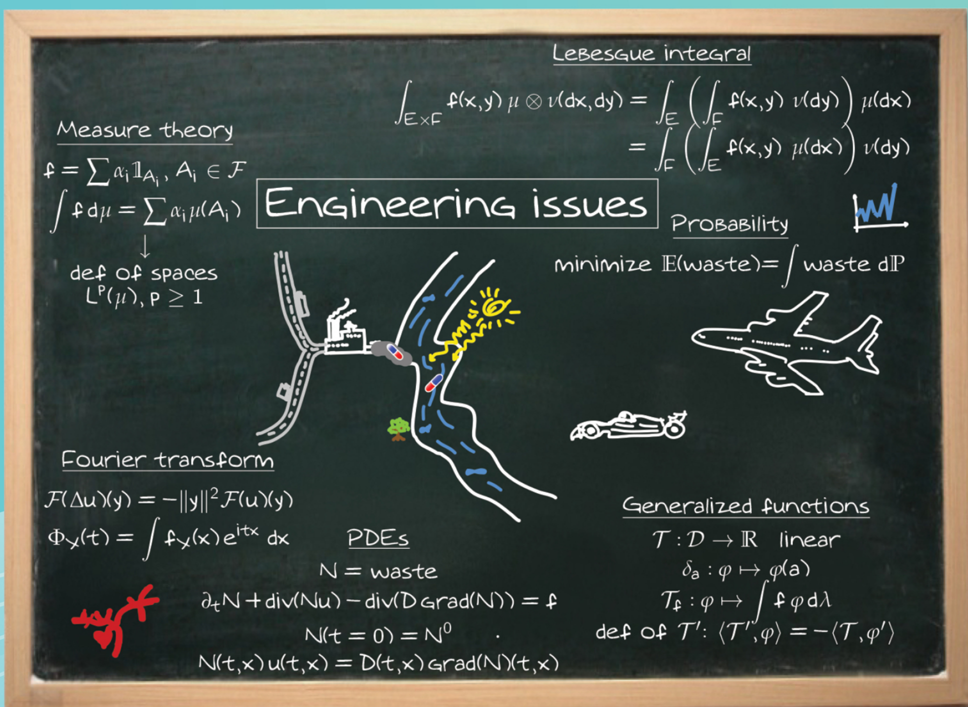
# MODERN MATHEMATICAL CONCEPTS FOR ENGINEERS

## Part I

## From Infinitesimal Calculus to Measure Theory





Erick Herbin • Pauline Lafitte



World Scientific

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**Erick Herbin • Pauline Lafitte**

*CentraleSupélec, France and Université Paris-Saclay, France*

**World Scientific**

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# About the Authors

**Erick Herbin**, a former engineer who was specializing in random modeling and fundamental research on stochastic processes at Dassault Aviation from 1995 to 2011, is now a full professor in mathematics at CentraleSupélec, where he conducts research on the local and geometric properties of random objects. He headed the mathematics department at CentraleSupélec, a leading French engineering school, from 2011 to 2025. He designed and taught the "Probabilités" course for 500 students from 2006 to 2017. He then designed and has been teaching the extensive "Convergence, integration, probabilities" course since 2018. He was heavily involved in the creation of CentraleSupélec's current curriculum, which trains 1,000 students a year over a three-year period. Erick Herbin is the author of several articles in international peer-reviewed journals in mathematics and is a member of the Laboratoire de Mathématiques d'Orsay at the Université Paris-Saclay.

**Pauline Lafitte**, a full professor at CentraleSupélec, specializes in mathematical modeling with partial differential equations. Her research focuses on performing the complete chain of modeling, from observation to validation using experimental data, through the analysis of mathematical models, the design of numerical schemes, their analysis, and their simulation. These models aim to reproduce phenomena appearing in various fields, including fluid mechanics, chemistry, and biology. She designed the "Partial Differential Equations (Theory and Numerics)" course at CentraleSupélec in 2012 and has been teaching it since.

Since 2016, she has been in charge of "Option Mathématiques Appliquées", which evolved into the "Dominante Mathématiques et Data Science", a key part of the mathematical studies in the third year of the CentraleSupélec curriculum.

Pauline Lafitte has authored several papers in international peer-reviewed journals in mathematics and is a member of the Laboratoire de Mathématiques d'Orsay at the Université Paris-Saclay.

# Acknowledgments

We acknowledge the privilege of having been entrusted with the responsibility for the courses Analysis, Probability, PDEs and, later, Convergence, Integration, Probability (CIP) at Ecole Centrale Paris and then at CentraleSupélec for over a decade. The extensive experience in this context affirmed our vision of teaching fundamental mathematics to engineers within a curriculum that is not exclusively mathematical. The rich interactions with over 500 students each year sharpened our approach to presenting each concept and the interplay between these concepts, that is, how to demonstrate and develop the edifice of mathematics. Among the individuals who entrusted us with these missions, we thank John Cagnol and Lionel Gabet, who successively held the position of dean of studies. With the conclusion of our responsibilities, the idea of writing a book emerged, and we aim to deliver this heritage to as many engineering and math students, young engineers, and even math instructors at these levels.

We thank the inventor of the style we adopted, Plato, and Jostein Gaarder, who more recently used the same style in his bestseller *Sophie's World*, for inspiring us to elucidate and demystify concepts through dialogue between imaginary characters.

Specifically, we would like to thank Brice Hannebicque, John Cagnol and Philippe Bouafia for their reading and remarks about the first chapters, as well as Mélanie, Stéphane and their staff – watchful and benevolent guardians of the place where we wrote most of this book. Finally, we would like to thank and recommend www.mathcha.io for the design of tikz figures.

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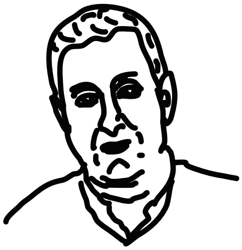
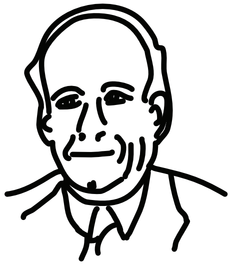
# The Characters



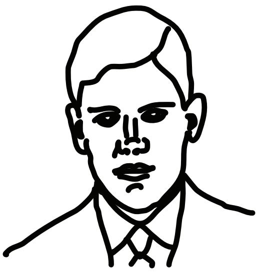
Pierre Namier

*Laurent's friend, Bernard's father*



Laurent Corps

*retired mathematician*



Bernard Namier

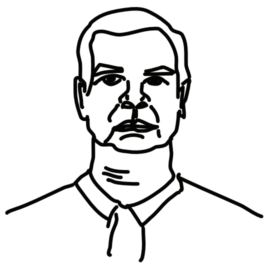
*young engineer*

Ann Winglett

*Bernard's boss*
*at SofterWorld Consulting*

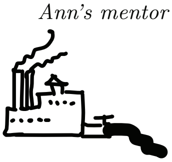
Manolis Papadiamantis

*Ann's mentor*



Harry Bannan

*emeritus medicine professor*



A pharmaceutical plant

$\mathcal{M}$

*Mathematics...*

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# The First Encounter

*Spring, 2023 – somewhere around Zermatt in Switzerland*

Two alpinists are traveling down from the Dufourspitze, the highest peak of Monte Rosa.

Laurent is enjoying his first year of retirement by spending as much time as possible in the high mountains. For his entire active life, this passion remained sidelined by mathematics, his other passion. Especially since his wife passed shortly after their wedding, and they had no children, he has given himself no chance at a family life. Due to his grief, he has allowed his two passions to consume him, and, as he has lived in Paris for a long time, math has taken precedence over his more distant passion.

They feel exhilarated at having climbed one of the highest summits of the Alps. Approaching the alpine hut, having left the dangers of the glacier behind, Pierre feels confident enough to open up to his old mathematician friend. These last few weeks, Pierre has been very preoccupied. His last son, Bernard, was hired a month ago as an engineer. Pierre is afraid that his son is not up to the task. When surrounded by the dangers of the high mountains, his worries had vanished. But, as he approaches solid ground, his son's professional situation is creeping up again, disturbing his euphoria.

Laurent and Pierre met long before they started mountaineering together. One coming from Annecy, the other from Grenoble – two big cities in the French Alps – they became very close friends during *mathématiques supérieures et spéciales* at a highly prestigious French institution in Paris. Laurent entered the *École normale supérieure de la rue d'Ulm*, initiating his brilliant career as a mathematician. Pierre entered the *École polytechnique* and has grown increasingly distant from math as he progressed rapidly in the industry. Even though their career choices could have separated them, they have remained close thanks to a continuing dedication to mountaineering – through shared weekends or longer periods – for over 40 years.

Walking downward from Monte Rosa, Pierre tells Laurent that his son has put himself at risk professionally:

— Bernard has been asked to study via numerical simulations the feasibility of building a pharmaceutical plant, taking into account current environmental contingencies. I am afraid he lied about his skills to get hired.
— You know, Pierre, I have been giving lectures to 25-year-old students for a long time. They brag that they find it hard to focus for more than seven minutes... I am not at all surprised.
— If you allow me, it is not only about young people. I stopped watching television because of stupid things that people wrongly infer from math and science. Remember what this "Prof." Harry Bannan said about the *inflection point of the exponential function (sic)* during the pandemic...

Thinking about his dear friend's rather harsh answer, Pierre reflects on his personal responsibility for his son's education and situation. Bernard was born when Pierre was 40, from his second wife. Bernard has qualities, real ones, but he has always been spoiled and doesn't realize that the more effort it takes to get things, the more valuable they are. Surely, he could have made something better of himself than entering into an engineering program whose selection is based on a math quiz.

Based on his own education and his experience in the industry, Pierre knows that mastering applied math is essential to solving Bernard's problem. He honestly thinks Bernard's curriculum is not serious enough to tackle it. Perhaps Laurent might be able to help him? He ventures...

— I'm not sure I'm the right person, you know. I am not particularly fond of this generation, says Laurent. As I have not had children, I do not really understand them. They speak too fast, too loud, they always zap from one thing to another, and they get bored so easily... It is so far from the life I chose for myself, where I can focus for days, weeks, or months even on a single math problem. And I am old and tired.

— You've known Bernard forever; you know he isn't a bad guy. True enough, he cannot focus for long. But he has the spirit of youth, and he wants to accomplish things. He wants to succeed in life. I honestly think you could help him acquire the necessary fundamentals. You would do me an immense favor by trying, at least.

*One week later – Café Le Rostand in the Quartier Latin in Paris*

Bernard is sitting inside a café. He is early for his 3:00 pm appointment. His father has told him that he absolutely had to be on time for his meeting with this exacting old man.

Laurent had agreed to meet his old friend's son (for a discussion) to assess his ability to be of help to him. Laurent would then decide whether he was ready to work to help Bernard with his pharmaceutical plant problem.

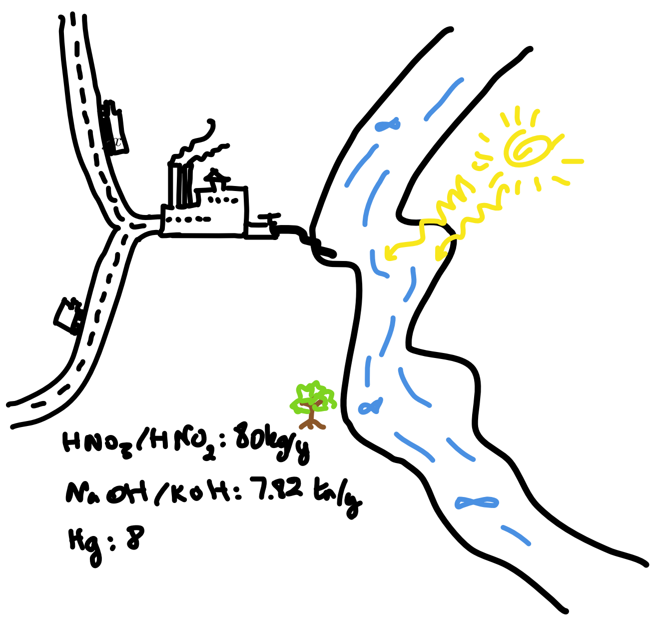
Bernard had just completed his curriculum at *CeePlus*, an engineering school that was sadly not among the best, said his *polytechnicien* father. To the latter's surprise, Bernard had managed to get a job at SofterWorld Consulting, one of the most demanding international companies.

Together with all the other students of his school, he thought that math was merely a selection tool and was absolutely useless in real life. What a (capital) mistake! He was just starting to realize...

Upon seeing Bernard, Laurent wants to understand precisely the situation in which Bernard finds himself. Bernard tells him that he had glorified himself throughout the interviews as being a *French engineer*, whose curriculum is more theoretical than the vast majority of curricula. In particular, he had to explain that *French-trained engineers mastered math better than their counterparts from other countries*. The department in which he has been posted is headed by Ann Winglett, a brilliant individual who obtained her PhD in applied math at MIT, following an MSc in pure mathematics at Orsay in France. She intends to use Bernard's trial period to check whether his true skills match what he has told her. In particular, she wants to confront him with the engineers of the company who were not trained in France.

— She told me that I would not last long at SofterWorld Consulting if I were not able to say smart things about the environmental study concerning the construction of a plant. It is commissioned by an international pharmaceutical company.
— Indeed, this sounds like a true engineering issue, Bernard. You must have been trained to answer questions of this kind.
— I have to admit that I may have oversold myself with respect to the other engineers, so I can't expect any help inside the company. I said that, as a French guy, I knew everything about modeling and using math in the industry.
— Anyway, what is right is that mathematics is the basis of modeling. We need to assess the true extent of your knowledge, as far as the precise problem you were given is concerned.

Laurent is warming up to the idea that he might be able to help this young engineer discover the mathematics that underlie the modeling of the impact that this plant would have on the environment. He tries to determine more precisely the problem at hand. Bernard draws a picture on his iPad and starts writing precise figures of the required performances for the plant; he also states the standards of release for substances into the river.



Laurent laughs at the figures...

— You need to formalize mathematically the discharge in the river. You need to describe the relations that exist between the physical quantities. I see that you are going to need the basics of functional analysis and probability to be able to write down the equations that govern the problem.
— Ah! My father has told me that I need to strengthen my mathematical knowledge and understand new stuff if I want to be relevant regarding the modeling of the problem.
— Certainly. You will even get further than constructing a plant. You will understand that formalizing this problem is generic in nature. A lot of other problems are formalized in the same way.

Laurent gets the notebook he always carries around with him and proceeds to mathematically formalize the problem of discharging particles into a river $\mathcal{U}$. The velocity of the water being $\mathbf{u}(t, \mathbf{x})$, the density of particles $N(t, \mathbf{x})$ satisfies the following: for $t > 0$ and $\mathbf{x} \in \mathcal{U}$,

$$\partial_t N(t, \mathbf{x}) + \operatorname{div}(N\mathbf{u})(t, \mathbf{x}) - \operatorname{div}(D\,\mathbf{grad}(N))(t, \mathbf{x}) = f(t, \mathbf{x}).$$

The existence and uniqueness of a solution to this equation is related to the boundary condition. Here, it is natural to assume that there is no flux at the boundary. This translates into the following: for $t > 0$ and $\mathbf{x} \in \partial\mathcal{U}$, the boundary of $\mathcal{U}$,

$$N(t, \mathbf{x})\,\mathbf{u}(t, \mathbf{x}) = D(t, \mathbf{x})\,\mathbf{grad}(N)(t, \mathbf{x}).$$

— The description of this problem does not only apply to discharging particles. If one wants to describe the evolution of the temperature in a river into which a plant discharges hot water, the mathematical description is exactly the same, replacing $N(t, \mathbf{x})$ with the temperature $\Theta(t, \mathbf{x})$.

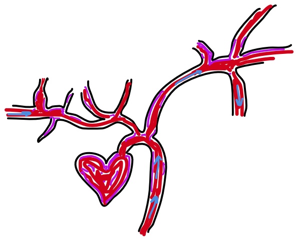
— Excuse me, sir. I have to admit that I don't understand the equations you just wrote down.
— Ah, maybe the operators are disturbing to you. I will write these equations in a one-dimensional setting in space: for $t > 0$ and $x \in [0, L]$ for $L > 0$,

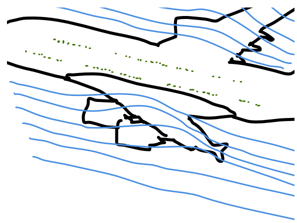
$$\frac{\partial N}{\partial t}(t, x) + \frac{\partial (Nu)}{\partial x}(t, x) - \frac{\partial}{\partial x}\left(D\,\frac{\partial N}{\partial x}\right)(t, x) = f(t, x), \qquad (*)$$

and

$$N(t,0)\,u(t,0) = D(t,0)\,\frac{\partial N}{\partial x}(t,0) \quad \text{and}$$

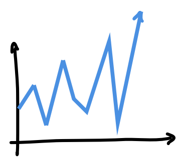
$$N(t,L)\,u(t,L) = D(t,L)\,\frac{\partial N}{\partial x}(t,L).$$





To emphasize the generic character of the mathematical formalization, to study e.g. a malfunctioning organ leaking sick cells into the blood flow, you use the *Navier–Stokes equation*, as is used to describe the airflow around a plane wing. One last example, which looks like it comes from another world, the Black–Scholes equation, usually used in finance, describes the price $V$ of an option as a function of an underlying asset $S$:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$



Up to a transformation of variables, you can see how close this equation is to $(*)$.

In the end, industrial problems that look very different, when transposed into the world of mathematics, can lead to very similar problems. And these problems are treated using very general mathematical methods.

After listening to such a speech, Bernard knows for sure that he cannot solve his problem on his own. The old researcher finally consents to help him on the sole condition that Bernard accept Laurent's rules and working methods. They agree to meet again the following week at the Institut Henri Poincaré, the famous IHP close to the *École normale supérieure* in Paris.

*Working at the IHP (Paris, Quartier Latin)*

To give a good impression to his future mentor, Bernard arrived very early, contrary to his habits, to visit the library. When Laurent arrives, he sees Bernard sitting at a table, with books he has picked somewhat randomly, highly oriented toward engineering: a book of probability and statistics and one of numerical simulation for mechanics.

— Did you know, Bernard, that the art of the engineer consists in designing, starting from models that approximate reality? And that the prediction of these models is as relevant as the engineer masters the approximation that is made.
— Do you mean I picked the wrong books?

The young engineer feels caught in the act and is very uncomfortable. He had thought that his curriculum would allow him, at least, to pick the appropriate books to tackle his problem. He wanted to impress Laurent, but here he was, feeling a bit humiliated.

The retired professor corrects the naive visions of the young graduate. Approximating models requires probability and topology. To consider ordinary and partial differential equations (ODEs and PDEs), the functional spaces arising from measure theory are fundamental. ODEs and PDEs cannot be solved explicitly in general, contrary to what is done in *classes préparatoires*.[1] They are approximated via discretizations of the continuous spaces. And these discretized problems are the ones that are studied. But engineers know that ODEs and PDEs are not the reality and that they merely approximate reality. A lot of unknowns appear and are usually described in a stochastic framework.

Laurent gets up and disappears behind the shelves. After half an hour, he shows up with the following books[2]:

[1]In France, students enter most engineering schools by taking competitive entrance exams, after having spent two years studying in so-called *classes préparatoires*, dedicated to math and physics.
[2]Dear Reader, do not look for these books in your library, nor on the Internet: they are fictional!

| | |
|---|---|
| $\int f\, d\mu$ 515.BRO.m | *Measure Theory* (Henri Brolle) |
| $\mathbf{E}[\varphi(X)]$ 519.FRA.f | *Foundations of Probability* (André François) |
| $\partial_x f$ 515.LOU.l | *Linear PDEs: Theory and Numerics* (Jean-Jacques Louis) |
| $x'$ 515.ZOI.d | *Dynamical Systems* (Jean-Christophe Zoïde) |
| $\{f\}$ 515.COR.f | *Functional Analysis* (Laurent Corps) |
|  514.COR.t | *Topology and Real Analysis* (Jacqueline Cordonnier) |

Reading their titles, Bernard is very dubious; how can these theoretical books be of use in his plant-building problem?

— At first, stop using the verb "use". Balance "What's the use?" with the fact that we sometimes do things simply "pour l'honneur de l'esprit humain" (for the honor of the human mind), as Carl Jacobi said. And stop annoying me. It is incredible to witness how you think you know everything while at the same time recognizing that you know nothing at all. If you want to solve your problem in a relevant way, your mathematical knowledge is far too limited. You will find that abstraction provides a devilishly efficient way to solve your engineering problems. You want my help; here are my rules:

- you shall not talk of our sessions to anyone;
- we shall meet once a week here at IHP;
- you shall be punctual, otherwise I leave at once;
- during each session, we shall study a theme through the chapters of these books;
- you shall take notes in a dedicated notebook, as if you are following a lecture;
- in between two sessions, you should be able to go forward and show your boss that you are working hard on the problem;
- prepare for very hard work: "I have nothing to offer you but blood, toil, tears and sweat", as someone said....

Bernard feels panic overcoming him, but he is determined not to let it show. His hands are slightly shaking, though...

# Contents

## Chapter 1

# Topology and Vector Spaces

*May 2nd at 7:45 am in front of the Centre Emile Borel in Paris.*

*The young engineer is waiting for the building to open. He made sure to arrive well in advance for his meeting, following the rules.*

*Laurent arrives on time as the building opens. The two men go up to the second floor, where several blackboards hung in the hall. Bernard puts six books on the table in front of one of them.*

*During their very first working session, they intend to immerse themselves in the issue of approximating models, which can be described as functions, in the fundamental notion of topology on functional spaces.*

**Bernard:** Let me open the book of *Measure Theory* by Henri Brolle at the starting point of the construction. I have the page... $\sigma$-algebras, you say...

**Laurent:** Bernard, what does the word "convergence" mean to you?

**Bernard:** In my *classes préparatoires* in France, when I studied advanced mathematics during two intensive years, I saw a technical definition with $\varepsilon$ and the symbols $\exists$ and $\forall$. It was a bit hard for us at first glance, but we got used to it.

**Laurent:** Advanced, you say? I am not so sure... We will see about that later on. Anyway. Could you try to give a definition, to be more precise? To start with, what sort of objects are involved?

**Bernard:** In my memory, it was a sequence or a function and their limits. But... I'm sorry, Sir. My problem is an engineer's problem, a real problem! Why are you talking about convergence? To me, it is merely a mathematical notion I needed to pass an entrance examination. Besides, I almost never heard about it afterward.

**Laurent:** Do I need to remind you that you came to me, Bernard? You do not know how to solve your engineering problem, and that is precisely the reason why! So now, listen to me and recall my rules. The convergence is the precise notion that governs the approximation. And the approximation is at the heart of any engineering issue.

**Bernard:** OK, don't get mad at me. You are right, Sir. I don't know how to tackle my problem. And I admit that I don't know enough.

**Laurent:** I have ideas to solve your problem. But I think it is best that you learn the theoretical foundations required to describe and solve your problem.

**Bernard:** I think I know what happens when the sequence is real: "a sequence $(u_n)$ converges to 0" means that you can make $u_n$ as small as you want with $n$ large enough.

**Laurent:** The idea for real sequences is almost correct, provided that you precisely define the meaning of "small". You need to be much more precise mathematically to treat your engineering problem. You know perfectly well that $(N(t, \mathbf{x}))_{t \geq 0, \mathbf{x} \in \mathcal{U}}$ is not a simple real number!

To start with, you need to realize that the question of approximation, which constantly arises in engineering problems, lies within the domain of convergence and, more generally, *topology*.

Do not be afraid of this word! Going back to $\mathbf{R}$, the starting questions can be expressed in the following way:

(i) What meaning can be given to $u_n \to \ell$ as $n \to \infty$ if $(u_n)_{n \in \mathbf{N}}$ is a sequence of real numbers and $\ell \in \mathbf{R}$?
(ii) What meaning can be given to $f(x) \to \ell$ as $x \to a$ if $f$ is a real-valued function on $\mathbf{R}$, $a \in \mathbf{R}$ and $\ell \in \mathbf{R}$?

**Bernard:** Wait a minute, Sir! If I understand you, my problem is not in $\mathbf{R}$, rather we should look for a function $N$ from $E = \mathbf{R}_+ \times \mathcal{U}$ to $F = \mathbf{R}$...

So, I imagine it is necessary to equip the sets $E$ and $F$ with a notion of distance in order to be able to consider the convergence of a sequence $(u_n)_{n\in\mathbf{N}}$ with values in $F$ or the continuity of a function $f\colon E \to F$?

**Laurent:** At this point, you can call me Laurent. Actually, the answer to this last question is, no. In order to be able to study the convergence of a sequence or a function, metrics are not required. What is really needed are *open sets*. Let us move slowly in this direction: we start with metric spaces, and we will see how we abstract the concepts.

**Bernard:** I get it, Sir... uhm... Laurent. Let me open my *Topology and Real Analysis* by Jacqueline Cordonnier. And I am ready to fill my brand-new notebook that I bought specially. I thought about what you said last time, and I understand that I need to write down math to understand them.

## 1.1 Metric Spaces


In the common sense (and etymologically), *topology* means the study of the location of all the points in relation to each other in the ambient space.

Besides, several technical concepts of topology originate from this meaning: neighborhood, accumulation, density, compact and connected sets, etc.

In this respect, the notion of metric can be seen as unavoidable when starting to study topology. It is not rigorously true, but it is for sure a good starting point.

**Definition 1.1.** A *metric* (or *distance function*) on a set $E$ is a function $d\colon E \times E \to \mathbf{R}_+$ such that:

(d1) for all $x, y \in E$, we have $d(x, y) = 0$ iff $x = y$ (the identity of indiscernibles);
(d2) for all $x, y \in E$, we have $d(x, y) = d(y, x)$ (symmetry);
(d3) for all $x, y, z \in E$, we have $d(x, y) \le d(x, z) + d(z, y)$ (triangle inequality).

A *metric space* is an ordered pair $(E, d)$, where $E$ is a set and $d$ a metric on $E$.

A direct application of the triangle inequality shows that for all $x, y, z$ in $E$, $|d(x,z) - d(z,y)| \leq d(x,y)$, which can be called the *left-hand side of the triangle inequality.*

**Example.** The function $d \colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}_+$, defined by $d(x,y) = |x - y|$ for $x, y \in \mathbf{R}$, is a metric on $\mathbf{R}$. The pair $(\mathbf{R}, d)$ is a metric space.

**Bernard:** Of course! To prove the left-hand side of the triangle inequality, I remember that it suffices to substitute twice $(x, y, z)$, once with $(x, z, y)$ and once with $(z, y, x)$. But what is the relation to topology?

### 1.1.1 *Open sets and neighborhoods*

**Laurent:** The starting point of topology is the concept of *open sets* and, subsequently, of *neighborhood*.


The basic element of the *topology of metric spaces* is the notion of an *open ball.*

Let $(E, d)$ be a metric space. We denote by $B_d(a, r) = \{x \in E : d(a, x) < r\}$ the open ball of center $a \in E$ and radius $r > 0$. When there is no ambiguity, we omit the subscript $_d$ and write $B(a, r)$.

**Definition 1.2.** A subset $A$ of $E$ is called *open* if one of the following two equivalent conditions is satisfied:

(i) $A$ is the union of open balls.
(ii) For all $a \in A$, there exists $r > 0$ such that $B(a, r) \subset A$.

**Justification.** We prove the equivalence of Conditions (i) and (ii).

(i)$\Rightarrow$(ii) Consider $A = \cup_{i \in \mathcal{I}} B(a_i, r_i)$, where $\mathcal{I}$ is any set, and let $a \in A$.

There exists $j \in \mathcal{I}$ such that $a \in B(a_j, r_j)$, and by the definition of $A$, $B(a_j, r_j) \subset A$.

The number $r = r_j - d(a, a_j)$ is positive and the triangle inequality implies $B(a, r) \subset B(a_j, r_j) \subset A$, which proves (ii).

(ii) $\Rightarrow$(i) Assume that for all $a \in A$, there exists $r_a > 0$ such that $B(a, r_a) \subset A$.

We have $\cup_{a \in A} B(a, r_a) \subset A$.

Since the converse inclusion $A \subset \cup_{a \in A} B(a, r_a)$ is obvious, we can conclude that $A = \cup_{a \in A} B(a, r_a)$, which proves (i). □

**Definition 1.3.** A subset $B$ of $E$ is called *closed* if its complement $E \setminus B$ is open.

We denote by $\overline{B_d}(a, r) = \{x \in E : d(a, x) \leq r\}$ the *closed ball* of center $a \in E$ and radius $r \geq 0$.

**Example.** ⋆ According to Definition 1.2, any open ball and any union of open balls is an open set.
⋆ Any closed ball is a closed set.

**Bernard:** The second point is not so clear to me...

**Laurent:** Excellent technical exercise, Bernard. Let me see you try. Do not be afraid. Take a point $x$ outside $\overline{B}(a, r)$.

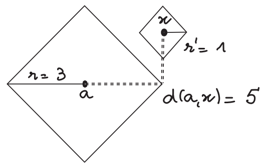
**Bernard:** All that I can say is that $d(a, x) > r$. How can I see that there is an open ball centered at $x$ that does not intersect $\overline{B}(a, r)$?

I have the feeling that this open ball would then be included in the complement of $\overline{B}(a, r)$. And that is what we want to prove so that the complement of $\overline{B}(a, r)$ is open.

**Laurent:** Take $r' = d(a, x) - r$ for the metric $d$: $(x, y) \mapsto |x_1 - y_1| + |x_2 - y_2|$ of $\mathbf{R}^2$, for instance. You simply have to consider $B(x, r'/2)$.

