $a > 1$. They have one x-intercept and one vertical asymptote. Notice how slowly the graph rises for $x > 1$. The following summarizes the basic characteristics of logarithmic graphs.



Graph of $y = \log_a x$, $a > 1$

- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- x-intercept: $(1, 0)$
- Increasing
- One-to-one, therefore has an inverse function
- y-axis is a vertical asymptote ($\log_a x \to -\infty$ as $x \to 0^+$).
- Continuous
- Reflection of graph of $y = a^x$ in the line $y = x$

The basic characteristics of the graph of $f(x) = a^x$ are shown below to illustrate the inverse relation between $f(x) = a^x$ and $g(x) = \log_a x$.

- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- y-intercept: $(0, 1)$
- x-axis is a horizontal asymptote ($a^x \to 0$ as $x \to -\infty$).

The next example uses the graph of $y = \log_a x$ to sketch the graphs of functions of the form $f(x) = b \pm \log_a(x + c)$. Notice how a horizontal shift of the graph results in a horizontal shift of the vertical asymptote.

EXAMPLE 7 Shifting Graphs of Logarithmic Functions

The graph of each of the functions is similar to the graph of $f(x) = \log x$.

a. Because $g(x) = \log(x - 1) = f(x - 1)$, the graph of g can be obtained by shifting the graph of f one unit to the right, as shown in Figure 5.12.

b. Because $h(x) = 2 + \log x = 2 + f(x)$, the graph of h can be obtained by shifting the graph of f two units up, as shown in Figure 5.13.

REMARK Use your understanding of transformations to identify vertical asymptotes of logarithmic functions. For instance, in Example 7(a), the graph of $g(x) = f(x - 1)$ shifts the graph of $f(x)$ one unit to the right. So, the vertical asymptote of the graph of $g(x)$ is $x = 1$, one unit to the right of the vertical asymptote of the graph of $f(x)$.

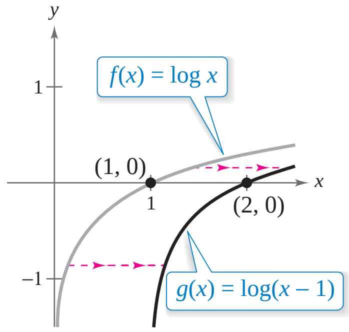


Figure 5.12

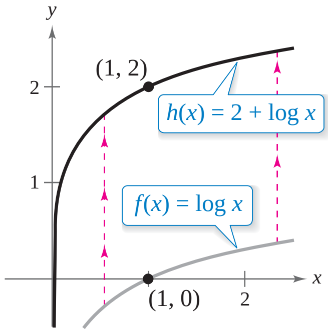


Figure 5.13

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use the graph of $f(x) = \log_3 x$ to sketch the graph of each function.

a. $g(x) = -1 + \log_3 x$

b. $h(x) = \log_3(x + 3)$

ALGEBRA HELP You can review the techniques for shifting, reflecting, and stretching graphs in Section 2.5.

The Natural Logarithmic Function

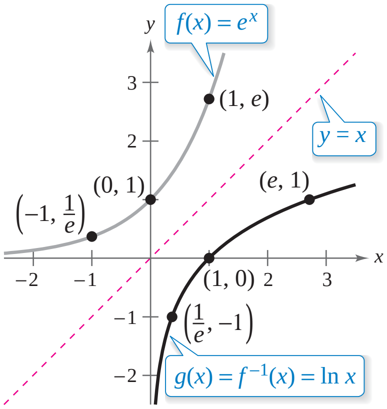
By looking back at the graph of the natural exponential function introduced on page 366 in Section 5.1, you will see that $f(x) = e^x$ is one-to-one and so has an inverse function. This inverse function is called the **natural logarithmic function** and is denoted by the special symbol ln x, read as "the natural log of x" or "el en of x." Note that the natural logarithm is written without a base. The base is understood to be e.

The Natural Logarithmic Function

The function defined by

$$f(x) = \log_e x = \ln x, \quad x > 0$$

is called the **natural logarithmic function.**



Reflection of graph of $f(x) = e^x$ in the line $y = x$
Figure 5.14

The above definition implies that the natural logarithmic function and the natural exponential function are inverse functions of each other. So, every logarithmic equation can be written in an equivalent exponential form, and every exponential equation can be written in an equivalent logarithmic form. That is, $y = \ln x$ and $x = e^y$ are equivalent equations.

Because the functions $f(x) = e^x$ and $g(x) = \ln x$ are inverse functions of each other, their graphs are reflections of each other in the line $y = x$. Figure 5.14 illustrates this reflective property.

On most calculators, LN denotes the natural logarithm, as illustrated in Example 8.

EXAMPLE 8 Evaluating the Natural Logarithmic Function

Use a calculator to evaluate the function $f(x) = \ln x$ at each value of x.

a. $x = 2$

b. $x = 0.3$

c. $x = -1$

d. $x = 1 + \sqrt{2}$

Solution

Function Value	Graphing Calculator Keystrokes	Display
a. $f(2) = \ln 2$	LN 2 ENTER	0.6931472
b. $f(0.3) = \ln 0.3$	LN .3 ENTER	–1.2039728
c. $f(-1) = \ln(-1)$	LN (−) 1 ENTER	ERROR
d. $f(1 + \sqrt{2}) = \ln(1 + \sqrt{2})$	LN (1 + √ 2) ENTER	0.8813736

REMARK Notice that as with every other logarithmic function, the domain of the natural logarithmic function is the set of *positive real numbers*—be sure you see that ln x is not defined for zero or for negative numbers.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use a calculator to evaluate the function $f(x) = \ln x$ at each value of x.

a. $x = 0.01$ **b.** $x = 4$ **c.** $x = \sqrt{3} + 2$ **d.** $x = \sqrt{3} - 2$

In Example 8, be sure you see that $\ln(-1)$ gives an error message on most calculators. (Some calculators may display a complex number.) This occurs because the domain of ln x is the set of positive real numbers (see Figure 5.14). So, $\ln(-1)$ is undefined.

The four properties of logarithms listed on page 374 are also valid for natural logarithms.

Properties of Natural Logarithms

1. $\ln 1 = 0$ because $e^0 = 1$.
2. $\ln e = 1$ because $e^1 = e$.
3. $\ln e^x = x$ and $e^{\ln x} = x$ Inverse Properties
4. If $\ln x = \ln y$, then $x = y$. One-to-One Property

EXAMPLE 9 Using Properties of Natural Logarithms

Use the properties of natural logarithms to simplify each expression.

a. $\ln \dfrac{1}{e}$ **b.** $e^{\ln 5}$ **c.** $\dfrac{\ln 1}{3}$ **d.** $2 \ln e$

Solution

a. $\ln \dfrac{1}{e} = \ln e^{-1} = -1$ Inverse Property

b. $e^{\ln 5} = 5$ Inverse Property

c. $\dfrac{\ln 1}{3} = \dfrac{0}{3} = 0$ Property 1

d. $2 \ln e = 2(1) = 2$ Property 2

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use the properties of natural logarithms to simplify each expression.

a. $\ln e^{1/3}$ **b.** $5 \ln 1$ **c.** $\frac{3}{4} \ln e$ **d.** $e^{\ln 7}$

EXAMPLE 10 Finding the Domains of Logarithmic Functions

Find the domain of each function.

a. $f(x) = \ln(x - 2)$ **b.** $g(x) = \ln(2 - x)$ **c.** $h(x) = \ln x^2$

Solution

a. Because $\ln(x - 2)$ is defined only when $x - 2 > 0$, it follows that the domain of f is $(2, \infty)$. Figure 5.15 shows the graph of f.

b. Because $\ln(2 - x)$ is defined only when $2 - x > 0$, it follows that the domain of g is $(-\infty, 2)$. Figure 5.16 shows the graph of g.

c. Because $\ln x^2$ is defined only when $x^2 > 0$, it follows that the domain of h is all real numbers except $x = 0$. Figure 5.17 shows the graph of h.

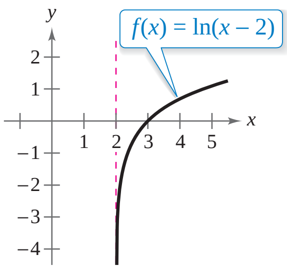


Figure 5.15

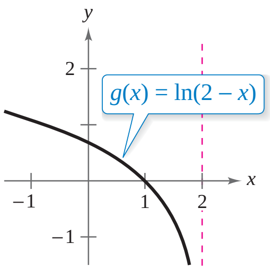


Figure 5.16

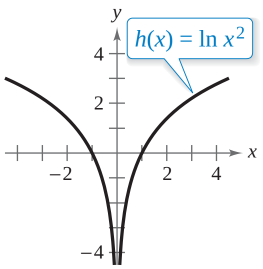


Figure 5.17

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the domain of $f(x) = \ln(x + 3)$.

Application

EXAMPLE 11 Human Memory Model

Students participating in a psychology experiment attended several lectures on a subject and took an exam. Every month for a year after the exam, the students took a retest to see how much of the material they remembered. The average scores for the group are given by the *human memory model* $f(t) = 75 - 6\ln(t + 1)$, $0 \le t \le 12$, where t is the time in months.

a. What was the average score on the original exam $(t = 0)$?

b. What was the average score at the end of $t = 2$ months?

c. What was the average score at the end of $t = 6$ months?

Algebraic Solution

a. The original average score was

$$\begin{aligned} f(0) &= 75 - 6\ln(0 + 1) && \text{Substitute 0 for } t. \\ &= 75 - 6\ln 1 && \text{Simplify.} \\ &= 75 - 6(0) && \text{Property of natural logarithms} \\ &= 75. && \text{Solution} \end{aligned}$$

b. After 2 months, the average score was

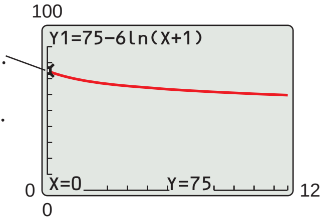
$$\begin{aligned} f(2) &= 75 - 6\ln(2 + 1) && \text{Substitute 2 for } t. \\ &= 75 - 6\ln 3 && \text{Simplify.} \\ &\approx 75 - 6(1.0986) && \text{Use a calculator.} \\ &\approx 68.41. && \text{Solution} \end{aligned}$$

c. After 6 months, the average score was

$$\begin{aligned} f(6) &= 75 - 6\ln(6 + 1) && \text{Substitute 6 for } t. \\ &= 75 - 6\ln 7 && \text{Simplify.} \\ &\approx 75 - 6(1.9459) && \text{Use a calculator.} \\ &\approx 63.32. && \text{Solution} \end{aligned}$$

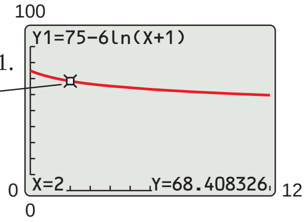
Graphical Solution

a.



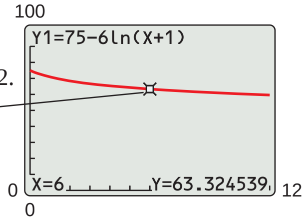
When $t = 0$, $f(0) = 75$. So, the original average score was 75.

b.



When $t = 2$, $f(2) \approx 68.41$. So, the average score after 2 months was about 68.41.

c.



When $t = 6$, $f(6) \approx 63.32$. So, the average score after 6 months was about 63.32.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

In Example 11, find the average score at the end of (a) $t = 1$ month, (b) $t = 9$ months, and (c) $t = 12$ months.

Summarize (Section 5.2)

1. State the definition of a logarithmic function with base *a (page 373)* and make a list of the properties of logarithms *(page 374)*. For examples of evaluating logarithmic functions and using the properties of logarithms, see Examples 1–4.
2. Explain how to graph a logarithmic function *(pages 375 and 376)*. For examples of graphing logarithmic functions, see Examples 5–7.
3. State the definition of the natural logarithmic function *(page 377)* and make a list of the properties of natural logarithms *(page 378)*. For examples of evaluating natural logarithmic functions and using the properties of natural logarithms, see Examples 8 and 9.
4. Describe an example of how to use a logarithmic function to model and solve a real-life problem *(page 379, Example 11)*.

5.2 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. The inverse function of the exponential function $f(x) = a^x$ is called the ________ function with base a.
2. The common logarithmic function has base ________ .
3. The logarithmic function $f(x) = \ln x$ is called the ________ logarithmic function and has base ________.
4. The Inverse Properties of logarithms state that $\log_a a^x = x$ and ________.
5. The One-to-One Property of natural logarithms states that if $\ln x = \ln y$, then ________.
6. The domain of the natural logarithmic function is the set of ________ ________ ________ .

Skills and Applications

Writing an Exponential Equation

In Exercises 7–10, write the logarithmic equation in exponential form. For example, the exponential form of $\log_5 25 = 2$ is $5^2 = 25$.

7. $\log_4 16 = 2$

8. $\log_9 \frac{1}{81} = -2$

9. $\log_{32} 4 = \frac{2}{5}$

10. $\log_{64} 8 = \frac{1}{2}$

Writing a Logarithmic Equation

In Exercises 11–14, write the exponential equation in logarithmic form. For example, the logarithmic form of $2^3 = 8$ is $\log_2 8 = 3$.

11. $5^3 = 125$

12. $9^{3/2} = 27$

13. $4^{-3} = \frac{1}{64}$

14. $24^0 = 1$

Evaluating a Logarithmic Function

In Exercises 15–20, evaluate the function at the indicated value of x without using a calculator.

Function	Value
15. $f(x) = \log_2 x$	$x = 64$
16. $f(x) = \log_{25} x$	$x = 5$
17. $f(x) = \log_8 x$	$x = 1$
18. $f(x) = \log x$	$x = 10$
19. $g(x) = \log_a x$	$x = a^2$
20. $g(x) = \log_b x$	$x = b^{-3}$

Evaluating a Common Logarithm on a Calculator

In Exercises 21–24, use a calculator to evaluate $f(x) = \log x$ at the indicated value of x. Round your result to three decimal places.

21. $x = \frac{7}{8}$

22. $x = \frac{1}{500}$

23. $x = 12.5$

24. $x = 96.75$

Using Properties of Logarithms

In Exercises 25–28, use the properties of logarithms to simplify the expression.

25. $\log_{11} 11^7$

26. $\log_{3.2} 1$

27. $\log_\pi \pi$

28. $9^{\log_9 15}$

Using the One-to-One Property

In Exercises 29–32, use the One-to-One Property to solve the equation for x.

29. $\log_5(x + 1) = \log_5 6$

30. $\log_2(x - 3) = \log_2 9$

31. $\log(2x + 1) = \log 15$

32. $\log(5x + 3) = \log 12$

Graphs of Exponential and Logarithmic Functions

In Exercises 33–36, sketch the graphs of f and g in the same coordinate plane.

33. $f(x) = 7^x, \ g(x) = \log_7 x$

34. $f(x) = 5^x, \ g(x) = \log_5 x$

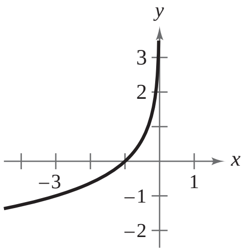
35. $f(x) = 6^x, \ g(x) = \log_6 x$

36. $f(x) = 10^x, \ g(x) = \log x$

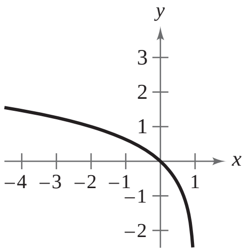
Matching a Logarithmic Function with Its Graph

In Exercises 37–40, use the graph of $g(x) = \log_3 x$ to match the given function with its graph. Then describe the relationship between the graphs of f and g. [The graphs are labeled (a), (b), (c), and (d).]

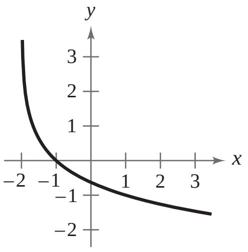
(a)



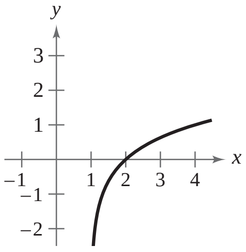
(b)



(c)



(d)



37. $f(x) = -\log_3(x + 2)$

38. $f(x) = \log_3(x - 1)$

39. $f(x) = \log_3(1 - x)$

40. $f(x) = -\log_3(-x)$

Sketching the Graph of a Logarithmic Function

In Exercises 41–48, find the domain, x-intercept, and vertical asymptote of the logarithmic function and sketch its graph.

41. $f(x) = \log_4 x$ **42.** $g(x) = \log_6 x$

43. $y = -\log_3 x + 2$ **44.** $h(x) = \log_4(x - 3)$

45. $f(x) = -\log_6(x + 2)$ **46.** $y = \log_5(x - 1) + 4$

47. $y = \log\left(\frac{x}{7}\right)$ **48.** $y = \log(-x)$

Writing a Natural Exponential Equation

In Exercises 49–52, write the logarithmic equation in exponential form.

49. $\ln \frac{1}{2} = -0.693 \ldots$ **50.** $\ln 7 = 1.945 \ldots$

51. $\ln 250 = 5.521 \ldots$ **52.** $\ln 1 = 0$

Writing a Natural Logarithmic Equation

In Exercises 53–56, write the exponential equation in logarithmic form.

53. $e^2 = 7.3890 \ldots$ **54.** $e^{1/2} = 1.6487 \ldots$

55. $e^{-0.9} = 0.406 \ldots$ **56.** $e^{2x} = 3$

Evaluating a Logarithmic Function on a Calculator

In Exercises 57–60, use a calculator to evaluate the function at the indicated value of x. Round your result to three decimal places.

Function	Value
57. $f(x) = \ln x$	$x = 18.42$
58. $f(x) = 3 \ln x$	$x = 0.74$
59. $g(x) = 8 \ln x$	$x = 0.05$
60. $g(x) = -\ln x$	$x = \frac{1}{2}$

Evaluating a Natural Logarithm

In Exercises 61–64, evaluate $g(x) = \ln x$ at the indicated value of x without using a calculator.

61. $x = e^5$ **62.** $x = e^{-4}$

63. $x = e^{-5/6}$ **64.** $x = e^{-5/2}$

Graphing a Natural Logarithmic Function

In Exercises 65–68, find the domain, x-intercept, and vertical asymptote of the logarithmic function and sketch its graph.

65. $f(x) = \ln(x - 4)$ **66.** $h(x) = \ln(x + 5)$

67. $g(x) = \ln(-x)$ **68.** $f(x) = \ln(3 - x)$

Graphing a Natural Logarithmic Function

In Exercises 69–72, use a graphing utility to graph the function. Be sure to use an appropriate viewing window.

69. $f(x) = \ln(x - 1)$ **70.** $f(x) = \ln(x + 2)$

71. $f(x) = \ln x + 8$ **72.** $f(x) = 3 \ln x - 1$

Using the One-to-One Property

In Exercises 73–76, use the One-to-One Property to solve the equation for x.

73. $\ln(x + 4) = \ln 12$ **74.** $\ln(x - 7) = \ln 7$

75. $\ln(x^2 - 2) = \ln 23$ **76.** $\ln(x^2 - x) = \ln 6$

77. Monthly Payment The model

$$t = 16.625 \ln\left(\frac{x}{x - 750}\right), \quad x > 750$$

approximates the length of a home mortgage of \$150,000 at 6% in terms of the monthly payment. In the model, t is the length of the mortgage in years and x is the monthly payment in dollars.

(a) Use the model to approximate the lengths of a \$150,000 mortgage at 6% when the monthly payment is \$897.72 and when the monthly payment is \$1659.24.

(b) Approximate the total amounts paid over the term of the mortgage with a monthly payment of \$897.72 and with a monthly payment of \$1659.24.

(c) Approximate the total interest charges for a monthly payment of \$897.72 and for a monthly payment of \$1659.24.

(d) What is the vertical asymptote for the model? Interpret its meaning in the context of the problem.

78. Wireless Only The percents P of households in the United States with wireless-only telephone service from 2005 through 2011 can be approximated by the model

$$P = -4.00 + 1.335t \ln t, \quad 5 \le t \le 11$$

where t represents the year, with $t = 5$ corresponding to 2005. *(Source: National Center for Health Statistics)*

(a) Complete the table.

t	5	6	7	8	9	10	11
P							

(b) Use a graphing utility to graph the function.

(c) Can the model be used to predict the percents of households with wireless-only telephone service beyond 2011? Explain.

79. Population The time t (in years) for the world population to double when it is increasing at a continuous rate of r is given by $t = (\ln 2)/r$.

(a) Complete the table and interpret your results.

r	0.005	0.010	0.015	0.020	0.025	0.030
t						

(b) Use a graphing utility to graph the function.

80. Compound Interest A principal P, invested at $5\frac{1}{2}\%$ and compounded continuously, increases to an amount K times the original principal after t years, where $t = (\ln K)/0.055$.

(a) Complete the table and interpret your results.

K	1	2	4	6	8	10	12
t							

(b) Sketch a graph of the function.

81. Human Memory Model

Students in a mathematics class took an exam and then took a retest monthly with an equivalent exam. The average scores for the class are given by the human memory model

$$f(t) = 80 - 17\log(t + 1), \quad 0 \le t \le 12$$

where t is the time in months.

(a) Use a graphing utility to graph the model over the specified domain.

(b) What was the average score on the original exam $(t = 0)$?

(c) What was the average score after 4 months?

(d) What was the average score after 10 months?

82. Sound Intensity The relationship between the number of decibels β and the intensity of a sound I in watts per square meter is

$$\beta = 10\log\left(\frac{I}{10^{-12}}\right).$$

(a) Determine the number of decibels of a sound with an intensity of 1 watt per square meter.

(b) Determine the number of decibels of a sound with an intensity of 10^{-2} watt per square meter.

(c) The intensity of the sound in part (a) is 100 times as great as that in part (b). Is the number of decibels 100 times as great? Explain.

Exploration

True or False? **In Exercises 83 and 84, determine whether the statement is true or false. Justify your answer.**

83. The graph of $f(x) = \log_6 x$ is a reflection of the graph of $g(x) = 6^x$ in the x-axis.

84. The graph of $f(x) = \log_3 x$ contains the point (27, 3).

85. Graphical Analysis Use a graphing utility to graph f and g in the same viewing window and determine which is increasing at the greater rate as x approaches $+\infty$. What can you conclude about the rate of growth of the natural logarithmic function?

(a) $f(x) = \ln x, \quad g(x) = \sqrt{x}$

(b) $f(x) = \ln x, \quad g(x) = \sqrt[4]{x}$

86. Limit of a Function

(a) Complete the table for the function

$$f(x) = (\ln x)/x.$$

x	1	5	10	10^2	10^4	10^6
$f(x)$						

(b) Use the table in part (a) to determine what value $f(x)$ approaches as x increases without bound.

(c) Use a graphing utility to confirm the result of part (b).

87. Think About It A student obtained the following table of values by evaluating a function. Determine which of the statements may be true and which must be false.

x	1	2	8
y	0	1	3

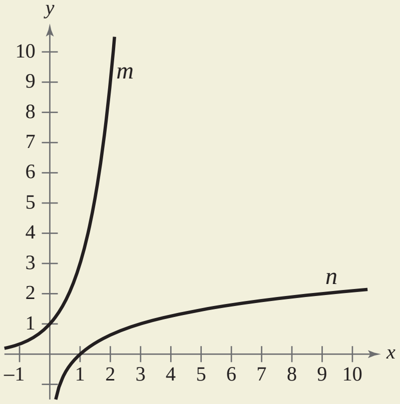
(a) y is an exponential function of x.

(b) y is a logarithmic function of x.

(c) x is an exponential function of y.

(d) y is a linear function of x.

88. HOW DO YOU SEE IT? The figure shows the graphs of $f(x) = 3^x$ and $g(x) = \log_3 x$. [The graphs are labeled m and n.]



(a) Match each function with its graph.

(b) Given that $f(a) = b$, what is $g(b)$? Explain.

89. Writing Explain why $\log_a x$ is defined only for $0 < a < 1$ and $a > 1$.

YAKOBCHUK VASYL/Shutterstock.com

5.3 Properties of Logarithms

Logarithmic functions can help you model and solve real-life problems. For instance, Exercises 85–88 on page 388 use a logarithmic function to model the relationship between the number of decibels and the intensity of a sound.

- **Use the change-of-base formula to rewrite and evaluate logarithmic expressions.**
- **Use properties of logarithms to evaluate or rewrite logarithmic expressions.**
- **Use properties of logarithms to expand or condense logarithmic expressions.**
- **Use logarithmic functions to model and solve real-life problems.**

Change of Base

Most calculators have only two types of log keys, one for common logarithms (base 10) and one for natural logarithms (base e). Although common logarithms and natural logarithms are the most frequently used, you may occasionally need to evaluate logarithms with other bases. To do this, use the following **change-of-base formula.**

Change-of-Base Formula

Let a, b, and x be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then $\log_a x$ can be converted to a different base as follows.

Base b	**Base 10**	**Base e**
$\log_a x = \dfrac{\log_b x}{\log_b a}$	$\log_a x = \dfrac{\log x}{\log a}$	$\log_a x = \dfrac{\ln x}{\ln a}$

One way to look at the change-of-base formula is that logarithms with base a are *constant multiples* of logarithms with base b. The constant multiplier is

$$\frac{1}{\log_b a}.$$

EXAMPLE 1 **Changing Bases Using Common Logarithms**

$$\log_4 25 = \frac{\log 25}{\log 4} \qquad \log_a x = \frac{\log x}{\log a}$$

$$\approx \frac{1.39794}{0.60206} \qquad \text{Use a calculator.}$$

$$\approx 2.3219 \qquad \text{Simplify.}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Evaluate $\log_2 12$ using the change-of-base formula and common logarithms.

EXAMPLE 2 **Changing Bases Using Natural Logarithms**

$$\log_4 25 = \frac{\ln 25}{\ln 4} \qquad \log_a x = \frac{\ln x}{\ln a}$$

$$\approx \frac{3.21888}{1.38629} \qquad \text{Use a calculator.}$$

$$\approx 2.3219 \qquad \text{Simplify.}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Evaluate $\log_2 12$ using the change-of-base formula and natural logarithms. ■

kentoh/Shutterstock.com

Properties of Logarithms

You know from the preceding section that the logarithmic function with base a is the *inverse function* of the exponential function with base a. So, it makes sense that the properties of exponents should have corresponding properties involving logarithms. For instance, the exponential property $a^u a^v = a^{u+v}$ has the corresponding logarithmic property $\log_a(uv) = \log_a u + \log_a v$.

REMARK There is no property that can be used to rewrite $\log_a(u \pm v)$. Specifically, $\log_a(u + v)$ is *not* equal to $\log_a u + \log_a v$.

Properties of Logarithms

Let a be a positive number such that $a \neq 1$, and let n be a real number. If u and v are positive real numbers, then the following properties are true.

	Logarithm with Base a	**Natural Logarithm**
1. Product Property:	$\log_a(uv) = \log_a u + \log_a v$	$\ln(uv) = \ln u + \ln v$
2. Quotient Property:	$\log_a \frac{u}{v} = \log_a u - \log_a v$	$\ln \frac{u}{v} = \ln u - \ln v$
3. Power Property:	$\log_a u^n = n \log_a u$	$\ln u^n = n \ln u$

For proofs of the properties listed above, see Proofs in Mathematics on page 420.

EXAMPLE 3 Using Properties of Logarithms

Write each logarithm in terms of ln 2 and ln 3.

a. $\ln 6$ **b.** $\ln \frac{2}{27}$

Solution

$$\begin{aligned} \textbf{a.}\quad \ln 6 &= \ln(2 \cdot 3) && \text{Rewrite 6 as } 2 \cdot 3. \\ &= \ln 2 + \ln 3 && \text{Product Property} \end{aligned}$$

$$\begin{aligned} \textbf{b.}\quad \ln \frac{2}{27} &= \ln 2 - \ln 27 && \text{Quotient Property} \\ &= \ln 2 - \ln 3^3 && \text{Rewrite 27 as } 3^3. \\ &= \ln 2 - 3 \ln 3 && \text{Power Property} \end{aligned}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write each logarithm in terms of log 3 and log 5.

a. $\log 75$ **b.** $\log \frac{9}{125}$

EXAMPLE 4 Using Properties of Logarithms

Find the exact value of $\log_5 \sqrt[3]{5}$ without using a calculator.

Solution

$$\log_5 \sqrt[3]{5} = \log_5 5^{1/3} = \tfrac{1}{3} \log_5 5 = \tfrac{1}{3}(1) = \tfrac{1}{3}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the exact value of $\ln e^6 - \ln e^2$ without using a calculator.

HISTORICAL NOTE

John Napier, a Scottish mathematician, developed logarithms as a way to simplify tedious calculations. Beginning in 1594, Napier worked about 20 years on the development of logarithms. Napier only partially succeeded in his quest to simplify tedious calculations. Nonetheless, the development of logarithms was a step forward and received immediate recognition.

Mary Evans Picture Library

Rewriting Logarithmic Expressions

The properties of logarithms are useful for rewriting logarithmic expressions in forms that simplify the operations of algebra. This is true because these properties convert complicated products, quotients, and exponential forms into simpler sums, differences, and products, respectively.

EXAMPLE 5 Expanding Logarithmic Expressions

Expand each logarithmic expression.

a. $\log_4 5x^3y$ **b.** $\ln \dfrac{\sqrt{3x-5}}{7}$

Solution

a.
$$\begin{aligned}\log_4 5x^3y &= \log_4 5 + \log_4 x^3 + \log_4 y && \text{Product Property}\\ &= \log_4 5 + 3\log_4 x + \log_4 y && \text{Power Property}\end{aligned}$$

b.
$$\begin{aligned}\ln \frac{\sqrt{3x-5}}{7} &= \ln \frac{(3x-5)^{1/2}}{7} && \text{Rewrite using rational exponent.}\\ &= \ln(3x-5)^{1/2} - \ln 7 && \text{Quotient Property}\\ &= \frac{1}{2}\ln(3x-5) - \ln 7 && \text{Power Property}\end{aligned}$$

▷ **ALGEBRA HELP** You can review rewriting radicals and rational exponents in Section P.2.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Expand the expression $\log_3 \dfrac{4x^2}{\sqrt{y}}$.

Example 5 uses the properties of logarithms to *expand* logarithmic expressions. Example 6 reverses this procedure and uses the properties of logarithms to *condense* logarithmic expressions.

EXAMPLE 6 Condensing Logarithmic Expressions

Condense each logarithmic expression.

a. $\frac{1}{2}\log x + 3\log(x+1)$ **b.** $2\ln(x+2) - \ln x$ **c.** $\frac{1}{3}[\log_2 x + \log_2(x+1)]$

Solution

a.
$$\begin{aligned}\tfrac{1}{2}\log x + 3\log(x+1) &= \log x^{1/2} + \log(x+1)^3 && \text{Power Property}\\ &= \log\left[\sqrt{x}(x+1)^3\right] && \text{Product Property}\end{aligned}$$

b.
$$\begin{aligned}2\ln(x+2) - \ln x &= \ln(x+2)^2 - \ln x && \text{Power Property}\\ &= \ln \frac{(x+2)^2}{x} && \text{Quotient Property}\end{aligned}$$

c.
$$\begin{aligned}\tfrac{1}{3}[\log_2 x + \log_2(x+1)] &= \tfrac{1}{3}\log_2[x(x+1)] && \text{Product Property}\\ &= \log_2[x(x+1)]^{1/3} && \text{Power Property}\\ &= \log_2 \sqrt[3]{x(x+1)} && \text{Rewrite with a radical.}\end{aligned}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Condense the expression $2[\log(x+3) - 2\log(x-2)]$.

Application

One method of determining how the x- and y-values for a set of nonlinear data are related is to take the natural logarithm of each of the x- and y-values. If the points, when graphed, fall on a line, then you can determine that the x- and y-values are related by the equation $\ln y = m \ln x$ where m is the slope of the line.

EXAMPLE 7 Finding a Mathematical Model

The table shows the mean distance from the sun x and the period y (the time it takes a planet to orbit the sun) for each of the six planets that are closest to the sun. In the table, the mean distance is given in terms of astronomical units (where Earth's mean distance is defined as 1.0), and the period is given in years. Find an equation that relates y and x.

DATA

Spreadsheet at LarsonPrecalculus.com

Planet	Mean Distance, x	Period, y
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.524	1.881
Jupiter	5.203	11.860
Saturn	9.555	29.420

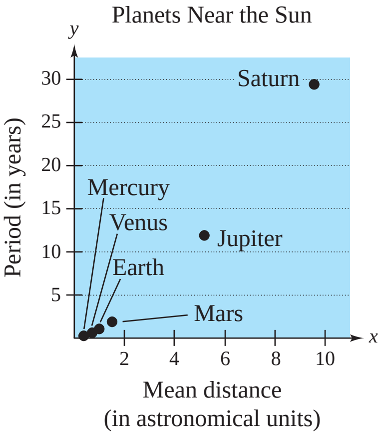


Figure 5.18

Solution Figure 5.18 shows the plots of the points given by the above table. From this figure, it is not clear how to find an equation that relates y and x. To solve this problem, take the natural logarithm of each of the x- and y-values, as shown in the table at the left. Now, by plotting the points in the table at the left, you can see that all six of the points appear to lie in a line (see Figure 5.19). Choose any two points to determine the slope of the line. Using the points $(0.421, 0.632)$ and $(0, 0)$, the slope of the line is

$$m = \frac{0.632 - 0}{0.421 - 0} \approx 1.5 = \frac{3}{2}.$$

By the point-slope form, the equation of the line is $Y = \frac{3}{2}X$, where $Y = \ln y$ and $X = \ln x$. So, $\ln y = \frac{3}{2} \ln x$.

Planet	$\ln x$	$\ln y$
Mercury	-0.949	-1.423
Venus	-0.324	-0.486
Earth	0.000	0.000
Mars	0.421	0.632
Jupiter	1.649	2.473
Saturn	2.257	3.382

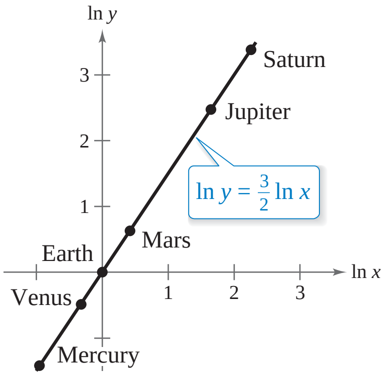


Figure 5.19

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find a logarithmic equation that relates y and x for the following ordered pairs.

$(0.37, 0.51)$, $(1.00, 1.00)$, $(2.72, 1.95)$ $(7.39, 3.79)$, $(20.09, 7.39)$

Summarize (Section 5.3)

1. State the change-of-base formula *(page 383)*. For examples that use the change-of-base formula to rewrite and evaluate logarithmic expressions, see Examples 1 and 2.
2. Make a list of the properties of logarithms *(page 384)*. For examples that use the properties of logarithms to evaluate or rewrite logarithmic expressions, see Examples 3 and 4.
3. Explain how to use the properties of logarithms to expand or condense logarithmic expressions *(page 385)*. For examples of expanding and condensing logarithmic expressions, see Examples 5 and 6.
4. Describe an example of how to use a logarithmic function to model and solve a real-life problem *(page 386, Example 7)*.

5.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary

In Exercises 1–3, fill in the blanks.

1. To evaluate a logarithm to any base, use the ________ formula.

2. The change-of-base formula for base e is given by $\log_a x =$ ________.

3. You can consider $\log_a x$ to be a constant multiple of $\log_b x$; the constant multiplier is ________.

In Exercises 4–6, match the property of logarithms with its name.

4. $\log_a(uv) = \log_a u + \log_a v$ (a) Power Property

5. $\ln u^n = n \ln u$ (b) Quotient Property

6. $\log_a \dfrac{u}{v} = \log_a u - \log_a v$ (c) Product Property

Skills and Applications

Rewriting a Logarithm **In Exercises 7–10, rewrite the logarithm as a ratio of (a) common logarithms and (b) natural logarithms.**

7. $\log_5 16$

8. $\log_{1/5} x$

9. $\log_x \frac{3}{10}$

10. $\log_{2.6} x$

Using the Change-of-Base Formula **In Exercises 11–14, evaluate the logarithm using the change-of-base formula. Round your result to three decimal places.**

11. $\log_3 7$

12. $\log_{1/2} 4$

13. $\log_9 0.1$

14. $\log_3 0.015$

Using Properties of Logarithms **In Exercises 15–20, use the properties of logarithms to rewrite and simplify the logarithmic expression.**

15. $\log_4 8$

16. $\log_2(4^2 \cdot 3^4)$

17. $\log_5 \frac{1}{250}$

18. $\log \frac{9}{300}$

19. $\ln(5e^6)$

20. $\ln \dfrac{6}{e^2}$

Using Properties of Logarithms **In Exercises 21–36, find the exact value of the logarithmic expression without using a calculator. (If this is not possible, then state the reason.)**

21. $\log_3 9$

22. $\log_5 \frac{1}{125}$

23. $\log_2 \sqrt[4]{8}$

24. $\log_6 \sqrt[3]{6}$

25. $\log_4 16^2$

26. $\log_3 81^{-3}$

27. $\log_2(-2)$

28. $\log_3(-27)$

29. $\ln e^{4.5}$

30. $3 \ln e^4$

31. $\ln \dfrac{1}{\sqrt{e}}$

32. $\ln \sqrt[4]{e^3}$

33. $\ln e^2 + \ln e^5$

34. $2 \ln e^6 - \ln e^5$

35. $\log_5 75 - \log_5 3$

36. $\log_4 2 + \log_4 32$

Expanding a Logarithmic Expression **In Exercises 37–58, use the properties of logarithms to expand the expression as a sum, difference, and/or constant multiple of logarithms. (Assume all variables are positive.)**

37. $\ln 4x$

38. $\log_3 10z$

39. $\log_8 x^4$

40. $\log_{10} \dfrac{y}{2}$

41. $\log_5 \dfrac{5}{x}$

42. $\log_6 \dfrac{1}{z^3}$

43. $\ln \sqrt{z}$

44. $\ln \sqrt[3]{t}$

45. $\ln xyz^2$

46. $\log 4x^2 y$

47. $\ln z(z-1)^2, \ z > 1$

48. $\ln\left(\dfrac{x^2-1}{x^3}\right), \ x > 1$

49. $\log_2 \dfrac{\sqrt{a-1}}{9}, \ a > 1$

50. $\ln \dfrac{6}{\sqrt{x^2+1}}$

51. $\ln \sqrt[3]{\dfrac{x}{y}}$

52. $\ln \sqrt{\dfrac{x^2}{y^3}}$

53. $\ln x^2 \sqrt{\dfrac{y}{z}}$

54. $\log_2 x^4 \sqrt{\dfrac{y}{z^3}}$

55. $\log_5 \dfrac{x^2}{y^2 z^3}$

56. $\log_{10} \dfrac{xy^4}{z^5}$

57. $\ln \sqrt[4]{x^3(x^2+3)}$

58. $\ln \sqrt{x^2(x+2)}$

Using Properties of Logarithms **In Exercises 59–66, approximate the logarithm using the properties of logarithms, given $\log_b 2 \approx 0.3562$, $\log_b 3 \approx 0.5646$, and $\log_b 5 \approx 0.8271$.**

59. $\log_b 10$

60. $\log_b \frac{2}{3}$

61. $\log_b 8$

62. $\log_b \sqrt{2}$

63. $\log_b 45$

64. $\log_b(2b)^{-2}$

65. $\log_b(3b^2)$

66. $\log_b \sqrt[3]{3b}$

Condensing a Logarithmic Expression **In Exercises 67–82, condense the expression to the logarithm of a single quantity.**

67. $\ln 2 + \ln x$

68. $\log_5 8 - \log_5 t$

69. $2 \log_2 x + 4 \log_2 y$

70. $\frac{2}{3} \log_7(z - 2)$

71. $\frac{1}{4} \log_3 5x$

72. $-4 \log_6 2x$

73. $\log x - 2 \log(x + 1)$

74. $2 \ln 8 + 5 \ln(z - 4)$

75. $\log x - 2 \log y + 3 \log z$

76. $3 \log_3 x + 4 \log_3 y - 4 \log_3 z$

77. $\ln x - [\ln(x + 1) + \ln(x - 1)]$

78. $4[\ln z + \ln(z + 5)] - 2 \ln(z - 5)$

79. $\frac{1}{3}[2 \ln(x + 3) + \ln x - \ln(x^2 - 1)]$

80. $2[3 \ln x - \ln(x + 1) - \ln(x - 1)]$

81. $\frac{1}{3}[\log_8 y + 2 \log_8(y + 4)] - \log_8(y - 1)$

82. $\frac{1}{2}[\log_4(x + 1) + 2 \log_4(x - 1)] + 6 \log_4 x$

Comparing Logarithmic Quantities **In Exercises 83 and 84, compare the logarithmic quantities. If two are equal, then explain why.**

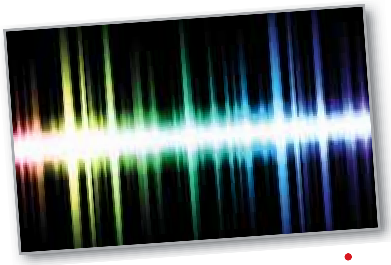
83. $\dfrac{\log_2 32}{\log_2 4}, \quad \log_2 \dfrac{32}{4}, \quad \log_2 32 - \log_2 4$

84. $\log_7 \sqrt{70}, \quad \log_7 35, \quad \frac{1}{2} + \log_7 \sqrt{10}$

Sound Intensity

In Exercises 85–88, use the following information. The relationship between the number of decibels β and the intensity of a sound I in watts per square meter is given by

$$\beta = 10 \log\left(\frac{I}{10^{-12}}\right).$$



85. Use the properties of logarithms to write the formula in simpler form, and determine the number of decibels of a sound with an intensity of 10^{-6} watt per square meter.

86. Find the difference in loudness between an average office with an intensity of 1.26×10^{-7} watt per square meter and a broadcast studio with an intensity of 3.16×10^{-10} watt per square meter.

87. Find the difference in loudness between a vacuum cleaner with an intensity of 10^{-4} watt per square meter and rustling leaves with an intensity of 10^{-11} watt per square meter.

88. You and your roommate are playing your stereos at the same time and at the same intensity. How much louder is the music when both stereos are playing compared with just one stereo playing?

Curve Fitting **In Exercises 89–92, find a logarithmic equation that relates y and x. Explain the steps used to find the equation.**

89.

x	1	2	3	4	5	6
y	1	1.189	1.316	1.414	1.495	1.565

90.

x	1	2	3	4	5	6
y	1	1.587	2.080	2.520	2.924	3.302

91.

x	1	2	3	4	5	6
y	2.5	2.102	1.9	1.768	1.672	1.597

92.

x	1	2	3	4	5	6
y	0.5	2.828	7.794	16	27.951	44.091

93. Galloping Speeds of Animals Four-legged animals run with two different types of motion: trotting and galloping. An animal that is trotting has at least one foot on the ground at all times, whereas an animal that is galloping has all four feet off the ground at some point in its stride. The number of strides per minute at which an animal breaks from a trot to a gallop depends on the weight of the animal. Use the table to find a logarithmic equation that relates an animal's weight x (in pounds) and its lowest galloping speed y (in strides per minute).

DATA

Weight, x	Galloping Speed, y
25	191.5
35	182.7
50	173.8
75	164.2
500	125.9
1000	114.2

Spreadsheet at LarsonPrecalculus.com

94. Nail Length The approximate lengths and diameters (in inches) of common nails are shown in the table. Find a logarithmic equation that relates the diameter y of a common nail to its length x.

Length, x	Diameter, y
1	0.072
2	0.120
3	0.148
4	0.203
5	0.238

kentoh/Shutterstock.com

95. Comparing Models A cup of water at an initial temperature of 78°C is placed in a room at a constant temperature of 21°C. The temperature of the water is measured every 5 minutes during a half-hour period. The results are recorded as ordered pairs of the form (t, T), where t is the time (in minutes) and T is the temperature (in degrees Celsius).

$(0, 78.0°)$, $(5, 66.0°)$, $(10, 57.5°)$, $(15, 51.2°)$,
$(20, 46.3°)$, $(25, 42.4°)$, $(30, 39.6°)$

(a) The graph of the model for the data should be asymptotic with the graph of the temperature of the room. Subtract the room temperature from each of the temperatures in the ordered pairs. Use a graphing utility to plot the data points (t, T) and $(t, T - 21)$.

(b) An exponential model for the data $(t, T - 21)$ is given by $T - 21 = 54.4(0.964)^t$. Solve for T and graph the model. Compare the result with the plot of the original data.

(c) Take the natural logarithms of the revised temperatures. Use the graphing utility to plot the points $(t, \ln(T - 21))$ and observe that the points appear to be linear. Use the *regression* feature of the graphing utility to fit a line to these data. This resulting line has the form $\ln(T - 21) = at + b$. Solve for T, and verify that the result is equivalent to the model in part (b).

(d) Fit a rational model to the data. Take the reciprocals of the y-coordinates of the revised data points to generate the points

$$\left(t, \frac{1}{T - 21}\right).$$

Use the graphing utility to graph these points and observe that they appear to be linear. Use the *regression* feature of the graphing utility to fit a line to these data. The resulting line has the form

$$\frac{1}{T - 21} = at + b.$$

Solve for T, and use the graphing utility to graph the rational function and the original data points.

(e) Why did taking the logarithms of the temperatures lead to a linear scatter plot? Why did taking the reciprocals of the temperatures lead to a linear scatter plot?

Exploration

96. Graphical Analysis Use a graphing utility to graph the functions $y_1 = \ln x - \ln(x - 3)$ and $y_2 = \ln \dfrac{x}{x - 3}$ in the same viewing window. Does the graphing utility show the functions with the same domain? If so, should it? Explain your reasoning.

True or False? **In Exercises 97–102, determine whether the statement is true or false given that $f(x) = \ln x$. Justify your answer.**

97. $f(0) = 0$

98. $f(ax) = f(a) + f(x), \quad a > 0, \quad x > 0$

99. $f(x - 2) = f(x) - f(2), \quad x > 2$

100. $\sqrt{f(x)} = \frac{1}{2}f(x)$

101. If $f(u) = 2f(v)$, then $v = u^2$.

102. If $f(x) < 0$, then $0 < x < 1$.

Using the Change-of-Base Formula **In Exercises 103–106, use the change-of-base formula to rewrite the logarithm as a ratio of logarithms. Then use a graphing utility to graph the ratio.**

103. $f(x) = \log_2 x$

104. $f(x) = \log_{1/2} x$

105. $f(x) = \log_{1/4} x$

106. $f(x) = \log_{11.8} x$

107. Discussion A classmate claims that the following are true.

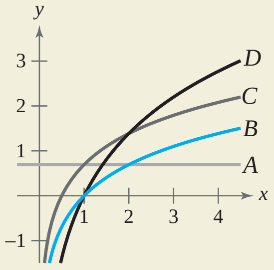
(a) $\ln(u + v) = \ln u + \ln v = \ln(uv)$

(b) $\ln(u - v) = \ln u - \ln v = \ln \dfrac{u}{v}$

(c) $(\ln u)^n = n(\ln u) = \ln u^n$

Discuss how you would demonstrate that these claims are not true.

108. HOW DO YOU SEE IT? The figure shows the graphs of $y = \ln x$, $y = \ln x^2$, $y = \ln 2x$, and $y = \ln 2$. Match each function with its graph. (The graphs are labeled A through D.) Explain your reasoning.



109. Think About It For how many integers between 1 and 20 can you approximate natural logarithms, given the values $\ln 2 \approx 0.6931$, $\ln 3 \approx 1.0986$, and $\ln 5 \approx 1.6094$? Approximate these logarithms (do not use a calculator).

5.4 Exponential and Logarithmic Equations

Exponential and logarithmic equations can help you model and solve life science applications. For instance, Exercise 83 on page 398 uses an exponential function to model the number of trees per acre given the average diameter of the trees.

- **Solve simple exponential and logarithmic equations.**
- **Solve more complicated exponential equations.**
- **Solve more complicated logarithmic equations.**
- **Use exponential and logarithmic equations to model and solve real-life problems.**

Introduction

So far in this chapter, you have studied the definitions, graphs, and properties of exponential and logarithmic functions. In this section, you will study procedures for *solving equations* involving these exponential and logarithmic functions.

There are two basic strategies for solving exponential or logarithmic equations. The first is based on the One-to-One Properties and was used to solve simple exponential and logarithmic equations in Sections 5.1 and 5.2. The second is based on the Inverse Properties. For $a > 0$ and $a \neq 1$, the following properties are true for all x and y for which $\log_a x$ and $\log_a y$ are defined.

One-to-One Properties	**Inverse Properties**
$a^x = a^y$ if and only if $x = y$.	$a^{\log_a x} = x$
$\log_a x = \log_a y$ if and only if $x = y$.	$\log_a a^x = x$

EXAMPLE 1 Solving Simple Equations

Original Equation	**Rewritten Equation**	**Solution**	**Property**
a. $2^x = 32$	$2^x = 2^5$	$x = 5$	One-to-One
b. $\ln x - \ln 3 = 0$	$\ln x = \ln 3$	$x = 3$	One-to-One
c. $\left(\frac{1}{3}\right)^x = 9$	$3^{-x} = 3^2$	$x = -2$	One-to-One
d. $e^x = 7$	$\ln e^x = \ln 7$	$x = \ln 7$	Inverse
e. $\ln x = -3$	$e^{\ln x} = e^{-3}$	$x = e^{-3}$	Inverse
f. $\log x = -1$	$10^{\log x} = 10^{-1}$	$x = 10^{-1} = \frac{1}{10}$	Inverse
g. $\log_3 x = 4$	$3^{\log_3 x} = 3^4$	$x = 81$	Inverse

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve each equation for x.

a. $2^x = 512$ **b.** $\log_6 x = 3$ **c.** $5 - e^x = 0$ **d.** $9^x = \frac{1}{3}$

The strategies used in Example 1 are summarized as follows.

Strategies for Solving Exponential and Logarithmic Equations

1. Rewrite the original equation in a form that allows the use of the One-to-One Properties of exponential or logarithmic functions.
2. Rewrite an *exponential* equation in logarithmic form and apply the Inverse Property of logarithmic functions.
3. Rewrite a *logarithmic* equation in exponential form and apply the Inverse Property of exponential functions.

goran cakmazovic/Shutterstock.com

Solving Exponential Equations

EXAMPLE 2 Solving Exponential Equations

Solve each equation and approximate the result to three decimal places, if necessary.

a. $e^{-x^2} = e^{-3x-4}$ **b.** $3(2^x) = 42$

Solution

a.

$$e^{-x^2} = e^{-3x-4} \qquad \text{Write original equation.}$$

$$-x^2 = -3x - 4 \qquad \text{One-to-One Property}$$

$$x^2 - 3x - 4 = 0 \qquad \text{Write in general form.}$$

$$(x + 1)(x - 4) = 0 \qquad \text{Factor.}$$

$$(x + 1) = 0 \Rightarrow x = -1 \qquad \text{Set 1st factor equal to 0.}$$

$$(x - 4) = 0 \Rightarrow x = 4 \qquad \text{Set 2nd factor equal to 0.}$$

The solutions are $x = -1$ and $x = 4$. Check these in the original equation.

b.

$$3(2^x) = 42 \qquad \text{Write original equation.}$$

$$2^x = 14 \qquad \text{Divide each side by 3.}$$

$$\log_2 2^x = \log_2 14 \qquad \text{Take log (base 2) of each side.}$$

$$x = \log_2 14 \qquad \text{Inverse Property}$$

$$x = \frac{\ln 14}{\ln 2} \approx 3.807 \qquad \text{Change-of-base formula}$$

The solution is $x = \log_2 14 \approx 3.807$. Check this in the original equation.

REMARK Another way to solve Example 2(b) is by taking the natural log of each side and then applying the Power Property, as follows.

$$3(2^x) = 42$$

$$2^x = 14$$

$$\ln 2^x = \ln 14$$

$$x \ln 2 = \ln 14$$

$$x = \frac{\ln 14}{\ln 2} \approx 3.807$$

Notice that you obtain the same result as in Example 2(b).

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve each equation and approximate the result to three decimal places, if necessary.

a. $e^{2x} = e^{x^2-8}$ **b.** $2(5^x) = 32$

In Example 2(b), the exact solution is $x = \log_2 14$, and the approximate solution is $x \approx 3.807$. An exact answer is preferred when the solution is an intermediate step in a larger problem. For a final answer, an approximate solution is easier to comprehend.

EXAMPLE 3 Solving an Exponential Equation

Solve $e^x + 5 = 60$ and approximate the result to three decimal places.

Solution

$$e^x + 5 = 60 \qquad \text{Write original equation.}$$

$$e^x = 55 \qquad \text{Subtract 5 from each side.}$$

$$\ln e^x = \ln 55 \qquad \text{Take natural log of each side.}$$

$$x = \ln 55 \approx 4.007 \qquad \text{Inverse Property}$$

The solution is $x = \ln 55 \approx 4.007$. Check this in the original equation.

REMARK Remember that the natural logarithmic function has a base of e.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $e^x - 7 = 23$ and approximate the result to three decimal places.

EXAMPLE 4 Solving an Exponential Equation

Solve $2(3^{2t-5}) - 4 = 11$ and approximate the result to three decimal places.

Solution

$$\begin{aligned} 2(3^{2t-5}) - 4 &= 11 && \text{Write original equation.} \\ 2(3^{2t-5}) &= 15 && \text{Add 4 to each side.} \\ 3^{2t-5} &= \frac{15}{2} && \text{Divide each side by 2.} \\ \log_3 3^{2t-5} &= \log_3 \frac{15}{2} && \text{Take log (base 3) of each side.} \\ 2t - 5 &= \log_3 \frac{15}{2} && \text{Inverse Property} \\ 2t &= 5 + \log_3 7.5 && \text{Add 5 to each side.} \\ t &= \frac{5}{2} + \frac{1}{2}\log_3 7.5 && \text{Divide each side by 2.} \\ t &\approx 3.417 && \text{Use a calculator.} \end{aligned}$$

The solution is $t = \frac{5}{2} + \frac{1}{2}\log_3 7.5 \approx 3.417$. Check this in the original equation.

REMARK Remember that to evaluate a logarithm such as $\log_3 7.5$, you need to use the change-of-base formula.

$$\log_3 7.5 = \frac{\ln 7.5}{\ln 3} \approx 1.834$$

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $6(2^{t+5}) + 4 = 11$ and approximate the result to three decimal places.

When an equation involves two or more exponential expressions, you can still use a procedure similar to that demonstrated in Examples 2, 3, and 4. However, the algebra is a bit more complicated.

EXAMPLE 5 Solving an Exponential Equation of Quadratic Type

Solve $e^{2x} - 3e^x + 2 = 0$.

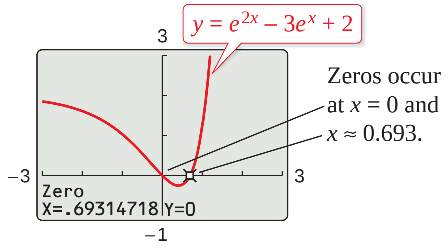
Algebraic Solution

$$\begin{aligned} e^{2x} - 3e^x + 2 &= 0 && \text{Write original equation.} \\ (e^x)^2 - 3e^x + 2 &= 0 && \text{Write in quadratic form.} \\ (e^x - 2)(e^x - 1) &= 0 && \text{Factor.} \\ e^x - 2 &= 0 && \text{Set 1st factor equal to 0.} \\ x &= \ln 2 && \text{Solution} \\ e^x - 1 &= 0 && \text{Set 2nd factor equal to 0.} \\ x &= 0 && \text{Solution} \end{aligned}$$

The solutions are $x = \ln 2 \approx 0.693$ and $x = 0$. Check these in the original equation.

Graphical Solution

Use a graphing utility to graph $y = e^{2x} - 3e^x + 2$ and then find the zeros.



So, you can conclude that the solutions are $x = 0$ and $x \approx 0.693$.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $e^{2x} - 7e^x + 12 = 0$.

Solving Logarithmic Equations

To solve a logarithmic equation, you can write it in exponential form.

$\ln x = 3$ Logarithmic form

$e^{\ln x} = e^3$ Exponentiate each side.

$x = e^3$ Exponential form

This procedure is called *exponentiating* each side of an equation.

REMARK Remember to check your solutions in the original equation when solving equations to verify that the answer is correct and to make sure that the answer is in the domain of the original equation.

EXAMPLE 6 Solving Logarithmic Equations

a. $\ln x = 2$ Original equation

$e^{\ln x} = e^2$ Exponentiate each side.

$x = e^2$ Inverse Property

b. $\log_3(5x - 1) = \log_3(x + 7)$ Original equation

$5x - 1 = x + 7$ One-to-One Property

$x = 2$ Solution

c. $\log_6(3x + 14) - \log_6 5 = \log_6 2x$ Original equation

$\log_6\left(\frac{3x + 14}{5}\right) = \log_6 2x$ Quotient Property of Logarithms

$\frac{3x + 14}{5} = 2x$ One-to-One Property

$3x + 14 = 10x$ Multiply each side by 5.

$x = 2$ Solution

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve each equation.

a. $\ln x = \frac{2}{3}$ **b.** $\log_2(2x - 3) = \log_2(x + 4)$ **c.** $\log 4x - \log(12 + x) = \log 2$

EXAMPLE 7 Solving a Logarithmic Equation

Solve $5 + 2 \ln x = 4$ and approximate the result to three decimal places.

Algebraic Solution

$5 + 2 \ln x = 4$ Write original equation.

$2 \ln x = -1$ Subtract 5 from each side.

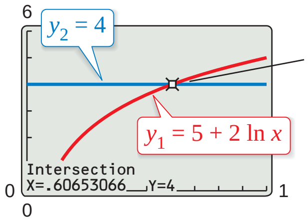
$\ln x = -\frac{1}{2}$ Divide each side by 2.

$e^{\ln x} = e^{-1/2}$ Exponentiate each side.

$x = e^{-1/2}$ Inverse Property

$x \approx 0.607$ Use a calculator.

Graphical Solution



The intersection point is about (0.607, 4). So, the solution is $x \approx 0.607$.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $7 + 3 \ln x = 5$ and approximate the result to three decimal places. ■

EXAMPLE 8 Solving a Logarithmic Equation

Solve $2 \log_5 3x = 4$.

Solution

$$\begin{aligned} 2 \log_5 3x &= 4 && \text{Write original equation.} \\ \log_5 3x &= 2 && \text{Divide each side by 2.} \\ 5^{\log_5 3x} &= 5^2 && \text{Exponentiate each side (base 5).} \\ 3x &= 25 && \text{Inverse Property} \\ x &= \frac{25}{3} && \text{Divide each side by 3.} \end{aligned}$$

The solution is $x = \frac{25}{3}$. Check this in the original equation.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $3 \log_4 6x = 9$.

Because the domain of a logarithmic function generally does not include all real numbers, you should be sure to check for extraneous solutions of logarithmic equations.

EXAMPLE 9 Checking for Extraneous Solutions

Solve $\log 5x + \log(x - 1) = 2$.

Algebraic Solution

$$\begin{aligned} \log 5x + \log(x - 1) &= 2 && \text{Write original equation.} \\ \log[5x(x - 1)] &= 2 && \text{Product Property of Logarithms} \\ 10^{\log(5x^2 - 5x)} &= 10^2 && \text{Exponentiate each side (base 10).} \\ 5x^2 - 5x &= 100 && \text{Inverse Property} \\ x^2 - x - 20 &= 0 && \text{Write in general form.} \\ (x - 5)(x + 4) &= 0 && \text{Factor.} \\ x - 5 &= 0 && \text{Set 1st factor equal to 0.} \\ x &= 5 && \text{Solution} \\ x + 4 &= 0 && \text{Set 2nd factor equal to 0.} \\ x &= -4 && \text{Solution} \end{aligned}$$

The solutions appear to be $x = 5$ and $x = -4$. However, when you check these in the original equation, you can see that $x = 5$ is the only solution.

Graphical Solution

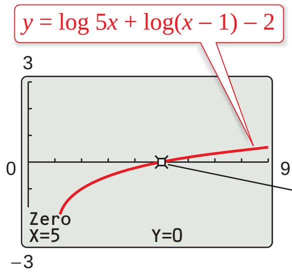
First, rewrite the original equation as

$$\log 5x + \log(x - 1) - 2 = 0.$$

Then use a graphing utility to graph the equation

$$y = \log 5x + \log(x - 1) - 2$$

and find the zero(s).



A zero occurs at $x = 5$. So, the solution is $x = 5$.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $\log x + \log(x - 9) = 1$.

REMARK Notice in Example 9 that the logarithmic part of the equation is condensed into a single logarithm before exponentiating each side of the equation.

In Example 9, the domain of $\log 5x$ is $x > 0$ and the domain of $\log(x - 1)$ is $x > 1$, so the domain of the original equation is

$$x > 1.$$

Because the domain is all real numbers greater than 1, the solution $x = -4$ is extraneous. The graphical solution verifies this conclusion.

Applications

EXAMPLE 10 Doubling an Investment

You invest \$500 at an annual interest rate of 6.75%, compounded continuously. How long will it take your money to double?

Solution Using the formula for continuous compounding, the balance is

$$A = Pe^{rt}$$

$$A = 500e^{0.0675t}.$$

To find the time required for the balance to double, let $A = 1000$ and solve the resulting equation for t.

$$500e^{0.0675t} = 1000 \qquad \text{Let } A = 1000.$$

$$e^{0.0675t} = 2 \qquad \text{Divide each side by 500.}$$

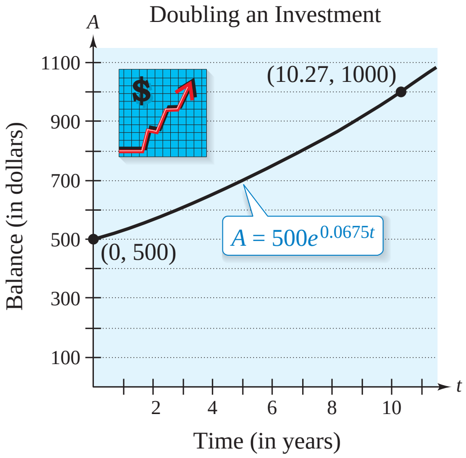
$$\ln e^{0.0675t} = \ln 2 \qquad \text{Take natural log of each side.}$$

$$0.0675t = \ln 2 \qquad \text{Inverse Property}$$

$$t = \frac{\ln 2}{0.0675} \qquad \text{Divide each side by 0.0675.}$$

$$t \approx 10.27 \qquad \text{Use a calculator.}$$

The balance in the account will double after approximately 10.27 years. This result is demonstrated graphically below.



✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

You invest \$500 at an annual interest rate of 5.25%, compounded continuously. How long will it take your money to double? Compare your result with that of Example 10.

In Example 10, an approximate answer of 10.27 years is given. Within the context of the problem, the exact solution

$$t = \frac{\ln 2}{0.0675}$$

does not make sense as an answer.

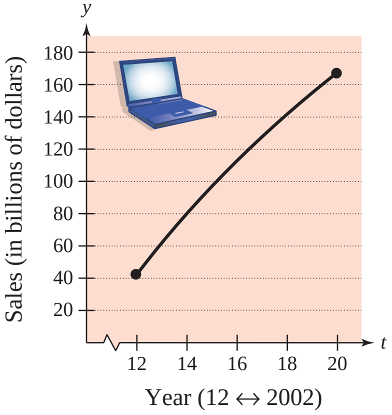
EXAMPLE 11 Retail Sales

The retail sales y (in billions of dollars) of e-commerce companies in the United States from 2002 through 2010 can be modeled by

$$y = -566 + 244.7 \ln t, \quad 12 \le t \le 20$$

where t represents the year, with $t = 12$ corresponding to 2002 (see figure). During which year did the sales reach $141 billion? *(Source: U.S. Census Bureau)*

Retail Sales of e-Commerce Companies



Solution

$$-566 + 244.7 \ln t = y \qquad \text{Write original equation.}$$

$$-566 + 244.7 \ln t = 141 \qquad \text{Substitute 141 for } y.$$

$$244.7 \ln t = 707 \qquad \text{Add 566 to each side.}$$

$$\ln t = \frac{707}{244.7} \qquad \text{Divide each side by 244.7.}$$

$$e^{\ln t} = e^{707/244.7} \qquad \text{Exponentiate each side.}$$

$$t = e^{707/244.7} \qquad \text{Inverse Property}$$

$$t \approx 18 \qquad \text{Use a calculator.}$$

The solution is $t \approx 18$. Because $t = 12$ represents 2002, it follows that the sales reached $141 billion in 2008.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

In Example 11, during which year did the sales reach $80 billion?

Summarize (Section 5.4)

1. State the One-to-One Properties and the Inverse Properties that can help you solve simple exponential and logarithmic equations *(page 390)*.
   For an example of solving simple exponential and logarithmic equations, see Example 1.
2. Describe strategies for solving exponential equations *(pages 391 and 392)*.
   For examples of solving exponential equations, see Examples 2–5.
3. Describe strategies for solving logarithmic equations *(pages 393 and 394)*.
   For examples of solving logarithmic equations, see Examples 6–9.
4. Describe examples of how to use exponential and logarithmic equations to model and solve real-life problems *(pages 395 and 396, Examples 10 and 11)*.

5.4 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: **Fill in the blanks.**

1. To solve exponential and logarithmic equations, you can use the following One-to-One and Inverse Properties.

(a) $a^x = a^y$ if and only if ________.

(b) $\log_a x = \log_a y$ if and only if ________.

(c) $a^{\log_a x} =$ ________

(d) $\log_a a^x =$ ________

2. An ________ solution does not satisfy the original equation.

Skills and Applications

Determining Solutions **In Exercises 3–6, determine whether each x-value is a solution (or an approximate solution) of the equation.**

3. $4^{2x-7} = 64$

(a) $x = 5$

(b) $x = 2$

4. $4e^{x-1} = 60$

(a) $x = 1 + \ln 15$

(b) $x = \ln 16$

5. $\log_2(x + 3) = 10$

(a) $x = 1021$

(b) $x = 17$

(c) $x = 10^2 - 3$

6. $\ln(2x + 3) = 5.8$

(a) $x = \frac{1}{2}(-3 + \ln 5.8)$

(b) $x = \frac{1}{2}(-3 + e^{5.8})$

(c) $x \approx 163.650$

Solving a Simple Equation **In Exercises 7–14, solve for x.**

7. $4^x = 16$

8. $\left(\frac{1}{2}\right)^x = 32$

9. $\ln x - \ln 2 = 0$

10. $e^x = 2$

11. $\ln x = -1$

12. $\log x = -2$

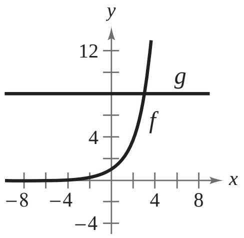
13. $\log_4 x = 3$

14. $\log_5 x = \frac{1}{2}$

Approximating a Point of Intersection **In Exercises 15 and 16, approximate the point of intersection of the graphs of f and g. Then solve the equation $f(x) = g(x)$ algebraically to verify your approximation.**

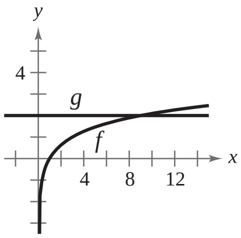
15. $f(x) = 2^x$

$g(x) = 8$



16. $f(x) = \log_3 x$

$g(x) = 2$



Solving an Exponential Equation **In Exercises 17–44, solve the exponential equation algebraically. Approximate the result to three decimal places.**

17. $e^x = e^{x^2 - 2}$

18. $e^{x^2 - 3} = e^{x - 2}$

19. $4(3^x) = 20$

20. $4e^x = 91$

21. $e^x - 9 = 19$

22. $6^x + 10 = 47$

23. $3^{2x} = 80$

24. $4^{-3t} = 0.10$

25. $2^{3-x} = 565$

26. $8^{-2-x} = 431$

27. $8(10^{3x}) = 12$

28. $8(3^{6-x}) = 40$

29. $e^{3x} = 12$

30. $1000e^{-4x} = 75$

31. $7 - 2e^x = 5$

32. $-14 + 3e^x = 11$

33. $6(2^{3x-1}) - 7 = 9$

34. $8(4^{6-2x}) + 13 = 41$

35. $2^x = 3^{x+1}$

36. $2^{x+1} = e^{1-x}$

37. $4^x = 5^{x^2}$

38. $3^{x^2} = 7^{6-x}$

39. $e^{2x} - 4e^x - 5 = 0$

40. $e^{2x} - 5e^x + 6 = 0$

41. $\dfrac{500}{100 - e^{x/2}} = 20$

42. $\dfrac{400}{1 + e^{-x}} = 350$

43. $\left(1 + \dfrac{0.065}{365}\right)^{365t} = 4$

44. $\left(1 + \dfrac{0.10}{12}\right)^{12t} = 2$

Solving a Logarithmic Equation **In Exercises 45–62, solve the logarithmic equation algebraically. Approximate the result to three decimal places.**

45. $\ln x = -3$

46. $\ln x - 7 = 0$

47. $2.1 = \ln 6x$

48. $\log 3z = 2$

49. $3 \ln 5x = 10$

50. $\ln\sqrt{x - 8} = 5$

51. $2 - 6 \ln x = 10$

52. $2 + 3 \ln x = 12$

53. $6 \log_3(0.5x) = 11$

54. $4 \log(x - 6) = 11$

55. $\ln x - \ln(x + 1) = 2$

56. $\ln x + \ln(x + 1) = 1$

57. $\ln(x + 5) = \ln(x - 1) - \ln(x + 1)$

58. $\ln(x + 1) - \ln(x - 2) = \ln x$

59. $\log(3x + 4) = \log(x - 10)$

60. $\log_2 x + \log_2(x + 2) = \log_2(x + 6)$

61. $\log_4 x - \log_4(x - 1) = \frac{1}{2}$

62. $\log 8x - \log\left(1 + \sqrt{x}\right) = 2$

Graphing and Solving an Equation **In Exercises 63–70, use a graphing utility to graph and solve the equation. Approximate the result to three decimal places. Verify your result algebraically.**

63. $5^x = 212$

64. $6e^{1-x} = 25$

65. $8e^{-2x/3} = 11$

66. $e^{0.09t} = 3$

67. $3 - \ln x = 0$

68. $10 - 4\ln(x - 2) = 0$

69. $2\ln(x + 3) = 3$

70. $\ln(x + 1) = 2 - \ln x$

Compound Interest **In Exercises 71 and 72, you invest \$2500 in an account at interest rate *r*, compounded continuously. Find the time required for the amount to (a) double and (b) triple.**

71. $r = 0.025$

72. $r = 0.0375$

Algebra of Calculus **In Exercises 73–80, solve the equation algebraically. Round your result to three decimal places. Verify your answer using a graphing utility.**

73. $2x^2e^{2x} + 2xe^{2x} = 0$

74. $-x^2e^{-x} + 2xe^{-x} = 0$

75. $-xe^{-x} + e^{-x} = 0$

76. $e^{-2x} - 2xe^{-2x} = 0$

77. $2x\ln x + x = 0$

78. $\dfrac{1 - \ln x}{x^2} = 0$

79. $\dfrac{1 + \ln x}{2} = 0$

80. $2x\ln\left(\dfrac{1}{x}\right) - x = 0$

81. Average Heights The percent m of American males between the ages of 20 and 29 who are under x inches tall is modeled by

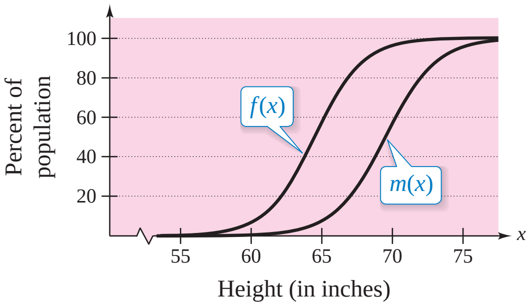
$$m(x) = \frac{100}{1 + e^{-0.5536(x - 69.51)}}, \quad 64 \le x \le 78$$

and the percent f of American females between the ages of 20 and 29 who are under x inches tall is modeled by

$$f(x) = \frac{100}{1 + e^{-0.5834(x - 64.49)}}, \quad 60 \le x \le 78.$$

(Source: U.S. National Center for Health Statistics)

(a) Use the graph to determine any horizontal asymptotes of the graphs of the functions. Interpret the meaning in the context of the problem.



(b) What is the average height of each sex?

James Marshall/CORBIS

82. U.S. Currency The values y (in billions of dollars) of U.S. currency in circulation in the years 2000 through 2010 can be modeled by $y = -611 + 507\ln t$, $10 \le t \le 20$, where t represents the year, with $t = 10$ corresponding to 2000. During which year did the value of U.S. currency in circulation exceed \$690 billion? *(Source: Board of Governors of the Federal Reserve System)*

83. Trees per Acre The number N of trees of a given species per acre is approximated by the model $N = 68(10^{-0.04x})$, $5 \le x \le 40$, where x is the average diameter of the trees (in inches) 3 feet above the ground. Use the model to approximate the average diameter of the trees in a test plot when $N = 21$.

84. Demand The demand equation for a smart phone is

$$p = 5000\left(1 - \frac{4}{4 + e^{-0.002x}}\right).$$

Find the demand x for a price of (a) $p = \$169$ and (b) $p = \$299$.

85. Automobiles Engineers design automobiles with crumple zones that help protect their occupants in crashes. The crumple zones allow the occupants to move short distances when the automobiles come to abrupt stops. The greater the distance moved, the fewer g's the crash victims experience. (One g is equal to the acceleration due to gravity.) In crash tests with vehicles moving at 90 kilometers per hour, analysts measured the numbers of g's experienced during deceleration by crash dummies that were permitted to move x meters during impact. The table shows the data. A model for the data is given by $y = -3.00 + 11.88\ln x + (36.94/x)$, where y is the number of g's.

x	0.2	0.4	0.6	0.8	1.0
g's	158	80	53	40	32

(a) Complete the table using the model.

x	0.2	0.4	0.6	0.8	1.0
y					

(b) Use a graphing utility to graph the data points and the model in the same viewing window. How do they compare?

(c) Use the model to estimate the distance traveled during impact, assuming that the passenger deceleration must not exceed 30 g's.

(d) Do you think it is practical to lower the number of g's experienced during impact to fewer than 23? Explain your reasoning.

86. **Data Analysis** An object at a temperature of 160°C was removed from a furnace and placed in a room at 20°C. The temperature T of the object was measured each hour h and recorded in the table. A model for the data is given by $T = 20[1 + 7(2^{-h})]$. The figure shows the graph of this model.

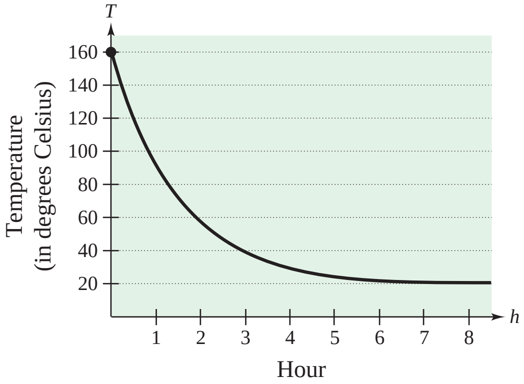
DATA

Hour, h	Temperature, T
0	160°
1	90°
2	56°
3	38°
4	29°
5	24°

Spreadsheet at LarsonPrecalculus.com

(a) Use the graph to identify the horizontal asymptote of the model and interpret the asymptote in the context of the problem.

(b) Use the model to approximate the time when the temperature of the object was 100°C.



Exploration

True or False? **In Exercises 87–90, rewrite each verbal statement as an equation. Then decide whether the statement is true or false. Justify your answer.**

87. The logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers.

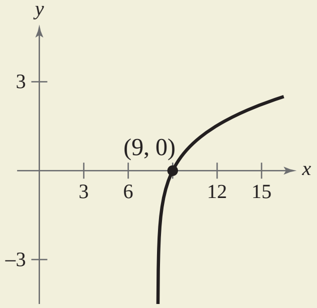
88. The logarithm of the sum of two numbers is equal to the product of the logarithms of the numbers.

89. The logarithm of the difference of two numbers is equal to the difference of the logarithms of the numbers.

90. The logarithm of the quotient of two numbers is equal to the difference of the logarithms of the numbers.

91. **Think About It** Is it possible for a logarithmic equation to have more than one extraneous solution? Explain.

92. **HOW DO YOU SEE IT?** Solving $\log_3 x + \log_3(x - 8) = 2$ algebraically, the solutions appear to be $x = 9$ and $x = -1$. Use the graph of $y = \log_3 x + \log_3(x - 8) - 2$ to determine whether each value is an actual solution of the equation. Explain your reasoning.



93. **Finance** You are investing P dollars at an annual interest rate of r, compounded continuously, for t years. Which of the following would result in the highest value of the investment? Explain your reasoning.

(a) Double the amount you invest.

(b) Double your interest rate.

(c) Double the number of years.

94. **Think About It** Are the times required for the investments in Exercises 71 and 72 to quadruple twice as long as the times for them to double? Give a reason for your answer and verify your answer algebraically.

95. **Effective Yield** The *effective yield* of an investment plan is the percent increase in the balance after 1 year. Find the effective yield for each investment plan. Which investment plan has the greatest effective yield? Which investment plan will have the highest balance after 5 years?

(a) 7% annual interest rate, compounded annually

(b) 7% annual interest rate, compounded continuously

(c) 7% annual interest rate, compounded quarterly

(d) 7.25% annual interest rate, compounded quarterly

96. **Graphical Analysis** Let $f(x) = \log_a x$ and $g(x) = a^x$, where $a > 1$.

(a) Let $a = 1.2$ and use a graphing utility to graph the two functions in the same viewing window. What do you observe? Approximate any points of intersection of the two graphs.

(b) Determine the value(s) of a for which the two graphs have one point of intersection.

(c) Determine the value(s) of a for which the two graphs have two points of intersection.

5.5 Exponential and Logarithmic Models

Exponential growth and decay models can often represent the populations of countries. For instance, in Exercise 30 on page 408, you will use exponential growth and decay models to compare the populations of several countries.

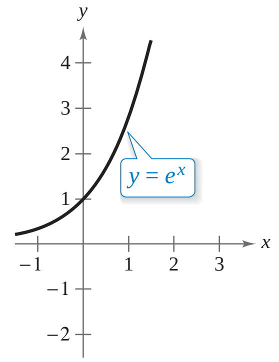
- **Recognize the five most common types of models involving exponential and logarithmic functions.**
- **Use exponential growth and decay functions to model and solve real-life problems.**
- **Use Gaussian functions to model and solve real-life problems.**
- **Use logistic growth functions to model and solve real-life problems.**
- **Use logarithmic functions to model and solve real-life problems.**

Introduction

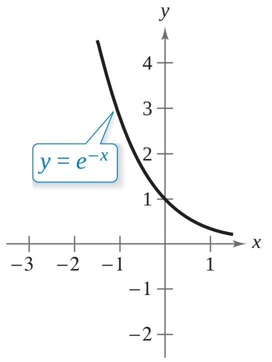
The five most common types of mathematical models involving exponential functions and logarithmic functions are as follows.

1. **Exponential growth model:** $y = ae^{bx}, \quad b > 0$
2. **Exponential decay model:** $y = ae^{-bx}, \quad b > 0$
3. **Gaussian model:** $y = ae^{-(x-b)^2/c}$
4. **Logistic growth model:** $y = \dfrac{a}{1 + be^{-rx}}$
5. **Logarithmic models:** $y = a + b \ln x, \quad y = a + b \log x$

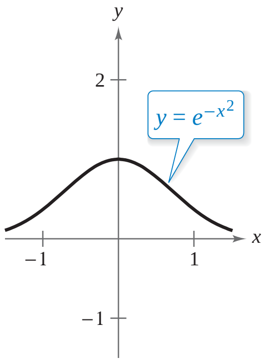
The basic shapes of the graphs of these functions are as follows.



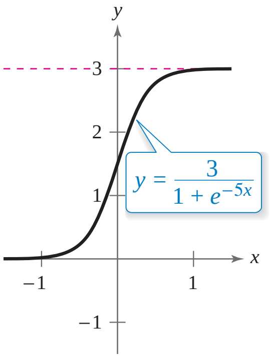
Exponential growth model



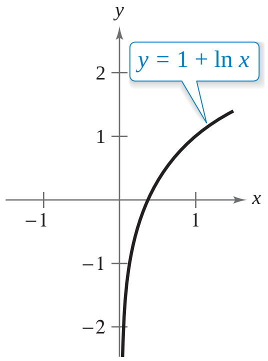
Exponential decay model



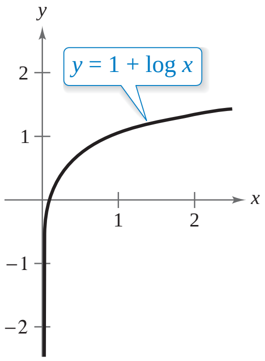
Gaussian model



Logistic growth model



Natural logarithmic model



Common logarithmic model

You often gain insight into a situation modeled by an exponential or logarithmic function by identifying and interpreting the asymptotes of the graph of the function. Identify the asymptote(s) of the graph of each function shown above.

Alan Becker/Stone/Getty Images

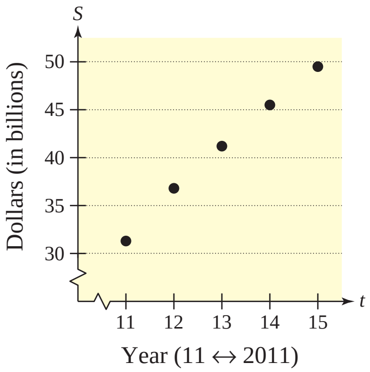
Exponential Growth and Decay

EXAMPLE 1 Online Advertising

Estimates of the amounts (in billions of dollars) of U.S. online advertising spending from 2011 through 2015 are shown in the table. A scatter plot of the data is shown at the right. *(Source: eMarketer)*

Year	2011	2012	2013	2014	2015
Advertising Spending	31.3	36.8	41.2	45.5	49.5

Online Advertising Spending



An exponential growth model that approximates the data is given by

$$S = 9.30e^{0.1129t}, \quad 11 \le t \le 15$$

where S is the amount of spending (in billions of dollars) and $t = 11$ represents 2011. Compare the values given by the model with the estimates shown in the table. According to this model, when will the amount of U.S. online advertising spending reach \$80 billion?

Algebraic Solution

The following table compares the two sets of advertising spending amounts.

Year	2011	2012	2013	2014	2015
Advertising Spending	31.3	36.8	41.2	45.5	49.5
Model	32.2	36.0	40.4	45.2	50.6

To find when the amount of U.S. online advertising spending will reach \$80 billion, let $S = 80$ in the model and solve for t.

$9.30e^{0.1129t} = S$ Write original model.

$9.30e^{0.1129t} = 80$ Substitute 80 for S.

$e^{0.1129t} \approx 8.6022$ Divide each side by 9.30.

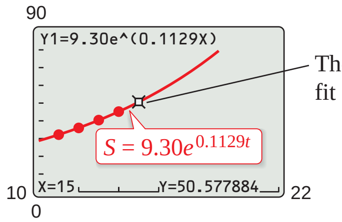
$\ln e^{0.1129t} \approx \ln 8.6022$ Take natural log of each side.

$0.1129t \approx 2.1520$ Inverse Property

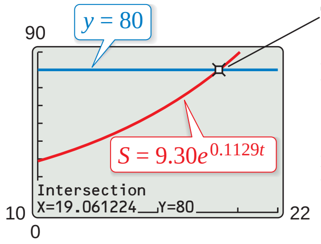
$t \approx 19.1$ Divide each side by 0.1129.

According to the model, the amount of U.S. online advertising spending will reach \$80 billion in 2019.

Graphical Solution



The model appears to fit the data closely.



The intersection point of the model and the line $y = 80$ is about (19.1, 80). So, according to the model, the amount of U.S. online advertising spending will reach \$80 billion in 2019.

✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

In Example 1, when will the amount of U.S. online advertising spending reach \$100 billion?

TECHNOLOGY Some graphing utilities have an *exponential regression* feature that can help you find exponential models to represent data. If you have such a graphing utility, try using it to find an exponential model for the data given in Example 1. How does your model compare with the model given in Example 1?

In Example 1, the exponential growth model is given. But if this model were not given, then how would you find such a model? Example 2 demonstrates one technique.

EXAMPLE 2 Modeling Population Growth

In a research experiment, a population of fruit flies is increasing according to the law of exponential growth. After 2 days there are 100 flies, and after 4 days there are 300 flies. How many flies will there be after 5 days?

Solution Let y be the number of flies at time t. From the given information, you know that $y = 100$ when $t = 2$ and $y = 300$ when $t = 4$. Substituting this information into the model $y = ae^{bt}$ produces

$$100 = ae^{2b} \quad \text{and} \quad 300 = ae^{4b}.$$

To solve for b, solve for a in the first equation.

$$\begin{aligned} 100 &= ae^{2b} && \text{Write first equation.} \\ \frac{100}{e^{2b}} &= a && \text{Solve for } a. \end{aligned}$$

Then substitute the result into the second equation.

$$\begin{aligned} 300 &= ae^{4b} && \text{Write second equation.} \\ 300 &= \left(\frac{100}{e^{2b}}\right)e^{4b} && \text{Substitute } \frac{100}{e^{2b}} \text{ for } a. \\ \frac{300}{100} &= e^{2b} && \text{Simplify, and divide each side by 100.} \\ \ln 3 &= 2b && \text{Take natural log of each side.} \\ \frac{1}{2}\ln 3 &= b && \text{Solve for } b. \end{aligned}$$

Using $b = \frac{1}{2}\ln 3$ and the equation you found for a,

$$\begin{aligned} a &= \frac{100}{e^{2[(1/2)\ln 3]}} && \text{Substitute } \tfrac{1}{2}\ln 3 \text{ for } b. \\ &= \frac{100}{e^{\ln 3}} && \text{Simplify.} \\ &= \frac{100}{3} && \text{Inverse Property} \\ &\approx 33.33. && \text{Divide.} \end{aligned}$$

So, with $a \approx 33.33$ and $b = \frac{1}{2}\ln 3 \approx 0.5493$, the exponential growth model is

$$y = 33.33e^{0.5493t}$$

as shown in Figure 5.20. After 5 days, the population will be

$$\begin{aligned} y &= 33.33e^{0.5493(5)} \\ &\approx 520 \text{ flies.} \end{aligned}$$

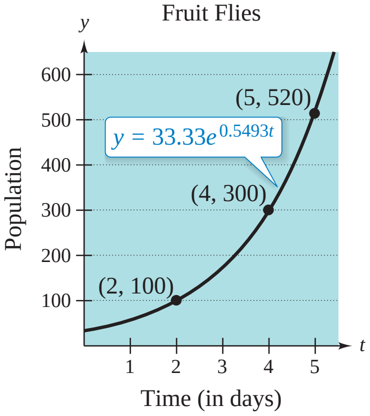
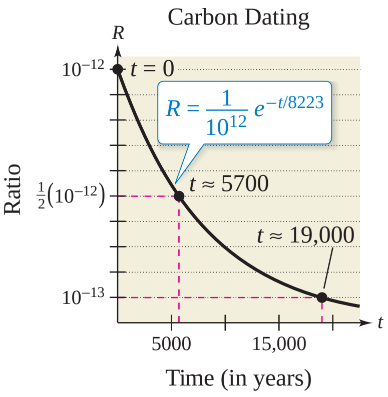


Figure 5.20

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The number of bacteria in a culture is increasing according to the law of exponential growth. After 1 hour there are 100 bacteria, and after 2 hours there are 200 bacteria. How many bacteria will there be after 3 hours?

In living organic material, the ratio of the number of radioactive carbon isotopes (carbon 14) to the number of nonradioactive carbon isotopes (carbon 12) is about 1 to 10^{12}. When organic material dies, its carbon 12 content remains fixed, whereas its radioactive carbon 14 begins to decay with a half-life of about 5700 years. To estimate the age of dead organic material, scientists use the following formula, which denotes the ratio of carbon 14 to carbon 12 present at any time t (in years).



$$R = \frac{1}{10^{12}}e^{-t/8223} \qquad \text{Carbon dating model}$$

The graph of R is shown at the right. Note that R decreases as t increases.

EXAMPLE 3 Carbon Dating

Estimate the age of a newly discovered fossil for which the ratio of carbon 14 to carbon 12 is $R = 1/10^{13}$.

Algebraic Solution

In the carbon dating model, substitute the given value of R to obtain the following.

$$\frac{1}{10^{12}}e^{-t/8223} = R \qquad \text{Write original model.}$$

$$\frac{e^{-t/8223}}{10^{12}} = \frac{1}{10^{13}} \qquad \text{Substitute } \frac{1}{10^{13}} \text{ for } R.$$

$$e^{-t/8223} = \frac{1}{10} \qquad \text{Multiply each side by } 10^{12}.$$

$$\ln e^{-t/8223} = \ln\frac{1}{10} \qquad \text{Take natural log of each side.}$$

$$-\frac{t}{8223} \approx -2.3026 \qquad \text{Inverse Property}$$

$$t \approx 18{,}934 \qquad \text{Multiply each side by } -8223.$$

So, to the nearest thousand years, the age of the fossil is about 19,000 years.

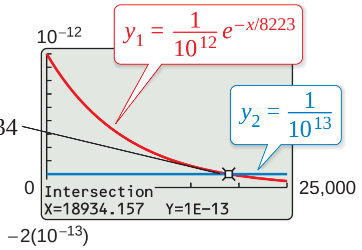
Graphical Solution

Use a graphing utility to graph the formula for the ratio of carbon 14 to carbon 12 at any time t as

$$y_1 = \frac{1}{10^{12}}e^{-x/8223}.$$

In the same viewing window, graph $y_2 = 1/10^{13}$.

Use the *intersect* feature to estimate that $x \approx 18{,}934$ when $y = 1/10^{13}$.



```
Intersection
X=18934.157   Y=1E-13
```

So, to the nearest thousand years, the age of the fossil is about 19,000 years.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Estimate the age of a newly discovered fossil for which the ratio of carbon 14 to carbon 12 is $R = 1/10^{14}$.

The value of b in the exponential decay model $y = ae^{-bt}$ determines the *decay* of radioactive isotopes. For instance, to find how much of an initial 10 grams of ^{226}Ra isotope with a half-life of 1599 years is left after 500 years, substitute this information into the model $y = ae^{-bt}$.

$$\frac{1}{2}(10) = 10e^{-b(1599)} \quad \Longrightarrow \quad \ln\frac{1}{2} = -1599b \quad \Longrightarrow \quad b = -\frac{\ln\frac{1}{2}}{1599}$$

Using the value of b found above and $a = 10$, the amount left is

$$y = 10e^{-[-\ln(1/2)/1599](500)} \approx 8.05 \text{ grams.}$$

Gaussian Models

As mentioned at the beginning of this section, Gaussian models are of the form

$$y = ae^{-(x-b)^2/c}.$$

In probability and statistics, this type of model commonly represents populations that are **normally distributed.** The graph of a Gaussian model is called a **bell-shaped curve.** Try graphing the normal distribution with a graphing utility. Can you see why it is called a bell-shaped curve?

For *standard* normal distributions, the model takes the form

$$y = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

The **average value** of a population can be found from the bell-shaped curve by observing where the maximum y-value of the function occurs. The x-value corresponding to the maximum y-value of the function represents the average value of the independent variable—in this case, x.

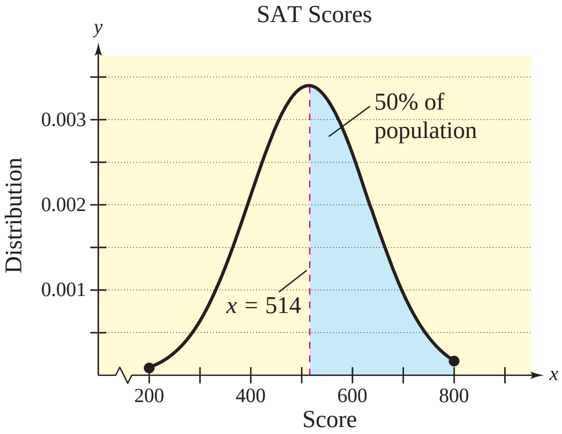
EXAMPLE 4 SAT Scores

In 2011, the SAT mathematics scores for high school graduates in the United States roughly followed the normal distribution given by

$$y = 0.0034e^{-(x-514)^2/27{,}378}, \quad 200 \le x \le 800$$

where x is the SAT score for mathematics. Sketch the graph of this function. From the graph, estimate the average SAT mathematics score. *(Source: The College Board)*

Solution The graph of the function is shown below. On this bell-shaped curve, the maximum value of the curve represents the average score. From the graph, you can estimate that the average mathematics score for high school graduates in 2011 was 514.



Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

In 2011, the SAT critical reading scores for high school graduates in the United States roughly followed the normal distribution given by

$$y = 0.0035e^{-(x-497)^2/25{,}992}, \quad 200 \le x \le 800$$

where x is the SAT score for critical reading. Sketch the graph of this function. From the graph, estimate the average SAT critical reading score. *(Source: The College Board)*

Logistic Growth Models

Some populations initially have rapid growth, followed by a declining rate of growth, as indicated by the graph in Figure 5.21. One model for describing this type of growth pattern is the **logistic curve** given by the function

$$y = \frac{a}{1 + be^{-rx}}$$

where y is the population size and x is the time. An example is a bacteria culture that is initially allowed to grow under ideal conditions and then under less favorable conditions that inhibit growth. A logistic growth curve is also called a **sigmoidal curve.**

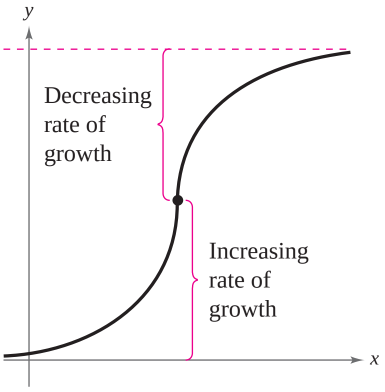


Figure 5.21

EXAMPLE 5 Spread of a Virus

On a college campus of 5000 students, one student returns from vacation with a contagious and long-lasting flu virus. The spread of the virus is modeled by

$$y = \frac{5000}{1 + 4999e^{-0.8t}}, \quad t \geq 0$$

where y is the total number of students infected after t days. The college will cancel classes when 40% or more of the students are infected.

a. How many students are infected after 5 days?

b. After how many days will the college cancel classes?

Algebraic Solution

a. After 5 days, the number of students infected is

$$y = \frac{5000}{1 + 4999e^{-0.8(5)}} = \frac{5000}{1 + 4999e^{-4}} \approx 54.$$

b. The college will cancel classes when the number of infected students is $(0.40)(5000) = 2000$.

$$2000 = \frac{5000}{1 + 4999e^{-0.8t}}$$

$$1 + 4999e^{-0.8t} = 2.5$$

$$e^{-0.8t} = \frac{1.5}{4999}$$

$$-0.8t = \ln \frac{1.5}{4999}$$

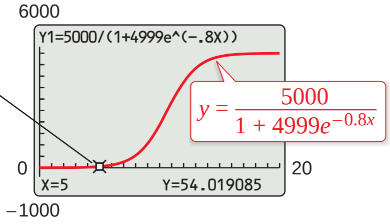
$$t = -\frac{1}{0.8} \ln \frac{1.5}{4999}$$

$$t \approx 10.14$$

So, after about 10 days, at least 40% of the students will be infected, and the college will cancel classes.

Graphical Solution

a.

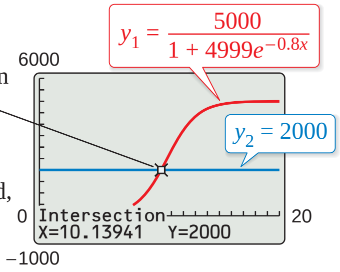


Use the *value* feature to estimate that $y \approx 54$ when $x = 5$. So, after 5 days, about 54 students are infected.

b. The college will cancel classes when the number of infected students is $(0.40)(5000) = 2000$. Use a graphing utility to graph

$$y_1 = \frac{5000}{1 + 4999e^{-0.8x}} \quad \text{and} \quad y_2 = 2000$$

in the same viewing window. Use the *intersect* feature of the graphing utility to find the point of intersection of the graphs.



The point of intersection occurs near $x \approx 10.14$. So, after about 10 days, at least 40% of the students will be infected, and the college will cancel classes.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

In Example 5, after how many days are 250 students infected?

Logarithmic Models

EXAMPLE 6 Magnitudes of Earthquakes

On the Richter scale, the magnitude R of an earthquake of intensity I is given by

$$R = \log \frac{I}{I_0}$$

where $I_0 = 1$ is the minimum intensity used for comparison. Find the intensity of each earthquake. (Intensity is a measure of the wave energy of an earthquake.)

a. Alaska in 2012: $R = 4.0$ **b.** Christchurch, New Zealand, in 2011: $R = 6.3$

On February 22, 2011, an earthquake of magnitude 6.3 struck Christchurch, New Zealand. The total economic loss was estimated at 15.5 billion U.S. dollars.

Solution

a. Because $I_0 = 1$ and $R = 4.0$, you have

$$4.0 = \log \frac{I}{1} \qquad \text{Substitute 1 for } I_0 \text{ and 4.0 for } R.$$

$$10^{4.0} = 10^{\log I} \qquad \text{Exponentiate each side.}$$

$$10^{4.0} = I \qquad \text{Inverse Property}$$

$$10{,}000 = I. \qquad \text{Simplify.}$$

b. For $R = 6.3$, you have

$$6.3 = \log \frac{I}{1} \qquad \text{Substitute 1 for } I_0 \text{ and 6.3 for } R.$$

$$10^{6.3} = 10^{\log I} \qquad \text{Exponentiate each side.}$$

$$10^{6.3} = I \qquad \text{Inverse Property}$$

$$2{,}000{,}000 \approx I. \qquad \text{Use a calculator.}$$

Note that an increase of 2.3 units on the Richter scale (from 4.0 to 6.3) represents an increase in intensity by a factor of $2{,}000{,}000/10{,}000 = 200$. In other words, the intensity of the earthquake in Christchurch was about 200 times as great as that of the earthquake in Alaska.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the intensities of earthquakes whose magnitudes are (a) $R = 6.0$ and (b) $R = 7.9$.

Summarize (Section 5.5)

1. State the five most common types of models involving exponential and logarithmic functions *(page 400)*.
2. Describe examples of how to use exponential growth and decay functions to model and solve real-life problems *(pages 401–403, Examples 1–3)*.
3. Describe an example of how to use a Gaussian function to model and solve a real-life problem *(page 404, Example 4)*.
4. Describe an example of how to use a logistic growth function to model and solve a real-life problem *(page 405, Example 5)*.
5. Describe an example of how to use a logarithmic function to model and solve a real-life problem *(page 406, Example 6)*.

Darrenp/Shutterstock.com

5.5 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: **Fill in the blanks.**

1. An exponential growth model has the form ________, and an exponential decay model has the form ________.

2. A logarithmic model has the form ________ or ________.

3. In probability and statistics, Gaussian models commonly represent populations that are ________ ________.

4. A logistic growth model has the form ________.

Skills and Applications

Solving for a Variable **In Exercises 5 and 6, (a) solve for *P* and (b) solve for *t*.**

5. $A = Pe^{rt}$

6. $A = P\left(1 + \frac{r}{n}\right)^{nt}$

Compound Interest **In Exercises 7–12, complete the table assuming continuously compounded interest.**

	Initial Investment	Annual % Rate	Time to Double	Amount After 10 Years
7.	\$1000	3.5%		
8.	\$750	$10\frac{1}{2}\%$		
9.	\$750		$7\frac{3}{4}$ yr	
10.	\$500			\$1505.00
11.		4.5%		\$10,000.00
12.			12 yr	\$2000.00

Compound Interest **In Exercises 13 and 14, determine the principal *P* that must be invested at rate *r*, compounded monthly, so that \$500,000 will be available for retirement in *t* years.**

13. $r = 5\%, t = 10$

14. $r = 3\frac{1}{2}\%, t = 15$

Compound Interest **In Exercises 15 and 16, determine the time necessary for *P* dollars to double when it is invested at interest rate *r* compounded (a) annually, (b) monthly, (c) daily, and (d) continuously.**

15. $r = 10\%$

16. $r = 6.5\%$

17. Compound Interest Complete the table for the time t (in years) necessary for P dollars to triple when interest is compounded (a) continuously and (b) annually at rate r.

r	2%	4%	6%	8%	10%	12%
t						

18. Modeling Data Draw scatter plots of the data in Exercise 17. Use the *regression* feature of a graphing utility to find models for the data.

19. Comparing Models If \$1 is invested over a 10-year period, then the balance A, where t represents the time in years, is given by $A = 1 + 0.075\llbracket t \rrbracket$ or $A = e^{0.07t}$ depending on whether the interest is simple interest at $7\frac{1}{2}\%$ or continuous compound interest at 7%. Graph each function on the same set of axes. Which grows at a greater rate? (Remember that $\llbracket t \rrbracket$ is the greatest integer function discussed in Section 2.4.)

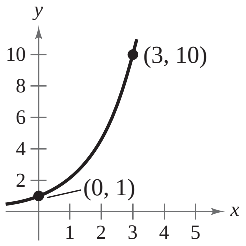
20. Comparing Models If \$1 is invested over a 10-year period, then the balance A, where t represents the time in years, is given by $A = 1 + 0.06\llbracket t \rrbracket$ or $A = [1 + (0.055/365)]^{\llbracket 365t \rrbracket}$ depending on whether the interest is simple interest at 6% or compound interest at $5\frac{1}{2}\%$ compounded daily. Use a graphing utility to graph each function in the same viewing window. Which grows at a greater rate?

Radioactive Decay **In Exercises 21–24, complete the table for the radioactive isotope.**

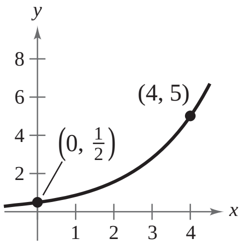
	Isotope	Half-life (years)	Initial Quantity	Amount After 1000 Years
21.	^{226}Ra	1599	10 g	
22.	^{14}C	5715	6.5 g	
23.	^{14}C	5715		2 g
24.	^{239}Pu	24,100		0.4 g

Finding an Exponential Model **In Exercises 25–28, find the exponential model that fits the points shown in the graph or table.**

25.



26.



27.

x	0	4
y	5	1

28.

x	0	3
y	1	$\frac{1}{4}$

29. Population The populations P (in thousands) of Horry County, South Carolina, from 1980 through 2010 can be modeled by

$$P = 20.6 + 85.5e^{0.0360t}$$

where t represents the year, with $t = 0$ corresponding to 1980. *(Source: U.S. Census Bureau)*

(a) Use the model to complete the table.

Year	1980	1990	2000	2010
Population				

(b) According to the model, when will the population of Horry County reach 350,000?

(c) Do you think the model is valid for long-term predictions of the population? Explain.

30. Population

The table shows the mid-year populations (in millions) of five countries in 2010 and the projected populations (in millions) for the year 2020. *(Source: U.S. Census Bureau)*

Country	2010	2020
Bulgaria	7.1	6.6
Canada	33.8	36.4
China	1330.1	1384.5
United Kingdom	62.3	65.8
United States	310.2	341.4

(a) Find the exponential growth or decay model $y = ae^{bt}$ or $y = ae^{-bt}$ for the population of each country by letting $t = 10$ correspond to 2010. Use the model to predict the population of each country in 2030.

Alan Becker/Stone/Getty Images

(b) You can see that the populations of the United States and the United Kingdom are growing at different rates. What constant in the equation $y = ae^{bt}$ gives the growth rate? Discuss the relationship between the different growth rates and the magnitude of the constant.

(c) You can see that the population of China is increasing, whereas the population of Bulgaria is decreasing. What constant in the equation $y = ae^{bt}$ reflects this difference? Explain.

31. Website Growth The number y of hits a new website receives each month can be modeled by $y = 4080e^{kt}$, where t represents the number of months the website has been operating. In the website's third month, there were 10,000 hits. Find the value of k, and use this value to predict the number of hits the website will receive after 24 months.

32. Population The populations P (in thousands) of Tallahassee, Florida, from 2005 through 2010 can be modeled by $P = 319.2e^{kt}$, where t represents the year, with $t = 5$ corresponding to 2005. In 2006, the population of Tallahassee was about 347,000. *(Source: U.S. Census Bureau)*

(a) Find the value of k. Is the population increasing or decreasing? Explain.

(b) Use the model to predict the populations of Tallahassee in 2015 and 2020. Are the results reasonable? Explain.

(c) According to the model, during what year will the population reach 410,000?

33. Bacteria Growth The number of bacteria in a culture is increasing according to the law of exponential growth. After 3 hours there are 100 bacteria, and after 5 hours there are 400 bacteria. How many bacteria will there be after 6 hours?

34. Bacteria Growth The number of bacteria in a culture is increasing according to the law of exponential growth. The initial population is 250 bacteria, and the population after 10 hours is double the population after 1 hour. How many bacteria will there be after 6 hours?

35. Depreciation A laptop computer that costs \$1150 new has a book value of \$550 after 2 years.

(a) Find the linear model $V = mt + b$.

(b) Find the exponential model $V = ae^{kt}$.

(c) Use a graphing utility to graph the two models in the same viewing window. Which model depreciates faster in the first 2 years?

(d) Find the book values of the computer after 1 year and after 3 years using each model.

(e) Explain the advantages and disadvantages of using each model to a buyer and a seller.

36. Learning Curve The management at a plastics factory has found that the maximum number of units a worker can produce in a day is 30. The learning curve for the number N of units produced per day after a new employee has worked t days is modeled by $N = 30(1 - e^{kt})$. After 20 days on the job, a new employee produces 19 units.

(a) Find the learning curve for this employee (first, find the value of k).

(b) How many days should pass before this employee is producing 25 units per day?

37. Carbon Dating

(a) The ratio of carbon 14 to carbon 12 in a piece of wood discovered in a cave is $R = 1/8^{14}$. Estimate the age of the piece of wood.

(b) The ratio of carbon 14 to carbon 12 in a piece of paper buried in a tomb is $R = 1/13^{11}$. Estimate the age of the piece of paper.

38. Radioactive Decay Carbon 14 dating assumes that the carbon dioxide on Earth today has the same radioactive content as it did centuries ago. If this is true, then the amount of ^{14}C absorbed by a tree that grew several centuries ago should be the same as the amount of ^{14}C absorbed by a tree growing today. A piece of ancient charcoal contains only 15% as much radioactive carbon as a piece of modern charcoal. How long ago was the tree burned to make the ancient charcoal, assuming that the half-life of ^{14}C is 5715 years?

39. IQ Scores The IQ scores for a sample of a class of returning adult students at a small northeastern college roughly follow the normal distribution

$$y = 0.0266e^{-(x-100)^2/450}, \quad 70 \le x \le 115$$

where x is the IQ score.

(a) Use a graphing utility to graph the function.

(b) From the graph in part (a), estimate the average IQ score of an adult student.

40. Education The amount of time (in hours per week) a student utilizes a math-tutoring center roughly follows the normal distribution

$$y = 0.7979e^{-(x-5.4)^2/0.5}, \quad 4 \le x \le 7$$

where x is the number of hours.

(a) Use a graphing utility to graph the function.

(b) From the graph in part (a), estimate the average number of hours per week a student uses the tutoring center.

41. Cell Sites A cell site is a site where electronic communications equipment is placed in a cellular network for the use of mobile phones. The numbers y of cell sites from 1985 through 2011 can be modeled by

$$y = \frac{269{,}573}{1 + 985e^{-0.308t}}$$

where t represents the year, with $t = 5$ corresponding to 1985. *(Source: CTIA-The Wireless Association)*

(a) Use the model to find the numbers of cell sites in the years 1998, 2003, and 2006.

(b) Use a graphing utility to graph the function.

(c) Use the graph to determine the year in which the number of cell sites reached 250,000.

(d) Confirm your answer to part (c) algebraically.

42. Population The populations P (in thousands) of a city from 2000 through 2010 can be modeled by

$$P = \frac{2632}{1 + 0.083e^{0.050t}}$$

where t represents the year, with $t = 0$ corresponding to 2000.

(a) Use the model to find the populations of the city in the years 2000, 2005, and 2010.

(b) Use a graphing utility to graph the function.

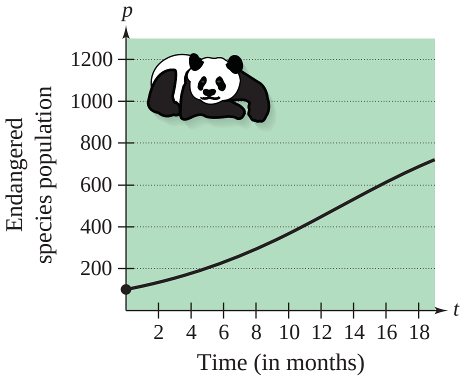
(c) Use the graph to determine the year in which the population will reach 2.2 million.

(d) Confirm your answer to part (c) algebraically.

43. Population Growth A conservation organization released 100 animals of an endangered species into a game preserve. The preserve has a carrying capacity of 1000 animals. The growth of the pack is modeled by the logistic curve

$$p(t) = \frac{1000}{1 + 9e^{-0.1656t}}$$

where t is measured in months (see figure).



(a) Estimate the population after 5 months.

(b) After how many months is the population 500?

(c) Use a graphing utility to graph the function. Use the graph to determine the horizontal asymptotes, and interpret the meaning of the asymptotes in the context of the problem.

44. Sales After discontinuing all advertising for a tool kit in 2007, the manufacturer noted that sales began to drop according to the model

$$S = \frac{500{,}000}{1 + 0.4e^{kt}}$$

where S represents the number of units sold and $t = 7$ represents 2007. In 2011, 300,000 units were sold.

(a) Complete the model by solving for k.

(b) Estimate sales in 2015.

Geology **In Exercises 45 and 46, use the Richter scale**

$$R = \log \frac{I}{I_0}$$

for measuring the magnitude R of an earthquake.

45. Find the intensity I of an earthquake measuring R on the Richter scale (let $I_0 = 1$).

(a) South Shetland Islands in 2012: $R = 6.6$

(b) Oklahoma in 2011: $R = 5.6$

(c) Papua New Guinea in 2011: $R = 7.1$

46. Find the magnitude R of each earthquake of intensity I (let $I_0 = 1$).

(a) $I = 199{,}500{,}000$

(b) $I = 48{,}275{,}000$

(c) $I = 17{,}000$

Intensity of Sound **In Exercises 47–50, use the following information for determining sound intensity. The level of sound β, in decibels, with an intensity of I, is given by $\beta = 10 \log(I/I_0)$, where I_0 is an intensity of 10^{-12} watt per square meter, corresponding roughly to the faintest sound that can be heard by the human ear. In Exercises 47 and 48, find the level of sound β.**

47. (a) $I = 10^{-10}$ watt per m^2 (quiet room)

(b) $I = 10^{-5}$ watt per m^2 (busy street corner)

(c) $I = 10^{-8}$ watt per m^2 (quiet radio)

(d) $I = 10^{0}$ watt per m^2 (threshold of pain)

48. (a) $I = 10^{-11}$ watt per m^2 (rustle of leaves)

(b) $I = 10^{2}$ watt per m^2 (jet at 30 meters)

(c) $I = 10^{-4}$ watt per m^2 (door slamming)

(d) $I = 10^{-2}$ watt per m^2 (siren at 30 meters)

49. Due to the installation of noise suppression materials, the noise level in an auditorium decreased from 93 to 80 decibels. Find the percent decrease in the intensity level of the noise as a result of the installation of these materials.

50. Due to the installation of a muffler, the noise level of an engine decreased from 88 to 72 decibels. Find the percent decrease in the intensity level of the noise as a result of the installation of the muffler.

pH Levels **In Exercises 51–56, use the acidity model given by $\mathrm{pH} = -\log[\mathrm{H}^+]$, where acidity (pH) is a measure of the hydrogen ion concentration $[\mathrm{H}^+]$ (measured in moles of hydrogen per liter) of a solution.**

51. Find the pH when $[\mathrm{H}^+] = 2.3 \times 10^{-5}$.

52. Find the pH when $[\mathrm{H}^+] = 1.13 \times 10^{-5}$.

53. Compute $[\mathrm{H}^+]$ for a solution in which $\mathrm{pH} = 5.8$.

54. Compute $[\mathrm{H}^+]$ for a solution in which $\mathrm{pH} = 3.2$.

55. Apple juice has a pH of 2.9 and drinking water has a pH of 8.0. The hydrogen ion concentration of the apple juice is how many times the concentration of drinking water?

56. The pH of a solution decreases by one unit. By what factor does the hydrogen ion concentration increase?

57. **Forensics** At 8:30 A.M., a coroner went to the home of a person who had died during the night. In order to estimate the time of death, the coroner took the person's temperature twice. At 9:00 A.M. the temperature was 85.7°F, and at 11:00 A.M. the temperature was 82.8°F. From these two temperatures, the coroner was able to determine that the time elapsed since death and the body temperature were related by the formula

$$t = -10 \ln \frac{T - 70}{98.6 - 70}$$

where t is the time in hours elapsed since the person died and T is the temperature (in degrees Fahrenheit) of the person's body. (This formula comes from a general cooling principle called *Newton's Law of Cooling*. It uses the assumptions that the person had a normal body temperature of 98.6°F at death and that the room temperature was a constant 70°F.) Use the formula to estimate the time of death of the person.

58. **Home Mortgage** A \$120,000 home mortgage for 30 years at $7\frac{1}{2}\%$ has a monthly payment of \$839.06. Part of the monthly payment covers the interest charge on the unpaid balance, and the remainder of the payment reduces the principal. The amount paid toward the interest is

$$u = M - \left(M - \frac{Pr}{12}\right)\left(1 + \frac{r}{12}\right)^{12t}$$

and the amount paid toward the reduction of the principal is

$$v = \left(M - \frac{Pr}{12}\right)\left(1 + \frac{r}{12}\right)^{12t}.$$

In these formulas, P is the size of the mortgage, r is the interest rate, M is the monthly payment, and t is the time (in years).

(a) Use a graphing utility to graph each function in the same viewing window. (The viewing window should show all 30 years of mortgage payments.)

(b) In the early years of the mortgage, is the greater part of the monthly payment paid toward the interest or the principal? Approximate the time when the monthly payment is evenly divided between interest and principal reduction.

(c) Repeat parts (a) and (b) for a repayment period of 20 years ($M = \$966.71$). What can you conclude?

59. Home Mortgage The total interest u paid on a home mortgage of P dollars at interest rate r for t years is

$$u = P\left[\frac{rt}{1-\left(\dfrac{1}{1+r/12}\right)^{12t}} - 1\right].$$

Consider a \$120,000 home mortgage at $7\frac{1}{2}\%$.

(a) Use a graphing utility to graph the total interest function.

(b) Approximate the length of the mortgage for which the total interest paid is the same as the size of the mortgage. Is it possible that some people are paying twice as much in interest charges as the size of the mortgage?

60. Data Analysis The table shows the time t (in seconds) required for a car to attain a speed of s miles per hour from a standing start.

Speed, s	Time, t
30	3.4
40	5.0
50	7.0
60	9.3
70	12.0
80	15.8
90	20.0

Spreadsheet at LarsonPrecalculus.com

Two models for these data are as follows.

$$t_1 = 40.757 + 0.556s - 15.817 \ln s$$

$$t_2 = 1.2259 + 0.0023s^2$$

(a) Use the *regression* feature of a graphing utility to find a linear model t_3 and an exponential model t_4 for the data.

(b) Use the graphing utility to graph the data and each model in the same viewing window.

(c) Create a table comparing the data with estimates obtained from each model.

(d) Use the results of part (c) to find the sum of the absolute values of the differences between the data and the estimated values given by each model. Based on the four sums, which model do you think best fits the data? Explain.

Exploration

True or False? **In Exercises 61–64, determine whether the statement is true or false. Justify your answer.**

61. The domain of a logistic growth function cannot be the set of real numbers.

62. A logistic growth function will always have an x-intercept.

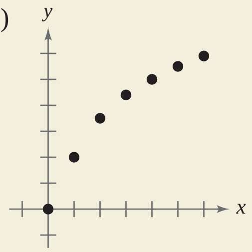
63. The graph of $f(x) = \dfrac{4}{1 + 6e^{-2x}} + 5$ is the graph of $g(x) = \dfrac{4}{1 + 6e^{-2x}}$ shifted to the right five units.

64. The graph of a Gaussian model will never have an x-intercept.

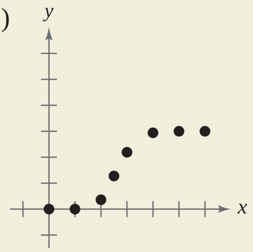
65. Writing Use your school's library, the Internet, or some other reference source to write a paper describing John Napier's work with logarithms.

66. HOW DO YOU SEE IT? Identify each model as exponential growth, exponential decay, Gaussian, linear, logarithmic, logistic growth, quadratic, or none of the above. Explain your reasoning.

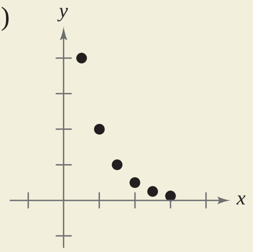
(a)



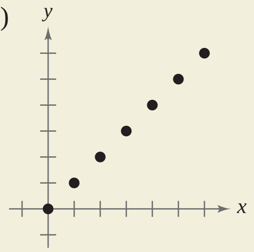
(b)



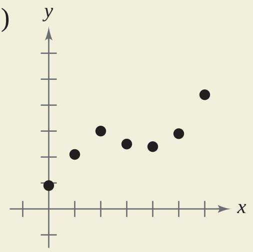
(c)



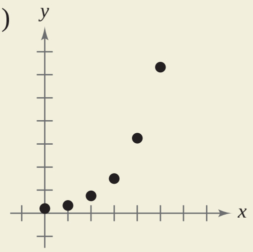
(d)



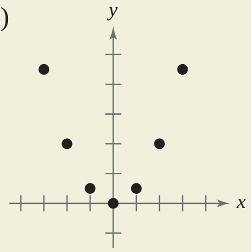
(e)



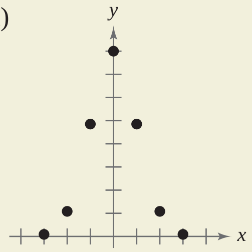
(f)



(g)

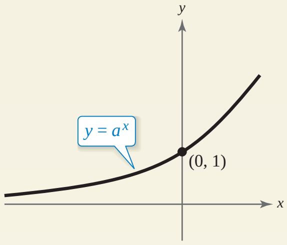
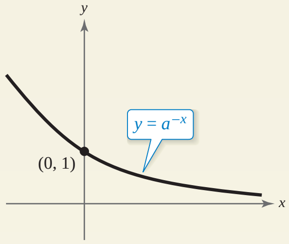
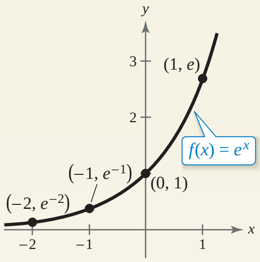
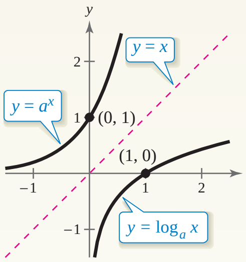
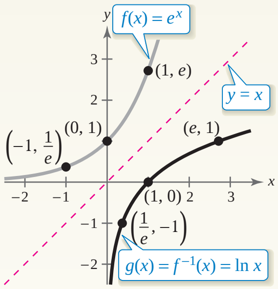


(h)



Project: Sales per Share To work an extended application analyzing the sales per share for Kohl's Corporation from 1995 through 2010, visit this text's website at *LarsonPrecalculus.com*. *(Source: Kohl's Corporation)*

Chapter Summary

	What Did You Learn?	Explanation/Examples	Review Exercises
Section 5.1	Recognize and evaluate exponential functions with base a *(p. 362)*.	The exponential function f with base a is denoted by $f(x) = a^x$, where $a > 0$, $a \neq 1$, and x is any real number.	1–6
	Graph exponential functions and use the One-to-One Property *(p. 363)*.	  **One-to-One Property:** For $a > 0$ and $a \neq 1$, $a^x = a^y$ if and only if $x = y$.	7–20
	Recognize, evaluate, and graph exponential functions with base e *(p. 366)*.	The function $f(x) = e^x$ is called the natural exponential function. 	21–28
	Use exponential functions to model and solve real-life problems *(p. 367)*.	Exponential functions are used in compound interest formulas (see Example 8) and in radioactive decay models (see Example 9).	29–32
Section 5.2	Recognize and evaluate logarithmic functions with base a *(p. 373)*.	For $x > 0$, $a > 0$, and $a \neq 1$, $y = \log_a x$ if and only if $x = a^y$. The function $f(x) = \log_a x$ is called the logarithmic function with base a. The logarithmic function with base 10 is the common logarithmic function. It is denoted by $\log_{10}$ or log.	33–44
	Graph logarithmic functions *(p. 375)*, and recognize, evaluate, and graph natural logarithmic functions *(p. 377)*.	The graph of $y = \log_a x$ is a reflection of the graph of $y = a^x$ in the line $y = x$. The function $f(x) = \ln x$, $x > 0$, is the natural logarithmic function. Its graph is a reflection of the graph of $f(x) = e^x$ in the line $y = x$.  	45–56
	Use logarithmic functions to model and solve real-life problems *(p. 379)*.	A logarithmic function can model human memory. (See Example 11.)	57, 58

	What Did You Learn?	Explanation/Examples	Review Exercises
Section 5.3	Use the change-of-base formula to rewrite and evaluate logarithmic expressions *(p. 383)*.	Let a, b, and x be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then $\log_a x$ can be converted to a different base as follows. **Base b**: $\log_a x = \dfrac{\log_b x}{\log_b a}$ **Base 10**: $\log_a x = \dfrac{\log x}{\log a}$ **Base e**: $\log_a x = \dfrac{\ln x}{\ln a}$	59–62
	Use properties of logarithms to evaluate, rewrite, expand, or condense logarithmic expressions *(p. 384)*.	Let a be a positive number such that $a \neq 1$, let n be a real number, and let u and v be positive real numbers. **1. Product Property:** $\log_a(uv) = \log_a u + \log_a v$ $\ln(uv) = \ln u + \ln v$ **2. Quotient Property:** $\log_a(u/v) = \log_a u - \log_a v$ $\ln(u/v) = \ln u - \ln v$ **3. Power Property:** $\log_a u^n = n \log_a u$, $\ln u^n = n \ln u$	63–78
	Use logarithmic functions to model and solve real-life problems *(p. 386)*.	Logarithmic functions can help you find an equation that relates the periods of several planets and their distances from the sun. (See Example 7.)	79, 80
Section 5.4	Solve simple exponential and logarithmic equations *(p. 390)*.	One-to-One Properties and Inverse Properties of exponential or logarithmic functions can help you solve exponential or logarithmic equations.	81–86
	Solve more complicated exponential equations *(p. 391)* and logarithmic equations *(p. 393)*.	To solve more complicated equations, rewrite the equations to allow the use of the One-to-One Properties or Inverse Properties of exponential or logarithmic functions. (See Examples 2–9.)	87–104
	Use exponential and logarithmic equations to model and solve real-life problems *(p. 395)*.	Exponential and logarithmic equations can help you find how long it will take to double an investment (see Example 10) and find the year in which companies reached a given amount of sales (see Example 11).	105, 106
Section 5.5	Recognize the five most common types of models involving exponential and logarithmic functions *(p. 400)*.	**1. Exponential growth model:** $y = ae^{bx}$, $b > 0$ **2. Exponential decay model:** $y = ae^{-bx}$, $b > 0$ **3. Gaussian model:** $y = ae^{-(x-b)^2/c}$ **4. Logistic growth model:** $y = \dfrac{a}{1 + be^{-rx}}$ **5. Logarithmic models:** $y = a + b \ln x$, $y = a + b \log x$	107–112
	Use exponential growth and decay functions to model and solve real-life problems *(p. 401)*.	An exponential growth function can help you model a population of fruit flies (see Example 2), and an exponential decay function can help you estimate the age of a fossil (see Example 3).	113, 114
	Use Gaussian functions *(p. 404)*, logistic growth functions *(p. 405)*, and logarithmic functions *(p. 406)* to model and solve real-life problems.	A Gaussian function can help you model SAT mathematics scores for high school graduates. (See Example 4.) A logistic growth function can help you model the spread of a flu virus. (See Example 5.) A logarithmic function can help you find the intensity of an earthquake given its magnitude. (See Example 6.)	115–117

Review Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

5.1 Evaluating an Exponential Function **In Exercises 1–6, evaluate the function at the indicated value of x. Round your result to three decimal places.**

1. $f(x) = 0.3^x, \quad x = 1.5$

2. $f(x) = 30^x, \quad x = \sqrt{3}$

3. $f(x) = 2^{-0.5x}, \quad x = \pi$

4. $f(x) = 1278^{x/5}, \quad x = 1$

5. $f(x) = 7(0.2^x), \quad x = -\sqrt{11}$

6. $f(x) = -14(5^x), \quad x = -0.8$

Transforming the Graph of an Exponential Function **In Exercises 7–10, use the graph of f to describe the transformation that yields the graph of g.**

7. $f(x) = 5^x, \quad g(x) = 5^x + 1$

8. $f(x) = 6^x, \quad g(x) = 6^{x+1}$

9. $f(x) = 3^x, \quad g(x) = 1 - 3^x$

10. $f(x) = \left(\frac{1}{2}\right)^x, \quad g(x) = -\left(\frac{1}{2}\right)^{x+2}$

Graphing an Exponential Function **In Exercises 11–16, use a graphing utility to construct a table of values for the function. Then sketch the graph of the function.**

11. $f(x) = 4^{-x} + 4$

12. $f(x) = 2.65^{x-1}$

13. $f(x) = 5^{x-2} + 4$

14. $f(x) = 2^{x-6} - 5$

15. $f(x) = \left(\frac{1}{2}\right)^{-x} + 3$

16. $f(x) = \left(\frac{1}{8}\right)^{x+2} - 5$

Using the One-to-One Property **In Exercises 17–20, use the One-to-One Property to solve the equation for x.**

17. $\left(\frac{1}{3}\right)^{x-3} = 9$

18. $3^{x+3} = \frac{1}{81}$

19. $e^{3x-5} = e^7$

20. $e^{8-2x} = e^{-3}$

Evaluating the Natural Exponential Function **In Exercises 21–24, evaluate $f(x) = e^x$ at the indicated value of x. Round your result to three decimal places.**

21. $x = 8$

22. $x = \frac{5}{8}$

23. $x = -1.7$

24. $x = 0.278$

Graphing a Natural Exponential Function **In Exercises 25–28, use a graphing utility to construct a table of values for the function. Then sketch the graph of the function.**

25. $h(x) = e^{-x/2}$

26. $h(x) = 2 - e^{-x/2}$

27. $f(x) = e^{x+2}$

28. $s(t) = 4e^{-2/t}, \quad t > 0$

29. Waiting Times The average time between incoming calls at a switchboard is 3 minutes. The probability F of waiting less than t minutes until the next incoming call is approximated by the model $F(t) = 1 - e^{-t/3}$. The switchboard has just received a call. Find the probability that the next call will be within

(a) $\frac{1}{2}$ minute. (b) 2 minutes. (c) 5 minutes.

30. Depreciation After t years, the value V of a car that originally cost \$23,970 is given by $V(t) = 23{,}970\left(\frac{3}{4}\right)^t$.

(a) Use a graphing utility to graph the function.

(b) Find the value of the car 2 years after it was purchased.

(c) According to the model, when does the car depreciate most rapidly? Is this realistic? Explain.

(d) According to the model, when will the car have no value?

Compound Interest **In Exercises 31 and 32, complete the table to determine the balance A for P dollars invested at rate r for t years and compounded n times per year.**

n	1	2	4	12	365	Continuous
A						

31. $P = \$5000, \quad r = 3\%, \quad t = 10$ years

32. $P = \$4500, \quad r = 2.5\%, \quad t = 30$ years

5.2 Writing a Logarithmic Equation **In Exercises 33–36, write the exponential equation in logarithmic form. For example, the logarithmic form of $2^3 = 8$ is $\log_2 8 = 3$.**

33. $3^3 = 27$

34. $25^{3/2} = 125$

35. $e^{0.8} = 2.2255\ldots$

36. $e^0 = 1$

Evaluating a Logarithmic Function **In Exercises 37–40, evaluate the function at the indicated value of x without using a calculator.**

37. $f(x) = \log x, x = 1000$

38. $g(x) = \log_9 x, x = 3$

39. $g(x) = \log_2 x, x = \frac{1}{4}$

40. $f(x) = \log_3 x, x = \frac{1}{81}$

Using the One-to-One Property **In Exercises 41–44, use the One-to-One Property to solve the equation for x.**

41. $\log_4(x + 7) = \log_4 14$

42. $\log_8(3x - 10) = \log_8 5$

43. $\ln(x + 9) = \ln 4$

44. $\ln(2x - 1) = \ln 11$

Sketching the Graph of a Logarithmic Function **In Exercises 45–48, find the domain, x-intercept, and vertical asymptote of the logarithmic function and sketch its graph.**

45. $g(x) = \log_7 x$

46. $f(x) = \log\left(\dfrac{x}{3}\right)$

47. $f(x) = 4 - \log(x + 5)$

48. $f(x) = \log(x - 3) + 1$

Evaluating a Logarithmic Function on a Calculator **In Exercises 49–52, use a calculator to evaluate the function at the indicated value of x. Round your result to three decimal places, if necessary.**

49. $f(x) = \ln x, \quad x = 22.6$ **50.** $f(x) = \ln x, \quad x = e^{-12}$

51. $f(x) = \frac{1}{2}\ln x, \quad x = \sqrt{e}$

52. $f(x) = 5\ln x, \quad x = 0.98$

Graphing a Natural Logarithmic Function **In Exercises 53–56, find the domain, x-intercept, and vertical asymptote of the logarithmic function and sketch its graph.**

53. $f(x) = \ln x + 3$ **54.** $f(x) = \ln(x - 3)$

55. $h(x) = \ln(x^2)$ **56.** $f(x) = \frac{1}{4}\ln x$

57. Antler Spread The antler spread a (in inches) and shoulder height h (in inches) of an adult male American elk are related by the model

$$h = 116\log(a + 40) - 176.$$

Approximate the shoulder height of a male American elk with an antler spread of 55 inches.

58. Snow Removal The number of miles s of roads cleared of snow is approximated by the model

$$s = 25 - \frac{13\ln(h/12)}{\ln 3}, \quad 2 \le h \le 15$$

where h is the depth of the snow in inches. Use this model to find s when $h = 10$ inches.

5.3 Using the Change-of-Base Formula **In Exercises 59–62, evaluate the logarithm using the change-of-base formula (a) with common logarithms and (b) with natural logarithms. Round your results to three decimal places.**

59. $\log_2 6$ **60.** $\log_{12} 200$

61. $\log_{1/2} 5$ **62.** $\log_3 0.28$

Using Properties of Logarithms **In Exercises 63–66, use the properties of logarithms to rewrite and simplify the logarithmic expression.**

63. $\log 18$ **64.** $\log_2\left(\frac{1}{12}\right)$

65. $\ln 20$ **66.** $\ln(3e^{-4})$

Expanding a Logarithmic Expression **In Exercises 67–72, use the properties of logarithms to expand the expression as a sum, difference, and/or constant multiple of logarithms. (Assume all variables are positive.)**

67. $\log_5 5x^2$ **68.** $\log 7x^4$

69. $\log_3 \frac{9}{\sqrt{x}}$ **70.** $\log_7 \frac{\sqrt[3]{x}}{14}$

71. $\ln x^2y^2z$ **72.** $\ln\left(\frac{y - 1}{4}\right)^2, \quad y > 1$

Condensing a Logarithmic Expression **In Exercises 73–78, condense the expression to the logarithm of a single quantity.**

73. $\log_2 5 + \log_2 x$

74. $\log_6 y - 2\log_6 z$

75. $\ln x - \frac{1}{4}\ln y$

76. $3\ln x + 2\ln(x + 1)$

77. $\frac{1}{2}\log_3 x - 2\log_3(y + 8)$

78. $5\ln(x - 2) - \ln(x + 2) - 3\ln x$

79. Climb Rate The time t (in minutes) for a small plane to climb to an altitude of h feet is modeled by $t = 50\log[18{,}000/(18{,}000 - h)]$, where 18,000 feet is the plane's absolute ceiling.

(a) Determine the domain of the function in the context of the problem.

(b) Use a graphing utility to graph the function and identify any asymptotes.

(c) As the plane approaches its absolute ceiling, what can be said about the time required to increase its altitude?

(d) Find the time for the plane to climb to an altitude of 4000 feet.

80. Human Memory Model Students in a learning theory study took an exam and then retested monthly for 6 months with an equivalent exam. The data obtained in the study are given by the ordered pairs (t, s), where t is the time in months after the initial exam and s is the average score for the class. Use the data to find a logarithmic equation that relates t and s.

$(1, 84.2), (2, 78.4), (3, 72.1),$
$(4, 68.5), (5, 67.1), (6, 65.3)$

5.4 Solving a Simple Equation **In Exercises 81–86, solve for x.**

81. $5^x = 125$ **82.** $6^x = \frac{1}{216}$

83. $e^x = 3$ **84.** $\log_6 x = -1$

85. $\ln x = 4$ **86.** $\ln x = -1.6$

Solving an Exponential Equation **In Exercises 87–90, solve the exponential equation algebraically. Approximate the result to three decimal places.**

87. $e^{4x} = e^{x^2+3}$ **88.** $e^{3x} = 25$

89. $2^x - 3 = 29$ **90.** $e^{2x} - 6e^x + 8 = 0$

Graphing and Solving an Exponential Equation **In Exercises 91 and 92, use a graphing utility to graph and solve the equation. Approximate the result to three decimal places. Verify your result algebraically.**

91. $25e^{-0.3x} = 12$

92. $2^x = 3 + x - e^x$

Solving a Logarithmic Equation **In Exercises 93–100, solve the logarithmic equation algebraically. Approximate the result to three decimal places.**

93. $\ln 3x = 8.2$

94. $4 \ln 3x = 15$

95. $\ln x - \ln 3 = 2$

96. $\ln \sqrt{x + 8} = 3$

97. $\log_8(x - 1) = \log_8(x - 2) - \log_8(x + 2)$

98. $\log_6(x + 2) - \log_6 x = \log_6(x + 5)$

99. $\log(1 - x) = -1$

100. $\log(-x - 4) = 2$

Graphing and Solving a Logarithmic Equation **In Exercises 101–104, use a graphing utility to graph and solve the equation. Approximate the result to three decimal places. Verify your result algebraically.**

101. $2 \ln(x + 3) - 3 = 0$

102. $x - 2 \log(x + 4) = 0$

103. $6 \log(x^2 + 1) - x = 0$

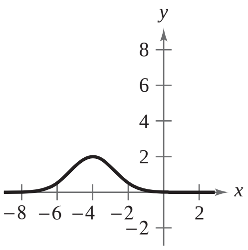
104. $3 \ln x + 2 \log x = e^x - 25$

105. **Compound Interest** You deposit \$8500 in an account that pays 1.5% interest, compounded continuously. How long will it take for the money to triple?

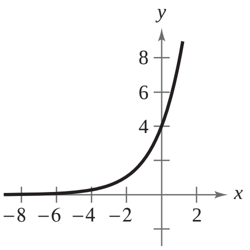
106. **Meteorology** The speed of the wind S (in miles per hour) near the center of a tornado and the distance d (in miles) the tornado travels are related by the model $S = 93 \log d + 65$. On March 18, 1925, a large tornado struck portions of Missouri, Illinois, and Indiana with a wind speed at the center of about 283 miles per hour. Approximate the distance traveled by this tornado.

5.5 **Matching a Function with Its Graph** **In Exercises 107–112, match the function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]**

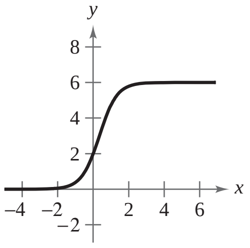
(a)



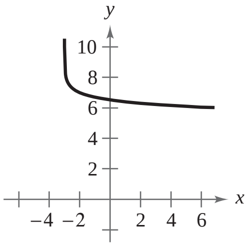
(b)



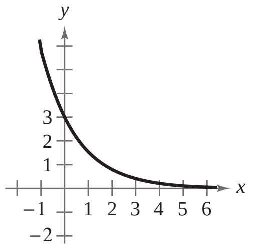
(c)



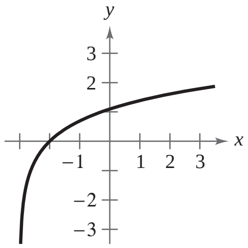
(d)



(e)



(f)



107. $y = 3e^{-2x/3}$

108. $y = 4e^{2x/3}$

109. $y = \ln(x + 3)$

110. $y = 7 - \log(x + 3)$

111. $y = 2e^{-(x+4)^2/3}$

112. $y = \dfrac{6}{1 + 2e^{-2x}}$

113. **Finding an Exponential Model** Find the exponential model $y = ae^{bx}$ that fits the points $(0, 2)$ and $(4, 3)$.

114. **Wildlife Population** A species of bat is in danger of becoming extinct. Five years ago, the total population of the species was 2000. Two years ago, the total population of the species was 1400. What was the total population of the species one year ago?

115. **Test Scores** The test scores for a biology test follow a normal distribution modeled by

$$y = 0.0499e^{-(x-71)^2/128}, \quad 40 \le x \le 100$$

where x is the test score. Use a graphing utility to graph the equation and estimate the average test score.

116. **Typing Speed** In a typing class, the average number N of words per minute typed after t weeks of lessons is

$$N = 157/(1 + 5.4e^{-0.12t}).$$

Find the time necessary to type (a) 50 words per minute and (b) 75 words per minute.

117. **Sound Intensity** The relationship between the number of decibels β and the intensity of a sound I in watts per square meter is

$$\beta = 10 \log(I/10^{-12}).$$

Find I for each decibel level β.

(a) $\beta = 60$ (b) $\beta = 135$ (c) $\beta = 1$

Exploration

118. **Graph of an Exponential Function** Consider the graph of $y = e^{kt}$. Describe the characteristics of the graph when k is positive and when k is negative.

True or False? **In Exercises 119 and 120, determine whether the equation is true or false. Justify your answer.**

119. $\log_b b^{2x} = 2x$

120. $\ln(x + y) = \ln x + \ln y$

Chapter Test

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

In Exercises 1–4, evaluate the expression. Round your result to three decimal places.

1. $4.2^{0.6}$ **2.** $4^{3\pi/2}$ **3.** $e^{-7/10}$ **4.** $e^{3.1}$

In Exercises 5–7, construct a table of values for the function. Then sketch the graph of the function.

5. $f(x) = 10^{-x}$ **6.** $f(x) = -6^{x-2}$ **7.** $f(x) = 1 - e^{2x}$

8. Evaluate (a) $\log_7 7^{-0.89}$ and (b) $4.6 \ln e^2$.

In Exercises 9–11, find the domain, *x*-intercept, and vertical asymptote of the logarithmic function and sketch its graph.

9. $f(x) = -\log x - 6$ **10.** $f(x) = \ln(x - 4)$ **11.** $f(x) = 1 + \ln(x + 6)$

In Exercises 12–14, evaluate the logarithm using the change-of-base formula. Round your result to three decimal places.

12. $\log_7 44$ **13.** $\log_{16} 0.63$ **14.** $\log_{3/4} 24$

In Exercises 15–17, use the properties of logarithms to expand the expression as a sum, difference, and/or constant multiple of logarithms. (Assume all variables are positive.)

15. $\log_2 3a^4$ **16.** $\ln \dfrac{5\sqrt{x}}{6}$ **17.** $\log \dfrac{(x-1)^3}{y^2 z}$

In Exercises 18–20, condense the expression to the logarithm of a single quantity.

18. $\log_3 13 + \log_3 y$ **19.** $4 \ln x - 4 \ln y$

20. $3 \ln x - \ln(x + 3) + 2 \ln y$

In Exercises 21–26, solve the equation algebraically. Approximate the result to three decimal places, if necessary.

21. $5^x = \dfrac{1}{25}$ **22.** $3e^{-5x} = 132$

23. $\dfrac{1025}{8 + e^{4x}} = 5$ **24.** $\ln x = \dfrac{1}{2}$

25. $18 + 4 \ln x = 7$ **26.** $\log x + \log(x - 15) = 2$

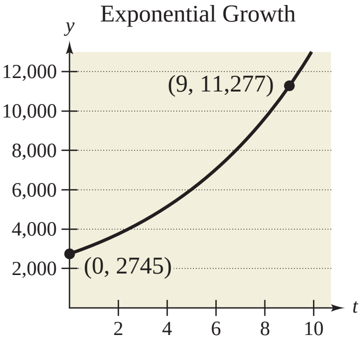


Figure for 27

27. Find the exponential growth model that fits the points shown in the graph.

28. The half-life of radioactive actinium (^{227}Ac) is 21.77 years. What percent of a present amount of radioactive actinium will remain after 19 years?

29. A model that can predict the height H (in centimeters) of a child based on his or her age is $H = 70.228 + 5.104x + 9.222 \ln x$, $\frac{1}{4} \le x \le 6$, where x is the age of the child in years. *(Source: Snapshots of Applications in Mathematics)*

(a) Construct a table of values for the model. Then sketch the graph of the model.

(b) Use the graph from part (a) to predict the height of a child when he or she is four years old. Then confirm your prediction algebraically.

Cumulative Test for Chapters 3–5

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

1. Find the quadratic function whose graph has a vertex at $(-8, 5)$ and passes through the point $(-4, -7)$.

In Exercises 2–4, sketch the graph of the function.

2. $h(x) = -(x^2 + 4x)$ **3.** $f(t) = \frac{1}{4}t(t - 2)^2$ **4.** $g(s) = s^2 + 2s + 9$

In Exercises 5 and 6, find all the zeros of the function.

5. $f(x) = x^3 + 2x^2 + 4x + 8$ **6.** $f(x) = x^4 + 4x^3 - 21x^2$

7. Use long division to divide: $\dfrac{6x^3 - 4x^2}{2x^2 + 1}$.

8. Use synthetic division to divide $3x^4 + 2x^2 - 5x + 3$ by $x - 2$.

9. Use a graphing utility to approximate (to two decimal places) the real zero of the function $g(x) = x^3 + 3x^2 - 6$.

10. Find a polynomial function with real coefficients that has -5, -2, and $2 + \sqrt{3}i$ as its zeros. (There are many correct answers.)

In Exercises 11 and 12, find the domain of the function and identify all asymptotes. Sketch the graph of the function.

11. $f(x) = \dfrac{2x}{x - 3}$ **12.** $f(x) = \dfrac{4x^2}{x - 5}$

In Exercises 13–15, sketch the graph of the rational function. Identify all intercepts and asymptotes.

13. $f(x) = \dfrac{2x}{x^2 + 2x - 3}$ **14.** $f(x) = \dfrac{x^2 - 4}{x^2 + x - 2}$

15. $f(x) = \dfrac{x^3 - 2x^2 - 9x + 18}{x^2 + 4x + 3}$

In Exercises 16 and 17, sketch the graph of the conic.

16. $\dfrac{(x + 3)^2}{16} - \dfrac{(y + 4)^2}{25} = 1$ **17.** $\dfrac{(x - 2)^2}{4} + \dfrac{(y + 1)^2}{9} = 1$

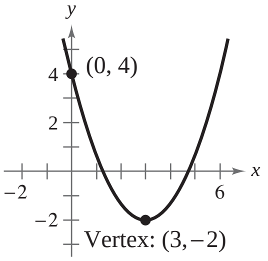


Figure for 18

18. Find the standard form of the equation of the parabola shown in the figure.

19. Find the standard form of the equation of the hyperbola with foci $(0, 0)$ and $(0, 4)$ and asymptotes $y = \pm\frac{1}{2}x + 2$.

In Exercises 20 and 21, use the graph of *f* to describe the transformation that yields the graph of *g*. Use a graphing utility to graph both functions in the same viewing window.

20. $f(x) = \left(\frac{2}{5}\right)^x$, $g(x) = -\left(\frac{2}{5}\right)^{-x+3}$ **21.** $f(x) = 2.2^x$, $g(x) = -2.2^x + 4$

In Exercises 22–25, use a calculator to evaluate the expression. Round your result to three decimal places.

22. $\log 98$ **23.** $\log \frac{6}{7}$

24. $\ln\sqrt{31}$ **25.** $\ln\left(\sqrt{30} - 4\right)$

In Exercises 26–28, evaluate the logarithm using the change-of-base formula. Round your result to three decimal places.

26. $\log_5 4.3$

27. $\log_3 0.149$

28. $\log_{1/2} 17$

29. Use the properties of logarithms to expand $\ln\left(\frac{x^2 - 16}{x^4}\right)$, where $x > 4$.

30. Condense $2 \ln x - \frac{1}{2} \ln(x + 5)$ to the logarithm of a single quantity.

In Exercises 31–36, solve the equation algebraically. Approximate the result to three decimal places.

31. $6e^{2x} = 72$

32. $4^{x-5} + 21 = 30$

33. $e^{2x} - 13e^x + 42 = 0$

34. $\log_2 x + \log_2 5 = 6$

35. $\ln 4x - \ln 2 = 8$

36. $\ln \sqrt{x + 2} = 3$

37. Use a graphing utility to graph

$$f(x) = \frac{1000}{1 + 4e^{-0.2x}}$$

and determine the horizontal asymptotes.

38. The sales S (in billions of dollars) of lottery tickets in the United States from 2000 through 2010 are shown in the table. *(Source: TLF Publications, Inc.)*

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 0$ corresponding to 2000.

(b) Use the *regression* feature of the graphing utility to find a cubic model for the data.

(c) Use the graphing utility to graph the model in the same viewing window used for the scatter plot. How well does the model fit the data?

(d) Use the model to predict the sales of lottery tickets in 2018. Does your answer seem reasonable? Explain.

DATA

Year	Sales, S
2000	37.2
2001	38.4
2002	42.0
2003	43.5
2004	47.7
2005	47.4
2006	51.6
2007	52.4
2008	53.4
2009	53.1
2010	54.2

Spreadsheet at LarsonPrecalculus.com

Table for 38

39. On the day a grandchild is born, a grandparent deposits \$2500 in a fund earning 7.5%, compounded continuously. Determine the balance in the account on the grandchild's 25th birthday.

40. The number N of bacteria in a culture is given by the model $N = 175e^{kt}$, where t is the time in hours. If $N = 420$ when $t = 8$, then estimate the time required for the population to double in size.

41. The populations P of Texas (in millions) from 2001 through 2010 can be approximated by the model $P = 20.871e^{0.0188t}$, where t represents the year, with $t = 1$ corresponding to 2001. According to this model, when will the population reach 30 million? *(Source: U.S. Census Bureau)*

42. The population p of a species of bird t years after it is introduced into a new habitat is given by

$$p = \frac{1200}{1 + 3e^{-t/5}}.$$

(a) Determine the population size that was introduced into the habitat.

(b) Determine the population after 5 years.

(c) After how many years will the population be 800?

Proofs in Mathematics

Each of the following three properties of logarithms can be proved by using properties of exponential functions.

SLIDE RULES

William Oughtred (1574–1660) invented the slide rule in 1625. The slide rule is a computational device with a sliding portion and a fixed portion. A slide rule enables you to perform multiplication by using the Product Property of Logarithms. There are other slide rules that allow for the calculation of roots and trigonometric functions. Mathematicians and engineers used slide rules until the invention of the hand-held calculator in 1972.

Properties of Logarithms ***(p. 384)***

Let a be a positive number such that $a \neq 1$, and let n be a real number. If u and v are positive real numbers, then the following properties are true.

	Logarithm with Base a	**Natural Logarithm**
1. Product Property:	$\log_a(uv) = \log_a u + \log_a v$	$\ln(uv) = \ln u + \ln v$
2. Quotient Property:	$\log_a \frac{u}{v} = \log_a u - \log_a v$	$\ln \frac{u}{v} = \ln u - \ln v$
3. Power Property:	$\log_a u^n = n \log_a u$	$\ln u^n = n \ln u$

Proof

Let

$$x = \log_a u \quad \text{and} \quad y = \log_a v.$$

The corresponding exponential forms of these two equations are

$$a^x = u \quad \text{and} \quad a^y = v.$$

To prove the Product Property, multiply u and v to obtain

$$\begin{aligned} uv &= a^x a^y \\ &= a^{x+y}. \end{aligned}$$

The corresponding logarithmic form of $uv = a^{x+y}$ is $\log_a(uv) = x + y$. So,

$$\log_a(uv) = \log_a u + \log_a v.$$

To prove the Quotient Property, divide u by v to obtain

$$\begin{aligned} \frac{u}{v} &= \frac{a^x}{a^y} \\ &= a^{x-y}. \end{aligned}$$

The corresponding logarithmic form of $\frac{u}{v} = a^{x-y}$ is $\log_a \frac{u}{v} = x - y$. So,

$$\log_a \frac{u}{v} = \log_a u - \log_a v.$$

To prove the Power Property, substitute a^x for u in the expression $\log_a u^n$, as follows.

$$\begin{aligned} \log_a u^n &= \log_a(a^x)^n && \text{Substitute } a^x \text{ for } u. \\ &= \log_a a^{nx} && \text{Property of Exponents} \\ &= nx && \text{Inverse Property of Logarithms} \\ &= n \log_a u && \text{Substitute } \log_a u \text{ for } x. \end{aligned}$$

So, $\log_a u^n = n \log_a u$. ■

P.S. Problem Solving

1. Graphical Analysis Graph the exponential function $y = a^x$ for $a = 0.5$, 1.2, and 2.0. Which of these curves intersects the line $y = x$? Determine all positive numbers a for which the curve $y = a^x$ intersects the line $y = x$.

2. Graphical Analysis Use a graphing utility to graph $y_1 = e^x$ and each of the functions $y_2 = x^2$, $y_3 = x^3$, $y_4 = \sqrt{x}$, and $y_5 = |x|$. Which function increases at the greatest rate as x approaches $+\infty$?

3. Conjecture Use the result of Exercise 2 to make a conjecture about the rate of growth of $y_1 = e^x$ and $y = x^n$, where n is a natural number and x approaches $+\infty$.

4. Implication of "Growing Exponentially" Use the results of Exercises 2 and 3 to describe what is implied when it is stated that a quantity is growing exponentially.

5. Exponential Function Given the exponential function

$$f(x) = a^x$$

show that

(a) $f(u + v) = f(u) \cdot f(v)$. (b) $f(2x) = [f(x)]^2$.

6. Hyperbolic Functions Given that

$$f(x) = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad g(x) = \frac{e^x - e^{-x}}{2}$$

show that

$$[f(x)]^2 - [g(x)]^2 = 1.$$

7. Graphical Analysis Use a graphing utility to compare the graph of the function $y = e^x$ with the graph of each given function. [$n!$ (read "n factorial") is defined as $n! = 1 \cdot 2 \cdot 3 \cdot \cdot \cdot (n - 1) \cdot n$.]

(a) $y_1 = 1 + \dfrac{x}{1!}$

(b) $y_2 = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!}$

(c) $y_3 = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}$

8. Identifying a Pattern Identify the pattern of successive polynomials given in Exercise 7. Extend the pattern one more term and compare the graph of the resulting polynomial function with the graph of $y = e^x$. What do you think this pattern implies?

9. Finding an Inverse Function Graph the function

$$f(x) = e^x - e^{-x}.$$

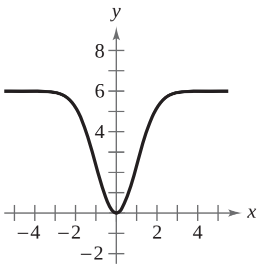
From the graph, the function appears to be one-to-one. Assuming that the function has an inverse function, find $f^{-1}(x)$.

10. Finding a Pattern for an Inverse Function Find a pattern for $f^{-1}(x)$ when

$$f(x) = \frac{a^x + 1}{a^x - 1}$$

where $a > 0$, $a \neq 1$.

11. Determining the Equation of a Graph By observation, determine whether equation (a), (b), or (c) corresponds to the graph. Explain your reasoning.



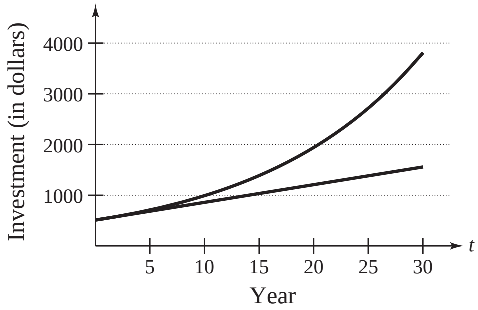
(a) $y = 6e^{-x^2/2}$

(b) $y = \dfrac{6}{1 + e^{-x/2}}$

(c) $y = 6(1 - e^{-x^2/2})$

12. Simple and Compound Interest You have two options for investing \$500. The first earns 7% compounded annually, and the second earns 7% simple interest. The figure shows the growth of each investment over a 30-year period.

(a) Identify which graph represents each type of investment. Explain your reasoning.



(b) Verify your answer in part (a) by finding the equations that model the investment growth and by graphing the models.

(c) Which option would you choose? Explain your reasoning.

13. Radioactive Decay Two different samples of radioactive isotopes are decaying. The isotopes have initial amounts of c_1 and c_2, as well as half-lives of k_1 and k_2, respectively. Find the time t required for the samples to decay to equal amounts.

14. **Bacteria Decay** A lab culture initially contains 500 bacteria. Two hours later, the number of bacteria decreases to 200. Find the exponential decay model of the form

$$B = B_0 a^{kt}$$

that approximates the number of bacteria after t hours.

15. **Colonial Population** The table shows the colonial population estimates of the American colonies from 1700 through 1780. *(Source: U.S. Census Bureau)*

Year	Population
1700	250,900
1710	331,700
1720	466,200
1730	629,400
1740	905,600
1750	1,170,800
1760	1,593,600
1770	2,148,400
1780	2,780,400

Spreadsheet at LarsonPrecalculus.com

In each of the following, let y represent the population in the year t, with $t = 0$ corresponding to 1700.

(a) Use the *regression* feature of a graphing utility to find an exponential model for the data.

(b) Use the *regression* feature of the graphing utility to find a quadratic model for the data.

(c) Use the graphing utility to plot the data and the models from parts (a) and (b) in the same viewing window.

(d) Which model is a better fit for the data? Would you use this model to predict the population of the United States in 2018? Explain your reasoning.

16. **Ratio of Logarithms** Show that

$$\frac{\log_a x}{\log_{a/b} x} = 1 + \log_a \frac{1}{b}.$$

17. **Solving a Logarithmic Equation** Solve

$$(\ln x)^2 = \ln x^2.$$

18. **Graphical Analysis** Use a graphing utility to compare the graph of the function $y = \ln x$ with the graph of each given function.

(a) $y_1 = x - 1$

(b) $y_2 = (x - 1) - \frac{1}{2}(x - 1)^2$

(c) $y_3 = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3$

19. **Identifying a Pattern** Identify the pattern of successive polynomials given in Exercise 18. Extend the pattern one more term and compare the graph of the resulting polynomial function with the graph of $y = \ln x$. What do you think the pattern implies?

20. **Finding Slope and *y*-Intercept** Using

$$y = ab^x \quad \text{and} \quad y = ax^b$$

take the natural logarithm of each side of each equation. What are the slope and y-intercept of the line relating x and $\ln y$ for $y = ab^x$? What are the slope and y-intercept of the line relating $\ln x$ and $\ln y$ for $y = ax^b$?

Ventilation Rate In Exercises 21 and 22, use the model

$$y = 80.4 - 11 \ln x, \quad 100 \le x \le 1500$$

which approximates the minimum required ventilation rate in terms of the air space per child in a public school classroom. In the model, *x* is the air space per child in cubic feet and *y* is the ventilation rate per child in cubic feet per minute.

21. Use a graphing utility to graph the model and approximate the required ventilation rate when there are 300 cubic feet of air space per child.

22. In a classroom designed for 30 students, the air conditioning system can move 450 cubic feet of air per minute.

(a) Determine the ventilation rate per child in a full classroom.

(b) Estimate the air space required per child.

(c) Determine the minimum number of square feet of floor space required for the room when the ceiling height is 30 feet.

Data Analysis In Exercises 23–26, (a) use a graphing utility to create a scatter plot of the data, (b) decide whether the data could best be modeled by a linear model, an exponential model, or a logarithmic model, (c) explain why you chose the model you did in part (b), (d) use the *regression* feature of the graphing utility to find the model you chose in part (b) for the data and graph the model with the scatter plot, and (e) determine how well the model you chose fits the data.

23. (1, 2.0), (1.5, 3.5), (2, 4.0), (4, 5.8), (6, 7.0), (8, 7.8)

24. (1, 4.4), (1.5, 4.7), (2, 5.5), (4, 9.9), (6, 18.1), (8, 33.0)

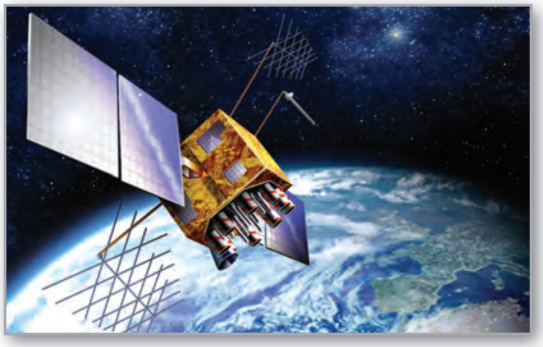
25. (1, 7.5), (1.5, 7.0), (2, 6.8), (4, 5.0), (6, 3.5), (8, 2.0)

26. (1, 5.0), (1.5, 6.0), (2, 6.4), (4, 7.8), (6, 8.6), (8, 9.0)

6 Systems of Equations and Inequalities

Nutrition *(Example 9, page 472)*

Waste Water *(Exercise 59, page 465)*



Global Positioning System *(page 451)*

Fuel Mixture
(Exercise 50, page 443)

Traffic Flow *(page 425)*

Clockwise from top left, tlorna/Shutterstock.com; Kinetic Imagery/Shutterstock.com;
Vlad61/Shutterstock.com; Taras Vyshnya/Shutterstock.com; edobric/Shutterstock.com

6.1 Linear and Nonlinear Systems of Equations

Graphs of systems of equations can help you solve real-life problems. For instance, in Exercise 64 on page 433, you will use the graph of a system of equations to approximate when the consumption of wind energy surpassed the consumption of geothermal energy.

- **Use the method of substitution to solve systems of linear equations in two variables.**
- **Use the method of substitution to solve systems of nonlinear equations in two variables.**
- **Use a graphical approach to solve systems of equations in two variables.**
- **Use systems of equations to model and solve real-life problems.**

The Method of Substitution

Up to this point in the text, most problems have involved either a function of one variable or a single equation in two variables. However, many problems in science, business, and engineering involve two or more equations in two or more variables. To solve such a problem, you need to find solutions of a **system of equations.** Here is an example of a system of two equations in two unknowns.

$$\begin{cases} 2x + y = 5 & \text{Equation 1} \\ 3x - 2y = 4 & \text{Equation 2} \end{cases}$$

A **solution** of this system is an ordered pair that satisfies each equation in the system. Finding the set of all solutions is called **solving the system of equations.** For instance, the ordered pair $(2, 1)$ is a solution of this system. To check this, substitute 2 for x and 1 for y in *each* equation.