**Bernard:** Oh, I see! The triangle inequality allows us to conclude that

$$B(x, r'/2) \cap \overline{B}(a, r) = \emptyset.$$




**Proposition 1.4.** *The open sets of a metric space $(E, d)$ satisfy the following properties:*

(O1) *The sets $\emptyset$ and $E$ are open.*
(O2) *For all open subsets $A$ and $B$ of $E$, the subset $A \cap B$ is open.*
(O3) *For any family of open subsets $(A_i)_{i \in \mathcal{I}}$ of $E$, the subset $\cup_{i \in \mathcal{I}} A_i$ is open.*

**Proof.** (O1) This point is obvious from Condition (ii) of Definition 1.2.

(O2) Let $a \in A \cap B$. Since $a$ is in the open subset $A$, there exists $r > 0$ such that $B(a, r) \subset A$.

And since $a$ is in the open subset $B$, there exists $r' > 0$ such that $B(a, r') \subset B$.

Let $r'' = \min(r, r') > 0$. We have $B(a, r'') \subset B(a, r) \cap B(a, r') \subset A \cap B$. This implies that $A \cap B$ is open.

(O3) We use Condition (i) of Definition 1.2: if each $A_i$ is a union of open balls, then $\cup_{i \in \mathcal{I}} A_i$ is a union of open balls. □

**Laurent:** Keep in mind that these essential properties of open sets, which will allow you to get beyond the metric spaces. We will see it later.

Now, let us have a look at how these open sets help us define the central concept of topology: the *neighborhoods*.

The concept of *neighborhood*, which is very much attached to that of open subsets, makes it possible to abstract (and therefore generalize) the notion of convergence.

**Definition 1.5.** We call the *neighborhood* of $a \in E$ any subset $V$ of $E$ such that there exists an open subset $U$ of $E$ that satisfies $\{a\} \subset U \subset V$.

We denote by $\mathcal{N}(a)$ the set of all the neighborhoods of $a$.

**Example.** In $\mathbf{R}$, intervals of the form $(a-\alpha, a+\alpha)$, where $\alpha \in \mathbf{R} \setminus \{0\}$, are (open) neighborhoods of $a$.

**Laurent:** Note that you do not specifically need a metric to define a neighborhood. You only need to know the collection of open sets.

**Bernard:** I have to admit I feel that you're playing with words. To define open sets, didn't you use the metric?

**Laurent:** Sure, in this precise case, you are right, Bernard. But you will see later on that a metric is not necessarily required to define the open subsets of an set. You are in for a surprise...

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The notion of neighborhood allows us to characterize the open subsets of metric spaces.

**Proposition 1.6.** *A subset $A$ is open if and only if $A$ is a neighborhood of each of its points.*

**Proof.** $\star$ If $A$ is open, then for all $a \in A$, the inclusions $\{a\} \subset A \subset A$ show that $A$ is a neighborhood of $a$.

$\star$ Conversely, if $A$ is a neighborhood of each of its points, then for all $a \in A$, there exists an open subset $U_a$ such that $\{a\} \subset U_a \subset A$.

This implies $A \subset \cup_{a \in A} U_a \subset A$, and thus $A = \cup_{a \in A} U_a$, which is open (as a union of open subsets). □

**Remark.** Definition 1.5 and Proposition 1.6 do not require a metric space structure. We will see later on that they remain valid within the general framework of topological spaces.

**Bernard:** Indeed, I am starting to see that, in Definition 1.5 and Proposition 1.6, I do not need a metric. If, let's say, an oracle, was able to give me a collection of all the open subsets of my set, I would be able to apply them straightforwardly.

Anyway, if I understand well Proposition 1.6, giving myself the collection of open sets or the collection of neighborhoods seems equivalent to me.

**Laurent:** I see that you are on the right path to what we will be calling *General topology*. Meanwhile, let us examine how open subsets and neighborhoods help us formulate the notions of limits and continuity.

**Bernard:** I am glad to see more concrete problems, at least closer to my industrial problem. Approximation issues may appear, then?

### 1.1.2 *Limits and continuity*

**Laurent:** You said earlier that *"a real sequence $u_n$ converges to 0" means that you can make $u_n$ as small as you want with $n$ large enough*. Let us formalize it in a metric space, which is more general than $\mathbf{R}$.

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**Definition 1.7 (Limit of a sequence).** A sequence $(u_n)_{n\in\mathbf{N}}$ in the metric space $(E, d)$ is said *to converge to $\ell \in E$ as $n$ goes to infinity* if

$$\forall \varepsilon > 0, \exists N \in \mathbf{N} : \quad \forall n \in \mathbf{N},\ n \geq N \Rightarrow d(u_n, \ell) < \varepsilon. \tag{1.1.1}$$

Using neighborhoods, the convergence can be equivalently written as

$$\forall V \in \mathcal{N}(\ell), \exists N \in \mathbf{N} : \quad \forall n \in \mathbf{N},\ n \geq N \Rightarrow u_n \in V. \tag{1.1.2}$$

**Proof of the equivalence** ⋆ By the definition of the open ball $B(\ell, \varepsilon)$, the inequality $d(u_n, \ell) < \varepsilon$ is equivalent to $u_n \in B(\ell, \varepsilon)$. Then, we can claim that (1.1.1) is equivalent to

$$\forall \varepsilon > 0, \exists N \in \mathbf{N} : \quad \forall n \in \mathbf{N},\ n \geq N \Rightarrow u_n \in B(\ell, \varepsilon).$$

⋆ In order to prove that (1.1.1) implies (1.1.2), consider any neighborhood $V$ of $\ell$.

There exists $\varepsilon > 0$ such that $B(\ell, \varepsilon) \subset V$.

By (1.1.1), there exists $N \in \mathbf{N}$ such that for all $n \geq N$, $u_n \in B(\ell, \varepsilon) \subset V$. Then, (1.1.2) holds.

⋆ Conversely, assuming that (1.1.2) holds, consider any $\varepsilon > 0$.

We apply (1.1.2) to the particular case of $V = B(\ell, \varepsilon)$. Then, (1.1.1) holds. $\square$

**Remark.** Since $\overline{B}(\ell, \varepsilon)$ is a neighborhood of $\ell$, a direct consequence of this last equivalence is that (1.1.1) is equivalent to

$$\forall \varepsilon > 0, \exists N \in \mathbf{N} : \quad \forall n \in \mathbf{N},\ n \geq N \Rightarrow d(u_n, \ell) \leq \varepsilon.$$

**Bernard:** Yes, I see the formalism with neighborhoods made the notion of distance disappear. It's magic!

**Laurent:** It is not magic at all. It is incredible to see how people who do not understand things call them magic. You just observed the power of abstraction, Bernard. And, thanks to it, we will free ourselves later from metric spaces.

Before that, let us see what happens for limits of functions.

**Definition 1.8 (Limit of a function).** For a function $f : E \to F$, where $(E, d)$ and $(F, d')$ are metric spaces, $f(x)$ is said *to converge to* $\ell \in F$ *as* $x$ *tends to* $a$ if

$$\forall \varepsilon > 0, \exists \eta > 0 : \quad \forall x \in E,\ d(x, a) < \eta \Rightarrow d'(f(x), \ell) < \varepsilon. \quad (1.1.3)$$

Using neighborhoods, the convergence can be equivalently written as

$$\forall V \in \mathcal{N}(\ell), \exists U \in \mathcal{N}(a) : \quad f(U) \subset V. \quad (1.1.4)$$

**Proof of the equivalence** $\star$ By the definitions of the open balls $B(a, \eta)$ and $B(\ell, \varepsilon)$, the inequality $d(x, a) < \eta$ is equivalent to $x \in B(a, \eta)$ and $d'(f(x), \ell) < \varepsilon$ is equivalent to $f(x) \in B(\ell, \varepsilon)$. Then, we can claim that (1.1.3) is equivalent to

$$\forall \varepsilon > 0, \exists \eta > 0 : \quad \forall x \in B(a, \eta), f(x) \in B(\ell, \varepsilon).$$

$\star$ In order to prove that (1.1.3) implies (1.1.4), consider any neighborhood $V$ of $\ell$ and $\varepsilon > 0$ such that $B(\ell, \varepsilon) \subset V$.

The assertion (1.1.3) implies the existence of $\eta > 0$ such that for all $x \in B(a,\eta)$, $f(x) \in B(\ell,\varepsilon)$. Then, (1.1.4) holds for $U = B(a,\eta)$.

$\star$ Conversely, assuming that (1.1.4) holds, consider any $\varepsilon > 0$ and apply (1.1.4) to $V = B(\ell,\varepsilon)$.

There exists $U \in \mathcal{N}(a)$ such that $f(U) \subset V$.

Since $U$ is a neighborhood of $a$, there exists $\eta > 0$ such that $B(a,\eta) \subset U$.

Then, for all $x \in B(a,\eta)$, we have $f(x) \in V = B(\ell,\varepsilon)$. $\square$

**Remark.** Since $\overline{B_{d'}}(\ell,\varepsilon)$ is a neighborhood of $\ell$, a direct consequence of this last equivalence is that (1.1.3) is equivalent to

$$\forall \varepsilon > 0, \exists \eta > 0 : \quad \forall x \in E,\ d(x,a) < \eta \Rightarrow d'(f(x),\ell) \leq \varepsilon.$$

**Bernard:** In **R**, can we define a neighborhood of $a = +\infty$?

**Laurent:** Sure. If you think about it, any subset containing an interval $(b,+\infty)$ is a neighborhood of $+\infty$. With this convention, (1.1.4) remains valid for $a = +\infty$.

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We recall the definition of continuity of a function at a point in metric spaces.

**Definition 1.9 (Continuity of a function at a point).** For a function $f$ from a metric space $(E,d)$ to another metric space $(F,d')$, $f$ is said *to be continuous at* $a \in E$ if

$$\forall \varepsilon > 0, \exists \eta > 0 : \quad \forall x \in E,\ d(x,a) < \eta \Rightarrow d'(f(x),f(a)) < \varepsilon. \tag{1.1.5}$$

The function $f$ is said to be *continuous on* $U \subset E$ if $f$ is continuous at every point $a \in U$.

We strengthen this definition in the uniform continuity property:

**Definition 1.10 (Uniform continuity).** A function $f : E \to F$, where $(E,d)$ and $(F,d')$ are metric spaces, is said *to be uniformly continuous on* $U \subset E$ if

$$\forall \varepsilon > 0, \exists \eta > 0 : \quad \forall x,y \in U,\ d(x,y) < \eta \Rightarrow d'(f(x),f(y)) < \varepsilon. \tag{1.1.6}$$

**Bernard:** Why another concept... again? Does it ever stop, in this book?

**Laurent:** Generally speaking, the uniform continuity property is stronger than simple continuity. The real $\eta$ of (1.1.5) depends on $\varepsilon$ and $a$. However, the $\eta$ of (1.1.6) only depends on $\varepsilon$.

And the answer to your question is no; it never stops. Each step makes you smarter...

However, on a compact set, the two definitions are equivalent. You should have learned the Heine theorem in your so famous *classes préparatoires.*

### 1.1.3 *Compactness in metric spaces*

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**Definition 1.11 (Compact subsets).** Let $(E, d)$ be a metric space.

⋆ A subset $A \subset E$ is said to satisfy the *Bolzano–Weierstrass property* if every sequence in $A$ admits a subsequence that converges in $A$.

⋆ Such a limit is called a *subsequential limit* of the sequence.

⋆ A subset $K \subset E$ is said to be *compact* if it satisfies the Bolzano–Weierstrass property.

**Remark.** In General topology, the compactness property is defined by the *Borel–Lebesgue property*:

*A subset $A$ is said to be compact if from any cover of $A$ by open subsets, $A \subset \cup_{i\in\mathcal{I}} O_i$, we can extract a finite subcover, $A \subset O_{i_1} \cup \cdots \cup O_{i_n}$.*

In a metric space, the Bolzano–Weierstrass and Borel–Lebesgue properties are equivalent (see [Ramis *et al.* (1982)]).

**Proposition 1.12.** *In* $\mathbf{R}^d$ *and, more generally, in any finite-dimensional vector space endowed with any norm, the compact subsets are exactly the bounded closed subsets.*

**Proof.** For the sake of avoiding difficulties of notations in the multi-dimensional cases, we prove the result in the case of $\mathbf{R}$.

(1) We first prove that every compact subset is closed and bounded.

$\star$ Let $K$ be a compact subset of $\mathbf{R}$ and $(u_n)_{n\in\mathbf{N}}$ be a sequence in $K$ that converges to $\ell \in \mathbf{R}$. The Bolzano–Weierstrass property implies that $(u_n)_{n\in\mathbf{N}}$ admits a subsequence $(u_{k_n})_{n\in\mathbf{N}}$ that converges in $K$. We have, obviously, $\lim_{n\to\infty} u_{k_n} = \ell$, and we can claim that $\ell \in K$.

We conclude that any sequence in $K$ that converges in $\mathbf{R}$ has its limit in $K$. This characterizes the fact that $K$ is closed.

$\star$ In order to prove that $K$ is bounded, we proceed by contradiction. Let us assume that $K$ is not bounded.

There exists an unbounded sequence $(u_n)_{n\in\mathbf{N}}$ in $K$ such that $|u_n| \to \infty$ as $n \to \infty$ (Ask yourself why?...).

But, since $K$ is compact, there exists a subsequence $(u_{k_n})_{n\in\mathbf{N}}$ that converges. This converging subsequence is necessarily bounded, which contradicts the fact that $|u_{k_n}| \to \infty$. We conclude that $K$ is bounded.

(2) Conversely, assume that $A$ is a bounded closed subset of $\mathbf{R}$. In order to prove that $A$ is compact, let us consider any sequence $(u_n)_{n\in\mathbf{N}}$ in $A$, and let us prove that $(u_n)_{n\in\mathbf{N}}$ admits a subsequence that converges in $A$.

$\star$ Since $A$ is closed, if a subsequence of $(u_n)_{n\in\mathbf{N}}$ converges, then its limit is necessarily in $A$. It only remains to prove that $(u_n)_{n\in\mathbf{N}}$ admits a converging subsequence.

$\star$ Since $A$ is bounded in $\mathbf{R}$, the quantities $\inf_{n\in\mathbf{N}} u_n$, $\sup_{n\in\mathbf{N}} u_n$, $\inf_{k\geq n} u_k$ and $\sup_{k\geq n} u_k$ are finite. We can also claim that $\inf_{n\in\mathbf{N}} \sup_{k\geq n} u_k$ is finite.

For all $n \in \mathbf{N}$, there exists $k_n$ such that $\sup_{k\geq n} u_k - 2^{-n} \leq u_{k_n} \leq \sup_{k\geq n} u_k$. We can choose $k_n > k_{n-1}$ for all $n \in \mathbf{N}$.

The sequence $(u_{k_n})_{n\in\mathbf{N}}$ converges to $\inf_{n\in\mathbf{N}} \sup_{k\geq n} u_k$, which completes the proof. $\square$

**Laurent:** I think it is important to recall the concept of closure:

**Definition.** *If $A$ is a subset of a topological space $E$, its closure, denoted by $\overline{A}$, is the smallest closed subset containing $A$ (for the inclusion) or, equivalently, the intersection of all the closed subsets containing $A$.*

And

**Proposition.** *If $E$ is a metric space, a subset $A$ is closed iff any convergent sequence $(u_n)_{n\in\mathbf{N}}$ of $E$ such that $u_n \in A$ for every $n \in \mathbf{N}$ has its limit in $A$.*

As an example: $\forall n \geq 1$, $u_n = 1 - 1/n \in [0,1)$ and the limit does not belong to $[0,1)$, so $[0,1)$ is not closed.

I let you refer to your *classes préparatoires* textbooks...

**Bernard:** I remember very well, thank you. And the closure of a subset $A$ is the union of $A$ and the set of the limits of the convergent sequences of elements of $A$. And a subset is closed iff it is equal to its closure. And the closed ball $\overline{B}(a,r)$ is the closure of $B(a,r)$, so the overline notation for the ball is consistent. I also remember perfectly well the link between compactness and uniform continuity:

**Theorem (Heine).** *A continuous function $f\colon A \to F$ with $A$ compact is uniformly continuous.*

But I see no link to my problem of pharmaceutical plant. Are we moving forward?

514.COR.t

One of the main interests in the concept of compactness is to be able to consider the image of a compact subset by a continuous function.

**Proposition 1.13.** *In the metric space $(E,d)$, let $(u_n)_{n\in\mathbf{N}}$ be a sequence that converges to $\ell$. And let $f$ be a continuous function from $(E,d)$ to the metric space $(F,d')$.*

*Then, the sequence $\big(f(u_n)\big)_{n\in\mathbf{N}}$ converges to $f(\ell)$ in $(F,d')$.*

Actually, Proposition 1.13 can be extended as a *sequential characterization* of the continuity property in metric spaces.

**Proof.** On the one hand, the continuity of the function $f$ at $\ell$ allows us to write

$$\forall \varepsilon > 0, \exists \eta > 0 : \quad \forall x \in E,\ d(x,\ell) < \eta \Rightarrow d'\big(f(x), f(\ell)\big) < \varepsilon.$$

On the other hand, since the sequence $(u_n)_{n\in\mathbf{N}}$ converges to $\ell$, we have

$$\forall \eta > 0, \exists N \in \mathbf{N} : \quad \forall n \in \mathbf{N},\ n \geq N \Rightarrow d(u_n,\ell) < \eta.$$

From these two assertions, we deduce that

$$\forall \varepsilon > 0, \exists N \in \mathbf{N}: \quad \forall n \in \mathbf{N},\, n \geq N \Rightarrow d(u_n, \ell) < \eta$$
$$\Rightarrow d'(f(u_n), f(\ell)) < \varepsilon.$$

This expresses exactly the fact that the sequence $(f(u_n))_{n\in\mathbf{N}}$ converges to $f(\ell)$ in $(F, d')$. □

**Theorem 1.14.** *Let $f$ be a continuous function from the metric space $(E, d)$ to the metric space $(F, d')$.*

*If $A$ is a compact subset of $(E, d)$, then $f(A)$ is a compact subset of $(F, d')$.*

**Proof.** Consider any sequence $(v_n)_{n\in\mathbf{N}}$ in $f(A)$. For all $n \in \mathbf{N}$, there exists $u_n \in A$ such that $v_n = f(u_n)$.

Since $A$ is compact, the Bolzano–Weierstrass property implies that the sequence $(u_n)_{n\in\mathbf{N}}$ admits a subsequence that converges in $A$.

From Proposition 1.13, we can conclude that the sequence $(v_n)_{n\in\mathbf{N}}$ converges in $f(A)$. Then, $f(A)$ is compact. □

**Laurent:** A direct consequence is that the maximum of a continuous function is reached at some point.

As an example, take $f\colon x \in [0,1] \mapsto \begin{cases} x\,|\sin(1/x)| & \text{if}\, x > 0, \\ 0 & \text{otherwise.} \end{cases}$

Without any consideration of the monotonicity, we can claim that the function $f$ is continuous and admits bounds which are reached.

## 1.2 More about Convergence in Metric Spaces

**Laurent:** I guess that your *classes préparatoires* were not in the same institution your father attended. And I guess that Ramis' books[1] were not your bedside books. So, there is a lot about metric spaces that you do not know about.

[1] See [Ramis *et al.* (1982)] and other volumes.

**Bernard:** Please, don't add to my pain. This point has been at the heart of my relationship with my father for a few years now.

I can't see what I could be missing about metric spaces. The program of *classes préparatoires* is extensive.

**Laurent:** I cannot know what happened in your *classes préparatoires*. Have you heard about the convergence of a Cauchy sequence?

**Bernard:** Absolutely not. Is it some useless, very theoretical, notion?

### 1.2.1 *Complete spaces and Cauchy sequences*

**Laurent:** The convergence of the Cauchy sequences is the key to the proof of many theorems that rely on a fixed point argument.

Without that, you cannot claim you understood the proofs of all the results you pretended to have seen when preparing for entrance examinations to engineering schools.

Actually, this notion must have disappeared from the program because people must have thought it was too difficult for you. To me, without these notions, you cannot understand the real math you need to tackle your pharmaceutical problem.

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**Definition 1.15.** A *Cauchy sequence* in a metric space $(E, d)$ is a sequence $(u_n)_{n\in\mathbf{N}}$ such that

$$\forall \varepsilon > 0, \exists N \in \mathbf{N} : \quad \forall n, p \in \mathbf{N}, [n \geq N, p \geq N \Rightarrow d(u_n, u_p) < \varepsilon].$$

**Bernard:** I admit you could be right. Formally, it looks like the definition of the convergence in a metric space... But I read no mention of a limit. Is it hidden?

**Laurent:** That is exactly the point: the idea is to have a convergence criterion without an explicit limit. Up to this point, you noted that you were not able show the convergence of a sequence without previously guessing the value of its limit.

**Bernard:** I have the intuition that all the values of a Cauchy sequence get closer to each other as $n$ goes to infinity and seem to accumulate, as is the case for a converging sequence.

**Laurent:** Indeed. Let us examine the link between being a converging sequence and being a Cauchy sequence.

**Proposition 1.16.** *In a metric space $(E, d)$, any sequence that converges is a Cauchy sequence.*

**Proof.** Consider any sequence $(u_n)_{n\in\mathbf{N}}$ that converges to $\ell \in E$.

For all $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that $\forall n \in \mathbf{N}$, $n \geq N \Rightarrow d(u_n, \ell) < \varepsilon/2$.

Then, for all $n, p \in \mathbf{N}$, $n \geq N$ and $p \geq N$, we have $d(u_n, u_p) \leq d(u_n, \ell) + d(u_p, \ell) < \varepsilon$.

We can conclude that $(u_n)_{n\in\mathbf{N}}$ is a Cauchy sequence. □

**Bernard:** A converging sequence is a Cauchy sequence. But finally, what can we say about a general Cauchy sequence?

**Laurent:** Let us see how to formulate your intuition of accumulation of the $u_n$ for a Cauchy sequence.

**Proposition 1.17.** *In a metric space $(E, d)$, any Cauchy sequence is bounded, that is, included in some closed ball.*

**Proof.** Let $(u_n)_{n\in\mathbf{N}}$ be a Cauchy sequence.

Set $U_n = \{u_k;\ k \geq n\}$ and $\delta(U_n) = \sup_{a,b\in U_n} d(a, b)$ for all $n \in \mathbf{N}$.

By the definition of a Cauchy sequence, we have $\lim_{n\to\infty} \delta(U_n) = 0$.

Then, there exists $m \in \mathbf{N}$ such that $\delta(U_m) < \infty$.

The fact that $U_0 = \{u_0, \ldots, u_{m-1}\} \cup U_m$ allows us to conclude that $\delta(U_0) < \infty$ and that $(u_n)_{n\in\mathbf{N}}$ is included in the ball $\overline{B}\big(u_0, \delta(U_0)\big)$. □

**Bernard:** We want to show that a sequence is bounded. We use a supremum. My teacher of *classes préparatoires* told me that sup cannot be written if the quantity inside is not bounded. So, the book assumes that something is bounded to show it is indeed bounded. It is ridiculous!

**Laurent:** Come on, Bernard. Get out of your *classes préparatoires*. You are annoying and arrogant.

There is no problem with the definition of sup if you accept the value $+\infty$.

So, $\delta(U_n) \in \mathbf{R}_+ \cup \{+\infty\}$. The point of the proof is that, since $\lim_{n\to\infty} \delta(U_n) = 0$, the quantity $\delta(U_n)$ cannot be infinite.

**Bernard:** OK, OK. I have the intuition that $u_n = \sum_{k=1}^{n} \frac{1}{k^2}$ is a Cauchy sequence of rational numbers. Using the fact that $t \mapsto 1/t^2$ is decreasing, for all $1 \le p < q$, we can write

$$0 \le u_q - u_p = \sum_{k=p+1}^{q} \frac{1}{k^2} \le \sum_{k=p+1}^{q} \int_{k-1}^{k} \frac{dt}{t^2} \le \int_{p}^{+\infty} \frac{dt}{t^2} = \frac{1}{p}.$$

Then, if I fix $\varepsilon > 0$ and $N = \lfloor 1/\varepsilon \rfloor + 1$, as soon as $p, q \ge N$, $|u_p - u_q| \le \varepsilon$. So, it is a Cauchy sequence in $\mathbf{Q}$.

But if I remember well, this sequence in $\mathbf{R}$ converges to $\pi^2/6$, which is not rational. I am lost.

**Laurent:** Ah! You pointed out the correct problem, surprisingly. There exist Cauchy sequences in $\mathbf{Q}$ that do not converge in $\mathbf{Q}$. This point is connected to the notion of a complete space.

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**Definition 1.18.** A metric space $(E, d)$ is said to be *complete* if every Cauchy sequence in $E$ converges.

**Cauchy criterion**

In a complete metric space $(E, d)$, a sequence $(u_n)_{n\in\mathbf{N}}$ converges if and only if it satisfies the following criterion:

$$\forall \varepsilon > 0, \exists N \in \mathbf{N} : \quad \forall n, p \in \mathbf{N},\ [n \ge N, p \ge N \Rightarrow d(u_n, u_p) < \varepsilon].$$

**Example.** ⋆ The space $\mathbf{Q}$ is not complete, but by the construction of $\mathbf{R}$, the spaces $\mathbf{R}^d$ are complete for all $d \ge 1$.

⋆ $[0, 1)$ is not complete even if $[0, 1]$ is.

⋆ The closed subsets of complete spaces are complete.

**Bernard:** To sum up, in a complete metric space, if I want to show that a sequence converges, it is enough for me to show that it is a Cauchy sequence, without knowing the explicit value of the limit.

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The concept of Cauchy sequences allows us to prove readily that the intersection of a nonincreasing sequence of nonempty closed sets is not empty.

**Theorem 1.19 (Cantor intersection theorem).** *Let* $(E, d)$ *be a complete metric space and* $(F_n)_{n\in\mathbf{N}}$ *be a sequence of nonempty closed subsets of* $E$ *such that*:

(i) $\forall n \in \mathbf{N},\ F_{n+1} \subset F_n$;
(ii) $\delta(F_n) = \sup\{d(a,b);\ a, b \in F_n\}$ *converges to* $0$ *as* $n \to \infty$.

*Then,* $\bigcap_{n\in\mathbf{N}} F_n$ *is reduced to a single point of* $E$.

**Proof.** $\star$ We construct a sequence $(u_n)_{n\in\mathbf{N}}$ such that for all $n \in \mathbf{N}$, $u_n \in F_n$.

Since the sequence $(F_n)_{n\in\mathbf{N}}$ is nonincreasing, we have $U_n = \{u_k;\ k \geq n\} \subset F_n$.

We deduce that $\delta(U_n) \leq \delta(F_n)$ and then, $\lim_{n\to\infty} \delta(U_n) = 0$, which implies that $(u_n)_{n\in\mathbf{N}}$ is a Cauchy sequence.

Since $E$ is complete, the sequence $(u_n)_{n\in\mathbf{N}}$ converges to some limit $\ell \in E$.

For all $n \in \mathbf{N}$, we have $\{u_k;\ k \geq n\} \subset F_n$. Since $F_n$ is closed, we can claim that $\ell \in F_n$ for all $n \in \mathbf{N}$. Finally, $\{\ell\} \subset \bigcap_{n\in\mathbf{N}} F_n$.
$\star$ It remains to prove that there can be only one point in $\bigcap_{n\in\mathbf{N}} F_n$. If $a$ and $b$ are in $\cap_{n\in\mathbf{N}} F_n$, we have $d(a,b) \leq \delta(F_n)$ for all $n \in \mathbf{N}$. This shows that $a = b$.

We conclude that $\{\ell\} = \cap_{n\in\mathbf{N}} F_n$. □

### 1.2.2 *Limit superior and limit inferior*

**Laurent:** You must have noted that, to derive conclusions on the limits of the results of manipulations on sequences, you must have proved beforehand that all these limits exist.

**Bernard:** True. And, we used to make these manipulations in order to prove the existence of the resulting limit. It seems there is a problem of logic here.

**Laurent:** It is true that it is sometimes tedious and not that easy to prove the existence of limits. The superior and inferior limits, which are always defined for real sequences, will help us.

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In this section, we assume that $E = \overline{\mathbf{R}} = \mathbf{R} \cup \{\pm\infty\}$.

**Definition 1.20 (Limit superior and limit inferior).** Let $(u_n)_{n\in\mathbf{N}}$ be a sequence in $\overline{\mathbf{R}}$ and $U_n = \{u_k;\ k \geq n\}$ for all $n \in \mathbf{N}$.

The sequences $(\sup U_n)_{n\in\mathbf{N}}$ and $(\inf U_n)_{n\in\mathbf{N}}$ converge in $\overline{\mathbf{R}}$. Their limits are denoted by $\limsup u_n$ and $\liminf u_n$, respectively.

Since $(\sup U_n)_{n\in\mathbf{N}}$ is nonincreasing and $\big(\inf U_n\big)_{n\in\mathbf{N}}$ is nondecreasing, we have

$$\limsup u_n = \inf_{n\in\mathbf{N}} \sup_{k\geq n} u_k \quad \text{and} \quad \liminf u_n = \sup_{n\in\mathbf{N}} \inf_{k\geq n} u_k.$$

**Bernard:** Hum. I see that $(\sup_{k\geq n} u_k)_{n\in\mathbf{N}}$ is nonincreasing. But I'm embarrassed to admit that I can't see why it has a limit.

**Laurent:** Think it over, Bernard! You know that if $(x_n)_{n\in\mathbf{N}}$ is a nondecreasing sequence in $\mathbf{R}$, either it admits a finite upper bound, which implies that the sequence converges, or it grows to $+\infty$.

In $\overline{\mathbf{R}}$, the values $+\infty$ and $-\infty$ are allowed, so nonincreasing and nondecreasing sequences always converge.

**Bernard:** We just studied the convergence of sequences in a metric space. I have difficulties seeing $\overline{\mathbf{R}}$ as a metric space since the distance between any real number and $+\infty$ should be infinite.

**Laurent:** I understand your point. The function $(x, y) \mapsto |x - y|$ is not a metric on $\overline{\mathbf{R}}$. We could define a metric on $\overline{\mathbf{R}}$, but it is not necessary here. All these questions will become clear in the context of General topology, where convergence is defined within the sole concept of neighborhoods.

**Bernard:** OK. I will be patient.

**Laurent:** Any sequence with values in $\overline{\mathbf{R}}$ admits a limit superior and a limit inferior, which can be different. As an example, take $v_n = (-1)^n$. What can you say?

**Bernard:** That one is easy: $\liminf v_n = -1$ and $\limsup v_n = 1$. But what is the connection to the convergence of the sequence $(v_n)_{n\in\mathbf{N}}$?

**Laurent:** In order to prove that the sequence does not converge, it is enough to check that its limits superior and inferior are not equal. Actually, this is equivalent.

**Theorem 1.21.** *A sequence* $(u_n)_{n\in\mathbf{N}}$ *in* $\overline{\mathbf{R}}$ *converges if and only if its limit superior and limit inferior are equal.*

*In that case, we have* $\lim_{n\to\infty} u_n = \limsup u_n = \liminf u_n$.

**Proof.** $\star$ Let us prove the direct implication.

At first, we assume that $(u_n)_{n\in\mathbf{N}}$ converges to $\ell \in \mathbf{R}$ :for $\varepsilon > 0$, then there exists $N \in \mathbf{N}$ such that $\forall n \in \mathbf{N}$, $n \geq N \Rightarrow |u_n - \ell| \leq \varepsilon$.

Then, for all $n \in \mathbf{N}$, $n \geq N$, we have

$$\ell - \varepsilon \leq \inf_{k\geq n} u_k \leq \sup_{k\geq n} u_k \leq \ell + \varepsilon.$$

This proves that $\lim_{n\to\infty} \inf_{k\geq n} u_k = \ell$ and $\lim_{n\to\infty} \sup_{k\geq n} u_k = \ell$.

Assume now that $(u_n)_{n\in\mathbf{N}}$ converges to $+\infty$; for $A > 0$, there exists $N \in \mathbf{N}$ such that for all $n \in \mathbf{N}$, $n \geq N \Rightarrow u_n \geq A$.

Then, for all $n \in \mathbf{N}$, $n \geq N$, we have

$$A \leq \inf_{k\geq n} u_k \leq \sup_{k\geq n} u_k.$$

This proves that $\lim_{n\to\infty} \inf_{k\geq n} u_k = +\infty$ and $\lim_{n\to\infty} \sup_{k\geq n} u_k = +\infty$.

$\star$ Let us turn to the other implication: let $\liminf u_k = \limsup u_k$.

For all $n \in \mathbf{N}$, we have $\inf_{k\geq n} u_k \leq u_n \leq \sup_{k\geq n} u_k$.

Since $(\inf_{k\geq n} u_k)_{n\in\mathbf{N}}$ converges to $\liminf u_n$ and $(\sup_{k\geq n} u_k)_{n\in\mathbf{N}}$ converges to $\limsup u_n$, the fact that $\liminf u_n = \limsup u_n$ implies that $(u_n)_{n\in\mathbf{N}}$ converges to $\ell = \liminf u_n = \limsup u_n$. $\square$

## 1.3 Elements of General Topology

**Bernard:** You have used the word *topology* many times. But basically, what is a topology?

**Laurent:** Many of us are used to calling *topology* the part of mathematics which studies the convergence of sequences or the regularity of functions.

But we have just seen that it only relies on the definition of open sets. More precisely, you have understood that everything about convergence is well defined using only neighborhoods. And the notion of neighborhood relies only on the notion of open set.

In fact, mathematically, we call *topology of a set* the collection of all its open subsets. This is the prime idea of General topology.

### 1.3.1 *Topological spaces*

The goal of General topology is to find the minimal structure that allows us to consider the notion of limit.

**Definition 1.22 (Topology, open subsets).** We call *topology* on a set $E$ any collection $\mathcal{T}$ of the subsets of $E$ that satisfies the following conditions:

(O1) $\emptyset$ and $E$ are in $\mathcal{T}$.
(O2) For all subsets $A, B$ in $\mathcal{T}$, the subset $A \cap B$ is in $\mathcal{T}$.
(O3) For all subsets $(A_i)_{i \in \mathcal{I}}$ in $\mathcal{T}$, the subset $\cup_{i \in \mathcal{I}} A_i$ is in $\mathcal{T}$.

The elements of $\mathcal{T}$ are called *open sets* of $\mathcal{T}$. The pair $(E, \mathcal{T})$ is called a *topological space*.

**Bernard:** Reflecting about it, I recognize the properties of Proposition 1.4 concerning the collection of the open sets of a metric space. But there is no mention of a metric here...

**Laurent:** Indeed, Bernard. We do not need anything else to study the convergence. Giving ourselves all the open sets, we do not need a metric to define them.

**Definition 1.23 (Closed sets).** A subset $A$ of $E$ is called *closed* if its complement $E \setminus A$ is open.

By taking the complement in Definition 1.22, we obtain the following result for the closed sets.

**Proposition 1.24.** *The collection of closed subsets of* $(E, \mathcal{T})$ *satisfies the following properties:*

(F1) $\emptyset$ *and* $E$ *are closed.*

(F2) *For all closed subsets $A$ and $B$, the subset $A \cup B$ is closed.*
(F3) *For all closed subsets $(A_i)_{i \in \mathcal{I}}$, the subset $\cap_{i \in \mathcal{I}} A_i$ is closed.*

Definition 1.5 and Proposition 1.6 concerning neighborhoods still hold for topological spaces:

- ⋆ In the topological space $E$, we call the *neighborhood* of $a \in E$ any subset $V$ of $E$ such that there exists an open subset $U$ of $E$ that satisfies $\{a\} \subset U \subset V$.
  We denote by $\mathcal{N}(a)$ the set of all the neighborhoods of $a$.
- ⋆ In the topological space $E$, a subset $A$ is open if and only if $A$ is a neighborhood of each of its points.

**Definition 1.25 (Separated space).** The topological space $(E, \mathcal{T})$ is called *separated or Hausdorff* if for all $a \neq b$,

$$\exists U \in \mathcal{N}(a), \exists V \in \mathcal{N}(b) : \quad U \cap V = \emptyset.$$

**Bernard:** I'm sorry, Laurent, but all this seems very theoretical to me. Are there really topological spaces that are not metric? Otherwise, we are putting a lot of effort uselessly.

**Laurent:** Maybe the most striking example is the one that shows that a topology can be defined as soon as we are able to order the set, without any intervention of some metric.

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**Example. (Topology of the order).** Let $(E, \preccurlyeq)$ be a (totally or partially) ordered space. We define, for all $a, b \in E$, what we will call *open intervals* by

$$(\leftarrow, b) = \{x \in E : x \prec b\}, \quad (a, \rightarrow) = \{x \in E : a \prec x\}$$
$$\text{and} \quad (a, b) = \{x \in E : a \prec x \prec b\}.$$

The collection of all unions of such intervals defines a topology (satisfying the conditions of Definition 1.22), called the *topology of the order*. Any open set is then some union (possibly infinite) of open intervals.

**Laurent:** You can see easily that the usual topology of $\mathbf{R}$ is the topology of the order.

**Bernard:** How do you mean?...

**Laurent:** Precisely, if you consider the topology of $\mathbf{R}$ that is generated by the open balls, that is, for which the open sets are the arbitrary unions of open balls, we obtain the same topology as the topology of the order of $(\mathbf{R}, \leq)$.

This is because open balls are open intervals in the sense of the order of $(\mathbf{R}, \leq)$.

**Bernard:** Since $\overline{\mathbf{R}} = \mathbf{R} \cup \{\pm\infty\}$ is totally ordered, I understand that we could endow it with the topology of the order. Right?

**Laurent:** You are right, Bernard. Besides, this topology is compatible with the usual topology of $\mathbf{R}$. This is connected to the notion of induced topology.

### 1.3.2 *Induced topology*

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Consider $(E, \mathcal{T})$ to be a topological space. Let $F$ be a subset of $E$.

**Definition 1.26.** The collection of subsets $F \cap U$, where $U \in \mathcal{T}$, is a topology on $F$, which is called the *topology induced by* $(E, \mathcal{T})$ *on* $F$.

**Justification.** The collection of subsets $\{F \cap U;\ U \in \mathcal{T}\}$ satisfies the three conditions of Definition 1.22. □

**Example.** The space $\overline{\mathbf{R}} = \mathbf{R} \cup \{\pm\infty\}$ is totally ordered by $\leq$. So, we can consider the topology $\mathcal{T}$ of its order. The restriction to $\mathbf{R}$ of any open interval of $\overline{\mathbf{R}}$ is an open interval of $\mathbf{R}$.

We can conclude that the topology induced by $(\overline{\mathbf{R}}, \mathcal{T})$ on $\mathbf{R}$ is the usual topology of $\mathbf{R}$.

**Bernard:** Wow. There is a buildup of concepts to understand these topologies of $\mathbf{R}$ and $\overline{\mathbf{R}}$.

**Laurent:** Do not panic, Bernard. We can see it in a simple way. Finally, the open intervals of $\overline{\mathbf{R}}$ are the open intervals of $\mathbf{R}$ plus $(a, +\infty]$, $[-\infty, b)$ and $[-\infty, +\infty]$ for $a, b \in \mathbf{R}$.

Once you understand that, you understand the difference between the open sets of $\mathbf{R}$ and the open sets of $\overline{\mathbf{R}}$, that is, possibly adding the points $+\infty$, $-\infty$, or both.

### 1.3.3 *Topology induced by a metric*

**Bernard:** In a metric space, we saw that the open sets were defined starting from open balls. Is this definition consistent with the ones we just saw in General topology?

**Laurent:** It looks like it was done on purpose. The conditions defining a topology are precisely the properties satisfied by the open sets of a metric space.

**Bernard:** Ah... I put the pieces together. If we call an open set any arbitrary union of open balls, then the collection of open sets satisfies the conditions to be a topology. So, the topology of metric spaces is a particular case of a topology.

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The general definition of a topology is consistent with the definition of open subsets of metric spaces.

**Definition 1.27 (Topology induced by a metric).** The collection of all unions of open balls $B(a, r)$ (where $a \in E$ and $r > 0$) defines a topology called the *topology induced by the metric d*.

The set $E$ equipped with this topology is a separated space.

### 1.3.4 *Convergence in topological spaces*

**Bernard:** Let me sum up my understanding. By definition, in a topological space, the starting point is to know all the open sets. Then, if we know the open sets, we know the neighborhoods. Right?

So, we must be able now to define in this more general case the notions of convergence that we saw before, in (1.1.2) and (1.1.4).

**Laurent:** Great, Bernard! You use precisely the same words as in that book we are reading.

In the context of metric spaces, the convergence of sequences and functions has been expressed in terms of neighborhoods (see Definitions 1.7 and 1.8). We recall these definitions in the context of General topology.

**Definition 1.28 (Limit of a sequence).** A sequence $(u_n)_{n\in\mathbf{N}}$ in the separated space $E$ is said *to converge to* $\ell \in E$ if

$$\forall V \in \mathcal{N}(\ell), \exists N \in \mathbf{N} : \quad \forall n \in \mathbf{N},\, n \geq N \Rightarrow u_n \in V.$$

**Definition 1.29 (Limit of a function).** Let $f$ be a function from the topological space $E$ to the separated space $F$. We say that $f(x)$ *converges to* $\ell \in F$ *as* $x \to a$ if

$$\forall V \in \mathcal{N}(\ell), \exists U \in \mathcal{N}(a) : \quad f(U) \subset V.$$

We note that the space of values of the sequence and of the function needs to be separated for the limit to be unique.

**Bernard:** I imagine very well that, for the limit to be unique, I need to ensure that, if two points are distinct, there exist two nonintersecting neighborhoods...

**Laurent:** That is precisely what the word *separated* means. Let us pursue with the notion of continuity.

**Theorem 1.30 (Continuity on a topological space).** *Let* $(E, \mathcal{T}_E)$ *and* $(F, \mathcal{T}_F)$ *be two topological spaces and* $f : E \to F$*.*

*The two following assertions are equivalent:*

(i) *$f$ is continuous at any point of $E$.*
(ii) *The inverse image of any open subset of $F$ under $f$ is an open set of $E$,* i.e.

$$\forall V \in \mathcal{T}_F, \quad f^{-1}(V) \in \mathcal{T}_E.$$

**Laurent:** I point out to you the fact that continuity, defined with neighborhoods, will appear as very similar to *measurability of functions*, where the conceptual notion of topology will be substituted with the concept of $\sigma$-*algebra*...

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**Proof.** $\star$ (i)$\Rightarrow$(ii): Let us assume that (i) holds and let $V$ be an open set of $F$.

In order to prove that $f^{-1}(V)$ is open, we prove that $f^{-1}(V)$ is a neighborhood of all its points.

Let $a \in f^{-1}(V)$. We have $f(a) \in V$, so $V \in \mathcal{N}\big(f(a)\big)$.

Since $f(x)$ converges to $f(a)$ as $x \to a$, there exists an open set $U \in \mathcal{N}(a)$ such that $f(U) \subset V$.

From $U \subset f^{-1}(V)$, we conclude that $f^{-1}(V)$ is a neighborhood of $a$.

$\star$ (ii)$\Rightarrow$(i): Let us assume that (ii) holds and consider $a \in E$. In order to prove that $f(x)$ converges to $f(a)$ as $x \to a$, we consider an open set $V \in \mathcal{N}\big(f(a)\big)$.

Assertion (ii) implies that $U = f^{-1}(V)$ is an open set such that $a \in U$ and $f(U) \subset V$. The result follows. □

## 1.4 Normed Vector Spaces

**Laurent:** What we have seen until now is very general. We are still far away from your pharmaceutical plant problem, Bernard. As we discussed earlier, solving an engineering problem relies on the control of approximations of mathematical models.

Schematically, a mathematical model is something that predicts values in the function of some parameters (initial conditions, environment and so on). In the end, it is a function, with values in $\mathbf{R}$ or $\mathbf{R}^d$. For example, the flow velocity is a three-component function of its initial conditions or the shape of its boundaries.

And then, approximating a model, the art of the engineer as one says, rigorously needs to study topology on a space of functions.

**Bernard:** That is quite clear! I mostly understand the link between topology and my concrete problem, Laurent. But since you said we need to consider *spaces* of functions, I imagine there is a vector structure on these spaces, as can be said about $C([a, b])$.

**Laurent:** You are absolutely right, and we will see that now. It motivates the study of the *normed vector spaces*.

But before we start, I would like to plant a seed in your mind. The mathematics necessary to solve "real problems" must leave behind the infinitesimal world, where the functions are studied pointwise, wondering about the regularity, differentiability, pointwise versus uniform convergence and so on, to reach the holy world where we study spaces whose elements are functions...

In this framework, studying the convergence of a sequence of functions will be treated as a topological issue in the space where the functions live.

### 1.4.1 *Norms on vector spaces*

In a functional analysis framework, functions are studied as the elements of a *functional space* in which the convergence of a sequence is expressed through the chosen topology of that space.

Normed functional spaces are an important type of functional spaces (even if it is only a special case of topological functional space) in which the convergence of a sequence of functions is defined by the considered norm.

Until now, no vector structure has been considered in our study of the topological (or metric) spaces. When we want to study the convergence of objects belonging to sets that have a vector space structure, as is often the case with functional spaces, we can consider a topology defined by a *norm*.

We assume here that $E$ is a vector space over the field $\mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$.

**Definition 1.31 (Norm).** A *norm on* $E$ is a function $N\colon E \to \mathbf{R}_+$ that satisfies the following conditions:

(N1) $\forall x \in E,\ (N(x) = 0 \Rightarrow x = 0)$;
(N2) $\forall x \in E, \forall \alpha \in \mathbf{K},\ N(\alpha x) = |\alpha|\ N(x)$;
(N3) $\forall x, y \in E,\ N(x + y) \leq N(x) + N(y)$.

**Property 1.32.** We have at once

- ⋆ $N(0) = 0$ (take $\alpha = 0$ in (ii)),
- ⋆ and for $x, y \in E$, $|N(x) - N(y)| \leq N(x - y)$ (apply (iii) to $x - y$ and $y$).

**Example.** ⋆ For all $d \in \mathbf{N}^*$, the functions $N_1$, $N_2$ and $N_\infty$, defined by the following,
for all $x \in \mathbf{K}^d$,

$$N_1(x) = \sum_{i=1}^{d} |x_i|, \quad N_2(x) = \left( \sum_{i=1}^{d} |x_i|^2 \right)^{1/2}, \quad N_\infty(x) = \sup_{1 \leq i \leq d} |x_i|,$$

are norms on the vector space $\mathbf{K}^d$, which are usually denoted by $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$, respectively.
⋆ Consider the space $E$ of all continuous functions $f\colon [0,1] \to \mathbf{K}$. The functions $N_1$, $N_2$ and $N_\infty$, defined by the following, for all $f \in E$,

$$N_1(f) = \int_0^1 |f(t)|\, dt, \quad N_2(f) = \left( \int_0^1 |f(t)|^2\, dt \right)^{1/2},$$
$$N_\infty(f) = \sup_{t \in [0,1]} |f(t)|,$$

are norms on the vector space $E$.

**Bernard:** At last! I note that we see our first functional space. I confess that I am curious to see how the convergence of a sequence of functions in a normed functional space could improve what I already know about pointwise and uniform convergence.

### 1.4.2 *Topologies of normed spaces*

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**Definition 1.33.** A *normed vector space (or normed space) over* $\mathbf{K}$ is any pair $(E, N)$, where $E$ is a vector space over the field $\mathbf{K}$ and $N$ is a norm on $E$.

We often note that $N(x) = \|x\|_E$ or $\|x\|$, for $x \in E$.

**Definition 1.34.** Let $\big(E, \|\cdot\|\big)$ be a normed vector space.

The topology of the metric space $E$, equipped with the metric $(x, y) \mapsto \|x - y\|$, is called *the topology induced by the norm* $\|\cdot\|$.

The following properties come directly from the definition of the metric induced by a norm.

**Proposition 1.35.**

(i) *The metric induced by a norm is invariant by any translation.*
(ii) *The set of open (or closed) balls is invariant by any homothety.*
(iii) *The function $x \mapsto \|x\|$ is a Lipschitz continuous function of constant* 1, i.e.

$$\forall x, y \in E, \quad \big|\|x\| - \|y\|\big| \leq \|x - y\|.$$

**Bernard:** If I understand well, the open balls of $E$ are $\{y \in E : \|y - x\| < r\}$ for $x \in E$, $r > 0$. But since convergence relies on neighborhoods, I wonder if we can say something about them?

**Laurent:** There is at least one obvious property: it should be clear to you that $\mathcal{N}(x) = x + \mathcal{N}(0) = \{x + y; y \in \mathcal{N}(0)\}$ thanks to translation invariance.

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**Definition 1.36 (Banach space).** A *Banach space* is any normed vector space that is complete (for the topology induced by its norm).

**Example.** The spaces $\mathbf{R}$, $\mathbf{R}^d$ and $C\big([0, 1]\big)$ with the uniform convergence are Banach spaces.

**Bernard:** Wow. This is not precise enough for me. What are the norms that are natural for these spaces?

**Laurent:** Oh, come on, Bernard! Be faster, please. It is obvious what norms to put on $\mathbf{R}$ and $\mathbf{R}^d$.

**Bernard:** For $\mathbf{R}$, you mean the absolute value, I guess. We get the usual topology of $\mathbf{R}$ (as a metric space), which makes it complete.

In the finite-dimensional case, I would say that it is enough to have a look componentwise. So, $\mathbf{R}^d$ equipped with $N_\infty$ is complete.

And since all norms are equivalent in a finite-dimensional space, it is OK. So, any would do.

But what can we do with infinite-dimensional spaces?

**Laurent:** This is precisely the general case for functional spaces. You can understand by yourself what happens for $C([0,1])$. What could "the topology of the uniform convergence" mean, in your opinion?

**Bernard:** I can see that a sequence of functions that converges for $N_\infty$ converges pointwise as well...

**Laurent:** Your small world of pointwise convergence is very limited, Bernard. We are looking at the bigger picture here.

Consider the following fundamental result:

> **Lemma.** *A Cauchy sequence $(f_n)_{n\in\mathbf{N}}$ in $C([0,1],\mathbf{R})$ converges in $C([0,1],\mathbf{R})$ in the sense of the norm $N_\infty$ to a continuous function.*

You cannot claim that fact with your pointwise convergence.

**Bernard:** Let me try to prove it. We fix $\varepsilon > 0$ and $N \in \mathbf{N}$ such that $p, q \geq N$ implies $N_\infty(f_p - f_q) \leq \varepsilon$, i.e.

$$\forall p, q \geq N,\ \forall x \in [0,1], \quad |f_p(x) - f_q(x)| \leq \varepsilon. \tag{*}$$

I observe that for all $x$ fixed in $[0,1]$, $\big(f_n(x)\big)_{n\in\mathbf{N}}$ is a Cauchy sequence in $\mathbf{R}$, which is complete, so it converges to a limit that we denote by $f(x)$.

Let $q \to +\infty$ in (*); we get $\forall p \geq N$, $\forall x \in [0,1], |f_p(x) - f(x)| \leq \varepsilon$.

We fix $p \geq N$. Let $x_0 \in [0,1]$ and $\eta > 0$ such that $y \in B(x_0,\eta) \cap [0,1]$ implies $|f_p(x_0) - f_p(y)| \leq \varepsilon$ by the continuity of $f_p$. Then, for $x \in B(x_0,\eta) \cap [0,1]$,

$$|f(x_0) - f(x)| \leq |f(x_0) - f_p(x_0)| + |f_p(x_0) - f_p(x)| \\ + |f_p(x) - f(x)| \leq 3\varepsilon.$$

This proves that $f$ is continuous at $x_0$ and for all $x_0 \in [0,1]$.

I conclude that $(C([0,1]), N_\infty)$ is a Banach space.

**Laurent:** Bravo for this rigorous proof, and I am happy to see that you specified the norm that makes $C([0,1])$ complete.

The completeness property depends on the norm, and in an infinite-dimensional space, all the norms are not equivalent. Indeed, in the list of norms we mentioned previously, there are some that do not make $C([0,1])$ complete, e.g. $N_2$.

### 1.4.3 *Continuous linear maps*

**Bernard:** What next, Laurent? I guess we have to consider functions defined on these Banach spaces.

**Laurent:** In a vector space, the most natural functions to consider are the ones that respect the vector structure. What is your opinion about them?

**Bernard:** I would say that these are the linear maps. I saw characterizations of their continuity in finite-dimensional spaces in terms of the boundedness on the unit sphere. I don't recall the proof being based on the finiteness of the dimension, but I expect it is the same in infinite-dimensional spaces.

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Let $E$ and $F$ be two normed vector spaces over $\mathbf{K}$. We denote by:

- ⋆ $L(E,F)$ the set of linear maps from $E$ to $F$;
- ⋆ $\mathcal{L}(E,F)$ the set of continuous linear maps from $E$ to $F$;
- ⋆ $E' = \mathcal{L}(E,\mathbf{K})$ the set of continuous linear functionals on $E$, called *the continuous dual space of* $E$ (or, sometimes, *the topological dual space*).

**Theorem 1.37 (Characterization of continuous linear maps).** *Let $u \in L(E,F)$. The following assertions are equivalent:*

(i) *$u$ is continuous on $E$.*
(ii) *$u$ is continuous at $0$.*
(iii) *$u$ is bounded on the closed unit ball $\overline{B} = \{x \in E : \|x\| \le 1\}$.*
(iv) *$u$ is bounded on the unit sphere $S = \{x \in E : \|x\| = 1\}$.*

(v) *There exists a constant* $C > 0$ *such that* $\forall x \in E, \quad \|u(x)\| \leq C\,\|x\|$.

(vi) $u$ *is a Lipschitz function on* $E$.

(vii) $u$ *is uniformly continuous on* $E$.

**Proof.** $\star$ The implication (i)⇒(ii) is obvious since (ii) is a particular case of (i).

$\star$ In order to prove that (ii)⇒(iii), we assume that $u$ is continuous at 0.

Then, for $\varepsilon = 1$, there exists $\eta > 0$ such that $\|x\| \leq \eta \Rightarrow \|u(x)\| \leq 1$.

For all $x \in \overline{B}$, we have $\|\eta x\| \leq \eta$, and then $\|u(\eta x)\| \leq 1$.

We deduce that $\|u(x)\| \leq 1/\eta$, which gives (iii).

$\star$ Assertion (iv) is a particular case of (iii).

$\star$ In order to prove that (iv)⇒(v), we assume the existence of $C > 0$ such that $\forall x \in S, \|u(x)\| \leq C$.

For $x \in E \setminus \{0\}$, we have $\frac{x}{\|x\|} \in S$. Then, $\|u(\frac{x}{\|x\|})\| \leq C$, so $\|u(x)\| \leq C\,\|x\|$. Then, (v) holds (the case $x = 0$ is obvious).

$\star$ In order to prove that (v)⇒(vi), we note that

$$\forall x, y \in E, \quad \|u(x) - u(y)\| = \|u(x - y)\| \leq C\,\|x - y\|.$$

$\star$ The implications (vi)⇒(vii)⇒(i) are obvious. □

**Laurent:** I reckon you did not learn anything new with this proof. You were right about the fact that it does not rely on the dimension being finite or not.

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For any normed vector spaces $E$ and $F$, $\mathcal{L}(E, F)$ is a subspace of $L(E, F)$.

The function $u \mapsto \|u\| = \sup_{x \in B} \|u(x)\|$ is a norm on $\mathcal{L}(E, F)$.

If $E \neq \{0\}$, we have

$$\forall u \in \mathcal{L}(E, F), \quad \|u\| = \sup_{x \neq 0} \frac{\|u(x)\|}{\|x\|} = \sup_{x \in S} \|u(x)\|.$$

**Example.** Any linear map from a finite-dimensional vector space $E$ to any normed space $F$ is continuous.

**Laurent:** Recall what the compacts of a finite-dimensional vector space are.

**Bernard:** We saw that in $\mathbf{R}^d$, the compacts are the closed and bounded subsets. I guess that, by (continuous) isomorphism, it is the same for any space of dimension $d$. So, we can claim that, for any $d \geq 1$, the compacts of a $d$-dimensional vector space are exactly the closed and bounded subsets.

**Laurent:** So, you see that the unit sphere is compact.

**Bernard:** Oh, I see what you want me to say! I consider any norm $N$ on $F$ and any $x = \sum_{1 \leq k \leq d} x_k \, e_k$ in $E$ of dimension $d$, and for $u \colon E \to F$, I write

$$N(u(x)) \leq \sum_{k=1}^{d} |x_k| \, N(u(e_k)) \leq C \max_{1 \leq k \leq d} |x_k| = C \, N_\infty(x).$$

Then, I conclude that the map $u : E \to F$ is continuous for any norm $N$ on $F$, its dimension being finite or not.

One of the simplest examples of continuous mappings is to consider the equality case with $C = 1$ in Theorem 1.37(v). That leads to the concept of *isometry*.

**Definition 1.38 (Isometry).** Let $E$ and $F$ be two normed vector spaces.

An *isometry* is a linear map $u \colon E \to F$ that preserves the norm i.e. such that for all $x \in E$, $\|u(x)\|_F = \|x\|_E$.

Continuity is a very constraining property for a function. It is so constraining that its values on a set $E$ are completely determined by its values on a *dense subset* $D$, that is, such that its closure $\overline{D}$ is the whole space $E$.

Moreover, the following theorem claims that any continuous function defined on this dense subspace $D$ extends uniquely to a continuous function on the whole space $E = \overline{D}$.

**Theorem 1.39 (Extension of a linear map).** *Let $E$ be a Banach space and $F$ a normed space. Let $D$ be a subspace of $E$ such that $\overline{D} = E$ and $u \colon D \to F$ be a continuous linear map.*

*Then, $u$ admits a unique extension as a continuous linear map $\widetilde{u}\colon E \to F$.*

*Moreover, if $u\colon D \to F$ is an isometry, then $\widetilde{u}$ is also an isometry.*

**Bernard:** I understand easily that, any $x \in E$ being the limit of a sequence $(x_n)_{n\in\mathbf{N}}$ in the dense subspace $D$ of $E$, for any continuous function $u$ from the normed space $E$ to the normed space $F$, $u(x)$ is the limit of $u(x_n)$ as $n$ goes to $\infty$.

So, I see that $u(x)$ is completely determined by the values taken by $u$ on $D$, as stated before the theorem.

However, the fact that a continuous linear map from a dense subspace $D$ of $E$ can be extended to a continuous linear map on $E$ is unclear to me.

**Laurent:** But you need exactly the arguments you mentioned! You just need to use them rigorously to show that the mapping is uniquely determined, linear and continuous. You will even see how we use the Cauchy criterion in practice.

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**Proof.** Let $x \in E$. Since $D$ is dense in $E$, there exists a sequence $(x_n)_{n\in\mathbf{N}}$ in $D$ that converges to $x$ in $E$: $\|x_n - x\|_E \to 0$ as $n \to \infty$.