Check (2, 1) in Equation 1 and Equation 2:

$$2x + y = 5 \qquad \text{Write Equation 1.}$$

$$2(2) + 1 \stackrel{?}{=} 5 \qquad \text{Substitute 2 for } x \text{ and 1 for } y.$$

$$4 + 1 = 5 \qquad \text{Solution checks in Equation 1.} \checkmark$$

$$3x - 2y = 4 \qquad \text{Write Equation 2.}$$

$$3(2) - 2(1) \stackrel{?}{=} 4 \qquad \text{Substitute 2 for } x \text{ and 1 for } y.$$

$$6 - 2 = 4 \qquad \text{Solution checks in Equation 2.} \checkmark$$

In this chapter, you will study four ways to solve systems of equations, beginning with the **method of substitution.**

Method	Section	Type of System
1. Substitution	6.1	Linear or nonlinear, two variables
2. Graphical method	6.1	Linear or nonlinear, two variables
3. Elimination	6.2	Linear, two variables
4. Gaussian elimination	6.3	Linear, three or more variables

Method of Substitution

1. *Solve* one of the equations for one variable in terms of the other.
2. *Substitute* the expression found in Step 1 into the other equation to obtain an equation in one variable.
3. *Solve* the equation obtained in Step 2.
4. *Back-substitute* the value obtained in Step 3 into the expression obtained in Step 1 to find the value of the other variable.
5. *Check* that the solution satisfies *each* of the original equations.

ssuaphotos/Shutterstock.com

Systems of equations have a wide variety of real-life applications. For instance, researchers in Italy studying the acoustical noise levels from vehicular traffic at a busy three-way intersection used a system of equations to model the traffic flow at the intersection. To help formulate the system of equations, "operators" stationed themselves at various locations along the intersection and counted the numbers of vehicles that passed them. *(Source: Proceedings of the 11th WSEAS International Conference on Acoustics & Music: Theory & Applications)*

EXAMPLE 1 Solving a System of Equations by Substitution

Solve the system of equations.

$$\begin{cases} x + y = 4 & \text{Equation 1} \\ x - y = 2 & \text{Equation 2} \end{cases}$$

Solution Begin by solving for y in Equation 1.

$y = 4 - x$ Solve for y in Equation 1.

Next, substitute this expression for y into Equation 2 and solve the resulting single-variable equation for x.

$x - y = 2$ Write Equation 2.

$x - (4 - x) = 2$ Substitute $4 - x$ for y.

$x - 4 + x = 2$ Distributive Property

$2x = 6$ Combine like terms.

$x = 3$ Divide each side by 2.

Finally, solve for y by *back-substituting* $x = 3$ into the equation $y = 4 - x$, to obtain

$y = 4 - x$ Write revised Equation 1.

$y = 4 - 3$ Substitute 3 for x.

$y = 1.$ Solve for y.

The solution is the ordered pair

$(3, 1).$

Check this solution as follows.

REMARK Because many steps are required to solve a system of equations, it is very easy to make errors in arithmetic. So, you should always check your solution by substituting it into *each* equation in the original system.

Check

Substitute (3, 1) into Equation 1:

$x + y = 4$ Write Equation 1.

$3 + 1 \overset{?}{=} 4$ Substitute for x and y.

$4 = 4$ Solution checks in Equation 1. ✓

Substitute (3, 1) into Equation 2:

$x - y = 2$ Write Equation 2.

$3 - 1 \overset{?}{=} 2$ Substitute for x and y.

$2 = 2$ Solution checks in Equation 2. ✓

Because (3, 1) satisfies both equations in the system, it is a solution of the system of equations.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve the system of equations.

$$\begin{cases} x - y = 0 \\ 5x - 3y = 6 \end{cases}$$

The term *back-substitution* implies that you work *backwards*. First you solve for one of the variables, and then you substitute that value *back* into one of the equations in the system to find the value of the other variable.

Taras Vyshnya/Shutterstock.com

EXAMPLE 2 Solving a System by Substitution

A total of \$12,000 is invested in two funds paying 5% and 3% simple interest. (Recall that the formula for simple interest is $I = Prt$, where P is the principal, r is the annual interest rate, and t is the time.) The yearly interest is \$500. How much is invested at each rate?

Solution

Verbal Model:

Amount in 5% fund	+	Amount in 3% fund	=	Total investment
Interest for 5% fund	+	Interest for 3% fund	=	Total interest

Labels:
Amount in 5% fund $= x$ (dollars)
Interest for 5% fund $= 0.05x$ (dollars)
Amount in 3% fund $= y$ (dollars)
Interest for 3% fund $= 0.03y$ (dollars)
Total investment $= 12{,}000$ (dollars)
Total interest $= 500$ (dollars)

System:

$$\begin{cases} x + y = 12{,}000 & \text{Equation 1} \\ 0.05x + 0.03y = 500 & \text{Equation 2} \end{cases}$$

REMARK When using the method of substitution, it does not matter which variable you choose to solve for first. Whether you solve for y first or x first, you will obtain the same solution. You should choose the variable and equation that make your work easier. For instance, in Example 2, solving for x in Equation 1 is easier than solving for x in Equation 2.

To begin, it is convenient to multiply each side of Equation 2 by 100. This eliminates the need to work with decimals.

$$\begin{aligned} 100(0.05x + 0.03y) &= 100(500) && \text{Multiply each side of Equation 2 by 100.} \\ 5x + 3y &= 50{,}000 && \text{Revised Equation 2} \end{aligned}$$

To solve this system, you can solve for x in Equation 1.

$$x = 12{,}000 - y \qquad \text{Revised Equation 1}$$

Then, substitute this expression for x into revised Equation 2 and solve the resulting equation for y.

$$\begin{aligned} 5x + 3y &= 50{,}000 && \text{Write revised Equation 2.} \\ 5(12{,}000 - y) + 3y &= 50{,}000 && \text{Substitute } 12{,}000 - y \text{ for } x. \\ 60{,}000 - 5y + 3y &= 50{,}000 && \text{Distributive Property} \\ -2y &= -10{,}000 && \text{Combine like terms.} \\ y &= 5000 && \text{Divide each side by } -2. \end{aligned}$$

Next, back-substitute $y = 5000$ to solve for x.

$$\begin{aligned} x &= 12{,}000 - y && \text{Write revised Equation 1.} \\ x &= 12{,}000 - 5000 && \text{Substitute 5000 for } y. \\ x &= 7000 && \text{Subtract.} \end{aligned}$$

The solution is (7000, 5000). So, \$7000 is invested at 5% and \$5000 is invested at 3%. Check this in the original system.

TECHNOLOGY One way to check the answers you obtain in this section is to use a graphing utility. For instance, graph the two equations in Example 2

$$y_1 = 12{,}000 - x$$

$$y_2 = \frac{500 - 0.05x}{0.03}$$

and find the point of intersection. Does this point agree with the solution obtained at the right?

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

A total of \$25,000 is invested in two funds paying 6.5% and 8.5% simple interest. The yearly interest is \$2000. How much is invested at each rate?

Nonlinear Systems of Equations

The equations in Examples 1 and 2 are linear. The method of substitution can also be used to solve systems in which one or both of the equations are nonlinear.

▷ **ALGEBRA HELP** You can review the techniques for factoring in Section P.4.

EXAMPLE 3 Substitution: Two-Solution Case

Solve the system of equations.

$$\begin{cases} 3x^2 + 4x - y = 7 & \text{Equation 1} \\ 2x - y = -1 & \text{Equation 2} \end{cases}$$

Solution Begin by solving for y in Equation 2 to obtain $y = 2x + 1$. Next, substitute this expression for y into Equation 1 and solve for x.

$$3x^2 + 4x - (2x + 1) = 7 \qquad \text{Substitute } 2x + 1 \text{ for } y \text{ in Equation 1.}$$

$$3x^2 + 2x - 1 = 7 \qquad \text{Simplify.}$$

$$3x^2 + 2x - 8 = 0 \qquad \text{Write in general form.}$$

$$(3x - 4)(x + 2) = 0 \qquad \text{Factor.}$$

$$x = \frac{4}{3}, -2 \qquad \text{Solve for } x.$$

Back-substituting these values of x to solve for the corresponding values of y produces the solutions $\left(\frac{4}{3}, \frac{11}{3}\right)$ and $(-2, -3)$. Check these in the original system.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve the system of equations.

$$\begin{cases} -2x + y = 5 \\ x^2 - y + 3x = 1 \end{cases}$$

EXAMPLE 4 Substitution: No-Real-Solution Case

Solve the system of equations.

$$\begin{cases} -x + y = 4 & \text{Equation 1} \\ x^2 + y = 3 & \text{Equation 2} \end{cases}$$

Solution Begin by solving for y in Equation 1 to obtain $y = x + 4$. Next, substitute this expression for y into Equation 2 and solve for x.

$$x^2 + (x + 4) = 3 \qquad \text{Substitute } x + 4 \text{ for } y \text{ in Equation 2.}$$

$$x^2 + x + 1 = 0 \qquad \text{Write in general form.}$$

$$x = \frac{-1 \pm \sqrt{-3}}{2} \qquad \text{Use the Quadratic Formula.}$$

Because the discriminant is negative, the equation

$$x^2 + x + 1 = 0$$

has no (real) solution. So, the original system has no (real) solution.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve the system of equations.

$$\begin{cases} 2x - y = -3 \\ 2x^2 + 4x - y^2 = 0 \end{cases}$$

TECHNOLOGY Most graphing utilities have built-in features that approximate the point(s) of intersection of two graphs. Use a graphing utility to find the points of intersection of the graphs in Figures 6.1 through 6.3. Be sure to adjust the viewing window so that you see all the points of intersection.

ALGEBRA HELP You can review the techniques for graphing equations in Section 1.1.

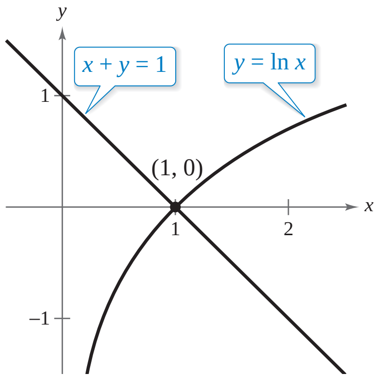
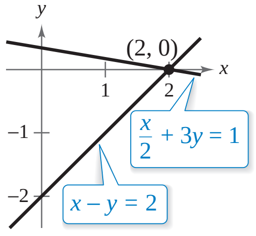


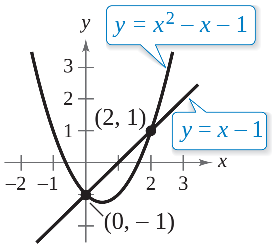
Figure 6.4

Graphical Approach to Finding Solutions

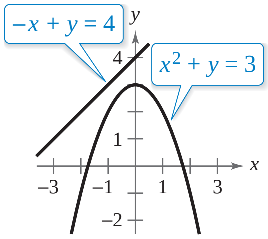
Notice from Examples 2, 3, and 4 that a system of two equations in two unknowns can have exactly one solution, more than one solution, or no solution. By using a **graphical method,** you can gain insight about the number of solutions and the location(s) of the solution(s) of a system of equations by graphing each of the equations in the same coordinate plane. The solutions of the system correspond to the **points of intersection** of the graphs. For instance, the two equations in Figure 6.1 graph as two lines with a *single point* of intersection; the two equations in Figure 6.2 graph as a parabola and a line with *two points* of intersection; and the two equations in Figure 6.3 graph as a parabola and a line with *no points* of intersection.



One intersection point
Figure 6.1



Two intersection points
Figure 6.2



No intersection points
Figure 6.3

EXAMPLE 5 Solving a System of Equations Graphically

Solve the system of equations.

$$\begin{cases} y = \ln x & \text{Equation 1} \\ x + y = 1 & \text{Equation 2} \end{cases}$$

Solution There is only one point of intersection of the graphs of the two equations, and (1, 0) is the solution point (see Figure 6.4). Check this solution as follows.

Check (1, 0) in Equation 1:

$y = \ln x$ Write Equation 1.

$0 = \ln 1$ Substitute for x and y.

$0 = 0$ Solution checks in Equation 1. ✓

Check (1, 0) in Equation 2:

$x + y = 1$ Write Equation 2.

$1 + 0 = 1$ Substitute for x and y.

$1 = 1$ Solution checks in Equation 2. ✓

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve the system of equations.

$$\begin{cases} y = 3 - \log x \\ -2x + y = 1 \end{cases}$$

Example 5 shows the benefit of a graphical approach to solving systems of equations in two variables. Notice that by trying only the substitution method in Example 5, you would obtain the equation $x + \ln x = 1$. It would be difficult to solve this equation for x using standard algebraic techniques.

Applications

The total cost C of producing x units of a product typically has two components—the initial cost and the cost per unit. When enough units have been sold so that the total revenue R equals the total cost C, the sales are said to have reached the **break-even point.** You will find that the break-even point corresponds to the point of intersection of the cost and revenue curves.

EXAMPLE 6 Break-Even Analysis

A shoe company invests \$300,000 in equipment to produce a new line of athletic footwear. Each pair of shoes costs \$5 to produce and sells for \$60. How many pairs of shoes must the company sell to break even?

Algebraic Solution

The total cost of producing x units is

Total cost = Cost per unit · Number of units + Initial cost

$C = 5x + 300{,}000.$ Equation 1

The revenue obtained by selling x units is

Total revenue = Price per unit · Number of units

$R = 60x.$ Equation 2

Because the break-even point occurs when $R = C$, you have $C = 60x$, and the system of equations to solve is

$$\begin{cases} C = 5x + 300{,}000 \\ C = 60x \end{cases}.$$

Solve by substitution.

$60x = 5x + 300{,}000$ Substitute $60x$ for C in Equation 1.

$55x = 300{,}000$ Subtract $5x$ from each side.

$x \approx 5455$ Divide each side by 55.

So, the company must sell about 5455 pairs of shoes to break even.

Graphical Solution

The total cost of producing x units is

Total cost = Cost per unit · Number of units + Initial cost

$C = 5x + 300{,}000.$ Equation 1

The revenue obtained by selling x units is

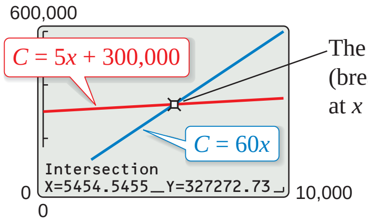
Total revenue = Price per unit · Number of units

$R = 60x.$ Equation 2

Because the break-even point occurs when $R = C$, you have $C = 60x$, and the system of equations to solve is

$$\begin{cases} C = 5x + 300{,}000 \\ C = 60x \end{cases}.$$

Use a graphing utility to graph $y_1 = 5x + 300{,}000$ and $y_2 = 60x$ in the same viewing window.



The point of intersection (break-even point) occurs at $x \approx 5455$.

So, the company must sell about 5455 pairs of shoes to break even.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

In Example 6, each pair of shoes costs \$12 to produce. How many pairs of shoes must the company sell to break even?

Another way to view the solution in Example 6 is to consider the profit function $P = R - C$. The break-even point occurs when the profit is 0, which is the same as saying that $R = C$.

EXAMPLE 7 Movie Ticket Sales

The weekly ticket sales for a new comedy movie decreased each week. At the same time, the weekly ticket sales for a new drama movie increased each week. Models that approximate the weekly ticket sales S (in millions of dollars) for the movies are

$$\begin{cases} S = 60 - 8x & \text{Comedy} \\ S = 10 + 4.5x & \text{Drama} \end{cases}$$

where x represents the number of weeks each movie was in theaters, with $x = 0$ corresponding to the opening weekend. After how many weeks will the ticket sales for the two movies be equal?

Algebraic Solution

Because both equations are already solved for S in terms of x, substitute either expression for S into the other equation and solve for x.

$$10 + 4.5x = 60 - 8x \qquad \text{Substitute for } S \text{ in Equation 1.}$$

$$4.5x + 8x = 60 - 10 \qquad \text{Add } 8x \text{ and } -10 \text{ to each side.}$$

$$12.5x = 50 \qquad \text{Combine like terms.}$$

$$x = 4 \qquad \text{Divide each side by 12.5.}$$

So, the weekly ticket sales for the two movies will be equal after 4 weeks.

Numerical Solution

You can create a table of values for each model to determine when the ticket sales for the two movies will be equal.

Number of Weeks, x	0	1	2	3	4	5	6
Sales, S (comedy)	60	52	44	36	28	20	12
Sales, S (drama)	10	14.5	19	23.5	28	32.5	37

So, from the above table, the weekly ticket sales for the two movies will be equal after 4 weeks.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The weekly ticket sales for a new animated movie decreased each week. At the same time, the weekly ticket sales for a new horror movie increased each week. Models that approximate the weekly ticket sales S (in millions of dollars) for the movies are

$$\begin{cases} S = 108 - 9.4x & \text{Animated} \\ S = 16 + 9x & \text{Horror} \end{cases}$$

where x represents the number of weeks each movie was in theaters, with $x = 0$ corresponding to the opening weekend. After how many weeks will the ticket sales for the two movies be equal?

Summarize (Section 6.1)

1. Explain how to use the method of substitution to solve a system of linear equations in two variables *(page 424)*. For examples of using the method of substitution to solve systems of linear equations in two variables, see Examples 1 and 2.
2. Explain how to use the method of substitution to solve a system of nonlinear equations in two variables *(page 427)*. For examples of using the method of substitution to solve systems of nonlinear equations in two variables, see Examples 3 and 4.
3. Explain how to use a graphical approach to solve a system of equations in two variables *(page 428)*. For an example of using a graphical approach to solve a system of equations in two variables, see Example 5.
4. Describe examples of how to use systems of equations to model and solve real-life problems *(pages 429 and 430, Examples 6 and 7)*.

6.1 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. A ________ of a system of equations is an ordered pair that satisfies each equation in the system.

2. The first step in solving a system of equations by the method of ________ is to solve one of the equations for one variable in terms of the other variable.

3. Graphically, the solution of a system of two equations is the ________ of ________ of the graphs of the two equations.

4. In business applications, the point at which the revenue equals costs is called the ________ point.

Skills and Applications

Checking Solutions

In Exercises 5 and 6, determine whether each ordered pair is a solution of the system.

5. $\begin{cases} 2x - y = 4 \\ 8x + y = -9 \end{cases}$

(a) $(0, -4)$ (b) $(-2, 7)$

(c) $\left(\frac{3}{2}, -1\right)$ (d) $\left(-\frac{1}{2}, -5\right)$

6. $\begin{cases} 4x^2 + y = 3 \\ -x - y = 11 \end{cases}$

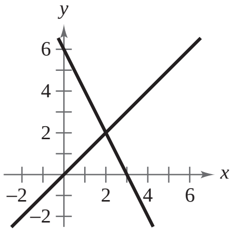
(a) $(2, -13)$ (b) $(2, -9)$

(c) $\left(-\frac{3}{2}, -\frac{31}{3}\right)$ (d) $\left(-\frac{7}{4}, -\frac{37}{4}\right)$

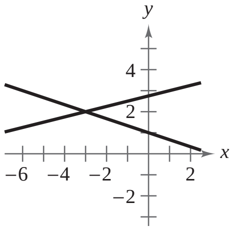
Solving a System by Substitution

In Exercises 7–14, solve the system by the method of substitution. Check your solution(s) graphically.

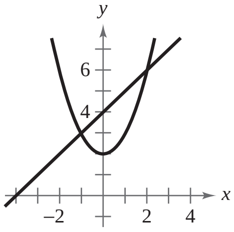
7. $\begin{cases} 2x + y = 6 \\ -x + y = 0 \end{cases}$



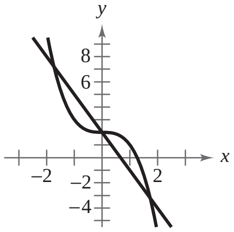
8. $\begin{cases} x - 4y = -11 \\ x + 3y = 3 \end{cases}$



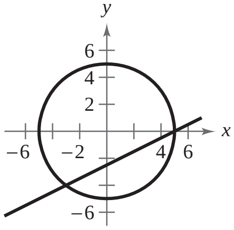
9. $\begin{cases} x - y = -4 \\ x^2 - y = -2 \end{cases}$



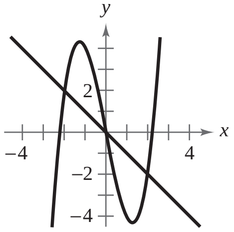
10. $\begin{cases} 3x + y = 2 \\ x^3 - 2 + y = 0 \end{cases}$



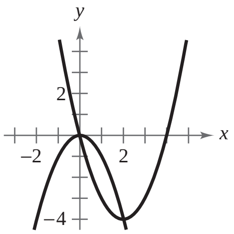
11. $\begin{cases} -\frac{1}{2}x + y = -\frac{5}{2} \\ x^2 + y^2 = 25 \end{cases}$



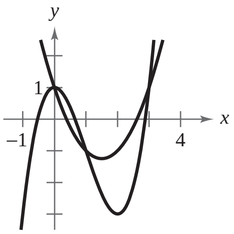
12. $\begin{cases} x + y = 0 \\ x^3 - 5x - y = 0 \end{cases}$



13. $\begin{cases} x^2 + y = 0 \\ x^2 - 4x - y = 0 \end{cases}$



14. $\begin{cases} y = x^3 - 3x^2 + 1 \\ y = x^2 - 3x + 1 \end{cases}$



Solving a System by Substitution

In Exercises 15–24, solve the system by the method of substitution.

15. $\begin{cases} x - y = 2 \\ 6x - 5y = 16 \end{cases}$

16. $\begin{cases} x + 4y = 3 \\ 2x - 7y = -24 \end{cases}$

17. $\begin{cases} 2x - y + 2 = 0 \\ 4x + y - 5 = 0 \end{cases}$

18. $\begin{cases} 6x - 3y - 4 = 0 \\ x + 2y - 4 = 0 \end{cases}$

19. $\begin{cases} 1.5x + 0.8y = 2.3 \\ 0.3x - 0.2y = 0.1 \end{cases}$

20. $\begin{cases} 0.5x + 3.2y = 9.0 \\ 0.2x - 1.6y = -3.6 \end{cases}$

21. $\begin{cases} \frac{1}{5}x + \frac{1}{2}y = 8 \\ x + y = 20 \end{cases}$

22. $\begin{cases} \frac{1}{2}x + \frac{3}{4}y = 10 \\ \frac{3}{4}x - y = 4 \end{cases}$

23. $\begin{cases} 6x + 5y = -3 \\ -x - \frac{5}{6}y = -7 \end{cases}$

24. $\begin{cases} -\frac{2}{3}x + y = 2 \\ 2x - 3y = 6 \end{cases}$

Solving a System by Substitution

In Exercises 25–28, you are given the total annual interest earned from a total of $12,000 invested in two funds paying the given rates of simple interest. Write and solve a system of equations to find the amount invested at each rate.

	Annual Interest	Rate 1	Rate 2
25.	$500	3%	5%
26.	$630	4%	6%
27.	$396	2.8%	3.8%
28.	$254	1.75%	2.25%

Solving a System with a Nonlinear Equation **In Exercises 29–32, solve the system by the method of substitution.**

29. $\begin{cases} x^2 - y = 0 \\ 2x + y = 0 \end{cases}$ **30.** $\begin{cases} x - 2y = 0 \\ 3x - y^2 = 0 \end{cases}$

31. $\begin{cases} x - y = -1 \\ x^2 - y = -4 \end{cases}$ **32.** $\begin{cases} y = -x \\ y = x^3 + 3x^2 + 2x \end{cases}$

Solving a System of Equations Graphically **In Exercises 33–44, solve the system graphically.**

33. $\begin{cases} -x + 2y = -2 \\ 3x + y = 20 \end{cases}$ **34.** $\begin{cases} x + y = 0 \\ 3x - 2y = 5 \end{cases}$

35. $\begin{cases} x - 3y = -3 \\ 5x + 3y = -6 \end{cases}$ **36.** $\begin{cases} -x + 2y = -7 \\ x - y = 2 \end{cases}$

37. $\begin{cases} x + y = 4 \\ x^2 + y^2 - 4x = 0 \end{cases}$ **38.** $\begin{cases} -x + y = 3 \\ x^2 - 6x - 27 + y^2 = 0 \end{cases}$

39. $\begin{cases} x - y + 3 = 0 \\ x^2 - 4x + 7 = y \end{cases}$ **40.** $\begin{cases} y^2 - 4x + 11 = 0 \\ -\frac{1}{2}x + y = -\frac{1}{2} \end{cases}$

41. $\begin{cases} 3x - 2y = 0 \\ x^2 - y^2 = 4 \end{cases}$ **42.** $\begin{cases} 2x - y + 3 = 0 \\ x^2 + y^2 - 4x = 0 \end{cases}$

43. $\begin{cases} x^2 + y^2 = 25 \\ 3x^2 - 16y = 0 \end{cases}$ **44.** $\begin{cases} x^2 + y^2 = 25 \\ (x - 8)^2 + y^2 = 41 \end{cases}$

Solving a System of Equations Graphically **In Exercises 45–48, use a graphing utility to solve the system of equations. Find the solution(s) accurate to two decimal places.**

45. $\begin{cases} y = e^x \\ x - y + 1 = 0 \end{cases}$ **46.** $\begin{cases} y = -4e^{-x} \\ y + 3x + 8 = 0 \end{cases}$

47. $\begin{cases} y + 2 = \ln(x - 1) \\ 3y + 2x = 9 \end{cases}$ **48.** $\begin{cases} x^2 + y^2 = 4 \\ 2x^2 - y = 2 \end{cases}$

Choosing a Solution Method **In Exercises 49–56, solve the system graphically or algebraically. Explain your choice of method.**

49. $\begin{cases} y = 2x \\ y = x^2 + 1 \end{cases}$ **50.** $\begin{cases} x^2 + y^2 = 25 \\ 2x + y = 10 \end{cases}$

51. $\begin{cases} x - 2y = 4 \\ x^2 - y = 0 \end{cases}$ **52.** $\begin{cases} y = x^3 - 2x^2 + x - 1 \\ y = -x^2 + 3x - 1 \end{cases}$

53. $\begin{cases} y - e^{-x} = 1 \\ y - \ln x = 3 \end{cases}$ **54.** $\begin{cases} x^2 + y = 4 \\ e^x - y = 0 \end{cases}$

55. $\begin{cases} xy - 1 = 0 \\ 2x - 4y + 7 = 0 \end{cases}$ **56.** $\begin{cases} x - 2y = 1 \\ y = \sqrt{x - 1} \end{cases}$

Break-Even Analysis **In Exercises 57 and 58, find the sales necessary to break even ($R = C$) for the total cost C of producing x units and the revenue R obtained by selling x units. (Round to the nearest whole unit.)**

57. $C = 8650x + 250{,}000, \quad R = 9950x$

58. $C = 5.5\sqrt{x} + 10{,}000, \quad R = 3.29x$

59. **Break-Even Analysis** A small software company invests \$16,000 to produce a software package that will sell for \$55.95. Each unit costs \$9.45 to produce.

(a) How many units must the company sell to break even?

(b) How many units must the company sell to make a profit of \$100,000?

60. **Choice of Two Jobs** You receive two sales job offers. One company offers a straight commission of 6% of sales. The other company offers a salary of \$500 per week plus 3% of sales. How much would you have to sell in a week in order to make the straight commission job offer better?

61. **DVD Rentals** The number of rentals of a newly released DVD of a horror film at a movie rental store decreased each week. At the same time, the number of rentals of a newly released DVD of a comedy film increased each week. Models that approximate the numbers N of DVDs rented are

$$\begin{cases} N = 360 - 24x & \text{Horror film} \\ N = 24 + 18x & \text{Comedy film} \end{cases}$$

where x represents the week, with $x = 1$ corresponding to the first week of release.

(a) After how many weeks will the numbers of DVDs rented for the two films be equal?

(b) Use a table to solve the system of equations numerically. Compare your result with that of part (a).

62. **Supply and Demand** The supply and demand curves for a business dealing with wheat are

Supply: $p = 1.45 + 0.00014x^2$

Demand: $p = (2.388 - 0.007x)^2$

where p is the price in dollars per bushel and x is the quantity in bushels per day. Use a graphing utility to graph the supply and demand equations and find the market equilibrium. (The market equilibrium is the point of intersection of the graphs for $x > 0$.)

63. **Log Volume** Two rules for estimating the number of board feet in a log include the *Doyle Log Rule* and the *Scribner Log Rule*. (A board foot is a unit of measure for lumber equal to a board 1 foot square and 1 inch thick.) For a 16-foot log, the Doyle Log Rule is modeled by $V_1 = (D - 4)^2, 5 \le D \le 40$, and the Scribner Log Rule is modeled by $V_2 = 0.79D^2 - 2D - 4, 5 \le D \le 40$, where D is the diameter (in inches) of the log and V is its volume (in board feet).

(a) Use a graphing utility to graph the two log rules in the same viewing window.

(b) For what diameter do the two rules agree?

(c) You are selling large logs by the board foot. Which rule would you use? Explain your reasoning.

64. Data Analysis: Renewable Energy The table shows the consumption C (in trillions of Btus) of geothermal energy and wind energy in the United States from 2001 through 2009. *(Source: U.S. Energy Information Administration)*

Year	Geothermal, C	Wind, C
2001	164	70
2002	171	105
2003	175	115
2004	178	142
2005	181	178
2006	181	264
2007	186	341
2008	192	546
2009	200	721

Spreadsheet at LarsonPrecalculus.com

(a) Use a graphing utility to find a cubic model for the geothermal energy consumption data and a quadratic model for the wind energy consumption data. Let t represent the year, with $t = 1$ corresponding to 2001.

(b) Use the graphing utility to graph the data and the two models in the same viewing window.

(c) Use the graph from part (b) to approximate the point of intersection of the graphs of the models. Interpret your answer in the context of the problem.

(d) Describe the behavior of each model. Do you think the models can accurately predict consumption of geothermal energy and wind energy in the United States for future years? Explain.

(e) Use your school's library, the Internet, or some other reference source to research the advantages and disadvantages of using renewable energy.

Geometry **In Exercises 65 and 66, find the dimensions of the rectangle meeting the specified conditions.**

65. The perimeter is 56 meters and the length is 4 meters greater than the width.

66. The perimeter is 42 inches and the width is three-fourths the length.

67. Geometry What are the dimensions of a rectangular tract of land when its perimeter is 44 kilometers and its area is 120 square kilometers?

68. Geometry What are the dimensions of a right triangle with a two-inch hypotenuse and an area of 1 square inch?

Exploration

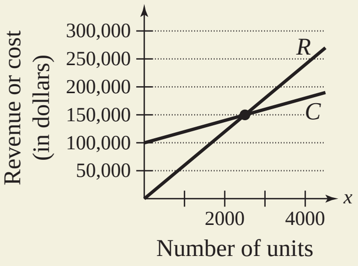
True or False? **In Exercises 69 and 70, determine whether the statement is true or false. Justify your answer.**

69. In order to solve a system of equations by substitution, you must always solve for y in one of the two equations and then back-substitute.

70. If a system consists of a parabola and a circle, then the system can have at most two solutions.

71. Think About It When solving a system of equations by substitution, how do you recognize that the system has no solution?

72. HOW DO YOU SEE IT? The cost C of producing x units and the revenue R obtained by selling x units are shown in the figure.



(a) Estimate the point of intersection. What does this point represent?

(b) Use the figure to identify the x-values that correspond to (i) an overall loss and (ii) a profit. Explain your reasoning.

73. Think About It Consider the system of equations

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}.$$

(a) Find values for a, b, c, d, e, and f so that the system has one distinct solution. (There is more than one correct answer.)

(b) Explain how to solve the system in part (a) by the method of substitution and graphically.

(c) Write a brief paragraph describing any advantages of the method of substitution over the graphical method of solving a system of equations.

ssuaphotos/Shutterstock.com

6.2 Two-Variable Linear Systems

Systems of equations in two variables can help you model and solve real-life problems. For instance, in Exercise 50 on page 443, you will write, graph, and solve a system of equations to find the numbers of gallons of 87- and 92-octane gasoline that must be mixed to obtain 500 gallons of 89-octane gasoline.

- **Use the method of elimination to solve systems of linear equations in two variables.**
- **Interpret graphically the numbers of solutions of systems of linear equations in two variables.**
- **Use systems of linear equations in two variables to model and solve real-life problems.**

The Method of Elimination

In Section 6.1, you studied two methods for solving a system of equations: substitution and graphing. Now, you will study the **method of elimination.** The key step in this method is to obtain, for one of the variables, coefficients that differ only in sign so that *adding* the equations eliminates the variable.

$$\begin{array}{rll} 3x + 5y = & 7 & \text{Equation 1} \\ \underline{-3x - 2y = } & \underline{-1} & \text{Equation 2} \\ 3y = & 6 & \text{Add equations.} \end{array}$$

Note that by adding the two equations, you eliminate the x-terms and obtain a single equation in y. Solving this equation for y produces $y = 2$, which you can then back-substitute into one of the original equations to solve for x.

EXAMPLE 1 Solving a System of Equations by Elimination

Solve the system of linear equations.

$$\begin{cases} 3x + 2y = 4 & \text{Equation 1} \\ 5x - 2y = 12 & \text{Equation 2} \end{cases}$$

Solution Because the coefficients of y differ only in sign, eliminate the y-terms by adding the two equations.

$$\begin{array}{rll} 3x + 2y = & 4 & \text{Write Equation 1.} \\ \underline{5x - 2y =} & \underline{12} & \text{Write Equation 2.} \\ 8x \qquad = & 16 & \text{Add equations.} \\ x \qquad = & 2 & \text{Solve for } x. \end{array}$$

Solve for y by back-substituting $x = 2$ into Equation 1.

$$\begin{array}{rll} 3x + 2y = 4 & & \text{Write Equation 1.} \\ 3(2) + 2y = 4 & & \text{Substitute 2 for } x. \\ y = -1 & & \text{Solve for } y. \end{array}$$

The solution is $(2, -1)$. Check this in the original system, as follows.

Check

$$\begin{array}{ll} 3(2) + 2(-1) = 4 & \text{Solution checks in Equation 1. ✓} \\ 5(2) - 2(-1) = 12 & \text{Solution checks in Equation 2. ✓} \end{array}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve the system of linear equations.

$$\begin{cases} 2x + y = 4 \\ 2x - y = -1 \end{cases}$$

Vlad61/Shutterstock.com

Method of Elimination

To use the **method of elimination** to solve a system of two linear equations in x and y, perform the following steps.

1. *Obtain coefficients* for x (or y) that differ only in sign by multiplying all terms of one or both equations by suitably chosen constants.
2. *Add* the equations to eliminate one variable.
3. *Solve* the equation obtained in Step 2.
4. *Back-substitute* the value obtained in Step 3 into either of the original equations and solve for the other variable.
5. *Check* that the solution satisfies *each* of the original equations.

EXAMPLE 2 Solving a System of Equations by Elimination

Solve the system of linear equations.

$$\begin{cases} 2x - 4y = -7 & \text{Equation 1} \\ 5x + y = -1 & \text{Equation 2} \end{cases}$$

Solution To obtain coefficients that differ only in sign, multiply Equation 2 by 4.

$$\begin{aligned} 2x - 4y &= -7 \quad \Rightarrow & 2x - 4y &= -7 && \text{Write Equation 1.} \\ \underline{5x + y} &\underline{= -1} \quad \Rightarrow & \underline{20x + 4y} &\underline{= -4} && \text{Multiply Equation 2 by 4.} \\ && 22x &= -11 && \text{Add equations.} \\ && x &= -\tfrac{1}{2} && \text{Solve for } x. \end{aligned}$$

Solve for y by back-substituting $x = -\frac{1}{2}$ into Equation 1.

$$\begin{aligned} 2x - 4y &= -7 && \text{Write Equation 1.} \\ 2\left(-\tfrac{1}{2}\right) - 4y &= -7 && \text{Substitute } -\tfrac{1}{2} \text{ for } x. \\ -4y &= -6 && \text{Simplify.} \\ y &= \tfrac{3}{2} && \text{Solve for } y. \end{aligned}$$

The solution is $\left(-\frac{1}{2}, \frac{3}{2}\right)$. Check this in the original system, as follows.

Check

$$\begin{aligned} 2x - 4y &= -7 && \text{Write Equation 1.} \\ 2\left(-\tfrac{1}{2}\right) - 4\left(\tfrac{3}{2}\right) &\stackrel{?}{=} -7 && \text{Substitute for } x \text{ and } y. \\ -1 - 6 &= -7 && \text{Solution checks in Equation 1. } \checkmark \\ 5x + y &= -1 && \text{Write Equation 2.} \\ 5\left(-\tfrac{1}{2}\right) + \tfrac{3}{2} &\stackrel{?}{=} -1 && \text{Substitute for } x \text{ and } y. \\ -\tfrac{5}{2} + \tfrac{3}{2} &= -1 && \text{Solution checks in Equation 2. } \checkmark \end{aligned}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve the system of linear equations.

$$\begin{cases} 2x + 3y = 17 \\ 5x - y = 17 \end{cases}$$

In Example 2, the two systems of linear equations (the original system and the system obtained by multiplying by a constant)

$$\begin{cases} 2x - 4y = -7 \\ 5x + y = -1 \end{cases} \quad \text{and} \quad \begin{cases} 2x - 4y = -7 \\ 20x + 4y = -4 \end{cases}$$

are called **equivalent systems** because they have precisely the same solution set. The operations that can be performed on a system of linear equations to produce an equivalent system are (1) interchanging any two equations, (2) multiplying an equation by a nonzero constant, and (3) adding a multiple of one equation to any other equation in the system.

EXAMPLE 3 Solving the System of Equations by Elimination

Solve the system of linear equations.

$$\begin{cases} 5x + 3y = 9 & \text{Equation 1} \\ 2x - 4y = 14 & \text{Equation 2} \end{cases}$$

Algebraic Solution

You can obtain coefficients that differ only in sign by multiplying Equation 1 by 4 and multiplying Equation 2 by 3.

$$\begin{aligned}
5x + 3y &= 9 \quad \Rightarrow \quad & 20x + 12y &= 36 && \text{Multiply Equation 1 by 4.}\\
\underline{2x - 4y} &\underline{= 14} \quad \Rightarrow \quad & \underline{6x - 12y} &\underline{= 42} && \text{Multiply Equation 2 by 3.}\\
& & 26x &= 78 && \text{Add equations.}\\
& & x &= 3 && \text{Solve for } x.
\end{aligned}$$

Solve for y by back-substituting $x = 3$ into Equation 2.

$$\begin{aligned}
2x - 4y &= 14 && \text{Write Equation 2.}\\
2(3) - 4y &= 14 && \text{Substitute 3 for } x.\\
-4y &= 8 && \text{Simplify.}\\
y &= -2 && \text{Solve for } y.
\end{aligned}$$

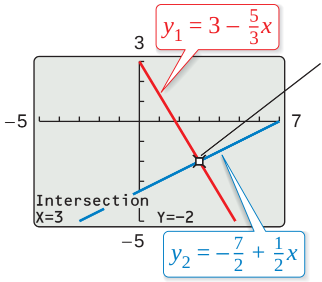
The solution is

$(3, -2)$.

Check this in the original system.

Graphical Solution

Solve each equation for y and use a graphing utility to graph the equations in the same viewing window.



$y_1 = 3 - \frac{5}{3}x$

$y_2 = -\frac{7}{2} + \frac{1}{2}x$

The point of intersection is (3, −2).

From the figure, the solution is

$(3, -2)$.

Check this in the original system.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve the system of linear equations.

$$\begin{cases} 3x + 2y = 7 \\ 2x + 5y = 1 \end{cases}$$

Check the solution from Example 3, as follows.

$$\begin{aligned}
5(3) + 3(-2) &\overset{?}{=} 9 && \text{Substitute 3 for } x \text{ and } -2 \text{ for } y \text{ in Equation 1.}\\
15 - 6 &= 9 && \text{Solution checks in Equation 1. ✓}\\
2(3) - 4(-2) &\overset{?}{=} 14 && \text{Substitute 3 for } x \text{ and } -2 \text{ for } y \text{ in Equation 2.}\\
6 + 8 &= 14 && \text{Solution checks in Equation 2. ✓}
\end{aligned}$$

Keep in mind that the terminology and methods discussed in this section apply only to systems of *linear* equations.

▷ **TECHNOLOGY** The general solution of the linear system

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

is

$$x = \frac{ce - bf}{ae - bd}$$

and

$$y = \frac{af - cd}{ae - bd}.$$

If $ae - bd = 0$, then the system does not have a unique solution. A graphing utility program (called Systems of Linear Equations) for solving such a system can be found at the website for this text at *LarsonPrecalculus.com*. Try using the program to solve the system in Example 4.

Example 4 illustrates a strategy for solving a system of linear equations that has decimal coefficients.

EXAMPLE 4 A Linear System Having Decimal Coefficients

Solve the system of linear equations.

$$\begin{cases} 0.02x - 0.05y = -0.38 \\ 0.03x + 0.04y = 1.04 \end{cases} \qquad \begin{array}{l} \text{Equation 1} \\ \text{Equation 2} \end{array}$$

Solution Because the coefficients in this system have two decimal places, you can begin by multiplying each equation by 100. This produces a system in which the coefficients are all integers.

$$\begin{cases} 2x - 5y = -38 \\ 3x + 4y = 104 \end{cases} \qquad \begin{array}{l} \text{Revised Equation 1} \\ \text{Revised Equation 2} \end{array}$$

Now, to obtain coefficients that differ only in sign, multiply Equation 1 by 3 and multiply Equation 2 by -2.

$$\begin{array}{rcrl} 2x - 5y = -38 & \Rightarrow & 6x - 15y = -114 & \text{Multiply Equation 1 by 3.} \\ \underline{3x + 4y = 104} & \Rightarrow & \underline{-6x - 8y = -208} & \text{Multiply Equation 2 by } -2. \\ & & -23y = -322 & \text{Add equations.} \end{array}$$

So,

$$\begin{aligned} y &= \frac{-322}{-23} \\ &= 14. \end{aligned}$$

Back-substituting $y = 14$ into revised Equation 2 produces the following.

$$\begin{array}{rl} 3x + 4y = 104 & \text{Write revised Equation 2.} \\ 3x + 4(14) = 104 & \text{Substitute 14 for } y. \\ 3x = 48 & \text{Simplify.} \\ x = 16 & \text{Solve for } x. \end{array}$$

The solution is

$(16, 14)$.

Check this in the original system, as follows.

Check

$$\begin{array}{rl} 0.02x - 0.05y = -0.38 & \text{Write Equation 1.} \\ 0.02(16) - 0.05(14) \overset{?}{=} -0.38 & \text{Substitute for } x \text{ and } y. \\ 0.32 - 0.70 = -0.38 & \text{Solution checks in Equation 1. } \checkmark \\ 0.03x + 0.04y = 1.04 & \text{Write Equation 2.} \\ 0.03(16) + 0.04(14) \overset{?}{=} 1.04 & \text{Substitute for } x \text{ and } y. \\ 0.48 + 0.56 = 1.04 & \text{Solution checks in Equation 2. } \checkmark \end{array}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve the system of linear equations.

$$\begin{cases} 0.03x + 0.04y = 0.75 \\ 0.02x + 0.06y = 0.90 \end{cases}$$

Graphical Interpretation of Solutions

It is possible for a *general* system of equations to have exactly one solution, two or more solutions, or no solution. If a system of *linear* equations has two different solutions, then it must have an *infinite* number of solutions.

Graphical Interpretations of Solutions

For a system of two linear equations in two variables, the number of solutions is one of the following.

Number of Solutions	Graphical Interpretation	Slopes of Lines
1. Exactly one solution	The two lines intersect at one point.	The slopes of the two lines are not equal.
2. Infinitely many solutions	The two lines coincide (are identical).	The slopes of the two lines are equal.
3. No solution	The two lines are parallel.	The slopes of the two lines are equal.

A system of linear equations is **consistent** when it has at least one solution. A consistent system with exactly one solution is *independent,* whereas a consistent system with infinitely many solutions is *dependent*. A system is **inconsistent** when it has no solution.

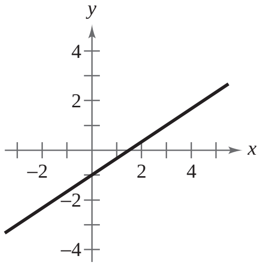
REMARK A comparison of the slopes of two lines gives useful information about the number of solutions of the corresponding system of equations. To solve a system of equations graphically, it helps to begin by writing the equations in slope-intercept form. Try doing this for the systems in Example 5.

EXAMPLE 5 Recognizing Graphs of Linear Systems

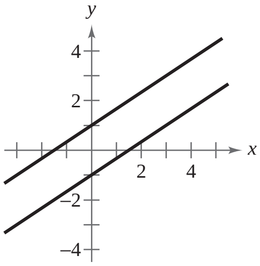
Match each system of linear equations with its graph. Describe the number of solutions and state whether the system is consistent or inconsistent.

a. $\begin{cases} 2x - 3y = 3 \\ -4x + 6y = 6 \end{cases}$ **b.** $\begin{cases} 2x - 3y = 3 \\ x + 2y = 5 \end{cases}$ **c.** $\begin{cases} 2x - 3y = 3 \\ -4x + 6y = -6 \end{cases}$

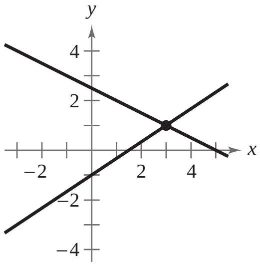
i.



ii.



iii.



Solution

a. The graph of system (a) is a pair of parallel lines (ii). The lines have no point of intersection, so the system has no solution. The system is inconsistent.

b. The graph of system (b) is a pair of intersecting lines (iii). The lines have one point of intersection, so the system has exactly one solution. The system is consistent.

c. The graph of system (c) is a pair of lines that coincide (i). The lines have infinitely many points of intersection, so the system has infinitely many solutions. The system is consistent.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Sketch the graph of the system of linear equations. Then describe the number of solutions and state whether the system is consistent or inconsistent.

$$\begin{cases} -2x + 3y = 6 \\ 4x - 6y = -9 \end{cases}$$

In Examples 6 and 7, note how you can use the method of elimination to determine that a system of linear equations has no solution or infinitely many solutions.

EXAMPLE 6 No-Solution Case: Method of Elimination

Solve the system of linear equations.

$$\begin{cases} x - 2y = 3 & \text{Equation 1} \\ -2x + 4y = 1 & \text{Equation 2} \end{cases}$$

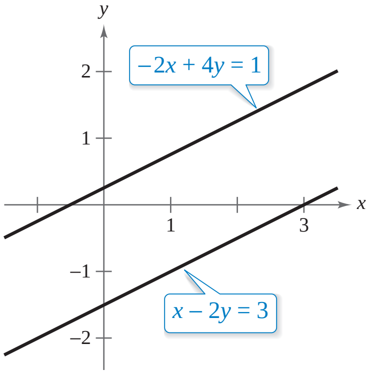


Figure 6.5

Solution To obtain coefficients that differ only in sign, multiply Equation 1 by 2.

$$\begin{array}{rcrl} x - 2y = 3 & \Rightarrow & 2x - 4y = 6 & \text{Multiply Equation 1 by 2.} \\ \underline{-2x + 4y = 1} & \Rightarrow & \underline{-2x + 4y = 1} & \text{Write Equation 2.} \\ & & 0 = 7 & \text{Add equations.} \end{array}$$

Because there are no values of x and y for which $0 = 7$, you can conclude that the system is inconsistent and has no solution. Figure 6.5 shows the lines corresponding to the two equations in this system. Note that the two lines are parallel, so they have no point of intersection.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve the system of linear equations.

$$\begin{cases} 6x - 5y = 3 \\ -12x + 10y = 5 \end{cases}$$

In Example 6, note that the occurrence of a false statement, such as $0 = 7$, indicates that the system has no solution. In the next example, note that the occurrence of a statement that is true for all values of the variables, such as $0 = 0$, indicates that the system has infinitely many solutions.

EXAMPLE 7 Many-Solution Case: Method of Elimination

Solve the system of linear equations.

$$\begin{cases} 2x - y = 1 & \text{Equation 1} \\ 4x - 2y = 2 & \text{Equation 2} \end{cases}$$

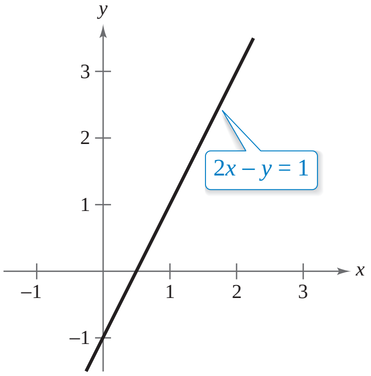


Figure 6.6

Solution To obtain coefficients that differ only in sign, multiply Equation 1 by -2.

$$\begin{array}{rcrl} 2x - y = 1 & \Rightarrow & -4x + 2y = -2 & \text{Multiply Equation 1 by } -2. \\ \underline{4x - 2y = 2} & \Rightarrow & \underline{4x - 2y = 2} & \text{Write Equation 2.} \\ & & 0 = 0 & \text{Add equations.} \end{array}$$

Because the two equations are equivalent (have the same solution set), the system has infinitely many solutions. The solution set consists of all points (x, y) lying on the line $2x - y = 1$, as shown in Figure 6.6. Letting $x = a$, where a is any real number, the solutions of the system are $(a, 2a - 1)$.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve the system of linear equations.

$$\begin{cases} \frac{1}{2}x - \frac{1}{8}y = -\frac{3}{8} \\ -4x + y = 3 \end{cases}$$

Applications

At this point, you may be asking the question “How can I tell whether I can solve an application problem using a system of linear equations?” The answer comes from the following considerations.

1. Does the problem involve more than one unknown quantity?
2. Are there two (or more) equations or conditions to be satisfied?

If the answer to one or both of these questions is “yes,” then the appropriate model for the problem may be a system of linear equations.

EXAMPLE 8 An Application of a Linear System

An airplane flying into a headwind travels the 2000-mile flying distance between Chicopee, Massachusetts, and Salt Lake City, Utah, in 4 hours and 24 minutes. On the return flight, the airplane travels this distance in 4 hours. Find the airspeed of the plane and the speed of the wind, assuming that both remain constant.

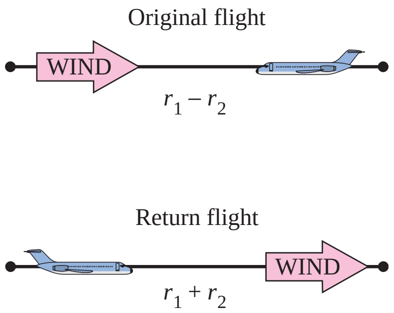


Figure 6.7

Solution The two unknown quantities are the speeds of the wind and the plane. If r_1 is the speed of the plane and r_2 is the speed of the wind, then

$r_1 - r_2 =$ speed of the plane against the wind

$r_1 + r_2 =$ speed of the plane with the wind

as shown in Figure 6.7. Using the formula

distance $=$ (rate)(time)

for these two speeds, you obtain the following equations.

$$2000 = (r_1 - r_2)\left(4 + \frac{24}{60}\right)$$

$$2000 = (r_1 + r_2)(4)$$

These two equations simplify as follows.

$$\begin{cases} 5000 = 11r_1 - 11r_2 & \qquad\text{Equation 1} \\ 500 = r_1 + r_2 & \qquad\text{Equation 2} \end{cases}$$

To solve this system by elimination, multiply Equation 2 by 11.

$$\begin{array}{rcl} 5000 = 11r_1 - 11r_2 & \Rightarrow & 5000 = 11r_1 - 11r_2 \qquad \text{Write Equation 1.} \\ \underline{500 = r_1 + r_2} & \Rightarrow & \underline{5500 = 11r_1 + 11r_2} \qquad \text{Multiply Equation 2 by 11.} \\ & & 10{,}500 = 22r_1 \qquad \text{Add equations.} \end{array}$$

So,

$$r_1 = \frac{10{,}500}{22} = \frac{5250}{11} \approx 477.27 \text{ miles per hour} \qquad \text{Speed of plane}$$

and

$$r_2 = 500 - \frac{5250}{11} = \frac{250}{11} \approx 22.73 \text{ miles per hour.} \qquad \text{Speed of wind}$$

Check this solution in the original statement of the problem.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

In Example 8, the return flight takes 4 hours and 6 minutes. Find the airspeed of the plane and the speed of the wind, assuming that both remain constant.

In a free market, the demands for many products are related to the prices of the products. As the prices decrease, the demands by consumers increase and the amounts that producers are able or willing to supply decrease.

EXAMPLE 9 Finding the Equilibrium Point

The demand and supply equations for a video game console are

$$\begin{cases} p = 180 - 0.00001x & \text{Demand equation} \\ p = \ \ 90 + 0.00002x & \text{Supply equation} \end{cases}$$

where p is the price per unit (in dollars) and x is the number of units. Find the equilibrium point for this market. **The equilibrium point** is the price p and number of units x that satisfy both the demand and supply equations.

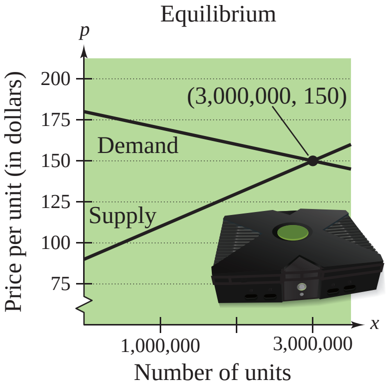


Figure 6.8

Solution Because p is written in terms of x, begin by substituting the value of p given in the supply equation into the demand equation.

$$p = 180 - 0.00001x \qquad \text{Write demand equation.}$$

$$90 + 0.00002x = 180 - 0.00001x \qquad \text{Substitute } 90 + 0.00002x \text{ for } p.$$

$$0.00003x = 90 \qquad \text{Combine like terms.}$$

$$x = 3{,}000{,}000 \qquad \text{Solve for } x.$$

So, the equilibrium point occurs when the demand and supply are each 3 million units. (See Figure 6.8.) Obtain the price that corresponds to this x-value by back-substituting $x = 3{,}000{,}000$ into either of the original equations. For instance, back-substituting into the demand equation produces

$$\begin{aligned} p &= 180 - 0.00001(3{,}000{,}000) \\ &= 180 - 30 \\ &= \$150. \end{aligned}$$

The solution is $(3{,}000{,}000, 150)$. Check this by substituting $(3{,}000{,}000, 150)$ into the demand and supply equations.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The demand and supply equations for a flat-screen television are

$$\begin{cases} p = 567 - 0.00002x & \text{Demand equation} \\ p = 492 + 0.00003x & \text{Supply equation} \end{cases}$$

where p is the price per unit (in dollars) and x is the number of units. Find the equilibrium point for this market.

Summarize (Section 6.2)

1. Explain how to use the method of elimination to solve a system of linear equations in two variables *(page 434)*. For examples of using the method of elimination to solve systems of linear equations in two variables, see Examples 1–4.
2. Explain how to interpret graphically the numbers of solutions of systems of linear equations in two variables *(page 438)*. For examples of interpreting graphically the numbers of solutions of systems of linear equations in two variables, see Examples 5–7.
3. Describe examples of how to use systems of linear equations in two variables to model and solve real-life problems *(pages 440 and 441, Examples 8 and 9)*.

6.2 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. The first step in solving a system of equations by the method of ________ is to obtain coefficients for *x* (or *y*) that differ only in sign.

2. Two systems of equations that have the same solution set are called ________ systems.

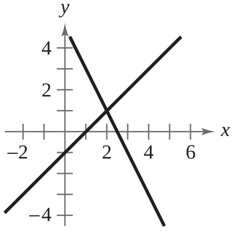
3. A system of linear equations that has at least one solution is called ________, whereas a system of linear equations that has no solution is called ________.

4. In business applications, the ________ ________ is defined as the price *p* and the number of units *x* that satisfy both the demand and supply equations.

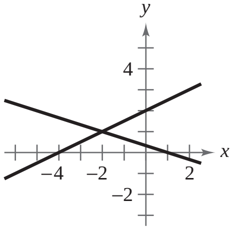
Skills and Applications

Solving a System by Elimination **In Exercises 5–12, solve the system by the method of elimination. Label each line with its equation. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

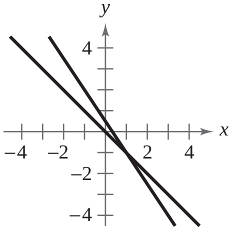
5. $\begin{cases} 2x + y = 5 \\ x - y = 1 \end{cases}$



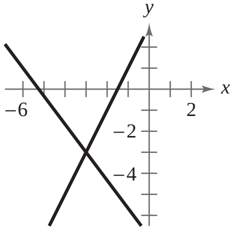
6. $\begin{cases} x + 3y = 1 \\ -x + 2y = 4 \end{cases}$



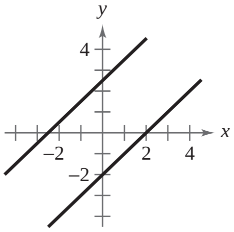
7. $\begin{cases} x + y = 0 \\ 3x + 2y = 1 \end{cases}$



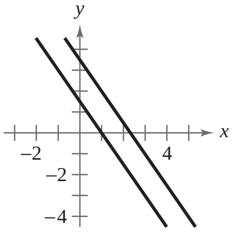
8. $\begin{cases} 2x - y = -3 \\ 4x + 3y = -21 \end{cases}$



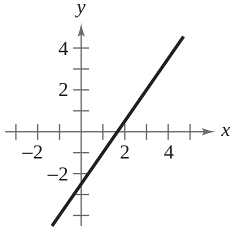
9. $\begin{cases} x - y = 2 \\ -2x + 2y = 5 \end{cases}$



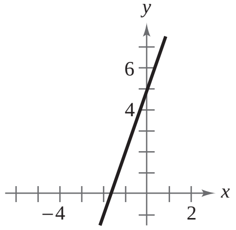
10. $\begin{cases} 3x + 2y = 3 \\ 6x + 4y = 14 \end{cases}$



11. $\begin{cases} 3x - 2y = 5 \\ -6x + 4y = -10 \end{cases}$



12. $\begin{cases} 9x - 3y = -15 \\ -3x + y = 5 \end{cases}$



Solving a System by Elimination **In Exercises 13–30, solve the system by the method of elimination and check any solutions algebraically.**

13. $\begin{cases} x + 2y = 6 \\ x - 2y = 2 \end{cases}$

14. $\begin{cases} 3x - 5y = 8 \\ 2x + 5y = 22 \end{cases}$

15. $\begin{cases} 5x + 3y = 6 \\ 3x - y = 5 \end{cases}$

16. $\begin{cases} x + 5y = 10 \\ 3x - 10y = -5 \end{cases}$

17. $\begin{cases} 3x + 2y = 10 \\ 2x + 5y = 3 \end{cases}$

18. $\begin{cases} 2r + 4s = 5 \\ 16r + 50s = 55 \end{cases}$

19. $\begin{cases} 5u + 6v = 24 \\ 3u + 5v = 18 \end{cases}$

20. $\begin{cases} 3x + 11y = 4 \\ -2x - 5y = 9 \end{cases}$

21. $\begin{cases} \frac{9}{5}x + \frac{6}{5}y = 4 \\ 9x + 6y = 3 \end{cases}$

22. $\begin{cases} \frac{3}{4}x + y = \frac{1}{8} \\ \frac{9}{4}x + 3y = \frac{3}{8} \end{cases}$

23. $\begin{cases} -5x + 6y = -3 \\ 20x - 24y = 12 \end{cases}$

24. $\begin{cases} 7x + 8y = 6 \\ -14x - 16y = -12 \end{cases}$

25. $\begin{cases} 0.2x - 0.5y = -27.8 \\ 0.3x + 0.4y = 68.7 \end{cases}$

26. $\begin{cases} 0.05x - 0.03y = 0.21 \\ 0.07x + 0.02y = 0.16 \end{cases}$

27. $\begin{cases} 4b + 3m = 3 \\ 3b + 11m = 13 \end{cases}$

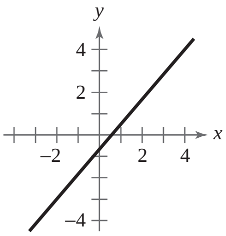
28. $\begin{cases} 2x + 5y = 8 \\ 5x + 8y = 10 \end{cases}$

29. $\begin{cases} \dfrac{x+3}{4} + \dfrac{y-1}{3} = 1 \\ 2x - y = 12 \end{cases}$

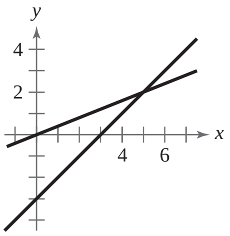
30. $\begin{cases} \dfrac{x-1}{2} + \dfrac{y+2}{3} = 4 \\ x - 2y = 5 \end{cases}$

Matching a System with Its Graph **In Exercises 31–34, match the system of linear equations with its graph. Describe the number of solutions and state whether the system is consistent or inconsistent. [The graphs are labeled (a), (b), (c) and (d).]**

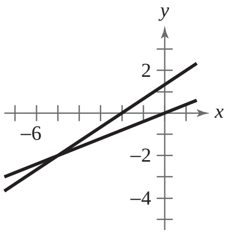
(a)



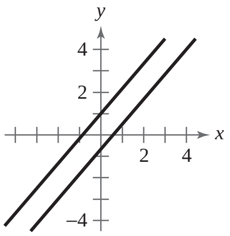
(b)



(c)



(d)



31. $\begin{cases} 2x - 5y = 0 \\ x - y = 3 \end{cases}$

32. $\begin{cases} 2x - 5y = 0 \\ 2x - 3y = -4 \end{cases}$

33. $\begin{cases} -7x + 6y = -4 \\ 14x - 12y = 8 \end{cases}$

34. $\begin{cases} 7x - 6y = -6 \\ -7x + 6y = -4 \end{cases}$

Solving a System **In Exercises 35–40, use any method to solve the system.**

35. $\begin{cases} 3x - 5y = 7 \\ 2x + y = 9 \end{cases}$

36. $\begin{cases} -x + 3y = 17 \\ 4x + 3y = 7 \end{cases}$

37. $\begin{cases} y = 2x - 5 \\ y = 5x - 11 \end{cases}$

38. $\begin{cases} 7x + 3y = 16 \\ y = x + 2 \end{cases}$

39. $\begin{cases} x - 5y = 21 \\ 6x + 5y = 21 \end{cases}$

40. $\begin{cases} 4x - 3y = 6 \\ -5x + 7y = -1 \end{cases}$

41. **Airplane Speed** An airplane flying into a headwind travels the 1800-mile flying distance between Pittsburgh, Pennsylvania, and Phoenix, Arizona, in 3 hours and 36 minutes. On the return flight, the airplane travels this distance in 3 hours. Find the airspeed of the plane and the speed of the wind, assuming that both remain constant.

42. **Airplane Speed** Two planes start from Los Angeles International Airport and fly in opposite directions. The second plane starts $\frac{1}{2}$ hour after the first plane, but its speed is 80 kilometers per hour faster. Find the airspeed of each plane when 2 hours after the first plane departs the planes are 3200 kilometers apart.

43. **Nutrition** Two cheeseburgers and one small order of French fries from a fast-food restaurant contain a total of 830 calories. Three cheeseburgers and two small orders of French fries contain a total of 1360 calories. Find the caloric content of each item.

Vlad61/Shutterstock.com

44. **Nutrition** One eight-ounce glass of apple juice and one eight-ounce glass of orange juice contain a total of 179.2 milligrams of vitamin C. Two eight-ounce glasses of apple juice and three eight-ounce glasses of orange juice contain a total of 442.1 milligrams of vitamin C. How much vitamin C is in an eight-ounce glass of each type of juice?

Finding the Equilibrium Point **In Exercises 45–48, find the equilibrium point of the demand and supply equations.**

	Demand	Supply
45.	$p = 500 - 0.4x$	$p = 380 + 0.1x$
46.	$p = 100 - 0.05x$	$p = 25 + 0.1x$
47.	$p = 140 - 0.00002x$	$p = 80 + 0.00001x$
48.	$p = 400 - 0.0002x$	$p = 225 + 0.0005x$

49. **Acid Mixture** Thirty liters of a 40% acid solution is obtained by mixing a 25% solution with a 50% solution.

(a) Write a system of equations in which one equation represents the amount of final mixture required and the other represents the percent of acid in the final mixture. Let x and y represent the amounts of the 25% and 50% solutions, respectively.

(b) Use a graphing utility to graph the two equations in part (a) in the same viewing window. As the amount of the 25% solution increases, how does the amount of the 50% solution change?

(c) How much of each solution is required to obtain the specified concentration of the final mixture?

50. **Fuel Mixture**

Five hundred gallons of 89-octane gasoline is obtained by mixing 87-octane gasoline with 92-octane gasoline.

(a) Write a system of equations in which one equation represents the amount of final mixture required and the other represents the amounts of 87- and 92-octane gasolines in the final mixture. Let x and y represent the numbers of gallons of 87-octane and 92-octane gasolines, respectively.

(b) Use a graphing utility to graph the two equations in part (a) in the same viewing window. As the amount of 87-octane gasoline increases, how does the amount of 92-octane gasoline change?

(c) How much of each type of gasoline is required to obtain the 500 gallons of 89-octane gasoline?

51. Investment Portfolio A total of \$24,000 is invested in two corporate bonds that pay 3.5% and 5% simple interest. The investor wants an annual interest income of \$930 from the investments. What amount should be invested in the 3.5% bond?

52. Investment Portfolio A total of \$32,000 is invested in two municipal bonds that pay 5.75% and 6.25% simple interest. The investor wants an annual interest income of \$1900 from the investments. What amount should be invested in the 5.75% bond?

53. Prescriptions The numbers of prescriptions P (in thousands) filled at two pharmacies from 2009 through 2013 are shown in the table.

Year	Pharmacy A	Pharmacy B
2009	19.2	20.4
2010	19.6	20.8
2011	20.0	21.1
2012	20.6	21.5
2013	21.3	22.0

(a) Use a graphing utility to create a scatter plot of the data for pharmacy A and find a linear model. Let t represent the year, with $t = 9$ corresponding to 2009. Repeat the procedure for pharmacy B.

(b) Assuming that the numbers for the given five years are representative of future years, will the number of prescriptions filled at pharmacy A ever exceed the number of prescriptions filled at pharmacy B? If so, then when?

54. Data Analysis A store manager wants to know the demand for a product as a function of the price. The table shows the daily sales y for different prices x of the product.

Price, x	Demand, y
\$1.00	45
\$1.20	37
\$1.50	23

(a) Find the least squares regression line $y = ax + b$ for the data by solving the system for a and b.

$$\begin{cases} 3.00b + 3.70a = 105.00 \\ 3.70b + 4.69a = 123.90 \end{cases}$$

(b) Use a graphing utility to confirm the result of part (a).

(c) Use the linear model from part (a) to predict the demand when the price is \$1.75.

Fitting a Line to Data **To find the least squares regression line $y = ax + b$ for a set of points**

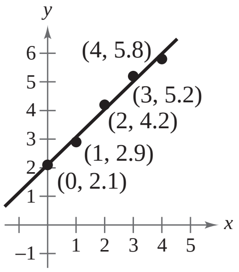
$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

you can solve the following system for a and b.

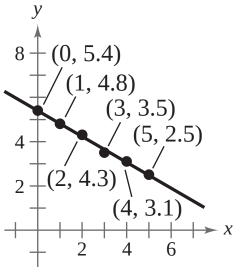
$$\begin{cases} nb + \left(\sum_{i=1}^{n} x_i\right)a = \left(\sum_{i=1}^{n} y_i\right) \\ \left(\sum_{i=1}^{n} x_i\right)b + \left(\sum_{i=1}^{n} x_i^2\right)a = \left(\sum_{i=1}^{n} x_i y_i\right) \end{cases}$$

In Exercises 55 and 56, the sums have been evaluated. Solve the given system for a and b to find the least squares regression line for the points. Use a graphing utility to confirm the result.

55. $\begin{cases} 5b + 10a = 20.2 \\ 10b + 30a = 50.1 \end{cases}$



56. $\begin{cases} 6b + 15a = 23.6 \\ 15b + 55a = 48.8 \end{cases}$



57. Data Analysis An agricultural scientist used four test plots to determine the relationship between wheat yield y (in bushels per acre) and the amount of fertilizer x (in hundreds of pounds per acre). The table shows the results.

Fertilizer, x	Yield, y
1.0	32
1.5	41
2.0	48
2.5	53

(a) Find the least squares regression line $y = ax + b$ for the data by solving the system for a and b.

$$\begin{cases} 4b + 7.0a = 174 \\ 7b + 13.5a = 322 \end{cases}$$

(b) Use the linear model from part (a) to estimate the yield for a fertilizer application of 160 pounds per acre.

58. **Defense Department Outlays** The table shows the total national outlays y for defense functions (in billions of dollars) for the years 2004 through 2011. *(Source: U.S. Office of Management and Budget)*

Spreadsheet at LarsonPrecalculus.com

Year	Outlays, y
2004	455.8
2005	495.3
2006	521.8
2007	551.3
2008	616.1
2009	661.0
2010	693.6
2011	705.6

(a) Find the least squares regression line $y = at + b$ for the data, where t represents the year with $t = 4$ corresponding to 2004, by solving the system for a and b.

$$\begin{cases} 8b + 60a = 4700.5 \\ 60b + 492a = 36{,}865.0 \end{cases}$$

(b) Use the *regression* feature of a graphing utility to confirm the result of part (a).

(c) Use the linear model to create a table of estimated values of y. Compare the estimated values with the actual data.

(d) Use the linear model to estimate the total national outlay for 2012.

(e) Use the Internet, your school's library, or some other reference source to find the total national outlay for 2012. How does this value compare with your answer in part (d)?

(f) Is the linear model valid for long-term predictions of total national outlays? Explain.

Exploration

True or False? **In Exercises 59 and 60, determine whether the statement is true or false. Justify your answer.**

59. If two lines do not have exactly one point of intersection, then they must be parallel.

60. Solving a system of equations graphically will always give an exact solution.

Finding the Value of a Constant **In Exercises 61 and 62, find the value of k such that the system of linear equations is inconsistent.**

61. $\begin{cases} 4x - 8y = -3 \\ 2x + ky = 16 \end{cases}$

62. $\begin{cases} 15x + 3y = 6 \\ -10x + ky = 9 \end{cases}$

63. **Writing** Briefly explain whether it is possible for a consistent system of linear equations to have exactly two solutions.

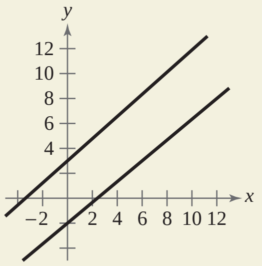
64. **Think About It** Give examples of a system of linear equations that has (a) no solution and (b) an infinite number of solutions.

65. **Comparing Methods** Use the method of substitution to solve the system in Example 1. Is the method of substitution or the method of elimination easier? Explain.

66. **HOW DO YOU SEE IT?** Use the graphs of the two equations shown below.

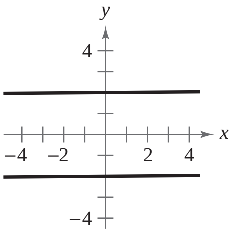
(a) Describe the graphs of the two equations.

(b) Can you conclude that the system of equations shown in the graph is inconsistent? Explain.

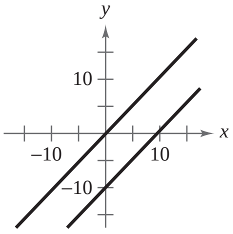


Think About It **In Exercises 67 and 68, the graphs of the two equations appear to be parallel. Yet, when you solve the system algebraically, you find that the system does have a solution. Find the solution and explain why it does not appear on the portion of the graph shown.**

67. $\begin{cases} 100y - x = 200 \\ 99y - x = -198 \end{cases}$

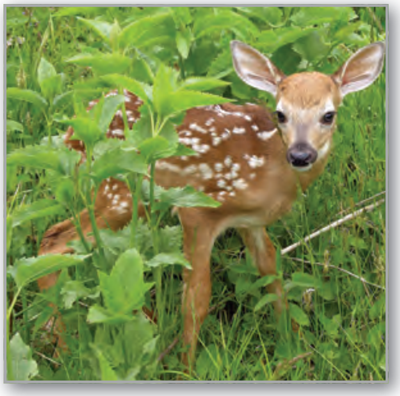


68. $\begin{cases} 21x - 20y = 0 \\ 13x - 12y = 120 \end{cases}$



Project: College Expenses To work an extended application analyzing the average undergraduate tuition, room, and board charges at private degree-granting institutions in the United States from 1990 through 2010, visit this text's website at *LarsonPrecalculus.com.* *(Source: U.S. Department of Education)*

6.3 Multivariable Linear Systems



Systems of linear equations in three or more variables can help you model and solve real-life problems. For instance, in Exercise 72 on page 457, you will use a system of linear equations in three variables to analyze the reproductive rates of deer in a wildlife preserve.

- **Use back-substitution to solve linear systems in row-echelon form.**
- **Use Gaussian elimination to solve systems of linear equations.**
- **Solve nonsquare systems of linear equations.**
- **Use systems of linear equations in three or more variables to model and solve real-life problems.**

Row-Echelon Form and Back-Substitution

The method of elimination can be applied to a system of linear equations in more than two variables. In fact, this method adapts to computer use for solving linear systems with dozens of variables.

When using the method of elimination to solve a system of linear equations, the goal is to rewrite the system in a form to which back-substitution can be applied. To see how this works, consider the following two systems of linear equations.

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y = -4 \\ 2x - 5y + 5z = 17 \end{cases}$$

System of three linear equations in three variables (See Example 3.)

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ z = 2 \end{cases}$$

Equivalent system in row-echelon form (See Example 1.)

The second system is in **row-echelon form,** which means that it has a “stair-step” pattern with leading coefficients of 1. After comparing the two systems, it should be clear that it is easier to solve the system in row-echelon form, using back-substitution.

EXAMPLE 1 Using Back-Substitution in Row-Echelon Form

Solve the system of linear equations.

$$\begin{cases} x - 2y + 3z = 9 & \text{Equation 1} \\ y + 3z = 5 & \text{Equation 2} \\ z = 2 & \text{Equation 3} \end{cases}$$

Solution From Equation 3, you know the value of z. To solve for y, back-substitute $z = 2$ into Equation 2 to obtain

$$y + 3(2) = 5 \qquad \text{Substitute 2 for } z.$$

$$y = -1. \qquad \text{Solve for } y.$$

Then back-substitute $y = -1$ and $z = 2$ into Equation 1 to obtain

$$x - 2(-1) + 3(2) = 9 \qquad \text{Substitute } -1 \text{ for } y \text{ and 2 for } z.$$

$$x = 1. \qquad \text{Solve for } x.$$

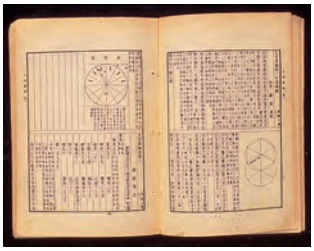
The solution is $x = 1$, $y = -1$, and $z = 2$, which can be written as the **ordered triple** $(1, -1, 2)$. Check this in the original system of equations.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve the system of linear equations.

$$\begin{cases} 2x - y + 5z = 22 \\ y + 3z = 6 \\ z = 3 \end{cases}$$

Bruce MacQueen/Shutterstock.com



One of the most influential Chinese mathematics books was the *Chui-chang suan-shu* or *Nine Chapters on the Mathematical Art* (written in approximately 250 B.C.). Chapter Eight of the *Nine Chapters* contained solutions of systems of linear equations such as

$$\begin{cases} 3x + 2y + z = 39 \\ 2x + 3y + z = 34 \\ x + 2y + 3z = 26 \end{cases}$$

using positive and negative numbers. This system was solved using column operations on a matrix. Matrices (plural for matrix) are discussed in the next chapter.

Gaussian Elimination

Two systems of equations are *equivalent* when they have the same solution set. To solve a system that is not in row-echelon form, first convert it to an *equivalent* system that is in row-echelon form by using the following operations.

Operations That Produce Equivalent Systems

Each of the following **row operations** on a system of linear equations produces an *equivalent* system of linear equations.

1. Interchange two equations.

2. Multiply one of the equations by a nonzero constant.

3. Add a multiple of one of the equations to another equation to replace the latter equation.

To see how to use row operations, take another look at the method of elimination, as applied to a system of two linear equations.

EXAMPLE 2 Using Row Operations to Solve a System

Solve the system of linear equations.

$$\begin{cases} 3x - 2y = -1 & \text{Equation 1} \\ x - y = 0 & \text{Equation 2} \end{cases}$$

Solution There are two strategies that seem reasonable: eliminate the variable x or eliminate the variable y. The following steps show how to eliminate x.

$$\begin{cases} x - y = 0 \\ 3x - 2y = -1 \end{cases} \qquad \text{Interchange the two equations in the system.}$$

$$\begin{cases} -3x + 3y = 0 \\ 3x - 2y = -1 \end{cases} \qquad \text{Multiply the first equation by } -3.$$

$$\begin{array}{r} -3x + 3y = 0 \\ \underline{3x - 2y = -1} \\ y = -1 \end{array} \qquad \text{Add the multiple of the first equation to the second equation to obtain a new second equation.}$$

$$\begin{cases} x - y = 0 \\ y = -1 \end{cases} \qquad \text{New system in row-echelon form}$$

Notice in the first step that interchanging rows is a way of obtaining a leading coefficient of 1. Now back-substitute $y = -1$ into Equation 2 and solve for x.

$$x - (-1) = 0 \qquad \text{Substitute } -1 \text{ for } y.$$

$$x = -1 \qquad \text{Solve for } x.$$

The solution is $x = -1$ and $y = -1$, which can be written as the ordered pair $(-1, -1)$.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve the system of linear equations.

$$\begin{cases} 2x + y = 3 \\ x + 2y = 3 \end{cases}$$

Christopher Lui/China Stock

Rewriting a system of linear equations in row-echelon form usually involves a chain of equivalent systems, which you obtain by using one of the three basic row operations listed on the previous page. This process is called **Gaussian elimination,** after the German mathematician Carl Friedrich Gauss (1777–1855).

EXAMPLE 3 Using Gaussian Elimination to Solve a System

Solve the system of linear equations.

$$\begin{cases} x - 2y + 3z = 9 & \text{Equation 1} \\ -x + 3y = -4 & \text{Equation 2} \\ 2x - 5y + 5z = 17 & \text{Equation 3} \end{cases}$$

REMARK Arithmetic errors are common when performing row operations. You should note the operation performed in each step to make checking your work easier.

Solution Because the leading coefficient of the first equation is 1, begin by keeping the x in the upper left position and eliminating the other x-terms from the first column.

$$\begin{array}{rl} x - 2y + 3z = 9 & \text{Write Equation 1.} \\ -x + 3y \qquad = -4 & \text{Write Equation 2.} \\ \hline y + 3z = 5 & \text{Add Equation 1 to Equation 2.} \end{array}$$

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ 2x - 5y + 5z = 17 \end{cases}$$

Adding the first equation to the second equation produces a new second equation.

$$\begin{array}{rl} -2x + 4y - 6z = -18 & \text{Multiply Equation 1 by } -2. \\ 2x - 5y + 5z = 17 & \text{Write Equation 3.} \\ \hline -y - z = -1 & \text{Add revised Equation 1 to Equation 3.} \end{array}$$

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ -y - z = -1 \end{cases}$$

Adding -2 times the first equation to the third equation produces a new third equation.

Now that you have eliminated all but the x in the upper position of the first column, work on the second column. (You need to eliminate y from the third equation.)

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ 2z = 4 \end{cases}$$

Adding the second equation to the third equation produces a new third equation.

Finally, you need a coefficient of 1 for z in the third equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ z = 2 \end{cases}$$

Multiplying the third equation by $\frac{1}{2}$ produces a new third equation.

This is the same system in Example 1, and, as in that example, the solution is

$$x = 1, \quad y = -1, \quad \text{and} \quad z = 2.$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve the system of linear equations.

$$\begin{cases} x + y + z = 6 \\ 2x - y + z = 3 \\ 3x + y - z = 2 \end{cases}$$

The next example involves an inconsistent system—one that has no solution. The key to recognizing an inconsistent system is that at some stage in the elimination process, you obtain a false statement such as $0 = -2$.

EXAMPLE 4 An Inconsistent System

Solve the system of linear equations.

$$\begin{cases} x - 3y + z = 1 & \text{Equation 1} \\ 2x - y - 2z = 2 & \text{Equation 2} \\ x + 2y - 3z = -1 & \text{Equation 3} \end{cases}$$

Solution

$$\begin{cases} x - 3y + z = 1 \\ 5y - 4z = 0 \\ x + 2y - 3z = -1 \end{cases}$$

Adding -2 times the first equation to the second equation produces a new second equation.

$$\begin{cases} x - 3y + z = 1 \\ 5y - 4z = 0 \\ 5y - 4z = -2 \end{cases}$$

Adding -1 times the first equation to the third equation produces a new third equation.

$$\begin{cases} x - 3y + z = 1 \\ 5y - 4z = 0 \\ 0 = -2 \end{cases}$$

Adding -1 times the second equation to the third equation produces a new third equation.

Because $0 = -2$ is a false statement, this system is inconsistent and has no solution. Moreover, because this system is equivalent to the original system, the original system also has no solution.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve the system of linear equations.

$$\begin{cases} x + y - 2z = 3 \\ 3x - 2y + 4z = 1 \\ 2x - 3y + 6z = 8 \end{cases}$$

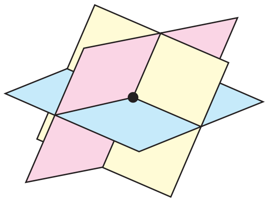
As with a system of linear equations in two variables, the solution(s) of a system of linear equations in more than two variables must fall into one of three categories.

The Number of Solutions of a Linear System

For a system of linear equations, exactly one of the following is true.

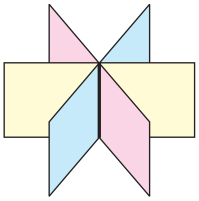
1. There is exactly one solution.
2. There are infinitely many solutions.
3. There is no solution.

In Section 6.2, you learned that a system of two linear equations in two variables can be represented graphically as a pair of lines that are intersecting, coincident, or parallel. A system of three linear equations in three variables has a similar graphical representation—it can be represented as three planes in space that intersect in one point (exactly one solution, see Figure 6.9), intersect in a line or a plane (infinitely many solutions, see Figures 6.10 and 6.11), or have no points common to all three planes (no solution, see Figures 6.12 and 6.13).



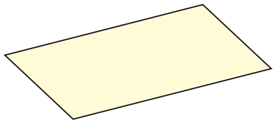
Solution: one point

Figure 6.9



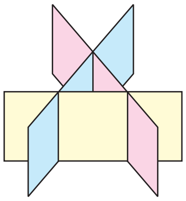
Solution: one line

Figure 6.10



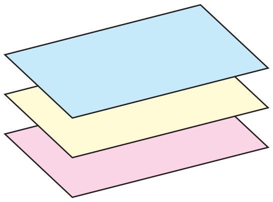
Solution: one plane

Figure 6.11



Solution: none

Figure 6.12



Solution: none

Figure 6.13

EXAMPLE 5 A System with Infinitely Many Solutions

Solve the system of linear equations.

$$\begin{cases} x + y - 3z = -1 & \text{Equation 1} \\ y - z = 0 & \text{Equation 2} \\ -x + 2y = 1 & \text{Equation 3} \end{cases}$$

Solution

$$\begin{cases} x + y - 3z = -1 \\ y - z = 0 \\ 3y - 3z = 0 \end{cases}$$

Adding the first equation to the third equation produces a new third equation.

$$\begin{cases} x + y - 3z = -1 \\ y - z = 0 \\ 0 = 0 \end{cases}$$

Adding -3 times the second equation to the third equation produces a new third equation.

This result means that Equation 3 depends on Equations 1 and 2 in the sense that it gives no additional information about the variables. Because $0 = 0$ is a true statement, this system has infinitely many solutions. However, it is incorrect to say that the solution is "infinite." You must also specify the correct form of the solution. So, the original system is equivalent to the system

$$\begin{cases} x + y - 3z = -1 \\ y - z = 0 \end{cases}.$$

In the second equation, solve for y in terms of z to obtain $y = z$. Back-substituting $y = z$ in the first equation produces $x = 2z - 1$. Finally, letting $z = a$, where a is a real number, the solutions of the given system are all of the form $x = 2a - 1$, $y = a$, and $z = a$. So, every ordered triple of the form

$$(2a - 1, a, a) \qquad a \text{ is a real number.}$$

is a solution of the system.

✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve the system of linear equations.

$$\begin{cases} x + 2y - 7z = -4 \\ 2x + 3y + z = 5 \\ 3x + 7y - 36z = -25 \end{cases}$$

In Example 5, there are other ways to write the same infinite set of solutions. For instance, letting $x = b$, the solutions could have been written as

$$\left(b, \tfrac{1}{2}(b + 1), \tfrac{1}{2}(b + 1)\right). \qquad b \text{ is a real number.}$$

To convince yourself that this description produces the same set of solutions, consider the following.

Substitution	**Solution**	
$a = 0$	$(2(0) - 1, 0, 0) = (-1, 0, 0)$	Same solution
$b = -1$	$\left(-1, \tfrac{1}{2}(-1 + 1), \tfrac{1}{2}(-1 + 1)\right) = (-1, 0, 0)$	
$a = 1$	$(2(1) - 1, 1, 1) = (1, 1, 1)$	Same solution
$b = 1$	$\left(1, \tfrac{1}{2}(1 + 1), \tfrac{1}{2}(1 + 1)\right) = (1, 1, 1)$	
$a = 2$	$(2(2) - 1, 2, 2) = (3, 2, 2)$	Same solution
$b = 3$	$\left(3, \tfrac{1}{2}(3 + 1), \tfrac{1}{2}(3 + 1)\right) = (3, 2, 2)$	

The Global Positioning System (GPS) is a network of 24 satellites originally developed by the U.S. military as a navigational tool. Civilian applications of GPS receivers include determining directions, locating vessels lost at sea, and monitoring earthquakes. A GPS receiver works by using satellite readings to calculate its location. In a simplified mathematical model, nonsquare systems of three linear equations in four variables (three dimensions and time) determine the coordinates of the receiver as a function of time.

Nonsquare Systems

So far, each system of linear equations you have looked at has been *square*, which means that the number of equations is equal to the number of variables. In a **nonsquare** system, the number of equations differs from the number of variables. A system of linear equations cannot have a unique solution unless there are at least as many equations as there are variables in the system.

EXAMPLE 6 A System with Fewer Equations than Variables

Solve the system of linear equations.

$$\begin{cases} x - 2y + z = 2 \\ 2x - y - z = 1 \end{cases}$$ Equation 1, Equation 2

Solution Begin by rewriting the system in row-echelon form.

$$\begin{cases} x - 2y + z = 2 \\ 3y - 3z = -3 \end{cases}$$

Adding -2 times the first equation to the second equation produces a new second equation.

$$\begin{cases} x - 2y + z = 2 \\ y - z = -1 \end{cases}$$

Multiplying the second equation by $\frac{1}{3}$ produces a new second equation.

Solve for y in terms of z to obtain

$$y = z - 1.$$

Solve for x by back-substituting $y = z - 1$ into Equation 1.

$$x - 2y + z = 2$$ Write Equation 1.

$$x - 2(z - 1) + z = 2$$ Substitute $z - 1$ for y.

$$x - 2z + 2 + z = 2$$ Distributive Property

$$x = z$$ Solve for x.

Finally, by letting $z = a$, where a is a real number, you have the solution

$$x = a, \quad y = a - 1, \quad \text{and} \quad z = a.$$

So, every ordered triple of the form

$$(a, a - 1, a)$$ a is a real number.

is a solution of the system. Because there were originally three variables and only two equations, the system cannot have a unique solution.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve the system of linear equations.

$$\begin{cases} x - y + 4z = 3 \\ 4x - z = 0 \end{cases}$$

In Example 6, try choosing some values of a to obtain different solutions of the system, such as

$$(1, 0, 1), \quad (2, 1, 2), \quad \text{and} \quad (3, 2, 3).$$

Then check each ordered triple in the original system to verify that it is a solution of the system.

edobric/Shutterstock.com

Applications

EXAMPLE 7 Vertical Motion

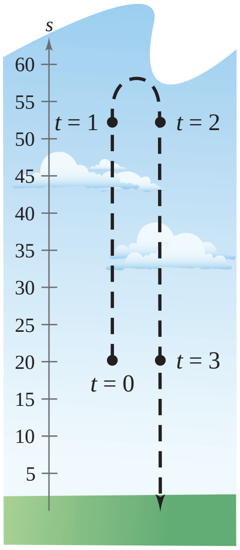


Figure 6.14

The height at time t of an object that is moving in a (vertical) line with constant acceleration a is given by the **position equation**

$$s = \frac{1}{2}at^2 + v_0 t + s_0$$

where s is the height in feet, a is the acceleration in feet per second squared, t is the time in seconds, v_0 is the initial velocity (at $t = 0$), and s_0 is the initial height. Find the values of a, v_0, and s_0 when $s = 52$ at $t = 1$, $s = 52$ at $t = 2$, and $s = 20$ at $t = 3$, and interpret the result. (See Figure 6.14.)

Solution By substituting the three values of t and s into the position equation, you can obtain three linear equations in a, v_0, and s_0.

When $t = 1$: $\frac{1}{2}a(1)^2 + v_0(1) + s_0 = 52 \Rightarrow a + 2v_0 + 2s_0 = 104$

When $t = 2$: $\frac{1}{2}a(2)^2 + v_0(2) + s_0 = 52 \Rightarrow 2a + 2v_0 + s_0 = 52$

When $t = 3$: $\frac{1}{2}a(3)^2 + v_0(3) + s_0 = 20 \Rightarrow 9a + 6v_0 + 2s_0 = 40$

This produces the following system of linear equations.

$$\begin{cases} a + 2v_0 + 2s_0 = 104 \\ 2a + 2v_0 + s_0 = 52 \\ 9a + 6v_0 + 2s_0 = 40 \end{cases}$$

Now solve the system using Gaussian elimination.

$$\begin{cases} a + 2v_0 + 2s_0 = 104 \\ \quad -2v_0 - 3s_0 = -156 \\ 9a + 6v_0 + 2s_0 = 40 \end{cases}$$

Adding -2 times the first equation to the second equation produces a new second equation.

$$\begin{cases} a + 2v_0 + 2s_0 = 104 \\ \quad -2v_0 - 3s_0 = -156 \\ \quad -12v_0 - 16s_0 = -896 \end{cases}$$

Adding -9 times the first equation to the third equation produces a new third equation.

$$\begin{cases} a + 2v_0 + 2s_0 = 104 \\ \quad -2v_0 - 3s_0 = -156 \\ \qquad 2s_0 = 40 \end{cases}$$

Adding -6 times the second equation to the third equation produces a new third equation.

$$\begin{cases} a + 2v_0 + 2s_0 = 104 \\ \quad v_0 + \frac{3}{2}s_0 = 78 \\ \qquad s_0 = 20 \end{cases}$$

Multiplying the second equation by $-\frac{1}{2}$ produces a new second equation and multiplying the third equation by $\frac{1}{2}$ produces a new third equation.

So, the solution of this system is $a = -32$, $v_0 = 48$, and $s_0 = 20$, which can be written as $(-32, 48, 20)$. This solution results in a position equation of $s = -16t^2 + 48t + 20$ and implies that the object was thrown upward at a velocity of 48 feet per second from a height of 20 feet.

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

For the position equation

$$s = \frac{1}{2}at^2 + v_0 t + s_0$$

given in Example 7, find the values of a, v_0, and s_0 when $s = 104$ at $t = 1$, $s = 76$ at $t = 2$, and $s = 16$ at $t = 3$, and interpret the result.

EXAMPLE 8 Data Analysis: Curve-Fitting

Find a quadratic equation $y = ax^2 + bx + c$ whose graph passes through the points $(-1, 3)$, $(1, 1)$, and $(2, 6)$.

Solution Because the graph of $y = ax^2 + bx + c$ passes through the points $(-1, 3)$, $(1, 1)$, and $(2, 6)$, you can write the following.

When $x = -1, y = 3$: $a(-1)^2 + b(-1) + c = 3$

When $x = 1, y = 1$: $a(1)^2 + b(1) + c = 1$

When $x = 2, y = 6$: $a(2)^2 + b(2) + c = 6$

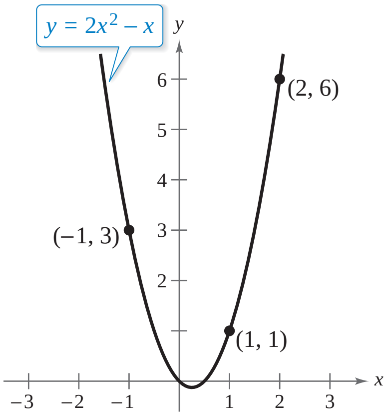
This produces the following system of linear equations.

$$\begin{cases} a - b + c = 3 & \text{Equation 1} \\ a + b + c = 1 & \text{Equation 2} \\ 4a + 2b + c = 6 & \text{Equation 3} \end{cases}$$

The solution of this system is $a = 2$, $b = -1$, and $c = 0$. So, the equation of the parabola is

$$y = 2x^2 - x$$

as shown below.



✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find a quadratic equation $y = ax^2 + bx + c$ whose graph passes through the points $(0, 0)$, $(3, -3)$, and $(6, 0)$.

Summarize (Section 6.3)

1. State the definition of row-echelon form *(page 446)*. For an example of solving a linear system in row-echelon form, see Example 1.
2. Describe the process of Gaussian elimination *(pages 447 and 448)*. For examples of using Gaussian elimination to solve systems of linear equations, see Examples 2–5.
3. Explain the difference between a square system of linear equations and a nonsquare system of linear equations *(page 451)*. For an example of solving a nonsquare system of linear equations, see Example 6.
4. Describe examples of how to use systems of linear equations in three or more variables to model and solve real-life problems *(pages 452 and 453, Examples 7 and 8)*.

6.3 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. A system of equations in ________ form has a "stair-step" pattern with leading coefficients of 1.
2. A solution of a system of three linear equations in three unknowns can be written as an ________ ________, which has the form (x, y, z).
3. The process used to write a system of linear equations in row-echelon form is called ________ elimination.
4. Interchanging two equations of a system of linear equations is a ________ ________ that produces an equivalent system.
5. A system of equations is called ________ when the number of equations differs from the number of variables in the system.
6. The equation $s = \frac{1}{2}at^2 + v_0t + s_0$ is called the ________ equation, and it models the height s of an object at time t that is moving in a vertical line with a constant acceleration a.

Skills and Applications

Checking Solutions **In Exercises 7–10, determine whether each ordered triple is a solution of the system of equations.**

7. $\begin{cases} 6x - y + z = -1 \\ 4x - 3z = -19 \\ 2y + 5z = 25 \end{cases}$

(a) $(2, 0, -2)$ (b) $(-3, 0, 5)$

(c) $(0, -1, 4)$ (d) $(-1, 0, 5)$

8. $\begin{cases} 3x + 4y - z = 17 \\ 5x - y + 2z = -2 \\ 2x - 3y + 7z = -21 \end{cases}$

(a) $(3, -1, 2)$ (b) $(1, 3, -2)$

(c) $(4, 1, -3)$ (d) $(1, -2, 2)$

9. $\begin{cases} 4x + y - z = 0 \\ -8x - 6y + z = -\frac{7}{4} \\ 3x - y = -\frac{9}{4} \end{cases}$

(a) $\left(\frac{1}{2}, -\frac{3}{4}, -\frac{7}{4}\right)$ (b) $\left(-\frac{3}{2}, \frac{5}{4}, -\frac{5}{4}\right)$

(c) $\left(-\frac{1}{2}, \frac{3}{4}, -\frac{5}{4}\right)$ (d) $\left(-\frac{1}{2}, \frac{1}{6}, -\frac{3}{4}\right)$

10. $\begin{cases} -4x - y - 8z = -6 \\ y + z = 0 \\ 4x - 7y = 6 \end{cases}$

(a) $(-2, -2, 2)$ (b) $\left(-\frac{33}{2}, -10, 10\right)$

(c) $\left(\frac{1}{8}, -\frac{1}{2}, \frac{1}{2}\right)$ (d) $\left(-\frac{11}{2}, -4, 4\right)$

Using Back-Substitution **In Exercises 11–16, use back-substitution to solve the system of linear equations.**

11. $\begin{cases} 2x - y + 5z = 24 \\ y + 2z = 6 \\ z = 8 \end{cases}$

12. $\begin{cases} 4x - 3y - 2z = 21 \\ 6y - 5z = -8 \\ z = -2 \end{cases}$

13. $\begin{cases} 2x + y - 3z = 10 \\ y + z = 12 \\ z = 2 \end{cases}$

14. $\begin{cases} x - y + 2z = 22 \\ 3y - 8z = -9 \\ z = -3 \end{cases}$

15. $\begin{cases} 4x - 2y + z = 8 \\ -y + z = 4 \\ z = 11 \end{cases}$

16. $\begin{cases} 5x - 8z = 22 \\ 3y - 5z = 10 \\ z = -4 \end{cases}$

Performing Row Operations **In Exercises 17 and 18, perform the row operation and write the equivalent system.**

17. Add Equation 1 to Equation 2.

$$\begin{cases} x - 2y + 3z = 5 & \text{Equation 1} \\ -x + 3y - 5z = 4 & \text{Equation 2} \\ 2x - 3z = 0 & \text{Equation 3} \end{cases}$$

What did this operation accomplish?

18. Add -2 times Equation 1 to Equation 3.

$$\begin{cases} x - 2y + 3z = 5 & \text{Equation 1} \\ -x + 3y - 5z = 4 & \text{Equation 2} \\ 2x - 3z = 0 & \text{Equation 3} \end{cases}$$

What did this operation accomplish?

Solving a System of Linear Equations **In Exercises 19–24, solve the system of linear equations and check any solutions algebraically.**

19. $\begin{cases} -2x + 3y = 10 \\ x + y = 0 \end{cases}$

20. $\begin{cases} 2x - y = 0 \\ x - y = 7 \end{cases}$

21. $\begin{cases} 3x - y = 9 \\ x - 2y = -2 \end{cases}$

22. $\begin{cases} x + 2y = 1 \\ 5x - 4y = -23 \end{cases}$

23. $\begin{cases} 1.5x + 0.8y = -0.1 \\ -0.3x + 0.2y = -0.7 \end{cases}$

24. $\begin{cases} \frac{1}{5}x + \frac{1}{2}y = -13 \\ x + y = -35 \end{cases}$

Solving a System of Linear Equations **In Exercises 25–46, solve the system of linear equations and check any solutions algebraically.**

25. $\begin{cases} x + y + z = 7 \\ 2x - y + z = 9 \\ 3x - z = 10 \end{cases}$

26. $\begin{cases} x + y + z = 5 \\ x - 2y + 4z = 13 \\ 3y + 4z = 13 \end{cases}$

27. $\begin{cases} 2x + 2z = 2 \\ 5x + 3y = 4 \\ 3y - 4z = 4 \end{cases}$

28. $\begin{cases} 2x + 4y + z = 1 \\ x - 2y - 3z = 2 \\ x + y - z = -1 \end{cases}$

29. $\begin{cases} 2x + y - z = 7 \\ x - 2y + 2z = -9 \\ 3x - y + z = 5 \end{cases}$

30. $\begin{cases} 5x - 3y + 2z = 3 \\ 2x + 4y - z = 7 \\ x - 11y + 4z = 3 \end{cases}$

31. $\begin{cases} 3x - 5y + 5z = 1 \\ 5x - 2y + 3z = 0 \\ 7x - y + 3z = 0 \end{cases}$

32. $\begin{cases} 2x + y + 3z = 1 \\ 2x + 6y + 8z = 3 \\ 6x + 8y + 18z = 5 \end{cases}$

33. $\begin{cases} 2x + 3y = 0 \\ 4x + 3y - z = 0 \\ 8x + 3y + 3z = 0 \end{cases}$

34. $\begin{cases} 4x + 3y + 17z = 0 \\ 5x + 4y + 22z = 0 \\ 4x + 2y + 19z = 0 \end{cases}$

35. $\begin{cases} x + 4z = 1 \\ x + y + 10z = 10 \\ 2x - y + 2z = -5 \end{cases}$

36. $\begin{cases} 2x - 2y - 6z = -4 \\ -3x + 2y + 6z = 1 \\ x - y - 5z = -3 \end{cases}$

37. $\begin{cases} 3x - 3y + 6z = 6 \\ x + 2y - z = 5 \\ 5x - 8y + 13z = 7 \end{cases}$

38. $\begin{cases} x + 2z = 5 \\ 3x - y - z = 1 \\ 6x - y + 5z = 16 \end{cases}$

39. $\begin{cases} x - 2y + 5z = 2 \\ 4x - z = 0 \end{cases}$

40. $\begin{cases} x - 3y + 2z = 18 \\ 5x - 13y + 12z = 80 \end{cases}$

41. $\begin{cases} x + 2y - 7z = -4 \\ 2x + y + z = 13 \\ 3x + 9y - 36z = -33 \end{cases}$

42. $\begin{cases} 2x + y - 3z = 4 \\ 4x + 2z = 10 \\ -2x + 3y - 13z = -8 \end{cases}$

43. $\begin{cases} 2x - 3y + z = -2 \\ -4x + 9y = 7 \end{cases}$

44. $\begin{cases} 2x + 3y + 3z = 7 \\ 4x + 18y + 15z = 44 \end{cases}$

45. $\begin{cases} x + 3w = 4 \\ 2y - z - w = 0 \\ 3y - 2w = 1 \\ 2x - y + 4z = 5 \end{cases}$

46. $\begin{cases} x + y + z + w = 6 \\ 2x + 3y - w = 0 \\ -3x + 4y + z + 2w = 4 \\ x + 2y - z + w = 0 \end{cases}$

Vertical Motion **In Exercises 47 and 48, an object moving vertically is at the given heights at the specified times. Find the position equation $s = \frac{1}{2}at^2 + v_0t + s_0$ for the object.**

47. At $t = 1$ second, $s = 128$ feet
At $t = 2$ seconds, $s = 80$ feet
At $t = 3$ seconds, $s = 0$ feet

48. At $t = 1$ second, $s = 132$ feet
At $t = 2$ seconds, $s = 100$ feet
At $t = 3$ seconds, $s = 36$ feet

Finding the Equation of a Parabola **In Exercises 49–54, find the equation of the parabola**

$$y = ax^2 + bx + c$$

that passes through the points. To verify your result, use a graphing utility to plot the points and graph the parabola.

49. $(0, 0), (2, -2), (4, 0)$

50. $(0, 3), (1, 4), (2, 3)$

51. $(2, 0), (3, -1), (4, 0)$

52. $(1, 3), (2, 2), (3, -3)$

53. $\left(\frac{1}{2}, 1\right), (1, 3), (2, 13)$

54. $(-2, -3), (-1, 0), \left(\frac{1}{2}, -3\right)$

Finding the Equation of a Circle **In Exercises 55–58, find the equation of the circle**

$$x^2 + y^2 + Dx + Ey + F = 0$$

that passes through the points. To verify your result, use a graphing utility to plot the points and graph the circle.

55. $(0, 0), (5, 5), (10, 0)$

56. $(0, 0), (0, 6), (3, 3)$

57. $(-3, -1), (2, 4), (-6, 8)$

58. $(0, 0), (0, -2), (3, 0)$

59. **Sports** In Super Bowl I, on January 15, 1967, the Green Bay Packers defeated the Kansas City Chiefs by a score of 35 to 10. The total points scored came from a combination of touchdowns, extra-point kicks, and field goals, worth 6, 1, and 3 points, respectively. The numbers of touchdowns and extra-point kicks were equal. There were six times as many touchdowns as field goals. Find the numbers of touchdowns, extra-point kicks, and field goals scored. *(Source: National Football League)*

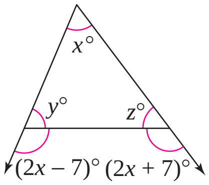
60. **Agriculture** A mixture of 5 pounds of fertilizer A, 13 pounds of fertilizer B, and 4 pounds of fertilizer C provides the optimal nutrients for a plant. Commercial brand X contains equal parts of fertilizer B and fertilizer C. Commercial brand Y contains one part of fertilizer A and two parts of fertilizer B. Commercial brand Z contains two parts of fertilizer A, five parts of fertilizer B, and two parts of fertilizer C. How much of each fertilizer brand is needed to obtain the desired mixture?

61. Finance A small corporation borrowed \$775,000 to expand its clothing line. Some of the money was borrowed at 8%, some at 9%, and some at 10%. How much was borrowed at each rate when the annual interest owed was \$67,500 and the amount borrowed at 8% was four times the amount borrowed at 10%?

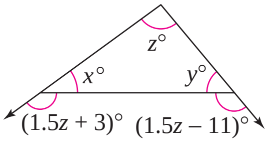
62. Advertising A health insurance company advertises on television, on radio, and in the local newspaper. The marketing department has an advertising budget of \$42,000 per month. A television ad costs \$1000, a radio ad costs \$200, and a newspaper ad costs \$500. The department wants to run 60 ads per month and have as many television ads as radio and newspaper ads combined. How many of each type of ad can the department run each month?

Geometry **In Exercises 63 and 64, find the values of x, y, and z in the figure.**

63.



64.



65. Geometry The perimeter of a triangle is 180 feet. The longest side of the triangle is 9 feet shorter than twice the shortest side. The sum of the lengths of the two shorter sides is 30 feet more than the length of the longest side. Find the lengths of the sides of the triangle.

66. Acid Mixture A chemist needs 10 liters of a 25% acid solution. The solution is to be mixed from three solutions whose concentrations are 10%, 20%, and 50%. How many liters of each solution will satisfy each condition?

(a) Use 2 liters of the 50% solution.

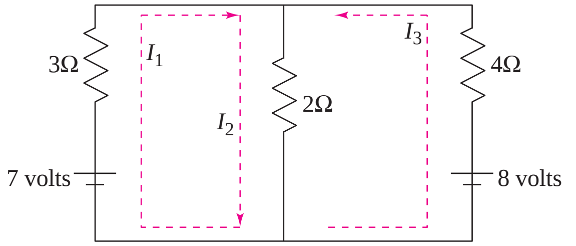
(b) Use as little as possible of the 50% solution.

(c) Use as much as possible of the 50% solution.

67. Electrical Network Applying Kirchhoff's Laws to the electrical network in the figure, the currents I_1, I_2, and I_3 are the solution of the system

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 3I_1 + 2I_2 \phantom{{}+ 4I_3} = 7. \\ \phantom{3I_1 +{}} 2I_2 + 4I_3 = 8 \end{cases}$$

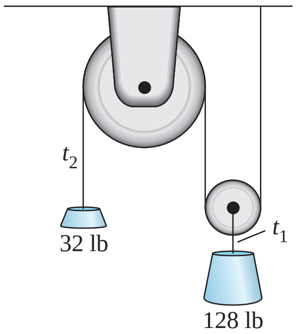
Find the currents.



68. Pulley System A system of pulleys is loaded with 128-pound and 32-pound weights (see figure). The tensions t_1 and t_2 in the ropes and the acceleration a of the 32-pound weight are found by solving the system of equations

$$\begin{cases} t_1 - 2t_2 \phantom{{}- 2a} = 0 \\ t_1 \phantom{{}- 2t_2} - 2a = 128 \\ \phantom{t_1 -{}} t_2 + a = 32 \end{cases}$$

where t_1 and t_2 are in pounds and a is in feet per second squared. Solve this system.



Fitting a Parabola **To find the least squares regression parabola $y = ax^2 + bx + c$ for a set of points**

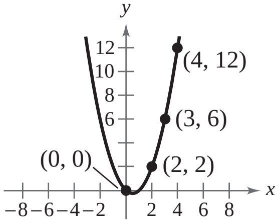
$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

you can solve the following system of linear equations for a, b, and c.

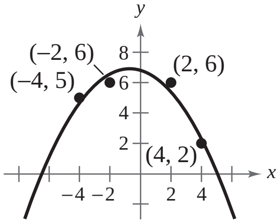
$$\begin{cases} nc + \left(\sum_{i=1}^{n} x_i\right)b + \left(\sum_{i=1}^{n} x_i^2\right)a = \sum_{i=1}^{n} y_i \\ \left(\sum_{i=1}^{n} x_i\right)c + \left(\sum_{i=1}^{n} x_i^2\right)b + \left(\sum_{i=1}^{n} x_i^3\right)a = \sum_{i=1}^{n} x_i y_i \\ \left(\sum_{i=1}^{n} x_i^2\right)c + \left(\sum_{i=1}^{n} x_i^3\right)b + \left(\sum_{i=1}^{n} x_i^4\right)a = \sum_{i=1}^{n} x_i^2 y_i \end{cases}$$

In Exercises 69 and 70, the sums have been evaluated. Solve the given system for a, b, and c to find the least squares regression parabola for the points. Use a graphing utility to confirm the result.

69. $$\begin{cases} 4c + 9b + 29a = 20 \\ 9c + 29b + 99a = 70 \\ 29c + 99b + 353a = 254 \end{cases}$$



70. $$\begin{cases} 4c \phantom{{}+ 40b} + 40a = 19 \\ \phantom{4c + {}} 40b \phantom{{}+ 40a} = -12 \\ 40c \phantom{{}+ 40b} + 544a = 160 \end{cases}$$



71. Data Analysis: Stopping Distance In testing a new automobile braking system, engineers recorded the speed x (in miles per hour) and the stopping distance y (in feet) in the following table.

Speed, x	30	40	50
Stopping Distance, y	55	105	188

(a) Find the least squares regression parabola $y = ax^2 + bx + c$ for the data by solving the system.

$$\left\{\begin{array}{rcrcrcl} 3c & + & 120b & + & 5000a & = & 348 \\ 120c & + & 5000b & + & 216{,}000a & = & 15{,}250 \\ 5000c & + & 216{,}000b & + & 9{,}620{,}000a & = & 687{,}500 \end{array}\right.$$

(b) Graph the parabola and the data on the same set of axes.

(c) Use the model to estimate the stopping distance when the speed is 70 miles per hour.

72. Data Analysis: Wildlife

A wildlife management team studied the reproductive rates of deer in three tracts of a wildlife preserve. Each tract contained 5 acres. In each tract, the number of females x, and the percent of females y, that had offspring the following year were recorded. The table shows the results.

Number, x	100	120	140
Percent, y	75	68	55

(a) Use the data to create a system of linear equations. Then find the least squares regression parabola for the data by solving the system.

(b) Use a graphing utility to graph the parabola and the data in the same viewing window.

(c) Use the model to estimate the percent of females that had offspring when there were 170 females.

(d) Use the model to estimate the number of females when 40% of the females had offspring.

Advanced Applications **In Exercises 73 and 74, find values of x, y, and λ that satisfy the system. These systems arise in certain optimization problems in calculus, and λ is called a Lagrange multiplier.**

73. $\left\{\begin{array}{l} 2x - 2x\lambda = 0 \\ -2y + \lambda = 0 \\ y - x^2 = 0 \end{array}\right.$

74. $\left\{\begin{array}{r} 2 + 2y + 2\lambda = 0 \\ 2x + 1 + \lambda = 0 \\ 2x + y - 100 = 0 \end{array}\right.$

Exploration

True or False? **In Exercises 75 and 76, determine whether the statement is true or false. Justify your answer.**

75. The system

$$\left\{\begin{array}{rcrcrcr} x & + & 3y & - & 6z & = & -16 \\ & & 2y & - & z & = & -1 \\ & & & & z & = & 3 \end{array}\right.$$

is in row-echelon form.

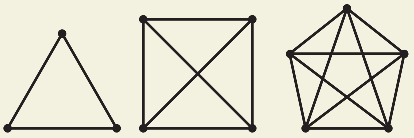
76. If a system of three linear equations is inconsistent, then its graph has no points common to all three equations.

77. Think About It Are the following two systems of equations equivalent? Give reasons for your answer.

$$\left\{\begin{array}{rcrcrcr} x & + & 3y & - & z & = & 6 \\ 2x & - & y & + & 2z & = & 1 \\ 3x & + & 2y & - & z & = & 2 \end{array}\right.$$

$$\left\{\begin{array}{rcrcrcr} x & + & 3y & - & z & = & 6 \\ & & -7y & + & 4z & = & 1 \\ & & -7y & - & 4z & = & -16 \end{array}\right.$$

78. HOW DO YOU SEE IT? The number of sides x and the combined number of sides and diagonals y for each of three regular polygons are shown below. Write a system of linear equations to find an equation of the form $y = ax^2 + bx + c$ that represents the relationship between x and y for the three polygons.



$x = 3, y = 3$ $x = 4, y = 6$ $x = 5, y = 10$

Finding Systems of Linear Equations **In Exercises 79–82, find two systems of linear equations that have the ordered triple as a solution. (There are many correct answers.)**

79. $(3, -4, 2)$

80. $(-5, -2, 1)$

81. $\left(-6, -\frac{1}{2}, -\frac{7}{4}\right)$

82. $\left(-\frac{3}{2}, 4, -7\right)$

Project: Earnings per Share To work an extended application analyzing the earnings per share for Wal-Mart Stores, Inc., from 1995 through 2010, visit this text's website at *LarsonPrecalculus.com*. *(Source: Wal-Mart Stores, Inc.)*

Bruce MacQueen/Shutterstock.com

6.4 Partial Fractions

Partial fractions can help you analyze the behavior of a rational function. For instance, in Exercise 59 on page 465, you will use partial fractions to analyze the predicted cost for a company to remove a percentage of a chemical from its waste water.

tlorna/Shutterstock.com

- **Recognize partial fraction decompositions of rational expressions.**
- **Find partial fraction decompositions of rational expressions.**

Introduction

In this section, you will learn to write a rational expression as the sum of two or more simpler rational expressions. For example, the rational expression

$$\frac{x+7}{x^2-x-6}$$

can be written as the sum of two fractions with first-degree denominators. That is,

$$\frac{x+7}{x^2-x-6} = \overbrace{\underbrace{\frac{2}{x-3}}_{\text{Partial fraction}} + \underbrace{\frac{-1}{x+2}}_{\text{Partial fraction}}}^{\text{Partial fraction decomposition of } \frac{x+7}{x^2-x-6}}.$$

Each fraction on the right side of the equation is a **partial fraction,** and together they make up the **partial fraction decomposition** of the left side.

ALGEBRA HELP You can review how to find the degree of a polynomial (such as $x-3$ and $x+2$) in Section P.3.

REMARK Section P.5 shows you how to combine expressions such as

$$\frac{1}{x-2} + \frac{-1}{x+3} = \frac{5}{(x-2)(x+3)}.$$

The method of partial fraction decomposition shows you how to reverse this process and write

$$\frac{5}{(x-2)(x+3)} = \frac{1}{x-2} + \frac{-1}{x+3}.$$

Decomposition of $N(x)/D(x)$ into Partial Fractions

1. *Divide when improper:* When $N(x)/D(x)$ is an improper fraction [degree of $N(x) \geq$ degree of $D(x)$], divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)} = (\text{polynomial}) + \frac{N_1(x)}{D(x)}$$

and apply Steps 2, 3, and 4 below to the proper rational expression

$$\frac{N_1(x)}{D(x)}.$$

Note that $N_1(x)$ is the remainder from the division of $N(x)$ by $D(x)$.

2. *Factor the denominator:* Completely factor the denominator into factors of the form

$$(px+q)^m \quad \text{and} \quad (ax^2+bx+c)^n$$

where (ax^2+bx+c) is irreducible.

3. *Linear factors:* For *each* factor of the form $(px+q)^m$, the partial fraction decomposition must include the following sum of m fractions.

$$\frac{A_1}{(px+q)} + \frac{A_2}{(px+q)^2} + \cdot \cdot \cdot + \frac{A_m}{(px+q)^m}$$

4. *Quadratic factors:* For *each* factor of the form $(ax^2+bx+c)^n$, the partial fraction decomposition must include the following sum of n fractions.

$$\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdot \cdot \cdot + \frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

Partial Fraction Decomposition

The following examples demonstrate algebraic techniques for determining the constants in the numerators of partial fractions. Note that the techniques vary slightly, depending on the type of factors of the denominator: linear or quadratic, distinct or repeated.

EXAMPLE 1 Distinct Linear Factors

Write the partial fraction decomposition of

$$\frac{x+7}{x^2-x-6}.$$

Solution The expression is proper, so you should begin by factoring the denominator. Because

$$x^2-x-6=(x-3)(x+2)$$

you should include one partial fraction with a constant numerator for each linear factor of the denominator. Write the form of the decomposition as follows.

$$\frac{x+7}{x^2-x-6}=\frac{A}{x-3}+\frac{B}{x+2} \qquad \text{Write form of decomposition.}$$

Multiplying each side of this equation by the least common denominator, $(x-3)(x+2)$, leads to the **basic equation**

$$x+7=A(x+2)+B(x-3). \qquad \text{Basic equation}$$

Because this equation is true for all x, substitute any *convenient* values of x that will help determine the constants A and B. Values of x that are especially convenient are those that make the factors $(x+2)$ and $(x-3)$ equal to zero. For instance, to solve for B, let $x=-2$. Then

$$\begin{aligned}
-2+7&=A(-2+2)+B(-2-3) && \text{Substitute } -2 \text{ for } x.\\
5&=A(0)+B(-5)\\
5&=-5B\\
-1&=B.
\end{aligned}$$

To solve for A, let $x=3$ and obtain

$$\begin{aligned}
3+7&=A(3+2)+B(3-3) && \text{Substitute 3 for } x.\\
10&=A(5)+B(0)\\
10&=5A\\
2&=A.
\end{aligned}$$

So, the partial fraction decomposition is

$$\frac{x+7}{x^2-x-6}=\frac{2}{x-3}+\frac{-1}{x+2}.$$

Check this result by combining the two partial fractions on the right side of the equation, or by using your graphing utility.

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Write the partial fraction decomposition of

$$\frac{x+5}{2x^2-x-1}.$$

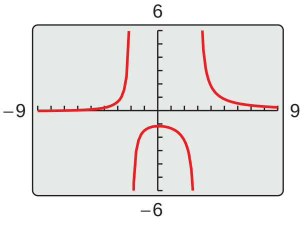
TECHNOLOGY To use a graphing utility to check the decomposition found in Example 1, graph

$$y_1=\frac{x+7}{x^2-x-6}$$

and

$$y_2=\frac{2}{x-3}+\frac{-1}{x+2}$$

in the same viewing window. The graphs should be identical, as shown below.



The next example shows how to find the partial fraction decomposition of a rational expression whose denominator has a *repeated* linear factor.

EXAMPLE 2 Repeated Linear Factors

Write the partial fraction decomposition of $\dfrac{x^4 + 2x^3 + 6x^2 + 20x + 6}{x^3 + 2x^2 + x}$.

ALGEBRA HELP You can review long division of polynomials in Section 3.3. You can review factoring of polynomials in Section P.4.

Solution This rational expression is improper, so you should begin by dividing the numerator by the denominator to obtain

$$x + \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}.$$

Because the denominator of the remainder factors as

$$x^3 + 2x^2 + x = x(x^2 + 2x + 1) = x(x + 1)^2$$

you should include one partial fraction with a constant numerator for each power of x and $(x + 1)$, and write the form of the decomposition as follows.

$$\frac{5x^2 + 20x + 6}{x(x + 1)^2} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2} \qquad \text{Write form of decomposition.}$$

Multiplying each side by the LCD, $x(x + 1)^2$, leads to the basic equation

$$5x^2 + 20x + 6 = A(x + 1)^2 + Bx(x + 1) + Cx. \qquad \text{Basic equation}$$

REMARK To obtain the basic equation, be sure to multiply *each* fraction in the form of the decomposition by the LCD.

Letting $x = -1$ eliminates the A- and B-terms and yields

$$\begin{aligned} 5(-1)^2 + 20(-1) + 6 &= A(-1 + 1)^2 + B(-1)(-1 + 1) + C(-1) \\ 5 - 20 + 6 &= 0 + 0 - C \\ C &= 9. \end{aligned}$$

Letting $x = 0$ eliminates the B- and C-terms and yields

$$\begin{aligned} 5(0)^2 + 20(0) + 6 &= A(0 + 1)^2 + B(0)(0 + 1) + C(0) \\ 6 &= A(1) + 0 + 0 \\ 6 &= A. \end{aligned}$$

At this point, you have exhausted the most convenient values of x, so to find the value of B, use *any other value* of x along with the known values of A and C. So, using $x = 1$, $A = 6$, and $C = 9$,

$$\begin{aligned} 5(1)^2 + 20(1) + 6 &= 6(1 + 1)^2 + B(1)(1 + 1) + 9(1) \\ 31 &= 6(4) + 2B + 9 \\ -2 &= 2B \\ -1 &= B. \end{aligned}$$

So, the partial fraction decomposition is

$$\frac{x^4 + 2x^3 + 6x^2 + 20x + 6}{x^3 + 2x^2 + x} = x + \frac{6}{x} + \frac{-1}{x + 1} + \frac{9}{(x + 1)^2}.$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write the partial fraction decomposition of

$$\frac{x^4 + x^3 + x + 4}{x^3 + x^2}.$$

The procedure used to solve for the constants in Examples 1 and 2 works well when the factors of the denominator are linear. However, when the denominator contains irreducible quadratic factors, you should use a different procedure, which involves writing the right side of the basic equation in polynomial form, *equating the coefficients* of like terms, and using a system of equations to solve for the coefficients.

EXAMPLE 3 Distinct Linear and Quadratic Factors

Write the partial fraction decomposition of

$$\frac{3x^2 + 4x + 4}{x^3 + 4x}.$$

John Bernoulli (1667–1748), a Swiss mathematician, introduced the method of partial fractions and was instrumental in the early development of calculus. Bernoulli was a professor at the University of Basel and taught many outstanding students, the most famous of whom was Leonhard Euler.

Solution This expression is proper, so begin by factoring the denominator. Because the denominator factors as

$$x^3 + 4x = x(x^2 + 4)$$

you should include one partial fraction with a constant numerator and one partial fraction with a linear numerator, and write the form of the decomposition as follows.

$$\frac{3x^2 + 4x + 4}{x^3 + 4x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4} \qquad \text{Write form of decomposition.}$$

Multiplying each side by the LCD, $x(x^2 + 4)$, yields the basic equation

$$3x^2 + 4x + 4 = A(x^2 + 4) + (Bx + C)x. \qquad \text{Basic equation}$$

Expanding this basic equation and collecting like terms produces

$$\begin{aligned} 3x^2 + 4x + 4 &= Ax^2 + 4A + Bx^2 + Cx \\ &= (A + B)x^2 + Cx + 4A. \qquad \text{Polynomial form} \end{aligned}$$

Finally, because two polynomials are equal if and only if the coefficients of like terms are equal, equate the coefficients of like terms on opposite sides of the equation.

$$3x^2 + 4x + 4 = (A + B)x^2 + Cx + 4A \qquad \text{Equate coefficients of like terms.}$$

Now write the following system of linear equations.

$$\begin{cases} A + B = 3 & \text{Equation 1} \\ C = 4 & \text{Equation 2} \\ 4A = 4 & \text{Equation 3} \end{cases}$$

From this system, you can see that

$$A = 1 \quad \text{and} \quad C = 4.$$

Moreover, back-substituting $A = 1$ into Equation 1 yields

$$1 + B = 3 \quad \Longrightarrow \quad B = 2.$$

So, the partial fraction decomposition is

$$\frac{3x^2 + 4x + 4}{x^3 + 4x} = \frac{1}{x} + \frac{2x + 4}{x^2 + 4}.$$

✓ **Checkpoint** 🔊 *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write the partial fraction decomposition of

$$\frac{2x^2 - 5}{x^3 + x}.$$

Mary Evans Picture Library

The next example shows how to find the partial fraction decomposition of a rational expression whose denominator has a *repeated* quadratic factor.

EXAMPLE 4 Repeated Quadratic Factors

Write the partial fraction decomposition of

$$\frac{8x^3 + 13x}{(x^2 + 2)^2}.$$

Solution Include one partial fraction with a linear numerator for each power of $(x^2 + 2)$.

$$\frac{8x^3 + 13x}{(x^2 + 2)^2} = \frac{Ax + B}{x^2 + 2} + \frac{Cx + D}{(x^2 + 2)^2}$$ Write form of decomposition.

Multiplying each side by the LCD, $(x^2 + 2)^2$, yields the basic equation

$$8x^3 + 13x = (Ax + B)(x^2 + 2) + Cx + D$$ Basic equation

$$= Ax^3 + 2Ax + Bx^2 + 2B + Cx + D$$

$$= Ax^3 + Bx^2 + (2A + C)x + (2B + D).$$ Polynomial form

Equating coefficients of like terms on opposite sides of the equation

$$8x^3 + 0x^2 + 13x + 0 = Ax^3 + Bx^2 + (2A + C)x + (2B + D)$$

produces the following system of linear equations.

$$\begin{cases} A \qquad\qquad\qquad = 8 & \text{Equation 1} \\ \qquad B \qquad\qquad = 0 & \text{Equation 2} \\ 2A + \qquad C \qquad = 13 & \text{Equation 3} \\ \qquad 2B + \qquad D = 0 & \text{Equation 4} \end{cases}$$

Use the values $A = 8$ and $B = 0$ to obtain the following.

$$2(8) + C = 13$$ Substitute 8 for A in Equation 3.

$$C = -3$$

$$2(0) + D = 0$$ Substitute 0 for B in Equation 4.

$$D = 0$$

So, using

$$A = 8, \quad B = 0, \quad C = -3, \quad \text{and} \quad D = 0$$

the partial fraction decomposition is

$$\frac{8x^3 + 13x}{(x^2 + 2)^2} = \frac{8x}{x^2 + 2} + \frac{-3x}{(x^2 + 2)^2}.$$

Check this result by combining the two partial fractions on the right side of the equation, or by using your graphing utility.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write the partial fraction decomposition of

$$\frac{x^3 + 3x^2 - 2x + 7}{(x^2 + 4)^2}.$$

EXAMPLE 5 Repeated Linear and Quadratic Factors

Write the partial fraction decomposition of $\dfrac{x+5}{x^2(x^2+1)^2}$.

Solution Include one partial fraction with a constant numerator for each power of x and one partial fraction with a linear numerator for each power of (x^2+1).

$$\frac{x+5}{x^2(x^2+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}$$

Write form of decomposition.

Multiplying each side by the LCD, $x^2(x^2+1)^2$, yields the basic equation

$$\begin{aligned} x+5 &= Ax(x^2+1)^2 + B(x^2+1)^2 + (Cx+D)x^2(x^2+1) + (Ex+F)x^2 \\ &= (A+C)x^5 + (B+D)x^4 + (2A+C+E)x^3 + (2B+D+F)x^2 + Ax + B. \end{aligned}$$

Basic equation

Verify, by equating coefficients of like terms on opposite sides of the equation and solving the resulting system of linear equations, that $A = 1$, $B = 5$, $C = -1$, $D = -5$, $E = -1$, and $F = -5$, and that the partial fraction decomposition is

$$\frac{x+5}{x^2(x^2+1)^2} = \frac{1}{x} + \frac{5}{x^2} - \frac{x+5}{x^2+1} - \frac{x+5}{(x^2+1)^2}.$$

✓ ***Checkpoint*** ***Audio-video solution in English & Spanish at LarsonPrecalculus.com.***

Write the partial fraction decomposition of $\dfrac{4x-8}{x^2(x^2+2)^2}$.

Guidelines for Solving the Basic Equation

Linear Factors

1. Substitute the *zeros* of the distinct linear factors into the basic equation.
2. For repeated linear factors, use the coefficients determined in Step 1 to rewrite the basic equation. Then substitute *other* convenient values of x and solve for the remaining coefficients.

Quadratic Factors

1. Expand the basic equation.
2. Collect terms according to powers of x.
3. Equate the coefficients of like terms to obtain equations involving A, B, C, and so on.
4. Use a system of linear equations to solve for A, B, C, and so on.

Keep in mind that for *improper* rational expressions, you must first divide before applying partial fraction decomposition.

Summarize (Section 6.4)

1. Explain what is meant by the partial fraction decomposition of a rational expression *(page 458)*.
2. Explain how to find the partial fraction decomposition of a rational expression *(pages 458–463)*. For examples of finding partial fraction decompositions of rational expressions, see Examples 1–5.

6.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. The process of writing a rational expression as the sum of two or more simpler rational expressions is called _________ _________ _________.
2. If the degree of the numerator of a rational expression is greater than or equal to the degree of the denominator, then the fraction is called _________.
3. Each fraction on the right side of the equation $\frac{x-1}{x^2-8x+15} = \frac{-1}{x-3} + \frac{2}{x-5}$ is a _________ _________.
4. You obtain the _________ _________ after multiplying each side of the partial fraction decomposition form by the least common denominator.

Skills and Applications

Matching **In Exercises 5–8, match the rational expression with the form of its decomposition. [The decompositions are labeled (a), (b), (c), and (d).]**

(a) $\frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2}$

(b) $\frac{A}{x} + \frac{B}{x-4}$

(c) $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-4}$

(d) $\frac{A}{x} + \frac{Bx+C}{x^2+4}$

5. $\frac{3x-1}{x(x-4)}$

6. $\frac{3x-1}{x^2(x-4)}$

7. $\frac{3x-1}{x(x^2+4)}$

8. $\frac{3x-1}{x(x^2-4)}$

Writing the Form of the Decomposition **In Exercises 9–16, write the form of the partial fraction decomposition of the rational expression. Do not solve for the constants.**

9. $\frac{3}{x^2-2x}$

10. $\frac{x-2}{x^2+4x+3}$

11. $\frac{9}{x^3-7x^2}$

12. $\frac{2x-3}{x^3+10x}$

13. $\frac{4x^2+3}{(x-5)^3}$

14. $\frac{6x+5}{(x+2)^4}$

15. $\frac{x-1}{x(x^2+1)^2}$

16. $\frac{x+4}{x^2(3x-1)^2}$

Writing the Partial Fraction Decomposition **In Exercises 17–42, write the partial fraction decomposition of the rational expression. Check your result algebraically.**

17. $\frac{1}{x^2+x}$

18. $\frac{3}{x^2-3x}$

19. $\frac{1}{2x^2+x}$

20. $\frac{5}{x^2+x-6}$

21. $\frac{3}{x^2+x-2}$

22. $\frac{x+1}{x^2-x-6}$

23. $\frac{1}{x^2-1}$

24. $\frac{1}{4x^2-9}$

25. $\frac{x^2+12x+12}{x^3-4x}$

26. $\frac{x+2}{x(x^2-9)}$

27. $\frac{3x}{(x-3)^2}$

28. $\frac{2x-3}{(x-1)^2}$

29. $\frac{4x^2+2x-1}{x^2(x+1)}$

30. $\frac{6x^2+1}{x^2(x-1)^2}$

31. $\frac{x^2+2x+3}{x^3+x}$

32. $\frac{2x}{x^3-1}$

33. $\frac{x}{x^3-x^2-2x+2}$

34. $\frac{x+6}{x^3-3x^2-4x+12}$

35. $\frac{2x^2+x+8}{(x^2+4)^2}$

36. $\frac{x^2}{x^4-2x^2-8}$

37. $\frac{x}{16x^4-1}$

38. $\frac{3}{x^4+x}$

39. $\frac{x^2+5}{(x+1)(x^2-2x+3)}$

40. $\frac{x^2-4x+7}{(x+1)(x^2-2x+3)}$

41. $\frac{8x-12}{x^2(x^2+2)^2}$

42. $\frac{x+1}{x^3(x^2+1)^2}$

Improper Rational Expression Decomposition **In Exercises 43–50, write the partial fraction decomposition of the improper rational expression.**

43. $\frac{x^2-x}{x^2+x+1}$

44. $\frac{x^2-4x}{x^2+x+6}$

45. $\frac{2x^3-x^2+x+5}{x^2+3x+2}$

46. $\frac{x^3+2x^2-x+1}{x^2+3x-4}$

47. $\frac{x^4}{(x-1)^3}$

48. $\frac{16x^4}{(2x-1)^3}$

49. $\frac{x^4+2x^3+4x^2+8x+2}{x^3+2x^2+x}$

50. $\frac{2x^4+8x^3+7x^2-7x-12}{x^3+4x^2+4x}$

Writing the Partial Fraction Decomposition **In Exercises 51–58, write the partial fraction decomposition of the rational expression. Use a graphing utility to check your result.**

51. $\dfrac{5 - x}{2x^2 + x - 1}$

52. $\dfrac{3x^2 - 7x - 2}{x^3 - x}$

53. $\dfrac{4x^2 - 1}{2x(x + 1)^2}$

54. $\dfrac{3x + 1}{2x^3 + 3x^2}$

55. $\dfrac{x^2 + x + 2}{(x^2 + 2)^2}$

56. $\dfrac{x^3}{(x + 2)^2(x - 2)^2}$

57. $\dfrac{2x^3 - 4x^2 - 15x + 5}{x^2 - 2x - 8}$

58. $\dfrac{x^3 - x + 3}{x^2 + x - 2}$

59. Waste Water

The predicted cost C (in thousands of dollars) for a company to remove p% of a chemical from its waste water is given by the model

$$C = \frac{120p}{10{,}000 - p^2}, \quad 0 \le p < 100.$$

Write the partial fraction decomposition for the rational function. Verify your result by using a graphing utility to create a table comparing the original function with the partial fractions.

60. Thermodynamics The magnitude of the range R of exhaust temperatures (in degrees Fahrenheit) in an experimental diesel engine is approximated by the model

$$R = \frac{5000(4 - 3x)}{(11 - 7x)(7 - 4x)}, \quad 0 < x \le 1$$

where x is the relative load (in foot-pounds).

(a) Write the partial fraction decomposition of the equation.

(b) The decomposition in part (a) is the difference of two fractions. The absolute values of the terms give the expected maximum and minimum temperatures of the exhaust gases for different loads.

Ymax = |1st term| Ymin = |2nd term|

Write the equations for Ymax and Ymin.

(c) Use a graphing utility to graph each equation from part (b) in the same viewing window.

(d) Determine the expected maximum and minimum temperatures for a relative load of 0.5.

Exploration

True or False? **In Exercises 61–63, determine whether the statement is true or false. Justify your answer.**

61. For the rational expression $\dfrac{x}{(x + 10)(x - 10)^2}$, the partial fraction decomposition is of the form

$$\frac{A}{x + 10} + \frac{B}{(x - 10)^2}.$$

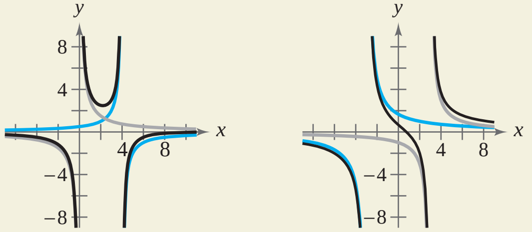
62. For the rational expression $\dfrac{2x + 3}{x^2(x + 2)^2}$, the partial fraction decomposition is of the form $\dfrac{Ax + B}{x^2} + \dfrac{Cx + D}{(x + 2)^2}$.

63. When writing the partial fraction decomposition of the expression $\dfrac{x^3 + x - 2}{x^2 - 5x - 14}$, the first step is to divide the numerator by the denominator.

64. HOW DO YOU SEE IT? Identify the graph of the rational function and the graph representing each partial fraction of its partial fraction decomposition. Then state any relationship between the vertical asymptotes of the graph of the rational function and the vertical asymptotes of the graphs representing the partial fractions of the decomposition. To print an enlarged copy of the graph, go to *MathGraphs.com*.

(a) $y = \dfrac{x - 12}{x(x - 4)}$
$= \dfrac{3}{x} - \dfrac{2}{x - 4}$

(b) $y = \dfrac{2(4x - 3)}{x^2 - 9}$
$= \dfrac{3}{x - 3} + \dfrac{5}{x + 3}$



Writing the Partial Fraction Decomposition **In Exercises 65–68, write the partial fraction decomposition of the rational expression. Then assign a value to the constant *a* to check the result algebraically and graphically.**

65. $\dfrac{1}{a^2 - x^2}$

66. $\dfrac{1}{x(x + a)}$

67. $\dfrac{1}{y(a - y)}$

68. $\dfrac{1}{(x + 1)(a - x)}$

69. Writing Describe two ways of solving for the constants in a partial fraction decomposition.

tlorna/Shutterstock.com

6.5 Systems of Inequalities

Systems of inequalities in two variables can help you model and solve real-life problems. For instance, in Exercise 70 on page 474, you will use a system of inequalities to analyze the nutritional content of a dietary supplement.

- **Sketch the graphs of inequalities in two variables.**
- **Solve systems of inequalities.**
- **Use systems of inequalities in two variables to model and solve real-life problems.**

The Graph of an Inequality

The statements

$$3x - 2y < 6 \quad \text{and} \quad 2x^2 + 3y^2 \geq 6$$

are inequalities in two variables. An ordered pair (a, b) is a **solution of an inequality** in x and y when the inequality is true after a and b are substituted for x and y, respectively. The **graph of an inequality** is the collection of all solutions of the inequality. To sketch the graph of an inequality, begin by sketching the graph of the *corresponding equation*. The graph of the equation will usually separate the plane into two or more regions. In each such region, one of the following must be true.

1. *All* points in the region are solutions of the inequality.
2. *No* point in the region is a solution of the inequality.

So, you can determine whether the points in an entire region satisfy the inequality by testing *one* point in the region.

Sketching the Graph of an Inequality in Two Variables

1. Replace the inequality sign by an equal sign and sketch the graph of the resulting equation. (Use a dashed line for $<$ or $>$ and a solid line for $\leq$ or $\geq$.)
2. Test one point in each of the regions formed by the graph in Step 1. If the point satisfies the inequality, then shade the entire region to denote that every point in the region satisfies the inequality.

REMARK Be careful when you are sketching the graph of an inequality in two variables. A dashed line means that all points on the line or curve *are not* solutions of the inequality. A solid line means that all points on the line or curve *are* solutions of the inequality.

EXAMPLE 1 Sketching the Graph of an Inequality

Sketch the graph of $y \geq x^2 - 1$.

Solution Begin by graphing the corresponding equation $y = x^2 - 1$, which is a parabola, as shown at the right. Test a point *above* the parabola, such as $(0, 0)$, and a point *below* the parabola, such as $(0, -2)$.

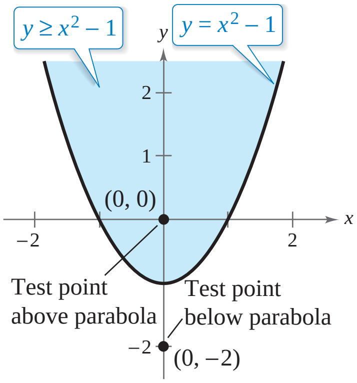
The points that satisfy the inequality are those lying above (or on) the parabola.

$$(0, 0):\ 0 \overset{?}{\geq} 0^2 - 1$$

$$0 \geq -1 \qquad (0, 0) \text{ is a solution.}$$

$$(0, -2):\ -2 \overset{?}{\geq} 0^2 - 1$$

$$-2 \not\geq -1 \qquad (0, -2) \text{ is not a solution.}$$



Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Sketch the graph of $(x + 2)^2 + (y - 2)^2 < 16$.

alexskopje/Shutterstock.com

The inequality in Example 1 is a nonlinear inequality in two variables. Most of the following examples involve **linear inequalities** such as $ax + by < c$ (where a and b are not both zero). The graph of a linear inequality is a half-plane lying on one side of the line $ax + by = c$.

EXAMPLE 2 Sketching the Graph of a Linear Inequality

Sketch the graph of each linear inequality.

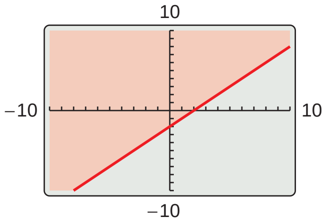
a. $x > -2$ **b.** $y \leq 3$

Solution

a. The graph of the corresponding equation $x = -2$ is a vertical line. The points that satisfy the inequality $x > -2$ are those lying to the right of this line, as shown in Figure 6.15.

b. The graph of the corresponding equation $y = 3$ is a horizontal line. The points that satisfy the inequality $y \leq 3$ are those lying below (or on) this line, as shown in Figure 6.16.

TECHNOLOGY A graphing utility can be used to graph an inequality or a system of inequalities. For instance, to graph $y \geq x - 2$, enter $y = x - 2$ and use the *shade* feature of the graphing utility to shade the correct part of the graph. You should obtain the graph shown below. Consult the user's guide for your graphing utility for specific keystrokes.



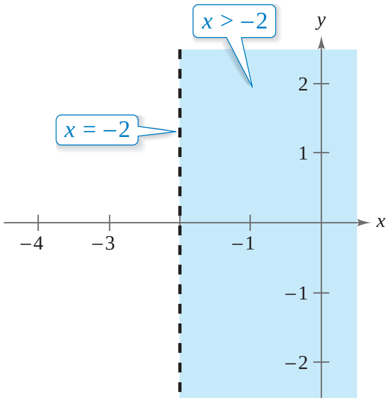


Figure 6.15

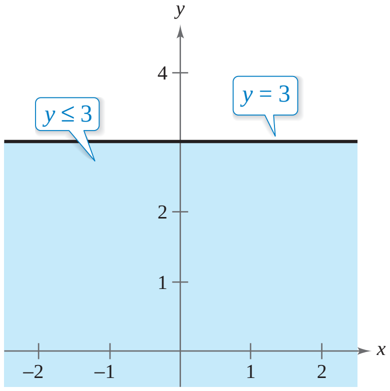


Figure 6.16

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Sketch the graph of $x \geq 3$.

EXAMPLE 3 Sketching the Graph of a Linear Inequality

Sketch the graph of $x - y < 2$.

Solution The graph of the corresponding equation $x - y = 2$ is a line, as shown in Figure 6.17. Because the origin $(0, 0)$ satisfies the inequality, the graph consists of the half-plane lying above the line. (Try checking a point below the line. Regardless of which point you choose, you will see that it does not satisfy the inequality.)

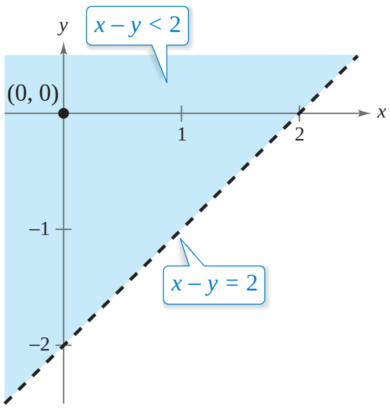


Figure 6.17

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Sketch the graph of $x + y > -2$.

To graph a linear inequality, it can help to write the inequality in slope-intercept form. For instance, by writing $x - y < 2$ in the form

$$y > x - 2$$

you can see that the solution points lie *above* the line $x - y = 2$ (or $y = x - 2$), as shown in Figure 6.17.

Systems of Inequalities

Many practical problems in business, science, and engineering involve systems of linear inequalities. A **solution** of a system of inequalities in x and y is a point

$$(x, y)$$

that satisfies each inequality in the system.

To sketch the graph of a system of inequalities in two variables, first sketch the graph of each individual inequality (on the same coordinate system) and then find the region that is *common* to every graph in the system. This region represents the **solution set** of the system. For systems of *linear inequalities*, it is helpful to find the vertices of the solution region.

EXAMPLE 4 Solving a System of Inequalities

Sketch the graph (and label the vertices) of the solution set of the system.

$$\begin{cases} x - y < 2 & \text{Inequality 1} \\ x > -2 & \text{Inequality 2} \\ y \le 3 & \text{Inequality 3} \end{cases}$$

Solution The graphs of these inequalities are shown in Figures 6.17, 6.15, and 6.16, respectively, on page 467. The triangular region common to all three graphs can be found by superimposing the graphs on the same coordinate system, as shown below. To find the vertices of the region, solve the three systems of corresponding equations obtained by taking *pairs* of equations representing the boundaries of the individual regions.

Vertex A: $(-2, -4)$

$$\begin{cases} x - y = 2 \\ x = -2 \end{cases}$$

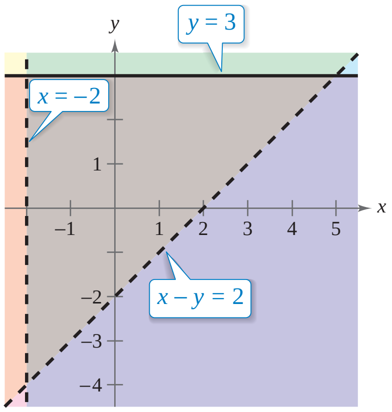
Vertex B: $(5, 3)$

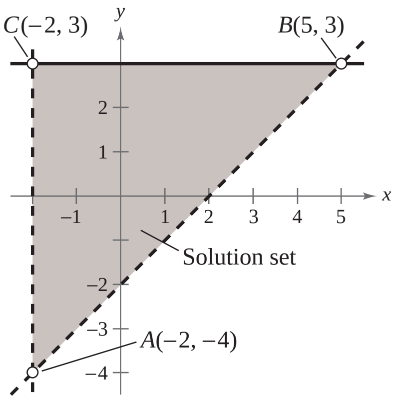
$$\begin{cases} x - y = 2 \\ y = 3 \end{cases}$$

Vertex C: $(-2, 3)$

$$\begin{cases} x = -2 \\ y = 3 \end{cases}$$

REMARK Using a different colored pencil to shade the solution of each inequality in a system will make identifying the solution of the system of inequalities easier.





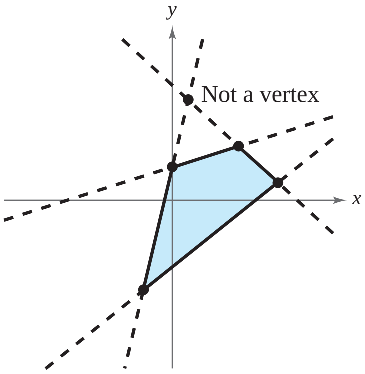
Note that the vertices of the region are represented by open dots. This means that the vertices *are not* solutions of the system of inequalities.

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Sketch the graph (and label the vertices) of the solution set of the system.

$$\begin{cases} x + y \ge 1 \\ -x + y \ge 1 \\ y \le 2 \end{cases}$$

For the triangular region shown on page 468, each point of intersection of a pair of boundary lines corresponds to a vertex. With more complicated regions, two border lines can sometimes intersect at a point that is not a vertex of the region, as shown below. To keep track of which points of intersection are actually vertices of the region, you should sketch the region and refer to your sketch as you find each point of intersection.



EXAMPLE 5 Solving a System of Inequalities

Sketch the region containing all points that satisfy the system of inequalities.

$$\begin{cases} x^2 - y \le 1 & \text{Inequality 1} \\ -x + y \le 1 & \text{Inequality 2} \end{cases}$$

Solution As shown in the figure below, the points that satisfy the inequality

$$x^2 - y \le 1 \qquad \text{Inequality 1}$$

are the points lying above (or on) the parabola

$$y = x^2 - 1. \qquad \text{Parabola}$$

The points satisfying the inequality

$$-x + y \le 1 \qquad \text{Inequality 2}$$

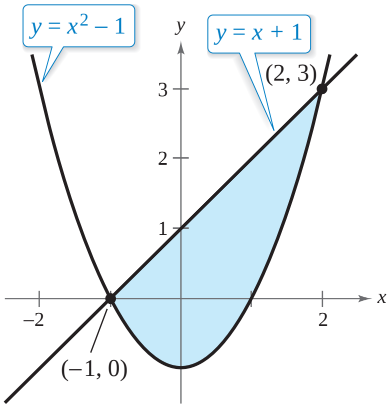
are the points lying below (or on) the line

$$y = x + 1. \qquad \text{Line}$$

To find the points of intersection of the parabola and the line, solve the system of corresponding equations.

$$\begin{cases} x^2 - y = 1 \\ -x + y = 1 \end{cases}$$

Using the method of substitution, you can find the solutions to be $(-1, 0)$ and $(2, 3)$. So, the region containing all points that satisfy the system is indicated by the shaded region at the right.



Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Sketch the region containing all points that satisfy the system of inequalities.

$$\begin{cases} x - y^2 > 0 \\ x + \ y < 2 \end{cases}$$

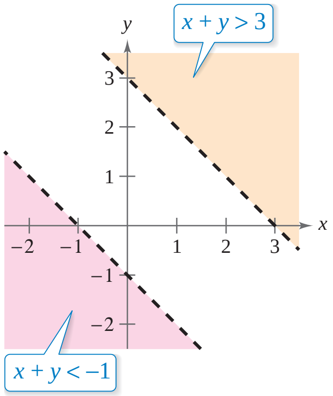
When solving a system of inequalities, you should be aware that the system might have no solution *or* its graph might be an unbounded region in the plane. Examples 6 and 7 show these two possibilities.

EXAMPLE 6 A System with No Solution

Sketch the solution set of the system of inequalities.

$$\begin{cases} x + y > 3 & \text{Inequality 1} \\ x + y < -1 & \text{Inequality 2} \end{cases}$$

Solution From the way the system is written, it should be clear that the system has no solution because the quantity $(x + y)$ cannot be both less than -1 and greater than 3. The graph of the inequality $x + y > 3$ is the half-plane lying above the line $x + y = 3$, and the graph of the inequality $x + y < -1$ is the half-plane lying below the line $x + y = -1$, as shown below. These two half-planes have no points in common. So, the system of inequalities has no solution.



Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Sketch the solution set of the system of inequalities.

$$\begin{cases} 2x - y < -3 \\ 2x - y > 1 \end{cases}$$

EXAMPLE 7 An Unbounded Solution Set

Sketch the solution set of the system of inequalities.

$$\begin{cases} x + y < 3 & \text{Inequality 1} \\ x + 2y > 3 & \text{Inequality 2} \end{cases}$$

Solution The graph of the inequality $x + y < 3$ is the half-plane that lies below the line $x + y = 3$. The graph of the inequality $x + 2y > 3$ is the half-plane that lies above the line $x + 2y = 3$. The intersection of these two half-planes is an *infinite wedge* that has a vertex at $(3, 0)$, as shown in Figure 6.18. So, the solution set of the system of inequalities is unbounded.

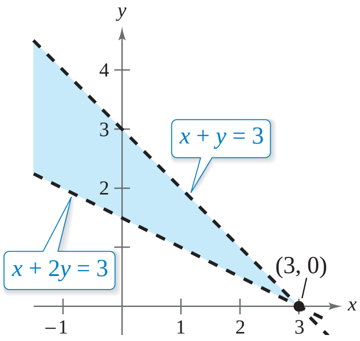


Figure 6.18

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Sketch the solution set of the system of inequalities.

$$\begin{cases} x^2 - y < 0 \\ x - y < -2 \end{cases}$$

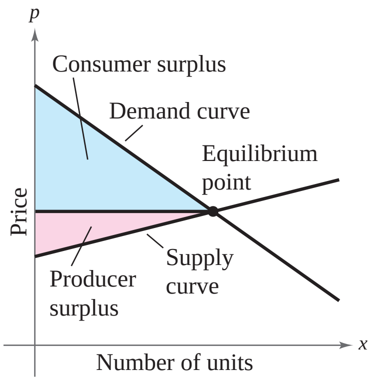


Figure 6.19

Applications

Example 9 in Section 6.2 discussed the *equilibrium point* for a system of demand and supply equations. The next example discusses two related concepts that economists call **consumer surplus** and **producer surplus.** As shown in Figure 6.19, the consumer surplus is defined as the area of the region formed by the *demand* curve, the horizontal line passing through the equilibrium point, and the p-axis. Similarly, the producer surplus is defined as the area of the region formed by the *supply* curve, the horizontal line passing through the equilibrium point, and the p-axis. The consumer surplus is a measure of the amount that consumers would have been willing to pay *above what they actually paid*, whereas the producer surplus is a measure of the amount that producers would have been willing to receive *below what they actually received.*

EXAMPLE 8 Consumer Surplus and Producer Surplus

The demand and supply equations for a new type of video game console are

$$\begin{cases} p = 180 - 0.00001x & \text{Demand equation} \\ p = 90 + 0.00002x & \text{Supply equation} \end{cases}$$

where p is the price per unit (in dollars) and x is the number of units. Find the consumer surplus and producer surplus for these two equations.

Solution Begin by finding the equilibrium point (when supply and demand are equal) by solving the equation

$$90 + 0.00002x = 180 - 0.00001x.$$

In Example 9 in Section 6.2, you saw that the solution is $x = 3{,}000{,}000$ units, which corresponds to an equilibrium price of $p = \$150$. So, the consumer surplus and producer surplus are the areas of the following triangular regions.

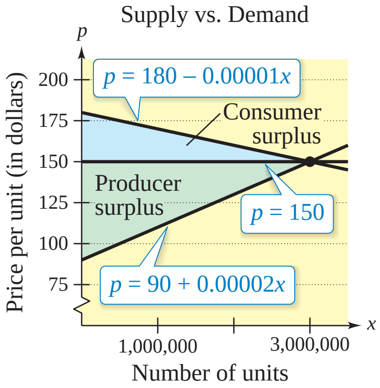


Figure 6.20

Consumer Surplus

$$\begin{cases} p \le 180 - 0.00001x \\ p \ge 150 \\ x \ge 0 \end{cases}$$

Producer Surplus

$$\begin{cases} p \ge 90 + 0.00002x \\ p \le 150 \\ x \ge 0 \end{cases}$$

The consumer and producer surpluses are the areas of the shaded triangles shown in Figure 6.20.

$$\begin{aligned} \text{Consumer surplus} &= \tfrac{1}{2}(\text{base})(\text{height}) \\ &= \tfrac{1}{2}(3{,}000{,}000)(30) \\ &= \$45{,}000{,}000 \end{aligned}$$

$$\begin{aligned} \text{Producer surplus} &= \tfrac{1}{2}(\text{base})(\text{height}) \\ &= \tfrac{1}{2}(3{,}000{,}000)(60) \\ &= \$90{,}000{,}000 \end{aligned}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The demand and supply equations for a flat-screen television are

$$\begin{cases} p = 567 - 0.00002x & \text{Demand equation} \\ p = 492 + 0.00003x & \text{Supply equation} \end{cases}$$

where p is the price per unit (in dollars) and x is the number of units. Find the consumer surplus and producer surplus for these two equations.

A system of linear inequalities can help you determine amounts of dietary drinks to consume to meet or exceed daily requirements for calories and vitamins.

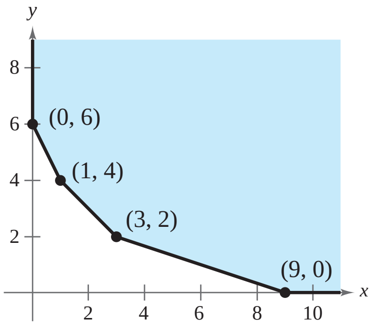
EXAMPLE 9 Nutrition

The liquid portion of a diet is to provide at least 300 calories, 36 units of vitamin A, and 90 units of vitamin C. A cup of dietary drink X provides 60 calories, 12 units of vitamin A, and 10 units of vitamin C. A cup of dietary drink Y provides 60 calories, 6 units of vitamin A, and 30 units of vitamin C. Set up a system of linear inequalities that describes how many cups of each drink should be consumed each day to meet or exceed the minimum daily requirements for calories and vitamins.

Solution Begin by letting x represent the number of cups of dietary drink X and y represent the number of cups of dietary drink Y. To meet or exceed the minimum daily requirements, the following inequalities must be satisfied.

$$\begin{cases} 60x + 60y \geq 300 & \text{Calories} \\ 12x + 6y \geq 36 & \text{Vitamin A} \\ 10x + 30y \geq 90 & \text{Vitamin C} \\ x \geq 0 \\ y \geq 0 \end{cases}$$

The last two inequalities are included because x and y cannot be negative. The graph of this system of inequalities is shown below. (More is said about this application in Example 6 in Section 6.6.)



Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

A pet supply company mixes two brands of dry dog food. A bag of brand X contains 8 units of nutrient A, 1 unit of nutrient B, and 2 units of nutrient C. A bag of brand Y contains 2 units of nutrient A, 1 unit of nutrient B, and 7 units of nutrient C. The minimum required amounts of nutrients A, B, and C are 16 units, 5 units, and 20 units, respectively. Set up a system of linear inequalities that describes how many bags of each brand should be mixed to meet or exceed the minimum required amounts of nutrients.

Summarize (Section 6.5)

1. Explain how to sketch the graph of an inequality in two variables *(page 466)*. For examples of sketching the graphs of inequalities in two variables, see Examples 1–3.
2. Explain how to solve a system of inequalities *(page 468)*. For examples of solving systems of inequalities, see Examples 4–7.
3. Describe examples of how to use systems of inequalities in two variables to model and solve real-life problems *(pages 471 and 472, Examples 8 and 9)*.

Kinetic Imagery/Shutterstock.com

6.5 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: **Fill in the blanks.**

1. An ordered pair (a, b) is a ________ of an inequality in x and y when the inequality is true after a and b are substituted for x and y, respectively.

2. The ________ of an inequality is the collection of all solutions of the inequality.

3. A ________ of a system of inequalities in x and y is a point (x, y) that satisfies each inequality in the system.

4. A ________ ________ of a system of inequalities in two variables is the region common to the graphs of every inequality in the system.

Skills and Applications

Graphing an Inequality

In Exercises 5–18, sketch the graph of the inequality.

5. $y < 5 - x^2$

6. $y^2 - x < 0$

7. $x \ge 6$

8. $x < -4$

9. $y > -7$

10. $10 \ge y$

11. $y < 2 - x$

12. $y > 4x - 3$

13. $2y - x \ge 4$

14. $5x + 3y \ge -15$

15. $(x + 1)^2 + (y - 2)^2 < 9$

16. $(x - 1)^2 + (y - 4)^2 > 9$

17. $y \le \dfrac{1}{1 + x^2}$

18. $y > \dfrac{-15}{x^2 + x + 4}$

Graphing an Inequality

In Exercises 19–28, use a graphing utility to graph the inequality.

19. $y < \ln x$

20. $y \ge -2 - \ln(x + 3)$

21. $y < 4^{-x-5}$

22. $y \le 2^{2x-0.5} - 7$

23. $y \le 6 - \frac{3}{2}x$

24. $y < -3.8x + 1.1$

25. $x^2 + 5y - 10 \le 0$

26. $2x^2 - y - 3 > 0$

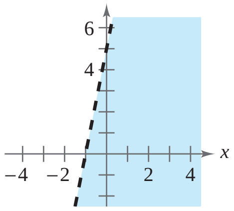
27. $\frac{5}{2}y - 3x^2 - 6 \ge 0$

28. $-\frac{1}{10}x^2 - \frac{3}{8}y < -\frac{1}{4}$

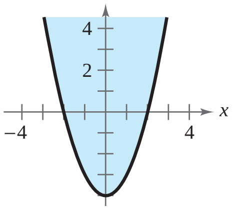
Writing an Inequality

In Exercises 29–32, write an inequality for the shaded region shown in the figure.

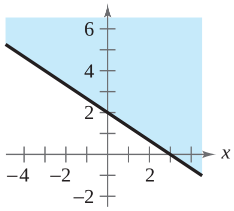
29.



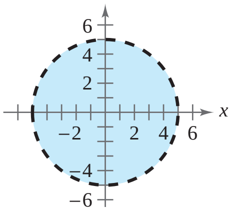
30.



31.



32.



Solving a System of Inequalities

In Exercises 33–46, sketch the graph (and label the vertices) of the solution set of the system of inequalities.

33. $\begin{cases} x + y \le 1 \\ -x + y \le 1 \\ y \ge 0 \end{cases}$

34. $\begin{cases} 3x + 4y < 12 \\ x > 0 \\ y > 0 \end{cases}$

35. $\begin{cases} x^2 + y \le 7 \\ x \ge -2 \\ y \ge 0 \end{cases}$

36. $\begin{cases} 4x^2 + y \ge 2 \\ x \le 1 \\ y \le 1 \end{cases}$

37. $\begin{cases} 2x + y > 2 \\ 6x + 3y < 2 \end{cases}$

38. $\begin{cases} x - 7y > -36 \\ 5x + 2y > 5 \\ 6x - 5y > 6 \end{cases}$

39. $\begin{cases} -3x + 2y < 6 \\ x - 4y > -2 \\ 2x + y < 3 \end{cases}$

40. $\begin{cases} x - 2y < -6 \\ 5x - 3y > -9 \end{cases}$

41. $\begin{cases} x > y^2 \\ x < y + 2 \end{cases}$

42. $\begin{cases} x - y^2 > 0 \\ x - y > 2 \end{cases}$

43. $\begin{cases} x^2 + y^2 \le 36 \\ x^2 + y^2 \ge 9 \end{cases}$

44. $\begin{cases} x^2 + y^2 \le 25 \\ 4x - 3y \le 0 \end{cases}$

45. $\begin{cases} 3x + 4 \ge y^2 \\ x - y < 0 \end{cases}$

46. $\begin{cases} x < 2y - y^2 \\ 0 < x + y \end{cases}$

Solving a System of Inequalities

In Exercises 47–52, use a graphing utility to graph the solution set of the system of inequalities.