The continuity of $u$ implies the existence of a constant $C > 0$ such that for all $y \in D$, $\|u(y)\|_F \leq C\ \|y\|_E$.

For all $n, m \in \mathbf{N}$, we have

$$\|u(x_n) - u(x_m)\|_F = \|u(x_n - x_m)\|_F \leq C\ \|x_n - x_m\|_E.$$

Using the fact that $(x_n)_{n\in\mathbf{N}}$ is a Cauchy sequence, we deduce that $\big(u(x_n)\big)_{n\in\mathbf{N}}$ is a Cauchy sequence in the complete space $E$. Then, $\big(u(x_n)\big)_{n\in\mathbf{N}}$ converges to a limit $\ell_x \in F$.

Let us check that the limit $\ell_x$ does not depend on the sequence $(x_n)_{n\in\mathbf{N}}$.

Assume that $(y_n)_{n\in\mathbf{N}}$ is another sequence in $D$ that converges to $x$ in $E$. We define the sequence $(z_n)_{n\in\mathbf{N}}$ as follows: for all $n \in \mathbf{N}$, $z_{2n} = x_n$ and $z_{2n+1} = y_n$. The sequence $(z_n)_{n\in\mathbf{N}}$ also converges to $x$.

The sequence $\big(u(z_n)\big)_{n\in\mathbf{N}}$ is also a Cauchy sequence and then converges. The two subsequences $\big(u(x_n)\big)_{n\in\mathbf{N}}$ and $\big(u(y_n)\big)_{n\in\mathbf{N}}$ thus converge to the same limit $\ell_x$.

We can define $\widetilde{u}(x) = \ell_x$.

Let us check the properties satisfied by the function $\widetilde{u}$.

⋆ For all $x \in D$, we have $\widetilde{u}(x) = u(x)$ (it suffices to consider the constant sequence equal to $x$).

⋆ For $x, y \in E$ and $\alpha \in \mathbf{K}$, we consider a sequence $(x_n)_{n\in\mathbf{N}}$ in $D$ that converges to $x$ and a sequence $(y_n)_{n\in\mathbf{N}}$ that converges to $y$. The sequence $(x_n + \alpha y_n)_{n\in\mathbf{N}}$ converges to $x + \alpha y$ and

$$\begin{aligned}\widetilde{u}(x+\alpha y) &= \lim u(x_n + \alpha y_n) = \lim u(x_n) + \alpha \lim u(y_n)\\ &= \widetilde{u}(x) + \alpha \widetilde{u}(y).\end{aligned}$$

Then, $\widetilde{u}$ is a linear map.

⋆ For $x \in E$, we consider a sequence $(x_n)_{n\in\mathbf{N}}$ in $D$ that converges to $x$. Thanks to the triangle inequality, $\|x_n\|_E$ converges to $\|x\|_E$. Moreover, we have

$$\big|\|u(x_n)\|_F - \|\widetilde{u}(x)\|_F\big| \leq \|u(x_n) - \widetilde{u}(x)\|_F \to 0 \quad \text{as } n \to \infty.$$

Then, $\|u(x_n)\|_F$ converges to $\|\widetilde{u}(x)\|_F$.

The continuity of $u$ on $D$ implies that $\|u(x_n)\|_F \leq C\,\|x_n\|_E$ for all $n \in \mathbf{N}$. Taking $n \to \infty$ in this inequality, we get $\|\widetilde{u}(x)\|_F \leq C\,\|x\|_E$. Then, $\widetilde{u}$ is continuous.

In the case where $u$ is an isometry, we write $\|u(x_n)\|_F = \|x_n\|_E$ for all $n \in \mathbf{N}$.

Taking $n \to \infty$, we get $\|\widetilde{u}(x)\|_F = \|x\|_E$. □

## 1.5 Hilbert Spaces

**Laurent:** Normally, you would have realized that the topological issues for the functional spaces (usually of infinite-dimensional spaces) are not addressed in the same way as for $\mathbf{R}^d$ (of finite-dimensional spaces). One can sometimes find some likeness of the topological properties in the two cases.

It may appear for some particular norm of a functional space, but, as two norms are usually not equivalent, these properties can disappear when changing the norm.

There is a particular case of the normed vector space that behaves closely to a finite-dimensional vector space: that is when the norm comes from an inner product.

### 1.5.1 *Inner products and normed vector spaces*

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In the sequel, we denote by $\mathcal{H}$ a vector space over the field $\mathbf{C}$ or $\mathbf{R}$.

**Definition 1.40 (Inner product).** We call *complex (resp., real) inner product* any function $\mathcal{H} \times \mathcal{H} \to \mathbf{C}$ (resp., $\mathbf{R}$), denoted by $(x, y) \mapsto \langle x, y \rangle$ such that:

(i) for all $y \in \mathcal{H}$, the function $x \mapsto \langle x, y \rangle$ is a linear map;
(ii) for all $x, y \in \mathcal{H}$, $\langle x, y \rangle = \overline{\langle y, x \rangle}$ in $\mathbf{C}$ (resp. $\langle x, y \rangle = \langle y, x \rangle$ in $\mathbf{R}$);
(iii) for all $x \in \mathcal{H}$, $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.

**Laurent:** In the complex case, the complicated points arise from the fact that the inner product is neither symmetric nor linear with respect to the second variable: for $x, y, z \in \mathcal{H}$ and $\alpha \in \mathbf{C}$, we have

$$\langle x, y + \alpha z \rangle = \overline{\langle y + \alpha z, x \rangle} = \overline{\langle y, x \rangle} + \overline{\alpha \langle z, x \rangle} = \langle x, y \rangle + \overline{\alpha} \langle x, z \rangle.$$

**Bernard:** But in the real case, the inner product is bilinear, right? So, it behaves like the Euclidean inner product I am familiar with in finite-dimensional spaces. Anyway. I guess that, from that definition, we could construct a norm, as in the Euclidean case.

**Laurent:** If you remember very well the finite-dimensional case, the key argument for the triangle inequality of such a norm is the Cauchy–Schwarz inequality.

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**Theorem 1.41 (Cauchy–Schwarz inequality).** *Given the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$, for any $x, y \in \mathcal{H}$, we have $|\langle x, y\rangle| \leq \sqrt{\langle x, x\rangle}\,\sqrt{\langle y, y\rangle}$.*

**Proof.** We set $\langle x, y\rangle = \rho e^{i\theta}$, and we study the function $t \mapsto \langle x + te^{i\theta}y, x + te^{i\theta}y\rangle$:

$$\begin{aligned}\langle x + te^{i\theta}y, x + te^{i\theta}y\rangle &= \langle x, x\rangle + te^{-i\theta}\langle x, y\rangle + te^{i\theta}\langle y, x\rangle + t^2\langle y, y\rangle \\ &= \langle x, x\rangle + 2\rho t + t^2\langle y, y\rangle.\end{aligned}$$

In order to ensure that this quadratic expression (in $t$) only takes nonnegative values, the discriminant needs to be nonpositive.

This gives $\rho^2 \leq \langle x, x\rangle\ \langle y, y\rangle$, and the result follows. □

**Proposition 1.42 (Norm/topology induced by an inner product).** *Given the inner product $\langle \cdot, \cdot\rangle$ on $\mathcal{H}$, we define $\|x\| = \sqrt{\langle x, x\rangle}$ for all $x \in \mathcal{H}$.*

*Then, the function $x \mapsto \|x\|$ is a norm on $\mathcal{H}$. The topology induced by this norm is called the topology induced by the inner product.*

**Proof.** We check that $\|\cdot\|$ satisfies the three conditions of a norm:

⋆ From Condition (iii) of Definition 1.40, we have $\|x\| \geq 0$ for all $x \in \mathcal{H}$.
  Moreover, $\|x\| = 0$ if and only if $x = 0$.

⋆ For all $\alpha \in \mathbf{C}$ and all $x \in \mathcal{H}$,

$$\|\alpha x\| = \sqrt{\langle \alpha x, \alpha x\rangle} = \sqrt{\alpha\langle x, \alpha x\rangle} = \sqrt{\alpha\overline{\alpha}\langle x, x\rangle} = |\alpha|\,\|x\|.$$

⋆ For the triangle inequality, we consider $x, y \in \mathcal{H}$, and we write

$$\begin{aligned}\|x + y\|^2 = \langle x + y, y + y\rangle &= \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle \\ &= \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}(\langle x, y\rangle).\end{aligned}$$

  By the Cauchy–Schwarz inequality, we have

$$\mathrm{Re}(\langle x, y\rangle) \leq |\langle x, y\rangle| \leq \|x\|\,\|y\|.$$

  We deduce that: $\|x+y\|^2 \leq \|x\|^2+\|y\|^2+2\,\|x\|\,\|y\| = (\|x\|+\|y\|)^2$.

□

**Definition 1.43 (Hilbert space).** Any vector space $\mathcal{H}$ that is complete for the topology induced by its inner product is called *a Hilbert space.*

**Bernard:** Here comes the completeness again...

**Laurent:** There are really more complex spaces than your $\mathbf{R}^d$ from your *classes préparatoires*... Completeness did not appear in your finite-dimensional spaces like $\mathbf{R}^d$ since all of them are complete.

The completeness condition is not artificial; it is a property that constrains the topologies and proposes representations that look like their counterparts in finite-dimensional vector spaces.

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**Example.** ⋆ Any Euclidean vector space is a Hilbert space, e.g. the space $\mathbf{C}^d$ equipped with the inner product $\langle \cdot, \cdot \rangle$, defined by $\forall x, y \in \mathbf{C}^d,\ \langle x, y \rangle = \sum_{n=1}^{d} x_n \overline{y_n}$.
⋆ We consider the set

$$\ell_0(\mathbf{N}) = \{(u_n)_{n\in\mathbf{N}} \in \mathbf{C}^{\mathbf{N}} : u_n = 0 \text{ except for a finite number of values of } n\}.$$

The set $\ell_0(\mathbf{N})$ is a vector space over $\mathbf{C}$, and we can define

$$\forall u, v \in \ell_0(\mathbf{N}), \quad \langle u, v \rangle = \sum_{n=0}^{\infty} u_n \overline{v_n}.$$

We now define $\ell^2(\mathbf{N})$ the completion of the (metric) space $\ell_0(\mathbf{N})$ such that $\ell^2(\mathbf{N}) = \overline{\ell_0(\mathbf{N})}$. We obtain in that way a Hilbert space.

We can observe that $\ell^2(\mathbf{N}) = \{u \in \mathbf{C}^{\mathbf{N}} : \langle u, u \rangle < \infty\}$.

**Bernard:** Something is really unclear to me. I am uneasy with writing $\ell^2(\mathbf{N}) = \overline{\ell_0(\mathbf{N})}$. In my understanding, it assumes that $\ell^2(\mathbf{N})$ exists on its own with a topology and that the closure $\overline{\ell_0(\mathbf{N})}$ is defined for this topology of $\ell^2(\mathbf{N})$.

To sum up, $\ell^2(\mathbf{N}) = \overline{\ell_0(\mathbf{N})}$ cannot be the definition of $\ell^2(\mathbf{N})$ and its topology.

**Laurent:** You are absolutely right, Bernard. It is not a justification but you already encountered this when we sum up the fact that $\mathbf{R}$ is the completion of $\mathbf{Q}$ with the relation $\mathbf{R} = \overline{\mathbf{Q}}$.

Rigorously, you know that $\mathbf{R}$ is constructed independently, as the quotient space of the Cauchy sequences in $\mathbf{Q}$ by the equivalence relation $(u_n)_{n\in\mathbf{N}} \sim (v_n)_{n\in\mathbf{N}}$, defined by $|u_n - v_n| \to 0$ as $n \to \infty$.

The construction of $\ell^2(\mathbf{N})$ derives from the same procedure, substituting $\mathbf{Q}$ with $\ell_0(\mathbf{N})$.

**Bernard:** Many thanks, Laurent. At last, I understand how $\mathbf{R}$ is defined, and the procedure to embed $\mathbf{Q}$ in a larger complete topological space called $\mathbf{R}$ is very general. I admit I hadn't understood it in my *classes préparatoires.*

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In addition to the Cauchy–Schwarz inequality, here are some geometric properties linking the inner product and the norm. In particular, the following Proposition 1.44(iv) shows that the inner product is characterized by the norm induced by it.

**Proposition 1.44.** *Denoting by $\|.\|$ the norm induced by the inner product $\langle .,.\rangle$ on the Hilbert space $\mathcal{H}$, the following assertions hold for all $x, y \in \mathcal{H}$:*

(i) $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}(\langle x, y\rangle)$ (*Pythagoras relation*);
(ii) $\|x + y\| \leq \|x\| + \|y\|$ (*Minkowski inequality*);
(iii) $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$ (*parallelogram law*);
(iv) $\langle x, y\rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2)$ (*polarization identity*).

When $\mathcal{H}$ is a Hilbert space over the field $\mathbf{R}$, endowed with the real inner product $\langle .,.\rangle : \mathcal{H} \times \mathcal{H} \to \mathbf{R}$, the polarization identity (Property (iv)) reads

(v) $\langle x, y\rangle = \dfrac{1}{4}\big(\|x + y\|^2 - \|x - y\|^2\big)$.

**Proof.** (i) For the Pythagoras relation, we write

$$\begin{aligned}\|x + y\|^2 = \langle x + y, x + y\rangle &= \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2 \\ &= \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}(\langle x, y\rangle).\end{aligned}$$

(ii) This is the triangle inequality of the norm induced by an inner product.

(iii) It suffices to use the Pythagoras relations $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}(\langle x, y\rangle)$ and $\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\,\mathrm{Re}(\langle x, y\rangle)$.

(iv) We also have

$$\begin{aligned}\|x+iy\|^2 &= \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}(\langle x, iy\rangle) \\ &= \|x\|^2 + \|y\|^2 + 2\,\mathrm{Im}(\langle x, y\rangle),\end{aligned}$$

and $\|x-iy\|^2 = \|x\|^2 + \|y\|^2 - 2\,\mathrm{Im}(\langle x, y\rangle)$. The result follows from these relations and the two relations in (iv) above. □

**Remark.** These properties are true only for norms that are induced by an inner product. Besides, it is important to note that not all norms allow us to define an inner product.

**Bernard:** This last remark is a bit mysterious to me. What does it mean?

**Laurent:** Well, for any norm, you can define the function

$$(x, y) \mapsto \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right).$$

But, in general, the defined function will not satisfy the conditions of Definition 1.40 and will not be an inner product.

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The results concerning orthogonality in a Euclidean space like $\mathbf{R}^d$ can be extended quite naturally to Hilbert spaces, even if they are $\mathbf{C}$-vector spaces.

**Definition 1.45 (Orthogonal subspace).** Let $F$ be a vector subspace of the Hilbert space $\mathcal{H}$. We define the *orthogonal of* $F$ as the closed vector subspace of $\mathcal{H}$: $F^\perp = \{x \in \mathcal{H} : \forall y \in F,\ \langle x, y\rangle = 0\}$.

**Justification.** ⋆ We first check that $F^\perp$ is a vector subspace of $\mathcal{H}$.

For all $u, v \in F^\perp$ and all $\alpha \in \mathbf{C}$, we have

$$\forall y \in F, \quad \langle u + \alpha v, y\rangle = \langle u, y\rangle + \alpha\langle v, y\rangle = 0,$$

which implies that $u + \alpha v \in F^\perp$.

⋆ In order to show that $F^{\perp}$ is closed, we consider any sequence $(x_n)_{n\in\mathbf{N}}$ in $F^{\perp}$ that converges in $\mathcal{H}$, and we prove that the limit is in $F^{\perp}$.

For all $n \in \mathbf{N}$ and all $y \in F$, we have $\langle x_n, y\rangle = 0$.

For $y$ fixed in $F$, the function $x \mapsto \langle x, y\rangle$ is continuous (as a direct consequence of the Cauchy–Schwarz inequality). This implies that $\langle \lim x_n, y\rangle = \lim\langle x_n, y\rangle = 0$.

Then, $\lim x_n \in F^{\perp}$ and $F^{\perp}$ is closed. □

**Bernard:** The definition doesn't seem to require the completeness property.

**Laurent:** For the sole definition of the orthogonal of a subspace, you are right. But we will need the orthogonal projection to define the orthogonal supplementary. And you will see that this projection cannot avoid the completeness properties.

Moreover, the fact that the orthogonal is closed is important. This point is clear in finite-dimensional spaces, but in infinite-dimensional spaces...

**Bernard:** I know! In $\mathbf{R}^d$ and, more generally, in any $d$-dimensional $\mathbf{R}$-vector space, all vector subspaces are closed. I don't know what happens in the infinite-dimensional case. I guess that my pharmaceutical plant issue is not finite-dimensional, is it?

**Laurent:** Before discretizing, your problem of finding the admissible density of particles $N(t, \mathbf{x})$ with respect to the pollutant injection $f(t, \mathbf{x})$ is infinite-dimensional. So, you cannot avoid these questions of completeness.

### 1.5.2 *Some remarkable linear maps on Hilbert spaces*

**Laurent:** Before examining very general cases of linear maps on Hilbert spaces, let us study the extension to the infinite-dimensional case of two geometric mappings: isometries and orthogonal projections.

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**1 – Isometry**

Recall that for two normed spaces $E$ and $F$, an isometry is a linear map $u\colon E \to F$ that preserves the norm, i.e. such that $\forall x \in E,\ \|u(x)\|_F = \|x\|_E$.

In the specific case of norms deriving from inner products, isometries can be defined as maps that preserve the inner product. The following result is a direct consequence of Proposition 1.44, Condition (v).

**Proposition 1.46.** *Let $(E, \langle .,.\rangle_E)$ and $(F, \langle .,.\rangle_F)$ be two Hilbert spaces.*

*Any isometry $u\colon E \to F$ preserves the inner product:*

$$\forall x, y \in E, \quad \langle u(x), u(y)\rangle_F = \langle x, y\rangle_E. \tag{1.5.1}$$

We can observe that the linearity of $u$ is a consequence of Relation (1.5.1). Actually, if $u\colon E \to F$ is a function that preserves the inner product, then $u$ is an isometry.

**Bernard:** Let me check this latter assertion. Consider $x, y \in \mathcal{H}$ and $\alpha \in \mathbf{C}$. In order to prove that $u(x+\alpha y) - u(x) - \alpha u(y) = 0$, I compute

$$\begin{aligned}
&\|u(x+\alpha y) - u(x) - \alpha u(y)\|^2 \\
&\quad = \|u(x+\alpha y)\|^2 + \|u(x)\|^2 + |\alpha|^2\|u(y)\|^2 \\
&\qquad - 2\operatorname{Re}\big(\langle u(x+\alpha y), u(x)\rangle\big) - 2\operatorname{Re}\big(\langle u(x+\alpha y), \alpha u(y)\rangle\big) \\
&\qquad + 2\operatorname{Re}\big(\langle u(x), \alpha u(y)\rangle\big) \\
&\quad = \|x+\alpha y\|^2 + \|x\|^2 + |\alpha|^2\|y\|^2 - 2\operatorname{Re}\big(\langle x+\alpha y, x\rangle \\
&\qquad + \overline{\alpha}\langle x+\alpha y, y\rangle - \overline{\alpha}\langle x, y\rangle\big) \\
&\quad = \|(x+\alpha y) - x - \alpha y\|^2 = 0.
\end{aligned}$$

So, I conclude indeed that a mapping that preserves the inner product, which is the case of an isometry between two Hilbert spaces, is necessarily linear.

**2 – Orthogonal projection**

**Definition 1.47 (Convex subset).** A subset $C$ of a vector space $E$ is said to be *convex* if $\forall x, y \in C$, $\forall t \in [0,1],\ tx + (1-t)y \in C$.

Recall that for $x \in E$ and $F$ being a nonempty subset of $E$, we note

$$d(x, F) = \inf_{y \in F} d(x, y) = \inf_{y \in F} \|x - y\|.$$

**Laurent:** Let us extend to the infinite-dimensional spaces what we know very well about projection infinite-dimensional spaces.

**Theorem 1.48 (Projection onto closed convex sets).** *Let $\mathcal{H}$ be a Hilbert space and $C \subset \mathcal{H}$ be a nonempty closed convex subset.*

*Then, for $x \in \mathcal{H}$, there exists a unique $x_0 \in C$ such that $d(x, C) = \|x - x_0\|$.*

*We note that $x_0 = p_C(x)$, which is called* the projection of $x$ onto $C$.

**Laurent:** A little more could be said about the projection onto closed convex sets. I am thinking about the fact that $p_C(x)$ is the only element of $C$ such that $\mathrm{Re}\big(\langle x - p_C(x), y - p_C(x)\rangle\big) \geq 0$ for all $y \in C$. But maybe we will not need that characterization here. So, I will leave it at that.

**Proof.** ⋆ By the definition of the infimum, we can construct a sequence $(y_n)_{n\in\mathbf{N}}$ in $C$ such that $d(x, y_n) = \|x - y_n\|$ converges to $d = d(x, C) = \inf_{y\in C} \|x - y\|$, as $n \to \infty$.

Let us prove that $(y_n)_{n\in\mathbf{N}}$ is a Cauchy sequence.

For all $n, m \in \mathbf{N}$, the parallelogram law (Proposition 1.44(iii)) implies

$$\begin{aligned}\|y_n - y_m\|^2 &= \|(y_n - x) - (y_m - x)\|^2 \\ &= -\|(y_n - x) + (y_m - x)\|^2 + 2\|y_n - x\|^2 + 2\|y_m - x\|^2 \\ &= -4\left\|\frac{y_n + y_m}{2} - x\right\|^2 + 2\|y_n - x\|^2 + 2\|y_m - x\|^2.\end{aligned}$$

By convexity, $\frac{y_n+y_m}{2} \in C$. This implies $\|\frac{y_n+y_m}{2} - x\| \geq d$.

For all $\varepsilon > 0$, there exists $N \in \mathbf{N}$, such that $\forall n \in \mathbf{N}$, $n \geq N \Rightarrow \|y_n - x\|^2 \leq d^2 + \varepsilon/4$. Thus, for all $n, m \in \mathbf{N}$, $n \geq N$ and $m \geq N$,

$$\|y_n - y_m\|^2 \leq -4d^2 + 2\left(d^2 + \frac{\varepsilon}{4}\right) + 2\left(d^2 + \frac{\varepsilon}{4}\right) = \varepsilon.$$

Thus, $(y_n)_{n \in \mathbf{N}}$ is a Cauchy sequence.

Since $\mathcal{H}$ is complete, $(y_n)_{n \in \mathbf{N}}$ converges to some $y$ in $\mathcal{H}$. And since $C$ is closed, $y \in C$.

Using the continuity of the function $z \mapsto \|z\|$, we deduce that $\|y_n - x\|$ converges to $\|y - x\|$ as $n \to \infty$. Then, $\|y - x\| = d(x, C)$.

$\star$ Let us prove that $x_0$ is unique.

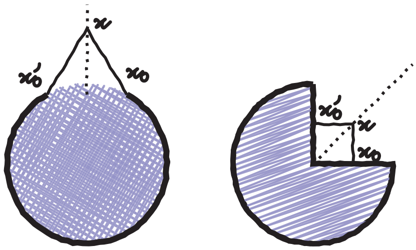
Assume that $x_0$ and $x_0'$ are in $C$ and such that $d(x_0, x) = d(x_0', x) = d$. We have

$$\begin{aligned}\|x_0 - x_0'\|^2 &= -4\left\|\frac{x_0 + x_0'}{2} - x\right\|^2 + 2\|x_0 - x\|^2 + 2\|x_0' - x\|^2 \\ &\leq -4d^2 + 2d^2 + 2d^2 = 0.\end{aligned}$$

Then, $x_0 = x_0'$. □

**Remark.** The two aspects – $C$ closed and $C$ convex – are necessary to prove this result. The existence is ensured by the closedness of $C$ and the uniqueness by the convexity.

**Bernard:** OK, Laurent. I can see that this proof doesn't work if the set $C$ isn't convex or closed. Now, I really see the importance of these properties!




**Laurent:** In fact, it is not only the proof that becomes false without these properties. The result itself becomes false.

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**Corollary 1.49 (Orthogonal projection).** *If $F$ is a closed vector subspace, then $p_F$ is a continuous linear map that satisfies $x - p_F(x) \in F^\perp$ for all $x \in \mathcal{H}$,* i.e.

$$\forall x \in \mathcal{H},\ \forall y \in F, \quad \langle x - p_F(x), y\rangle = 0.$$

**Proof.** $\star$ The subspace $F$ is obviously convex. Then, Theorem 1.48 can be applied to the closed and convex subset $F$.

For all $y \in F$, $\alpha \in \mathbf{C}$ and $t \in \mathbf{R}$, we have $p_F(x) + t\,\alpha\,y \in F$ and

$$\begin{aligned} &\|x - (p_F(x) + t\,\alpha\,y)\|^2 \\ &\quad = \|x - p_F(x)\|^2 - 2t\,\mathrm{Re}\big(\langle x - p_F(x), \alpha\,y\rangle\big) + t^2\,|\alpha|^2\|y\|^2. \end{aligned}$$

By definition of $p_F(x)$, the infimum of this quadratic expression in $t$ (for any fixed $y \in F$ and $\alpha \in \mathbf{C}$) should be reached at $t = 0$. This implies $\mathrm{Re}\big(\langle x - p_F(x), \alpha\,y\rangle\big) = 0$.

Taking the particular cases of $\alpha = 1$ and $\alpha = i$ (with $\mathrm{Re}(iz) = -\mathrm{Im}(z)$), we get $\mathrm{Re}\big(\langle x - p_F(x), y\rangle\big) = 0$ and $\mathrm{Im}\big(\langle x - p_F(x), y\rangle\big) = 0$.

$\star$ For all $x \in \mathcal{H}$ and $\alpha \in \mathbf{C}$, we have

$$\begin{aligned} &\|p_F(\alpha\,x) - \alpha\,p_F(x)\|^2 \\ &\quad = \|p_F(\alpha\,x) - \alpha\,x\|^2 + 2\langle p_F(\alpha\,x) - \alpha\,x, \alpha\,x - \alpha\,p_F(x)\rangle \\ &\qquad + \alpha^2\|x - p_F(x)\|^2. \end{aligned}$$

However, since $p_F(\alpha\, x)$ and $p_F(x)$ belong to $F$ and $p_F(\alpha\, x) - \alpha\, x$ belongs to $F^\perp$,

$$\langle p_F(\alpha\, x) - \alpha\, x, -\alpha\, p_F(x)\rangle = 0 = \langle p_F(\alpha\, x) - \alpha\, x, -p_F(\alpha\, x)\rangle$$

and, similarly,

$$\langle p_F(\alpha\, x), \alpha\, x - \alpha\, p_F(x)\rangle = 0 = \langle p_F(x), \alpha\, x - \alpha\, p_F(x)\rangle,$$

so

$$\begin{aligned} &2\langle p_F(\alpha\, x) - \alpha\, x, \alpha\, x - \alpha\, p_F(x)\rangle \\ &\quad = \langle p_F(\alpha\, x) - \alpha\, x, \alpha\, x - p_F(\alpha\, x)\rangle \\ &\qquad + \langle \alpha\, p_F(x) - \alpha\, x, \alpha\, x - \alpha\, p_F(x)\rangle. \end{aligned}$$

We conclude that $\|p_F(\alpha\, x) - \alpha\, p_F(x)\|^2 = 0$.

For all $x, z \in \mathcal{H}$, we have

$$\begin{aligned} &\|p_F(x+z) - p_F(x) - p_F(z)\|^2 \\ &\quad = \|p_F(x+z) - x - z\|^2 + 2\langle p_F(x+z) - x - z, \\ &\qquad x + z - p_F(x) - p_F(z)\rangle + \|x + z - p_F(x) - p_F(z)\|^2. \end{aligned}$$

Using again orthogonality, we note that

$$\begin{aligned} &\langle p_F(x+z) - x - z, p_F(x) + p_F(z)\rangle \\ &\quad = 0 = \langle p_F(x+z) - x - z, p_F(x+z)\rangle \\ &\langle p_F(x+z), x + z - p_F(x) - p_F(z)\rangle \\ &\quad = 0 = \langle p_F(x) + p_F(z), x + z - p_F(x) - p_F(z)\rangle. \end{aligned}$$

We conclude that $\|p_F(x+z) - p_F(x) - p_F(z)\|^2 = 0$.

Thus, $p_F$ is a linear map.

$\star$ For all $x$ in $\mathcal{H}$, we have, thanks to the Cauchy–Schwarz inequality,

$$\begin{aligned} \|p_F(x)\|^2 &= \langle p_F(x) - x, p_F(x)\rangle + \langle x, p_F(x)\rangle \\ &= \langle x, p_F(x)\rangle \le \|x\|\, \|p_F(x)\|. \end{aligned}$$

We conclude that $p_F$, being linear, is continuous. $\square$

**Proposition 1.50 (Orthogonal complement).** *Let $F$ be a closed linear subspace of $\mathcal{H}$. Then, $F \oplus F^{\perp} = \mathcal{H}$.*

**Proof.** Let $x \in \mathcal{H}$. We decompose $x$ into the form $x = p_F(x) + (x - p_F(x))$, with $p_F(x) \in F$ and $x - p_F(x) \in F^{\perp}$. This shows that $F + F^{\perp} = \mathcal{H}$.

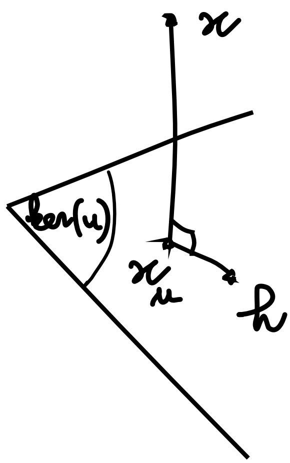
In order to prove that this sum is direct, we check that $F \cap F^{\perp} = \{0\}$.

For $y \in F \cap F^{\perp}$, we have $\langle y, y \rangle = 0$, which implies $y = 0$. □

**Laurent:** I know that you know it, Bernard, but I will tell it all the same: the word "complement" does not mean here complement in the sense of the set theory. Formally, it is written as $F^{\perp} \neq \mathcal{H} \setminus F$.

### 1.5.3 *Representation of continuous linear functionals on Hilbert spaces*

**Laurent:** You remember that in $\mathbf{R}^d$, any linear functional $u$ can be written in the form $u(x) = \langle x_u, x \rangle$ for a unique $x_u \in \mathbf{R}^d$ and that $\ker(u)$ is a hyperplane of $\mathbf{R}^d = \ker(u) \oplus \mathbf{R} \cdot x_u$?




**Bernard:** Yes, I remember. It allowed us to identify the space of linear functionals over $\mathbf{R}^d$ to the space $\mathbf{R}^d$ itself.

**Laurent:** Actually, this *duality property* also occurs in the infinite-dimensional case as long as the orthogonal projection can be defined, that is, if $u$ is continuous. The proof clearly relies on the orthogonal projection being well defined and continuous.

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Recall that we denote by $\mathcal{H}' = \mathcal{L}(\mathcal{H}, \mathbf{C})$ the space of continuous linear functionals (also called linear forms) on $\mathcal{H}$.

**Theorem 1.51 (The Riesz–Fréchet representation theorem).** *Let $\mathcal{H}$ be a Hilbert space and $u \in \mathcal{H}'$. Then, there exists a unique $x_u \in \mathcal{H}$ such that*

$$\forall x \in \mathcal{H}, \quad u(x) = \langle x, x_u \rangle.$$

**Laurent:** Note that $\mathcal{H}' \cong \mathcal{H}$ since $u \mapsto x_u$ is an isometry. Actually, $(u, v) \mapsto \langle x_u, x_v \rangle$ is the inner product on $\mathcal{H}' \times \mathcal{H}'$ associated with the induced norm $\mathcal{H}'$. In a word,: $\mathcal{H}'$ can be identified with $\mathcal{H}$.

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**Proof.** The result is obvious in the case $u = 0$. Now, we assume that $u \neq 0$.

$\star$ Set $F = \ker u = u^{-1}(\{0\})$. Since $u$ is continuous, $F$ is closed. Then, according to Proposition 1.50, we have $F \oplus F^{\perp} = \mathcal{H}$.

Since $u \neq 0$, we have $F \subsetneq \mathcal{H}$. Then, there exists $\widetilde{x_u}$ in $F^{\perp}$ such that $\|\widetilde{x_u}\| = 1$.

For all $x \in \mathcal{H}$, we have $u(x - \frac{u(x)}{u(\widetilde{x_u})} \cdot \widetilde{x_u}) = 0$, i.e. $x - \frac{u(x)}{u(\widetilde{x_u})} \cdot \widetilde{x_u} \in F$.

Since $\mathcal{H} = F \oplus F^{\perp}$, we can claim that the decomposition of $x$ on $F \oplus F^{\perp}$,

$$x = \left( x - \frac{u(x)}{u(\widetilde{x_u})} \cdot \widetilde{x_u} \right) + \frac{u(x)}{u(\widetilde{x_u})} \cdot \widetilde{x_u},$$

is unique and then $F^{\perp} = \mathbf{C} \cdot \widetilde{x_u}$.

Let us define $x_u = \overline{u(\widetilde{x_u})}\ \widetilde{x_u}$. We have

$$\begin{aligned}\forall x \in \mathcal{H}, \quad \langle x, x_u \rangle &= u(\widetilde{x_u})\ \langle x, \widetilde{x_u} \rangle \\ &= u(\widetilde{x_u})\ \left\langle x - \frac{u(x)}{u(\widetilde{x_u})} \cdot \widetilde{x_u} + \frac{u(x)}{u(\widetilde{x_u})} \cdot \widetilde{x_u}, \widetilde{x_u} \right\rangle \\ &= u(x).\end{aligned}$$

⋆ We now prove the uniqueness of such a $x_u$.

If $x_u$ and $x'_u$ are such that

$$\forall x \in \mathcal{H}, \quad u(x) = \langle x, x_u \rangle = \langle x, x'_u \rangle,$$

then, for all $x \in \mathcal{H}$, $\langle x, x_u - x'_u \rangle = 0$. We conclude that $x_u = x'_u$. □

**Bernard:** What a generalization... With respect to what I knew, you only changed **R** into **C**!

**Laurent:** What? You cannot be serious! You think you know everything, Bernard. Maybe your teacher of *classes préparatoires* put that idea into your mind.

Let me show you an extension to bilinear forms that you have no idea of. Note that the inner product is a very special bilinear form...

### 1.5.4 *Extension of the Riesz–Fréchet theorem*

Actually, one can go further by generalizing to bilinear forms instead of merely a representation by the inner product (which is obviously bilinear). For the sake of simplicity, in this section, the Hilbert space will be real. The extension to a complex Hilbert space is left as an exercise to the reader.

**Definition 1.52.** Let $E$ be a real vector space. A function $B \colon E \times E \to \mathbf{R}$ is said to be a *bilinear form* if:

⋆ for every $x \in E$, $y \mapsto B(x, y)$ is a linear functional,
⋆ for every $y \in E$, $x \mapsto B(x, y)$ is a linear functional.

A bilinear form is said to be *symmetric* if, for all $x, y \in E$, $B(x, y) = B(y, x)$.

**Remark.** Any inner product is a symmetric bilinear form. But we do not assume a bilinear form is necessarily symmetric!

**Proposition 1.53.** *A bilinear form $B$ over $E \times E$, where $(E, \|\cdot\|)$ is a normed real vector space, is continuous if and only if there exists* *some constant* $C > 0$ *such that*

$$\forall x, y \in E, \qquad |B(x,y)| \leq C\,\|x\|\,\|y\|.$$

*Such a constant* $C$ *is sometimes called a* continuity constant *of* $B$*.*

**Proof.** This follows directly from the definition of continuity in the vector space $E \times E$ equipped with the topology of the product. It suffices to prove that the continuity of $B$ is equivalent to the boundedness of $B$ on the unit ball

$$\{(x,y) \in E \times E : \max(\|x\|, \|y\|) \leq 1\},$$

and to write that, for all $x, y \in E \setminus \{0\}$, $B(x,y) = B\big(x/\|x\|, y/\|y\|\big)\,\|x\|\,\|y\|$ (see [Ramis *et al.* (1982), tome 2] for a detailed proof).

$\square$

**Definition 1.54.** A bilinear form $B$ over $E \times E$, where $(E, \|\cdot\|)$ is a normed real vector space, is said to be *coercive* if there exists $\alpha > 0$ such that

$$\forall x \in E, \qquad B(x,x) \geq \alpha\,\|x\|^2. \tag{1.5.2}$$

Such a constant $\alpha$ is sometimes called a *coercivity constant* of $B$.

**Remark.** Since $\alpha$ is positive, the inequality (1.5.2) implies that the quantity $B(x,x)$ is necessarily positive when $x \neq 0$.

**Bernard:** You pretend you generalize, but you did not replace the inner product of $\mathcal{H}$ with $B$ since the norm appears everywhere! It is confusing.

**Laurent:** Listen, Bernard. We want a result on a normed vector space, whose topology derives from a norm. The results we obtain concern bilinear forms that have some regularity properties: continuity and coercivity.

You should not be surprised that these topological properties are expressed in terms of the norm.

The following theorem is a generalized version of the Riesz–Fréchet theorem where the inner product is substituted with a (not necessarily symmetric) continuous and coercive bilinear form $B$.

**Theorem 1.55 (The Lax–Milgram theorem).** *Let $\mathcal{H}$ be a real Hilbert space and $B$ be a continuous and coercive bilinear form over $\mathcal{H} \times \mathcal{H}$ and $u \in \mathcal{H}'$.*

*Then, there exists one and only one element $x_u \in \mathcal{H}$ such that*

$$\forall y \in \mathcal{H}, \qquad u(y) = B(x_u, y). \tag{1.5.3}$$

*Furthermore, if $B$ is symmetric, $x_u$ is characterized as being the only minimizer of the functional $y \mapsto B(y, y) - 2u(y)$:*

$$B(x_u, x_u) - 2u(x_u) = \min \{B(y, y) - 2u(y);\ y \in \mathcal{H}\}.$$

**Laurent:** Did you note that, if $B$ is symmetric, $B$ is an inner product?

**Bernard:** Sure. And I can say even more: its associated norm $x \mapsto \sqrt{B(x, x)}$ is equivalent to the norm associated with the inner product of $\mathcal{H}$ since, for all $x \in \mathcal{H}$, $\alpha \|x\|^2 \leq B(x, x) \leq C\|x\|^2$.

**Laurent:** Great, Bernard!

**Proof.** Let $C$ be a continuity constant and $\alpha$ be a coercivity constant of $B$.

$\star$ Let us formulate the problem in another way.

Thanks to the Riesz–Fréchet theorem 1.51, there exists one and only one $x_u \in \mathcal{H}$ such that $\forall y \in \mathcal{H}$, $u(y) = \langle x_u, y\rangle$.

Let $x \in \mathcal{H}$. The map $y \in \mathcal{H} \mapsto B(x, y)$ is a continuous linear functional over $\mathcal{H}$, and thanks to the Riesz–Fréchet theorem 1.51, there exists a unique element $A(x)$ of $\mathcal{H}$ such that for all $y \in \mathcal{H}$, $B(x, y) = \langle A(x), y\rangle$.

The bilinearity of $B$ implies directly that $x \mapsto A(x)$ is a linear map from $\mathcal{H}$ to $\mathcal{H}$. Moreover, for all $x \in \mathcal{H}$,

$$\|A(x)\| \leq C\,\|x\| \qquad \text{and} \qquad \langle A(x), x\rangle \geq \alpha\,\|x\|^2.$$

We see at once from the first inequality that $x \mapsto A(x)$ is a continuous function and from the second inequality that it is injective.

Thus, finding a solution of (1.5.3) is equivalent to finding $x \in \mathcal{H}$ such that

$$\forall y \in \mathcal{H}, \qquad \langle A(x), y\rangle = \langle x_u, y\rangle, \tag{1.5.4}$$

which, in turn, is equivalent to showing that the map $A$ is surjective.

⋆ To do that, let us show that (i) $\mathrm{Im}(A)$ is closed and (ii) $\mathrm{Im}(A)^{\perp} = \{0\}$. Proposition 1.50 then allows us to conclude.

(i) Let $(x_n)_{n\geq 0}$ a sequence of elements of $\mathcal{H}$ such that $(A(x_n))_{n\geq 0}$ converges to $y \in \mathcal{H}$. Then, for all $p, q \in \mathbf{N}$, using the Cauchy–Schwarz inequality,

$$\begin{aligned} \alpha\,\|x_p - x_q\|^2 &\leq B(x_p - x_q, x_p - x_q) \\ = \langle A(x_p - x_q), x_p - x_q\rangle &\leq C\,\|A(x_p) - A(x_q)\|\,\|x_p - x_q\|. \end{aligned}$$

Since $(A(x_n))_{n\geq 0}$ is a Cauchy sequence, so is $(x_n)_{n\geq 0}$.

Thus, $\mathcal{H}$ being complete, there exists $x \in \mathcal{H}$ such that $x_n \to x$ as $n \to \infty$.

So, by the continuity of $A$, $A(x) = y$. This shows that $\mathrm{Im}(A)$ is a closed subspace of $\mathcal{H}$.

(ii) Let $z$ be in $\mathrm{Im}(A)^{\perp}$. Then, for all $x \in \mathcal{H}$, $0 = \langle A(x), z\rangle = B(x, z)$.

For $x = z$, we have $B(z, z) = 0$, which implies that $z = 0$, with $B$ being coercive.

So, $\mathrm{Im}(A)^{\perp} = \{0\}$.

$\star$ Assume that $B$ is symmetric. Let $J\colon y \mapsto B(y,y) - 2u(y)$.

Assume that $x$ solves (1.5.3). Let $y \in \mathcal{H}$ and $t \in \mathbf{R}$. Then,

$$J(x+ty) - J(x) = t^2 B(y,y) + 2t\big(B(x,y) - u(y)\big) = t^2 B(y,y) \geq 0,$$

so $J(x)$ is the minimum of $J$ over $\mathcal{H}$.

Conversely, let $x$ be such that for all $y \in \mathcal{H}$, $J(y) \geq J(x)$.

Then, for all $t \in \mathbf{R}$, $t^2 B(y,y) + 2t\big(B(x,y) - u(y)\big) \geq 0$. Hence, if $t > 0$ and $t \to 0^+$, $B(x,y) - u(y) \geq 0$.

In the same way, taking $t < 0$ and $t \to 0^-$, we get $B(x,y) - u(y) \leq 0$.

Then, we conclude that $B(x,y) - u(y) = 0$. $\square$

### 1.5.5 *Separability and Hilbert bases*

**Laurent:** Mark my words, Bernard: there is essentially one and only one infinite-dimensional Hilbert space. Since the only one you know is $\ell^2(\mathbf{N})$, it is the one.

**Bernard:** I have to admit it's difficult for me to believe it... I imagine it relies on a mysterious property you will introduce now.

**Laurent:** I have to admit it. Indeed, it derives from the existence of Hilbert bases, also called orthonormal bases in the literature. Before we go into the details, I warn you: you must be careful not to mistake them with bases of vector spaces.


The full power of the Hilbert spaces appears through the use of Hilbert bases, which allow us to describe them with a flavor of finite-dimensional spaces. However, any Hilbert space does not have a Hilbert basis. This only occurs if the space satisfies the topological *separability* property.

Recall that a basis of a vector space $E$ is a subset $\mathcal{B}$ of $E$ such that:

$\star$ For all $x \in E$, there exist a finite subset $(e_k)_{1 \leq k \leq n}$ of $\mathcal{B}$ and $(\alpha_k)_{1 \leq k \leq n}$ in $\mathbf{C}$ such that $x = \alpha_1 e_1 + \cdots + \alpha_n e_n$. This property is denoted by $x \in \mathrm{span}(\mathcal{B})$.

$\star$ For any finite subset $(e_k)_{1 \leq k \leq n}$ of $\mathcal{B}$ and for all $(\alpha_k)_{1 \leq k \leq n}$ in $\mathbf{C}$,

$$\alpha_1 e_1 + \cdots + \alpha_n e_n = 0 \quad \Longleftrightarrow \quad \alpha_1 = \cdots = \alpha_n = 0.$$

**Example.** The family $(X^n)_{n \in \mathbf{N}}$ is a basis of the space $\mathbf{C}[X]$ of polynomials with complex coefficients.

In the frame of Hilbert spaces, we consider a different definition of basis. To avoid any confusion, this new basis will be called *a Hilbert basis.*

**Definition 1.56 (Hilbert basis).** We call *a Hilbert basis (or orthonormal basis)* of $\mathcal{H}$ any family $(e_n)_{n \in \mathbf{N}} \subset \mathcal{H}$ that satisfies:

(i) $\forall n, m \in \mathbf{N}, \quad \langle e_n, e_m \rangle = \mathbf{1}_{n=m} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$

(ii) $\overline{\mathrm{span}\{e_n;\ n \in \mathbf{N}\}} = \mathcal{H}$.

**Remark.** In the specific case of a finite-dimensional Hilbert space $\mathcal{H}$, a Hilbert basis is an orthonormal basis of the vector space $\mathcal{H}$. This is false in general.

**Laurent:** To build a counterexample to this remark, we need to consider an infinite-dimensional space. If you knew more about Fourier series for instance, we could do that. But no matter, Bernard, we will do that later on with functions whose square is integrable.

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**Definition 1.57 (Separable space).** A topological space $E$ is said to be *separable* if it contains a countable dense subset $F$, i.e. $E = \overline{F}$, where $F \subset E$ and $F$ is countable.

**Laurent:** To go from the separability property to the existence of a Hilbert space, you certainly feel that there is something to be created from this countable dense subset to build a Hilbert basis.

By the way, in the case of an Euclidean space, can you remember how to change any basis to an orthonormal basis?

**Bernard:** I remember indeed. It was a process we called the Gram–Schmidt process. We would make linear combinations of the vectors of the original basis to build an orthonormal basis.

**Laurent:** It is exactly the same idea for separable Hilbert spaces. Let us see.

**Theorem 1.58 (Existence of a Hilbert basis).** *Any separable Hilbert space $\mathcal{H}$ admits a Hilbert basis $(e_n)_{n\in\mathbf{N}}$.*

**Proof.** The construction of a Hilbert basis relies on the *Gram–Schmidt process.*

Since $\mathcal{H}$ is separable, there exists a countable family $(f_n)_{n\in\mathbf{N}}$ in $\mathcal{H}$ such that $\overline{\{f_n;\ n\in\mathbf{N}\}}=\mathcal{H}$ (i.e. $\{f_n;\ n\in\mathbf{N}\}$ is a countable dense subset).

By induction, we construct a family $(e_n)_{n\in\mathbf{N}}$ in $\mathcal{H}$ such that:

(C1) for all $n,m\in\mathbf{N}$, $\langle e_n,e_m\rangle=\mathbf{1}_{n=m}$;
(C2) for all $n\in\mathbf{N}$, $\mathrm{span}(f_0,\dots,f_n)\subset\mathrm{span}(e_0,\dots,e_n)$, which leads finally to $\overline{\mathrm{span}\{e_n;\ n\in\mathbf{N}\}}=\mathcal{H}$.

We assume that $\mathcal{H}$ is an infinite-dimensional vector space (the finite-dimensional case is already known) and that $f_n\neq 0$ for all $n\in\mathbf{N}$.

⋆ $n=0$: we set $e_0=\frac{f_0}{\|f_0\|}$.
⋆ Assuming that $(e_k)_{0\le k\le n}$ are defined, we define $e_{n+1}$.

Since $\mathcal{H}$ is an infinite-dimensional vector space, there exists $f_k\in\mathcal{H}$ such that $f_k\notin\mathrm{span}(e_0,\dots,e_n)$. We set $\varphi(n+1)=\inf\{k\ge n+1 : f_k\notin\mathrm{span}(e_0,\dots,e_n)\}$ and $\widetilde{e}_{n+1}=f_{\varphi(n+1)}-\sum_{j=0}^{n}\langle f_{\varphi(n+1)},e_j\rangle\cdot e_j\neq 0$.

Then, for all $j=0,\dots,n$, we have $\langle\widetilde{e}_{n+1},e_j\rangle=0$, and we set $e_{n+1}=\frac{\widetilde{e}_{n+1}}{\|\widetilde{e}_{n+1}\|}$.

The two conditions (C1) and (C2) are satisfied for $(e_k)_{0\leq k\leq n+1}$.

The family $(e_n)_{n\in\mathbf{N}}$ constructed by induction is then a Hilbert basis of $\mathcal{H}$. □

**Bernard:** I understood the definition. But how can you decompose an element of $\mathcal{H}$ in this basis?

**Laurent:** We are getting there, Bernard. In the case of a possibly infinite-dimensional space, we will need the topology since it will take the form of a series...

It is not surprising since, generally, $\operatorname{span}\{e_n;\ n\in\mathbf{N}\} \subsetneq \overline{\operatorname{span}\{e_n;\ n\in\mathbf{N}\}} = \mathcal{H}$. Hence, an element of $\mathcal{H}$ cannot be written, generally, as a (finite) linear combination of $\{e_n;\ n\in\mathbf{N}\}$.

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**Theorem 1.59 (Parseval identity).** *Let $\mathcal{H}$ be a separable Hilbert space and $(e_n)_{n\in\mathbf{N}}$ a Hilbert basis of $\mathcal{H}$.*

*Then, for all $x\in\mathcal{H}$, we have $x=\lim_{N\to\infty}\sum_{n=0}^{N}\langle x,e_n\rangle\, e_n$ in $\mathcal{H}$, which is denoted $x=\sum_{n=0}^{\infty}\langle x,e_n\rangle\, e_n$. Moreover, for all $x,y\in\mathcal{H}$, we have $\langle x,y\rangle=\sum_{n=0}^{\infty}\langle x,e_n\rangle\,\overline{\langle y,e_n\rangle}$ and in particular, for all $x\in\mathcal{H}$, $\|x\|^2=\sum_{n=0}^{\infty}|\langle x,e_n\rangle|^2$.*

**Proof.** ⋆ For all $n\in\mathbf{N}$, we note that $F_n=\operatorname{span}(\{e_0,\dots,e_n\})$. As a finite-dimensional vector subspace, $F_n$ is closed. Then, we can consider that, for all $x\in\mathcal{H}$, $x_n=p_{F_n}(x)$.

By the decomposition $x=(x-x_n)+x_n$, with $x_n\in F_n$ and $x-x_n\in F_n^{\perp}$, we obtain

$$\forall k\leq n,\quad \langle x,e_k\rangle=\langle x_n,e_k\rangle.$$

Then, $x_n=\sum_{k=0}^{n}\langle x_n,e_k\rangle\, e_k=\sum_{k=0}^{n}\langle x,e_k\rangle\, e_k$.

Let us prove that $(x_n)_{n\in\mathbf{N}}$ converges to $x$ in $\mathcal{H}$, i.e. $\|x-x_n\|\to 0$ as $n\to\infty$.

By definition, for all $n\in\mathbf{N}$, we have $\|x-x_n\|=d(x,F_n)$. Since $F_n\subset F_{n+1}$ for all $n\in\mathbf{N}$, the sequence $\big(d(x,F_n)\big)_{n\in\mathbf{N}}$ is nonincreasing, and it converges to some limit $\ell\geq 0$.

Since $(e_n)_{n\in\mathbf{N}}$ is a Hilbert basis, the subset $\cup_{n\in\mathbf{N}}F_n$ is dense in $\mathcal{H}$.

Then, for all $\varepsilon > 0$, there exists $\widetilde{x} \in \cup_{n\in\mathbf{N}} F_n$ such that $\|x-\widetilde{x}\| < \varepsilon$. Let $n_0 \in \mathbf{N}$ be such that $\widetilde{x} \in F_{n_0}$. We have

$$\forall n \in \mathbf{N},\, n \geq n_0, \quad d(x, F_n) \leq d(x, F_{n_0}) \leq \|x - \widetilde{x}\| < \varepsilon.$$

Taking the limit $n \to \infty$, we deduce that $\ell \leq \varepsilon$ for all $\varepsilon > 0$. Then, we can conclude that $\ell = 0$ and $\|x - x_n\| \to 0$ as $n \to \infty$.

⋆ For all $x, y \in \mathcal{H}$, we have

$$\forall n \in \mathbf{N}, \quad \langle x_n, y_n\rangle = \left\langle \sum_{k=0}^{n} \langle x, e_k\rangle\, e_k, \sum_{j=0}^{n} \langle y, e_j\rangle\, e_j \right\rangle$$
$$= \sum_{k=0}^{n} \sum_{j=0}^{n} \langle x, e_k\rangle\, \overline{\langle y, e_j\rangle}\, \langle e_k, e_j\rangle = \sum_{k=0}^{n} \langle x, e_k\rangle\, \overline{\langle y, e_k\rangle}.$$

The Cauchy–Schwarz inequality implies that the function $(x, y) \mapsto \langle x, y\rangle$ is continuous from $\mathcal{H} \times \mathcal{H}$ to $\mathbf{C}$. Then, since $x_n \to x$ in $\mathcal{H}$ and $y_n \to y$ in $\mathcal{H}$, we have $\langle x_n, y_n\rangle \to \langle x, y\rangle$ as $n \to \infty$.

We deduce that $\langle x, y\rangle = \sum_{n=0}^{\infty} \langle x, e_n\rangle\, \overline{\langle y, e_n\rangle}$. □

**Laurent:** As we said earlier, generally, for $x \in \mathcal{H}$, there does not exist $N \in \mathbf{N}$ such that $x = \sum_{n=0}^{N} \langle x_n, e_n\rangle\, e_n$. It means that $(e_n)_{n\in\mathbf{N}}$ is not an algebraic basis of $\mathcal{H}$.

And it is crucial to understand that $x = \sum \langle x_n, e_n\rangle\, e_n$ is to be taken as a shorthand for the limit of the sequence $(\sum_{n=0}^{N} \langle x_n, e_n\rangle\, e_n)_{N\in\mathbf{N}}$ in $\mathcal{H}$, for the topology of $\mathcal{H}$.

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**Proposition 1.60.** *If $\mathcal{H}$ is a separable Hilbert space, then $\mathcal{H} \cong \ell^2(\mathbf{N})$.*

**Proof.** Let $(e_n)_{n\in\mathbf{N}}$ be a Hilbert basis of $\mathcal{H}$. Define the linear mapping $\varphi\colon x \in \mathcal{H} \mapsto (\langle x, e_n\rangle)_{n\in\mathbf{N}} \in \ell^2(\mathbf{N})$. The Parseval identity states that $\varphi$ is an isometry. Hence, it is one-to-one.

And for $(u_n)_{n\in\mathbf{N}} \in \ell^2(\mathbf{N})$, $x = \sum u_n\, e_n \in \mathcal{H}$. Hence, $\varphi$ is onto. □

**Bernard:** I understand that any separable can be identified with $\ell^2(\mathbf{N})$, algebraically as well as topologically. In some sense, there is only one separable Hilbert space.

Anyway, I am certainly looking forward to seeing how all these notions apply to concrete functional spaces that, as I understood it, will be defined thanks to the theory of integration.

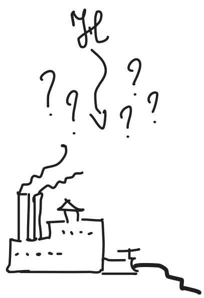
*After this intense session of general study of the concept of topology, and more specifically of the case of normed vector spaces, Bernard feels puzzled.*

*His first emotion is euphoria at the nature of the exchanges he has managed to have with the brilliant researcher, for whom his father was so full of praise. He cannot deny that he has enjoyed very much this first work session.*

*But at the same time, he feels worried, or perhaps rather frustrated. Of course, he is well aware of the following:*

- *The general question of approximation is part of topology.*
- *The first equations, which seem to formalize the question of river pollution, cannot be solved directly, and their resolution would require this approximation step.*
- *Measuring the approximation depends on the properties of the spaces in which the problem solutions live (distances, norms, scalar products, etc.).*

*But he does not fully grasp the concrete contribution of topology, in its theoretical expression, to providing relevant elements for the environmental study of the positioning of his pharmaceutical plant.*



*The specter of Ann Winglett looms over him, and he is afraid he will not be able to give her any convincing evidence. Having understood things better still does not seem enough to reassure her...*

*Bernard has heard that his boss has left the path of fundamental mathematics in France after meeting his future mentor. The latter had convinced her that* the only mathematics worth anything is the one you can use*. To be clear, understand that the mathematics addressing issues that are the focus of significant funding.*

*The two men part ways, agreeing to meet the following week (May 9th) at the same location (the Centre Emile Borel). Laurent promised that they would get to the heart of the matter through measure theory. But what does he mean?*

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## Chapter 2

# Measure Spaces

*On the early morning of May 9th, Laurent is drinking an espresso seated in a café in the* Quartier Latin*.*

*He is leafing through a thick, yellow book when his gaze is attracted by the curious shoes of a bystander on the opposite sidewalk. After hesitating for a while, Bernard decides to join him. Certainly, the Centre Emile Borel just opened.*

*Back in the same place as last week, the two men intend to construct the integral of functions with respect to a measure, then to define functional spaces that would form the framework for the mathematical modeling of the positioning of the pharmaceutical plant.*

*Laurent places his hand on the two books he brought with him:* Measure Theory *by Henri Brolle and* Foundations of Probability *by André François.*

## 2.1 Measurable Functions, Measures

**Laurent:** Dear Bernard, I would like to discuss an important subject concerning the study of functions...

**Bernard:** What else could there be, besides the infinitesimal properties we studied in Chapter 1 through the concept of continuity?

Oh, I see! You want to talk about differential calculus, or something of the sort.

**Laurent:** What do you think of the integration of functions?

**Bernard:** What could be new? It is just the reciprocal operation of differentiation: we have

$$f(y) - f(x) = \int_x^y f'(z)\, dz,$$

and denoting $F(x) = \int_0^x f(z)\, dz$, we also have $F'(x) = f(x)$.

**Laurent:** You are talking about the Riemann Integration of continuous functions, then. Does that mean you are not interested in functions which are not continuous, not even those that are piecewise?

**Bernard:** Of course, I think functions which are not continuous are interesting. I retain from my engineering studies that it is interesting to integrate linear combinations of indicator functions. I also remember the use of a weird function, called the *Dirac function*. But I have to admit that I never understood if it was really a function.

**Laurent:** You raise a very interesting question, Bernard, which we will answer later.

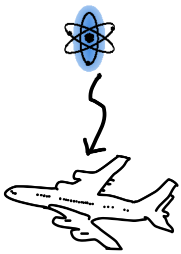
Using math in our society is closely linked to the description of physical phenomena. The infinitesimal descriptions of quantities, as well as their derivatives, were once considered sufficient for the mechanics of points and, more generally, the classical mechanics of rigid bodies. Things changed in the 20th century.

**Bernard:** I see what you are getting at. You are talking about soft matter or large interacting systems. It also reminds me of the fact that, now, the state of a particle is no longer considered certain but is described by the so-called wavefunction, which represents more or less the probability of being in a given state.