47. $\begin{cases} y \le \sqrt{3x} + 1 \\ y \ge x^2 + 1 \end{cases}$

48. $\begin{cases} y < -x^2 + 2x + 3 \\ y > x^2 - 4x + 3 \end{cases}$

49. $\begin{cases} y < x^3 - 2x + 1 \\ y > -2x \\ x \le 1 \end{cases}$

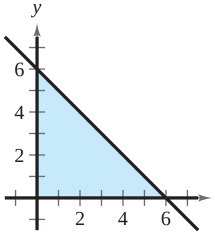
50. $\begin{cases} y \ge x^4 - 2x^2 + 1 \\ y \le 1 - x^2 \end{cases}$

51. $\begin{cases} x^2y \ge 1 \\ 0 < x \le 4 \\ y \le 4 \end{cases}$

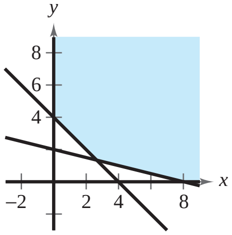
52. $\begin{cases} y \le e^{-x^2/2} \\ y \ge 0 \\ -2 \le x \le 2 \end{cases}$

Writing a System of Inequalities **In Exercises 53–60, write a system of inequalities to describe the region.**

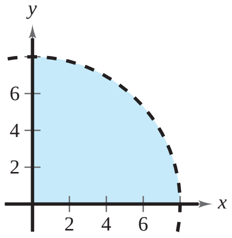
53.



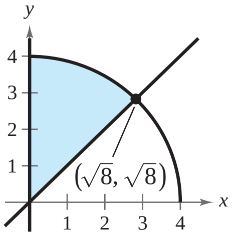
54.



55.



56.



57. Rectangle: vertices at $(4, 3)$, $(9, 3)$, $(9, 9)$, $(4, 9)$

58. Parallelogram: vertices at $(0, 0)$, $(4, 0)$, $(1, 4)$, $(5, 4)$

59. Triangle: vertices at $(0, 0)$, $(6, 0)$, $(1, 5)$

60. Triangle: vertices at $(-1, 0)$, $(1, 0)$, $(0, 1)$

Consumer Surplus and Producer Surplus **In Exercises 61–64, (a) graph the systems representing the consumer surplus and producer surplus for the supply and demand equations and (b) find the consumer surplus and producer surplus.**

	Demand	Supply
61.	$p = 50 - 0.5x$	$p = 0.125x$
62.	$p = 100 - 0.05x$	$p = 25 + 0.1x$
63.	$p = 140 - 0.00002x$	$p = 80 + 0.00001x$
64.	$p = 400 - 0.0002x$	$p = 225 + 0.0005x$

65. Production A furniture company produces tables and chairs. Each table requires 1 hour in the assembly center and $1\frac{1}{3}$ hours in the finishing center. Each chair requires $1\frac{1}{2}$ hours in the assembly center and $1\frac{1}{2}$ hours in the finishing center. The assembly center is available 12 hours per day, and the finishing center is available 15 hours per day. Find and graph a system of inequalities describing all possible production levels.

66. Inventory A store sells two models of laptop computers. Because of the demand, the store stocks at least twice as many units of model A as of model B. The costs to the store for the two models are $800 and $1200, respectively. The management does not want more than $20,000 in computer inventory at any one time, and it wants at least four model A laptop computers and two model B laptop computers in inventory at all times. Find and graph a system of inequalities describing all possible inventory levels.

alexskopje/Shutterstock.com

67. Investment Analysis A person plans to invest up to $20,000 in two different interest-bearing accounts. Each account is to contain at least $5000. Moreover, the amount in one account should be at least twice the amount in the other account. Find and graph a system of inequalities to describe the various amounts that can be deposited in each account.

68. Ticket Sales For a concert event, there are $30 reserved seat tickets and $20 general admission tickets. There are 2000 reserved seats available, and fire regulations limit the number of paid ticket holders to 3000. The promoter must take in at least $75,000 in ticket sales. Find and graph a system of inequalities describing the different numbers of tickets that can be sold.

69. Truck Scheduling A small company that manufactures two models of exercise machines has an order for 15 units of the standard model and 16 units of the deluxe model. The company has trucks of two different sizes that can haul the products, as shown in the table.

Truck	Standard	Deluxe
Large	6	3
Medium	4	6

Find and graph a system of inequalities describing the numbers of trucks of each size that are needed to deliver the order.

70. Nutrition

A dietitian designs a special dietary supplement using two different foods. Each ounce of food X contains 20 units of calcium, 15 units of iron, and 10 units of vitamin B. Each ounce of food Y contains 10 units of calcium, 10 units of iron, and 20 units of vitamin B. The minimum daily requirements of the diet are 300 units of calcium, 150 units of iron, and 200 units of vitamin B.

(a) Write a system of inequalities describing the different amounts of food X and food Y that can be used.

(b) Sketch a graph of the region corresponding to the system in part (a).

(c) Find two solutions of the system and interpret their meanings in the context of the problem.

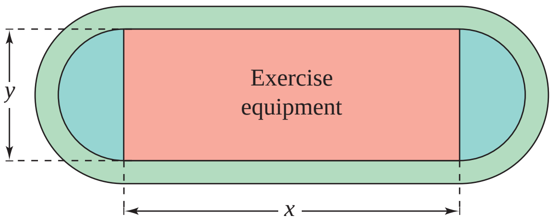
71. **Health** A person's maximum heart rate is $220 - x$, where x is the person's age in years for $20 \le x \le 70$. The American Heart Association recommends that when a person exercises, the person should strive for a heart rate that is at least 50% of the maximum and at most 85% of the maximum. *(Source: American Heart Association)*

(a) Write a system of inequalities that describes the exercise target heart rate region.

(b) Sketch a graph of the region in part (a).

(c) Find two solutions to the system and interpret their meanings in the context of the problem.

72. **Physical Fitness Facility** An indoor running track is to be constructed with a space for exercise equipment inside the track (see figure). The track must be at least 125 meters long, and the exercise space must have an area of at least 500 square meters.



(a) Find a system of inequalities describing the requirements of the facility.

(b) Graph the system from part (a).

Exploration

True or False? **In Exercises 73 and 74, determine whether the statement is true or false. Justify your answer.**

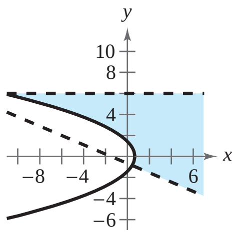
73. The area of the figure defined by the system

$$\begin{cases} x \ge -3 \\ x \le 6 \\ y \le 5 \\ y \ge -6 \end{cases}$$

is 99 square units.

74. The following graph shows the solution of the system

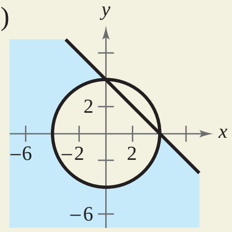
$$\begin{cases} y \le 6 \\ -4x - 9y > 6. \\ 3x + y^2 \ge 2 \end{cases}$$



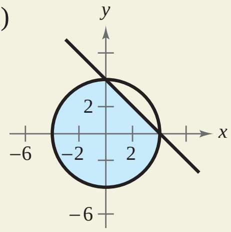
75. **Think About It** After graphing the boundary of the inequality $x + y < 3$, explain how you decide on which side of the boundary the solution set of the inequality lies.

76. **HOW DO YOU SEE IT?** Match the system of inequalities with the graph of its solution. [The graphs are labeled (i), (ii), (iii), and (iv).]

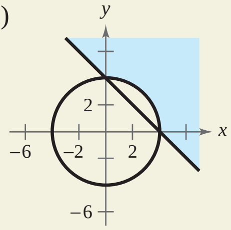
(i)



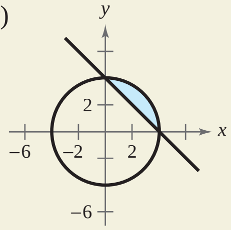
(ii)



(iii)



(iv)



(a) $\begin{cases} x^2 + y^2 \le 16 \\ x + y \ge 4 \end{cases}$ (b) $\begin{cases} x^2 + y^2 \le 16 \\ x + y \le 4 \end{cases}$

(c) $\begin{cases} x^2 + y^2 \ge 16 \\ x + y \ge 4 \end{cases}$ (d) $\begin{cases} x^2 + y^2 \ge 16 \\ x + y \le 4 \end{cases}$

77. **Graphical Reasoning** Two concentric circles have radii x and y, where $y > x$. The area between the circles is at least 10 square units.

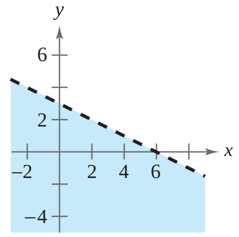
(a) Find a system of inequalities describing the constraints on the circles.

(b) Use a graphing utility to graph the system of inequalities in part (a). Graph the line $y = x$ in the same viewing window.

(c) Identify the graph of the line in relation to the boundary of the inequality. Explain its meaning in the context of the problem.

78. **Changing the Inequality Symbol** The graph of the solution of the inequality $x + 2y < 6$ is shown in the figure. Describe how the solution set would change for each of the following.

(a) $x + 2y \le 6$ (b) $x + 2y > 6$



6.6 Linear Programming

Linear programming is often useful in making real-life decisions. For instance, in Exercise 43 on page 484, you will use linear programming to determine the optimal acreage and yield for two fruit crops.

- **Solve linear programming problems.**
- **Use linear programming to model and solve real-life problems.**

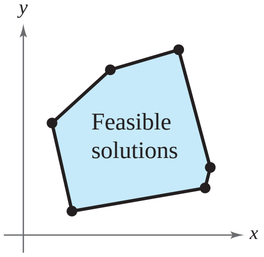
Linear Programming: A Graphical Approach

Many applications in business and economics involve a process called **optimization,** in which you find the minimum or maximum value of a quantity. In this section, you will study an optimization strategy called **linear programming.**

A two-dimensional linear programming problem consists of a linear **objective function** and a system of linear inequalities called **constraints.** The objective function gives the quantity to be maximized (or minimized), and the constraints determine the set of **feasible solutions.** For instance, suppose you want to maximize the value of

$$z = ax + by \qquad \text{Objective function}$$

subject to a set of constraints that determines the shaded region shown below. Because every point in the shaded region satisfies each constraint, it is not clear how you should find the point that yields a maximum value of z. Fortunately, it can be shown that when there is an optimal solution, it must occur at one of the vertices. This means that *to find the maximum value of z, test z at each of the vertices.*



Optimal Solution of a Linear Programming Problem

If a linear programming problem has a solution, then it must occur at a vertex of the set of feasible solutions. If there is more than one solution, then at least one solution must occur at such a vertex. In either case, the value of the objective function is unique.

A linear programming problem can include hundreds, and sometimes even thousands, of variables. However, in this section, you will solve linear programming problems that involve only two variables. The guidelines for solving a linear programming problem in two variables are listed below.

Solving a Linear Programming Problem

1. Sketch the region corresponding to the system of constraints. (The points inside or on the boundary of the region are *feasible solutions.*)
2. Find the vertices of the region.
3. Test the objective function at each of the vertices and select the values of the variables that optimize the objective function. For a bounded region, both a minimum and a maximum value will exist. (For an unbounded region, *if* an optimal solution exists, then it will occur at a vertex.)

Alison McC/Shutterstock.com

EXAMPLE 1 Solving a Linear Programming Problem

Find the maximum value of

$$z = 3x + 2y$$ Objective function

subject to the following constraints.

$$\left.\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + 2y &\leq 4 \\ x - y &\leq 1 \end{aligned}\right\}$$ Constraints

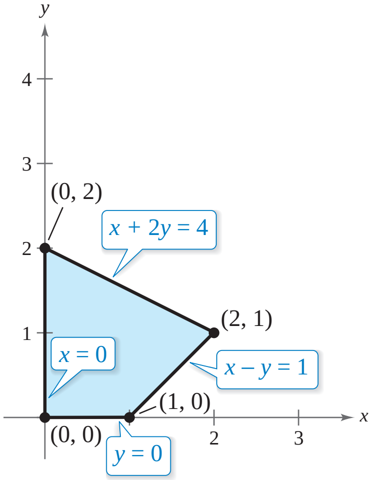


Figure 6.21

Solution The constraints form the region shown in Figure 6.21. At the four vertices of this region, the objective function has the following values.

At (0, 0): $z = 3(0) + 2(0) = 0$

At (0, 2): $z = 3(0) + 2(2) = 4$

At (2, 1): $z = 3(2) + 2(1) = 8$ Maximum value of z

At (1, 0): $z = 3(1) + 2(0) = 3$

So, the maximum value of z is 8, and this occurs when $x = 2$ and $y = 1$.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the maximum value of

$$z = 4x + 5y$$

subject to the following constraints.

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + y &\leq 6 \end{aligned}$$

In Example 1, try testing some of the *interior* points in the region. You will see that the corresponding values of z are less than 8. Here are some examples.

At (1, 1): $z = 3(1) + 2(1) = 5$

At $\left(1, \frac{1}{2}\right)$: $z = 3(1) + 2\left(\frac{1}{2}\right) = 4$

At $\left(\frac{1}{2}, \frac{3}{2}\right)$: $z = 3\left(\frac{1}{2}\right) + 2\left(\frac{3}{2}\right) = \frac{9}{2}$

At $\left(\frac{3}{2}, 1\right)$: $z = 3\left(\frac{3}{2}\right) + 2(1) = \frac{13}{2}$

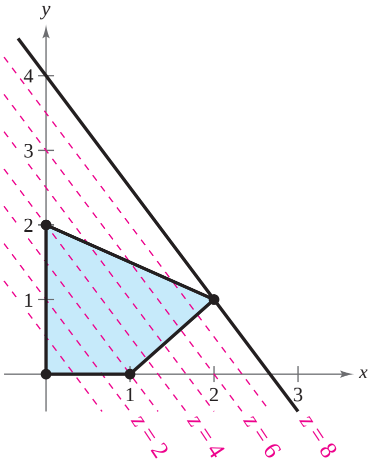


Figure 6.22

To see why the maximum value of the objective function in Example 1 must occur at a vertex, consider writing the objective function in slope-intercept form

$$y = -\frac{3}{2}x + \frac{z}{2}$$ Family of lines

where

$$\frac{z}{2}$$

is the y-intercept of the objective function. This equation represents a family of lines, each of slope $-\frac{3}{2}$. Of these infinitely many lines, you want the one that has the largest z-value while still intersecting the region determined by the constraints. In other words, of all the lines whose slope is $-\frac{3}{2}$, you want the one that has the largest y-intercept *and* intersects the given region, as shown in Figure 6.22. Notice from the graph that such a line will pass through one (or more) of the vertices of the region.

The next example shows that the same basic procedure can be used to solve a problem in which the objective function is to be *minimized*.

EXAMPLE 2 Minimizing an Objective Function

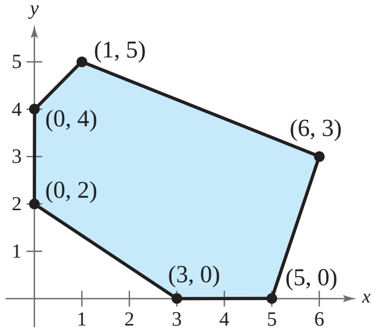


Figure 6.23

Find the minimum value of

$$z = 5x + 7y \qquad \text{Objective function}$$

where $x \geq 0$ and $y \geq 0$, subject to the following constraints.

$$\left.\begin{aligned} 2x + 3y &\geq 6 \\ 3x - y &\leq 15 \\ -x + y &\leq 4 \\ 2x + 5y &\leq 27 \end{aligned}\right\} \qquad \text{Constraints}$$

Solution Figure 6.23 shows the region bounded by the constraints. By testing the objective function at each vertex, you obtain the following.

$$\begin{aligned} &\text{At } (0, 2): && z = 5(0) + 7(2) = 14 && \text{Minimum value of } z \\ &\text{At } (0, 4): && z = 5(0) + 7(4) = 28 \\ &\text{At } (1, 5): && z = 5(1) + 7(5) = 40 \\ &\text{At } (6, 3): && z = 5(6) + 7(3) = 51 \\ &\text{At } (5, 0): && z = 5(5) + 7(0) = 25 \\ &\text{At } (3, 0): && z = 5(3) + 7(0) = 15 \end{aligned}$$

So, the minimum value of z is 14, and this occurs when $x = 0$ and $y = 2$.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the minimum value of

$$z = 12x + 8y$$

where $x \geq 0$ and $y \geq 0$, subject to the following constraints.

$$\begin{aligned} 5x + 6y &\leq 420 \\ -x + 6y &\leq 240 \\ -2x + y &\geq -100 \end{aligned}$$

EXAMPLE 3 Maximizing an Objective Function

Find the maximum value of

$$z = 5x + 7y \qquad \text{Objective function}$$

where $x \geq 0$ and $y \geq 0$, subject to the constraints given in Example 2.

Solution Using the values of z at the vertices shown in Example 2, the maximum value of z is

$$\begin{aligned} z &= 5(6) + 7(3) \\ &= 51 \end{aligned}$$

and occurs when $x = 6$ and $y = 3$.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the maximum value of $z = 12x + 8y$, where $x \geq 0$ and $y \geq 0$, subject to the constraints given in the Checkpoint with Example 2.

George Dantzig (1914–2005) was the first to propose the simplex method, or linear programming, in 1947. This technique defined the steps needed to find the optimal solution to a complex multivariable problem.

Edward W. Souza/News Services/Stanford University

It is possible for the maximum (or minimum) value in a linear programming problem to occur at *two* different vertices. For instance, at the vertices of the region shown in Figure 6.24, the objective function

$z = 2x + 2y$ Objective function

has the following values.

At (0, 0): $z = 2(0) + 2(0) = 0$

At (0, 4): $z = 2(0) + 2(4) = 8$

At (2, 4): $z = 2(2) + 2(4) = 12$ Maximum value of z

At (5, 1): $z = 2(5) + 2(1) = 12$ Maximum value of z

At (5, 0): $z = 2(5) + 2(0) = 10$

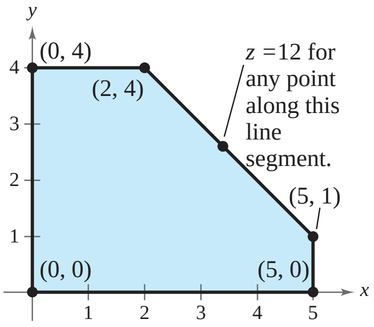


Figure 6.24

In this case, the objective function has a maximum value not only at the vertices (2, 4) and (5, 1); it also has a maximum value (of 12) at *any point on the line segment connecting these two vertices.* Note that the objective function in slope-intercept form $y = -x + \frac{1}{2}z$ has the same slope as the line through the vertices (2, 4) and (5, 1).

Some linear programming problems have no optimal solutions. This can occur when the region determined by the constraints is *unbounded*. Example 4 illustrates such a problem.

▷ **ALGEBRA HELP** The slope m of the nonvertical line through the points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

where $x_1 \neq x_2$.

EXAMPLE 4 An Unbounded Region

Find the maximum value of

$z = 4x + 2y$ Objective function

where $x \geq 0$ and $y \geq 0$, subject to the following constraints.

$$\left.\begin{aligned} x + 2y &\geq 4 \\ 3x + y &\geq 7 \\ -x + 2y &\leq 7 \end{aligned}\right\}$$ Constraints

Solution Figure 6.25 shows the region determined by the constraints. For this unbounded region, there is no maximum value of z. To see this, note that the point $(x, 0)$ lies in the region for all values of $x \geq 4$. Substituting this point into the objective function, you get

$z = 4(x) + 2(0) = 4x.$

By choosing x to be large, you can obtain values of z that are as large as you want. So, there is no maximum value of z. However, there *is* a minimum value of z.

At (1, 4): $z = 4(1) + 2(4) = 12$

At (2, 1): $z = 4(2) + 2(1) = 10$ Minimum value of z

At (4, 0): $z = 4(4) + 2(0) = 16$

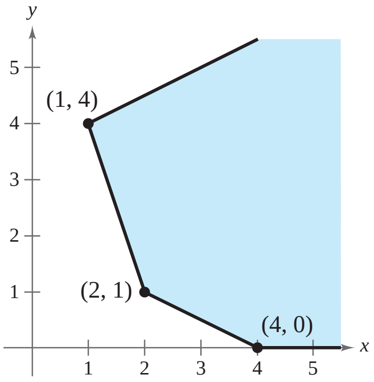


Figure 6.25

So, the minimum value of z is 10, and this occurs when $x = 2$ and $y = 1$.

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Find the minimum value of

$z = 3x + 7y$

where $x \geq 0$ and $y \geq 0$, subject to the following constraints.

$$\begin{aligned} x + y &\geq 8 \\ 3x + 5y &\geq 30 \end{aligned}$$

Applications

Example 5 shows how linear programming can help you find the maximum profit in a business application.

EXAMPLE 5 Optimal Profit

A candy manufacturer wants to maximize the combined profit for two types of boxed chocolates. A box of chocolate-covered creams yields a profit of \$1.50 per box, and a box of chocolate-covered nuts yields a profit of \$2.00 per box. Market tests and available resources have indicated the following constraints.

1. The combined production level should not exceed 1200 boxes per month.
2. The demand for a box of chocolate-covered nuts is no more than half the demand for a box of chocolate-covered creams.
3. The production level for chocolate-covered creams should be less than or equal to 600 boxes plus three times the production level for chocolate-covered nuts.

What is the maximum monthly profit? How many boxes of each type should be produced per month to yield the maximum profit?

Solution Let x be the number of boxes of chocolate-covered creams and let y be the number of boxes of chocolate-covered nuts. So, the objective function (for the combined profit) is

$$P = 1.5x + 2y.$$ Objective function

The three constraints translate into the following linear inequalities.

1. $x + y \le 1200$ ⇨ $x + y \le 1200$

2. $y \le \frac{1}{2}x$ ⇨ $-x + 2y \le 0$

3. $x \le 600 + 3y$ ⇨ $x - 3y \le 600$

Because neither x nor y can be negative, you also have the two additional constraints of

$$x \ge 0 \quad \text{and} \quad y \ge 0.$$

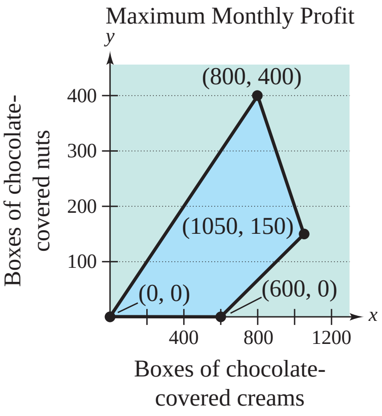


Figure 6.26

Figure 6.26 shows the region determined by the constraints. To find the maximum monthly profit, test the values of P at the vertices of the region.

At $(0, 0)$: $P = 1.5(0) + 2(0) = 0$

At $(800, 400)$: $P = 1.5(800) + 2(400) = 2000$ Maximum profit

At $(1050, 150)$: $P = 1.5(1050) + 2(150) = 1875$

At $(600, 0)$: $P = 1.5(600) + 2(0) = 900$

So, the maximum monthly profit is \$2000, and it occurs when the monthly production consists of 800 boxes of chocolate-covered creams and 400 boxes of chocolate-covered nuts.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

In Example 5, the candy manufacturer improves the production of chocolate-covered creams so that the profit is \$2.50 per box. What is the maximum monthly profit? How many boxes of each type should be produced per month to yield the maximum profit?

Example 6 shows how linear programming can help you find the optimal cost in a real-life application.

EXAMPLE 6 Optimal Cost

The liquid portion of a diet is to provide at least 300 calories, 36 units of vitamin A, and 90 units of vitamin C. A cup of dietary drink X costs \$0.12 and provides 60 calories, 12 units of vitamin A, and 10 units of vitamin C. A cup of dietary drink Y costs \$0.15 and provides 60 calories, 6 units of vitamin A, and 30 units of vitamin C. How many cups of each drink should be consumed each day to obtain an optimal cost and still meet the daily requirements?

Solution As in Example 9 in Section 6.5, let x be the number of cups of dietary drink X and let y be the number of cups of dietary drink Y.

$$\left.\begin{array}{lrcl} \text{For calories:} & 60x + 60y & \geq & 300 \\ \text{For vitamin A:} & 12x + 6y & \geq & 36 \\ \text{For vitamin C:} & 10x + 30y & \geq & 90 \\ & x & \geq & 0 \\ & y & \geq & 0 \end{array}\right\} \quad \text{Constraints}$$

The cost C is given by $C = 0.12x + 0.15y$. Objective function

Figure 6.27 shows the graph of the region corresponding to the constraints. Because you want to incur as little cost as possible, you want to determine the *minimum* cost. To determine the minimum cost, test C at each vertex of the region.

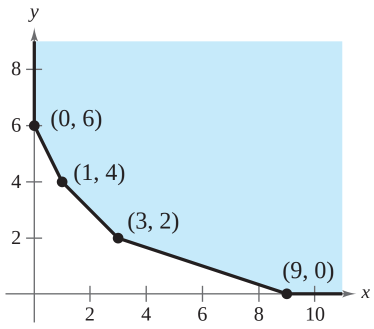


Figure 6.27

At (0, 6): $C = 0.12(0) + 0.15(6) = 0.90$

At (1, 4): $C = 0.12(1) + 0.15(4) = 0.72$

At (3, 2): $C = 0.12(3) + 0.15(2) = 0.66$ Minimum value of C

At (9, 0): $C = 0.12(9) + 0.15(0) = 1.08$

So, the minimum cost is \$0.66 per day, and this occurs when 3 cups of drink X and 2 cups of drink Y are consumed each day.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

A pet supply company mixes two brands of dry dog food. Brand X costs \$15 per bag and contains eight units of nutrient A, one unit of nutrient B, and two units of nutrient C. Brand Y costs \$30 per bag and contains two units of nutrient A, one unit of nutrient B, and seven units of nutrient C. The minimum required amounts of nutrients A, B, and C are 16 units, 5 units, and 20 units, respectively. How many bags of each brand should be mixed to obtain an optimal cost and still meet the minimum required amounts of nutrients?

Summarize (Section 6.6)

1. State the guidelines for solving a linear programming problem *(page 476)*. For an example of solving a linear programming problem, see Example 1.
2. Explain how to find the minimum and maximum values of an objective function of a linear programming problem *(pages 477 and 478)*. For an example of minimizing an objective function, see Example 2. For examples of maximizing objective functions, see Examples 1 and 3.
3. Describe a situation in which a linear programming problem has no optimal solution *(page 479)*. For an example of solving a linear programming problem that has no optimal solution, see Example 4.
4. Describe real-life examples of how to use linear programming to find the maximum profit in a business application *(page 480, Example 5)* and the optimal cost of diet drinks *(page 481, Example 6)*.

6.6 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: **Fill in the blanks.**

1. In the process called ________, you find the maximum or minimum value of a quantity.
2. One type of optimization strategy is called ________ ________.
3. The ________ function of a linear programming problem gives the quantity to be maximized or minimized.
4. The ________ of a linear programming problem determine the set of ________ ________.
5. The feasible solutions are ________ or ________ the boundary of the region corresponding to a system of constraints.
6. If a linear programming problem has a solution, then it must occur at a ________ of the set of feasible solutions.

Skills and Applications

Solving a Linear Programming Problem

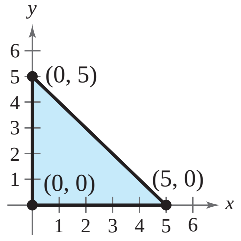
In Exercises 7–12, find the minimum and maximum values of the objective function and where they occur, subject to the indicated constraints. (For each exercise, the graph of the region determined by the constraints is provided.)

7. Objective function:

$z = 4x + 3y$

Constraints:

$$\begin{aligned} x &\ge 0 \\ y &\ge 0 \\ x + y &\le 5 \end{aligned}$$

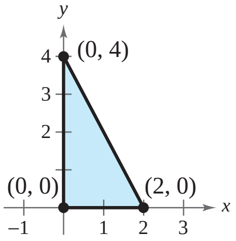


8. Objective function:

$z = 2x + 8y$

Constraints:

$$\begin{aligned} x &\ge 0 \\ y &\ge 0 \\ 2x + y &\le 4 \end{aligned}$$

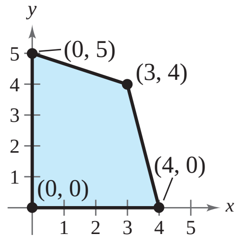


9. Objective function:

$z = 2x + 5y$

Constraints:

$$\begin{aligned} x &\ge 0 \\ y &\ge 0 \\ x + 3y &\le 15 \\ 4x + y &\le 16 \end{aligned}$$

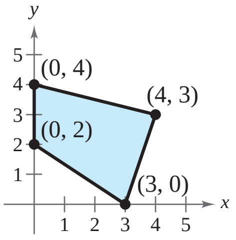


10. Objective function:

$z = 4x + 5y$

Constraints:

$$\begin{aligned} x &\ge 0 \\ 2x + 3y &\ge 6 \\ 3x - y &\le 9 \\ x + 4y &\le 16 \end{aligned}$$

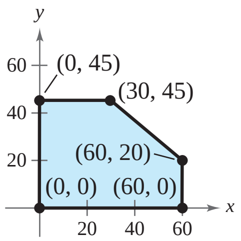


11. Objective function:

$z = 10x + 7y$

Constraints:

$$\begin{aligned} 0 \le x &\le 60 \\ 0 \le y &\le 45 \\ 5x + 6y &\le 420 \end{aligned}$$

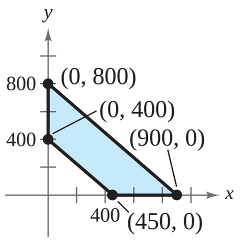


12. Objective function:

$z = 40x + 45y$

Constraints:

$$\begin{aligned} x &\ge 0 \\ y &\ge 0 \\ 8x + 9y &\le 7200 \\ 8x + 9y &\ge 3600 \end{aligned}$$



Solving a Linear Programming Problem

In Exercises 13–16, sketch the region determined by the constraints. Then find the minimum and maximum values of the objective function (if possible) and where they occur, subject to the indicated constraints.

13. Objective function:

$z = 3x + 2y$

Constraints:

$$\begin{aligned} x &\ge 0 \\ y &\ge 0 \\ 5x + 2y &\le 20 \\ 5x + y &\ge 10 \end{aligned}$$

14. Objective function:

$z = 5x + \frac{1}{2}y$

Constraints:

$$\begin{aligned} x &\ge 0 \\ y &\ge 0 \\ \tfrac{1}{2}x + y &\le 8 \\ x + \tfrac{1}{2}y &\ge 4 \end{aligned}$$

15. Objective function:

$z = 4x + 5y$

Constraints:

$$\begin{aligned} x &\ge 0 \\ y &\ge 0 \\ x + y &\ge 8 \\ 3x + 5y &\ge 30 \end{aligned}$$

16. Objective function:

$z = 5x + 4y$

Constraints:

$$\begin{aligned} x &\ge 0 \\ y &\ge 0 \\ 2x + 2y &\ge 10 \\ x + 2y &\ge 6 \end{aligned}$$

Solving a Linear Programming Problem **In Exercises 17–20, use a graphing utility to graph the region determined by the constraints. Then find the minimum and maximum values of the objective function and where they occur, subject to the constraints.**

17. Objective function:

$z = 3x + y$

Constraints:

$x \geq 0$

$y \geq 0$

$x + 4y \leq 60$

$3x + 2y \geq 48$

18. Objective function:

$z = x$

Constraints:

$x \geq 0$

$y \geq 0$

$2x + 3y \leq 60$

$2x + y \leq 28$

$4x + y \leq 48$

19. Objective function:

$z = x + 4y$

Constraints:

(See Exercise 17.)

20. Objective function:

$z = y$

Constraints:

(See Exercise 18.)

Finding Minimum and Maximum Values **In Exercises 21–24, find the minimum and maximum values of the objective function and where they occur, subject to the constraints $x \geq 0$, $y \geq 0$, $3x + y \leq 15$, and $4x + 3y \leq 30$.**

21. $z = 2x + y$

22. $z = 5x + y$

23. $z = x + y$

24. $z = 3x + y$

Finding Minimum and Maximum Values **In Exercises 25–28, find the minimum and maximum values of the objective function and where they occur, subject to the constraints $x \geq 0$, $y \geq 0$, $x + 4y \leq 20$, $x + y \leq 18$, and $2x + 2y \leq 21$.**

25. $z = x + 5y$

26. $z = 2x + 4y$

27. $z = 4x + 5y$

28. $z = 4x + y$

Describing an Unusual Characteristic **In Exercises 29–36, the linear programming problem has an unusual characteristic. Sketch a graph of the solution region for the problem and describe the unusual characteristic. Find the minimum and maximum values of the objective function (if possible) and where they occur.**

29. Objective function:

$z = 2.5x + y$

Constraints:

$x \geq 0$

$y \geq 0$

$3x + 5y \leq 15$

$5x + 2y \leq 10$

30. Objective function:

$z = x + y$

Constraints:

$x \geq 0$

$y \geq 0$

$-x + y \leq 1$

$-x + 2y \leq 4$

31. Objective function:

$z = -x + 2y$

Constraints:

$x \geq 0$

$y \geq 0$

$x \leq 10$

$x + y \leq 7$

32. Objective function:

$z = x + y$

Constraints:

$x \geq 0$

$y \geq 0$

$-x + y \leq 0$

$-3x + y \geq 3$

33. Objective function:

$z = 3x + 4y$

Constraints:

$x \geq 0$

$y \geq 0$

$x + y \leq 1$

$2x + y \leq 4$

34. Objective function:

$z = x + 2y$

Constraints:

$x \geq 0$

$y \geq 0$

$x + 2y \leq 4$

$2x + y \leq 4$

35. Objective function:

$z = x + y$

Constraints:

(See Exercise 33.)

36. Objective function:

$z = 2x + y$

Constraints:

(See Exercise 34.)

37. **Optimal Profit** A merchant plans to sell two models of MP3 players at prices of \$225 and \$250. The \$225 model yields a profit of \$30 per unit and the \$250 model yields a profit of \$31 per unit. The merchant estimates that the total monthly demand will not exceed 275 units. The merchant does not want to invest more than \$63,000 in inventory for these products. What is the optimal inventory level for each model? What is the optimal profit?

38. **Optimal Profit** A manufacturer produces two models of elliptical cross-training exercise machines. The times for assembling, finishing, and packaging model X are 3 hours, 3 hours, and 0.8 hour, respectively. The times for model Y are 4 hours, 2.5 hours, and 0.4 hour. The total times available for assembling, finishing, and packaging are 6000 hours, 4200 hours, and 950 hours, respectively. The profits per unit are \$300 for model X and \$375 for model Y. What is the optimal production level for each model? What is the optimal profit?

39. **Optimal Cost** An animal shelter mixes two brands of dry dog food. Brand X costs \$25 per bag and contains two units of nutrient A, two units of nutrient B, and two units of nutrient C. Brand Y costs \$20 per bag and contains one unit of nutrient A, nine units of nutrient B, and three units of nutrient C. The minimum required amounts of nutrients A, B, and C are 12 units, 36 units, and 24 units, respectively. What is the optimal number of bags of each brand that should be mixed? What is the optimal cost?

40. **Optimal Cost** A humanitarian agency can use two models of vehicles for a refugee rescue mission. Each model A vehicle costs $1000 and each model B vehicle costs $1500. Mission strategies and objectives indicate the following constraints.

- The agency must use a total of at least 20 vehicles.
- A model A vehicle can hold 45 boxes of supplies. A model B vehicle can hold 30 boxes of supplies. The agency must deliver at least 690 boxes of supplies to the refugee camp.
- A model A vehicle can hold 20 refugees. A model B vehicle can hold 32 refugees. The agency must rescue at least 520 refugees.

What is the optimal number of vehicles of each model that should be used? What is the optimal cost?

41. **Optimal Revenue** An accounting firm has 780 hours of staff time and 272 hours of reviewing time available each week. The firm charges $1600 for an audit and $250 for a tax return. Each audit requires 60 hours of staff time and 16 hours of review time. Each tax return requires 10 hours of staff time and 4 hours of review time. What numbers of audits and tax returns will yield an optimal revenue? What is the optimal revenue?

42. **Optimal Revenue** The accounting firm in Exercise 41 lowers its charge for an audit to $1400. What numbers of audits and tax returns will yield an optimal revenue? What is the optimal revenue?

43. **Optimal Yield** A fruit grower raises crops A and B. The yield is 300 bushels per acre for crop A and 500 bushels per acre for crop B. Research and available resources indicate the following constraints.

- The fruit grower has 150 acres of land for raising the crops.
- It takes 1 day to trim an acre of crop A and 2 days to trim an acre of crop B, and there are 240 days per year available for trimming.
- It takes 0.3 day to pick an acre of crop A and 0.1 day to pick an acre of crop B, and there are 30 days per year available for picking.

What is the optimal acreage for each fruit? What is the optimal yield?

44. **Optimal Profit** In Exercise 43, the profit is $185 per acre for crop A and $245 per acre for crop B. What is the optimal profit?

45. **Media Selection** A company has budgeted a maximum of $1,000,000 for national advertising of an allergy medication. Each minute of television time costs $100,000 and each one-page newspaper ad costs $20,000. Each television ad is expected to be viewed by 20 million viewers, and each newspaper ad is expected to be seen by 5 million readers. The company's market research department recommends that at most 80% of the advertising budget be spent on television ads. What is the optimal amount that should be spent on each type of ad? What is the optimal total audience?

46. **Investment Portfolio** An investor has up to $450,000 to invest in two types of investments. Type A pays 6% annually and type B pays 10% annually. To have a well-balanced portfolio, the investor imposes the following conditions. At least one-half of the total portfolio is to be allocated to type A investments and at least one-fourth of the portfolio is to be allocated to type B investments. What is the optimal amount that should be invested in each type of investment? What is the optimal return?

Exploration

True or False? **In Exercises 47–49, determine whether the statement is true or false. Justify your answer.**

47. If an objective function has a maximum value at the vertices (4, 7) and (8, 3), then it also has a maximum value at the points (4.5, 6.5) and (7.8, 3.2).

48. If an objective function has a minimum value at the vertex (20, 0), then it also has a minimum value at the point (0, 0).

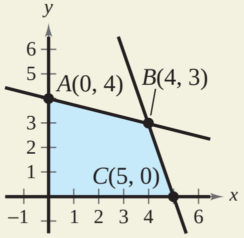
49. When solving a linear programming problem, if the objective function has a maximum value at more than one vertex, then there are an infinite number of points that will produce the maximum value.

50. **HOW DO YOU SEE IT?** Using the constraint region shown below, determine which of the following objective functions has (a) a maximum at vertex *A*, (b) a maximum at vertex *B*, (c) a maximum at vertex *C*, and (d) a minimum at vertex *C*.

(i) $z = 2x + y$

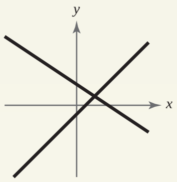
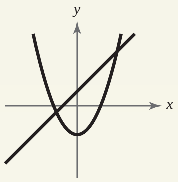
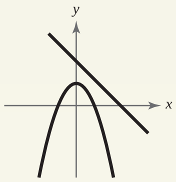
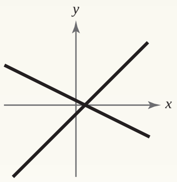
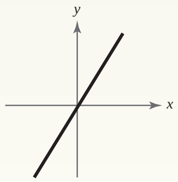
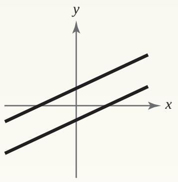
(ii) $z = 2x - y$

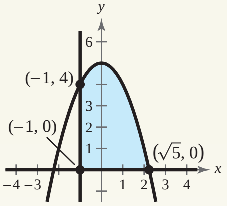
(iii) $z = -x + 2y$



Alison McC/Shutterstock.com

Chapter Summary

	What Did You Learn?	Explanation/Examples	Review Exercises
Section 6.1	Use the method of substitution to solve systems of linear equations in two variables *(p. 424)*.	**Method of Substitution** **1.** *Solve* one of the equations for one variable in terms of the other. **2.** *Substitute* the expression found in Step 1 into the other equation to obtain an equation in one variable. **3.** *Solve* the equation obtained in Step 2. **4.** *Back-substitute* the value obtained in Step 3 into the expression obtained in Step 1 to find the value of the other variable. **5.** *Check* that the solution satisfies *each* of the original equations.	1–6
	Use the method of substitution to solve systems of nonlinear equations in two variables *(p. 427)*.	The method of substitution (see steps above) can be used to solve systems in which one or both of the equations are nonlinear. (See Examples 3 and 4.)	7–10
	Use a graphical approach to solve systems of equations in two variables *(p. 428)*.	 One intersection point  Two intersection points  No intersection points	11–18
	Use systems of equations to model and solve real-life problems *(p. 429)*.	A system of equations can help you find the break-even point for a company. (See Example 6.)	19–22
Section 6.2	Use the method of elimination to solve systems of linear equations in two variables *(p. 434)*.	**Method of Elimination** **1.** *Obtain coefficients* for x (or y) that differ only in sign. **2.** *Add* the equations to eliminate one variable. **3.** *Solve* the equation obtained in Step 2. **4.** *Back-substitute* the value obtained in Step 3 into either of the original equations and solve for the other variable. **5.** *Check* that the solution satisfies *each* of the original equations.	23–28
	Interpret graphically the numbers of solutions of systems of linear equations in two variables *(p. 438)*.	 Exactly one solution  Infinitely many solutions  No solution	29–32
	Use systems of linear equations in two variables to model and solve real-life problems *(p. 440)*.	A system of linear equations in two variables can help you find the equilibrium point for a market. (See Example 9.)	33, 34

	What Did You Learn?	Explanation/Examples	Review Exercises
Section 6.3	Use back-substitution to solve linear systems in row-echelon form *(p. 446)*.	$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y = -4 \\ 2x - 5y + 5z = 17 \end{cases} \Rightarrow$ **Row-Echelon Form** $\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ z = 2 \end{cases}$	35–38
	Use Gaussian elimination to solve systems of linear equations *(p. 447)*.	To produce an equivalent system of linear equations, use row operations by (1) interchanging two equations, (2) multiplying one equation by a nonzero constant, or (3) adding a multiple of one equation to another equation to replace the latter equation.	39–42
	Solve nonsquare systems of linear equations *(p. 451)*.	In a nonsquare system, the number of equations differs from the number of variables. A system of linear equations cannot have a unique solution unless there are at least as many equations as there are variables.	43, 44
	Use systems of linear equations in three or more variables to model and solve real-life problems *(p. 452)*.	A system of linear equations in three variables can help you find the position equation of an object that is moving in a (vertical) line with constant acceleration. (See Example 7.)	45–54
Section 6.4	Recognize partial fraction decompositions of rational expressions *(p. 458)*.	$\frac{9}{x^3 - 6x^2} = \frac{9}{x^2(x - 6)}$ $= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 6}$	55–58
	Find partial fraction decompositions of rational expressions *(p. 459)*.	The techniques used for determining constants in the numerators of partial fractions vary slightly, depending on the type of factors of the denominator: linear or quadratic, distinct or repeated.	59–66
Section 6.5	Sketch the graphs of inequalities in two variables *(p. 466)*, and solve systems of inequalities *(p. 468)*.	$\begin{cases} x^2 + y \le 5 \\ x \ge -1 \\ y \ge 0 \end{cases}$ 	67–78
	Use systems of inequalities in two variables to model and solve real-life problems *(p. 471)*.	A system of inequalities in two variables can help you find the consumer surplus and producer surplus for given demand and supply equations. (See Example 8.)	79–84
Section 6.6	Solve linear programming problems *(p. 476)*.	To solve a linear programming problem, (1) sketch the region corresponding to the system of constraints, (2) find the vertices of the region, and (3) test the objective function at each of the vertices and select the values of the variables that optimize the objective function.	85–88
	Use linear programming to model and solve real-life problems *(p. 480)*.	Linear programming can help you find the maximum profit in business applications. (See Example 5.)	89, 90

Review Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

6.1 Solving a System by Substitution **In Exercises 1–10, solve the system by the method of substitution.**

1. $\begin{cases} x + y = 2 \\ x - y = 0 \end{cases}$

2. $\begin{cases} 2x - 3y = 3 \\ x - y = 0 \end{cases}$

3. $\begin{cases} 4x - y - 1 = 0 \\ 8x + y - 17 = 0 \end{cases}$

4. $\begin{cases} 10x + 6y + 14 = 0 \\ x + 9y + 7 = 0 \end{cases}$

5. $\begin{cases} 0.5x + y = 0.75 \\ 1.25x - 4.5y = -2.5 \end{cases}$

6. $\begin{cases} -x + \frac{2}{5}y = \frac{3}{5} \\ -x + \frac{1}{5}y = -\frac{4}{5} \end{cases}$

7. $\begin{cases} x^2 - y^2 = 9 \\ x - y = 1 \end{cases}$

8. $\begin{cases} x^2 + y^2 = 169 \\ 3x + 2y = 39 \end{cases}$

9. $\begin{cases} y = 2x^2 \\ y = x^4 - 2x^2 \end{cases}$

10. $\begin{cases} x = y + 3 \\ x = y^2 + 1 \end{cases}$

Solving a System of Equations Graphically **In Exercises 11–14, solve the system graphically.**

11. $\begin{cases} 2x - y = 10 \\ x + 5y = -6 \end{cases}$

12. $\begin{cases} 8x - 3y = -3 \\ 2x + 5y = 28 \end{cases}$

13. $\begin{cases} y = 2x^2 - 4x + 1 \\ y = x^2 - 4x + 3 \end{cases}$

14. $\begin{cases} y^2 - 2y + x = 0 \\ x + y = 0 \end{cases}$

Solving a System of Equations Graphically **In Exercises 15–18, use a graphing utility to solve the systems of equations. Find the solution(s) accurate to two decimal places.**

15. $\begin{cases} y = -2e^{-x} \\ 2e^{x} + y = 0 \end{cases}$

16. $\begin{cases} x^2 + y^2 = 100 \\ 2x - 3y = -12 \end{cases}$

17. $\begin{cases} y = 2 + \log x \\ y = \frac{3}{4}x + 5 \end{cases}$

18. $\begin{cases} y = \ln(x - 1) - 3 \\ y = 4 - \frac{1}{2}x \end{cases}$

19. Body Mass Index Body Mass Index (BMI) is a measure of body fat based on height and weight. The 85th percentile BMI for females, ages 9 to 20, is growing more slowly than that for males of the same age range. Models that represent the 85th percentile BMI for males and females, ages 9 to 20, are

$$\begin{cases} B = 0.77a + 11.7 & \text{Males} \\ B = 0.68a + 13.5 & \text{Females} \end{cases}$$

where B is the BMI $\left(\text{kg/m}^2\right)$ and a represents the age, with $a = 9$ corresponding to 9 years old. Use a graphing utility to determine whether the BMI for males exceeds the BMI for females. *(Source: Centers for Disease Control and Prevention)*

20. Choice of Two Jobs You receive two sales job offers. One company offers an annual salary of $55,000 plus a year-end bonus of 1.5% of your total sales. The other company offers an annual salary of $52,000 plus a year-end bonus of 2% of your total sales. What amount of sales will make the second job offer better? Explain.

21. Geometry The perimeter of a rectangle is 68 feet and its width is $\frac{8}{9}$ times its length. Find the dimensions of the rectangle.

22. Geometry The perimeter of a rectangle is 40 inches. The area of the rectangle is 96 square inches. Find the dimensions of the rectangle.

6.2 Solving a System by Elimination **In Exercises 23–28, solve the system by the method of elimination and check any solutions algebraically.**

23. $\begin{cases} 2x - y = 2 \\ 6x + 8y = 39 \end{cases}$

24. $\begin{cases} 12x + 42y = -17 \\ 30x - 18y = 19 \end{cases}$

25. $\begin{cases} 3x - 2y = 0 \\ 3x + 2(y + 5) = 10 \end{cases}$

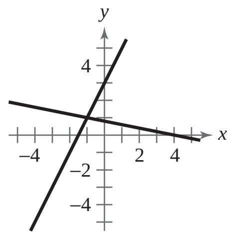
26. $\begin{cases} 7x + 12y = 63 \\ 2x + 3(y + 2) = 21 \end{cases}$

27. $\begin{cases} 1.25x - 2y = 3.5 \\ 5x - 8y = 14 \end{cases}$

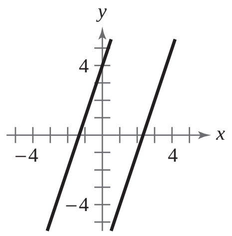
28. $\begin{cases} 1.5x + 2.5y = 8.5 \\ 6x + 10y = 24 \end{cases}$

Matching a System with Its Graph **In Exercises 29–32, match the system of linear equations with its graph. Describe the number of solutions and state whether the system is consistent or inconsistent. [The graphs are labeled (a), (b), (c), and (d).]**

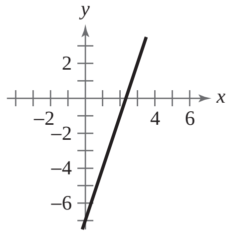
(a)



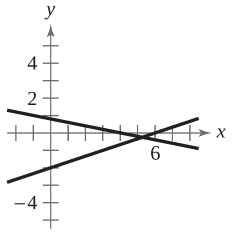
(b)



(c)



(d)



29. $\begin{cases} x + 5y = 4 \\ x - 3y = 6 \end{cases}$

30. $\begin{cases} -3x + y = -7 \\ 9x - 3y = 21 \end{cases}$

31. $\begin{cases} 3x - y = 7 \\ -6x + 2y = 8 \end{cases}$

32. $\begin{cases} 2x - y = -3 \\ x + 5y = 4 \end{cases}$

Finding the Equilibrium Point

In Exercises 33 and 34, find the equilibrium point of the demand and supply equations.

	Demand	Supply
33.	$p = 43 - 0.0002x$	$p = 22 + 0.00001x$
34.	$p = 120 - 0.0001x$	$p = 45 + 0.0002x$

6.3 Using Back-Substitution

In Exercises 35–38, use back-substitution to solve the system of linear equations.

35. $\begin{cases} x - 4y + 3z = 3 \\ -y + z = -1 \\ z = -5 \end{cases}$

36. $\begin{cases} x - 7y + 8z = 85 \\ y - 9z = -35 \\ z = 3 \end{cases}$

37. $\begin{cases} 4x - 3y - 2z = -65 \\ 8y - 7z = -14 \\ z = 10 \end{cases}$

38. $\begin{cases} 5x - 7z = 9 \\ 3y - 8z = -4 \\ z = -7 \end{cases}$

Using Gaussian Elimination

In Exercises 39–42, use Gaussian elimination to solve the system of linear equations and check any solutions algebraically.

39. $\begin{cases} x + 2y + 6z = 4 \\ -3x + 2y - z = -4 \\ 4x + 2z = 16 \end{cases}$

40. $\begin{cases} x - 2y + z = -6 \\ 2x - 3y = -7 \\ -x + 3y - 3z = 11 \end{cases}$

41. $\begin{cases} 2x + 6z = -9 \\ 3x - 2y + 11z = -16 \\ 3x - y + 7z = -11 \end{cases}$

42. $\begin{cases} x + 4w = 1 \\ 3y + z - w = 4 \\ 2y - 3w = 2 \\ 4x - y + 2z = 5 \end{cases}$

Solving a Nonsquare System

In Exercises 43 and 44, solve the nonsquare system of equations.

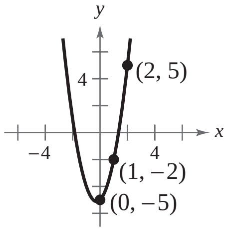
43. $\begin{cases} 5x - 12y + 7z = 16 \\ 3x - 7y + 4z = 9 \end{cases}$

44. $\begin{cases} 2x + 5y - 19z = 34 \\ 3x + 8y - 31z = 54 \end{cases}$

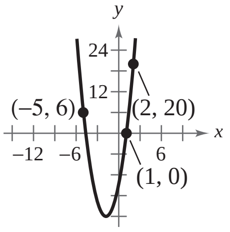
Finding the Equation of a Parabola

In Exercises 45 and 46, find the equation of the parabola $y = ax^2 + bx + c$ that passes through the points. Verify your result using a graphing utility.

45.



46.



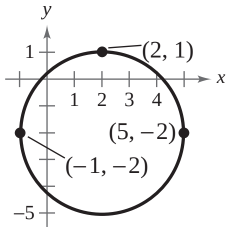
Finding the Equation of a Circle

In Exercises 47 and 48, find the equation of the circle

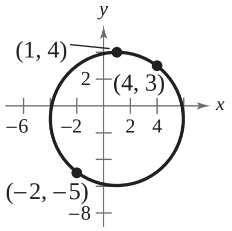
$$x^2 + y^2 + Dx + Ey + F = 0$$

that passes through the points. Verify your result using a graphing utility.

47.



48.



49. Data Analysis: E-Commerce The table shows the retail e-commerce sales y (in billions of dollars) in the United States from 2007 through 2009. *(Source: U.S. Census Bureau)*

Year	2007	2008	2009
Retail E-Commerce Sales, y	126.7	141.9	145.2

(a) Find the least squares regression parabola $y = at^2 + bt + c$ for the data, where t represents the year with $t = 7$ corresponding to 2007, by solving the system for a, b, and c.

$$\begin{cases} 3c + 24b + 194a = 413.8 \\ 24c + 194b + 1584a = 3328.9 \\ 194c + 1584b + 13{,}058a = 27{,}051.1 \end{cases}$$

(b) Use a graphing utility to graph the parabola and the data in the same viewing window. How well does the model fit the data?

(c) Use the model to estimate the retail e-commerce sales in the United States in 2015. Does your answer seem reasonable?

50. Agriculture A mixture of 6 gallons of chemical A, 8 gallons of chemical B, and 13 gallons of chemical C is required to kill a destructive crop insect. Commercial spray X contains 1, 2, and 2 parts, respectively, of these chemicals. Commercial spray Y contains only chemical C. Commercial spray Z contains chemicals A, B, and C in equal amounts. How much of each type of commercial spray gives the desired mixture?

51. Investment Analysis An inheritance of \$40,000 was divided among three investments yielding \$3500 in interest per year. The interest rates for the three investments were 7%, 9%, and 11%. Find the amount placed in each investment when the second and third were \$3000 and \$5000 less than the first, respectively.

52. **Sports** The Old Course at St Andrews Links in St Andrews, Scotland, is one of the oldest golf courses in the world. It is an 18-hole course that consists of par-3 holes, par-4 holes, and par-5 holes. There are seven times as many par-4 holes as par-5 holes, and the sum of the numbers of par-3 and par-5 holes is four. Find the numbers of par-3, par-4, and par-5 holes on the course. *(Source: St Andrews Links Trust)*

Vertical Motion **In Exercises 53 and 54, an object moving vertically is at the given heights at the specified times. Find the position equation**

$$s = \tfrac{1}{2}at^2 + v_0t + s_0$$

for the object.

53. At $t = 1$ second, $s = 134$ feet
At $t = 2$ seconds, $s = 86$ feet
At $t = 3$ seconds, $s = 6$ feet

54. At $t = 1$ second, $s = 184$ feet
At $t = 2$ seconds, $s = 116$ feet
At $t = 3$ seconds, $s = 16$ feet

6.4 **Writing the Form of the Decomposition** **In Exercises 55–58, write the form of the partial fraction decomposition of the rational expression. Do not solve for the constants.**

55. $\dfrac{3}{x^2 + 20x}$ **56.** $\dfrac{x - 8}{x^2 - 3x - 28}$

57. $\dfrac{3x - 4}{x^3 - 5x^2}$ **58.** $\dfrac{x - 2}{x(x^2 + 2)^2}$

Writing the Partial Fraction Decomposition **In Exercises 59–66, write the partial fraction decomposition of the rational expression. Check your result algebraically.**

59. $\dfrac{4 - x}{x^2 + 6x + 8}$ **60.** $\dfrac{-x}{x^2 + 3x + 2}$

61. $\dfrac{x^2}{x^2 + 2x - 15}$ **62.** $\dfrac{9}{x^2 - 9}$

63. $\dfrac{x^2 + 2x}{x^3 - x^2 + x - 1}$ **64.** $\dfrac{4x}{3(x - 1)^2}$

65. $\dfrac{3x^2 + 4x}{(x^2 + 1)^2}$ **66.** $\dfrac{4x^2}{(x - 1)(x^2 + 1)}$

6.5 **Graphing an Inequality** **In Exercises 67–72, sketch the graph of the inequality.**

67. $y \le 5 - \frac{1}{2}x$ **68.** $3y - x \ge 7$

69. $y - 4x^2 > -1$ **70.** $y \le \dfrac{3}{x^2 + 2}$

71. $(x - 1)^2 + (y - 3)^2 < 16$

72. $x^2 + (y + 5)^2 > 1$

Solving a System of Inequalities **In Exercises 73–78, sketch the graph (and label the vertices) of the solution set of the system of inequalities.**

73. $\begin{cases} x + 2y \le 160 \\ 3x + y \le 180 \\ x \ge 0 \\ y \ge 0 \end{cases}$ **74.** $\begin{cases} 3x + 2y \ge 24 \\ x + 2y \ge 12 \\ 2 \le x \le 15 \\ y \le 15 \end{cases}$

75. $\begin{cases} y < x + 1 \\ y > x^2 - 1 \end{cases}$ **76.** $\begin{cases} y \le 6 - 2x - x^2 \\ y \ge x + 6 \end{cases}$

77. $\begin{cases} 2x - 3y \ge 0 \\ 2x - y \le 8 \\ y \ge 0 \end{cases}$ **78.** $\begin{cases} x^2 + y^2 \le 9 \\ (x - 3)^2 + y^2 \le 9 \end{cases}$

Consumer Surplus and Producer Surplus **In Exercises 79 and 80, (a) graph the systems representing the consumer surplus and producer surplus for the supply and demand equations and (b) find the consumer surplus and producer surplus.**

	Demand	**Supply**
79.	$p = 160 - 0.0001x$	$p = 70 + 0.0002x$
80.	$p = 130 - 0.0002x$	$p = 30 + 0.0003x$

81. **Geometry** Write a system of inequalities to describe the region of a rectangle with vertices at (3, 1), (7, 1), (7, 10), and (3, 10).

82. **Data Analysis: Sales** The table shows the sales y (in millions of dollars) for Aéropostale, Inc., from 2003 through 2010. *(Source: Aéropostale, Inc.)*

DATA

Spreadsheet at LarsonPrecalculus.com

Year	Sales, y
2003	734.9
2004	964.2
2005	1204.3
2006	1413.2
2007	1590.9
2008	1885.5
2009	2230.1
2010	2400.4

(a) Use the *regression* feature of a graphing utility to find a linear model for the data. Let t represent the year, with $t = 3$ corresponding to 2003.

(b) The total sales for Aéropostale during this eight-year period can be approximated by finding the area of the trapezoid bounded by the linear model you found in part (a) and the lines $y = 0$, $t = 2.5$, and $t = 10.5$. Use the graphing utility to graph this region.

(c) Use the formula for the area of a trapezoid to approximate the total sales for Aéropostale.

83. Inventory Costs A warehouse operator has 24,000 square feet of floor space in which to store two products. Each unit of product I requires 20 square feet of floor space and costs \$12 per day to store. Each unit of product II requires 30 square feet of floor space and costs \$8 per day to store. The total storage cost per day cannot exceed \$12,400. Find and graph a system of inequalities describing all possible inventory levels.

84. Nutrition A dietitian designs a special dietary supplement using two different foods. Each ounce of food X contains 12 units of calcium, 10 units of iron, and 20 units of vitamin B. Each ounce of food Y contains 15 units of calcium, 20 units of iron, and 12 units of vitamin B. The minimum daily requirements of the diet are 300 units of calcium, 280 units of iron, and 300 units of vitamin B.

(a) Write a system of inequalities describing the different amounts of food X and food Y that can be used.

(b) Sketch a graph of the region in part (a).

(c) Find two solutions to the system and interpret their meanings in the context of the problem.

6.6 Solving a Linear Programming Problem In Exercises 85–88, sketch the region determined by the constraints. Then find the minimum and maximum values of the objective function (if possible) and where they occur, subject to the indicated constraints.

85. Objective function:

$z = 3x + 4y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 2x + 5y &\leq 50 \\ 4x + y &\leq 28 \end{aligned}$$

86. Objective function:

$z = 10x + 7y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 2x + y &\geq 100 \\ x + y &\geq 75 \end{aligned}$$

87. Objective function:

$z = 1.75x + 2.25y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 2x + y &\geq 25 \\ 3x + 2y &\geq 45 \end{aligned}$$

88. Objective function:

$z = 50x + 70y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + 2y &\leq 1500 \\ 5x + 2y &\leq 3500 \end{aligned}$$

89. Optimal Revenue A student is working part time as a hairdresser to pay college expenses. The student may work no more than 24 hours per week. Haircuts cost \$25 and require an average of 20 minutes, and permanents cost \$70 and require an average of 1 hour and 10 minutes. How many haircuts and/or permanents will yield an optimal revenue? What is the optimal revenue?

90. Optimal Profit A manufacturer produces two models of bicycles. The table shows the times (in hours) required for assembling, painting, and packaging each model.

Process	Hours, Model A	Hours, Model B
Assembling	2	2.5
Painting	4	1
Packaging	1	0.75

The total times available for assembling, painting, and packaging are 4000 hours, 4800 hours, and 1500 hours, respectively. The profits per unit are \$45 for model A and \$50 for model B. What is the optimal production level for each model? What is the optimal profit?

Exploration

True or False? In Exercises 91 and 92, determine whether the statement is true or false. Justify your answer.

91. The system

$$\begin{cases} y \leq 5 \\ y \geq -2 \\ y \leq \frac{7}{2}x - 9 \\ y \leq -\frac{7}{2}x + 26 \end{cases}$$

represents the region covered by an isosceles trapezoid.

92. It is possible for an objective function of a linear programming problem to have exactly 10 maximum value points.

Finding a System of Linear Equations In Exercises 93–96, find a system of linear equations having the ordered pair as a solution. (There are many correct answers.)

93. $(-8, 10)$

94. $(5, -4)$

95. $\left(\frac{4}{3}, 3\right)$

96. $\left(-2, \frac{11}{5}\right)$

Finding a System of Linear Equations In Exercises 97–100, find a system of linear equations having the ordered triple as a solution. (There are many correct answers.)

97. $(4, -1, 3)$

98. $(-3, 5, 6)$

99. $\left(5, \frac{3}{2}, 2\right)$

100. $\left(-\frac{1}{2}, -2, -\frac{3}{4}\right)$

101. Writing Explain what is meant by an inconsistent system of linear equations.

102. Graphical Reasoning How can you tell graphically that a system of linear equations in two variables has no solution? Give an example.

Chapter Test

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

In Exercises 1–3, solve the system by the method of substitution.

1. $\begin{cases} x + y = -9 \\ 5x - 8y = 20 \end{cases}$ **2.** $\begin{cases} y = x - 1 \\ y = (x - 1)^3 \end{cases}$ **3.** $\begin{cases} 2x - y^2 = 0 \\ x - y = 4 \end{cases}$

In Exercises 4–6, solve the system graphically.

4. $\begin{cases} 3x - 6y = 0 \\ 3x + 6y = 18 \end{cases}$ **5.** $\begin{cases} y = 9 - x^2 \\ y = x + 3 \end{cases}$ **6.** $\begin{cases} y - \ln x = 12 \\ 7x - 2y + 11 = -6 \end{cases}$

In Exercises 7 and 8, solve the system by the method of elimination.

7. $\begin{cases} 3x + 4y = -26 \\ 7x - 5y = 11 \end{cases}$ **8.** $\begin{cases} 1.4x - y = 17 \\ 0.8x + 6y = -10 \end{cases}$

In Exercises 9 and 10, solve the system of linear equations and check any solutions algebraically.

9. $\begin{cases} x - 2y + 3z = 11 \\ 2x - z = 3 \\ 3y + z = -8 \end{cases}$ **10.** $\begin{cases} 3x + 2y + z = 17 \\ -x + y + z = 4 \\ x - y - z = 3 \end{cases}$

In Exercises 11–14, write the partial fraction decomposition of the rational expression. Check your result algebraically.

11. $\dfrac{2x + 5}{x^2 - x - 2}$ **12.** $\dfrac{3x^2 - 2x + 4}{x^2(2 - x)}$ **13.** $\dfrac{x^2 + 5}{x^3 - x}$ **14.** $\dfrac{x^2 - 4}{x^3 + 2x}$

In Exercises 15–17, sketch the graph (and label the vertices) of the solution set of the system of inequalities.

15. $\begin{cases} 2x + y \le 4 \\ 2x - y \ge 0 \\ x \ge 0 \end{cases}$ **16.** $\begin{cases} y < -x^2 + x + 4 \\ y > 4x \end{cases}$ **17.** $\begin{cases} x^2 + y^2 \le 36 \\ x \ge 2 \\ y \ge -4 \end{cases}$

18. Find the minimum and maximum values of the objective function $z = 20x + 12y$ and where they occur, subject to the following constraints.

$$\left.\begin{aligned} x &\ge 0 \\ y &\ge 0 \\ x + 4y &\le 32 \\ 3x + 2y &\le 36 \end{aligned}\right\} \quad \text{Constraints}$$

19. A total of $50,000 is invested in two funds that pay 4% and 5.5% simple interest. The yearly interest is $2390. How much is invested at each rate?

20. Find the equation of the parabola $y = ax^2 + bx + c$ that passes through the points $(0, 6)$, $(-2, 2)$, and $\left(3, \frac{9}{2}\right)$.

21. A manufacturer produces two models of television stands. The table at the left shows the times (in hours) required for assembling, staining, and packaging the two models. The total times available for assembling, staining, and packaging are 4000 hours, 8950 hours, and 2650 hours, respectively. The profits per unit are $30 for model I and $40 for model II. What is the optimal inventory level for each model? What is the optimal profit?

	Model I	Model II
Assembling	0.5	0.75
Staining	2.0	1.5
Packaging	0.5	0.5

Table for 21

Proofs in Mathematics

An **indirect proof** can be useful in proving statements of the form "p implies q." Recall that the conditional statement $p \rightarrow q$ is false only when p is true and q is false. To prove a conditional statement indirectly, assume that p is true and q is false. If this assumption leads to an impossibility, then you have proved that the conditional statement is true. An indirect proof is also called a **proof by contradiction.**

To use an indirect proof to prove the following conditional statement,

"If a is a positive integer and a^2 is divisible by 2, then a is divisible by 2,"

proceed as follows. First, assume that p, "a is a positive integer and a^2 is divisible by 2," is true and q, "a is divisible by 2," is false. This means that a is not divisible by 2. If so, then a is odd and can be written as $a = 2n + 1$, where n is an integer.

$$a = 2n + 1 \qquad \text{Definition of an odd integer}$$

$$a^2 = 4n^2 + 4n + 1 \qquad \text{Square each side.}$$

$$a^2 = 2(2n^2 + 2n) + 1 \qquad \text{Distributive Property}$$

So, by the definition of an odd integer, a^2 is odd. This contradicts the assumption, and you can conclude that a is divisible by 2.

EXAMPLE Using an Indirect Proof

Use an indirect proof to prove that $\sqrt{2}$ is an irrational number.

Solution Begin by assuming that $\sqrt{2}$ is *not* an irrational number. Then $\sqrt{2}$ can be written as the quotient of two integers a and $b (b \neq 0)$ that have no common factors.

$$\sqrt{2} = \frac{a}{b} \qquad \text{Assume that } \sqrt{2} \text{ is a rational number.}$$

$$2 = \frac{a^2}{b^2} \qquad \text{Square each side.}$$

$$2b^2 = a^2 \qquad \text{Multiply each side by } b^2.$$

This implies that 2 is a factor of a^2. So, 2 is also a factor of a, and a can be written as $2c$, where c is an integer.

$$2b^2 = (2c)^2 \qquad \text{Substitute } 2c \text{ for } a.$$

$$2b^2 = 4c^2 \qquad \text{Simplify.}$$

$$b^2 = 2c^2 \qquad \text{Divide each side by 2.}$$

This implies that 2 is a factor of b^2 and also a factor of b. So, 2 is a factor of both a and b. This contradicts the assumption that a and b have no common factors. So, you can conclude that $\sqrt{2}$ is an irrational number. ■

P.S. Problem Solving

1. **Geometry** A theorem from geometry states that if a triangle is inscribed in a circle such that one side of the triangle is a diameter of the circle, then the triangle is a right triangle. Show that this theorem is true for the circle

$$x^2 + y^2 = 100$$

and the triangle formed by the lines

$$y = 0,\ y = \tfrac{1}{2}x + 5, \text{ and } y = -2x + 20.$$

2. **Finding Values of Constants** Find values of k_1 and k_2 such that the system of equations has an infinite number of solutions.

$$\begin{cases} 3x - 5y = 8 \\ 2x + k_1y = k_2 \end{cases}$$

3. **Finding Conditions on Constants** Consider the following system of linear equations in x and y.

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

Under what condition(s) will the system have exactly one solution?

4. **Finding Values of Constants** Find values of a, b, and c (if possible) such that the system of linear equations has (a) a unique solution, (b) no solution, and (c) an infinite number of solutions.

$$\begin{cases} x + y = 2 \\ y + z = 2 \\ x + z = 2 \\ ax + by + cz = 0 \end{cases}$$

5. **Graphical Analysis** Graph the lines determined by each system of linear equations. Then use Gaussian elimination to solve each system. At each step of the elimination process, graph the corresponding lines. What do you observe?

(a) $\begin{cases} x - 4y = -3 \\ 5x - 6y = 13 \end{cases}$

(b) $\begin{cases} 2x - 3y = 7 \\ -4x + 6y = -14 \end{cases}$

6. **Maximum Numbers of Solutions** A system of two equations in two unknowns has a finite number of solutions. Determine the maximum number of solutions of the system satisfying each condition.

(a) Both equations are linear.

(b) One equation is linear and the other is quadratic.

(c) Both equations are quadratic.

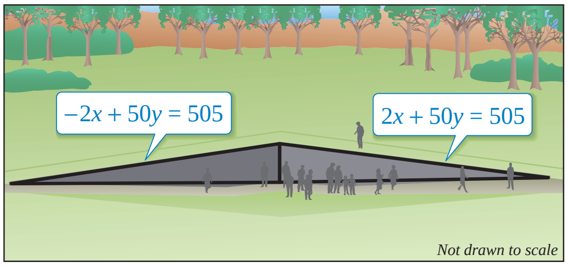
7. **Vietnam Veterans Memorial** The Vietnam Veterans Memorial (or "The Wall") in Washington, D.C., was designed by Maya Ying Lin when she was a student at Yale University. This monument has two vertical, triangular sections of black granite with a common side (see figure). The bottom of each section is level with the ground. The tops of the two sections can be approximately modeled by the equations

$$-2x + 50y = 505$$

and

$$2x + 50y = 505$$

when the x-axis is superimposed at the base of the wall. Each unit in the coordinate system represents 1 foot. How high is the memorial at the point where the two sections meet? How long is each section?



8. **Finding Atomic Weights** Weights of atoms and molecules are measured in atomic mass units (u). A molecule of C_2H_6 (ethane) is made up of two carbon atoms and six hydrogen atoms and weighs 30.070 u. A molecule of C_3H_8 (propane) is made up of three carbon atoms and eight hydrogen atoms and weighs 44.097 u. Find the weights of a carbon atom and a hydrogen atom.

9. **DVD Connector Cables** Connecting a DVD player to a television set requires a cable with special connectors at both ends. You buy a six-foot cable for \$15.50 and a three-foot cable for \$10.25. Assuming that the cost of a cable is the sum of the cost of the two connectors and the cost of the cable itself, what is the cost of a four-foot cable? Explain your reasoning.

10. **Distance** A hotel 35 miles from an airport runs a shuttle service to and from the airport. The 9:00 A.M. bus leaves for the airport traveling at 30 miles per hour. The 9:15 A.M. bus leaves for the airport traveling at 40 miles per hour. Write a system of linear equations that represents distance as a function of time for the buses. Graph and solve the system. How far from the airport will the 9:15 A.M. bus catch up to the 9:00 A.M. bus?

11. **Systems with Rational Expressions** Solve each system of equations by letting $X = 1/x$, $Y = 1/y$, and $Z = 1/z$.

(a) $$\begin{cases} \dfrac{12}{x} - \dfrac{12}{y} = 7 \\ \dfrac{3}{x} + \dfrac{4}{y} = 0 \end{cases}$$

(b) $$\begin{cases} \dfrac{2}{x} + \dfrac{1}{y} - \dfrac{3}{z} = 4 \\ \dfrac{4}{x} \qquad\; + \dfrac{2}{z} = 10 \\ -\dfrac{2}{x} + \dfrac{3}{y} - \dfrac{13}{z} = -8 \end{cases}$$

12. **Finding Values of Constants** What values should be given to a, b, and c so that the following linear system has $(-1, 2, -3)$ as its only solution?

$$\begin{cases} x + 2y - 3z = a & \text{Equation 1} \\ -x - y + z = b & \text{Equation 2} \\ 2x + 3y - 2z = c & \text{Equation 3} \end{cases}$$

13. **System of Linear Equations** The following system has one solution: $x = 1$, $y = -1$, and $z = 2$.

$$\begin{cases} 4x - 2y + 5z = 16 & \text{Equation 1} \\ x + y \qquad = 0 & \text{Equation 2} \\ -x - 3y + 2z = 6 & \text{Equation 3} \end{cases}$$

Solve the system given by (a) Equation 1 and Equation 2, (b) Equation 1 and Equation 3, and (c) Equation 2 and Equation 3. (d) How many solutions does each of these systems have?

14. **System of Linear Equations** Solve the system of linear equations algebraically.

$$\begin{cases} x_1 - x_2 + 2x_3 + 2x_4 + 6x_5 = 6 \\ 3x_1 - 2x_2 + 4x_3 + 4x_4 + 12x_5 = 14 \\ \quad - x_2 - x_3 - x_4 - 3x_5 = -3 \\ 2x_1 - 2x_2 + 4x_3 + 5x_4 + 15x_5 = 10 \\ 2x_1 - 2x_2 + 4x_3 + 4x_4 + 13x_5 = 13 \end{cases}$$

15. **Biology** Each day, an average adult moose can process about 32 kilograms of terrestrial vegetation (twigs and leaves) and aquatic vegetation. From this food, it needs to obtain about 1.9 grams of sodium and 11,000 calories of energy. Aquatic vegetation has about 0.15 gram of sodium per kilogram and about 193 calories of energy per kilogram, whereas terrestrial vegetation has minimal sodium and about four times as much energy as aquatic vegetation. Write and graph a system of inequalities that describes the amounts t and a of terrestrial and aquatic vegetation, respectively, for the daily diet of an average adult moose. *(Source: Biology by Numbers)*

16. **Height and Weight** For a healthy person who is 4 feet 10 inches tall, the recommended minimum weight is about 91 pounds and increases by about 3.6 pounds for each additional inch of height. The recommended maximum weight is about 115 pounds and increases by about 4.5 pounds for each additional inch of height. *(Source: National Institutes of Health)*

(a) Let x be the number of inches by which a person's height exceeds 4 feet 10 inches and let y be the person's weight in pounds. Write a system of inequalities that describes the possible values of x and y for a healthy person.

(b) Use a graphing utility to graph the system of inequalities from part (a).

(c) What is the recommended weight range for a healthy person who is 6 feet tall?

17. **Cholesterol** Cholesterol in human blood is necessary, but too much can lead to health problems. There are three main types of cholesterol: HDL (high-density lipoproteins), LDL (low-density lipoproteins), and VLDL (very low-density lipoproteins). HDL is considered "good" cholesterol; LDL and VLDL are considered "bad" cholesterol.

A standard fasting cholesterol blood test measures total cholesterol, HDL cholesterol, and triglycerides. These numbers are used to estimate LDL and VLDL, which are difficult to measure directly. It is recommended that your combined LDL/VLDL cholesterol level be less than 130 milligrams per deciliter, your HDL cholesterol level be at least 40 milligrams per deciliter, and your total cholesterol level be no more than 200 milligrams per deciliter. *(Source: American Heart Association)*

(a) Write a system of linear inequalities for the recommended cholesterol levels. Let x represent the HDL cholesterol level, and let y represent the combined LDL/VLDL cholesterol level.

(b) Graph the system of inequalities from part (a). Label any vertices of the solution region.

(c) Are the following cholesterol levels within recommendations? Explain your reasoning.

LDL/VLDL: 120 milligrams per deciliter

HDL: 90 milligrams per deciliter

Total: 210 milligrams per deciliter

(d) Give an example of cholesterol levels in which the LDL/VLDL cholesterol level is too high but the HDL cholesterol level is acceptable.

(e) Another recommendation is that the ratio of total cholesterol to HDL cholesterol be less than 4 (that is, less than 4 to 1). Find a point in the solution region from part (b) that meets this recommendation, and explain why it meets the recommendation.

7 Matrices and Determinants

Sudoku *(page 536)*

Data Encryption *(page 546)*

Beam Deflection *(page 524)*

Flight Crew Scheduling
(page 517)

Health and Wellness *(Exercise 101, page 508)*

Clockwise from top left, James E. Knopf/Shutterstock.com; Andrea Danti/Shutterstock.com;
© Sean Locke/iStockphoto.com; Tyler Olson/Shutterstock.com; nostal6ie/Shutterstock.com

7.1 Matrices and Systems of Equations

You can use matrices to solve systems of linear equations in two or more variables. For instance, in Exercise 101 on page 508, you will use a matrix to find a model for the numbers of new cases of a waterborne disease in a small city.

- **Write matrices and identify their orders.**
- **Perform elementary row operations on matrices.**
- **Use matrices and Gaussian elimination to solve systems of linear equations.**
- **Use matrices and Gauss-Jordan elimination to solve systems of linear equations.**

Matrices

In this section, you will study a streamlined technique for solving systems of linear equations. This technique involves the use of a rectangular array of real numbers called a **matrix.** The plural of matrix is *matrices.*

Definition of Matrix

If m and n are positive integers, then an $m \times n$ (read "m by n") matrix is a rectangular array

$$\begin{array}{c} \\ \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \\ \vdots \\ \text{Row } m \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Column 1} & \text{Column 2} & \text{Column 3} & . \; . \; . & \text{Column } n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

in which each **entry** a_{ij} of the matrix is a number. An $m \times n$ matrix has m rows and n columns. Matrices are usually denoted by capital letters.

The entry in the ith row and jth column is denoted by the *double subscript* notation a_{ij}. For instance, a_{23} refers to the entry in the second row, third column. A matrix that has only one row is called a **row matrix,** and a matrix that has only one column is called a **column matrix.** A matrix having m rows and n columns is said to be of **order** $m \times n$. If $m = n$, then the matrix is **square** of order $m \times m$ (or $n \times n$). For a square matrix, the entries $a_{11}, a_{22}, a_{33}, . \; . \; .$ are the **main diagonal** entries.

EXAMPLE 1 **Orders of Matrices**

Determine the order of each matrix.

a. $\begin{bmatrix} 2 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & -3 & 0 & \frac{1}{2} \end{bmatrix}$ **c.** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **d.** $\begin{bmatrix} 5 & 0 \\ 2 & -2 \\ -7 & 4 \end{bmatrix}$

Solution

a. This matrix has *one* row and *one* column. The order of the matrix is 1 × 1.

b. This matrix has *one* row and *four* columns. The order of the matrix is 1 × 4.

c. This matrix has *two* rows and *two* columns. The order of the matrix is 2 × 2.

d. This matrix has *three* rows and *two* columns. The order of the matrix is 3 × 2.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Determine the order of the matrix $\begin{bmatrix} 14 & 7 & 10 \\ -2 & -3 & -8 \end{bmatrix}$.

Tyler Olson/Shutterstock.com

A matrix derived from a system of linear equations (each written in standard form with the constant term on the right) is the **augmented matrix** of the system. Moreover, the matrix derived from the coefficients of the system (but not including the constant terms) is the **coefficient matrix** of the system.

$$\textit{System:}\quad \begin{cases} x - 4y + 3z = 5 \\ -x + 3y - z = -3 \\ 2x \qquad - 4z = 6 \end{cases}$$

$$\textit{Augmented Matrix:}\quad \begin{bmatrix} 1 & -4 & 3 & \vdots & 5 \\ -1 & 3 & -1 & \vdots & -3 \\ 2 & 0 & -4 & \vdots & 6 \end{bmatrix} \qquad \textit{Coefficient Matrix:}\quad \begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}$$

REMARK The vertical dots in an augmented matrix separate the coefficients of the linear system from the constant terms.

Note the use of 0 for the coefficient of the missing y-variable in the third equation, and also note the fourth column of constant terms in the augmented matrix.

When forming either the coefficient matrix or the augmented matrix of a system, you should begin by vertically aligning the variables in the equations and using zeros for the coefficients of the missing variables.

EXAMPLE 2 Writing an Augmented Matrix

Write the augmented matrix for the system of linear equations.

$$\begin{cases} x + 3y - w = 9 \\ -y + 4z + 2w = -2 \\ x - 5z - 6w = 0 \\ 2x + 4y - 3z = 4 \end{cases}$$

What is the order of the augmented matrix?

Solution

Begin by rewriting the linear system and aligning the variables.

$$\begin{cases} x + 3y \qquad\quad - w = 9 \\ \quad\; -y + 4z + 2w = -2 \\ x \qquad\quad - 5z - 6w = 0 \\ 2x + 4y - 3z \qquad\quad = 4 \end{cases}$$

Next, use the coefficients and constant terms as the matrix entries. Include zeros for the coefficients of the missing variables.

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{bmatrix} 1 & 3 & 0 & -1 & \vdots & 9 \\ 0 & -1 & 4 & 2 & \vdots & -2 \\ 1 & 0 & -5 & -6 & \vdots & 0 \\ 2 & 4 & -3 & 0 & \vdots & 4 \end{bmatrix}$$

The augmented matrix has four rows and five columns, so it is a 4×5 matrix. The notation R_n is used to designate each row in the matrix. For example, Row 1 is represented by R_1.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write the augmented matrix for the system of linear equations.

$$\begin{cases} x + y + z = 2 \\ 2x - y + 3z = -1 \\ -x + 2y - z = 4 \end{cases}$$

Elementary Row Operations

In Section 6.3, you studied three operations that can be used on a system of linear equations to produce an equivalent system.

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of an equation to another equation.

In matrix terminology, these three operations correspond to **elementary row operations.** An elementary row operation on an augmented matrix of a given system of linear equations produces a new augmented matrix corresponding to a new (but equivalent) system of linear equations. Two matrices are **row-equivalent** when one can be obtained from the other by a sequence of elementary row operations.

REMARK Although elementary row operations are simple to perform, they involve a lot of arithmetic. Because it is easy to make a mistake, you should get in the habit of noting the elementary row operations performed in each step so that you can go back and check your work.

Elementary Row Operations

Operation	Notation
1. Interchange two rows.	$R_a \leftrightarrow R_b$
2. Multiply a row by a nonzero constant.	$cR_a \quad (c \neq 0)$
3. Add a multiple of a row to another row.	$cR_a + R_b$

EXAMPLE 3 Elementary Row Operations

a. Interchange the first and second rows of the original matrix.

Original Matrix **New Row-Equivalent Matrix**

$$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix} \qquad \begin{array}{r@{}c} \begin{matrix} R_2 \\ R_1 \\ \end{matrix} & \begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix} \end{array}$$

b. Multiply the first row of the original matrix by $\frac{1}{2}$.

Original Matrix **New Row-Equivalent Matrix**

$$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix} \qquad \begin{array}{r@{}c} \begin{matrix} \frac{1}{2}R_1 \rightarrow \\ \\ \end{matrix} & \begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix} \end{array}$$

c. Add -2 times the first row of the original matrix to the third row.

Original Matrix **New Row-Equivalent Matrix**

$$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix} \qquad \begin{array}{r@{}c} \begin{matrix} \\ \\ -2R_1 + R_3 \rightarrow \end{matrix} & \begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix} \end{array}$$

Note that the elementary row operation is written beside the row that is *changed.*

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Identify the elementary row operation being performed to obtain the new row-equivalent matrix.

Original Matrix **New Row-Equivalent Matrix**

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 7 \\ 2 & -6 & 14 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & -6 & 14 \end{bmatrix}$$

TECHNOLOGY Most graphing utilities can perform elementary row operations on matrices. Consult the user's guide for your graphing utility for specific keystrokes.

After performing a row operation, the new row-equivalent matrix that is displayed on your graphing utility is stored in the *answer* variable. You should use the *answer* variable and not the original matrix for subsequent row operations.

Gaussian Elimination with Back-Substitution

In Example 3 in Section 6.3, you used Gaussian elimination with back-substitution to solve a system of linear equations. The next example demonstrates the matrix version of Gaussian elimination. The two methods are essentially the same. The basic difference is that with matrices you do not need to keep writing the variables.

EXAMPLE 4 Comparing Linear Systems and Matrix Operations

Linear System

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y \phantom{{}+3z} = -4 \\ 2x - 5y + 5z = 17 \end{cases}$$

Associated Augmented Matrix

$$\left[\begin{array}{rrrcr} 1 & -2 & 3 & \vdots & 9 \\ -1 & 3 & 0 & \vdots & -4 \\ 2 & -5 & 5 & \vdots & 17 \end{array}\right]$$

Add the first equation to the second equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ 2x - 5y + 5z = 17 \end{cases}$$

Add the first row to the second row ($R_1 + R_2$).

$$R_1 + R_2 \rightarrow \left[\begin{array}{rrrcr} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 2 & -5 & 5 & \vdots & 17 \end{array}\right]$$

Add -2 times the first equation to the third equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ -y - z = -1 \end{cases}$$

Add -2 times the first row to the third row ($-2R_1 + R_3$).

$$-2R_1 + R_3 \rightarrow \left[\begin{array}{rrrcr} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & -1 & -1 & \vdots & -1 \end{array}\right]$$

Add the second equation to the third equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ 2z = 4 \end{cases}$$

Add the second row to the third row ($R_2 + R_3$).

$$R_2 + R_3 \rightarrow \left[\begin{array}{rrrcr} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 0 & 2 & \vdots & 4 \end{array}\right]$$

Multiply the third equation by $\frac{1}{2}$.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ z = 2 \end{cases}$$

Multiply the third row by $\frac{1}{2}$ ($\frac{1}{2}R_3$).

$$\tfrac{1}{2}R_3 \rightarrow \left[\begin{array}{rrrcr} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 0 & 1 & \vdots & 2 \end{array}\right]$$

At this point, you can use back-substitution to find x and y.

$$\begin{aligned} y + 3(2) &= 5 && \text{Substitute 2 for } z. \\ y &= -1 && \text{Solve for } y. \\ x - 2(-1) + 3(2) &= 9 && \text{Substitute } -1 \text{ for } y \text{ and 2 for } z. \\ x &= 1 && \text{Solve for } x. \end{aligned}$$

The solution is $x = 1$, $y = -1$, and $z = 2$.

REMARK Remember that you should check a solution by substituting the values of x, y, and z into each equation of the original system. For example, you can check the solution to Example 4 as follows.

Equation 1:
$1 - 2(-1) + 3(2) = 9$ ✓

Equation 2:
$-1 + 3(-1) = -4$ ✓

Equation 3:
$2(1) - 5(-1) + 5(2) = 17$ ✓

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Compare solving the linear system below to solving it using its associated augmented matrix.

$$\begin{cases} 2x + y - z = -3 \\ 4x - 2y + 2z = -2 \\ -6x + 5y + 4z = 10 \end{cases}$$

The last matrix in Example 4 is said to be in **row-echelon form.** The term *echelon* refers to the stair-step pattern formed by the nonzero entries of the matrix. To be in this form, a matrix must have the following properties.

Row-Echelon Form and Reduced Row-Echelon Form

A matrix in **row-echelon form** has the following properties.

1. Any rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a **leading 1**).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

A matrix in *row-echelon form* is in **reduced row-echelon form** when every column that has a leading 1 has zeros in every position above and below its leading 1.

It is worth noting that the row-echelon form of a matrix is not unique. That is, two different sequences of elementary row operations may yield different row-echelon forms. However, the *reduced* row-echelon form of a given matrix is unique.

EXAMPLE 5 Row-Echelon Form

Determine whether each matrix is in row-echelon form. If it is, determine whether the matrix is in reduced row-echelon form.

a. $$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

b. $$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

c. $$\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

d. $$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

e. $$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

f. $$\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution The matrices in (a), (c), (d), and (f) are in row-echelon form. The matrices in (d) and (f) are in *reduced* row-echelon form because every column that has a leading 1 has zeros in every position above and below its leading 1. The matrix in (b) is not in row-echelon form because a row of all zeros does not occur at the bottom of the matrix. The matrix in (e) is not in row-echelon form because the first nonzero entry in Row 2 is not a leading 1.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Determine whether the matrix is in row-echelon form. If it is, determine whether it is in reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & 11 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Every matrix is row-equivalent to a matrix in row-echelon form. For instance, in Example 5, you can change the matrix in part (e) to row-echelon form by multiplying its second row by $\frac{1}{2}$.

Gaussian elimination with back-substitution works well for solving systems of linear equations by hand or with a computer. For this algorithm, the order in which the elementary row operations are performed is important. You should operate from left to right by columns, using elementary row operations to obtain zeros in all entries directly below the leading 1's.

EXAMPLE 6 Gaussian Elimination with Back-Substitution

Solve the system $\begin{cases} y + z - 2w = -3 \\ x + 2y - z = 2 \\ 2x + 4y + z - 3w = -2 \\ x - 4y - 7z - w = -19 \end{cases}$.

Solution

$$\left[\begin{array}{rrrrcr} 0 & 1 & 1 & -2 & \vdots & -3 \\ 1 & 2 & -1 & 0 & \vdots & 2 \\ 2 & 4 & 1 & -3 & \vdots & -2 \\ 1 & -4 & -7 & -1 & \vdots & -19 \end{array}\right]$$

Write augmented matrix.

$$\begin{array}{r} R_2 \\ R_1 \\ \\ \\ \end{array}\left[\begin{array}{rrrrcr} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & -3 \\ 2 & 4 & 1 & -3 & \vdots & -2 \\ 1 & -4 & -7 & -1 & \vdots & -19 \end{array}\right]$$

Interchange R_1 and R_2 so first column has leading 1 in upper left corner.

$$\begin{array}{r} \\ \\ -2R_1 + R_3 \rightarrow \\ -R_1 + R_4 \rightarrow \end{array}\left[\begin{array}{rrrrcr} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & -3 \\ 0 & 0 & 3 & -3 & \vdots & -6 \\ 0 & -6 & -6 & -1 & \vdots & -21 \end{array}\right]$$

Perform operations on R_3 and R_4 so first column has zeros below its leading 1.

$$\begin{array}{r} \\ \\ \\ 6R_2 + R_4 \rightarrow \end{array}\left[\begin{array}{rrrrcr} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & -3 \\ 0 & 0 & 3 & -3 & \vdots & -6 \\ 0 & 0 & 0 & -13 & \vdots & -39 \end{array}\right]$$

Perform operations on R_4 so second column has zeros below its leading 1.

$$\begin{array}{r} \\ \\ \frac{1}{3}R_3 \rightarrow \\ -\frac{1}{13}R_4 \rightarrow \end{array}\left[\begin{array}{rrrrcr} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & -3 \\ 0 & 0 & 1 & -1 & \vdots & -2 \\ 0 & 0 & 0 & 1 & \vdots & 3 \end{array}\right]$$

Perform operations on R_3 and R_4 so third and fourth columns have leading 1's.

The matrix is now in row-echelon form, and the corresponding system is

$$\begin{cases} x + 2y - z = 2 \\ y + z - 2w = -3 \\ z - w = -2 \\ w = 3 \end{cases}.$$

Using back-substitution, you can determine that the solution is

$x = -1, \quad y = 2, \quad z = 1, \quad$ and $\quad w = 3$.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve the system $\begin{cases} -3x + 5y + 3z = -19 \\ 3x + 4y + 4z = 8 \\ 4x - 8y - 6z = 26 \end{cases}$.

The procedure for using Gaussian elimination with back-substitution is summarized below.

Gaussian Elimination with Back-Substitution

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to rewrite the augmented matrix in row-echelon form.
3. Write the system of linear equations corresponding to the matrix in row-echelon form and use back-substitution to find the solution.

When solving a system of linear equations, remember that it is possible for the system to have no solution. If, in the elimination process, you obtain a row of all zeros except for the last entry, then the system has no solution, or is *inconsistent*.

EXAMPLE 7 A System with No Solution

Solve the system $\begin{cases} x - y + 2z = 4 \\ x + z = 6 \\ 2x - 3y + 5z = 4 \\ 3x + 2y - z = 1 \end{cases}$.

Solution

$$\left[\begin{array}{rrrcr} 1 & -1 & 2 & \vdots & 4 \\ 1 & 0 & 1 & \vdots & 6 \\ 2 & -3 & 5 & \vdots & 4 \\ 3 & 2 & -1 & \vdots & 1 \end{array}\right]$$

Write augmented matrix.

$$\begin{array}{r} \\ -R_1 + R_2 \rightarrow \\ -2R_1 + R_3 \rightarrow \\ -3R_1 + R_4 \rightarrow \end{array} \left[\begin{array}{rrrcr} 1 & -1 & 2 & \vdots & 4 \\ 0 & 1 & -1 & \vdots & 2 \\ 0 & -1 & 1 & \vdots & -4 \\ 0 & 5 & -7 & \vdots & -11 \end{array}\right]$$

Perform row operations.

$$\begin{array}{r} \\ \\ R_2 + R_3 \rightarrow \\ \\ \end{array} \left[\begin{array}{rrrcr} 1 & -1 & 2 & \vdots & 4 \\ 0 & 1 & -1 & \vdots & 2 \\ 0 & 0 & 0 & \vdots & -2 \\ 0 & 5 & -7 & \vdots & -11 \end{array}\right]$$

Perform row operations.

Note that the third row of this matrix consists entirely of zeros except for the last entry. This means that the original system of linear equations is inconsistent. You can see why this is true by converting back to a system of linear equations.

$$\begin{cases} x - y + 2z = 4 \\ y - z = 2 \\ 0 = -2 \\ 5y - 7z = -11 \end{cases}$$

Because the third equation is not possible, the system has no solution.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve the system $\begin{cases} x + y + z = 1 \\ x + 2y + 2z = 2 \\ x - y - z = 1 \end{cases}$.

Gauss-Jordan Elimination

With Gaussian elimination, elementary row operations are applied to a matrix to obtain a (row-equivalent) row-echelon form of the matrix. A second method of elimination, called **Gauss-Jordan elimination,** after Carl Friedrich Gauss and Wilhelm Jordan (1842–1899), continues the reduction process until a *reduced* row-echelon form is obtained. This procedure is demonstrated in Example 8.

EXAMPLE 8 Gauss-Jordan Elimination

▷ **TECHNOLOGY** For a demonstration of a graphical approach to Gauss-Jordan elimination on a 2×3 matrix, see the Visualizing Row Operations Program available for several models of graphing calculators at the website for this text at *LarsonPrecalculus.com.*

Use Gauss-Jordan elimination to solve the system $\begin{cases} x - 2y + 3z = 9 \\ -x + 3y = -4. \\ 2x - 5y + 5z = 17 \end{cases}$

Solution In Example 4, Gaussian elimination was used to obtain the row-echelon form of the linear system above.

$$\left[\begin{array}{rrrcr} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 0 & 1 & \vdots & 2 \end{array}\right]$$

Now, apply elementary row operations until you obtain zeros above each of the leading 1's, as follows.

$$\begin{array}{r} 2R_2 + R_1 \rightarrow \\ \\ \\ \end{array} \left[\begin{array}{rrrcr} 1 & 0 & 9 & \vdots & 19 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 0 & 1 & \vdots & 2 \end{array}\right]$$

Perform operations on R_1 so second column has a zero above its leading 1.

$$\begin{array}{r} -9R_3 + R_1 \rightarrow \\ -3R_3 + R_2 \rightarrow \\ \\ \end{array} \left[\begin{array}{rrrcr} 1 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & -1 \\ 0 & 0 & 1 & \vdots & 2 \end{array}\right]$$

Perform operations on R_1 and R_2 so third column has zeros above its leading 1.

The matrix is now in reduced row-echelon form. Converting back to a system of linear equations, you have

$$\begin{cases} x = 1 \\ y = -1. \\ z = 2 \end{cases}$$

REMARK The advantage of using Gauss-Jordan elimination to solve a system of linear equations is that the solution of the system is easily found without using back-substitution, as illustrated in Example 8. ▷

Now you can simply read the solution, $x = 1$, $y = -1$, and $z = 2$, which can be written as the ordered triple $(1, -1, 2)$.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use Gauss-Jordan elimination to solve the system $\begin{cases} -3x + 7y + 2z = 1 \\ -5x + 3y - 5z = -8. \\ 2x - 2y - 3z = 15 \end{cases}$

The elimination procedures described in this section sometimes result in fractional coefficients. For instance, in the elimination procedure for the system

$$\begin{cases} 2x - 5y + 5z = 17 \\ 3x - 2y + 3z = 11 \\ -3x + 3y = -6 \end{cases}$$

you may be inclined to multiply the first row by $\frac{1}{2}$ to produce a leading 1, which will result in working with fractional coefficients. You can sometimes avoid fractions by judiciously choosing the order in which you apply elementary row operations.

EXAMPLE 9 A System with an Infinite Number of Solutions

Solve the system $\begin{cases} 2x + 4y - 2z = 0 \\ 3x + 5y \qquad\;\; = 1 \end{cases}$.

Solution

$$\begin{bmatrix} 2 & 4 & -2 & \vdots & 0 \\ 3 & 5 & 0 & \vdots & 1 \end{bmatrix}$$

$$\begin{array}{r} \tfrac{1}{2}R_1 \rightarrow \\ \end{array}\begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 3 & 5 & 0 & \vdots & 1 \end{bmatrix}$$

$$\begin{array}{r} \\ -3R_1 + R_2 \rightarrow \end{array}\begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 0 & -1 & 3 & \vdots & 1 \end{bmatrix}$$

$$\begin{array}{r} \\ -R_2 \rightarrow \end{array}\begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 0 & 1 & -3 & \vdots & -1 \end{bmatrix}$$

$$\begin{array}{r} -2R_2 + R_1 \rightarrow \\ \end{array}\begin{bmatrix} 1 & 0 & 5 & \vdots & 2 \\ 0 & 1 & -3 & \vdots & -1 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{cases} x + 5z = 2 \\ y - 3z = -1 \end{cases}.$$

Solving for x and y in terms of z, you have

$$x = -5z + 2 \quad \text{and} \quad y = 3z - 1.$$

To write a solution of the system that does not use any of the three variables of the system, let a represent any real number and let $z = a$. Substituting a for z in the equations for x and y, you have

$$x = -5z + 2 = -5a + 2 \quad \text{and} \quad y = 3z - 1 = 3a - 1.$$

So, the solution set can be written as an ordered triple of the form

$$(-5a + 2, 3a - 1, a)$$

where a is any real number. Remember that a solution set of this form represents an infinite number of solutions. Try substituting values for a to obtain a few solutions. Then check each solution in the original system of equations.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve the system $\begin{cases} 2x - 6y + 6z = 46 \\ 2x - 3y \qquad\;\; = 31 \end{cases}$.

Summarize (Section 7.1)

1. State the definition of a matrix *(page 496)*. For examples of writing matrices and determining their orders, see Examples 1 and 2.
2. List the elementary row operations *(page 498)*. For examples of performing elementary row operations, see Examples 3 and 4.
3. State the definitions of row-echelon form and reduced row-echelon form *(page 500)*. For an example of matrices in these forms, see Example 5.
4. Describe Gaussian elimination with back-substitution *(pages 501 and 502)*. For examples of using this procedure, see Examples 6 and 7.
5. Describe Gauss-Jordan elimination *(page 503)*. For examples of using this procedure, see Examples 8 and 9.

7.1 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. A rectangular array of real numbers that can be used to solve a system of linear equations is called a ________.
2. A matrix is ________ when the number of rows equals the number of columns.
3. For a square matrix, the entries $a_{11}, a_{22}, a_{33}, . . .$ are the ________ ________ entries.
4. A matrix with only one row is called a ________ matrix, and a matrix with only one column is called a ________ matrix.
5. The matrix derived from a system of linear equations is called the ________ matrix of the system.
6. The matrix derived from the coefficients of a system of linear equations is called the ________ matrix of the system.
7. Two matrices are called ________ when one of the matrices can be obtained from the other by a sequence of elementary row operations.
8. A matrix in row-echelon form is in ________ ________ ________ when every column that has a leading 1 has zeros in every position above and below its leading 1.

Skills and Applications

Order of a Matrix **In Exercises 9–16, determine the order of the matrix.**

9. $\begin{bmatrix} 7 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 5 & -3 & 8 & 7 \end{bmatrix}$

11. $\begin{bmatrix} 2 \\ 36 \\ 3 \end{bmatrix}$

12. $\begin{bmatrix} -3 & 7 & 15 & 0 \\ 0 & 0 & 3 & 3 \\ 1 & 1 & 6 & 7 \end{bmatrix}$

13. $\begin{bmatrix} 33 & 45 \\ -9 & 20 \end{bmatrix}$

14. $\begin{bmatrix} -7 & 6 & 4 \\ 0 & -5 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 6 & -1 \\ 8 & 0 & 3 \\ 3 & -9 & 9 \end{bmatrix}$

16. $\begin{bmatrix} 3 & -1 \\ 4 & 1 \\ -5 & 9 \end{bmatrix}$

Writing an Augmented Matrix **In Exercises 17–22, write the augmented matrix for the system of linear equations.**

17. $\begin{cases} 4x - 3y = -5 \\ -x + 3y = 12 \end{cases}$

18. $\begin{cases} 7x + 4y = 22 \\ 5x - 9y = 15 \end{cases}$

19. $\begin{cases} x + 10y - 2z = 2 \\ 5x - 3y + 4z = 0 \\ 2x + y = 6 \end{cases}$

20. $\begin{cases} -x - 8y + 5z = 8 \\ -7x - 15z = -38 \\ 3x - y + 8z = 20 \end{cases}$

21. $\begin{cases} 7x - 5y + z = 13 \\ 19x - 8z = 10 \end{cases}$

22. $\begin{cases} 9x + 2y - 3z = 20 \\ -25y + 11z = -5 \end{cases}$

Writing a System of Equations **In Exercises 23–28, write the system of linear equations represented by the augmented matrix. (Use variables *x*, *y*, *z*, and *w*, if applicable.)**

23. $\left[\begin{array}{cccc} 1 & 2 & \vdots & 7 \\ 2 & -3 & \vdots & 4 \end{array}\right]$

24. $\left[\begin{array}{cccc} 7 & -5 & \vdots & 0 \\ 8 & 3 & \vdots & -2 \end{array}\right]$

25. $\left[\begin{array}{ccccc} 2 & 0 & 5 & \vdots & -12 \\ 0 & 1 & -2 & \vdots & 7 \\ 6 & 3 & 0 & \vdots & 2 \end{array}\right]$

26. $\left[\begin{array}{ccccc} 4 & -5 & -1 & \vdots & 18 \\ -11 & 0 & 6 & \vdots & 25 \\ 3 & 8 & 0 & \vdots & -29 \end{array}\right]$

27. $\left[\begin{array}{cccccc} 9 & 12 & 3 & 0 & \vdots & 0 \\ -2 & 18 & 5 & 2 & \vdots & 10 \\ 1 & 7 & -8 & 0 & \vdots & -4 \\ 3 & 0 & 2 & 0 & \vdots & -10 \end{array}\right]$

28. $\left[\begin{array}{cccccc} 6 & 2 & -1 & -5 & \vdots & -25 \\ -1 & 0 & 7 & 3 & \vdots & 7 \\ 4 & -1 & -10 & 6 & \vdots & 23 \\ 0 & 8 & 1 & -11 & \vdots & -21 \end{array}\right]$

Identifying an Elementary Row Operation **In Exercises 29–32, identify the elementary row operation(s) being performed to obtain the new row-equivalent matrix.**

	Original Matrix	New Row-Equivalent Matrix
29.	$\begin{bmatrix} -2 & 5 & 1 \\ 3 & -1 & -8 \end{bmatrix}$	$\begin{bmatrix} 13 & 0 & -39 \\ 3 & -1 & -8 \end{bmatrix}$
30.	$\begin{bmatrix} 3 & -1 & -4 \\ -4 & 3 & 7 \end{bmatrix}$	$\begin{bmatrix} 3 & -1 & -4 \\ 5 & 0 & -5 \end{bmatrix}$
31.	$\begin{bmatrix} 0 & -1 & -5 & 5 \\ -1 & 3 & -7 & 6 \\ 4 & -5 & 1 & 3 \end{bmatrix}$	$\begin{bmatrix} -1 & 3 & -7 & 6 \\ 0 & -1 & -5 & 5 \\ 0 & 7 & -27 & 27 \end{bmatrix}$
32.	$\begin{bmatrix} -1 & -2 & 3 & -2 \\ 2 & -5 & 1 & -7 \\ 5 & 4 & -7 & 6 \end{bmatrix}$	$\begin{bmatrix} -1 & -2 & 3 & -2 \\ 0 & -9 & 7 & -11 \\ 0 & -6 & 8 & -4 \end{bmatrix}$

Elementary Row Operations **In Exercises 33–40, fill in the blank(s) using elementary row operations to form a row-equivalent matrix.**

33. $\begin{bmatrix} 1 & 4 & 3 \\ 2 & 10 & 5 \end{bmatrix}$

$\begin{bmatrix} 1 & 4 & 3 \\ 0 & \square & -1 \end{bmatrix}$

34. $\begin{bmatrix} 3 & 6 & 8 \\ 4 & -3 & 6 \end{bmatrix}$

$\begin{bmatrix} 1 & \square & \frac{8}{3} \\ 4 & -3 & 6 \end{bmatrix}$

35. $\begin{bmatrix} 1 & 1 & 1 \\ 5 & -2 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & \square & -1 \end{bmatrix}$

36. $\begin{bmatrix} -3 & 3 & 12 \\ 18 & -8 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 & \square \\ 18 & -8 & 4 \end{bmatrix}$

37. $\begin{bmatrix} 1 & 5 & 4 & -1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -7 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & \square & \square \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -7 \end{bmatrix}$

38. $\begin{bmatrix} 1 & 0 & 6 & 1 \\ 0 & -1 & 0 & 7 \\ 0 & 0 & -1 & 3 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 6 & 1 \\ 0 & 1 & 0 & \square \\ 0 & 0 & 1 & \square \end{bmatrix}$

39. $\begin{bmatrix} 1 & 1 & 4 & -1 \\ 3 & 8 & 10 & 3 \\ -2 & 1 & 12 & 6 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 4 & -1 \\ 0 & 5 & \square & \square \\ 0 & 3 & \square & \square \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 4 & -1 \\ 0 & 1 & -\frac{2}{5} & \frac{6}{5} \\ 0 & 3 & \square & \square \end{bmatrix}$

40. $\begin{bmatrix} 2 & 4 & 8 & 3 \\ 1 & -1 & -3 & 2 \\ 2 & 6 & 4 & 9 \end{bmatrix}$

$\begin{bmatrix} 1 & \square & \square & \square \\ 1 & -1 & -3 & 2 \\ 2 & 6 & 4 & 9 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 4 & \frac{3}{2} \\ 0 & \square & -7 & \frac{1}{2} \\ 0 & 2 & \square & \square \end{bmatrix}$

Comparing Linear Systems and Matrix Operations **In Exercises 41 and 42, (a) perform the row operations to solve the augmented matrix, (b) write and solve the system of linear equations represented by the augmented matrix, and (c) compare the two solution methods. Which do you prefer?**

41. $\left[\begin{array}{rrcr} -3 & 4 & \vdots & 22 \\ 6 & -4 & \vdots & -28 \end{array}\right]$

(i) Add R_2 to R_1.

(ii) Add -2 times R_1 to R_2.

(iii) Multiply R_2 by $-\frac{1}{4}$.

(iv) Multiply R_1 by $\frac{1}{3}$.

42. $\left[\begin{array}{rrrcr} 7 & 13 & 1 & \vdots & -4 \\ -3 & -5 & -1 & \vdots & -4 \\ 3 & 6 & 1 & \vdots & -2 \end{array}\right]$

(i) Add R_2 to R_1.

(ii) Multiply R_1 by $\frac{1}{4}$.

(iii) Add R_3 to R_2.

(iv) Add -3 times R_1 to R_3.

(v) Add -2 times R_2 to R_1.

Row-Echelon Form **In Exercises 43–46, determine whether the matrix is in row-echelon form. If it is, determine whether it is in reduced row-echelon form.**

43. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

44. $\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

45. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

46. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Writing a Matrix in Row-Echelon Form **In Exercises 47–50, write the matrix in row-echelon form. (Remember that the row-echelon form of a matrix is not unique.)**

47. $\begin{bmatrix} 1 & 1 & 0 & 5 \\ -2 & -1 & 2 & -10 \\ 3 & 6 & 7 & 14 \end{bmatrix}$

48. $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 7 & -5 & 14 \\ -2 & -1 & -3 & 8 \end{bmatrix}$

49. $\begin{bmatrix} 1 & -1 & -1 & 1 \\ 5 & -4 & 1 & 8 \\ -6 & 8 & 18 & 0 \end{bmatrix}$

50. $\begin{bmatrix} 1 & -3 & 0 & -7 \\ -3 & 10 & 1 & 23 \\ 4 & -10 & 2 & -24 \end{bmatrix}$

Using a Graphing Utility **In Exercises 51–56, use the matrix capabilities of a graphing utility to write the matrix in *reduced* row-echelon form.**

51. $\begin{bmatrix} 3 & 3 & 3 \\ -1 & 0 & -4 \\ 2 & 4 & -2 \end{bmatrix}$

52. $\begin{bmatrix} 1 & 3 & 2 \\ 5 & 15 & 9 \\ 2 & 6 & 10 \end{bmatrix}$

53. $\begin{bmatrix} 1 & 2 & 3 & -5 \\ 1 & 2 & 4 & -9 \\ -2 & -4 & -4 & 3 \\ 4 & 8 & 11 & -14 \end{bmatrix}$

54. $\begin{bmatrix} -2 & 3 & -1 & -2 \\ 4 & -2 & 5 & 8 \\ 1 & 5 & -2 & 0 \\ 3 & 8 & -10 & -30 \end{bmatrix}$

55. $\begin{bmatrix} -3 & 5 & 1 & 12 \\ 1 & -1 & 1 & 4 \end{bmatrix}$

56. $\begin{bmatrix} 5 & 1 & 2 & 4 \\ -1 & 5 & 10 & -32 \end{bmatrix}$

Using Back-Substitution **In Exercises 57–60, write the system of linear equations represented by the augmented matrix. Then use back-substitution to solve. (Use variables *x*, *y*, and *z*, if applicable.)**

57. $\left[\begin{array}{rrcr} 1 & -2 & \vdots & 4 \\ 0 & 1 & \vdots & -3 \end{array}\right]$

58. $\left[\begin{array}{rrcr} 1 & 5 & \vdots & 0 \\ 0 & 1 & \vdots & -1 \end{array}\right]$

59. $\left[\begin{array}{rrrcr} 1 & -1 & 2 & \vdots & 4 \\ 0 & 1 & -1 & \vdots & 2 \\ 0 & 0 & 1 & \vdots & -2 \end{array}\right]$

60. $\left[\begin{array}{rrrcr} 1 & 2 & -2 & \vdots & -1 \\ 0 & 1 & 1 & \vdots & 9 \\ 0 & 0 & 1 & \vdots & -3 \end{array}\right]$

Interpreting Reduced Row-Echelon Form **In Exercises 61–64, an augmented matrix that represents a system of linear equations (in variables *x*, *y*, and *z*, if applicable) has been reduced using Gauss-Jordan elimination. Write the solution represented by the augmented matrix.**

61. $\begin{bmatrix} 1 & 0 & \vdots & 3 \\ 0 & 1 & \vdots & -4 \end{bmatrix}$ **62.** $\begin{bmatrix} 1 & 0 & \vdots & -6 \\ 0 & 1 & \vdots & 10 \end{bmatrix}$

63. $\begin{bmatrix} 1 & 0 & 0 & \vdots & -4 \\ 0 & 1 & 0 & \vdots & -10 \\ 0 & 0 & 1 & \vdots & 4 \end{bmatrix}$ **64.** $\begin{bmatrix} 1 & 0 & 0 & \vdots & 5 \\ 0 & 1 & 0 & \vdots & -3 \\ 0 & 0 & 1 & \vdots & 0 \end{bmatrix}$

Gaussian Elimination with Back-Substitution **In Exercises 65–74, use matrices to solve the system of equations (if possible). Use Gaussian elimination with back-substitution.**

65. $\begin{cases} x + 2y = 7 \\ 2x + y = 8 \end{cases}$ **66.** $\begin{cases} 2x + 6y = 16 \\ 2x + 3y = 7 \end{cases}$

67. $\begin{cases} 3x - 2y = -27 \\ x + 3y = 13 \end{cases}$ **68.** $\begin{cases} -x + y = 4 \\ 2x - 4y = -34 \end{cases}$

69. $\begin{cases} x + 2y - 3z = -28 \\ 4y + 2z = 0 \\ -x + y - z = -5 \end{cases}$

70. $\begin{cases} 3x - 2y + z = 15 \\ -x + y + 2z = -10 \\ x - y - 4z = 14 \end{cases}$

71. $\begin{cases} -x + y = -22 \\ 3x + 4y = 4 \\ 4x - 8y = 32 \end{cases}$ **72.** $\begin{cases} x + 2y = 0 \\ x + y = 6 \\ 3x - 2y = 8 \end{cases}$

73. $\begin{cases} 3x + 2y - z + w = 0 \\ x - y + 4z + 2w = 25 \\ -2x + y + 2z - w = 2 \\ x + y + z + w = 6 \end{cases}$

74. $\begin{cases} x - 4y + 3z - 2w = 9 \\ 3x - 2y + z - 4w = -13 \\ -4x + 3y - 2z + w = -4 \\ -2x + y - 4z + 3w = -10 \end{cases}$

Gauss-Jordan Elimination **In Exercises 75–84, use matrices to solve the system of equations (if possible). Use Gauss-Jordan elimination.**

75. $\begin{cases} -2x + 6y = -22 \\ x + 2y = -9 \end{cases}$ **76.** $\begin{cases} 5x - 5y = -5 \\ -2x - 3y = 7 \end{cases}$

77. $\begin{cases} 8x - 4y = 7 \\ 5x + 2y = 1 \end{cases}$ **78.** $\begin{cases} x - 3y = 5 \\ -2x + 6y = -10 \end{cases}$

79. $\begin{cases} x + 2y + z = 8 \\ 3x + 7y + 6z = 26 \end{cases}$ **80.** $\begin{cases} x + y + 4z = 5 \\ 2x + y - z = 9 \end{cases}$

81. $\begin{cases} x - 3z = -2 \\ 3x + y - 2z = 5 \\ 2x + 2y + z = 4 \end{cases}$ **82.** $\begin{cases} 2x - y + 3z = 24 \\ 2y - z = 14 \\ 7x - 5y = 6 \end{cases}$

83. $\begin{cases} -x + y - z = -14 \\ 2x - y + z = 21 \\ 3x + 2y + z = 19 \end{cases}$ **84.** $\begin{cases} 2x + 2y - z = 2 \\ x - 3y + z = -28 \\ -x + y = 14 \end{cases}$

Using a Graphing Utility **In Exercises 85–90, use the matrix capabilities of a graphing utility to write the augmented matrix corresponding to the system of equations in reduced row-echelon form. Then solve the system.**

85. $\begin{cases} 3x + 3y + 12z = 6 \\ x + y + 4z = 2 \\ 2x + 5y + 20z = 10 \\ -x + 2y + 8z = 4 \end{cases}$

86. $\begin{cases} 2x + 10y + 2z = 6 \\ x + 5y + 2z = 6 \\ x + 5y + z = 3 \\ -3x - 15y - 3z = -9 \end{cases}$

87. $\begin{cases} 2x + y - z + 2w = -6 \\ 3x + 4y + w = 1 \\ x + 5y + 2z + 6w = -3 \\ 5x + 2y - z - w = 3 \end{cases}$

88. $\begin{cases} x + 2y + 2z + 4w = 11 \\ 3x + 6y + 5z + 12w = 30 \\ x + 3y - 3z + 2w = -5 \\ 6x - y - z + w = -9 \end{cases}$

89. $\begin{cases} x + y + z + w = 0 \\ 2x + 3y + z - 2w = 0 \\ 3x + 5y + z = 0 \end{cases}$

90. $\begin{cases} x + 2y + z + 3w = 0 \\ x - y + w = 0 \\ y - z + 2w = 0 \end{cases}$

Comparing Solutions of Two Systems **In Exercises 91–94, determine whether the two systems of linear equations yield the same solution. If so, find the solution using matrices.**

91. (a) $\begin{cases} x - 2y + z = -6 \\ y - 5z = 16 \\ z = -3 \end{cases}$ (b) $\begin{cases} x + y - 2z = 6 \\ y + 3z = -8 \\ z = -3 \end{cases}$

92. (a) $\begin{cases} x - 3y + 4z = -11 \\ y - z = -4 \\ z = 2 \end{cases}$ (b) $\begin{cases} x + 4y = -11 \\ y + 3z = 4 \\ z = 2 \end{cases}$

93. (a) $\begin{cases} x - 4y + 5z = 27 \\ y - 7z = -54 \\ z = 8 \end{cases}$ (b) $\begin{cases} x - 6y + z = 15 \\ y + 5z = 42 \\ z = 8 \end{cases}$

94. (a) $\begin{cases} x + 3y - z = 19 \\ y + 6z = -18 \\ z = -4 \end{cases}$ (b) $\begin{cases} x - y + 3z = -15 \\ y - 2z = 14 \\ z = -4 \end{cases}$

Curve Fitting **In Exercises 95–100, use a system of equations to find the quadratic function $f(x) = ax^2 + bx + c$ that satisfies the given conditions. Solve the system using matrices.**

95. $f(1) = 1, f(2) = -1, f(3) = -5$

96. $f(1) = 2, f(2) = 9, f(3) = 20$

97. $f(-2) = -15, f(-1) = 7, f(1) = -3$

98. $f(-2) = -3, f(1) = -3, f(2) = -11$

99. $f(1) = 8,\ f(2) = 13,\ f(3) = 20$

100. $f(1) = 9,\ f(2) = 8,\ f(3) = 5$

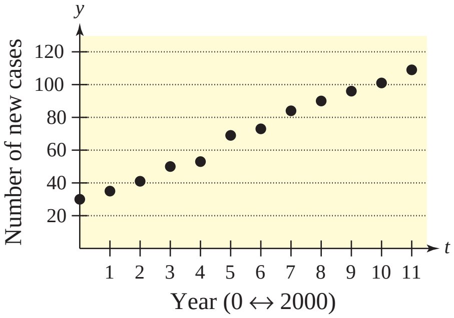
101. Health and Wellness From 2000 through 2011, the numbers of new cases of a waterborne disease in a small city increased in a pattern that was approximately linear (see figure). Find the least squares regression line

$y = at + b$

for the data shown in the figure by solving the following system using matrices. Let t represent the year, with $t = 0$ corresponding to 2000.

$$\begin{cases} 12b + 66a = 831 \\ 66b + 506a = 5643 \end{cases}$$

Use the result to predict the number of new cases of the waterborne disease in 2014. Is the estimate reasonable? Explain.



102. Breeding Facility A city zoo borrowed \$2,000,000 at simple annual interest to construct a breeding facility. Some of the money was borrowed at 8%, some at 9%, and some at 12%. Use a system of linear equations to determine how much was borrowed at each rate given that the total annual interest was \$186,000 and the amount borrowed at 8% was twice the amount borrowed at 12%. Solve the system of linear equations using matrices.

103. Museum A natural history museum borrowed \$2,000,000 at simple annual interest to purchase new exhibits. Some of the money was borrowed at 7%, some at 8.5%, and some at 9.5%. Use a system of linear equations to determine how much was borrowed at each rate given that the total annual interest was \$169,750 and the amount borrowed at 8.5% was four times the amount borrowed at 9.5%. Solve the system of linear equations using matrices.

104. Mathematical Modeling A video of the path of a ball thrown by a baseball player was analyzed with a grid covering the TV screen. The tape was paused three times, and the position of the ball was measured each time. The coordinates obtained are shown in the table. (x and y are measured in feet.)

Horizontal Distance, x	0	15	30
Height, y	5.0	9.6	12.4

(a) Use a system of equations to find the equation of the parabola $y = ax^2 + bx + c$ that passes through the three points. Solve the system using matrices.

(b) Use a graphing utility to graph the parabola.

(c) Graphically approximate the maximum height of the ball and the point at which the ball struck the ground.

(d) Analytically find the maximum height of the ball and the point at which the ball struck the ground.

(e) Compare your results from parts (c) and (d).

Exploration

True or False? **In Exercises 105 and 106, determine whether the statement is true or false. Justify your answer.**

105. $\begin{bmatrix} 5 & 0 & -2 & 7 \\ -1 & 3 & -6 & 0 \end{bmatrix}$ is a 4×2 matrix.

106. The method of Gaussian elimination reduces a matrix until a reduced row-echelon form is obtained.

107. Think About It What is the relationship between the three elementary row operations performed on an augmented matrix and the operations that lead to equivalent systems of equations?

108. HOW DO YOU SEE IT? Determine whether the matrix below is in row-echelon form, reduced row-echelon form, or neither when it satisfies the given conditions.

$$\begin{bmatrix} 1 & b \\ c & 1 \end{bmatrix}$$

(a) $b = 0, c = 0$ (b) $b \neq 0, c = 0$

(c) $b = 0, c \neq 0$ (d) $b \neq 0, c \neq 0$

Tyler Olson/Shutterstock.com

7.2 Operations with Matrices

You can use matrix operations to model and solve real-life problems. For instance, in Exercise 74 on page 522, you will use matrix multiplication to determine the numbers of calories burned by individuals of different body weights while performing different types of exercises.

- **Decide whether two matrices are equal.**
- **Add and subtract matrices, and multiply matrices by scalars.**
- **Multiply two matrices.**
- **Use matrix operations to model and solve real-life problems.**

Equality of Matrices

In Section 7.1, you used matrices to solve systems of linear equations. There is a rich mathematical theory of matrices, and its applications are numerous. This section and the next two introduce some fundamentals of matrix theory. It is standard mathematical convention to represent matrices in any of the following three ways.

Representation of Matrices

1. A matrix can be denoted by an uppercase letter such as A, B, or C.
2. A matrix can be denoted by a representative element enclosed in brackets, such as $[a_{ij}]$, $[b_{ij}]$, or $[c_{ij}]$.
3. A matrix can be denoted by a rectangular array of numbers such as

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal** when they have the same order $(m \times n)$ and $a_{ij} = b_{ij}$ for $1 \le i \le m$ and $1 \le j \le n$. In other words, two matrices are equal when their corresponding entries are equal.

EXAMPLE 1 Equality of Matrices

Solve for a_{11}, a_{12}, a_{21}, and a_{22} in the following matrix equation.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix}$$

Solution Because two matrices are equal when their corresponding entries are equal, you can conclude that $a_{11} = 2$, $a_{12} = -1$, $a_{21} = -3$, and $a_{22} = 0$.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve for a_{11}, a_{12}, a_{21}, and a_{22} in the following matrix equation.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ -2 & 4 \end{bmatrix}$$

Be sure you see that for two matrices to be equal, they must have the same order *and* their corresponding entries must be equal. For instance,

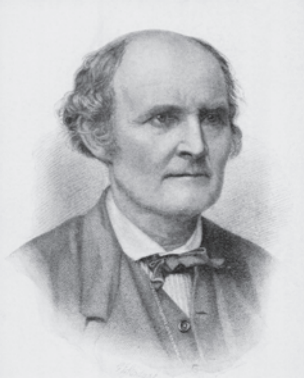
$$\begin{bmatrix} 2 & -1 \\ \sqrt{4} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & 0.5 \end{bmatrix} \quad \text{but} \quad \begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & 0 \end{bmatrix} \neq \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}.$$

Simone van den Berg/Shutterstock.com

Matrix Addition and Scalar Multiplication

In this section, three basic matrix operations will be covered. The first two are matrix addition and scalar multiplication. With matrix addition, you can add two matrices (of the same order) by adding their corresponding entries.

HISTORICAL NOTE



Arthur Cayley (1821–1895), a British mathematician, invented matrices around 1858. Cayley was a Cambridge University graduate and a lawyer by profession. His groundbreaking work on matrices was begun as he studied the theory of transformations. Cayley also was instrumental in the development of determinants. Cayley and two American mathematicians, Benjamin Peirce (1809–1880) and his son Charles S. Peirce (1839–1914), are credited with developing "matrix algebra."

Definition of Matrix Addition

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of order $m \times n$, then their sum is the $m \times n$ matrix given by

$$A + B = [a_{ij} + b_{ij}].$$

The sum of two matrices of different orders is undefined.

EXAMPLE 2 Addition of Matrices

a. $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1+1 & 2+3 \\ 0+(-1) & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix}$

c. $\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

d. The sum of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & -1 \\ 3 & -2 & 2 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 3 \\ 2 & 4 \end{bmatrix}$$

is undefined because A is of order 3×3 and B is of order 3×2.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Evaluate the expression.

$$\begin{bmatrix} 4 & -1 \\ 2 & -3 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 0 & 6 \end{bmatrix}$$

In operations with matrices, numbers are usually referred to as **scalars.** In this text, scalars will always be real numbers. You can multiply a matrix A by a scalar c by multiplying each entry in A by c.

Definition of Scalar Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and c is a scalar, then the **scalar multiple** of A by c is the $m \times n$ matrix given by

$$cA = [ca_{ij}].$$

Bettmann/CORBIS

The symbol $-A$ represents the negation of A, which is the scalar product $(-1)A$. Moreover, if A and B are of the same order, then $A - B$ represents the sum of A and $(-1)B$. That is,

$$A - B = A + (-1)B.$$

Subtraction of matrices

EXAMPLE 3 Scalar Multiplication and Matrix Subtraction

For the following matrices, find (a) $3A$, (b) $-B$, and (c) $3A - B$.

$$A = \begin{bmatrix} 2 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

Solution

a. $$3A = 3\begin{bmatrix} 2 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}$$

Scalar multiplication

$$= \begin{bmatrix} 3(2) & 3(2) & 3(4) \\ 3(-3) & 3(0) & 3(-1) \\ 3(2) & 3(1) & 3(2) \end{bmatrix}$$

Multiply each entry by 3.

$$= \begin{bmatrix} 6 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$$

Simplify.

b. $$-B = (-1)\begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

Definition of negation

$$= \begin{bmatrix} -2 & 0 & 0 \\ -1 & 4 & -3 \\ 1 & -3 & -2 \end{bmatrix}$$

Multiply each entry by -1.

c. $$3A - B = \begin{bmatrix} 6 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

Perform scalar multiplication first.

$$= \begin{bmatrix} 4 & 6 & 12 \\ -10 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix}$$

Subtract corresponding entries.

REMARK The order of operations for matrix expressions is similar to that for real numbers. In particular, you perform scalar multiplication before matrix addition and subtraction, as shown in Example 3(c).

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

For the following matrices, find (a) $A + B$, (b) $A - B$, (c) $3A$, and (d) $3A - 2B$.

$$A = \begin{bmatrix} 4 & -1 \\ 0 & 4 \\ -3 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 4 \\ -1 & 3 \\ 1 & 7 \end{bmatrix}$$

It is often convenient to rewrite the scalar multiple cA by factoring c out of every entry in the matrix. For instance, in the following example, the scalar $\frac{1}{2}$ has been factored out of the matrix.

$$\begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ \frac{5}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1) & \frac{1}{2}(-3) \\ \frac{1}{2}(5) & \frac{1}{2}(1) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -3 \\ 5 & 1 \end{bmatrix}$$

ALGEBRA HELP You can review the properties of addition and multiplication of real numbers (and other properties of real numbers) in Section P.1.

The properties of matrix addition and scalar multiplication are similar to those of addition and multiplication of real numbers.

Properties of Matrix Addition and Scalar Multiplication

Let A, B, and C be $m \times n$ matrices and let c and d be scalars.

1. $A + B = B + A$	Commutative Property of Matrix Addition
2. $A + (B + C) = (A + B) + C$	Associative Property of Matrix Addition
3. $(cd)A = c(dA)$	Associative Property of Scalar Multiplication
4. $1A = A$	Scalar Identity Property
5. $c(A + B) = cA + cB$	Distributive Property
6. $(c + d)A = cA + dA$	Distributive Property

Note that the Associative Property of Matrix Addition allows you to write expressions such as $A + B + C$ without ambiguity because the same sum occurs no matter how the matrices are grouped. This same reasoning applies to sums of four or more matrices.

TECHNOLOGY Most graphing utilities have the capability of performing matrix operations. Consult the user's guide for your graphing utility for specific keystrokes. Try using a graphing utility to find the sum of the matrices

$$A = \begin{bmatrix} 2 & -3 \\ -1 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} -1 & 4 \\ 2 & -5 \end{bmatrix}.$$

EXAMPLE 4 Addition of More than Two Matrices

By adding corresponding entries, you obtain the following sum of four matrices.

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Evaluate the expression.

$$\begin{bmatrix} 3 & -8 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 6 & -5 \end{bmatrix} + \begin{bmatrix} 0 & 7 \\ 4 & -1 \end{bmatrix}$$

EXAMPLE 5 Using the Distributive Property

$$\begin{aligned} 3\left(\begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -2 \\ 3 & 7 \end{bmatrix}\right) &= 3\begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix} + 3\begin{bmatrix} 4 & -2 \\ 3 & 7 \end{bmatrix} \\ &= \begin{bmatrix} -6 & 0 \\ 12 & 3 \end{bmatrix} + \begin{bmatrix} 12 & -6 \\ 9 & 21 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -6 \\ 21 & 24 \end{bmatrix} \end{aligned}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Evaluate the expression using the Distributive Property.

$$2\left(\begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix} + \begin{bmatrix} -4 & 0 \\ -3 & 1 \end{bmatrix}\right)$$

In Example 5, you could add the two matrices first and then multiply the resulting matrix by 3. The result would be the same.

One important property of addition of real numbers is that the number 0 is the additive identity. That is, $c + 0 = c$ for any real number c. For matrices, a similar property holds. That is, if A is an $m \times n$ matrix and O is the $m \times n$ **zero matrix** consisting entirely of zeros, then $A + O = A$.

In other words, O is the **additive identity** for the set of all $m \times n$ matrices. For example, the following matrices are the additive identities for the sets of all 2×3 and 2×2 matrices.

$$O = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{2 \times 3 \text{ zero matrix}} \quad \text{and} \quad O = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{2 \times 2 \text{ zero matrix}}$$

The algebra of real numbers and the algebra of matrices have many similarities. For example, compare the following solutions.

REMARK Remember that matrices are denoted by capital letters. So, when you solve for X, you are solving for a *matrix* that makes the matrix equation true.

Real Numbers (Solve for *x*.)	**$m \times n$ Matrices (Solve for *X*.)**
$x + a = b$	$X + A = B$
$x + a + (-a) = b + (-a)$	$X + A + (-A) = B + (-A)$
$x + 0 = b - a$	$X + O = B - A$
$x = b - a$	$X = B - A$

The algebra of real numbers and the algebra of matrices also have important differences (see Example 10 and Exercises 77–82).

EXAMPLE 6 Solving a Matrix Equation

Solve for X in the equation $3X + A = B$, where

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}.$$

Solution Begin by solving the matrix equation for X to obtain

$$\begin{aligned} 3X + A &= B \\ 3X &= B - A \\ X &= \frac{1}{3}(B - A). \end{aligned}$$

Now, using the matrices A and B, you have

$$\begin{aligned} X &= \frac{1}{3}\left(\begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}\right) && \text{Substitute the matrices.} \\ &= \frac{1}{3}\begin{bmatrix} -4 & 6 \\ 2 & -2 \end{bmatrix} && \text{Subtract matrix } A \text{ from matrix } B. \\ &= \begin{bmatrix} -\frac{4}{3} & 2 \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix}. && \text{Multiply the resulting matrix by } \tfrac{1}{3}. \end{aligned}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve for X in the equation $2X - A = B$, where

$$A = \begin{bmatrix} 6 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & -1 \\ -2 & 5 \end{bmatrix}.$$

Matrix Multiplication

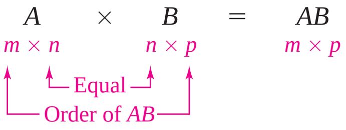
Another basic matrix operation is **matrix multiplication.** At first glance, the definition may seem unusual. You will see later, however, that this definition of the product of two matrices has many practical applications.

Definition of Matrix Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, then the product AB is an $m \times p$ matrix

$$AB = [c_{ij}]$$

where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdot \cdot \cdot + a_{in}b_{nj}$.



The definition of matrix multiplication indicates a *row-by-column* multiplication, where the entry in the ith row and jth column of the product AB is obtained by multiplying the entries in the ith row of A by the corresponding entries in the jth column of B and then adding the results. So, for the product of two matrices to be defined, the number of columns of the first matrix must equal the number of rows of the second matrix. That is, the middle two indices must be the same. The outside two indices give the order of the product, as shown at the left. The general pattern for matrix multiplication is as follows.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdot\cdot\cdot & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdot\cdot\cdot & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdot\cdot\cdot & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdot\cdot\cdot & a_{in} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdot\cdot\cdot & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdot\cdot\cdot & b_{1j} & \cdot\cdot\cdot & b_{1p} \\ b_{21} & b_{22} & \cdot\cdot\cdot & b_{2j} & \cdot\cdot\cdot & b_{2p} \\ b_{31} & b_{32} & \cdot\cdot\cdot & b_{3j} & \cdot\cdot\cdot & b_{3p} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdot\cdot\cdot & b_{nj} & \cdot\cdot\cdot & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdot\cdot\cdot & c_{1j} & \cdot\cdot\cdot & c_{1p} \\ c_{21} & c_{22} & \cdot\cdot\cdot & c_{2j} & \cdot\cdot\cdot & c_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{i1} & c_{i2} & \cdot\cdot\cdot & c_{ij} & \cdot\cdot\cdot & c_{ip} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdot\cdot\cdot & c_{mj} & \cdot\cdot\cdot & c_{mp} \end{bmatrix}$$

$$a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdot \cdot \cdot + a_{in}b_{nj} = c_{ij}$$

EXAMPLE 7 Finding the Product of Two Matrices

Find the product AB using $A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$.

Solution To find the entries of the product, multiply each row of A by each column of B.

$$AB = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)(-3) + (3)(-4) & (-1)(2) + (3)(1) \\ (4)(-3) + (-2)(-4) & (4)(2) + (-2)(1) \\ (5)(-3) + (0)(-4) & (5)(2) + (0)(1) \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{bmatrix}$$

REMARK In Example 7, the product AB is defined because the number of columns of A is equal to the number of rows of B. Also, note that the product AB has order 3×2.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the product AB using $A = \begin{bmatrix} -1 & 4 \\ 2 & 0 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 0 & 7 \end{bmatrix}$.

EXAMPLE 8 Finding the Product of Two Matrices

Find the product AB using $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 4 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}$.

Solution Note that the order of A is 2×3 and the order of B is 3×2. So, the product AB has order 2×2.

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 1 & 0 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1(-2) + 0(1) + 3(-1) & 1(4) + 0(0) + 3(1) \\ 2(-2) + (-1)(1) + (-2)(-1) & 2(4) + (-1)(0) + (-2)(1) \end{bmatrix} \\ &= \begin{bmatrix} -5 & 7 \\ -3 & 6 \end{bmatrix} \end{aligned}$$

✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the product AB using $A = \begin{bmatrix} 0 & 4 & -3 \\ 2 & 1 & 7 \\ 3 & -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 0 \\ 0 & -4 \\ 1 & 2 \end{bmatrix}$.

EXAMPLE 9 Patterns in Matrix Multiplication

a. $$\underset{2 \times 2}{\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}}\underset{2 \times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \underset{2 \times 2}{\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}}$$

b. $$\underset{3 \times 3}{\begin{bmatrix} 6 & 2 & 0 \\ 3 & -1 & 2 \\ 1 & 4 & 6 \end{bmatrix}}\underset{3 \times 1}{\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}} = \underset{3 \times 1}{\begin{bmatrix} 10 \\ -5 \\ -9 \end{bmatrix}}$$

c. The product AB for the following matrices is not defined.

$$A = \underset{3 \times 2}{\begin{bmatrix} -2 & 1 \\ 1 & -3 \\ 1 & 4 \end{bmatrix}} \quad \text{and} \quad B = \underset{3 \times 4}{\begin{bmatrix} -2 & 3 & 1 & 4 \\ 0 & 1 & -1 & 2 \\ 2 & -1 & 0 & 1 \end{bmatrix}}$$

✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find, if possible, the product AB using $A = \begin{bmatrix} 3 & 1 & 2 \\ 7 & 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 4 \\ 2 & -1 \end{bmatrix}$.

EXAMPLE 10 Patterns in Matrix Multiplication

a. $$\underset{1 \times 3}{\begin{bmatrix} 1 & -2 & -3 \end{bmatrix}}\underset{3 \times 1}{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}} = \underset{1 \times 1}{\begin{bmatrix} 1 \end{bmatrix}}$$

b. $$\underset{3 \times 1}{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}}\underset{1 \times 3}{\begin{bmatrix} 1 & -2 & -3 \end{bmatrix}} = \underset{3 \times 3}{\begin{bmatrix} 2 & -4 & -6 \\ -1 & 2 & 3 \\ 1 & -2 & -3 \end{bmatrix}}$$

✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find AB and BA using $A = \begin{bmatrix} 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

REMARK In Example 10, note that the two products are different. Even when both AB and BA are defined, matrix multiplication is not, in general, commutative. That is, for most matrices, $AB \neq BA$. This is one way in which the algebra of real numbers and the algebra of matrices differ.

EXAMPLE 11 Squaring a Matrix

Find A^2, where $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$. (*Note:* $A^2 = AA$.)

Solution

$$A^2 = AA = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3(3) + 1(-1) & 3(1) + 1(2) \\ -1(3) + 2(-1) & -1(1) + 2(2) \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find A^2, where $A = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$.

Properties of Matrix Multiplication

Let A, B, and C be matrices and let c be a scalar.

1. $A(BC) = (AB)C$ Associative Property of Matrix Multiplication

2. $A(B + C) = AB + AC$ Distributive Property

3. $(A + B)C = AC + BC$ Distributive Property

4. $c(AB) = (cA)B = A(cB)$ Associative Property of Scalar Multiplication

Definition of Identity Matrix

The $n \times n$ matrix that consists of 1's on its main diagonal and 0's elsewhere is called the **identity matrix of order $n \times n$** and is denoted by

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \quad \text{Identity matrix}$$

Note that an identity matrix must be *square*. When the order is understood to be $n \times n$, you can denote I_n simply by I.

If A is an $n \times n$ matrix, then the identity matrix has the property that $AI_n = A$ and $I_nA = A$. For example,

$$\begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix} \quad AI = A$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix}. \quad IA = A$$

Many real-life applications of linear systems involve enormous numbers of equations and variables. For example, a flight crew scheduling problem for American Airlines required the manipulation of matrices with 837 rows and 12,753,313 columns. *(Source: Very Large-Scale Linear Programming. A Case Study in Combining Interior Point and Simplex Methods, Bixby, Robert E., et al., Operations Research, 40, no. 5)*

Applications

Matrix multiplication can be used to represent a system of linear equations. Note how the system below can be written as the matrix equation $AX = B$, where A is the *coefficient matrix* of the system and X and B are column matrices. The column matrix B is also called a *constant matrix*. Its entries are the constant terms in the system of equations.

System

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

Matrix Equation $AX = B$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} \underset{\times}{} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} \underset{=}{=} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{B}$$

In Example 12, $[A \vdots B]$ represents the augmented matrix formed when matrix B is *adjoined* to matrix A. Also, $[I \vdots X]$ represents the reduced row-echelon form of the augmented matrix that yields the solution of the system.

EXAMPLE 12 Solving a System of Linear Equations

For the system of linear equations, (a) write the system as a matrix equation, $AX = B$, and (b) use Gauss-Jordan elimination on $[A \vdots B]$ to solve for the matrix X.

$$\begin{cases} x_1 - 2x_2 + x_3 = -4 \\ x_2 + 2x_3 = 4 \\ 2x_1 + 3x_2 - 2x_3 = 2 \end{cases}$$

Solution

a. In matrix form, $AX = B$, the system can be written as follows.

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}$$

b. The augmented matrix is formed by adjoining matrix B to matrix A.

$$[A \vdots B] = \left[\begin{array}{ccccc} 1 & -2 & 1 & \vdots & -4 \\ 0 & 1 & 2 & \vdots & 4 \\ 2 & 3 & -2 & \vdots & 2 \end{array}\right]$$

Using Gauss-Jordan elimination, you can rewrite this matrix as

$$[I \vdots X] = \left[\begin{array}{ccccc} 1 & 0 & 0 & \vdots & -1 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 1 & \vdots & 1 \end{array}\right].$$

So, the solution of the matrix equation is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

For the system of linear equations, (a) write the system as a matrix equation, $AX = B$, and (b) use Gauss-Jordan elimination on $[A \vdots B]$ to solve for the matrix X.

$$\begin{cases} -2x_1 - 3x_2 = -4 \\ 6x_1 + x_2 = -36 \end{cases}$$

© Sean Locke/iStockphoto.com

EXAMPLE 13 Softball Team Expenses

Two softball teams submit equipment lists to their sponsors.

Equipment	**Women's Team**	**Men's Team**
Bats	12	15
Balls	45	38
Gloves	15	17

Each bat costs \$80, each ball costs \$6, and each glove costs \$60. Use matrices to find the total cost of equipment for each team.

Solution The equipment lists E and the costs per item C can be written in matrix form as

$$E = \begin{bmatrix} 12 & 15 \\ 45 & 38 \\ 15 & 17 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 80 & 6 & 60 \end{bmatrix}.$$

The total cost of equipment for each team is given by the product

$$\begin{aligned} CE &= \begin{bmatrix} 80 & 6 & 60 \end{bmatrix} \begin{bmatrix} 12 & 15 \\ 45 & 38 \\ 15 & 17 \end{bmatrix} \\ &= \begin{bmatrix} 80(12) + 6(45) + 60(15) & 80(15) + 6(38) + 60(17) \end{bmatrix} \\ &= \begin{bmatrix} 2130 & 2448 \end{bmatrix}. \end{aligned}$$

So, the total cost of equipment for the women's team is \$2130, and the total cost of equipment for the men's team is \$2448.

REMARK Notice in Example 13 that you cannot find the total cost using the product EC because EC is not defined. That is, the number of columns of E (2 columns) does not equal the number of rows of C (1 row).

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Repeat Example 13 when each bat costs \$100, each ball costs \$7, and each glove costs \$65.

Summarize (Section 7.2)

1. State the conditions under which two matrices are equal *(page 509)*. For an example of the equality of matrices, see Example 1.
2. State the definition of matrix addition *(page 510)*. For an example of matrix addition, see Example 2.
3. State the definition of scalar multiplication *(page 510)*. For an example of scalar multiplication, see Example 3.
4. List the properties of matrix addition and scalar multiplication *(page 512)*. For examples of using these properties, see Examples 4, 5, and 6.
5. State the definition of matrix multiplication *(page 514)*. For examples of matrix multiplication, see Examples 7–11.
6. Describe applications of matrix multiplication *(pages 517 and 518, Examples 12 and 13)*.

7.2 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: **In Exercises 1–4, fill in the blanks.**

1. Two matrices are ________ when their corresponding entries are equal.

2. When performing matrix operations, real numbers are often referred to as ________.

3. A matrix consisting entirely of zeros is called a ________ matrix and is denoted by ________.

4. The $n \times n$ matrix consisting of 1's on its main diagonal and 0's elsewhere is called the ________ matrix of order $n \times n$.

Skills and Applications

Equality of Matrices

In Exercises 5–8, find x and y.

5. $\begin{bmatrix} x & -2 \\ 7 & y \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ 7 & 22 \end{bmatrix}$

6. $\begin{bmatrix} -5 & x \\ y & 8 \end{bmatrix} = \begin{bmatrix} -5 & 13 \\ 12 & 8 \end{bmatrix}$

7. $\begin{bmatrix} 16 & 4 & 5 & 4 \\ -3 & 13 & 15 & 6 \\ 0 & 2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 4 & 2x+1 & 4 \\ -3 & 13 & 15 & 3x \\ 0 & 2 & 3y-5 & 0 \end{bmatrix}$

8. $\begin{bmatrix} x+2 & 8 & -3 \\ 1 & 2y & 2x \\ 7 & -2 & y+2 \end{bmatrix} = \begin{bmatrix} 2x+6 & 8 & -3 \\ 1 & 18 & -8 \\ 7 & -2 & 11 \end{bmatrix}$

Operations with Matrices

In Exercises 9–16, if possible, find (a) $A + B$, (b) $A - B$, (c) $3A$, and (d) $3A - 2B$.

9. $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix}$

10. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & -2 \\ 4 & 2 \end{bmatrix}$

11. $A = \begin{bmatrix} 8 & -1 \\ 2 & 3 \\ -4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 6 \\ -1 & -5 \\ 1 & 10 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 6 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 0 & -5 \\ -3 & 4 & -7 \end{bmatrix}$

13. $A = \begin{bmatrix} 4 & 5 & -1 & 3 & 4 \\ 1 & 2 & -2 & -1 & 0 \end{bmatrix},$

$B = \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ -6 & 8 & 2 & -3 & -7 \end{bmatrix}$

14. $A = \begin{bmatrix} -1 & 4 & 0 \\ 3 & -2 & 2 \\ 5 & 4 & -1 \\ 0 & 8 & -6 \\ -4 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 5 & 1 \\ 2 & -4 & -7 \\ 10 & -9 & -1 \\ 3 & 2 & -4 \\ 0 & 1 & -2 \end{bmatrix}$

15. $A = \begin{bmatrix} 6 & 0 & 3 \\ -1 & -4 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & -1 \\ 4 & -3 \end{bmatrix}$

16. $A = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 6 & 2 \end{bmatrix}$

Evaluating an Expression

In Exercises 17–22, evaluate the expression.

17. $\begin{bmatrix} -5 & 0 \\ 3 & -6 \end{bmatrix} + \begin{bmatrix} 7 & 1 \\ -2 & -1 \end{bmatrix} + \begin{bmatrix} -10 & -8 \\ 14 & 6 \end{bmatrix}$

18. $\begin{bmatrix} 6 & 8 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 5 \\ -3 & -1 \end{bmatrix} + \begin{bmatrix} -11 & -7 \\ 2 & -1 \end{bmatrix}$

19. $4\left(\begin{bmatrix} -4 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -2 \\ 3 & -6 & 0 \end{bmatrix}\right)$

20. $\frac{1}{2}(\begin{bmatrix} 5 & -2 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 14 & 6 & -18 & 9 \end{bmatrix})$

21. $-3\left(\begin{bmatrix} 0 & -3 \\ 7 & 2 \end{bmatrix} + \begin{bmatrix} -6 & 3 \\ 8 & 1 \end{bmatrix}\right) - 2\begin{bmatrix} 4 & -4 \\ 7 & -9 \end{bmatrix}$

22. $-1\begin{bmatrix} 4 & 11 \\ -2 & -1 \\ 9 & 3 \end{bmatrix} + \frac{1}{6}\left(\begin{bmatrix} -5 & -1 \\ 3 & 4 \\ 0 & 13 \end{bmatrix} + \begin{bmatrix} 7 & 5 \\ -9 & -1 \\ 6 & -1 \end{bmatrix}\right)$

Operations with Matrices

In Exercises 23–26, use the matrix capabilities of a graphing utility to evaluate the expression.

23. $\frac{11}{25}\begin{bmatrix} 2 & 5 \\ -1 & -4 \end{bmatrix} + 6\begin{bmatrix} -3 & 0 \\ 2 & 2 \end{bmatrix}$

24. $55\left(\begin{bmatrix} 14 & -11 \\ -22 & 19 \end{bmatrix} + \begin{bmatrix} -22 & 20 \\ 13 & 6 \end{bmatrix}\right)$

25. $-1\begin{bmatrix} 3.211 & 6.829 \\ -1.004 & 4.914 \\ 0.055 & -3.889 \end{bmatrix} - \begin{bmatrix} -1.630 & -3.090 \\ 5.256 & 8.335 \\ -9.768 & 4.251 \end{bmatrix}$

26. $-1\begin{bmatrix} 10 & 15 \\ -20 & 10 \\ 12 & 4 \end{bmatrix} + \frac{1}{8}\left(\begin{bmatrix} -13 & 11 \\ 7 & 0 \\ 6 & 9 \end{bmatrix} + \begin{bmatrix} -3 & 13 \\ -3 & 8 \\ -14 & 15 \end{bmatrix}\right)$

Solving a Matrix Equation

In Exercises 27–34, solve for X in the equation, where

$$A = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix}.$$

27. $X = 2A + 2B$

28. $X = 3A - 2B$

29. $2X = A + B$

30. $2X = 2A - B$

31. $2X + 3A = B$

32. $3X - 4A = 2B$

33. $2A + 4B = -2X$

34. $5A - 6B = -3X$

Finding the Product of Two Matrices **In Exercises 35–42, if possible, find *AB* and state the order of the result.**

35. $A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \\ 1 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{bmatrix}$

36. $A = \begin{bmatrix} 0 & -1 & 2 \\ 6 & 0 & 3 \\ 7 & -1 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ 4 & -5 \\ 1 & 6 \end{bmatrix}$

37. $A = \begin{bmatrix} -1 & 6 \\ -4 & 5 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 0 & 9 \end{bmatrix}$

38. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

39. $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & -\frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

40. $A = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 0 & -3 \\ 0 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & -11 & 4 \\ 8 & 16 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

41. $A = \begin{bmatrix} 10 \\ 12 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & -2 & 1 & 6 \end{bmatrix}$

42. $A = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 6 & 13 & 8 & -17 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 6 \\ 4 & 2 \end{bmatrix}$

Matrix Multiplication **In Exercises 43–46, use the matrix capabilities of a graphing utility to find *AB*, if possible.**

43. $A = \begin{bmatrix} 7 & 5 & -4 \\ -2 & 5 & 1 \\ 10 & -4 & -7 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -2 & 3 \\ 8 & 1 & 4 \\ -4 & 2 & -8 \end{bmatrix}$

44. $A = \begin{bmatrix} 11 & -12 & 4 \\ 14 & 10 & 12 \\ 6 & -2 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 12 & 10 \\ -5 & 12 \\ 15 & 16 \end{bmatrix}$

45. $A = \begin{bmatrix} -3 & 8 & -6 & 8 \\ -12 & 15 & 9 & 6 \\ 5 & -1 & 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 & 6 \\ 24 & 15 & 14 \\ 16 & 10 & 21 \\ 8 & -4 & 10 \end{bmatrix}$

46. $A = \begin{bmatrix} -2 & 4 & 8 \\ 21 & 5 & 6 \\ 13 & 2 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ -7 & 15 \\ 32 & 14 \\ 0.5 & 1.6 \end{bmatrix}$

Operations with Matrices **In Exercises 47–52, if possible, find (a) *AB*, (b) *BA*, and (c) A^2.**

47. $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix}$

48. $A = \begin{bmatrix} 6 & 3 \\ -2 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 0 \\ 2 & 4 \end{bmatrix}$

49. $A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$

50. $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$

51. $A = \begin{bmatrix} 7 \\ 8 \\ -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$

52. $A = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

Operations with Matrices **In Exercises 53–56, evaluate the expression. Use the matrix capabilities of a graphing utility to verify your answer.**

53. $\begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$

54. $-3\left(\begin{bmatrix} 6 & 5 & -1 \\ 1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ -1 & -3 \\ 4 & 1 \end{bmatrix}\right)$

55. $\begin{bmatrix} 0 & 2 & -2 \\ 4 & 1 & 2 \end{bmatrix}\left(\begin{bmatrix} 4 & 0 \\ 0 & -1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ -3 & 5 \\ 0 & -3 \end{bmatrix}\right)$

56. $\begin{bmatrix} 3 \\ -1 \\ 5 \\ 7 \end{bmatrix}\left(\begin{bmatrix} 5 & -6 \end{bmatrix} + \begin{bmatrix} 7 & -1 \end{bmatrix} + \begin{bmatrix} -8 & 9 \end{bmatrix}\right)$

Solving a System of Linear Equations **In Exercises 57–64, (a) write the system of linear equations as a matrix equation, $AX = B$, and (b) use Gauss-Jordan elimination on $[A \vdots B]$ to solve for the matrix *X*.**

57. $\begin{cases} -x_1 + x_2 = 4 \\ -2x_1 + x_2 = 0 \end{cases}$

58. $\begin{cases} 2x_1 + 3x_2 = 5 \\ x_1 + 4x_2 = 10 \end{cases}$

59. $\begin{cases} -2x_1 - 3x_2 = -4 \\ 6x_1 + x_2 = -36 \end{cases}$

60. $\begin{cases} -4x_1 + 9x_2 = -13 \\ x_1 - 3x_2 = 12 \end{cases}$

61. $\begin{cases} x_1 - 2x_2 + 3x_3 = 9 \\ -x_1 + 3x_2 - x_3 = -6 \\ 2x_1 - 5x_2 + 5x_3 = 17 \end{cases}$

62. $\begin{cases} x_1 + x_2 - 3x_3 = -1 \\ -x_1 + 2x_2 = 1 \\ x_1 - x_2 + x_3 = 2 \end{cases}$

63. $\begin{cases} x_1 - 5x_2 + 2x_3 = -20 \\ -3x_1 + x_2 - x_3 = 8 \\ -2x_2 + 5x_3 = -16 \end{cases}$

64. $\begin{cases} x_1 - x_2 + 4x_3 = 17 \\ x_1 + 3x_2 = -11 \\ -6x_2 + 5x_3 = 40 \end{cases}$

65. Manufacturing A corporation has four factories, each of which manufactures sport utility vehicles and pickup trucks. The production levels are represented by A.

$$A = \begin{array}{ccccll} \multicolumn{4}{c}{\overbrace{\text{Factory}}} & & \\ 1 & 2 & 3 & 4 & & \\ 100 & 90 & 70 & 30 & \text{SUV} & \text{Vehicle} \\ 40 & 20 & 60 & 60 & \text{Pickup} & \text{Type} \end{array}$$

Find the production levels when production is increased by 10%.

66. Vacation Packages A vacation service has identified four resort hotels with a special all-inclusive package. The quoted room rates are for double and family (maximum of four people) occupancy for 5 days and 4 nights. The current rates for the two types of rooms at the four hotels are represented by A.

$$A = \begin{array}{ccccll} \text{Hotel} & \text{Hotel} & \text{Hotel} & \text{Hotel} & & \\ w & x & y & z & & \\ 615 & 670 & 740 & 990 & \text{Double} & \text{Occupancy} \\ 995 & 1030 & 1180 & 1105 & \text{Family} & \end{array}$$

Room rates are guaranteed not to increase by more than 12% by next season. What is the maximum rate per package per hotel?

67. Agriculture A fruit grower raised two crops, apples and peaches. Each of these crops is shipped to three different outlets. The shipment levels are represented by A.

$$A = \begin{array}{cccll} \multicolumn{3}{c}{\overbrace{\text{Outlet}}} & & \\ 1 & 2 & 3 & & \\ 125 & 100 & 75 & \text{Apples} & \text{Crop} \\ 100 & 175 & 125 & \text{Peaches} & \end{array}$$

The profits per unit are represented by the matrix

$B = [\$3.50 \quad \$6.00]$.

Compute BA and interpret the result.

68. Inventory A company sells five models of computers through three retail outlets. The inventories are represented by S. The wholesale and retail prices are represented by T. Compute ST and interpret the result.

$$S = \begin{array}{cccccll} \multicolumn{5}{c}{\overbrace{\text{Model}}} & & \\ A & B & C & D & E & & \\ 3 & 2 & 2 & 3 & 0 & 1 & \\ 0 & 2 & 3 & 4 & 3 & 2 & \text{Outlet} \\ 4 & 2 & 1 & 3 & 2 & 3 & \end{array}$$

$$T = \begin{array}{ccll} \multicolumn{2}{c}{\overbrace{\text{Price}}} & & \\ \text{Wholesale} & \text{Retail} & & \\ \$840 & \$1100 & A & \\ \$1200 & \$1350 & B & \\ \$1450 & \$1650 & C & \text{Model} \\ \$2650 & \$3000 & D & \\ \$3050 & \$3200 & E & \end{array}$$

69. Revenue An electronics manufacturer produces three models of LCD televisions, which are shipped to two warehouses. The shipment levels are represented by A.

$$A = \begin{array}{ccll} \multicolumn{2}{c}{\overbrace{\text{Warehouse}}} & & \\ 1 & 2 & & \\ 5{,}000 & 4{,}000 & A & \\ 6{,}000 & 10{,}000 & B & \text{Model} \\ 8{,}000 & 5{,}000 & C & \end{array}$$

The prices per unit are represented by the matrix

$B = [\$699.95 \quad \$899.95 \quad \$1099.95]$.

Compute BA and interpret the result.

70. Labor/Wage Requirements A company that manufactures boats has the following labor-hour and wage requirements. Compute ST and interpret the result.

Labor per boat

$$S = \begin{array}{cccll} \multicolumn{3}{c}{\overbrace{\text{Department}}} & & \\ \text{Cutting} & \text{Assembly} & \text{Packaging} & & \\ 1.0\text{ h} & 0.5\text{ h} & 0.2\text{ h} & \text{Small} & \\ 1.6\text{ h} & 1.0\text{ h} & 0.2\text{ h} & \text{Medium} & \text{Boat size} \\ 2.5\text{ h} & 2.0\text{ h} & 1.4\text{ h} & \text{Large} & \end{array}$$

Wages per hour

$$T = \begin{array}{ccll} \multicolumn{2}{c}{\overbrace{\text{Plant}}} & & \\ A & B & & \\ \$15 & \$13 & \text{Cutting} & \\ \$12 & \$11 & \text{Assembly} & \text{Department} \\ \$11 & \$10 & \text{Packaging} & \end{array}$$

71. Profit At a local dairy mart, the numbers of gallons of skim milk, 2% milk, and whole milk sold over the weekend are represented by A.

$$A = \begin{array}{cccl} \text{Skim milk} & 2\%\text{ milk} & \text{Whole milk} & \\ 40 & 64 & 52 & \text{Friday} \\ 60 & 82 & 76 & \text{Saturday} \\ 76 & 96 & 84 & \text{Sunday} \end{array}$$

The selling prices (in dollars per gallon) and the profits (in dollars per gallon) for the three types of milk sold by the dairy mart are represented by B.

$$B = \begin{array}{ccl} \text{Selling price} & \text{Profit} & \\ \$3.45 & \$1.20 & \text{Skim milk} \\ \$3.65 & \$1.30 & 2\%\text{ milk} \\ \$3.85 & \$1.45 & \text{Whole milk} \end{array}$$

(a) Compute AB and interpret the result.

(b) Find the dairy mart's total profit from milk sales for the weekend.

72. Voting Preferences The matrix

$$P = \begin{array}{c} \overbrace{\begin{array}{ccc} \text{R} & \text{D} & \text{I} \end{array}}^{\text{From}} \\ \left[\begin{array}{ccc} 0.6 & 0.1 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.2 & 0.8 \end{array}\right] \end{array} \left.\begin{array}{l} \text{R} \\ \text{D} \\ \text{I} \end{array}\right\} \text{To}$$

is called a *stochastic matrix*. Each entry p_{ij} $(i \neq j)$ represents the proportion of the voting population that changes from party i to party j, and p_{ii} represents the proportion that remains loyal to the party from one election to the next. Compute and interpret P^2.

73. Voting Preferences Use a graphing utility to find P^3, P^4, P^5, P^6, P^7, and P^8 for the matrix given in Exercise 72. Can you detect a pattern as P is raised to higher powers?

74. Exercise

The numbers of calories burned by individuals of different body weights while performing different types of exercises for a one-hour time period are represented by A.

$$A = \begin{array}{c} \overbrace{\begin{array}{cc} \text{130-lb person} & \text{155-lb person} \end{array}}^{\text{Calories burned}} \\ \left[\begin{array}{cc} 472 & 563 \\ 590 & 704 \\ 177 & 211 \end{array}\right] \end{array} \begin{array}{l} \text{Basketball} \\ \text{Jumping rope} \\ \text{Weight lifting} \end{array}$$

(a) A 130-pound person and a 155-pound person played basketball for 2 hours, jumped rope for 15 minutes, and lifted weights for 30 minutes. Organize the times spent exercising in a matrix B.

(b) Compute BA and interpret the result.

Exploration

True or False? **In Exercises 75 and 76, determine whether the statement is true or false. Justify your answer.**

75. Two matrices can be added only when they have the same order.

76. Matrix multiplication is commutative.

Think About It **In Exercises 77–80, use the matrices**

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}.$$

77. Show that $(A + B)^2 \neq A^2 + 2AB + B^2$.

78. Show that $(A - B)^2 \neq A^2 - 2AB + B^2$.

79. Show that $(A + B)(A - B) \neq A^2 - B^2$.

80. Show that $(A + B)^2 = A^2 + AB + BA + B^2$.

81. Think About It If a, b, and c are real numbers such that $c \neq 0$ and $ac = bc$, then $a = b$. However, if A, B, and C are nonzero matrices such that $AC = BC$, then A is *not necessarily* equal to B. Illustrate this using the following matrices.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$$

82. Think About It If a and b are real numbers such that $ab = 0$, then $a = 0$ or $b = 0$. However, if A and B are matrices such that $AB = O$, it is *not necessarily* true that $A = O$ or $B = O$. Illustrate this using the following matrices.

$$A = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

83. Conjecture Let A and B be unequal diagonal matrices of the same order. (A **diagonal matrix** is a square matrix in which each entry not on the main diagonal is zero.) Determine the products AB for several pairs of such matrices. Make a conjecture about a quick rule for such products.

84. Matrices with Complex Entries Let $i = \sqrt{-1}$ and let

$$A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

(a) Find A^2, A^3, and A^4. Identify any similarities with i^2, i^3, and i^4.

(b) Find and identify B^2.

85. Finding Matrices Find two matrices A and B such that $AB = BA$.

86. HOW DO YOU SEE IT? A corporation has three factories, each of which manufactures acoustic guitars and electric guitars. The production levels are represented by A.

$$A = \begin{array}{c} \overbrace{\begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array}}^{\text{Factory}} \\ \left[\begin{array}{ccc} 70 & 50 & 25 \\ 35 & 100 & 70 \end{array}\right] \end{array} \left.\begin{array}{l} \text{Acoustic} \\ \text{Electric} \end{array}\right\} \text{Guitar type}$$

(a) Interpret the value of a_{22}.

(b) How could you find the production levels when production is increased by 20%?

(c) Each acoustic guitar sells for \$80 and each electric guitar sells for \$120. How could you use matrices to find the total sales value of the guitars produced at each factory?

Simone van den Berg/Shutterstock.com

7.3 The Inverse of a Square Matrix

You can use inverse matrices to model and solve real-life problems. For instance, in Exercises 65–68 on pages 530 and 531, you will use an inverse matrix to find the currents in a circuit.

- **Verify that two matrices are inverses of each other.**
- **Use Gauss-Jordan elimination to find the inverses of matrices.**
- **Use a formula to find the inverses of 2 × 2 matrices.**
- **Use inverse matrices to solve systems of linear equations.**

The Inverse of a Matrix

This section further develops the algebra of matrices. To begin, consider the real number equation $ax = b$. To solve this equation for x, multiply each side of the equation by a^{-1} (provided that $a \neq 0$).

$$\begin{aligned} ax &= b \\ (a^{-1}a)x &= a^{-1}b \\ (1)x &= a^{-1}b \\ x &= a^{-1}b \end{aligned}$$

The number a^{-1} is called the *multiplicative inverse* of a because $a^{-1}a = 1$. The definition of the multiplicative **inverse of a matrix** is similar.

Definition of the Inverse of a Square Matrix

Let A be an $n \times n$ matrix and let I_n be the $n \times n$ identity matrix. If there exists a matrix A^{-1} such that

$$AA^{-1} = I_n = A^{-1}A$$

then A^{-1} is called the **inverse** of A. The symbol A^{-1} is read "A inverse."

EXAMPLE 1 The Inverse of a Matrix

Show that B is the inverse of A, where

$$A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}.$$

Solution To show that B is the inverse of A, show that $AB = I = BA$, as follows.

$$AB = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

As you can see, $AB = I = BA$. This is an example of a square matrix that has an inverse. Note that not all square matrices have inverses.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Show that B is the inverse of A, where

$$A = \begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & -1 \\ -3 & -2 \end{bmatrix}.$$ ■

Recall that it is not always true that $AB = BA$, even when both products are defined. However, if A and B are both square matrices and $AB = I_n$, then it can be shown that $BA = I_n$. So, in Example 1, you need only to check that $AB = I_2$.

Ingrid Prats/Shutterstock.com

One real-life application of inverse matrices is in the study of beam deflection. In a simply supported elastic beam subjected to multiple forces, deflection **d** is related to force **w** by the matrix equation

$$\mathbf{d} = F\mathbf{w}$$

where F is a *flexibility matrix* whose entries depend on the material of the beam. The inverse of the flexibility matrix, F^{-1}, is called the *stiffness matrix*.

Finding Inverse Matrices

If a matrix A has an inverse, then A is called **invertible** (or **nonsingular**); otherwise, A is called **singular.** A nonsquare matrix cannot have an inverse. To see this, note that when A is of order $m \times n$ and B is of order $n \times m$ (where $m \neq n$), the products AB and BA are of different orders and so cannot be equal to each other. Not all square matrices have inverses (see the matrix at the bottom of page 526). When, however, a matrix does have an inverse, that inverse is unique. Example 2 shows how to use a system of equations to find the inverse of a matrix.

EXAMPLE 2 Finding the Inverse of a Matrix

Find the inverse of

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}.$$

Solution To find the inverse of A, solve the matrix equation $AX = I$ for X.

$$\overset{A}{\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}} \overset{X}{\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}} \overset{=}{=} \overset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \qquad \text{Write matrix equation.}$$

$$\begin{bmatrix} x_{11} + 4x_{21} & x_{12} + 4x_{22} \\ -x_{11} - 3x_{21} & -x_{12} - 3x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{Multiply.}$$

Equating corresponding entries, you obtain two systems of linear equations.

$$\begin{cases} x_{11} + 4x_{21} = 1 \\ -x_{11} - 3x_{21} = 0 \end{cases} \qquad \begin{cases} x_{12} + 4x_{22} = 0 \\ -x_{12} - 3x_{22} = 1 \end{cases}$$

Solve the first system using elementary row operations to determine that

$$x_{11} = -3 \quad \text{and} \quad x_{21} = 1.$$

From the second system, you can determine that

$$x_{12} = -4 \quad \text{and} \quad x_{22} = 1.$$

So, the inverse of A is

$$\begin{aligned} X &= A^{-1} \\ &= \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}. \end{aligned}$$

You can use matrix multiplication to check this result.

Check

$$AA^{-1} = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$A^{-1}A = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the inverse of

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}.$$

nostal6ie/Shutterstock.com

In Example 2, note that the two systems of linear equations have the *same coefficient matrix A*. Rather than solve the two systems represented by

$$\left[\begin{array}{rr|r} 1 & 4 & 1 \\ -1 & -3 & 0 \end{array}\right]$$

and

$$\left[\begin{array}{rr|r} 1 & 4 & 0 \\ -1 & -3 & 1 \end{array}\right]$$

separately, you can solve them *simultaneously* by *adjoining* the identity matrix to the coefficient matrix to obtain

$$\begin{array}{c} \begin{array}{cc} \quad A \quad & \quad\quad I \end{array} \\ \left[\begin{array}{rr|rr} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right]. \end{array}$$

This "doubly augmented" matrix can be represented as

$$[A \ \vdots\ I].$$

By applying Gauss-Jordan elimination to this matrix, you can solve *both* systems with a single elimination process.

TECHNOLOGY Most graphing utilities can find the inverse of a square matrix. To do so, you may have to use the inverse key $\boxed{x^{-1}}$. Consult the user's guide for your graphing utility for specific keystrokes.

$$\begin{array}{r} \\ R_1 + R_2 \rightarrow \\ -4R_2 + R_1 \rightarrow \end{array} \begin{array}{c} \left[\begin{array}{rr|rr} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right] \\ \left[\begin{array}{rr|rr} 1 & 4 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right] \\ \left[\begin{array}{rr|rr} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 1 \end{array}\right] \end{array}$$

So, from the "doubly augmented" matrix $[A \ \vdots\ I]$, you obtain the matrix $[I \ \vdots\ A^{-1}]$.

$$\begin{array}{c} \begin{array}{cc} \quad A \quad & \quad\quad I \end{array} \\ \left[\begin{array}{rr|rr} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right] \end{array} \Rightarrow \begin{array}{c} \begin{array}{cc} \quad I \quad & \quad\quad A^{-1} \end{array} \\ \left[\begin{array}{rr|rr} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 1 \end{array}\right] \end{array}$$

This procedure (or algorithm) works for any square matrix that has an inverse.

Finding an Inverse Matrix

Let A be a square matrix of order $n \times n$.

1. Write the $n \times 2n$ matrix that consists of the given matrix A on the left and the $n \times n$ identity matrix I on the right to obtain

   $$[A \ \vdots\ I].$$

2. If possible, row reduce A to I using elementary row operations on the *entire* matrix

   $$[A \ \vdots\ I].$$

   The result will be the matrix

   $$[I \ \vdots\ A^{-1}].$$

   If this is not possible, then A is not invertible.

3. Check your work by multiplying to see that

   $$AA^{-1} = I = A^{-1}A.$$

EXAMPLE 3 Finding the Inverse of a Matrix

Find the inverse of

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}.$$

Solution Begin by adjoining the identity matrix to A to form the matrix

$$[A \ \vdots \ I] = \left[\begin{array}{rrrcrrr} 1 & -1 & 0 & \vdots & 1 & 0 & 0 \\ 1 & 0 & -1 & \vdots & 0 & 1 & 0 \\ 6 & -2 & -3 & \vdots & 0 & 0 & 1 \end{array}\right].$$

Use elementary row operations to obtain the form $[I \ \vdots \ A^{-1}]$, as follows.

$$\begin{array}{r} \\ -R_1 + R_2 \rightarrow \\ -6R_1 + R_3 \rightarrow \end{array}\left[\begin{array}{rrrcrrr} 1 & -1 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 1 & -1 & \vdots & -1 & 1 & 0 \\ 0 & 4 & -3 & \vdots & -6 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} R_2 + R_1 \rightarrow \\ \\ -4R_2 + R_3 \rightarrow \end{array}\left[\begin{array}{rrrcrrr} 1 & 0 & -1 & \vdots & 0 & 1 & 0 \\ 0 & 1 & -1 & \vdots & -1 & 1 & 0 \\ 0 & 0 & 1 & \vdots & -2 & -4 & 1 \end{array}\right]$$

$$\begin{array}{r} R_3 + R_1 \rightarrow \\ R_3 + R_2 \rightarrow \\ \\ \end{array}\left[\begin{array}{rrrcrrr} 1 & 0 & 0 & \vdots & -2 & -3 & 1 \\ 0 & 1 & 0 & \vdots & -3 & -3 & 1 \\ 0 & 0 & 1 & \vdots & -2 & -4 & 1 \end{array}\right] = [I \ \vdots \ A^{-1}]$$

So, the matrix A is invertible and its inverse is $A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}.$

Confirm this result by multiplying A and A^{-1} to obtain I, as follows.

Check

$$AA^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

REMARK Be sure to check your solution because it is easy to make algebraic errors when using elementary row operations.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the inverse of

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 0 & -1 & 2 \\ 1 & -2 & 0 \end{bmatrix}.$$

The process shown in Example 3 applies to any $n \times n$ matrix A. When using this algorithm, if the matrix A does not reduce to the identity matrix, then A does not have an inverse. For instance, the following matrix has no inverse.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}$$

To confirm that matrix A above has no inverse, adjoin the identity matrix to A to form $[A \ \vdots \ I]$ and perform elementary row operations on the matrix. After doing so, you will see that it is impossible to obtain the identity matrix I on the left. So, A is not invertible.

The Inverse of a 2 × 2 Matrix

Using Gauss-Jordan elimination to find the inverse of a matrix works well (even as a computer technique) for matrices of order 3×3 or greater. For 2×2 matrices, however, many people prefer to use a formula for the inverse rather than Gauss-Jordan elimination. This simple formula, which works *only* for 2×2 matrices, is explained as follows. If A is a 2×2 matrix given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then A is invertible if and only if $ad - bc \neq 0$. Moreover, if $ad - bc \neq 0$, then the inverse is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Formula for the inverse of a 2×2 matrix

The denominator $ad - bc$ is called the **determinant** of the 2×2 matrix A. You will study determinants in the next section.

EXAMPLE 4 Finding the Inverse of a 2 × 2 Matrix

If possible, find the inverse of each matrix.

a. $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$ **b.** $B = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$

Solution

a. For the matrix A, apply the formula for the inverse of a 2×2 matrix to obtain

$$ad - bc = 3(2) - (-1)(-2) = 4.$$

Because this quantity is not zero, the matrix is invertible. The inverse is formed by interchanging the entries on the main diagonal, changing the signs of the other two entries, and multiplying by the scalar $\frac{1}{4}$, as follows.

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Formula for the inverse of a 2×2 matrix

$$= \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$

Substitute for a, b, c, d, and the determinant.

$$= \begin{bmatrix} \frac{1}{4}(2) & \frac{1}{4}(1) \\ \frac{1}{4}(2) & \frac{1}{4}(3) \end{bmatrix}$$

Multiply by the scalar $\frac{1}{4}$.

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

Simplify.

b. For the matrix B, you have

$$ad - bc = 3(2) - (-1)(-6) = 0.$$

Because $ad - bc = 0$, B is not invertible.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

If possible, find the inverse of $A = \begin{bmatrix} 5 & -1 \\ 3 & 4 \end{bmatrix}$.

Systems of Linear Equations

You know that a system of linear equations can have exactly one solution, infinitely many solutions, or no solution. If the coefficient matrix A of a *square* system (a system that has the same number of equations as variables) is invertible, then the system has a unique solution, which is defined as follows.

A System of Equations with a Unique Solution

If A is an invertible matrix, then the system of linear equations represented by $AX = B$ has a unique solution given by $X = A^{-1}B$.

EXAMPLE 5 Solving a System Using an Inverse Matrix

Use an inverse matrix to solve the system

$$\begin{cases} x + y + z = 10{,}000 \\ 0.06x + 0.075y + 0.095z = 730. \\ x - 2z = 0 \end{cases}$$

Solution Begin by writing the system in the matrix form $AX = B$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0.06 & 0.075 & 0.095 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10{,}000 \\ 730 \\ 0 \end{bmatrix}$$

Then, use Gauss-Jordan elimination to find A^{-1}.

$$A^{-1} = \begin{bmatrix} 15 & -200 & -2 \\ -21.5 & 300 & 3.5 \\ 7.5 & -100 & -1.5 \end{bmatrix}$$

Finally, multiply B by A^{-1} on the left to obtain the solution.

$$X = A^{-1}B = \begin{bmatrix} 15 & -200 & -2 \\ -21.5 & 300 & 3.5 \\ 7.5 & -100 & -1.5 \end{bmatrix} \begin{bmatrix} 10{,}000 \\ 730 \\ 0 \end{bmatrix} = \begin{bmatrix} 4000 \\ 4000 \\ 2000 \end{bmatrix}$$

The solution of the system is $x = 4000$, $y = 4000$, and $z = 2000$.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use an inverse matrix to solve the system $\begin{cases} 2x + 3y + z = -1 \\ 3x + 3y + z = 1. \\ 2x + 4y + z = -2 \end{cases}$

TECHNOLOGY To solve a system of equations with a graphing utility, enter the matrices A and B in the matrix editor. Then, using the inverse key, solve for X.

A x^{-1} B ENTER

The screen will display the solution, matrix X.

Summarize (Section 7.3)

1. State the definition of the inverse of a square matrix *(page 523)*. For an example of how to show that a matrix is the inverse of another matrix, see Example 1.
2. Explain how to find an inverse matrix *(pages 524 and 525)*. For examples of finding inverse matrices, see Examples 2 and 3.
3. Give the formula for finding the inverse of a 2 × 2 matrix *(page 527)*. For an example of using this formula, see Example 4.
4. Describe how to use an inverse matrix to solve a system of linear equations *(page 528)*. For an example of using an inverse matrix to solve a system of linear equations, see Example 5.

7.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. In a ________ matrix, the number of rows equals the number of columns.
2. If there exists an $n \times n$ matrix A^{-1} such that $AA^{-1} = I_n = A^{-1}A$, then A^{-1} is called the ________ of A.
3. If a matrix A has an inverse, then it is called invertible or ________; if it does not have an inverse, then it is called ________.
4. If A is an invertible matrix, then the system of linear equations represented by $AX = B$ has a unique solution given by $X =$ ________.

Skills and Applications

The Inverse of a Matrix **In Exercises 5–12, show that B is the inverse of A.**

5. $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

7. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \frac{1}{2}\begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$

8. $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, B = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$

9. $A = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$

10. $A = \begin{bmatrix} -4 & 1 & 5 \\ -1 & 2 & 4 \\ 0 & -1 & -1 \end{bmatrix}, B = \frac{1}{4}\begin{bmatrix} -2 & 4 & 6 \\ 1 & -4 & -11 \\ -1 & 4 & 7 \end{bmatrix}$

11. $A = \begin{bmatrix} 2 & 0 & 2 & 1 \\ 3 & 0 & 0 & 1 \\ -1 & 1 & -2 & 1 \\ 3 & -1 & 1 & 0 \end{bmatrix},$

$B = \frac{1}{3}\begin{bmatrix} -1 & 3 & -2 & -2 \\ -2 & 9 & -7 & -10 \\ 1 & 0 & -1 & -1 \\ 3 & -6 & 6 & 6 \end{bmatrix}$

12. $A = \begin{bmatrix} -1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 2 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix},$

$B = \frac{1}{3}\begin{bmatrix} -3 & 1 & 1 & -3 \\ -3 & -1 & 2 & -3 \\ 0 & 1 & 1 & 0 \\ -3 & -2 & 1 & 0 \end{bmatrix}$

Finding the Inverse of a Matrix **In Exercises 13–24, find the inverse of the matrix (if it exists).**

13. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$

15. $\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$

16. $\begin{bmatrix} -7 & 33 \\ 4 & -19 \end{bmatrix}$

17. $\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$

18. $\begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$

19. $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{bmatrix}$

20. $\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$

21. $\begin{bmatrix} -5 & 0 & 0 \\ 2 & 0 & 0 \\ -1 & 5 & 7 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 5 & 5 \end{bmatrix}$

23. $\begin{bmatrix} -8 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}$

24. $\begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

Finding the Inverse of a Matrix **In Exercises 25–34, use the matrix capabilities of a graphing utility to find the inverse of the matrix (if it exists).**

25. $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -10 \\ -5 & -7 & -15 \end{bmatrix}$

26. $\begin{bmatrix} 10 & 5 & -7 \\ -5 & 1 & 4 \\ 3 & 2 & -2 \end{bmatrix}$

27. $\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \\ -2 & 0 & 3 \end{bmatrix}$

28. $\begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 2 \\ -4 & 4 & 3 \end{bmatrix}$

29. $\begin{bmatrix} -\frac{1}{2} & \frac{3}{4} & \frac{1}{4} \\ 1 & 0 & -\frac{3}{2} \\ 0 & -1 & \frac{1}{2} \end{bmatrix}$

30. $\begin{bmatrix} -\frac{5}{6} & \frac{1}{3} & \frac{11}{6} \\ 0 & \frac{2}{3} & 2 \\ 1 & -\frac{1}{2} & -\frac{5}{2} \end{bmatrix}$

31. $\begin{bmatrix} 0.1 & 0.2 & 0.3 \\ -0.3 & 0.2 & 0.2 \\ 0.5 & 0.4 & 0.4 \end{bmatrix}$

32. $\begin{bmatrix} 0.6 & 0 & -0.3 \\ 0.7 & -1 & 0.2 \\ 1 & 0 & -0.9 \end{bmatrix}$

33. $\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

34. $\begin{bmatrix} 1 & -2 & -1 & -2 \\ 3 & -5 & -2 & -3 \\ 2 & -5 & -2 & -5 \\ -1 & 4 & 4 & 11 \end{bmatrix}$

Finding the Inverse of a 2 × 2 Matrix **In Exercises 35–40, use the formula on page 527 to find the inverse of the 2 × 2 matrix (if it exists).**

35. $\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$ **36.** $\begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix}$

37. $\begin{bmatrix} -4 & -6 \\ 2 & 3 \end{bmatrix}$ **38.** $\begin{bmatrix} -12 & 3 \\ 5 & -2 \end{bmatrix}$

39. $\begin{bmatrix} \frac{7}{2} & -\frac{3}{4} \\ \frac{1}{5} & \frac{4}{5} \end{bmatrix}$ **40.** $\begin{bmatrix} -\frac{1}{4} & \frac{9}{4} \\ \frac{5}{3} & \frac{8}{9} \end{bmatrix}$

Solving a System Using an Inverse Matrix **In Exercises 41–44, use the inverse matrix found in Exercise 15 to solve the system of linear equations.**

41. $\begin{cases} x - 2y = 5 \\ 2x - 3y = 10 \end{cases}$ **42.** $\begin{cases} x - 2y = 0 \\ 2x - 3y = 3 \end{cases}$

43. $\begin{cases} x - 2y = 4 \\ 2x - 3y = 2 \end{cases}$ **44.** $\begin{cases} x - 2y = 1 \\ 2x - 3y = -2 \end{cases}$

Solving a System Using an Inverse Matrix **In Exercises 45 and 46, use the inverse matrix found in Exercise 19 to solve the system of linear equations.**

45. $\begin{cases} x + y + z = 0 \\ 3x + 5y + 4z = 5 \\ 3x + 6y + 5z = 2 \end{cases}$ **46.** $\begin{cases} x + y + z = -1 \\ 3x + 5y + 4z = 2 \\ 3x + 6y + 5z = 0 \end{cases}$

Solving a System Using an Inverse Matrix **In Exercises 47 and 48, use the inverse matrix found in Exercise 34 to solve the system of linear equations.**

47. $\begin{cases} x_1 - 2x_2 - x_3 - 2x_4 = 0 \\ 3x_1 - 5x_2 - 2x_3 - 3x_4 = 1 \\ 2x_1 - 5x_2 - 2x_3 - 5x_4 = -1 \\ -x_1 + 4x_2 + 4x_3 + 11x_4 = 2 \end{cases}$

48. $\begin{cases} x_1 - 2x_2 - x_3 - 2x_4 = 1 \\ 3x_1 - 5x_2 - 2x_3 - 3x_4 = -2 \\ 2x_1 - 5x_2 - 2x_3 - 5x_4 = 0 \\ -x_1 + 4x_2 + 4x_3 + 11x_4 = -3 \end{cases}$

Solving a System Using an Inverse Matrix **In Exercises 49–56, use an inverse matrix to solve (if possible) the system of linear equations.**

49. $\begin{cases} 3x + 4y = -2 \\ 5x + 3y = 4 \end{cases}$ **50.** $\begin{cases} 18x + 12y = 13 \\ 30x + 24y = 23 \end{cases}$

51. $\begin{cases} -0.4x + 0.8y = 1.6 \\ 2x - 4y = 5 \end{cases}$ **52.** $\begin{cases} 0.2x - 0.6y = 2.4 \\ -x + 1.4y = -8.8 \end{cases}$

53. $\begin{cases} -\frac{1}{4}x + \frac{3}{8}y = -2 \\ \frac{3}{2}x + \frac{3}{4}y = -12 \end{cases}$ **54.** $\begin{cases} \frac{5}{6}x - y = -20 \\ \frac{4}{3}x - \frac{7}{2}y = -51 \end{cases}$

55. $\begin{cases} 4x - y + z = -5 \\ 2x + 2y + 3z = 10 \\ 5x - 2y + 6z = 1 \end{cases}$ **56.** $\begin{cases} 4x - 2y + 3z = -2 \\ 2x + 2y + 5z = 16 \\ 8x - 5y - 2z = 4 \end{cases}$

Using a Graphing Utility **In Exercises 57–60, use the matrix capabilities of a graphing utility to solve (if possible) the system of linear equations.**

57. $\begin{cases} 5x - 3y + 2z = 2 \\ 2x + 2y - 3z = 3 \\ x - 7y + 8z = -4 \end{cases}$ **58.** $\begin{cases} 2x + 3y + 5z = 4 \\ 3x + 5y + 9z = 7 \\ 5x + 9y + 17z = 13 \end{cases}$

59. $\begin{cases} 3x - 2y + z = -29 \\ -4x + y - 3z = 37 \\ x - 5y + z = -24 \end{cases}$

60. $\begin{cases} -8x + 7y - 10z = -151 \\ 12x + 3y - 5z = 86 \\ 15x - 9y + 2z = 187 \end{cases}$

Investment Portfolio **In Exercises 61–64, you invest in AAA-rated bonds, A-rated bonds, and B-rated bonds. The average yields are 6.5% on AAA bonds, 7% on A bonds, and 9% on B bonds. You invest twice as much in B bonds as in A bonds. Let *x*, *y*, and *z* represent the amounts invested in AAA, A, and B bonds, respectively.**

$$\begin{cases} x + y + z = \text{(total investment)} \\ 0.065x + 0.07y + 0.09z = \text{(annual return)} \\ 2y - z = 0 \end{cases}$$

Use the inverse of the coefficient matrix of this system to find the amount invested in each type of bond.

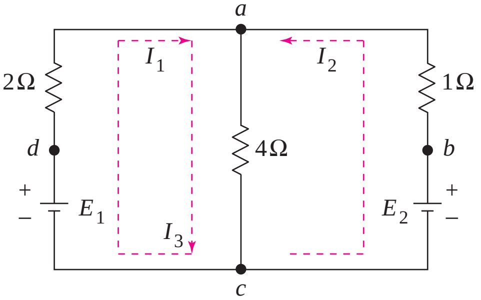
	Total Investment	Annual Return
61.	$10,000	$705
62.	$10,000	$760
63.	$12,000	$835
64.	$500,000	$38,000

Circuit Analysis

In Exercises 65–68, consider the circuit shown in the figure. The currents I_1, I_2, and I_3, in amperes, are the solution of the system of linear equations.

$$\begin{cases} 2I_1 + 4I_3 = E_1 \\ I_2 + 4I_3 = E_2 \\ I_1 + I_2 - I_3 = 0 \end{cases}$$

where E_1 and E_2 are voltages. Use the inverse of the coefficient matrix of this system to find the unknown currents for the given voltages.



Circuit Analysis —continued—

65. $E_1 = 15$ volts,
$E_2 = 17$ volts

66. $E_1 = 10$ volts,
$E_2 = 10$ volts

67. $E_1 = 28$ volts,
$E_2 = 21$ volts

68. $E_1 = 24$ volts,
$E_2 = 23$ volts

Raw Materials **In Exercises 69 and 70, find the numbers of bags of potting soil that a company can produce for seedlings, general potting, and hardwood plants with the given amounts of raw materials. The raw materials used in one bag of each type of potting soil are shown below.**

	Sand	Loam	Peat Moss
Seedlings	2 units	1 units	1 unit
General	1 unit	2 units	1 unit
Hardwoods	2 units	2 units	2 units

69. 500 units of sand
500 units of loam
400 units of peat moss

70. 500 units of sand
750 units of loam
450 units of peat moss

71. Floral Design A florist is creating 10 centerpieces for the tables at a wedding reception. Roses cost \$2.50 each, lilies cost \$4 each, and irises cost \$2 each. The customer has a budget of \$300 allocated for the centerpieces and wants each centerpiece to contain 12 flowers, with twice as many roses as the number of irises and lilies combined.

(a) Write a system of linear equations that represents the situation.

(b) Write a matrix equation that corresponds to your system.

(c) Solve your system of linear equations using an inverse matrix. Find the number of flowers of each type that the florist can use to create the 10 centerpieces.

72. International Travel The table shows the numbers of international travelers y (in thousands) to the United States from South America from 2008 through 2010. *(Source: U.S. Department of Commerce)*

Year	Travelers, y (in thousands)
2008	2556
2009	2742
2010	3250

(a) The data can be modeled by the quadratic function $y = at^2 + bt + c$. Create a system of linear equations for the data. Let t represent the year, with $t = 8$ corresponding to 2008.

(b) Use the matrix capabilities of a graphing utility to find the inverse matrix to solve the system from part (a) and find the least squares regression parabola $y = at^2 + bt + c$.

(c) Use the graphing utility to graph the parabola with the data.

(d) Do you believe the model is a reasonable predictor of future numbers of travelers? Explain.

Exploration

True or False? **In Exercises 73 and 74, determine whether the statement is true or false. Justify your answer.**

73. Multiplication of an invertible matrix and its inverse is commutative.

74. When the product of two square matrices is the identity matrix, the matrices are inverses of one another.

75. Writing Explain how to determine whether the inverse of a 2 × 2 matrix exists. If so, explain how to find the inverse.

76. Writing Explain in your own words how to write a system of three linear equations in three variables as a matrix equation, $AX = B$, as well as how to solve the system using an inverse matrix.

77. Conjecture Consider matrices of the form

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}.$$

(a) Write a 2 × 2 matrix and a 3 × 3 matrix in the form of A. Find the inverse of each.

(b) Use the result of part (a) to make a conjecture about the inverses of matrices in the form of A.

78. HOW DO YOU SEE IT? Let A be the 2 × 2 matrix

$$A = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}.$$

Use the determinant of A to state the conditions for which (a) A^{-1} exists and (b) $A^{-1} = A$.

Project: Viewing Television To work an extended application analyzing the average amounts of time spent viewing television in the United States, visit this text's website at *LarsonPrecalculus.com.* *(Source: The Nielsen Company)*

Ingrid Prats/Shutterstock.com

7.4 The Determinant of a Square Matrix

Determinants are often used in other branches of mathematics. For instance, Exercises 89–94 on page 539 show some types of determinants that are useful when changes of variables are made in calculus.

- **Find the determinants of 2 × 2 matrices.**
- **Find minors and cofactors of square matrices.**
- **Find the determinants of square matrices.**

The Determinant of a 2 × 2 Matrix

Every *square* matrix can be associated with a real number called its **determinant.** Determinants have many uses, and several will be discussed in this and the next section. Historically, the use of determinants arose from special number patterns that occur when systems of linear equations are solved. For instance, the system

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

has a solution

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \quad \text{and} \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

provided that $a_1b_2 - a_2b_1 \neq 0$. Note that the denominators of the two fractions are the same. This denominator is called the *determinant* of the coefficient matrix of the system.

Coefficient Matrix	**Determinant**
$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$	$\det(A) = a_1b_2 - a_2b_1$

The determinant of the matrix A can also be denoted by vertical bars on both sides of the matrix, as indicated in the following definition.

Definition of the Determinant of a 2 × 2 Matrix

The **determinant** of the matrix

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

is given by

$$\det(A) = |A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

In this text, $\det(A)$ and $|A|$ are used interchangeably to represent the determinant of A. Although vertical bars are also used to denote the absolute value of a real number, the context will show which use is intended.

A convenient method for remembering the formula for the determinant of a 2 × 2 matrix is shown in the following diagram.

$$\det(A) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

Note that the determinant is the difference of the products of the two diagonals of the matrix.

lightpoet/Shutterstock.com

In Example 1, you will see that the determinant of a matrix can be positive, zero, or negative.

EXAMPLE 1 The Determinant of a 2 × 2 Matrix

Find the determinant of each matrix.

a. $A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$

b. $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$

c. $C = \begin{bmatrix} 0 & \frac{3}{2} \\ 2 & 4 \end{bmatrix}$

Solution

a. $$\det(A) = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2(2) - 1(-3) = 4 + 3 = 7$$

b. $$\det(B) = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 2(2) - 4(1) = 4 - 4 = 0$$

c. $$\det(C) = \begin{vmatrix} 0 & \frac{3}{2} \\ 2 & 4 \end{vmatrix} = 0(4) - 2\left(\tfrac{3}{2}\right) = 0 - 3 = -3$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the determinant of each matrix.

a. $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ **b.** $B = \begin{bmatrix} 5 & 0 \\ -4 & 2 \end{bmatrix}$ **c.** $C = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$ ■

The determinant of a matrix of order 1 × 1 is defined simply as the entry of the matrix. For instance, if $A = [-2]$, then $\det(A) = -2$.

TECHNOLOGY Most graphing utilities can evaluate the determinant of a matrix. For instance, you can evaluate the determinant of

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$$

by entering the matrix as $[A]$ and then choosing the *determinant* feature. The result should be 7, as in Example 1(a). Try evaluating the determinants of other matrices. Consult the user's guide for your graphing utility for specific keystrokes.

Minors and Cofactors

To define the determinant of a square matrix of order 3 × 3 or higher, it is convenient to introduce the concepts of **minors** and **cofactors.**

Sign Pattern for Cofactors

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

3 × 3 matrix

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

4 × 4 matrix

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

$n \times n$ matrix

Minors and Cofactors of a Square Matrix

If A is a square matrix, then the **minor** M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The **cofactor** C_{ij} of the entry a_{ij} is

$$C_{ij} = (-1)^{i+j}M_{ij}.$$

In the sign pattern for cofactors at the left, notice that *odd* positions (where $i + j$ is odd) have negative signs and *even* positions (where $i + j$ is even) have positive signs.

EXAMPLE 2 Finding the Minors and Cofactors of a Matrix

Find all the minors and cofactors of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}.$$

Solution To find the minor M_{11}, delete the first row and first column of A and evaluate the determinant of the resulting matrix.

$$\begin{bmatrix} \cancel{0} & \cancel{2} & \cancel{1} \\ \cancel{3} & -1 & 2 \\ \cancel{4} & 0 & 1 \end{bmatrix}, \quad M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1(1) - 0(2) = -1$$

Similarly, to find M_{12}, delete the first row and second column.

$$\begin{bmatrix} \cancel{0} & \cancel{2} & \cancel{1} \\ 3 & \cancel{-1} & 2 \\ 4 & \cancel{0} & 1 \end{bmatrix}, \quad M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 3(1) - 4(2) = -5$$

Continuing this pattern, you obtain the minors.

$$\begin{array}{lll} M_{11} = -1 & M_{12} = -5 & M_{13} = 4 \\ M_{21} = 2 & M_{22} = -4 & M_{23} = -8 \\ M_{31} = 5 & M_{32} = -3 & M_{33} = -6 \end{array}$$

Now, to find the cofactors, combine these minors with the checkerboard pattern of signs for a 3 × 3 matrix shown at the upper left.

$$\begin{array}{lll} C_{11} = -1 & C_{12} = 5 & C_{13} = 4 \\ C_{21} = -2 & C_{22} = -4 & C_{23} = 8 \\ C_{31} = 5 & C_{32} = 3 & C_{33} = -6 \end{array}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find all the minors and cofactors of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}.$$

The Determinant of a Square Matrix

The definition below is called *inductive* because it uses determinants of matrices of order $n - 1$ to define determinants of matrices of order n.

Determinant of a Square Matrix

If A is a square matrix (of order 2×2 or greater), then the determinant of A is the sum of the entries in any row (or column) of A multiplied by their respective cofactors. For instance, expanding along the first row yields

$$|A| = a_{11}C_{11} + a_{12}C_{12} + \cdot \cdot \cdot + a_{1n}C_{1n}.$$

Applying this definition to find a determinant is called **expanding by cofactors.**

Try checking that for a 2 × 2 matrix

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

this definition of the determinant yields

$$|A| = a_1b_2 - a_2b_1$$

as previously defined.

EXAMPLE 3 **The Determinant of a 3 × 3 Matrix**

Find the determinant of $A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$.

Solution Note that this is the same matrix that was given in Example 2. There you found the cofactors of the entries in the first row to be

$$C_{11} = -1, \quad C_{12} = 5, \quad \text{and} \quad C_{13} = 4.$$

So, by the definition of a determinant, you have

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} && \text{First-row expansion} \\ &= 0(-1) + 2(5) + 1(4) \\ &= 14. \end{aligned}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the determinant of $A = \begin{bmatrix} 3 & 4 & -2 \\ 3 & 5 & 0 \\ -1 & 4 & 1 \end{bmatrix}$.

In Example 3, the determinant was found by expanding by cofactors in the first row. You could have used any row or column. For instance, you could have expanded along the second row to obtain

$$\begin{aligned} |A| &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} && \text{Second-row expansion} \\ &= 3(-2) + (-1)(-4) + 2(8) \\ &= 14. \end{aligned}$$

In the number-placement puzzle Sudoku, the object is to fill out a partially completed 9 × 9 grid of boxes with numbers from 1 to 9 so that each column, row, and 3 × 3 sub-grid contains each of these numbers without repetition. For a completed Sudoku grid to be valid, no two rows (or columns) will have the numbers in the same order. If this should happen in a row or column, then the determinant of the matrix formed by the numbers in the grid will be zero.

When expanding by cofactors, you do not need to find cofactors of zero entries, because zero times its cofactor is zero. So, the row (or column) containing the most zeros is usually the best choice for expansion by cofactors. This is demonstrated in the next example.

EXAMPLE 4 The Determinant of a 4 × 4 Matrix

Find the determinant of $A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & 2 \end{bmatrix}$.

Solution After inspecting this matrix, you can see that three of the entries in the third column are zeros. So, you can eliminate some of the work in the expansion by using the third column.

$$|A| = 3(C_{13}) + 0(C_{23}) + 0(C_{33}) + 0(C_{43})$$

Because C_{23}, C_{33}, and C_{43} have zero coefficients, you need only find the cofactor C_{13}. To do this, delete the first row and third column of A and evaluate the determinant of the resulting matrix.

$$C_{13} = (-1)^{1+3}\begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & 2 \end{vmatrix} \qquad \text{Delete 1st row and 3rd column.}$$

$$= \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & 2 \end{vmatrix} \qquad \text{Simplify.}$$

Expanding by cofactors in the second row yields

$$C_{13} = 0(-1)^3\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} + 2(-1)^4\begin{vmatrix} -1 & 2 \\ 3 & 2 \end{vmatrix} + 3(-1)^5\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix}$$

$$= 0 + 2(1)(-8) + 3(-1)(-7)$$

$$= 5.$$

So, $|A| = 3C_{13} = 3(5) = 15$.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the determinant of $A = \begin{bmatrix} 2 & 6 & -4 & 2 \\ 2 & -2 & 3 & 6 \\ 1 & 5 & 0 & 1 \\ 3 & 1 & 0 & -5 \end{bmatrix}$.

Summarize (Section 7.4)

1. State the definition of the determinant of a 2 × 2 matrix *(page 532)*. For an example of finding the determinants of 2 × 2 matrices, see Example 1.
2. State the definitions of minors and cofactors of a square matrix *(page 534)*. For an example of finding the minors and cofactors of a square matrix, see Example 2.
3. State the definition of the determinant of a square matrix using expansion by cofactors *(page 535)*. For examples of finding determinants using this definition, see Examples 3 and 4.

James E. Knopf/Shutterstock.com

7.4 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. Both $\det(A)$ and $|A|$ represent the ________ of the matrix A.
2. The ________ M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of the square matrix A.
3. The ________ C_{ij} of the entry a_{ij} of the square matrix A is given by $(-1)^{i+j}M_{ij}$.
4. The method of finding the determinant of a matrix of order 2×2 or greater is ________ by ________.

Skills and Applications

Finding the Determinant of a Matrix In Exercises 5–22, find the determinant of the matrix.

5. $[4]$
6. $[-10]$
7. $\begin{bmatrix} 8 & 4 \\ 2 & 3 \end{bmatrix}$
8. $\begin{bmatrix} -9 & 0 \\ 6 & 2 \end{bmatrix}$
9. $\begin{bmatrix} 6 & 2 \\ -5 & 3 \end{bmatrix}$
10. $\begin{bmatrix} 3 & -3 \\ 4 & -8 \end{bmatrix}$
11. $\begin{bmatrix} -7 & 0 \\ 3 & 0 \end{bmatrix}$
12. $\begin{bmatrix} 4 & -3 \\ 0 & 0 \end{bmatrix}$
13. $\begin{bmatrix} 2 & 6 \\ 0 & 3 \end{bmatrix}$
14. $\begin{bmatrix} 2 & -3 \\ -6 & 9 \end{bmatrix}$
15. $\begin{bmatrix} -3 & -2 \\ -6 & -1 \end{bmatrix}$
16. $\begin{bmatrix} 4 & 7 \\ -2 & 5 \end{bmatrix}$
17. $\begin{bmatrix} -2 & -7 \\ -3 & 1 \end{bmatrix}$
18. $\begin{bmatrix} 2 & -5 \\ -4 & -1 \end{bmatrix}$
19. $\begin{bmatrix} -7 & 6 \\ \frac{1}{2} & 3 \end{bmatrix}$
20. $\begin{bmatrix} 0 & 6 \\ -3 & 2 \end{bmatrix}$
21. $\begin{bmatrix} -\frac{1}{2} & \frac{1}{3} \\ -6 & \frac{1}{3} \end{bmatrix}$
22. $\begin{bmatrix} \frac{2}{3} & \frac{4}{3} \\ -1 & -\frac{1}{3} \end{bmatrix}$

Using a Graphing Utility In Exercises 23–30, use the matrix capabilities of a graphing utility to find the determinant of the matrix.

23. $\begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}$
24. $\begin{bmatrix} 5 & -9 \\ 7 & 16 \end{bmatrix}$
25. $\begin{bmatrix} 19 & -20 \\ 43 & 56 \end{bmatrix}$
26. $\begin{bmatrix} 101 & 197 \\ -253 & 172 \end{bmatrix}$
27. $\begin{bmatrix} 0.3 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}$
28. $\begin{bmatrix} 0.1 & 0.2 \\ -0.3 & 0.2 \end{bmatrix}$
29. $\begin{bmatrix} 0.9 & 0.7 \\ -0.1 & 0.3 \end{bmatrix}$
30. $\begin{bmatrix} 0.1 & 0.1 \\ 7.5 & 6.2 \end{bmatrix}$

Finding the Minors and Cofactors of a Matrix In Exercises 31–38, find all the (a) minors and (b) cofactors of the matrix.

31. $\begin{bmatrix} 4 & 5 \\ 3 & -6 \end{bmatrix}$
32. $\begin{bmatrix} 0 & 10 \\ 3 & -4 \end{bmatrix}$
33. $\begin{bmatrix} 3 & 1 \\ -2 & -4 \end{bmatrix}$
34. $\begin{bmatrix} -6 & 5 \\ 7 & -2 \end{bmatrix}$
35. $\begin{bmatrix} 4 & 0 & 2 \\ -3 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$
36. $\begin{bmatrix} 1 & -1 & 0 \\ 3 & 2 & 5 \\ 4 & -6 & 4 \end{bmatrix}$
37. $\begin{bmatrix} -4 & 6 & 3 \\ 7 & -2 & 8 \\ 1 & 0 & -5 \end{bmatrix}$
38. $\begin{bmatrix} -2 & 9 & 4 \\ 7 & -6 & 0 \\ 6 & 7 & -6 \end{bmatrix}$

Finding the Determinant of a Matrix In Exercises 39–46, find the determinant of the matrix. Expand by cofactors using the indicated row or column.

39. $\begin{bmatrix} -3 & 2 & 1 \\ 4 & 5 & 6 \\ 2 & -3 & 1 \end{bmatrix}$
    (a) Row 1
    (b) Column 2
40. $\begin{bmatrix} -3 & 4 & 2 \\ 6 & 3 & 1 \\ 4 & -7 & -8 \end{bmatrix}$
    (a) Row 2
    (b) Column 3
41. $\begin{bmatrix} 5 & 0 & -3 \\ 0 & 12 & 4 \\ 1 & 6 & 3 \end{bmatrix}$
    (a) Row 2
    (b) Column 2
42. $\begin{bmatrix} 10 & -5 & 5 \\ 30 & 0 & 10 \\ 0 & 10 & 1 \end{bmatrix}$
    (a) Row 3
    (b) Column 1
43. $\begin{bmatrix} -2 & 4 & 7 & 1 \\ 3 & 0 & 0 & 0 \\ 8 & 5 & 10 & 5 \\ 6 & 0 & 5 & 0 \end{bmatrix}$
    (a) Row 2
    (b) Column 4
44. $\begin{bmatrix} 7 & 0 & 0 & -6 \\ 6 & 0 & 1 & -2 \\ 1 & -2 & 3 & 2 \\ -3 & 0 & -1 & 4 \end{bmatrix}$
    (a) Row 4
    (b) Column 2
45. $\begin{bmatrix} 6 & 0 & -3 & 5 \\ 4 & 13 & 6 & -8 \\ -1 & 0 & 7 & 4 \\ 8 & 6 & 0 & 2 \end{bmatrix}$
    (a) Row 2
    (b) Column 2
46. $\begin{bmatrix} 10 & 8 & 3 & -7 \\ 4 & 0 & 5 & -6 \\ 0 & 3 & 2 & 7 \\ 1 & 0 & -3 & 2 \end{bmatrix}$
    (a) Row 3
    (b) Column 1

Finding the Determinant of a Matrix **In Exercises 47–62, find the determinant of the matrix. Expand by cofactors using the row or column that appears to make the computations easiest.**

47. $\begin{bmatrix} 2 & -1 & 0 \\ 4 & 2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$

48. $\begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$

49. $\begin{bmatrix} 6 & 3 & -7 \\ 0 & 0 & 0 \\ 4 & -6 & 3 \end{bmatrix}$

50. $\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \\ -2 & 0 & 3 \end{bmatrix}$

51. $\begin{bmatrix} -1 & 8 & -3 \\ 0 & 3 & -6 \\ 0 & 0 & 3 \end{bmatrix}$

52. $\begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 4 & 11 & 5 \end{bmatrix}$

53. $\begin{bmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{bmatrix}$

54. $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & 4 \\ 1 & 0 & 2 \end{bmatrix}$

55. $\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 1 \\ 0 & 0 & -5 \end{bmatrix}$

56. $\begin{bmatrix} -3 & 0 & 0 \\ 7 & 11 & 0 \\ 1 & 2 & 2 \end{bmatrix}$

57. $\begin{bmatrix} 2 & 6 & 6 & 2 \\ 2 & 7 & 3 & 6 \\ 1 & 5 & 0 & 1 \\ 3 & 7 & 0 & 7 \end{bmatrix}$

58. $\begin{bmatrix} 3 & 6 & -5 & 4 \\ -2 & 0 & 6 & 0 \\ 1 & 1 & 2 & 2 \\ 0 & 3 & -1 & -1 \end{bmatrix}$

59. $\begin{bmatrix} 5 & 3 & 0 & 6 \\ 4 & 6 & 4 & 12 \\ 0 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \end{bmatrix}$

60. $\begin{bmatrix} 1 & 4 & 3 & 2 \\ -5 & 6 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 3 & -2 & 1 & 5 \end{bmatrix}$

61. $\begin{bmatrix} 3 & 2 & 4 & -1 & 5 \\ -2 & 0 & 1 & 3 & 2 \\ 1 & 0 & 0 & 4 & 0 \\ 6 & 0 & 2 & -1 & 0 \\ 3 & 0 & 5 & 1 & 0 \end{bmatrix}$

62. $\begin{bmatrix} 5 & 2 & 0 & 0 & -2 \\ 0 & 1 & 4 & 3 & 2 \\ 0 & 0 & 2 & 6 & 3 \\ 0 & 0 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

Using a Graphing Utility **In Exercises 63–66, use the matrix capabilities of a graphing utility to evaluate the determinant.**

63. $\begin{vmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 8 & 1 & 6 \end{vmatrix}$

64. $\begin{vmatrix} 5 & -8 & 0 \\ 9 & 7 & 4 \\ -8 & 7 & 1 \end{vmatrix}$

65. $\begin{vmatrix} 1 & -1 & 8 & 4 \\ 2 & 6 & 0 & -4 \\ 2 & 0 & 2 & 6 \\ 0 & 2 & 8 & 0 \end{vmatrix}$

66. $\begin{vmatrix} 0 & -3 & 8 & 2 \\ 8 & 1 & -1 & 6 \\ -4 & 6 & 0 & 9 \\ -7 & 0 & 0 & 14 \end{vmatrix}$

The Determinant of a Matrix Product **In Exercises 67–74, find (a) $|A|$, (b) $|B|$, (c) AB, and (d) $|AB|$.**

67. $A = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

68. $A = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

69. $A = \begin{bmatrix} 4 & 0 \\ 3 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$

70. $A = \begin{bmatrix} 5 & 4 \\ 3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 6 \\ 1 & -2 \end{bmatrix}$

71. $A = \begin{bmatrix} 0 & 1 & 2 \\ -3 & -2 & 1 \\ 0 & 4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -2 & 0 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix}$

72. $A = \begin{bmatrix} 3 & 2 & 0 \\ -1 & -3 & 4 \\ -2 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 0 & 1 \\ 0 & 2 & -1 \\ -2 & -1 & 1 \end{bmatrix}$

73. $A = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

74. $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$

Verifying an Equation **In Exercises 75–80, evaluate the determinant(s) to verify the equation.**

75. $\begin{vmatrix} w & x \\ y & z \end{vmatrix} = -\begin{vmatrix} y & z \\ w & x \end{vmatrix}$

76. $\begin{vmatrix} w & cx \\ y & cz \end{vmatrix} = c\begin{vmatrix} w & x \\ y & z \end{vmatrix}$

77. $\begin{vmatrix} w & x \\ y & z \end{vmatrix} = \begin{vmatrix} w & x + cw \\ y & z + cy \end{vmatrix}$

78. $\begin{vmatrix} w & x \\ cw & cx \end{vmatrix} = 0$

79. $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (y - x)(z - x)(z - y)$

80. $\begin{vmatrix} a + b & a & a \\ a & a + b & a \\ a & a & a + b \end{vmatrix} = b^2(3a + b)$

Solving an Equation **In Exercises 81–88, solve for x.**

81. $\begin{vmatrix} x & 2 \\ 1 & x \end{vmatrix} = 2$

82. $\begin{vmatrix} x & 4 \\ -1 & x \end{vmatrix} = 20$

83. $\begin{vmatrix} x & 1 \\ 2 & x - 2 \end{vmatrix} = -1$

84. $\begin{vmatrix} x + 1 & 2 \\ -1 & x \end{vmatrix} = 4$

85. $\begin{vmatrix} x - 1 & 2 \\ 3 & x - 2 \end{vmatrix} = 0$

86. $\begin{vmatrix} x - 2 & -1 \\ -3 & x \end{vmatrix} = 0$

87. $\begin{vmatrix} x + 3 & 2 \\ 1 & x + 2 \end{vmatrix} = 0$

88. $\begin{vmatrix} x + 4 & -2 \\ 7 & x - 5 \end{vmatrix} = 0$

Entries Involving Expressions

In Exercises 89–94, evaluate the determinant in which the entries are functions. Determinants of this type occur when changes of variables are made in calculus.

89. $\begin{vmatrix} 4u & -1 \\ -1 & 2v \end{vmatrix}$

90. $\begin{vmatrix} 3x^2 & -3y^2 \\ 1 & 1 \end{vmatrix}$

91. $\begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix}$

92. $\begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & (1-x)e^{-x} \end{vmatrix}$

93. $\begin{vmatrix} x & \ln x \\ 1 & 1/x \end{vmatrix}$

94. $\begin{vmatrix} x & x \ln x \\ 1 & 1 + \ln x \end{vmatrix}$

Exploration

True or False? **In Exercises 95 and 96, determine whether the statement is true or false. Justify your answer.**

95. If a square matrix has an entire row of zeros, then the determinant will always be zero.

96. If two columns of a square matrix are the same, then the determinant of the matrix will be zero.

97. Providing a Counterexample Find square matrices A and B to demonstrate that $|A + B| \neq |A| + |B|$.

98. Conjecture Consider square matrices in which the entries are consecutive integers. An example of such a matrix is

$$\begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}.$$

(a) Use a graphing utility to evaluate the determinants of four matrices of this type. Make a conjecture based on the results.

(b) Verify your conjecture.

99. Writing Write a brief paragraph explaining the difference between a square matrix and its determinant.

100. Think About It Let A be a 3×3 matrix such that $|A| = 5$. Is it possible to find $|2A|$? Explain.

Properties of Determinants **In Exercises 101–103, a property of determinants is given (A and B are square matrices). State how the property has been applied to the given determinants and use a graphing utility to verify the results.**

101. If B is obtained from A by interchanging two rows of A or interchanging two columns of A, then $|B| = -|A|$.

(a) $\begin{vmatrix} 1 & 3 & 4 \\ -7 & 2 & -5 \\ 6 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 4 & 3 \\ -7 & -5 & 2 \\ 6 & 2 & 1 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 3 & 4 \\ -2 & 2 & 0 \\ 1 & 6 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 6 & 2 \\ -2 & 2 & 0 \\ 1 & 3 & 4 \end{vmatrix}$

102. If B is obtained from A by adding a multiple of a row of A to another row of A or by adding a multiple of a column of A to another column of A, then $|B| = |A|$.

(a) $\begin{vmatrix} 1 & -3 \\ 5 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 \\ 0 & 17 \end{vmatrix}$

(b) $\begin{vmatrix} 5 & 4 & 2 \\ 2 & -3 & 4 \\ 7 & 6 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 10 & -6 \\ 2 & -3 & 4 \\ 7 & 6 & 3 \end{vmatrix}$

103. If B is obtained from A by multiplying a row by a nonzero constant c or by multiplying a column by a nonzero constant c, then $|B| = c|A|$.

(a) $\begin{vmatrix} 5 & 10 \\ 2 & -3 \end{vmatrix} = 5\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 8 & -3 \\ 3 & -12 & 6 \\ 7 & 4 & 9 \end{vmatrix} = 12\begin{vmatrix} 1 & 2 & -1 \\ 3 & -3 & 2 \\ 7 & 1 & 3 \end{vmatrix}$

104. HOW DO YOU SEE IT? Explain why the determinant of the matrix is equal to zero.

(a) $\begin{bmatrix} 3 & 4 & -2 & 7 \\ 1 & 3 & -1 & 2 \\ 0 & 5 & 7 & 1 \\ 1 & 3 & -1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 2 & -1 \\ -6 & -4 & 2 \\ 5 & -7 & 9 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & -4 & 5 \\ 1 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & -4 & 5 & 7 \\ 2 & -2 & 3 & 1 \\ 4 & -4 & 5 & 7 \\ 6 & 1 & -3 & -3 \end{bmatrix}$

105. Conjecture A **diagonal matrix** is a square matrix with all zero entries above and below its main diagonal. Evaluate the determinant of each diagonal matrix. Make a conjecture based on your results.

(a) $\begin{bmatrix} 7 & 0 \\ 0 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

lightpoet/Shutterstock.com

7.5 Applications of Matrices and Determinants

You can use determinants to solve real-life problems. For instance, in Exercise 33 on page 549, you will use a determinant to find the area of a region of forest infested with gypsy moths.

- **Use Cramer's Rule to solve systems of linear equations.**
- **Use determinants to find the areas of triangles.**
- **Use a determinant to test for collinear points and find an equation of a line passing through two points.**
- **Use matrices to encode and decode messages.**

Cramer's Rule

So far, you have studied three methods for solving a system of linear equations: substitution, elimination with equations, and elimination with matrices. In this section, you will study one more method, **Cramer's Rule,** named after Gabriel Cramer (1704–1752). This rule uses determinants to write the solution of a system of linear equations. To see how Cramer's Rule works, take another look at the solution described at the beginning of Section 7.4. There, it was pointed out that the system

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

has a solution

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \quad \text{and} \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

provided that

$$a_1b_2 - a_2b_1 \neq 0.$$

Each numerator and denominator in this solution can be expressed as a determinant, as follows.

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Relative to the original system, the denominators for x and y are simply the determinant of the *coefficient* matrix of the system. This determinant is denoted by D. The numerators for x and y are denoted by D_x and D_y, respectively. They are formed by using the column of constants as replacements for the coefficients of x and y, as follows.

Coefficient Matrix	D	D_x	D_y
$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$	$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$	$\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$	$\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

For example, given the system

$$\begin{cases} 2x - 5y = 3 \\ -4x + 3y = 8 \end{cases}$$

the coefficient matrix, D, D_x, and D_y are as follows.

Coefficient Matrix	D	D_x	D_y
$\begin{bmatrix} 2 & -5 \\ -4 & 3 \end{bmatrix}$	$\begin{vmatrix} 2 & -5 \\ -4 & 3 \end{vmatrix}$	$\begin{vmatrix} 3 & -5 \\ 8 & 3 \end{vmatrix}$	$\begin{vmatrix} 2 & 3 \\ -4 & 8 \end{vmatrix}$

Martynova Anna/Shutterstock.com

Cramer's Rule generalizes easily to systems of n equations in n variables. The value of each variable is given as the quotient of two determinants. The denominator is the determinant of the coefficient matrix, and the numerator is the determinant of the matrix formed by replacing the column corresponding to the variable (being solved for) with the column representing the constants. For instance, the solution for x_3 in the following system is shown.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases} \qquad x_3 = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

Cramer's Rule

If a system of n linear equations in n variables has a coefficient matrix A with a nonzero determinant $|A|$, then the solution of the system is

$$x_1 = \frac{|A_1|}{|A|}, \quad x_2 = \frac{|A_2|}{|A|}, \quad . . . , \quad x_n = \frac{|A_n|}{|A|}$$

where the ith column of A_i is the column of constants in the system of equations. If the determinant of the coefficient matrix is zero, then the system has either no solution or infinitely many solutions.

EXAMPLE 1 Using Cramer's Rule for a 2 × 2 System

Use Cramer's Rule to solve the system

$$\begin{cases} 4x - 2y = 10 \\ 3x - 5y = 11 \end{cases}.$$

Solution To begin, find the determinant of the coefficient matrix.

$$D = \begin{vmatrix} 4 & -2 \\ 3 & -5 \end{vmatrix} = -20 - (-6) = -14$$

Because this determinant is not zero, you can apply Cramer's Rule.

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 10 & -2 \\ 11 & -5 \end{vmatrix}}{-14} = \frac{-50 - (-22)}{-14} = \frac{-28}{-14} = 2$$

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} 4 & 10 \\ 3 & 11 \end{vmatrix}}{-14} = \frac{44 - 30}{-14} = \frac{14}{-14} = -1$$

So, the solution is $x = 2$ and $y = -1$. Check this in the original system.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use Cramer's Rule to solve the system

$$\begin{cases} 3x + 4y = 1 \\ 5x + 3y = 9 \end{cases}.$$

EXAMPLE 2 Using Cramer's Rule for a 3 × 3 System

Use Cramer's Rule to solve the system $\begin{cases} -x + 2y - 3z = 1 \\ 2x \qquad\quad + \; z = 0. \\ 3x - 4y + 4z = 2 \end{cases}$

Solution To find the determinant of the coefficient matrix

$$\begin{bmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{bmatrix}$$

expand along the second row, as follows.

$$\begin{aligned} D &= 2(-1)^3\begin{vmatrix} 2 & -3 \\ -4 & 4 \end{vmatrix} + 0(-1)^4\begin{vmatrix} -1 & -3 \\ 3 & 4 \end{vmatrix} + 1(-1)^5\begin{vmatrix} -1 & 2 \\ 3 & -4 \end{vmatrix} \\ &= -2(-4) + 0 - 1(-2) \\ &= 10 \end{aligned}$$

Because this determinant is not zero, you can apply Cramer's Rule.

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix}}{10} = \frac{8}{10} = \frac{4}{5}$$

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} -1 & 1 & -3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{vmatrix}}{10} = \frac{-15}{10} = -\frac{3}{2}$$

$$z = \frac{D_z}{D} = \frac{\begin{vmatrix} -1 & 2 & 1 \\ 2 & 0 & 0 \\ 3 & -4 & 2 \end{vmatrix}}{10} = \frac{-16}{10} = -\frac{8}{5}$$

The solution is $\left(\frac{4}{5}, -\frac{3}{2}, -\frac{8}{5}\right)$. Check this in the original system, as follows.

Check

$$\begin{array}{rcrcrcll} -\left(\frac{4}{5}\right) & + & 2\left(-\frac{3}{2}\right) & - & 3\left(-\frac{8}{5}\right) & \overset{?}{=} & 1 & \quad\text{Substitute into Equation 1.} \\ -\frac{4}{5} & - & 3 & + & \frac{24}{5} & = & 1 & \quad\text{Equation 1 checks. ✓} \\ 2\left(\frac{4}{5}\right) & & & + & \left(-\frac{8}{5}\right) & \overset{?}{=} & 0 & \quad\text{Substitute into Equation 2.} \\ \frac{8}{5} & & & - & \frac{8}{5} & = & 0 & \quad\text{Equation 2 checks. ✓} \\ 3\left(\frac{4}{5}\right) & - & 4\left(-\frac{3}{2}\right) & + & 4\left(-\frac{8}{5}\right) & \overset{?}{=} & 2 & \quad\text{Substitute into Equation 3.} \\ \frac{12}{5} & + & 6 & - & \frac{32}{5} & = & 2 & \quad\text{Equation 3 checks. ✓} \end{array}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use Cramer's Rule to solve the system $\begin{cases} 4x - \;\; y + \;\; z = 12 \\ 2x + 2y + 3z = \;\; 1. \\ 5x - 2y + 6z = 22 \end{cases}$

Remember that Cramer's Rule does not apply when the determinant of the coefficient matrix is zero. This would create division by zero, which is undefined.

Area of a Triangle

Another application of matrices and determinants is finding the area of a triangle whose vertices are given as points in a coordinate plane.

Area of a Triangle

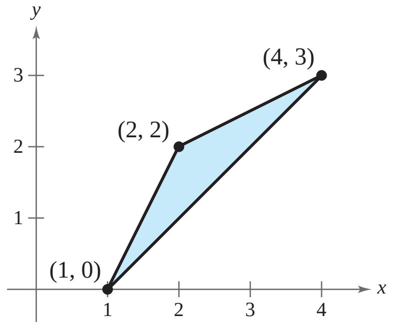
The area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is

$$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

where the symbol ± indicates that the appropriate sign should be chosen to yield a positive area.

EXAMPLE 3 Finding the Area of a Triangle

Find the area of a triangle whose vertices are (1, 0), (2, 2), and (4, 3), as shown below.



Solution Let $(x_1, y_1) = (1, 0)$, $(x_2, y_2) = (2, 2)$, and $(x_3, y_3) = (4, 3)$. Then, to find the area of the triangle, evaluate the determinant.

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

$$= 1(-1)^2\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} + 0(-1)^3\begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} + 1(-1)^4\begin{vmatrix} 2 & 2 \\ 4 & 3 \end{vmatrix}$$

$$= 1(-1) + 0 + 1(-2)$$

$$= -3$$

Using this value, you can conclude that the area of the triangle is

$$\text{Area} = -\frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$ Choose (−) so that the area is positive.

$$= -\frac{1}{2}(-3)$$

$$= \frac{3}{2} \text{ square units.}$$

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Find the area of a triangle whose vertices are (0, 0), (4, 1), and (2, 5).

Lines in a Plane

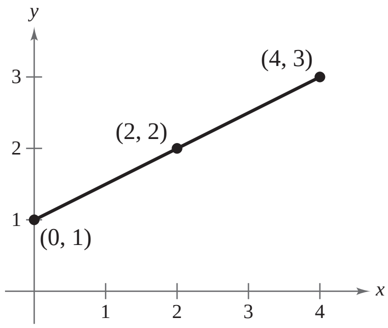


Figure 7.1

Suppose the three points in Example 3 had been on the same line. What would have happened had the area formula been applied to three such points? The answer is that the determinant would have been zero. Consider, for instance, the three collinear points (0, 1), (2, 2), and (4, 3), as shown in Figure 7.1. The area of the "triangle" that has these three points as vertices is

$$\frac{1}{2}\begin{vmatrix} 0 & 1 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = \frac{1}{2}\left[0(-1)^2\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} + 1(-1)^3\begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} + 1(-1)^4\begin{vmatrix} 2 & 2 \\ 4 & 3 \end{vmatrix}\right]$$

$$= \frac{1}{2}[0 - 1(-2) + 1(-2)]$$

$$= 0.$$

This result is generalized as follows.

Test for Collinear Points

Three points

$$(x_1, y_1), \quad (x_2, y_2), \quad \text{and} \quad (x_3, y_3)$$

are **collinear** (lie on the same line) if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

EXAMPLE 4 Testing for Collinear Points

Determine whether the points $(-2, -2)$, $(1, 1)$, and $(7, 5)$ are collinear. (See Figure 7.2.)

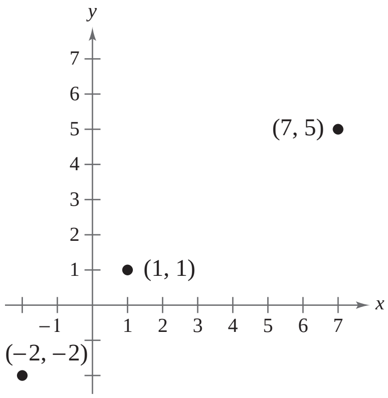


Figure 7.2

Solution Letting $(x_1, y_1) = (-2, -2)$, $(x_2, y_2) = (1, 1)$, and $(x_3, y_3) = (7, 5)$, you have

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} -2 & -2 & 1 \\ 1 & 1 & 1 \\ 7 & 5 & 1 \end{vmatrix}$$

$$= -2(-1)^2\begin{vmatrix} 1 & 1 \\ 5 & 1 \end{vmatrix} + (-2)(-1)^3\begin{vmatrix} 1 & 1 \\ 7 & 1 \end{vmatrix} + 1(-1)^4\begin{vmatrix} 1 & 1 \\ 7 & 5 \end{vmatrix}$$

$$= -2(-4) + 2(-6) + 1(-2)$$

$$= -6.$$

Because the value of this determinant is *not* zero, you can conclude that the three points do not lie on the same line. Moreover, the area of the triangle with vertices at these points is $\left(-\frac{1}{2}\right)(-6) = 3$ square units.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Determine whether the points

$$(-2, 4), \quad (3, -1), \quad \text{and} \quad (6, -4)$$

are collinear.

The test for collinear points can be adapted for another use. Given two points on a rectangular coordinate system, you can find an equation of the line passing through the two points.

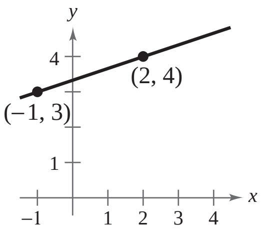
Two-Point Form of the Equation of a Line

An equation of the line passing through the distinct points (x_1, y_1) and (x_2, y_2) is given by

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

EXAMPLE 5 Finding an Equation of a Line

Find an equation of the line passing through the two points $(2, 4)$ and $(-1, 3)$, as shown below.



Solution Let $(x_1, y_1) = (2, 4)$ and $(x_2, y_2) = (-1, 3)$. Applying the determinant formula for the equation of a line produces

$$\begin{vmatrix} x & y & 1 \\ 2 & 4 & 1 \\ -1 & 3 & 1 \end{vmatrix} = 0.$$

To evaluate this determinant, you can expand by cofactors along the first row to find an equation of the line.

$$\begin{aligned} x(-1)^2\begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} + y(-1)^3\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} + 1(-1)^4\begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix} &= 0 \\ x(1) - y(3) + (1)(10) &= 0 \\ x - 3y + 10 &= 0 \end{aligned}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find an equation of the line passing through the two points $(-3, -1)$ and $(3, 5)$. ■

Note that this method of finding the equation of a line works for all lines, including horizontal and vertical lines. For instance, the equation of the vertical line passing through $(2, 0)$ and $(2, 2)$ is

$$\begin{aligned} \begin{vmatrix} x & y & 1 \\ 2 & 0 & 1 \\ 2 & 2 & 1 \end{vmatrix} &= 0 \\ 4 - 2x &= 0 \\ x &= 2. \end{aligned}$$

Because of the heavy use of the Internet to conduct business, Internet security is of the utmost importance. If a malicious party should receive confidential information such as passwords, personal identification numbers, credit card numbers, social security numbers, bank account details, or corporate secrets, the effects can be damaging. To protect the confidentiality and integrity of such information, the most popular forms of Internet security use data *encryption,* the process of encoding information so that the only way to decode it, apart from a brute force "exhaustion attack," is to use a *key.* Data encryption technology uses algorithms based on the material presented here, but on a much more sophisticated level, to prevent malicious parties from discovering the key.

Cryptography

A **cryptogram** is a message written according to a secret code. (The Greek word *kryptos* means "hidden.") Matrix multiplication can be used to encode and decode messages. To begin, you need to assign a number to each letter in the alphabet (with 0 assigned to a blank space), as follows.

0 = _
1 = A
2 = B
3 = C
4 = D
5 = E
6 = F
7 = G
8 = H
9 = I
10 = J
11 = K
12 = L
13 = M
14 = N
15 = O
16 = P
17 = Q
18 = R
19 = S
20 = T
21 = U
22 = V
23 = W
24 = X
25 = Y
26 = Z

Then the message is converted to numbers and partitioned into **uncoded row matrices,** each having n entries, as demonstrated in Example 6.

EXAMPLE 6 Forming Uncoded Row Matrices

Write the uncoded row matrices of order 1 × 3 for the message

MEET ME MONDAY.

Solution Partitioning the message (including blank spaces, but ignoring punctuation) into groups of three produces the following uncoded row matrices.

$$\begin{array}{ccccc} [13\quad 5\quad 5] & [20\quad 0\quad 13] & [5\quad 0\quad 13] & [15\quad 14\quad 4] & [1\quad 25\quad 0] \\ \text{M}\quad\text{E}\quad\text{E} & \text{T}\qquad\quad\text{M} & \text{E}\qquad\quad\text{M} & \text{O}\quad\text{N}\quad\text{D} & \text{A}\quad\text{Y}\qquad \end{array}$$

Note that a blank space is used to fill out the last uncoded row matrix.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write the uncoded row matrices of order 1 × 3 for the message

OWLS ARE NOCTURNAL.

To encode a message, use the techniques demonstrated in Section 7.3 to choose an $n \times n$ invertible matrix such as

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$$

and multiply the uncoded row matrices by A (on the right) to obtain **coded row matrices.** Here is an example.

$$\begin{array}{ccccc} \textbf{Uncoded Matrix} & \textbf{Encoding Matrix } \boldsymbol{A} & & \textbf{Coded Matrix} \\ [13\quad 5\quad 5] & \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} & = & [13\quad -26\quad 21] \end{array}$$

Andrea Danti/Shutterstock.com

EXAMPLE 7 Encoding a Message

Use the following invertible matrix to encode the message MEET ME MONDAY.

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$$

Solution The coded row matrices are obtained by multiplying each of the uncoded row matrices found in Example 6 by the matrix A, as follows.

Uncoded Matrix Encoding Matrix *A* Coded Matrix

$$[13 \quad 5 \quad 5] \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [13 \quad -26 \quad 21]$$

$$[20 \quad 0 \quad 13] \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [33 \quad -53 \quad -12]$$

$$[5 \quad 0 \quad 13] \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [18 \quad -23 \quad -42]$$

$$[15 \quad 14 \quad 4] \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [5 \quad -20 \quad 56]$$

$$[1 \quad 25 \quad 0] \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [-24 \quad 23 \quad 77]$$

So, the sequence of coded row matrices is

$$[13 \quad -26 \quad 21]\,[33 \quad -53 \quad -12]\,[18 \quad -23 \quad -42]\,[5 \quad -20 \quad 56]\,[-24 \quad 23 \quad 77].$$

Finally, removing the matrix notation produces the following cryptogram.

13 −26 21 33 −53 −12 18 −23 −42 5 −20 56 −24 23 77

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use the following invertible matrix to encode the message OWLS ARE NOCTURNAL.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$$

For those who do not know the encoding matrix A, decoding the cryptogram found in Example 7 is difficult. But for an authorized receiver who knows the encoding matrix A, decoding is simple. The receiver just needs to multiply the coded row matrices by A^{-1} (on the right) to retrieve the uncoded row matrices. Here is an example.

$$\underbrace{[13 \quad -26 \quad 21]}_{\text{Coded}} \underbrace{\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}}_{A^{-1}} = \underbrace{[13 \quad 5 \quad 5]}_{\text{Uncoded}}$$

HISTORICAL NOTE

During World War II, Navajo soldiers created a code using their native language to send messages between battalions. Native words were assigned to represent characters in the English alphabet, and the soldiers created a number of expressions for important military terms, such as *iron-fish* to mean *submarine*. Without the Navajo Code Talkers, the Second World War might have had a very different outcome.

EXAMPLE 8 Decoding a Message

Use the inverse of the matrix A in Example 7 to decode the cryptogram

$$13 \quad -26 \quad 21 \quad 33 \quad -53 \quad -12 \quad 18 \quad -23 \quad -42 \quad 5 \quad -20 \quad 56 \quad -24 \quad 23 \quad 77.$$

Solution First find the decoding matrix A^{-1} by using the techniques demonstrated in Section 7.3. Then partition the message into groups of three to form the coded row matrices. Finally, multiply each coded row matrix by A^{-1} (on the right).

Coded Matrix **Decoding Matrix A^{-1}** **Decoded Matrix**

$$\begin{bmatrix} 13 & -26 & 21 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 13 & 5 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 33 & -53 & -12 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 20 & 0 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 18 & -23 & -42 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -20 & 56 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 15 & 14 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -24 & 23 & 77 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 25 & 0 \end{bmatrix}$$

So, the message is as follows.

$$\begin{array}{ccccccccccccccc} [13 & 5 & 5] & [20 & 0 & 13] & [5 & 0 & 13] & [15 & 14 & 4] & [1 & 25 & 0] \\ \text{M} & \text{E} & \text{E} & \text{T} & & \text{M} & \text{E} & & \text{M} & \text{O} & \text{N} & \text{D} & \text{A} & \text{Y} & \end{array}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use the inverse of the matrix A in the Checkpoint with Example 7 to decode the cryptogram

$$\begin{array}{cccccccccccc} 110 & -39 & -59 & 25 & -21 & -3 & 23 & -18 & -5 & 47 & -20 & -24 \\ 149 & -56 & -75 & 87 & -38 & -37. \end{array}$$

Summarize (Section 7.5)

1. State Cramer's Rule *(page 541)*. For examples of using Cramer's Rule to solve systems of linear equations, see Examples 1 and 2.
2. State the formula for finding the area of a triangle using a determinant *(page 543)*. For an example of finding the area of a triangle, see Example 3.
3. State the test for collinear points *(page 544)*. For an example of testing for collinear points, see Example 4.
4. Explain how to use a determinant to find an equation of a line *(page 545)*. For an example of finding an equation of a line, see Example 5.
5. Describe how to use matrices to encode and decode a message *(pages 546–548, Examples 6–8)*.

Bettmann/Corbis

7.5 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. The method of using determinants to solve a system of linear equations is called ________ ________.
2. Three points are ________ when the points lie on the same line.
3. The area A of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is given by ________.
4. A message written according to a secret code is called a ________.
5. To encode a message, choose an invertible matrix A and multiply the ________ row matrices by A (on the right) to obtain ________ row matrices.
6. If a message is encoded using an invertible matrix A, then the message can be decoded by multiplying the coded row matrices by ________ (on the right).

Skills and Applications

Using Cramer's Rule

In Exercises 7–16, use Cramer's Rule to solve (if possible) the system of equations.

7. $\begin{cases} -7x + 11y = -1 \\ 3x - 9y = 9 \end{cases}$ **8.** $\begin{cases} 4x - 3y = -10 \\ 6x + 9y = 12 \end{cases}$

9. $\begin{cases} 3x + 2y = -2 \\ 6x + 4y = 4 \end{cases}$ **10.** $\begin{cases} 6x - 5y = 17 \\ -13x + 3y = -76 \end{cases}$

11. $\begin{cases} -0.4x + 0.8y = 1.6 \\ 0.2x + 0.3y = 2.2 \end{cases}$ **12.** $\begin{cases} 2.4x - 1.3y = 14.63 \\ -4.6x + 0.5y = -11.51 \end{cases}$

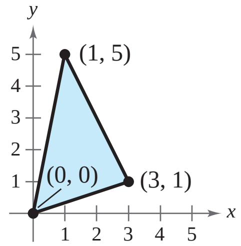
13. $\begin{cases} 4x - y + z = -5 \\ 2x + 2y + 3z = 10 \\ 5x - 2y + 6z = 1 \end{cases}$ **14.** $\begin{cases} 4x - 2y + 3z = -2 \\ 2x + 2y + 5z = 16 \\ 8x - 5y - 2z = 4 \end{cases}$

15. $\begin{cases} x + 2y + 3z = -3 \\ -2x + y - z = 6 \\ 3x - 3y + 2z = -11 \end{cases}$ **16.** $\begin{cases} 5x - 4y + z = -14 \\ -x + 2y - 2z = 10 \\ 3x + y + z = 1 \end{cases}$

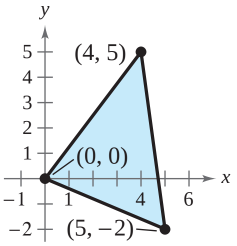
Finding the Area of a Triangle

In Exercises 17–28, use a determinant to find the area with the given vertices.

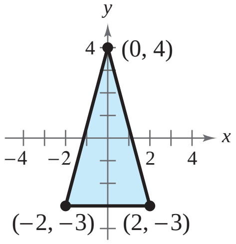
17.



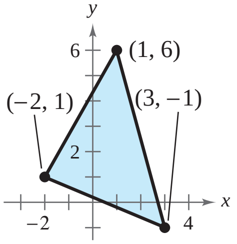
18.



19.



20.



21. $\left(0, \frac{1}{2}\right), (4, 3), \left(\frac{5}{2}, 0\right)$ **22.** $(-4, -5), (6, 10), (6, -1)$

23. $(-2, 4), (2, 3), (-1, 5)$ **24.** $(0, -2), (-1, 4), (3, 5)$

25. $(-3, 5), (2, 6), (3, -5)$ **26.** $(-2, 4), (1, 5), (3, -2)$

27. $(-4, 2), \left(0, \frac{7}{2}\right), \left(3, -\frac{1}{2}\right)$ **28.** $\left(\frac{9}{2}, 0\right), (2, 6), \left(0, -\frac{3}{2}\right)$

Finding a Coordinate

In Exercises 29 and 30, find a value of y such that the triangle with the given vertices has an area of 4 square units.

29. $(-5, 1), (0, 2), (-2, y)$ **30.** $(-4, 2), (-3, 5), (-1, y)$

Finding a Coordinate

In Exercises 31 and 32, find a value of y such that the triangle with the given vertices has an area of 6 square units.

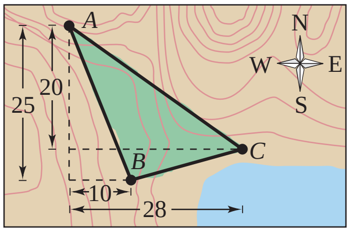
31. $(-2, -3), (1, -1), (-8, y)$

32. $(1, 0), (5, -3), (-3, y)$

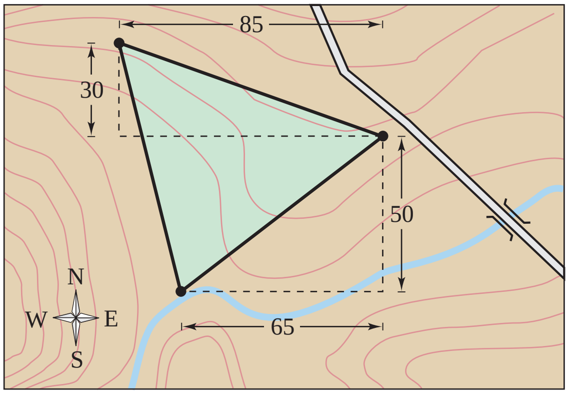
33. Area of Infestation

A large region of forest has been infested with gypsy moths. The region is roughly triangular, as shown in the figure. From the northernmost vertex A of the region, the distances to the other vertices are 25 miles south and 10 miles east (for vertex B), and 20 miles south and 28 miles east (for vertex C). Use a graphing utility to approximate the number of square miles in this region.

Martynova Anna/Shutterstock.com



34. **Botany** A botanist is studying the plants growing in a triangular tract of land, as shown in the figure. To estimate the number of square feet in the tract, the botanist starts at one vertex, walks 65 feet east and 50 feet north to the second vertex, and then walks 85 feet west and 30 feet north to the third vertex. Use a graphing utility to determine how many square feet there are in the tract of land.



Testing for Collinear Points **In Exercises 35–40, use a determinant to determine whether the points are collinear.**

35. $(3, -1), (0, -3), (12, 5)$

36. $(3, -5), (6, 1), (4, 2)$

37. $\left(2, -\frac{1}{2}\right), (-4, 4), (6, -3)$

38. $\left(0, \frac{1}{2}\right), (2, -1), \left(-4, \frac{7}{2}\right)$

39. $(0, 2), (1, 2.4), (-1, 1.6)$

40. $(2, 3), (3, 3.5), (-1, 2)$

Finding a Coordinate **In Exercises 41 and 42, find y such that the points are collinear.**

41. $(2, -5), (4, y), (5, -2)$ **42.** $(-6, 2), (-5, y), (-3, 5)$

Finding an Equation of a Line **In Exercises 43–48, use a determinant to find an equation of the line passing through the points.**

43. $(0, 0), (5, 3)$ **44.** $(0, 0), (-2, 2)$

45. $(-4, 3), (2, 1)$ **46.** $(10, 7), (-2, -7)$

47. $\left(-\frac{1}{2}, 3\right), \left(\frac{5}{2}, 1\right)$ **48.** $\left(\frac{2}{3}, 4\right), (6, 12)$

Encoding a Message **In Exercises 49 and 50, (a) write the uncoded 1 × 2 row matrices for the message, and then (b) encode the message using the encoding matrix.**

Message	Encoding Matrix
49. COME HOME SOON	$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$
50. HELP IS ON THE WAY	$\begin{bmatrix} -2 & 3 \\ -1 & 1 \end{bmatrix}$

Encoding a Message **In Exercises 51 and 52, (a) write the uncoded 1 × 3 row matrices for the message, and then (b) encode the message using the encoding matrix.**

Message	Encoding Matrix
51. CALL ME TOMORROW	$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$
52. PLEASE SEND MONEY	$\begin{bmatrix} 4 & 2 & 1 \\ -3 & -3 & -1 \\ 3 & 2 & 1 \end{bmatrix}$

Encoding a Message **In Exercises 53–56, write a cryptogram for the message using the matrix**

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}.$$

53. LANDING SUCCESSFUL

54. ICEBERG DEAD AHEAD

55. HAPPY BIRTHDAY

56. OPERATION OVERLOAD

Decoding a Message **In Exercises 57–60, use A^{-1} to decode the cryptogram.**

57. $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

11 21 64 112 25 50 29 53 23 46
40 75 55 92

58. $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

85 120 6 8 10 15 84 117 42 56 90
125 60 80 30 45 19 26

59. $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$

9 −1 −9 38 −19 −19 28 −9 −19
−80 25 41 −64 21 31 9 −5 −4

60. $A = \begin{bmatrix} 3 & -4 & 2 \\ 0 & 2 & 1 \\ 4 & -5 & 3 \end{bmatrix}$

112 −140 83 19 −25 13 72 −76 61 95
−118 71 20 21 38 35 −23 36 42 −48 32

Decoding a Message **In Exercises 61 and 62, decode the cryptogram by using the inverse of the matrix A in Exercises 53–56.**

61. 20 17 −15 −12 −56 −104 1 −25 −65
62 143 181

62. 13 −9 −59 61 112 106 −17 −73 −131 11
24 29 65 144 172

63. **Decoding a Message** The following cryptogram was encoded with a 2×2 matrix.

8 21 −15 −10 −13 −13 5 10 5 25 5 19
−1 6 20 40 −18 −18 1 16

The last word of the message is _RON. What is the message?

64. **Decoding a Message** The following cryptogram was encoded with a 2×2 matrix.

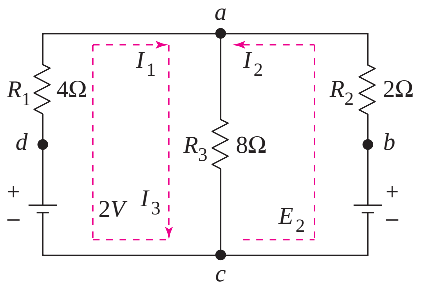
5 2 25 11 −2 −7 −15 −15 32 14 −8
−13 38 19 −19 −19 37 16

The last word of the message is _SUE. What is the message?

65. **Circuit Analysis** Consider the circuit in the figure. The currents I_1, I_2, and I_3 in amperes are given by the solution of the system of linear equations.

$$\begin{cases} 4I_1 \qquad\quad + 8I_3 = 2 \\ \qquad 2I_2 + 8I_3 = 6 \\ I_1 + I_2 - I_3 = 0. \end{cases}$$

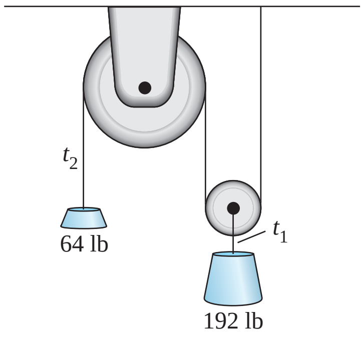
Use Cramer's Rule to find the three currents.



66. **Pulley System** A system of pulleys that is assumed frictionless and without mass is loaded with 192-pound and 64-pound weights (see figure). The tensions t_1 and t_2 in the ropes and the acceleration a of the 64-pound weight are found by solving the system

$$\begin{cases} t_1 - 2t_2 \qquad = 0 \\ t_1 \qquad - 3a = 192 \\ \qquad t_2 + 2a = 64 \end{cases}$$

where t_1 and t_2 are measured in pounds and a is in feet per second squared. Use Cramer's Rule to find t_1, t_2, and a.



Exploration

True or False? **In Exercises 67–70, determine whether the statement is true or false. Justify your answer.**

67. In Cramer's Rule, the numerator is the determinant of the coefficient matrix.

68. You cannot use Cramer's Rule to solve a system of linear equations when the determinant of the coefficient matrix is zero.

69. In a system of linear equations, when the determinant of the coefficient matrix is zero, the system has no solution.

70. The points $(-5, -13)$, $(0, 2)$, and $(3, 11)$ are collinear.

71. **Writing** Use your school's library, the Internet, or some other reference source to research a few current real-life uses of cryptography. Write a short summary of these uses. Include a description of how messages are encoded and decoded in each case.

72. **Writing**

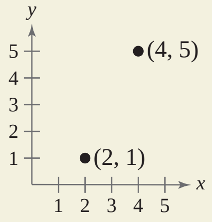
(a) State Cramer's Rule for solving a system of linear equations.

(b) At this point in the text, you have learned several methods for solving systems of linear equations. Briefly describe which method(s) you find easiest to use and which method(s) you find most difficult to use.

73. **Reasoning** Use determinants to find the area of a triangle with vertices $(3, -1)$, $(7, -1)$, and $(7, 5)$. Confirm your answer by plotting the points in a coordinate plane and using the formula

$$\text{Area} = \tfrac{1}{2}(\text{base})(\text{height}).$$

74. **HOW DO YOU SEE IT?** At this point in the text, you have learned several methods for finding an equation of a line that passes through two given points. Briefly describe the methods that can be used to find the equation of the line that passes through the two points shown. Discuss the advantages and disadvantages of each method.



Chapter Summary

	What Did You Learn?	Explanation/Examples	Review Exercises
Section 7.1	Write matrices and identify their orders *(p. 496)*.	$\underset{2 \times 2}{\begin{bmatrix}-1 & 1\\ 4 & 7\end{bmatrix}}$ $\underset{1 \times 3}{\begin{bmatrix}-2 & 3 & 0\end{bmatrix}}$ $\underset{3 \times 2}{\begin{bmatrix}4 & -3\\ 5 & 0\\ -2 & 1\end{bmatrix}}$ $\underset{2 \times 1}{\begin{bmatrix}8\\ -8\end{bmatrix}}$	1–8
	Perform elementary row operations on matrices *(p. 498)*.	**Elementary Row Operations** **1.** Interchange two rows. **2.** Multiply a row by a nonzero constant. **3.** Add a multiple of a row to another row.	9, 10
	Use matrices and Gaussian elimination to solve systems of linear equations *(p. 499)*.	**Gaussian Elimination with Back-Substitution** **1.** Write the augmented matrix of the system of linear equations. **2.** Use elementary row operations to rewrite the augmented matrix in row-echelon form. **3.** Write the system of linear equations corresponding to the matrix in row-echelon form and use back-substitution to find the solution.	11–28
	Use matrices and Gauss-Jordan elimination to solve systems of linear equations *(p. 503)*.	Gauss-Jordan elimination continues the reduction process on a matrix in row-echelon form until a *reduced* row-echelon form is obtained. (See Example 8.)	29–36
Section 7.2	Decide whether two matrices are equal *(p. 509)*.	Two matrices are equal when their corresponding entries are equal.	37–40
	Add and subtract matrices, and multiply matrices by scalars *(p. 510)*.	**Definition of Matrix Addition** If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of order $m \times n$, then their sum is the $m \times n$ matrix given by $A + B = [a_{ij} + b_{ij}]$. **Definition of Scalar Multiplication** If $A = [a_{ij}]$ is an $m \times n$ matrix and c is a scalar, then the scalar multiple of A by c is the $m \times n$ matrix given by $cA = [ca_{ij}]$.	41–54
	Multiply two matrices *(p. 514)*.	**Definition of Matrix Multiplication** If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, then the product AB is an $m \times p$ matrix $AB = [c_{ij}]$ where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdot \cdot \cdot + a_{in}b_{nj}$.	55–66
	Use matrix operations to model and solve real-life problems *(p. 517)*.	You can use matrix operations to find the total cost of equipment for two softball teams. (See Example 13.)	67–70
Section 7.3	Verify that two matrices are inverses of each other *(p. 523)*.	**Definition of the Inverse of a Square Matrix** Let A be an $n \times n$ matrix and let I_n be the $n \times n$ identity matrix. If there exists a matrix A^{-1} such that $AA^{-1} = I_n = A^{-1}A$ then A^{-1} is the inverse of A.	71–74

	What Did You Learn?	Explanation/Examples	Review Exercises
Section 7.3	Use Gauss-Jordan elimination to find the inverses of matrices *(p. 524).*	**Finding an Inverse Matrix** Let A be a square matrix of order $n \times n$. **1.** Write the $n \times 2n$ matrix that consists of the given matrix A on the left and the $n \times n$ identity matrix I on the right to obtain $[A \vdots I]$. **2.** If possible, row reduce A to I using elementary row operations on the *entire* matrix $[A \vdots I]$. The result will be the matrix $[I \vdots A^{-1}]$. If this is not possible, then A is not invertible. **3.** Check your work by multiplying to see that $AA^{-1} = I = A^{-1}A$.	75–82
	Use a formula to find the inverses of 2×2 matrices *(p. 527).*	If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - bc \neq 0$, then $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.	83–90
	Use inverse matrices to solve systems of linear equations *(p. 528).*	If A is an invertible matrix, then the system of linear equations represented by $AX = B$ has a unique solution given by $X = A^{-1}B$.	91–108
Section 7.4	Find the determinants of 2×2 matrices *(p. 532).*	The determinant of the matrix $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ is given by $\det(A) = \lvert A \rvert = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$.	109–112
	Find minors and cofactors of square matrices *(p. 534).*	If A is a square matrix, then the minor M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The cofactor C_{ij} of the entry a_{ij} is $C_{ij} = (-1)^{i+j}M_{ij}$.	113–116
	Find the determinants of square matrices *(p. 535).*	If A is a square matrix (of order 2×2 or greater), then the determinant of A is the sum of the entries in any row (or column) of A multiplied by their respective cofactors.	117–126
Section 7.5	Use Cramer's Rule to solve systems of linear equations *(p. 540).*	Cramer's Rule uses determinants to write the solution of a system of linear equations.	127–130
	Use determinants to find the areas of triangles *(p. 543).*	The area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is $\text{Area} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ where the symbol $\pm$ indicates that the appropriate sign should be chosen to yield a positive area.	131–134
	Use a determinant to test for collinear points and find an equation of a line passing through two points *(p. 544).*	Three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear (lie on the same line) if and only if $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$.	135–140
	Use matrices to encode and decode messages *(p. 546).*	You can use the inverse of a matrix to decode a cryptogram. (See Example 8.)	141–144

Review Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

7.1 Order of a Matrix

In Exercises 1–4, determine the order of the matrix.

1. $\begin{bmatrix} -4 \\ 0 \\ 5 \end{bmatrix}$

2. $\begin{bmatrix} 3 & -1 & 0 & 6 \\ -2 & 7 & 1 & 4 \end{bmatrix}$

3. $[3]$

4. $[6 \quad 2 \quad -5 \quad 8 \quad 0]$

Writing an Augmented Matrix

In Exercises 5 and 6, write the augmented matrix for the system of linear equations.

5. $\left\{\begin{aligned} 3x - 10y &= 15 \\ 5x + 4y &= 22 \end{aligned}\right.$

6. $\left\{\begin{aligned} 8x - 7y + 4z &= 12 \\ 3x - 5y + 2z &= 20 \end{aligned}\right.$

Writing a System of Equations

In Exercises 7 and 8, write the system of linear equations represented by the augmented matrix. (Use variables *x*, *y*, *z*, and *w*, if applicable.)

7. $\left[\begin{array}{ccccc} 5 & 1 & 7 & \vdots & -9 \\ 4 & 2 & 0 & \vdots & 10 \\ 9 & 4 & 2 & \vdots & 3 \end{array}\right]$

8. $\left[\begin{array}{cccccc} 13 & 16 & 7 & 3 & \vdots & 2 \\ 4 & 10 & -4 & 3 & \vdots & -1 \end{array}\right]$

Writing a Matrix in Row-Echelon Form

In Exercises 9 and 10, write the matrix in row-echelon form. (Remember that the row-echelon form of a matrix is not unique.)

9. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix}$

10. $\begin{bmatrix} 4 & 8 & 16 \\ 3 & -1 & 2 \\ -2 & 10 & 12 \end{bmatrix}$

Using Back-Substitution

In Exercises 11–14, write the system of linear equations represented by the augmented matrix. Then use back-substitution to solve the system. (Use variables *x*, *y*, and *z*, if applicable.)

11. $\left[\begin{array}{ccccc} 1 & 2 & 3 & \vdots & 9 \\ 0 & 1 & -2 & \vdots & 2 \\ 0 & 0 & 1 & \vdots & 0 \end{array}\right]$

12. $\left[\begin{array}{ccccc} 1 & 3 & -9 & \vdots & 4 \\ 0 & 1 & -1 & \vdots & 10 \\ 0 & 0 & 1 & \vdots & -2 \end{array}\right]$

13. $\left[\begin{array}{ccccc} 1 & -5 & 4 & \vdots & 1 \\ 0 & 1 & 2 & \vdots & 3 \\ 0 & 0 & 1 & \vdots & 4 \end{array}\right]$

14. $\left[\begin{array}{ccccc} 1 & -8 & 0 & \vdots & -2 \\ 0 & 1 & -1 & \vdots & -7 \\ 0 & 0 & 1 & \vdots & 1 \end{array}\right]$

Gaussian Elimination with Back-Substitution

In Exercises 15–28, use matrices to solve the system of equations (if possible). Use Gaussian elimination with back-substitution.

15. $\left\{\begin{aligned} 5x + 4y &= 2 \\ -x + y &= -22 \end{aligned}\right.$

16. $\left\{\begin{aligned} 2x - 5y &= 2 \\ 3x - 7y &= 1 \end{aligned}\right.$

17. $\left\{\begin{aligned} 0.3x - 0.1y &= -0.13 \\ 0.2x - 0.3y &= -0.25 \end{aligned}\right.$

18. $\left\{\begin{aligned} 0.2x - 0.1y &= 0.07 \\ 0.4x - 0.5y &= -0.01 \end{aligned}\right.$

19. $\left\{\begin{aligned} -x + 2y &= 3 \\ 2x - 4y &= 6 \end{aligned}\right.$

20. $\left\{\begin{aligned} -x + 2y &= 3 \\ 2x - 4y &= -6 \end{aligned}\right.$

21. $\left\{\begin{aligned} x - 2y + z &= 7 \\ 2x + y - 2z &= -4 \\ -x + 3y + 2z &= -3 \end{aligned}\right.$

22. $\left\{\begin{aligned} x - 2y + z &= 4 \\ 2x + y - 2z &= -24 \\ -x + 3y + 2z &= 20 \end{aligned}\right.$

23. $\left\{\begin{aligned} 2x + y + 2z &= 4 \\ 2x + 2y \phantom{{}+ 2z} &= 5 \\ 2x - y + 6z &= 2 \end{aligned}\right.$

24. $\left\{\begin{aligned} x + 2y + 6z &= 1 \\ 2x + 5y + 15z &= 4 \\ 3x + y + 3z &= -6 \end{aligned}\right.$

25. $\left\{\begin{aligned} 2x + 3y + z &= 10 \\ 2x - 3y - 3z &= 22 \\ 4x - 2y + 3z &= -2 \end{aligned}\right.$

26. $\left\{\begin{aligned} 2x + 3y + 3z &= 3 \\ 6x + 6y + 12z &= 13 \\ 12x + 9y - z &= 2 \end{aligned}\right.$

27. $\left\{\begin{aligned} 2x + y + z \phantom{{}- 2w} &= 6 \\ -2y + 3z - w &= 9 \\ 3x + 3y - 2z - 2w &= -11 \\ x + z + 3w &= 14 \end{aligned}\right.$

28. $\left\{\begin{aligned} x + 2y + w &= 3 \\ -3y + 3z &= 0 \\ 4x + 4y + z + 2w &= 0 \\ 2x + z &= 3 \end{aligned}\right.$

Gauss-Jordan Elimination

In Exercises 29–34, use matrices to solve the system of equations (if possible). Use Gauss-Jordan elimination.

29. $\left\{\begin{aligned} x + 2y - z &= 3 \\ x - y - z &= -3 \\ 2x + y + 3z &= 10 \end{aligned}\right.$

30. $\left\{\begin{aligned} x - 3y + z &= 2 \\ 3x - y - z &= -6 \\ -x + y - 3z &= -2 \end{aligned}\right.$

31. $\left\{\begin{aligned} -x + y + 2z &= 1 \\ 2x + 3y + z &= -2 \\ 5x + 4y + 2z &= 4 \end{aligned}\right.$

32. $\left\{\begin{aligned} 4x + 4y + 4z &= 5 \\ 4x - 2y - 8z &= 1 \\ 5x + 3y + 8z &= 6 \end{aligned}\right.$

33. $\left\{\begin{aligned} 2x - y + 9z &= -8 \\ -x - 3y + 4z &= -15 \\ 5x + 2y - z &= 17 \end{aligned}\right.$

34. $\left\{\begin{aligned} -3x + y + 7z &= -20 \\ 5x - 2y - z &= 34 \\ -x + y + 4z &= -8 \end{aligned}\right.$

Using a Graphing Utility **In Exercises 35 and 36, use the matrix capabilities of a graphing utility to write the augmented matrix corresponding to the system of equations in reduced row-echelon form. Then solve the system.**

35. $$\begin{cases} 3x - y + 5z - 2w = -44 \\ x + 6y + 4z - w = 1 \\ 5x - y + z + 3w = -15 \\ 4y - z - 8w = 58 \end{cases}$$

36. $$\begin{cases} 4x + 12y + 2z = 20 \\ x + 6y + 4z = 12 \\ x + 6y + z = 8 \\ -2x - 10y - 2z = -10 \end{cases}$$

7.2 **Equality of Matrices** **In Exercises 37–40, find *x* and *y*.**

37. $\begin{bmatrix} -1 & x \\ y & 9 \end{bmatrix} = \begin{bmatrix} -1 & 12 \\ -7 & 9 \end{bmatrix}$

38. $\begin{bmatrix} -1 & 0 \\ x & 5 \\ -4 & y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 8 & 5 \\ -4 & 0 \end{bmatrix}$

39. $\begin{bmatrix} x+3 & -4 & 4y \\ 0 & -3 & 2 \\ -2 & y+5 & 6x \end{bmatrix} = \begin{bmatrix} 5x-1 & -4 & 44 \\ 0 & -3 & 2 \\ -2 & 16 & 6 \end{bmatrix}$

40. $\begin{bmatrix} -9 & 4 & 2 & -5 \\ 0 & -3 & 7 & -4 \\ 6 & -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -9 & 4 & x-10 & -5 \\ 0 & -3 & 7 & 2y \\ \frac{1}{2}x & -1 & 1 & 0 \end{bmatrix}$

Operations with Matrices **In Exercises 41–44, if possible, find (a) $A + B$, (b) $A - B$, (c) $4A$, and (d) $A + 3B$.**

41. $A = \begin{bmatrix} 2 & -2 \\ 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 10 \\ 12 & 8 \end{bmatrix}$

42. $A = \begin{bmatrix} 4 & 3 \\ -6 & 1 \\ 10 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 11 \\ 15 & 25 \\ 20 & 29 \end{bmatrix}$

43. $A = \begin{bmatrix} 5 & 4 \\ -7 & 2 \\ 11 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 3 \\ 4 & 12 \\ 20 & 40 \end{bmatrix}$

44. $A = \begin{bmatrix} 6 & -5 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 4 \\ 8 \end{bmatrix}$

Evaluating an Expression **In Exercises 45–50, evaluate the expression. If it is not possible, explain why.**

45. $\begin{bmatrix} 7 & 3 \\ -1 & 5 \end{bmatrix} + \begin{bmatrix} 10 & -20 \\ 14 & -3 \end{bmatrix}$

46. $\begin{bmatrix} -11 & 16 & 19 \\ -7 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 8 & -4 \\ -2 & 10 \end{bmatrix}$

47. $-2\begin{bmatrix} 1 & 2 \\ 5 & -4 \\ 6 & 0 \end{bmatrix} + 8\begin{bmatrix} 7 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}$

48. $-\begin{bmatrix} 8 & -1 & 8 \\ -2 & 4 & 12 \\ 0 & -6 & 0 \end{bmatrix} - 5\begin{bmatrix} -2 & 0 & -4 \\ 3 & -1 & 1 \\ 6 & 12 & -8 \end{bmatrix}$

49. $3\begin{bmatrix} 8 & -2 & 5 \\ 1 & 3 & -1 \end{bmatrix} + 6\begin{bmatrix} 4 & -2 & -3 \\ 2 & 7 & 6 \end{bmatrix}$

50. $-5\begin{bmatrix} 2 & 0 \\ 7 & -2 \\ 8 & 2 \end{bmatrix} + 4\begin{bmatrix} 4 & -2 \\ 6 & 11 \\ -1 & 3 \end{bmatrix}$

Solving a Matrix Equation **In Exercises 51–54, solve for *X* in the equation, where**

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{-4} & \mathbf{0} \\ \mathbf{1} & \mathbf{-5} \\ \mathbf{-3} & \mathbf{2} \end{bmatrix} \quad \textbf{and} \quad \boldsymbol{B} = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{-2} & \mathbf{1} \\ \mathbf{4} & \mathbf{4} \end{bmatrix}.$$

51. $X = 2A - 3B$

52. $6X = 4A + 3B$

53. $3X + 2A = B$

54. $2A - 5B = 3X$

Finding the Product of Two Matrices **In Exercises 55–58, find *AB*, if possible.**

55. $A = \begin{bmatrix} 2 & -2 \\ 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 10 \\ 12 & 8 \end{bmatrix}$

56. $A = \begin{bmatrix} 5 & 4 \\ -7 & 2 \\ 11 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 12 \\ 20 & 40 \\ 15 & 30 \end{bmatrix}$

57. $A = \begin{bmatrix} 5 & 4 \\ -7 & 2 \\ 11 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 12 \\ 20 & 40 \end{bmatrix}$

58. $A = \begin{bmatrix} 6 & -5 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 4 \\ 8 \end{bmatrix}$

Evaluating an Expression **In Exercises 59–62, evaluate the expression. If it is not possible, explain why.**

59. $\begin{bmatrix} 1 & 2 \\ 5 & -4 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} 6 & -2 & 8 \\ 4 & 0 & 0 \end{bmatrix}$

60. $\begin{bmatrix} 1 & 5 & 6 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} 6 & -2 & 8 \\ 4 & 0 & 0 \end{bmatrix}$

61. $\begin{bmatrix} 1 & 5 & 6 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} 6 & 4 \\ -2 & 0 \\ 8 & 0 \end{bmatrix}$

62. $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & -4 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 4 & -3 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

Matrix Multiplication **In Exercises 63–66, use the matrix capabilities of a graphing utility to find the product, if possible.**

63. $\begin{bmatrix} 4 & 1 \\ 11 & -7 \\ 12 & 3 \end{bmatrix}\begin{bmatrix} 3 & -5 & 6 \\ 2 & -2 & -2 \end{bmatrix}$

64. $\begin{bmatrix} -2 & 3 & 10 \\ 4 & -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -5 & 2 \\ 3 & 2 \end{bmatrix}$

65. $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & -2 \\ 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$

66. $\begin{bmatrix} 4 & -2 & 6 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 0 & -3 \\ 2 & 0 \end{bmatrix}$

67. **Manufacturing** A tire corporation has three factories, each of which manufactures two models of tires. The production levels are represented by A.

$$A = \begin{array}{cc} \overbrace{\begin{array}{ccc} 1 & 2 & 3 \end{array}}^{\text{Factory}} & \\ \begin{bmatrix} 80 & 120 & 140 \\ 40 & 100 & 80 \end{bmatrix} & \left.\begin{matrix} \text{A} \\ \text{B} \end{matrix}\right\} \text{Model} \end{array}$$

Find the production levels when production is decreased by 5%.

68. **Manufacturing** A power tool company has four manufacturing plants, each of which produces three types of cordless power tools. The production levels are represented by A.

$$A = \begin{array}{cc} \overbrace{\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}}^{\text{Plant}} & \\ \begin{bmatrix} 80 & 70 & 90 & 40 \\ 50 & 30 & 80 & 20 \\ 90 & 60 & 100 & 50 \end{bmatrix} & \left.\begin{matrix} \text{A} \\ \text{B} \\ \text{C} \end{matrix}\right\} \text{Type} \end{array}$$

Find the production levels when production is increased by 20%.

69. **Manufacturing** An electronics manufacturing company produces three different models of headphones that are shipped to two warehouses. The shipment levels are represented by A.

$$A = \begin{array}{cc} \overbrace{\begin{array}{cc} 1 & 2 \end{array}}^{\text{Warehouse}} & \\ \begin{bmatrix} 8200 & 7400 \\ 6500 & 9800 \\ 5400 & 4800 \end{bmatrix} & \left.\begin{matrix} \text{A} \\ \text{B} \\ \text{C} \end{matrix}\right\} \text{Model} \end{array}$$

The prices per unit are represented by the matrix

$B = [\$79.99 \quad \$109.95 \quad \$189.99]$.

Compute BA and interpret the result.

70. **Cell Phone Charges** The pay-as-you-go charges (in dollars per minute) of two cellular telephone companies for calls inside the coverage area, regional roaming calls, and calls outside the coverage area are represented by C.

$$C = \begin{array}{cc} \overbrace{\begin{array}{cc} \text{A} & \text{B} \end{array}}^{\text{Company}} & \\ \begin{bmatrix} 0.07 & 0.095 \\ 0.10 & 0.08 \\ 0.28 & 0.25 \end{bmatrix} & \left.\begin{matrix} \text{Inside} \\ \text{Regional roaming} \\ \text{Outside} \end{matrix}\right\} \text{Coverage area} \end{array}$$

The numbers of minutes you plan to use in the coverage areas per month are represented by the matrix

$T = [120 \quad 80 \quad 20]$.

Compute TC and interpret the result.

7.3 **The Inverse of a Matrix** **In Exercises 71–74, show that B is the inverse of A.**

71. $A = \begin{bmatrix} -4 & -1 \\ 7 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & -1 \\ 7 & 4 \end{bmatrix}$

72. $A = \begin{bmatrix} 5 & -1 \\ 11 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 1 \\ -11 & 5 \end{bmatrix}$

73. $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 6 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & -3 & 1 \\ 3 & 3 & -1 \\ 2 & 4 & -1 \end{bmatrix}$

74. $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ 8 & -4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 1 & \frac{1}{2} \\ -3 & 1 & \frac{1}{2} \\ 2 & -2 & -\frac{1}{2} \end{bmatrix}$

Finding the Inverse of a Matrix **In Exercises 75–78, find the inverse of the matrix (if it exists).**

75. $\begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix}$

76. $\begin{bmatrix} -3 & -5 \\ 2 & 3 \end{bmatrix}$

77. $\begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$

78. $\begin{bmatrix} 0 & -2 & 1 \\ -5 & -2 & -3 \\ 7 & 3 & 4 \end{bmatrix}$

Finding the Inverse of a Matrix **In Exercises 79–82, use the matrix capabilities of a graphing utility to find the inverse of the matrix (if it exists).**

79. $\begin{bmatrix} -1 & -2 & -2 \\ 3 & 7 & 9 \\ 1 & 4 & 7 \end{bmatrix}$

80. $\begin{bmatrix} 1 & 4 & 6 \\ 2 & -3 & 1 \\ -1 & 18 & 16 \end{bmatrix}$

81. $\begin{bmatrix} 1 & 3 & 1 & 6 \\ 4 & 4 & 2 & 6 \\ 3 & 4 & 1 & 2 \\ -1 & 2 & -1 & -2 \end{bmatrix}$

82. $\begin{bmatrix} 8 & 0 & 2 & 8 \\ 4 & -2 & 0 & -2 \\ 1 & 2 & 1 & 4 \\ -1 & 4 & 1 & 1 \end{bmatrix}$

Finding the Inverse of a 2 × 2 Matrix

In Exercises 83–90, use the formula on page 527 to find the inverse of the 2 × 2 matrix (if it exists).

83. $\begin{bmatrix} -7 & 2 \\ -8 & 2 \end{bmatrix}$

84. $\begin{bmatrix} 10 & 4 \\ 7 & 3 \end{bmatrix}$

85. $\begin{bmatrix} -12 & 6 \\ 10 & -5 \end{bmatrix}$

86. $\begin{bmatrix} -18 & -15 \\ -6 & -5 \end{bmatrix}$

87. $\begin{bmatrix} -\frac{1}{2} & 20 \\ \frac{3}{10} & -6 \end{bmatrix}$

88. $\begin{bmatrix} -\frac{3}{4} & \frac{5}{2} \\ -\frac{4}{5} & -\frac{8}{3} \end{bmatrix}$

89. $\begin{bmatrix} 0.5 & 0.1 \\ -0.2 & -0.4 \end{bmatrix}$

90. $\begin{bmatrix} 1.6 & -3.2 \\ 1.2 & -2.4 \end{bmatrix}$

Solving a System Using an Inverse Matrix

In Exercises 91–102, use an inverse matrix to solve (if possible) the system of linear equations.

91. $\begin{cases} -x + 4y = 8 \\ 2x - 7y = -5 \end{cases}$

92. $\begin{cases} 5x - y = 13 \\ -9x + 2y = -24 \end{cases}$

93. $\begin{cases} -3x + 10y = 8 \\ 5x - 17y = -13 \end{cases}$

94. $\begin{cases} 4x - 2y = -10 \\ -19x + 9y = 47 \end{cases}$

95. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 2 \\ -3x + 2y = 0 \end{cases}$

96. $\begin{cases} -\frac{5}{6}x + \frac{3}{8}y = -2 \\ 4x - 3y = 0 \end{cases}$

97. $\begin{cases} 0.3x + 0.7y = 10.2 \\ 0.4x + 0.6y = 7.6 \end{cases}$

98. $\begin{cases} 3.5x - 4.5y = 8 \\ 2.5x - 7.5y = 25 \end{cases}$

99. $\begin{cases} 3x + 2y - z = 6 \\ x - y + 2z = -1 \\ 5x + y + z = 7 \end{cases}$

100. $\begin{cases} 4x + 5y - 6z = -6 \\ 3x + 2y + 2z = 8 \\ 2x + y + z = 3 \end{cases}$

101. $\begin{cases} -2x + y + 2z = -13 \\ -x - 4y + z = -11 \\ -y - z = 0 \end{cases}$

102. $\begin{cases} 3x - y + 5z = -14 \\ -x + y + 6z = 8 \\ -8x + 4y - z = 44 \end{cases}$

Using a Graphing Utility

In Exercises 103–108, use the matrix capabilities of a graphing utility to solve (if possible) the system of linear equations.

103. $\begin{cases} x + 2y = -1 \\ 3x + 4y = -5 \end{cases}$

104. $\begin{cases} x + 3y = 23 \\ -6x + 2y = -18 \end{cases}$

105. $\begin{cases} \frac{6}{5}x - \frac{4}{7}y = \frac{6}{5} \\ -\frac{12}{5}x + \frac{12}{7}y = -\frac{17}{5} \end{cases}$

106. $\begin{cases} 5x + 10y = 7 \\ 2x + y = -98 \end{cases}$

107. $\begin{cases} -3x - 3y - 4z = 2 \\ y + z = -1 \\ 4x + 3y + 4z = -1 \end{cases}$

108. $\begin{cases} x - 3y - 2z = 8 \\ -2x + 7y + 3z = -19 \\ x - y - 3z = 3 \end{cases}$

7.4 Finding the Determinant of a Matrix

In Exercises 109–112, find the determinant of the matrix.

109. $\begin{bmatrix} 8 & 5 \\ 2 & -4 \end{bmatrix}$

110. $\begin{bmatrix} -9 & 11 \\ 7 & -4 \end{bmatrix}$

111. $\begin{bmatrix} 50 & -30 \\ 10 & 5 \end{bmatrix}$

112. $\begin{bmatrix} 14 & -24 \\ 12 & -15 \end{bmatrix}$

Finding the Minors and Cofactors of a Matrix

In Exercises 113–116, find all the (a) minors and (b) cofactors of the matrix.

113. $\begin{bmatrix} 2 & -1 \\ 7 & 4 \end{bmatrix}$

114. $\begin{bmatrix} 3 & 6 \\ 5 & -4 \end{bmatrix}$

115. $\begin{bmatrix} 3 & 2 & -1 \\ -2 & 5 & 0 \\ 1 & 8 & 6 \end{bmatrix}$

116. $\begin{bmatrix} 8 & 3 & 4 \\ 6 & 5 & -9 \\ -4 & 1 & 2 \end{bmatrix}$

Finding the Determinant of a Matrix

In Exercises 117–126, find the determinant of the matrix. Expand by cofactors using the row or column that appears to make the computations easiest.

117. $\begin{bmatrix} -2 & 0 & 0 \\ 2 & -1 & 0 \\ -1 & 1 & -3 \end{bmatrix}$

118. $\begin{bmatrix} 0 & 1 & -2 \\ 0 & 1 & 2 \\ -1 & -1 & 3 \end{bmatrix}$

119. $\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & 2 \\ 1 & -1 & 0 \end{bmatrix}$

120. $\begin{bmatrix} -1 & -2 & 1 \\ 2 & 3 & 0 \\ -5 & -1 & 3 \end{bmatrix}$

121. $\begin{bmatrix} -2 & 4 & 1 \\ -6 & 0 & 2 \\ 5 & 3 & 4 \end{bmatrix}$

122. $\begin{bmatrix} 1 & 1 & 4 \\ -4 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$

123. $\begin{bmatrix} 1 & 2 & -1 & 0 \\ 1 & 2 & -4 & 1 \\ 2 & -4 & -3 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix}$

124. $\begin{bmatrix} 1 & -2 & 1 & 2 \\ 4 & 1 & 4 & 1 \\ 2 & 3 & 3 & 0 \\ 0 & -2 & -4 & 2 \end{bmatrix}$

125. $\begin{bmatrix} 3 & 0 & -4 & 0 \\ 0 & 8 & 1 & 2 \\ 6 & 1 & 8 & 2 \\ 0 & 3 & -4 & 1 \end{bmatrix}$

126. $\begin{bmatrix} -5 & 6 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ -3 & 4 & -5 & 1 \\ 1 & 6 & 0 & 3 \end{bmatrix}$

7.5 Using Cramer's Rule

In Exercises 127–130, use Cramer's Rule to solve (if possible) the system of equations.

127. $\begin{cases} 5x - 2y = 6 \\ -11x + 3y = -23 \end{cases}$

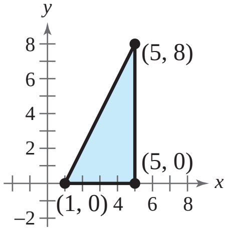
128. $\begin{cases} 3x + 8y = -7 \\ 9x - 5y = 37 \end{cases}$

129. $\begin{cases} -2x + 3y - 5z = -11 \\ 4x - y + z = -3 \\ -x - 4y + 6z = 15 \end{cases}$

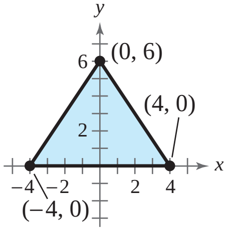
130. $\begin{cases} 5x - 2y + z = 15 \\ 3x - 3y - z = -7 \\ 2x - y - 7z = -3 \end{cases}$

Finding the Area of a Triangle **In Exercises 131–134, use a determinant to find the area of the triangle with the given vertices.**

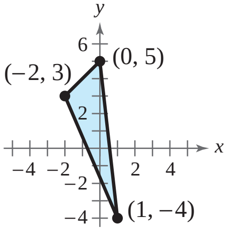
131.



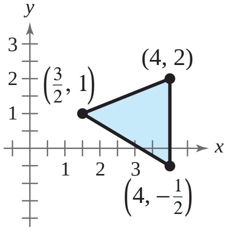
132.



133.



134.



Testing for Collinear Points **In Exercises 135 and 136, use a determinant to determine whether the points are collinear.**

135. $(-1, 7), (3, -9), (-3, 15)$

136. $(0, -5), (-2, -6), (8, -1)$

Finding an Equation of a Line **In Exercises 137–140, use a determinant to find an equation of the line passing through the points.**

137. $(-4, 0), (4, 4)$

138. $(2, 5), (6, -1)$

139. $\left(-\frac{5}{2}, 3\right), \left(\frac{7}{2}, 1\right)$

140. $(-0.8, 0.2), (0.7, 3.2)$

Encoding a Message **In Exercises 141 and 142, (a) write the uncoded 1 × 3 row matrices for the message and then (b) encode the message using the encoding matrix.**

	Message	**Encoding Matrix**
141.	LOOK OUT BELOW	$\begin{bmatrix} 2 & -2 & 0 \\ 3 & 0 & -3 \\ -6 & 2 & 3 \end{bmatrix}$
142.	HEAD DUE WEST	$\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$

Decoding a Message **In Exercises 143 and 144, decode the cryptogram by using the inverse of the matrix**

$$\boldsymbol{A = \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix}.}$$

143. −5 11 −2 370 −265 225 −57 48 −33 32 −15 20 245 −171 147

144. 145 −105 92 264 −188 160 23 −16 15 129 −84 78 −9 8 −5 159 −118 100 219 −152 133 370 −265 225 −105 84 −63

Exploration

True or False? **In Exercises 145 and 146, determine whether the statement is true or false. Justify your answer.**

145. It is possible to find the determinant of a 4 × 5 matrix.

146. $$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} + c_1 & a_{32} + c_2 & a_{33} + c_3 \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ c_1 & c_2 & c_3 \end{vmatrix}$$

147. **Using a Graphing Utility** Use the matrix capabilities of a graphing utility to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}.$$

What message appears on the screen? Why does the graphing utility display this message?

148. **Invertible Matrices** Under what conditions does a matrix have an inverse?

149. **Writing** What is meant by the cofactor of an entry of a matrix? How are cofactors used to find the determinant of the matrix?

150. **Think About It** Three people were asked to solve a system of equations using an augmented matrix. Each person reduced the matrix to row-echelon form. The reduced matrices were

$$\begin{bmatrix} 1 & 2 & \vdots & 3 \\ 0 & 1 & \vdots & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & \vdots & 1 \\ 0 & 1 & \vdots & 1 \end{bmatrix},$$

and

$$\begin{bmatrix} 1 & 2 & \vdots & 3 \\ 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Can all three be right? Explain.

151. **Think About It** Describe the row-echelon form of an augmented matrix that corresponds to a system of linear equations that has a unique solution.

152. **Solving an Equation** Solve the equation for λ.

$$\begin{vmatrix} 2 - \lambda & 5 \\ 3 & -8 - \lambda \end{vmatrix} = 0$$

Chapter Test

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

In Exercises 1 and 2, write the matrix in reduced row-echelon form.

1. $\begin{bmatrix} 1 & -1 & 5 \\ 6 & 2 & 3 \\ 5 & 3 & -3 \end{bmatrix}$ **2.** $\begin{bmatrix} 1 & 0 & -1 & 2 \\ -1 & 1 & 1 & -3 \\ 1 & 1 & -1 & 1 \\ 3 & 2 & -3 & 4 \end{bmatrix}$

3. Write the augmented matrix for the system of equations and solve the system.

$$\begin{cases} 4x + 3y - 2z = 14 \\ -x - y + 2z = -5 \\ 3x + y - 4z = 8 \end{cases}$$

4. Find (a) $A - B$, (b) $3A$, (c) $3A - 2B$, and (d) AB (if possible).

$$A = \begin{bmatrix} 6 & 5 \\ -5 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 0 \\ -5 & -1 \end{bmatrix}$$

In Exercises 5 and 6, find the inverse of the matrix (if it exists).

5. $\begin{bmatrix} -4 & 3 \\ 5 & -2 \end{bmatrix}$ **6.** $\begin{bmatrix} -2 & 4 & -6 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$

7. Use the result of Exercise 5 to solve the system.

$$\begin{cases} -4x + 3y = 6 \\ 5x - 2y = 24 \end{cases}$$

In Exercises 8–10, find the determinant of the matrix.

8. $\begin{bmatrix} -6 & 4 \\ 10 & 12 \end{bmatrix}$ **9.** $\begin{bmatrix} \frac{5}{2} & \frac{13}{4} \\ -8 & \frac{6}{5} \end{bmatrix}$ **10.** $\begin{bmatrix} 6 & -7 & 2 \\ 3 & -2 & 0 \\ 1 & 5 & 1 \end{bmatrix}$

In Exercises 11 and 12, use Cramer's Rule to solve (if possible) the system of equations.

11. $\begin{cases} 7x + 6y = 9 \\ -2x - 11y = -49 \end{cases}$ **12.** $\begin{cases} 6x - y + 2z = -4 \\ -2x + 3y - z = 10 \\ 4x - 4y + z = -18 \end{cases}$

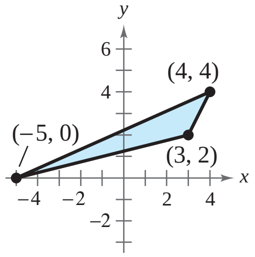


Figure for 13

13. Use a determinant to find the area of the triangle at the left.

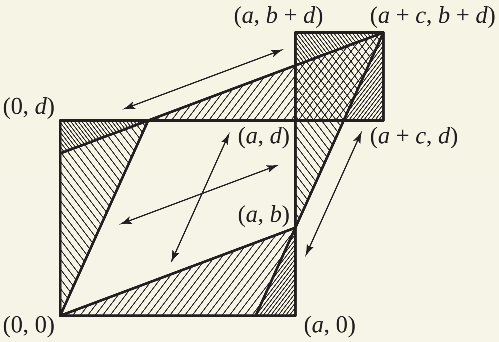
14. Write the uncoded 1 × 3 row matrices for the message KNOCK ON WOOD. Then encode the message using the encoding matrix A below.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$$

15. One hundred liters of a 50% solution is obtained by mixing a 60% solution with a 20% solution. How many liters of each solution must be used to obtain the desired mixture?

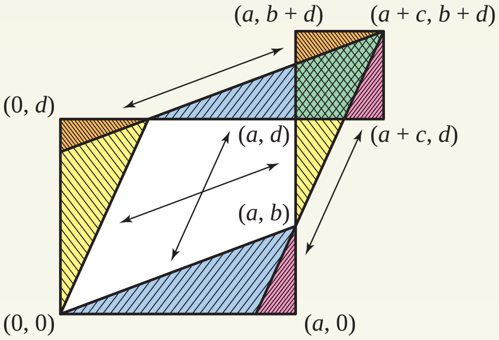
Proofs in Mathematics

A proof without words is a picture or diagram that gives a visual understanding of why a theorem or statement is true. It can also provide a starting point for writing a formal proof. The following proof shows that a 2 × 2 determinant is the area of a parallelogram.



$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \|\text{▭}\| - \|\text{□}\| = \|\text{▱}\|$$

The following is a color-coded version of the proof along with a brief explanation of why this proof works.



$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \|\text{▭}\| - \|\text{□}\| = \|\text{▱}\|$$

Area of ▭ = Area of orange △ + Area of yellow △ + Area of blue △ +
Area of pink △ + Area of white quadrilateral

Area of □ = Area of orange △ + Area of pink △ + Area of green quadrilateral

Area of ▱ = Area of white quadrilateral + Area of blue △ + Area of yellow △ −
Area of green quadrilateral

= Area of ▭ − Area of □ ■

From "Proof Without Words" by Solomon W. Golomb, *Mathematics Magazine,* March 1985, Vol. 58, No. 2, pg. 107. Reprinted with permission.

P.S. Problem Solving

1. **Multiplying by a Transformation Matrix** The columns of matrix T show the coordinates of the vertices of a triangle. Matrix A is a transformation matrix.

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \end{bmatrix}$$

   (a) Find AT and AAT. Then sketch the original triangle and the two images of the triangle. What transformation does A represent?

   (b) Given the triangle determined by AAT, describe the transformation that produces the triangle determined by AT and then the triangle determined by T.

2. **Population** The matrices show the number of people (in thousands) who lived in each region of the United States in 2000 and the number of people (in thousands) who lived in each region in 2010. The regional populations are separated into three age groups. *(Source: U.S. Census Bureau)*

$$\begin{array}{l} \\ \text{Northeast} \\ \text{Midwest} \\ \text{South} \\ \text{West} \end{array}
\begin{array}{c} \begin{array}{ccc} & 2000 & \\ \text{0–17} & \text{18–64} & \text{65+} \end{array} \\ \begin{bmatrix} 13{,}048 & 33{,}174 & 7{,}372 \\ 16{,}648 & 39{,}486 & 8{,}259 \\ 25{,}567 & 62{,}232 & 12{,}438 \\ 17{,}031 & 39{,}244 & 6{,}922 \end{bmatrix} \end{array}$$

$$\begin{array}{l} \\ \text{Northeast} \\ \text{Midwest} \\ \text{South} \\ \text{West} \end{array}
\begin{array}{c} \begin{array}{ccc} & 2010 & \\ \text{0–17} & \text{18–64} & \text{65+} \end{array} \\ \begin{bmatrix} 12{,}333 & 35{,}179 & 7{,}805 \\ 16{,}128 & 41{,}777 & 9{,}022 \\ 27{,}789 & 71{,}873 & 14{,}894 \\ 17{,}931 & 45{,}467 & 8{,}547 \end{bmatrix} \end{array}$$

   (a) The total population in 2000 was approximately 281,422,000 and the total population in 2010 was approximately 308,746,000. Rewrite the matrices to give the information as percents of the total population.

   (b) Write a matrix that gives the change in the percent of the population in each region and age group from 2000 to 2010.

   (c) Based on the result of part (b), which region(s) and age group(s) had percents that decreased from 2000 to 2010?

3. **Determining Whether Matrices are Idempotent** Determine whether the matrix is idempotent. A square matrix is **idempotent** when $A^2 = A$.

   (a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

   (c) $\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

4. **Quadratic Matrix Equation** Let $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

   (a) Show that $A^2 - 2A + 5I = O$, where I is the identity matrix of order 2×2.

   (b) Show that $A^{-1} = \frac{1}{5}(2I - A)$.

   (c) Show in general that for any square matrix satisfying

$$A^2 - 2A + 5I = O$$

   the inverse of A is given by

$$A^{-1} = \tfrac{1}{5}(2I - A).$$

5. **Satellite Television** Two competing companies offer satellite television to a city with 100,000 households. Gold Satellite System has 25,000 subscribers and Galaxy Satellite Network has 30,000 subscribers. (The other 45,000 households do not subscribe.) The percent changes in satellite subscriptions each year are shown in the matrix below.

$$\begin{array}{cc} & \overbrace{\begin{array}{ccc} \text{From Gold} & \text{From Galaxy} & \text{From Nonsubscriber} \end{array}}^{\text{Percent Changes}} \\ \text{Percent Changes} \left\{ \begin{array}{l} \text{To Gold} \\ \text{To Galaxy} \\ \text{To Nonsubscriber} \end{array} \right. & \begin{bmatrix} 0.70 & 0.15 & 0.15 \\ 0.20 & 0.80 & 0.15 \\ 0.10 & 0.05 & 0.70 \end{bmatrix} \end{array}$$

   (a) Find the number of subscribers each company will have in 1 year using matrix multiplication. Explain how you obtained your answer.

   (b) Find the number of subscribers each company will have in 2 years using matrix multiplication. Explain how you obtained your answer.

   (c) Find the number of subscribers each company will have in 3 years using matrix multiplication. Explain how you obtained your answer.

   (d) What is happening to the number of subscribers to each company? What is happening to the number of nonsubscribers?

6. **The Transpose of a Matrix** The **transpose** of a matrix, denoted A^T, is formed by writing its columns as rows. Find the transpose of each matrix and verify that $(AB)^T = B^T A^T$.

$$A = \begin{bmatrix} -1 & 1 & -2 \\ 2 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 0 \\ 1 & 2 \\ 1 & -1 \end{bmatrix}$$

7. **Finding a Value** Find x such that the matrix is equal to its own inverse.

$$A = \begin{bmatrix} 3 & x \\ -2 & -3 \end{bmatrix}$$

8. Finding a Value Find x such that the matrix is singular.

$$A = \begin{bmatrix} 4 & x \\ -2 & -3 \end{bmatrix}$$

9. Finding a Matrix Find a singular 2×2 matrix satisfying $A^2 = A$.

10. Verifying an Equation Verify the following equation.

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a - b)(b - c)(c - a)$$

11. Verifying an Equation Verify the following equation.

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a - b)(b - c)(c - a)(a + b + c)$$

12. Verifying an Equation Verify the following equation.

$$\begin{vmatrix} x & 0 & c \\ -1 & x & b \\ 0 & -1 & a \end{vmatrix} = ax^2 + bx + c$$

13. Finding a Determinant Use the equation given in Exercise 12 as a model to find a determinant that is equal to $ax^3 + bx^2 + cx + d$.

14. Finding Atomic Masses The atomic masses of three compounds are shown in the table. Use a linear system and Cramer's Rule to find the atomic masses of sulfur (S), nitrogen (N), and fluorine (F).

DATA

Compound	Formula	Atomic Mass
Tetrasulfur tetranitride	S_4N_4	184
Sulfur hexafluoride	SF_6	146
Dinitrogen tetrafluoride	N_2F_4	104

Spreadsheet at LarsonPrecalculus.com

15. Finding the Costs of Items A walkway lighting package includes a transformer, a certain length of wire, and a certain number of lights on the wire. The price of each lighting package depends on the length of wire and the number of lights on the wire. Use the following information to find the cost of a transformer, the cost per foot of wire, and the cost of a light. Assume that the cost of each item is the same in each lighting package.

- A package that contains a transformer, 25 feet of wire, and 5 lights costs \$20.
- A package that contains a transformer, 50 feet of wire, and 15 lights costs \$35.
- A package that contains a transformer, 100 feet of wire, and 20 lights costs \$50.

16. Decoding a Message Use the inverse of matrix A to decode the cryptogram.

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 1 & -3 \\ 1 & -1 & 4 \end{bmatrix}$$

23 13 −34 31 −34 63 25 −17 61
24 14 −37 41 −17 −8 20 −29 40
38 −56 116 13 −11 1 22 −3 −6
41 −53 85 28 −32 16

17. Decoding a Message A code breaker intercepted the encoded message below.

45 −35 38 −30 18 −18 35 −30 81 −60
42 −28 75 −55 2 −2 22 −21 15 −10

Let $A^{-1} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$.

(a) You know that $[45 \quad -35]A^{-1} = [10 \quad 15]$ and that $[38 \quad -30]A^{-1} = [8 \quad 14]$, where A^{-1} is the inverse of the encoding matrix A. Write and solve two systems of equations to find w, x, y, and z.

(b) Decode the message.

18. Conjecture Let

$$A = \begin{bmatrix} 6 & 4 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$

Use a graphing utility to find A^{-1}. Compare $|A^{-1}|$ with $|A|$. Make a conjecture about the determinant of the inverse of a matrix.

19. Finding the Determinant of a Matrix Let A be an $n \times n$ matrix each of whose rows adds up to zero. Find $|A|$.

20. Conjecture Consider matrices of the form

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ 0 & 0 & a_{23} & a_{24} & \cdots & a_{2n} \\ 0 & 0 & 0 & a_{34} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{(n-1)n} \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

(a) Write a 2×2 matrix and a 3×3 matrix in the form of A.

(b) Use a graphing utility to raise each of the matrices to higher powers. Describe the result.

(c) Use the result of part (b) to make a conjecture about powers of A when A is a 4×4 matrix. Use the graphing utility to test your conjecture.

(d) Use the results of parts (b) and (c) to make a conjecture about powers of A when A is an $n \times n$ matrix.

8 Sequences, Series, and Probability

Tossing Dice *(Example 3, page 622)*

Horse Racing *(Example 6, page 613)*

Electricity *(Exercise 92, page 609)*

Dominoes *(page 593)*

AIDS Cases *(Exercise 98, page 573)*

Clockwise from top left, Mikhail Pogosov/Shutterstock.com; Sergey Mironov/Shutterstock.com; © mark wragg/istockphoto.com; michaeljung/Shutterstock.com; wang song/Shutterstock.com

8.1 Sequences and Series

- **Use sequence notation to write the terms of sequences.**
- **Use factorial notation.**
- **Use summation notation to write sums.**
- **Find the sums of series.**
- **Use sequences and series to model and solve real-life problems.**

Sequences and series can help you model real-life problems. For instance, in Exercise 98 on page 573, a sequence models the numbers of AIDS cases reported from 2003 through 2010.

Sequences

In mathematics, the word *sequence* is used in much the same way as in ordinary English. Saying that a collection is listed in *sequence* means that it is ordered so that it has a first member, a second member, a third member, and so on. Two examples are 1, 2, 3, 4, . . . and 1, 3, 5, 7, ...

Mathematically, you can think of a sequence as a *function* whose domain is the set of positive integers. Rather than using function notation, however, sequences are usually written using subscript notation, as indicated in the following definition.

Definition of Sequence

An **infinite sequence** is a function whose domain is the set of positive integers. The function values

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

are the **terms** of the sequence. When the domain of the function consists of the first n positive integers only, the sequence is a **finite sequence.**

On occasion, it is convenient to begin subscripting a sequence with 0 instead of 1 so that the terms of the sequence become

$$a_0, a_1, a_2, a_3, \ldots.$$

When this is the case, the domain includes 0.

REMARK The subscripts of a sequence make up the domain of the sequence and serve to identify the locations of terms within the sequence. For example, a_4 is the fourth term of the sequence, and a_n is the nth term of the sequence. Any variable can be a subscript. The most commonly used variable subscripts in sequence and series notation are i, j, k, and n.

EXAMPLE 1 Writing the Terms of a Sequence

a. The first four terms of the sequence given by $a_n = 3n - 2$ are

$a_1 = 3(1) - 2 = 1$ 1st term

$a_2 = 3(2) - 2 = 4$ 2nd term

$a_3 = 3(3) - 2 = 7$ 3rd term

$a_4 = 3(4) - 2 = 10.$ 4th term

b. The first four terms of the sequence given by $a_n = 3 + (-1)^n$ are

$a_1 = 3 + (-1)^1 = 3 - 1 = 2$ 1st term

$a_2 = 3 + (-1)^2 = 3 + 1 = 4$ 2nd term

$a_3 = 3 + (-1)^3 = 3 - 1 = 2$ 3rd term

$a_4 = 3 + (-1)^4 = 3 + 1 = 4.$ 4th term

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write the first four terms of the sequence given by $a_n = 2n + 1$.

michaeljung/Shutterstock.com

REMARK Write the first four terms of the sequence whose nth term is

$$a_n = \frac{(-1)^{n+1}}{2n+1}.$$

Are they the same as the first four terms of the sequence in Example 2? If not, then how do they differ?

EXAMPLE 2 A Sequence Whose Terms Alternate in Sign

Write the first four terms of the sequence given by $a_n = \dfrac{(-1)^n}{2n+1}$.

Solution The first four terms of the sequence are as follows.

$$a_1 = \frac{(-1)^1}{2(1)+1} = \frac{-1}{2+1} = -\frac{1}{3} \qquad \text{1st term}$$

$$a_2 = \frac{(-1)^2}{2(2)+1} = \frac{1}{4+1} = \frac{1}{5} \qquad \text{2nd term}$$

$$a_3 = \frac{(-1)^3}{2(3)+1} = \frac{-1}{6+1} = -\frac{1}{7} \qquad \text{3rd term}$$

$$a_4 = \frac{(-1)^4}{2(4)+1} = \frac{1}{8+1} = \frac{1}{9} \qquad \text{4th term}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

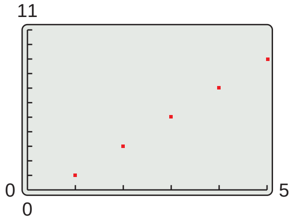
Write the first four terms of the sequence given by $a_n = \dfrac{2+(-1)^n}{n}$.

Simply listing the first few terms is not sufficient to define a unique sequence—the nth term *must be given*. To see this, consider the following sequences, both of which have the same first three terms.

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$$

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{15}, \ldots, \frac{6}{(n+1)(n^2-n+6)}, \ldots$$

TECHNOLOGY
To graph a sequence using a graphing utility, set the mode to *sequence* and *dot* and enter the sequence. The graph of the sequence in Example 3(a) is below. To identify the terms, use the *trace* feature or *value* feature.



EXAMPLE 3 Finding the nth Term of a Sequence

Write an expression for the apparent nth term (a_n) of each sequence.

a. 1, 3, 5, 7, . . . **b.** 2, −5, 10, −17, . . .

Solution

a.

n:	1	2	3	4 . . . n
Terms:	1	3	5	7 . . . a_n

Apparent pattern: Each term is 1 less than twice n, which implies that

$$a_n = 2n - 1.$$

b.

n:	1	2	3	4 . . . n
Terms:	2	−5	10	−17 . . . a_n

Apparent pattern: The absolute value of each term is 1 more than the square of n, and the terms have alternating signs, with those in the even positions being negative. This implies that

$$a_n = (-1)^{n+1}(n^2 + 1).$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write an expression for the apparent nth term (a_n) of each sequence.

a. 1, 5, 9, 13, . . . **b.** 2, −4, 6, −8, . . .

Some sequences are defined **recursively.** To define a sequence recursively, you need to be given one or more of the first few terms. All other terms of the sequence are then defined using previous terms.

EXAMPLE 4 A Recursive Sequence

Write the first five terms of the sequence defined recursively as

$$a_1 = 3, \quad a_k = 2a_{k-1} + 1, \quad \text{where } k \geq 2.$$

Solution

$$a_1 = 3 \qquad \text{1st term is given.}$$

$$a_2 = 2a_{2-1} + 1 = 2a_1 + 1 = 2(3) + 1 = 7 \qquad \text{Use recursion formula.}$$

$$a_3 = 2a_{3-1} + 1 = 2a_2 + 1 = 2(7) + 1 = 15 \qquad \text{Use recursion formula.}$$

$$a_4 = 2a_{4-1} + 1 = 2a_3 + 1 = 2(15) + 1 = 31 \qquad \text{Use recursion formula.}$$

$$a_5 = 2a_{5-1} + 1 = 2a_4 + 1 = 2(31) + 1 = 63 \qquad \text{Use recursion formula.}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write the first five terms of the sequence defined recursively as

$$a_1 = 6, \quad a_{k+1} = a_k + 1, \quad \text{where} \quad k \geq 2.$$ ■

In the next example, you will study a well-known recursive sequence, the Fibonacci sequence.

EXAMPLE 5 The Fibonacci Sequence: A Recursive Sequence

The Fibonacci sequence is defined recursively, as follows.

$$a_0 = 1, \quad a_1 = 1, \quad a_k = a_{k-2} + a_{k-1}, \quad \text{where} \quad k \geq 2$$

Write the first six terms of this sequence.

Solution

$$a_0 = 1 \qquad \text{0th term is given.}$$

$$a_1 = 1 \qquad \text{1st term is given.}$$

$$a_2 = a_{2-2} + a_{2-1} = a_0 + a_1 = 1 + 1 = 2 \qquad \text{Use recursion formula.}$$

$$a_3 = a_{3-2} + a_{3-1} = a_1 + a_2 = 1 + 2 = 3 \qquad \text{Use recursion formula.}$$

$$a_4 = a_{4-2} + a_{4-1} = a_2 + a_3 = 2 + 3 = 5 \qquad \text{Use recursion formula.}$$

$$a_5 = a_{5-2} + a_{5-1} = a_3 + a_4 = 3 + 5 = 8 \qquad \text{Use recursion formula.}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write the first five terms of the sequence defined recursively as

$$a_0 = 1, \quad a_1 = 3, \quad a_k = a_{k-2} + a_{k-1}, \quad \text{where} \quad k \geq 2.$$ ■

Factorial Notation

Some very important sequences in mathematics involve terms that are defined with special types of products called **factorials.**

Definition of Factorial

If n is a positive integer, then ***n* factorial** is defined as

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \cdot (n-1) \cdot n.$$

As a special case, zero factorial is defined as $0! = 1$.

REMARK The value of n does not have to be very large before the value of $n!$ becomes extremely large. For instance, $10! = 3{,}628{,}800$.

Notice that $0! = 1$ and $1! = 1$. Here are some other values of $n!$.

$$2! = 1 \cdot 2 = 2 \qquad 3! = 1 \cdot 2 \cdot 3 = 6 \qquad 4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$$

Factorials follow the same conventions for order of operations as do exponents. For instance,

$$2n! = 2(n!) = 2(1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \cdot n), \quad \text{whereas} \quad (2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \cdot 2n.$$

EXAMPLE 6 Writing the Terms of a Sequence Involving Factorials

Write the first five terms of the sequence given by $a_n = \dfrac{2^n}{n!}$. Begin with $n = 0$.

Algebraic Solution

$a_0 = \dfrac{2^0}{0!} = \dfrac{1}{1} = 1$ 0th term

$a_1 = \dfrac{2^1}{1!} = \dfrac{2}{1} = 2$ 1st term

$a_2 = \dfrac{2^2}{2!} = \dfrac{4}{2} = 2$ 2nd term

$a_3 = \dfrac{2^3}{3!} = \dfrac{8}{6} = \dfrac{4}{3}$ 3rd term

$a_4 = \dfrac{2^4}{4!} = \dfrac{16}{24} = \dfrac{2}{3}$ 4th term

Graphical Solution

Using a graphing utility set to *dot* and *sequence* modes, enter the sequence. Next, graph the sequence. You can estimate the first five terms of the sequence as follows.

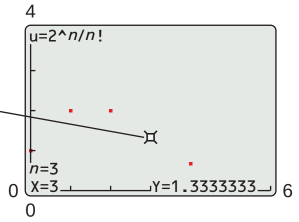
$$u_0 = 1$$

$$u_1 = 2$$

$$u_2 = 2$$

$$u_3 \approx 1.333 \approx \tfrac{4}{3}$$

$$u_4 \approx 0.667 \approx \tfrac{2}{3}$$



Use the *trace* feature to approximate the first five terms.

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Write the first five terms of the sequence given by $a_n = \dfrac{3^n + 1}{n!}$. Begin with $n = 0$.

When working with fractions involving factorials, you will often be able to reduce the fractions to simplify the computations.

EXAMPLE 7 Simplifying Factorial Expressions

a. $\dfrac{8!}{2! \cdot 6!} = \dfrac{\cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot 7 \cdot 8}{1 \cdot 2 \cdot \cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}} = \dfrac{7 \cdot 8}{2} = 28$

b. $\dfrac{n!}{(n-1)!} = \dfrac{\cancel{1 \cdot 2 \cdot 3 \cdot \cdot \cdot (n-1)} \cdot n}{\cancel{1 \cdot 2 \cdot 3 \cdot \cdot \cdot (n-1)}} = n$

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Simplify the factorial expression $\dfrac{4!(n+1)!}{3!n!}$.

ALGEBRA HELP You can also simplify the expression in Example 7(a) as follows.

$$\frac{8!}{2! \cdot 6!} = \frac{8 \cdot 7 \cdot \cancel{6!}}{2! \cdot \cancel{6!}}$$

$$= \frac{8 \cdot 7}{2 \cdot 1} = 28$$

TECHNOLOGY Most graphing utilities can sum the first n terms of a sequence. Check your user's guide for a *sum sequence* feature or a *series* feature.

Summation Notation

A convenient notation for the sum of the terms of a finite sequence is called **summation notation** or **sigma notation.** It involves the use of the uppercase Greek letter sigma, written as Σ.

Definition of Summation Notation

The sum of the first n terms of a sequence is represented by

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \cdot \cdot \cdot + a_n$$

where i is called the **index of summation,** n is the **upper limit of summation,** and 1 is the **lower limit of summation.**

REMARK Summation notation is an instruction to add the terms of a sequence. From the definition at the right, the upper limit of summation tells you the last term of the sum. Summation notation helps you generate the terms of the sequence prior to finding the sum.

REMARK In Example 8, note that the lower limit of a summation does not have to be 1. Also note that the index of summation does not have to be the letter i. For instance, in part (b), the letter k is the index of summation.

EXAMPLE 8 Summation Notation for a Sum

a. $$\sum_{i=1}^{5} 3i = 3(1) + 3(2) + 3(3) + 3(4) + 3(5)$$
$$= 45$$

b. $$\sum_{k=3}^{6} (1 + k^2) = (1 + 3^2) + (1 + 4^2) + (1 + 5^2) + (1 + 6^2)$$
$$= 10 + 17 + 26 + 37$$
$$= 90$$

c. $$\sum_{i=0}^{8} \frac{1}{i!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!}$$
$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40,320}$$
$$\approx 2.71828$$

For this summation, note that the sum is very close to the irrational number

$$e \approx 2.718281828.$$

It can be shown that as more terms of the sequence whose nth term is $1/n!$ are added, the sum becomes closer and closer to e.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the sum $\sum_{i=1}^{4} (4i + 1)$. ■

Properties of Sums

1. $\sum_{i=1}^{n} c = cn,$ c is a constant.
2. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i,$ c is a constant.
3. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
4. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

For proofs of these properties, see Proofs in Mathematics on page 640.

Series

Many applications involve the sum of the terms of a finite or infinite sequence. Such a sum is called a **series.**

Definition of Series

Consider the infinite sequence $a_1, a_2, a_3, \ldots, a_i, \ldots$

1. The sum of the first n terms of the sequence is called a **finite series** or the ***n*th partial sum** of the sequence and is denoted by

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i.$$

2. The sum of all the terms of the infinite sequence is called an **infinite series** and is denoted by

$$a_1 + a_2 + a_3 + \cdots + a_i + \cdots = \sum_{i=1}^{\infty} a_i.$$

EXAMPLE 9 Finding the Sum of a Series

For the series

$$\sum_{i=1}^{\infty} \frac{3}{10^i}$$

find (a) the third partial sum and (b) the sum.

Solution

a. The third partial sum is

$$\sum_{i=1}^{3} \frac{3}{10^i} = \frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3}$$

$$= 0.3 + 0.03 + 0.003$$

$$= 0.333.$$

b. The sum of the series is

$$\sum_{i=1}^{\infty} \frac{3}{10^i} = \frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \frac{3}{10^5} + \cdots$$

$$= 0.3 + 0.03 + 0.003 + 0.0003 + 0.00003 + \cdots$$

$$= 0.33333. \; . \; .$$

$$= \frac{1}{3}.$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

For the series

$$\sum_{i=1}^{\infty} \frac{5}{10^i}$$

find (a) the fourth partial sum and (b) the sum.

Notice in Example 9(b) that the sum of an infinite series can be a finite number.

Application

Sequences have many applications in business and science. Example 10 illustrates such an application.

EXAMPLE 10 Compound Interest

An investor deposits \$5000 in an account that earns 3% interest compounded quarterly. The balance in the account after n quarters is given by

$$A_n = 5000\left(1 + \frac{0.03}{4}\right)^n, \quad n = 0, 1, 2, \ldots .$$

a. Write the first three terms of the sequence.

b. Find the balance in the account after 10 years by computing the 40th term of the sequence.

Solution

a. The first three terms of the sequence are as follows.

$$A_0 = 5000\left(1 + \frac{0.03}{4}\right)^0 = \$5000.00 \qquad \text{Original deposit}$$

$$A_1 = 5000\left(1 + \frac{0.03}{4}\right)^1 = \$5037.50 \qquad \text{First-quarter balance}$$

$$A_2 = 5000\left(1 + \frac{0.03}{4}\right)^2 \approx \$5075.28 \qquad \text{Second-quarter balance}$$

b. The 40th term of the sequence is

$$A_{40} = 5000\left(1 + \frac{0.03}{4}\right)^{40} \approx \$6741.74. \qquad \text{Ten-year balance}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

An investor deposits \$1000 in an account that earns 3% interest compounded monthly. The balance in the account after n months is given by

$$A_n = 1000\left(1 + \frac{0.03}{12}\right)^n, \quad n = 0, 1, 2, \ldots .$$

a. Write the first three terms of the sequence.

b. Find the balance in the account after four years by computing the 48th term of the sequence.

Summarize (Section 8.1)

1. State the definition of a sequence *(page 564)*. For examples of writing the terms of sequences, see Examples 1–5.
2. State the definition of a factorial *(page 567)*. For examples of using factorial notation, see Examples 6 and 7.
3. State the definition of summation notation *(page 568)*. For an example of using summation notation, see Example 8.
4. State the definition of a series *(page 569)*. For an example of finding the sum of a series, see Example 9.
5. Describe an example of how to use a sequence to model and solve a real-life problem *(page 570, Example 10)*.

8.1 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. An ________ ________ is a function whose domain is the set of positive integers.
2. A sequence is a ________ sequence when the domain of the function consists only of the first n positive integers.
3. If you are given one or more of the first few terms of a sequence, and all other terms of the sequence are defined using previous terms, then the sequence is said to be defined ________.
4. If n is a positive integer, then n ________ is defined as $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n$.
5. For the sum $\sum_{i=1}^{n} a_i$, i is called the ________ of summation, n is the ________ limit of summation, and 1 is the ________ limit of summation.
6. The sum of the terms of a finite or infinite sequence is called a ________.

Skills and Applications

Writing the Terms of a Sequence **In Exercises 7–22, write the first five terms of the sequence. (Assume that n begins with 1.)**

7. $a_n = 4n - 7$
8. $a_n = 2 - \dfrac{1}{3^n}$
9. $a_n = (-2)^n$
10. $a_n = \left(\frac{1}{2}\right)^n$
11. $a_n = \dfrac{n}{n+2}$
12. $a_n = \dfrac{6n}{3n^2 - 1}$
13. $a_n = \dfrac{1 + (-1)^n}{n}$
14. $a_n = \dfrac{(-1)^n}{n^2}$
15. $a_n = \dfrac{2^n}{3^n}$
16. $a_n = \dfrac{1}{n^{3/2}}$
17. $a_n = \frac{2}{3}$
18. $a_n = 1 + (-1)^n$
19. $a_n = n(n-1)(n-2)$
20. $a_n = n(n^2 - 6)$
21. $a_n = (-1)^n\left(\dfrac{n}{n+1}\right)$
22. $a_n = \dfrac{(-1)^{n+1}}{n^2 + 1}$

Finding a Term of a Sequence **In Exercises 23–26, find the indicated term of the sequence.**

23. $a_n = (-1)^n(3n - 2)$

    $a_{25} = \square$
24. $a_n = (-1)^{n-1}[n(n-1)]$

    $a_{16} = \square$
25. $a_n = \dfrac{4n}{2n^2 - 3}$

    $a_{11} = \square$
26. $a_n = \dfrac{4n^2 - n + 3}{n(n-1)(n+2)}$

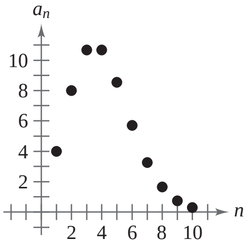
    $a_{13} = \square$

Graphing the Terms of a Sequence **In Exercises 27–32, use a graphing utility to graph the first 10 terms of the sequence. (Assume that n begins with 1.)**

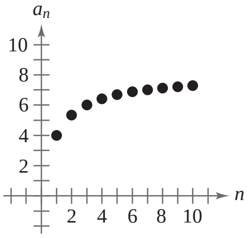
27. $a_n = \dfrac{2}{3}n$
28. $a_n = 2 - \dfrac{4}{n}$
29. $a_n = 16(-0.5)^{n-1}$
30. $a_n = 8(0.75)^{n-1}$
31. $a_n = \dfrac{2n}{n+1}$
32. $a_n = \dfrac{3n^2}{n^2 + 1}$

Matching a Sequence with a Graph **In Exercises 33–36, match the sequence with the graph of its first 10 terms. [The graphs are labeled (a), (b), (c), and (d).]**

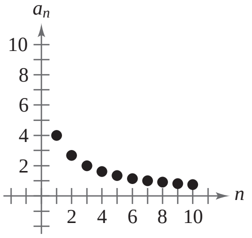
(a)



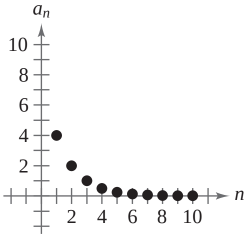
(b)



(c)



(d)



33. $a_n = \dfrac{8}{n+1}$
34. $a_n = \dfrac{8n}{n+1}$
35. $a_n = 4(0.5)^{n-1}$
36. $a_n = \dfrac{4^n}{n!}$

Finding the nth Term of a Sequence **In Exercises 37–48, write an expression for the apparent nth term (a_n) of the sequence. (Assume that n begins with 1.)**

37. 3, 7, 11, 15, 19, . . .
38. 0, 3, 8, 15, 24, . . .
39. $-\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, -\frac{6}{7}, \ldots$
40. $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \ldots$
41. $\frac{2}{1}, \frac{3}{3}, \frac{4}{5}, \frac{5}{7}, \frac{6}{9}, \ldots$
42. $\frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \frac{8}{81}, \ldots$
43. $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$
44. $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots$
45. $1, -1, 1, -1, 1, \ldots$
46. 1, 3, 1, 3, 1, . . .
47. $1, 3, \dfrac{3^2}{2}, \dfrac{3^3}{6}, \dfrac{3^4}{24}, \dfrac{3^5}{120}, \ldots$
48. $1 + \frac{1}{2}, 1 + \frac{3}{4}, 1 + \frac{7}{8}, 1 + \frac{15}{16}, 1 + \frac{31}{32}, \ldots$

Writing the Terms of a Recursive Sequence **In Exercises 49–52, write the first five terms of the sequence defined recursively.**

49. $a_1 = 28, \quad a_{k+1} = a_k - 4$

50. $a_1 = 3, \quad a_{k+1} = 2(a_k - 1)$

51. $a_0 = 1,\ a_1 = 2, \quad a_k = a_{k-2} + \frac{1}{2}a_{k-1}$

52. $a_0 = -1,\ a_1 = 1, \quad a_k = a_{k-2} + a_{k-1}$

Writing the *n*th Term of a Recursive Sequence **In Exercises 53–56, write the first five terms of the sequence defined recursively. Use the pattern to write the *n*th term of the sequence as a function of *n*.**

53. $a_1 = 6, \quad a_{k+1} = a_k + 2$

54. $a_1 = 25, \quad a_{k+1} = a_k - 5$

55. $a_1 = 81, \quad a_{k+1} = \frac{1}{3}a_k$

56. $a_1 = 14, \quad a_{k+1} = (-2)a_k$

Fibonacci Sequence **In Exercises 57 and 58, use the Fibonacci sequence. (See Example 5.)**

57. Write the first 12 terms of the Fibonacci sequence a_n and the first 10 terms of the sequence given by

$$b_n = \frac{a_{n+1}}{a_n}, \quad n \geq 1.$$

58. Using the definition for b_n in Exercise 57, show that b_n can be defined recursively by

$$b_n = 1 + \frac{1}{b_{n-1}}.$$

Writing the Terms of a Sequence Involving Factorials **In Exercises 59–62, write the first five terms of the sequence. (Assume that *n* begins with 0.)**

59. $a_n = \dfrac{5}{n!}$

60. $a_n = \dfrac{n!}{2n + 1}$

61. $a_n = \dfrac{1}{(n + 1)!}$

62. $a_n = \dfrac{(-1)^{2n+1}}{(2n + 1)!}$

Simplifying a Factorial Expression **In Exercises 63–66, simplify the factorial expression.**

63. $\dfrac{4!}{6!}$

64. $\dfrac{12!}{4! \cdot 8!}$

65. $\dfrac{(n + 1)!}{n!}$

66. $\dfrac{(2n - 1)!}{(2n + 1)!}$

Finding a Sum **In Exercises 67–74, find the sum.**

67. $\displaystyle\sum_{i=1}^{5} (2i + 1)$

68. $\displaystyle\sum_{j=3}^{5} \frac{1}{j^2 - 3}$

69. $\displaystyle\sum_{k=1}^{4} 10$

70. $\displaystyle\sum_{i=0}^{4} i^2$

71. $\displaystyle\sum_{k=2}^{5} (k + 1)^2(k - 3)$

72. $\displaystyle\sum_{i=1}^{4} [(i - 1)^2 + (i + 1)^3]$

73. $\displaystyle\sum_{i=1}^{4} 2^i$

74. $\displaystyle\sum_{j=0}^{4} (-2)^j$

Finding a Sum **In Exercises 75–78, use a graphing utility to find the sum.**

75. $\displaystyle\sum_{n=0}^{5} \frac{1}{2n + 1}$

76. $\displaystyle\sum_{k=0}^{4} \frac{(-1)^k}{k + 1}$

77. $\displaystyle\sum_{k=0}^{4} \frac{(-1)^k}{k!}$

78. $\displaystyle\sum_{n=0}^{25} \frac{1}{4^n}$

Using Sigma Notation to Write a Sum **In Exercises 79–88, use sigma notation to write the sum.**

79. $\dfrac{1}{3(1)} + \dfrac{1}{3(2)} + \dfrac{1}{3(3)} + \cdots + \dfrac{1}{3(9)}$

80. $\dfrac{5}{1 + 1} + \dfrac{5}{1 + 2} + \dfrac{5}{1 + 3} + \cdots + \dfrac{5}{1 + 15}$

81. $\left[2\left(\frac{1}{8}\right) + 3\right] + \left[2\left(\frac{2}{8}\right) + 3\right] + \cdots + \left[2\left(\frac{8}{8}\right) + 3\right]$

82. $\left[1 - \left(\frac{1}{6}\right)^2\right] + \left[1 - \left(\frac{2}{6}\right)^2\right] + \cdots + \left[1 - \left(\frac{6}{6}\right)^2\right]$

83. $3 - 9 + 27 - 81 + 243 - 729$

84. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots - \frac{1}{128}$

85. $\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \cdots - \dfrac{1}{20^2}$

86. $\dfrac{1}{1 \cdot 3} + \dfrac{1}{2 \cdot 4} + \dfrac{1}{3 \cdot 5} + \cdots + \dfrac{1}{10 \cdot 12}$

87. $\frac{1}{4} + \frac{3}{8} + \frac{7}{16} + \frac{15}{32} + \frac{31}{64}$

88. $\frac{1}{2} + \frac{2}{4} + \frac{6}{8} + \frac{24}{16} + \frac{120}{32} + \frac{720}{64}$

Finding a Partial Sum of a Series **In Exercises 89–92, find the indicated partial sum of the series.**

89. $\displaystyle\sum_{i=1}^{\infty} 5\left(\frac{1}{2}\right)^i$ Fourth partial sum

90. $\displaystyle\sum_{i=1}^{\infty} 2\left(\frac{1}{3}\right)^i$ Fifth partial sum

91. $\displaystyle\sum_{n=1}^{\infty} 4\left(-\frac{1}{2}\right)^n$ Third partial sum

92. $\displaystyle\sum_{n=1}^{\infty} 8\left(-\frac{1}{4}\right)^n$ Fourth partial sum

Finding the Sum of an Infinite Series **In Exercises 93–96, find the sum of the infinite series.**

93. $\displaystyle\sum_{i=1}^{\infty} \frac{6}{10^i}$

94. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{10}\right)^k$

95. $\displaystyle\sum_{k=1}^{\infty} 7\left(\frac{1}{10}\right)^k$

96. $\displaystyle\sum_{i=1}^{\infty} \frac{2}{10^i}$

97. Compound Interest An investor deposits \$10,000 in an account that earns 3.5% interest compounded quarterly. The balance in the account after n quarters is given by

$$A_n = 10{,}000\left(1 + \frac{0.035}{4}\right)^n, \quad n = 1, 2, 3, \ldots.$$

(a) Write the first eight terms of the sequence.