**Laurent:** Indeed, Bernard, it means going from classical mechanics theory to *statistical and quantum mechanics theories*. And, in this framework, infinitesimal calculus is no longer sufficient...

**Bernard:** Besides, it seems to me that what we are discussing is not limited to theoretical physics. I am under the impression that the objects engineers manipulate inherit the same characteristics.

A modern engineering object consists of a large number of interacting components. I can think of planes, nuclear plants, biological organs,... all of which must be described, in the end, at many scales.



**Laurent:** This is the reason why modern mathematics occupies an increasingly important place for engineers. At some point, it was believed that useful mathematics for engineers could not be improved as all functions were continuous and that differential equations could be solved numerically. Cross-multiplication was sufficient.

In the context we just described, the world is not at all linear and, sometimes, not even continuous. Come on, let's talk about measure theory. This means integrating functions that may not be smooth, at any point. In that case, integration has nothing to do with the regularity of the function.

**Bernard:** I can't wait, Laurent! How do we begin?

**Laurent:** The starting point is the concept of $\sigma$*-algebras*.

### 2.1.1 *$\sigma$-algebras or $\sigma$-fields*

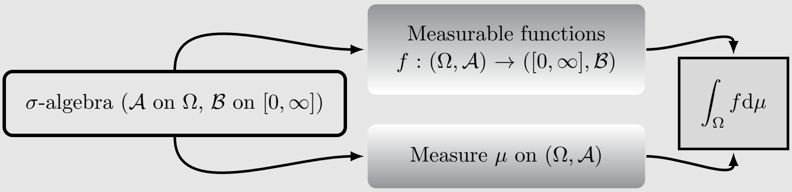
**Bernard:** Let me open your book, *Measure Theory* by Henri Brolle, at the starting point of the construction. I have the page... $\sigma$-algebras, you said...

In this chapter, the goal is the construction of the integral of certain wide classes of functions. Two different issues are considered in this first step:

(1) What are the minimal properties which should be satisfied by a function in order to be integrated?

(2) How to construct the integral?

For the first point, the functions to be integrated can be more general than the continuous functions on $\mathbf{R}$. Moreover, they can be defined on a more general space than $\mathbf{R}$. The definition of the integral relies on the concept of *measure*, which is defined on a *$\sigma$-algebra*.




**Definition 2.1.** A *$\sigma$-algebra* $\mathcal{A}$ on a set $\Omega$ is a nonempty collection of subsets of $\Omega$ which satisfies the following:

(i) $\emptyset \in \mathcal{A}$.
(ii) $\mathcal{A}$ is stable by complementing: if $A \in \mathcal{A}$, then $\Omega \backslash A \in \mathcal{A}$.
(iii) $\mathcal{A}$ is stable by countable unions: if $(A_n)_{n \in \mathbf{N}}$ belong to $\mathcal{A}$, then $\bigcup_{n \in \mathbf{N}} A_n \in \mathcal{A}$.

These conditions can be expressed in the following equivalent form:

**Lemma 2.2.** *A collection $\mathcal{A}$ of subsets of $\Omega$ is a $\sigma$-algebra if and only if the following four assertions hold:*

(a) $\emptyset \in \mathcal{A}$.
(b) *If $A$ belongs to $\mathcal{A}$, then $\Omega \backslash A$ belongs to $\mathcal{A}$.*
(c) *If $A$ and $B$ belong to $\mathcal{A}$, then $A \cup B$ belongs to $\mathcal{A}$.*
(d) *If $(A_n)_{n \in \mathbf{N}}$ are pairwise disjoint elements of $\mathcal{A}$, then $\bigcup_{n \in \mathbf{N}} A_n$ belongs to $\mathcal{A}$.*

**Proof.** $\star$ Following Definition 2.1, the four assertions obviously hold if $\mathcal{A}$ is a $\sigma$-algebra.
$\star$ Conversely, let us assume that these four assertions hold. We prove that $\mathcal{A}$ is a $\sigma$-algebra.

According to (i) and (ii), the only thing to prove is that for any $(A_n)_{n \in \mathbf{N}}$ in $\mathcal{A}$, the subset $\bigcup_{n \in \mathbf{N}} A_n$ remains in $\mathcal{A}$.

We study the second member of the equality:

$$\bigcup_{n\in\mathbf{N}} A_n = \bigcup_{n\in\mathbf{N}} \left[ A_n \cap \left( \Omega \backslash \bigcup_{i<n} A_i \right) \right]. \tag{2.1.1}$$

By induction, we can see that (iii) implies that any finite union in $\mathcal{A}$ remains in $\mathcal{A}$. Then, we can claim that $\bigcup_{i<n} A_i$ is in $\mathcal{A}$. And from (ii), $\Omega\backslash \bigcup_{i<n} A_i$ is in $\mathcal{A}$.

A combination of (b) and (c) implies that $A_n \cap \left(\Omega\backslash \bigcup_{i<n} A_i\right)$ is in $\mathcal{A}$ for all $n \in \mathbf{N}$.

Finally, from (d), the disjoint union $\bigcup_{n\in\mathbf{N}} \left[A_n \cap \left(\Omega\backslash \bigcup_{i<n} A_i\right)\right]$ is in $\mathcal{A}$, and (2.1.1) gives the result. □

**Bernard:** If I understand the definition well, one can combine these conditions to obtain other stability properties satisfied by $\sigma$-algebras.

**Laurent:** Definitely, Bernard. For instance, we can apply Condition (ii) of Definition 2.1 to $A = \emptyset$: we obtain $\Omega = \Omega\backslash\emptyset \in \mathcal{A}$.

**Remark.** The $\sigma$-algebra $\mathcal{A}$ also satisfies $\Omega \in \mathcal{A}$ (using both Conditions (i) and (ii) of Definition 2.1), and if $A_n \in \mathcal{A}$ for all $n \in \mathbf{N}$, then $\bigcap_{n\in\mathbf{N}} A_n \in \mathcal{A}$ (using both Conditions (ii) and (iii)).

**Proposition 2.3.** *If $\mathcal{A}$ is a $\sigma$-algebra on $\Omega$, then $\mathcal{A}$ is stable by differences: if $A \in \mathcal{A}$ and $B \in \mathcal{A}$, then $B\backslash A \in \mathcal{A}$.*

**Proof.** First, we note that $B\backslash A = B \cap (\Omega\backslash A)$.

Since $A \in \mathcal{A}$ implies $\Omega\backslash A \in \mathcal{A}$ by Condition (ii) of Definition 2.1, the previous remark implies that $B\backslash A$ is in $\mathcal{A}$. □

**Definition 2.4.** Any couple $(\Omega, \mathcal{A})$, where $\mathcal{A}$ is a $\sigma$-algebra on $\Omega$, is called *a measurable space.*

**Bernard:** The stability conditions in the definition of a $\sigma$-algebra seem familiar to me. Even if they are not identical, they recall the definition of a topology we discussed not so long ago.

**Laurent:** You are absolutely correct, Bernard. Actually, the similarity that you noticed will be useful in showing that the construction of the integral can be applied to continuous functions... when the $\sigma$-algebra considered is properly chosen.

**Bernard:** That is the very least!

**Laurent:** Before seeing that in detail, could you point out an important difference between $\sigma$-algebra conditions and topology conditions?

**Bernard:** You mean the fact that Condition (iii) here only stands for countable unions, I guess. Of course, it is a serious reduction in the stability conditions of a topology with respect to arbitrary unions.

**Laurent:** Very good, Bernard. Let us continue with some examples of $\sigma$-algebras.

**Example.** Here are some $\sigma$-algebras on the set $\Omega = \{a, b, c\}$:

- ⋆ $\{\emptyset, \Omega\}$,
- ⋆ $\{\emptyset, \{a\}, \{b, c\}, \Omega\}$,
- ⋆ $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b\}, \Omega\}$.

For a finite or countable set, such as $\mathbf{N}$ or $\mathbf{Z}$, we are familiar with considering the *discrete $\sigma$-algebra*, which is the collection of all the subsets.

In the case of $\mathbf{R}$ or $\mathbf{R}^d$, we often consider the $\sigma$-algebra generated by open sets, called the *Borel $\sigma$-algebra*.

**Definition 2.5 (Generated $\sigma$-algebra).** For any collection $\mathcal{C}$ of subsets of $\Omega$, the intersection of all the $\sigma$-algebras which contain $\mathcal{C}$ is a $\sigma$-algebra, called *$\sigma$-algebra generated by* $\mathcal{C}$ and denoted by $\sigma(\mathcal{C})$.

**Justification.** In order to check the validity of this definition, consider any family $(\mathcal{A}_t)_{t\in\mathcal{T}}$ of $\sigma$-algebras on $\Omega$ (where $\mathcal{T}$ may be uncountable).

(i) For all $t \in \mathcal{T}$, $\emptyset \in \mathcal{A}_t$. Then, $\emptyset \in \bigcap_{t\in\mathcal{T}} \mathcal{A}_t$.

(ii) Assume $A \in \bigcap_{t\in\mathcal{T}} \mathcal{A}_t$. For all $t \in \mathcal{T}$, we have $A \in \mathcal{A}_t$, and since $\mathcal{A}_t$ is a $\sigma$-algebra, $\Omega \backslash A \in \mathcal{A}_t$. Then, $\Omega \backslash A \in \bigcap_{t\in\mathcal{T}} \mathcal{A}_t$.

(iii) Assume $(A_n)_{n\in\mathbf{N}}$ are in $\bigcap_{t\in\mathcal{T}} \mathcal{A}_t$. For each $t \in \mathcal{T}$, $A_n \in \mathcal{A}_t$ for all $n \in \mathbf{N}$, and since $\mathcal{A}_t$ is a $\sigma$-algebra, we have $\bigcup_{n\in\mathbf{N}} A_n \in \mathcal{A}_t$. Then, $\bigcup_{n\in\mathbf{N}} A_n \in \bigcap_{t\in\mathcal{T}} \mathcal{A}_t$.

This shows that $\bigcap_{t \in \mathcal{T}} \mathcal{A}_t$ is a $\sigma$-algebra. $\square$

**Definition 2.6 (Borel $\sigma$-algebra).** The $\sigma$-algebra generated by all the open sets of a topological space $\mathcal{T}$ is called the Borel $\sigma$-algebra of $\mathcal{T}$ and is denoted by $\mathcal{B}(\mathcal{T})$.

The elements of $\mathcal{B}(\mathcal{T})$ are called the Borel sets of $\mathcal{T}$.

**Bernard:** Although I understand the idea of generated mathematical structures, it seems difficult to comprehend what a Borel set is. Even the idea of an open set is not obvious in all cases. Does some kind of characterization exist for it?

**Laurent:** I'm afraid not, in general. However, in the particular case of $\mathbf{R}$ or $\mathbf{R}^d$, a more precise characterization can be stated: the Borel $\sigma$-algebra is generated by intervals or rectangles.

In the particular case of $\mathcal{T} = \mathbf{R}$, the Borel $\sigma$-algebra $\mathcal{B}(\mathbf{R})$ contains all the intervals of the form $(a, b)$, $(a, +\infty)$ and $(-\infty, a)$, where $a, b \in \mathbf{R}$, since these intervals are open sets.

By stability under implementation, $\mathcal{B}(\mathbf{R})$ is generated by the closed subsets of $\mathbf{R}$. So, it contains all the intervals of the form $[a, b]$, $[a, +\infty)$ and $(-\infty, a]$, where $a, b \in \mathbf{R}$. Consequently, $\mathbf{N} = \bigcup_{n \in \mathbf{N}} \{n\}$ and $\mathbf{Z} = \bigcup_{n \in \mathbf{N}} (\{n\} \cup \{-n\})$ are in $\mathcal{B}(\mathbf{R})$.

Moreover, since we know that the set $\mathbf{Q}$ is countable, we can write $\mathbf{Q} = \bigcup_{n \in \mathbf{N}} \{q_n\}$ and conclude that $\mathbf{Q} \in \mathcal{B}(\mathbf{R})$.

For all $a, b \in \mathbf{R}$ such that $a \leq b$, the semi-open intervals $[a, b)$ and $(a, b]$ belong to $\mathcal{B}(\mathbf{R})$. In order to show that, we write

$$[a, b) = (-\infty, b) \backslash (-\infty, a) \quad \text{and} \quad (a, b] = (a, +\infty) \backslash (b, +\infty),$$

and note that open intervals $(-\infty, a)$, $(-\infty, b)$, $(a, +\infty)$ and $(b, +\infty)$ are in $\mathcal{B}(\mathbf{R})$.

**Laurent:** We are almost at the point where the $\sigma$-algebra generated by open sets of $\mathbf{R}$ is actually generated by the open intervals... Before that, let me insist on the fact that $\mathcal{B}(\mathbf{R})$ contains any countable union of (not necessarily open) intervals.

**Bernard:** Yes, it's clear by what we've just read and the fact that a $\sigma$-algebra is closed under countable unions.

**Lemma 2.7.** *The Borel $\sigma$-algebra $\mathcal{B}(\mathbf{R})$ is generated by each of the following collections:*

(i) *the collection of open intervals $(a, b)$, with $a, b \in \mathbf{R}$ and $a \leq b$;*
(ii) *the collection of open intervals $(-\infty, a)$, with $a \in \mathbf{R}$;*
(iii) *the collection of closed intervals $[a, +\infty)$, with $a \in \mathbf{R}$;*
(iv) *the collection of open intervals $(a, +\infty)$, with $a \in \mathbf{R}$;*
(v) *the collection of closed intervals $(-\infty, a]$, with $a \in \mathbf{R}$;*
(vi) *the collection of closed intervals $[a, b]$, with $a, b \in \mathbf{R}$ and $a \leq b$.*

**Proof.** (i) We denote $\mathcal{I}_O = \sigma\big((a,b);\ a \leq b\big)$. Since each interval $(a, b)$ is an open subset of $\mathbf{R}$, we can claim that $\mathcal{I}_O \subset \mathcal{B}(\mathbf{R})$.

Conversely, for any open subset $A \subset \mathbf{R}$, for all $a \in A$, there exists $\eta_a \in \mathbf{Q}_+^*$ such that $(a - \eta_a, a + \eta_a) \subset A$. Hence, we can write $A = \bigcup_{a \in A}(a - \eta_a, a + \eta_a)$.

Moreover, for all $a \in A$, there exists $q_a \in \mathbf{Q}$ such that $a \in (q_a - \eta_a/2, q_a + \eta_a/2)$, and this interval is included in $(a - \eta_a, a + \eta_a)$.

Hence, we have $A = \bigcup_{\substack{q,\eta \in \mathbf{Q} \\ \exists a \in A: q = q_a, \eta = \eta_a}} (q - \eta/2, q + \eta/2)$.

We deduce that $\mathcal{B}(\mathbf{R}) \subset \mathcal{I}_O$, and finally $\mathcal{B}(\mathbf{R}) = \mathcal{I}_O$.

(ii) We first remark that the $\sigma$-algebra generated by the intervals $(-\infty, a)$ is included in $\mathcal{B}(\mathbf{R})$ because $(-\infty, a)$ is an open subset of $\mathbf{R}$.

Conversely, each open interval $(a, b)$ can be written as

$$(a,b) = \bigcup_{n \in \mathbf{N}^*} \left[a + \frac{1}{n}, b\right) = \bigcup_{n \in \mathbf{N}^*} (-\infty, b) \backslash \left(-\infty, a + \frac{1}{n}\right).$$

This allows us to claim that $\mathcal{B}(\mathbf{R})$ is included in the $\sigma$-algebra generated by the intervals $(-\infty, a)$. And finally, the two $\sigma$-algebras are equal.

(iii) This point follows (ii) by implementation.

Points (iv), (v) and (vi) are proved in a similar manner. □

We can also endow $\overline{\mathbf{R}}_+ = [0, \infty]$ with a Borel $\sigma$-algebra, denoted by $\mathcal{B}\big([0, \infty]\big)$. It is generated by the elements of the topology of $[0, \infty]$, that is, the open sets of $\mathbf{R}_+$ possibly augmented with $\{+\infty\}$.

**Bernard:** This book does not seem very pedagogical, Laurent. I am not sure that any reader could understand this sentence. Is it really obvious that the open sets of $[0, \infty]$ are precisely the open sets of $\mathbf{R}_+$ augmented with $\{\infty\}$?

**Laurent:** This point is important in the following. I am not sure why we need to talk about this again; we discussed it just a week ago.

You better remember the topology of the order on $\overline{\mathbf{R}}$ that we discussed together last week. In the case of $[0, \infty]$, it is consistent with what you can find in Schwartz's book,[1] corresponding to the course at *Ecole polytechnique* so dear to your father:

> *In $\overline{\mathbf{R}}_+$, the collection of all the arbitrary unions of open intervals of the form $(x, y)$, $(x, \infty]$, $[0, y)$, $[0, \infty]$, $x, y \in \overline{\mathbf{R}}_+$, is the so-called order topology.*

**Bernard:** I remember. And since $\mathbf{R}_+$ is a subset of $\overline{\mathbf{R}}_+$, the induced topology on $\mathbf{R}_+$ is obviously the order topology of $\mathbf{R}_+$.

And the order topology of $\mathbf{R}_+$ is the usual topology of $\mathbf{R}_+$ as a metric space generated by the open balls $B(x, \varepsilon) = \{y \in \mathbf{R}_+ : |y - x| < \varepsilon\}$, $x > 0$, $\varepsilon > 0$.

**Laurent:** Precisely! It is important to obtain a topology of $\overline{\mathbf{R}}_+$ which is consistent with the topology of $\mathbf{R}$. In the same spirit as the $\overline{\mathbf{R}}$ case, the open sets of $\overline{\mathbf{R}}_+$ are exactly the open sets of $\mathbf{R}_+$ together with the half-lines $[0, \infty]$, $(x, \infty]$, $x \geq 0$.

Eventually, once the open sets are defined, the Borel $\sigma$-algebra is generated by them.

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[1]See Schwartz (1991, p. 140).

### 2.1.2 *Measurable functions*

**Bernard:** All this stuff is beautiful, but what is their link with functions that we need to integrate?

**Laurent:** This is precisely where we are, Bernard. Let us continue reading the book to understand the concept of *measurable functions*.

**Definition 2.8.** Let $(\Omega, \mathcal{A})$ and $(E, \mathcal{E})$ be two measurable spaces.

A function $f\colon (\Omega, \mathcal{A}) \to (E, \mathcal{E})$ is said to be *measurable* if $f^{-1}(\mathcal{E}) \subset \mathcal{A}$, i.e. if

$$\forall B \in \mathcal{E}, \quad f^{-1}(B) \in \mathcal{A},$$

where $f^{-1}(B) = \{x \in \Omega : f(x) \in B\}$.

**Bernard:** The similarity with General topology continues...

**Laurent:** Definitely. As you noticed, the definition of a measurable function has the same flavor as that of a continuous function, provided the substitution of topologies with $\sigma$-algebras. This remark will be developed later on.

**Bernard:** What happens when you change the $\sigma$-algebra on one of the spaces?

**Laurent:** It will change the measurability status.

**Bernard:** All right. But a constant function is measurable, no matter the $\sigma$-algebra, right?

**Laurent:** You are right. It comes from the fact that $\emptyset$ and $\Omega$ belong to any $\sigma$-algebra on $\Omega$. Precisely, consider the constant function

$$\begin{aligned} f\colon (\Omega, \mathcal{A}) &\longrightarrow (E, \mathcal{E}) \\ x &\longmapsto \kappa. \end{aligned}$$

Then, $f$ is measurable for all choices of the $\sigma$-algebras $\mathcal{A}$ and $\mathcal{E}$ on the sets $\Omega$ and $E$, respectively. Indeed, for any $C \in \mathcal{E}$, $f^{-1}(C) = \emptyset$ if $\kappa \notin C$, and $f^{-1}(C) = \Omega$ otherwise. And $\emptyset$ and $\Omega$ belong to $\mathcal{A}$ by the definition of a $\sigma$-algebra.

In the case of a $\sigma$-algebra $\mathcal{E}$ generated by some subsets of $E$, the measurability of $f$ can be deduced by considering only $f^{-1}(B)$, where $B$ spans these subsets.

**Proposition 2.9.** *If $\mathcal{E} = \sigma(\mathcal{C})$, where $\mathcal{C}$ is a collection of subsets of $E$, then the function $f\colon (\Omega, \mathcal{A}) \to (E, \mathcal{E})$ is measurable if and only if $f^{-1}(\mathcal{C}) \subset \mathcal{A}$.*

**Proof.** Let $\mathcal{G} = \{B \in \mathcal{E} : f^{-1}(B) \in \mathcal{A}\}$. We check that $\mathcal{G}$ is a $\sigma$-algebra:
(i) Since $f^{-1}(\emptyset) = \emptyset$, we have $\emptyset \in \mathcal{G}$.
(ii) For any $B \in \mathcal{G}$, $f^{-1}(B) \in \mathcal{A}$ implies $f^{-1}(\Omega \backslash B) = \Omega \backslash f^{-1}(B) \in \mathcal{A}$. Then, $\Omega \backslash B \in \mathcal{G}$.
(iii) For all $(B_n)_{n \in \mathbf{N}}$ in $\mathcal{G}$, the fact that $f^{-1}(B_n) \in \mathcal{G}$ for all $n \in \mathbf{N}$ implies $f^{-1}\Big(\bigcup_{n \in \mathbf{N}} B_n\Big) = \bigcup_{n \in \mathbf{N}} f^{-1}(B_n) \in \mathcal{A}$.

Then, $\bigcup_{n \in \mathbf{N}} B_n \in \mathcal{G}$.

By assumption, we have $\mathcal{C} \subset \mathcal{G}$. We deduce that $\sigma(\mathcal{C}) \subset \mathcal{G}$ (because $\sigma(\mathcal{C})$ is the intersection of all the $\sigma$-algebras containing $\mathcal{C}$ and $\mathcal{G}$ is one of them).

Therefore, for all $B \in \mathcal{E} = \sigma(\mathcal{C})$, $f^{-1}(B) \in \mathcal{A}$, which states that $f$ is measurable. □

The previous proof uses a classic argument that is often used to prove results involving measurability or $\sigma$-algebras. Proposition 2.9 can also be proved by the equality

$$f^{-1}\big(\sigma(\mathcal{C})\big) = \sigma\big(f^{-1}(\mathcal{C})\big),$$

whose proof is left to the reader.

**Definition 2.10 (Borel function).** Let $E$ and $F$ be two topological spaces, endowed with their Borel $\sigma$-algebras $\mathcal{B}(E)$ and $\mathcal{B}(F)$, respectively.

Any measurable function $f\colon \big(E, \mathcal{B}(E)\big) \to \big(F, \mathcal{B}(F)\big)$ is called a *Borel function*.

An important particular case of Borel functions appears when $E = \mathbf{R}^d$ and $F = \mathbf{R}^m$: a function $f\colon \big(\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d)\big) \to \big(\mathbf{R}^m, \mathcal{B}(\mathbf{R}^m)\big)$ is a Borel function if it is measurable.

**Proposition 2.11.** *Let $E$ and $F$ be two topological spaces, endowed with their Borel $\sigma$-algebras $\mathcal{B}(E)$ and $\mathcal{B}(F)$, respectively.*

*Any continuous function $f\colon E \to F$ is a Borel function from the measure space $\big(E, \mathcal{B}(E)\big)$ to $\big(F, \mathcal{B}(F)\big)$.*

When $E = \mathbf{R}^d$ and $F = \mathbf{R}^m$, Proposition 2.11 reads as follows: if the function $f\colon \mathbf{R}^d \to \mathbf{R}^m$ is continuous, then it is a Borel function.

**Proof.** Let $\mathcal{O}_E$ (resp., $\mathcal{O}_F$) denote the open sets of $E$ (resp., $F$).

Since $f$ is continuous, we have $f^{-1}(\mathcal{O}_F) \subset \mathcal{O}_E$ and, consequently,

$$f^{-1}(\mathcal{O}_F) \subset \sigma(\mathcal{O}_E) = \mathcal{B}(E).$$

From Proposition 2.9, we can conclude that $f^{-1}\big(\mathcal{B}(F)\big) \subset \mathcal{B}(E)$ and $f$ is a Borel function. □

**Bernard:** I find this proof to be very elegant. It makes use of the fact that the definitions of continuous functions, on the one hand, and measurable functions, on the other, are similar.

**Laurent:** You observe here the depth of abstraction in mathematics. The parallels between open sets and Borel sets, whose definitions are quite abstract, allows us to obtain a simple proof of the fact that continuous functions are measurable for the Borel $\sigma$-algebra.

Without that, it would be more technically involved to prove the result using the definition of continuity with $\varepsilon$: $\forall \varepsilon > 0, \exists \delta > 0$ such that...

**Bernard:** Okay, okay. And $E$ and $F$ would need to be metric. But what about piecewise continuous functions (which can be integrated using the Riemann integral)?

**Laurent:** Piecewise continuous functions are also measurable for the Borel $\sigma$-algebra...

**Bernard:** How would I know practically that a function is measurable?

**Laurent:** The best way is to use the following properties: the chain rule, addition, multiplication and limits.

**Bernard:** As an example, can you say that $f(x) = \sin(1/x^2)$ for $x \neq 0$, $f(0) = 0$, is measurable? Clearly, it is not continuous at $x = 0$...

**Laurent:** Indeed, there is a discontinuity at $x = 0$, and we need to approach this with ingenuity. The idea in this case is to approximate $f$ by a sequence of measurable functions. We will see later on that the limit of a converging sequence of measurable functions is measurable.

**Proposition 2.12.** *Let $(\Omega, \mathcal{A})$, $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be three measurable spaces.*

*If $f\colon (\Omega, \mathcal{A}) \to (E, \mathcal{E})$ and $g\colon (E, \mathcal{E}) \to (F, \mathcal{F})$ are two measurable functions, then $g \circ f$ is a measurable function from $(\Omega, \mathcal{A})$ to $(F, \mathcal{F})$.*

**Proof.** Since $f\colon (\Omega, \mathcal{A}) \to (E, \mathcal{E})$ is measurable, we have $f^{-1}(\mathcal{E}) \subset \mathcal{A}$. In the same way, $g\colon (E, \mathcal{E}) \to (F, \mathcal{F})$ is measurable, and then $g^{-1}(\mathcal{F}) \subset \mathcal{E}$. We deduce that

$$(g \circ f)^{-1}(\mathcal{F}) = f^{-1}\big(g^{-1}(\mathcal{F})\big) \subset \mathcal{A},$$

and $g \circ f$ is measurable. □

**Laurent:** What do you think of the function $f\colon x \mapsto \big\lfloor \exp\big(\sqrt{|x|}\big) \big\rfloor$, where $\lfloor y \rfloor$ denotes the integer part of $y$?

**Bernard:** Let me see: $x \mapsto \exp\big(\sqrt{|x|}\big)$ is continuous, defined over $\mathbf{R}$, so it is a Borel function. $x \mapsto \lfloor x \rfloor$ is piecewise constant. Following Proposition 2.12, it is enough to check that every piecewise constant function is measurable to conclude that $f$ is a Borel function.

**Laurent:** Be careful, Bernard. For a piecewise function $g$ to be measurable with respect to $\mathcal{B}(\mathbf{R})$, every subset over which $g$ is constant must be in $\mathcal{B}(\mathbf{R})$.

**Bernard:** We need the integer part $I\colon x \mapsto \lfloor x \rfloor$ to be measurable. I know that it is constant on the intervals $[k, k+1)$, $k \in \mathbf{N}$, which are in $\mathcal{B}(\mathbf{R})$, as we saw previously.

So, $I$ should be measurable as you just said. Unfortunately, I have to admit that it is not so obvious to me; let me check. Let $B \in \mathcal{B}(\mathbf{R})$:

- either $B \cap \mathbf{Z} = \emptyset$, which implies that $I^{-1}(B) = \emptyset \in \mathcal{B}(\mathbf{R})$,
- or $B \cap \mathbf{Z} = \bigcup_{j \in J} \{j\}$, which implies that $I^{-1}(B) = \bigcup_{j \in J} [j, j+1) \in \mathcal{B}(\mathbf{R})$, $J$ being a finite or countable subset of $\mathbf{Z}$.

**Laurent:** Very good! Let us move on.

**Proposition 2.13.** *In the following statements, the considered functions map* $(\Omega, \mathcal{A})$ *to* $\big(\mathbf{R}, \mathcal{B}(\mathbf{R})\big)$ *or* $\big(\overline{\mathbf{R}}_+, \mathcal{B}(\overline{\mathbf{R}}_+)\big)$*, where* $\mathcal{A}$ *is a* $\sigma$*-algebra on* $\Omega$*.*

(i) $\mathbf{1}_A$ *is measurable if and only if* $A \in \mathcal{A}$*.*
(ii) *If* $f$ *and* $g$ *are measurable, then the functions* $f + g$ *and* $fg$ *are measurable.*
(iii) *Let* $(f_n)_{n \in \mathbf{N}}$ *be measurable functions. The functions*

$$\sup_{n \in \mathbf{N}} f_n,\ \inf_{n \in \mathbf{N}} f_n,\ \limsup_{n \to \infty} f_n,\ \liminf_{n \to \infty} f_n,\ \lim_{n \to \infty} f_n \text{ (if it exists)},$$

$$\sum_{n=0}^{\infty} f_n \text{ (if it exists)}$$

*are all measurable.*
(iv) *If* $f$ *is measurable, then* $|f|$ *is measurable.*

**Proof.** (i) First, since $(\mathbf{1}_A)^{-1}(\{1\}) = A$, if $\mathbf{1}_A$ is measurable, then $A \in \mathcal{A}$.

Conversely, assume that $A \in \mathcal{A}$. For $B \in \mathcal{B}(\mathbf{R})$, the following cases must be distinguished:

⋆ If $0 \notin B$ and $1 \notin B$, then $(\mathbf{1}_A)^{-1}(B) = \emptyset$.
⋆ If $0 \notin B$ and $1 \in B$, then $(\mathbf{1}_A)^{-1}(B) = A$.
⋆ If $0 \in B$ and $1 \notin B$, then $(\mathbf{1}_A)^{-1}(B) = \Omega \backslash A$.
⋆ If $0 \in B$ and $1 \in B$, then $(\mathbf{1}_A)^{-1}(B) = \Omega$.