(b) Find the balance in the account after 10 years by computing the 40th term of the sequence.

(c) Is the balance after 20 years twice the balance after 10 years? Explain.

98. AIDS Cases

The numbers a_n (in thousands) of AIDS cases reported from 2003 through 2010 can be approximated by

$$a_n = -0.0126n^3 + 0.391n^2 - 4.21n + 48.5,$$

$$n = 3, 4, \ldots, 10$$

where n is the year, with $n = 3$ corresponding to 2003. *(Source: U.S. Centers for Disease Control and Prevention)*

(a) Write the terms of this finite sequence. Use a graphing utility to construct a bar graph that represents the sequence.

(b) What does the graph in part (a) say about reported cases of AIDS?

Exploration

True or False? **In Exercises 99 and 100, determine whether the statement is true or false. Justify your answer.**

99. $\displaystyle\sum_{i=1}^{4}(i^2 + 2i) = \sum_{i=1}^{4} i^2 + 2\sum_{i=1}^{4} i$

100. $\displaystyle\sum_{j=1}^{4} 2^j = \sum_{j=3}^{6} 2^{j-2}$

Arithmetic Mean **In Exercises 101–103, use the following definition of the arithmetic mean $\bar{x}$ of a set of n measurements $x_1, x_2, x_3, \ldots, x_n$.**

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

101. Find the arithmetic mean of the six checking account balances \$327.15, \$785.69, \$433.04, \$265.38, \$604.12, and \$590.30. Use the statistical capabilities of a graphing utility to verify your result.

102. Proof Prove that $\displaystyle\sum_{i=1}^{n}(x_i - \bar{x}) = 0.$

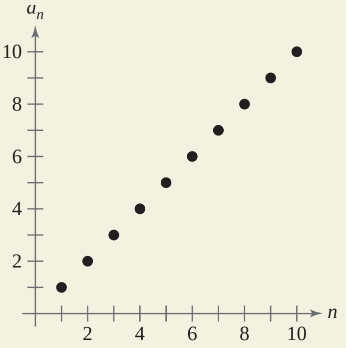
103. Proof Prove that $\displaystyle\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2.$

104. HOW DO YOU SEE IT? The graph represents the first 10 terms of a sequence. Complete each expression for the apparent nth term a_n of the sequence. Which expressions are appropriate to represent the cost a_n to buy n MP3 songs at a cost of \$1 per song? Explain.

(a) $a_n = 1 \cdot \boxed{}$

(b) $a_n = \dfrac{\boxed{}\,!}{(n-1)!}$

(c) $a_n = \displaystyle\sum_{k=1}^{n} \boxed{}$

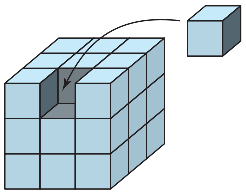


Finding the Terms of a Sequence **In Exercises 105 and 106, find the first five terms of the sequence.**

105. $a_n = \dfrac{x^n}{n!}$

106. $a_n = \dfrac{(-1)^n x^{2n+1}}{2n+1}$

107. Cube A $3 \times 3 \times 3$ cube is made up of 27 unit cubes (a unit cube has a length, width, and height of 1 unit), and only the faces of each cube that are visible are painted blue, as shown in the figure.



(a) Complete the table to determine how many unit cubes of the $3 \times 3 \times 3$ cube have 0 blue faces, 1 blue face, 2 blue faces, and 3 blue faces.

Number of Blue Cube Faces	0	1	2	3
$3 \times 3 \times 3$				

(b) Repeat part (a) for a $4 \times 4 \times 4$ cube, a $5 \times 5 \times 5$ cube, and a $6 \times 6 \times 6$ cube.

(c) What type of pattern do you observe?

(d) Write formulas you could use to repeat part (a) for an $n \times n \times n$ cube.

michaeljung/Shutterstock.com

8.2 Arithmetic Sequences and Partial Sums

- **Recognize, write, and find the *n*th terms of arithmetic sequences.**
- **Find *n*th partial sums of arithmetic sequences.**
- **Use arithmetic sequences to model and solve real-life problems.**

Arithmetic sequences have many real-life applications. For instance, in Exercise 81 on page 581, you will use an arithmetic sequence to determine how far an object falls in 7 seconds from the top of the Willis Tower in Chicago.

Arithmetic Sequences

A sequence whose consecutive terms have a common difference is called an **arithmetic sequence.**

Definition of Arithmetic Sequence

A sequence is **arithmetic** when the differences between consecutive terms are the same. So, the sequence

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

is arithmetic when there is a number d such that

$$a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \cdots = d.$$

The number d is the **common difference** of the arithmetic sequence.

EXAMPLE 1 Examples of Arithmetic Sequences

a. The sequence whose nth term is $4n + 3$ is arithmetic. For this sequence, the common difference between consecutive terms is 4.

$$\underbrace{7, 11}_{11 - 7 = 4}, 15, 19, \ldots, 4n + 3, \ldots \qquad \text{Begin with } n = 1.$$

b. The sequence whose nth term is $7 - 5n$ is arithmetic. For this sequence, the common difference between consecutive terms is -5.

$$\underbrace{2, -3}_{-3 - 2 = -5}, -8, -13, \ldots, 7 - 5n, \ldots \qquad \text{Begin with } n = 1.$$

c. The sequence whose nth term is $\frac{1}{4}(n + 3)$ is arithmetic. For this sequence, the common difference between consecutive terms is $\frac{1}{4}$.

$$\underbrace{1, \frac{5}{4}}_{\frac{5}{4} - 1 = \frac{1}{4}}, \frac{3}{2}, \frac{7}{4}, \ldots, \frac{n + 3}{4}, \ldots \qquad \text{Begin with } n = 1.$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write the first four terms of the arithmetic sequence whose nth term is $3n - 1$. Then find the common difference between consecutive terms.

The sequence 1, 4, 9, 16, . . . , whose nth term is n^2, is *not* arithmetic. The difference between the first two terms is

$$a_2 - a_1 = 4 - 1 = 3$$

but the difference between the second and third terms is

$$a_3 - a_2 = 9 - 4 = 5.$$

Eugene Moerman/Shutterstock.com

The *n*th term of an arithmetic sequence can be derived from the following pattern.

$$a_1 = a_1 \qquad \text{1st term}$$

$$a_2 = a_1 + d \qquad \text{2nd term}$$

$$a_3 = a_1 + 2d \qquad \text{3rd term}$$

$$a_4 = a_1 + 3d \qquad \text{4th term}$$

$$a_5 = a_1 + 4d \qquad \text{5th term}$$

1 less

$$\vdots$$

$$a_n = a_1 + (n - 1)d \qquad n\text{th term}$$

1 less

The following definition summarizes this result.

The *n*th Term of an Arithmetic Sequence

The *n*th term of an arithmetic sequence has the form

$$a_n = a_1 + (n - 1)d$$

where d is the common difference between consecutive terms of the sequence and a_1 is the first term.

EXAMPLE 2 Finding the *n*th Term

Find a formula for the *n*th term of the arithmetic sequence whose common difference is 3 and whose first term is 2.

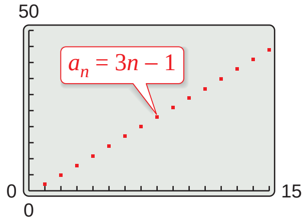
Solution You know that the formula for the *n*th term is of the form $a_n = a_1 + (n - 1)d$. Moreover, because the common difference is $d = 3$ and the first term is $a_1 = 2$, the formula must have the form

$$a_n = 2 + 3(n - 1). \qquad \text{Substitute 2 for } a_1 \text{ and 3 for } d.$$

So, the formula for the *n*th term is $a_n = 3n - 1$. The sequence therefore has the following form.

$$2, 5, 8, 11, 14, \ldots, 3n - 1, \ldots$$

The figure below shows a graph of the first 15 terms of the sequence. Notice that the points lie on a line. This makes sense because a_n is a linear function of n. In other words, the terms "arithmetic" and "linear" are closely connected.



Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find a formula for the *n*th term of the arithmetic sequence whose common difference is 5 and whose first term is -1.

REMARK You can find a_1 in Example 3 by using the nth term of an arithmetic sequence, as follows.

$$\begin{aligned} a_n &= a_1 + (n - 1)d \\ a_4 &= a_1 + (4 - 1)d \\ 20 &= a_1 + (4 - 1)5 \\ 20 &= a_1 + 15 \\ 5 &= a_1 \end{aligned}$$

EXAMPLE 3 Writing the Terms of an Arithmetic Sequence

The fourth term of an arithmetic sequence is 20, and the 13th term is 65. Write the first 11 terms of this sequence.

Solution You know that $a_4 = 20$ and $a_{13} = 65$. So, you must add the common difference d nine times to the fourth term to obtain the 13th term. Therefore, the fourth and 13th terms of the sequence are related by

$$a_{13} = a_4 + 9d. \qquad a_4 \text{ and } a_{13} \text{ are nine terms apart.}$$

Using $a_4 = 20$ and $a_{13} = 65$, solve for d to find that $d = 5$, which implies that the sequence is as follows.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11} . . .
5	10	15	20	25	30	35	40	45	50	55 . . .

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The eighth term of an arithmetic sequence is 25, and the 12th term is 41. Write the first 11 terms of this sequence. ■

When you know the nth term of an arithmetic sequence *and* you know the common difference of the sequence, you can find the $(n + 1)$th term by using the *recursion formula*

$$a_{n+1} = a_n + d. \qquad \text{Recursion formula}$$

With this formula, you can find any term of an arithmetic sequence, *provided* that you know the preceding term. For instance, when you know the first term, you can find the second term. Then, knowing the second term, you can find the third term, and so on.

EXAMPLE 4 Using a Recursion Formula

Find the ninth term of the arithmetic sequence that begins with 2 and 9.

Solution For this sequence, the common difference is

$$d = 9 - 2 = 7.$$

There are two ways to find the ninth term. One way is to write the first nine terms (by repeatedly adding 7).

2, 9, 16, 23, 30, 37, 44, 51, 58

Another way to find the ninth term is to first find a formula for the nth term. Because the common difference is $d = 7$ and the first term is $a_1 = 2$, the formula must have the form

$$a_n = 2 + 7(n - 1). \qquad \text{Substitute 2 for } a_1 \text{ and 7 for } d.$$

Therefore, a formula for the nth term is

$$a_n = 7n - 5$$

which implies that the ninth term is

$$\begin{aligned} a_9 &= 7(9) - 5 \\ &= 58. \end{aligned}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the tenth term of the arithmetic sequence that begins with 7 and 15. ■

The Sum of a Finite Arithmetic Sequence

There is a formula for the *sum* of a finite arithmetic sequence.

REMARK Note that this formula works only for *arithmetic* sequences.

The Sum of a Finite Arithmetic Sequence

The sum of a finite arithmetic sequence with n terms is $S_n = \frac{n}{2}(a_1 + a_n)$.

For a proof of this formula for the sum of a finite arithmetic sequence, see Proofs in Mathematics on page 641.

EXAMPLE 5 Sum of a Finite Arithmetic Sequence

Find the sum: $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19$.

Solution To begin, notice that the sequence is arithmetic (with a common difference of 2). Moreover, the sequence has 10 terms. So, the sum of the sequence is

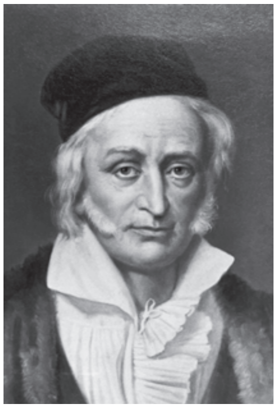
$$S_n = \frac{n}{2}(a_1 + a_n) \qquad \text{Sum of a finite arithmetic sequence}$$

$$= \frac{10}{2}(1 + 19) \qquad \text{Substitute 10 for } n, \text{ 1 for } a_1, \text{ and 19 for } a_n.$$

$$= 100. \qquad \text{Simplify.}$$

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Find the sum: $40 + 37 + 34 + 31 + 28 + 25 + 22$.



A teacher of Carl Friedrich Gauss (1777–1855) asked him to add all the integers from 1 to 100. When Gauss returned with the correct answer after only a few moments, the teacher could only look at him in astounded silence. This is what Gauss did:

$$\begin{array}{rcl} S_n & = & 1 + \ \ 2 + \ \ 3 + \cdots + 100 \\ S_n & = & 100 + \ 99 + \ 98 + \cdots + \ \ 1 \\ \hline 2S_n & = & 101 + 101 + 101 + \cdots + 101 \\ S_n & = & \dfrac{100 \times 101}{2} = 5050 \end{array}$$

Bettmann/Corbis

EXAMPLE 6 Sum of a Finite Arithmetic Sequence

Find the sum of the integers (a) from 1 to 100 and (b) from 1 to N.

Solution

a. The integers from 1 to 100 form an arithmetic sequence that has 100 terms. So, you can use the formula for the sum of a finite arithmetic sequence, as follows.

$$S_n = 1 + 2 + 3 + 4 + 5 + 6 + \cdot \cdot \cdot + 99 + 100$$

$$= \frac{n}{2}(a_1 + a_n) \qquad \text{Sum of a finite arithmetic sequence}$$

$$= \frac{100}{2}(1 + 100) \qquad \text{Substitute 100 for } n, \text{ 1 for } a_1, \text{ and 100 for } a_n.$$

$$= 5050 \qquad \text{Simplify.}$$

b. $S_n = 1 + 2 + 3 + 4 + \cdot \cdot \cdot + N$

$$= \frac{n}{2}(a_1 + a_n) \qquad \text{Sum of a finite arithmetic sequence}$$

$$= \frac{N}{2}(1 + N) \qquad \text{Substitute } N \text{ for } n, \text{ 1 for } a_1, \text{ and } N \text{ for } a_n.$$

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Find the sum of the integers (a) from 1 to 35 and (b) from 1 to $2N$.

The sum of the first n terms of an infinite sequence is the *nth partial sum*. The nth partial sum can be found by using the formula for the sum of a finite arithmetic sequence.

EXAMPLE 7 Partial Sum of an Arithmetic Sequence

Find the 150th partial sum of the arithmetic sequence

$5, 16, 27, 38, 49, \ldots .$

Solution For this arithmetic sequence, $a_1 = 5$ and $d = 16 - 5 = 11$. So,

$$a_n = 5 + 11(n - 1)$$

and the nth term is

$$a_n = 11n - 6.$$

Therefore, $a_{150} = 11(150) - 6 = 1644$, and the sum of the first 150 terms is

$$\begin{aligned} S_{150} &= \frac{n}{2}(a_1 + a_{150}) && n\text{th partial sum formula} \\ &= \frac{150}{2}(5 + 1644) && \text{Substitute 150 for } n\text{, 5 for } a_1\text{, and 1644 for } a_{150}. \\ &= 75(1649) && \text{Simplify.} \\ &= 123{,}675. && n\text{th partial sum} \end{aligned}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the 120th partial sum of the arithmetic sequence

$6, 12, 18, 24, 30, \ldots .$

EXAMPLE 8 Partial Sum of an Arithmetic Sequence

Find the 16th partial sum of the arithmetic sequence

$100, 95, 90, 85, 80, \ldots .$

Solution For this arithmetic sequence, $a_1 = 100$ and $d = 95 - 100 = -5$. So,

$$a_n = 100 + (-5)(n - 1)$$

and the nth term is

$$a_n = -5n + 105.$$

Therefore, $a_{16} = -5(16) + 105 = 25$, and the sum of the first 16 terms is

$$\begin{aligned} S_{16} &= \frac{n}{2}(a_1 + a_{16}) && n\text{th partial sum formula} \\ &= \frac{16}{2}(100 + 25) && \text{Substitute 16 for } n\text{, 100 for } a_1\text{, and 25 for } a_{16}. \\ &= 8(125) && \text{Simplify.} \\ &= 1000. && n\text{th partial sum} \end{aligned}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the 30th partial sum of the arithmetic sequence

$78, 76, 74, 72, 70, \ldots .$

Application

EXAMPLE 9 Total Sales

A small business sells \$10,000 worth of skin care products during its first year. The owner of the business has set a goal of increasing annual sales by \$7500 each year for 9 years. Assuming that this goal is met, find the total sales during the first 10 years this business is in operation.

Solution The annual sales form an arithmetic sequence in which

$$a_1 = 10{,}000 \quad \text{and} \quad d = 7500.$$

So,

$$a_n = 10{,}000 + 7500(n - 1)$$

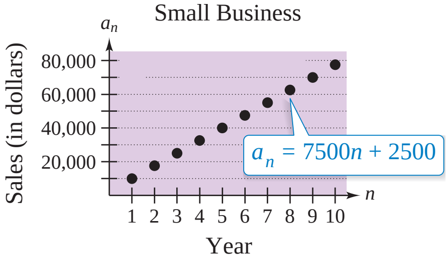
and the nth term of the sequence is

$$a_n = 7500n + 2500.$$

Therefore, the 10th term of the sequence is

$$a_{10} = 7500(10) + 2500$$

$$= 77{,}500. \qquad \text{See figure.}$$



The sum of the first 10 terms of the sequence is

$$S_{10} = \frac{n}{2}(a_1 + a_{10}) \qquad n\text{th partial sum formula}$$

$$= \frac{10}{2}(10{,}000 + 77{,}500) \qquad \text{Substitute 10 for } n\text{, 10,000 for } a_1\text{, and 77,500 for } a_{10}.$$

$$= 5(87{,}500) \qquad \text{Simplify.}$$

$$= 437{,}500. \qquad \text{Multiply.}$$

So, the total sales for the first 10 years will be \$437,500.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

A company sells \$160,000 worth of printing paper during its first year. The sales manager has set a goal of increasing annual sales of printing paper by \$20,000 each year for 9 years. Assuming that this goal is met, find the total sales of printing paper during the first 10 years this company is in operation.

Summarize (Section 8.2)

1. State the definition of an arithmetic sequence *(page 574)*, and state the formula for the nth term of an arithmetic sequence *(page 575)*. For examples of recognizing, writing, and finding the nth terms of arithmetic sequences, see Examples 1–4.
2. State the formula for the sum of a finite arithmetic sequence and explain how to use it to find an nth partial sum *(pages 577 and 578)*. For examples of finding sums of arithmetic sequences, see Examples 5–8.
3. Describe an example of how to use an arithmetic sequence to model and solve a real-life problem *(page 579, Example 9)*.

8.2 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. A sequence is called an ________ sequence when the differences between consecutive terms are the same. This difference is called the ________ difference.
2. The nth term of an arithmetic sequence has the form ________.
3. When you know the nth term of an arithmetic sequence *and* you know the common difference of the sequence, you can find the $(n + 1)$th term by using the ________ formula $a_{n+1} = a_n + d$.
4. You can use the formula $S_n = \frac{n}{2}(a_1 + a_n)$ to find the sum of the first n terms of an arithmetic sequence, which is called the ________ of a ________ ________ ________.

Skills and Applications

Determining Whether a Sequence Is Arithmetic

In Exercises 5–12, determine whether the sequence is arithmetic. If so, then find the common difference.

5. 10, 8, 6, 4, 2, . . .

6. 4, 9, 14, 19, 24, . . .

7. 1, 2, 4, 8, 16, . . .

8. 80, 40, 20, 10, 5, . . .

9. $\frac{9}{4}, 2, \frac{7}{4}, \frac{3}{2}, \frac{5}{4}, \ldots$

10. 5.3, 5.7, 6.1, 6.5, 6.9, . . .

11. ln 1, ln 2, ln 3, ln 4, ln 5, . . .

12. $1^2, 2^2, 3^2, 4^2, 5^2, \ldots$

Writing the Terms of a Sequence

In Exercises 13–20, write the first five terms of the sequence. Determine whether the sequence is arithmetic. If so, then find the common difference. (Assume that n begins with 1.)

13. $a_n = 5 + 3n$

14. $a_n = 100 - 3n$

15. $a_n = 3 - 4(n - 2)$

16. $a_n = 1 + (n - 1)4$

17. $a_n = (-1)^n$

18. $a_n = 2^{n-1}$

19. $a_n = \dfrac{(-1)^n 3}{n}$

20. $a_n = (2^n)n$

Finding the nth Term

In Exercises 21–30, find a formula for a_n for the arithmetic sequence.

21. $a_1 = 1, d = 3$

22. $a_1 = 15, d = 4$

23. $a_1 = 100, d = -8$

24. $a_1 = 0, d = -\frac{2}{3}$

25. $4, \frac{3}{2}, -1, -\frac{7}{2}, \ldots$

26. $10, 5, 0, -5, -10, \ldots$

27. $a_1 = 5, a_4 = 15$

28. $a_1 = -4, a_5 = 16$

29. $a_3 = 94, a_6 = 85$

30. $a_5 = 190, a_{10} = 115$

Writing the Terms of an Arithmetic Sequence

In Exercises 31–38, write the first five terms of the arithmetic sequence.

31. $a_1 = 5, d = 6$

32. $a_1 = 5, d = -\frac{3}{4}$

33. $a_1 = -\frac{13}{5}, d = -\frac{2}{5}$

34. $a_1 = 16.5, d = 0.25$

35. $a_1 = 2, a_{12} = 46$

36. $a_4 = 16, a_{10} = 46$

37. $a_8 = 26, a_{12} = 42$

38. $a_3 = 19, a_{15} = -1.7$

Writing the Terms of an Arithmetic Sequence

In Exercises 39–42, write the first five terms of the arithmetic sequence defined recursively.

39. $a_1 = 15, a_{n+1} = a_n + 4$

40. $a_1 = 200, a_{n+1} = a_n - 10$

41. $a_1 = \frac{5}{8}, a_{n+1} = a_n - \frac{1}{8}$

42. $a_1 = 0.375, a_{n+1} = a_n + 0.25$

Using a Recursion Formula

In Exercises 43–46, the first two terms of the arithmetic sequence are given. Find the missing term.

43. $a_1 = 5, a_2 = 11, a_{10} = \square$

44. $a_1 = 3, a_2 = 13, a_9 = \square$

45. $a_1 = 4.2, a_2 = 6.6, a_7 = \square$

46. $a_1 = -0.7, a_2 = -13.8, a_8 = \square$

Sum of a Finite Arithmetic Sequence

In Exercises 47–52, find the sum of the finite arithmetic sequence.

47. $2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + 20$

48. $1 + 4 + 7 + 10 + 13 + 16 + 19$

49. $-1 + (-3) + (-5) + (-7) + (-9)$

50. $-5 + (-3) + (-1) + 1 + 3 + 5$

51. Sum of the first 100 positive odd integers

52. Sum of the integers from -100 to 30

Partial Sum of an Arithmetic Sequence

In Exercises 53–58, find the indicated nth partial sum of the arithmetic sequence.

53. $8, 20, 32, 44, \ldots, \quad n = 10$

54. $-6, -2, 2, 6, \ldots, \quad n = 50$

55. $4.2, 3.7, 3.2, 2.7, \ldots, \quad n = 12$

56. $75, 70, 65, 60, \ldots, \quad n = 25$

57. $a_1 = 100, \quad a_{25} = 220, \quad n = 25$

58. $a_1 = 15, \quad a_{100} = 307, \quad n = 100$

Finding a Partial Sum **In Exercises 59–64, find the partial sum.**

59. $\sum_{n=1}^{50} n$

60. $\sum_{n=51}^{100} 7n$

61. $\sum_{n=11}^{30} n - \sum_{n=1}^{10} n$

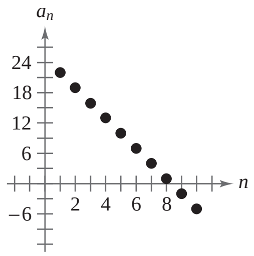
62. $\sum_{n=51}^{100} n - \sum_{n=1}^{50} n$

63. $\sum_{n=1}^{500} (n + 8)$

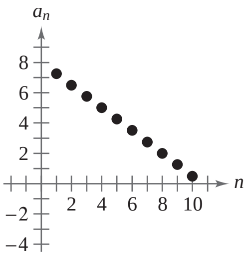
64. $\sum_{n=1}^{250} (1000 - n)$

Matching an Arithmetic Sequence with Its Graph **In Exercises 65–68, match the arithmetic sequence with its graph. [The graphs are labeled (a), (b), (c), and (d).]**

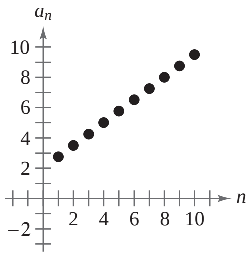
(a)



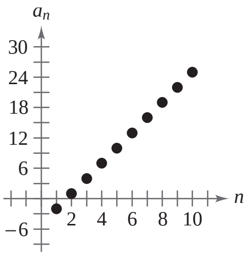
(b)



(c)



(d)



65. $a_n = -\frac{3}{4}n + 8$

66. $a_n = 3n - 5$

67. $a_n = 2 + \frac{3}{4}n$

68. $a_n = 25 - 3n$

Graphing the Terms of a Sequence **In Exercises 69–72, use a graphing utility to graph the first 10 terms of the sequence. (Assume that *n* begins with 1.)**

69. $a_n = 15 - \frac{3}{2}n$

70. $a_n = -5 + 2n$

71. $a_n = 0.2n + 3$

72. $a_n = -0.3n + 8$

Finding a Partial Sum **In Exercises 73–76, use a graphing utility to find the partial sum.**

73. $\sum_{n=0}^{50} (50 - 2n)$

74. $\sum_{n=1}^{100} \frac{n + 1}{2}$

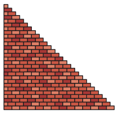
75. $\sum_{i=1}^{60} \left(250 - \frac{2}{5}i\right)$

76. $\sum_{j=1}^{200} (10.5 + 0.025j)$

77. **Seating Capacity** Determine the seating capacity of an auditorium with 36 rows of seats when there are 15 seats in the first row, 18 seats in the second row, 21 seats in the third row, and so on.

Eugene Moerman/Shutterstock.com

78. **Brick Pattern** A triangular brick wall is made by cutting some bricks in half to use in the first column of every other row (see figure). The wall has 28 rows. The top row is one-half brick wide and the bottom row is 14 bricks wide. How many bricks are in the finished wall?



Job Offer **In Exercises 79 and 80, consider a job offer with the given starting salary and the given annual raise.**

(a) Determine the salary during the sixth year of employment.

(b) Determine the total compensation from the company through six full years of employment.

	Starting Salary	**Annual Raise**
79.	\$32,500	\$1500
80.	\$36,800	\$1750

81. **Falling Object** An object with negligible air resistance is dropped from the top of the Willis Tower in Chicago at a height of 1450 feet. During the first second of fall, the object falls 16 feet; during the second second, it falls 48 feet; during the third second, it falls 80 feet; during the fourth second, it falls 112 feet. If this arithmetic pattern continues, then how many feet will the object fall in 7 seconds?

82. **Prize Money** A county fair is holding a baked goods competition in which the top eight bakers receive cash prizes. First place receives a cash prize of \$200, second place receives \$175, third place receives \$150, and so on.

(a) Write a sequence a_n that represents the cash prize awarded in terms of the place n in which the baked good places.

(b) Find the total amount of prize money awarded at the competition.

83. **Total Sales** An entrepreneur sells \$15,000 worth of sports memorabilia during one year and sets a goal of increasing annual sales by \$5000 each year for 9 years. Assuming that the entrepreneur meets this goal, find the total sales during the first 10 years of this business. What kinds of economic factors could prevent the business from meeting its goals?

84. Borrowing Money You borrow \$5000 from your parents to purchase a used car. The arrangements of the loan are such that you make payments of \$250 per month plus 1% interest on the unpaid balance.

(a) Find the first year's monthly payments and the unpaid balance after each month.

(b) Find the total amount of interest paid over the term of the loan.

85. Data Analysis: Sales The table shows the sales a_n (in billions of dollars) for Eli Lilly and Co. from 2004 through 2011. *(Source: Eli Lilly and Co.)*

DATA

Spreadsheet at LarsonPrecalculus.com

Year	Sales, a_n
2004	13.9
2005	14.6
2006	15.7
2007	18.6
2008	20.4
2009	21.8
2010	23.1
2011	24.3

(a) Construct a bar graph showing the annual sales from 2004 through 2011.

(b) Find an arithmetic sequence that models the data. Let n represent the year, with $n = 4$ corresponding to 2004.

(c) Use a graphing utility to graph the terms of the finite sequence you found in part (a).

(d) Use summation notation to represent the *total* sales from 2004 through 2011. Find the total sales.

Exploration

86. Writing Explain how to use the first two terms of an arithmetic sequence to find the nth term.

True or False? In Exercises 87 and 88, determine whether the statement is true or false. Justify your answer.

87. Given an arithmetic sequence for which only the first two terms are known, it is possible to find the nth term.

88. If the only known information about a finite arithmetic sequence is its first term and its last term, then it is possible to find the sum of the sequence.

Finding the Terms of a Sequence In Exercises 89 and 90, find the first 10 terms of the sequence.

89. $a_1 = x, d = 2x$

90. $a_1 = -y, d = 5y$

91. Comparing Graphs of a Sequence and a Line

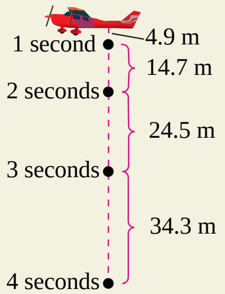
(a) Graph the first 10 terms of the arithmetic sequence $a_n = 2 + 3n$.

(b) Graph the equation of the line $y = 3x + 2$.

(c) Discuss any differences between the graph of $a_n = 2 + 3n$ and the graph of $y = 3x + 2$.

(d) Compare the slope of the line in part (b) with the common difference of the sequence in part (a). What can you conclude about the slope of a line and the common difference of an arithmetic sequence?

92. HOW DO YOU SEE IT? A steel ball with negligible air resistance is dropped from a plane. The figure shows the distance that the ball falls during each of the first four seconds after it is dropped.



(a) Describe a pattern of the distances shown. Explain why the distances form a finite arithmetic sequence.

(b) Assume the pattern described in part (a) continues. Describe the steps and formulas for using the sum of a finite sequence to find the total distance the ball falls in a given whole number of seconds n.

93. Pattern Recognition

(a) Compute the following sums of consecutive positive odd integers.

$1 + 3 = \square$

$1 + 3 + 5 = \square$

$1 + 3 + 5 + 7 = \square$

$1 + 3 + 5 + 7 + 9 = \square$

$1 + 3 + 5 + 7 + 9 + 11 = \square$

(b) Use the sums in part (a) to make a conjecture about the sums of consecutive positive odd integers. Check your conjecture for the sum

$1 + 3 + 5 + 7 + 9 + 11 + 13 = \square.$

(c) Verify your conjecture algebraically.

Project: Municipal Waste To work an extended application analyzing the amounts of municipal waste recovered in the United States from 1991 through 2010, visit this text's website at *LarsonPrecalculus.com.* *(Source: U.S. Environmental Protection Agency)*

8.3 Geometric Sequences and Series

Geometric sequences can help you model and solve real-life problems. For instance, in Exercise 86 on page 590, you will use a geometric sequence to model the populations of China from 2004 through 2010.

- **Recognize, write, and find the *n*th terms of geometric sequences.**
- **Find the sum of a finite geometric sequence.**
- **Find the sum of an infinite geometric series.**
- **Use geometric sequences to model and solve real-life problems.**

Geometric Sequences

In Section 8.2, you learned that a sequence whose consecutive terms have a common *difference* is an arithmetic sequence. In this section, you will study another important type of sequence called a **geometric sequence.** Consecutive terms of a geometric sequence have a common *ratio.*

Definition of Geometric Sequence

A sequence is **geometric** when the ratios of consecutive terms are the same. So, the sequence $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$ is geometric when there is a number r such that

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \cdots = r, \quad r \neq 0.$$

The number r is the **common ratio** of the geometric sequence.

REMARK Be sure you understand that a sequence such as 1, 4, 9, 16, . . . , whose nth term is n^2, is *not* geometric. The ratio of the second term to the first term is

$$\frac{a_2}{a_1} = \frac{4}{1} = 4$$

but the ratio of the third term to the second term is

$$\frac{a_3}{a_2} = \frac{9}{4}.$$

EXAMPLE 1 Examples of Geometric Sequences

a. The sequence whose nth term is 2^n is geometric. For this sequence, the common ratio of consecutive terms is 2.

$$\underbrace{2, 4}_{\frac{4}{2} = 2}, 8, 16, \ldots, 2^n, \ldots$$

Begin with $n = 1$.

b. The sequence whose nth term is $4(3^n)$ is geometric. For this sequence, the common ratio of consecutive terms is 3.

$$\underbrace{12, 36}_{\frac{36}{12} = 3}, 108, 324, \ldots, 4(3^n), \ldots$$

Begin with $n = 1$.

c. The sequence whose nth term is $\left(-\frac{1}{3}\right)^n$ is geometric. For this sequence, the common ratio of consecutive terms is $-\frac{1}{3}$.

$$\underbrace{-\frac{1}{3}, \frac{1}{9}}_{\frac{1/9}{-1/3} = -\frac{1}{3}}, -\frac{1}{27}, \frac{1}{81}, \ldots, \left(-\frac{1}{3}\right)^n, \ldots$$

Begin with $n = 1$.

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Write the first four terms of the geometric sequence whose nth term is $6(-2)^n$. Then find the common ratio of the consecutive terms.

In Example 1, notice that each of the geometric sequences has an nth term that is of the form ar^n, where the common ratio of the sequence is r. A geometric sequence may be thought of as an exponential function whose domain is the set of natural numbers.

TonyV3112/Shutterstock.com

The *n*th Term of a Geometric Sequence

The *n*th term of a geometric sequence has the form

$$a_n = a_1 r^{n-1}$$

where r is the common ratio of consecutive terms of the sequence. So, every geometric sequence can be written in the following form.

$$\begin{array}{cccccccc} a_1, & a_2, & a_3, & a_4, & a_5, & \ldots\ldots, & a_n, & \ldots\ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \\ a_1, & a_1 r, & a_1 r^2, & a_1 r^3, & a_1 r^4, & \ldots, & a_1 r^{n-1}, & \ldots \end{array}$$

When you know the *n*th term of a geometric sequence, you can find the $(n + 1)$th term by multiplying by r. That is, $a_{n+1} = a_n r$.

EXAMPLE 2 Writing the Terms of a Geometric Sequence

Write the first five terms of the geometric sequence whose first term is $a_1 = 3$ and whose common ratio is $r = 2$. Then graph the terms on a set of coordinate axes.

Solution Starting with 3, repeatedly multiply by 2 to obtain the following.

$a_1 = 3$ 1st term

$a_2 = 3(2^1) = 6$ 2nd term

$a_3 = 3(2^2) = 12$ 3rd term

$a_4 = 3(2^3) = 24$ 4th term

$a_5 = 3(2^4) = 48$ 5th term

Figure 8.1 shows the first five terms of this geometric sequence.

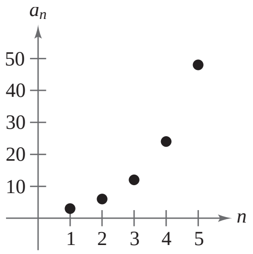


Figure 8.1

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Write the first five terms of the geometric sequence whose first term is $a_1 = 2$ and whose common ratio is $r = 4$. Then graph the terms on a set of coordinate axes.

EXAMPLE 3 Finding a Term of a Geometric Sequence

Find the 15th term of the geometric sequence whose first term is 20 and whose common ratio is 1.05.

Algebraic Solution

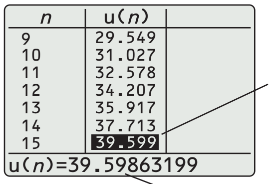
$a_n = a_1 r^{n-1}$ Formula for *n*th term of a geometric sequence

$a_{15} = 20(1.05)^{15-1}$ Substitute 20 for a_1, 1.05 for r, and 15 for n.

≈ 39.60 Use a calculator.

Numerical Solution

For this sequence, $r = 1.05$ and $a_1 = 20$. So, $a_n = 20(1.05)^{n-1}$. Use a graphing utility to create a table that shows the terms of the sequence.



```
 n    u(n)
 9    29.549
 10   31.027
 11   32.578
 12   34.207
 13   35.917
 14   37.713
 15   39.599
u(n)=39.59863199
```

The number in the 15th row is the 15th term of the sequence.

So, $a_{15} \approx 39.60$.

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Find the 12th term of the geometric sequence whose first term is 14 and whose common ratio is 1.2.

EXAMPLE 4 Finding a Term of a Geometric Sequence

Find the 12th term of the geometric sequence

5, 15, 45, ...

Solution The common ratio of this sequence is $r = 15/5 = 3$. Because the first term is $a_1 = 5$, the 12th term $(n = 12)$ is

$$a_n = a_1r^{n-1}$$ Formula for *n*th term of a geometric sequence

$$a_{12} = 5(3)^{12-1}$$ Substitute 5 for a_1, 3 for r, and 12 for n.

$$= 5(177{,}147)$$ Use a calculator.

$$= 885{,}735.$$ Multiply.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the 12th term of the geometric sequence

4, 20, 100, ...

When you know *any* two terms of a geometric sequence, you can use that information to find *any other* term of the sequence.

EXAMPLE 5 Finding a Term of a Geometric Sequence

The fourth term of a geometric sequence is 125, and the 10th term is 125/64. Find the 14th term. (Assume that the terms of the sequence are positive.)

ALGEBRA HELP Remember that r is the common ratio of consecutive terms of a geometric sequence. So, in Example 5

$$a_{10} = a_1r^9$$
$$= a_1 \cdot r \cdot r \cdot r \cdot r^6$$
$$= a_1 \cdot \frac{a_2}{a_1} \cdot \frac{a_3}{a_2} \cdot \frac{a_4}{a_3} \cdot r^6$$
$$= a_4r^6.$$

Solution The 10th term is related to the fourth term by the equation

$$a_{10} = a_4r^6.$$ Multiply fourth term by r^{10-4}.

Because $a_{10} = 125/64$ and $a_4 = 125$, you can solve for r as follows.

$$\frac{125}{64} = 125r^6$$ Substitute $\frac{125}{64}$ for a_{10} and 125 for a_4.

$$\frac{1}{64} = r^6$$ Divide each side by 125.

$$\frac{1}{2} = r$$ Take the sixth root of each side.

You can obtain the 14th term by multiplying the 10th term by r^4.

$$a_{14} = a_{10}r^4$$ Multiply the 10th term by r^{14-10}.

$$= \frac{125}{64}\left(\frac{1}{2}\right)^4$$ Substitute $\frac{125}{64}$ for a_{10} and $\frac{1}{2}$ for r.

$$= \frac{125}{64}\left(\frac{1}{16}\right)$$ Evaluate power.

$$= \frac{125}{1024}$$ Multiply fractions.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The second term of a geometric sequence is 6, and the fifth term is 81/4. Find the eighth term. (Assume that the terms of the sequence are positive.)

The Sum of a Finite Geometric Sequence

The formula for the sum of a *finite* geometric sequence is as follows.

The Sum of a Finite Geometric Sequence

The sum of the finite geometric sequence

$$a_1,\ a_1r,\ a_1r^2,\ a_1r^3,\ a_1r^4, \ldots, a_1r^{n-1}$$

with common ratio $r \neq 1$ is given by $S_n = \sum_{i=1}^{n} a_1r^{i-1} = a_1\left(\frac{1-r^n}{1-r}\right)$.

For a proof of this formula for the sum of a finite geometric sequence, see Proofs in Mathematics on page 641.

EXAMPLE 6 Sum of a Finite Geometric Sequence

Find the sum $\sum_{i=1}^{12} 4(0.3)^{i-1}$.

Solution You have

$$\sum_{i=1}^{12} 4(0.3)^{i-1} = 4(0.3)^0 + 4(0.3)^1 + 4(0.3)^2 + \cdots + 4(0.3)^{11}.$$

Now, $a_1 = 4$, $r = 0.3$, and $n = 12$, so applying the formula for the sum of a finite geometric sequence, you obtain

$$\begin{aligned} S_n &= a_1\left(\frac{1-r^n}{1-r}\right) && \text{Sum of a finite geometric sequence} \\ \sum_{i=1}^{12} 4(0.3)^{i-1} &= 4\left[\frac{1-(0.3)^{12}}{1-0.3}\right] && \text{Substitute 4 for } a_1\text{, 0.3 for } r\text{, and 12 for } n. \\ &\approx 5.714. && \text{Use a calculator.} \end{aligned}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the sum $\sum_{i-1}^{10} 2(0.25)^{i-1}$. ■

When using the formula for the sum of a finite geometric sequence, be careful to check that the sum is of the form

$$\sum_{i=1}^{n} a_1r^{i-1}. \qquad \text{Exponent for } r \text{ is } i-1.$$

For a sum that is not of this form, you must adjust the formula. For instance, if the sum in Example 6 were $\sum_{i=1}^{12} 4(0.3)^i$, then you would evaluate the sum as follows.

$$\begin{aligned} \sum_{i=1}^{12} 4(0.3)^i &= 4(0.3) + 4(0.3)^2 + 4(0.3)^3 + \cdots + 4(0.3)^{12} \\ &= 4(0.3) + [4(0.3)](0.3) + [4(0.3)](0.3)^2 + \cdots + [4(0.3)](0.3)^{11} \\ &= 4(0.3)\left[\frac{1-(0.3)^{12}}{1-0.3}\right] && a_1 = 4(0.3),\ r = 0.3,\ n = 12 \\ &\approx 1.714 \end{aligned}$$

Geometric Series

The summation of the terms of an infinite geometric *sequence* is called an **infinite geometric series** or simply a **geometric series.**

The formula for the sum of a *finite* geometric *sequence* can, depending on the value of r, be extended to produce a formula for the sum of an *infinite* geometric *series*. Specifically, if the common ratio r has the property that $|r| < 1$, then it can be shown that r^n approaches zero as n increases without bound. Consequently,

$$a_1\left(\frac{1 - r^n}{1 - r}\right) \longrightarrow a_1\left(\frac{1 - 0}{1 - r}\right) \quad \text{as} \quad n \longrightarrow \infty.$$

The following summarizes this result.

The Sum of an Infinite Geometric Series

If $|r| < 1$, then the infinite geometric series

$$a_1 + a_1r + a_1r^2 + a_1r^3 + \cdot \cdot \cdot + a_1r^{n-1} + \cdot \cdot \cdot$$

has the sum

$$S = \sum_{i=0}^{\infty} a_1r^i = \frac{a_1}{1 - r}.$$

Note that when $|r| \geq 1$, the series does not have a sum.

EXAMPLE 7 Finding the Sum of an Infinite Geometric Series

Find each sum.

a. $\displaystyle\sum_{n=0}^{\infty} 4(0.6)^n$

b. $3 + 0.3 + 0.03 + 0.003 + \cdot \cdot \cdot$

Solution

a. $$\sum_{n=0}^{\infty} 4(0.6)^n = 4 + 4(0.6) + 4(0.6)^2 + 4(0.6)^3 + \cdot \cdot \cdot + 4(0.6)^n + \cdot \cdot \cdot$$

$$= \frac{4}{1 - 0.6} \qquad \frac{a_1}{1 - r}$$

$$= 10$$

b. $$3 + 0.3 + 0.03 + 0.003 + \cdot \cdot \cdot = 3 + 3(0.1) + 3(0.1)^2 + 3(0.1)^3 + \cdot \cdot \cdot$$

$$= \frac{3}{1 - 0.1} \qquad \frac{a_1}{1 - r}$$

$$= \frac{10}{3}$$

$$\approx 3.33$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find each sum.

a. $\displaystyle\sum_{n=0}^{\infty} 5(0.5)^n$

b. $5 + 1 + 0.2 + 0.04 + \cdot \cdot \cdot$

Application

EXAMPLE 8 Increasing Annuity

An investor deposits \$50 on the first day of each month in an account that pays 3% interest, compounded monthly. What is the balance at the end of 2 years? (This type of investment plan is called an **increasing annuity.**)

Solution To find the balance in the account after 24 months, consider each of the 24 deposits separately. The first deposit will gain interest for 24 months, and its balance will be

$$\begin{aligned}A_{24} &= 50\left(1 + \frac{0.03}{12}\right)^{24}\\ &= 50(1.0025)^{24}.\end{aligned}$$

The second deposit will gain interest for 23 months, and its balance will be

$$\begin{aligned}A_{23} &= 50\left(1 + \frac{0.03}{12}\right)^{23}\\ &= 50(1.0025)^{23}.\end{aligned}$$

The last deposit will gain interest for only 1 month, and its balance will be

$$\begin{aligned}A_{1} &= 50\left(1 + \frac{0.03}{12}\right)^{1}\\ &= 50(1.0025).\end{aligned}$$

The total balance in the annuity will be the sum of the balances of the 24 deposits. Using the formula for the sum of a finite geometric sequence, with $A_1 = 50(1.0025)$, $r = 1.0025$, and $n = 24$, you have

$$S_n = A_1\left(\frac{1 - r^n}{1 - r}\right) \qquad \text{Sum of a finite geometric sequence}$$

$$S_{24} = 50(1.0025)\left[\frac{1 - (1.0025)^{24}}{1 - 1.0025}\right] \qquad \text{Substitute } 50(1.0025) \text{ for } A_1, 1.0025 \text{ for } r, \text{ and } 24 \text{ for } n.$$

$$\approx \$1238.23. \qquad \text{Use a calculator.}$$

REMARK Recall from Section 5.1 that the formula for compound interest (for n compoundings per year) is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

So, in Example 8, \$50 is the principal P, 0.03 is the annual interest rate r, 12 is the number n of compoundings per year, and 2 is the time t in years. When you substitute these values into the formula, you obtain

$$\begin{aligned}A &= 50\left(1 + \frac{0.03}{12}\right)^{12(2)}\\ &= 50\left(1 + \frac{0.03}{12}\right)^{24}.\end{aligned}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

An investor deposits \$70 on the first day of each month in an account that pays 2% interest, compounded monthly. What is the balance at the end of 4 years?

Summarize (Section 8.3)

1. State the definition of a geometric sequence *(page 583)* and state the formula for the nth term of a geometric sequence *(page 584)*. For examples of recognizing, writing, and finding the nth terms of geometric sequences, see Examples 1–5.
2. State the formula for the sum of a finite geometric sequence *(page 586)*. For an example of finding the sum of a finite geometric sequence, see Example 6.
3. State the formula for the sum of an infinite geometric series *(page 587)*. For an example of finding the sums of infinite geometric series, see Example 7.
4. Describe an example of how to use a geometric series to model and solve a real-life problem *(page 588, Example 8)*.

8.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: **Fill in the blanks.**

1. A sequence is called a ________ sequence when the ratios of consecutive terms are the same. This ratio is called the ________ ratio.
2. The nth term of a geometric sequence has the form ________.
3. The sum of a finite geometric sequence with common ratio $r \neq 1$ is given by ________.
4. The sum of the terms of an infinite geometric sequence is called a ________ ________.

Skills and Applications

Determining Whether a Sequence Is Geometric

In Exercises 5–12, determine whether the sequence is geometric. If so, then find the common ratio.

5. $2, 10, 50, 250, \ldots$

6. $3, 12, 21, 30, \ldots$

7. $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, \ldots$

8. $9, -6, 4, -\frac{8}{3}, \ldots$

9. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

10. $5, 1, 0.2, 0.04, \ldots$

11. $1, -\sqrt{7}, 7, -7\sqrt{7}, \ldots$

12. $2, \frac{4}{\sqrt{3}}, \frac{8}{3}, \frac{16}{3\sqrt{3}}, \ldots$

Writing the Terms of a Geometric Sequence

In Exercises 13–22, write the first five terms of the geometric sequence.

13. $a_1 = 4, r = 3$

14. $a_1 = 8, r = 2$

15. $a_1 = 1, r = \frac{1}{2}$

16. $a_1 = 6, r = -\frac{1}{4}$

17. $a_1 = 1, r = e$

18. $a_1 = 2, r = \pi$

19. $a_1 = 3, r = \sqrt{5}$

20. $a_1 = 4, r = -\frac{1}{\sqrt{2}}$

21. $a_1 = 2, r = \frac{x}{4}$

22. $a_1 = 5, r = 2x$

Writing the *n*th Term of a Geometric Sequence

In Exercises 23–28, write the first five terms of the geometric sequence. Determine the common ratio and write the *n*th term of the sequence as a function of *n*.

23. $a_1 = 64, \quad a_{k+1} = \frac{1}{2}a_k$

24. $a_1 = 81, \quad a_{k+1} = \frac{1}{3}a_k$

25. $a_1 = 9, \quad a_{k+1} = 2a_k$

26. $a_1 = 5, \quad a_{k+1} = -2a_k$

27. $a_1 = 6, \quad a_{k+1} = -\frac{3}{2}a_k$

28. $a_1 = 80, \quad a_{k+1} = -\frac{1}{2}a_k$

Finding a Term of a Geometric Sequence

In Exercises 29–38, write an expression for the *n*th term of the geometric sequence. Then find the indicated term.

29. $a_1 = 4, r = \frac{1}{2}, n = 10$

30. $a_1 = 5, r = \frac{7}{2}, n = 8$

31. $a_1 = 6, r = -\frac{1}{3}, n = 12$

32. $a_1 = 64, r = -\frac{1}{4}, n = 10$

33. $a_1 = 100, r = e^x, n = 9$

34. $a_1 = 1, r = e^{-x}, n = 4$

35. $a_1 = 1, r = \sqrt{2}, n = 12$

36. $a_1 = 1, r = \sqrt{3}, n = 8$

37. $a_1 = 500, r = 1.02, n = 40$

38. $a_1 = 1000, r = 1.005, n = 60$

Finding a Term of a Geometric Sequence

In Exercises 39–46, find the indicated term of the geometric sequence.

39. 9th term: $11, 33, 99, \ldots$

40. 7th term: $3, 36, 432, \ldots$

41. 8th term: $\frac{1}{2}, -\frac{1}{8}, \frac{1}{32}, -\frac{1}{128}, \ldots$

42. 7th term: $\frac{8}{5}, -\frac{16}{25}, \frac{32}{125}, -\frac{64}{625}, \ldots$

43. 3rd term: $a_1 = 16, a_4 = \frac{27}{4}$

44. 1st term: $a_2 = 3, a_5 = \frac{3}{64}$

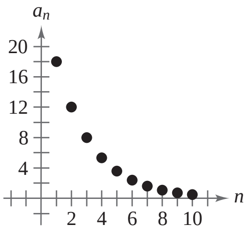
45. 6th term: $a_4 = -18, a_7 = \frac{2}{3}$

46. 5th term: $a_2 = 2, a_3 = -\sqrt{2}$

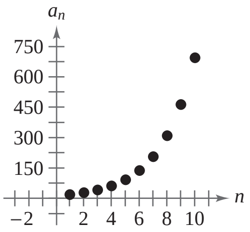
Matching a Geometric Sequence with Its Graph

In Exercises 47–50, match the geometric sequence with its graph. [The graphs are labeled (a), (b), (c), and (d).]

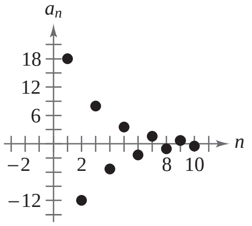
(a)



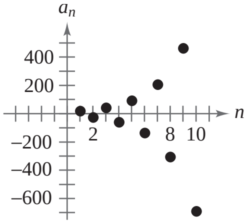
(b)



(c)



(d)



47. $a_n = 18\left(\frac{2}{3}\right)^{n-1}$

48. $a_n = 18\left(-\frac{2}{3}\right)^{n-1}$

49. $a_n = 18\left(\frac{3}{2}\right)^{n-1}$

50. $a_n = 18\left(-\frac{3}{2}\right)^{n-1}$

Graphing the Terms of a Sequence

In Exercises 51–54, use a graphing utility to graph the first 10 terms of the sequence.

51. $a_n = 10(1.5)^{n-1}$

52. $a_n = 12(-0.4)^{n-1}$

53. $a_n = 20(-1.25)^{n-1}$

54. $a_n = 2(1.3)^{n-1}$

Sum of a Finite Geometric Sequence **In Exercises 55–68, find the sum of the finite geometric sequence.**

55. $\sum_{n=1}^{7} 4^{n-1}$ **56.** $\sum_{n=1}^{10} \left(\frac{3}{2}\right)^{n-1}$

57. $\sum_{n=1}^{6} (-7)^{n-1}$ **58.** $\sum_{n=1}^{8} 5\left(-\frac{5}{2}\right)^{n-1}$

59. $\sum_{i=1}^{7} 64\left(-\frac{1}{2}\right)^{i-1}$ **60.** $\sum_{i=1}^{12} 16\left(\frac{1}{2}\right)^{i-1}$

61. $\sum_{n=0}^{20} 3\left(\frac{3}{2}\right)^{n}$ **62.** $\sum_{n=0}^{40} 5\left(\frac{3}{5}\right)^{n}$

63. $\sum_{n=0}^{20} 10\left(\frac{1}{5}\right)^{n}$ **64.** $\sum_{n=0}^{6} 500(1.04)^{n}$

65. $\sum_{n=0}^{40} 2\left(-\frac{1}{4}\right)^{n}$ **66.** $\sum_{n=0}^{50} 10\left(\frac{2}{3}\right)^{n-1}$

67. $\sum_{i=1}^{100} 15\left(\frac{2}{3}\right)^{i-1}$ **68.** $\sum_{i=1}^{10} 8\left(-\frac{1}{4}\right)^{i-1}$

Using Summation Notation **In Exercises 69–72, use summation notation to write the sum.**

69. $10 + 30 + 90 + \cdot \cdot \cdot + 7290$

70. $15 - 3 + \frac{3}{5} - \cdot \cdot \cdot - \frac{3}{625}$

71. $0.1 + 0.4 + 1.6 + \cdot \cdot \cdot + 102.4$

72. $32 + 24 + 18 + \cdot \cdot \cdot + 10.125$

Sum of an Infinite Geometric Series **In Exercises 73–80, find the sum of the infinite geometric series.**

73. $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n}$ **74.** $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{n}$

75. $\sum_{n=0}^{\infty} 2\left(-\frac{2}{3}\right)^{n}$ **76.** $\sum_{n=0}^{\infty} 4(0.2)^{n}$

77. $8 + 6 + \frac{9}{2} + \frac{27}{8} + \cdot \cdot \cdot$

78. $9 + 6 + 4 + \frac{8}{3} + \cdot \cdot \cdot$

79. $\frac{1}{9} - \frac{1}{3} + 1 - 3 + \cdot \cdot \cdot$

80. $-\frac{125}{36} + \frac{25}{6} - 5 + 6 - \cdot \cdot \cdot$

Writing a Repeating Decimal as a Rational Number **In Exercises 81 and 82, find the rational number representation of the repeating decimal.**

81. $0.\overline{36}$ **82.** $0.3\overline{18}$

Graphical Reasoning **In Exercises 83 and 84, use a graphing utility to graph the function. Identify the horizontal asymptote of the graph and determine its relationship to the sum.**

83. $f(x) = 6\left[\dfrac{1-(0.5)^{x}}{1-(0.5)}\right], \quad \sum_{n=0}^{\infty} 6\left(\dfrac{1}{2}\right)^{n}$

84. $f(x) = 2\left[\dfrac{1-(0.8)^{x}}{1-(0.8)}\right], \quad \sum_{n=0}^{\infty} 2\left(\dfrac{4}{5}\right)^{n}$

85. **Depreciation** A tool and die company buys a machine for $175,000 and it depreciates at a rate of 30% per year. (In other words, at the end of each year the depreciated value is 70% of what it was at the beginning of the year.) Find the depreciated value of the machine after 5 full years.

86. **Data Analysis: Population**

The table shows the mid-year populations a_n of China (in millions) from 2004 through 2010. *(Source: U.S. Census Bureau)*

DATA

Spreadsheet at LarsonPrecalculus.com

Year	Population, a_n
2004	1291.0
2005	1297.8
2006	1304.3
2007	1310.6
2008	1317.1
2009	1323.6
2010	1330.1

(a) Use the *exponential regression* feature of a graphing utility to find a geometric sequence that models the data. Let n represent the year, with $n = 4$ corresponding to 2004.

(b) Use the sequence from part (a) to describe the rate at which the population of China is growing.

(c) Use the sequence from part (a) to predict the population of China in 2017. The U.S. Census Bureau predicts the population of China will be 1372.1 million in 2017. How does this value compare with your prediction?

(d) Use the sequence from part (a) to determine when the population of China will reach 1.4 billion.

87. **Annuity** An investor deposits P dollars on the first day of each month in an account with an annual interest rate r, compounded monthly. The balance A after t years is

$$A = P\left(1 + \frac{r}{12}\right) + P\left(1 + \frac{r}{12}\right)^{2} + \cdot \cdot \cdot + P\left(1 + \frac{r}{12}\right)^{12t}.$$

Show that the balance is

$$A = P\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]\left(1 + \frac{12}{r}\right).$$

TonyV3112/Shutterstock.com

88. Annuity An investor deposits \$100 on the first day of each month in an account that pays 2% interest, compounded monthly. The balance A in the account at the end of 5 years is

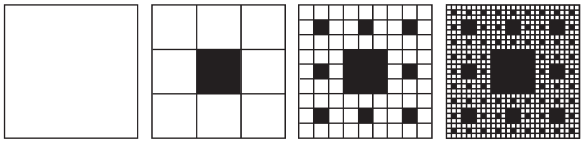
$$A = 100\left(1 + \frac{0.02}{12}\right)^1 + \cdot \cdot \cdot + 100\left(1 + \frac{0.02}{12}\right)^{60}.$$

Use the result of Exercise 87 to find A.

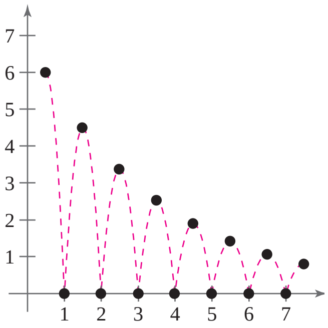
Multiplier Effect In Exercises 89 and 90, use the following information. A state government gives property owners a tax rebate with the anticipation that each property owner will spend approximately p% of the rebate, and in turn each recipient of this amount will spend p% of what he or she receives, and so on. Economists refer to this exchange of money and its circulation within the economy as the "multiplier effect." The multiplier effect operates on the idea that the expenditures of one individual become the income of another individual. For the given tax rebate, find the total amount put back into the state's economy, assuming that this effect continues without end.

	Tax rebate	$p\%$
89.	\$400	75%
90.	\$600	72.5%

91. Geometry The sides of a square are 27 inches in length. New squares are formed by dividing the original square into nine squares. The center square is then shaded (see figure). If this process is repeated three more times, then determine the total area of the shaded region.



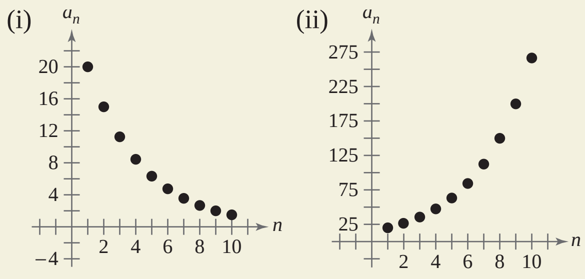
92. Distance A ball is dropped from a height of 6 feet and begins bouncing as shown in the figure. The height of each bounce is three-fourths the height of the previous bounce. Find the total vertical distance the ball travels before coming to rest.



93. Salary An investment firm has a job opening with a salary of \$45,000 for the first year. During the next 39 years, there is a 5% raise each year. Find the total compensation over the 40-year period.

Exploration

94. HOW DO YOU SEE IT? Use the figures shown below.



(a) Without performing any calculations, determine which figure shows the terms of the sequence $a_n = 20\left(\frac{4}{3}\right)^{n-1}$ and which shows the terms of $a_n = 20\left(\frac{3}{4}\right)^{n-1}$. Explain your reasoning.

(b) Which sequence has terms that can be summed? Explain your reasoning.

True or False? In Exercises 95 and 96, determine whether the statement is true or false. Justify your answer.

95. A sequence is geometric when the ratios of consecutive differences of consecutive terms are the same.

96. To find the nth term of a geometric sequence, multiply its common ratio by the first term of the sequence raised to the $(n - 1)$th power.

97. Graphical Reasoning Consider the graph of

$$y = \left(\frac{1 - r^x}{1 - r}\right).$$

(a) Use a graphing utility to graph y for $r = \frac{1}{2}, \frac{2}{3}$, and $\frac{4}{5}$. What happens as $x \to \infty$?

(b) Use the graphing utility to graph y for $r = 1.5$, 2, and 3. What happens as $x \to \infty$?

98. Writing Write a brief paragraph explaining why the terms of a geometric sequence decrease in magnitude when $-1 < r < 1$.

Project: Housing Vacancies To work an extended application analyzing the numbers of vacant houses in the United States from 1990 through 2011, visit this text's website at *LarsonPrecalculus.com. (Source: U.S. Census Bureau)*

8.4 Mathematical Induction

- **Use mathematical induction to prove statements involving a positive integer *n*.**
- **Use pattern recognition and mathematical induction to write a formula for the *n*th term of a sequence.**
- **Find the sums of powers of integers.**
- **Find finite differences of sequences.**

Finite differences can help you determine what type of model to use to represent a sequence. For instance, in Exercise 75 on page 601, you will use finite differences to find a model that represents the numbers of Alaskan residents from 2005 through 2010.

Introduction

In this section, you will study a form of mathematical proof called **mathematical induction.** It is important that you see clearly the logical need for it, so take a closer look at the problem discussed in Example 5 in Section 8.2.

$$S_1 = 1 = 1^2$$
$$S_2 = 1 + 3 = 2^2$$
$$S_3 = 1 + 3 + 5 = 3^2$$
$$S_4 = 1 + 3 + 5 + 7 = 4^2$$
$$S_5 = 1 + 3 + 5 + 7 + 9 = 5^2$$
$$S_6 = 1 + 3 + 5 + 7 + 9 + 11 = 6^2$$

Judging from the pattern formed by these first six sums, it appears that the sum of the first n odd integers is

$$S_n = 1 + 3 + 5 + 7 + 9 + 11 + \cdot \cdot \cdot + (2n - 1) = n^2.$$

Although this particular formula *is* valid, it is important for you to see that recognizing a pattern and then simply *jumping to the conclusion* that the pattern must be true for all values of n is *not* a logically valid method of proof. There are many examples in which a pattern appears to be developing for small values of n, but then at some point the pattern fails. One of the most famous cases of this was the conjecture by the French mathematician Pierre de Fermat (1601–1665), who speculated that all numbers of the form

$$F_n = 2^{2^n} + 1, \quad n = 0, 1, 2, . . .$$

are prime. For $n = 0, 1, 2, 3,$ and 4, the conjecture is true.

$$F_0 = 3$$
$$F_1 = 5$$
$$F_2 = 17$$
$$F_3 = 257$$
$$F_4 = 65{,}537$$

The size of the next Fermat number $(F_5 = 4{,}294{,}967{,}297)$ is so great that it was difficult for Fermat to determine whether it was prime or not. However, another well-known mathematician, Leonhard Euler (1707–1783), later found the factorization

$$\begin{aligned} F_5 &= 4{,}294{,}967{,}297 \\ &= 641(6{,}700{,}417) \end{aligned}$$

which proved that F_5 is not prime and therefore Fermat's conjecture was false.

Just because a rule, pattern, or formula seems to work for several values of n, you cannot simply decide that it is valid for all values of n without going through a *legitimate proof.* Mathematical induction is one method of proof.

Mark Herreid/Shutterstock.com

REMARK It is important to recognize that in order to prove a statement by induction, both parts of the Principle of Mathematical Induction are necessary.

The Principle of Mathematical Induction

Let P_n be a statement involving the positive integer n. If

1. P_1 is true, and
2. for every positive integer k, the truth of P_k implies the truth of P_{k+1}

then the statement P_n must be true for all positive integers n.

To apply the Principle of Mathematical Induction, you need to be able to determine the statement P_{k+1} for a given statement P_k. To determine P_{k+1}, substitute the quantity $k + 1$ for k in the statement P_k.

EXAMPLE 1 A Preliminary Example

Find the statement P_{k+1} for each given statement P_k.

a. $P_k: \; S_k = \dfrac{k^2(k+1)^2}{4}$

b. $P_k: \; S_k = 1 + 5 + 9 + \cdot \cdot \cdot + [4(k-1) - 3] + (4k - 3)$

c. $P_k: \; k + 3 < 5k^2$

d. $P_k: \; 3^k \geq 2k + 1$

Solution

a. $$P_{k+1}: \; S_{k+1} = \frac{(k+1)^2(k+1+1)^2}{4} \qquad \text{Replace } k \text{ with } k+1.$$
$$= \frac{(k+1)^2(k+2)^2}{4} \qquad \text{Simplify.}$$

b. $$P_{k+1}: \; S_{k+1} = 1 + 5 + 9 + \cdot \cdot \cdot + \{4[(k+1) - 1] - 3\} + [4(k+1) - 3]$$
$$= 1 + 5 + 9 + \cdot \cdot \cdot + (4k - 3) + (4k + 1)$$

c. $$P_{k+1}: \; (k+1) + 3 < 5(k+1)^2$$
$$k + 4 < 5(k^2 + 2k + 1)$$

d. $$P_{k+1}: \; 3^{k+1} \geq 2(k+1) + 1$$
$$3^{k+1} \geq 2k + 3$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the statement P_{k+1} for each given statement P_k.

a. P_k: $S_k = \dfrac{6}{k(k+3)}$ **b.** P_k: $k + 2 \leq 3(k-1)^2$ **c.** P_k: $2^{4k-2} + 1 > 5k$ ■

A well-known illustration used to explain why the Principle of Mathematical Induction works is that of an unending line of dominoes. If the line actually contains infinitely many dominoes, then it is clear that you could not knock the entire line down by knocking down only *one domino* at a time. However, suppose it were true that each domino would knock down the next one as it fell. Then you could knock them all down by pushing the first one and starting a chain reaction. Mathematical induction works in the same way. If the truth of P_k implies the truth of P_{k+1} and if P_1 is true, then the chain reaction proceeds as follows: P_1 implies P_2, P_2 implies P_3, P_3 implies P_4, and so on.

An unending line of dominoes can illustrate why the Principle of Mathematical Induction works.

© mark wragg/istockphoto.com

When using mathematical induction to prove a *summation* formula (such as the one in Example 2), it is helpful to think of S_{k+1} as $S_{k+1} = S_k + a_{k+1}$, where a_{k+1} is the $(k + 1)$th term of the original sum.

EXAMPLE 2 Using Mathematical Induction

Use mathematical induction to prove the formula

$$\begin{aligned} S_n &= 1 + 3 + 5 + 7 + \cdot \cdot \cdot + (2n - 1) \\ &= n^2 \end{aligned}$$

for all integers $n \geq 1$.

Solution Mathematical induction consists of two distinct parts.

1. First, you must show that the formula is true when $n = 1$. When $n = 1$, the formula is valid, because

$$\begin{aligned} S_1 &= 1 \\ &= 1^2. \end{aligned}$$

2. The second part of mathematical induction has two steps. The first step is to *assume* that the formula is valid for some integer k. The second step is to use this assumption to prove that the formula is valid for the *next* integer, $k + 1$. Assuming that the formula

$$\begin{aligned} S_k &= 1 + 3 + 5 + 7 + \cdot \cdot \cdot + (2k - 1) \\ &= k^2 \end{aligned}$$

is true, you must show that the formula $S_{k+1} = (k + 1)^2$ is true.

$$\begin{aligned} S_{k+1} &= 1 + 3 + 5 + 7 + \cdot \cdot \cdot + (2k - 1) \\ &\qquad + [2(k + 1) - 1] && S_{k+1} = S_k + a_{k+1} \\ &= [1 + 3 + 5 + 7 + \cdot \cdot \cdot + (2k - 1)] + (2k + 2 - 1) \\ &= S_k + (2k + 1) && \text{Group terms to form } S_k. \\ &= k^2 + 2k + 1 && \text{Substitute } k^2 \text{ for } S_k. \\ &= (k + 1)^2 \end{aligned}$$

Combining the results of parts (1) and (2), you can conclude by mathematical induction that the formula is valid for all integers $n \geq 1$.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use mathematical induction to prove the formula

$$S_n = 5 + 7 + 9 + 11 + \cdot \cdot \cdot + (2n + 3) = n(n + 4)$$

for all integers $n \geq 1$.

It occasionally happens that a statement involving natural numbers is not true for the first $k - 1$ positive integers but is true for all values of $n \geq k$. In these instances, you use a slight variation of the Principle of Mathematical Induction in which you verify P_k rather than P_1. This variation is called the *Extended Principle of Mathematical Induction.* To see the validity of this, note in the unending line of dominoes discussed on page 593 that all but the first $k - 1$ dominoes can be knocked down by knocking over the kth domino. This suggests that you can prove a statement P_n to be true for $n \geq k$ by showing that P_k is true and that P_k implies P_{k+1}. In Exercises 25–30 of this section, you will apply this extension of mathematical induction.

EXAMPLE 3 Using Mathematical Induction

Use mathematical induction to prove the formula

$$S_n = 1^2 + 2^2 + 3^2 + 4^2 + \cdot \cdot \cdot + n^2$$
$$= \frac{n(n+1)(2n+1)}{6}$$

for all integers $n \geq 1$.

Solution

1. When $n = 1$, the formula is valid, because

$$S_1 = 1^2$$
$$= \frac{1(2)(3)}{6}.$$

2. Assuming that

$$S_k = 1^2 + 2^2 + 3^2 + 4^2 + \cdot \cdot \cdot + k^2 \qquad a_k = k^2$$
$$= \frac{k(k+1)(2k+1)}{6}$$

you must show that

$$S_{k+1} = \frac{(k+1)(k+1+1)[2(k+1)+1]}{6}$$
$$= \frac{(k+1)(k+2)(2k+3)}{6}.$$

To do this, write the following.

$$S_{k+1} = S_k + a_{k+1}$$
$$= (1^2 + 2^2 + 3^2 + 4^2 + \cdot \cdot \cdot + k^2) + (k+1)^2 \qquad \text{Substitute for } S_k,\ a_{k+1} = (k+1)^2.$$
$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \qquad \text{By assumption}$$
$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \qquad \text{Combine fractions.}$$
$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \qquad \text{Factor.}$$
$$= \frac{(k+1)(2k^2 + 7k + 6)}{6} \qquad \text{Simplify.}$$
$$= \frac{(k+1)(k+2)(2k+3)}{6} \qquad S_k \text{ implies } S_{k+1}.$$

REMARK Remember that when adding rational expressions, you must first find a common denominator. Example 3 uses the *least* common denominator of 6.

Combining the results of parts (1) and (2), you can conclude by mathematical induction that the formula is valid for all positive integers n.

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Use mathematical induction to prove the formula

$$S_n = 1(1-1) + 2(2-1) + 3(3-1) + \cdot \cdot \cdot + n(n-1) = \frac{n(n-1)(n+1)}{3}$$

for all integers $n \geq 1$.

When proving a formula using mathematical induction, the only statement that you *need* to verify is P_1. As a check, however, it is a good idea to try verifying some of the other statements. For instance, in Example 3, try verifying P_2 and P_3.

EXAMPLE 4 Proving an Inequality

Prove that

$$n < 2^n$$

for all positive integers n.

Solution

1. For $n = 1$ and $n = 2$, the statement is true because

$$1 < 2^1 \quad \text{and} \quad 2 < 2^2.$$

2. Assuming that

$$k < 2^k$$

you need to show that $k + 1 < 2^{k+1}$. For $n = k$, you have

$$2^{k+1} = 2(2^k) > 2(k) = 2k. \qquad \text{By assumption}$$

Because $2k = k + k > k + 1$ for all $k > 1$, it follows that

$$2^{k+1} > 2k > k + 1 \quad \text{or} \quad k + 1 < 2^{k+1}.$$

Combining the results of parts (1) and (2), you can conclude by mathematical induction that $n < 2^n$ for all integers $n \geq 1$.

REMARK To check a result that you have proved by mathematical induction, it helps to list the statement for several values of n. For instance, in Example 4, you could list

$$1 < 2^1 = 2, \quad 2 < 2^2 = 4,$$
$$3 < 2^3 = 8, \quad 4 < 2^4 = 16,$$
$$5 < 2^5 = 32, \quad 6 < 2^6 = 64.$$

From this list, your intuition confirms that the statement $n < 2^n$ is reasonable.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Prove that

$$n! \geq n$$

for all positive integers n.

EXAMPLE 5 Proving a Factor

Prove that 3 is a factor of $4^n - 1$ for all positive integers n.

Solution

1. For $n = 1$, the statement is true because

$$4^1 - 1 = 3.$$

So, 3 is a factor.

2. Assuming that 3 is a factor of $4^k - 1$, you must show that 3 is a factor of $4^{k+1} - 1$. To do this, write the following.

$$\begin{aligned} 4^{k+1} - 1 &= 4^{k+1} - 4^k + 4^k - 1 && \text{Subtract and add } 4^k. \\ &= 4^k(4 - 1) + (4^k - 1) && \text{Regroup terms.} \\ &= 4^k \cdot 3 + (4^k - 1) && \text{Simplify.} \end{aligned}$$

Because 3 is a factor of $4^k \cdot 3$ and 3 is also a factor of $4^k - 1$, it follows that 3 is a factor of $4^{k+1} - 1$. Combining the results of parts (1) and (2), you can conclude by mathematical induction that 3 is a factor of $4^n - 1$ for all positive integers n.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Prove that 2 is a factor of $3^n + 1$ for all positive integers n.

Pattern Recognition

Although choosing a formula on the basis of a few observations does *not* guarantee the validity of the formula, pattern recognition *is* important. Once you have a pattern or formula that you think works, try using mathematical induction to prove your formula.

Finding a Formula for the *n*th Term of a Sequence

To find a formula for the nth term of a sequence, consider these guidelines.

1. Calculate the first several terms of the sequence. It is often a good idea to write the terms in both simplified and factored forms.

2. Try to find a recognizable pattern for the terms and write a formula for the nth term of the sequence. This is your *hypothesis* or *conjecture*. You might try computing one or two more terms in the sequence to test your hypothesis.

3. Use mathematical induction to prove your hypothesis.

EXAMPLE 6 Finding a Formula for a Finite Sum

Find a formula for the finite sum and prove its validity.

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdot\cdot\cdot+\frac{1}{n(n+1)}$$

Solution Begin by writing the first few sums.

$$S_1=\frac{1}{1\cdot 2}=\frac{1}{2}=\frac{1}{1+1}$$

$$S_2=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}=\frac{4}{6}=\frac{2}{3}=\frac{2}{2+1}$$

$$S_3=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}=\frac{9}{12}=\frac{3}{4}=\frac{3}{3+1}$$

From this sequence, it appears that the formula for the kth sum is

$$S_k=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdot\cdot\cdot+\frac{1}{k(k+1)}=\frac{k}{k+1}.$$

To prove the validity of this hypothesis, use mathematical induction. Note that you have already verified the formula for $n = 1$, so begin by assuming that the formula is valid for $n = k$ and trying to show that it is valid for $n = k + 1$.

$$\begin{aligned}S_{k+1}&=\left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdot\cdot\cdot+\frac{1}{k(k+1)}\right]+\frac{1}{(k+1)(k+2)}\\&=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}\qquad\text{By assumption}\\&=\frac{k(k+2)+1}{(k+1)(k+2)}=\frac{k^2+2k+1}{(k+1)(k+2)}=\frac{(k+1)^2}{(k+1)(k+2)}=\frac{k+1}{k+2}\end{aligned}$$

So, by mathematical induction the hypothesis is valid.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find a formula for the finite sum and prove its validity.

$$3+7+11+15+\cdot\cdot\cdot+4n-1$$

Sums of Powers of Integers

The formula in Example 3 is one of a collection of useful summation formulas. This and other formulas dealing with the sums of various powers of the first n positive integers are as follows.

Sums of Powers of Integers

1. $1 + 2 + 3 + 4 + \cdot \cdot \cdot + n = \dfrac{n(n + 1)}{2}$

2. $1^2 + 2^2 + 3^2 + 4^2 + \cdot \cdot \cdot + n^2 = \dfrac{n(n + 1)(2n + 1)}{6}$

3. $1^3 + 2^3 + 3^3 + 4^3 + \cdot \cdot \cdot + n^3 = \dfrac{n^2(n + 1)^2}{4}$

4. $1^4 + 2^4 + 3^4 + 4^4 + \cdot \cdot \cdot + n^4 = \dfrac{n(n + 1)(2n + 1)(3n^2 + 3n - 1)}{30}$

5. $1^5 + 2^5 + 3^5 + 4^5 + \cdot \cdot \cdot + n^5 = \dfrac{n^2(n + 1)^2(2n^2 + 2n - 1)}{12}$

EXAMPLE 7 Finding Sums

Find each sum.

a. $\displaystyle\sum_{i=1}^{7} i^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3$ **b.** $\displaystyle\sum_{i=1}^{4} (6i - 4i^2)$

Solution

a. Using the formula for the sum of the cubes of the first n positive integers, you obtain

$$\begin{aligned}\sum_{i=1}^{7} i^3 &= 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3\\ &= \frac{7^2(7 + 1)^2}{4} && \text{Formula 3}\\ &= \frac{49(64)}{4}\\ &= 784.\end{aligned}$$

b.
$$\begin{aligned}\sum_{i=1}^{4} (6i - 4i^2) &= \sum_{i=1}^{4} 6i - \sum_{i=1}^{4} 4i^2\\ &= 6\sum_{i=1}^{4} i - 4\sum_{i=1}^{4} i^2\\ &= 6\left[\frac{4(4 + 1)}{2}\right] - 4\left[\frac{4(4 + 1)(2 \cdot 4 + 1)}{6}\right] && \text{Formulas 1 and 2}\\ &= 6(10) - 4(30)\\ &= -60\end{aligned}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find each sum.

a. $\displaystyle\sum_{i=1}^{20} i$ **b.** $\displaystyle\sum_{i=1}^{5} (2i^2 + 3i^3)$ ■

Finite Differences

The **first differences** of a sequence are found by subtracting consecutive terms. The **second differences** are found by subtracting consecutive first differences. The first and second differences of the sequence 3, 5, 8, 12, 17, 23, . . . are as follows.

n:	1		2		3		4		5		6
a_n:	3		5		8		12		17		23
First differences:		2		3		4		5		6	
Second differences:			1		1		1		1		

For this sequence, the second differences are all the same. When this happens, the sequence has a perfect *quadratic* model. When the first differences are all the same, the sequence has a perfect *linear* model. That is, it is arithmetic.

EXAMPLE 8 Finding a Quadratic Model

Find the quadratic model for the sequence

$3, 5, 8, 12, 17, 23, \ldots.$

Solution You know from the second differences shown above that the model is quadratic and has the form $a_n = an^2 + bn + c$. By substituting 1, 2, and 3 for n, you can obtain a system of three linear equations in three variables.

$a_1 = a(1)^2 + b(1) + c = 3$ Substitute 1 for n.

$a_2 = a(2)^2 + b(2) + c = 5$ Substitute 2 for n.

$a_3 = a(3)^2 + b(3) + c = 8$ Substitute 3 for n.

You now have a system of three equations in a, b, and c.

$$\begin{cases} a + b + c = 3 & \text{Equation 1} \\ 4a + 2b + c = 5 & \text{Equation 2} \\ 9a + 3b + c = 8 & \text{Equation 3} \end{cases}$$

Using the techniques discussed in Chapter 6, you can find the solution to be $a = \frac{1}{2}$, $b = \frac{1}{2}$, and $c = 2$. So, the quadratic model is $a_n = \frac{1}{2}n^2 + \frac{1}{2}n + 2$. Try checking the values of a_1, a_2, and a_3.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find a quadratic model for the sequence $-2, 0, 4, 10, 18, 28, \ldots.$ ■

Summarize (Section 8.4)

1. State the Principle of Mathematical Induction *(page 593)*. For examples of using mathematical induction to prove statements involving a positive integer n, see Examples 2–5.
2. Explain how to use pattern recognition and mathematical induction to write a formula for the nth term of a sequence *(page 597)*. For an example of using pattern recognition and mathematical induction to write a formula for the nth term of a sequence, see Example 6.
3. State the formulas for the sums of powers of integers *(page 598)*. For an example of finding sums of powers of integers, see Example 7.
4. Explain how to find finite differences of sequences *(page 599)*. For an example of using finite differences to find a quadratic model, see Example 8.

8.4 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. The first step in proving a formula by ________ ________ is to show that the formula is true when $n = 1$.
2. To find the ________ differences of a sequence, subtract consecutive terms.
3. A sequence is an ________ sequence when the first differences are all the same nonzero number.
4. If the ________ differences of a sequence are all the same nonzero number, then the sequence has a perfect quadratic model.

Skills and Applications

Finding P_{k+1} Given P_k **In Exercises 5–10, find P_{k+1} for the given P_k.**

5. $P_k = \dfrac{5}{k(k+1)}$

6. $P_k = \dfrac{1}{2(k+2)}$

7. $P_k = \dfrac{k^2(k+3)^2}{6}$

8. $P_k = \dfrac{k}{3}(2k+1)$

9. $P_k = \dfrac{3}{(k+2)(k+3)}$

10. $P_k = \dfrac{k^2}{2(k+1)^2}$

Using Mathematical Induction **In Exercises 11–24, use mathematical induction to prove the formula for every positive integer n.**

11. $2 + 4 + 6 + 8 + \cdot \cdot \cdot + 2n = n(n+1)$

12. $3 + 6 + 9 + 12 + \cdot \cdot \cdot + 3n = \dfrac{3n}{2}(n+1)$

13. $2 + 7 + 12 + 17 + \cdot \cdot \cdot + (5n-3) = \dfrac{n}{2}(5n-1)$

14. $1 + 4 + 7 + 10 + \cdot \cdot \cdot + (3n-2) = \dfrac{n}{2}(3n-1)$

15. $1 + 2 + 2^2 + 2^3 + \cdot \cdot \cdot + 2^{n-1} = 2^n - 1$

16. $2(1 + 3 + 3^2 + 3^3 + \cdot \cdot \cdot + 3^{n-1}) = 3^n - 1$

17. $1 + 2 + 3 + 4 + \cdot \cdot \cdot + n = \dfrac{n(n+1)}{2}$

18. $1^3 + 2^3 + 3^3 + 4^3 + \cdot \cdot \cdot + n^3 = \dfrac{n^2(n+1)^2}{4}$

19. $1^2 + 3^2 + 5^2 + \cdot \cdot \cdot + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$

20. $\left(1 + \dfrac{1}{1}\right)\left(1 + \dfrac{1}{2}\right)\left(1 + \dfrac{1}{3}\right) \cdot \cdot \cdot \left(1 + \dfrac{1}{n}\right) = n + 1$

21. $\displaystyle\sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$

22. $\displaystyle\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

23. $\displaystyle\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$

24. $\displaystyle\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$

Proving an Inequality **In Exercises 25–30, use mathematical induction to prove the inequality for the indicated integer values of n.**

25. $n! > 2^n, \quad n \geq 4$

26. $\left(\frac{4}{3}\right)^n > n, \quad n \geq 7$

27. $\dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdot \cdot \cdot + \dfrac{1}{\sqrt{n}} > \sqrt{n}, \quad n \geq 2$

28. $\left(\dfrac{x}{y}\right)^{n+1} < \left(\dfrac{x}{y}\right)^n, \quad n \geq 1 \text{ and } 0 < x < y$

29. $(1 + a)^n \geq na, \quad n \geq 1 \text{ and } a > 0$

30. $2n^2 > (n+1)^2, \quad n \geq 3$

Proving a Property **In Exercises 31–40, use mathematical induction to prove the property for all positive integers n.**

31. $(ab)^n = a^n b^n$

32. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

33. If $x_1 \neq 0,\ x_2 \neq 0, \ldots, x_n \neq 0$, then
$(x_1 x_2 x_3 \cdot \cdot \cdot x_n)^{-1} = x_1^{-1} x_2^{-1} x_3^{-1} \cdot \cdot \cdot x_n^{-1}$.

34. If $x_1 > 0,\ x_2 > 0, \ldots, x_n > 0$, then
$\ln(x_1 x_2 \cdot \cdot \cdot x_n) = \ln x_1 + \ln x_2 + \cdot \cdot \cdot + \ln x_n$.

35. Generalized Distributive Law:
$x(y_1 + y_2 + \cdot \cdot \cdot + y_n) = xy_1 + xy_2 + \cdot \cdot \cdot + xy_n$

36. $(a + bi)^n$ and $(a - bi)^n$ are complex conjugates for all $n \geq 1$.

37. A factor of $(n^3 + 3n^2 + 2n)$ is 3.

38. A factor of $(n^4 - n + 4)$ is 2.

39. A factor of $(2^{2n+1} + 1)$ is 3.

40. A factor of $(2^{2n-1} + 3^{2n-1})$ is 5.

Finding a Formula for a Sum **In Exercises 41–44, use mathematical induction to find a formula for the sum of the first n terms of the sequence.**

41. $1, 5, 9, 13, \ldots$

42. $3, -\frac{9}{2}, \frac{27}{4}, -\frac{81}{8}, \ldots$

43. $\dfrac{1}{4}, \dfrac{1}{12}, \dfrac{1}{24}, \dfrac{1}{40}, \ldots, \dfrac{1}{2n(n+1)}, \ldots$

44. $\dfrac{1}{2 \cdot 3}, \dfrac{1}{3 \cdot 4}, \dfrac{1}{4 \cdot 5}, \dfrac{1}{5 \cdot 6}, \ldots, \dfrac{1}{(n+1)(n+2)}, \ldots$

Finding a Sum **In Exercises 45–54, find the sum using the formulas for the sums of powers of integers.**

45. $\sum_{n=1}^{15} n$

46. $\sum_{n=1}^{30} n$

47. $\sum_{n=1}^{6} n^2$

48. $\sum_{n=1}^{10} n^3$

49. $\sum_{n=1}^{5} n^4$

50. $\sum_{n=1}^{8} n^5$

51. $\sum_{n=1}^{6} (n^2 - n)$

52. $\sum_{n=1}^{20} (n^3 - n)$

53. $\sum_{i=1}^{6} (6i - 8i^3)$

54. $\sum_{j=1}^{10} \left(3 - \frac{1}{2}j + \frac{1}{2}j^2\right)$

Finding a Linear or Quadratic Model **In Exercises 55–60, decide whether the sequence can be represented perfectly by a linear or a quadratic model. If so, then find the model.**

55. 5, 13, 21, 29, 37, 45, . . .

56. 2, 9, 16, 23, 30, 37, . . .

57. 6, 15, 30, 51, 78, 111, . . .

58. 0, 6, 16, 30, 48, 70, . . .

59. −2, 1, 6, 13, 22, 33, . . .

60. −1, 8, 23, 44, 71, 104, . . .

Linear Model, Quadratic Model, or Neither? **In Exercises 61–68, write the first six terms of the sequence beginning with the given term. Then calculate the first and second differences of the sequence. State whether the sequence has a perfect linear model, a perfect quadratic model, or neither.**

61. $a_1 = 0$
$a_n = a_{n-1} + 3$

62. $a_1 = 2$
$a_n = a_{n-1} + 2$

63. $a_1 = 3$
$a_n = a_{n-1} - n$

64. $a_2 = -3$
$a_n = -2a_{n-1}$

65. $a_0 = 2$
$a_n = (a_{n-1})^2$

66. $a_0 = 0$
$a_n = a_{n-1} + n$

67. $a_1 = 2$
$a_n = n - a_{n-1}$

68. $a_1 = 0$
$a_n = a_{n-1} + 2n$

Finding a Quadratic Model **In Exercises 69–74, find a quadratic model for the sequence with the indicated terms.**

69. $a_0 = 3,\ a_1 = 3,\ a_4 = 15$

70. $a_0 = 7,\ a_1 = 6,\ a_3 = 10$

71. $a_0 = -3,\ a_2 = 1,\ a_4 = 9$

72. $a_0 = 3,\ a_2 = 0,\ a_6 = 36$

73. $a_1 = 0,\ a_2 = 8,\ a_4 = 30$

74. $a_0 = -3,\ a_2 = -5,\ a_6 = -57$

Mark Herreid/Shutterstock.com

75. Data Analysis: Residents

The table shows the numbers a_n (in thousands) of Alaskan residents from 2005 through 2010. *(Source: U.S. Census Bureau)*

DATA

Spreadsheet at LarsonPrecalculus.com

Year	Number of Residents, a_n
2005	664
2006	671
2007	676
2008	682
2009	691
2010	705

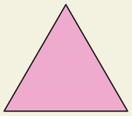
(a) Find the first differences of the data shown in the table. Then find a linear model that approximates the data. Let n represent the year, with $n = 5$ corresponding to 2005.

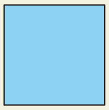
(b) Use a graphing utility to find a linear model for the data. Compare this model with the model from part (a).

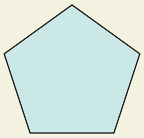
(c) Use the models found in parts (a) and (b) to predict the number of residents in 2016. How do these values compare?

Exploration

76. **HOW DO YOU SEE IT?** Find a formula for the sum of the angles (in degrees) of a regular polygon. Then use mathematical induction to prove this formula for a general n-sided polygon.


Equilateral triangle (180°)


Square (360°)


Regular pentagon (540°)

True or False? **In Exercises 77 and 78, determine whether the statement is true or false. Justify your answer.**

77. If the statement P_k is true and P_k implies P_{k+1}, then P_1 is also true.

78. A sequence with n terms has $n - 1$ second differences.

8.5 The Binomial Theorem

Binomial coefficients can help you model and solve real-life problems. For instance, in Exercise 92 on page 609, you will use binomial coefficients to write the expansion of a model that represents the average prices of residential electricity in the United States from 2003 through 2010.

- **Use the Binomial Theorem to calculate binomial coefficients.**
- **Use Pascal's Triangle to calculate binomial coefficients.**
- **Use binomial coefficients to write binomial expansions.**

Binomial Coefficients

Recall that a *binomial* is a polynomial that has two terms. In this section, you will study a formula that provides a quick method of raising a binomial to a power. To begin, look at the expansion of

$$(x+y)^n$$

for several values of n.

$$(x+y)^0 = 1$$

$$(x+y)^1 = x+y$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

There are several observations you can make about these expansions.

1. In each expansion, there are $n + 1$ terms.
2. In each expansion, x and y have symmetrical roles. The powers of x decrease by 1 in successive terms, whereas the powers of y increase by 1.
3. The sum of the powers of each term is n. For instance, in the expansion of $(x+y)^5$, the sum of the powers of each term is 5.

$$(x+y)^5 = x^5 + 5\overbrace{x^4y^1}^{4+1=5} + 10\overbrace{x^3y^2}^{3+2=5} + 10x^2y^3 + 5x^1y^4 + y^5$$

4. The coefficients increase and then decrease in a symmetric pattern.

The coefficients of a binomial expansion are called **binomial coefficients.** To find them, you can use the **Binomial Theorem.**

The Binomial Theorem

In the expansion of $(x+y)^n$

$$(x+y)^n = x^n + nx^{n-1}y + \cdots + {}_nC_r\, x^{n-r}y^r + \cdots + nxy^{n-1} + y^n$$

the coefficient of $x^{n-r}y^r$ is

$${}_nC_r = \frac{n!}{(n-r)!r!}.$$

The symbol $\binom{n}{r}$ is often used in place of ${}_nC_r$ to denote binomial coefficients.

For a proof of the Binomial Theorem, see Proofs in Mathematics on page 642.

wang song/Shutterstock.com

▷ TECHNOLOGY

Most graphing calculators have the capability of evaluating ${}_nC_r$. If yours does, then evaluate ${}_8C_5$. You should get an answer of 56.

EXAMPLE 1 Finding Binomial Coefficients

Find each binomial coefficient.

a. ${}_8C_2$ **b.** $\binom{10}{3}$ **c.** ${}_7C_0$ **d.** $\binom{8}{8}$

Solution

a. $${}_8C_2 = \frac{8!}{6! \cdot 2!} = \frac{(8 \cdot 7) \cdot \cancel{6!}}{\cancel{6!} \cdot 2!} = \frac{8 \cdot 7}{2 \cdot 1} = 28$$

b. $$\binom{10}{3} = \frac{10!}{7! \cdot 3!} = \frac{(10 \cdot 9 \cdot 8) \cdot \cancel{7!}}{\cancel{7!} \cdot 3!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$$

c. $${}_7C_0 = \frac{\cancel{7!}}{\cancel{7!} \cdot 0!} = 1$$ **d.** $$\binom{8}{8} = \frac{\cancel{8!}}{0! \cdot \cancel{8!}} = 1$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find each binomial coefficient.

a. $\binom{11}{5}$ **b.** ${}_9C_2$ **c.** $\binom{5}{0}$ **d.** ${}_{15}C_{15}$ ■

When $r \neq 0$ and $r \neq n$, as in parts (a) and (b) above, there is a pattern for evaluating binomial coefficients that works because there will always be factorial terms that divide out from the expression.

$${}_8C_2 = \frac{\overbrace{8 \cdot 7}^{\text{2 factors}}}{\underbrace{2 \cdot 1}_{\text{2 factors}}} \quad \text{and} \quad \binom{10}{3} = \frac{\overbrace{10 \cdot 9 \cdot 8}^{\text{3 factors}}}{\underbrace{3 \cdot 2 \cdot 1}_{\text{3 factors}}}$$

EXAMPLE 2 Finding Binomial Coefficients

a. $${}_7C_3 = \frac{7 \cdot \cancel{6} \cdot 5}{\cancel{3} \cdot \cancel{2} \cdot 1} = 35$$

b. $$\binom{7}{4} = \frac{7 \cdot \cancel{6} \cdot 5 \cdot \cancel{4}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 35$$

c. $${}_{12}C_1 = \frac{12}{1} = 12$$

d. $$\binom{12}{11} = \frac{12!}{1! \cdot 11!} = \frac{(12) \cdot \cancel{11!}}{1! \cdot \cancel{11!}} = \frac{12}{1} = 12$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find each binomial coefficient.

a. ${}_7C_5$ **b.** $\binom{7}{2}$ **c.** ${}_{14}C_{13}$ **d.** $\binom{14}{1}$ ■

It is not a coincidence that the results in parts (a) and (b) of Example 2 are the same and that the results in parts (c) and (d) are the same. In general, it is true that ${}_nC_r = {}_nC_{n-r}$. This shows the symmetric property of binomial coefficients identified earlier.

Pascal's Triangle

There is a convenient way to remember the pattern for binomial coefficients. By arranging the coefficients in a triangular pattern, you obtain the following array, which is called **Pascal's Triangle.** This triangle is named after the famous French mathematician Blaise Pascal (1623–1662).

$$\begin{array}{ccccccccccccccc|l}
 & & & & & & & 1 & & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & 4+6=10 \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & 15+6=21
\end{array}$$

The first and last numbers in each row of Pascal's Triangle are 1. Every other number in each row is formed by adding the two numbers immediately above the number. Pascal noticed that numbers in this triangle are precisely the same numbers that are the coefficients of binomial expansions, as follows.

$$\begin{array}{rl}
(x+y)^0 = 1 & \text{0th row} \\
(x+y)^1 = 1x + 1y & \text{1st row} \\
(x+y)^2 = 1x^2 + 2xy + 1y^2 & \text{2nd row} \\
(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3 & \text{3rd row} \\
(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4 & \vdots \\
(x+y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5 & \\
(x+y)^6 = 1x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1y^6 & \\
(x+y)^7 = 1x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + 1y^7 &
\end{array}$$

The top row in Pascal's Triangle is called the *zeroth row* because it corresponds to the binomial expansion $(x + y)^0 = 1$. Similarly, the next row is called the *first row* because it corresponds to the binomial expansion

$$(x + y)^1 = 1(x) + 1(y).$$

In general, the *nth row* in Pascal's Triangle gives the coefficients of $(x + y)^n$.

EXAMPLE 3 Using Pascal's Triangle

Use the seventh row of Pascal's Triangle to find the binomial coefficients.

$${}_8C_0,\ {}_8C_1,\ {}_8C_2,\ {}_8C_3,\ {}_8C_4,\ {}_8C_5,\ {}_8C_6,\ {}_8C_7,\ {}_8C_8$$

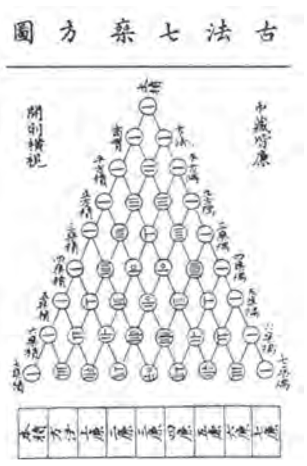
Solution

$$\begin{array}{ccccccccccccccccc}
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1 \\
| & & | & & | & & | & & | & & | & & | & & | & & | \\
{}_8C_0 & & {}_8C_1 & & {}_8C_2 & & {}_8C_3 & & {}_8C_4 & & {}_8C_5 & & {}_8C_6 & & {}_8C_7 & & {}_8C_8
\end{array}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use the eighth row of Pascal's Triangle to find the binomial coefficients.

$${}_9C_0,\ {}_9C_1,\ {}_9C_2,\ {}_9C_3,\ {}_9C_4,\ {}_9C_5,\ {}_9C_6,\ {}_9C_7,\ {}_9C_8,\ {}_9C_9$$



PRECIOUS MIRROR OF THE FOUR ELEMENTS

Eastern cultures were familiar with "Pascal's" Triangle and forms of the Binomial Theorem prior to the Western "discovery" of the theorem. A Chinese text entitled *Precious Mirror of the Four Elements* contains a triangle of binomial expansions through the eighth power.

Binomial Expansions

As mentioned at the beginning of this section, when you write the coefficients for a binomial that is raised to a power, you are **expanding a binomial.** The formula for binomial coefficients and Pascal's Triangle give you a relatively easy way to expand binomials, as demonstrated in the next four examples.

EXAMPLE 4 Expanding a Binomial

Write the expansion of the expression

$$(x+1)^3.$$

Solution The binomial coefficients from the third row of Pascal's Triangle are

1, 3, 3, 1.

So, the expansion is as follows.

$$\begin{aligned}(x+1)^3 &= (1)x^3 + (3)x^2(1) + (3)x(1^2) + (1)(1^3)\\ &= x^3 + 3x^2 + 3x + 1\end{aligned}$$

✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write the expansion of the expression

$$(x+2)^4.$$

To expand binomials representing *differences* rather than sums, you alternate signs. Here are two examples.

$$(x-1)^3 = x^3 - 3x^2 + 3x - 1$$

$$(x-1)^4 = x^4 - 4x^3 + 6x^2 - 4x + 1$$

ALGEBRA HELP The solutions to Example 5 use the property of exponents

$$(ab)^m = a^m b^m.$$

For instance, in Example 5(a),

$$(2x)^4 = 2^4x^4 = 16x^4.$$

You can review properties of exponents in Section P.2.

EXAMPLE 5 Expanding a Binomial

Write the expansion of each expression.

a. $(2x-3)^4$

b. $(x-2y)^4$

Solution The binomial coefficients from the fourth row of Pascal's Triangle are

1, 4, 6, 4, 1.

So, the expansions are as follows.

a. $$\begin{aligned}(2x-3)^4 &= (1)(2x)^4 - (4)(2x)^3(3) + (6)(2x)^2(3^2) - (4)(2x)(3^3) + (1)(3^4)\\ &= 16x^4 - 96x^3 + 216x^2 - 216x + 81\end{aligned}$$

b. $$\begin{aligned}(x-2y)^4 &= (1)x^4 - (4)x^3(2y) + (6)x^2(2y)^2 - (4)x(2y)^3 + (1)(2y)^4\\ &= x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4\end{aligned}$$

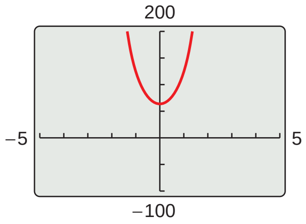
✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write the expansion of each expression.

a. $(y-2)^4$

b. $(2x-y)^5$

TECHNOLOGY Use a graphing utility to check the expansion in Example 6. Graph the original binomial expression and the expansion in the same viewing window. The graphs should coincide, as shown below.



EXAMPLE 6 Expanding a Binomial

Write the expansion of $(x^2 + 4)^3$.

Solution Use the third row of Pascal's Triangle, as follows.

$$(x^2 + 4)^3 = (1)(x^2)^3 + (3)(x^2)^2(4) + (3)x^2(4^2) + (1)(4^3)$$
$$= x^6 + 12x^4 + 48x^2 + 64$$

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Write the expansion of $(5 + y^2)^3$.

Sometimes you will need to find a specific term in a binomial expansion. Instead of writing the entire expansion, use the fact that, from the Binomial Theorem, the $(r + 1)$th term is ${}_nC_r\, x^{n-r} y^r$.

EXAMPLE 7 Finding a Term or Coefficient

a. Find the sixth term of $(a + 2b)^8$.

b. Find the coefficient of the term a^6b^5 in the expansion of $(3a - 2b)^{11}$.

Solution

a. Remember that the formula is for the $(r + 1)$th term, so r is one less than the number of the term you need. So, to find the sixth term in this binomial expansion, use $r = 5$, $n = 8$, $x = a$, and $y = 2b$, as shown.

$${}_8C_5\, a^{8-5}(2b)^5 = 56 \cdot a^3 \cdot (2b)^5$$
$$= 56(2^5)a^3b^5$$
$$= 1792a^3b^5$$

b. In this case, $n = 11$, $r = 5$, $x = 3a$, and $y = -2b$. Substitute these values to obtain

$${}_nC_r\, x^{n-r} y^r = {}_{11}C_5(3a)^6(-2b)^5$$
$$= (462)(729a^6)(-32b^5)$$
$$= -10{,}777{,}536a^6b^5.$$

So, the coefficient is $-10{,}777{,}536$.

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

a. Find the fifth term of $(a + 2b)^8$.

b. Find the coefficient of the term a^4b^7 in the expansion of $(3a - 2b)^{11}$.

Summarize (Section 8.5)

1. State the Binomial Theorem *(page 602)*. For examples of using the Binomial Theorem to calculate binomial coefficients, see Examples 1 and 2.
2. Explain how to use Pascal's Triangle to calculate binomial coefficients *(page 604)*. For an example of using Pascal's Triangle to calculate binomial coefficients, see Example 3.
3. Explain how to use binomial coefficients to write a binomial expansion *(page 605)*. For examples of using binomial coefficients to write binomial expansions, see Examples 4–6.

8.5 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. The coefficients of a binomial expansion are called ________ ________.
2. To find binomial coefficients, you can use the ________ ________ or ________ ________.
3. The symbol used to denote a binomial coefficient is ________ or ________.
4. When you write the coefficients for a binomial that is raised to a power, you are ________ a ________.

Skills and Applications

Finding a Binomial Coefficient In Exercises 5–14, find the binomial coefficient.

5. $_5C_3$
6. $_8C_6$
7. $_{12}C_0$
8. $_{20}C_{20}$
9. $_{20}C_{15}$
10. $_{12}C_5$
11. $\binom{10}{4}$
12. $\binom{10}{6}$
13. $\binom{100}{98}$
14. $\binom{100}{2}$

Using Pascal's Triangle In Exercises 15–18, evaluate using Pascal's Triangle.

15. $\binom{6}{5}$
16. $\binom{5}{2}$
17. $_7C_4$
18. $_{10}C_2$

Expanding an Expression In Exercises 19–40, use the Binomial Theorem to expand and simplify the expression.

19. $(x + 1)^4$
20. $(x + 1)^6$
21. $(a + 6)^4$
22. $(a + 5)^5$
23. $(y - 4)^3$
24. $(y - 2)^5$
25. $(x + y)^5$
26. $(c + d)^3$
27. $(2x + y)^3$
28. $(7a + b)^3$
29. $(r + 3s)^6$
30. $(x + 2y)^4$
31. $(3a - 4b)^5$
32. $(2x - 5y)^5$
33. $(x^2 + y^2)^4$
34. $(x^2 + y^2)^6$
35. $\left(\frac{1}{x} + y\right)^5$
36. $\left(\frac{1}{x} + 2y\right)^6$
37. $\left(\frac{2}{x} - y\right)^4$
38. $\left(\frac{2}{x} - 3y\right)^5$
39. $2(x - 3)^4 + 5(x - 3)^2$
40. $(4x - 1)^3 - 2(4x - 1)^4$

Expanding a Binomial In Exercises 41–44, expand the binomial by using Pascal's Triangle to determine the coefficients.

41. $(2t - s)^5$
42. $(3 - 2z)^4$
43. $(x + 2y)^5$
44. $(3v + 2)^6$

Finding a Term in a Binomial Expansion In Exercises 45–52, find the specified *n*th term in the expansion of the binomial.

45. $(x + y)^{10}, \quad n = 4$
46. $(x - y)^6, \quad n = 7$
47. $(x - 6y)^5, \quad n = 3$
48. $(x - 10z)^7, \quad n = 4$
49. $(4x + 3y)^9, \quad n = 8$
50. $(5a + 6b)^5, \quad n = 5$
51. $(10x - 3y)^{12}, \quad n = 10$
52. $(7x + 2y)^{15}, \quad n = 7$

Finding a Coefficient In Exercises 53–60, find the coefficient *a* of the term in the expansion of the binomial.

	Binomial	Term
53.	$(x + 3)^{12}$	ax^5
54.	$(x^2 + 3)^{12}$	ax^8
55.	$(4x - y)^{10}$	ax^2y^8
56.	$(x - 2y)^{10}$	ax^8y^2
57.	$(2x - 5y)^9$	ax^4y^5
58.	$(3x - 4y)^8$	ax^6y^2
59.	$(x^2 + y)^{10}$	ax^8y^6
60.	$(z^2 - t)^{10}$	az^4t^8

Expanding an Expression In Exercises 61–66, use the Binomial Theorem to expand and simplify the expression.

61. $\left(\sqrt{x} + 5\right)^3$
62. $\left(2\sqrt{t} - 1\right)^3$
63. $(x^{2/3} - y^{1/3})^3$
64. $(u^{3/5} + 2)^5$
65. $\left(3\sqrt{t} + \sqrt[4]{t}\right)^4$
66. $(x^{3/4} - 2x^{5/4})^4$

Simplifying a Difference Quotient In Exercises 67–72, simplify the difference quotient, using the Binomial Theorem if necessary.

$$\frac{f(x + h) - f(x)}{h}$$

Difference quotient

67. $f(x) = x^3$
68. $f(x) = x^4$
69. $f(x) = x^6$
70. $f(x) = x^8$
71. $f(x) = \sqrt{x}$
72. $f(x) = \frac{1}{x}$

Expanding a Complex Number **In Exercises 73–78, use the Binomial Theorem to expand the complex number. Simplify your result.**

73. $(1 + i)^4$

74. $(2 - i)^5$

75. $(2 - 3i)^6$

76. $\left(5 + \sqrt{-9}\right)^3$

77. $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3$

78. $\left(5 - \sqrt{3}i\right)^4$

Approximation **In Exercises 79–82, use the Binomial Theorem to approximate the quantity accurate to three decimal places. For example, in Exercise 79, use the expansion**

$$\mathbf{(1.02)^8 = (1 + 0.02)^8}$$
$$\mathbf{= 1 + 8(0.02) + 28(0.02)^2 + \cdots + (0.02)^8.}$$

79. $(1.02)^8$

80. $(2.005)^{10}$

81. $(2.99)^{12}$

82. $(1.98)^9$

Graphical Reasoning **In Exercises 83 and 84, use a graphing utility to graph f and g in the same viewing window. What is the relationship between the two graphs? Use the Binomial Theorem to write the polynomial function g in standard form.**

83. $f(x) = x^3 - 4x, \quad g(x) = f(x + 4)$

84. $f(x) = -x^4 + 4x^2 - 1, \quad g(x) = f(x - 3)$

Probability **In Exercises 85–88, consider n independent trials of an experiment in which each trial has two possible outcomes: "success" or "failure." The probability of a success on each trial is p, and the probability of a failure is $q = 1 - p$. In this context, the term ${}_nC_k p^k q^{n-k}$ in the expansion of $(p + q)^n$ gives the probability of k successes in the n trials of the experiment.**

85. You toss a fair coin seven times. To find the probability of obtaining four heads, evaluate the term

$${}_7C_4\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^3$$

in the expansion of $\left(\frac{1}{2} + \frac{1}{2}\right)^7$.

86. The probability of a baseball player getting a hit during any given time at bat is $\frac{1}{4}$. To find the probability that the player gets three hits during the next 10 times at bat, evaluate the term

$${}_{10}C_3\left(\tfrac{1}{4}\right)^3\left(\tfrac{3}{4}\right)^7$$

in the expansion of $\left(\frac{1}{4} + \frac{3}{4}\right)^{10}$.

87. The probability of a sales representative making a sale with any one customer is $\frac{1}{3}$. The sales representative makes eight contacts a day. To find the probability of making four sales, evaluate the term

$${}_8C_4\left(\tfrac{1}{3}\right)^4\left(\tfrac{2}{3}\right)^4$$

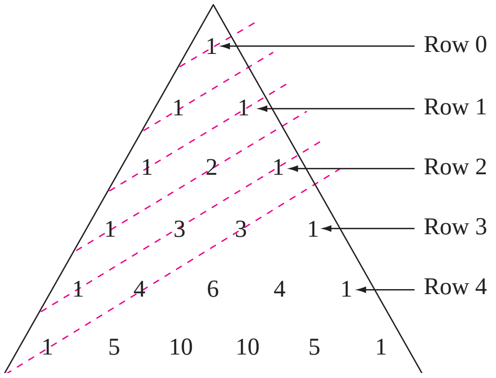
in the expansion of $\left(\frac{1}{3} + \frac{2}{3}\right)^8$.

88. To find the probability that the sales representative in Exercise 87 makes four sales when the probability of a sale with any one customer is $\frac{1}{2}$, evaluate the term

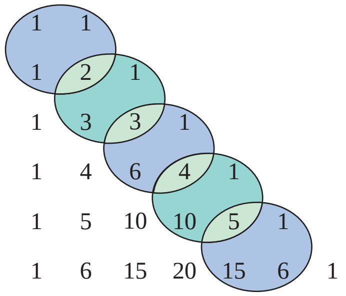
$${}_8C_4\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^4$$

in the expansion of $\left(\frac{1}{2} + \frac{1}{2}\right)^8$.

89. Finding a Pattern Describe the pattern formed by the sums of the numbers along the diagonal line segments shown in Pascal's Triangle (see figure).



90. Finding a Pattern Use each of the encircled groups of numbers in the figure to form a 2×2 matrix. Find the determinant of each matrix. Describe the pattern.



91. Child Support The amounts $f(t)$ (in billions of dollars) of child support collected in the United States from 2002 through 2009 can be approximated by the model

$$f(t) = -0.009t^2 + 1.05t + 18.0, \quad 2 \le t \le 9$$

where t represents the year, with $t = 2$ corresponding to 2002. *(Source: U.S. Department of Health and Human Services)*

(a) You want to adjust the model so that $t = 2$ corresponds to 2007 rather than 2002. To do this, you shift the graph of f five units *to the left* to obtain $g(t) = f(t + 5)$. Use binomial coefficients to write $g(t)$ in standard form.

(b) Use a graphing utility to graph f and g in the same viewing window.

(c) Use the graphs to estimate when the child support collections exceeded $25 billion.

92. Data Analysis: Electricity The table shows the average prices $f(t)$ (in cents per kilowatt hour) of residential electricity in the United States from 2003 through 2010. *(Source: U.S. Energy Information Administration)*

Year	Average Price, $f(t)$
2003	8.72
2004	8.95
2005	9.45
2006	10.40
2007	10.65
2008	11.26
2009	11.51
2010	11.58

Spreadsheet at LarsonPrecalculus.com

(a) Use the *regression* feature of a graphing utility to find a cubic model for the data. Let t represent the year, with $t = 3$ corresponding to 2003.

(b) Use the graphing utility to plot the data and the model in the same viewing window.

(c) You want to adjust the model so that $t = 3$ corresponds to 2008 rather than 2003. To do this, you shift the graph of f five units *to the left* to obtain $g(t) = f(t + 5)$. Use binomial coefficients to write $g(t)$ in standard form.

(d) Use the graphing utility to graph g in the same viewing window as f.

(e) Use both models to predict the average price in 2011. Do you obtain the same answer?

(f) Do your answers to part (e) seem reasonable? Explain.

(g) What factors do you think may have contributed to the change in the average price?

Exploration

True or False? **In Exercises 93 and 94, determine whether the statement is true or false. Justify your answer.**

93. The Binomial Theorem could be used to produce each row of Pascal's Triangle.

94. A binomial that represents a difference cannot always be accurately expanded using the Binomial Theorem.

95. Writing In your own words, explain how to form the rows of Pascal's Triangle.

96. Forming Rows of Pascal's Triangle Form rows 8–10 of Pascal's Triangle.

97. Graphical Reasoning Which two functions have identical graphs, and why? Use a graphing utility to graph the functions in the given order and in the same viewing window. Compare the graphs.

(a) $f(x) = (1 - x)^3$

(b) $g(x) = 1 - x^3$

(c) $h(x) = 1 + 3x + 3x^2 + x^3$

(d) $k(x) = 1 - 3x + 3x^2 - x^3$

(e) $p(x) = 1 + 3x - 3x^2 + x^3$

98. HOW DO YOU SEE IT? The expansions of $(x + y)^4$, $(x + y)^5$, and $(x + y)^6$ are as follows.

$$(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

$$(x + y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$$

$$(x + y)^6 = 1x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1y^6$$

(a) Explain how the exponent of a binomial is related to the number of terms in its expansion.

(b) How many terms are in the expansion of $(x + y)^n$?

Proof **In Exercises 99–102, prove the property for all integers r and n where $0 \le r \le n$.**

99. ${}_nC_r = {}_nC_{n-r}$

100. ${}_nC_0 - {}_nC_1 + {}_nC_2 - \cdots \pm {}_nC_n = 0$

101. ${}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}$

102. The sum of the numbers in the nth row of Pascal's Triangle is 2^n.

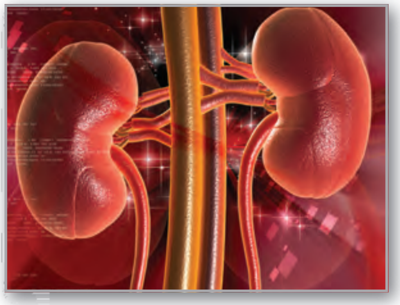
103. Binomial Coefficients and Pascal's Triangle Complete the table and describe the result.

n	r	${}_nC_r$	${}_nC_{n-r}$
9	5		
7	1		
12	4		
6	0		
10	7		

What characteristic of Pascal's Triangle does this table illustrate?

wang song/Shutterstock.com

8.6 Counting Principles



Counting principles can help you solve counting problems that occur in real life. For instance, in Exercise 39 on page 618, you will use counting principles to determine the numbers of possible orders there are for best match, second-best match, and third-best match kidney donors.

- **Solve simple counting problems.**
- **Use the Fundamental Counting Principle to solve counting problems.**
- **Use permutations to solve counting problems.**
- **Use combinations to solve counting problems.**

Simple Counting Problems

This section and Section 8.7 present a brief introduction to some of the basic counting principles and their applications to probability. In Section 8.7, you will see that much of probability has to do with counting the number of ways an event can occur. The following two examples describe simple counting problems.

EXAMPLE 1 Selecting Pairs of Numbers at Random

You place eight pieces of paper, numbered from 1 to 8, in a box. You draw one piece of paper at random from the box, record its number, and *replace* the paper in the box. Then, you draw a second piece of paper at random from the box and record its number. Finally, you add the two numbers. How many different ways can you obtain a sum of 12?

Solution To solve this problem, count the different ways to obtain a sum of 12 using two numbers from 1 to 8.

First number	4	5	6	7	8
Second number	8	7	6	5	4

So, a sum of 12 can occur in five different ways.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

In Example 1, how many different ways can you obtain a sum of 14?

EXAMPLE 2 Selecting Pairs of Numbers at Random

You place eight pieces of paper, numbered from 1 to 8, in a box. You draw one piece of paper at random from the box, record its number, and *do not* replace the paper in the box. Then, you draw a second piece of paper at random from the box and record its number. Finally, you add the two numbers. How many different ways can you obtain a sum of 12?

Solution To solve this problem, count the different ways to obtain a sum of 12 *using two different numbers* from 1 to 8.

First number	4	5	7	8
Second number	8	7	5	4

So, a sum of 12 can occur in four different ways.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Repeat Example 2 for drawing *three* pieces of paper. ■

The difference between the counting problems in Examples 1 and 2 can be described by saying that the random selection in Example 1 occurs **with replacement,** whereas the random selection in Example 2 occurs **without replacement,** which eliminates the possibility of choosing two 6's.

Kannanimages/Shutterstock.com

The Fundamental Counting Principle

Examples 1 and 2 describe simple counting problems in which you can *list* each possible way that an event can occur. When it is possible, this is always the best way to solve a counting problem. However, some events can occur in so many different ways that it is not feasible to write out the entire list. In such cases, you must rely on formulas and counting principles. The most important of these is the **Fundamental Counting Principle.**

Fundamental Counting Principle

Let E_1 and E_2 be two events. The first event E_1 can occur in m_1 different ways. After E_1 has occurred, E_2 can occur in m_2 different ways. The number of ways that the two events can occur is $m_1 \cdot m_2$.

The Fundamental Counting Principle can be extended to three or more events. For instance, the number of ways that three events E_1, E_2, and E_3 can occur is $m_1 \cdot m_2 \cdot m_3$.

EXAMPLE 3 Using the Fundamental Counting Principle

How many different pairs of letters from the English alphabet are possible?

Solution There are two events in this situation. The first event is the choice of the first letter, and the second event is the choice of the second letter. Because the English alphabet contains 26 letters, it follows that the number of two-letter pairs is $26 \cdot 26 = 676$.

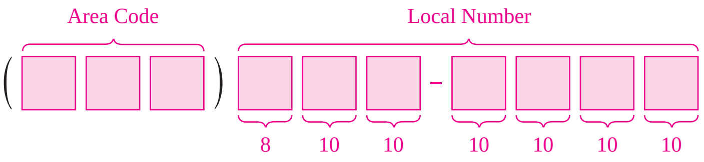
✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

A combination lock will open when you select the right choice of three numbers (from 1 to 30, inclusive). How many different lock combinations are possible?

EXAMPLE 4 Using the Fundamental Counting Principle

Telephone numbers in the United States have 10 digits. The first three digits are the *area code* and the next seven digits are the *local telephone number.* How many different telephone numbers are possible within each area code? (Note that a local telephone number cannot begin with 0 or 1.)

Solution Because the first digit of a local telephone number cannot be 0 or 1, there are only eight choices for the first digit. For each of the other six digits, there are 10 choices.



So, the number of telephone numbers that are possible within each area code is

$$8 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 8{,}000{,}000.$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

A product's catalog number is made up of one letter from the English alphabet followed by a five-digit number. How many different catalog numbers are possible? ■

Permutations

One important application of the Fundamental Counting Principle is in determining the number of ways that n elements can be arranged (in order). An ordering of n elements is called a **permutation** of the elements.

Definition of Permutation

A **permutation** of n different elements is an ordering of the elements such that one element is first, one is second, one is third, and so on.

EXAMPLE 5 Finding the Number of Permutations

How many permutations of the letters

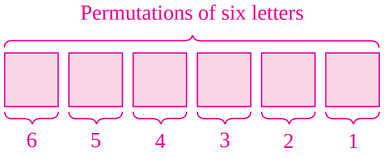
A, B, C, D, E, and F

are possible?

Solution Consider the following reasoning.

First position: Any of the *six* letters
Second position: Any of the remaining *five* letters
Third position: Any of the remaining *four* letters
Fourth position: Any of the remaining *three* letters
Fifth position: Either of the remaining *two* letters
Sixth position: The *one* remaining letter

So, the numbers of choices for the six positions are as follows.



The total number of permutations of the six letters is

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$
$$= 720.$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

How many permutations of the letters

W, X, Y, and Z

are possible?

Number of Permutations of *n* Elements

The number of permutations of n elements is

$$n \cdot (n - 1) \cdot \cdot \cdot 4 \cdot 3 \cdot 2 \cdot 1 = n!.$$

In other words, there are $n!$ different ways that n elements can be ordered.

Eleven thoroughbred racehorses hold the title of Triple Crown winner for winning the Kentucky Derby, the Preakness, and the Belmont Stakes in the same year. Forty-nine horses have won two out of the three races.

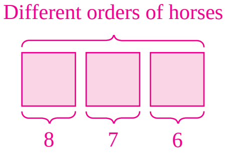
EXAMPLE 6 Counting Horse Race Finishes

Eight horses are running in a race. In how many different ways can these horses come in first, second, and third? (Assume that there are no ties.)

Solution Here are the different possibilities.

Win (first position): *Eight* choices
Place (second position): *Seven* choices
Show (third position): *Six* choices

Using the Fundamental Counting Principle, multiply these three numbers to obtain the following.



So, there are

$$8 \cdot 7 \cdot 6 = 336$$

different orders.

✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

A coin club has five members. In how many ways can there be a president and a vice-president?

It is useful, on occasion, to order a *subset* of a collection of elements rather than the entire collection. For example, you might want to order r elements out of a collection of n elements. Such an ordering is called a **permutation of n elements taken r at a time.**

▷ **TECHNOLOGY**

Most graphing calculators have the capability of evaluating ${}_nP_r$. If yours does, then evaluate ${}_8P_5$. You should get an answer of 6720.

Permutations of n Elements Taken r at a Time

The number of permutations of n elements taken r at a time is

$${}_nP_r = \frac{n!}{(n-r)!} = n(n-1)(n-2) \cdot \cdot \cdot (n-r+1).$$

Using this formula, you can rework Example 6 to find that the number of permutations of eight horses taken three at a time is

$$\begin{aligned} {}_8P_3 &= \frac{8!}{(8-3)!} \\ &= \frac{8!}{5!} \\ &= \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}} \\ &= 336 \end{aligned}$$

which is the same answer obtained in the example.

Mikhail Pogosov/Shutterstock.com

Remember that for permutations, order is important. So, to find the possible permutations of the letters A, B, C, and D taken three at a time, you count the permutations (A, B, D) and (B, A, D) as different because the *order* of the elements is different.

Suppose, however, that you want to find the possible permutations of the letters A, A, B, and C. The total number of permutations of the four letters would be ${}_4P_4 = 4!$. However, not all of these arrangements would be *distinguishable* because there are two A's in the list. To find the number of distinguishable permutations, you can use the following formula.

Distinguishable Permutations

Suppose a set of n objects has n_1 of one kind of object, n_2 of a second kind, n_3 of a third kind, and so on, with

$$n = n_1 + n_2 + n_3 + \cdot \cdot \cdot + n_k.$$

Then the number of **distinguishable permutations** of the n objects is

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdot \cdot \cdot \cdot \cdot n_k!}.$$

EXAMPLE 7 Distinguishable Permutations

In how many distinguishable ways can the letters in BANANA be written?

Solution This word has six letters, of which three are A's, two are N's, and one is a B. So, the number of distinguishable ways the letters can be written is

$$\begin{aligned}\frac{n!}{n_1! \cdot n_2! \cdot n_3!} &= \frac{6!}{3! \cdot 2! \cdot 1!}\\ &= \frac{6 \cdot 5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!} \cdot 2!}\\ &= 60.\end{aligned}$$

The 60 different distinguishable permutations are as follows.

AAABNN	AAANBN	AAANNB	AABANN
AABNAN	AABNNA	AANABN	AANANB
AANBAN	AANBNA	AANNAB	AANNBA
ABAANN	ABANAN	ABANNA	ABNAAN
ABNANA	ABNNAA	ANAABN	ANAANB
ANABAN	ANABNA	ANANAB	ANANBA
ANBAAN	ANBANA	ANBNAA	ANNAAB
ANNABA	ANNBAA	BAAANN	BAANAN
BAANNA	BANAAN	BANANA	BANNAA
BNAAAN	BNAANA	BNANAA	BNNAAA
NAAABN	NAAANB	NAABAN	NAABNA
NAANAB	NAANBA	NABAAN	NABANA
NABNAA	NANAAB	NANABA	NANBAA
NBAAAN	NBAANA	NBANAA	NBNAAA
NNAAAB	NNAABA	NNABAA	NNBAAA

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

In how many distinguishable ways can the letters in MITOSIS be written?

Combinations

When you count the number of possible permutations of a set of elements, order is important. As a final topic in this section, you will look at a method of selecting subsets of a larger set in which order is *not* important. Such subsets are called **combinations of *n* elements taken *r* at a time.** For instance, the combinations

$$\{A, B, C\} \qquad \text{and} \qquad \{B, A, C\}$$

are equivalent because both sets contain the same three elements, and the order in which the elements are listed is not important. So, you would count only one of the two sets. A common example of how a combination occurs is in a card game in which players are free to reorder the cards after they have been dealt.

EXAMPLE 8 Combinations of *n* Elements Taken *r* at a Time

In how many different ways can three letters be chosen from the letters

A, B, C, D, and E?

(The order of the three letters is not important.)

Solution The following subsets represent the different combinations of three letters that can be chosen from the five letters.

{A, B, C} {A, B, D}
{A, B, E} {A, C, D}
{A, C, E} {A, D, E}
{B, C, D} {B, C, E}
{B, D, E} {C, D, E}

From this list, you can conclude that there are 10 different ways that three letters can be chosen from five letters.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

In how many different ways can two letters be chosen from the letters A, B, C, D, E, F, and G? (The order of the two letters is not important.)

Combinations of *n* Elements Taken *r* at a Time

The number of combinations of n elements taken r at a time is

$$_nC_r = \frac{n!}{(n - r)!r!}$$

which is equivalent to $_nC_r = \frac{_nP_r}{r!}$.

Note that the formula for $_nC_r$ is the same one given for binomial coefficients. To see how to use this formula, solve the counting problem in Example 8. In that problem, you want to find the number of combinations of five elements taken three at a time. So, $n = 5$, $r = 3$, and the number of combinations is

$$_5C_3 = \frac{5!}{2!3!} = \frac{5 \cdot \overset{2}{\cancel{4}} \cdot \cancel{3!}}{\cancel{2} \cdot 1 \cdot \cancel{3!}} = 10$$

which is the same answer obtained in Example 8.

EXAMPLE 9 Counting Card Hands

A standard poker hand consists of five cards dealt from a deck of 52 (see Figure 8.2). How many different poker hands are possible? (After the cards are dealt, the player may reorder them, and so order is not important.)

A ♥	A ♦	A ♣	A ♠
2 ♥	2 ♦	2 ♣	2 ♠
3 ♥	3 ♦	3 ♣	3 ♠
4 ♥	4 ♦	4 ♣	4 ♠
5 ♥	5 ♦	5 ♣	5 ♠
6 ♥	6 ♦	6 ♣	6 ♠
7 ♥	7 ♦	7 ♣	7 ♠
8 ♥	8 ♦	8 ♣	8 ♠
9 ♥	9 ♦	9 ♣	9 ♠
10 ♥	10 ♦	10 ♣	10 ♠
J ♥	J ♦	J ♣	J ♠
Q ♥	Q ♦	Q ♣	Q ♠
K ♥	K ♦	K ♣	K ♠

Ranks and suits in a standard deck of playing cards
Figure 8.2

Solution To find the number of different poker hands, use the formula for the number of combinations of 52 elements taken five at a time, as follows.

$$\begin{aligned} {}_{52}C_5 &= \frac{52!}{(52-5)!5!} \\ &= \frac{52!}{47!5!} \\ &= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47!}}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot \cancel{47!}} \\ &= 2{,}598{,}960 \end{aligned}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

In three-card poker, each player is dealt three cards from a deck of 52. How many different three-card poker hands are possible? (Order is not important.)

EXAMPLE 10 Forming a Team

You are forming a 12-member swim team from 10 girls and 15 boys. The team must consist of five girls and seven boys. How many different 12-member teams are possible?

Solution There are ${}_{10}C_5$ ways of choosing five girls. There are ${}_{15}C_7$ ways of choosing seven boys. By the Fundamental Counting Principle, there are ${}_{10}C_5 \cdot {}_{15}C_7$ ways of choosing five girls and seven boys.

$${}_{10}C_5 \cdot {}_{15}C_7 = \frac{10!}{5! \cdot 5!} \cdot \frac{15!}{8! \cdot 7!} = 252 \cdot 6435 = 1{,}621{,}620$$

So, there are 1,621,620 12-member swim teams possible.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

In Example 10, the team must consist of six boys and six girls. How many different 12-member teams are possible?

REMARK When solving problems involving counting principles, you need to be able to distinguish among the various counting principles in order to determine which is necessary to solve the problem correctly. To do this, ask yourself the following questions.

1. Is the order of the elements important? *Permutation*
2. Are the chosen elements a subset of a larger set in which order is not important? *Combination*
3. Does the problem involve two or more separate events? *Fundamental Counting Principle*

Summarize (Section 8.6)

1. Explain how to solve a simple counting problem *(page 610)*. For examples of solving simple counting problems, see Examples 1 and 2.
2. State the Fundamental Counting Principle *(page 611)*. For examples of using the Fundamental Counting Principle to solve counting problems, see Examples 3 and 4.
3. Explain how to find the number of permutations of n elements *(page 612)*, the number of permutations of n elements taken r at a time *(page 613)*, and the number of distinguishable permutations *(page 614)*. For examples of using permutations to solve counting problems, see Examples 5–7.
4. Explain how to find the number of combinations of n elements taken r at a time *(page 615)*. For examples of using combinations to solve counting problems, see Examples 8–10.

8.6 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. The ________ ________ ________ states that when there are m_1 different ways for one event to occur and m_2 different ways for a second event to occur, there are $m_1 \cdot m_2$ ways for both events to occur.
2. An ordering of n elements is called a ________ of the elements.
3. The number of permutations of n elements taken r at a time is given by ________.
4. The number of ________ ________ of n objects is given by $\dfrac{n!}{n_1!n_2!n_3! \cdot \cdot \cdot n_k!}$.
5. When selecting subsets of a larger set in which order is not important, you are finding the number of ________ of n elements taken r at a time.
6. The number of combinations of n elements taken r at a time is given by ________.

Skills and Applications

Random Selection **In Exercises 7–14, determine the number of ways a computer can randomly generate one or more such integers from 1 through 12.**

7. An odd integer
8. An even integer
9. A prime integer
10. An integer that is greater than 9
11. An integer that is divisible by 4
12. An integer that is divisible by 3
13. Two *distinct* integers whose sum is 9
14. Two *distinct* integers whose sum is 8

15. **Entertainment Systems** A customer can choose one of three amplifiers, one of two compact disc players, and one of five speaker models for an entertainment system. Determine the number of possible system configurations.
16. **Job Applicants** A small college needs two additional faculty members: a chemist and a statistician. There are five applicants for the chemistry position and three applicants for the statistics position. In how many ways can the college fill these positions?
17. **Course Schedule** A college student is preparing a course schedule for the next semester. The student may select one of two mathematics courses, one of three science courses, and one of five courses from the social sciences and humanities. How many schedules are possible?
18. **Aircraft Boarding** Eight people are boarding an aircraft. Two have tickets for first class and board before those in the economy class. In how many ways can the eight people board the aircraft?
19. **True-False Exam** In how many ways can you answer a six-question true-false exam? (Assume that you do not omit any questions.)
20. **True-False Exam** In how many ways can you answer a 12-question true-false exam? (Assume that you do not omit any questions.)
21. **License Plate Numbers** In the state of Pennsylvania, each standard automobile license plate number consists of three letters followed by a four-digit number. How many distinct license plate numbers can be formed in Pennsylvania?
22. **License Plate Numbers** In a certain state, each automobile license plate number consists of two letters followed by a four-digit number. To avoid confusion between “O” and “zero” and between “I” and “one,” the letters “O” and “I” are not used. How many distinct license plate numbers can be formed in this state?
23. **Three-Digit Numbers** How many three-digit numbers can you form under each condition?
    (a) The leading digit cannot be zero.
    (b) The leading digit cannot be zero and no repetition of digits is allowed.
    (c) The leading digit cannot be zero and the number must be a multiple of 5.
    (d) The number is at least 400.
24. **Four-Digit Numbers** How many four-digit numbers can you form under each condition?
    (a) The leading digit cannot be zero.
    (b) The leading digit cannot be zero and no repetition of digits is allowed.
    (c) The leading digit cannot be zero and the number must be less than 5000.
    (d) The leading digit cannot be zero and the number must be even.
25. **Combination Lock** A combination lock will open when you select the right choice of three numbers (from 1 to 40, inclusive). How many different lock combinations are possible?

26. **Combination Lock** A combination lock will open when you select the right choice of three numbers (from 1 to 50, inclusive). How many different lock combinations are possible?

27. **Concert Seats** Four couples have reserved seats in one row for a concert. In how many different ways can they sit when

(a) there are no seating restrictions?

(b) the two members of each couple wish to sit together?

28. **Single File** In how many orders can four girls and four boys walk through a doorway single file when

(a) there are no restrictions?

(b) the girls walk through before the boys?

29. **Posing for a Photograph** In how many ways can five children posing for a photograph line up in a row?

30. **Riding in a Car** In how many ways can six people sit in a six-passenger car?

Evaluating $_nP_r$ **In Exercises 31–34, evaluate $_nP_r$.**

31. $_4P_4$	**32.** $_8P_3$
33. $_{20}P_2$	**34.** $_5P_4$

Evaluating $_nP_r$ **In Exercises 35–38, evaluate $_nP_r$ using a graphing utility.**

35. $_{20}P_5$	**36.** $_{100}P_5$
37. $_{100}P_3$	**38.** $_{10}P_8$

39. **Kidney Donors**

A patient with end-stage kidney disease has nine family members who are potential kidney donors. How many possible orders are there for a best match, a second-best match, and a third-best match?

40. **Choosing Officers** From a pool of 12 candidates, the offices of president, vice-president, secretary, and treasurer will be filled. In how many different ways can the offices be filled?

41. **Batting Order** A baseball coach is creating a nine-player batting order by selecting from a team of 15 players. How many different batting orders are possible?

42. **Athletics** Eight sprinters have qualified for the finals in the 100-meter dash at the NCAA national track meet. In how many ways can the sprinters come in first, second, and third? (Assume there are no ties.)

Number of Distinguishable Permutations **In Exercises 43–46, find the number of distinguishable permutations of the group of letters.**

43. A, A, G, E, E, E, M	**44.** B, B, B, T, T, T, T, T
45. A, L, G, E, B, R, A	**46.** M, I, S, S, I, S, S, I, P, P, I

47. **Writing Permutations** Write all permutations of the letters A, B, C, and D.

48. **Writing Permutations** Write all permutations of the letters A, B, C, and D when letters B and C must remain between A and D.

Evaluating $_nC_r$ **In Exercises 49–52, evaluate $_nC_r$ using the formula from this section.**

49. $_5C_2$	**50.** $_6C_3$
51. $_4C_1$	**52.** $_{25}C_0$

Evaluating $_nC_r$ **In Exercises 53–56, evaluate $_nC_r$ using a graphing utility.**

53. $_{20}C_4$	**54.** $_{10}C_7$
55. $_{42}C_5$	**56.** $_{50}C_6$

57. **Writing Combinations** Write all combinations of two letters that you can form from the letters A, B, C, D, E, and F. (The order of the two letters is not important.)

58. **Forming an Experimental Group** In order to conduct an experiment, researchers randomly select five students from a class of 20. How many different groups of five students are possible?

59. **Jury Selection** In how many different ways can a jury of 12 people be randomly selected from a group of 40 people?

60. **Committee Members** A U.S. Senate Committee has 14 members. Assuming party affiliation is not a factor in selection, how many different committees are possible from the 100 U.S. senators?

61. **Lottery Choices** In the Massachusetts Mass Cash game, a player randomly chooses five distinct numbers from 1 to 35. In how many ways can a player select the five numbers?

62. **Lottery Choices** In the Louisiana Lotto game, a player randomly chooses six distinct numbers from 1 to 40. In how many ways can a player select the six numbers?

63. **Defective Units** A shipment of 25 television sets contains three defective units. In how many ways can a vending company purchase four of these units and receive (a) all good units, (b) two good units, and (c) at least two good units?

64. **Interpersonal Relationships** The complexity of interpersonal relationships increases dramatically as the size of a group increases. Determine the numbers of different two-person relationships in groups of people of sizes (a) 3, (b) 8, (c) 12, and (d) 20.

Kannanimages/Shutterstock.com

65. **Poker Hand** You are dealt five cards from a standard deck of 52 playing cards. In how many ways can you get (a) a full house and (b) a five-card combination containing two jacks and three aces? (A full house consists of three of one kind and two of another. For example, A-A-A-5-5 and K-K-K-10-10 are full houses.)

66. **Job Applicants** An employer interviews 12 people for four openings at a company. Five of the 12 people are women. All 12 applicants are qualified. In how many ways can the employer fill the four positions when (a) the selection is random and (b) exactly two selections are women?

67. **Forming a Committee** A local college is forming a six-member research committee having one administrator, three faculty members, and two students. There are seven administrators, 12 faculty members, and 20 students in contention for the committee. How many six-member committees are possible?

68. **Law Enforcement** A police department uses computer imaging to create digital photographs of alleged perpetrators from eyewitness accounts. One software package contains 195 hairlines, 99 sets of eyes and eyebrows, 89 noses, 105 mouths, and 74 chins and cheek structures.

(a) Find the possible number of different faces that the software could create.

(b) An eyewitness can clearly recall the hairline and eyes and eyebrows of a suspect. How many different faces can be produced with this information?

Geometry **In Exercises 69–72, find the number of diagonals of the polygon. (A line segment connecting any two nonadjacent vertices is called a *diagonal* of the polygon.)**

69. Pentagon

70. Hexagon

71. Octagon

72. Decagon (10 sides)

73. **Geometry** Three points that are not collinear determine three lines. How many lines are determined by nine points, no three of which are collinear?

74. **Lottery** Powerball is a lottery game that is operated by the Multi-State Lottery Association and is played in 42 states, Washington D.C., and the U.S. Virgin Islands. The game is played by drawing five white balls out of a drum of 59 white balls (numbered 1–59) and one red powerball out of a drum of 35 red balls (numbered 1–35). The jackpot is won by matching all five white balls in any order and the red powerball.

(a) Find the possible number of winning Powerball numbers.

(b) Find the possible number of winning Powerball numbers when you win the jackpot by matching all five white balls in order and the red powerball.

Solving an Equation **In Exercises 75–82, solve for *n*.**

75. $4 \cdot {}_{n+1}P_2 = {}_{n+2}P_3$

76. $5 \cdot {}_{n-1}P_1 = {}_nP_2$

77. ${}_{n+1}P_3 = 4 \cdot {}_nP_2$

78. ${}_{n+2}P_3 = 6 \cdot {}_{n+2}P_1$

79. $14 \cdot {}_nP_3 = {}_{n+2}P_4$

80. ${}_nP_5 = 18 \cdot {}_{n-2}P_4$

81. ${}_nP_4 = 10 \cdot {}_{n-1}P_3$

82. ${}_nP_6 = 12 \cdot {}_{n-1}P_5$

Exploration

True or False? **In Exercises 83 and 84, determine whether the statement is true or false. Justify your answer.**

83. The number of letter pairs that can be formed in any order from any two of the first 13 letters in the alphabet (A–M) is an example of a permutation.

84. The number of permutations of n elements can be determined by using the Fundamental Counting Principle.

85. **Think About It** Without calculating the numbers, determine which of the following is greater. Explain.

(a) The number of combinations of 10 elements taken six at a time

(b) The number of permutations of 10 elements taken six at a time

86. **HOW DO YOU SEE IT?** Without calculating, determine whether the value of ${}_nP_r$ is greater than the value of ${}_nC_r$ for the values of n and r given in the table. Complete the table using yes (Y) or no (N). Is the value of ${}_nP_r$ always greater than the value of ${}_nC_r$? Explain.

r n	0	1	2	3	4	5	6	7
1								
2								
3								
4								
5								
6								
7								

Proof **In Exercises 87–90, prove the identity.**

87. ${}_nP_{n-1} = {}_nP_n$

88. ${}_nC_n = {}_nC_0$

89. ${}_nC_{n-1} = {}_nC_1$

90. ${}_nC_r = \dfrac{{}_nP_r}{r!}$

91. **Think About It** Can your graphing utility evaluate ${}_{100}P_{80}$? If not, then explain why.

8.7 Probability

- **Find the probabilities of events.**
- **Find the probabilities of mutually exclusive events.**
- **Find the probabilities of independent events.**
- **Find the probability of the complement of an event.**

Probability applies to many real-life applications. For instance, in Exercise 59 on page 630, you will calculate probabilities that relate to a communication network and an independent backup system for a space vehicle.

The Probability of an Event

Any happening for which the result is uncertain is called an **experiment.** The possible results of the experiment are **outcomes,** the set of all possible outcomes of the experiment is the **sample space** of the experiment, and any subcollection of a sample space is an **event.**

For instance, when you toss a six-sided die, the numbers 1 through 6 can represent the sample space. For this experiment, each of the outcomes is *equally likely.*

To describe sample spaces in such a way that each outcome is equally likely, you must sometimes distinguish between or among various outcomes in ways that appear artificial. Example 1 illustrates such a situation.

EXAMPLE 1 Finding a Sample Space

Find the sample space for each of the following experiments.

a. You toss one coin.

b. You toss two coins.

c. You toss three coins.

Solution

a. Because the coin will land either heads up (denoted by H) or tails up (denoted by T), the sample space is

$$S = \{H, T\}.$$

b. Because either coin can land heads up or tails up, the possible outcomes are as follows.

HH = heads up on both coins

HT = heads up on first coin and tails up on second coin

TH = tails up on first coin and heads up on second coin

TT = tails up on both coins

So, the sample space is

$$S = \{HH, HT, TH, TT\}.$$

Note that this list distinguishes between the two cases HT and TH, even though these two outcomes appear to be similar.

c. Following the notation of part (b), the sample space is

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Note that this list distinguishes among the cases HHT, HTH, and THH, and among the cases HTT, THT, and TTH.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the sample space for the following experiment.

You toss a coin twice and a six-sided die once.

1971yes/Shutterstock.com

To calculate the probability of an event, count the number of outcomes in the event and in the sample space. The *number of outcomes* in event E is denoted by $n(E)$, and the number of outcomes in the sample space S is denoted by $n(S)$. The probability that event E will occur is given by $n(E)/n(S)$.

The Probability of an Event

If an event E has $n(E)$ equally likely outcomes and its sample space S has $n(S)$ equally likely outcomes, then the **probability** of event E is

$$P(E) = \frac{n(E)}{n(S)}.$$

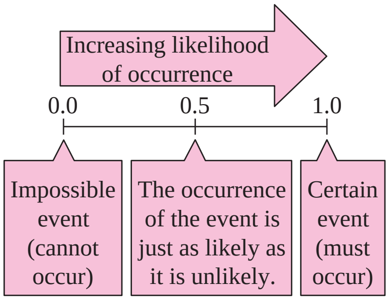


Figure 8.3

Because the number of outcomes in an event must be less than or equal to the number of outcomes in the sample space, the probability of an event must be a number between 0 and 1. That is,

$$0 \leq P(E) \leq 1$$

as indicated in Figure 8.3. If $P(E) = 0$, then event E *cannot occur,* and E is called an **impossible event.** If $P(E) = 1$, then event E *must occur,* and E is called a **certain event.**

EXAMPLE 2 Finding the Probability of an Event

a. You toss two coins. What is the probability that both land heads up?

b. You draw one card at random from a standard deck of playing cards. What is the probability that it is an ace?

Solution

a. Following the procedure in Example 1(b), let

$$E = \{HH\}$$

and

$$S = \{HH, HT, TH, TT\}.$$

The probability of getting two heads is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{4}.$$

REMARK You can write a probability as a fraction, a decimal, or a percent. For instance, in Example 2(a), the probability of getting two heads can be written as $\frac{1}{4}$, 0.25, or 25%.

b. Because there are 52 cards in a standard deck of playing cards and there are four aces (one in each suit), the probability of drawing an ace is

$$P(E) = \frac{n(E)}{n(S)}$$
$$= \frac{4}{52}$$
$$= \frac{1}{13}.$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

a. You toss three coins. What is the probability that all three land tails up?

b. You draw one card at random from a standard deck of playing cards. What is the probability that it is a diamond?

As shown in Example 3, when you toss two six-sided dice, the probability that the total of the two dice is 7 is $\frac{1}{6}$.

EXAMPLE 3 Finding the Probability of an Event

You toss two six-sided dice. What is the probability that the total of the two dice is 7?

Solution Because there are six possible outcomes on each die, use the Fundamental Counting Principle to conclude that there are $6 \cdot 6$ or 36 different outcomes when you toss two dice. To find the probability of rolling a total of 7, you must first count the number of ways in which this can occur.

First Die	Second Die
1	6
2	5
3	4
4	3
5	2
6	1

So, a total of 7 can be rolled in six ways, which means that the probability of rolling a 7 is

$$P(E) = \frac{n(E)}{n(S)} = \frac{6}{36} = \frac{1}{6}.$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

You toss two six-sided dice. What is the probability that the total of the two dice is 5?

EXAMPLE 4 Finding the Probability of an Event

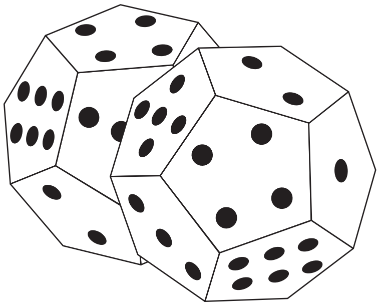


Figure 8.4

Twelve-sided dice, as shown in Figure 8.4, can be constructed (in the shape of regular dodecahedrons) such that each of the numbers from 1 to 6 appears twice on each die. Prove that these dice can be used in any game requiring ordinary six-sided dice without changing the probabilities of the various events.

Solution For an ordinary six-sided die, each of the numbers

1, 2, 3, 4, 5, and 6

occurs only once, so the probability of rolling any particular number is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{6}.$$

For one of the 12-sided dice, each number occurs twice, so the probability of rolling any particular number is

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{12} = \frac{1}{6}.$$

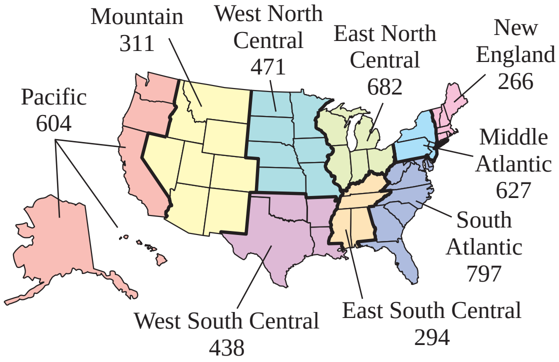
✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Show that the probability of drawing a club at random from a standard deck of playing cards is the same as the probability of drawing the ace of hearts at random from a set consisting of the aces of hearts, diamonds, clubs, and spades. ■

Sergey Mironov/Shutterstock.com

EXAMPLE 5 Random Selection

The following figure shows the numbers of colleges and universities in various regions of the United States in 2010. What is the probability that an institution selected at random is in one of the three southern regions? *(Source: National Center for Education Statistics)*



Solution From the figure, the total number of colleges and universities is 4490. Because there are 797 + 294 + 438 = 1529 colleges and universities in the three southern regions, the probability that the institution is in one of these regions is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1529}{4490} \approx 0.341.$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

In Example 5, what is the probability that the randomly selected institution is in the Pacific region?

EXAMPLE 6 The Probability of Winning a Lottery

In Arizona's The Pick game, a player chooses six different numbers from 1 to 44. If these six numbers match the six numbers drawn (in any order) by the lottery commission, then the player wins (or shares) the top prize. What is the probability of winning the top prize when the player buys one ticket?

Solution To find the number of elements in the sample space, use the formula for the number of combinations of 44 elements taken six at a time.

$$\begin{aligned} n(S) &= {}_{44}C_6 \\ &= \frac{44 \cdot 43 \cdot 42 \cdot 41 \cdot 40 \cdot 39}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\ &= 7{,}059{,}052 \end{aligned}$$

When a player buys one ticket, the probability of winning is

$$P(E) = \frac{1}{7{,}059{,}052}.$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

In Pennsylvania's Cash 5 game, a player chooses five different numbers from 1 to 43. If these five numbers match the five numbers drawn (in any order) by the lottery commission, then the player wins (or shares) the top prize. What is the probability of winning the top prize when the player buys one ticket?

Mutually Exclusive Events

Two events A and B (from the same sample space) are **mutually exclusive** when A and B have no outcomes in common. In the terminology of sets, the intersection of A and B is the empty set, which implies that

$$P(A \cap B) = 0.$$

For instance, when you toss two dice, the event A of rolling a total of 6 and the event B of rolling a total of 9 are mutually exclusive. To find the probability that one or the other of two mutually exclusive events will occur, *add* their individual probabilities.

Probability of the Union of Two Events

If A and B are events in the same sample space, then the probability of A or B occurring is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

If A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

EXAMPLE 7 The Probability of a Union of Events

You draw one card at random from a standard deck of 52 playing cards. What is the probability that the card is either a heart or a face card?

Solution Because the deck has 13 hearts, the probability of drawing a heart (event A) is

$$P(A) = \frac{13}{52}.$$

Similarly, because the deck has 12 face cards, the probability of drawing a face card (event B) is

$$P(B) = \frac{12}{52}.$$

Because three of the cards are hearts *and* face cards (see Figure 8.5), it follows that

$$P(A \cap B) = \frac{3}{52}.$$

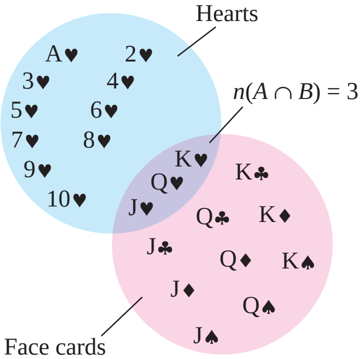


Figure 8.5

Finally, applying the formula for the probability of the union of two events, the probability of drawing a heart or a face card is

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} \\ &= \frac{22}{52} \\ &\approx 0.423. \end{aligned}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

You draw one card at random from a standard deck of 52 playing cards. What is the probability that the card is either an ace or a spade?

EXAMPLE 8 Probability of Mutually Exclusive Events

The human resources department of a company has compiled data showing the number of years of service for each employee. The table shows the results.

DATA

Spreadsheet at LarsonPrecalculus.com

Years of Service	Number of Employees
0–4	157
5–9	89
10–14	74
15–19	63
20–24	42
25–29	38
30–34	35
35–39	21
40–44	8
45 or more	2

a. What is the probability that an employee chosen at random has 4 or fewer years of service?

b. What is the probability that an employee chosen at random has 9 or fewer years of service?

Solution

a. To begin, add the number of employees to find that the total is 529. Next, let event A represent choosing an employee with 0 to 4 years of service. Then the probability of choosing an employee who has 4 or fewer years of service is

$$\begin{aligned} P(A) &= \frac{157}{529} \\ &\approx 0.297. \end{aligned}$$

b. Let event B represent choosing an employee with 5 to 9 years of service. Then

$$P(B) = \frac{89}{529}.$$

Because event A from part (a) and event B have no outcomes in common, these two events are mutually exclusive and

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) \\ &= \frac{157}{529} + \frac{89}{529} \\ &= \frac{246}{529} \\ &\approx 0.465. \end{aligned}$$

So, the probability of choosing an employee who has 9 or fewer years of service is about 0.465.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

In Example 8, what is the probability that an employee chosen at random has 30 or more years of service?

Independent Events

Two events are **independent** when the occurrence of one has no effect on the occurrence of the other. For instance, rolling a total of 12 with two six-sided dice has no effect on the outcome of future rolls of the dice. To find the probability that two independent events will occur, *multiply* the probabilities of each.

Probability of Independent Events

If A and B are independent events, then the probability that both A and B will occur is

$$P(A \text{ and } B) = P(A) \cdot P(B).$$

EXAMPLE 9 Probability of Independent Events

A random number generator on a computer selects three integers from 1 to 20. What is the probability that all three numbers are less than or equal to 5?

Solution The probability of selecting a number from 1 to 5 is

$$P(A) = \frac{5}{20} = \frac{1}{4}.$$

So, the probability that all three numbers are less than or equal to 5 is

$$P(A) \cdot P(A) \cdot P(A) = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{64}.$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

A random number generator on a computer selects two integers from 1 to 30. What is the probability that both numbers are less than 12?

EXAMPLE 10 Probability of Independent Events

In 2012, approximately 27% of the population of the United States got their news from mobile devices. In a survey, researchers selected 10 people at random from the population. What is the probability that all 10 got their news from mobile devices? *(Source: Pew Research Center)*

Solution Let A represent choosing a person who gets his or her news from a mobile device. The probability of choosing a person who got his or her news from a mobile device is 0.27, the probability of choosing a second person who got his or her news from a mobile device is 0.27, and so on. Because these events are independent, the probability that all 10 people got their news from mobile devices is

$$[P(A)]^{10} = (0.27)^{10} \approx 0.000002059.$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

In Example 10, researchers selected five people at random from the population. What is the probability that all five got their news from mobile devices?

The Complement of an Event

The **complement of an event** A is the collection of all outcomes in the sample space that are *not* in A. The complement of event A is denoted by A'. Because $P(A \text{ or } A') = 1$ and because A and A' are mutually exclusive, it follows that $P(A) + P(A') = 1$. So, the probability of A' is

$$P(A') = 1 - P(A).$$

For instance, if the probability of *winning* a certain game is $P(A) = \frac{1}{4}$, then the probability of *losing* the game is $P(A') = 1 - \frac{1}{4} = \frac{3}{4}$.

Probability of a Complement

Let A be an event and let A' be its complement. If the probability of A is $P(A)$, then the probability of the complement is

$$P(A') = 1 - P(A).$$

EXAMPLE 11 Finding the Probability of a Complement

A manufacturer has determined that a machine averages one faulty unit for every 1000 it produces. What is the probability that an order of 200 units will have one or more faulty units?

Solution To solve this problem as stated, you would need to find the probabilities of having exactly one faulty unit, exactly two faulty units, exactly three faulty units, and so on. However, using complements, you can find the probability that all units are perfect and then subtract this value from 1. Because the probability that any given unit is perfect is 999/1000, the probability that all 200 units are perfect is

$$P(A) = \left(\frac{999}{1000}\right)^{200} \approx 0.819.$$

So, the probability that at least one unit is faulty is

$$P(A') = 1 - P(A) \approx 1 - 0.819 = 0.181.$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

A manufacturer has determined that a machine averages one faulty unit for every 500 it produces. What is the probability that an order of 300 units will have one or more faulty units? ■

Summarize (Section 8.7)

1. State the definition of the probability of an event *(page 621)*. For examples of finding the probabilities of events, see Examples 2–6.
2. State the definition of mutually exclusive events and explain how to find the probability of the union of two events *(page 624)*. For examples of finding the probabilities of the unions of two events, see Examples 7 and 8.
3. State the definition of, and explain how to find the probability of, independent events *(page 626)*. For examples of finding the probabilities of independent events, see Examples 9 and 10.
4. State the definition of, and explain how to find the probability of, the complement of an event *(page 627)*. For an example of finding the probability of the complement of an event, see Example 11.

8.7 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary

In Exercises 1–7, fill in the blanks.

1. An ________ is any happening for which the result is uncertain, and the possible results are called ________.

2. The set of all possible outcomes of an experiment is called the ________ ________.

3. To determine the ________ of an event, you can use the formula $P(E) = \dfrac{n(E)}{n(S)}$, where $n(E)$ is the number of outcomes in the event and $n(S)$ is the number of outcomes in the sample space.

4. If $P(E) = 0$, then E is an ________ event, and if $P(E) = 1$, then E is a ________ event.

5. If two events from the same sample space have no outcomes in common, then the two events are ________ ________.

6. If the occurrence of one event has no effect on the occurrence of a second event, then the events are ________.

7. The ________ of an event A is the collection of all outcomes in the sample space that are not in A.

8. Match the probability formula with the correct probability name.

(a) Probability of the union of two events	(i) $P(A \cup B) = P(A) + P(B)$
(b) Probability of mutually exclusive events	(ii) $P(A') = 1 - P(A)$
(c) Probability of independent events	(iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
(d) Probability of a complement	(iv) $P(A \text{ and } B) = P(A) \cdot P(B)$

Skills and Applications

Finding a Sample Space

In Exercises 9–14, find the sample space for the experiment.

9. You toss a coin and a six-sided die.

10. You toss a six-sided die twice and record the sum of the results.

11. A taste tester ranks three varieties of yogurt, A, B, and C, according to preference.

12. You select two marbles (without replacement) from a bag containing two red marbles, two blue marbles, and one yellow marble. You record the color of each marble.

13. Two county supervisors are selected from five supervisors, A, B, C, D, and E, to study a recycling plan.

14. A sales representative makes presentations about a product in three homes per day. In each home, there may be a sale (denote by S) or there may be no sale (denote by F).

Tossing a Coin

In Exercises 15–20, find the probability for the experiment of tossing a coin three times. Use the sample space

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

15. The probability of getting exactly one tail

16. The probability of getting exactly two tails

17. The probability of getting a head on the first toss

18. The probability of getting a tail on the last toss

19. The probability of getting at least one head

20. The probability of getting at least two heads

Drawing a Card

In Exercises 21–24, find the probability for the experiment of drawing a card at random from a standard deck of 52 playing cards.

21. The card is a face card.

22. The card is not a face card.

23. The card is a red face card.

24. The card is a 9 or lower. (Aces are low.)

Tossing a Die

In Exercises 25–30, find the probability for the experiment of tossing a six-sided die twice.

25. The sum is 6.

26. The sum is at least 8.

27. The sum is less than 11.

28. The sum is 2, 3, or 12.

29. The sum is odd and no more than 7.

30. The sum is odd or prime.

Drawing Marbles

In Exercises 31–34, find the probability for the experiment of drawing two marbles at random (without replacement) from a bag containing one green, two yellow, and three red marbles.

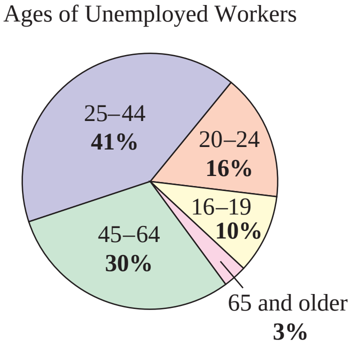
31. Both marbles are red.

32. Both marbles are yellow.

33. Neither marble is yellow.

34. The marbles are different colors.

35. **Graphical Reasoning** In 2011, there were approximately 13.75 million unemployed workers in the United States. The circle graph shows the age profile of these unemployed workers. *(Source: U.S. Bureau of Labor Statistics)*



(a) Estimate the number of unemployed workers in the 16–19 age group.

(b) What is the probability that a person selected at random from the population of unemployed workers is in the 25–44 age group?

(c) What is the probability that a person selected at random from the population of unemployed workers is in the 45–64 age group?

(d) What is the probability that a person selected at random from the population of unemployed workers is 45 or older?

36. **Data Analysis** An independent polling organization interviews one hundred college students to determine their political party affiliations and whether they favor a balanced-budget amendment to the Constitution. The table lists the results of the study. In the table, *D* represents Democrat and *R* represents Republican.

	Favor	Not Favor	Unsure	Total
D	23	25	7	55
R	32	9	4	45
Total	55	34	11	100

Find the probability that a person selected at random from the sample is as described.

(a) A person who does not favor the amendment

(b) A Republican

(c) A Democrat who favors the amendment

37. **Education** In a high school graduating class of 128 students, 52 are on the honor roll. Of these, 48 are going on to college; of the other 76 students, 56 are going on to college. What is the probability that a student selected at random from the class is (a) going to college, (b) not going to college, and (c) not going to college and on the honor roll?

38. **Alumni Association** A college sends a survey to members of the class of 2012. Of the 1254 people who graduated that year, 672 are women, of whom 124 went on to graduate school. Of the 582 male graduates, 198 went on to graduate school. What is the probability that a class of 2012 alumnus selected at random is (a) female, (b) male, and (c) female and did not attend graduate school?

39. **Winning an Election** Three people are running for president of a class. The results of a poll indicate that the first candidate has an estimated 37% chance of winning and the second candidate has an estimated 44% chance of winning. What is the probability that the third candidate will win?