Consequently, the function $\mathbf{1}_A$ is measurable.

(ii) According to Lemma 2.7 and Proposition 2.9, in order to prove the measurability of $f + g$, it suffices to show that $(f + g)^{-1}\big((-\infty, a)\big) \in \mathcal{A}$ for all $a \in \mathbf{R}$.

Let us fix $a \in \mathbf{R}$ and consider

$$\begin{aligned}(f+g)^{-1}\big((-\infty,a)\big) &= \{x \in \Omega : f(x)+g(x) < a\} \\ &= \bigcup_{\substack{r,s\in\mathbf{Q} \\ r+s<a}} \{x \in \Omega : f(x) < r\} \cap \{x \in \Omega : g(x) < s\}.\end{aligned}$$

Since $f$ and $g$ are measurable, we have

$$\{x \in \Omega : f(x) < r\} \in \mathcal{A} \quad \text{and} \quad \{x \in \Omega : g(x) < s\} \in \mathcal{A}.$$

Then, $\{x \in \Omega : f(x)+g(x) < a\} \in \mathcal{A}$, and $f+g$ is measurable.

For the measurability of $fg$, we first assume that $f \geq 0$ and $g \geq 0$. In that case, for $a \in \mathbf{R}_+$, we have

$$\begin{aligned}(fg)^{-1}\big((-\infty,a)\big) &= \{x \in \Omega : f(x)g(x) < a\} \\ &= \bigcup_{\substack{r,s\in\mathbf{Q}_+ \\ rs<a}} \{x \in \Omega : f(x) < r\} \cap \{x \in \Omega : g(x) < s\}.\end{aligned}$$

We deduce, as above, that $\{x \in \Omega : f(x)g(x) < a\} \in \mathcal{A}$ and $fg$ is measurable.

In the general case (when $f$ and $g$ can take negative values), we decompose $f = f_+ - f_-$, where $f_+ = \sup(f,0)$ and $f_- = \sup(-f,0)$, and $g = g_+ - g_-$, where $g_+ = \sup(g,0)$ and $g_- = \sup(-g,0)$.

We have $fg = f_+g_+ - f_+g_- - f_-g_+ + f_-g_-$.

Since $f_+, f_-, g_+, g_-$ are nonnegative, the products $f_+g_+$, $f_+g_-$, $f_-g_+$, $f_-g_-$ are measurable. We can conclude that $fg$ is measurable.

(iii) For $a \in \mathbf{R}$,

$$\left\{x \in \Omega : \sup_{n\in\mathbf{N}} f_n(x) \leq a\right\} = \bigcap_{n\in\mathbf{N}} \{x \in \Omega : f_n(x) \leq a\} \in \mathcal{A}$$

because $\{x \in \Omega : f_n(x) \leq a\} \in \mathcal{A}$ for all $n \in \mathbf{N}$.

By Lemma 2.7 and Proposition 2.9, we deduce that $\sup_{n\in\mathbf{N}} f_n$ is measurable.

In the same way, we can show that $\inf_{n\in\mathbf{N}} f_n$ is measurable.

For the limit superior and limit inferior, we have

$$\liminf_{n\to\infty} f_n = \sup_{n\in\mathbf{N}} \inf_{k\geq n} f_k \quad \text{and} \quad \limsup_{n\to\infty} f_n = \inf_{n\in\mathbf{N}} \sup_{k\geq n} f_k.$$

Then, $\liminf_{n\to\infty} f_n$ and $\limsup_{n\to\infty} f_n$ are measurable.

If $\lim_{n\to\infty} f_n$ exists, its measurability follows from the measurability of $\limsup_{n\to\infty} f_n$. The case of $\sum_{n=0}^{\infty} f_n$ follows.

(iv) We have $f = f_+ - f_-$, where $f_+ = \sup(f, 0)$ and $f_- = \sup(-f, 0)$. By the previous point, $f_+$ and $f_-$ are measurable. We deduce that $|f| = f_+ + f_-$ is measurable. □

**Bernard:** Can we go back to our example: how do these properties allow me to show that $g$ such that $g(x) = \sin(1/x^2)$ if $x \neq 0$ and $g(0) = 0$ is measurable?

**Laurent:** Don't you think this function is the limit of a sequence of continuous functions?

**Bernard:** Let me think. Don't tell me. I can try $g_n : x \mapsto \sin\big(1/(x^2 + 1/n\pi)\big)$ for $n \geq 1$. All these functions look continuous to me, so they are measurable with respect to the Borel $\sigma$-algebra. They converge pointwise to $g$.

**Laurent:** So, you see that (iii) allows you to conclude.

**Bernard:** Indeed. I begin to see how powerful these properties can be in showing the measurability of functions. But I have a question that you might find strange: what would happen if we considered the idea of using $\mathbf{1}_A : \Omega \to \{0,1\}$, that is, the set $\{0,1\}$ instead of $\mathbf{R}$ in the first point?

**Laurent:** Your question is very interesting, Bernard. But I notice that you did not specify the $\sigma$-algebra in your question...

If you consider the finite set $\{0,1\}$ as the set of values of $\mathbf{1}_A$, the "good" $\sigma$-algebra to consider is $\big\{\emptyset, \{0\}, \{1\}, \{0,1\}\big\}$.

If the function $\mathbf{1}_A$ is measurable, then $A = (\mathbf{1}_A)^{-1}(\{1\}) \in \mathcal{A}$.

Conversely, if $A \in \mathcal{A}$, the cases to be considered are $(\mathbf{1}_A)^{-1}(\emptyset) = \emptyset$, $(\mathbf{1}_A)^{-1}(\{0\}) = \Omega \backslash A$, $(\mathbf{1}_A)^{-1}(\{1\}) = A$ and $(\mathbf{1}_A)^{-1}(\{0,1\}) = \Omega$. Then, the measurability of $\mathbf{1}_A$ is obvious.

**Bernard:** We saw that piecewise constant functions belong to your class of measurable functions. I have to say that I'm not totally convinced that your class of integrable functions is larger than the Riemann-integrable class. Does it at least contain the piecewise continuous functions?

**Laurent:** Sure, they belong to this class. You just need to apply the previous proposition: let $f$ be piecewise continuous – it can even be defined over all $\mathbf{R}$.

Let $(x_k)_{k\in\mathbf{Z}}$ be an increasing sequence of real numbers, $(\alpha_k)_{k\in\mathbf{Z}}$ be some real numbers and $(f_k)_{k\in\mathbf{Z}}$ be a sequence of continuous functions over $\mathbf{R}$ such that

$$\forall x \in \mathbf{R}, \quad f(x) = \sum_{k\in\mathbf{Z}} f_k(x)\,\mathbf{1}_{(x_k,x_{k+1})}(x) + \sum_{k\in\mathbf{Z}} \alpha_k\,\mathbf{1}_{\{x_k\}}(x).$$

Since $(x_k, x_{k+1})$ and $\{x_k\}$ are in $\mathcal{B}(\mathbf{R})$ for all $k \in \mathbf{Z}$, and since the functions $f_k$ are Borel functions for all $k \in \mathbf{Z}$, Proposition 2.13 implies that $f$ is a Borel function.

**Bernard:** I'm beginning to feel better. And if $f$ is a differentiable function on $\mathbf{R}$, can we say something about its derivative $f'$?

**Laurent:** Definitely. $f'$ is also a Borel function. It is a simple consequence of Proposition 2.13 again.

For all integer $n \geq 1$, consider the function $f_n\colon \mathbf{R} \to \mathbf{R}$ defined by

$$\forall x \in \mathbf{R}, \quad f_n(x) = \frac{f(x+1/n) - f(x)}{1/n}.$$

We remark that the function $f_n$ is continuous on $\mathbf{R}$, and then $f_n$ is a Borel function.

Moreover, $f$ is differentiable on $\mathbf{R}$, so for all $x \in \mathbf{R}$, $f_n(x)$ converges to $f'(x)$ as $n$ goes to $\infty$.

We deduce that $f'$ is a Borel function.

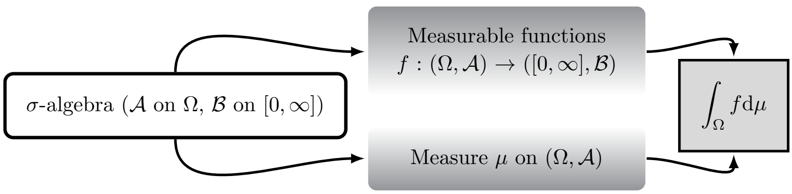
### 2.1.3 *Measure spaces*

**Bernard:** I understand now what a measurable function is. But why is it called *measurable*? I don't understand the use of the word *measure*? And how does it relate to the need for integrating a function?

**Laurent:** Saying that a function $f$ is measurable means we are able to consider its image through a *measure*. This point will be essential to *integrate $f$ against a measure*.

**Bernard:** (smiling) I'm beginning to wonder whether you are testing my patience. What is a measure yet? I imagine it is connected to $\sigma$-algebras, but how?

**Laurent:** Keep in mind the big picture of integration:




The main idea of a positive measure is to generalize the intuitive notions of length, area and volume on $\mathbf{R}$ or $\mathbf{R}^N$ and the cardinality of a discrete set.

**Definition 2.14.** Let $\mathcal{A}$ be a $\sigma$-algebra on a set $\Omega$.
A (positive) *measure* over $(\Omega, \mathcal{A})$ is a function $\mu\colon \mathcal{A} \to [0, \infty]$ which satisfies:

(i) $\mu(\emptyset) = 0$ ;
(ii) for all $(A_n)_{n\in\mathbf{N}}$ pairwise disjoint elements of $\mathcal{A}$,

$$\mu\Big(\bigcup_{n\in\mathbf{N}} A_n\Big) = \sum_{n\in\mathbf{N}} \mu(A_n).$$

This property is called the *$\sigma$-additivity* of the measure $\mu$.

**Bernard:** This additivity property can certainly be written in a simpler way. Why doesn't Henri Brolle simply write $\mu(A \cup B) = \mu(A) + \mu(B)$ when $A, B \in \mathcal{A}$ and $A \cap B = \emptyset$?

**Laurent:** Certainly because your property is not equivalent to $\sigma$-additivity.

Actually, the $\sigma$-additivity property implies the *additivity property* (sometimes called *finite additivity* to avoid confusion), whose statement is: for all $n \in \mathbf{N}^*$ and all $A_1, \dots, A_n$ pairwise disjoint elements of $\mathcal{A}$,

$$\mu\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mu(A_k).$$

Could you prove, by writing on this sheet of paper, that your property for any disjoint $A$ and $B$ in $\mathcal{A}$ is equivalent to finite additivity?

**Bernard:** First, I remark that for $n = 2$, the finite additivity statement reduces to $\mu(A \cup B) = \mu(A) + \mu(B)$ when $A, B \in \mathcal{A}$ and $A \cap B = \emptyset$.

In order to prove the converse implication, I suggest to proceed by induction:

- For $n = 1$, there is nothing to prove.
- For $n = 2$, additivity is precisely the property $\mu(A \cup B) = \mu(A) + \mu(B)$ when $A, B \in \mathcal{A}$ and $A \cap B = \emptyset$.
- Assume that additivity holds for $n \geq 1$, and consider $A_1, \dots, A_{n+1}$ to be pairwise disjoint elements of $\mathcal{A}$. We have $\mu\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mu(A_k)$. Then,

$$\mu\left(\bigcup_{k=1}^{n+1} A_k\right) = \mu\left(\left(\bigcup_{k=1}^{n} A_k\right) \cup A_{n+1}\right),$$

  and I can use the fact that $\bigcup_{k=1}^{n} A_k$ and $A_{n+1}$ are disjoint to write

$$\mu\left(\left(\bigcup_{k=1}^{n} A_k\right) \cup A_{n+1}\right) = \mu\left(\bigcup_{k=1}^{n} A_k\right) + \mu(A_{n+1}).$$

  We can conclude that

$$\mu\left(\bigcup_{k=1}^{n+1} A_k\right) = \sum_{k=1}^{n} \mu(A_k) + \mu(A_{n+1}),$$

  which is the additivity property for $n + 1$.

By induction, the additivity is proved for any $n \geq 1$.

**Laurent:** Very good. Now you can observe the difference between

$$\forall n \in \mathbf{N}^*, \quad \mu\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mu(A_k),$$

which is equivalent to the statement you suggested, and the $\sigma$-additivity

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k).$$

**Bernard:** Okay, Okay. I can see that the second one implies the first one and I accept that my statement is weaker than the $\sigma$-additivity property.

**Laurent:** Let me insist on a final point. The collection $(A_n)_{n\in\mathbf{N}}$ considered in this property really needs to be countable...

**Bernard:** Hey, I did not make any mistake here, since I noticed that $\bigcup_{n\in\mathbf{N}} A_n$ needs to be in $\mathcal{A}$. That's the reason why it is imposed by the definition of a $\sigma$-algebra.

**Example.**

(1) For $\Omega = \mathbf{N}$ and $\mathcal{A} = \mathcal{P}(\mathbf{N})$, the collection of all the subsets of $\mathbf{N}$, the *counting measure* $\mu$ is defined by

$$\forall A \subset \mathbf{N}, \quad \mu(A) = \#A = \sum_{k\in\mathbf{N}} \mathbf{1}_A(k).$$

(2) For any measurable space $(\Omega, \mathcal{A})$ and $x \in \Omega$, the map $\delta_x \colon \mathcal{A} \to \mathbf{R}_+$, defined by

$$\forall A \in \mathcal{A}, \quad \delta_x(A) = \mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$$

is a measure called the *Dirac measure* at point $x$.

More generally, for $(x_n)_{n\in\mathbf{N}}$ in $\Omega$ and $(\alpha_n)_{n\in\mathbf{N}}$ in $[0, \infty]$, we can consider the measure $\nu = \sum_n \alpha_n\, \delta_{x_n}$, defined by

$$\forall A \in \mathcal{A}, \quad \nu(A) = \sum_n \alpha_n\, \delta_{x_n}(A) = \sum_n \alpha_n\, \mathbf{1}_A(x_n).$$

**Bernard:** I understand quite well the first example. The cardinality of $\emptyset$ is equal to zero, and if $(A_n)_{n\in\mathbf{N}}$ are pairwise disjoint subsets of $\mathbf{N}$, the cardinality of their union is equal to the sum of the cardinality of the $A_n$'s. Formally, we can write

$$\#\emptyset = 0 \quad \text{and} \quad \#\left(\bigcup_{n\in\mathbf{N}} A_n\right) = \sum_{n\in\mathbf{N}} \#A_n.$$

However, the second example puzzles me.

**Laurent:** Come on, Bernard! You should not give up in the face of abstraction. Do I need to remind you of its benefits?

**Bernard:** No no. Okay, let me try. I consider that $x$ is fixed in $\Omega$, and I study the function $\delta_x$. The first step is to check that $\delta_x(\emptyset) = 0$, which is clear since $x \notin \emptyset$.

Then, I consider the pairwise disjoint elements $(A_n)_{n\in\mathbf{N}}$ of $\mathcal{A}$. Let me write

$$\delta_x\left(\bigcup_{n\in\mathbf{N}} A_n\right) = \mathbf{1}_{\bigcup_{n\in\mathbf{N}} A_n}(x) = \begin{cases} 1 & \text{if } x \in \bigcup_{n\in\mathbf{N}} A_n, \\ 0 & \text{otherwise.} \end{cases}$$

This leads to

$$\delta_x\left(\bigcup_{n\in\mathbf{N}} A_n\right) = \begin{cases} 1 & \text{if } x \in A_n \text{ for some } n \in \mathbf{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Besides, I study the value of $\sum_{n\in\mathbf{N}} \delta_x(A_n)$,

$$\sum_{n\in\mathbf{N}} \delta_x(A_n) = \sum_{n\in\mathbf{N}} \mathbf{1}_{A_n}(x) = \begin{cases} 1 & \text{if } x \in A_n \text{ for some } n \in \mathbf{N}, \\ 0 & \text{otherwise,} \end{cases}$$

using the fact that the $A_n$'s are disjoint, which implies that only one term of the sum can be different from zero.

**Laurent:** Don't you see, Bernard, that you are already familiar with the use of measures? Let us continue with their properties.

**Proposition 2.15.** (i) *For $A, B \in \mathcal{A}$, if $A \subset B$, then $\mu(A) \leq \mu(B)$.*

*Moreover, if $\mu(A) < \infty$, then $\mu(B \backslash A) = \mu(B) - \mu(A)$.*

(ii) *For $A, B \in \mathcal{A}$, $\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B)$.*

(iii) *For any $(A_n)_{n\in\mathbf{N}}$ in $\mathcal{A}$, $\mu\left(\bigcup_{n\in\mathbf{N}} A_n\right) \leq \sum_{n=0}^{\infty} \mu(A_n)$.*

(iv) *For $(A_n)_{n\in\mathbf{N}}$ in $\mathcal{A}$ such that $A_n \subset A_{n+1}$ for all $n \in \mathbf{N}$,*

$$\mu\left(\bigcup_{n\in\mathbf{N}} A_n\right) = \lim_{n\to\infty} \mu(A_n) = \sup_{n\in\mathbf{N}} \mu(A_n).$$

(v) *For $(A_n)_{n\in\mathbf{N}}$ in $\mathcal{A}$ such that $A_{n+1} \subset A_n$ for all $n \in \mathbf{N}$, and $\mu(A_0) < \infty$,*

$$\mu\left(\bigcap_{n\in\mathbf{N}} A_n\right) = \lim_{n\to\infty} \mu(A_n) = \inf_{n\in\mathbf{N}} \mu(A_n).$$

**Proof.** (i) As the sets $A$ and $B\backslash A$ are disjoint, we have

$$\mu\big(A \cup (B\backslash A)\big) = \mu(A) + \mu(B\backslash A).$$

But, since $A \subset B$, we also have

$$\mu\big(A \cup (B\backslash A)\big) = \mu(A \cup B) = \mu(B).$$

Then, these two equalities imply $\mu(B) = \mu(A) + \mu(B\backslash A)$, and the result follows.

(ii) The equality $A \cup B = A \cup \big(B\backslash(A \cap B)\big)$, where the sets $A$ and $B\backslash(A \cap B)$ are disjoint, yields

$$\mu(A \cup B) = \mu(A) + \mu\big(B\backslash(A \cap B)\big).$$

The previous point implies $\mu(B) = \mu(A \cap B) + \mu\big(B\backslash(A \cap B)\big)$.

We deduce $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$ in the following two cases:

- ⋆ $\mu(B) < \infty$, which implies $\mu\big(B\backslash(A \cap B)\big) < \infty$;
- ⋆ $\mu(B) = \infty$, which implies $\mu(A \cup B) = \infty$.

(iii) Let us consider $B_0 = A_0$ and, for all $n \geq 1$, $B_n = A_n \backslash \bigcup_{k=0}^{n-1} A_k$.

The sets $(B_n)_{n\in\mathbf{N}}$ are pairwise disjoint and $\bigcup_{n\in\mathbf{N}} B_n = \bigcup_{n\in\mathbf{N}} A_n$. Then,

$$\mu\left(\bigcup_{n\in\mathbf{N}} A_n\right) = \mu\left(\bigcup_{n\in\mathbf{N}} B_n\right) = \sum_{n=0}^{\infty} \mu(B_n) \leq \sum_{n=0}^{\infty} \mu(A_n),$$

using the fact that $B_n \subset A_n$ for all $n \in \mathbf{N}$.

(iv) Consider $C_0 = A_0$ and, for all $n \geq 1$, $C_n = A_n \backslash A_{n-1}$.

We have $\bigcup_{n\in\mathbf{N}} A_n = \bigcup_{n\in\mathbf{N}} C_n$, and as the sets $(C_n)_{n\in\mathbf{N}}$ are disjoint,

$$\mu\left(\bigcup_{n\in\mathbf{N}} A_n\right) = \mu\left(\bigcup_{n\in\mathbf{N}} C_n\right) = \sum_{n=1}^{\infty} \mu(C_n) = \lim_{N\to\infty} \sum_{n=0}^{N} \mu(C_n)$$
$$= \lim_{N\to\infty} \mu(A_N).$$

(v) Consider $D_n = A_0 \backslash A_n$ for all $n \in \mathbf{N}$.

The sequence $(D_n)_{n\in\mathbf{N}}$ satisfies $D_n \subset D_{n+1}$ for all $n \in \mathbf{N}$. Then,

$$\mu\left(\bigcup_{n\in\mathbf{N}} D_n\right) = \lim_{n\to\infty} \mu(D_n) = \sup_{n\in\mathbf{N}} \mu(D_n).$$

However,

$$\mu\left(\bigcup_{n\in\mathbf{N}} D_n\right) = \mu\left(A_0 \backslash \bigcap_{n\in\mathbf{N}} A_n\right) = \mu(A_0) - \mu\left(\bigcap_{n\in\mathbf{N}} A_n\right).$$

Hence,

$$\mu(A_0) - \mu\left(\bigcap_{n\in\mathbf{N}} A_n\right) = \lim_{n\to\infty} \mu(A_0 \backslash A_n) = \lim_{n\to\infty} \big(\mu(A_0) - \mu(A_n)\big).$$

We deduce that $\mu\left(\bigcap_{n\in\mathbf{N}} A_n\right) = \lim_{n\to\infty} \mu(A_n) = \inf_{n\in\mathbf{N}} \mu(A_n)$. $\square$

**Bernard:** I think I grasp what a measure is. But I need to understand: how do you construct a measure over a measurable space?

**Laurent:** The answer to this question is very complicated. Over countable sets, we will see that it is quite easy since points are isolated. But for a continuous space, I mean not countable, it relies on the Caratheodory theory.

We will go over this issue when we get to probability theory and then later integration over $\mathbf{R}$ and the Lebesgue measure.

Before tackling this issue, let us state the particular properties of some specific measures, which are related.

**Definition 2.16.** Let $\mu$ be a measure over $(\Omega, \mathcal{A})$.

⋆ The measure $\mu$ is said to be *finite* if $\mu(\Omega) < \infty$.
⋆ $\mu$ is called a *probability measure* if $\mu(\Omega) = 1$.
⋆ $\mu$ is said to be *σ-finite* if there exists an increasing sequence $(E_n)_{n\in\mathbf{N}}$ in $\mathcal{A}$ such that $\Omega = \bigcup_{n\in\mathbf{N}} E_n$ and $\mu(E_n) < \infty$, for all $n \in \mathbf{N}$.
⋆ $x \in \Omega$ is an *atom* of the measure $\mu$ if $\{x\} \in \mathcal{A}$ and $\mu(\{x\}) > 0$.
⋆ $\mu$ is said to be *diffuse* if $\mu(\{x\}) = 0$ for all $x \in \Omega$.

## 2.2 A Particular Case: Probability Spaces and Random Variables

**Laurent:** One of the main outcomes of measure theory in early 20th century was the establishment of probability theory.

**Bernard:** You are kidding, Laurent. Probability is merely a matter of computation, that is, counting the favorable cases and dividing them by the total number of cases.

I remember that the beginning of probability involved a gambling issue. When a game was stopped before its end, the problem was, how to divide the transitional gains. It was the subject of a correspondence between Fermat and Pascal in the 17th century.

If $x_k$ is the $k$th realization of $X$, taking values in $\{a_i;\ i \in \mathbf{N}\}$, they defined

- the probability $p_i \approx \frac{\#\{k \le n : x_k = a_i\}}{n}$, for large $n$,
- and the expectation $\mathbf{E}[X] = \sum_{i=0}^{+\infty} a_i\, p_i$.

**Laurent:** You are speaking of the 19th-, I mean, 17th-century frequentist vision of probability. Now, probability is a mathematical theory in its own right. It comes naturally from measure theory.

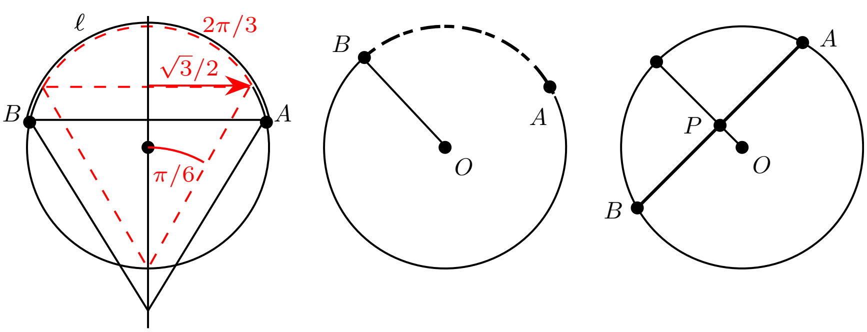
**Bernard:** (beginning to worry) What is the benefit, particularly for engineering? I'm sorry, Laurent, all these theoretical ideas are interesting, but I really need to consider my concrete pharmaceutical plant problem.

**Laurent:** Your problem is really related to the *stochastic modeling* issue. Don't be so impatient, Bernard. You really need to understand the theoretical framework of probability.

Thanks to the modern vision of probability as defined in measure theory, a number of ambiguities that were due to incorrect problem formulations and hence considered paradoxical have been explained.

**Bernard:** I think I see what you mean. I remember a so-called Bertrand paradox.

The goal was to randomly draw a string $[A, B]$ from the unit circle $\mathcal{U}$ and evaluate the probability that an equilateral triangle built on $[A, B]$ would be entirely contained within the disk. As I recall, depending on how $[A, B]$ was randomly drawn, the probability calculation led to different results.




- In a first construction, we fix the point $A$ and *draw the point $B$ uniformly on the circle*.

One of the two equilateral triangles constructed on $[A, B]$ is contained in the disc if and only if the length $\ell$ of the arc $\overset{\frown}{AB}$ is less than $2\pi/3$.

By rotational symmetry, the choice of the first point $A$ doesn't matter, and the probability can be calculated:

$$\mathbf{P}\left(\ell < \frac{2\pi}{3}\right) = \frac{1}{2\pi}\frac{2\pi}{3} = \frac{1}{3}.$$

- In another construction, we fix a radius of the circle $\mathcal{U}$ and *we uniformly draw a point $P$ on the radius.* The string is constructed so that $P$ is the midpoint of $[A, B]$.

  One of the two equilateral triangles built on $[A, B]$ is contained in the disk if and only if the length of $[A, B]$ is less than $\sqrt{3}$ (i.e. if and only if the distance $OP$ is greater than $1/2$).

  By rotational symmetry, the choice of radius doesn't matter, and the desired probability can be calculated:

$$\mathbf{P}\big(AB < \sqrt{3}\big) = \mathbf{P}\left(OP > \frac{1}{2}\right) = \frac{1}{2}.$$

Different constructions lead to different values of the probability. What is the real value: $1/3$ or $1/2$? It is the reason why probability does not look particularly scientific, in my opinion.

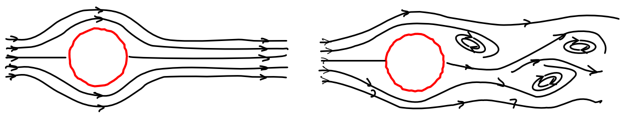
**Laurent:** The point you observed is not a problem of probability computation. With the modern formalism of probability, each construction corresponds to a certain law of the random variable. And, as we will see later, the value of the probability depends on the chosen distribution.

Before exploring probability theory, the first question to consider is the concept of a random phenomenon.

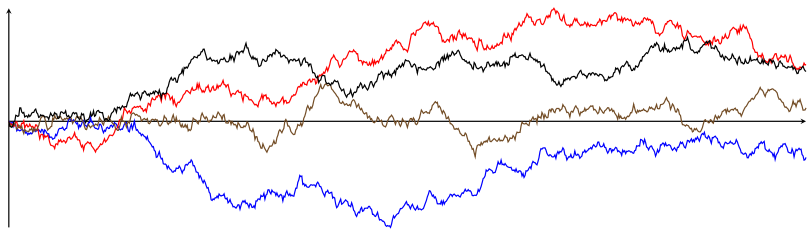
**Bernard:** For the engineer, random modeling is a very important issue. There are many definitions for randomness. In my point of view, the most important characteristics are: nonreproductibility and variability.

And the issue of random modeling is: what can we say about situations where results cannot be completely prescribed?

**Laurent:** By the way, the beginning of mathematical probability theory can be traced back to an engineering issue. Kolmogorov was interested in modeling the turbulence phenomenon. He developed in 1933 an axiom based on the emerging measure theory developed by Borel and Lebesgue.



Among all the benefits of this new theory, this axiom allows us to study time-varying random phenomena via the mathematical concept of *stochastic processes*, but that is another matter...



Now, I suggest that we start reading the book *Foundations of Probability* by André François and that you continue taking notes.

### 2.2.1 *From measure theory to probability spaces*

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**Events, probability**

Probability theory provides a mathematical formalism for describing *random phenomena*. The qualifier *random* is usually used to describe nonreproducible behaviors, i.e. behaviors that cannot be predicted in advance. For example, rolling a coin or dice is a random experiment, in the sense that we do not know how to predict its outcome. Here, we will not venture into the epistemological debate on the intrinsic existence of randomness. So, we confound *randomness* and *lack of knowledge*.