40. **Payroll Error** The employees of a company work in six departments: 31 are in sales, 54 are in research, 42 are in marketing, 20 are in engineering, 47 are in finance, and 58 are in production. The payroll department loses one employee's paycheck. What is the probability that the employee works in the research department?

Using Counting Principles **In Exercises 41–48, the sample spaces are large and you should use the counting principles discussed in Section 8.6.**

41. A class receives a list of 20 study problems, from which 10 will be part of an upcoming exam. A student knows how to solve 15 of the problems. Find the probability that the student will be able to answer (a) all 10 questions on the exam, (b) exactly eight questions on the exam, and (c) at least nine questions on the exam.

42. A payroll department addresses five paychecks and envelopes to five different people and randomly inserts the paychecks into the envelopes. What is the probability that (a) exactly one paycheck is inserted in the correct envelope and (b) at least one paycheck is in the correct envelope?

43. On a game show, you are given five digits to arrange in the proper order to form the price of a car. If you are correct, then you win the car. What is the probability of winning, given the following conditions?

(a) You guess the position of each digit.

(b) You know the first digit and guess the positions of the other digits.

44. The deck for a card game is made up of 108 cards. Twenty-five each are red, yellow, blue, and green, and eight are wild cards. Each player is randomly dealt a seven-card hand.

(a) What is the probability that a hand will contain exactly two wild cards?

(b) What is the probability that a hand will contain two wild cards, two red cards, and three blue cards?

45. You draw one card at random from a standard deck of 52 playing cards. Find the probability that (a) the card is an even-numbered card, (b) the card is a heart or a diamond, and (c) the card is a nine or a face card.

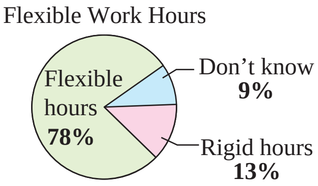
46. You draw five cards at random from a standard deck of 52 playing cards. What is the probability that the hand drawn is a full house? (A full house is a hand that consists of two of one kind and three of another kind.)

47. A shipment of 12 microwave ovens contains three defective units. A vending company has ordered four units, and because each has identical packaging, the selection will be random. What is the probability that (a) all four units are good, (b) exactly two units are good, and (c) at least two units are good?

48. ATM personal identification number (PIN) codes typically consist of four-digit sequences of numbers. Find the probability that if you forget your PIN, then you can guess the correct sequence (a) at random and (b) when you recall the first two digits.

49. Random Number Generator A random number generator on a computer selects two integers from 1 through 40. What is the probability that (a) both numbers are even, (b) one number is even and one number is odd, (c) both numbers are less than 30, and (d) the same number is selected twice?

50. Flexible Work Hours In a survey, people were asked whether they would prefer to work flexible hours—even when it meant slower career advancement—so they could spend more time with their families. The figure shows the results of the survey. What is the probability that three people chosen at random would prefer flexible work hours?



Finding the Probability of a Complement In Exercises 51–54, you are given the probability that an event *will* happen. Find the probability that the event *will not* happen.

51. $P(E) = 0.87$

52. $P(E) = 0.36$

53. $P(E) = \frac{1}{4}$

54. $P(E) = \frac{2}{3}$

Finding the Probability of a Complement In Exercises 55–58, you are given the probability that an event *will not* happen. Find the probability that the event *will* happen.

55. $P(E') = 0.23$

56. $P(E') = 0.92$

57. $P(E') = \frac{17}{35}$

58. $P(E') = \frac{61}{100}$

59. Backup System A space vehicle has an independent backup system for one of its communication networks. The probability that either system will function satisfactorily during a flight is 0.985. What is the probability that during a given flight (a) both systems function satisfactorily, (b) at least one system functions satisfactorily, and (c) both systems fail?

60. Backup Vehicle A fire company keeps two rescue vehicles. Because of the demand on the vehicles and the chance of mechanical failure, the probability that a specific vehicle is available when needed is 90%. The availability of one vehicle is independent of the availability of the other. Find the probability that (a) both vehicles are available at a given time, (b) neither vehicle is available at a given time, and (c) at least one vehicle is available at a given time.

61. Roulette American roulette is a game in which a wheel turns on a spindle and is divided into 38 pockets. Thirty-six of the pockets are numbered 1–36, of which half are red and half are black. Two of the pockets are green and are numbered 0 and 00 (see figure). The dealer spins the wheel and a small ball in opposite directions. As the ball slows to a stop, it has an equal probability of landing in any of the numbered pockets.

(a) Find the probability of landing in the number 00 pocket.

(b) Find the probability of landing in a red pocket.

(c) Find the probability of landing in a green pocket or a black pocket.

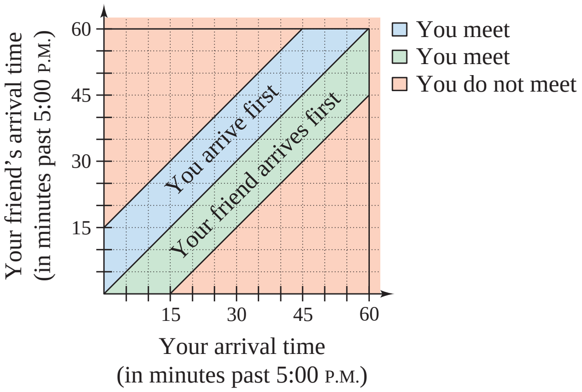
(d) Find the probability of landing in the number 14 pocket on two consecutive spins.

(e) Find the probability of landing in a red pocket on three consecutive spins.

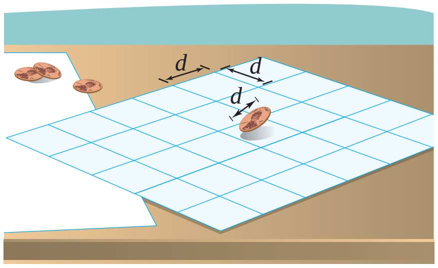
1971yes/Shutterstock.com

62. **A Boy or a Girl?** Assume that the probability of the birth of a child of a particular sex is 50%. In a family with four children, what is the probability that (a) all the children are boys, (b) all the children are the same sex, and (c) there is at least one boy?

63. **Geometry** You and a friend agree to meet at your favorite fast-food restaurant between 5:00 P.M. and 6:00 P.M. The one who arrives first will wait 15 minutes for the other, and then will leave (see figure). What is the probability that the two of you will actually meet, assuming that your arrival times are random within the hour?



64. **Estimating π** You drop a coin of diameter d onto a paper that contains a grid of squares d units on a side (see figure).



(a) Find the probability that the coin covers a vertex of one of the squares on the grid.

(b) Perform the experiment 100 times and use the results to approximate π.

Exploration

True or False? **In Exercises 65 and 66, determine whether the statement is true or false. Justify your answer.**

65. If A and B are independent events with nonzero probabilities, then A can occur when B occurs.

66. Rolling a number less than 3 on a normal six-sided die has a probability of $\frac{1}{3}$. The complement of this event is to roll a number greater than 3, and its probability is $\frac{1}{2}$.

67. **Pattern Recognition** Consider a group of n people.

(a) Explain why the following pattern gives the probabilities that the n people have distinct birthdays.

$$n = 2: \quad \frac{365}{365} \cdot \frac{364}{365} = \frac{365 \cdot 364}{365^2}$$

$$n = 3: \quad \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} = \frac{365 \cdot 364 \cdot 363}{365^3}$$

(b) Use the pattern in part (a) to write an expression for the probability that $n = 4$ people have distinct birthdays.

(c) Let P_n be the probability that the n people have distinct birthdays. Verify that this probability can be obtained recursively by

$$P_1 = 1 \text{ and } P_n = \frac{365 - (n - 1)}{365} P_{n-1}.$$

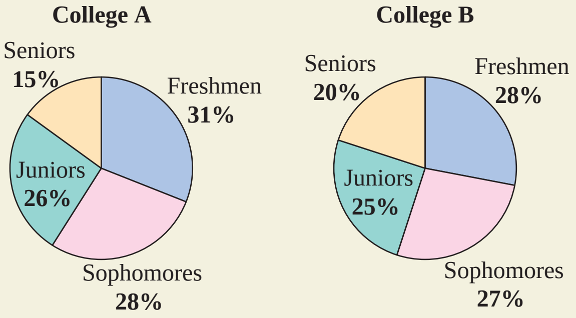
(d) Explain why $Q_n = 1 - P_n$ gives the probability that at least two people in a group of n people have the same birthday.

(e) Use the results of parts (c) and (d) to complete the table.

n	10	15	20	23	30	40	50
P_n							
Q_n							

(f) How many people must be in a group so that the probability of at least two of them having the same birthday is greater than $\frac{1}{2}$? Explain.

68. **HOW DO YOU SEE IT?** The circle graphs show the percents of undergraduate students by class level at two colleges. A student is chosen at random from the combined undergraduate population of the two colleges. The probability that the student is a freshman, sophomore, or junior is 81%. Which college has a greater number of undergraduate students? Explain your reasoning.



Chapter Summary

	What Did You Learn?	Explanation/Examples	Review Exercises
Section 8.1	Use sequence notation to write the terms of sequences *(p. 564).*	$a_n = 7n - 4$; $a_1 = 7(1) - 4 = 3$, $a_2 = 7(2) - 4 = 10$, $a_3 = 7(3) - 4 = 17$, $a_4 = 7(4) - 4 = 24$	1–8
	Use factorial notation *(p. 566).*	If n is a positive integer, then $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \cdot (n - 1) \cdot n$.	9–12
	Use summation notation to write sums *(p. 568).*	The sum of the first n terms of a sequence is represented by $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \cdot \cdot \cdot + a_n$.	13–16
	Find the sums of series *(p. 569).*	$\sum_{i=1}^{\infty} \frac{8}{10^i} = \frac{8}{10^1} + \frac{8}{10^2} + \frac{8}{10^3} + \frac{8}{10^4} + \frac{8}{10^5} + \cdot \cdot \cdot$ $= 0.8 + 0.08 + 0.008 + 0.0008 + 0.00008 + \cdot \cdot \cdot$ $= 0.88888 \ldots = \frac{8}{9}$	17, 18
	Use sequences and series to model and solve real-life problems *(p. 570).*	A sequence can help you model the balance in an account that earns compound interest. (See Example 10.)	19, 20
Section 8.2	Recognize, write, and find the *n*th terms of arithmetic sequences *(p. 574).*	$a_n = 9n + 5$; $a_1 = 9(1) + 5 = 14$, $a_2 = 9(2) + 5 = 23$, $a_3 = 9(3) + 5 = 32$, $a_4 = 9(4) + 5 = 41$	21–30
	Find *n*th partial sums of arithmetic sequences *(p. 577).*	The sum of a finite arithmetic sequence with n terms is $S_n = (n/2)(a_1 + a_n)$.	31–36
	Use arithmetic sequences to model and solve real-life problems *(p. 579).*	An arithmetic sequence can help you find the total sales of a small business. (See Example 9.)	37, 38
Section 8.3	Recognize, write, and find the *n*th terms of geometric sequences *(p. 583).*	$a_n = 3(4^n)$; $a_1 = 3(4^1) = 12$, $a_2 = 3(4^2) = 48$, $a_3 = 3(4^3) = 192$, $a_4 = 3(4^4) = 768$	39–50
	Find the sum of a finite geometric sequence *(p. 586).*	The sum of the finite geometric sequence $a_1, a_1r, a_1r^2, \ldots, a_1r^{n-1}$ with common ratio $r \neq 1$ is given by $S_n = \sum_{i=1}^{n} a_1 r^{i-1} = a_1\left(\frac{1 - r^n}{1 - r}\right)$.	51–58
	Find the sum of an infinite geometric series *(p. 587).*	If $\lvert r \rvert < 1$, then the infinite geometric series $a_1 + a_1r + a_1r^2 + \cdot \cdot \cdot + a_1r^{n-1} + \cdot \cdot \cdot$ has the sum $S = \sum_{i=0}^{\infty} a_1 r^i = \frac{a_1}{1 - r}$.	59–62
	Use geometric sequences to model and solve real-life problems *(p. 588).*	A finite geometric sequence can help you find the balance in an annuity at the end of two years. (See Example 8.)	63, 64

	What Did You Learn?	Explanation/Examples	Review Exercises
Section 8.4	Use mathematical induction to prove statements involving a positive integer *n* *(p. 592)*.	Let P_n be a statement involving the positive integer n. If (1) P_1 is true, and (2) for every positive integer k, the truth of P_k implies the truth of P_{k+1}, then the statement P_n must be true for all positive integers n.	65–68
	Use pattern recognition and mathematical induction to write the *n*th term of a sequence *(p. 597)*.	To find a formula for the *n*th term of a sequence, (1) calculate the first several terms of the sequence, (2) try to find a pattern for the terms and write a formula (hypothesis) for the *n*th term of the sequence, and (3) use mathematical induction to prove your hypothesis.	69–72
	Find the sums of powers of integers *(p. 598)*.	$\sum_{i=1}^{8} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{8(8+1)(16+1)}{6} = 204$	73, 74
	Find finite differences of sequences *(p. 599)*.	The first differences of a sequence are found by subtracting consecutive terms. The second differences are found by subtracting consecutive first differences.	75, 76
Section 8.5	Use the Binomial Theorem to calculate binomial coefficients *(p. 602)*.	**The Binomial Theorem:** In the expansion of $(x+y)^n = x^n + nx^{n-1}y + \cdot\cdot\cdot + {}_nC_r x^{n-r}y^r + \cdot\cdot\cdot + nxy^{n-1} + y^n$, the coefficient of $x^{n-r}y^r$ is ${}_nC_r = \frac{n!}{(n-r)!r!}$.	77, 78
	Use Pascal's Triangle to calculate binomial coefficients *(p. 604)*.	First several rows of Pascal's Triangle: 1 1 1 1 2 1 1 3 3 1 1 4 6 4 1	79, 80
	Use binomial coefficients to write binomial expansions *(p. 605)*.	$(x+1)^3 = x^3 + 3x^2 + 3x + 1$ $(x-1)^4 = x^4 - 4x^3 + 6x^2 - 4x + 1$	81–84
Section 8.6	Solve simple counting problems *(p. 610)*.	A computer randomly generates an integer from 1 through 15. The computer can generate an integer that is divisible by 3 in 5 ways (3, 6, 9, 12, and 15).	85, 86
	Use the Fundamental Counting Principle to solve counting problems *(p. 611)*.	**Fundamental Counting Principle:** Let E_1 and E_2 be two events. The first event E_1 can occur in m_1 different ways. After E_1 has occurred, E_2 can occur in m_2 different ways. The number of ways that the two events can occur is $m_1 \cdot m_2$.	87, 88
	Use permutations *(p. 612)* and combinations *(p. 615)* to solve counting problems.	The number of permutations of n elements taken r at a time is ${}_nP_r = n!/(n-r)!$. The number of combinations of n elements taken r at a time is ${}_nC_r = n!/[(n-r)!r!]$, or ${}_nC_r = {}_nP_r/r!$.	89–92
Section 8.7	Find the probabilities of events *(p. 620)*.	If an event E has $n(E)$ equally likely outcomes and its sample space S has $n(S)$ equally likely outcomes, then the probability of event E is $P(E) = n(E)/n(S)$.	93, 94
	Find the probabilities of mutually exclusive events *(p. 624)* and independent events *(p. 626)*, and find the probability of the complement of an event *(p. 627)*.	If A and B are events in the same sample space, then the probability of A or B occurring is $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. If A and B are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$. If A and B are independent, then $P(A \text{ and } B) = P(A) \cdot P(B)$. The probability of the complement is $P(A') = 1 - P(A)$.	95–100

Review Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

8.1 Writing the Terms of a Sequence **In Exercises 1–4, write the first five terms of the sequence. (Assume that n begins with 1.)**

1. $a_n = 2 + \frac{6}{n}$

2. $a_n = \frac{(-1)^n 5n}{2n - 1}$

3. $a_n = \frac{72}{n!}$

4. $a_n = n(n - 1)$

Finding the nth Term of a Sequence **In Exercises 5–8, write an expression for the apparent nth term (a_n) of the sequence. (Assume that n begins with 1.)**

5. $-2, 2, -2, 2, -2, . . .$

6. $-1, 2, 7, 14, 23, . . .$

7. $4, 2, \frac{4}{3}, 1, \frac{4}{5}, . . .$

8. $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, . . .$

Simplifying a Factorial Expression **In Exercises 9–12, simplify the factorial expression.**

9. $9!$

10. $4! \cdot 0!$

11. $\frac{3! \cdot 5!}{6!}$

12. $\frac{7! \cdot 6!}{6! \cdot 8!}$

Finding a Sum **In Exercises 13 and 14, find the sum.**

13. $\sum_{j=1}^{4} \frac{6}{j^2}$

14. $\sum_{k=1}^{10} 2k^3$

Using Sigma Notation to Write a Sum **In Exercises 15 and 16, use sigma notation to write the sum.**

15. $\frac{1}{2(1)} + \frac{1}{2(2)} + \frac{1}{2(3)} + \cdot \cdot \cdot + \frac{1}{2(20)}$

16. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdot \cdot \cdot + \frac{9}{10}$

Finding the Sum of an Infinite Series **In Exercises 17 and 18, find the sum of the infinite series.**

17. $\sum_{i=1}^{\infty} \frac{4}{10^i}$

18. $\sum_{k=1}^{\infty} 2\left(\frac{1}{100}\right)^k$

19. Compound Interest An investor deposits \$10,000 in an account that earns 2.25% interest compounded monthly. The balance in the account after n months is given by

$$A_n = 10{,}000\left(1 + \frac{0.0225}{12}\right)^n, \quad n = 1, 2, 3, . . .$$

(a) Write the first 10 terms of the sequence.

(b) Find the balance in the account after 10 years by computing the 120th term of the sequence.

20. Lottery Ticket Sales The total sales a_n (in billions of dollars) of lottery tickets in the United States from 2001 through 2010 can be approximated by the model

$$a_n = -0.18n^2 + 3.7n + 35, \quad n = 1, 2, . . ., 10$$

where n is the year, with $n = 1$ corresponding to 2001. Write the terms of this finite sequence. Use a graphing utility to construct a bar graph that represents the sequence. *(Source: TLF Publications, Inc.)*

8.2 Determining Whether a Sequence Is Arithmetic **In Exercises 21–24, determine whether the sequence is arithmetic. If so, then find the common difference.**

21. $6, -1, -8, -15, -22, . . .$

22. $0, 1, 3, 6, 10, . . .$

23. $\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, . . .$

24. $1, \frac{15}{16}, \frac{7}{8}, \frac{13}{16}, \frac{3}{4}, . . .$

Finding the nth Term **In Exercises 25–28, find a formula for a_n for the arithmetic sequence.**

25. $a_1 = 7, d = 12$

26. $a_1 = 28, d = -5$

27. $a_2 = 93, a_6 = 65$

28. $a_7 = 8, a_{13} = 6$

Writing the Terms of an Arithmetic Sequence **In Exercises 29 and 30, write the first five terms of the arithmetic sequence.**

29. $a_1 = 3, d = 11$

30. $a_1 = 25, a_{k+1} = a_k + 3$

31. Sum of a Finite Arithmetic Sequence Find the sum of the first 100 positive multiples of 7.

32. Sum of a Finite Arithmetic Sequence Find the sum of the integers from 40 to 90 (inclusive).

Finding a Partial Sum **In Exercises 33–36, find the partial sum.**

33. $\sum_{j=1}^{10} (2j - 3)$

34. $\sum_{j=1}^{8} (20 - 3j)$

35. $\sum_{k=1}^{11} \left(\frac{2}{3}k + 4\right)$

36. $\sum_{k=1}^{25} \left(\frac{3k + 1}{4}\right)$

37. Job Offer The starting salary for a job is \$43,800 with a guaranteed increase of \$1950 per year. Determine (a) the salary during the fifth year and (b) the total compensation through five full years of employment.

38. Baling Hay In the first two trips baling hay around a large field, a farmer obtains 123 bales and 112 bales, respectively. Because each round gets shorter, the farmer estimates that the same pattern will continue. Estimate the total number of bales made after the farmer takes another six trips around the field.

8.3 Determining Whether a Sequence Is Geometric **In Exercises 39–42, determine whether the sequence is geometric. If so, then find the common ratio.**

39. $6, 12, 24, 48, \ldots$

40. $54, -18, 6, -2, \ldots$

41. $\frac{1}{5}, -\frac{3}{5}, \frac{9}{5}, -\frac{27}{5}, \ldots$

42. $\frac{1}{4}, \frac{2}{5}, \frac{3}{6}, \frac{4}{7}, \ldots$

Writing the Terms of a Geometric Sequence **In Exercises 43–46, write the first five terms of the geometric sequence.**

43. $a_1 = 2, \ r = 15$

44. $a_1 = 4, \ r = -\frac{1}{4}$

45. $a_1 = 9, \ a_3 = 4$

46. $a_1 = 2, \ a_3 = 12$

Finding a Term of a Geometric Sequence **In Exercises 47–50, write an expression for the *n*th term of the geometric sequence. Then find the 10th term of the sequence.**

47. $a_1 = 18, a_2 = -9$

48. $a_3 = 6, a_4 = 1$

49. $a_1 = 100, r = 1.05$

50. $a_1 = 5, r = 0.2$

Sum of a Finite Geometric Sequence **In Exercises 51–56, find the sum of the finite geometric sequence.**

51. $\sum_{i=1}^{7} 2^{i-1}$

52. $\sum_{i=1}^{5} 3^{i-1}$

53. $\sum_{i=1}^{4} \left(\frac{1}{2}\right)^{i}$

54. $\sum_{i=1}^{6} \left(\frac{1}{3}\right)^{i-1}$

55. $\sum_{i=1}^{5} (2)^{i-1}$

56. $\sum_{i=1}^{4} 6(3)^{i}$

Sum of a Finite Geometric Sequence **In Exercises 57 and 58, use a graphing utility to find the sum of the finite geometric sequence.**

57. $\sum_{i=1}^{10} 10\left(\frac{3}{5}\right)^{i-1}$

58. $\sum_{i=1}^{15} 20(0.2)^{i-1}$

Sum of an Infinite Geometric Sequence **In Exercises 59–62, find the sum of the infinite geometric series.**

59. $\sum_{i=0}^{\infty} \left(\frac{7}{8}\right)^{i}$

60. $\sum_{i=0}^{\infty} (0.5)^{i}$

61. $\sum_{k=1}^{\infty} 4\left(\frac{2}{3}\right)^{k-1}$

62. $\sum_{k=1}^{\infty} 1.3\left(\frac{1}{10}\right)^{k-1}$

63. Depreciation A paper manufacturer buys a machine for $120,000. During the next 5 years, it will depreciate at a rate of 30% per year. (In other words, at the end of each year the depreciated value will be 70% of what it was at the beginning of the year.)

(a) Find the formula for the *n*th term of a geometric sequence that gives the value of the machine *t* full years after it was purchased.

(b) Find the depreciated value of the machine after 5 full years.

64. Annuity An investor deposits $800 in an account on the first day of each month for 10 years. The account pays 3%, compounded monthly. What will the balance be at the end of 10 years?

8.4 Using Mathematical Induction **In Exercises 65–68, use mathematical induction to prove the formula for every positive integer *n*.**

65. $3 + 5 + 7 + \cdot \cdot \cdot + (2n + 1) = n(n + 2)$

66. $1 + \frac{3}{2} + 2 + \frac{5}{2} + \cdot \cdot \cdot + \frac{1}{2}(n + 1) = \frac{n}{4}(n + 3)$

67. $\sum_{i=0}^{n-1} ar^{i} = \frac{a(1 - r^{n})}{1 - r}$

68. $\sum_{k=0}^{n-1} (a + kd) = \frac{n}{2}[2a + (n - 1)d]$

Finding a Formula for a Sum **In Exercises 69–72, use mathematical induction to find a formula for the sum of the first *n* terms of the sequence.**

69. $9, 13, 17, 21, \ldots$

70. $68, 60, 52, 44, \ldots$

71. $1, \frac{3}{5}, \frac{9}{25}, \frac{27}{125}, \ldots$

72. $12, -1, \frac{1}{12}, -\frac{1}{144}, \ldots$

Finding a Sum **In Exercises 73 and 74, find the sum using the formulas for the sums of powers of integers.**

73. $\sum_{n=1}^{75} n$

74. $\sum_{n=1}^{6} (n^5 - n^2)$

Linear Model, Quadratic Model, or Neither? **In Exercises 75 and 76, write the first five terms of the sequence beginning with the given term. Then calculate the first and second differences of the sequence. State whether the sequence has a perfect linear model, a perfect quadratic model, or neither.**

75. $a_1 = 5$
$a_n = a_{n-1} + 5$

76. $a_1 = -3$
$a_n = a_{n-1} - 2n$

8.5 Finding a Binomial Coefficient **In Exercises 77 and 78, find the binomial coefficient.**

77. ${}_6C_4$

78. ${}_{12}C_3$

Using Pascal's Triangle **In Exercises 79 and 80, evaluate using Pascal's Triangle.**

79. $\binom{7}{2}$

80. $\binom{10}{4}$

Expanding an Expression **In Exercises 81–84, use the Binomial Theorem to expand and simplify the expression. (Remember that $i = \sqrt{-1}$.)**

81. $(x + 4)^4$

82. $(a - 3b)^5$

83. $(5 + 2i)^4$

84. $(4 - 5i)^3$

8.6

85. **Numbers in a Hat** You place slips of paper numbered 1 through 14 in a hat. In how many ways can you draw two numbers at random with replacement that total 12?

86. **Shopping** A customer in an electronics store can choose one of six speaker systems, one of five DVD players, and one of six flat screen televisions to design a home theater system. How many systems can the customer design?

87. **Telephone Numbers** All of the land line telephone numbers in a small town use the same three-digit prefix. How many different telephone numbers are possible by changing only the last four digits?

88. **Course Schedule** A college student is preparing a course schedule for the next semester. The student may select one of three mathematics courses, one of four science courses, and one of six history courses. How many schedules are possible?

89. **Genetics** A geneticist is using gel electrophoresis to analyze five DNA samples. The geneticist treats each sample with a different restriction enzyme and then injects it into one of five wells formed in a bed of gel. In how many orders can the geneticist inject the five samples into the wells?

90. **Race** There are 10 bicyclists entered in a race. In how many different ways could the top 3 places be decided?

91. **Jury Selection** A group of potential jurors has been narrowed down to 32 people. In how many ways can a jury of 12 people be selected?

92. **Menu Choices** A local sub shop offers five different breads, four different meats, three different cheeses, and six different vegetables. You can choose one bread and any number of the other items. Find the total number of combinations of sandwiches possible.

8.7

93. **Apparel** A man has five pairs of socks, of which no two pairs are the same color. He randomly selects two socks from a drawer. What is the probability that he gets a matched pair?

94. **Bookshelf Order** A child returns a five-volume set of books to a bookshelf. The child is not able to read, and so cannot distinguish one volume from another. What is the probability that the child shelves the books in the correct order?

95. **Students by Class** At a university, 31% of the students are freshmen, 26% are sophomores, 25% are juniors, and 18% are seniors. One student receives a cash scholarship randomly by lottery. What is the probability that the scholarship winner is

(a) a junior or senior?

(b) a freshman, sophomore, or junior?

96. **Data Analysis** Interviewers asked a sample of college students, faculty members, and administrators whether they favored a proposed increase in the annual activity fee to enhance student life on campus. The table lists the results.

	Students	Faculty	Admin.	Total
Favor	237	37	18	292
Oppose	163	38	7	208
Total	400	75	25	500

Find the probability that a person selected at random from the sample is as described.

(a) Not in favor of the proposal

(b) A student

(c) A faculty member in favor of the proposal

97. **Tossing a Die** You toss a six-sided die four times. What is the probability of getting a 5 on each roll?

98. **Tossing a Die** You toss a six-sided die six times. What is the probability of getting each number exactly once?

99. **Drawing a Card** You randomly draw a card from a standard deck of 52 playing cards. What is the probability that the card is not a club?

100. **Tossing a Coin** Find the probability of obtaining at least one tail when you toss a coin five times.

Exploration

True or False? In Exercises 101–104, determine whether the statement is true or false. Justify your answer.

101. $\dfrac{(n+2)!}{n!} = \dfrac{n+2}{n}$

102. $\displaystyle\sum_{i=1}^{5}(i^3 + 2i) = \sum_{i=1}^{5} i^3 + \sum_{i=1}^{5} 2i$

103. $\displaystyle\sum_{k=1}^{8} 3k = 3\sum_{k=1}^{8} k$

104. $\displaystyle\sum_{j=1}^{6} 2^j = \sum_{j=3}^{8} 2^{j-2}$

105. **Think About It** An infinite sequence is a function. What is the domain of the function?

106. **Think About It** How do the two sequences differ?

(a) $a_n = \dfrac{(-1)^n}{n}$ (b) $a_n = \dfrac{(-1)^{n+1}}{n}$

107. **Writing** Explain what is meant by a recursion formula.

108. **Writing** Write a brief paragraph explaining how to identify the graph of an arithmetic sequence and the graph of a geometric sequence.

Chapter Test

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

1. Write the first five terms of the sequence $a_n = \frac{(-1)^n}{3n + 2}$. (Assume that n begins with 1.)

2. Write an expression for the apparent nth term (a_n) of the sequence.

$$\frac{3}{1!}, \frac{4}{2!}, \frac{5}{3!}, \frac{6}{4!}, \frac{7}{5!}, . . .$$

3. Write the next three terms of the series. Then find the sixth partial sum of the series.

$$8 + 21 + 34 + 47 + \cdot \cdot \cdot$$

4. The fifth term of an arithmetic sequence is 5.4, and the 12th term is 11.0. Find the nth term.

5. The second term of a geometric sequence is 28, and the sixth term is 7168. Find the nth term.

6. Write the first five terms of the geometric sequence $a_n = 5(2)^{n-1}$. (Assume that n begins with 1.)

In Exercises 7–9, find the sum.

7. $\displaystyle\sum_{i=1}^{50} (2i^2 + 5)$ **8.** $\displaystyle\sum_{n=1}^{9} (12n - 7)$ **9.** $\displaystyle\sum_{i=1}^{\infty} 4\left(\frac{1}{2}\right)^i$

10. Use mathematical induction to prove the formula for every positive integer n.

$$5 + 10 + 15 + \cdot \cdot \cdot + 5n = \frac{5n(n + 1)}{2}$$

11. Use the Binomial Theorem to expand and simplify (a) $(x + 6y)^4$ and (b) $3(x - 2)^5 + 4(x - 2)^3$.

12. Find the coefficient of the term a^4b^3 in the expansion of $(3a - 2b)^7$.

In Exercises 13 and 14, evaluate each expression.

13. (a) ${}_9P_2$ (b) ${}_{70}P_3$

14. (a) ${}_{11}C_4$ (b) ${}_{66}C_4$

15. How many distinct license plates can be issued consisting of one letter followed by a three-digit number?

16. Eight people are going for a ride in a boat that seats eight people. One person will drive, and only three of the remaining people are willing to ride in the two bow seats. How many seating arrangements are possible?

17. You attend a karaoke night and hope to hear your favorite song. The karaoke song book has 300 different songs (your favorite song is among them). Assuming that the singers are equally likely to pick any song and no song repeats, what is the probability that your favorite song is one of the 20 that you hear that night?

18. You are with three of your friends at a party. Names of all of the 30 guests are placed in a hat and drawn randomly to award four door prizes. Each guest can win only one prize. What is the probability that you and your friends win all four prizes?

19. The weather report calls for a 90% chance of snow. According to this report, what is the probability that it will *not* snow?

Cumulative Test for Chapters 6–8

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

In Exercises 1–4, solve the system by the specified method.

1. Substitution

$$\begin{cases} y = 3 - x^2 \\ 2(y-2) = x - 1 \end{cases}$$

2. Elimination

$$\begin{cases} x + 3y = -6 \\ 2x + 4y = -10 \end{cases}$$

3. Elimination

$$\begin{cases} -2x + 4y - z = -16 \\ x - 2y + 2z = 5 \\ x - 3y - z = 13 \end{cases}$$

4. Gauss-Jordan Elimination

$$\begin{cases} x + 3y - 2z = -7 \\ -2x + y - z = -5 \\ 4x + y + z = 3 \end{cases}$$

5. A custom-blend bird seed is to be mixed from seed mixtures costing \$0.75 per pound and \$1.25 per pound. How many pounds of each seed mixture are used to make 200 pounds of custom-blend bird seed costing \$0.95 per pound?

6. Find the equation of the parabola $y = ax^2 + bx + c$ passing through the points $(0, 6)$, $(2, 3)$, and $(4, 2)$.

In Exercises 7 and 8, sketch the graph (and label the vertices) of the solution set of the system of inequalities.

7. $\begin{cases} 2x + y \geq -3 \\ x - 3y \leq 2 \end{cases}$

8. $\begin{cases} x - y > 6 \\ 5x + 2y < 10 \end{cases}$

9. Sketch the region determined by the constraints. Then find the minimum and maximum values of the objective function $z = 3x + 2y$ and where they occur, subject to the indicated constraints.

$$\begin{aligned} x + 4y &\leq 20 \\ 2x + y &\leq 12 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

$$\begin{cases} -x + 2y - z = 9 \\ 2x - y + 2z = -9 \\ 3x + 3y - 4z = 7 \end{cases}$$

System for 10 and 11

In Exercises 10 and 11, use the system of linear equations shown at the left.

10. Write the augmented matrix for the system.

11. Solve the system using the matrix found in Exercise 10 and Gauss-Jordan elimination.

In Exercises 12–17, perform the operations using the following matrices.

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 5 \\ 0 & -1 \end{bmatrix}$$

12. $A + B$

13. $-8B$

14. $2A - 5B$

15. AB

16. A^2

17. $BA - B^2$

$$\begin{bmatrix} 7 & 1 & 0 \\ -2 & 4 & -1 \\ 3 & 8 & 5 \end{bmatrix}$$

Matrix for 19

18. Find the inverse of the matrix (if it exists): $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -10 \\ -5 & -7 & -15 \end{bmatrix}$.

19. Find the determinant of the matrix shown at the left.

$$\begin{array}{ll|ccc} & & \text{Gym shoes} & \text{Jogging shoes} & \text{Walking shoes} \\ \hline \text{Age group} & 14\text{–}17 & 0.079 & 0.064 & 0.029 \\ & 18\text{–}24 & 0.050 & 0.060 & 0.022 \\ & 25\text{–}34 & 0.103 & 0.259 & 0.085 \end{array}$$

Matrix for 20

20. The percents (in decimal form) of the total amounts spent on three types of footwear in a recent year are shown in the matrix at the left. The total amounts (in millions of dollars) spent by the age groups on the three types of footwear were \$479.88 (14–17 age group), \$365.88 (18–24 age group), and \$1248.89 (25–34 age group). How many dollars worth of gym shoes, jogging shoes, and walking shoes were sold that year? *(Source: National Sporting Goods Association)*

In Exercises 21 and 22, use Cramer's Rule to solve the system of equations.

21. $\begin{cases} 8x - 3y = -52 \\ 3x + 5y = 5 \end{cases}$

22. $\begin{cases} 5x + 4y + 3z = 7 \\ -3x - 8y + 7z = -9 \\ 7x - 5y - 6z = -53 \end{cases}$

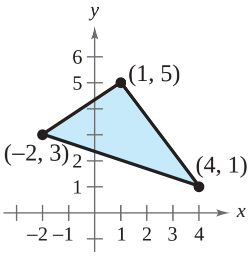


Figure for 23

23. Use a determinant to find the area of the triangle shown at the left.

24. Write the first five terms of the sequence $a_n = \dfrac{(-1)^{n+1}}{2n + 3}$. (Assume that n begins with 1.)

25. Write an expression for the apparent nth term (a_n) of the sequence.

$$\frac{2!}{4}, \frac{3!}{5}, \frac{4!}{6}, \frac{5!}{7}, \frac{6!}{8}, \ldots$$

26. Find the 16th partial sum of the arithmetic sequence 6, 18, 30, 42, ...

27. The sixth term of an arithmetic sequence is 20.6, and the ninth term is 30.2.

(a) Find the 20th term.

(b) Find the nth term.

28. Write the first five terms of the sequence $a_n = 3(2)^{n-1}$. (Assume that n begins with 1.)

29. Find the sum: $\displaystyle\sum_{i=0}^{\infty} 1.3\left(\tfrac{1}{10}\right)^{i-1}$.

30. Use mathematical induction to prove the inequality

$$(n + 1)! > 2^n, \quad n \geq 2.$$

31. Use the Binomial Theorem to expand and simplify $(w - 9)^4$.

In Exercises 32–35, evaluate the expression.

32. ${}_{14}P_3$

33. ${}_{25}P_2$

34. $\dbinom{8}{4}$

35. ${}_{11}C_6$

In Exercises 36 and 37, find the number of distinguishable permutations of the group of letters.

36. B, A, S, K, E, T, B, A, L, L

37. A, N, T, A, R, C, T, I, C, A

38. A personnel manager at a department store has 10 applicants to fill three different sales positions. In how many ways can this be done, assuming that all the applicants are qualified for any of the three positions?

39. On a game show, contestants must arrange the digits 3, 4, and 5 in the proper order to form the price of an appliance. When the contestant arranges the digits correctly, he or she wins the appliance. What is the probability of winning when the contestant knows that the price is at least \$400?

Proofs in Mathematics

Properties of Sums *(p. 568)*

1. $\sum_{i=1}^{n} c = cn, \quad c \text{ is a constant.}$

2. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i, \quad c \text{ is a constant.}$

3. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

4. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

INFINITE SERIES

People considered the study of infinite series a novelty in the fourteenth century. Logician Richard Suiseth, whose nickname was Calculator, solved this problem.

If throughout the first half of a given time interval a variation continues at a certain intensity; throughout the next quarter of the interval at double the intensity; throughout the following eighth at triple the intensity and so ad infinitum; The average intensity for the whole interval will be the intensity of the variation during the second subinterval (or double the intensity).

This is the same as saying that the sum of the infinite series

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdot \cdot \cdot + \frac{n}{2^n} + \cdot \cdot \cdot$$

is 2.

Proof

Each of these properties follows directly from the properties of real numbers.

$$1.\ \sum_{i=1}^{n} c = c + c + c + \cdot \cdot \cdot + c = cn \qquad n \text{ terms}$$

The proof of Property 2 uses the Distributive Property.

$$\begin{aligned} 2.\ \sum_{i=1}^{n} ca_i &= ca_1 + ca_2 + ca_3 + \cdot \cdot \cdot + ca_n \\ &= c(a_1 + a_2 + a_3 + \cdot \cdot \cdot + a_n) \\ &= c\sum_{i=1}^{n} a_i \end{aligned}$$

The proof of Property 3 uses the Commutative and Associative Properties of Addition.

$$\begin{aligned} 3.\ \sum_{i=1}^{n} (a_i + b_i) &= (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \cdot \cdot \cdot + (a_n + b_n) \\ &= (a_1 + a_2 + a_3 + \cdot \cdot \cdot + a_n) + (b_1 + b_2 + b_3 + \cdot \cdot \cdot + b_n) \\ &= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \end{aligned}$$

The proof of Property 4 uses the Commutative and Associative Properties of Addition and the Distributive Property.

$$\begin{aligned} 4.\ \sum_{i=1}^{n} (a_i - b_i) &= (a_1 - b_1) + (a_2 - b_2) + (a_3 - b_3) + \cdot \cdot \cdot + (a_n - b_n) \\ &= (a_1 + a_2 + a_3 + \cdot \cdot \cdot + a_n) + (-b_1 - b_2 - b_3 - \cdot \cdot \cdot - b_n) \\ &= (a_1 + a_2 + a_3 + \cdot \cdot \cdot + a_n) - (b_1 + b_2 + b_3 + \cdot \cdot \cdot + b_n) \\ &= \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \end{aligned}$$

■

The Sum of a Finite Arithmetic Sequence *(p. 577)*

The sum of a finite arithmetic sequence with n terms is

$$S_n = \frac{n}{2}(a_1 + a_n).$$

Proof

Begin by generating the terms of the arithmetic sequence in two ways. In the first way, repeatedly add d to the first term to obtain

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \cdot\cdot\cdot + a_{n-2} + a_{n-1} + a_n \\ &= a_1 + [a_1 + d] + [a_1 + 2d] + \cdot\cdot\cdot + [a_1 + (n-1)d]. \end{aligned}$$

In the second way, repeatedly subtract d from the nth term to obtain

$$\begin{aligned} S_n &= a_n + a_{n-1} + a_{n-2} + \cdot\cdot\cdot + a_3 + a_2 + a_1 \\ &= a_n + [a_n - d] + [a_n - 2d] + \cdot\cdot\cdot + [a_n - (n-1)d]. \end{aligned}$$

If you add these two versions of S_n, then the multiples of d sum to zero and you obtain

$$2S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdot\cdot\cdot + (a_1 + a_n) \qquad n \text{ terms}$$

$$2S_n = n(a_1 + a_n)$$

$$S_n = \frac{n}{2}(a_1 + a_n).$$

■

The Sum of a Finite Geometric Sequence *(p. 586)*

The sum of the finite geometric sequence

$$a_1,\ a_1r,\ a_1r^2,\ a_1r^3,\ a_1r^4,\ .\ .\ .\ ,\ a_1r^{n-1}$$

with common ratio $r \neq 1$ is given by $S_n = \sum_{i=1}^{n} a_1r^{i-1} = a_1\left(\frac{1-r^n}{1-r}\right)$.

Proof

$$S_n = a_1 + a_1r + a_1r^2 + \cdot\cdot\cdot + a_1r^{n-2} + a_1r^{n-1}$$

$$rS_n = a_1r + a_1r^2 + a_1r^3 + \cdot\cdot\cdot + a_1r^{n-1} + a_1r^n \qquad \text{Multiply by } r.$$

Subtracting the second equation from the first yields

$$S_n - rS_n = a_1 - a_1r^n.$$

So, $S_n(1 - r) = a_1(1 - r^n)$, and, because $r \neq 1$, you have $S_n = a_1\left(\frac{1-r^n}{1-r}\right)$. ■

The Binomial Theorem ***(p. 602)***

In the expansion of $(x + y)^n$

$$(x + y)^n = x^n + nx^{n-1}y + \cdots + {}_nC_r\, x^{n-r}y^r + \cdots + nxy^{n-1} + y^n$$

the coefficient of $x^{n-r}y^r$ is

$${}_nC_r = \frac{n!}{(n-r)!r!}.$$

Proof

The Binomial Theorem can be proved quite nicely using mathematical induction. The steps are straightforward but look a little messy, so only an outline of the proof follows.

1. For $n = 1$, you have $(x + y)^1 = x^1 + y^1 = {}_1C_0 x + {}_1C_1 y$, and the formula is valid.
2. Assuming that the formula is true for $n = k$, the coefficient of $x^{k-r}y^r$ is

$${}_kC_r = \frac{k!}{(k-r)!r!} = \frac{k(k-1)(k-2)\cdots(k-r+1)}{r!}.$$

To show that the formula is true for $n = k + 1$, look at the coefficient of $x^{k+1-r}y^r$ in the expansion of

$$(x + y)^{k+1} = (x + y)^k(x + y).$$

On the right-hand side, the term involving $x^{k+1-r}y^r$ is the sum of two products.

$$\begin{aligned}
&({}_kC_r x^{k-r}y^r)(x) + ({}_kC_{r-1}x^{k+1-r}y^{r-1})(y)\\
&\qquad = \left[\frac{k!}{(k-r)!r!} + \frac{k!}{(k+1-r)!(r-1)!}\right]x^{k+1-r}y^r\\
&\qquad = \left[\frac{(k+1-r)k!}{(k+1-r)!r!} + \frac{k!r}{(k+1-r)!r!}\right]x^{k+1-r}y^r\\
&\qquad = \left[\frac{k!(k+1-r+r)}{(k+1-r)!r!}\right]x^{k+1-r}y^r\\
&\qquad = \left[\frac{(k+1)!}{(k+1-r)!r!}\right]x^{k+1-r}y^r\\
&\qquad = {}_{k+1}C_r\, x^{k+1-r}y^r
\end{aligned}$$

So, by mathematical induction, the Binomial Theorem is valid for all positive integers n. ■

P.S. Problem Solving

1. Decreasing Sequence Consider the sequence

$$a_n = \frac{n+1}{n^2+1}.$$

(a) Use a graphing utility to graph the first 10 terms of the sequence.

(b) Use the graph from part (a) to estimate the value of a_n as n approaches infinity.

(c) Complete the table.

n	1	10	100	1000	10,000
a_n					

(d) Use the table from part (c) to determine (if possible) the value of a_n as n approaches infinity.

2. Alternating Sequence Consider the sequence

$$a_n = 3 + (-1)^n.$$

(a) Use a graphing utility to graph the first 10 terms of the sequence.

(b) Use the graph from part (a) to describe the behavior of the graph of the sequence.

(c) Complete the table.

n	1	10	101	1000	10,001
a_n					

(d) Use the table from part (c) to determine (if possible) the value of a_n as n approaches infinity.

3. Greek Mythology Can the Greek hero Achilles, running at 20 feet per second, ever catch a tortoise, starting 20 feet ahead of Achilles and running at 10 feet per second? The Greek mathematician Zeno said no. When Achilles runs 20 feet, the tortoise will be 10 feet ahead. Then, when Achilles runs 10 feet, the tortoise will be 5 feet ahead. Achilles will keep cutting the distance in half but will never catch the tortoise. The table shows Zeno's reasoning. From the table, you can see that both the distances and the times required to achieve them form infinite geometric series. Using the table, show that both series have finite sums. What do these sums represent?

DATA

Distance (in feet)	Time (in seconds)
20	1
10	0.5
5	0.25
2.5	0.125
1.25	0.0625
0.625	0.03125

Spreadsheet at LarsonPrecalculus.com

4. Conjecture Let $x_0 = 1$ and consider the sequence x_n given by

$$x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}, \quad n = 1, 2, \ldots$$

Use a graphing utility to compute the first 10 terms of the sequence and make a conjecture about the value of x_n as n approaches infinity.

5. Operations on an Arithmetic Sequence The following operations are performed on each term of an arithmetic sequence. Determine whether the resulting sequence is arithmetic. If so, then state the common difference.

(a) A constant C is added to each term.

(b) Each term is multiplied by a nonzero constant C.

(c) Each term is squared.

6. Sequences of Powers The following sequence of perfect squares is not arithmetic.

$$1, 4, 9, 16, 25, 36, 49, 64, 81, \ldots$$

However, you can form a related sequence that is arithmetic by finding the differences of consecutive terms.

(a) Write the first eight terms of the related arithmetic sequence described above. What is the nth term of this sequence?

(b) Describe how you can find an arithmetic sequence that is related to the following sequence of perfect cubes.

$$1, 8, 27, 64, 125, 216, 343, 512, 729, \ldots$$

(c) Write the first seven terms of the related sequence in part (b) and find the nth term of the sequence.

(d) Describe how you can find the arithmetic sequence that is related to the following sequence of perfect fourth powers.

$$1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, \ldots$$

(e) Write the first six terms of the related sequence in part (d) and find the nth term of the sequence.

7. Piecewise-Defined Sequence You can define a sequence using a piecewise formula. The following is an example of a piecewise-defined sequence.

$$a_1 = 7, \; a_n = \begin{cases} \dfrac{a_{n-1}}{2}, & \text{when } a_{n-1} \text{ is even.} \\ 3a_{n-1} + 1, & \text{when } a_{n-1} \text{ is odd.} \end{cases}$$

(a) Write the first 20 terms of the sequence.

(b) Write the first 10 terms of the sequences for which $a_1 = 4$, $a_1 = 5$, and $a_1 = 12$ (using a_n as defined above). What conclusion can you make about the behavior of each sequence?

8. **Fibonacci Sequence** Let $f_1, f_2, \ldots, f_n, \ldots$ be the Fibonacci sequence.

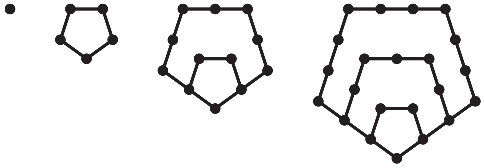
(a) Use mathematical induction to prove that

$$f_1 + f_2 + \cdot \cdot \cdot + f_n = f_{n+2} - 1.$$

(b) Find the sum of the first 20 terms of the Fibonacci sequence.

9. **Pentagonal Numbers** The numbers 1, 5, 12, 22, 35, 51, . . . are called pentagonal numbers because they represent the numbers of dots used to make pentagons, as shown below. Use mathematical induction to prove that the nth pentagonal number P_n is given by

$$P_n = \frac{n(3n - 1)}{2}.$$



10. **Think About It** What conclusion can be drawn from the following information about the sequence of statements P_n?

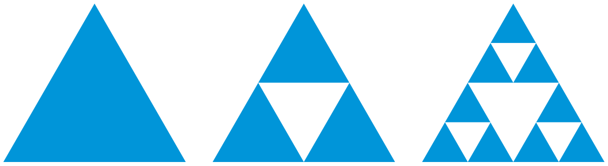
(a) P_3 is true and P_k implies P_{k+1}.

(b) $P_1, P_2, P_3, \ldots, P_{50}$ are all true.

(c) P_1, P_2, and P_3 are all true, but the truth of P_k does not imply that P_{k+1} is true.

(d) P_2 is true and P_{2k} implies P_{2k+2}.

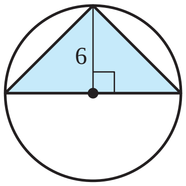
11. **Sierpinski Triangle** Recall that a *fractal* is a geometric figure that consists of a pattern that is repeated infinitely on a smaller and smaller scale. A well-known fractal is called the *Sierpinski Triangle*. In the first stage, the midpoints of the three sides are used to create the vertices of a new triangle, which is then removed, leaving three triangles. The figure below shows the first two stages. Note that each remaining triangle is similar to the original triangle. Assume that the length of each side of the original triangle is one unit. Write a formula that describes the side length of the triangles generated in the nth stage. Write a formula for the area of the triangles generated in the nth stage.



12. **Job Offer** You work for a company that pays \$0.01 the first day, \$0.02 the second day, \$0.04 the third day, and so on. If the daily wage keeps doubling, then what would your total income be for working 30 days?

13. **Multiple Choice** A multiple choice question has five possible answers. You know that the answer is not B or D, but you are not sure about answers A, C, and E. What is the probability that you will get the right answer when you take a guess?

14. **Throwing a Dart** You throw a dart at the circular target shown below. The dart is equally likely to hit any point inside the target. What is the probability that it hits the region outside the triangle?



15. **Odds** The odds in favor of an event occurring are the ratio of the probability that the event will occur to the probability that the event will not occur. The reciprocal of this ratio represents the odds against the event occurring.

(a) Six of the marbles in a bag are red. The odds against choosing a red marble are 4 to 1. How many marbles are in the bag?

(b) A bag contains three blue marbles and seven yellow marbles. What are the odds in favor of choosing a blue marble? What are the odds against choosing a blue marble?

(c) Write a formula for converting the odds in favor of an event to the probability of the event.

(d) Write a formula for converting the probability of an event to the odds in favor of the event.

16. **Expected Value** An event A has n possible outcomes, which have the values $x_1, x_2, \ldots, x_n$. The probabilities of the n outcomes occurring are $p_1, p_2, \ldots, p_n$. The **expected value** V of an event A is the sum of the products of the outcomes' probabilities and their values,

$$V = p_1x_1 + p_2x_2 + \cdot \cdot \cdot + p_nx_n.$$

(a) To win California's Super Lotto Plus game, you must match five different numbers chosen from the numbers 1 to 47, plus one MEGA number chosen from the numbers 1 to 27. You purchase a ticket for \$1. If the jackpot for the next drawing is \$12,000,000, then what is the expected value of the ticket?

(b) You are playing a dice game in which you need to score 60 points to win. On each turn, you toss two six-sided dice. Your score for the turn is 0 when the dice do not show the same number. Your score for the turn is the product of the numbers on the dice when they do show the same number. What is the expected value of each turn? How many turns will it take on average to score 60 points?

Appendix A Errors and the Algebra of Calculus

- **Avoid common algebraic errors.**
- **Recognize and use algebraic techniques that are common in calculus.**

Algebraic Errors to Avoid

This section contains five lists of common algebraic errors: errors involving parentheses, errors involving fractions, errors involving exponents, errors involving radicals, and errors involving dividing out. Many of these errors are made because they seem to be the *easiest* things to do. For instance, the operations of subtraction and division are often believed to be commutative and associative. The following examples illustrate the fact that subtraction and division are neither commutative nor associative.

Not commutative	**Not associative**
$4 - 3 \neq 3 - 4$	$8 - (6 - 2) \neq (8 - 6) - 2$
$15 \div 5 \neq 5 \div 15$	$20 \div (4 \div 2) \neq (20 \div 4) \div 2$

Errors Involving Parentheses

Potential Error	**Correct Form**	**Comment**
$\xcancel{a - (x - b) = a - x - b}$	$a - (x - b) = a - x + b$	Distribute to each term in parentheses.
$\xcancel{(a + b)^2 = a^2 + b^2}$	$(a + b)^2 = a^2 + 2ab + b^2$	Remember the middle term when squaring binomials.
$\xcancel{\left(\frac{1}{2}a\right)\left(\frac{1}{2}b\right) = \frac{1}{2}(ab)}$	$\left(\frac{1}{2}a\right)\left(\frac{1}{2}b\right) = \frac{1}{4}(ab) = \frac{ab}{4}$	$\frac{1}{2}$ occurs twice as a factor.
$\xcancel{(3x + 6)^2 = 3(x + 2)^2}$	$(3x + 6)^2 = [3(x + 2)]^2$ $= 3^2(x + 2)^2$	When factoring, raise all factors to the power.

Errors Involving Fractions

Potential Error	**Correct Form**	**Comment**
$\xcancel{\frac{2}{x + 4} = \frac{2}{x} + \frac{2}{4}}$	Leave as $\frac{2}{x + 4}$.	The fraction is already in simplest form.
$\xcancel{\frac{\left(\frac{x}{a}\right)}{b} = \frac{bx}{a}}$	$\frac{\left(\frac{x}{a}\right)}{b} = \left(\frac{x}{a}\right)\left(\frac{1}{b}\right) = \frac{x}{ab}$	Multiply by the reciprocal when dividing fractions.
$\xcancel{\frac{1}{a} + \frac{1}{b} = \frac{1}{a + b}}$	$\frac{1}{a} + \frac{1}{b} = \frac{b + a}{ab}$	Use the property for adding fractions.
$\xcancel{\frac{1}{3x} = \frac{1}{3}x}$	$\frac{1}{3x} = \frac{1}{3} \cdot \frac{1}{x}$	Use the property for multiplying fractions.
$\xcancel{(1/3)x = \frac{1}{3x}}$	$(1/3)x = \frac{1}{3} \cdot x = \frac{x}{3}$	Be careful when expressing fractions in the form $1/a$.
$\xcancel{(1/x) + 2 = \frac{1}{x + 2}}$	$(1/x) + 2 = \frac{1}{x} + 2 = \frac{1 + 2x}{x}$	Be careful when expressing fractions in the form $1/a$. Be sure to find a common denominator before adding fractions.

Errors Involving Exponents

Potential Error	Correct Form	Comment
$\xcancel{(x^2)^3 = x^5}$	$(x^2)^3 = x^{2 \cdot 3} = x^6$	Multiply exponents when raising a power to a power.
$\xcancel{x^2 \cdot x^3 = x^6}$	$x^2 \cdot x^3 = x^{2+3} = x^5$	Add exponents when multiplying powers with like bases.
$\xcancel{(2x)^3 = 2x^3}$	$(2x)^3 = 2^3x^3 = 8x^3$	Raise each factor to the power.
$\xcancel{\dfrac{1}{x^2 - x^3} = x^{-2} - x^{-3}}$	Leave as $\dfrac{1}{x^2 - x^3}$.	Do not move term-by-term from denominator to numerator.

Errors Involving Radicals

Potential Error	Correct Form	Comment
$\xcancel{\sqrt{5x} = 5\sqrt{x}}$	$\sqrt{5x} = \sqrt{5}\sqrt{x}$	Radicals apply to every factor inside the radical.
$\xcancel{\sqrt{x^2 + a^2} = x + a}$	Leave as $\sqrt{x^2 + a^2}$.	Do not apply radicals term-by-term when adding or subtracting terms.
$\xcancel{\sqrt{-x + a} = -\sqrt{x - a}}$	Leave as $\sqrt{-x + a}$.	Do not factor minus signs out of square roots.

Errors Involving Dividing Out

Potential Error	Correct Form	Comment
$\xcancel{\dfrac{a + bx}{a} = 1 + bx}$	$\dfrac{a + bx}{a} = \dfrac{a}{a} + \dfrac{bx}{a} = 1 + \dfrac{b}{a}x$	Divide out common factors, not common terms.
$\xcancel{\dfrac{a + ax}{a} = a + x}$	$\dfrac{a + ax}{a} = \dfrac{a(1 + x)}{a} = 1 + x$	Factor before dividing out common factors.
$\xcancel{1 + \dfrac{x}{2x} = 1 + \dfrac{1}{x}}$	$1 + \dfrac{x}{2x} = 1 + \dfrac{1}{2} = \dfrac{3}{2}$	Divide out common factors.

A good way to avoid errors is to *work slowly, write neatly,* and *talk to yourself.* Each time you write a step, ask yourself why the step is algebraically legitimate. You can justify the step below because *dividing the numerator and denominator by the same nonzero number produces an equivalent fraction.*

$$\frac{2x}{6} = \frac{\cancel{2} \cdot x}{\cancel{2} \cdot 3} = \frac{x}{3}$$

EXAMPLE 1 Describing and Correcting an Error

Describe and correct the error. $\xcancel{\dfrac{1}{2x} + \dfrac{1}{3x} = \dfrac{1}{5x}}$

Solution Use the property for adding fractions: $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{b + a}{ab}$.

$$\frac{1}{2x} + \frac{1}{3x} = \frac{3x + 2x}{6x^2} = \frac{5x}{6x^2} = \frac{5}{6x}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Describe and correct the error. $\xcancel{\sqrt{x^2 + 4} = x + 2}$

Some Algebra of Calculus

In calculus it is often necessary to take a simplified algebraic expression and rewrite it. See the following lists, taken from a standard calculus text.

Unusual Factoring

Expression	Useful Calculus Form	Comment
$\frac{5x^4}{8}$	$\frac{5}{8}x^4$	Write with fractional coefficient.
$\frac{x^2 + 3x}{-6}$	$-\frac{1}{6}(x^2 + 3x)$	Write with fractional coefficient.
$2x^2 - x - 3$	$2\left(x^2 - \frac{x}{2} - \frac{3}{2}\right)$	Factor out the leading coefficient.
$\frac{x}{2}(x + 1)^{-1/2} + (x + 1)^{1/2}$	$\frac{(x + 1)^{-1/2}}{2}[x + 2(x + 1)]$	Factor out the variable expression with the lesser exponent.

Writing with Negative Exponents

Expression	Useful Calculus Form	Comment
$\frac{9}{5x^3}$	$\frac{9}{5}x^{-3}$	Move the factor to the numerator and change the sign of the exponent.
$\frac{7}{\sqrt{2x - 3}}$	$7(2x - 3)^{-1/2}$	Move the factor to the numerator and change the sign of the exponent.

Writing a Fraction as a Sum

Expression	Useful Calculus Form	Comment
$\frac{x + 2x^2 + 1}{\sqrt{x}}$	$x^{1/2} + 2x^{3/2} + x^{-1/2}$	Divide each term of the numerator by $x^{1/2}$.
$\frac{1 + x}{x^2 + 1}$	$\frac{1}{x^2 + 1} + \frac{x}{x^2 + 1}$	Rewrite the fraction as a sum of fractions.
$\frac{2x}{x^2 + 2x + 1}$	$\frac{2x + 2 - 2}{x^2 + 2x + 1}$	Add and subtract the same term.
	$= \frac{2x + 2}{x^2 + 2x + 1} - \frac{2}{(x + 1)^2}$	Rewrite the fraction as a difference of fractions.
$\frac{x^2 - 2}{x + 1}$	$x - 1 - \frac{1}{x + 1}$	Use long division. (See Section 3.3.)
$\frac{x + 7}{x^2 - x - 6}$	$\frac{2}{x - 3} - \frac{1}{x + 2}$	Use the method of partial fractions. (See Section 6.4.)

Inserting Factors and Terms

Expression	**Useful Calculus Form**	**Comment**
$(2x-1)^3$	$\frac{1}{2}(2x-1)^3(2)$	Multiply and divide by 2.
$7x^2(4x^3-5)^{1/2}$	$\frac{7}{12}(4x^3-5)^{1/2}(12x^2)$	Multiply and divide by 12.
$\frac{4x^2}{9}-4y^2=1$	$\frac{x^2}{9/4}-\frac{y^2}{1/4}=1$	Write with fractional denominators.
$\frac{x}{x+1}$	$\frac{x+1-1}{x+1}=1-\frac{1}{x+1}$	Add and subtract the same term.

The next five examples demonstrate many of the steps in the preceding lists.

EXAMPLE 2 Factors Involving Negative Exponents

Factor $x(x+1)^{-1/2}+(x+1)^{1/2}$.

Solution When multiplying powers with like bases, you add exponents. When factoring, you are undoing multiplication, and so you *subtract* exponents.

$$\begin{aligned} x(x+1)^{-1/2}+(x+1)^{1/2} &= (x+1)^{-1/2}[x(x+1)^0+(x+1)^1] \\ &= (x+1)^{-1/2}[x+(x+1)] \\ &= (x+1)^{-1/2}(2x+1) \end{aligned}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Factor $x(x-2)^{-1/2}+3(x-2)^{1/2}$.

Another way to simplify the expression in Example 2 is to multiply the expression by a fractional form of 1 and then use the Distributive Property.

$$\begin{aligned} x(x+1)^{-1/2}+(x+1)^{1/2} &= [x(x+1)^{-1/2}+(x+1)^{1/2}]\cdot\frac{(x+1)^{1/2}}{(x+1)^{1/2}} \\ &= \frac{x(x+1)^0+(x+1)^1}{(x+1)^{1/2}}=\frac{2x+1}{\sqrt{x+1}} \end{aligned}$$

EXAMPLE 3 Inserting Factors in an Expression

Insert the required factor: $\frac{x+2}{(x^2+4x-3)^2}=(\quad)\frac{1}{(x^2+4x-3)^2}(2x+4)$.

Solution The expression on the right side of the equation is twice the expression on the left side. To make both sides equal, insert a factor of $\frac{1}{2}$.

$$\frac{x+2}{(x^2+4x-3)^2}=\left(\frac{1}{2}\right)\frac{1}{(x^2+4x-3)^2}(2x+4)$$

Right side is multiplied and divided by 2.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Insert the required factor: $\frac{6x-3}{(x^2-x+4)^2}=(\quad)\frac{1}{(x^2-x+4)^2}(2x-1)$.

EXAMPLE 4 Rewriting Fractions

Explain why the two expressions are equivalent.

$$\frac{4x^2}{9} - 4y^2 = \frac{x^2}{9/4} - \frac{y^2}{1/4}$$

Solution To write the expression on the left side of the equation in the form given on the right side, multiply the numerator and denominator of each term by $\frac{1}{4}$.

$$\frac{4x^2}{9} - 4y^2 = \frac{4x^2}{9}\left(\frac{1/4}{1/4}\right) - 4y^2\left(\frac{1/4}{1/4}\right) = \frac{x^2}{9/4} - \frac{y^2}{1/4}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Explain why the two expressions are equivalent.

$$\frac{9x^2}{16} + 25y^2 = \frac{x^2}{16/9} + \frac{y^2}{1/25}$$

EXAMPLE 5 Rewriting with Negative Exponents

Rewrite each expression using negative exponents.

a. $\dfrac{-4x}{(1-2x^2)^2}$ **b.** $\dfrac{2}{5x^3} - \dfrac{1}{\sqrt{x}} + \dfrac{3}{5(4x)^2}$

Solution

a. $\dfrac{-4x}{(1-2x^2)^2} = -4x(1-2x^2)^{-2}$

b.
$$\frac{2}{5x^3} - \frac{1}{\sqrt{x}} + \frac{3}{5(4x)^2} = \frac{2}{5x^3} - \frac{1}{x^{1/2}} + \frac{3}{5(4x)^2}$$
$$= \frac{2}{5}x^{-3} - x^{-1/2} + \frac{3}{5}(4x)^{-2}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Rewrite $\dfrac{-6x}{(1-3x^2)^2} + \dfrac{1}{\sqrt[3]{x}}$ using negative exponents.

EXAMPLE 6 Rewriting a Fraction as a Sum of Terms

Rewrite each fraction as the sum of three terms.

a. $\dfrac{x^2 - 4x + 8}{2x}$ **b.** $\dfrac{x + 2x^2 + 1}{\sqrt{x}}$

Solution

a.
$$\frac{x^2 - 4x + 8}{2x} = \frac{x^2}{2x} - \frac{4x}{2x} + \frac{8}{2x}$$
$$= \frac{x}{2} - 2 + \frac{4}{x}$$

b.
$$\frac{x + 2x^2 + 1}{\sqrt{x}} = \frac{x}{x^{1/2}} + \frac{2x^2}{x^{1/2}} + \frac{1}{x^{1/2}}$$
$$= x^{1/2} + 2x^{3/2} + x^{-1/2}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Rewrite $\dfrac{x^4 - 2x^3 + 5}{x^3}$ as the sum of three terms.

A Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. To rewrite the expression $\dfrac{3}{x^5}$ using negative exponents, move x^5 to the ________ and change the sign of the exponent.

2. When dividing fractions, multiply by the ________.

Skills and Applications

Describing and Correcting an Error

In Exercises 3–22, describe and correct the error.

3. $\xcancel{2x-(3y+4)=2x-3y+4}$

4. $\xcancel{5z+3(x-2)=5z+3x-2}$

5. $\xcancel{\dfrac{4}{16x-(2x+1)}=\dfrac{4}{14x+1}}$

6. $\xcancel{\dfrac{1-x}{(5-x)(-x)}=\dfrac{x-1}{x(x-5)}}$

7. $\xcancel{(5z)(6z)=30z}$

8. $\xcancel{x(yz)=(xy)(xz)}$

9. $\xcancel{a\left(\dfrac{x}{y}\right)=\dfrac{ax}{ay}}$

10. $\xcancel{(4x)^2=4x^2}$

11. $\xcancel{\sqrt{x+9}=\sqrt{x}+3}$

12. $\xcancel{\sqrt{25-x^2}=5-x}$

13. $\xcancel{\dfrac{2x^2+1}{5x}=\dfrac{2x+1}{5}}$

14. $\xcancel{\dfrac{6x+y}{6x-y}=\dfrac{x+y}{x-y}}$

15. $\xcancel{\dfrac{1}{a^{-1}+b^{-1}}=\left(\dfrac{1}{a+b}\right)^{-1}}$

16. $\xcancel{\dfrac{1}{x+y^{-1}}=\dfrac{y}{x+1}}$

17. $\xcancel{(x^2+5x)^{1/2}=x(x+5)^{1/2}}$

18. $\xcancel{x(2x-1)^2=(2x^2-x)^2}$

19. $\xcancel{\dfrac{3}{x}+\dfrac{4}{y}=\dfrac{7}{x+y}}$

20. $\xcancel{\dfrac{1}{2y}=(1/2)y}$

21. $\xcancel{\dfrac{x}{2y}+\dfrac{y}{3}=\dfrac{x+y}{2y+3}}$

22. $\xcancel{5+(1/y)=\dfrac{1}{5+y}}$

Inserting Factors in an Expression

In Exercises 23–44, insert the required factor in the parentheses.

23. $\dfrac{5x+3}{4}=\dfrac{1}{4}\left(\boxed{}\right)$

24. $\dfrac{7x^2}{10}=\dfrac{7}{10}\left(\boxed{}\right)$

25. $\frac{2}{3}x^2+\frac{1}{3}x+5=\frac{1}{3}\left(\boxed{}\right)$

26. $\frac{3}{4}x+\frac{1}{2}=\frac{1}{4}\left(\boxed{}\right)$

27. $x^2(x^3-1)^4=\left(\boxed{}\right)(x^3-1)^4(3x^2)$

28. $x(1-2x^2)^3=\left(\boxed{}\right)(1-2x^2)^3(-4x)$

29. $2(y-5)^{1/2}+y(y-5)^{-1/2}=(y-5)^{-1/2}\left(\boxed{}\right)$

30. $3t(6t+1)^{-1/2}+(6t+1)^{1/2}=(6t+1)^{-1/2}\left(\boxed{}\right)$

31. $\dfrac{4x+6}{(x^2+3x+7)^3}=\left(\boxed{}\right)\dfrac{1}{(x^2+3x+7)^3}(2x+3)$

32. $\dfrac{x+1}{(x^2+2x-3)^2}=\left(\boxed{}\right)\dfrac{1}{(x^2+2x-3)^2}(2x+2)$

33. $\dfrac{3}{x}+\dfrac{5}{2x^2}-\dfrac{3}{2}x=\left(\boxed{}\right)(6x+5-3x^3)$

34. $\dfrac{(x-1)^2}{169}+(y+5)^2=\dfrac{(x-1)^3}{169\left(\boxed{}\right)}+(y+5)^2$

35. $\dfrac{25x^2}{36}+\dfrac{4y^2}{9}=\dfrac{x^2}{\left(\boxed{}\right)}+\dfrac{y^2}{\left(\boxed{}\right)}$

36. $\dfrac{5x^2}{9}-\dfrac{16y^2}{49}=\dfrac{x^2}{\left(\boxed{}\right)}-\dfrac{y^2}{\left(\boxed{}\right)}$

37. $\dfrac{x^2}{3/10}-\dfrac{y^2}{4/5}=\dfrac{10x^2}{\left(\boxed{}\right)}-\dfrac{5y^2}{\left(\boxed{}\right)}$

38. $\dfrac{x^2}{5/8}+\dfrac{y^2}{6/11}=\dfrac{8x^2}{\left(\boxed{}\right)}+\dfrac{11y^2}{\left(\boxed{}\right)}$

39. $x^{1/3}-5x^{4/3}=x^{1/3}\left(\boxed{}\right)$

40. $3(2x+1)x^{1/2}+4x^{3/2}=x^{1/2}\left(\boxed{}\right)$

41. $(1-3x)^{4/3}-4x(1-3x)^{1/3}=(1-3x)^{1/3}\left(\boxed{}\right)$

42. $\dfrac{1}{2\sqrt{x}}+5x^{3/2}-10x^{5/2}=\dfrac{1}{2\sqrt{x}}\left(\boxed{}\right)$

43. $\dfrac{1}{10}(2x+1)^{5/2}-\dfrac{1}{6}(2x+1)^{3/2}=\dfrac{(2x+1)^{3/2}}{15}\left(\boxed{}\right)$

44. $\dfrac{3}{7}(t+1)^{7/3}-\dfrac{3}{4}(t+1)^{4/3}=\dfrac{3(t+1)^{4/3}}{28}\left(\boxed{}\right)$

Rewriting with Negative Exponents

In Exercises 45–50, rewrite the expression using negative exponents.

45. $\dfrac{7}{(x+3)^5}$

46. $\dfrac{2-x}{(x+1)^{3/2}}$

47. $\dfrac{2x^5}{(3x+5)^4}$

48. $\dfrac{x+1}{x(6-x)^{1/2}}$

49. $\dfrac{4}{3x}+\dfrac{4}{x^4}-\dfrac{7x}{\sqrt[3]{2x}}$

50. $\dfrac{x}{x-2}+\dfrac{1}{x^2}+\dfrac{8}{3(9x)^3}$

Rewriting a Fraction as a Sum of Terms

In Exercises 51–56, rewrite the fraction as a sum of two or more terms.

51. $\dfrac{x^2+6x+12}{3x}$

52. $\dfrac{x^3-5x^2+4}{x^2}$

53. $\dfrac{4x^3-7x^2+1}{x^{1/3}}$

54. $\dfrac{2x^5-3x^3+5x-1}{x^{3/2}}$

55. $\dfrac{3-5x^2-x^4}{\sqrt{x}}$

56. $\dfrac{x^3-5x^4}{3x^2}$

Simplifying an Expression **In Exercises 57–68, simplify the expression.**

57. $\dfrac{-2(x^2-3)^{-3}(2x)(x+1)^3 - 3(x+1)^2(x^2-3)^{-2}}{[(x+1)^3]^2}$

58. $\dfrac{x^5(-3)(x^2+1)^{-4}(2x) - (x^2+1)^{-3}(5)x^4}{(x^5)^2}$

59. $\dfrac{(6x+1)^3(27x^2+2) - (9x^3+2x)(3)(6x+1)^2(6)}{[(6x+1)^3]^2}$

60. $\dfrac{(4x^2+9)^{1/2}(2) - (2x+3)\left(\frac{1}{2}\right)(4x^2+9)^{-1/2}(8x)}{[(4x^2+9)^{1/2}]^2}$

61. $\dfrac{(x+2)^{3/4}(x+3)^{-2/3} - (x+3)^{1/3}(x+2)^{-1/4}}{[(x+2)^{3/4}]^2}$

62. $(2x-1)^{1/2} - (x+2)(2x-1)^{-1/2}$

63. $\dfrac{2(3x-1)^{1/3} - (2x+1)\left(\frac{1}{3}\right)(3x-1)^{-2/3}(3)}{(3x-1)^{2/3}}$

64. $\dfrac{(x+1)\left(\frac{1}{2}\right)(2x-3x^2)^{-1/2}(2-6x) - (2x-3x^2)^{1/2}}{(x+1)^2}$

65. $\dfrac{1}{(x^2+4)^{1/2}} \cdot \dfrac{1}{2}(x^2+4)^{-1/2}(2x)$

66. $\dfrac{1}{x^2-6}(2x) + \dfrac{1}{2x+5}(2)$

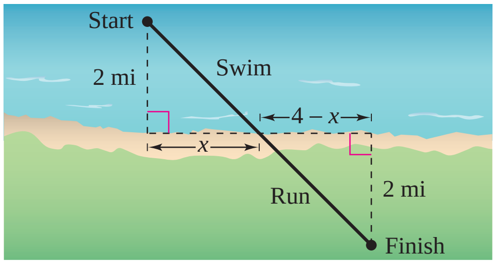
67. $(x^2+5)^{1/2}\left(\frac{3}{2}\right)(3x-2)^{1/2}(3)$
$+ (3x-2)^{3/2}\left(\frac{1}{2}\right)(x^2+5)^{-1/2}(2x)$

68. $(3x+2)^{-1/2}(3)(x-6)^{1/2}(1)$
$+ (x-6)^3\left(-\frac{1}{2}\right)(3x+2)^{-3/2}(3)$

69. Athletics An athlete has set up a course for training as part of her regimen in preparation for an upcoming triathlon. She is dropped off by a boat 2 miles from the nearest point on shore. The finish line is 4 miles down the coast and 2 miles inland (see figure). She can swim 2 miles per hour and run 6 miles per hour. The time t (in hours) required for her to reach the finish line can be approximated by the model

$$t = \frac{\sqrt{x^2+4}}{2} + \frac{\sqrt{(4-x)^2+4}}{6}$$

where x is the distance down the coast (in miles) to the point at which she swims and then leaves the water to start her run.



(a) Find the times required for the triathlete to finish when she swims to the points $x = 0.5$, $x = 1.0, \ldots, x = 3.5$, and $x = 4.0$ miles down the coast.

(b) Use your results from part (a) to determine the distance down the coast that will yield the minimum amount of time required for the triathlete to reach the finish line.

(c) The expression below was obtained using calculus. It can be used to find the minimum amount of time required for the triathlete to reach the finish line. Simplify the expression.

$$\tfrac{1}{2}x(x^2+4)^{-1/2} + \tfrac{1}{6}(x-4)(x^2-8x+20)^{-1/2}$$

70. Verifying an Equation

(a) Verify that $y_1 = y_2$ analytically.

$$y_1 = x^2\left(\frac{1}{3}\right)(x^2+1)^{-2/3}(2x) + (x^2+1)^{1/3}(2x)$$

$$y_2 = \frac{2x(4x^2+3)}{3(x^2+1)^{2/3}}$$

(b) Complete the table and demonstrate the equality in part (a) numerically.

x	-2	-1	$-\frac{1}{2}$	0	1	2	$\frac{5}{2}$
y_1							
y_2							

Exploration

71. Writing Write a paragraph explaining to a classmate why $\dfrac{1}{(x-2)^{1/2}+x^4} \neq (x-2)^{-1/2} + x^{-4}$.

72. Think About It You are taking a course in calculus, and for one of the homework problems you obtain the following answer.

$$\frac{1}{10}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{3/2}$$

The answer in the back of the book is

$$\frac{1}{15}(2x-1)^{3/2}(3x+1).$$

Show how the second answer can be obtained from the first. Then use the same technique to simplify each of the following expressions.

(a) $\dfrac{2}{3}x(2x-3)^{3/2} - \dfrac{2}{15}(2x-3)^{5/2}$

(b) $\dfrac{2}{3}x(4+x)^{3/2} - \dfrac{2}{15}(4+x)^{5/2}$

Answers to Odd-Numbered Exercises and Tests

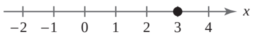
Chapter P

Section P.1 *(page 12)*

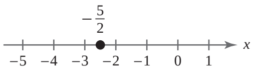
1. irrational **3.** absolute value **5.** terms

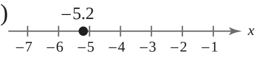
7. (a) 5, 1, 2 (b) 0, 5, 1, 2 (c) $-9, 5, 0, 1, -4, 2, -11$
(d) $-\frac{7}{2}, \frac{2}{3}, -9, 5, 0, 1, -4, 2, -11$ (e) $\sqrt{2}$

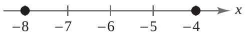
9. (a) 1 (b) 1 (c) $-13, 1, -6$
(d) $2.01, -13, 1, -6, 0.666 . . .$ (e) $0.010110111 . . .$

11. (a)


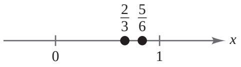
(b)


(c)


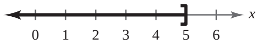
(d)


13.


$-4 > -8$

15.


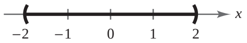
$\frac{5}{6} > \frac{2}{3}$

17. (a) $x \leq 5$ denotes the set of all real numbers less than or equal to 5.
(b)


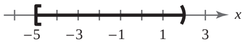
(c) Unbounded

19. (a) $[4, \infty)$ denotes the set of all real numbers greater than or equal to 4.
(b)


(c) Unbounded

21. (a) $-2 < x < 2$ denotes the set of all real numbers greater than -2 and less than 2.
(b)


(c) Bounded

23. (a) $[-5, 2)$ denotes the set of all real numbers greater than or equal to -5 and less than 2.
(b)


(c) Bounded

	Inequality	**Interval**
25.	$y \geq 0$	$[0, \infty)$
27.	$10 \leq t \leq 22$	$[10, 22]$
29.	$W > 65$	$(65, \infty)$

31. 10 **33.** 5 **35.** -1 **37.** -1 **39.** -1

41. $|-4| = |4|$ **43.** $-|-6| < |-6|$ **45.** 51 **47.** $\frac{5}{2}$

49. $\frac{128}{75}$ **51.** $|x - 5| \leq 3$ **53.** $|y - a| \leq 2$

55. \$1880.1 billion; \$412.7 billion

57. \$2524.0 billion; \$458.5 billion

59. $7x$ and 4 are the terms; 7 is the coefficient.

61. $4x^3$, $0.5x$, and -5 are the terms; 4 and 0.5 are the coefficients.

63. (a) -10 (b) -6 **65.** (a) -10 (b) 0

67. Multiplicative Inverse Property

69. Distributive Property

71. Associative and Commutative Properties of Multiplication

73. $\dfrac{3}{8}$ **75.** $\dfrac{5x}{12}$

77. (a) Negative (b) Negative (c) Positive (d) Positive

79. False. Zero is nonnegative, but not positive.

81. (a)

n	0.0001	0.01	1	100	10,000
$\frac{5}{n}$	50,000	500	5	0.05	0.0005

(b) (i) The value of $5/n$ approaches infinity as n approaches 0.
(ii) The value of $5/n$ approaches 0 as n increases without bound.

Section P.2 *(page 24)*

1. exponent; base **3.** square root **5.** like radicals

7. rationalizing **9.** (a) 81 (b) $\frac{1}{9}$

11. (a) 5184 (b) $-\frac{3}{5}$ **13.** (a) $\frac{16}{3}$ (b) 1

15. -24 **17.** 6 **19.** -48 **21.** (a) $-125z^3$ (b) $5x^6$

23. (a) $24y^2$ (b) $-3z^7$ **25.** (a) $\dfrac{5184}{y^7}$ (b) 1

27. (a) 1 (b) $\dfrac{1}{4x^4}$ **29.** (a) $\dfrac{125x^9}{y^{12}}$ (b) $\dfrac{b^5}{a^5}$

31. 1.02504×10^4 **33.** 3.937×10^{-5} in. **35.** 0.000314

37. 9,460,000,000,000 km **39.** (a) 6.8×10^5 (b) 6.0×10^4

41. (a) 3 (b) $\frac{3}{2}$ **43.** (a) 2 (b) $2x$

45. (a) $2\sqrt{5}$ (b) $4\sqrt[3]{2}$ **47.** (a) $6x\sqrt{2x}$ (b) $3y^2\sqrt{6x}$

49. (a) $2x\sqrt[3]{2x^2}$ (b) $\dfrac{5|x|\sqrt{3}}{y^2}$ **51.** (a) $22\sqrt{2}$ (b) $11\sqrt[3]{2}$

53. (a) $23|x|\sqrt{3}$ (b) $18\sqrt{5x}$ **55.** $\dfrac{\sqrt{3}}{3}$ **57.** $\dfrac{\sqrt{14} + 2}{2}$

59. $\dfrac{2}{\sqrt{2}}$ **61.** $\dfrac{2}{3(\sqrt{5} - \sqrt{3})}$ **63.** $64^{1/3}$ **65.** $\dfrac{3}{\sqrt[3]{x^2}}$

67. $(3x^3y)^{1/2}$ **69.** (a) $\dfrac{1}{8}$ (b) $\dfrac{27}{8}$ **71.** (a) $\dfrac{2}{|x|}$ (b) $xy^{1/3}$

73. (a) $\sqrt{3}$ (b) $\sqrt[3]{(x + 1)^2}$ **75.** (a) $2\sqrt[4]{2}$ (b) $\sqrt[8]{2x}$

77. (a) $x - 1$ (b) $\dfrac{1}{x - 1}$

79. (a)

h	0	1	2	3	4	5	6
t	0	2.93	5.48	7.67	9.53	11.08	12.32

h	7	8	9	10	11	12
t	13.29	14.00	14.50	14.80	14.93	14.96

(b) $t \to 8.64\sqrt{3} \approx 14.96$

81. False. When $x = 0$, the expressions are not equal.

83. False. For instance, $(3 + 5)^2 = 8^2 = 64 \neq 34 = 3^2 + 5^2$.

Section P.3 *(page 31)*

1. n; a_n; a_0 **3.** like terms

5. (a) $-\frac{1}{2}x^5 + 14x$ (b) Degree: 5; Leading coefficient: $-\frac{1}{2}$
(c) Binomial

7. (a) $-x^6 + 3$ (b) Degree: 6; Leading coefficient: -1
(c) Binomial

9. (a) 3 (b) Degree: 0; Leading coefficient: 3
(c) Monomial

CHAPTER P

11. (a) $-4x^5 + 6x^4 + 1$
(b) Degree: 5; Leading coefficient: -4
(c) Trinomial

13. (a) $4x^3y$
(b) Degree: 4; Leading coefficient: 4
(c) Monomial

15. Polynomial: $-3x^3 + 2x + 8$

17. Not a polynomial because it includes a term with a negative exponent

19. Polynomial: $-y^4 + y^3 + y^2$ **21.** $-2x - 10$

23. $5t^3 - 5t + 1$ **25.** $8.3x^3 + 29.7x^2 + 11$

27. $12z + 8$ **29.** $3x^3 - 6x^2 + 3x$ **31.** $-15z^2 + 5z$

33. $-4x^4 + 4x$ **35.** $-4.5t^3 - 15t$ **37.** $-0.2x^2 - 34x$

39. $4x^3 - 2x^2 + 4$ **41.** $5x^2 - 4x + 11$

43. $2x^2 + 17x + 21$ **45.** $x^2 + 7x + 12$

47. $6x^2 - 7x - 5$ **49.** $x^4 + x^2 + 1$ **51.** $x^2 - 100$

53. $x^2 - 4y^2$ **55.** $4x^2 + 12x + 9$ **57.** $16x^6 - 24x^3 + 9$

59. $x^3 + 3x^2 + 3x + 1$ **61.** $8x^3 - 12x^2y + 6xy^2 - y^3$

63. $m^2 - n^2 - 6m + 9$

65. $x^2 + 2xy + y^2 - 6x - 6y + 9$ **67.** $\frac{1}{25}x^2 - 9$

69. $2.25x^2 - 16$ **71.** $\frac{1}{16}x^2 - \frac{5}{2}x + 25$ **73.** $2x^2 + 2x$

75. $u^4 - 16$ **77.** $x - y$ **79.** $x^2 - 2\sqrt{5}x + 5$

81. (a) $P = 22x - 25{,}000$
(b) \$85,000

83. $2x^2 + 46x + 252$ **85.** $3x^2 + 8x$ **87.** $42x^2$

89. $15x^2 + 18x + 3$

91. (a) $V = 4x^3 - 88x^2 + 468x$
(b)

x (cm)	1	2	3
V (cm^3)	384	616	720

93. (a) Approximations will vary.
(b) The difference of the safe loads decreases in magnitude as the span increases.

95. False. $(4x^2 + 1)(3x + 1) = 12x^3 + 4x^2 + 3x + 1$

97. $m + n$

99. The student omitted the middle term when squaring the binomial. $(x - 3)^2 = x^2 - 6x + 9 \neq x^2 + 9$

101. No. $(x^2 + 1) + (-x^2 + 3) = 4$, which is not a second-degree polynomial. (Examples will vary.)

103. $(3 + 4)^2 = 49 \neq 25 = 3^2 + 4^2$.
If either x or y is zero, then $(x + y)^2 = x^2 + y^2$.

Section P.4 *(page 39)*

1. factoring **3.** perfect square trinomial **5.** $2x(x^2 - 3)$

7. $(x - 5)(3x + 8)$ **9.** $\frac{1}{2}x(x^2 + 4x - 10)$

11. $\frac{2}{3}(x - 6)(x - 3)$ **13.** $(x + 9)(x - 9)$

15. $3(4y - 3)(4y + 3)$ **17.** $\left(4x + \frac{1}{3}\right)\left(4x - \frac{1}{3}\right)$

19. $(x + 1)(x - 3)$ **21.** $(3u - 1)(3u + 1)(9u^2 + 1)$

23. $(x - 2)^2$ **25.** $(3u + 4v)^2$ **27.** $\left(z + \frac{1}{2}\right)^2$

29. $(x - 2)(x^2 + 2x + 4)$ **31.** $(y + 4)(y^2 - 4y + 16)$

33. $(2t - 1)(4t^2 + 2t + 1)$ **35.** $(u + 3v)(u^2 - 3uv + 9v^2)$

37. $(x + 2)(x - 1)$ **39.** $(s - 3)(s - 2)$

41. $-(y + 5)(y - 4)$ **43.** $(3x - 2)(x - 1)$

45. $(5x + 1)(x + 5)$ **47.** $(x - 1)(x^2 + 2)$

49. $(2x - 1)(x^2 - 3)$ **51.** $(x^2 + 2)(x + 1)(x^2 - x + 1)$

53. $(x + 3)(2x + 3)$ **55.** $(2x + 3)(3x - 5)$

57. $6(x + 3)(x - 3)$ **59.** $x^2(x - 1)$ **61.** $(x - 1)^2$

63. $(1 - 2x)^2$ **65.** $-2x(x + 1)(x - 2)$

67. $\frac{1}{96}(4x - 3)(3x + 2)$ **69.** $(x + 1)^2(x - 1)^2$

71. $(5 - x)(1 + x^2)$ **73.** $(x - 2)(x + 2)(2x + 1)$

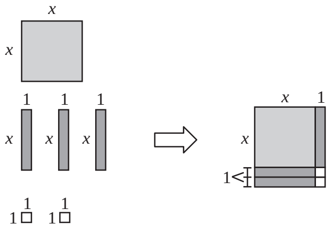
75. $2(t - 2)(t^2 + 2t + 4)$ **77.** $(3 - 4x)(23 - 60x)$

79. $5(1 - x)^2(3x + 2)(4x + 3)$

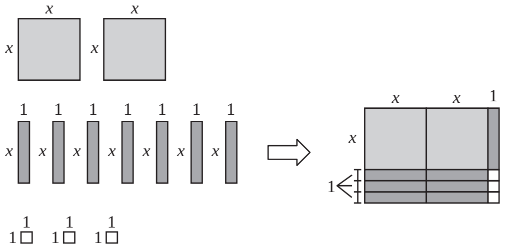
81. $4x^3(2x + 1)^3(2x^2 + 2x + 1)$

83. $(2x - 5)^3(5x - 4)^2(70x - 107)$

85.



87.



89. $4\pi(r + 1)$

91. (a) $\pi h(R + r)(R - r)$ (b) $V = 2\pi\left[\left(\frac{R + r}{2}\right)(R - r)\right]h$

93. $-14, 14, -2, 2$ **95.** Two possible answers: $2, -12$

97. True. $a^2 - b^2 = (a + b)(a - b)$

99. A 3 was not factored out of the second binomial.

101. $(x^n + y^n)(x^n - y^n)$

103. Answers will vary. *Sample answer:* $x^2 - 3$

105. $(u + v)(u - v)(u^2 + uv + v^2)(u^2 - uv + v^2)$
$(x - 1)(x + 1)(x^2 + x + 1)(x^2 - x + 1)$
$(x - 2)(x + 2)(x^2 - 2x + 4)(x^2 + 2x + 4)$

Section P.5 *(page 48)*

1. domain **3.** complex **5.** All real numbers x

7. All real numbers x such that $x \neq 3$

9. All real numbers x such that $x \neq 1$

11. All real numbers x such that $x \neq 3$

13. All real numbers x such that $x \leq 4$

15. All real numbers x such that $x > 3$

17. $\frac{3x}{2},\ x \neq 0$ **19.** $\frac{3y}{y + 1},\ x \neq 0$ **21.** $-\frac{1}{2},\ x \neq 5$

23. $y - 4,\ y \neq -4$ **25.** $\frac{x(x + 3)}{x - 2},\ x \neq -2$

27. $\frac{-(x^2 + 1)}{x + 2},\ x \neq 2$ **29.** $z - 2$

31. When simplifying fractions, you can only divide out common factors, not terms.

33. $\frac{1}{5(x - 2)},\ x \neq 1$ **35.** $-\frac{8}{5},\ y \neq -3, 4$

37. $\frac{x - y}{x(x + y)^2},\ x \neq -2y$ **39.** $\frac{6x + 13}{x + 3}$ **41.** $-\frac{2}{x - 2}$

43. $\frac{-2x^2 + 3x + 8}{(2x + 1)(x + 2)}$ **45.** $\frac{2 - x}{x^2 + 1},\ x \neq 0$

47. The error is incorrect subtraction in the numerator.

49. $\frac{1}{2}, \quad x \neq 2$ **51.** $x(x+1), \quad x \neq -1, 0$

53. $\frac{2x-1}{2x}, \quad x > 0$ **55.** $\frac{x^7-2}{x^2}$ **57.** $\frac{-1}{(x^2+1)^5}$

59. $\frac{2x^3-2x^2-5}{(x-1)^{1/2}}$ **61.** $\frac{3x-1}{3}, \quad x \neq 0$

63. $\frac{-1}{x(x+h)}, \quad h \neq 0$ **65.** $\frac{-1}{(x-4)(x+h-4)}, \quad h \neq 0$

67. $\frac{1}{\sqrt{x+2}+\sqrt{x}}$ **69.** $\frac{1}{\sqrt{t+3}+\sqrt{3}}, \quad t \neq 0$

71. $\frac{1}{\sqrt{x+h+1}+\sqrt{x+1}}, \quad h \neq 0$

73. (a)

t	0	2	4	6	8	10	12	14	16	18	20	22
T	75	55.9	48.3	45	43.3	42.3	41.7	41.3	41.1	40.9	40.7	40.6

(b) The model is approaching a T-value of 40.

75. $\frac{x}{2(2x+1)}, \quad x \neq 0$

77. (a)

Year, t	Banking, (millions)	Paying Bills (millions)
5	46.9	17
6	57.6	25.6
7	63.5	27.3
8	67.3	28.8
9	69.9	30.8
10	71.9	33.7

(b) The estimates are fairly close to the actual numbers of households.

(c) $\text{Ratio} = \frac{-0.1228t^3 + 2.923t^2 - 16.38t + 24.6}{-0.5061t^3 + 11.2742t^2 - 67.844t + 121.8}$

(d)

Year	Ratio
2005	0.3625
2006	0.4449
2007	0.4292
2008	0.4282
2009	0.4403
2010	0.4685

Answers will vary.

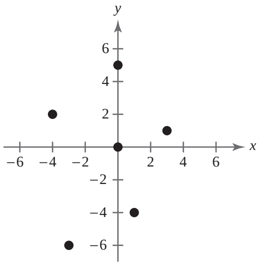
79. $\frac{R_1R_2}{R_2+R_1}$

81. False. In order for the simplified expression to be equivalent to the original expression, the domain of the simplified expression needs to be restricted. If n is even, $x \neq -1, 1$. If n is odd, $x \neq 1$.

Section P.6 *(page 57)*

1. Cartesian **3.** Distance Formula

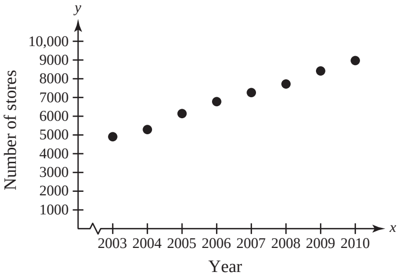
5.



7. $(-3, 4)$ **9.** Quadrant IV

11. Quadrant II **13.** Quadrant III

15.



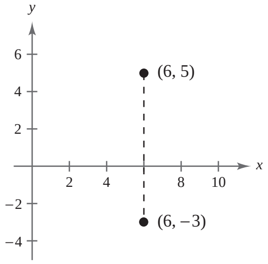
17. 13 **19.** $\sqrt{61}$ **21.** $\frac{\sqrt{277}}{6}$

23. (a) 5, 12, 13 (b) $5^2 + 12^2 = 13^2$

25. $(\sqrt{5})^2 + (\sqrt{45})^2 = (\sqrt{50})^2$

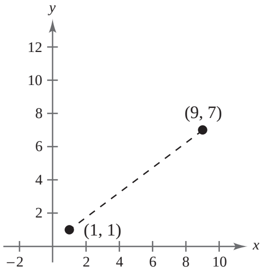
27. Distances between the points: $\sqrt{29}$, $\sqrt{58}$, $\sqrt{29}$

29. (a)



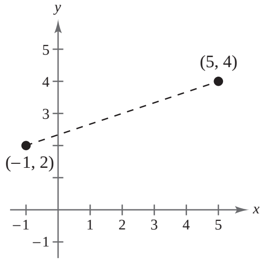
(b) 8 (c) $(6, 1)$

31. (a)



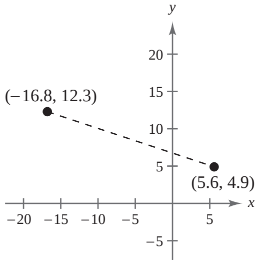
(b) 10 (c) $(5, 4)$

33. (a)



(b) $2\sqrt{10}$
(c) $(2, 3)$

35. (a)



(b) $\sqrt{556.52}$
(c) $(-5.6, 8.6)$

37. $30\sqrt{41} \approx 192$ km **39.** \$27,343.5 million

41. $(0, 1), (4, 2), (1, 4)$ **43.** $(-3, 6), (2, 10), (2, 4), (-3, 4)$

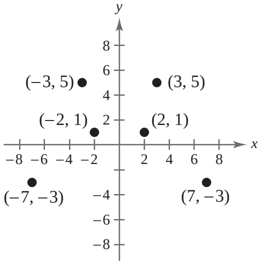
45. (a) 2000s (b) 11.8%; 70.6% (c) \$12.37

(d) Answers will vary. *Sample answer:* No, the prediction is too high because it is likely that the percent increase over a 7-year period (2009–2016) will be less than the percent increase over a 16-year period (1995–2011).

47. $(2x_m - x_1, 2y_m - y_1)$

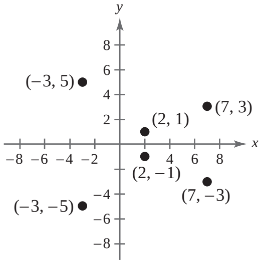
49. $\left(\dfrac{3x_1 + x_2}{4}, \dfrac{3y_1 + y_2}{4}\right), \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right),$
$\left(\dfrac{x_1 + 3x_2}{4}, \dfrac{y_1 + 3y_2}{4}\right)$

51. (a)



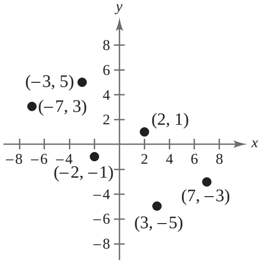
The point is reflected through the y-axis.

(b)



The point is reflected through the x-axis.

(c)



The point is reflected through the origin.

53. No. It depends on the magnitudes of the quantities measured.

55. False. The Midpoint Formula would be used 15 times.

57. False. The polygon could be a rhombus.

59. Use the Midpoint Formula to prove that the diagonals of the parallelogram bisect each other.

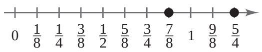
$$\left(\frac{b + a}{2}, \frac{c + 0}{2}\right) = \left(\frac{a + b}{2}, \frac{c}{2}\right)$$

$$\left(\frac{a + b + 0}{2}, \frac{c + 0}{2}\right) = \left(\frac{a + b}{2}, \frac{c}{2}\right)$$

Review Exercises *(page 62)*

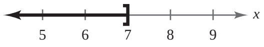
1. (a) 11 (b) 11 (c) 11, -14
(d) 11, -14, $-\frac{8}{9}$, $\frac{5}{2}$, 0.4 (e) $\sqrt{6}$

3.



$\frac{5}{4} > \frac{7}{8}$

5. The set consists of all real numbers less than or equal to 7.



7. 122 **9.** $|x - 7| \geq 4$ **11.** $|y + 30| < 5$

13. (a) -7 (b) -19 **15.** (a) -1 (b) -3

17. Associative Property of Addition

19. Additive Identity Property

21. Commutative Property of Addition

23. -11 **25.** $\dfrac{1}{12}$ **27.** -144 **29.** $\dfrac{47x}{60}$

31. (a) $192x^{11}$ (b) $\dfrac{y^5}{2}$, $y \neq 0$ **33.** (a) $-8z^3$ (b) $\dfrac{1}{y^2}$

35. (a) a^2b^2 (b) $3a^3b^2$ **37.** (a) $\dfrac{1}{625a^4}$ (b) $64x^2$

39. 1.685×10^8 **41.** 484,000,000 **43.** (a) 9 (b) 343

45. (a) 216 (b) 32 **47.** (a) $2\sqrt{2}$ (b) $26\sqrt{2}$

49. Radicals cannot be combined by addition or subtraction unless the index and the radicand are the same.

51. $\dfrac{\sqrt{3}}{4}$ **53.** $2 + \sqrt{3}$ **55.** $\dfrac{3}{\sqrt{7} - 1}$ **57.** 64

59. $6x^{9/10}$

61. $-11x^2 + 3$; Degree: 2; Leading coefficient: -11

63. $-12x^2 - 4$; Degree: 2; Leading coefficient: -12

65. $-3x^2 - 7x + 1$ **67.** $2x^3 - 10x^2 + 12x$

69. $2x^3 - x^2 + 3x - 9$ **71.** $15x^2 - 27x - 6$

73. $4x^2 - 12x + 9$ **75.** $45 - 4x^2$

77. $2500r^2 + 5000r + 2500$ **79.** $x^2 + 28x + 192$

81. $x(x + 1)(x - 1)$ **83.** $(5x + 7)(5x - 7)$

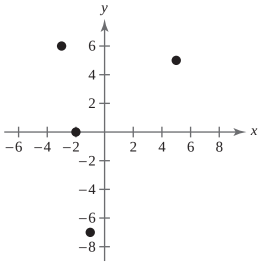
85. $(x - 4)(x^2 + 4x + 16)$ **87.** $(x + 10)(2x + 1)$

89. $(x - 1)(x^2 + 2)$ **91.** All real numbers except $x = -6$

93. $\dfrac{x - 8}{15}$, $x \neq -8$ **95.** $\dfrac{1}{x^2}$, $x \neq \pm 2$

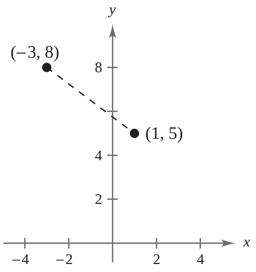
97. $\dfrac{3x}{(x - 1)(x^2 + x + 1)}$ **99.** $\dfrac{3ax^2}{(a^2 - x)(a - x)}$

101. $\dfrac{-1}{2x(x + h)}$, $h \neq 0$ **103.**



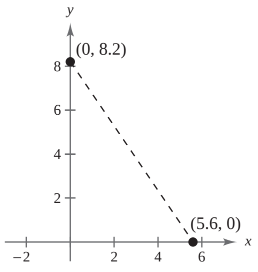
105. Quadrant IV

107. (a)



(b) 5 (c) $\left(-1, \frac{13}{2}\right)$

109. (a)



(b) $\sqrt{98.6}$ (c) (2.8, 4.1)

111. (0, 0), (2, 0), (0, -5), (2, -5) **113.** \$6.05 billion

115. False. There is also a cross-product term when a binomial sum is squared.
$(x + a)^2 = x^2 + 2ax + a^2$

117. No. When $x = b/a$, the expression is undefined.

Chapter Test *(page 65)*

1. $-\frac{10}{3} > -|-4|$ **2.** 9.15 **3.** Additive Identity Property

4. (a) -18 (b) $\frac{4}{9}$ (c) $-\frac{27}{125}$ (d) $\frac{8}{729}$

5. (a) 25 (b) $\dfrac{3\sqrt{6}}{2}$ (c) 1.8×10^5 (d) 2.7×10^{13}

6. (a) $12z^8$ (b) $(u-2)^{-7}$ (c) $\dfrac{3x^2}{y^2}$

7. (a) $15z\sqrt{2z}$ (b) $4x^{14/15}$ (c) $\dfrac{2\sqrt[3]{2v}}{v^2}$

8. $-2x^5 - x^4 + 3x^3 + 3$; Degree: 5; Leading coefficient: -2

9. $2x^2 - 3x - 5$ **10.** $x^2 - 5$ **11.** $5, x \neq 4$

12. $\dfrac{x-1}{2x},\ x \neq \pm 1$

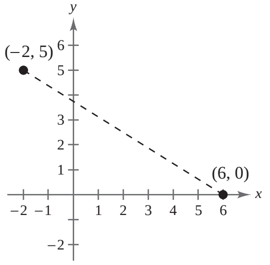
13. (a) $x^2(2x+1)(x-2)$ (b) $(x-2)(x+2)^2$

14. (a) $4\sqrt[3]{4}$ (b) $-4(1+\sqrt{2})$

15. All real numbers x except $x = 1$

16. $\dfrac{4}{y+4},\ y \neq 2$ **17.** $545

18.



Midpoint: $\left(2, \frac{5}{2}\right)$; Distance: $\sqrt{89}$

19. $\frac{5}{6}\sqrt{3}x^2$

Problem Solving *(page 67)*

1. (a) Men's: 1,150,347 mm³; 696,910 mm³
Women's: 696,910 mm³; 448,921 mm³
(b) Men's: 1.04×10^{-5} kg/mm³; 6.31×10^{-6} kg/mm³
Women's: 8.91×10^{-6} kg/mm³; 5.74×10^{-6} kg/mm³
(c) No. Iron has a greater density than cork.

3. 1.62 oz **5.** Answers will vary. **7.** $r \approx 0.28$

9. 9.57 ft² **11.** $y_1(0) = 0, y_2(0) = 2$

$$y_2 = \frac{x(2-3x^2)}{\sqrt{1-x^2}}$$

13. (a) $(2, -1), (3, 0)$ (b) $\left(-\frac{4}{3}, -2\right), \left(-\frac{2}{3}, -1\right)$

Chapter 1

Section 1.1 *(page 78)*

1. solution or solution point **3.** intercepts

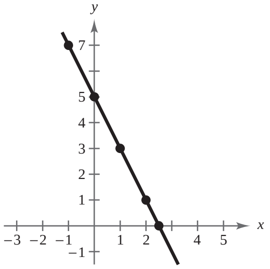
5. circle; (h, k); r **7.** (a) Yes (b) Yes

9. (a) Yes (b) No **11.** (a) Yes (b) No

13. (a) No (b) Yes

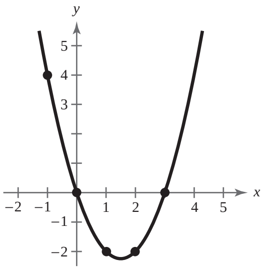
15.

x	-1	0	1	2	$\frac{5}{2}$
y	7	5	3	1	0
(x, y)	$(-1, 7)$	$(0, 5)$	$(1, 3)$	$(2, 1)$	$\left(\frac{5}{2}, 0\right)$



17.

x	-1	0	1	2	3
y	4	0	-2	-2	0
(x, y)	$(-1, 4)$	$(0, 0)$	$(1, -2)$	$(2, -2)$	$(3, 0)$



19. x-intercept: $(3, 0)$
y-intercept: $(0, 9)$

21. x-intercept: $(-2, 0)$
y-intercept: $(0, 2)$

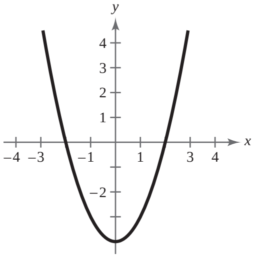
23. x-intercept: $(1, 0)$
y-intercept: $(0, 2)$

25. y-axis symmetry

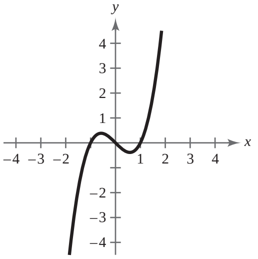
27. Origin symmetry **29.** Origin symmetry

31. x-axis symmetry

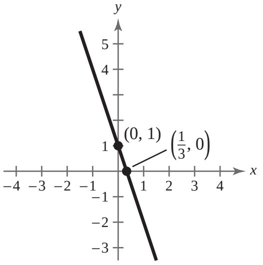
33.



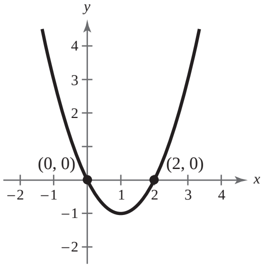
35.



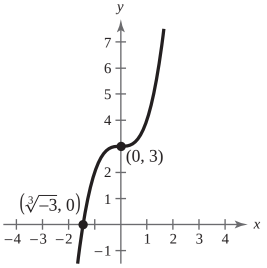
37. x-intercept: $\left(\frac{1}{3}, 0\right)$
y-intercept: $(0, 1)$
No symmetry



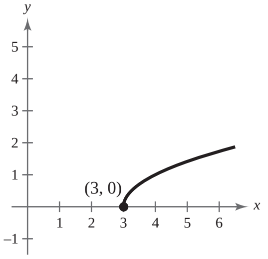
39. x-intercepts: $(0, 0), (2, 0)$
y-intercept: $(0, 0)$
No symmetry



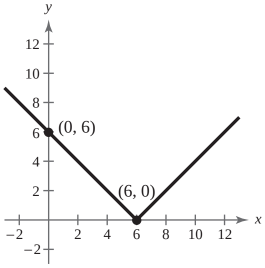
41. x-intercept: $\left(\sqrt[3]{-3}, 0\right)$
y-intercept: $(0, 3)$
No symmetry



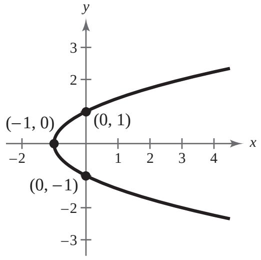
43. x-intercept: $(3, 0)$
y-intercept: None
No symmetry



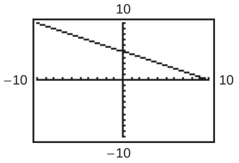
45. x-intercept: $(6, 0)$
y-intercept: $(0, 6)$
No symmetry



47. x-intercept: $(-1, 0)$
y-intercepts: $(0, \pm 1)$
x-axis symmetry

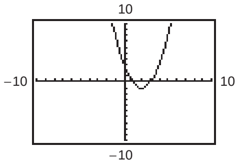


49.



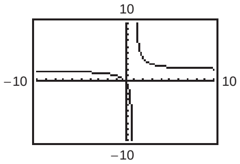
Intercepts: $(10, 0), (0, 5)$

51.



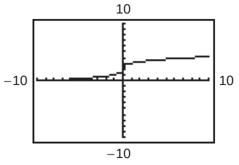
Intercepts:
$(3, 0), (1, 0), (0, 3)$

53.



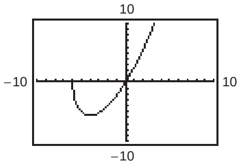
Intercept: $(0, 0)$

55.



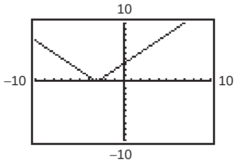
Intercepts: $(-8, 0), (0, 2)$

57.



Intercepts: $(0, 0), (-6, 0)$

59.



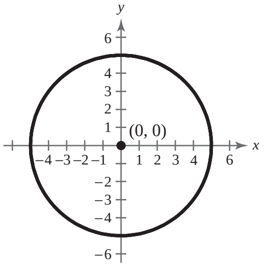
Intercepts: $(-3, 0), (0, 3)$

61. $x^2 + y^2 = 16$ **63.** $(x - 2)^2 + (y + 1)^2 = 16$

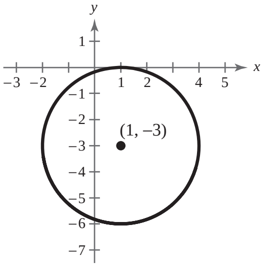
65. $(x + 1)^2 + (y - 2)^2 = 5$

67. $(x - 3)^2 + (y - 4)^2 = 25$

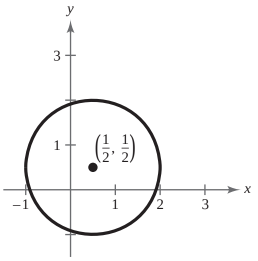
69. Center: $(0, 0)$; Radius: 5



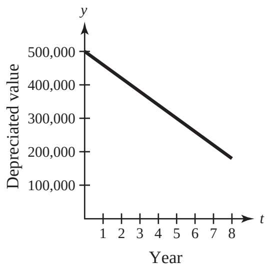
71. Center: $(1, -3)$; Radius: 3



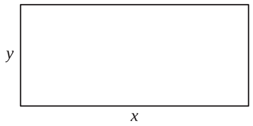
73. Center: $\left(\frac{1}{2}, \frac{1}{2}\right)$; Radius: $\frac{3}{2}$



75.

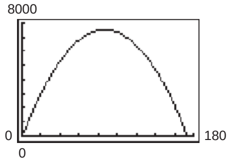


77. (a)



(b) Answers will vary.

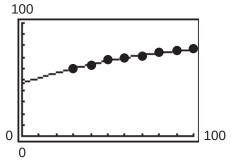
(c)



(d) $x = 86\frac{2}{3}, y = 86\frac{2}{3}$

(e) A regulation NFL playing field is 120 yards long and $53\frac{1}{3}$ yards wide. The actual area is 6400 square yards.

79. (a)



The model fits the data very well.

(b) 75.4 yr (c) 1994

(d) 77.65 yr; The projection given by the model is slightly less.

(e) Answers will vary.

81. (a) $a = 1, b = 0$ (b) $a = 0, b = 1$

Section 1.2 *(page 87)*

1. equation **3.** $ax + b = 0$ **5.** rational **7.** Identity
9. Conditional equation **11.** Identity **13.** Contradiction
15. 4 **17.** -9 **19.** 12 **21.** 1 **23.** No solution
25. 9 **27.** $-\frac{96}{23}$ **29.** -4 **31.** 4 **33.** 3
35. 0 **37.** No solution. The variable is divided out.
39. No solution. The solution is extraneous.
41. 5 **43.** No solution. The solution is extraneous.

45. x-intercept: $\left(\frac{12}{5}, 0\right)$
y-intercept: $(0, 12)$

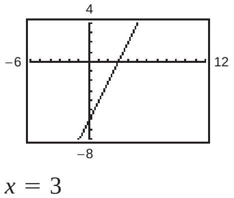
47. x-intercept: $\left(-\frac{1}{2}, 0\right)$
y-intercept: $(0, -3)$

49. x-intercept: $(5, 0)$
y-intercept: $\left(0, \frac{10}{3}\right)$

51. x-intercept: $(1.6, 0)$
y-intercept: $(0, -0.3)$

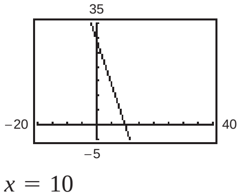
53. x-intercept: $(-20, 0)$
y-intercept: $\left(0, \frac{8}{3}\right)$

55.



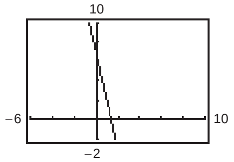
$x = 3$

57.



$x = 10$

59.



$x = \frac{7}{5}$

61. 138.889 **63.** 19.993 **65.** $h = 10$ ft **67.** 65.8 in.

69. (a) About $(0, 1480)$

(b) $(0, 1480.6)$; There were about 1481 newspapers in the United States in 2000.

(c) 2015; Answers will vary.

71. 20,000 mi

73. False. $x(3 - x) = 10$

$3x - x^2 = 10$

The equation cannot be written in the form $ax + b = 0$.

75. False. The equation is an identity.

77. False. $x = 4$ is a solution.

79. (a)

x	-1	0	1	2	3	4
$3.2x - 5.8$	-9	-5.8	-2.6	0.6	3.8	7

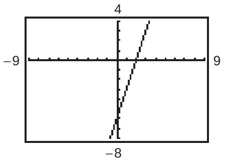
(b) $1 < x < 2$. The expression changes from negative to positive in this interval.

(c)

x	1.5	1.6	1.7	1.8	1.9	2
$3.2x - 5.8$	-1	-0.68	-0.36	-0.04	0.28	0.6

(d) $1.8 < x < 1.9$. To improve accuracy, evaluate the expression in this interval and determine where the sign changes.

81. (a)



(b) $(2, 0)$

(c) The x-intercept is the solution of the equation $3x - 6 = 0$.

83. (a) $\left(\frac{c}{a}, 0\right)$ (b) $\left(0, \frac{c}{b}\right)$

(c) x-intercept: $\left(\frac{11}{2}, 0\right)$

y-intercept: $\left(0, \frac{11}{7}\right)$

Section 1.3 *(page 97)*

1. mathematical modeling **3.** A number increased by 4

5. A number divided by 5

7. A number decreased by 4 is divided by 5.

9. Negative 3 is multiplied by a number increased by 2.

11. 4 is multiplied by a number decreased by 1 and the product is divided by that number.

13. $n + (n + 1) = 2n + 1$ **15.** $(2n - 1)(2n + 1) = 4n^2 - 1$

17. $55t$ **19.** $0.20x$ **21.** $6x$ **23.** $2500 + 40x$

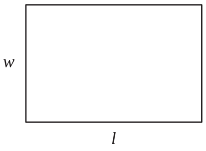
25. $0.30L$ **27.** $N = p(672)$ **29.** $4x + 8x = 12x$

31. 262, 263 **33.** 37, 185 **35.** $-5, -4$

37. First salesperson: \$516.89; Second salesperson: \$608.11

39. \$49,000

41. (a)

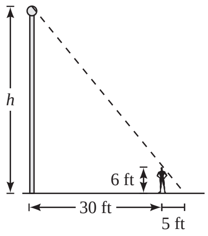


(b) $l = 1.5w$; $p = 5w$

(c) 7.5 m × 5 m

43. 97 **45.** 5 h **47.** About 8.33 min **49.** 945 ft

51. (a)



(b) 42 ft

53. \$4000 at $4\frac{1}{2}\%$, \$8000 at 5%

55. Red maple: \$25,000; Dogwood: \$15,000

57. About 1.09 gal **59.** $\frac{2A}{b}$ **61.** $\frac{S}{1 + R}$ **63.** $\frac{A - P}{Pt}$

65. $x = 6$ feet from the 50-pound child

67. $\sqrt[3]{\frac{4.47}{\pi}} \approx 1.12$ in. **69.** 18°C **71.** 122°F

73. False. The expression should be $\frac{z^3 - 8}{z^2 - 9}$.

75. False.

Cube: $V = s^3$

$= 9.5^3$

$= 857.375 \text{ in.}^3$

Sphere: $V = \frac{4}{3}\pi r^3$

$= \left(\frac{4}{3}\right)(\pi)(5.9)^3$

$\approx 860.290 \text{ in.}^3$

Section 1.4 *(page 110)*

1. quadratic equation

3. factoring; square roots; completing; square; Quadratic Formula

5. position equation **7.** $2x^2 + 5x - 3 = 0$

9. $x^2 - 6x + 6 = 0$ **11.** $3x^2 - 60x - 10 = 0$

13. $0, -\frac{1}{2}$ **15.** $4, -2$ **17.** -5 **19.** $3, -\frac{1}{2}$

21. $2, -6$ **23.** $-\frac{20}{3}, -4$ **25.** ± 7

27. $\pm\sqrt{11} \approx \pm 3.32$ **29.** $\pm 3\sqrt{3} \approx \pm 5.20$

31. 8, 16 **33.** $-2 \pm \sqrt{14} \approx 1.74, -5.74$

35. $\frac{1 \pm 3\sqrt{2}}{2} \approx 2.62, -1.62$ **37.** 2

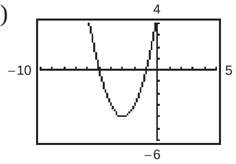
39. $4, -8$ **41.** $-3 \pm \sqrt{7}$ **43.** $1 \pm \frac{\sqrt{6}}{3}$

45. $1 \pm 2\sqrt{2}$ **47.** $\frac{-5 \pm \sqrt{89}}{4}$ **49.** $\frac{1}{(x + 1)^2 + 4}$

51. $\frac{4}{(x + 5)^2 + 49}$ **53.** $\frac{1}{\sqrt{4 - (x - 1)^2}}$

55. $\frac{1}{\sqrt{16 - (x - 2)^2}}$

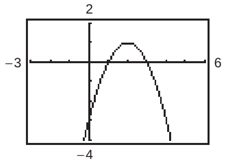
57. (a)



(b) and (c) $x = -1, -5$

(d) The answers are the same.

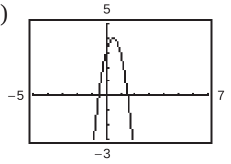
59.



(b) and (c) $x = 3, 1$

(d) The answers are the same.

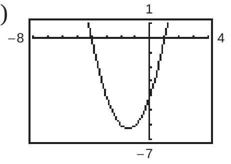
61. (a)



(b) and (c) $x = -\frac{1}{2}, \frac{3}{2}$

(d) The answers are the same.

63. (a)



(b) and (c) $x = 1, -4$

(d) The answers are the same.

65. No real solution **67.** Two real solutions

69. No real solution **71.** Two real solutions

73. $\frac{1}{2}, -1$ **75.** $\frac{1}{4}, -\frac{3}{4}$ **77.** $1 \pm \sqrt{3}$

79. $-6 \pm 2\sqrt{5}$

81. $-4 \pm 2\sqrt{5}$ **83.** $\dfrac{2}{3} \pm \dfrac{\sqrt{7}}{3}$ **85.** $-\dfrac{5}{3}$

87. $-\dfrac{1}{2} \pm \sqrt{2}$ **89.** $\dfrac{2}{7}$ **91.** $2 \pm \dfrac{\sqrt{6}}{2}$ **93.** $6 \pm \sqrt{11}$

95. $-\dfrac{3}{8} \pm \dfrac{\sqrt{265}}{8}$ **97.** $0.976, -0.643$

99. $1.355, -14.071$ **101.** $1.687, -0.488$

103. $-0.290, -2.200$ **105.** $1 \pm \sqrt{2}$ **107.** $6, -12$

109. $\frac{1}{2} \pm \sqrt{3}$ **111.** $-\frac{1}{2}$

113. (a) $w(w + 14) = 1632$ (b) $w = 34$ ft, $l = 48$ ft

115. 6 in. × 6 in. × 3 in. **117.** 19.936 ft; 10 trips

119. (a) About 5.86 sec (b) About 0.2 mi

121. (a) $s = -16t^2 + 146\frac{2}{3}t + 6.25$

(b) $s(3) = 302.25$ ft; $s(4) \approx 336.92$ ft; $s(5) \approx 339.58$ ft; During the interval $3 \le t \le 5$, the baseball's speed decreased due to gravity.

(c) Assuming the ball is not caught and drops to the ground, the baseball is in the air about 9.209 seconds.

123. (a)

t	5	6	7	8	9	10	11
D	7.7	8.0	8.6	9.5	10.7	12.2	14.0

The public debt reached $10 trillion sometime in 2008.

(b) $t \approx 8.5$ (c) $44.7 trillion; Answers will vary.

125. 4:00 P.M.

127. (a) $x^2 + 15^2 = l^2$ (b) $30\sqrt{6} \approx 73.5$ ft

129. False. $b^2 - 4ac < 0$, so the quadratic equation has no real solution.

131. Answers will vary. *Sample answer:* $x^2 + 2x - 15 = 0$

133. Answers will vary. *Sample answer:* $x^2 - 22x + 112 = 0$

135. Answers will vary. *Sample answer:* $x^2 - 2x - 1 = 0$

137. Yes, the vertex of the parabola would be on the x-axis.

Section 1.5 *(page 119)*

1. real **3.** pure imaginary **5.** principal square

7. $a = -12,\ b = 7$ **9.** $a = 6,\ b = 5$ **11.** $8 + 5i$

13. $2 - 3\sqrt{3}i$ **15.** $4\sqrt{5}i$ **17.** 14

19. $-1 - 10i$ **21.** $0.3i$ **23.** $10 - 3i$ **25.** 1

27. $3 - 3\sqrt{2}i$ **29.** $-14 + 20i$ **31.** $\frac{1}{6} + \frac{7}{6}i$

33. $5 + i$ **35.** $108 + 12i$ **37.** 24 **39.** $-13 + 84i$

41. -10 **43.** $9 - 2i, 85$ **45.** $-1 + \sqrt{5}i, 6$

47. $-2\sqrt{5}i, 20$ **49.** $\sqrt{6}, 6$ **51.** $-3i$ **53.** $\frac{8}{41} + \frac{10}{41}i$

55. $\frac{12}{13} + \frac{5}{13}i$ **57.** $-4 - 9i$ **59.** $-\frac{120}{1681} - \frac{27}{1681}i$

61. $-\frac{1}{2} - \frac{5}{2}i$ **63.** $\frac{62}{949} + \frac{297}{949}i$ **65.** $-2\sqrt{3}$ **67.** -15

69. $\left(21 + 5\sqrt{2}\right) + \left(7\sqrt{5} - 3\sqrt{10}\right)i$ **71.** $1 \pm i$

73. $-2 \pm \frac{1}{2}i$ **75.** $-\frac{5}{2}, -\frac{3}{2}$ **77.** $2 \pm \sqrt{2}i$

79. $\dfrac{5}{7} \pm \dfrac{5\sqrt{15}}{7}$ **81.** $-1 + 6i$ **83.** $-14i$

85. $-432\sqrt{2}i$ **87.** i **89.** 81

91. (a) $z_1 = 9 + 16i, z_2 = 20 - 10i$

(b) $z = \dfrac{11{,}240}{877} + \dfrac{4630}{877}i$

93. False. When the complex number is real, the number equals its conjugate.

95. False.

$i^{44} + i^{150} - i^{74} - i^{109} + i^{61} = 1 - 1 + 1 - i + i = 1$

97. $i, -1, -i, 1, i, -1, -i, 1$; The pattern repeats the first four results. Divide the exponent by 4.

When the remainder is 1, the result is i.

When the remainder is 2, the result is -1.

When the remainder is 3, the result is $-i$.

When the remainder is 0, the result is 1.

99. $\sqrt{-6}\sqrt{-6} = \sqrt{6}i\sqrt{6}i = 6i^2 = -6$

101. Proof

Section 1.6 *(page 128)*

1. polynomial **3.** radical **5.** $0, \pm\dfrac{\sqrt{21}}{3}$ **7.** $\pm3, \pm3i$

9. $-8, 4 \pm 4\sqrt{3}i$ **11.** $-3, 0$ **13.** $3, 1, -1$

15. $\pm1, \dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i$ **17.** $\pm\sqrt{3}, \pm1$ **19.** $\pm\dfrac{1}{2}, \pm4$

21. $1, -2, 1 \pm \sqrt{3}i, -\dfrac{1}{2} \pm \dfrac{\sqrt{3}i}{2}$ **23.** $-\dfrac{1}{5}, -\dfrac{1}{3}$

25. $-\frac{2}{3}, -4$ **27.** $\frac{1}{4}$ **29.** $1, -\frac{125}{8}$ **31.** 48 **33.** 26

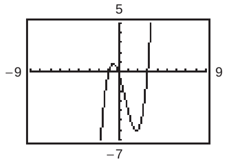
35. -16 **37.** $2,\ -5$ **39.** 9 **41.** $\frac{101}{4}$ **43.** $\frac{7}{4}$

45. 9 **47.** $\pm\sqrt{14}$ **49.** 1 **51.** $2, -\frac{3}{2}$

53. $\dfrac{-3 \pm \sqrt{21}}{6}$ **55.** $5, -6$ **57.** $\dfrac{1 \pm \sqrt{31}}{3}$

59. $8, -3$ **61.** $2\sqrt{6}, -6$ **63.** $3, \dfrac{-1 - \sqrt{17}}{2}$

65. (a)

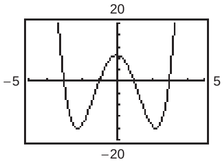


(b) $(0, 0), (3, 0), (-1, 0)$

(c) $x = 0, 3, -1$

(d) The x-intercepts and the solutions are the same.

67. (a)

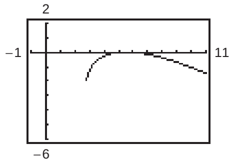


(b) $(\pm3, 0), (\pm1, 0)$

(c) $x = \pm3, \pm1$

(d) The x-intercepts and the solutions are the same.

69. (a)

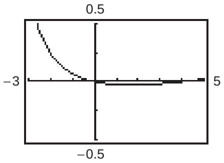


(b) $(5, 0), (6, 0)$

(c) $x = 5, 6$

(d) The x-intercepts and the solutions are the same.

71. (a)

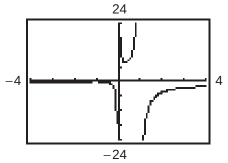


(b) $(0, 0), (4, 0)$

(c) $x = 0, 4$

(d) The x-intercepts and the solutions are the same.

73. (a)

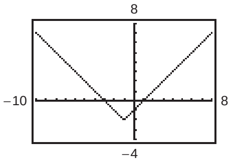


(b) $(-1, 0)$

(c) $x = -1$

(d) The x-intercept and the solution are the same.

75. (a)



(b) $(1, 0), (-3, 0)$

(c) $x = 1, -3$

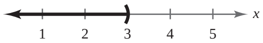
(d) The x-intercepts and the solutions are the same.

77. ± 1.038 **79.** $-1.143,\ 0.968$ **81.** 16.756
83. $-2.280,\ -0.320$ **85.** $x^2 - 3x - 28 = 0$
87. $21x^2 + 31x - 42 = 0$ **89.** $x^3 - 4x^2 - 3x + 12 = 0$
91. $x^2 + 1 = 0$ **93.** $x^4 - 1 = 0$ **95.** 34 students
97. 191.5 mi/h **99.** 4% **101.** 26,250 passengers
103. (a) About 15 lb/in.^{2} (b) Answers will vary.
105. 500 units **107.** 3 h **109.** $g = \dfrac{\mu s v^2}{R}$
111. False. See Example 7 on page 125. **113.** $6, -4$
115. ± 15 **117.** $a = 9, b = 9$ **119.** $a = 4, b = 24$
121. The quadratic equation was not written in general form. As a result, the substitutions in the Quadratic Formula are incorrect.

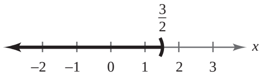
Section 1.7 *(page 137)*

1. solution set **3.** double **5.** $0 \le x < 9$; Bounded
7. $-1 \le x \le 5$; Bounded **9.** $x > 11$; Unbounded
11. $x < -2$; Unbounded

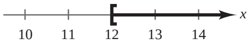
13. $x < 3$



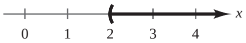
15. $x < \frac{3}{2}$



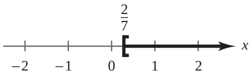
17. $x \ge 12$



19. $x > 2$



21. $x \ge \frac{2}{7}$

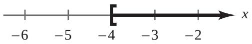


23. $x < 5$

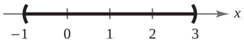
25. $x \ge 4$

27. $x \ge 2$

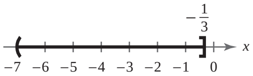
29. $x \ge -4$



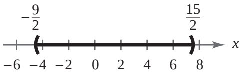
31. $-1 < x < 3$



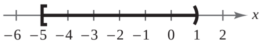
33. $-7 < x \le -\frac{1}{3}$



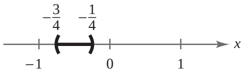
35. $-\frac{9}{2} < x < \frac{15}{2}$



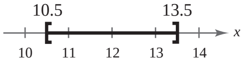
37. $-5 \le x < 1$



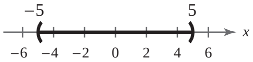
39. $-\frac{3}{4} < x < -\frac{1}{4}$



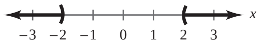
41. $10.5 \le x \le 13.5$



43. $-5 < x < 5$

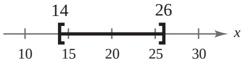


45. $x < -2, x > 2$



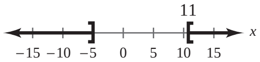
47. No solution

49. $14 \le x \le 26$



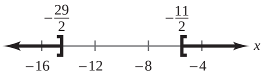
51. $x \le -\frac{3}{2}, x \ge 3$

53. $x \le -5,\ x \ge 11$

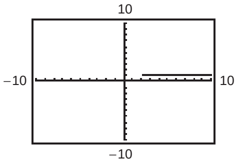


55. $4 < x < 5$

57. $x \le -\frac{29}{2},\ x \ge -\frac{11}{2}$

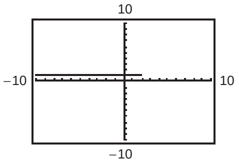


59.



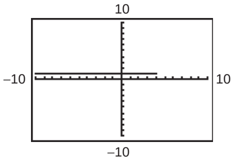
$x > 2$

61.



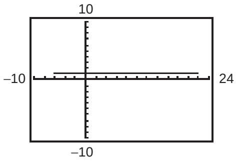
$x \le 2$

63.



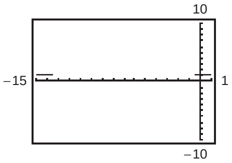
$x \le 4$

65.



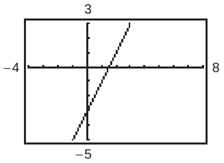
$-6 \le x \le 22$

67.



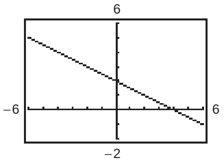
$x \le -\frac{27}{2}, x \ge -\frac{1}{2}$

69.



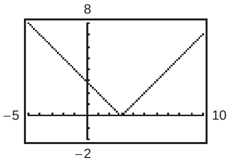
(a) $x \ge 2$ (b) $x \le \frac{3}{2}$

71.



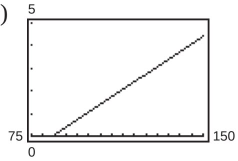
(a) $-2 \le x \le 4$
(b) $x \le 4$

73.



(a) $1 \le x \le 5$
(b) $x \le -1,\ x \ge 7$

75. $[5, \infty)$ **77.** $[-3, \infty)$ **79.** $\left(-\infty, \frac{7}{2}\right]$
81. All real numbers less than eight units from 10
83. $|x| \le 3$ **85.** $|x - 7| \ge 3$ **87.** $|x - 12| \ge 10$
89. $|x + 3| > 4$ **91.** $4.10 \le E \le 4.25$ **93.** $r < 0.08$
95. $100 \le r \le 170$ **97.** More than 6 units per hour
99. Greater than 3.125% **101.** $x \ge 36$
103. $87 \le x \le 210$
105. (a)



(b) $x \ge 129$
(c) $x \ge 128.93$
(d) Answers will vary.
107. (a) $2.87 \le t \le 6.54$ (Between 2002 and 2006)
(b) $t > 15.37$ (2015)
109. 106.864 in.^{2} $\le$ area $\le$ 109.464 in.^{2}
111. \$0.36 **113.** $13.7 < t < 17.5$
115. True by the Addition of a Constant Property of Inequalities.
117. False. If $-10 \le x \le 8$, then $10 \ge -x$ and $-x \ge -8$.

Section 1.8 *(page 147)*

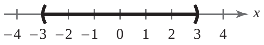
1. positive; negative **3.** zeros; undefined values

5. (a) No (b) Yes (c) Yes (d) No

7. (a) Yes (b) No (c) No (d) Yes

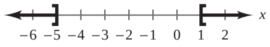
9. $-\frac{2}{3}, 1$ **11.** 4, 5

13. $(-3, 3)$

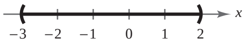


15. $[-7, 3]$

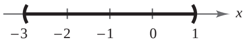
17. $(-\infty, -5] \cup [1, \infty)$



19. $(-3, 2)$



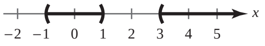
21. $(-3, 1)$



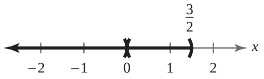
23. $\left(-\infty, -\frac{4}{3}\right) \cup (5, \infty)$

25. $(-\infty, -3) \cup (6, \infty)$

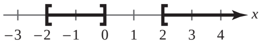
27. $(-1, 1) \cup (3, \infty)$



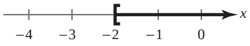
29. $(-\infty, 0) \cup \left(0, \frac{3}{2}\right)$



31. $[-2, 0] \cup [2, \infty)$



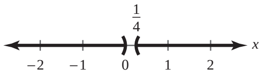
33. $[-2, \infty)$



35. The solution set consists of the single real number $\frac{1}{2}$.

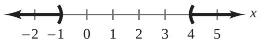
37. The solution set is empty.

39. $(-\infty, 0) \cup \left(\frac{1}{4}, \infty\right)$

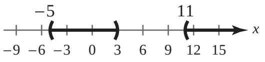


41. $\left(-\infty, \frac{5}{3}\right] \cup (5, \infty)$

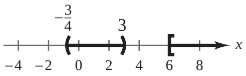
43. $(-\infty, -1) \cup (4, \infty)$



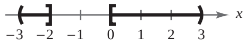
45. $(-5, 3) \cup (11, \infty)$



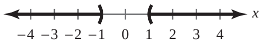
47. $\left(-\frac{3}{4}, 3\right) \cup [6, \infty)$



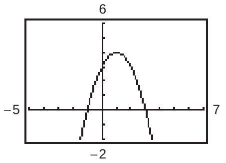
49. $(-3, -2] \cup [0, 3)$



51. $(-\infty, -1) \cup (1, \infty)$



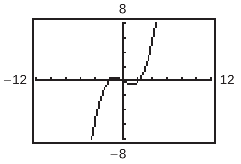
53.



(a) $x \le -1,\ x \ge 3$

(b) $0 \le x \le 2$

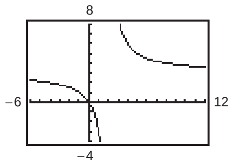
55.



(a) $-2 \le x \le 0, 2 \le x < \infty$

(b) $x \le 4$

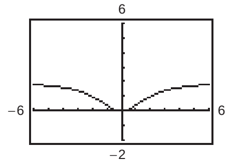
57.



(a) $0 \le x < 2$

(b) $2 < x \le 4$

59.



(a) $|x| \ge 2$

(b) $-\infty < x < \infty$

61. $[-2, 2]$ **63.** $(-\infty, 4] \cup [5, \infty)$ **65.** $(-5, 0] \cup (7, \infty)$

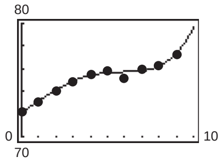
67. $(-3.51, 3.51)$ **69.** $(-0.13, 25.13)$ **71.** $(2.26, 2.39)$

73. (a) $t = 10$ sec (b) $4 \text{ sec} < t < 6 \text{ sec}$

75. $13.8 \text{ m} \le L \le 36.2 \text{ m}$

77. $40{,}000 \le x \le 50{,}000$; $\$50.00 \le p \le \$55.00$

79. (a) and (c)



The model fits the data well.

(b) $N = 0.00406t^4 - 0.0564t^3 + 0.147t^2 + 0.86t + 72.2$

(d) 2001

(e) No. The model can be used to predict enrollments for years close to those in its domain but when you project too far into the future, the numbers predicted by the model increase too rapidly to be considered reasonable.

81. $R_1 \ge 2$ ohms

83. True. The test intervals are $(-\infty, -3)$, $(-3, 1)$, $(1, 4)$, and $(4, \infty)$.

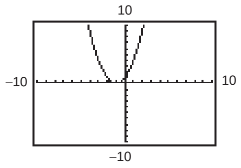
85. (a) $(-\infty, -4] \cup [4, \infty)$

(b) When $a > 0$ and $c > 0$, $b \le -2\sqrt{ac}$ or $b \ge 2\sqrt{ac}$.

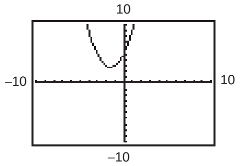
87. (a) $\left(-\infty, -2\sqrt{30}\right] \cup \left[2\sqrt{30}, \infty\right)$

(b) When $a > 0$ and $c > 0$, $b \le -2\sqrt{ac}$ or $b \ge 2\sqrt{ac}$.

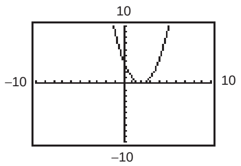
89.



For part (b), the y-values that are less than or equal to 0 occur only at $x = -1$.



For part (c), there are no y-values that are less than 0.

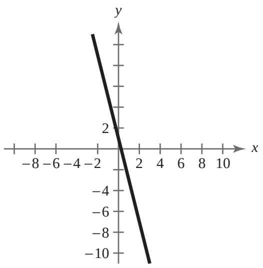


For part (d), the y-values that are greater than 0 occur for all values of x except 2.

Review Exercises *(page 152)*

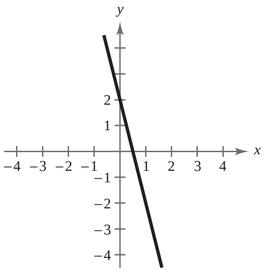
1.

x	-2	-1	0	1	2
y	9	5	1	-3	-7

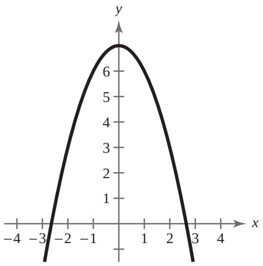


3. x-intercepts: $(1, 0), (5, 0)$
y-intercept: $(0, 5)$

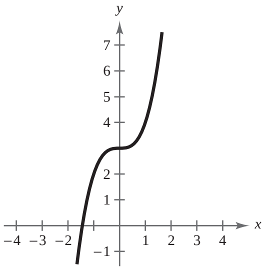
5. No symmetry



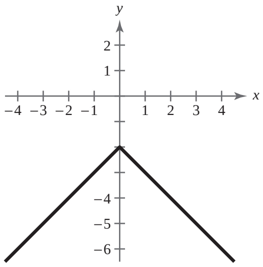
7. y-axis symmetry



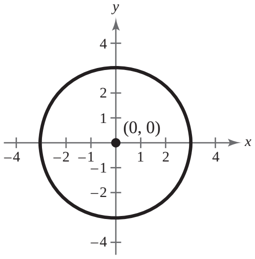
9. No symmetry



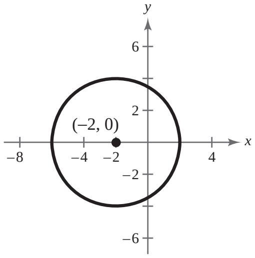
11. y-axis symmetry



13. Center: $(0, 0)$;
Radius: 3

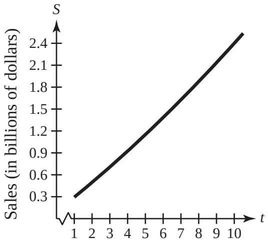


15. Center: $(-2, 0)$;
Radius: 4



17. $(x - 2)^2 + (y + 3)^2 = 13$

19. (a)



(b) 2005

21. Identity **23.** Contradiction **25.** 5 **27.** 9

29. -30

31. x-intercept: $\left(\frac{1}{3}, 0\right)$
y-intercept: $(0, -1)$

33. x-intercept: $(4, 0)$
y-intercept: $(0, -8)$

35. x-intercept: $\left(\frac{4}{3}, 0\right)$
y-intercept: $\left(0, \frac{2}{3}\right)$

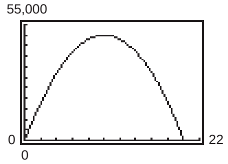
37. $h = 10$ in.

39. September: \$325,000; October: \$364,000 **41.** 9

43. $\frac{20}{7}$ L ≈ 2.857 L **45.** $\frac{3V}{\pi r^2}$ **47.** $-\frac{5}{2}, 3$ **49.** $\pm\sqrt{2}$

51. $-8, -18$ **53.** $-6 \pm \sqrt{11}$ **55.** $-\frac{5}{4} \pm \frac{\sqrt{241}}{4}$

57. (a) $x = 0, 20$
(b)



(c) $x = 10$

59. $4 + 3i$ **61.** $-1 + 3i$ **63.** $3 + 7i$ **65.** $12 + 30i$

67. $-5 - 6i$ **69.** $\frac{21}{13} - \frac{1}{13}i$ **71.** $1 \pm 3i$

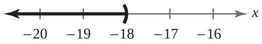
73. $-\frac{1}{2} \pm \frac{\sqrt{6}}{2}i$ **75.** $0, \frac{12}{5}$ **77.** $\pm\sqrt{2}, \pm\sqrt{3}$

79. No solution **81.** $-124, 126$ **83.** $\pm\sqrt{10}$

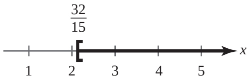
85. $-5, 15$ **87.** $1, 3$ **89.** 143,203 units

91. $-7 < x \leq 2$; Bounded **93.** $x \leq -10$; Unbounded

95. $x < -18$



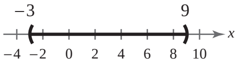
97. $x \geq \frac{32}{15}$



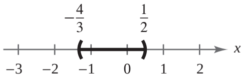
99. $(-\infty, -1) \cup (7, \infty)$

101. $353.44 \text{ cm}^2 \leq \text{area} \leq 392.04 \text{ cm}^2$

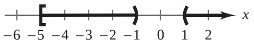
103. $(-3, 9)$



105. $\left(-\frac{4}{3}, \frac{1}{2}\right)$



107. $[-5, -1) \cup (1, \infty)$



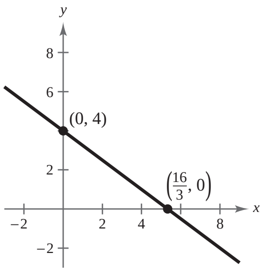
109. 4.9%

111. False. $\sqrt{-18}\sqrt{-2} = \left(3\sqrt{2}\,i\right)\left(\sqrt{2}i\right) = 6i^2 = -6$ and $\sqrt{(-18)(-2)} = \sqrt{36} = 6$

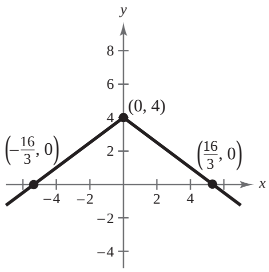
113. Some solutions to certain types of equations may be extraneous solutions, which do not satisfy the original equations. So, checking is crucial.

Chapter Test *(page 155)*

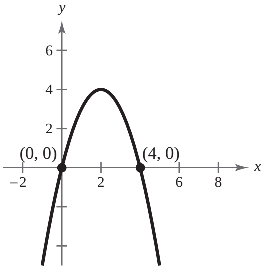
1. No symmetry



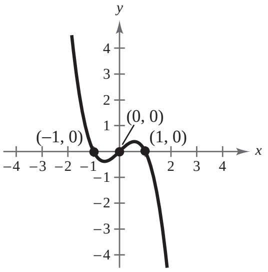
2. y-axis symmetry



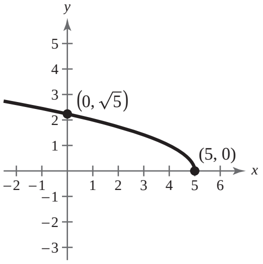
3. No symmetry



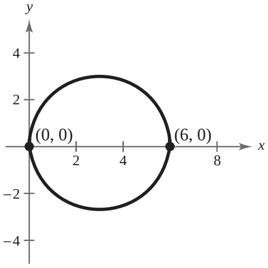
4. Origin symmetry



5. No symmetry



6. x-axis symmetry

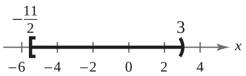


7. $\frac{128}{11}$ **8.** $-3, 5$

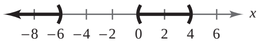
9. No solution. The variable is divided out.

10. $\pm\sqrt{2}, \pm\sqrt{3}i$ **11.** 4 **12.** $-2, \frac{8}{3}$

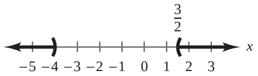
13. $-\frac{11}{2} \le x < 3$



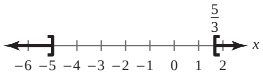
14. $x < -6$ or $0 < x < 4$



15. $x < -4$ or $x > \frac{3}{2}$

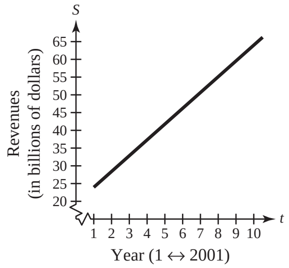


16. $x \le -5$ or $x \ge \frac{5}{3}$



17. (a) $-3 + 5i$ (b) 26 **18.** $2 - i$

19. (a)

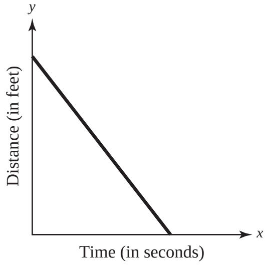


(b) and (c) \$86.25 billion

20. 4.774 in. **21.** $93\frac{3}{4}$ km/h **22.** $a = 80, b = 20$

Problem Solving *(page 157)*

1.



3. (a) Answers will vary. *Sample answer:*

$A = \pi ab$

$b = 20 - a$, since $a + b = 20$

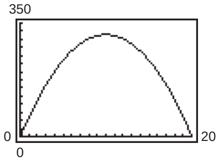
$A = \pi a(20 - a)$

(b)

a	4	7	10	13	16
A	64π	91π	100π	91π	64π

(c) $\dfrac{10\pi + 10\sqrt{\pi(\pi - 3)}}{\pi} \approx 12.12$ or $\dfrac{10\pi - 10\sqrt{\pi(\pi - 3)}}{\pi} \approx 7.88$

(d)



(e) (0, 0), (20, 0)

They represent the minimum and maximum values of a.

(f) 100π; $a = 10$; $b = 10$

5. (a) About 60.6 sec (b) About 146.2 sec

(c) The speed at which water drains decreases as the amount of water in the bathtub decreases.

7. (a) Answers will vary. *Sample answers:* 5, 12, 13; 8, 15, 17.

(b) Yes; yes; yes

(c) The product of the three numbers in a Pythagorean Triple is divisible by 60.

9. Proof

11. (a) $\frac{1}{2} - \frac{1}{2}i$ (b) $\frac{3}{10} + \frac{1}{10}i$ (c) $-\frac{1}{34} - \frac{2}{17}i$

13. (a) Yes (b) No (c) Yes

15. $(-\infty, -2) \cup (-1, 2) \cup (2, \infty)$

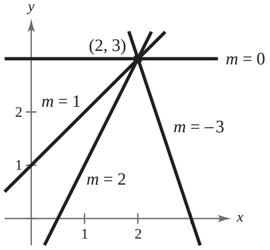
Chapter 2

Section 2.1 *(page 169)*

1. linear **3.** point-slope **5.** perpendicular

7. linear extrapolation **9.** (a) L_2 (b) L_3 (c) L_1

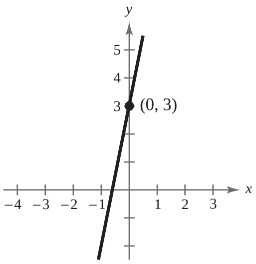
11.



13. $\frac{3}{2}$

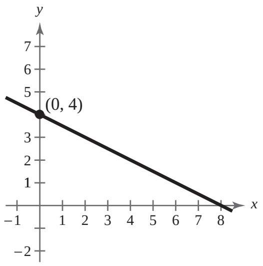
15. $m = 5$

y-intercept: (0, 3)

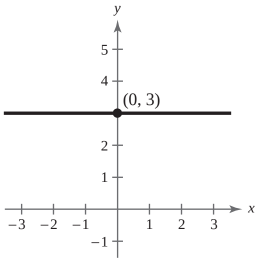


17. $m = -\frac{1}{2}$

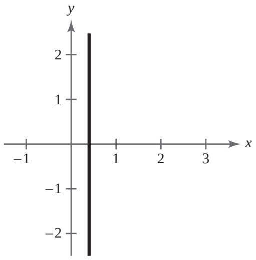
y-intercept: (0, 4)



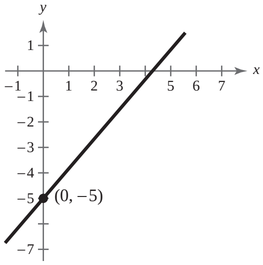
19. $m = 0$
y-intercept: $(0, 3)$



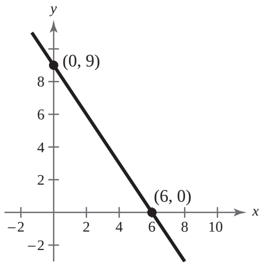
21. m is undefined.
There is no y-intercept.



23. $m = \frac{7}{6}$
y-intercept: $(0, -5)$

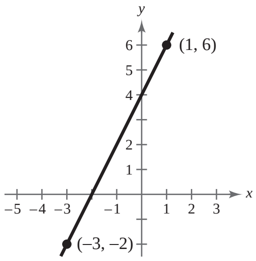


25.



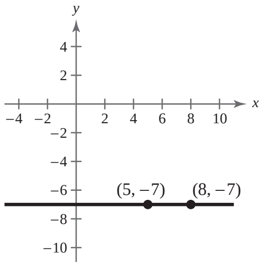
$m = -\frac{3}{2}$

27.



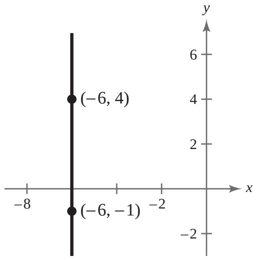
$m = 2$

29.



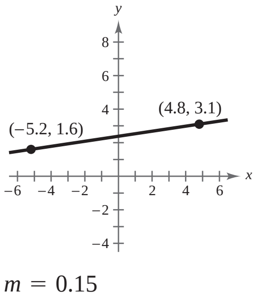
$m = 0$

31.



m is undefined.

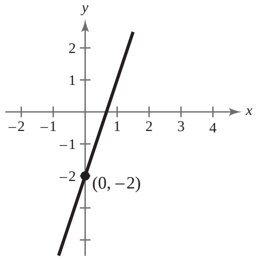
33.



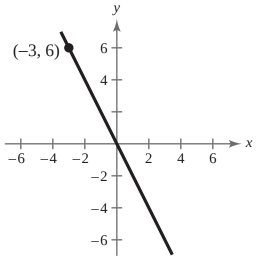
$m = 0.15$

35. $(0, 1), (3, 1), (-1, 1)$ **37.** $(-8, 0), (-8, 2), (-8, 3)$
39. $(-4, 6), (-3, 8), (-2, 10)$
41. $(-3, -5), (1, -7), (5, -9)$

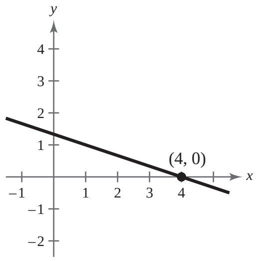
43. $y = 3x - 2$



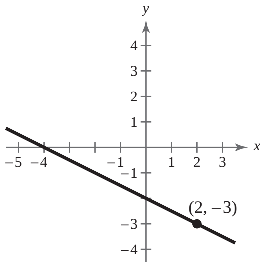
45. $y = -2x$



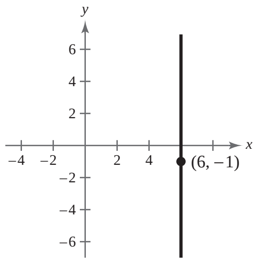
47. $y = -\frac{1}{3}x + \frac{4}{3}$



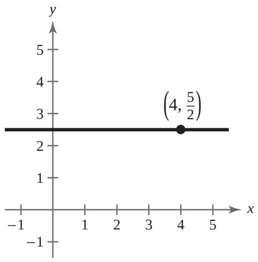
49. $y = -\frac{1}{2}x - 2$



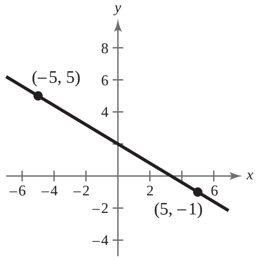
51. $x = 6$



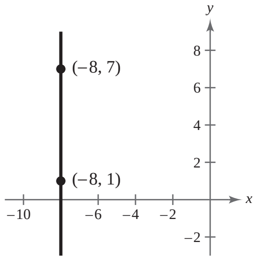
53. $y = \frac{5}{2}$



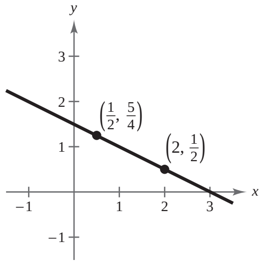
55. $y = -\frac{3}{5}x + 2$



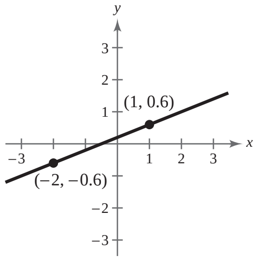
57. $x = -8$



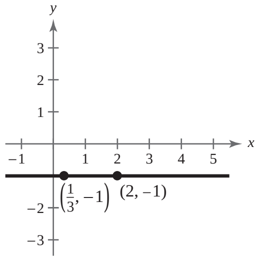
59. $y = -\frac{1}{2}x + \frac{3}{2}$



61. $y = 0.4x + 0.2$



63. $y = -1$



65. Parallel **67.** Neither **69.** Perpendicular

71. Parallel **73.** (a) $y = 2x - 3$ (b) $y = -\frac{1}{2}x + 2$

75. (a) $y = -\frac{3}{4}x + \frac{3}{8}$ (b) $y = \frac{4}{3}x + \frac{127}{72}$

77. (a) $y = 0$ (b) $x = -1$

79. (a) $y = x + 4.3$ (b) $y = -x + 9.3$

81. $3x + 2y - 6 = 0$ **83.** $12x + 3y + 2 = 0$

85. $x + y - 3 = 0$

87. (a) Sales increasing 135 units/yr
(b) No change in sales
(c) Sales decreasing 40 units/yr

89. 12 ft **91.** $V(t) = -125t + 4165,\ 13 \le t \le 18$

93. C-intercept: fixed initial cost; Slope: cost of producing an additional laptop bag

95. $V(t) = -175t + 875,\ 0 \le t \le 5$

97. $F = 1.8C + 32$ or $C = \frac{5}{9}F - \frac{160}{9}$

99. (a) $C = 21t + 42{,}000$ (b) $R = 45t$
(c) $P = 24t - 42{,}000$ (d) 1750 h

101. False. The slope with the greatest magnitude corresponds to the steepest line.

103. Find the slopes of the lines containing each two points and use the relationship $m_1 = -\dfrac{1}{m_2}$.

105. No. The slope cannot be determined without knowing the scale on the y-axis. The slopes could be the same.

107. No. The slopes of two perpendicular lines have opposite signs (assume that neither line is vertical or horizontal).

109. The line $y = 4x$ rises most quickly, and the line $y = -4x$ falls most quickly. The greater the magnitude of the slope (the absolute value of the slope), the faster the line rises or falls.

111. $3x - 2y - 1 = 0$ **113.** $80x + 12y + 139 = 0$

Section 2.2 *(page 182)*

1. domain; range; function **3.** implied domain

5. Yes, each input value has exactly one output value.

7. No, the input values 7 and 10 each have two different output values.

9. (a) Function
(b) Not a function, because the element 1 in A corresponds to two elements, -2 and 1, in B.
(c) Function
(d) Not a function, because not every element in A is matched with an element in B.

11. Not a function **13.** Function **15.** Function

17. Function **19.** Function

21. (a) -1 (b) -9 (c) $2x - 5$

23. (a) 15 (b) $4t^2 - 19t + 27$ (c) $4t^2 - 3t - 10$

25. (a) 1 (b) 2.5 (c) $3 - 2|x|$

27. (a) $-\dfrac{1}{9}$ (b) Undefined (c) $\dfrac{1}{y^2 + 6y}$

29. (a) 1 (b) -1 (c) $\dfrac{|x - 1|}{x - 1}$

31. (a) -1 (b) 2 (c) 6

33.

x	1	2	3	4	5
$f(x)$	8	5	0	1	2

35.

t	-5	-4	-3	-2	-1
$h(t)$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1

37. 5 **39.** $\frac{4}{3}$ **41.** ± 3 **43.** $0, \pm 1$ **45.** $-1, 2$

47. $0, \pm 2$ **49.** All real numbers x

51. All real numbers t except $t = 0$

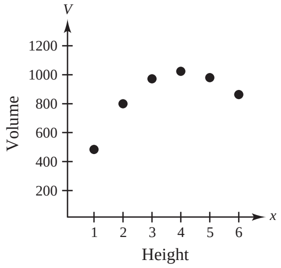
53. All real numbers y such that $y \ge 10$

55. All real numbers x except $x = 0, -2$

57. All real numbers s such that $s \ge 1$ except $s = 4$

59. All real numbers x such that $x > 0$

61. (a) The maximum volume is 1024 cubic centimeters.
(b) Yes, V is a function of x.



(c) $V = x(24 - 2x)^2,\quad 0 < x < 12$

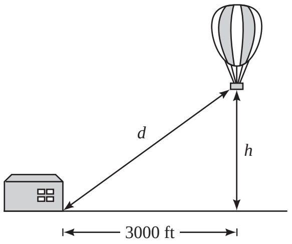
63. $A = \dfrac{P^2}{16}$ **65.** Yes, the ball will be at a height of 6 feet.

67. $A = \dfrac{x^2}{2(x - 2)},\quad x > 2$

69. 2004: 45.58%
2005: 50.15%
2006: 54.72%
2007: 59.29%
2008: 64.40%
2009: 67.75%
2010: 71.10%

71. (a) $C = 12.30x + 98{,}000$ (b) $R = 17.98x$
(c) $P = 5.68x - 98{,}000$

73. (a)



(b) $h = \sqrt{d^2 - 3000^2},\quad d \ge 3000$

75. (a) $R = \dfrac{240n - n^2}{20},\quad n \ge 80$

(b)

n	90	100	110	120	130	140	150
$R(n)$	\$675	\$700	\$715	\$720	\$715	\$700	\$675

The revenue is maximum when 120 people take the trip.

77. $3 + h,\ h \ne 0$ **79.** $3x^2 + 3xh + h^2 + 3,\ h \ne 0$

81. $-\dfrac{x + 3}{9x^2},\ x \ne 3$ **83.** $\dfrac{\sqrt{5x} - 5}{x - 5}$

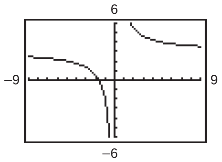
85. $g(x) = cx^2;\ c = -2$ **87.** $r(x) = \dfrac{c}{x};\ c = 32$

89. False. A function is a special type of relation.
91. False. The range is $[-1, \infty)$.
93. Domain of $f(x)$: all real numbers $x \geq 1$
Domain of $g(x)$: all real numbers $x > 1$
Notice that the domain of $f(x)$ includes $x = 1$ and the domain of $g(x)$ does not because you cannot divide by 0.
95. No; x is the independent variable, f is the name of the function.
97. (a) Yes. The amount you pay in sales tax will increase as the price of the item purchased increases.
(b) No. The length of time that you study will not necessarily determine how well you do on an exam.

Section 2.3 *(page 194)*

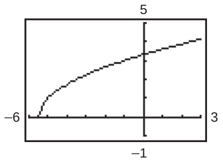
1. Vertical Line Test **3.** decreasing
5. average rate of change; secant
7. Domain: $(-\infty, \infty)$; Range: $[-4, \infty)$
(a) 0 (b) -1 (c) 0 (d) -2
9. Domain: $(-\infty, \infty)$; Range: $(-2, \infty)$
(a) 0 (b) 1 (c) 2 (d) 3
11. Function **13.** Not a function **15.** $-\frac{5}{2}, 6$ **17.** 0
19. $0, \pm\sqrt{2}$ **21.** $\pm 3, 4$ **23.** $\frac{1}{2}$

25.



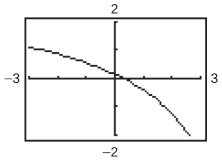
$-\frac{5}{3}$

27.



$-\frac{11}{2}$

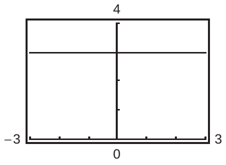
29.



$\frac{1}{3}$

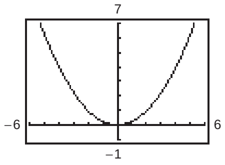
31. Increasing on $(-\infty, \infty)$
33. Increasing on $(-\infty, 0)$ and $(2, \infty)$; Decreasing on $(0, 2)$
35. Increasing on $(1, \infty)$; Decreasing on $(-\infty, -1)$
Constant on $(-1, 1)$
37. Increasing on $(-\infty, 0)$ and $(2, \infty)$; Constant on $(0, 2)$

39.



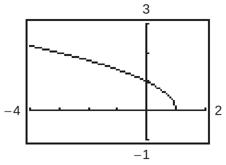
Constant on $(-\infty, \infty)$

41.



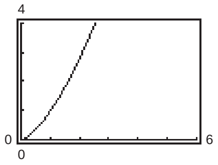
Decreasing on $(-\infty, 0)$
Increasing on $(0, \infty)$

43.



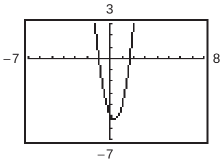
Decreasing on $(-\infty, 1)$

45.



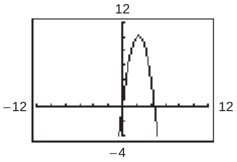
Increasing on $(0, \infty)$

47.



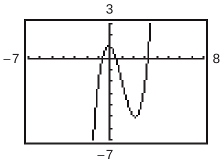
Relative minimum:
$\left(\frac{1}{3}, -\frac{16}{3}\right)$

49.



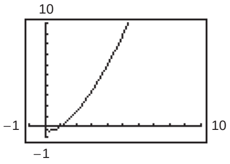
Relative maximum:
(2.25, 10.125)

51.



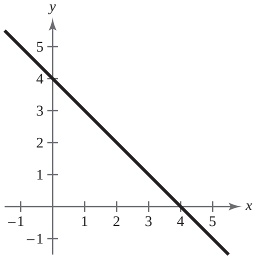
Relative maximum:
$(-0.15, 1.08)$
Relative minimum:
$(2.15, -5.08)$

53.



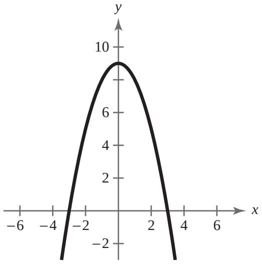
Relative minimum:
$(0.33, -0.38)$

55.



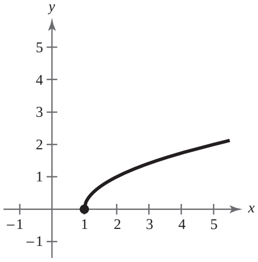
$(-\infty, 4]$

57.



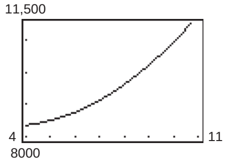
$[-3, 3]$

59.



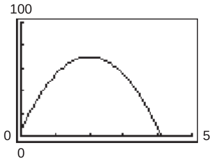
$[1, \infty)$

61. The average rate of change from $x_1 = 0$ to $x_2 = 3$ is -2.
63. The average rate of change from $x_1 = 1$ to $x_2 = 3$ is 0.
65. (a)

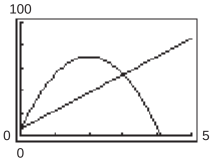


(b) 484.75 million; The amount the U.S. Department of Energy spent for research and development increased by about \$484.75 million each year from 2005 to 2010.

67. (a) $s = -16t^2 + 64t + 6$
(b)

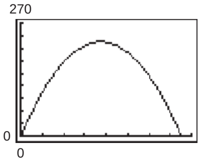


(c) Average rate of change $= 16$
(d) The slope of the secant line is positive.
(e) Secant line: $16t + 6$
(f)



69. (a) $s = -16t^2 + 120t$

(b)

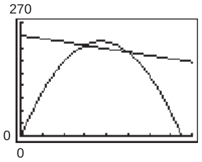


(c) Average rate of change $= -8$

(d) The slope of the secant line is negative.

(e) Secant line: $-8t + 240$

(f)

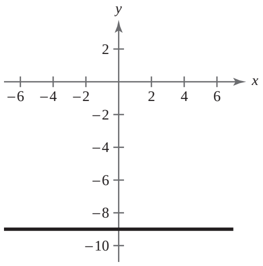


71. Even; y-axis symmetry

73. Odd; origin symmetry

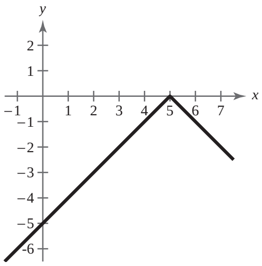
75. Neither; no symmetry

77.



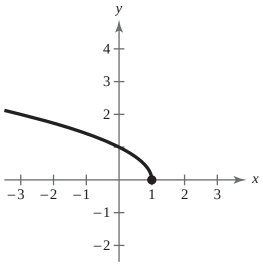
Even

79.



Neither

81.

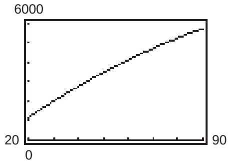


Neither

83. $h = 3 - 4x + x^2$

85. $L = 2 - \sqrt[3]{2y}$

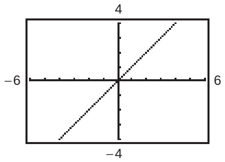
87. (a)



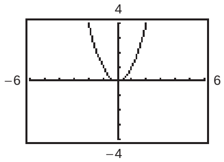
(b) 30 W

89. (a) Ten thousands (b) Ten millions (c) Percents

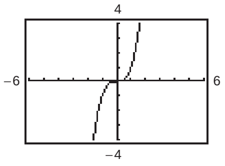
91. (a)



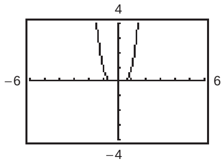
(b)



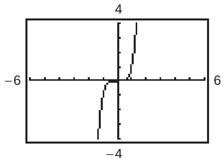
(c)



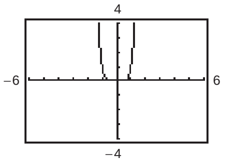
(d)



(e)



(f)

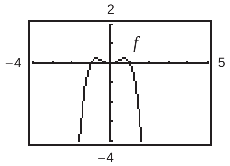


All the graphs pass through the origin. The graphs of the odd powers of x are symmetric with respect to the origin, and the graphs of the even powers are symmetric with respect to the y-axis. As the powers increase, the graphs become flatter in the interval $-1 < x < 1$.

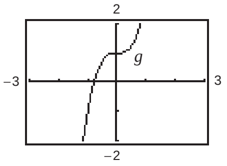
93. False. The function $f(x) = \sqrt{x^2 + 1}$ has a domain of all real numbers.

95. (a) $\left(\frac{5}{3}, -7\right)$ (b) $\left(\frac{5}{3}, 7\right)$

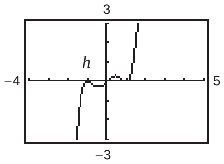
97.



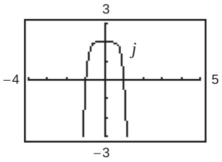
Even



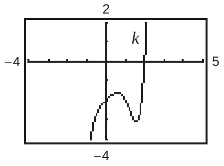
Neither



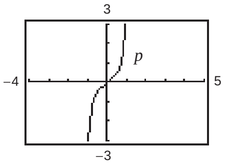
Odd



Even



Neither



Odd

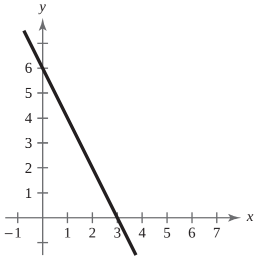
Equations of odd functions contain only odd powers of x. Equations of even functions contain only even powers of x. Odd functions have all variables raised to odd powers and even functions have all variables raised to even powers. A function that has variables raised to even and odd powers is neither odd nor even.

Section 2.4 *(page 203)*

1. g **2.** i **3.** h **4.** a **5.** b **6.** e **7.** f
8. c **9.** d

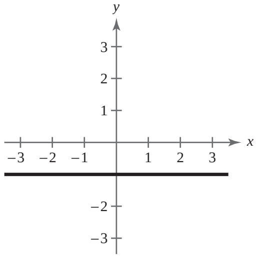
11. (a) $f(x) = -2x + 6$

(b)

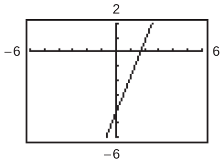


13. (a) $f(x) = -1$

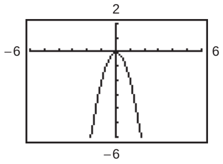
(b)



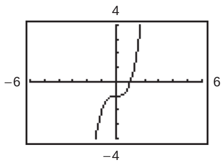
15.



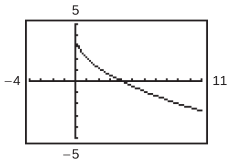
17.



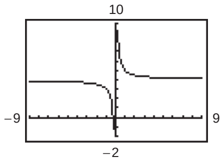
19.



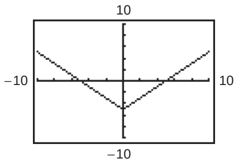
21.



23.



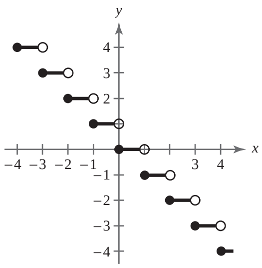
25.



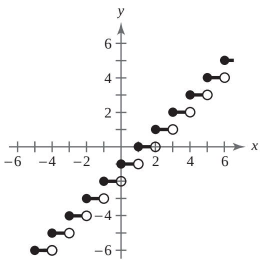
27. (a) 2 (b) 2 (c) −4 (d) 3

29. (a) 8 (b) 2 (c) 6 (d) 13

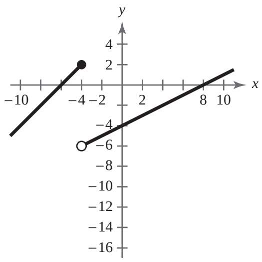
31.



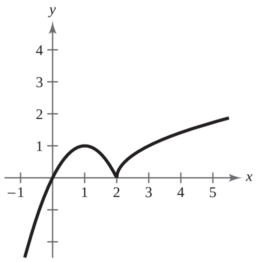
33.



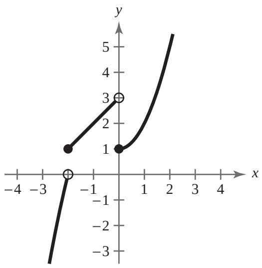
35.



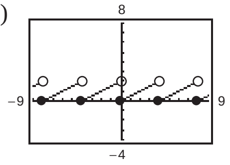
37.



39.



41. (a)



(b) Domain: $(-\infty, \infty)$

Range: $[0, 2)$

43. (a) $W(30) = 420$; $W(40) = 560$;

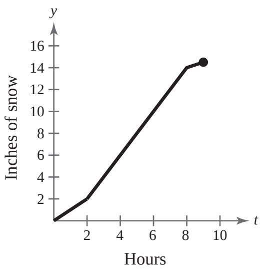
$W(45) = 665$; $W(50) = 770$

(b) $W(h) = \begin{cases} 14h, & 0 < h \le 45 \\ 21(h - 45) + 630, & h > 45 \end{cases}$

45.

Interval	**Input Pipe**	**Drain Pipe 1**	**Drain Pipe 2**
$[0, 5]$	Open	Closed	Closed
$[5, 10]$	Open	Open	Closed
$[10, 20]$	Closed	Closed	Closed
$[20, 30]$	Closed	Closed	Open
$[30, 40]$	Open	Open	Open
$[40, 45]$	Open	Closed	Open
$[45, 50]$	Open	Open	Open
$[50, 60]$	Open	Open	Closed

47. $f(t) = \begin{cases} t, & 0 \le t \le 2 \\ 2t - 2, & 2 < t \le 8 \\ \frac{1}{2}t + 10, & 8 < t \le 9 \end{cases}$



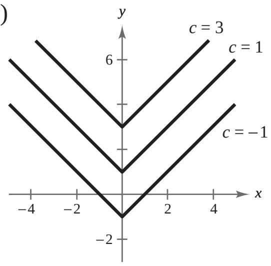
Total accumulation = 14.5 in.

49. False. A piecewise-defined function is a function that is defined by two or more equations over a specified domain. That domain may or may not include *x*- and *y*-intercepts.

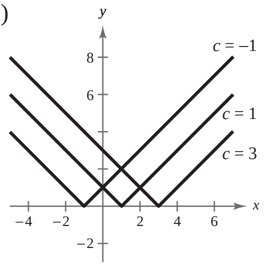
Section 2.5 *(page 210)*

1. rigid **3.** vertical stretch; vertical shrink

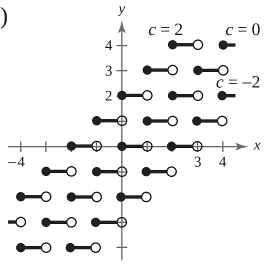
5. (a)



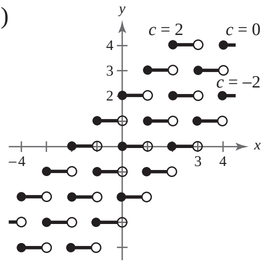
(b)



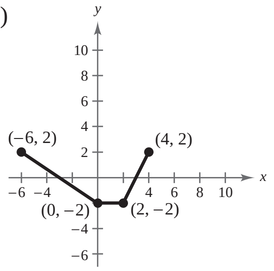
7. (a)



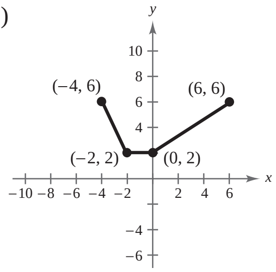
(b)



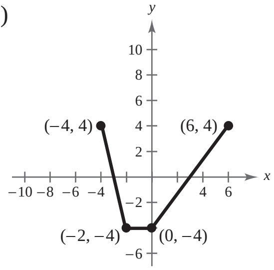
9. (a)



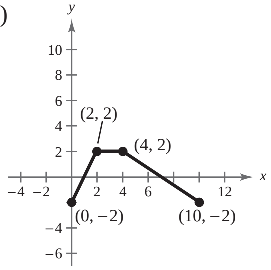
(b)



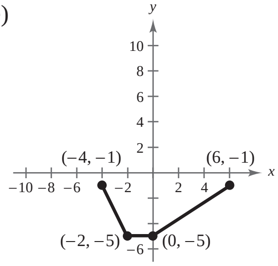
(c)



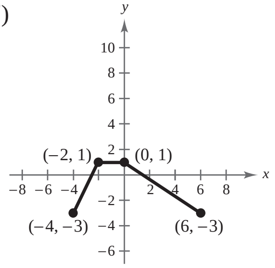
(d)



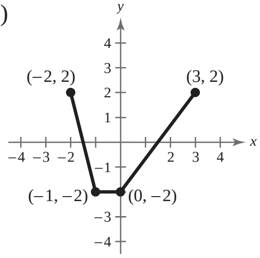
(e)



(f)



(g)



11. (a) $y = x^2 - 1$ (b) $y = 1 - (x + 1)^2$

13. (a) $y = -|x + 3|$ (b) $y = |x - 2| - 4$

15. Horizontal shift of $y = x^3$; $y = (x - 2)^3$

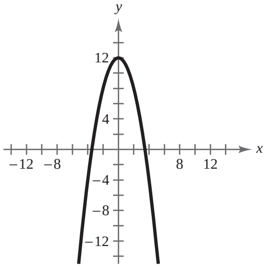
17. Reflection in the x-axis of $y = x^2$; $y = -x^2$

19. Reflection in the x-axis and vertical shift of $y = \sqrt{x}$; $y = 1 - \sqrt{x}$

21. (a) $f(x) = x^2$

(b) Reflection in the x-axis and vertical shift 12 units up

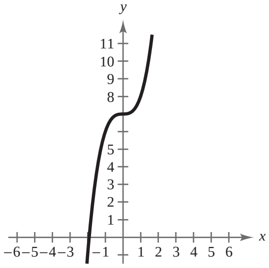
(c)



(d) $g(x) = 12 - f(x)$

23. (a) $f(x) = x^3$ (b) Vertical shift seven units up

(c)

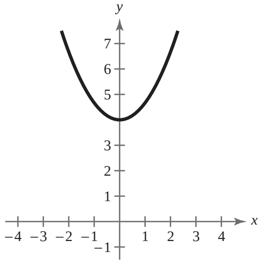


(d) $g(x) = f(x) + 7$

25. (a) $f(x) = x^2$

(b) Vertical shrink of two-thirds and vertical shift four units up

(c)

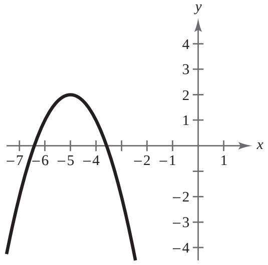


(d) $g(x) = \frac{2}{3}f(x) + 4$

27. (a) $f(x) = x^2$

(b) Reflection in the x-axis, horizontal shift five units to the left, and vertical shift two units up

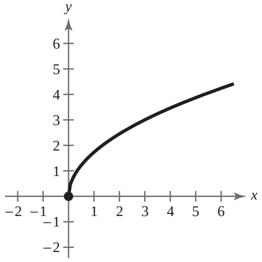
(c)



(d) $g(x) = 2 - f(x + 5)$

29. (a) $f(x) = \sqrt{x}$ (b) Horizontal shrink of one-third

(c)

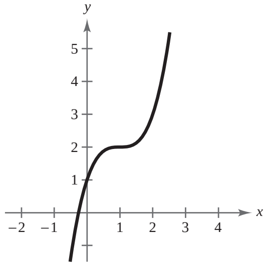


(d) $g(x) = f(3x)$

31. (a) $f(x) = x^3$

(b) Vertical shift two units up and horizontal shift one unit to the right

(c)

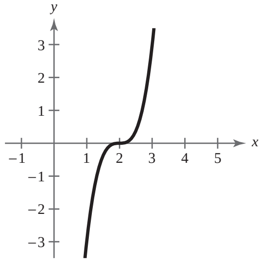


(d) $g(x) = f(x - 1) + 2$

33. (a) $f(x) = x^3$

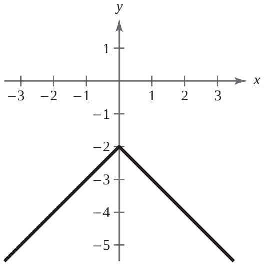
(b) Vertical stretch of three and horizontal shift two units to the right

(c)



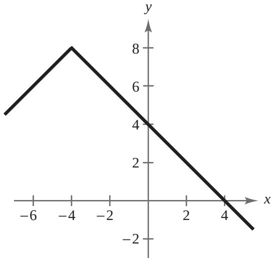
(d) $g(x) = 3f(x - 2)$

35. (a) $f(x) = |x|$
(b) Reflection in the x-axis and vertical shift two units down
(c)



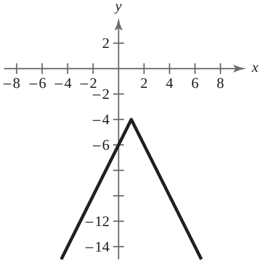
(d) $g(x) = -f(x) - 2$

37. (a) $f(x) = |x|$
(b) Reflection in the x-axis, horizontal shift four units to the left, and vertical shift eight units up
(c)



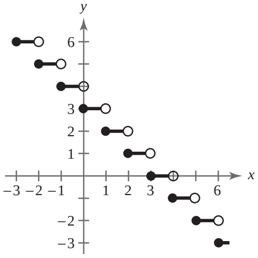
(d) $g(x) = -f(x + 4) + 8$

39. (a) $f(x) = |x|$
(b) Reflection in the x-axis, vertical stretch of two, horizontal shift one unit to the right, and vertical shift four units down
(c)



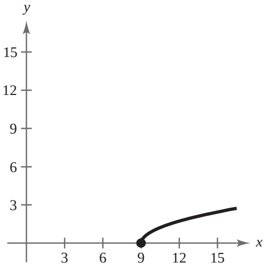
(d) $g(x) = -2f(x - 1) - 4$

41. (a) $f(x) = [\![x]\!]$
(b) Reflection in the x-axis and vertical shift three units up
(c)



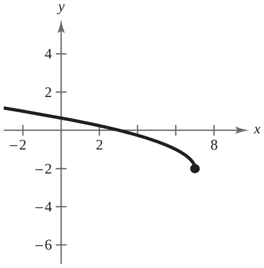
(d) $g(x) = 3 - f(x)$

43. (a) $f(x) = \sqrt{x}$ (b) Horizontal shift nine units to the right
(c)



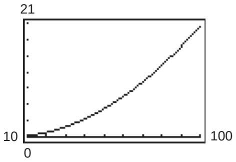
(d) $g(x) = f(x - 9)$

45. (a) $f(x) = \sqrt{x}$
(b) Reflection in the y-axis, horizontal shift seven units to the right, and vertical shift two units down
(c)



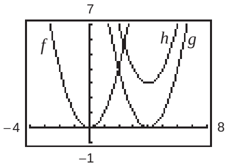
(d) $g(x) = f(7 - x) - 2$

47. $g(x) = (x - 3)^2 - 7$ **49.** $g(x) = (x - 13)^3$
51. $g(x) = -|x| + 12$ **53.** $g(x) = -\sqrt{-x + 6}$
55. (a) $y = -3x^2$ (b) $y = 4x^2 + 3$
57. (a) $y = -\frac{1}{2}|x|$ (b) $y = 3|x| - 3$
59. Vertical stretch of $y = x^3$; $y = 2x^3$
61. Reflection in the x-axis and vertical shrink of $y = x^2$; $y = -\frac{1}{2}x^2$
63. Reflection in the y-axis and vertical shrink of $y = \sqrt{x}$; $y = \frac{1}{2}\sqrt{-x}$
65. $y = -(x - 2)^3 + 2$
67. $y = -\sqrt{x} - 3$
69. (a)



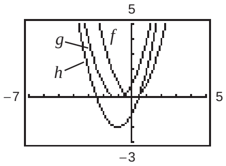
(b) $H\left(\dfrac{x}{1.6}\right) = 0.00078x^2 + 0.003x - 0.029,\ 16 \le x \le 160$;
Horizontal stretch

71. False. The graph of $y = f(-x)$ is a reflection of the graph of $f(x)$ in the y-axis.
73. True. $|-x| = |x|$
75. $(-2, 0), (-1, 1), (0, 2)$
77. (a)



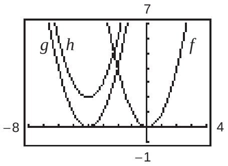
g is a right shift of four units. h is a right shift of four units and a shift of three units up.

(b)



g is a left shift of one unit. h is a left shift of one unit and a shift of two units down.

(c)



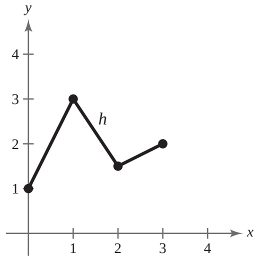
g is a left shift of four units. h is a left shift of four units and a shift of two units up.

79. (a) $g(t) = \frac{3}{4}f(t)$ (b) $g(t) = f(t) + 10{,}000$
(c) $g(t) = f(t - 2)$

Section 2.6 *(page 219)*

1. addition; subtraction; multiplication; division

3.



5. (a) $2x$ (b) 4 (c) $x^2 - 4$

(d) $\dfrac{x+2}{x-2}$; all real numbers x except $x = 2$

7. (a) $x^2 + 4x - 5$ (b) $x^2 - 4x + 5$ (c) $4x^3 - 5x^2$

(d) $\dfrac{x^2}{4x-5}$; all real numbers x except $x = \dfrac{5}{4}$

9. (a) $x^2 + 6 + \sqrt{1-x}$ (b) $x^2 + 6 - \sqrt{1-x}$

(c) $(x^2+6)\sqrt{1-x}$

(d) $\dfrac{(x^2+6)\sqrt{1-x}}{1-x}$; all real numbers x such that $x < 1$

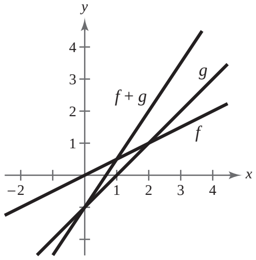
11. (a) $\dfrac{x+1}{x^2}$ (b) $\dfrac{x-1}{x^2}$ (c) $\dfrac{1}{x^3}$

(d) x; all real numbers x except $x = 0$

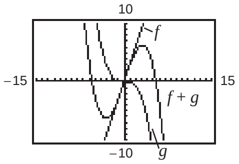
13. 3 **15.** 5 **17.** $9t^2 - 3t + 5$ **19.** 74

21. 26 **23.** $\frac{3}{5}$

25.

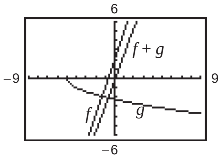


27.



$f(x)$, $g(x)$

29.



$f(x)$, $f(x)$

31. (a) $(x-1)^2$ (b) $x^2 - 1$ (c) $x - 2$

33. (a) x (b) x (c) $x^9 + 3x^6 + 3x^3 + 2$

35. (a) $\sqrt{x^2+4}$ (b) $x + 4$

Domains of f and $g \circ f$: all real numbers x such that $x \geq -4$

Domains of g and $f \circ g$: all real numbers x

37. (a) $x + 1$ (b) $\sqrt{x^2+1}$

Domains of f and $g \circ f$: all real numbers x

Domains of g and $f \circ g$: all real numbers x such that $x \geq 0$

39. (a) $|x+6|$ (b) $|x| + 6$

Domains of f, g, $f \circ g$, and $g \circ f$: all real numbers x

41. (a) $\dfrac{1}{x+3}$ (b) $\dfrac{1}{x} + 3$

Domains of f and $g \circ f$: all real numbers x except $x = 0$

Domain of g: all real numbers x

Domain of $f \circ g$: all real numbers x except $x = -3$

43. (a) 3 (b) 0 **45.** (a) 0 (b) 4

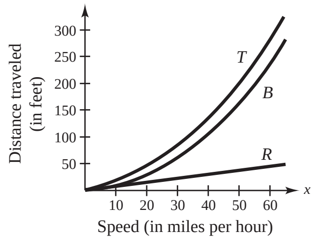
47. $f(x) = x^2$, $g(x) = 2x + 1$

49. $f(x) = \sqrt[3]{x}$, $g(x) = x^2 - 4$

51. $f(x) = \dfrac{1}{x}$, $g(x) = x + 2$ **53.** $f(x) = \dfrac{x+3}{4+x}$, $g(x) = -x^2$

55. (a) $T = \frac{3}{4}x + \frac{1}{15}x^2$

(b)



(c) The braking function $B(x)$. As x increases, $B(x)$ increases at a faster rate than $R(x)$.

57. (a) $p(t) = d(t) + c(t)$

(b) $p(13)$ is the number of dogs and cats in the year 2013.

(c) $h(t) = \dfrac{d(t) + c(t)}{n(t)}$;

The function $h(t)$ represents the number of dogs and cats per capita.

59. (a) $r(x) = \dfrac{x}{2}$

(b) $A(r) = \pi r^2$

(c) $(A \circ r)(x) = \pi\left(\dfrac{x}{2}\right)^2$;

$(A \circ r)(x)$ represents the area of the circular base of the tank on the square foundation with side length x.

61. $g(f(x))$ represents 3 percent of an amount over $500,000.

63. (a) $O(M(Y)) = 2\left(6 + \frac{1}{2}Y\right) = 12 + Y$

(b) Middle child is 8 years old; youngest child is 4 years old.

65. False. $(f \circ g)(x) = 6x + 1$ and $(g \circ f)(x) = 6x + 6$

67–69. Proofs

Section 2.7 *(page 228)*

1. inverse **3.** range; domain **5.** one-to-one

7. $f^{-1}(x) = \dfrac{1}{6}x$ **9.** $f^{-1}(x) = \dfrac{x-1}{3}$ **11.** $f^{-1}(x) = x^3$

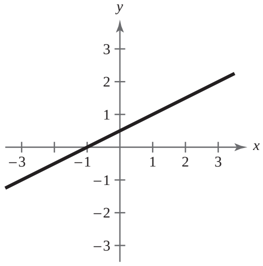
13. $f(g(x)) = f\left(-\dfrac{2x+6}{7}\right) = -\dfrac{7}{2}\left(-\dfrac{2x+6}{7}\right) - 3 = x$

$g(f(x)) = g\left(-\dfrac{7}{2}x - 3\right) = -\dfrac{2\left(-\frac{7}{2}x - 3\right) + 6}{7} = x$

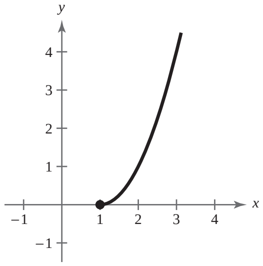
15. $f(g(x)) = f\left(\sqrt[3]{x-5}\right) = \left(\sqrt[3]{x-5}\right)^3 + 5 = x$

$g(f(x)) = g(x^3 + 5) = \sqrt[3]{(x^3+5) - 5} = x$

17.



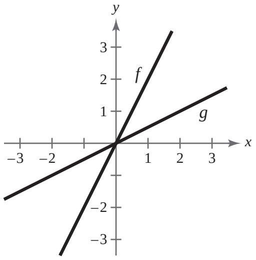
19.



21. (a) $f(g(x)) = f\left(\frac{x}{2}\right) = 2\left(\frac{x}{2}\right) = x$

$g(f(x)) = g(2x) = \frac{(2x)}{2} = x$

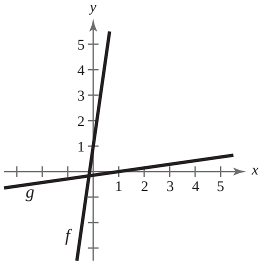
(b)



23. (a) $f(g(x)) = f\left(\frac{x-1}{7}\right) = 7\left(\frac{x-1}{7}\right) + 1 = x$

$g(f(x)) = g(7x+1) = \frac{(7x+1)-1}{7} = x$

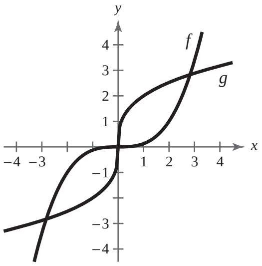
(b)



25. (a) $f(g(x)) = f\left(\sqrt[3]{8x}\right) = \frac{\left(\sqrt[3]{8x}\right)^3}{8} = x$

$g(f(x)) = g\left(\frac{x^3}{8}\right) = \sqrt[3]{8\left(\frac{x^3}{8}\right)} = x$

(b)



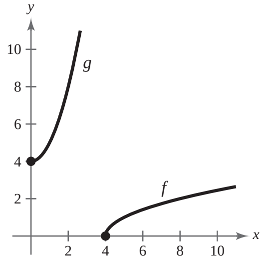
27. (a) $f(g(x)) = f(x^2 + 4),\ x \geq 0$

$= \sqrt{(x^2 + 4) - 4} = x$

$g(f(x)) = g\left(\sqrt{x-4}\right)$

$= \left(\sqrt{x-4}\right)^2 + 4 = x$

(b)



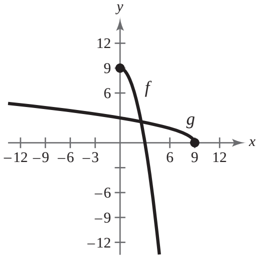
29. (a) $f(g(x)) = f\left(\sqrt{9-x}\right),\ x \leq 9$

$= 9 - \left(\sqrt{9-x}\right)^2 = x$

$g(f(x)) = g(9 - x^2),\ x \geq 0$

$= \sqrt{9 - (9 - x^2)} = x$

(b)



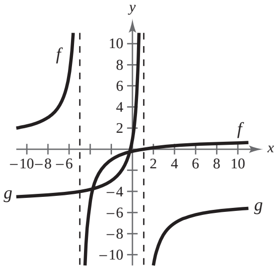
31. (a) $f(g(x)) = f\left(-\frac{5x+1}{x-1}\right) = \frac{-\left(\frac{5x+1}{x-1}\right) - 1}{-\left(\frac{5x+1}{x-1}\right) + 5}$

$= \frac{-5x - 1 - x + 1}{-5x - 1 + 5x - 5} = x$

$g(f(x)) = g\left(\frac{x-1}{x+5}\right) = \frac{-5\left(\frac{x-1}{x+5}\right) - 1}{\frac{x-1}{x+5} - 1}$

$= \frac{-5x + 5 - x - 5}{x - 1 - x - 5} = x$

(b)



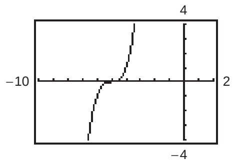
33. No

35.

x	−2	0	2	4	6	8
$f^{-1}(x)$	−2	−1	0	1	2	3

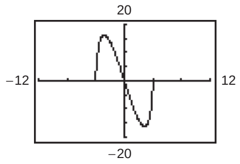
37. Yes **39.** No

41.



The function has an inverse function.

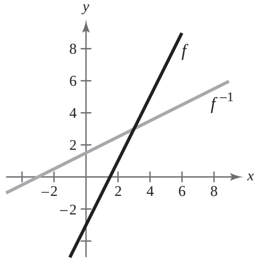
43.



The function does not have an inverse function.

45. (a) $f^{-1}(x) = \dfrac{x+3}{2}$

(b)

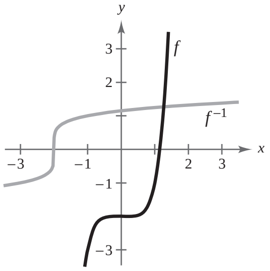


(c) The graph of f^{-1} is the reflection of the graph of f in the line $y = x$.

(d) The domains and ranges of f and f^{-1} are all real numbers.

47. (a) $f^{-1}(x) = \sqrt[5]{x+2}$

(b)

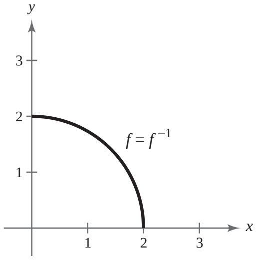


(c) The graph of f^{-1} is the reflection of the graph of f in the line $y = x$.

(d) The domains and ranges of f and f^{-1} are all real numbers.

49. (a) $f^{-1}(x) = \sqrt{4 - x^2}$, $0 \le x \le 2$

(b)

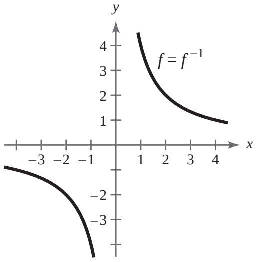


(c) The graph of f^{-1} is the same as the graph of f.

(d) The domains and ranges of f and f^{-1} are all real numbers x such that $0 \le x \le 2$.

51. (a) $f^{-1}(x) = \dfrac{4}{x}$

(b)

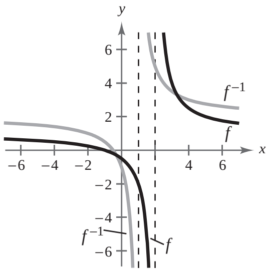


(c) The graph of f^{-1} is the same as the graph of f.

(d) The domains and ranges of f and f^{-1} are all real numbers x except $x = 0$.

53. (a) $f^{-1}(x) = \dfrac{2x+1}{x-1}$

(b)

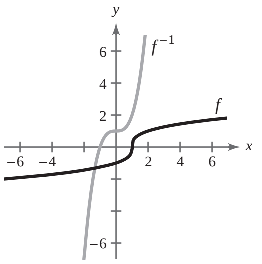


(c) The graph of f^{-1} is the reflection of the graph of f in the line $y = x$.

(d) The domain of f and the range of f^{-1} are all real numbers x except $x = 2$. The domain of f^{-1} and the range of f are all real numbers x except $x = 1$.

55. (a) $f^{-1}(x) = x^3 + 1$

(b)



(c) The graph of f^{-1} is the reflection of the graph of f in the line $y = x$.

(d) The domains and ranges of f and f^{-1} are all real numbers.

57. No inverse function **59.** $g^{-1}(x) = 8x$

61. No inverse function **63.** $f^{-1}(x) = \sqrt{x} - 3$

65. No inverse function **67.** No inverse function

69. $f^{-1}(x) = \dfrac{x^2 - 3}{2}$, $x \ge 0$ **71.** $f^{-1}(x) = \dfrac{5x - 4}{6 - 4x}$

73. $f^{-1}(x) = \sqrt{x} + 2$

The domain of f and the range of f^{-1} are all real numbers x such that $x \ge 2$. The domain of f^{-1} and the range of f are all real numbers x such that $x \ge 0$.

75. $f^{-1}(x) = x - 2$

The domain of f and the range of f^{-1} are all real numbers x such that $x \ge -2$. The domain of f^{-1} and the range of f are all real numbers x such that $x \ge 0$.

77. $f^{-1}(x) = \sqrt{x} - 6$

The domain of f and the range of f^{-1} are all real numbers x such that $x \ge -6$. The domain of f^{-1} and the range of f are all real numbers x such that $x \ge 0$.

79. $f^{-1}(x) = \dfrac{\sqrt{-2(x-5)}}{2}$

The domain of f and the range of f^{-1} are all real numbers x such that $x \ge 0$. The domain of f^{-1} and the range of f are all real numbers x such that $x \le 5$.

81. $f^{-1}(x) = x + 3$

The domain of f and the range of f^{-1} are all real numbers x such that $x \ge 4$. The domain of f^{-1} and the range of f are all real numbers x such that $x \ge 1$.

83. 32 **85.** 600 **87.** $2\sqrt[3]{x+3}$ **89.** $\dfrac{x+1}{2}$

91. $\dfrac{x+1}{2}$

93. (a) $y = \dfrac{x - 10}{0.75}$

x = hourly wage; y = number of units produced

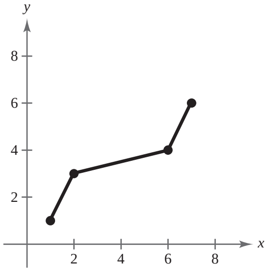
(b) 19 units

95. False. $f(x) = x^2$ has no inverse function.

97.

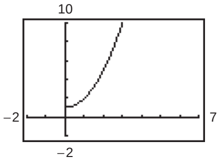
x	1	3	4	6
y	1	2	6	7

x	1	2	6	7
$f^{-1}(x)$	1	3	4	6



99. Proof **101.** $k = \frac{1}{4}$

103.

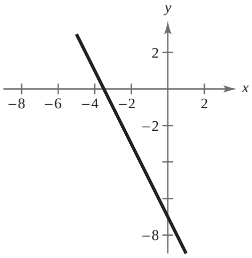


There is an inverse function $f^{-1}(x) = \sqrt{x - 1}$ because the domain of f is equal to the range of f^{-1} and the range of f is equal to the domain of f^{-1}.

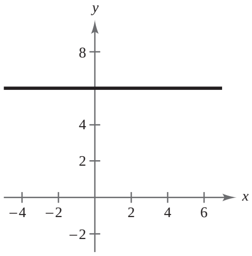
105. This situation could be represented by a one-to-one function if the runner does not stop to rest. The inverse function would represent the time in hours for a given number of miles completed.

Review Exercises *(page 233)*

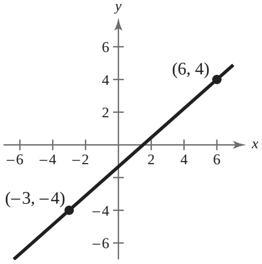
1. slope: -2
y-intercept: -7



3. slope: 0
y-intercept: 6

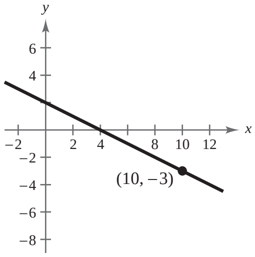


5.



$m = \frac{8}{9}$

7. $y = -\frac{1}{2}x + 2$

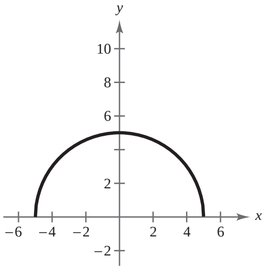


9. $y = \frac{2}{7}x + \frac{2}{7}$ **11.** (a) $y = \frac{5}{4}x - \frac{23}{4}$ (b) $y = -\frac{4}{5}x + \frac{2}{5}$

13. $S = 0.80L$ **15.** No **17.** Yes

19. (a) 16 (b) $(t + 1)^{4/3}$ (c) 81 (d) $x^{4/3}$

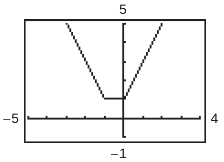
21. All real numbers x such that $-5 \le x \le 5$



23. 16 ft/sec **25.** $4x + 2h + 3, \quad h \neq 0$

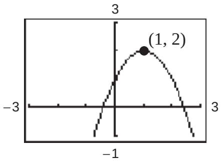
27. Function **29.** $-1, \frac{1}{5}$ **31.** $-\frac{1}{2}$

33.



Increasing on $(0, \infty)$
Decreasing on $(-\infty, -1)$
Constant on $(-1, 0)$

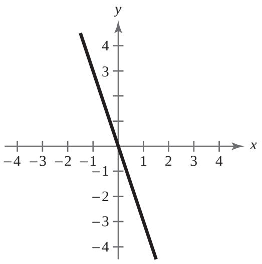
35.



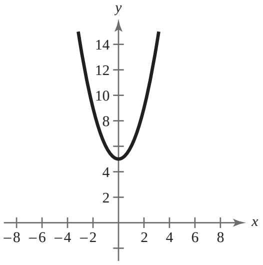
37. 4 **39.** Neither even nor odd; no symmetry

41. Odd; origin symmetry

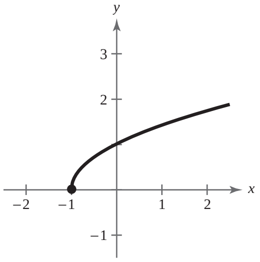
43. (a) $f(x) = -3x$
(b)



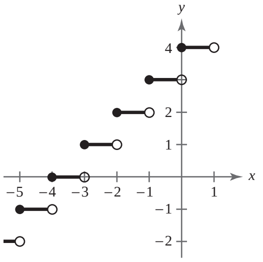
45.



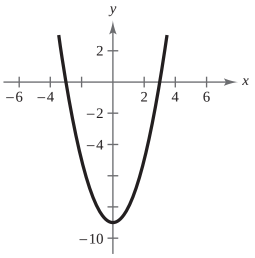
47.



49.



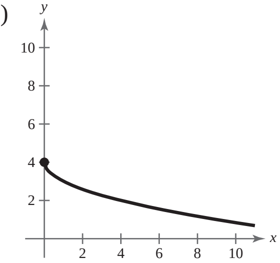
51. (a) $f(x) = x^2$ (b) Vertical shift nine units down
(c) (d) $h(x) = f(x) - 9$



53. (a) $f(x) = \sqrt{x}$

(b) Reflection in the x-axis and vertical shift four units up

(c)

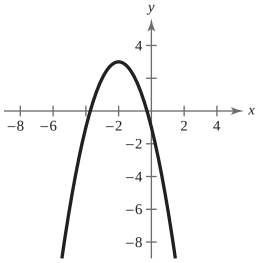


(d) $h(x) = -f(x) + 4$

55. (a) $f(x) = x^2$

(b) Reflection in the x-axis, horizontal shift two units to the left, and vertical shift three units up

(c)

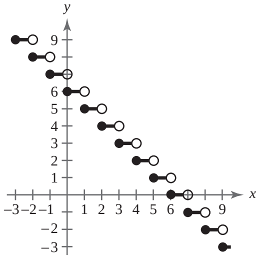


(d) $h(x) = -f(x + 2) + 3$

57. (a) $f(x) = [\![x]\!]$

(b) Reflection in the x-axis and vertical shift six units up

(c)

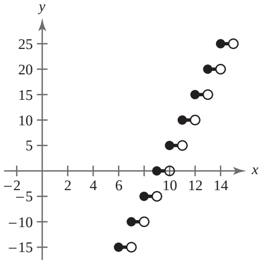


(d) $h(x) = -f(x) + 6$

59. (a) $f(x) = [\![x]\!]$

(b) Horizontal shift nine units to the right and vertical stretch

(c)



(d) $h(x) = 5\,f(x - 9)$

61. (a) $x^2 + 2x + 2$ (b) $x^2 - 2x + 4$

(c) $2x^3 - x^2 + 6x - 3$

(d) $\dfrac{x^2 + 3}{2x - 1}$; all real numbers x except $x = \dfrac{1}{2}$

63. (a) $x - \frac{8}{3}$ (b) $x - 8$

Domains of f, g, $f \circ g$, and $g \circ f$: all real numbers x

65. $N(T(t)) = 100t^2 + 275$

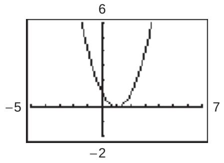
The composition function $N(T(t))$ represents the number of bacteria in the food as a function of time.

67. $f^{-1}(x) = 5x + 4$

$$f(f^{-1}(x)) = \frac{5x + 4 - 4}{5} = x$$

$$f^{-1}(f(x)) = 5\left(\frac{x - 4}{5}\right) + 4 = x$$

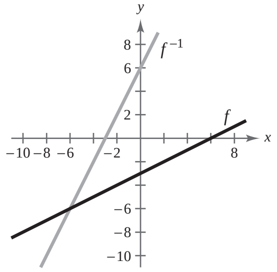
69.



The function does not have an inverse function.

71. (a) $f^{-1}(x) = 2x + 6$

(b)

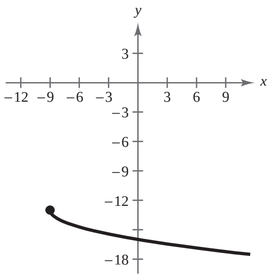


(c) The graphs are reflections of each other in the line $y = x$.

(d) Both f and f^{-1} have domains and ranges that are all real numbers.

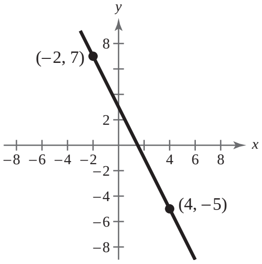
73. $x > 4$; $f^{-1}(x) = \sqrt{\dfrac{x}{2}} + 4, x \neq 0$

75. False. The graph is reflected in the x-axis, shifted 9 units to the left, and then shifted 13 units down.

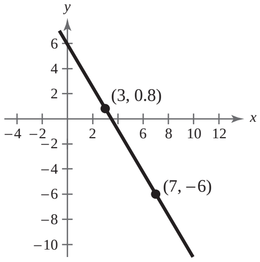


Chapter Test *(page 235)*

1. $y = -2x + 3$



2. $y = -1.7x + 5.9$



3. (a) $y = -\frac{5}{2}x + 4$ (b) $y = \frac{2}{5}x + 4$

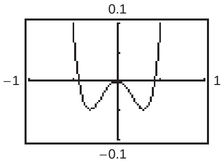
4. (a) -9 (b) 1 (c) $|x - 4| - 15$

5. (a) $-\dfrac{1}{8}$ (b) $-\dfrac{1}{28}$ (c) $\dfrac{\sqrt{x}}{x^2 - 18x}$

6. All real numbers x

7. All real numbers x such that $x \leq 3$

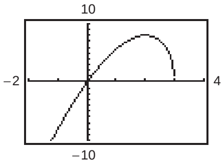
8. (a)



(b) Increasing on $(-0.31, 0)$, $(0.31, \infty)$
Decreasing on $(-\infty, -0.31)$, $(0, 0.31)$

(c) Even

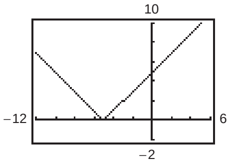
9. (a)



(b) Increasing on $(-\infty, 2)$
Decreasing on $(2, 3)$

(c) Neither even nor odd

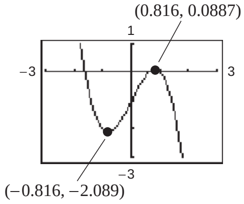
10. (a)



(b) Increasing on $(-5, \infty)$
Decreasing on $(-\infty, -5)$

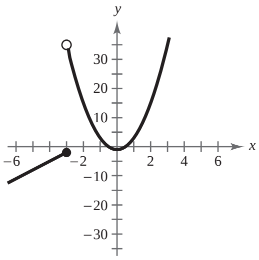
(c) Neither even nor odd

11.



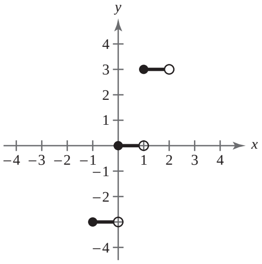
12. Average rate of change $= -3$

13.



14. (a) $f(x) = [\![x]\!]$ (b) Vertical stretch of $y = [\![x]\!]$

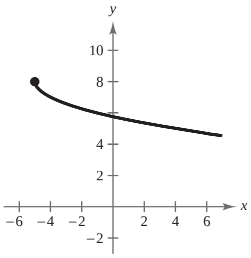
(c)



15. (a) $f(x) = \sqrt{x}$

(b) Reflection in the x-axis, horizontal shift, and vertical shift of $y = \sqrt{x}$

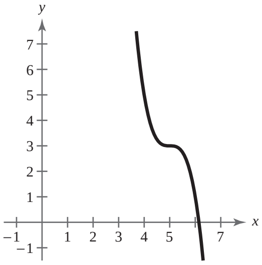
(c)



16. (a) $f(x) = x^3$

(b) Vertical stretch, reflections in the x-axis, vertical shift, and horizontal shift of $y = x^3$

(c)



17. (a) $2x^2 - 4x - 2$ (b) $4x^2 + 4x - 12$

(c) $-3x^4 - 12x^3 + 22x^2 + 28x - 35$

(d) $\dfrac{3x^2 - 7}{-x^2 - 4x + 5}$, $x \neq 1, -5$

(e) $3x^4 + 24x^3 + 18x^2 - 120x + 68$

(f) $-9x^4 + 30x^2 - 16$

18. (a) $\dfrac{1 + 2x^{3/2}}{x}$, $x > 0$ (b) $\dfrac{1 - 2x^{3/2}}{x}$, $x > 0$

(c) $\dfrac{2\sqrt{x}}{x}$, $x > 0$ (d) $\dfrac{1}{2x^{3/2}}$, $x > 0$

(e) $\dfrac{\sqrt{x}}{2x}$, $x > 0$ (f) $\dfrac{2\sqrt{x}}{x}$, $x > 0$

19. $f^{-1}(x) = \sqrt[3]{x - 8}$ **20.** No inverse function

21. $f^{-1}(x) = \left(\frac{1}{3}x\right)^{2/3}$, $x \geq 0$ **22.** \$153

Cumulative Test for Chapters P–2 *(page 236)*

1. $\dfrac{4x^3}{15y^5}$, $x \neq 0$ **2.** $3xy^2\sqrt{2x}$ **3.** $5x - 6$

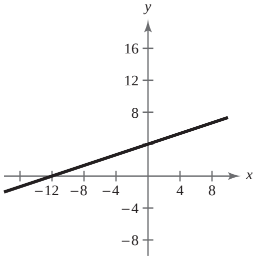
4. $x^3 - x^2 - 5x + 6$ **5.** $\dfrac{s - 1}{(s + 1)(s + 3)}$

6. $(x + 3)(7 - x)$ **7.** $x(x + 1)(1 - 6x)$

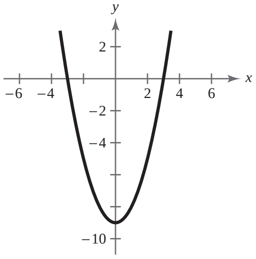
8. $2(3x + 2)(9x^2 - 6x + 4)$

9. $4x^2 + 12x$ **10.** $\frac{3}{2}x^2 + 8x + \frac{5}{2}$

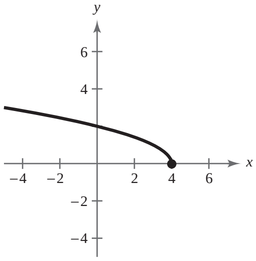
11.



12.



13.



14. $-\frac{13}{3}$ **15.** $\frac{27}{5}$ **16.** $\frac{23}{6}$ **17.** 1, 3 **18.** $2 \pm \sqrt{10}$

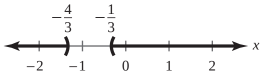
19. ± 6 **20.** $\dfrac{-5 \pm \sqrt{97}}{6}$ **21.** $-\dfrac{3}{2} \pm \dfrac{\sqrt{69}}{6}$

22. ± 8 **23.** $0, -12, \pm 2i$ **24.** 0, 3 **25.** ± 8

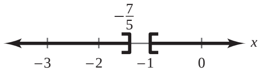
26. 6 **27.** 13, -5 **28.** No solution

29. $-7 \le x \le 5$

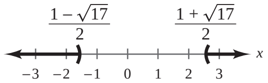
30. $x > -\frac{1}{3},\ x < -\frac{4}{3}$



31. $x \le -\frac{7}{5},\ x \ge -1$



32. $x < \dfrac{1-\sqrt{17}}{2},\ x > \dfrac{1+\sqrt{17}}{2}$



33. $y = 2x + 2$

34. For some values of x there correspond two values of y.

35. (a) $\dfrac{3}{2}$ (b) Division by 0 is undefined. (c) $\dfrac{s+2}{s}$

36. Neither **37.** Neither **38.** Even

39. (a) Vertical shrink by $\frac{1}{2}$ (b) Vertical shift two units up
(c) Horizontal shift two units to the left

40. (a) $4x - 3$ (b) $-2x - 5$ (c) $3x^2 - 11x - 4$
(d) $\dfrac{x-4}{3x+1}$; Domain: all real numbers x except $x = -\dfrac{1}{3}$

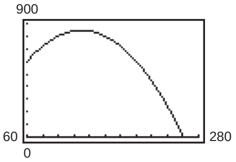
41. (a) $\sqrt{x-1} + x^2 + 1$ (b) $\sqrt{x-1} - x^2 - 1$
(c) $x^2\sqrt{x-1} + \sqrt{x-1}$
(d) $\dfrac{\sqrt{x-1}}{x^2+1}$; Domain: all real numbers x such that $x \ge 1$

42. (a) $2x + 12$ (b) $\sqrt{2x^2 + 6}$
Domain of $f \circ g$: all real numbers x such that $x \ge -6$
Domain of $g \circ f$: all real numbers x

43. (a) $|x| - 2$ (b) $|x - 2|$
Domains of $f \circ g$ and $g \circ f$: all real numbers x

44. $h(x)^{-1} = \frac{1}{3}(x + 4)$ **45.** $n = 9$

46. (a) $R(n) = -0.05n^2 + 13n,\quad n \ge 60$
(b)

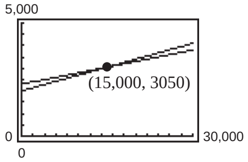


130 passengers

47. 4; Answers will vary.

Problem Solving *(page 239)*

1. (a) $W_1 = 2000 + 0.07S$ (b) $W_2 = 2300 + 0.05S$
(c)

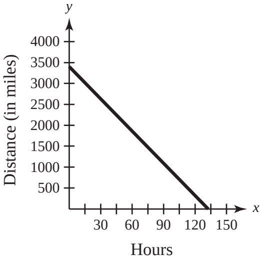


Both jobs pay the same monthly salary when sales equal \$15,000.
(d) No. Job 1 would pay \$3400 and job 2 would pay \$3300.

3. (a) The function will be even.
(b) The function will be odd.
(c) The function will be neither even nor odd.

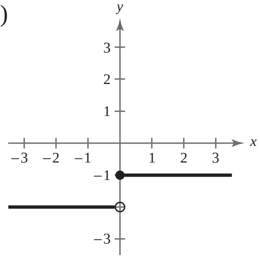
5.
$$f(x) = a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + \cdot \cdot \cdot + a_2x^2 + a_0$$
$$f(-x) = a_{2n}(-x)^{2n} + a_{2n-2}(-x)^{2n-2} + \cdot \cdot \cdot + a_2(-x)^2 + a_0$$
$$= f(x)$$

7. (a) $81\frac{2}{3}$ h
(b) $25\frac{5}{7}$ mi/h
(c) $y = \dfrac{-180}{7}x + 3400$
Domain: $0 \le x \le \dfrac{1190}{9}$
Range: $0 \le y \le 3400$
(d)

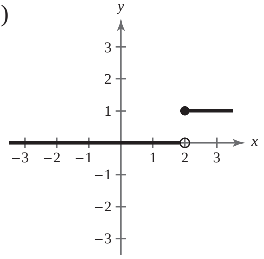


9. (a) $(f \circ g)(x) = 4x + 24$ (b) $(f \circ g)^{-1}(x) = \frac{1}{4}x - 6$
(c) $f^{-1}(x) = \frac{1}{4}x$; $g^{-1}(x) = x - 6$
(d) $(g^{-1} \circ f^{-1})(x) = \frac{1}{4}x - 6$; They are the same.
(e) $(f \circ g)(x) = 8x^3 + 1$; $(f \circ g)^{-1}(x) = \frac{1}{2}\sqrt[3]{x-1}$;
$f^{-1}(x) = \sqrt[3]{x-1}$; $g^{-1}(x) = \frac{1}{2}x$;
$(g^{-1} \circ f^{-1})(x) = \frac{1}{2}\sqrt[3]{x-1}$
(f) Answers will vary. (g) $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$

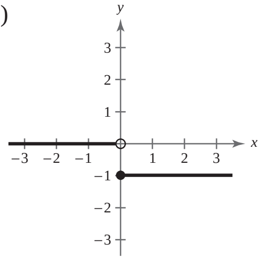
11. (a)



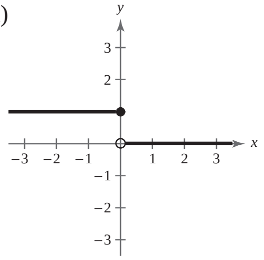
(b)



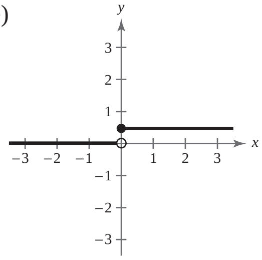
(c)



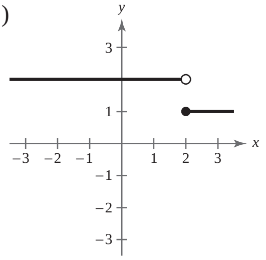
(d)



(e)



(f)



13. Proof

15. (a)

x	-4	-2	0	4
$f(f^{-1}(x))$	-4	-2	0	4

(b)

x	-3	-2	0	1
$(f + f^{-1})(x)$	5	1	-3	-5

(c)

x	-3	-2	0	1
$(f \cdot f^{-1})(x)$	4	0	2	6

(d)

x	-4	-3	0	4
$\lvert f^{-1}(x) \rvert$	2	1	1	3

Chapter 3

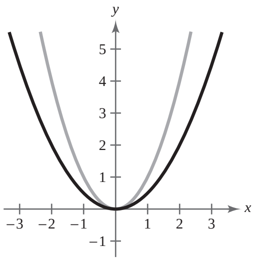
Section 3.1 *(page 248)*

1. polynomial **3.** quadratic; parabola

5. positive; minimum

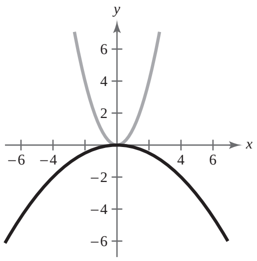
7. e **8.** c **9.** b **10.** a **11.** f **12.** d

13. (a)



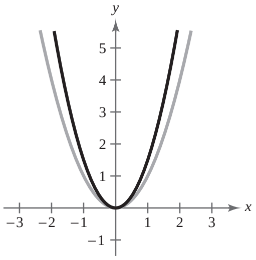
Vertical shrink

(b)



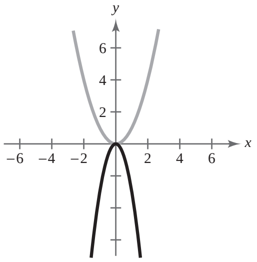
Vertical shrink and reflection in the x-axis

(c)



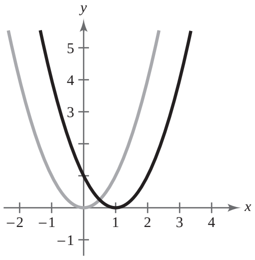
Vertical stretch

(d)



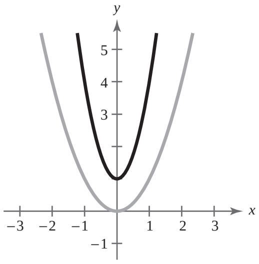
Vertical stretch and reflection in the x-axis

15. (a)



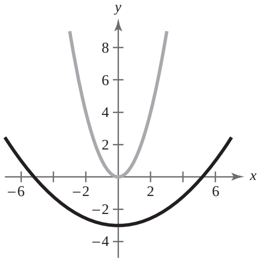
Horizontal shift one unit to the right

(b)



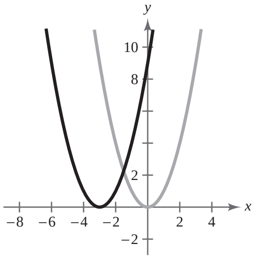
Horizontal shrink and vertical shift one unit up

(c)



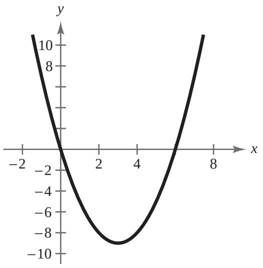
Horizontal stretch and vertical shift three units down

(d)



Horizontal shift three units to the left

17. $f(x) = (x - 3)^2 - 9$

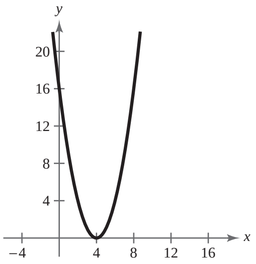


Vertex: $(3, -9)$

Axis of symmetry: $x = 3$

x-intercepts: $(0, 0), (6, 0)$

19. $h(x) = (x - 4)^2$

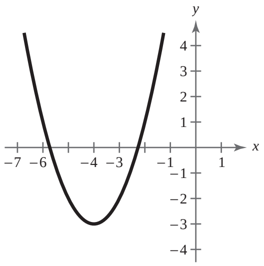


Vertex: $(4, 0)$

Axis of symmetry: $x = 4$

x-intercept: $(4, 0)$

21. $f(x) = (x + 4)^2 - 3$

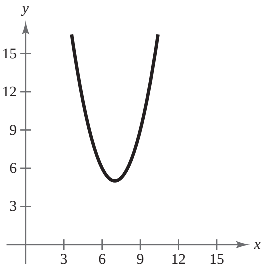


Vertex: $(-4, -3)$

Axis of symmetry: $x = -4$

x-intercepts: $\left(-4 \pm \sqrt{3}, 0\right)$

23. $f(x) = (x - 7)^2 + 5$

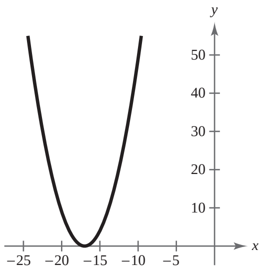


Vertex: $(7, 5)$

Axis of symmetry: $x = 7$

No x-intercept

25. $f(x) = (x + 17)^2$

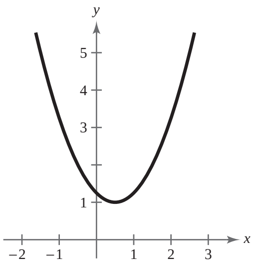


Vertex: $(-17, 0)$

Axis of symmetry: $x = -17$

x-intercept: $(-17, 0)$

27. $f(x) = \left(x - \frac{1}{2}\right)^2 + 1$

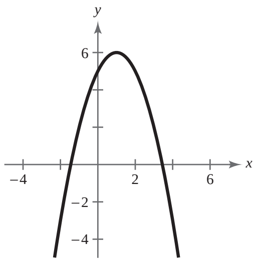


Vertex: $\left(\frac{1}{2}, 1\right)$

Axis of symmetry: $x = \frac{1}{2}$

No x-intercept

29. $f(x) = -(x - 1)^2 + 6$

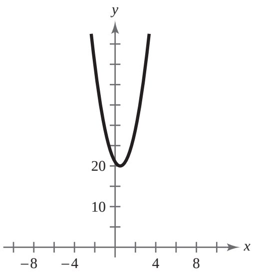


Vertex: $(1, 6)$

Axis of symmetry: $x = 1$

x-intercepts: $\left(1 \pm \sqrt{6}, 0\right)$

31. $h(x) = 4\left(x - \frac{1}{2}\right)^2 + 20$

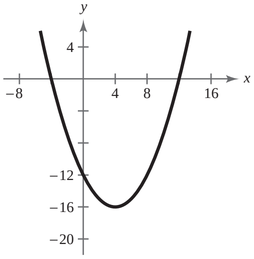


Vertex: $\left(\frac{1}{2}, 20\right)$

Axis of symmetry: $x = \frac{1}{2}$

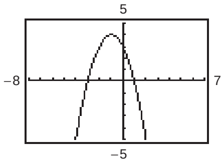
No x-intercept

33. $f(x) = \frac{1}{4}(x - 4)^2 - 16$



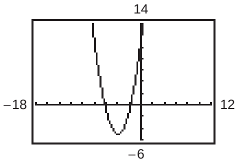
Vertex: $(4, -16)$
Axis of symmetry: $x = 4$
x-intercepts: $(-4, 0), (12, 0)$

35.



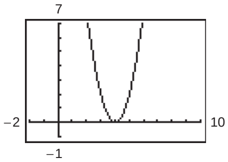
Vertex: $(-1, 4)$
Axis of symmetry: $x = -1$
x-intercepts: $(1, 0), (-3, 0)$

37.



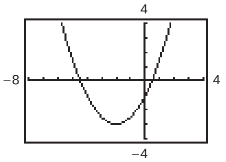
Vertex: $(-4, -5)$
Axis of symmetry: $x = -4$
x-intercepts: $\left(-4 \pm \sqrt{5}, 0\right)$

39.



Vertex: $(4, 0)$
Axis of symmetry: $x = 4$
x-intercept: $(4, 0)$

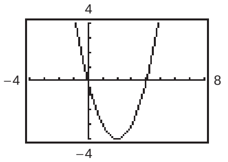
41.



Vertex: $(-2, -3)$
Axis of symmetry: $x = -2$
x-intercepts: $\left(-2 \pm \sqrt{6}, 0\right)$

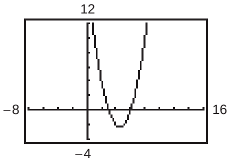
43. $y = -(x + 1)^2 + 4$ **45.** $y = -2(x + 2)^2 + 2$
47. $f(x) = (x + 2)^2 + 5$ **49.** $f(x) = 4(x - 1)^2 - 2$
51. $f(x) = \frac{3}{4}(x - 5)^2 + 12$ **53.** $f(x) = -\frac{24}{49}\left(x + \frac{1}{4}\right)^2 + \frac{3}{2}$
55. $f(x) = -\frac{16}{3}\left(x + \frac{5}{2}\right)^2$ **57.** $(5, 0), (-1, 0)$

59.



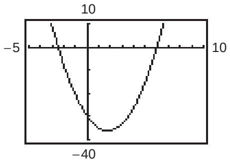
$(0, 0), (4, 0)$

61.



$(3, 0), (6, 0)$

63.



$\left(-\frac{5}{2}, 0\right), (6, 0)$

65. $f(x) = x^2 - 2x - 3$
$g(x) = -x^2 + 2x + 3$

67. $f(x) = x^2 - 10x$
$g(x) = -x^2 + 10x$

69. $f(x) = 2x^2 + 7x + 3$
$g(x) = -2x^2 - 7x - 3$

71. 55, 55 **73.** 12, 6 **75.** 16 ft **77.** 20 fixtures

79. (a) \$14,000,000; \$14,375,000; \$13,500,000
(b) \$24; \$14,400,000; Answers will vary.

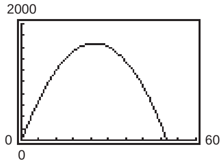
81. (a) $A = \dfrac{8x(50 - x)}{3}$

(b)

x	5	10	15	20	25	30
a	600	$1066\frac{2}{3}$	1400	1600	$1666\frac{2}{3}$	1600

$x = 25$ ft, $y = 33\frac{1}{3}$ ft

(c)

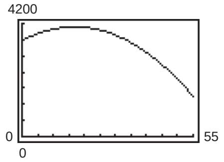


$x = 25$ ft, $y = 33\frac{1}{3}$ ft

(d) $A = -\frac{8}{3}(x - 25)^2 + \frac{5000}{3}$ (e) They are identical.

83. (a) $R = -100x^2 + 3500x, \quad 15 \le x \le 20$
(b) \$17.50; \$30,625

85. (a)



(b) 4075 cigarettes; Yes, the warning had an effect because the maximum consumption occurred in 1966.
(c) 7366 cigarettes per year; 20 cigarettes per day

87. True. The equation has no real solutions, so the graph has no x-intercepts.

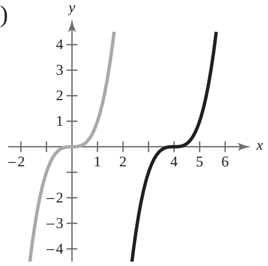
89. $b = \pm 20$ **91.** $b = \pm 8$

93. $f(x) = a\left(x + \dfrac{b}{2a}\right)^2 + \dfrac{4ac - b^2}{4a}$ **95.** Proof

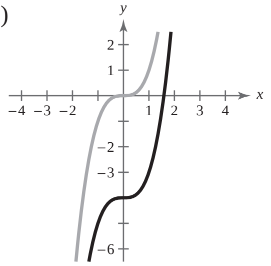
Section 3.2 *(page 261)*

1. continuous **3.** n; $n - 1$ **5.** touches; crosses
7. standard **9.** c **10.** f **11.** a **12.** e
13. d **14.** b

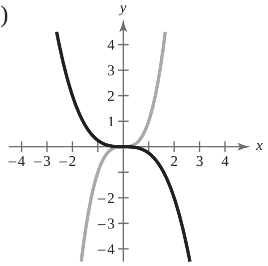
15. (a)



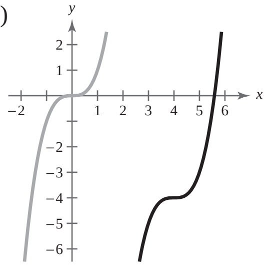
(b)



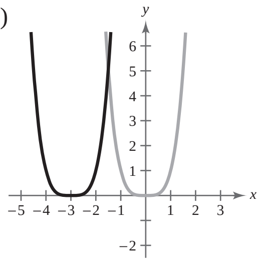
(c)



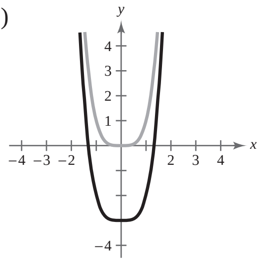
(d)



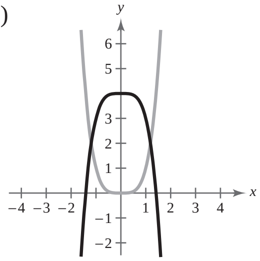
17. (a)



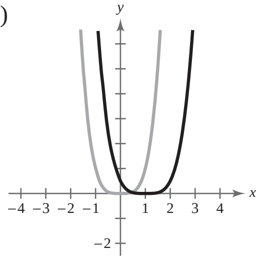
(b)



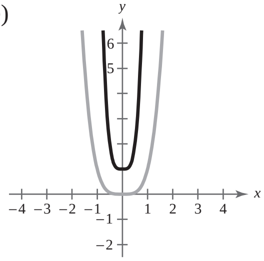
(c)



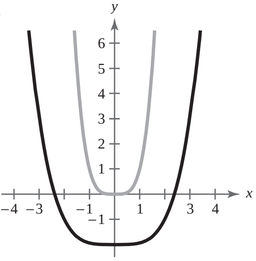
(d)



(e)



(f)



19. Falls to the left, rises to the right

21. Falls to the left, falls to the right

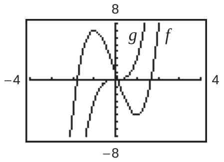
23. Rises to the left, falls to the right

25. Rises to the left, falls to the right

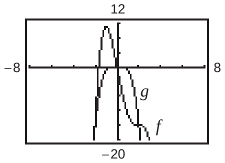
27. Rises to the left, falls to the right

29. Falls to the left, falls to the right

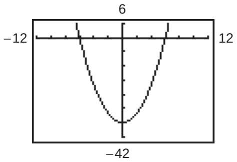
31.



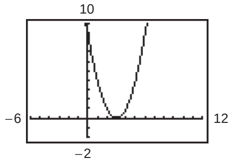
33.



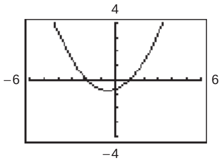
35. (a) ±6
(b) Odd multiplicity
(c) 1
(d)



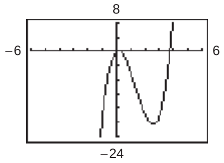
37. (a) 3
(b) Even multiplicity
(c) 1
(d)



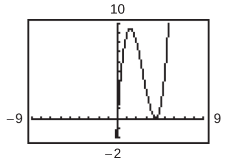
39. (a) −2, 1
(b) Odd multiplicity
(c) 1
(d)



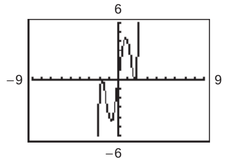
41. (a) $0, 2 \pm \sqrt{3}$
(b) Odd multiplicity
(c) 2
(d)



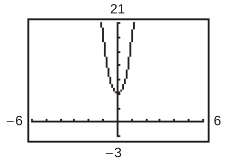
43. (a) 0, 4
(b) 0, odd multiplicity; 4, even multiplicity
(c) 2
(d)



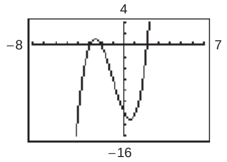
45. (a) $0, \pm\sqrt{3}$
(b) 0, odd multiplicity; $\pm\sqrt{3}$, even multiplicity
(c) 4
(d)



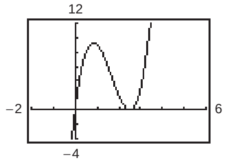
47. (a) No real zero
(b) No multiplicity
(c) 1
(d)



49. (a) ±2, −3
(b) Odd multiplicity
(c) 2
(d)

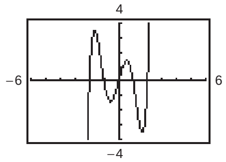


51. (a)



(b) x-intercepts: $(0, 0), \left(\frac{5}{2}, 0\right)$
(c) $x = 0, \frac{5}{2}$
(d) The answers in part (c) match the x-intercepts.

53. (a)



(b) x-intercepts: $(0, 0), (\pm 1, 0), (\pm 2, 0)$
(c) $x = 0, 1, -1, 2, -2$
(d) The answers in part (c) match the x-intercepts.

55. $f(x) = x^2 - 8x$ **57.** $f(x) = x^2 + 4x - 12$

59. $f(x) = x^3 + 9x^2 + 20x$

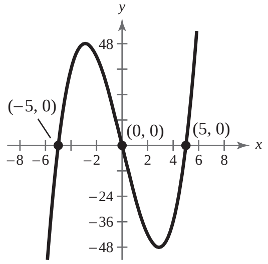
61. $f(x) = x^4 - 4x^3 - 9x^2 + 36x$ **63.** $f(x) = x^2 - 2x - 2$

65. $f(x) = x^2 + 6x + 9$ **67.** $f(x) = x^3 + 4x^2 - 5x$

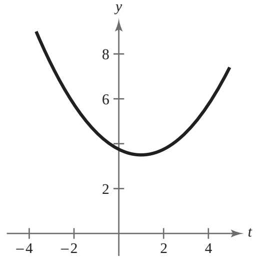
69. $f(x) = x^3 - 3x$ **71.** $f(x) = x^4 + x^3 - 15x^2 + 23x - 10$

73. $f(x) = x^5 + 16x^4 + 96x^3 + 256x^2 + 256x$

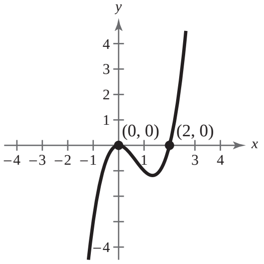
75. (a) Falls to the left, rises to the right
(b) 0, 5, −5 (c) Answers will vary.
(d)



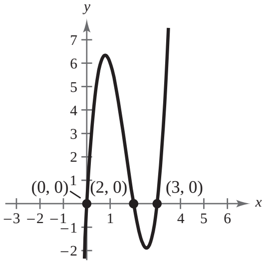
77. (a) Rises to the left, rises to the right
(b) No zeros (c) Answers will vary.
(d)



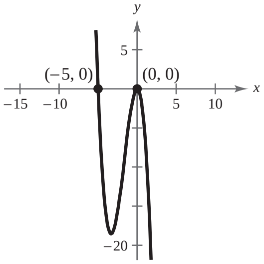
79. (a) Falls to the left, rises to the right
(b) 0, 2 (c) Answers will vary.
(d)



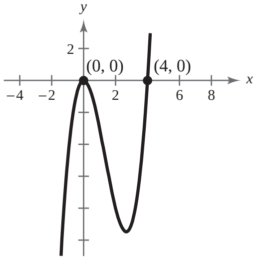
81. (a) Falls to the left, rises to the right
(b) 0, 2, 3 (c) Answers will vary.
(d)



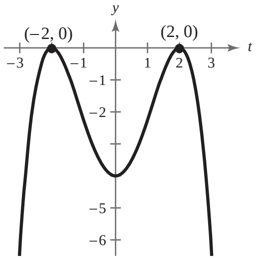
83. (a) Rises to the left, falls to the right
(b) $-5, 0$ (c) Answers will vary.
(d)



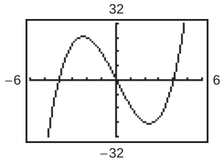
85. (a) Falls to the left, rises to the right
(b) 0, 4 (c) Answers will vary.
(d)



87. (a) Falls to the left, falls to the right
(b) ± 2 (c) Answers will vary.
(d)

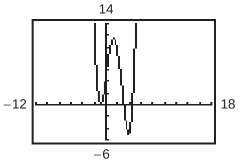


89.



Zeros: 0, ±4,
odd multiplicity

91.

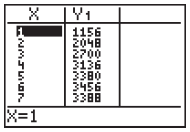


Zeros: -1,
even multiplicity;
$3, \frac{9}{2}$, odd multiplicity

93. (a) $[-1, 0], [1, 2], [2, 3]$
(b) About $-0.879, 1.347, 2.532$

95. (a) $[-2, -1], [0, 1]$ (b) About $-1.585, 0.779$

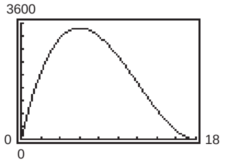
97. (a) $V(x) = x(36 - 2x)^2$ (b) Domain: $0 < x < 18$
(c)



```
X    Y1
1    1156
2    2048
3    2700
4    3136
5    3380
6    3456
7    3388
X=1
```

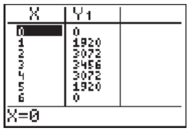
6 in. × 24 in. × 24 in.

(d)



$x = 6$; The results are the same.

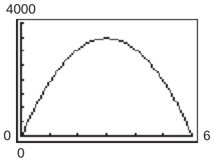
99. (a) $A = -2x^2 + 12x$ (b) $V = -384x^2 + 2304x$
(c) 0 in. $< x <$ 6 in.
(d)



```
X    Y1
0    0
1    1920
2    3072
3    3456
4    3072
5    1920
6    0
X=0
```

When $x = 3$, the volume is maximum at $V = 3456$; dimensions of gutter are 3 in. × 6 in. × 3 in.

(e)



The maximum value is the same.

(f) No. Answers will vary.

101. (a) Relative maximum: $(5.01, 655.75)$
Relative minimum: $(9.25, 417.42)$
(b) Increasing: $(3, 5.01), (9.25, 10)$
Decreasing: $(5.01, 9.25)$
(c) Answers will vary.

103. $x \approx 200$

105. False. A fifth-degree polynomial can have at most four turning points.

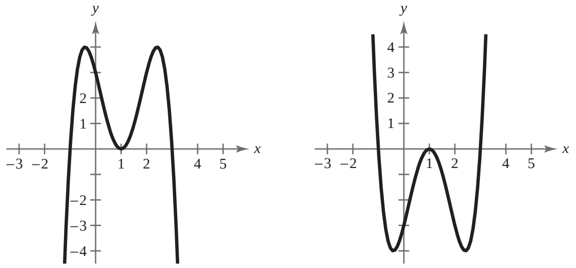
107. False. The function $f(x) = (x - 2)^2$ has one turning point and two real (repeated) zeros.

109. False. $f(x) = -x^3$ rises to the left.

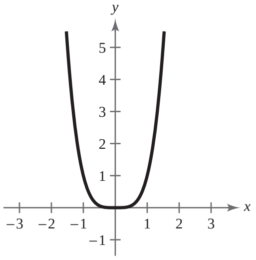
111. True. The leading coefficient is negative and the degree is odd.

113. Answers will vary. *Sample answers:*

$a_4 < 0$ $a_4 > 0$

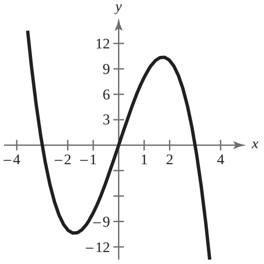


115.



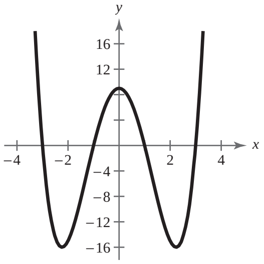
(a) Vertical shift of two units; Even
(b) Horizontal shift of two units; Neither
(c) Reflection in the y-axis; Even
(d) Reflection in the x-axis; Even
(e) Horizontal stretch; Even (f) Vertical shrink; Even
(g) $g(x) = x^3, x \geq 0$; Neither (h) $g(x) = x^{16}$; Even

117. (a)



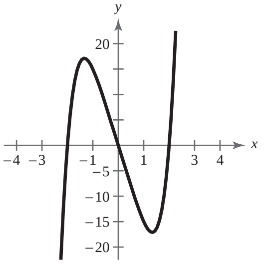
Zeros: 3
Relative minimum: 1
Relative maximum: 1
The number of zeros is the same as the degree, and the number of extrema is one less than the degree.

(b)



Zeros: 4
Relative minima: 2
Relative maximum: 1
The number of zeros is the same as the degree, and the number of extrema is one less than the degree.

(c)



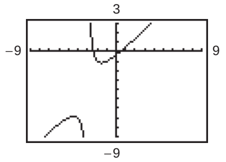
Zeros: 3
Relative minimum: 1
Relative maximum: 1
The number of zeros and the number of extrema are both less than the degree.

Section 3.3 *(page 272)*

1. $f(x)$: dividend; $d(x)$: divisor; $q(x)$: quotient; $r(x)$: remainder

3. improper **5.** Factor **7.** Answers will vary.

9. (a) and (b)



(c) Answers will vary.

11. $2x + 4, \quad x \neq -3$ **13.** $x^2 - 3x + 1, \quad x \neq -\frac{5}{4}$

15. $x^3 + 3x^2 - 1, \quad x \neq -2$ **17.** $x^2 + 3x + 9, \quad x \neq 3$

19. $7 - \dfrac{11}{x + 2}$ **21.** $x - \dfrac{x + 9}{x^2 + 1}$ **23.** $2x - 8 + \dfrac{x - 1}{x^2 + 1}$

25. $x + 3 + \dfrac{6x^2 - 8x + 3}{(x - 1)^3}$ **27.** $3x^2 - 2x + 5, \ x \neq 5$

29. $6x^2 + 25x + 74 + \dfrac{248}{x - 3}$ **31.** $4x^2 - 9, \quad x \neq -2$

33. $-x^2 + 10x - 25, \ x \neq -10$

35. $5x^2 + 14x + 56 + \dfrac{232}{x - 4}$

37. $10x^3 + 10x^2 + 60x + 360 + \dfrac{1360}{x - 6}$

39. $x^2 - 8x + 64, \quad x \neq -8$

41. $-3x^3 - 6x^2 - 12x - 24 - \dfrac{48}{x - 2}$

43. $-x^3 - 6x^2 - 36x - 36 - \dfrac{216}{x - 6}$

45. $4x^2 + 14x - 30, \ x \neq -\dfrac{1}{2}$

47. $f(x) = (x - 4)(x^2 + 3x - 2) + 3, \quad f(4) = 3$

49. $f(x) = \left(x + \frac{2}{3}\right)(15x^3 - 6x + 4) + \frac{34}{3}, \quad f\left(-\frac{2}{3}\right) = \frac{34}{3}$

51. $f(x) = \left(x - \sqrt{2}\right)\left[x^2 + \left(3 + \sqrt{2}\right)x + 3\sqrt{2}\right] - 8,$
$f\left(\sqrt{2}\right) = -8$

53. $f(x) = \left(x - 1 + \sqrt{3}\right)\left[-4x^2 + \left(2 + 4\sqrt{3}\right)x + \left(2 + 2\sqrt{3}\right)\right],$
$f\left(1 - \sqrt{3}\right) = 0$

55. (a) -2 (b) 1 (c) $-\frac{1}{4}$ (d) 5

57. (a) -35 (b) -22 (c) -10 (d) -211

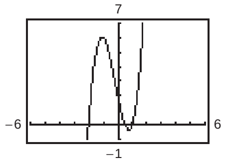
59. $(x - 2)(x + 3)(x - 1)$; Solutions: $2, -3, 1$

61. $(2x - 1)(x - 5)(x - 2)$; Solutions: $\frac{1}{2}, 5, 2$

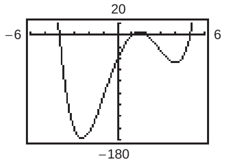
63. $\left(x + \sqrt{3}\right)\left(x - \sqrt{3}\right)(x + 2)$; Solutions: $-\sqrt{3}, \sqrt{3}, -2$

65. $(x - 1)\left(x - 1 - \sqrt{3}\right)\left(x - 1 + \sqrt{3}\right)$;
Solutions: $1, \ 1 + \sqrt{3}, \ 1 - \sqrt{3}$

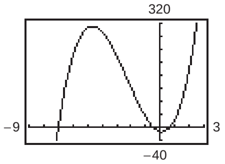
67. (a) Answers will vary. (b) $2x - 1$
(c) $f(x) = (2x - 1)(x + 2)(x - 1)$ (d) $\frac{1}{2}, -2, 1$
(e)



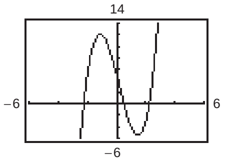
69. (a) Answers will vary. (b) $(x - 1), (x - 2)$
(c) $f(x) = (x - 1)(x - 2)(x - 5)(x + 4)$ (d) $1, 2, 5, -4$
(e)



71. (a) Answers will vary. (b) $x + 7$
(c) $f(x) = (x + 7)(2x + 1)(3x - 2)$ (d) $-7, -\frac{1}{2}, \frac{2}{3}$
(e)



73. (a) Answers will vary. (b) $x - \sqrt{5}$
(c) $f(x) = (x - \sqrt{5})(x + \sqrt{5})(2x - 1)$ (d) $\pm\sqrt{5}, \frac{1}{2}$
(e)



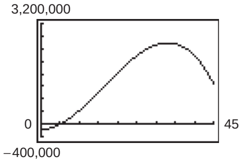
75. (a) Zeros are 2 and about ±2.236.
(b) $x = 2$ (c) $f(x) = (x - 2)(x - \sqrt{5})(x + \sqrt{5})$

77. (a) Zeros are -2, about 0.268, and about 3.732.
(b) $t = -2$
(c) $h(t) = (t + 2)[t - (2 + \sqrt{3})][t - (2 - \sqrt{3})]$

79. (a) Zeros are 0, 3, 4, and about ±1.414. (b) $x = 0$
(c) $h(x) = x(x - 4)(x - 3)(x + \sqrt{2})(x - \sqrt{2})$

81. $2x^2 - x - 1,\ x \neq \frac{3}{2}$ **83.** $x^2 + 3x,\ x \neq -2, -1$

85. (a)



(b) $250,366
(c) Answers will vary.

87. False. $-\frac{4}{7}$ is a zero of f.

89. True. The degree of the numerator is greater than the degree of the denominator.

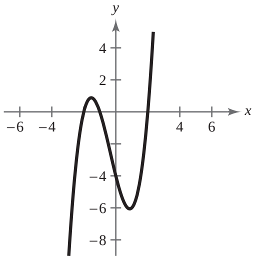
91. $x^{2n} + 6x^n + 9,\ x^n \neq -3$ **93.** The remainder is 0.

95. $c = -210$ **97.** $k = 7$

Section 3.4 *(page 285)*

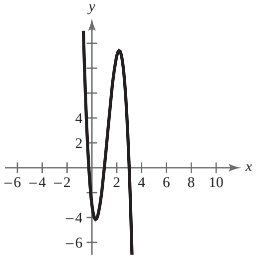
1. Fundamental Theorem of Algebra **3.** Rational Zero
5. linear; quadratic; quadratic **7.** Descartes's Rule of Signs
9. 1 **11.** 3 **13.** 2 **15.** ±1, ±2
17. $\pm1, \pm3, \pm5, \pm9, \pm15, \pm45, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{9}{2}, \pm\frac{15}{2}, \pm\frac{45}{2}$
19. $-2, -1, 3$ **21.** $1, -1, 4$ **23.** $-6, -1$ **25.** $\frac{1}{2}, -1$
27. $-2, 3, \pm\frac{2}{3}$ **29.** $-2, 1$ **31.** $-4, \frac{1}{2}, 1, 1$

33. (a) ±1, ±2, ±4
(b)



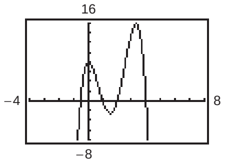
(c) $-2, -1, 2$

35. (a) $\pm1, \pm3, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}$
(b)



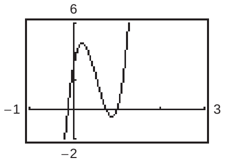
(c) $-\frac{1}{4}, 1, 3$

37. (a) $\pm1, \pm2, \pm4, \pm8, \pm\frac{1}{2}$
(b)



(c) $-\frac{1}{2}, 1, 2, 4$

39. (a) $\pm1, \pm3, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{1}{8}, \pm\frac{3}{8}, \pm\frac{1}{16}, \pm\frac{3}{16}, \pm\frac{1}{32}, \pm\frac{3}{32}$
(b)



(c) $1, \frac{3}{4}, -\frac{1}{8}$

41. (a) ±1, about ±1.414 (b) $\pm1, \pm\sqrt{2}$
(c) $f(x) = (x + 1)(x - 1)(x + \sqrt{2})(x - \sqrt{2})$

43. (a) 0, 3, 4, about ±1.414 (b) $0, 3, 4, \pm\sqrt{2}$
(c) $h(x) = x(x - 3)(x - 4)(x + \sqrt{2})(x - \sqrt{2})$

45. $x^3 - x^2 + 25x - 25$

47. $x^3 - 12x^2 + 46x - 52$

49. $3x^4 - 17x^3 + 25x^2 + 23x - 22$

51. (a) $(x^2 + 9)(x^2 - 3)$ (b) $(x^2 + 9)(x + \sqrt{3})(x - \sqrt{3})$
(c) $(x + 3i)(x - 3i)(x + \sqrt{3})(x - \sqrt{3})$

53. (a) $(x^2 - 2x - 2)(x^2 - 2x + 3)$
(b) $(x - 1 + \sqrt{3})(x - 1 - \sqrt{3})(x^2 - 2x + 3)$
(c) $(x - 1 + \sqrt{3})(x - 1 - \sqrt{3})(x - 1 + \sqrt{2}i)$
$(x - 1 - \sqrt{2}i)$

55. $\pm2i, 1$ **57.** $\pm5i, -\frac{1}{2}, 1$ **59.** $-3 \pm i, \frac{1}{4}$

61. $2, -3 \pm \sqrt{2}i, 1$ **63.** $(x + 6i)(x - 6i); \pm6i$

65. $(x - 1 - 4i)(x - 1 + 4i); 1 \pm 4i$

67. $(x - 2)(x + 2)(x - 2i)(x + 2i); \pm2, \pm2i$

69. $(z - 1 + i)(z - 1 - i); 1 \pm i$

71. $(x + 1)(x - 2 + i)(x - 2 - i); -1, 2 \pm i$

73. $(x + 2)(x - 1 + \sqrt{2}i)(x - 1 - \sqrt{2}i); -2, 1 \pm \sqrt{2}i$

75. $(5x + 1)(x - 1 + \sqrt{5}i)(x - 1 - \sqrt{5}i); -\frac{1}{5}, 1 \pm \sqrt{5}i$

77. $(x - 2)^2(x + 2i)(x - 2i); 2, \pm2i$

79. $(x + i)(x - i)(x + 3i)(x - 3i); \pm i, \pm3i$

81. $-10, -7 \pm 5i$ **83.** $-\frac{3}{4}, 1 \pm \frac{1}{2}i$ **85.** $-2, -\frac{1}{2}, \pm i$

87. One positive zero **89.** One negative zero

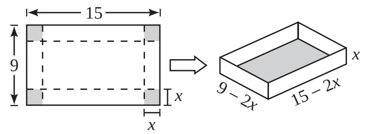
91. One positive zero, one negative zero

93. One or three positive zeros **95–97.** Answers will vary.

99. $1, -\frac{1}{2}$ **101.** $-\frac{3}{4}$ **103.** $\pm2, \pm\frac{3}{2}$ **105.** $\pm1, \frac{1}{4}$

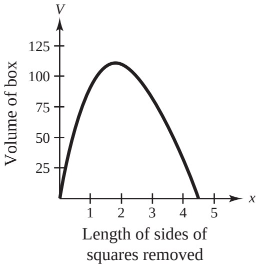
107. d **108.** a **109.** b **110.** c

111. (a)



(b) $V(x) = x(9 - 2x)(15 - 2x)$
Domain: $0 < x < \frac{9}{2}$

(c)



1.82 cm × 5.36 cm × 11.36 cm

(d) $\frac{1}{2}, \frac{7}{2}$, 8; 8 is not in the domain of V.

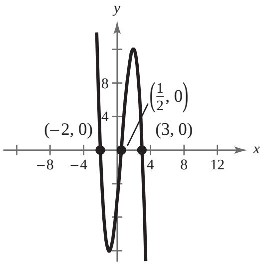
113. (a) $V(x) = x^3 + 9x^2 + 26x + 24 = 120$
(b) 4 ft × 5 ft × 6 ft

115. False. The most complex zeros it can have is two, and the Linear Factorization Theorem guarantees that there are three linear factors, so one zero must be real.

117. r_1, r_2, r_3 **119.** $5 + r_1, 5 + r_2, 5 + r_3$

121. The zeros cannot be determined.

123. Answers will vary. There are infinitely many possible functions for f. *Sample equation and graph:*
$f(x) = -2x^3 + 3x^2 + 11x - 6$



125. $f(x) = x^3 - 3x^2 + 4x - 2$ **127.** $f(x) = x^4 + 5x^2 + 4$

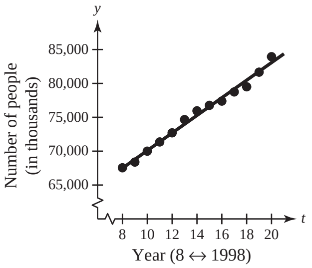
129. (a) $x^2 + b$ (b) $x^2 - 2ax + a^2 + b^2$

131. (a) Not correct because f has $(0, 0)$ as an intercept.
(b) Not correct because the function must be at least a fourth-degree polynomial.
(c) Correct function
(d) Not correct because k has $(-1, 0)$ as an intercept.

Section 3.5 *(page 296)*

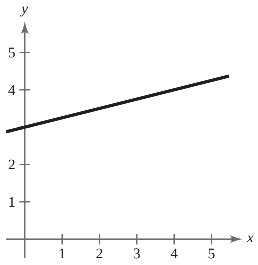
1. variation; regression **3.** least squares regression
5. directly proportional **7.** directly proportional
9. combined

11.



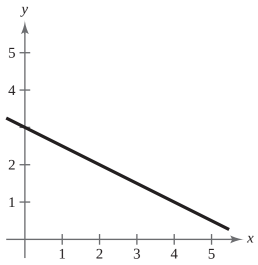
The model fits the data well.

13.



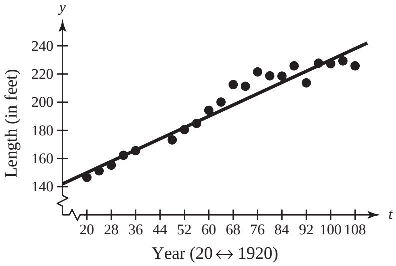
$y = \frac{1}{4}x + 3$

15.



$y = -\frac{1}{2}x + 3$

17. (a) and (b)

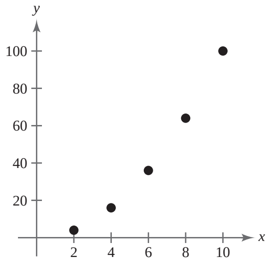


$y \approx t + 130$
(c) $y = 1.01t + 130.82$
(d) The models are similar.

19. $y = 7x$ **21.** $y = 205x$ **23.** $y = \frac{1}{5}x$ **25.** $y = 2\pi x$

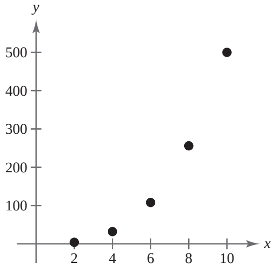
27.

x	2	4	6	8	10
$y = x^2$	4	16	36	64	100



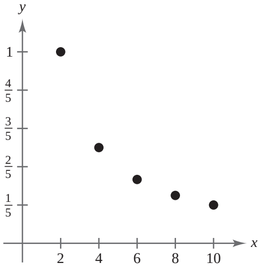
29.

x	2	4	6	8	10
$y = \frac{1}{2}x^3$	4	32	108	256	500



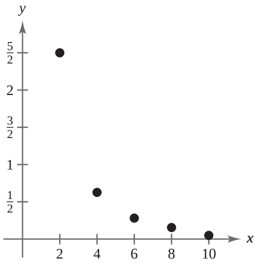
31.

x	2	4	6	8	10
$y = \frac{2}{x}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$



33.

x	2	4	6	8	10
$y = \frac{10}{x^2}$	$\frac{5}{2}$	$\frac{5}{8}$	$\frac{5}{18}$	$\frac{5}{32}$	$\frac{1}{10}$



35. Inversely **37.** $y = \frac{5}{x}$ **39.** $y = -\frac{7}{10}x$

41. $A = kr^2$ **43.** $y = \frac{k}{x^2}$ **45.** $F = \frac{kg}{r^2}$

47. $R = k(T - T_e)$ **49.** $R = kS(L - S)$

51. The surface area of a sphere varies directly as the square of its radius.

53. The area of a triangle is jointly proportional to its base and height.

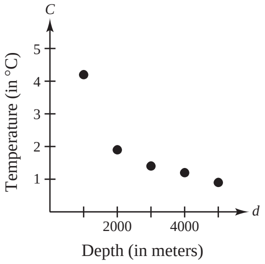
55. $A = \pi r^2$ **57.** $y = \frac{28}{x}$

59. $F = 14rs^3$ **61.** $z = \frac{2x^2}{3y}$ **63.** $I = 0.035P$

65. Model: $y = \frac{33}{13}x$; 25.4 cm, 50.8 cm **67.** $293\frac{1}{3}$ N

69. 39.47 lb **71.** About 0.61 mi/h **73.** 1470 J

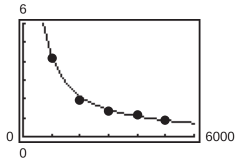
75. (a)



(b) Yes. $k_1 = 4200, k_2 = 3800, k_3 = 4200,$ $k_4 = 4800, k_5 = 4500$

(c) $C = \frac{4300}{d}$

(d)



(e) About 1433 m

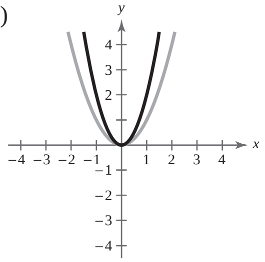
77. False. π is a constant, not a variable.

79. (a) y will change by a factor of four.

(b) y will change by a factor of one-fourth.

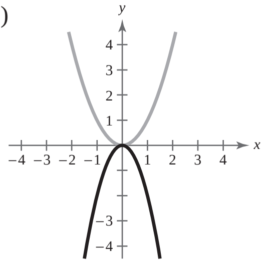
Review Exercises *(page 302)*

1. (a)



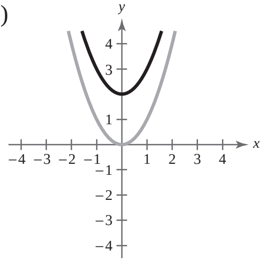
Vertical stretch

(b)



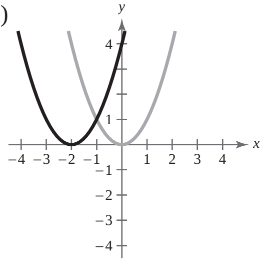
Vertical stretch and reflection in the x-axis

(c)



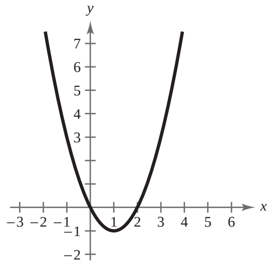
Vertical translation

(d)



Horizontal translation

3. $g(x) = (x - 1)^2 - 1$

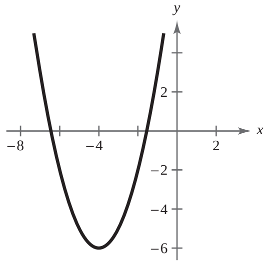


Vertex: $(1, -1)$

Axis of symmetry: $x = 1$

x-intercepts: $(0, 0), (2, 0)$

5. $f(x) = (x + 4)^2 - 6$

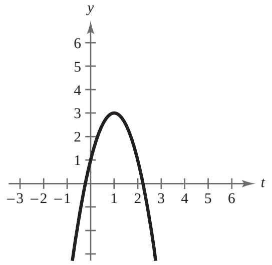


Vertex: $(-4, -6)$

Axis of symmetry: $x = -4$

x-intercepts: $(-4 \pm \sqrt{6}, 0)$

7. $f(t) = -2(t - 1)^2 + 3$

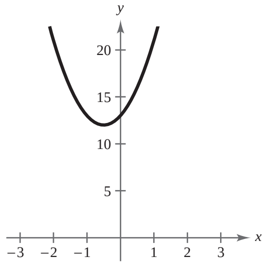


Vertex: $(1, 3)$

Axis of symmetry: $t = 1$

t-intercepts: $\left(1 \pm \frac{\sqrt{6}}{2}, 0\right)$

9. $h(x) = 4\left(x + \frac{1}{2}\right)^2 + 12$

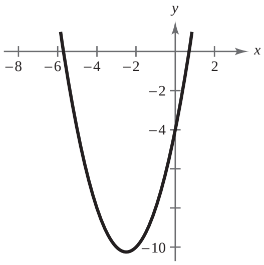


Vertex: $\left(-\frac{1}{2}, 12\right)$

Axis of symmetry: $x = -\frac{1}{2}$

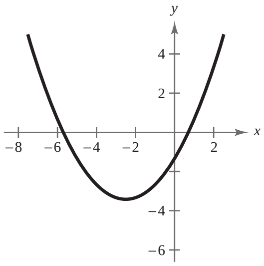
No x-intercept

11. $h(x) = \left(x + \frac{5}{2}\right)^2 - \frac{41}{4}$



Vertex: $\left(-\frac{5}{2}, -\frac{41}{4}\right)$
Axis of symmetry: $x = -\frac{5}{2}$
x-intercepts: $\left(\frac{\pm\sqrt{41} - 5}{2}, 0\right)$

13. $f(x) = \frac{1}{3}\left(x + \frac{5}{2}\right)^2 - \frac{41}{12}$



Vertex: $\left(-\frac{5}{2}, -\frac{41}{12}\right)$
Axis of symmetry: $x = -\frac{5}{2}$
x-intercepts: $\left(\frac{\pm\sqrt{41} - 5}{2}, 0\right)$

15. $f(x) = -\frac{1}{2}(x - 4)^2 + 1$ **17.** $f(x) = (x - 1)^2 - 4$

19. $y = -\frac{11}{36}\left(x + \frac{3}{2}\right)^2$

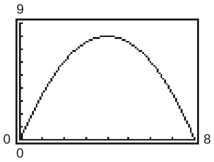
21. (a) $A = x\left(\frac{8 - x}{2}\right)$ (b) $0 < x < 8$

(c)

x	1	2	3	4	5	6
A	$\frac{7}{2}$	6	$\frac{15}{2}$	8	$\frac{15}{2}$	6

$x = 4, y = 2$

(d)



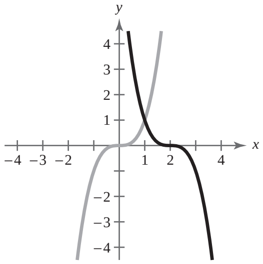
$x = 4, y = 2$

(e) $A = -\frac{1}{2}(x - 4)^2 + 8$; $x = 4, y = 2$

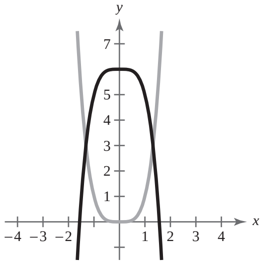
23. (a) \$12,000; \$13,750; \$15,000
(b) Maximum revenue at \$40; \$16,000; Any price greater or less than \$40 per unit will not yield as much revenue.

25. 1091 units

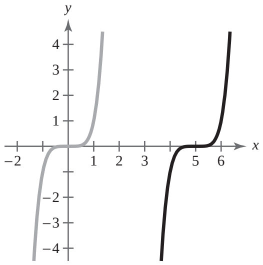
27.



29.



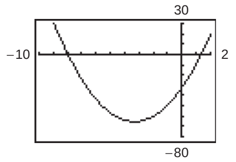
31.



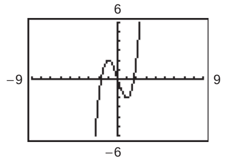
33. Falls to the left, falls to the right

35. Rises to the left, rises to the right

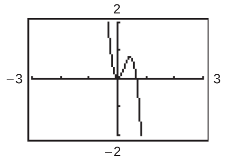
37. (a) $-8, \frac{4}{3}$ (b) Odd multiplicity (c) 1
(d)



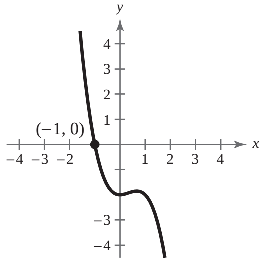
39. (a) $0, \pm\sqrt{3}$ (b) Odd multiplicity (c) 2
(d)



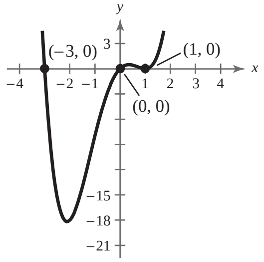
41. (a) $\frac{2}{3}, 0$ (b) $\frac{2}{3}$, odd multiplicity; 0, even multiplicity (c) 2
(d)



43. (a) Rises to the left, falls to the right (b) -1
(c) Answers will vary.
(d)



45. (a) Rises to the left, rises to the right (b) $-3, 0, 1$
(c) Answers will vary.
(d)



47. (a) $[-1, 0]$ (b) About -0.900

49. (a) $[-1, 0], [1, 2]$ (b) About -0.200, about 1.772

51. $6x + 3 + \frac{17}{5x - 3}$ **53.** $5x + 4,\ x \neq \frac{5}{2} \pm \frac{\sqrt{29}}{2}$

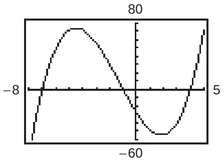
55. $x^2 - 3x + 2 - \frac{1}{x^2 + 2}$

57. $6x^3 + 8x^2 - 11x - 4 - \frac{8}{x - 2}$

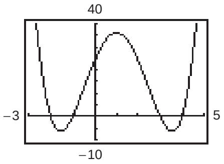
59. $2x^2 - 9x - 6,\ x \neq 8$ **61.** (a) -421 (b) -9

63. (a) Yes (b) Yes (c) Yes (d) No

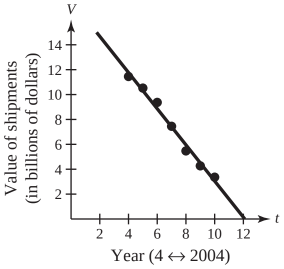
65. (a) Answers will vary. (b) $(x + 7), (x + 1)$
(c) $f(x) = (x + 7)(x + 1)(x - 4)$ (d) $-7, -1, 4$
(e)



67. (a) Answers will vary. (b) $(x + 1), (x - 4)$
(c) $f(x) = (x + 1)(x - 4)(x + 2)(x - 3)$
(d) $-2, -1, 3, 4$
(e)



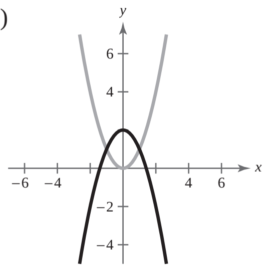
69. 1 **71.** 5 **73.** 3
75. $\pm1, \pm3, \pm5, \pm15, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{15}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{5}{4}, \pm\frac{15}{4}$
77. $-6, -2, 5$ **79.** 1, 8 **81.** $-4, 3$
83. $f(x) = 3x^4 - 14x^3 + 17x^2 - 42x + 24$ **85.** $4, \pm i$
87. $-3, \frac{1}{2}, 2 \pm i$ **89.** $f(x) = x(x - 1)(x + 5); 0, 1, -5$
91. $g(x) = (x + 4)^2(x - 2 - 3i)(x - 2 + 3i); -4, 2 \pm 3i$
93. $16, \pm i$ **95.** $-3, 6, \pm 2i$
97. Two or no positive zeros, one negative zero
99. Answers will vary.
101.



The model fits the data well.
103. Model: $y = \frac{8}{5}x$; 3.2 km, 16 km
105. A factor of 4 **107.** About 2 h, 26 min
109. False. A fourth-degree polynomial can have at most four zeros, and complex zeros occur in conjugate pairs.
111. Find the vertex of the quadratic function and write the function in standard form. If the leading coefficient is positive, the vertex is a minimum. If the leading coefficient is negative, the vertex is a maximum.

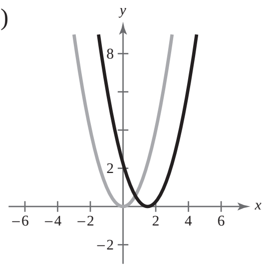
Chapter Test *(page 306)*

1. (a)



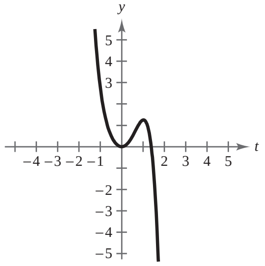
Reflection in the x-axis and vertical translation

(b)



Horizontal translation

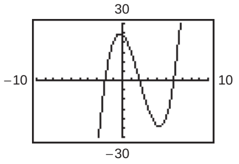
2. Vertex: $(-2, -1)$; Intercepts: $(0, 3), (-3, 0), (-1, 0)$
3. $y = (x - 3)^2 - 6$
4. (a) 50 ft
(b) 5. Yes, changing the constant term results in a vertical translation of the graph and therefore changes the maximum height.
5. Rises to the left, falls to the right



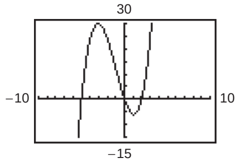
6. $3x + \dfrac{x - 1}{x^2 + 1}$ **7.** $2x^3 + 4x^2 + 3x + 6 + \dfrac{9}{x - 2}$
8. $(2x - 5)(x - \sqrt{3})(x + \sqrt{3})$; Zeros: $\pm\sqrt{3}, \frac{5}{2}$
9. $-2, \frac{3}{2}$ **10.** $\pm1, -\frac{2}{3}$
11. $f(x) = x^4 - 7x^3 + 17x^2 - 15x$
12. $f(x) = x^4 - 6x^3 + 16x^2 - 24x + 16$
13. $1, -5, -\frac{2}{3}$ **14.** $-2, 4, -1 \pm \sqrt{2}i$
15. $v = 6\sqrt{s}$ **16.** $A = \dfrac{25}{6}xy$ **17.** $b = \dfrac{48}{a}$
18. $y = -218.6t + 8777$; The model fits the data well.

Problem Solving *(page 309)*

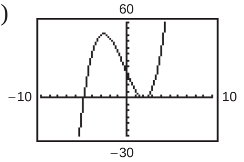
1. (a) (i) $6, -2$ (ii) $0, -5$ (iii) $-5, 2$
(iv) 2 (v) $1 \pm \sqrt{7}$ (vi) $\dfrac{-3 \pm \sqrt{7}i}{2}$
(b) (i)



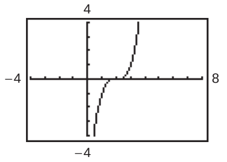
(ii)



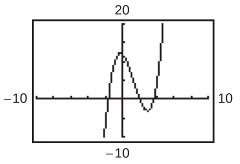
(iii)



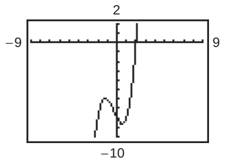
(iv)



(v)



(vi)



Graph (iii) touches the x-axis at $(2, 0)$, and all the other graphs pass through the x-axis at $(2, 0)$.
(c) (i) $(6, 0), (-2, 0)$ (iv) No other x-intercepts
(ii) $(0, 0), (-5, 0)$ (v) $(-1.6, 0), (3.6, 0)$
(iii) $(-5, 0)$ (vi) No other x-intercepts
(d) When the function has two real zeros, the results are the same. When the function has one real zero, the graph touches the x-axis at the zero. When there are no real zeros, there is no x-intercept.
3. Answers will vary. **5.** 2 in. × 2 in. × 5 in.

7. False. The statement would be true if $f(-1) = 2$.

9. (a) $m_1 = 5$; less than (b) $m_2 = 3$; greater than
(c) $m_3 = 4.1$; less than (d) $m_h = h + 4$
(e) $m_h = 3, 5, 4.1$; The values are the same.
(f) $m_{\tan} = 4$ because $h = 0$.

11. (a) $A(x) = \dfrac{x^2}{16} + \dfrac{1}{\pi}\left(\dfrac{100 - x}{2}\right)^2$ (b) $0 \le x \le 100$
(c) Maximum area at $x = 0$; Minimum area at $x = 56$
(d) Answers will vary.

Chapter 4

Section 4.1 *(page 317)*

1. rational functions **3.** horizontal asymptote

5. Domain: all real numbers x except $x = 1$
$f(x) \to -\infty$ as $x \to 1^-$, $f(x) \to \infty$ as $x \to 1^+$

7. Domain: all real numbers x except $x = -2$
$f(x) \to \infty$ as $x \to -2^-$, $f(x) \to -\infty$ as $x \to -2^+$

9. Domain: all real numbers x except $x = \pm 1$
$f(x) \to \infty$ as $x \to -1^-$ and as $x \to 1^+$,
$f(x) \to -\infty$ as $x \to -1^+$ and as $x \to 1^-$

11. Domain: all real numbers x except $x = 1$
$f(x) \to \infty$ as $x \to 1^-$ and as $x \to 1^+$

13. Vertical asymptote: $x = 0$
Horizontal asymptote: $y = 0$

15. Vertical asymptote: $x = 5$
Horizontal asymptote: $y = -1$

17. Vertical asymptotes: $x = \pm 1$

19. Horizontal asymptote: $y = 3$

21. Vertical asymptote: $x = -4$
Horizontal asymptote: $y = 0$

23. Vertical asymptote: $x = 3$
Horizontal asymptote: $y = 1$

25. Vertical asymptote: $x = \frac{1}{2}$
Horizontal asymptote: $y = \frac{1}{2}$

27. Vertical asymptote: $x = 2$
Horizontal asymptote: $y = 2$

29. g **30.** e **31.** c **32.** b
33. a **34.** d **35.** f **36.** h

37. (a) Domain of f: all real numbers x except $x = -2$
Domain of g: all real numbers x
(b) $x - 2$; Vertical asymptote: none
(c)

x	-4	-3	-2.5	-2	-1.5	-1	0
$f(x)$	-6	-5	-4.5	Undef.	-3.5	-3	-2
$g(x)$	-6	-5	-4.5	-4	-3.5	-3	-2

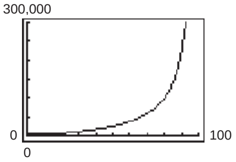
(d) The functions differ only at $x = -2$, where f is undefined.

39. (a) Domain of f: all real numbers x except $x = 0, \frac{1}{2}$
Domain of g: all real numbers x except $x = 0$
(b) $\dfrac{1}{x}$; Vertical asymptote: $x = 0$
(c)

x	-1	-0.5	0	0.5	2	3	4
$f(x)$	-1	-2	Undef.	Undef.	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
$g(x)$	-1	-2	Undef.	2	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$

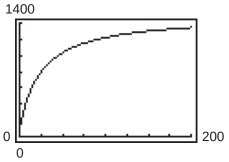
(d) The functions differ only at $x = 0.5$, where f is undefined.

41. (a)



(b) \$4411.76; \$25,000; \$225,000
(c) No. The function is undefined at $p = 100$.

43. (a)



(b) 333 deer, 500 deer, 800 deer (c) 1500 deer

45. (a)

n	1	2	3	4	5
P	0.50	0.74	0.82	0.86	0.89

n	6	7	8	9	10
P	0.91	0.92	0.93	0.94	0.95

P approaches 1 as n increases.
(b) 100%

47. False. Polynomials do not have vertical asymptotes.

49. (a) 4 (b) Less than (c) Greater than

51. (a) 2 (b) Greater than (c) Less than

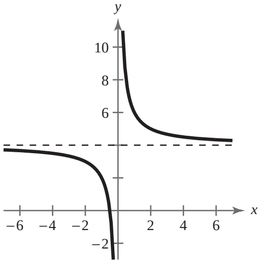
53. *Sample answer:* $f(x) = \dfrac{2x^2}{x^2 + 1}$

55. Answers will vary.
Sample answers: $f(x) = \dfrac{1}{x^2 + 15}$; $f(x) = \dfrac{1}{x - 15}$

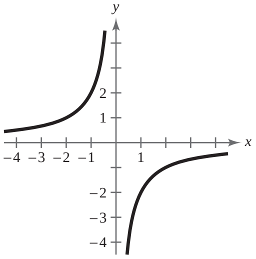
Section 4.2 *(page 325)*

1. slant asymptote

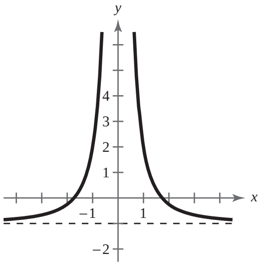
3.



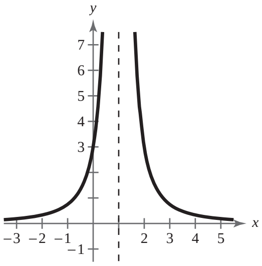
5.



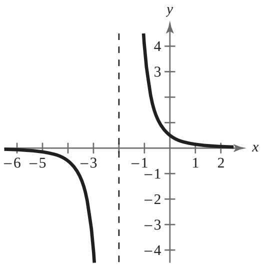
7.



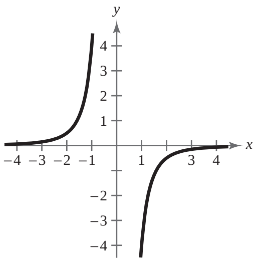
9.



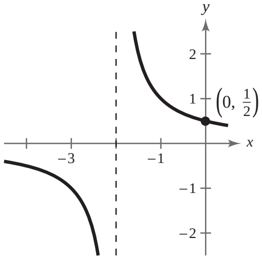
11.



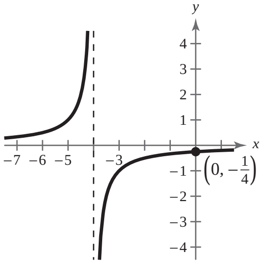
13.



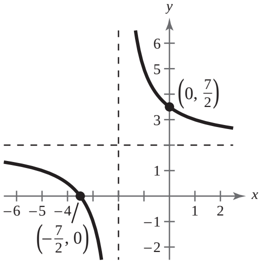
15. (a) The domain is all real numbers x except $x = -2$.
(b) y-intercept: $\left(0, \frac{1}{2}\right)$
(c) Vertical asymptote: $x = -2$
Horizontal asymptote: $y = 0$
(d)



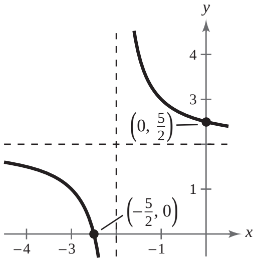
17. (a) The domain is all real numbers x except $x = -4$.
(b) y-intercept: $\left(0, -\frac{1}{4}\right)$
(c) Vertical asymptote: $x = -4$
Horizontal asymptote: $y = 0$
(d)



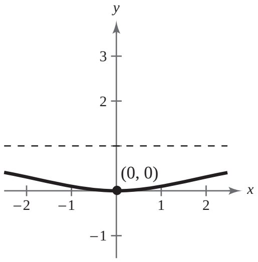
19. (a) The domain is all real numbers x except $x = -2$.
(b) x-intercept: $\left(-\frac{7}{2}, 0\right)$
y-intercept: $\left(0, \frac{7}{2}\right)$
(c) Vertical asymptote: $x = -2$
Horizontal asymptote: $y = 2$
(d)



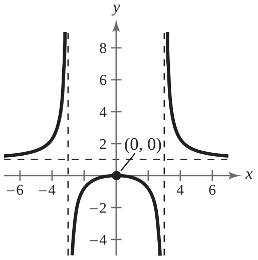
21. (a) The domain is all real numbers x except $x = -2$.
(b) x-intercept: $\left(-\frac{5}{2}, 0\right)$
y-intercept: $\left(0, \frac{5}{2}\right)$
(c) Vertical asymptote: $x = -2$
Horizontal asymptote: $y = 2$
(d)



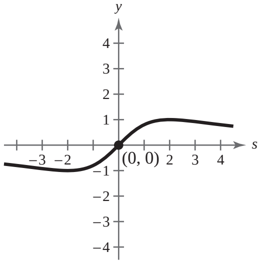
23. (a) The domain is all real numbers x.
(b) Intercept: $(0, 0)$
(c) Horizontal asymptote: $y = 1$
(d)



25. (a) The domain is all real numbers x except $x = \pm 3$.
(b) Intercept: $(0, 0)$
(c) Vertical asymptotes: $x = \pm 3$
Horizontal asymptote: $y = 1$
(d)



27. (a) The domain is all real numbers s.
(b) Intercept: $(0, 0)$
(c) Horizontal asymptote: $y = 0$
(d)



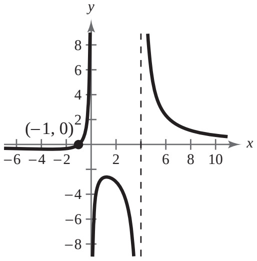
29. (a) The domain is all real numbers x except $x = 0, 4$.

(b) x-intercept: $(-1, 0)$

(c) Vertical asymptotes: $x = 0, x = 4$

Horizontal asymptote: $y = 0$

(d)



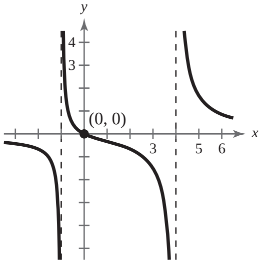
31. (a) The domain is all real numbers x except $x = 4, -1$.

(b) Intercept: $(0, 0)$

(c) Vertical asymptotes: $x = -1, x = 4$

Horizontal asymptote: $y = 0$

(d)



33. (a) The domain is all real numbers x except $x = \pm 2$.

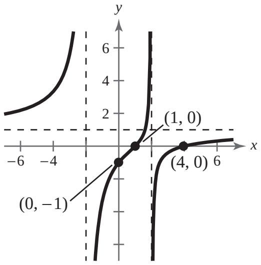
(b) x-intercepts: $(1, 0)$ and $(4, 0)$

y-intercept: $(0, -1)$

(c) Vertical asymptotes: $x = \pm 2$

Horizontal asymptote: $y = 1$

(d)



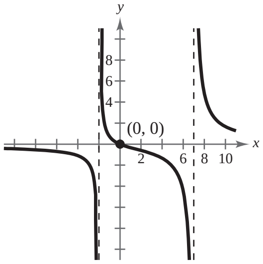
35. (a) The domain is all real numbers x except $x = -2, 7$.

(b) Intercept: $(0, 0)$

(c) Vertical asymptotes: $x = -2, x = 7$

Horizontal asymptote: $y = 0$

(d)



37. (a) The domain is all real numbers x except $x = \pm 1, 2$.

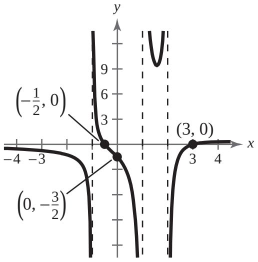
(b) x-intercepts: $(3, 0), \left(-\frac{1}{2}, 0\right)$

y-intercept: $\left(0, -\frac{3}{2}\right)$

(c) Vertical asymptotes: $x = 2, x = \pm 1$

Horizontal asymptote: $y = 0$

(d)



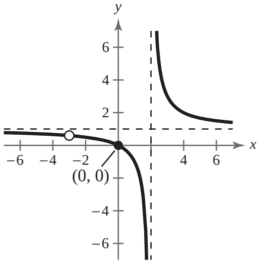
39. (a) The domain is all real numbers x except $x = 2, -3$.

(b) Intercept: $(0, 0)$

(c) Vertical asymptote: $x = 2$

Horizontal asymptote: $y = 1$

(d)



41. (a) The domain is all real numbers x except $x = -\frac{3}{2}, 2$.

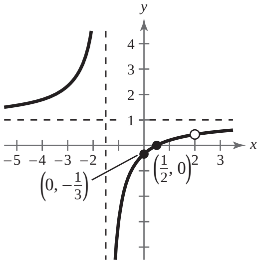
(b) x-intercept: $\left(\frac{1}{2}, 0\right)$

y-intercept: $\left(0, -\frac{1}{3}\right)$

(c) Vertical asymptote: $x = -\frac{3}{2}$

Horizontal asymptote: $y = 1$

(d)



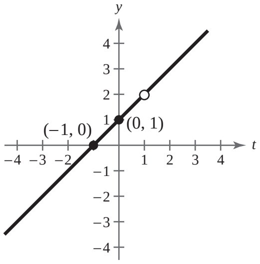
43. (a) The domain is all real numbers t except $t = 1$.

(b) t-intercept: $(-1, 0)$

y-intercept: $(0, 1)$

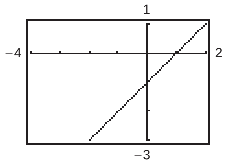
(c) No asymptotes

(d)



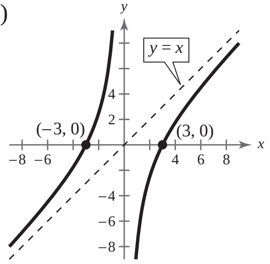
45. (a) Domain of f: all real numbers x except $x = -1$
Domain of g: all real numbers x

(b)

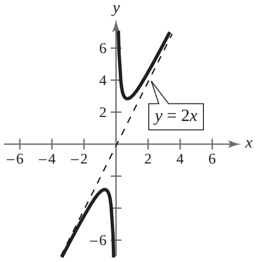


(c) Because there are only finitely many pixels, the graphing utility may not attempt to evaluate the function where it does not exist.

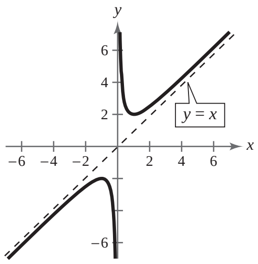
47. (a) Domain: all real numbers x except $x = 0$
(b) x-intercepts: $(\pm 3, 0)$
(c) Vertical asymptote: $x = 0$
Slant asymptote: $y = x$
(d)



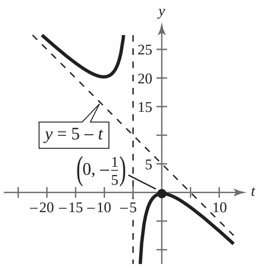
49. (a) Domain: all real numbers x except $x = 0$
(b) No intercepts
(c) Vertical asymptote: $x = 0$
Slant asymptote: $y = 2x$
(d)



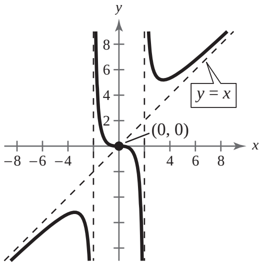
51. (a) Domain: all real numbers x except $x = 0$
(b) No intercepts
(c) Vertical asymptote: $x = 0$
Slant asymptote: $y = x$
(d)



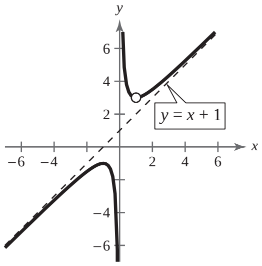
53. (a) Domain: all real numbers t except $t = -5$
(b) y-intercept: $\left(0, -\frac{1}{5}\right)$
(c) Vertical asymptote: $t = -5$
Slant asymptote: $y = -t + 5$
(d)



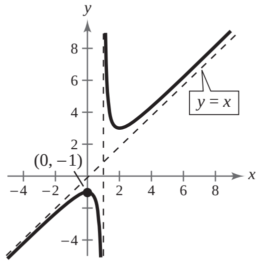
55. (a) Domain: all real numbers x except $x = \pm 2$
(b) Intercept: $(0, 0)$
(c) Vertical asymptotes: $x = \pm 2$
Slant asymptote: $y = x$
(d)



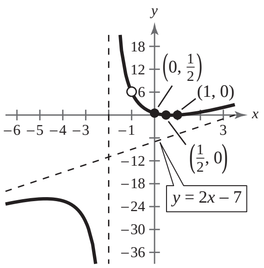
57. (a) Domain: all real numbers x except $x = 0, 1$
(b) No intercepts
(c) Vertical asymptote: $x = 0$
Slant asymptote: $y = x + 1$
(d)



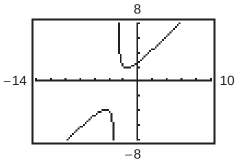
59. (a) Domain: all real numbers x except $x = 1$
(b) y-intercept: $(0, -1)$
(c) Vertical asymptote: $x = 1$
Slant asymptote: $y = x$
(d)



61. (a) Domain: all real numbers x except $x = -1, -2$
(b) y-intercept: $\left(0, \frac{1}{2}\right)$
x-intercepts: $\left(\frac{1}{2}, 0\right)$, $(1, 0)$
(c) Vertical asymptote: $x = -2$
Slant asymptote: $y = 2x - 7$
(d)

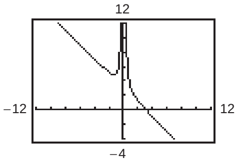


63.



Domain: all real numbers x except $x = -3$
Vertical asymptote: $x = -3$
Slant asymptote: $y = x + 2$
$y = x + 2$

65.

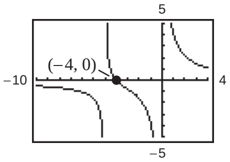


Domain: all real numbers x except $x = 0$
Vertical asymptote: $x = 0$
Slant asymptote: $y = -x + 3$
$y = -x + 3$

67. (a) $(-1, 0)$ (b) -1

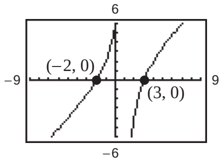
69. (a) $(1, 0), (-1, 0)$ (b) ± 1

71. (a)



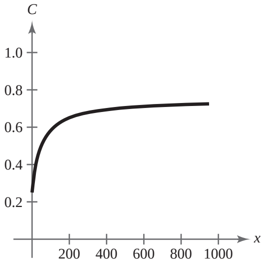
$(-4, 0)$
(b) -4

73. (a)



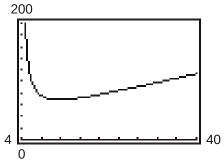
$(3, 0), (-2, 0)$
(b) $3, -2$

75. (a) Answers will vary.
(b) $[0, 950]$
(c)



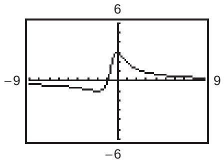
(d) Increases more slowly; 0.75

77. (a) Answers will vary.
(b) $(4, \infty)$
(c)



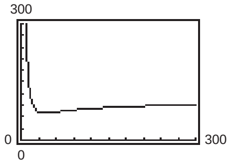
11.75 in. × 5.87 in.

79.



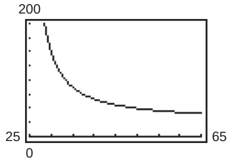
Minimum: $(-2, -1)$
Maximum: $(0, 3)$

81.



$x \approx 40.45$, or
4045 components

83. (a) Answers will vary.
(b) Vertical asymptote: $x = 25$
Horizontal asymptote: $y = 25$
(c)



(d)

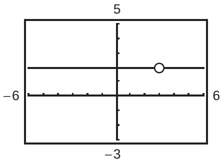
x	30	35	40	45	50	55	60
y	150	87.5	66.7	56.3	50	45.8	42.9

(e) *Sample answer:* No. You might expect the average speed for the round trip to be the average of the average speeds for the two parts of the trip.
(f) No. At 20 miles per hour you would use more time in one direction than is required for the round trip at an average speed of 50 miles per hour.

85. False. There are two distinct branches of the graph.

87. False. The degree of the numerator is 2 more than the degree of the denominator.

89.



The fraction is not reduced.

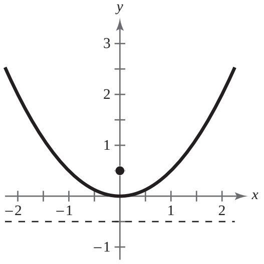
91. (a) If the degree of the numerator is exactly one more than the degree of the denominator, then the graph of the function has a slant asymptote.
(b) To find the equation of a slant asymptote, use long division to expand the function.

93. No. Given $f(x) = \dfrac{a_n x^n + \cdot \cdot \cdot + a_0}{b_m x^m + \cdot \cdot \cdot + b_0}$, when $n > m$, there is no horizontal asymptote, and n must be greater than m for a slant asymptote to occur.

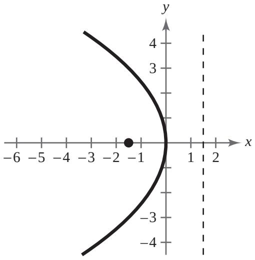
Section 4.3 *(page 339)*

1. conic or conic section **3.** parabola; directrix; focus
5. axis **7.** major axis; center **9.** hyberbola; foci
11. b **12.** c **13.** f **14.** d **15.** a **16.** e

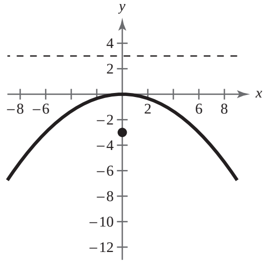
17. Focus: $\left(0, \frac{1}{2}\right)$
Directrix: $y = -\frac{1}{2}$



19. Focus: $\left(-\frac{3}{2}, 0\right)$
Directrix: $x = \frac{3}{2}$



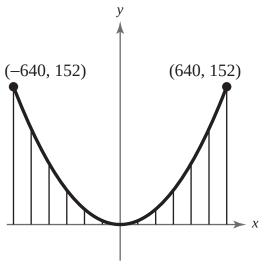
21. Focus: $(0, -3)$
Directrix: $y = 3$



23. $y^2 = -8x$ **25.** $x^2 = 8y$ **27.** $y^2 = 9x$

29. $x^2 = \frac{3}{2}y$; Focus: $\left(0, \frac{3}{8}\right)$ **31.** $y^2 = 6x$

33. (a)



(b) $y = \dfrac{19x^2}{51{,}200}$

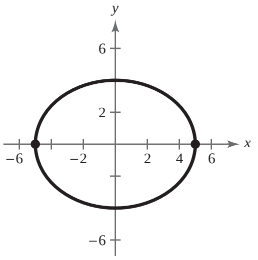
(c)

Distance, x	0	200	400	500	600
Height, y	0	$14\frac{27}{32}$	$59\frac{3}{8}$	$92\frac{99}{128}$	$133\frac{19}{32}$

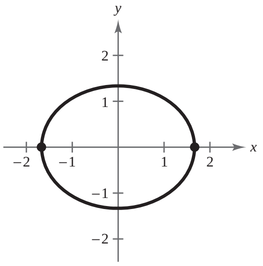
35. $\dfrac{x^2}{1} + \dfrac{y^2}{4} = 1$ **37.** $\dfrac{x^2}{25} + \dfrac{y^2}{21} = 1$ **39.** $\dfrac{x^2}{49} + \dfrac{y^2}{24} = 1$

41. $\dfrac{x^2}{81} + \dfrac{y^2}{9} = 1$ **43.** $\dfrac{21x^2}{400} + \dfrac{y^2}{25} = 1$

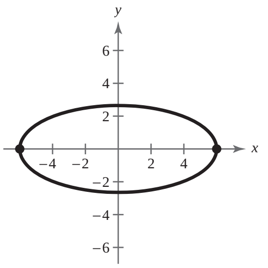
45. Vertices: $(\pm 5, 0)$



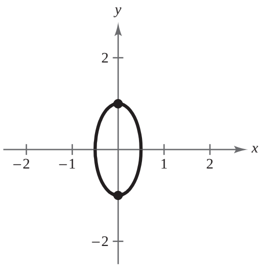
47. Vertices: $\left(\pm\frac{5}{3}, 0\right)$



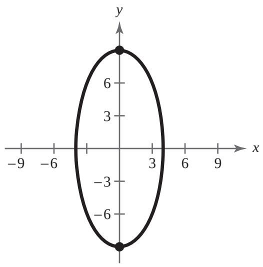
49. Vertices: $(\pm 6, 0)$



51. Vertices: $(0, \pm 1)$

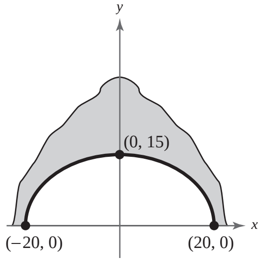


53. Vertices: $(0, \pm 9)$



55. $\left(\pm\sqrt{5}, 0\right)$; Length of string: 6 ft

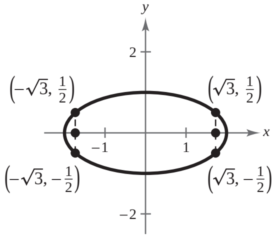
57. (a)



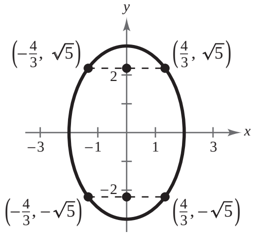
(b) $y = \frac{3}{4}\sqrt{400 - x^2}$

(c) Yes, with clearance of 0.52 foot.

59.



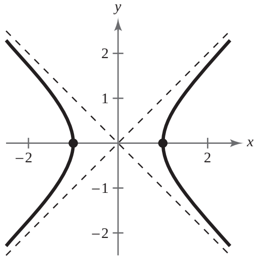
61.



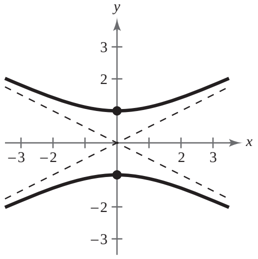
63. $\dfrac{y^2}{4} - \dfrac{x^2}{32} = 1$ **65.** $\dfrac{x^2}{1} - \dfrac{y^2}{9} = 1$

67. $\dfrac{17y^2}{1024} - \dfrac{17x^2}{64} = 1$ **69.** $\dfrac{y^2}{9} - \dfrac{x^2}{9/4} = 1$

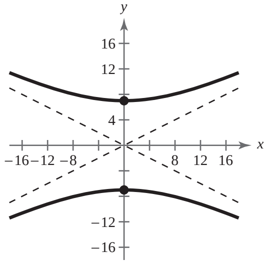
71. Vertices: $(\pm 1, 0)$



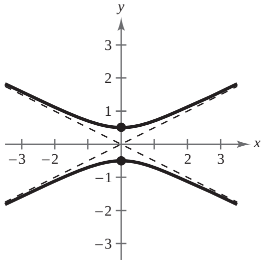
73. Vertices: $(0, \pm 1)$



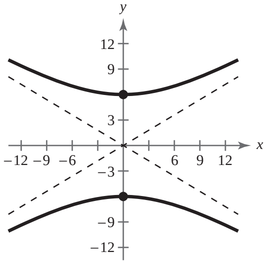
75. Vertices: $(0, \pm 7)$



77. Vertices: $\left(0, \pm\frac{1}{2}\right)$



79. Vertices: $(0, \pm 6)$



81. (a) $\dfrac{x^2}{1} - \dfrac{y^2}{169/3} = 1$ (b) 2.403 ft **83.** 10 mi

85. False. The equation represents a hyperbola:
$\dfrac{x^2}{144} - \dfrac{y^2}{144} = 1.$

87. False. If the graph crossed the directrix, there would exist points nearer the directrix than the focus.

89. (a) $A = \pi a(20 - a)$

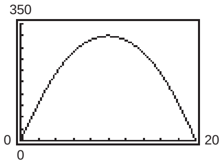
(b) $\dfrac{x^2}{196} + \dfrac{y^2}{36} = 1$

(c)

a	8	9	10	11	12	13
A	301.6	311.0	314.2	311.0	301.6	285.9

$a = 10$, circle

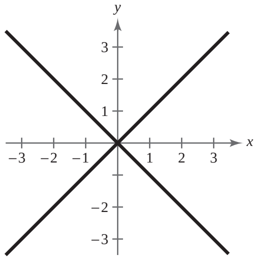
(d)



$a = 10$, circle

91. An ellipse is a circle when $a = b$.

93.



Two intersecting lines

95. Answers will vary. **97–99.** Answers will vary.

Section 4.4 *(page 349)*

1. b **2.** d **3.** e **4.** c **5.** a **6.** f

7. Circle; horizontal shift two units to the left and vertical shift one unit up

9. Parabola; reflection in the x-axis, horizontal shift one unit to the left, and vertical shift two units up

11. Hyperbola; horizontal shift one unit to the right and vertical shift three units down

13. Ellipse; horizontal shift four units to the left and vertical shift two units down

15. Center: $(0, 0)$
Radius: 7

17. Center: $(4, 5)$
Radius: 6

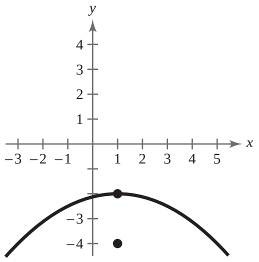
19. Center: $(1, 0)$
Radius: $\sqrt{10}$

21. $(x - 1)^2 + (y + 3)^2 = 1$
Center: $(1, -3)$
Radius: 1

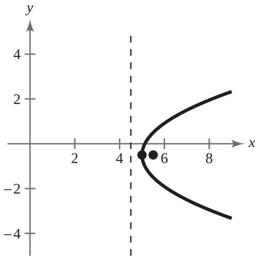
23. $(x - 4)^2 + y^2 = 16$
Center: $(4, 0)$
Radius: 4

25. $\left(x + \frac{3}{2}\right)^2 + (y - 3)^2 = 1$
Center: $\left(-\frac{3}{2}, 3\right)$
Radius: 1

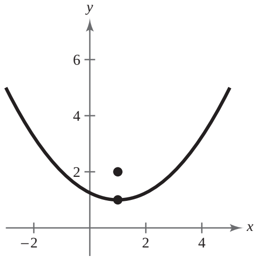
27. Vertex: $(1, -2)$
Focus: $(1, -4)$
Directrix: $y = 0$



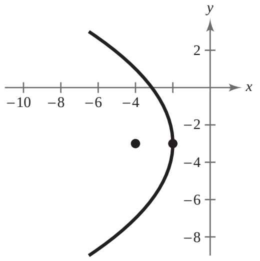
29. Vertex: $\left(5, -\frac{1}{2}\right)$
Focus: $\left(\frac{11}{2}, -\frac{1}{2}\right)$
Directrix: $x = \frac{9}{2}$



31. Vertex: $(1, 1)$
Focus: $(1, 2)$
Directrix: $y = 0$



33. Vertex: $(-2, -3)$
Focus: $(-4, -3)$
Directrix: $x = 0$



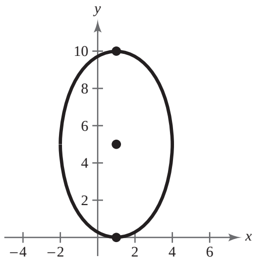
35. $(y - 2)^2 = -8(x - 3)$ **37.** $x^2 = 8(y - 4)$

39. $(y - 4)^2 = 16x$

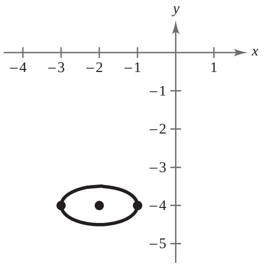
41. (a) $17{,}500\sqrt{2}$ mi/h (b) $x^2 = -16{,}400(y - 4100)$

43. (a) $x^2 = -49(y - 100)$ (b) 70 ft

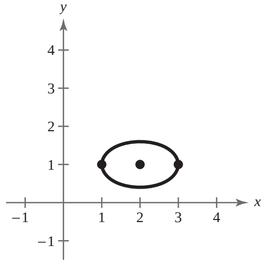
45. Center: $(1, 5)$
Foci: $(1, 9), (1, 1)$
Vertices: $(1, 10), (1, 0)$



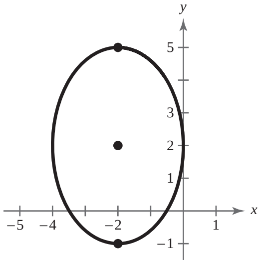
47. Center: $(-2, -4)$
Foci: $\left(\dfrac{-4 \pm \sqrt{3}}{2}, -4\right)$
Vertices: $(-3, -4), (-1, -4)$



49. Center: (2, 1)
Foci: $\left(\frac{14}{5}, 1\right), \left(\frac{6}{5}, 1\right)$
Vertices: (1, 1), (3, 1)



51. Center: $(-2, 2)$
Foci: $\left(-2, 2 \pm \sqrt{5}\right)$
Vertices: $(-2, -1), (-2, 5)$

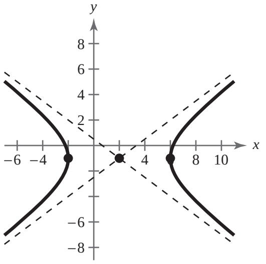


53. $\frac{(x-3)^2}{1} + \frac{y^2}{9} = 1$ **55.** $\frac{(x-2)^2}{16} + \frac{y^2}{12} = 1$

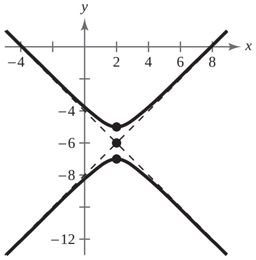
57. $\frac{x^2}{16} + \frac{(y-4)^2}{12} = 1$ **59.** $\frac{(x-2)^2}{25} + \frac{y^2}{21} = 1$

61. $\frac{x^2}{25} + \frac{y^2}{16} = 1$ **63.** 2,756,170,000 mi; 4,583,830,000 mi

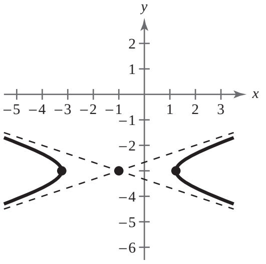
65. Center: $(2, -1)$
Foci: $(7, -1), (-3, -1)$
Vertices: $(6, -1), (-2, -1)$



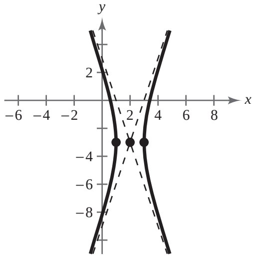
67. Center: $(2, -6)$
Foci: $\left(2, -6 \pm \sqrt{2}\right)$
Vertices: $(2, -5), (2, -7)$



69. Center: $(-1, -3)$
Foci: $\left(-1 \pm \frac{5\sqrt{2}}{3}, -3\right)$
Vertices: $\left(-1 \pm \sqrt{5}, -3\right)$



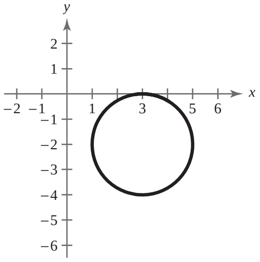
71. Center: $(2, -3)$
Foci: $\left(2 \pm \sqrt{10}, -3\right)$
Vertices: $(3, -3), (1, -3)$



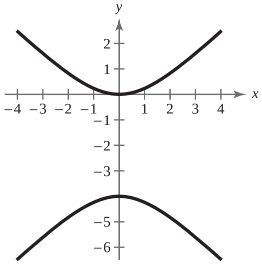
73. $\frac{(y-1)^2}{1} - \frac{x^2}{3} = 1$ **75.** $\frac{(x-4)^2}{4} - \frac{y^2}{12} = 1$

77. $\frac{y^2}{9} - \frac{4(x-2)^2}{9} = 1$ **79.** $\frac{(x-3)^2}{9} - \frac{(y-2)^2}{4} = 1$

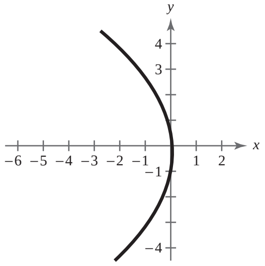
81. $(x-3)^2 + (y+2)^2 = 4$
Circle



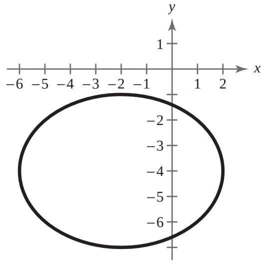
83. $\frac{(y+2)^2}{4} - \frac{x^2}{4} = 1$
Hyperbola



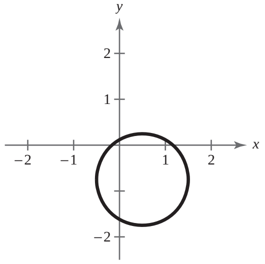
85. $\left(y + \frac{1}{4}\right)^2 = -8\left(x - \frac{1}{16}\right)$
Parabola



87. $\frac{(x+2)^2}{16} + \frac{(y+4)^2}{9} = 1$
Ellipse

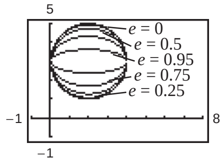


89. $\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{3}{4}\right)^2 = 1$
Circle



91. True. The conic is an ellipse.

93. (a) Answers will vary.
(b)



As e approaches 0, the ellipse becomes a circle.

Review Exercises *(page 354)*

1. Domain: all real numbers x except $x = -10$
$f(x) \to \infty$ as $x \to -10^-$, $f(x) \to -\infty$ as $x \to -10^+$

3. Domain: all real numbers x except $x = 6, 4$
$f(x) \to \infty$ as $x \to 4^-$ and as $x \to 6^+$,
$f(x) \to -\infty$ as $x \to 4^+$ and as $x \to 6^-$

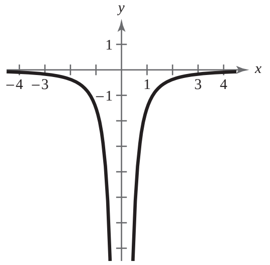
5. Vertical asymptote: $x = -3$
Horizontal asymptote: $y = 0$

7. Vertical asymptotes: $x = \pm 2$
Horizontal asymptote: $y = 1$

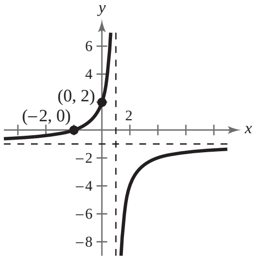
9. Vertical asymptote: $x = 6$
Horizontal asymptote: $y = 0$

11. \$0.50 is the horizontal asymptote of the function.

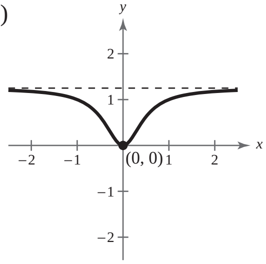
13. (a) Domain: all real numbers x except $x = 0$
(b) No intercepts
(c) Vertical asymptote: $x = 0$
Horizontal asymptote: $y = 0$
(d)



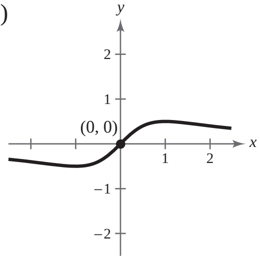
15. (a) Domain: all real numbers x except $x = 1$
(b) x-intercept: $(-2, 0)$
y-intercept: $(0, 2)$
(c) Vertical asymptote: $x = 1$
Horizontal asymptote: $y = -1$
(d)



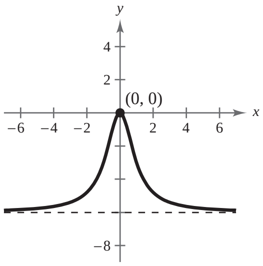
17. (a) Domain: all real numbers x
(b) Intercept: $(0, 0)$
(c) Horizontal asymptote: $y = \frac{5}{4}$
(d)



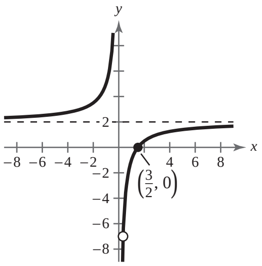
19. (a) Domain: all real numbers x
(b) Intercept: $(0, 0)$
(c) Horizontal asymptote: $y = 0$
(d)



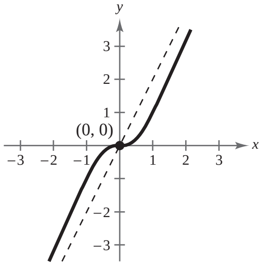
21. (a) Domain: all real numbers x
(b) Intercept: $(0, 0)$
(c) Horizontal asymptote: $y = -6$
(d)



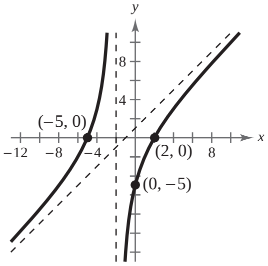
23. (a) Domain: all real numbers x except $x = 0, \frac{1}{3}$
(b) x-intercept: $\left(\frac{3}{2}, 0\right)$
(c) Vertical asymptote: $x = 0$
Horizontal asymptote: $y = 2$
(d)



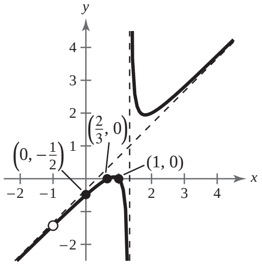
25. (a) Domain: all real numbers x
(b) Intercept: $(0, 0)$
(c) Slant asymptote: $y = 2x$
(d)



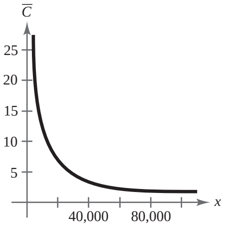
27. (a) Domain: all real numbers x except $x = -2$
(b) x-intercepts: $(2, 0), (-5, 0)$
y-intercept: $(0, -5)$
(c) Vertical asymptote: $x = -2$
Slant asymptote: $y = x + 1$
(d)



29. (a) Domain: all real numbers x except $x = \frac{4}{3}, -1$
(b) x-intercepts: $\left(\frac{2}{3}, 0\right), (1, 0)$
y-intercept: $\left(0, -\frac{1}{2}\right)$
(c) Vertical asymptote: $x = \frac{4}{3}$
Slant asymptote: $y = x - \frac{1}{3}$
(d)

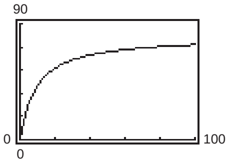


31. (a)



(b) \$100.90, \$10.90, \$1.90
(c) \$0.90 is the horizontal asymptote of the function.

33.



80.3 milligrams per square decimeter per hour

35. Parabola **37.** Hyperbola **39.** Parabola

41. Hyperbola **43.** $y^2 = -24x$ **45.** $x^2 = 12y$

47. $y^2 = 12x$ **49.** $(0, 50)$

51. $\frac{x^2}{25} + \frac{y^2}{36} = 1$ **53.** $\frac{x^2}{49} + \frac{y^2}{13} = 1$ **55.** $\frac{x^2}{221} + \frac{y^2}{25} = 1$

57. The foci should be placed 3 feet on either side of the center and have the same height as the pillars.

59. $\frac{y^2}{1} - \frac{x^2}{24} = 1$ **61.** $\frac{x^2}{1} - \frac{y^2}{4} = 1$

63. $(x - 4)^2 = -8(y - 2)$ **65.** $(x + 8)^2 = 28(y - 8)$

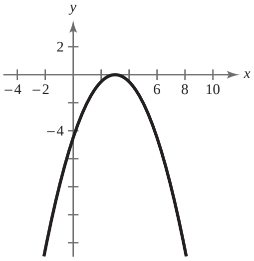
67. $\frac{(x-2)^2}{4} + \frac{(y-2)^2}{1} = 1$ **69.** $\frac{(x-2)^2}{16} + \frac{(y-3)^2}{25} = 1$

71. $\frac{(x+2)^2}{64} - \frac{(y-3)^2}{36} = 1$ **73.** $\frac{(y+2)^2}{4} - \frac{x^2}{1} = 1$

75. $\frac{5(x-4)^2}{16} - \frac{5y^2}{64} = 1$

77. $(x - 3)^2 = -2y$

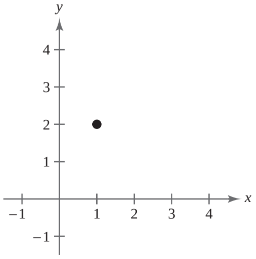
Parabola



Shifted three units to the right from the origin

79. $(1, 2)$

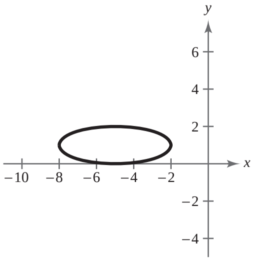
Degenerate conic (a point)



Shifted one unit to the right and two units up from the origin

81. $\frac{(x+5)^2}{9} + \frac{(y-1)^2}{1} = 1$

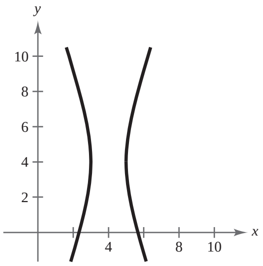
Ellipse



Shifted five units to the left and one unit up from the origin

83. $\frac{(x-4)^2}{1} - \frac{(y-4)^2}{9} = 1$

Hyperbola



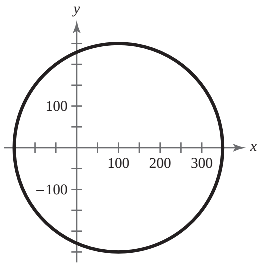
Shifted four units to the right and four units up from the origin

85. $8\sqrt{6}$ m

87. (a) Circle

(b) $(x - 100)^2 + y^2 = 62,500$

(c) Approximately 180.28 m



89. True. See Exercise 79.

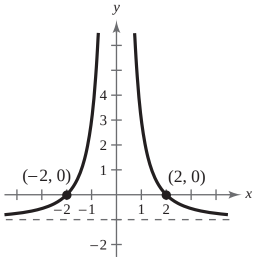
Chapter Test *(page 357)*

1. Domain: all real numbers x except $x = -1$
Vertical asymptote: $x = -1$
Horizontal asymptote: $y = 3$

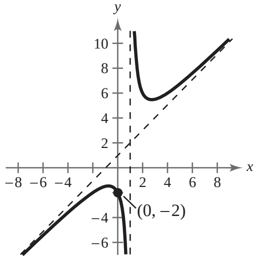
2. Domain: all real numbers x
Vertical asymptote: none
Horizontal asymptote: $y = -1$

3. Domain: all real numbers x except $x = 3$
No asymptotes

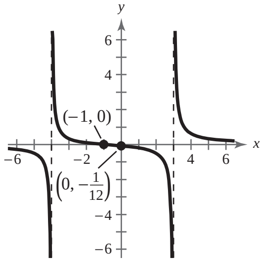
4. x-intercepts: $(-2, 0), (2, 0)$
Vertical asymptote: $x = 0$
Horizontal asymptote: $y = -1$



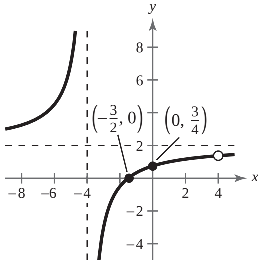
5. y-intercept: $(0, -2)$
Vertical asymptote: $x = 1$
Slant asymptote:
$y = x + 1$



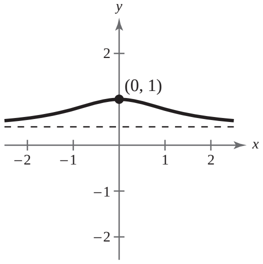
6. x-intercept: $(-1, 0)$
y-intercept: $\left(0, -\frac{1}{12}\right)$
Vertical asymptotes:
$x = 3, x = -4$
Horizontal asymptote:
$y = 0$



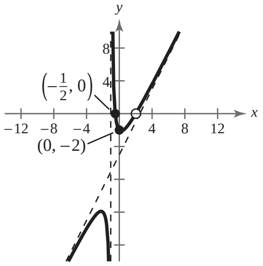
7. x-intercept: $\left(-\frac{3}{2}, 0\right)$
y-intercept: $\left(0, \frac{3}{4}\right)$
Vertical asymptote: $x = -4$
Horizontal asymptote: $y = 2$



8. y-intercept: $(0, 1)$
Horizontal asymptote:
$y = \frac{2}{5}$



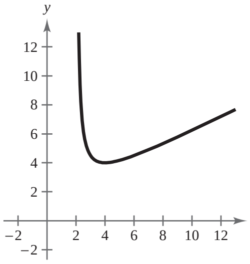
9. x-intercept: $\left(-\frac{1}{2}, 0\right)$
y-intercept: $(0, -2)$
Vertical asymptote: $x = -1$
Slant asymptote: $y = 2x - 5$



10. 6.24 in. × 12.49 in.

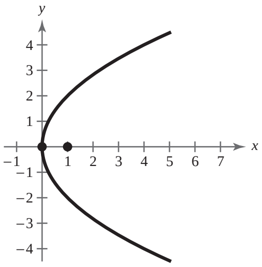
11. (a) Answers will vary. (b) $A = \frac{x^2}{2(x-2)},\ x > 2$

(c)



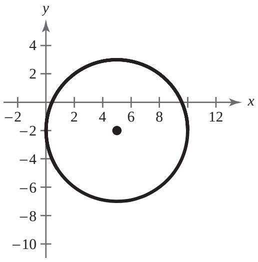
$A = 4$

12.



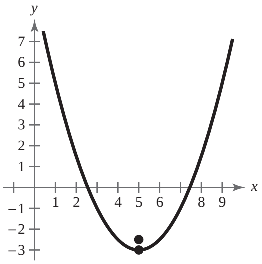
Vertex: $(0, 0)$
Focus: $(1, 0)$

13.



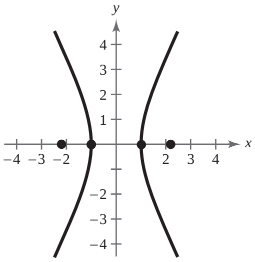
Center: $(5, -2)$

14.



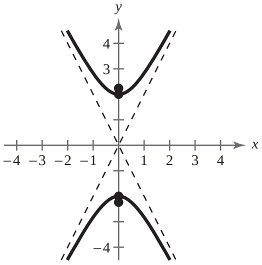
Vertex: $(5, -3)$
Focus: $\left(5, -\frac{5}{2}\right)$

15.



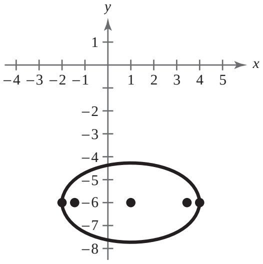
Vertices: $(\pm 1, 0)$
Foci: $\left(\pm\sqrt{5}, 0\right)$

16.



Vertices: $(0, \pm 2)$
Foci: $\left(0, \pm\sqrt{5}\right)$

17.



Center: $(1, -6)$
Vertices: $(4, -6), (-2, -6)$
Foci: $\left(1 \pm \sqrt{6}, -6\right)$

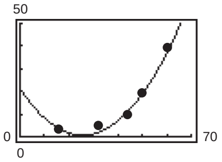
18. $\frac{(x-4)^2}{16} + \frac{(y-2)^2}{4} = 1$ **19.** $\frac{y^2}{9} - \frac{x^2}{4} = 1$ **20.** 34 m

21. Least distance: About 363,292 km
Greatest distance: About 405,508 km

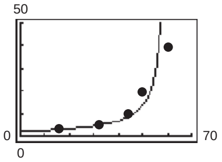
Problem Solving *(page 359)*

1. (a) iii (b) ii (c) iv (d) i

3. (a) $y_1 \approx 0.031x^2 - 1.59x + 21.0$



(b) $y_2 \approx \frac{1}{-0.007x + 0.44}$

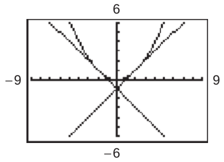


(c) The models are a good fit for the original data.
(d) $y_1(25) = 0.625$; $y_2(25) = 3.774$
The rational model is the better fit for the original data.
(e) The reciprocal model should not be used to predict the near point for a person who is 70 years old because a negative value is obtained. The quadratic model is a better fit.

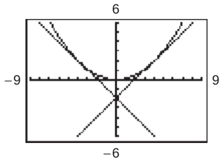
5. Answers will vary. **7.** $y^2 = 6x$; About 2.04 in.

9. (a) $\frac{x_1}{2p}$

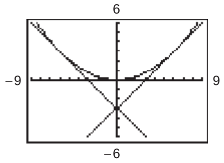
(b) $x^2 = 4y$
Tangent lines:
$y = x - 1$
$y = -x - 1$



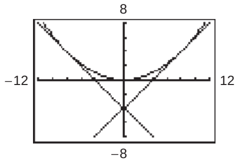
$x^2 = 8y$
Tangent lines:
$y = x - 2$
$y = -x - 2$



$x^2 = 12y$
Tangent lines:
$y = x - 3$
$y = -x - 3$



$x^2 = 16y$
Tangent lines:
$y = x - 4$
$y = -x - 4$



11. Proof

Chapter 5

Section 5.1 *(page 370)*

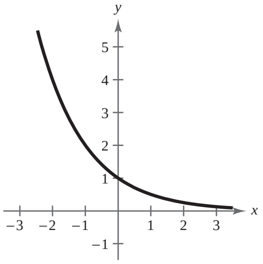
1. algebraic **3.** One-to-One **5.** $A = P\left(1 + \frac{r}{n}\right)^{nt}$

7. 0.863 **9.** 0.006 **11.** 1767.767

13. d **14.** c **15.** a **16.** b

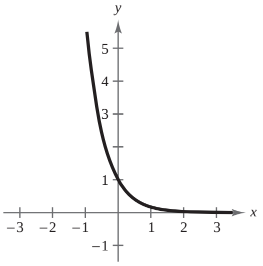
17.

x	-2	-1	0	1	2
$f(x)$	4	2	1	0.5	0.25



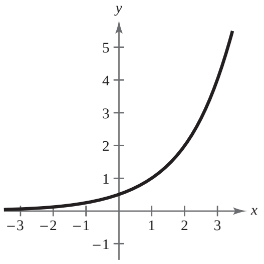
19.

x	-2	-1	0	1	2
$f(x)$	36	6	1	0.167	0.028



21.

x	-2	-1	0	1	2
$f(x)$	0.125	0.25	0.5	1	2

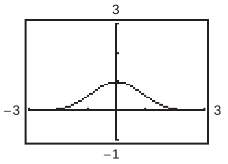


23. $x = 2$ **25.** $x = -5$

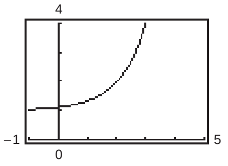
27. Shift the graph of f one unit up.

29. Reflect the graph of f in the origin.

31.



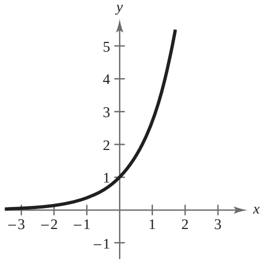
33.



35. 24.533 **37.** 7166.647

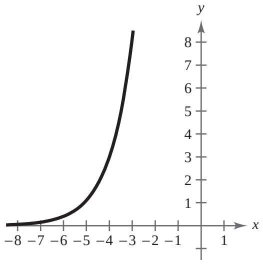
39.

x	-2	-1	0	1	2
$f(x)$	0.135	0.368	1	2.718	7.389



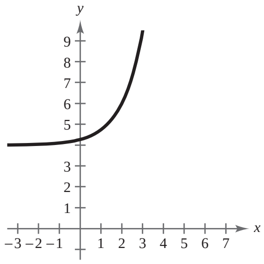
41.

x	-8	-7	-6	-5	-4
$f(x)$	0.055	0.149	0.406	1.104	3

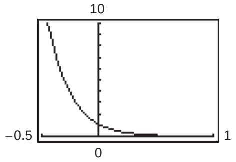


43.

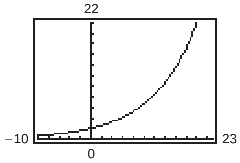
x	-2	-1	0	1	2
$f(x)$	4.037	4.100	4.271	4.736	6



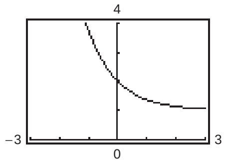
45.



47.



49.



51. $x = \frac{1}{3}$ **53.** $x = 3, -1$

55.

n	1	2	4	12
A	\$1828.49	\$1830.29	\$1831.19	\$1831.80

n	365	Continuous
A	\$1832.09	\$1832.10

57.

n	1	2	4	12
A	\$5477.81	\$5520.10	\$5541.79	\$5556.46

n	365	Continuous
A	\$5563.61	\$5563.85

59.

t	10	20	30
A	\$17,901.90	\$26,706.49	\$39,841.40

t	40	50
A	\$59,436.39	\$88,668.67

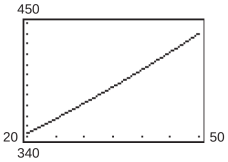
61.

t	10	20	30
A	\$22,986.49	\$44,031.56	\$84,344.25

t	40	50
A	\$161,564.86	\$309,484.08

63. \$104,710.29 **65.** \$35.45

67. (a)



(b)

t	20	21	22	23
P (in millions)	342.748	345.604	348.485	351.389

t	24	25	26	27
P (in millions)	354.318	357.271	360.249	363.251

t	28	29	30	31
P (in millions)	366.279	369.331	372.410	375.513

t	32	33	34	35
P (in millions)	378.643	381.799	384.981	388.190

t	36	37	38	39
P (in millions)	391.425	394.687	397.977	401.294

t	40	41	42	43
P (in millions)	404.639	408.011	411.412	414.840

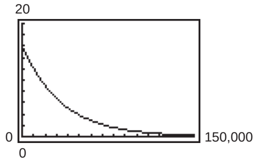
t	44	45	46	47
P (in millions)	418.298	421.784	425.300	428.844

t	48	49	50
P (in millions)	432.419	436.023	439.657

(c) 2038

69. (a) 16 g (b) 1.85 g

(c)

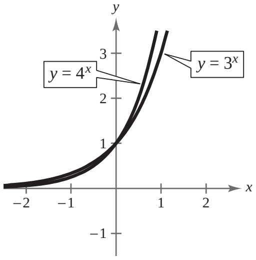


71. (a) $V(t) = 49{,}810\left(\frac{7}{8}\right)^t$ (b) \$29,197.71

73. True. As $x \to -\infty$, $f(x) \to -2$ but never reaches -2.

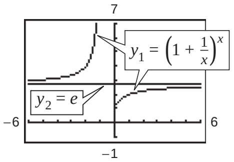
75. $f(x) = h(x)$ **77.** $f(x) = g(x) = h(x)$

79.



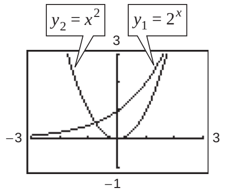
(a) $x < 0$ (b) $x > 0$

81.

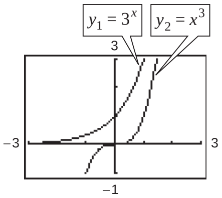


As the x-value increases, y_1 approaches the value of e.

83. (a)



(b)



In both viewing windows, the constant raised to a variable power increases more rapidly than the variable raised to a constant power.

85. c, d

Section 5.2 *(page 380)*

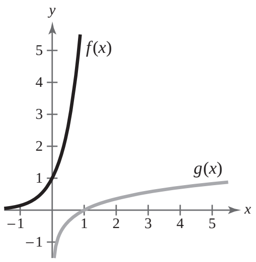
1. logarithmic **3.** natural; e **5.** $x = y$ **7.** $4^2 = 16$

9. $32^{2/5} = 4$ **11.** $\log_5 125 = 3$ **13.** $\log_4 \frac{1}{64} = -3$

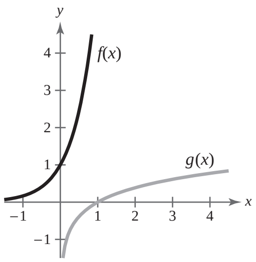
15. 6 **17.** 0 **19.** 2 **21.** -0.058 **23.** 1.097

25. 7 **27.** 1 **29.** $x = 5$ **31.** $x = 7$

33.

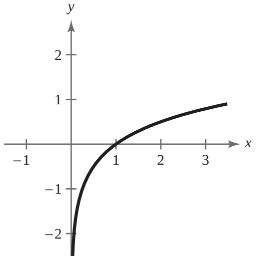


35.



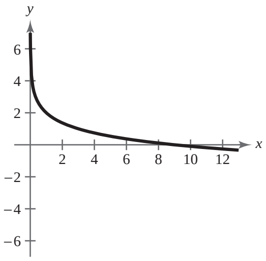
37. c **38.** d **39.** b **40.** a

41.



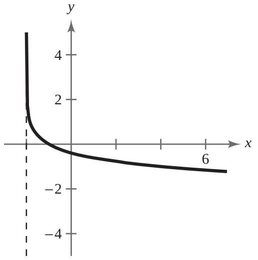
Domain: $(0, \infty)$
x-intercept: $(1, 0)$
Vertical asymptote: $x = 0$

43.



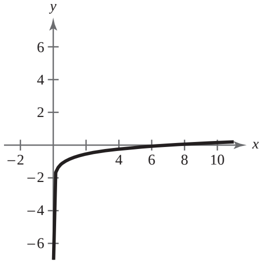
Domain: $(0, \infty)$
x-intercept: $(9, 0)$
Vertical asymptote: $x = 0$

45.



Domain: $(-2, \infty)$
x-intercept: $(-1, 0)$
Vertical asymptote: $x = -2$

47.



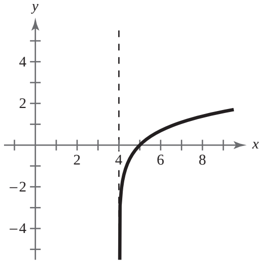
Domain: $(0, \infty)$
x-intercept: $(7, 0)$
Vertical asymptote: $x = 0$

49. $e^{-0.693\ldots} = \frac{1}{2}$ **51.** $e^{5.521\ldots} = 250$

53. $\ln 7.3890\ldots = 2$ **55.** $\ln 0.406\ldots = -0.9$

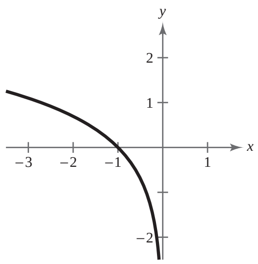
57. 2.913 **59.** −23.966 **61.** 5 **63.** $-\frac{5}{6}$

65.



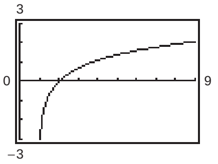
Domain: $(4, \infty)$
x-intercept: $(5, 0)$
Vertical asymptote: $x = 4$

67.

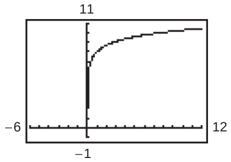


Domain: $(-\infty, 0)$
x-intercept: $(-1, 0)$
Vertical asymptote: $x = 0$

69.



71.



73. $x = 8$ **75.** $x = -5, 5$

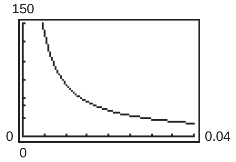
77. (a) 30 yr; 10 yr (b) \$323,179; \$199,109
(c) \$173,179; \$49,109
(d) $x = 750$; The monthly payment must be greater than \$750.

79. (a)

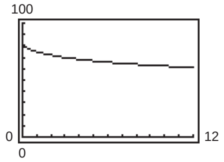
r	0.005	0.010	0.015	0.020	0.025	0.030
t	138.6	69.3	46.2	34.7	27.7	23.1

As the rate of increase r increases, the time t in years for the population to double decreases.

(b)



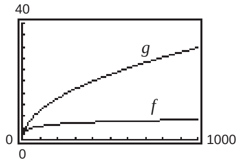
81. (a)



(b) 80 (c) 68.1 (d) 62.3

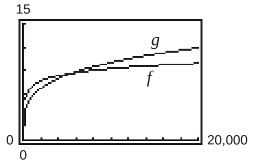
83. False. Reflecting $g(x)$ in the line $y = x$ will determine the graph of $f(x)$.

85. (a)



$g(x)$; The natural log function grows at a slower rate than the square root function.

(b)



$g(x)$; The natural log function grows at a slower rate than the fourth root function.

87. (a) False (b) True (c) True (d) False

89. Answers will vary.

Section 5.3 *(page 387)*

1. change-of-base **3.** $\dfrac{1}{\log_b a}$ **4.** c **5.** a **6.** b

7. (a) $\dfrac{\log 16}{\log 5}$ (b) $\dfrac{\ln 16}{\ln 5}$ **9.** (a) $\dfrac{\log \frac{3}{10}}{\log x}$ (b) $\dfrac{\ln \frac{3}{10}}{\ln x}$

11. 1.771 **13.** -1.048 **15.** $\frac{3}{2}$ **17.** $-3 - \log_5 2$

19. $6 + \ln 5$ **21.** 2 **23.** $\frac{3}{4}$ **25.** 4

27. -2 is not in the domain of $\log_2 x$. **29.** 4.5 **31.** $-\frac{1}{2}$

33. 7 **35.** 2 **37.** $\ln 4 + \ln x$ **39.** $4 \log_8 x$

41. $1 - \log_5 x$ **43.** $\frac{1}{2} \ln z$ **45.** $\ln x + \ln y + 2 \ln z$

47. $\ln z + 2 \ln(z - 1)$ **49.** $\frac{1}{2} \log_2(a - 1) - 2 \log_2 3$

51. $\frac{1}{3} \ln x - \frac{1}{3} \ln y$ **53.** $2 \ln x + \frac{1}{2} \ln y - \frac{1}{2} \ln z$

55. $2 \log_5 x - 2 \log_5 y - 3 \log_5 z$ **57.** $\frac{3}{4} \ln x + \frac{1}{4} \ln(x^2 + 3)$

59. 1.1833 **61.** 1.0686 **63.** 1.9563 **65.** 2.5646

67. $\ln 2x$ **69.** $\log_2 x^2 y^4$ **71.** $\log_3 \sqrt[4]{5x}$

73. $\log \dfrac{x}{(x + 1)^2}$ **75.** $\log \dfrac{xz^3}{y^2}$ **77.** $\ln \dfrac{x}{(x + 1)(x - 1)}$

79. $\ln \sqrt[3]{\dfrac{x(x + 3)^2}{x^2 - 1}}$ **81.** $\log_8 \dfrac{\sqrt[3]{y(y + 4)^2}}{y - 1}$

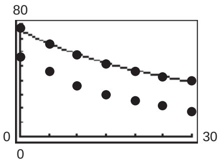
83. $\log_2 \frac{32}{4} = \log_2 32 - \log_2 4$; Property 2

85. $\beta = 10(\log I + 12)$; 60 dB **87.** 70 dB

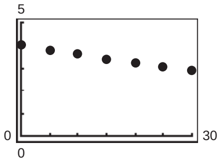
89. $\ln y = \frac{1}{4} \ln x$ **91.** $\ln y = -\frac{1}{4} \ln x + \ln \frac{5}{2}$

93. $y = 256.24 - 20.8 \ln x$

95. (a) and (b)



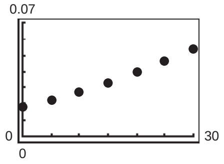
(c)



$T = 21 + e^{-0.037t + 3.997}$

The results are similar.

(d)



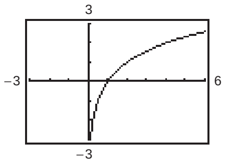
$$T = 21 + \frac{1}{0.001t + 0.016}$$

(e) Answers will vary.

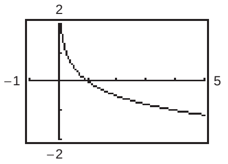
97. False; $\ln 1 = 0$ **99.** False; $\ln(x - 2) \neq \ln x - \ln 2$

101. False; $u = v^2$

103. $f(x) = \dfrac{\log x}{\log 2} = \dfrac{\ln x}{\ln 2}$



105. $f(x) = \dfrac{\log x}{\log \frac{1}{4}} = \dfrac{\ln x}{\ln \frac{1}{4}}$



107. *Sample answers:*

(a) $\ln(1 + 3) \overset{?}{=} \ln(1) + \ln(3)$

$1.39 \neq 0 + 1.10$

$\ln u + \ln v = \ln(uv)$, but $\ln(u + v) \neq \ln u + \ln v$

(b) $\ln(3 - 1) \overset{?}{=} \ln(3) - \ln(1)$

$0.69 \neq 1.10 - 0$

$\ln u - \ln v = \ln \dfrac{u}{v}$, but $\ln(u - v) \neq \ln u - \ln v$

(c) $(\ln 2)^3 \overset{?}{=} 3 \ln 2$

$0.33 \neq 2.08$

$n(\ln u) = \ln u^n$, but $(\ln u)^n \neq n(\ln u)$

109. $\ln 1 = 0$
$\ln 2 \approx 0.6931$
$\ln 3 \approx 1.0986$
$\ln 4 \approx 1.3862$
$\ln 5 \approx 1.6094$
$\ln 6 \approx 1.7917$
$\ln 8 \approx 2.0793$
$\ln 9 \approx 2.1972$
$\ln 10 \approx 2.3025$
$\ln 12 \approx 2.4848$
$\ln 15 \approx 2.7080$
$\ln 16 \approx 2.7724$
$\ln 18 \approx 2.8903$
$\ln 20 \approx 2.9956$

Section 5.4 *(page 397)*

1. (a) $x = y$ (b) $x = y$ (c) x (d) x

3. (a) Yes (b) No **5.** (a) Yes (b) No (c) No

7. 2 **9.** 2 **11.** $e^{-1} \approx 0.368$ **13.** 64

15. $(3, 8)$ **17.** $2, -1$ **19.** $\dfrac{\ln 5}{\ln 3} \approx 1.465$

21. $\ln 28 \approx 3.332$ **23.** $\dfrac{\ln 80}{2 \ln 3} \approx 1.994$

25. $3 - \dfrac{\ln 565}{\ln 2} \approx -6.142$ **27.** $\dfrac{1}{3} \log\left(\dfrac{3}{2}\right) \approx 0.059$

29. $\dfrac{\ln 12}{3} \approx 0.828$ **31.** 0 **33.** $\dfrac{\ln \frac{8}{3}}{3 \ln 2} + \dfrac{1}{3} \approx 0.805$

35. $\dfrac{\ln 3}{\ln 2 - \ln 3} \approx -2.710$ **37.** $0, \dfrac{\ln 4}{\ln 5} \approx 0.861$

39. $\ln 5 \approx 1.609$ **41.** $2 \ln 75 \approx 8.635$

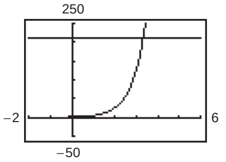
43. $\dfrac{\ln 4}{365 \ln\left(1 + \dfrac{0.065}{365}\right)} \approx 21.330$ **45.** $e^{-3} \approx 0.050$

47. $\dfrac{e^{2.1}}{6} \approx 1.361$ **49.** $\dfrac{e^{10/3}}{5} \approx 5.606$ **51.** $e^{-4/3} \approx 0.264$

53. $2(3^{11/6}) \approx 14.988$ **55.** No solution **57.** No solution

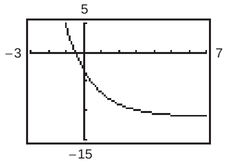
59. No solution **61.** 2

63.



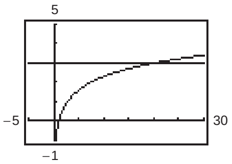
3.328

65.



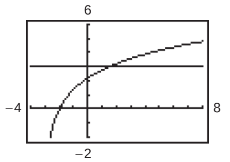
-0.478

67.



20.086

69.



1.482

71. (a) 27.73 yr (b) 43.94 yr **73.** $-1, 0$

75. 1 **77.** $e^{-1/2} \approx 0.607$ **79.** $e^{-1} \approx 0.368$

81. (a) $y = 100$ and $y = 0$; The range falls between 0% and 100%.

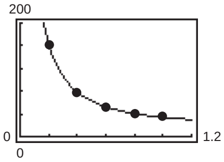
(b) Males: 69.51 in. Females: 64.49 in.

83. 12.76 in.

85. (a)

x	0.2	0.4	0.6	0.8	1.0
y	162.6	78.5	52.5	40.5	33.9

(b)



The model appears to fit the data well.

(c) 1.2 m

(d) No. According to the model, when the number of g's is less than 23, x is between 2.276 meters and 4.404 meters, which isn't realistic in most vehicles.

87. $\log_b uv = \log_b u + \log_b v$
True by Property 1 in Section 5.3.

89. $\log_b(u - v) = \log_b u - \log_b v$
False.
$1.95 \approx \log(100 - 10) \neq \log 100 - \log 10 = 1$

91. Yes. See Exercise 57.

93. For $rt < \ln 2$ years, double the amount you invest. For $rt > \ln 2$ years, double your interest rate or double the number of years, because either of these will double the exponent in the exponential function.

95. (a) 7% (b) 7.25% (c) 7.19% (d) 7.45%
The investment plan with the greatest effective yield and the highest balance after 5 years is plan (d).

Section 5.5 *(page 407)*

1. $y = ae^{bx}; y = ae^{-bx}$ **3.** normally distributed

5. (a) $P = \dfrac{A}{e^{rt}}$ (b) $t = \dfrac{\ln\left(\dfrac{A}{P}\right)}{r}$

	Initial Investment	Annual % Rate	Time to Double	Amount After 10 years
7.	$1000	3.5%	19.8 yr	$1419.07
9.	$750	8.9438%	7.75 yr	$1834.37
11.	$6376.28	4.5%	15.4 yr	$10,000.00
13.	$303,580.52			

15. (a) 7.27 yr (b) 6.96 yr (c) 6.93 yr (d) 6.93 yr

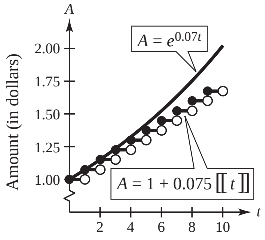
17. (a)

r	2%	4%	6%	8%	10%	12%
t	54.93	27.47	18.31	13.73	10.99	9.16

(b)

r	2%	4%	6%	8%	10%	12%
t	55.48	28.01	18.85	14.27	11.53	9.69

19.



Continuous compounding

	Half-life (years)	Initial Quantity	Amount After 1000 Years
21.	1599	10 g	6.48 g
23.	5715	2.26 g	2 g

25. $y = e^{0.7675x}$ **27.** $y = 5e^{-0.4024x}$

29. (a)

Year	1980	1990	2000	2010
Population	106.1	143.15	196.25	272.37

(b) 2017

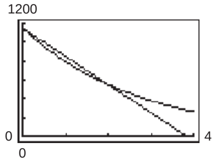
(c) No; The population will not continue to grow at such a quick rate.

31. $k = 0.2988$; About 5,309,734 hits

33. About 800 bacteria

35. (a) $V = -300t + 1150$ (b) $V = 1150e^{-0.368799t}$

(c)



The exponential model depreciates faster.

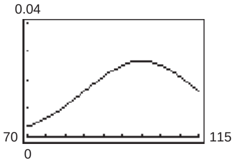
(d)

t	1 yr	3 yr
$V = -300t + 1150$	850	250
$V = 1150e^{-0.368799t}$	795	380

(e) Answers will vary.

37. (a) About 12,180 yr old (b) About 4797 yr old

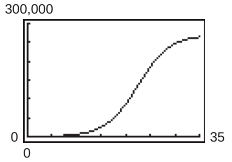
39. (a)



(b) 100

41. (a) 1998: 55,557 sites
2003: 147,644 sites
2006: 203,023 sites

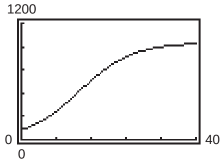
(b)



(c) and (d) 2010

43. (a) 203 animals (b) 13 mo

(c)



Horizontal asymptotes: $p = 0, p = 1000$. The population size will approach 1000 as time increases.

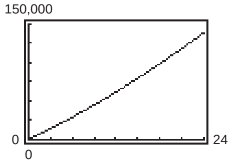
45. (a) $10^{6.6} \approx 3{,}981{,}072$ (b) $10^{5.6} \approx 398{,}107$
(c) $10^{7.1} \approx 12{,}589{,}254$

47. (a) 20 dB (b) 70 dB (c) 40 dB (d) 120 dB

49. 95% **51.** 4.64 **53.** 1.58×10^{-6} moles/L

55. $10^{5.1}$ **57.** 3:00 A.M.

59. (a)



(b) $t \approx 21$ yr; Yes

61. False. The domain can be the set of real numbers for a logistic growth function.

63. False. The graph of $f(x)$ is the graph of $g(x)$ shifted five units up.

65. Answers will vary.

Review Exercises *(page 414)*

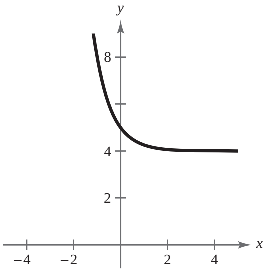
1. 0.164 **3.** 0.337 **5.** 1456.529

7. Shift the graph of f one unit up.

9. Reflect f in the x-axis and shift one unit up.

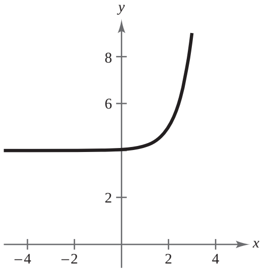
11.

x	-1	0	1	2	3
$f(x)$	8	5	4.25	4.063	4.016



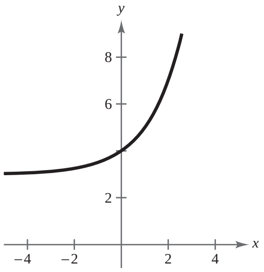
13.

x	-1	0	1	2	3
$f(x)$	4.008	4.04	4.2	5	9



15.

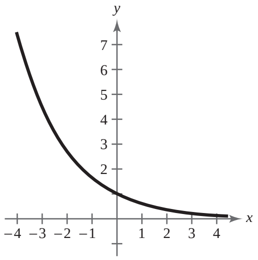
x	-2	-1	0	1	2
$f(x)$	3.25	3.5	4	5	7



17. $x = 1$ **19.** $x = 4$ **21.** 2980.958 **23.** 0.183

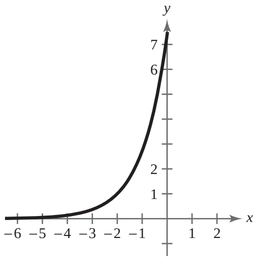
25.

x	-2	-1	0	1	2
$h(x)$	2.72	1.65	1	0.61	0.37



27.

x	-3	-2	-1	0	1
$f(x)$	0.37	1	2.72	7.39	20.09



29. (a) 0.154 (b) 0.487 (c) 0.811

31.

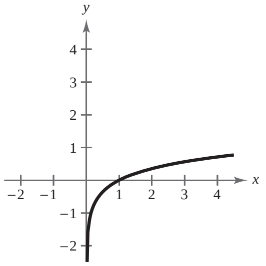
n	1	2	4	12
A	\$6719.58	\$6734.28	\$6741.74	\$6746.77

n	365	Continuous
A	\$6749.21	\$6749.29

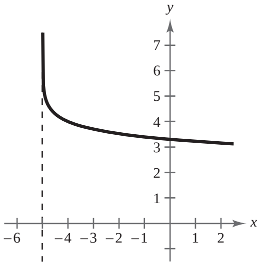
33. $\log_3 27 = 3$ **35.** $\ln 2.2255\ldots = 0.8$ **37.** 3

39. -2 **41.** $x = 7$ **43.** $x = -5$

45. Domain: $(0, \infty)$
x-intercept: $(1, 0)$
Vertical asymptote: $x = 0$

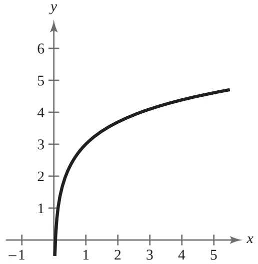


47. Domain: $(-5, \infty)$
x-intercept: $(9995, 0)$
Vertical asymptote: $x = -5$

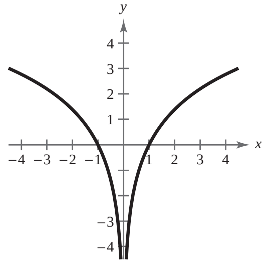


49. 3.118 **51.** 0.25

53. Domain: $(0, \infty)$
x-intercept: $(e^{-3}, 0)$
Vertical asymptote: $x = 0$



55. Domain: $(-\infty, 0), (0, \infty)$
x-intercepts: $(\pm 1, 0)$
Vertical asymptote: $x = 0$



57. 53.4 in. **59.** (a) and (b) 2.585

61. (a) and (b) -2.322 **63.** $\log 2 + 2 \log 3 \approx 1.255$

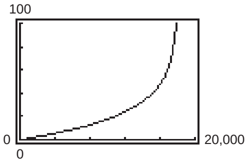
65. $2 \ln 2 + \ln 5 \approx 2.996$ **67.** $1 + 2 \log_5 x$

69. $2 - \frac{1}{2} \log_3 x$ **71.** $2 \ln x + 2 \ln y + \ln z$

73. $\log_2 5x$ **75.** $\ln \dfrac{x}{\sqrt[4]{y}}$ **77.** $\log_3 \dfrac{\sqrt{x}}{(y + 8)^2}$

79. (a) $0 \le h < 18{,}000$

(b)



Vertical asymptote: $h = 18{,}000$

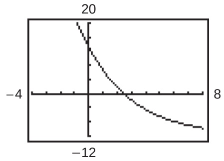
(c) The plane is climbing at a slower rate, so the time required increases.

(d) 5.46 min

81. 3 **83.** $\ln 3 \approx 1.099$ **85.** $e^4 \approx 54.598$ **87.** 1, 3

89. $\dfrac{\ln 32}{\ln 2} = 5$

91.

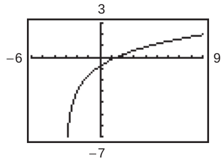


2.447

93. $\frac{1}{3}e^{8.2} \approx 1213.650$ **95.** $3e^2 \approx 22.167$

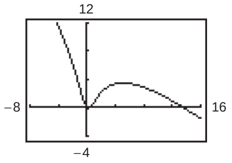
97. No solution **99.** 0.900

101.



1.482

103.

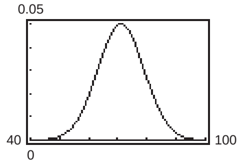


0, 0.416, 13.627

105. 73.2 yr **107.** e **108.** b **109.** f **110.** d

111. a **112.** c **113.** $y = 2e^{0.1014x}$

115.



71

117. (a) 10^{-6} W/m^2 (b) $10\sqrt{10}\text{ W/m}^2$
(c) $1.259 \times 10^{-12}\text{ W/m}^2$

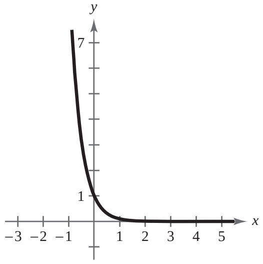
119. True by the inverse properties.

Chapter Test *(page 417)*

1. 2.366 **2.** 687.291 **3.** 0.497 **4.** 22.198

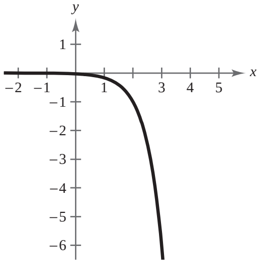
5.

x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
$f(x)$	10	3.162	1	0.316	0.1



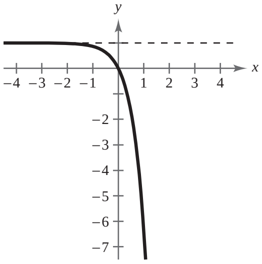
6.

x	-1	0	1	2	3
$f(x)$	-0.005	-0.028	-0.167	-1	-6



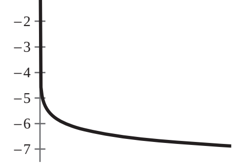
7.

x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
$f(x)$	0.865	0.632	0	-1.718	-6.389

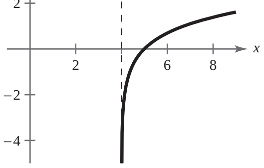


8. (a) -0.89 (b) 9.2

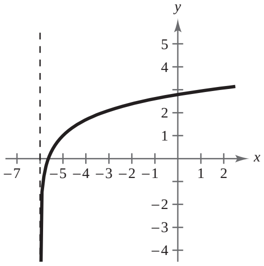
9. Domain: $(0, \infty)$
x-intercept: $(10^{-6}, 0)$
Vertical asymptote: $x = 0$



10. Domain: $(4, \infty)$
x-intercept: $(5, 0)$
Vertical asymptote: $x = 4$



11. Domain: $(-6, \infty)$
x-intercept: $(e^{-1} - 6, 0)$
Vertical asymptote: $x = -6$



12. 1.945 **13.** -0.167 **14.** -11.047

15. $\log_2 3 + 4\log_2 a$ **16.** $\ln 5 + \frac{1}{2}\ln x - \ln 6$

17. $3\log(x - 1) - 2\log y - \log z$ **18.** $\log_3 13y$

19. $\ln \dfrac{x^4}{y^4}$ **20.** $\ln \dfrac{x^3y^2}{x + 3}$ **21.** -2

22. $\dfrac{\ln 44}{-5} \approx -0.757$ **23.** $\dfrac{\ln 197}{4} \approx 1.321$

24. $e^{1/2} \approx 1.649$ **25.** $e^{-11/4} \approx 0.0639$ **26.** 20

27. $y = 2745e^{0.1570t}$ **28.** 55%

29. (a)

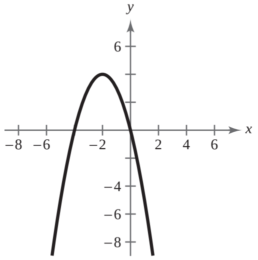
x	$\frac{1}{4}$	1	2	4	5	6
H	58.720	75.332	86.828	103.43	110.59	117.38

(b) 103 cm; 103.43 cm

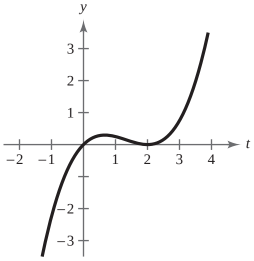
Cumulative Test for Chapters 3–5 *(page 418)*

1. $y = -\frac{3}{4}(x + 8)^2 + 5$

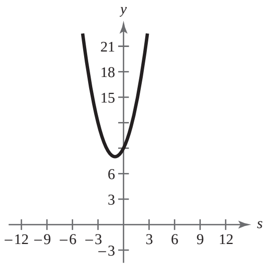
2.



3.



4.

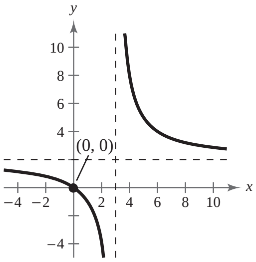


5. $-2, \pm 2i$ **6.** $-7, 0, 3$ **7.** $3x - 2 - \dfrac{3x - 2}{2x^2 + 1}$

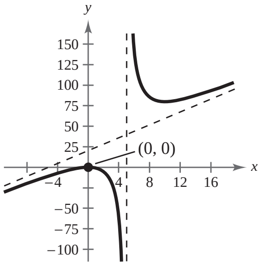
8. $3x^3 + 6x^2 + 14x + 23 + \dfrac{49}{x - 2}$ **9.** 1.20

10. $f(x) = x^4 + 3x^3 - 11x^2 + 9x + 70$

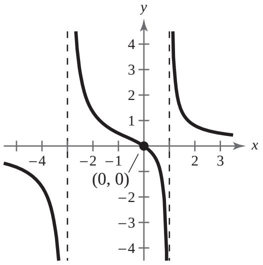
11. Domain: all real numbers x except $x = 3$
Vertical asymptote: $x = 3$
Horizontal asymptote:
$y = 2$



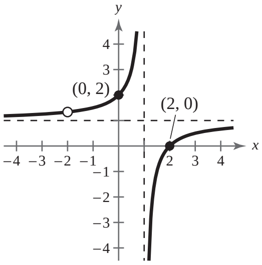
12. Domain: all real numbers x except $x = 5$
Vertical asymptote: $x = 5$
Slant asymptote:
$y = 4x + 20$



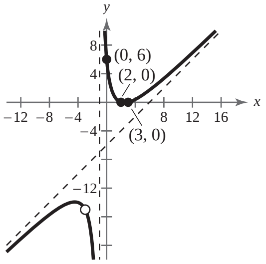
13. Intercept: $(0, 0)$
Vertical asymptotes:
$x = 1, -3$
Horizontal asymptote:
$y = 0$



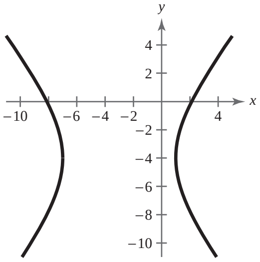
14. y-intercept: $(0, 2)$
x-intercept: $(2, 0)$
Vertical asymptote: $x = 1$
Horizontal asymptote:
$y = 1$



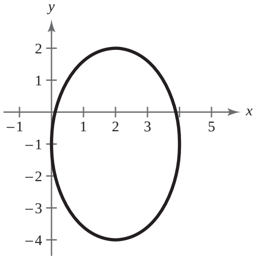
15. y-intercept: $(0, 6)$
x-intercepts: $(2, 0), (3, 0)$
Vertical asymptote: $x = -1$
Slant asymptote: $y = x - 6$



16.

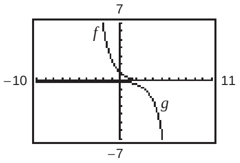


17.

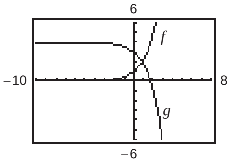


18. $(x - 3)^2 = \dfrac{3}{2}(y + 2)$ **19.** $\dfrac{(y - 2)^2}{\frac{4}{5}} - \dfrac{x^2}{\frac{16}{5}} = 1$

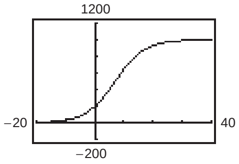
20. Reflect f in the x-axis and y-axis, and shift three units to the right.



21. Reflect f in the x-axis, and shift four units up.

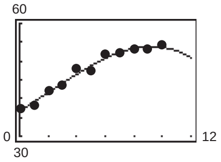


22. 1.991 **23.** -0.067 **24.** 1.717 **25.** 0.390
26. 0.906 **27.** -1.733 **28.** -4.087
29. $\ln(x + 4) + \ln(x - 4) - 4 \ln x,\ x > 4$
30. $\ln \dfrac{x^2}{\sqrt{x + 5}},\ x > 0$ **31.** $\dfrac{\ln 12}{2} \approx 1.242$
32. $\dfrac{\ln 9}{\ln 4} + 5 \approx 6.585$ **33.** $\ln 6 \approx 1.792$ or $\ln 7 \approx 1.946$
34. $\frac{64}{5} = 12.8$ **35.** $\frac{1}{2}e^8 \approx 1490.479$
36. $e^6 - 2 \approx 401.429$
37.



Horizontal asymptotes: $y = 0, y = 1000$

38. (a) and (c)

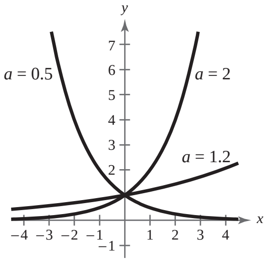


The model is a good fit for the data.
(b) $S = -0.0172t^3 + 0.119t^2 + 2.22t + 36.8$
(d) \$15.0 billion; No; The trend appears to show sales increasing.

39. \$16,302.05 **40.** 6.3 h **41.** 2019
42. (a) 300 (b) 570 (c) About 9 yr

Problem Solving *(page 421)*

1.

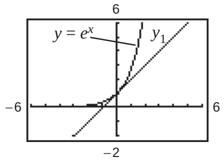


$y = 0.5^x$ and $y = 1.2^x$
$0 < a \le e^{1/e}$

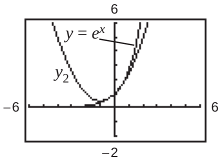
3. As $x \to \infty$, the graph of e^x increases at a greater rate than the graph of x^n.

5. Answers will vary.

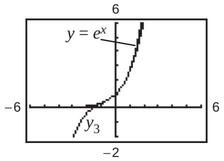
7. (a)



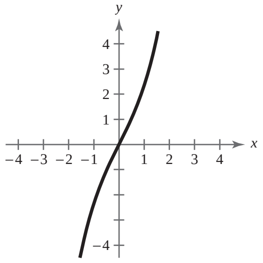
(b)



(c)



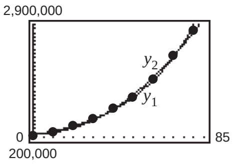
9.



$f^{-1}(x) = \ln\left(\dfrac{x + \sqrt{x^2 + 4}}{2}\right)$

11. c **13.** $t = \dfrac{\ln c_1 - \ln c_2}{\left(\dfrac{1}{k_2} - \dfrac{1}{k_1}\right)\ln \dfrac{1}{2}}$

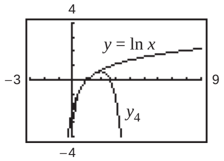
15. (a) $y_1 = 252{,}606(1.0310)^t$
(b) $y_2 = 400.88t^2 - 1464.6t + 291{,}782$
(c)



(d) The exponential model is a better fit. No, because the model is rapidly approaching infinity.

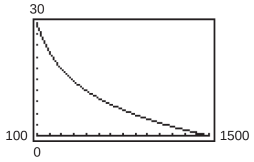
17. $1, e^2$

19. $y_4 = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4$



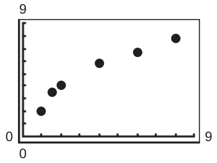
The pattern implies that
$\ln x = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \cdots.$

21.



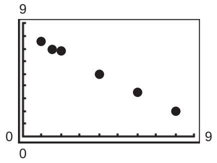
17.7 ft^3/min

23. (a)



(b)–(e) Answers will vary.

25. (a)



(b)–(e) Answers will vary.

Chapter 6

Section 6.1 *(page 431)*

1. solution **3.** point; intersection

5. (a) No (b) No (c) No (d) Yes

7. (2, 2) **9.** (2, 6), (−1, 3) **11.** (−3, −4), (5, 0)

13. (0, 0), (2, −4) **15.** (6, 4) **17.** $\left(\frac{1}{2}, 3\right)$ **19.** (1, 1)

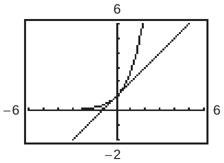
21. $\left(\frac{20}{3}, \frac{40}{3}\right)$ **23.** No solution **25.** \$5000; \$7000

27. \$6000; \$6000 **29.** (−2, 4), (0, 0) **31.** No solution

33. (6, 2) **35.** $\left(-\frac{3}{2}, \frac{1}{2}\right)$ **37.** (2, 2), (4, 0)

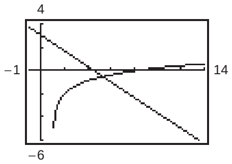
39. (1, 4), (4, 7) **41.** No solution **43.** (4, 3), (−4, 3)

45.



(0, 1)

47.



(5.31, −0.54)

49. (1, 2) **51.** No solution **53.** (0.287, 1.751)

55. $\left(\frac{1}{2}, 2\right), \left(-4, -\frac{1}{4}\right)$ **57.** 192 units

59. (a) 344 units (b) 2495 units

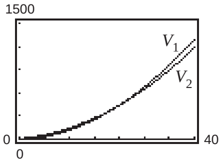
61. (a) 8 weeks

(b)

	1	2	3	4
$360 - 24x$	336	312	288	264
$24 + 18x$	42	60	78	96

	5	6	7	8
$360 - 24x$	240	216	192	168
$24 + 18x$	114	132	150	168

63. (a)



(b) 24.7 in.

(c) Doyle Log Rule; For large logs, the Doyle Log Rule gives a greater volume for a given diameter.

65. 12 m × 16 m **67.** 10 km × 12 km

69. False. To solve a system of equations by substitution, you can solve for either variable in one of the two equations and then back-substitute.

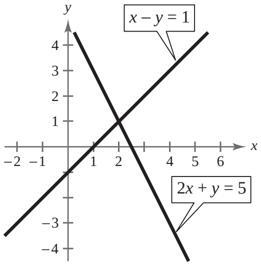
71. For a linear system, the result will be a contradictory equation such as $0 = N$, where N is a nonzero real number. For a nonlinear system, there may be an equation with imaginary solutions.

73. (a)–(c) Answers will vary.

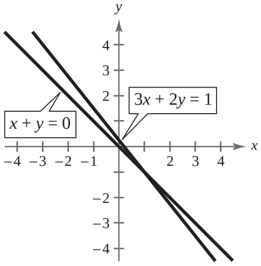
Section 6.2 *(page 442)*

1. elimination **3.** consistent; inconsistent

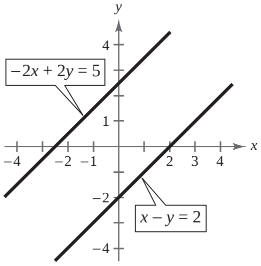
5. (2, 1)



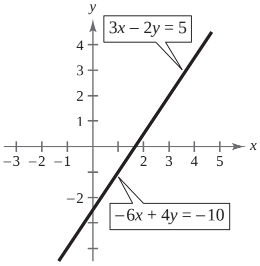
7. (1, −1)



9. No solution



11. $\left(a, \frac{3}{2}a - \frac{5}{2}\right)$



13. (4, 1) **15.** $\left(\frac{3}{2}, -\frac{1}{2}\right)$ **17.** (4, −1) **19.** $\left(\frac{12}{7}, \frac{18}{7}\right)$

21. No solution **23.** Infinitely many solutions: $\left(a, -\frac{1}{2} + \frac{5}{6}a\right)$

25. (101, 96) **27.** $\left(-\frac{6}{35}, \frac{43}{35}\right)$ **29.** (5, −2)

31. b; one solution; consistent **32.** c; one solution; consistent

33. a; infinitely many solutions; consistent

34. d; no solutions; inconsistent **35.** (4, 1) **37.** (2, −1)

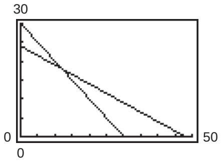
39. (6, −3) **41.** 550 mi/h, 50 mi/h

43. Cheeseburger: 300 calories; French fries: 230 calories

45. (240, 404) **47.** (2,000,000, 100)

49. (a) $\begin{cases} x + y = 30 \\ 0.25x + 0.5y = 12 \end{cases}$

(b)

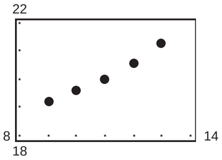


Decreases

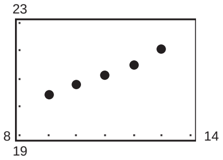
(c) 25% solution: 12 L; 50% solution: 18 L

51. \$18,000

53. (a)



Pharmacy A:
$P = 0.52t + 14.4$



Pharmacy B:
$P = 0.39t + 16.9$

(b) Yes. 2019

55. $y = 0.97x + 2.1$

57. (a) $y = 14x + 19$ (b) 41.4 bushels/acre

59. False. Two lines that coincide have infinitely many points of intersection.

61. $k = -4$

63. No. Two lines will intersect only once or will coincide, and if they coincide the system will have infinitely many solutions.

65. The method of elimination is much easier.

67. (39,600, 398). It is necessary to change the scale on the axes to see the point of intersection.

Section 6.3 *(page 454)*

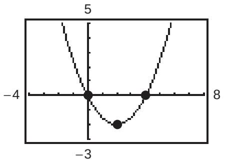
1. row-echelon **3.** Gaussian **5.** nonsquare
7. (a) No (b) No (c) No (d) Yes
9. (a) No (b) No (c) Yes (d) No
11. $(-13, -10, 8)$ **13.** $(3, 10, 2)$ **15.** $\left(\frac{11}{4}, 7, 11\right)$

17. $$\begin{cases} x - 2y + 3z = 5 \\ y - 2z = 9 \\ 2x - 3z = 0 \end{cases}$$

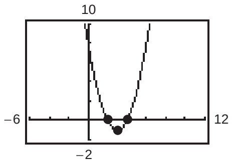
First step in putting the system in row-echelon form.

19. $(-2, 2)$ **21.** $(4, 3)$ **23.** $(1, -2)$ **25.** $(4, 1, 2)$
27. $(-4, 8, 5)$ **29.** No solution **31.** $\left(-\frac{1}{2}, 1, \frac{3}{2}\right)$
33. $(0, 0, 0)$ **35.** No solution **37.** $(-a + 3, a + 1, a)$
39. $(2a, 21a - 1, 8a)$ **41.** $(-3a + 10, 5a - 7, a)$
43. $\left(-\frac{3}{2}a + \frac{1}{2}, -\frac{2}{3}a + 1, a\right)$ **45.** $(1, 1, 1, 1)$
47. $s = -16t^2 + 144$

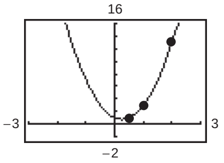
49. $y = \frac{1}{2}x^2 - 2x$



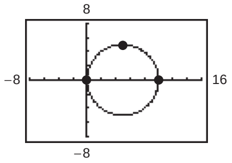
51. $y = x^2 - 6x + 8$



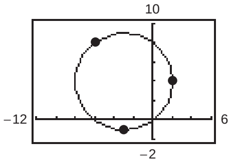
53. $y = 4x^2 - 2x + 1$



55. $x^2 + y^2 - 10x = 0$



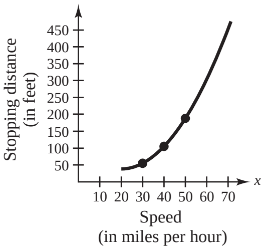
57. $x^2 + y^2 + 6x - 8y = 0$



59. 6 touchdowns
6 extra-point kicks
1 field goal

61. \$300,000 at 8%
\$400,000 at 9%
\$75,000 at 10%

63. $x = 60°, y = 67°, z = 53°$ **65.** 75 ft, 63 ft, 42 ft
67. $I_1 = 1, I_2 = 2, I_3 = 1$ **69.** $y = x^2 - x$
71. (a) $y = 0.165x^2 - 6.55x + 103$
(b)



(c) 453 ft

73. $x = \pm\frac{\sqrt{2}}{2}, y = \frac{1}{2}, \lambda = 1$ or $x = 0, y = 0, \lambda = 0$

75. False. Equation 2 does not have a leading coefficient of 1.
77. No. Answers will vary.
79. *Sample answers:*

$$\begin{cases} 2x + y - z = 0 \\ y + 2z = 0 \\ -x + 2y + z = -9 \end{cases} \qquad \begin{cases} x + y + z = 1 \\ 2x - z = 4 \\ 4y + 8z = 0 \end{cases}$$

81. *Sample answers:*

$$\begin{cases} x + 2y + 4z = -14 \\ x - 12y = 0 \\ x - 8z = 8 \end{cases} \qquad \begin{cases} 4x - 2y - 8z = -9 \\ -x + 4z = -1 \\ -7y + 2z = 0 \end{cases}$$

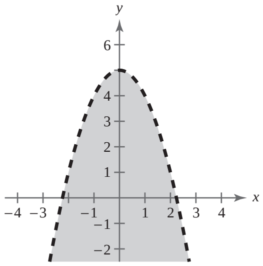
Section 6.4 *(page 464)*

1. partial fraction decomposition **3.** partial fraction
5. b **6.** c **7.** d **8.** a **9.** $\frac{A}{x} + \frac{B}{x - 2}$
11. $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 7}$ **13.** $\frac{A}{x - 5} + \frac{B}{(x - 5)^2} + \frac{C}{(x - 5)^3}$
15. $\frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}$ **17.** $\frac{1}{x} - \frac{1}{x + 1}$
19. $\frac{1}{x} - \frac{2}{2x + 1}$ **21.** $\frac{1}{x - 1} - \frac{1}{x + 2}$
23. $\frac{1}{2}\left(\frac{1}{x - 1} - \frac{1}{x + 1}\right)$ **25.** $-\frac{3}{x} - \frac{1}{x + 2} + \frac{5}{x - 2}$
27. $\frac{3}{x - 3} + \frac{9}{(x - 3)^2}$ **29.** $\frac{3}{x} - \frac{1}{x^2} + \frac{1}{x + 1}$
31. $\frac{3}{x} - \frac{2x - 2}{x^2 + 1}$ **33.** $-\frac{1}{x - 1} + \frac{x + 2}{x^2 - 2}$
35. $\frac{2}{x^2 + 4} + \frac{x}{(x^2 + 4)^2}$
37. $\frac{1}{8}\left(\frac{1}{2x + 1} + \frac{1}{2x - 1} - \frac{4x}{4x^2 + 1}\right)$
39. $\frac{1}{x + 1} + \frac{2}{x^2 - 2x + 3}$
41. $\frac{2}{x} - \frac{3}{x^2} - \frac{2x - 3}{x^2 + 2} - \frac{4x - 6}{(x^2 + 2)^2}$ **43.** $1 - \frac{2x + 1}{x^2 + x + 1}$
45. $2x - 7 + \frac{17}{x + 2} + \frac{1}{x + 1}$
47. $x + 3 + \frac{6}{x - 1} + \frac{4}{(x - 1)^2} + \frac{1}{(x - 1)^3}$
49. $x + \frac{2}{x} + \frac{1}{x + 1} + \frac{3}{(x + 1)^2}$ **51.** $\frac{3}{2x - 1} - \frac{2}{x + 1}$
53. $\frac{1}{2}\left[-\frac{1}{x} + \frac{5}{x + 1} - \frac{3}{(x + 1)^2}\right]$ **55.** $\frac{1}{x^2 + 2} + \frac{x}{(x^2 + 2)^2}$
57. $2x + \frac{1}{2}\left(\frac{3}{x - 4} - \frac{1}{x + 2}\right)$ **59.** $\frac{60}{100 - p} - \frac{60}{100 + p}$
61. False. The partial fraction decomposition is $\frac{A}{x + 10} + \frac{B}{x - 10} + \frac{C}{(x - 10)^2}$.
63. True. The expression is an improper rational expression.
65. $\frac{1}{2a}\left(\frac{1}{a + x} + \frac{1}{a - x}\right)$ **67.** $\frac{1}{a}\left(\frac{1}{y} + \frac{1}{a - y}\right)$
69. Answers will vary. *Sample answer:* You can substitute any convenient values of x that will help determine the constants. You can also find the basic equation, expand it, then equate coefficients of like terms.

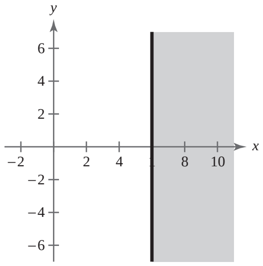
Section 6.5 *(page 473)*

1. solution **3.** solution

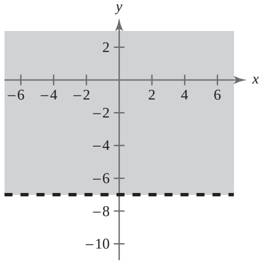
5.



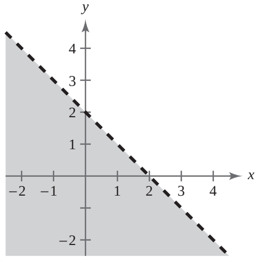
7.



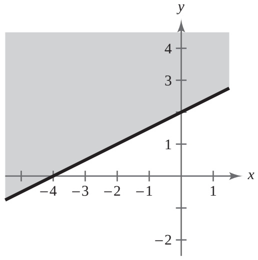
9.



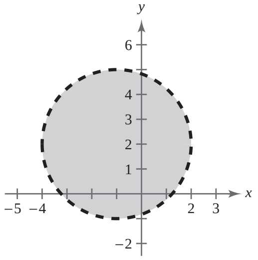
11.



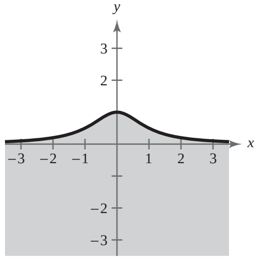
13.



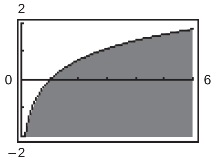
15.



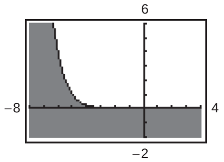
17.



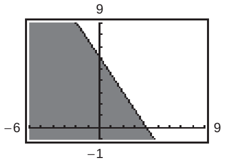
19.



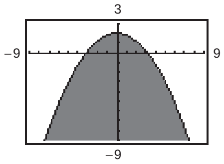
21.



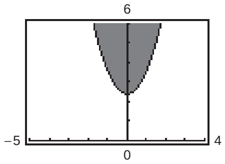
23.



25.

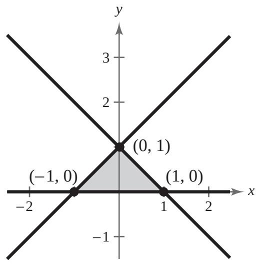


27.

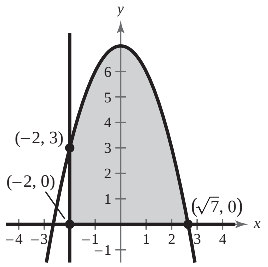


29. $y < 5x + 5$ **31.** $y \geq -\frac{2}{3}x + 2$

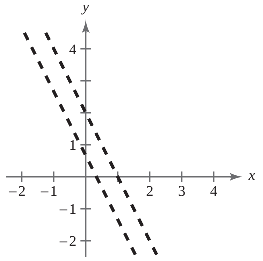
33.



35.

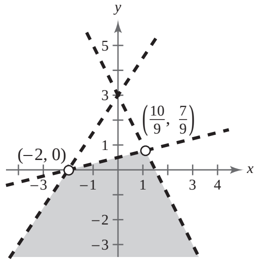


37.

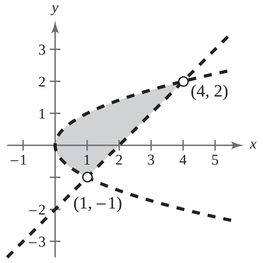


No solution

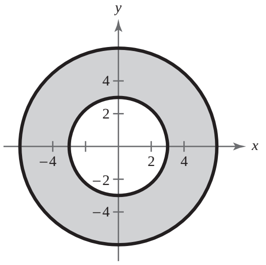
39.



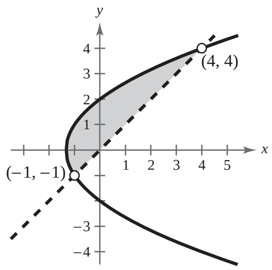
41.



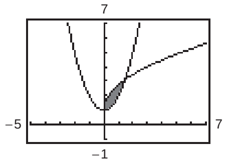
43.



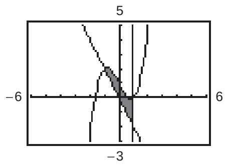
45.



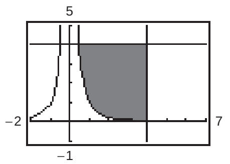
47.



49.



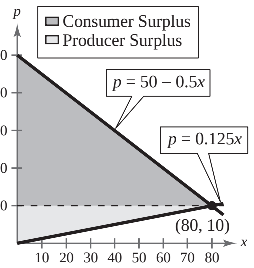
51.



53. $\begin{cases} x \geq 0 \\ y \geq 0 \\ y \leq 6 - x \end{cases}$ **55.** $\begin{cases} x \geq 0 \\ y \geq 0 \\ x^2 + y^2 < 64 \end{cases}$ **57.** $\begin{cases} x \geq 4 \\ x \leq 9 \\ y \geq 3 \\ y \leq 9 \end{cases}$

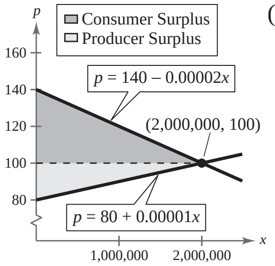
59. $\begin{cases} y \geq 0 \\ y \leq 5x \\ y \leq -x + 6 \end{cases}$

61. (a)



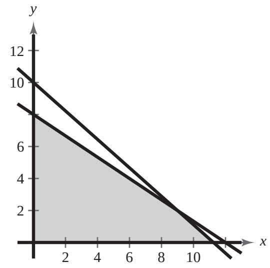
(b) Consumer surplus: $1600
Producer surplus: $400

63. (a)

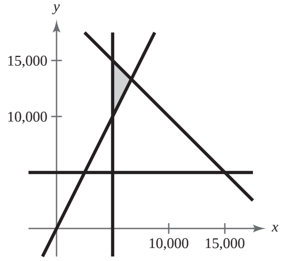


(b) Consumer surplus: \$40,000,000
Producer surplus: \$20,000,000

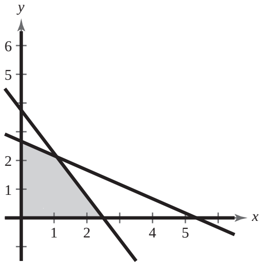
65. $\begin{cases} x + \frac{3}{2}y \le 12 \\ \frac{4}{3}x + \frac{3}{2}y \le 15 \\ x \ge 0 \\ y \ge 0 \end{cases}$



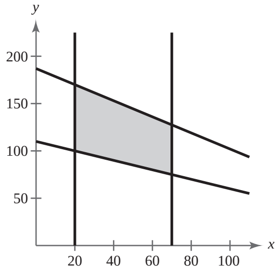
67. $\begin{cases} x + y \le 20{,}000 \\ y \ge 2x \\ x \ge 5{,}000 \\ y \ge 5{,}000 \end{cases}$



69. $\begin{cases} 6x + 4y \ge 15 \\ 3x + 6y \ge 16 \\ x \ge 0 \\ y \ge 0 \end{cases}$



71. (a) $\begin{cases} y \ge 0.5(220 - x) \\ y \le 0.85(220 - x) \\ x \ge 20 \\ x \le 70 \end{cases}$ (b)

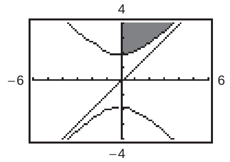


(c) Answers will vary.

73. True. The figure is a rectangle with a length of 9 units and a width of 11 units.

75. Test a point on each side of the line. Because the origin (0, 0) satisfies the inequality, the solution set of the inequality lies below the line.

77. (a) $\begin{cases} \pi y^2 - \pi x^2 \ge 10 \\ y > x \\ x > 0 \end{cases}$ (b)



(c) The line is an asymptote to the boundary. The larger the circles, the closer the radii can be while still satisfying the constraint.

Section 6.6 *(page 482)*

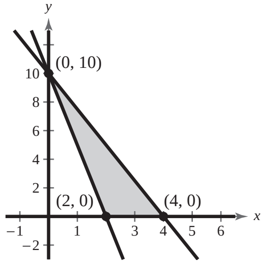
1. optimization **3.** objective **5.** inside; on

7. Minimum at (0, 0): 0
Maximum at (5, 0): 20

9. Minimum at (0, 0): 0
Maximum at (3, 4): 26

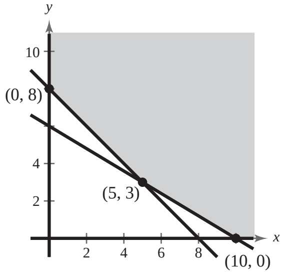
11. Minimum at (0, 0): 0
Maximum at (60, 20): 740

13.



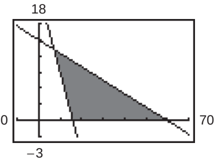
Minimum at (2, 0): 6
Maximum at (0, 10): 20

15.



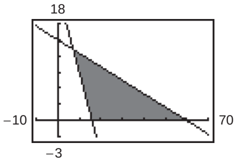
Minimum at (5, 3): 35
No maximum

17.



Minimum at (7.2, 13.2): 34.8
Maximum at (60, 0): 180

19.



Minimum at (16, 0): 16
Maximum at any point on the line segment connecting (7.2, 13.2) and (60, 0): 60

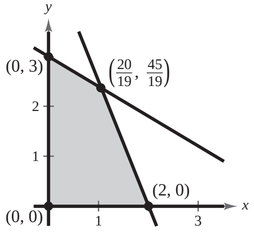
21. Minimum at (0, 0): 0
Maximum at (3, 6): 12

23. Minimum at (0, 0): 0
Maximum at (0, 10): 10

25. Minimum at (0, 0): 0
Maximum at (0, 5): 25

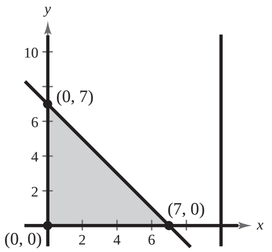
27. Minimum at (0, 0): 0
Maximum at $\left(\frac{22}{3}, \frac{19}{6}\right)$: $\frac{271}{6}$

29.



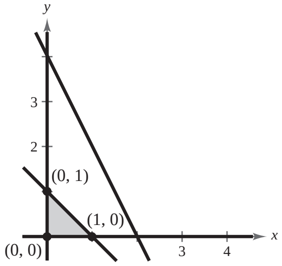
The maximum, 5, occurs at any point on the line segment connecting (2, 0) and $\left(\frac{20}{19}, \frac{45}{19}\right)$. Minimum at (0, 0): 0

31.



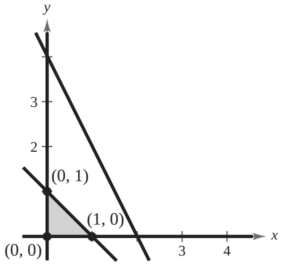
The constraint $x \leq 10$ is extraneous. Minimum at $(7, 0)$: -7; maximum at $(0, 7)$: 14

33.



The constraint $2x + y \leq 4$ is extraneous. Minimum at $(0, 0)$: 0; maximum at $(0, 1)$: 4

35.



The constraint $2x + y \leq 4$ is extraneous. Minimum at $(0, 0)$: 0; The maximum, 1, occurs at any point on the line segment connecting $(0, 1)$ and $(1, 0)$.

37. 230 units of the \$225 model
45 units of the \$250 model
Optimal profit: \$8295

39. 3 bags of brand X
6 bags of brand Y
Optimal cost: \$195

41. 13 audits
0 tax returns
Optimal revenue: \$20,800

43. 60 acres for crop A
90 acres for crop B
Optimal yield: 63,000 bushels

45. \$0 on TV ads
\$1,000,000 on newspaper ads
Optimal audience: 250 million people

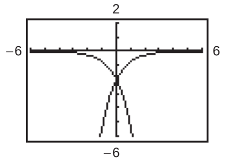
47. True. The objective function has a maximum value at any point on the line segment connecting the two vertices.

49. True. If an objective function has a maximum value at more than one vertex, then any point on the line segment connecting the points will produce the maximum value.

Review Exercises *(page 487)*

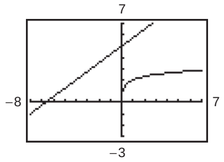
1. $(1, 1)$ **3.** $\left(\frac{3}{2}, 5\right)$ **5.** $(0.25, 0.625)$ **7.** $(5, 4)$
9. $(0, 0), (2, 8), (-2, 8)$ **11.** $(4, -2)$
13. $(1.41, -0.66), (-1.41, 10.66)$

15.



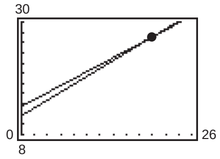
$(0, -2)$

17.



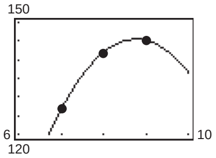
No solution

19.



The BMI for males exceeds the BMI for females after age 20.

21. 16 ft × 18 ft **23.** $\left(\frac{5}{2}, 3\right)$ **25.** $(0, 0)$
27. $\left(\frac{8}{5}a + \frac{14}{5}, a\right)$ **29.** d, one solution, consistent
30. c, infinitely many solutions, consistent
31. b, no solution, inconsistent
32. a, one solution, consistent **33.** $(100{,}000, 23)$
35. $(2, -4, -5)$ **37.** $(-6, 7, 10)$ **39.** $\left(\frac{24}{5}, \frac{22}{5}, -\frac{8}{5}\right)$
41. $\left(-\frac{3}{4}, 0, -\frac{5}{4}\right)$ **43.** $(a - 4, a - 3, a)$
45. $y = 2x^2 + x - 5$ **47.** $x^2 + y^2 - 4x + 4y - 1 = 0$
49. (a) $y = -5.950t^2 + 104.45t - 312.9$
(b)



The model is a good fit for the data.

(c) $-\$84.9$ billion; no

51. \$16,000 at 7%
\$13,000 at 9%
\$11,000 at 11%

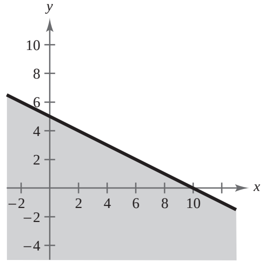
53. $s = -16t^2 + 150$

55. $\dfrac{A}{x} + \dfrac{B}{x + 20}$ **57.** $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x - 5}$

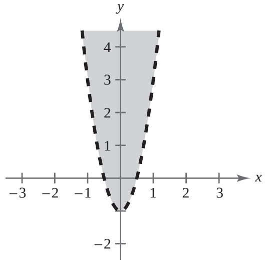
59. $\dfrac{3}{x + 2} - \dfrac{4}{x + 4}$ **61.** $1 - \dfrac{25}{8(x + 5)} + \dfrac{9}{8(x - 3)}$

63. $\dfrac{1}{2}\left(\dfrac{3}{x - 1} - \dfrac{x - 3}{x^2 + 1}\right)$ **65.** $\dfrac{3}{x^2 + 1} + \dfrac{4x - 3}{(x^2 + 1)^2}$

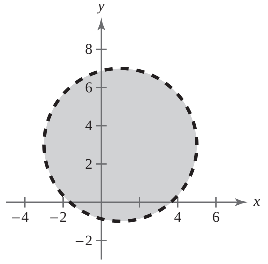
67.



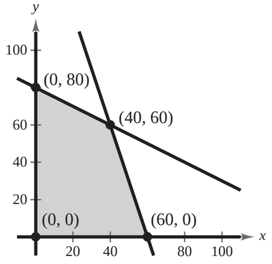
69.



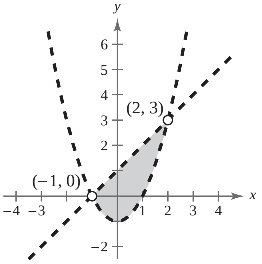
71.



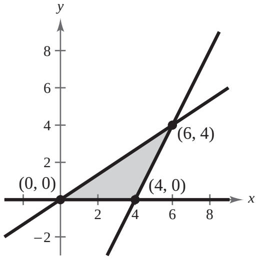
73.



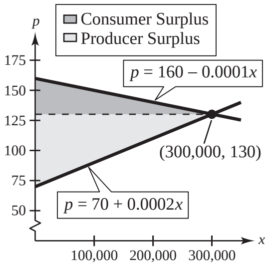
75.



77.



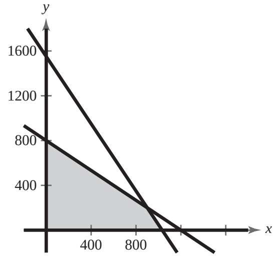
79. (a)



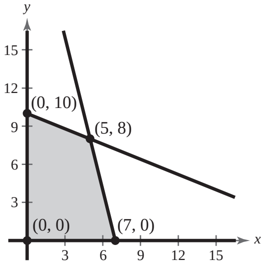
(b) Consumer surplus: \$4,500,000
Producer surplus: \$9,000,000

81. $\begin{cases} x \ge 3 \\ x \le 7 \\ y \ge 1 \\ y \le 10 \end{cases}$

83. $\begin{cases} 20x + 30y \le 24{,}000 \\ 12x + 8y \le 12{,}400 \\ x \ge 0 \\ y \ge 0 \end{cases}$

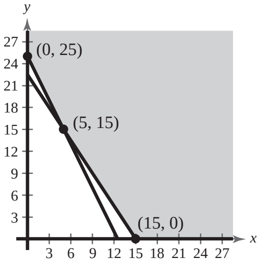


85.



Minimum at (0, 0): 0
Maximum at (5, 8): 47

87.



Minimum at (15, 0): 26.25
No maximum

89. 72 haircuts
0 permanents
Optimal revenue: \$1800

91. True. The nonparallel sides of the trapezoid are equal in length.

93. $\begin{cases} 4x + y = -22 \\ \frac{1}{2}x + y = 6 \end{cases}$

95. $\begin{cases} 3x + y = 7 \\ -6x + 3y = 1 \end{cases}$

97. $\begin{cases} x + y + z = 6 \\ x + y - z = 0 \\ x - y - z = 2 \end{cases}$

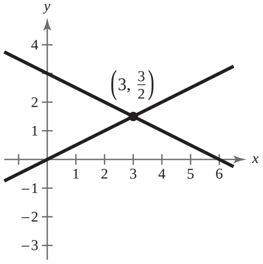
99. $\begin{cases} 2x + 2y - 3z = 7 \\ x - 2y + z = 4 \\ -x + 4y - z = -1 \end{cases}$

101. An inconsistent system of linear equations has no solution.

Chapter Test *(page 491)*

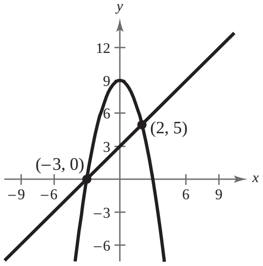
1. $(-4, -5)$ **2.** $(0, -1), (1, 0), (2, 1)$ **3.** $(8, 4), (2, -2)$

4.



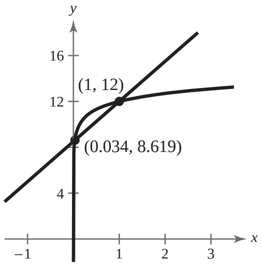
$\left(3, \frac{3}{2}\right)$

5.



$(-3, 0), (2, 5)$

6.



$(1, 12), (0.034, 8.619)$

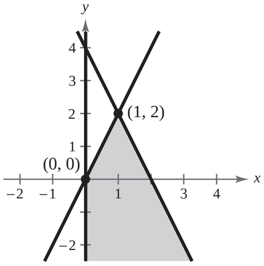
7. $(-2, -5)$ **8.** $(10, -3)$ **9.** $(2, -3, 1)$

10. No solution

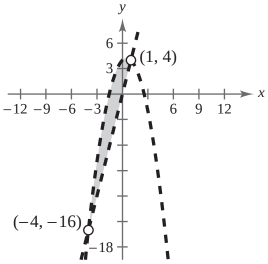
11. $-\frac{1}{x+1} + \frac{3}{x-2}$ **12.** $\frac{2}{x^2} + \frac{3}{2-x}$

13. $-\frac{5}{x} + \frac{3}{x+1} + \frac{3}{x-1}$ **14.** $-\frac{2}{x} + \frac{3x}{x^2+2}$

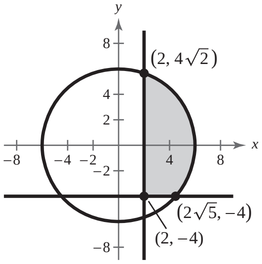
15.



16.



17.



18. Minimum at (0, 0): 0
Maximum at (12, 0): 240

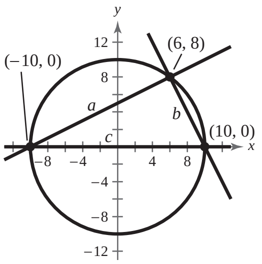
19. \$24,000 in 4% fund
\$26,000 in 5.5% fund

20. $y = -\frac{1}{2}x^2 + x + 6$

21. 0 units of model I
5300 units of model II
Optimal profit: \$212,000

Problem Solving *(page 493)*

1.



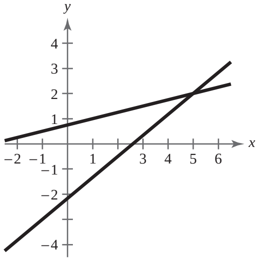
$a = 8\sqrt{5}, b = 4\sqrt{5}, c = 20$

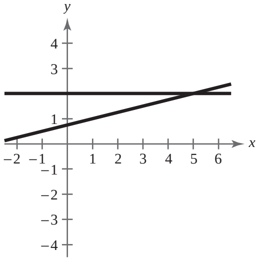
$\left(8\sqrt{5}\right)^2 + \left(4\sqrt{5}\right)^2 = 20^2$

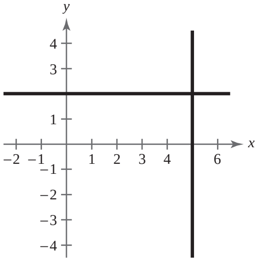
Therefore, the triangle is a right triangle.

3. $ad \neq bc$

5. (a)

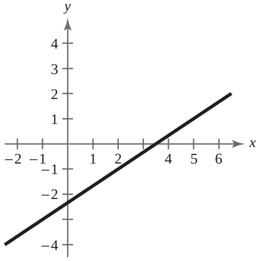






(5, 2)
Answers will vary.

(b)



$\left(\frac{3}{2}a + \frac{7}{2}, a\right)$
Answers will vary.

7. 10.1 ft; About 252.7 ft **9.** \$12.00

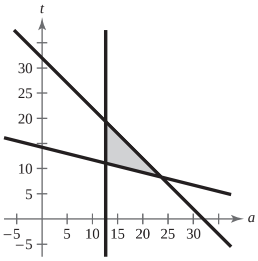
11. (a) $(3, -4)$ (b) $\left(\dfrac{2}{-a+5}, \dfrac{1}{4a-1}, \dfrac{1}{a}\right)$

13. (a) $\left(\dfrac{-5a+16}{6}, \dfrac{5a-16}{6}, a\right)$

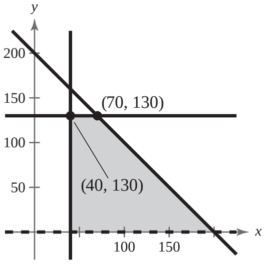
(b) $\left(\dfrac{-11a+36}{14}, \dfrac{13a-40}{14}, a\right)$

(c) $(-a+3, a-3, a)$ (d) Infinitely many

15. $\begin{cases} a + t \leq 32 \\ 0.15a \geq 1.9 \\ 193a + 772t \geq 11{,}000 \end{cases}$



17. (a) $\begin{cases} 0 < y \leq 130 \\ x \geq 40 \\ x + y \leq 200 \end{cases}$ (b)



(c) No. The point (90, 120) is not in the solution region.
(d) LDL/VLDL: 135 mg/dL
HDL: 65 mg/dL
(e) (75, 90); $\frac{165}{75} = 2.2 < 4$

Chapter 7

Section 7.1 *(page 505)*

1. matrix **3.** main diagonal **5.** augmented
7. row-equivalent **9.** 1×2 **11.** 3×1
13. 2×2 **15.** 3×3

17. $\begin{bmatrix} 4 & -3 & \vdots & -5 \\ -1 & 3 & \vdots & 12 \end{bmatrix}$ **19.** $\begin{bmatrix} 1 & 10 & -2 & \vdots & 2 \\ 5 & -3 & 4 & \vdots & 0 \\ 2 & 1 & 0 & \vdots & 6 \end{bmatrix}$

21. $\begin{bmatrix} 7 & -5 & 1 & \vdots & 13 \\ 19 & 0 & -8 & \vdots & 10 \end{bmatrix}$ **23.** $\begin{cases} x + 2y = 7 \\ 2x - 3y = 4 \end{cases}$

25. $\begin{cases} 2x + 5z = -12 \\ y - 2z = 7 \\ 6x + 3y = 2 \end{cases}$

27. $\begin{cases} 9x + 12y + 3z = 0 \\ -2x + 18y + 5z + 2w = 10 \\ x + 7y - 8z = -4 \\ 3x + 2z = -10 \end{cases}$

29. Add 5 times Row 2 to Row 1.

31. Interchange Row 1 and Row 2.
Add 4 times new Row 1 to Row 3.

33. $\begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & -1 \end{bmatrix}$ **35.** $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -7 & -1 \end{bmatrix}$

37. $\begin{bmatrix} 1 & 0 & 14 & -11 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -7 \end{bmatrix}$

39. $\begin{bmatrix} 1 & 1 & 4 & -1 \\ 0 & 5 & -2 & 6 \\ 0 & 3 & 20 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 4 & -1 \\ 0 & 1 & -\frac{2}{5} & \frac{6}{5} \\ 0 & 3 & 20 & 4 \end{bmatrix}$

41. (a) (i) $\begin{bmatrix} 3 & 0 & \vdots & -6 \\ 6 & -4 & \vdots & -28 \end{bmatrix}$

(ii) $\begin{bmatrix} 3 & 0 & \vdots & -6 \\ 0 & -4 & \vdots & -16 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & 0 & \vdots & -6 \\ 0 & 1 & \vdots & 4 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 0 & \vdots & -2 \\ 0 & 1 & \vdots & 4 \end{bmatrix}$

(b) $\begin{cases} -3x + 4y = 22 \\ 6x - 4y = -28 \end{cases}$ (c) Answers will vary.
Solution: $(-2, 4)$

43. Reduced row-echelon form **45.** Not in row-echelon form

47. $\begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ **49.** $\begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

51. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **53.** $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **55.** $\begin{bmatrix} 1 & 0 & 3 & 16 \\ 0 & 1 & 2 & 12 \end{bmatrix}$

57. $\begin{cases} x - 2y = 4 \\ y = -3 \end{cases}$ **59.** $\begin{cases} x - y + 2z = 4 \\ \phantom{x -{}} y - z = 2 \\ \phantom{x - y +{}} z = -2 \end{cases}$

$(-2, -3)$ $(8, 0, -2)$

61. $(3, -4)$ **63.** $(-4, -10, 4)$ **65.** $(3, 2)$

67. $(-5, 6)$ **69.** $(-4, -3, 6)$ **71.** Inconsistent

73. $(3, -2, 5, 0)$ **75.** $(-1, -4)$ **77.** $\left(\frac{1}{2}, -\frac{3}{4}\right)$

79. $(5a + 4, -3a + 2, a)$ **81.** $(4, -3, 2)$

83. $(7, -3, 4)$ **85.** $(0, 2 - 4a, a)$ **87.** $(1, 0, 4, -2)$

89. $(-2a, a, a, 0)$ **91.** Yes; $(-1, 1, -3)$ **93.** No

95. $f(x) = -x^2 + x + 1$ **97.** $f(x) = -9x^2 - 5x + 11$

99. $f(x) = x^2 + 2x + 5$

101. $y = 7.5t + 28$; 133 cases; Yes, because the data values increase in a linear pattern.

103. $250,000 at 7%
$1,400,000 at 8.5%
$350,000 at 9.5%

105. False. It is a 2 × 4 matrix.

107. They are the same.

Section 7.2 *(page 519)*

1. equal **3.** zero; O **5.** $x = -4, y = 22$

7. $x = 2, y = 3$

9. (a) $\begin{bmatrix} 3 & -2 \\ 1 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 0 \\ 3 & -9 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & -3 \\ 6 & -3 \end{bmatrix}$

(d) $\begin{bmatrix} -1 & -1 \\ 8 & -19 \end{bmatrix}$

11. (a) $\begin{bmatrix} 9 & 5 \\ 1 & -2 \\ -3 & 15 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & -7 \\ 3 & 8 \\ -5 & -5 \end{bmatrix}$ (c) $\begin{bmatrix} 24 & -3 \\ 6 & 9 \\ -12 & 15 \end{bmatrix}$

(d) $\begin{bmatrix} 22 & -15 \\ 8 & 19 \\ -14 & -5 \end{bmatrix}$

13. (a) $\begin{bmatrix} 5 & 5 & -2 & 4 & 4 \\ -5 & 10 & 0 & -4 & -7 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 5 & 0 & 2 & 4 \\ 7 & -6 & -4 & 2 & 7 \end{bmatrix}$

(c) $\begin{bmatrix} 12 & 15 & -3 & 9 & 12 \\ 3 & 6 & -6 & -3 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 10 & 15 & -1 & 7 & 12 \\ 15 & -10 & -10 & 3 & 14 \end{bmatrix}$

15. (a), (b), and (d) not possible (c) $\begin{bmatrix} 18 & 0 & 9 \\ -3 & -12 & 0 \end{bmatrix}$

17. $\begin{bmatrix} -8 & -7 \\ 15 & -1 \end{bmatrix}$ **19.** $\begin{bmatrix} -24 & -4 & 12 \\ -12 & 32 & 12 \end{bmatrix}$ **21.** $\begin{bmatrix} 10 & 8 \\ -59 & 9 \end{bmatrix}$

23. $\begin{bmatrix} -17.12 & 2.2 \\ 11.56 & 10.24 \end{bmatrix}$ **25.** $\begin{bmatrix} -1.581 & -3.739 \\ -4.252 & -13.249 \\ 9.713 & -0.362 \end{bmatrix}$

27. $\begin{bmatrix} -4 & 4 \\ 6 & 0 \\ -2 & -10 \end{bmatrix}$ **29.** $\begin{bmatrix} -1 & 1 \\ \frac{3}{2} & 0 \\ -\frac{1}{2} & -\frac{5}{2} \end{bmatrix}$ **31.** $\begin{bmatrix} 3 & 3 \\ -\frac{1}{2} & 0 \\ -\frac{13}{2} & \frac{11}{2} \end{bmatrix}$

33. $\begin{bmatrix} 2 & -5 \\ -5 & 0 \\ 5 & 6 \end{bmatrix}$ **35.** Not possible

37. $\begin{bmatrix} -2 & 51 \\ -8 & 33 \\ 0 & 27 \end{bmatrix}$ **39.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{7}{2} \end{bmatrix}$ **41.** $\begin{bmatrix} 60 & -20 & 10 & 60 \\ 72 & -24 & 12 & 72 \end{bmatrix}$

Order: 3 × 2 Order: 3 × 3 Order: 2 × 4

43. $\begin{bmatrix} 70 & -17 & 73 \\ 32 & 11 & 6 \\ 16 & -38 & 70 \end{bmatrix}$ **45.** $\begin{bmatrix} 151 & 25 & 48 \\ 516 & 279 & 387 \\ 47 & -20 & 87 \end{bmatrix}$

47. (a) $\begin{bmatrix} 0 & 15 \\ 6 & 12 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & 2 \\ 31 & 14 \end{bmatrix}$ (c) $\begin{bmatrix} 9 & 6 \\ 12 & 12 \end{bmatrix}$

49. (a) $\begin{bmatrix} 0 & -10 \\ 10 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -10 \\ 10 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 8 & -6 \\ 6 & 8 \end{bmatrix}$

51. (a) $\begin{bmatrix} 7 & 7 & 14 \\ 8 & 8 & 16 \\ -1 & -1 & -2 \end{bmatrix}$ (b) $[13]$ (c) Not possible

53. $\begin{bmatrix} 5 & 8 \\ -4 & -16 \end{bmatrix}$ **55.** $\begin{bmatrix} -4 & 10 \\ 3 & 14 \end{bmatrix}$

57. (a) $\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$

59. (a) $\begin{bmatrix} -2 & -3 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ -36 \end{bmatrix}$ (b) $\begin{bmatrix} -7 \\ 6 \end{bmatrix}$

61. (a) $\begin{bmatrix} 1 & -2 & 3 \\ -1 & 3 & -1 \\ 2 & -5 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \\ 17 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

63. (a) $\begin{bmatrix} 1 & -5 & 2 \\ -3 & 1 & -1 \\ 0 & -2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -20 \\ 8 \\ -16 \end{bmatrix}$ (b) $\begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$

65. $\begin{bmatrix} 110 & 99 & 77 & 33 \\ 44 & 22 & 66 & 66 \end{bmatrix}$

67. [$1037.50 $1400 $1012.50]
The entries represent the profits from both crops at each of the three outlets.

69. [$17,699,050 $17,299,050]
The entries represent the costs of the three models of the LCD televisions at the two warehouses.

71. (a) $\begin{array}{c} \begin{array}{cc} \text{Sales \$} & \text{Profit} \end{array} \\ \begin{bmatrix} 571.8 & 206.6 \\ 798.9 & 288.8 \\ 936 & 337.8 \end{bmatrix} \end{array}$

The entries represent the total sales and profits for milk on Friday, Saturday, and Sunday.

(b) $833.20

73. $P^3 = \begin{bmatrix} 0.300 & 0.175 & 0.175 \\ 0.308 & 0.433 & 0.217 \\ 0.392 & 0.392 & 0.608 \end{bmatrix}$

$P^4 = \begin{bmatrix} 0.250 & 0.188 & 0.188 \\ 0.315 & 0.377 & 0.248 \\ 0.435 & 0.435 & 0.565 \end{bmatrix}$

$P^5 = \begin{bmatrix} 0.225 & 0.194 & 0.194 \\ 0.314 & 0.345 & 0.267 \\ 0.461 & 0.461 & 0.539 \end{bmatrix}$

$P^6 = \begin{bmatrix} 0.213 & 0.197 & 0.197 \\ 0.311 & 0.326 & 0.280 \\ 0.477 & 0.477 & 0.523 \end{bmatrix}$

$P^7 = \begin{bmatrix} 0.206 & 0.198 & 0.198 \\ 0.308 & 0.316 & 0.288 \\ 0.486 & 0.486 & 0.514 \end{bmatrix}$

$P^8 = \begin{bmatrix} 0.203 & 0.199 & 0.199 \\ 0.305 & 0.309 & 0.292 \\ 0.492 & 0.492 & 0.508 \end{bmatrix}$

Approaches the matrix $\begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.3 \\ 0.5 & 0.5 & 0.5 \end{bmatrix}$

75. True. The sum of two matrices of different orders is undefined.

77. $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 3 & 2 \end{bmatrix}$ **79.** $\begin{bmatrix} 3 & -2 \\ 8 & 3 \end{bmatrix} \neq \begin{bmatrix} 2 & -2 \\ 5 & 4 \end{bmatrix}$

81. $AC = BC = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$

83. AB is a diagonal matrix whose entries are the products of the corresponding entries of A and B.

85. Answers will vary.

Section 7.3 *(page 529)*

1. square **3.** nonsingular; singular

5–11. $AB = I$ and $BA = I$

13. $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$ **15.** $\begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$ **17.** $\begin{bmatrix} 1 & -\frac{1}{2} \\ -2 & \frac{3}{2} \end{bmatrix}$

19. $\begin{bmatrix} 1 & 1 & -1 \\ -3 & 2 & -1 \\ 3 & -3 & 2 \end{bmatrix}$ **21.** Does not exist

23. $\begin{bmatrix} -\frac{1}{8} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & -\frac{1}{5} \end{bmatrix}$ **25.** $\begin{bmatrix} -175 & 37 & -13 \\ 95 & -20 & 7 \\ 14 & -3 & 1 \end{bmatrix}$

27. $\begin{bmatrix} -1.5 & 1.5 & 1 \\ 4.5 & -3.5 & -3 \\ -1 & 1 & 1 \end{bmatrix}$ **29.** $\begin{bmatrix} -12 & -5 & -9 \\ -4 & -2 & -4 \\ -8 & -4 & -6 \end{bmatrix}$

31. $\begin{bmatrix} 0 & -1.\overline{81} & 0.\overline{90} \\ -10 & 5 & 5 \\ 10 & -2.\overline{72} & -3.\overline{63} \end{bmatrix}$ **33.** $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

35. $\begin{bmatrix} \frac{5}{13} & -\frac{3}{13} \\ \frac{1}{13} & \frac{2}{13} \end{bmatrix}$ **37.** Does not exist **39.** $\begin{bmatrix} \frac{16}{59} & \frac{15}{59} \\ -\frac{4}{59} & \frac{70}{59} \end{bmatrix}$

41. $(5, 0)$ **43.** $(-8, -6)$ **45.** $(3, 8, -11)$

47. $(2, 1, 0, 0)$ **49.** $(2, -2)$ **51.** No solution

53. $(-4, -8)$ **55.** $(-1, 3, 2)$ **57.** $\left(\frac{5}{16}a + \frac{13}{16}, \frac{19}{16}a + \frac{11}{16}, a\right)$

59. $(-7, 3, -2)$

61. \$7000 in AAA-rated bonds
\$1000 in A-rated bonds
\$2000 in B-rated bonds

63. \$9000 in AAA-rated bonds
\$1000 in A-rated bonds
\$2000 in B-rated bonds

65. $I_1 = 0.5$ ampere
$I_2 = 3$ amperes
$I_3 = 3.5$ amperes

67. $I_1 = 4$ amperes
$I_2 = 1$ ampere
$I_3 = 5$ amperes

69. 100 bags of potting soil for seedlings
100 bags of potting soil for general potting
100 bags of potting soil for hardwood plants

71. (a) $\begin{cases} 2.5r + 4l + 2i = 300 \\ -r + 2l + 2i = 0 \\ r + l + i = 120 \end{cases}$

(b) $\begin{bmatrix} 2.5 & 4 & 2 \\ -1 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} r \\ l \\ i \end{bmatrix} = \begin{bmatrix} 300 \\ 0 \\ 120 \end{bmatrix}$

(c) 80 roses, 10 lilies, 30 irises

73. True. If B is the inverse of A, then $AB = I = BA$.

75. Answers will vary.

77. (a) Answers will vary.

(b) $A^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{a_{33}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{bmatrix}$

Section 7.4 *(page 537)*

1. determinant **3.** cofactor **5.** 4 **7.** 16 **9.** 28

11. 0 **13.** 6 **15.** -9 **17.** -23 **19.** -24

21. $\frac{11}{6}$ **23.** 11 **25.** 1924 **27.** 0.02 **29.** 0.34

31. (a) $M_{11} = -6, M_{12} = 3, M_{21} = 5, M_{22} = 4$
(b) $C_{11} = -6, C_{12} = -3, C_{21} = -5, C_{22} = 4$

33. (a) $M_{11} = -4, M_{12} = -2, M_{21} = 1, M_{22} = 3$
(b) $C_{11} = -4, C_{12} = 2, C_{21} = -1, C_{22} = 3$

35. (a) $M_{11} = 3, M_{12} = -4, M_{13} = 1, M_{21} = 2, M_{22} = 2, M_{23} = -4, M_{31} = -4, M_{32} = 10, M_{33} = 8$
(b) $C_{11} = 3, C_{12} = 4, C_{13} = 1, C_{21} = -2, C_{22} = 2, C_{23} = 4, C_{31} = -4, C_{32} = -10, C_{33} = 8$

37. (a) $M_{11} = 10, M_{12} = -43, M_{13} = 2, M_{21} = -30, M_{22} = 17, M_{23} = -6, M_{31} = 54, M_{32} = -53, M_{33} = -34$
(b) $C_{11} = 10, C_{12} = 43, C_{13} = 2, C_{21} = 30, C_{22} = 17, C_{23} = 6, C_{31} = 54, C_{32} = 53, C_{33} = -34$

39. (a) -75 (b) -75 **41.** (a) 96 (b) 96

43. (a) 225 (b) 225 **45.** (a) 170 (b) 170 **47.** 0

49. 0 **51.** -9 **53.** -58 **55.** -30 **57.** -168

59. 0 **61.** 412 **63.** -126 **65.** -336

67. (a) -3 (b) -2 (c) $\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$ (d) 6

69. (a) -8 (b) 0 (c) $\begin{bmatrix} -4 & 4 \\ 1 & -1 \end{bmatrix}$ (d) 0

71. (a) -21 (b) -19 (c) $\begin{bmatrix} 7 & 1 & 4 \\ -8 & 9 & -3 \\ 7 & -3 & 9 \end{bmatrix}$ (d) 399

73. (a) 2 (b) -6 (c) $\begin{bmatrix} 1 & 4 & 3 \\ -1 & 0 & 3 \\ 0 & 2 & 0 \end{bmatrix}$ (d) -12

75–79. Answers will vary. **81.** ± 2 **83.** $1 \pm \sqrt{2}$

85. $-1, 4$ **87.** $-1, -4$ **89.** $8uv - 1$ **91.** e^{5x}

93. $1 - \ln x$

95. True. If an entire row is zero, then each cofactor in the expansion is multiplied by zero.

97. Answers will vary.

99. A square matrix is a square array of numbers. The determinant of a square matrix is a real number.

101. (a) Columns 2 and 3 of A were interchanged.
$|A| = -115 = -|B|$
(b) Rows 1 and 3 of A were interchanged.
$|A| = -40 = -|B|$

103. (a) Multiply Row 1 by 5.
(b) Multiply Column 2 by 4 and Column 3 by 3.

105. (a) 28 (b) -10 (c) -12
The determinant of a diagonal matrix is the product of the entries on the main diagonal.

Section 7.5 *(page 549)*

1. Cramer's Rule **3.** $A = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

5. uncoded; coded **7.** $(-3, -2)$ **9.** Not possible

11. $\left(\frac{32}{7}, \frac{30}{7}\right)$ **13.** $(-1, 3, 2)$ **15.** $(-2, 1, -1)$ **17.** 7

19. 14 **21.** $\frac{33}{8}$ **23.** $\frac{5}{2}$ **25.** 28 **27.** $\frac{41}{4}$

29. $y = \frac{16}{5}$ or $y = 0$ **31.** $y = -3$ or $y = -11$

33. 250 mi^2 **35.** Collinear **37.** Not collinear

39. Collinear **41.** $y = -3$ **43.** $3x - 5y = 0$

45. $x + 3y - 5 = 0$ **47.** $2x + 3y - 8 = 0$

49. (a) Uncoded: [3 15], [13 5], [0 8], [15 13], [5 0], [19 15], [15 14]
(b) Encoded: 48 81 28 51 24 40 54 95 5 10 64 113 57 100

51. (a) Uncoded: [3 1 12], [12 0 13], [5 0 20], [15 13 15], [18 18 15], [23 0 0]
(b) Encoded: -68 21 35 -66 14 39 -115 35 60 -62 15 32 -54 12 27 23 -23 0

53. 1 -25 -65 17 15 -9 -12 -62 -119 27 51 48 43 67 48 57 111 117

55. -5 -41 -87 91 207 257 11 -5 -41 40 80 84 76 177 227

57. HAPPY NEW YEAR **59.** CLASS IS CANCELED

61. SEND PLANES **63.** MEET ME TONIGHT RON

65. $I_1 = -0.5$ ampere, $I_2 = 1$ ampere, $I_3 = 0.5$ ampere

67. False. The denominator is the determinant of the coefficient matrix.

69. False. When the determinant of the coefficient matrix is zero, the system has either no solution or infinitely many solutions.

71. Answers will vary. **73.** 12

Review Exercises *(page 554)*

1. 3×1 **3.** 1×1 **5.** $\left[\begin{array}{cc|c} 3 & -10 & 15 \\ 5 & 4 & 22 \end{array}\right]$

7. $\begin{cases} 5x + y + 7z = -9 \\ 4x + 2y = 10 \\ 9x + 4y + 2z = 3 \end{cases}$ **9.** $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

11. $\begin{cases} x + 2y + 3z = 9 \\ y - 2z = 2 \\ z = 0 \end{cases}$
$(5, 2, 0)$

13. $\begin{cases} x - 5y + 4z = 1 \\ y + 2z = 3 \\ z = 4 \end{cases}$
$(-40, -5, 4)$

15. $(10, -12)$ **17.** $\left(-\frac{1}{5}, \frac{7}{10}\right)$ **19.** Inconsistent

21. $(1, -2, 2)$ **23.** $\left(-2a + \frac{3}{2}, 2a + 1, a\right)$ **25.** $(5, 2, -6)$

27. $(1, 0, 4, 3)$ **29.** $(1, 2, 2)$ **31.** $(2, -3, 3)$

33. $(2, 3, -1)$ **35.** $(2, 6, -10, -3)$ **37.** $x = 12, y = -7$

39. $x = 1, y = 11$

41. (a) $\begin{bmatrix} -1 & 8 \\ 15 & 13 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & -12 \\ -9 & -3 \end{bmatrix}$
(c) $\begin{bmatrix} 8 & -8 \\ 12 & 20 \end{bmatrix}$ (d) $\begin{bmatrix} -7 & 28 \\ 39 & 29 \end{bmatrix}$

43. (a) $\begin{bmatrix} 5 & 7 \\ -3 & 14 \\ 31 & 42 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 1 \\ -11 & -10 \\ -9 & -38 \end{bmatrix}$
(c) $\begin{bmatrix} 20 & 16 \\ -28 & 8 \\ 44 & 8 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & 13 \\ 5 & 38 \\ 71 & 122 \end{bmatrix}$

45. $\begin{bmatrix} 17 & -17 \\ 13 & 2 \end{bmatrix}$ **47.** $\begin{bmatrix} 54 & 4 \\ -2 & 24 \\ -4 & 32 \end{bmatrix}$ **49.** $\begin{bmatrix} 48 & -18 & -3 \\ 15 & 51 & 33 \end{bmatrix}$

51. $\begin{bmatrix} -11 & -6 \\ 8 & -13 \\ -18 & -8 \end{bmatrix}$ **53.** $\begin{bmatrix} 3 & \frac{2}{3} \\ -\frac{4}{3} & \frac{11}{3} \\ \frac{10}{3} & 0 \end{bmatrix}$ **55.** $\begin{bmatrix} -30 & 4 \\ 51 & 70 \end{bmatrix}$

57. $\begin{bmatrix} 100 & 220 \\ 12 & -4 \\ 84 & 212 \end{bmatrix}$ **59.** $\begin{bmatrix} 14 & -2 & 8 \\ 14 & -10 & 40 \\ 36 & -12 & 48 \end{bmatrix}$

61. $\begin{bmatrix} 44 & 4 \\ 20 & 8 \end{bmatrix}$ **63.** $\begin{bmatrix} 14 & -22 & 22 \\ 19 & -41 & 80 \\ 42 & -66 & 66 \end{bmatrix}$

65. Not possible. The number of columns of the first matrix does not equal the number of rows of the second matrix.

67. $\begin{bmatrix} 76 & 114 & 133 \\ 38 & 95 & 76 \end{bmatrix}$

69. [\$2,396,539 \$2,581,388]
The merchandise shipped to warehouse 1 is worth \$2,396,539 and the merchandise shipped to warehouse 2 is worth \$2,581,388.

71–73. $AB = I$ and $BA = I$ **75.** $\begin{bmatrix} 4 & -5 \\ 5 & -6 \end{bmatrix}$

77. $\begin{bmatrix} \frac{1}{2} & -1 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{2}{3} & -\frac{5}{6} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$ **79.** $\begin{bmatrix} 13 & 6 & -4 \\ -12 & -5 & 3 \\ 5 & 2 & -1 \end{bmatrix}$

81. $\begin{bmatrix} -3 & 6 & -5.5 & 3.5 \\ 1 & -2 & 2 & -1 \\ 7 & -15 & 14.5 & -9.5 \\ -1 & 2.5 & -2.5 & 1.5 \end{bmatrix}$ **83.** $\begin{bmatrix} 1 & -1 \\ 4 & -\frac{7}{2} \end{bmatrix}$

85. Does not exist **87.** $\begin{bmatrix} 2 & \frac{20}{3} \\ \frac{1}{10} & \frac{1}{6} \end{bmatrix}$ **89.** $\begin{bmatrix} \frac{20}{9} & \frac{5}{9} \\ -\frac{10}{9} & -\frac{25}{9} \end{bmatrix}$

91. $(36, 11)$ **93.** $(-6, -1)$ **95.** $(2, 3)$ **97.** $(-8, 18)$

99. $(2, -1, -2)$ **101.** $(6, 1, -1)$ **103.** $(-3, 1)$

105. $\left(\frac{1}{6}, -\frac{7}{4}\right)$ **107.** $(1, 1, -2)$ **109.** -42 **111.** 550

113. (a) $M_{11} = 4, M_{12} = 7, M_{21} = -1, M_{22} = 2$
(b) $C_{11} = 4, C_{12} = -7, C_{21} = 1, C_{22} = 2$

115. (a) $M_{11} = 30, M_{12} = -12, M_{13} = -21, M_{21} = 20, M_{22} = 19, M_{23} = 22, M_{31} = 5, M_{32} = -2, M_{33} = 19$
(b) $C_{11} = 30, C_{12} = 12, C_{13} = -21, C_{21} = -20, C_{22} = 19, C_{23} = -22, C_{31} = 5, C_{32} = 2, C_{33} = 19$

117. -6 **119.** 15 **121.** 130 **123.** -8 **125.** 279

127. $(4, 7)$ **129.** $(-1, 4, 5)$ **131.** 16 **133.** 10

135. Collinear **137.** $x - 2y + 4 = 0$

139. $2x + 6y - 13 = 0$

141. (a) Uncoded: $[12 \quad 15 \quad 15], [11 \quad 0 \quad 15], [21 \quad 20 \quad 0], [2 \quad 5 \quad 12], [15 \quad 23 \quad 0]$
(b) Encoded: $-21 \quad 6 \quad 0 \quad -68 \quad 8 \quad 45 \quad 102 \quad -42 \quad -60 \quad -53 \quad 20 \quad 21 \quad 99 \quad -30 \quad -69$

143. SEE YOU FRIDAY

145. False. The matrix must be square.

147. An error message appears because $1(6) - (-2)(-3) = 0$.

149. If A is a square matrix, then the cofactor C_{ij} of the entry a_{ij} is $(-1)^{i+j}M_{ij}$, where M_{ij} is the determinant obtained by deleting the ith row and jth column of A. The determinant of A is the sum of the entries of any row or column of A multiplied by their respective cofactors.

151. The part of the matrix corresponding to the coefficients of the system reduces to a matrix in which the number of rows with nonzero entries is the same as the number of variables.

Chapter Test *(page 559)*

1. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

3. $\left[\begin{array}{rrr:r} 4 & 3 & -2 & 14 \\ -1 & -1 & 2 & -5 \\ 3 & 1 & -4 & 8 \end{array}\right], \left(1, 3, -\frac{1}{2}\right)$

4. (a) $\begin{bmatrix} 1 & 5 \\ 0 & -4 \end{bmatrix}$ (b) $\begin{bmatrix} 18 & 15 \\ -15 & -15 \end{bmatrix}$
(c) $\begin{bmatrix} 8 & 15 \\ -5 & -13 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & -5 \\ 0 & 5 \end{bmatrix}$

5. $\begin{bmatrix} \frac{2}{7} & \frac{3}{7} \\ \frac{5}{7} & \frac{4}{7} \end{bmatrix}$ **6.** $\begin{bmatrix} -\frac{5}{2} & 4 & -3 \\ 5 & -7 & 6 \\ 4 & -6 & 5 \end{bmatrix}$ **7.** $(12, 18)$ **8.** -112

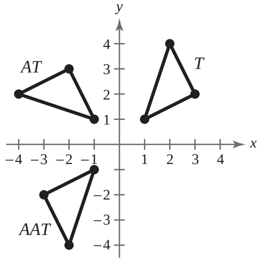
9. 29 **10.** 43 **11.** $(-3, 5)$ **12.** $(-2, 4, 6)$ **13.** 7

14. Uncoded: $[11 \;\; 14 \;\; 15], [3 \;\; 11 \;\; 0], [15 \;\; 14 \;\; 0], [23 \;\; 15 \;\; 15], [4 \;\; 0 \;\; 0]$
Encoded: $115 \quad -41 \quad -59 \quad 14 \quad -3 \quad -11 \quad 29 \quad -15 \quad -14 \quad 128 \quad -53 \quad -60 \quad 4 \quad -4 \quad 0$

15. 75 L of 60% solution, 25 L of 20% solution

Problem Solving *(page 561)*

1. (a) $AT = \begin{bmatrix} -1 & -4 & -2 \\ 1 & 2 & 3 \end{bmatrix}, AAT = \begin{bmatrix} -1 & -2 & -3 \\ -1 & -4 & -2 \end{bmatrix}$



A represents a counterclockwise rotation.

(b) AAT is rotated clockwise 90° to obtain AT. AT is then rotated clockwise 90° to obtain T.

3. (a) Yes (b) No (c) No (d) No

5. (a) Gold Satellite System: 28,750 subscribers
Galaxy Satellite Network: 35,750 subscribers
Nonsubscribers: 35,500
Answers will vary.
(b) Gold Satellite System: 30,813 subscribers
Galaxy Satellite Network: 39,675 subscribers
Nonsubscribers: 29,513
Answers will vary.
(c) Gold Satellite System: 31,947 subscribers
Galaxy Satellite Network: 42,329 subscribers
Nonsubscribers: 25,724
Answers will vary.
(d) Satellite companies are increasing the number of subscribers, while the nonsubscribers are decreasing.

7. $x = 4$

9. Answers will vary. For example: $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

11. Answers will vary.

13. $\begin{vmatrix} x & 0 & 0 & d \\ -1 & x & 0 & c \\ 0 & -1 & x & b \\ 0 & 0 & -1 & a \end{vmatrix}$

15. Transformer: $10.00
Foot of wire: $0.20
Light: $1.00

17. (a) $A^{-1} = \begin{bmatrix} 1 & -2 \\ 1 & -3 \end{bmatrix}$ (b) JOHN RETURN TO BASE

19. $|A| = 0$

Chapter 8

Section 8.1 *(page 571)*

1. infinite sequence **3.** recursively

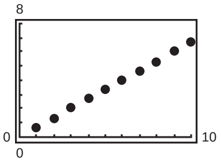
5. index; upper; lower **7.** $-3, 1, 5, 9, 13$

9. $-2, 4, -8, 16, -32$ **11.** $\frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}$ **13.** $0, 1, 0, \frac{1}{2}, 0$

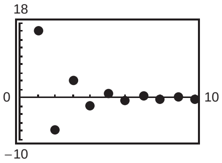
15. $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \frac{32}{243}$ **17.** $\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}$ **19.** $0, 0, 6, 24, 60$

21. $-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}$ **23.** -73 **25.** $\frac{44}{239}$

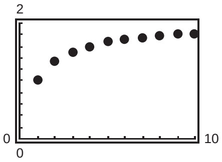
27.



29.



31.



33. c **34.** b **35.** d **36.** a **37.** $a_n = 4n - 1$

39. $a_n = \dfrac{(-1)^n(n + 1)}{n + 2}$ **41.** $a_n = \dfrac{n + 1}{2n - 1}$ **43.** $a_n = \dfrac{1}{n^2}$

45. $a_n = (-1)^{n+1}$ **47.** $a_n = \dfrac{3^{n-1}}{(n - 1)!}$

49. $28, 24, 20, 16, 12$ **51.** $1, 2, 2, 3, \frac{7}{2}$

53. $6, 8, 10, 12, 14$
$a_n = 2n + 4$

55. $81, 27, 9, 3, 1$
$a_n = \dfrac{243}{3^n}$

57. $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144$
$1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}$

59. $5, 5, \frac{5}{2}, \frac{5}{6}, \frac{5}{24}$ **61.** $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}$ **63.** $\frac{1}{30}$

65. $n + 1$ **67.** 35 **69.** 40 **71.** 88 **73.** 30

75. $\dfrac{6508}{3465}$ **77.** $\dfrac{3}{8}$ **79.** $\displaystyle\sum_{i=1}^{9} \frac{1}{3i}$ **81.** $\displaystyle\sum_{i=1}^{8} \left[2\left(\frac{i}{8}\right) + 3\right]$

83. $\displaystyle\sum_{i=1}^{6} (-1)^{i+1}3^i$ **85.** $\displaystyle\sum_{i=1}^{20} \frac{(-1)^{i+1}}{i^2}$ **87.** $\displaystyle\sum_{i=1}^{5} \frac{2^i - 1}{2^{i+1}}$

89. $\frac{75}{16}$ **91.** $-\frac{3}{2}$ **93.** $\frac{2}{3}$ **95.** $\frac{7}{9}$

97. (a) $A_1 = \$10{,}087.50$, $A_2 \approx \$10{,}175.77$, $A_3 \approx \$10{,}264.80$, $A_4 \approx \$10{,}354.62$, $A_5 \approx \$10{,}445.22$, $A_6 \approx \$10{,}536.62$, $A_7 \approx \$10{,}628.81$, $A_8 \approx \$10{,}721.82$
(b) \$14,169.09
(c) No. $A_{80} \approx \$20{,}076.31 \neq 2A_{40} \approx \$28{,}338.18$

99. True by the Properties of Sums.

101. \$500.95 **103.** Proof **105.** $x, \dfrac{x^2}{2}, \dfrac{x^3}{6}, \dfrac{x^4}{24}, \dfrac{x^5}{120}$

107. (a)

Number of blue cube faces	0	1	2	3
$3 \times 3 \times 3$	1	6	12	8

(b)

Number of blue cube faces	0	1	2	3
$4 \times 4 \times 4$	8	24	24	8
$5 \times 5 \times 5$	27	54	36	8
$6 \times 6 \times 6$	64	96	48	8

(c) The different columns change at different rates.
(d)

Number of blue cube faces	0	1	2	3
$n \times n \times n$	$(n - 2)^3$	$6(n - 2)^2$	$12(n - 2)$	8

Section 8.2 *(page 580)*

1. arithmetic; common **3.** recursion

5. Arithmetic sequence, $d = -2$

7. Not an arithmetic sequence

9. Arithmetic sequence, $d = -\frac{1}{4}$

11. Not an arithmetic sequence

13. $8, 11, 14, 17, 20$
Arithmetic sequence, $d = 3$

15. $7, 3, -1, -5, -9$
Arithmetic sequence, $d = -4$

17. $-1, 1, -1, 1, -1$
Not an arithmetic sequence

19. $-3, \frac{3}{2}, -1, \frac{3}{4}, -\frac{3}{5}$
Not an arithmetic sequence

21. $a_n = 3n - 2$ **23.** $a_n = -8n + 108$

25. $a_n = -\frac{5}{2}n + \frac{13}{2}$ **27.** $a_n = \frac{10}{3}n + \frac{5}{3}$

29. $a_n = -3n + 103$ **31.** $5, 11, 17, 23, 29$

33. $-\frac{13}{5}, -3, -\frac{17}{5}, -\frac{19}{5}, -\frac{21}{5}$ **35.** $2, 6, 10, 14, 18$

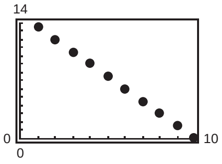
37. $-2, 2, 6, 10, 14$ **39.** $15, 19, 23, 27, 31$ **41.** $\frac{5}{8}, \frac{1}{2}, \frac{3}{8}, \frac{1}{4}, \frac{1}{8}$

43. 59 **45.** 18.6 **47.** 110 **49.** -25 **51.** 10,000

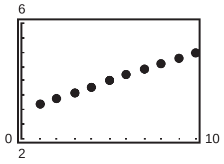
53. 620 **55.** 17.4 **57.** 4000 **59.** 1275 **61.** 355

63. 129,250 **65.** b **66.** d **67.** c **68.** a

69.



71.

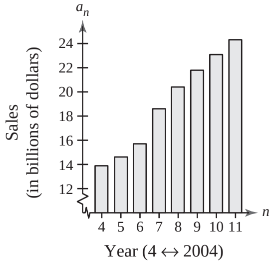


73. 0 **75.** 14,268 **77.** 2430 seats

79. (a) \$40,000 (b) \$217,500 **81.** 784 ft

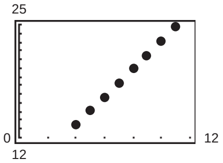
83. \$375,000; Answers will vary.

85. (a)



(b) $a_n = 7.0 + 1.61n$

(c)

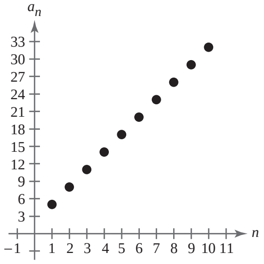


(d) $\displaystyle\sum_{n=4}^{11} a_n = \$152.4$ billion

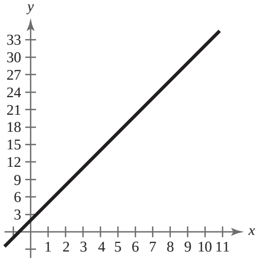
87. True. Given a_1 and a_2, $d = a_2 - a_1$ and $a_n = a_1 + (n - 1)d$.

89. $x, 3x, 5x, 7x, 9x, 11x, 13x, 15x, 17x, 19x$

91. (a)



(b)



(c) The graph of $y = 3x + 2$ contains all points on the line. The graph of $a_n = 2 + 3n$ contains only points at the positive integers.

(d) The slope of the line and the common difference of the arithmetic sequence are equal.

93. (a) 4, 9, 16, 25, 36 (b) $S_n = n^2$; $S_7 = 49 = 7^2$

(c) $\frac{n}{2}[1 + (2n - 1)] = n^2$

Section 8.3 *(page 589)*

1. geometric; common **3.** $S_n = a_1\left(\frac{1 - r^n}{1 - r}\right)$

5. Geometric sequence, $r = 5$ **7.** Geometric sequence, $r = 2$

9. Not a geometric sequence

11. Geometric sequence, $r = -\sqrt{7}$ **13.** 4, 12, 36, 108, 324

15. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$ **17.** $1, e, e^2, e^3, e^4$

19. $3, 3\sqrt{5}, 15, 15\sqrt{5}, 75$ **21.** $2, \frac{x}{2}, \frac{x^2}{8}, \frac{x^3}{32}, \frac{x^4}{128}$

23. 64, 32, 16, 8, 4; $r = \frac{1}{2}$; $a_n = 128\left(\frac{1}{2}\right)^n$

25. 9, 18, 36, 72, 144; $r = 2$; $a_n = \frac{9}{2}(2)^n$

27. $6, -9, \frac{27}{2}, -\frac{81}{4}, \frac{243}{8}$; $r = -\frac{3}{2}$; $a_n = -4\left(-\frac{3}{2}\right)^n$

29. $a_n = 4\left(\frac{1}{2}\right)^{n-1}$; $\frac{1}{128}$ **31.** $a_n = 6\left(-\frac{1}{3}\right)^{n-1}$; $-\frac{2}{59{,}049}$

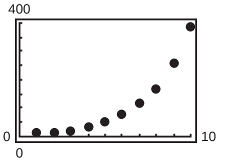
33. $a_n = 100e^{x(n-1)}$; $100e^{8x}$ **35.** $a_n = \left(\sqrt{2}\right)^{n-1}$; $32\sqrt{2}$

37. $a_n = 500(1.02)^{n-1}$; About 1082.372 **39.** $a_9 = 72{,}171$

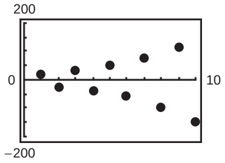
41. $a_8 = -\frac{1}{32{,}768}$ **43.** $a_3 = 9$ **45.** $a_6 = -2$

47. a **48.** c **49.** b **50.** d

51.



53.



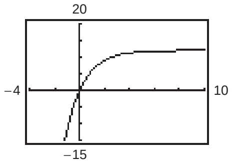
55. 5461 **57.** −14,706 **59.** 43 **61.** 29,921.311

63. 12.500 **65.** 1.600 **67.** 45.000 **69.** $\sum_{n=1}^{7} 10(3)^{n-1}$

71. $\sum_{n=1}^{5} 32\left(\frac{3}{4}\right)^{n-1}$ **73.** 2 **75.** $\frac{6}{5}$ **77.** 32

79. Undefined **81.** $\frac{4}{11}$

83.



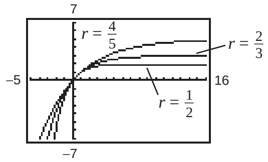
Horizontal asymptote: $y = 12$
Corresponds to the sum of the series

85. \$29,412.25 **87.** Answers will vary. **89.** \$1600

91. $273\frac{8}{9}$ in.^{2} **93.** \$5,435,989.84

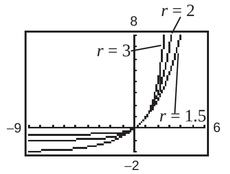
95. False. A sequence is geometric when the ratios of consecutive terms are the same.

97. (a)



As $x \to \infty$, $y \to \frac{1}{1 - r}$.

(b)



As $x \to \infty$, $y \to \infty$.

Section 8.4 *(page 600)*

1. mathematical induction **3.** arithmetic

5. $\frac{5}{(k + 1)(k + 2)}$ **7.** $\frac{(k + 1)^2(k + 4)^2}{6}$

9. $\frac{3}{(k + 3)(k + 4)}$ **11–39.** Proofs **41.** $S_n = n(2n - 1)$

43. $S_n = \frac{n}{2(n + 1)}$ **45.** 120 **47.** 91 **49.** 979

51. 70 **53.** −3402 **55.** Linear; $a_n = 8n - 3$

57. Quadratic; $a_n = 3n^2 + 3$ **59.** Quadratic; $a_n = n^2 - 3$

61. 0, 3, 6, 9, 12, 15
First differences: 3, 3, 3, 3, 3
Second differences: 0, 0, 0, 0
Linear

63. 3, 1, −2, −6, −11, −17
First differences: −2, −3, −4, −5, −6
Second differences: −1, −1, −1, −1
Quadratic

65. 2, 4, 16, 256, 65,536, 4,294,967,296
First differences: 2, 12, 240, 65,280, 4,294,901,760
Second differences: 10, 228, 65,040, 4,294,836,480
Neither

67. 2, 0, 3, 1, 4, 2
First differences: −2, 3, −2, 3, −2
Second differences: 5, −5, 5, −5
Neither

69. $a_n = n^2 - n + 3$ **71.** $a_n = \frac{1}{2}n^2 + n - 3$

73. $a_n = n^2 + 5n - 6$

75. (a) 7, 5, 6, 9, 14; *Sample answer:* $a_n = 7n + 629$
(b) $a_n = 7.7n + 623$; The models are similar.
(c) Part (a): 741,000; Part (b): 746,200
The values are similar.

77. False. P_1 must be proven to be true.

Section 8.5 *(page 607)*

1. binomial coefficients **3.** $\binom{n}{r}$; ${}_nC_r$ **5.** 10 **7.** 1

9. 15,504 **11.** 210 **13.** 4950 **15.** 6 **17.** 35

19. $x^4 + 4x^3 + 6x^2 + 4x + 1$

21. $a^4 + 24a^3 + 216a^2 + 864a + 1296$

23. $y^3 - 12y^2 + 48y - 64$

25. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

27. $8x^3 + 12x^2y + 6xy^2 + y^3$

29. $r^6 + 18r^5s + 135r^4s^2 + 540r^3s^3 + 1215r^2s^4$
$+ 1458rs^5 + 729s^6$

31. $243a^5 - 1620a^4b + 4320a^3b^2 - 5760a^2b^3$
$+ 3840ab^4 - 1024b^5$

33. $x^8 + 4x^6y^2 + 6x^4y^4 + 4x^2y^6 + y^8$

35. $\dfrac{1}{x^5} + \dfrac{5y}{x^4} + \dfrac{10y^2}{x^3} + \dfrac{10y^3}{x^2} + \dfrac{5y^4}{x} + y^5$

37. $\dfrac{16}{x^4} - \dfrac{32y}{x^3} + \dfrac{24y^2}{x^2} - \dfrac{8y^3}{x} + y^4$

39. $2x^4 - 24x^3 + 113x^2 - 246x + 207$

41. $32t^5 - 80t^4s + 80t^3s^2 - 40t^2s^3 + 10ts^4 - s^5$

43. $x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5$

45. $120x^7y^3$ **47.** $360x^3y^2$ **49.** $1{,}259{,}712x^2y^7$

51. $-4{,}330{,}260{,}000y^9x^3$ **53.** 1,732,104 **55.** 720

57. $-6{,}300{,}000$ **59.** 210 **61.** $x^{3/2} + 15x + 75x^{1/2} + 125$

63. $x^2 - 3x^{4/3}y^{1/3} + 3x^{2/3}y^{2/3} - y$

65. $81t^2 + 108t^{7/4} + 54t^{3/2} + 12t^{5/4} + t$

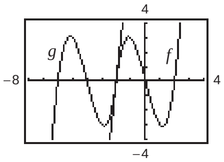
67. $3x^2 + 3xh + h^2,\ h \neq 0$

69. $6x^5 + 15x^4h + 20x^3h^2 + 15x^2h^3 + 6xh^4 + h^5,\ h \neq 0$

71. $\dfrac{1}{\sqrt{x+h} + \sqrt{x}},\ h \neq 0$ **73.** -4 **75.** $2035 + 828i$

77. 1 **79.** 1.172 **81.** 510,568.785

83.

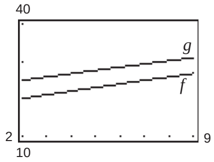


g is shifted four units to the left of f.
$g(x) = x^3 + 12x^2 + 44x + 48$

85. 0.273 **87.** 0.171 **89.** Fibonacci sequence

91. (a) $g(t) = -0.009t^2 + 0.96t + 23.025$

(b)

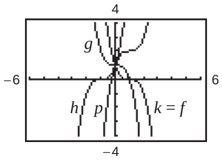


(c) 2007

93. True. The coefficients from the Binomial Theorem can be used to find the numbers in Pascal's Triangle.

95. The first and last numbers in each row are 1. Every other number in each row is formed by adding the two numbers immediately above the number.

97.



$k(x)$ is the expansion of $f(x)$.

99–101. Proofs

103.

n	r	${}_nC_r$	${}_nC_{n-r}$
9	5	126	126
7	1	7	7
12	4	495	495
6	0	1	1
10	7	120	120

${}_nC_r = {}_nC_{n-r}$
This illustrates the symmetry of Pascal's Triangle.

Section 8.6 *(page 617)*

1. Fundamental Counting Principle **3.** ${}_nP_r = \dfrac{n!}{(n-r)!}$

5. combinations **7.** 6 **9.** 5 **11.** 3 **13.** 8

15. 30 **17.** 30 **19.** 64 **21.** 175,760,000

23. (a) 900 (b) 648 (c) 180 (d) 600

25. 64,000 **27.** (a) 40,320 (b) 384 **29.** 120

31. 24 **33.** 380 **35.** 1,860,480 **37.** 970,200

39. 504 **41.** 1,816,214,400 **43.** 420 **45.** 2520

47. ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, CABD, CADB, DABC, DACB, BCAD, BDAC, CBAD, CDAB, DBAC, DCAB, BCDA, BDCA, CBDA, CDBA, DBCA, DCBA

49. 10 **51.** 4 **53.** 4845 **55.** 850,668

57. AB, AC, AD, AE, AF, BC, BD, BE, BF, CD, CE, CF, DE, DF, EF

59. 5,586,853,480 **61.** 324,632

63. (a) 7315 (b) 693 (c) 12,628

65. (a) 3744 (b) 24 **67.** 292,600 **69.** 5 **71.** 20

73. 36 **75.** $n = 2$ **77.** $n = 3$ **79.** $n = 5$ or $n = 6$

81. $n = 10$ **83.** False. It is an example of a combination.

85. ${}_{10}P_6 > {}_{10}C_6$. Changing the order of any of the six elements selected results in a different permutation but the same combination.

87–89. Proofs

91. No. For some calculators the number is too great.

Section 8.7 *(page 628)*

1. experiment; outcomes **3.** probability

5. mutually exclusive **7.** complement

9. $\{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6),$
$(T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$

11. {ABC, ACB, BAC, BCA, CAB, CBA}

13. {AB, AC, AD, AE, BC, BD, BE, CD, CE, DE}

15. $\frac{3}{8}$ **17.** $\frac{1}{2}$ **19.** $\frac{7}{8}$ **21.** $\frac{3}{13}$ **23.** $\frac{3}{26}$ **25.** $\frac{5}{36}$

27. $\frac{11}{12}$ **29.** $\frac{1}{3}$ **31.** $\frac{1}{5}$ **33.** $\frac{2}{5}$

35. (a) 1.375 million (b) $\frac{41}{100}$ (c) $\frac{3}{10}$ (d) $\frac{33}{100}$

37. (a) $\frac{13}{16}$ (b) $\frac{3}{16}$ (c) $\frac{1}{32}$ **39.** 19%

41. (a) $\frac{21}{1292}$ (b) $\frac{225}{646}$ (c) $\frac{49}{323}$ **43.** (a) $\frac{1}{120}$ (b) $\frac{1}{24}$

45. (a) $\frac{5}{13}$ (b) $\frac{1}{2}$ (c) $\frac{4}{13}$ **47.** (a) $\frac{14}{55}$ (b) $\frac{12}{55}$ (c) $\frac{54}{55}$

49. (a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{841}{1600}$ (d) $\frac{1}{40}$ **51.** 0.13

53. $\frac{3}{4}$ **55.** 0.77 **57.** $\frac{18}{35}$

59. (a) 0.9702 (b) 0.9998 (c) 0.0002

61. (a) $\frac{1}{38}$ (b) $\frac{9}{19}$ (c) $\frac{10}{19}$ (d) $\frac{1}{1444}$ (e) $\frac{729}{6859}$ **63.** $\frac{7}{16}$

65. True. Two events are independent when the occurrence of one has no effect on the occurrence of the other.

67. (a) As you consider successive people with distinct birthdays, the probabilities must decrease to take into account the birth dates already used. Because the birth dates of people are independent events, multiply the respective probabilities of distinct birthdays.

(b) $\frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365}$ (c) Answers will vary.

(d) Q_n is the probability that the birthdays are *not* distinct, which is equivalent to at least two people having the same birthday.

(e)

n	10	15	20	23	30	40	50
P_n	0.88	0.75	0.59	0.49	0.29	0.11	0.03
Q_n	0.12	0.25	0.41	0.51	0.71	0.89	0.97

(f) 23; $Q_n > 0.5$ for $n \geq 23$.

Review Exercises *(page 634)*

1. $8, 5, 4, \frac{7}{2}, \frac{16}{5}$ **3.** $72, 36, 12, 3, \frac{3}{5}$ **5.** $a_n = 2(-1)^n$

7. $a_n = \dfrac{4}{n}$ **9.** 362,880 **11.** 1 **13.** $\dfrac{205}{24}$

15. $\displaystyle\sum_{k=1}^{20} \frac{1}{2k}$ **17.** $\dfrac{4}{9}$

19. (a) $A_1 = \$10{,}018.75$
$A_2 \approx \$10{,}037.54$
$A_3 \approx \$10{,}056.36$
$A_4 \approx \$10{,}075.21$
$A_5 \approx \$10{,}094.10$
$A_6 \approx \$10{,}113.03$
$A_7 \approx \$10{,}131.99$
$A_8 \approx \$10{,}150.99$
$A_9 \approx \$10{,}170.02$
$A_{10} \approx \$10{,}189.09$
(b) \$12,520.59

21. Arithmetic sequence, $d = -7$

23. Arithmetic sequence, $d = \frac{1}{2}$ **25.** $a_n = 12n - 5$

27. $a_n = -7n + 107$ **29.** 3, 14, 25, 36, 47 **31.** 35,350

33. 80 **35.** 88 **37.** (a) \$51,600 (b) \$238,500

39. Geometric sequence, $r = 2$

41. Geometric sequence, $r = -3$

43. 2, 30, 450, 6750, 101,250

45. $9, 6, 4, \frac{8}{3}, \frac{16}{9}$ or $9, -6, 4, -\frac{8}{3}, \frac{16}{9}$

47. $a_n = 18\left(-\frac{1}{2}\right)^{n-1}; \frac{9}{512}$

49. $a_n = 100(1.05)^{n-1}$; About 155.133 **51.** 127 **53.** $\frac{15}{16}$

55. 31 **57.** 24.85 **59.** 8 **61.** 12

63. (a) $a_n = 120{,}000(0.7)^n$ (b) \$20,168.40

65–67. Proofs **69.** $S_n = n(2n + 7)$

71. $S_n = \frac{5}{2}\left[1 - \left(\frac{3}{5}\right)^n\right]$ **73.** 2850

75. 5, 10, 15, 20, 25
First differences: 5, 5, 5, 5
Second differences: 0, 0, 0
Linear

77. 15 **79.** 21 **81.** $x^4 + 16x^3 + 96x^2 + 256x + 256$

83. $41 + 840i$ **85.** 11 **87.** 10,000 **89.** 120

91. 225,792,840 **93.** $\frac{1}{9}$ **95.** (a) 43% (b) 82%

97. $\frac{1}{1296}$ **99.** $\frac{3}{4}$

101. False. $\dfrac{(n+2)!}{n!} = \dfrac{(n+2)(n+1)n!}{n!} = (n+2)(n+1)$

103. True by the Properties of Sums.

105. The set of natural numbers

107. Each term of the sequence is defined in terms of preceding terms.

Chapter Test *(page 637)*

1. $-\dfrac{1}{5}, \dfrac{1}{8}, -\dfrac{1}{11}, \dfrac{1}{14}, -\dfrac{1}{17}$ **2.** $a_n = \dfrac{n+2}{n!}$

3. 60, 73, 86; 243 **4.** $a_n = 0.8n + 1.4$

5. $a_n = 7(4)^{n-1}$ **6.** 5, 10, 20, 40, 80 **7.** 86,100

8. 477 **9.** 4 **10.** Proof

11. (a) $x^4 + 24x^3y + 216x^2y^2 + 864xy^3 + 1296y^4$
(b) $3x^5 - 30x^4 + 124x^3 - 264x^2 + 288x - 128$

12. $-22{,}680$ **13.** (a) 72 (b) 328,440

14. (a) 330 (b) 720,720 **15.** 26,000 **16.** 720

17. $\dfrac{1}{15}$ **18.** $\dfrac{1}{27{,}405}$ **19.** 10%

Cumulative Test for Chapters 6–8 *(page 638)*

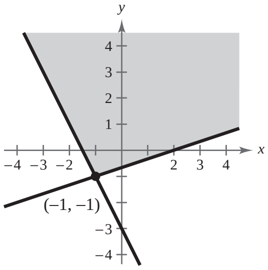
1. $(1, 2), \left(-\frac{3}{2}, \frac{3}{4}\right)$ **2.** $(-3, -1)$ **3.** $(5, -2, -2)$

4. $(1, -2, 1)$

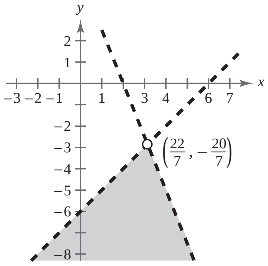
5. \$0.75 mixture: 120 lb; \$1.25 mixture: 80 lb

6. $y = \frac{1}{4}x^2 - 2x + 6$

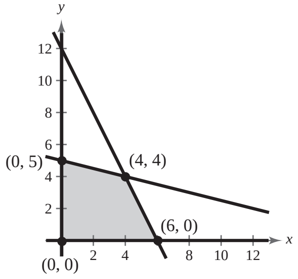
7.



8.



9.



Maximum at $(4, 4)$: $z = 20$
Minimum at $(0, 0)$: $z = 0$

10. $\begin{bmatrix} -1 & 2 & -1 & \vdots & 9 \\ 2 & -1 & 2 & \vdots & -9 \\ 3 & 3 & -4 & \vdots & 7 \end{bmatrix}$

11. $(-2, 3, -1)$ **12.** $\begin{bmatrix} 1 & 5 \\ -1 & 3 \end{bmatrix}$ **13.** $\begin{bmatrix} 16 & -40 \\ 0 & 8 \end{bmatrix}$

14. $\begin{bmatrix} 16 & -25 \\ -2 & 13 \end{bmatrix}$ **15.** $\begin{bmatrix} -6 & 15 \\ 2 & -9 \end{bmatrix}$ **16.** $\begin{bmatrix} 9 & 0 \\ -7 & 16 \end{bmatrix}$

17. $\begin{bmatrix} -15 & 35 \\ 1 & -5 \end{bmatrix}$ **18.** $\begin{bmatrix} -175 & 37 & -13 \\ 95 & -20 & 7 \\ 14 & -3 & 1 \end{bmatrix}$

19. 203

20. Gym shoes: \$2539 million
Jogging shoes: \$2362 million
Walking shoes: \$4418 million

21. $(-5, 4)$ **22.** $(-3, 4, 2)$ **23.** 9

24. $\frac{1}{5}, -\frac{1}{7}, \frac{1}{9}, -\frac{1}{11}, \frac{1}{13}$ **25.** $a_n = \frac{(n+1)!}{n+3}$

26. 1536 **27.** (a) 65.4 (b) $a_n = 3.2n + 1.4$

28. 3, 6, 12, 24, 48 **29.** $\frac{130}{9}$ **30.** Proof

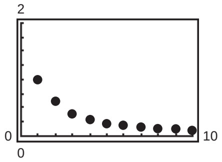
31. $w^4 - 36w^3 + 486w^2 - 2916w + 6561$ **32.** 2184

33. 600 **34.** 70 **35.** 462 **36.** 453,600

37. 151,200 **38.** 720 **39.** $\frac{1}{4}$

Problem Solving *(page 643)*

1. (a)



(b) 0

(c)

n	1	10	100	1000	10,000
a_n	1	0.1089	0.0101	0.0010	0.0001

(d) 0

3. $s_d = \frac{a_1}{1-r} = \frac{20}{1-\frac{1}{2}} = 40$

This represents the total distance Achilles ran.

$s_t = \frac{a_1}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$

This represents the total amount of time Achilles ran.

5. (a) Arithmetic sequence, difference $= d$
(b) Arithmetic sequence, difference $= dC$
(c) Not an arithmetic sequence

7. (a) 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1
(b) $a_1 = 4$: 4, 2, 1, 4, 2, 1, 4, 2, 1, 4
$a_1 = 5$: 5, 16, 8, 4, 2, 1, 4, 2, 1, 4
$a_1 = 12$: 12, 6, 3, 10, 5, 16, 8, 4, 2, 1
Eventually, the terms repeat: 4, 2, 1

9. Proof **11.** $S_n = \left(\frac{1}{2}\right)^{n-1}; A_n = \frac{\sqrt{3}}{4}S_n^2$ **13.** $\frac{1}{3}$

15. (a) 30 marbles (b) 3 to 7; 7 to 3
(c) $P(E) = \frac{\text{odds in favor of } E}{\text{odds in favor of } E + 1}$
(d) Odds in favor of event $E = \frac{P(E)}{P(E')}$

Appendix A *(page A6)*

1. numerator

3. Change all signs when distributing the minus sign.
$2x - (3y + 4) = 2x - 3y - 4$

5. Change all signs when distributing the minus sign.
$\frac{4}{16x - (2x+1)} = \frac{4}{14x - 1}$

7. z occurs twice as a factor. $(5z)(6z) = 30z^2$

9. The fraction as a whole is multiplied by a, not the numerator and denominator separately.
$a\left(\frac{x}{y}\right) = \frac{ax}{y}$

11. $\sqrt{x + 9}$ cannot be simplified.

13. Divide out common factors, not common terms.
$\frac{2x^2 + 1}{5x}$ cannot be simplified.

15. To get rid of negative exponents:
$\frac{1}{a^{-1} + b^{-1}} = \frac{1}{a^{-1} + b^{-1}} \cdot \frac{ab}{ab} = \frac{ab}{b + a}$.

17. Factor within grouping symbols before applying exponent to each factor.
$(x^2 + 5x)^{1/2} = [x(x + 5)]^{1/2} = x^{1/2}(x + 5)^{1/2}$

19. To add fractions, first find a common denominator.
$\frac{3}{x} + \frac{4}{y} = \frac{3y + 4x}{xy}$

21. To add fractions, first find a common denominator.
$\frac{x}{2y} + \frac{y}{3} = \frac{3x + 2y^2}{6y}$

23. $5x + 3$ **25.** $2x^2 + x + 15$ **27.** $\frac{1}{3}$ **29.** $3y - 10$

31. 2 **33.** $\frac{1}{2x^2}$ **35.** $\frac{36}{25}, \frac{9}{4}$ **37.** 3, 4 **39.** $1 - 5x$

41. $1 - 7x$ **43.** $3x - 1$ **45.** $7(x + 3)^{-5}$

47. $2x^5(3x + 5)^{-4}$ **49.** $\frac{4}{3}x^{-1} + 4x^{-4} - 7x(2x)^{-1/3}$

51. $\frac{x}{3} + 2 + \frac{4}{x}$ **53.** $4x^{8/3} - 7x^{5/3} + \frac{1}{x^{1/3}}$

55. $\frac{3}{x^{1/2}} - 5x^{3/2} - x^{7/2}$ **57.** $\frac{-7x^2 - 4x + 9}{(x^2 - 3)^3(x + 1)^4}$

59. $\frac{27x^2 - 24x + 2}{(6x + 1)^4}$ **61.** $\frac{-1}{(x + 3)^{2/3}(x + 2)^{7/4}}$

63. $\frac{4x - 3}{(3x - 1)^{4/3}}$ **65.** $\frac{x}{x^2 + 4}$

67. $\frac{(3x - 2)^{1/2}(15x^2 - 4x + 45)}{2(x^2 + 5)^{1/2}}$

69. (a)

x	0.50	1.0	1.5	2.0
t	1.70	1.72	1.78	1.89

x	2.5	3.0	3.5	4.0
t	2.02	2.18	2.36	2.57

(b) $x = 0.5$ mi

(c) $\frac{3x\sqrt{x^2 - 8x + 20} + (x - 4)\sqrt{x^2 + 4}}{6\sqrt{x^2 + 4}\sqrt{x^2 - 8x + 20}}$

71. You cannot move term-by-term from the denominator to the numerator.

Index

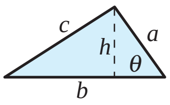
FORMULAS FROM GEOMETRY

Triangle:

$h = a \sin \theta$

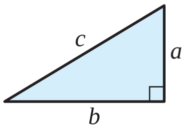
$\text{Area} = \frac{1}{2}bh$

$c^2 = a^2 + b^2 - 2ab \cos \theta$ (Law of Cosines)



Right Triangle:

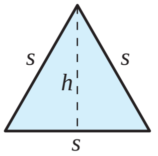
Pythagorean Theorem
$c^2 = a^2 + b^2$



Equilateral Triangle:

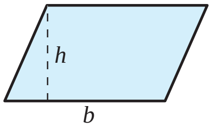
$h = \frac{\sqrt{3}s}{2}$

$\text{Area} = \frac{\sqrt{3}s^2}{4}$



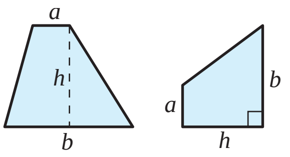
Parallelogram:

$\text{Area} = bh$



Trapezoid:

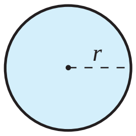
$\text{Area} = \frac{h}{2}(a + b)$



Circle:

$\text{Area} = \pi r^2$

$\text{Circumference} = 2\pi r$

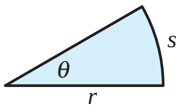


Sector of Circle:

$\text{Area} = \frac{\theta r^2}{2}$

$s = r\theta$

θ in radians



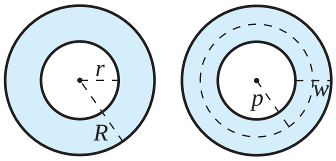
Circular Ring:

$\text{Area} = \pi(R^2 - r^2)$

$= 2\pi pw$

p = average radius,

w = width of ring



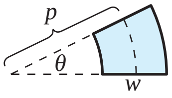
Sector of Circular Ring:

$\text{Area} = \theta pw$

p = average radius,

w = width of ring,

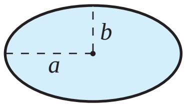
θ in radians



Ellipse:

$\text{Area} = \pi ab$

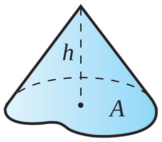
$\text{Circumference} \approx 2\pi\sqrt{\frac{a^2 + b^2}{2}}$



Cone:

$\text{Volume} = \frac{Ah}{3}$

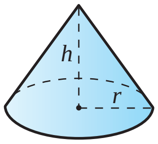
A = area of base



Right Circular Cone:

$\text{Volume} = \frac{\pi r^2 h}{3}$

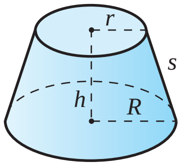
$\text{Lateral Surface Area} = \pi r\sqrt{r^2 + h^2}$



Frustum of Right Circular Cone:

$\text{Volume} = \frac{\pi(r^2 + rR + R^2)h}{3}$

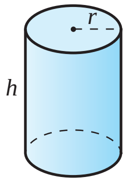
$\text{Lateral Surface Area} = \pi s(R + r)$



Right Circular Cylinder:

$\text{Volume} = \pi r^2 h$

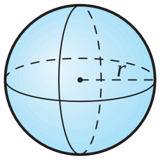
$\text{Lateral Surface Area} = 2\pi rh$



Sphere:

$\text{Volume} = \frac{4}{3}\pi r^3$

$\text{Surface Area} = 4\pi r^2$

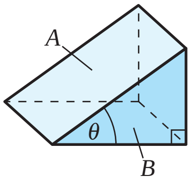


Wedge:

$A = B \sec \theta$

A = area of upper face,

B = area of base



ALGEBRA

Factors and Zeros of Polynomials:

Given the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_1 x + a_0$. If $p(b) = 0$, then b is a *zero* of the polynomial and a *solution* of the equation $p(x) = 0$. Furthermore, $(x - b)$ is a *factor* of the polynomial.

Fundamental Theorem of Algebra:

An nth degree polynomial has n (not necessarily distinct) zeros.

Quadratic Formula:

If $p(x) = ax^2 + bx + c$, $a \neq 0$ and $b^2 - 4ac \geq 0$, then the real zeros of p are $x = \left(-b \pm \sqrt{b^2 - 4ac}\right)/2a$.

Example

If $p(x) = x^2 + 3x - 1$, then $p(x) = 0$ if

$$x = \frac{-3 \pm \sqrt{13}}{2}.$$

Special Factors:

$x^2 - a^2 = (x - a)(x + a)$

$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$

$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$

$x^4 - a^4 = (x - a)(x + a)(x^2 + a^2)$

$x^4 + a^4 = \left(x^2 + \sqrt{2}ax + a^2\right)\left(x^2 - \sqrt{2}ax + a^2\right)$

$x^n - a^n = (x - a)(x^{n-1} + ax^{n-2} + \cdot \cdot \cdot + a^{n-1})$, for n odd

$x^n + a^n = (x + a)(x^{n-1} - ax^{n-2} + \cdot \cdot \cdot + a^{n-1})$, for n odd

$x^{2n} - a^{2n} = (x^n - a^n)(x^n + a^n)$

Examples

$x^2 - 9 = (x - 3)(x + 3)$

$x^3 - 8 = (x - 2)(x^2 + 2x + 4)$

$x^3 + 4 = \left(x + \sqrt[3]{4}\right)\left(x^2 - \sqrt[3]{4}x + \sqrt[3]{16}\right)$

$x^4 - 4 = \left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)(x^2 + 2)$

$x^4 + 4 = (x^2 + 2x + 2)(x^2 - 2x + 2)$

$x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1)$

$x^7 + 1 = (x + 1)(x^6 - x^5 + x^4 - x^3 + x^2 - x + 1)$

$x^6 - 1 = (x^3 - 1)(x^3 + 1)$

Binomial Theorem:

$(x + a)^2 = x^2 + 2ax + a^2$

$(x - a)^2 = x^2 - 2ax + a^2$

$(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$

$(x - a)^3 = x^3 - 3ax^2 + 3a^2x - a^3$

$(x + a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4$

$(x - a)^4 = x^4 - 4ax^3 + 6a^2x^2 - 4a^3x + a^4$

$$(x + a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{2!}a^2x^{n-2} + \cdot \cdot \cdot + na^{n-1}x + a^n$$

$$(x - a)^n = x^n - nax^{n-1} + \frac{n(n-1)}{2!}a^2x^{n-2} - \cdot \cdot \cdot \pm na^{n-1}x \mp a^n$$

Examples

$(x + 3)^2 = x^2 + 6x + 9$

$(x^2 - 5)^2 = x^4 - 10x^2 + 25$

$(x + 2)^3 = x^3 + 6x^2 + 12x + 8$

$(x - 1)^3 = x^3 - 3x^2 + 3x - 1$

$\left(x + \sqrt{2}\right)^4 = x^4 + 4\sqrt{2}x^3 + 12x^2 + 8\sqrt{2}x + 4$

$(x - 4)^4 = x^4 - 16x^3 + 96x^2 - 256x + 256$

$(x + 1)^5 = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$

$(x - 1)^6 = x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$

Rational Zero Test:

If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_1 x + a_0$ has integer coefficients, then every *rational* zero of $p(x) = 0$ is of the form $x = r/s$, where r is a factor of a_0 and s is a factor of a_n.

Example

If $p(x) = 2x^4 - 7x^3 + 5x^2 - 7x + 3$, then the only possible *rational* zeros are $x = \pm 1, \pm\frac{1}{2}, \pm 3$, and $\pm\frac{3}{2}$. By testing, you find the two rational zeros to be $\frac{1}{2}$ and 3.

Factoring by Grouping:

$$\begin{aligned} acx^3 + adx^2 + bcx + bd &= ax^2(cx + d) + b(cx + d) \\ &= (ax^2 + b)(cx + d) \end{aligned}$$

Example

$$\begin{aligned} 3x^3 - 2x^2 - 6x + 4 &= x^2(3x - 2) - 2(3x - 2) \\ &= (x^2 - 2)(3x - 2) \end{aligned}$$

Arithmetic Operations:

$$ab + ac = a(b + c) \qquad \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \frac{a + b}{c} = \frac{a}{c} + \frac{b}{c} \qquad \frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \frac{ad}{bc}$$

$$a\left(\frac{b}{c}\right) = \frac{ab}{c} \qquad \frac{a - b}{c - d} = \frac{b - a}{d - c} \qquad \frac{ab + ac}{a} = b + c,\ a \neq 0$$

$$\frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc} \qquad \frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b}$$

Exponents and Radicals:

$$a^0 = 1,\ a \neq 0 \qquad \frac{a^x}{a^y} = a^{x-y} \qquad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \qquad \sqrt[n]{a^m} = a^{m/n} = \left(\sqrt[n]{a}\right)^m$$

$$a^{-x} = \frac{1}{a^x} \qquad (a^x)^y = a^{xy} \qquad \sqrt{a} = a^{1/2} \qquad \sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$$

$$a^x a^y = a^{x+y} \qquad (ab)^x = a^x b^x \qquad \sqrt[n]{a} = a^{1/n} \qquad \sqrt[n]{\left(\frac{a}{b}\right)} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Algebraic Errors to Avoid:

$\frac{a}{x + b} \neq \frac{a}{x} + \frac{a}{b}$	(To see this error, let $a = b = x = 1$.)
$\sqrt{x^2 + a^2} \neq x + a$	(To see this error, let $x = 3$ and $a = 4$.)
$a - b(x - 1) \neq a - bx - b$	[Remember to distribute negative signs. The equation should be $a - b(x - 1) = a - bx + b$.]
$\frac{\left(\frac{x}{a}\right)}{b} \neq \frac{bx}{a}$	[To divide fractions, invert and multiply. The equation should be $\frac{\left(\frac{x}{a}\right)}{b} = \frac{\left(\frac{x}{a}\right)}{\left(\frac{b}{1}\right)} = \left(\frac{x}{a}\right)\left(\frac{1}{b}\right) = \frac{x}{ab}$.]
$\sqrt{-x^2 + a^2} \neq -\sqrt{x^2 - a^2}$	(The negative sign cannot be factored out of the square root.)
$\frac{\cancel{a} + bx}{\cancel{a}} \neq 1 + bx$	(This is one of many examples of incorrect dividing out. The equation should be $\frac{a + bx}{a} = \frac{a}{a} + \frac{bx}{a} = 1 + \frac{bx}{a}$.)
$\frac{1}{x^{1/2} - x^{1/3}} \neq x^{-1/2} - x^{-1/3}$	(This error is a more complex version of the first error.)
$(x^2)^3 \neq x^5$	[This equation should be $(x^2)^3 = x^2x^2x^2 = x^6$.]

Conversion Table:

1 centimeter ≈ 0.394 inch
1 meter ≈ 39.370 inches
≈ 3.281 feet
1 kilometer ≈ 0.621 mile
1 liter ≈ 0.264 gallon
1 newton ≈ 0.225 pound

1 joule ≈ 0.738 foot-pound
1 gram ≈ 0.035 ounce
1 kilogram ≈ 2.205 pounds
1 inch = 2.540 centimeters
1 foot = 30.480 centimeters
≈ 0.305 meter

1 mile ≈ 1.609 kilometers
1 gallon ≈ 3.785 liters
1 pound ≈ 4.448 newtons
1 foot-lb ≈ 1.356 joules
1 ounce ≈ 28.350 grams
1 pound ≈ 0.454 kilogram