The formalism of probability theory begins with the definition of three mathematical objects which form the *probability space* $(\Omega, \mathcal{F}, \mathbf{P})$:

- ⋆ the *sample space* $\Omega$: all the possible outcomes of a random phenomenon,
- ⋆ the *set of events* $\mathcal{F}$: each event is composed of outcomes,
- ⋆ the *probability measure* $\mathbf{P}$: to each event corresponds one value of probability.

*Events* are, then, properties whose validity – or lack thereof – can be determined from the results of random experiments.

The *probability* of an event is a number between 0 and 1, representing the likelihood of the event.

**$\sigma$-algebra of events**

An *event* is an element of the $\sigma$-algebra $\mathcal{F}$ on $\Omega$. From the definition of a $\sigma$-algebra, the following properties hold:

(i) $\emptyset \in \mathcal{F}$ and $\Omega \in \mathcal{F}$;

(ii) $\mathcal{F}$ is stable by complementing: if $A \in \mathcal{F}$, then $A^c = \Omega \backslash A \in \mathcal{F}$;

(iii) $\mathcal{F}$ is stable by countable intersections and unions: if $A_n \in \mathcal{F}$ for all $n \in \mathbf{N}$, then $\bigcap_{n\in\mathbf{N}} A_n \in \mathcal{F}$ and $\bigcup_{n\in\mathbf{N}} A_n \in \mathcal{F}$.

$\mathcal{F}$ is stable by difference: $A, B \in \mathcal{F} \Rightarrow A \backslash B \in \mathcal{F}$.

**Bernard:** Then, intuitively, for two events $A$ and $B$, the logical expression "$A$ and $B$" is the event $A \cap B$, whereas "$A$ or $B$" is the event $A \cup B$.

Finally, the stability conditions for events naturally lead to a $\sigma$-algebra structure.

**Laurent:** The most important benefit of this $\sigma$-algebra structure is the definition of the probability measure.

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**Probability measure**

A *probability measure* on the measurable space $(\Omega, \mathcal{F})$ is a function $\mathbf{P} \colon \mathcal{F} \to [0, 1]$ such that

(i) $\mathbf{P}(\emptyset) = 0$ and $\mathbf{P}(\Omega) = 1$;

(ii) $\mathbf{P}$ is $\sigma$-additive, i.e. for all pairwise disjoint $(A_n)_{n\in\mathbf{N}}$ in $\mathcal{F}$, $\mathbf{P}\left(\bigcup_{n\in\mathbf{N}} A_n\right) = \sum_{n=0}^{\infty} \mathbf{P}(A_n)$.

The assumption $\mathbf{P}(\Omega) < +\infty$ appears as an important distinction from the general definition of a measure (e.g. the Lebesgue measure $\lambda$ on $\mathbf{R}$ satisfies $\lambda(\mathbf{R}) = +\infty$).

**Remark.** The set of events of $\Omega$ is not necessarily the power set of $\Omega$. More precisely, if $\mathcal{F}$ is not equal to $\mathcal{P}(\Omega)$, there exists $A \subset \Omega$ which is not an event. In that case, the quantity $\mathbf{P}(A)$ is not defined.

**Two important examples of probability measures:**

(1) The *Equiprobability* is defined from the counting measure on $\Omega = \{\omega_1, \ldots, \omega_n\}$, with $\mathcal{F} = \mathcal{P}(\Omega)$, as the function

$$\begin{aligned} \mathbf{P}\colon \mathcal{P}(\Omega) &\to [0,1] \\ A &\mapsto \frac{1}{\#\Omega} \sum_{k=1}^{n} \mathbf{1}_A(\omega_k) = \frac{\#A}{\#\Omega}. \end{aligned}$$

This example shows that the intuitive notion of probability defined by a count is a natural part of Kolmogorov's framework.

(2) The *Dirac measure* at $a \in \Omega$, endowed with the $\sigma$-algebra $\mathcal{F}$, defined as

$$\begin{aligned} \delta_a\colon \mathcal{F} &\to [0,1] \\ A &\mapsto \begin{cases} 1 & \text{if } a \in A, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

is a probability measure on $(\Omega, \mathcal{F})$.

The Dirac measure will be used later in the definition of the expectation of discrete random variables.

**Definition 2.17.** An event $A$ of the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is said to be $\mathbf{P}$-*almost sure* (or, simply, *almost sure*) if $\mathbf{P}(A) = 1$.

**Properties arising from measure theory**
For $A, B \in \mathcal{F}$,

$$\mathbf{P}(A^c) = 1 - \mathbf{P}(A)$$

$$B \subset A \Rightarrow \mathbf{P}(A \backslash B) = \mathbf{P}(A) - \mathbf{P}(B)$$

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B).$$

If $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence in $\mathcal{F}$, then

$$\mathbf{P}\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mathbf{P}(A_n).$$

If $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence in $\mathcal{F}$, then

$$\mathbf{P}\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mathbf{P}(A_n).$$

### 2.2.2 *Conditional probability and independence*

**Bernard:** I understand that probability measures inherit the general properties of positive measures. But, does probability theory merely sums up as a restriction of measure theory?

**Laurent:** Not at all. The real starting point of probability theory is the concept of independence. Before explaining in detail this difficult concept regarding random variables, the first step is understanding the independence of events.

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**Definition 2.18.** Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and $B \in \mathcal{F}$ such that $\mathbf{P}(B) \neq 0$.

The *conditional probability of* $A \in \mathcal{F}$ *given* $B$ is defined as

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)}.$$

An immediate consequence of the definition is that $\mathbf{P}(\star \mid B)$ is a probability measure over $(\Omega, \mathcal{F})$.

**Proposition 2.19.** *If $A_1, \ldots, A_n$ (where $n \geq 2$) are events of the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that $\mathbf{P}(A_1 \cap \cdots \cap A_{n-1}) > 0$, then*

$$\mathbf{P}(A_1 \cap \cdots \cap A_n) = \mathbf{P}(A_1)\mathbf{P}(A_2 \mid A_1)\mathbf{P}(A_3 \mid A_1 \cap A_2) \ldots \mathbf{P}(A_n \mid A_1 \cap \cdots \cap A_{n-1}).$$

**Proof.** The result is proved by induction:
⋆ If $n = 2$, the equality follows from Definition 2.18.
⋆ If the equality holds for $n$, we write at the order $n + 1$

$$\mathbf{P}(A_1 \cap \cdots \cap A_{n+1}) = \mathbf{P}(A_1 \cap \cdots \cap A_n)\, \mathbf{P}(A_{n+1} \mid A_1 \cap \cdots \cap A_n).$$

The induction assumption allows us to conclude that the equality holds for $n + 1$. □

**Bernard:** Formally, I understand the objects and the definition. But I must admit that I don't grasp the purpose of conditioning a probability with respect to an event.

**Laurent:** Let me use the recent example of COVID-19 in France. At the end of 2021, it was said that among those admitted to hospitals, 50% had been vaccinated.

**Bernard:** I remember, Laurent. It was the reason why some people were saying, and I said that too, that the vaccine is useless.

**Laurent:** Here we are, Bernard. You have no idea of what conditioning is. You have to take into account the fact that, at that time, 75% of the French population had already been vaccinated.

To measure the true efficacy of the vaccine, you need to compare the probability of a vaccinated person catching the disease with the probability of an unvaccinated person catching the disease. And this is precisely a question of conditional probability.

**Bernard:** Can you give some concrete precision? With values?

**Laurent:** The exact values are not known, but let us imagine what is happening. Assume that the probability of being infected if you are not vaccinated is three times larger than if you are vaccinated.

**Bernard:** As an engineer, to figure this out, I need precise values. Assume that the probability of being infected if you are not vaccinated is 0.001. It corresponds, for a population like France's, which is about 65 million people with 25% of them not vaccinated, to roughly 16,000 infected unvaccinated people.

The numbers are certainly false, but the order of magnitude should be correct.

Besides, if I apply the same reasoning to the vaccinated population, knowing that the probability of being infected if one is vaccinated becomes one-third of the previous probability, there would be roughly 16,000 infected vaccinated people as well.

**Laurent:** With these data, among the infected population, it appears that one out of two people is vaccinated. And yet, the data we chose are such that the vaccine is very efficient since a vaccinated person has three times less chance of being infected than an unvaccinated person.

**Bernard:** I can't wait to read further on this, so that I can perform these computations...

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The two following results are sometimes called *Bayes' formulas.* Although their proofs are elementary, they are of huge importance to random modeling and statistics.

**Theorem 2.20 (Law of total probability).** *Let* $(E_n)_{n\in I}\in\mathcal{F}^{\mathbf{N}}$ *be a finite or countable partition of the probability space* $(\Omega, \mathcal{F}, \mathbf{P})$*.*

*Then, for all events* $A \in \mathcal{F}$*,*

$$\mathbf{P}(A) = \sum_{n\in I} \mathbf{P}(A \mid E_n)\,\mathbf{P}(E_n).$$

**Proof.** Writing

$$A = A \cap \left(\bigcup_{n\in I} E_n\right) = \bigcup_{n\in I} (A \cap E_n),$$

where the events $(A \cap E_n)_n$ are pairwise disjoint, we have

$$\mathbf{P}(A) = \mathbf{P}\left(\bigcup_{n \in I} (A \cap E_n)\right) = \sum_{n \in I} \mathbf{P}(A \cap E_n) = \sum_{n \in I} \mathbf{P}(A \mid E_n)\, \mathbf{P}(E_n).$$

□

**Theorem 2.21 (Bayes' theorem).** *Let $(E_n)_{n \in I} \in \mathcal{F}^{\mathbf{N}}$ be a finite or countable partition of the probability space $(\Omega, \mathcal{F}, \mathbf{P})$.*

*Then, for all events $A \in \mathcal{F}$ such that $\mathbf{P}(A) > 0$,*

$$\forall n \in I, \quad \mathbf{P}(E_n \mid A) = \frac{\mathbf{P}(A \mid E_n)\, \mathbf{P}(E_n)}{\sum_{m \in I} \mathbf{P}(A \mid E_m)\, \mathbf{P}(E_m)}.$$

**Proof.** By the definition of a conditional probability,

$$\mathbf{P}(E_n \mid A) = \frac{\mathbf{P}(A \cap E_n)}{\mathbf{P}(A)} = \frac{\mathbf{P}(A \mid E_n)\, \mathbf{P}(E_n)}{\mathbf{P}(A)}.$$

Theorem 2.20 allows us to write the denominator in the form

$$\mathbf{P}(A) = \sum_{m \in I} \mathbf{P}(A \mid E_m)\, \mathbf{P}(E_m).$$

□

**Bernard:** I imagine that I'm now able to perform the computation. Let's see. For an individual, we denote by:

- $\mathbf{P}(V) = 0.75$ the probability of being vaccinated,
- $\mathbf{P}(M \mid V^c) = 0.001$ the probability of being infected if not vaccinated,
- $\mathbf{P}(M \mid V) = \frac{1}{3} P(M \mid V^c)$ the probability of being infected if vaccinated.

With these data, I should be able to compute $\mathbf{P}(V \mid M)$, the proportion of vaccinated people among those infected:

$$\mathbf{P}(V \mid M) = \frac{\mathbf{P}(V \cap M)}{\mathbf{P}(M)} = \frac{\mathbf{P}(V \mid M)\ \mathbf{P}(V)}{\mathbf{P}(M \mid V)\ \mathbf{P}(V) + \mathbf{P}(M \mid V^c)\ \mathbf{P}(V^c)}$$
$$= \frac{\frac{1}{3}10^{-3}}{\frac{1}{3}10^{-3} \times 0.75 + 10^{-3} \times 0.25} = 0.5.$$

I realize that, since I was able to compute this proportion by myself, even with data which express that the vaccine is very effective, one can observe that one out of two infected individuals is vaccinated.

This observation doesn't cast doubt on the efficacy of the vaccine. My very first reasoning turned out to be completely wrong. Thank you, Laurent, for this important clarification.

**Laurent:** Definitely. Note that if $\mathbf{P}(M \mid V) = \mathbf{P}(M \mid V^c)$, which would mean that the vaccine is absolutely ineffective, the previous computation would lead to $\mathbf{P}(V \mid M) = 0.75$, that is, the proportion of vaccinated people.

We could also have considered the efficiency of a test, knowing the probability of it being a false positive.

Let us go further into the concept of independence.

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**Definition 2.22 (Independence of events).** In the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, two events $A$ and $B$ are said to be *independent* if $\mathbf{P}(A \cap B) = \mathbf{P}(A)\ \mathbf{P}(B)$.

Two events $A$ and $B$ are independent if and only if $A^c = \Omega \backslash A$ and $B$ are independent.
$\star$ If $A$ and $B$ are independent, $\mathbf{P}(A \cap B) = \mathbf{P}(A)\ \mathbf{P}(B)$. Then, we have

$$\begin{aligned}\mathbf{P}(A^c \cap B) &= \mathbf{P}(B \backslash A) = \mathbf{P}(B) - \mathbf{P}(A \cap B)\\ &= \mathbf{P}(B) - \mathbf{P}(A)\ \mathbf{P}(B) = \mathbf{P}(A^c)\ \mathbf{P}(B).\end{aligned}$$

This shows the independence of $A^c$ and $B$.
$\star$ Conversely, if $A^c$ and $B$ are independent, we replace $A$ with $A^c$ in the previous computation and show the independence of $A$ and $B$.

**Definition 2.23.** A collection $(A_i)_{i \in \mathcal{I}}$ of events of the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is said to be *independent* if, for all finite subsets $\mathcal{J}$ of $\mathcal{I}$,

$$\mathbf{P}\left(\bigcap_{i \in \mathcal{J}} A_i\right) = \prod_{i \in \mathcal{J}} \mathbf{P}(A_i).$$

Obviously, if $A_1, \dots, A_n$ are independent events, then they are *pairwise independent*. It suffices to consider the particular cases $\mathcal{J} = \{i, j\}$ for each couple $(i, j)$ of $\mathcal{I}$ such that $i \neq j$. The converse implication is false.

**Bernard:** I don't understand why it's false.

**Laurent:** Throw a dice twice in a row. Consider the following three events:

$$A_1 = \{\text{the result after throwing it the first time is } 6\},$$
$$A_2 = \{\text{the result after throwing it the second time is } 6\}, \text{and}$$
$$A_3 = \{\text{the results of both throws are equal}\}.$$

Are they independent?

**Bernard:** I can claim that, since the two throws are independent, the events $A_1$, $A_2$ and $A_3$ are pairwise independent and that

$$\mathbf{P}(A_1) = \mathbf{P}(A_2) = \mathbf{P}(A_3) = 1/6.$$

However, I observe that if $A_1$ and $A_2$ are satisfied, then is $A_3$ is also satisfied. It suggests to me that $A_1$, $A_2$ and $A_3$ are not independent.

**Laurent:** Right. We can check this rigorously by noting that

$$\mathbf{P}(A_1 \cap A_2 \cap A_3) = \mathbf{P}(A_1 \cap A_2) = \mathbf{P}(A_1)\,\mathbf{P}(A_2) = \frac{1}{36} \neq \frac{1}{216}$$
$$= \mathbf{P}(A_1)\,\mathbf{P}(A_2)\,\mathbf{P}(A_3).$$

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The independence of two events can be expressed using conditional probabilities.

**Proposition 2.24.** *In the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, let $A$ and $B$ be two events, with $\mathbf{P}(B) > 0$. Then, $A$ and $B$ are independent if and only if $\mathbf{P}(A \mid B) = \mathbf{P}(A)$.*

**Proof.** Since $\mathbf{P}(B) > 0$, the conditional probability $\mathbf{P}(A \mid B)$ is defined and

$$\mathbf{P}(A \cap B) = \mathbf{P}(A \mid B)\,\mathbf{P}(B).$$

Hence, Definition 2.22 implies that $A$ and $B$ are independent if and only if $\mathbf{P}(A \mid B) = \mathbf{P}(A)$. □

### 2.2.3 *Probability on a finite or countable space*

**Laurent:** Now that the framework of probability theory is settled, we can return to your question about constructing a measure. At first, let us consider the elementary case of a countable set $\Omega$, keeping in mind that the general case is the subject of the *Caratheodory theory.*

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In this section, let $\Omega$ be a countable set. Unless it is explicitly written, $\Omega$ is equipped with the $\sigma$-algebra $\mathcal{F} = \mathcal{P}(\Omega)$, which is the power set of $\Omega$.

**Theorem 2.25.** ⋆ *In the countable probability space,* $\big(\Omega, \mathcal{F} = \mathcal{P}(\Omega), \mathbf{P}\big)$, *the probability measure* $\mathbf{P}$ *is characterized by its values at every atom* $p_\omega = \mathbf{P}\big(\{\omega\}\big)$, $\omega \in \Omega$.
⋆ *Conversely, let* $\Omega = \{\omega_n;\ n \in \mathbf{N}\}$ *be a countable set and* $(p_n)_{n\in\mathbf{N}}$ *be a sequence of real values. Then, there exists a probability measure* $\mathbf{P}$ *on the* $\sigma$*-algebra* $\mathcal{F} = \mathcal{P}(\Omega)$ *satisfying* $\mathbf{P}\big(\{\omega_n\}\big) = p_n$ *(for all* $n \in \mathbf{N}$*) if and only if*

$$\forall n \in \mathbf{N}, \quad p_n \geq 0 \qquad \text{and} \quad \sum_{n=0}^{\infty} p_n = 1. \tag{2.2.1}$$

*Thus, given* $(p_n)_{n\in\mathbf{N}}$, *the probability* $\mathbf{P}$ *is unique.*

**Proof.** ⋆ Any $A \in \mathcal{F} = \mathcal{P}(\Omega)$ can be written as $A = \bigcup_{\omega\in A} \{\omega\}$. Since $A$ is finite or countable, the $\sigma$-additivity of the measure $\mathbf{P}$

yields

$$\mathbf{P}(A) = \mathbf{P}\left(\bigcup_{\omega \in A} \{\omega\}\right) = \sum_{\omega \in A} \mathbf{P}\big(\{\omega\}\big) = \sum_{\omega \in A} p_\omega.$$

This equality proves the first point.

⋆ Concerning the second point, note that if there exists a probability on $\mathcal{F} = \mathcal{P}(\Omega)$ such that $\mathbf{P}\big(\{\omega_n\}\big) = p_n$ (for all $n \in \mathbf{N}$), then $p_n \geq 0$, for all $n$, and

$$1 = \mathbf{P}(\Omega) = \mathbf{P}\left(\bigcup_{n \in \mathbf{N}} \{\omega_n\}\right) = \sum_{n \in \mathbf{N}} \mathbf{P}\big(\{\omega_n\}\big) = \sum_{n=0}^{\infty} p_n.$$

Conversely, if $(p_n)_{n \in \mathbf{N}}$ is a sequence of real values satisfying Conditions (2.2.1), then the mapping

$$\mathbf{P}\colon \mathcal{P}(\Omega) \to [0,1]$$
$$A \mapsto \mathbf{P}(A) = \sum_{n\colon\, \omega_n \in A} p_n$$

defines a probability measure.

Indeed, one can check that $\mathbf{P}(\emptyset) = 0$ and $\mathbf{P}(\Omega) = 1$. The $\sigma$-additivity results from the absolute convergence of the series $\sum p_n$. For all sequences $(A_n)_{n \in \mathbf{N}}$ of pairwise disjoint subsets of $\Omega$, one has

$$\mathbf{P}\left(\bigcup_{m \in \mathbf{N}} A_m\right) = \sum_{n\colon\, \omega_n \in \bigcup_m A_m} p_n = \sum_{m \in \mathbf{N}} \underbrace{\left(\sum_{n\colon\, \omega_n \in A_m} p_n\right)}_{\mathbf{P}(A_m)}$$

by splitting the sum over $\omega$ in $\bigcup_{m \in \mathbf{N}} A_m$ into the sum of sums over $\omega$ in each $A_m$.

The uniqueness of the probability measure $\mathbf{P}$ is a consequence of the first point. □

**Remark.** Theorem 2.25 only applies for $\mathcal{F} = \mathcal{P}(\Omega)$, with $\Omega$ being finite or countable. Indeed, one notes that if $\mathcal{F}$ contains all singletons $\{\omega\}$ $(\omega \in \Omega)$, then $\mathcal{F} = \mathcal{P}(\Omega)$.

**Bernard:** Wait a minute! It's not so clear to me... Don't forget, I'm a beginner in measure theory.

**Laurent:** The $\sigma$-algebra generated by the singletons, for a countable set, is the collection of all subsets.

In order to prove that, we first note that the collection of all subsets contains all singletons, which implies $\sigma\big(\{\omega\}; \omega \in \Omega\big) \subset \mathcal{P}(\Omega)$.

Conversely, for any $A \subset \Omega$, we write $A = \bigcup_{\omega \in A} \{\omega\}$, which belongs to $\sigma\big(\{\omega\}; \omega \in \Omega\big)$ since the union is countable.

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The following example illustrates the case where the $\sigma$-algebra $\mathcal{F}$ is not equal to $\mathcal{P}(\Omega)$.

**Example.** We do not assume any longer that $\Omega$ is finite or countable.

We equip $\Omega$ with the $\sigma$-algebra $\mathcal{F} = \{\emptyset, A, A^c, \Omega\}$ for some subset $A \subset \Omega$, $A \notin \{\emptyset, \Omega\}$.

Even in the particular case of a countable set $\Omega$, Theorem 2.25 does not apply since $\mathcal{F}$ does not contain all the singletons of $\Omega$.

However, we show that every probability measure $\mathbf{P}$ over $(\Omega, \mathcal{F})$ is characterized by the value of $\mathbf{P}(A)$:

- ⋆ If $\mathbf{P}$ is a probability measure on $\mathcal{F}$, then it is determined by $\mathbf{P}(\emptyset) = 0$, $\mathbf{P}(A) = p$ (with $p \in [0, 1]$), $\mathbf{P}(A^c) = 1 - p$ and $\mathbf{P}(\Omega) = 1$.
- ⋆ Conversely, if $p \in [0, 1]$, then there exists a probability measure over $(\Omega, \mathcal{F})$ such that $\mathbf{P}(A) = p$. Simply, let $\mathbf{P}(\emptyset) = 0$, $\mathbf{P}(A) = p$, $\mathbf{P}(A^c) = 1 - p$ and $\mathbf{P}(\Omega) = 1$ define a function $\mathbf{P}\colon \mathcal{F} \to [0, 1]$ that satisfies the conditions to be a probability measure.

**Laurent:** In the proof of Theorem 2.25 and in the previous example, we note that the determination of a probability measure on the $\sigma$-algebra $\mathcal{F}$ of a finite or countable set $\Omega$ follows from the property of $\sigma$-additivity of the measure.

Keep in mind that this is due to the cardinality of $\mathcal{F}$ and not that of $\Omega$.

We will see later on how to define a probability measure on a noncountable $\sigma$-algebra.

### 2.2.4 *Random variables and laws*

**Bernard:** How are all these things linked to the well-known notion of a random variable?

**Laurent:** It is precisely where we are now, Bernard. A *random variable* is a measurable function in the particular case of a probability space.

And the interest lies in describing the distribution of the values of a random variable $X$ through a specific measure related to $X$.

$\mathbf{E}[\varphi(X)]$ 519.FRA.f

The notion of a random variable $X$ is an elementary concept in probability theory. It represents the result of observing with respect to the area $\omega$. The fact that $X(\omega)$ is generally different for two different values of $\omega$ represents the random character of $X$.

Mathematically, the variability of the quantity $X(\omega)$ simply means that $X$ is a function from the state space $\Omega$ onto its space of values $E$. Actually, we are more interested in the distribution of the values of $X$ in $E$ than the precise values of $X$.

The framework of measure theory is suitable for this description. This is the reason why a random variable is the probabilistic counterpart of a measurable function.

**Definition 2.26.** Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and $(E, \mathcal{E})$ be a measurable space. A *random variable* with values in $E$ is a measurable function $X: \Omega \to E$, i.e. such that

$$\forall A \in \mathcal{E}, \quad X^{-1}(A) \in \mathcal{F}.$$

**Example.** Assume that we want to evaluate the altitude $h \in \mathbf{R}_+^*$ of a flying airplane. Several apparatus allow this physical measurement: radiosonde, GPS, altimeter, etc. Each of these sensors is marred with measurement errors. So, the result is a quantity $Z$ that can be considered as a random variable with values in $\mathbf{R}_+^*$.

**Remark.** When $E$ is countable and is equipped with the $\sigma$-algebra $\mathcal{E} = \mathcal{P}(E)$, Definition 2.26 is equivalent to $\forall e \in E, \quad X^{-1}(\{e\}) \in \mathcal{F}$.

**Bernard:** What's "physical measurement" supposed to mean? There are too many notions of measure or measurement for me...

**Laurent:** Here, a physical measurement is to be taken in the common sense. Did you understand well the remark concerning the countable case?

**Bernard:** I think so. At first, if $E$ is countable and equipped with $\mathcal{E} = \mathcal{P}(E)$ and if $X$ is a random variable, we can apply the definition to $A = \{e\}$ for all $e \in E$. This shows the direct implication of the equivalence.

For the converse case, if $X^{-1}(\{e\}) \in \mathcal{F}$ for all $e \in E$,

$$\forall A \subset E, \quad X^{-1}(A) = \bigcup_{e \in A} X^{-1}(\{e\}) \in \mathcal{F}.$$

**Laurent:** Excellent, Bernard! Let us turn to a crucial concept in probability theory: the law of a random variable.

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**Definition 2.27.** Let $X\colon (\Omega, \mathcal{F}) \to (E, \mathcal{E})$ be a random variable. The sub-$\sigma$-algebra $X^{-1}(\mathcal{E}) = \big\{X^{-1}(A);\ A \in \mathcal{E}\big\}$ of $\mathcal{F}$ is called *the $\sigma$-algebra generated by $X$* and is denoted by $\sigma(X)$.

We can extend this definition to a collection of random variables. We define the $\sigma$-algebra generated by the collection $(X_i)_{i \in I}$, for any set $I$, by

$$\sigma(X_i;\ i \in I) = \sigma\left(\big\{X_i^{-1}(A);\ A \in \mathcal{E}, i \in I\big\}\right).$$

Clearly, the $\sigma$-algebra $\sigma(X_i;\ i \in I)$ is the smallest $\sigma$-algebra on $\Omega$ such that, for each $i \in I$, $X_i$ is measurable.

**Bernard:** Yes, yes, I see: for all $i \in I$ and for all $B \in \mathcal{E}$, $X_i^{-1}(B) \in \sigma(X_i;\ i \in I)$. Thus, all the $X_i$'s are measurable with respect to $\sigma(X_i;\ i \in I)$.

Conversely, if all the $X_i$'s are measurable with respect to some $\sigma$-algebra $\mathcal{G}$, for all $i \in I$ and for all $B \in \mathcal{E}$, we have $X_i^{-1}(B) \in \mathcal{G}$. This implies that $\sigma(X_i;\ i \in I) \subset \mathcal{G}$.

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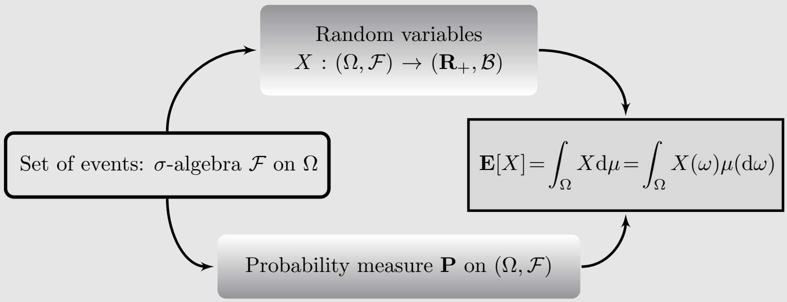
The measurability of $X$ allows one to transfer to $(E, \mathcal{E})$ the measure of probability defined on $(\Omega, \mathcal{F})$.




**Definition 2.28.** Let $X\colon (\Omega, \mathcal{F}) \to (E, \mathcal{E})$ be a random variable. The function $P_X\colon \mathcal{E} \to [0, 1]$, defined as

$$\forall A \in \mathcal{E}, \quad P_X(A) = \mathbf{P}\big(\{\omega : X(\omega) \in A\}\big) = \mathbf{P}\big(X^{-1}(A)\big),$$

is a probability measure on the space $(E, \mathcal{E})$, which is called the *pushforward measure* of $\mathbf{P}$ by $X$. Let $\mathbf{P}(X \in A)$ denote the quantity $P_X(A)$. The probability measure $P_X$ is called the *distribution* or *law of* $X$.

Let $X$ and $Y$ be two random variables over $(E, \mathcal{E})$.

The expression $X = Y$ $\mathbf{P}$-*almost surely* (or, simply, *almost surely* if there is no ambiguity about the measure, a.s. for short) means $\mathbf{P}(X = Y) = 1$, i.e. if $P_{X-Y}(\{0\}) = 1$.

**Laurent:** Listen to me, Bernard. We will see that the purpose of probability is not to study random variables as functions of $\omega$.

The purpose of probability is to study the distribution of the values of random variables. And the mathematical concept which allows us to do so is the laws of random variables.

You should be able to check that $\mathbf{P} \circ X^{-1}$ is a probability measure over $(E, \mathcal{E})$.

**Bernard:** It is clear to me. But the interest in this notion of "almost surely" is less clear to me.

**Laurent:** Since the question pertains to the distribution of the values of the random variable $X$, if $X = Y$ a.s., the values of $X$ are the same as those of $Y$, except for $\omega$ belonging to a set of measure zero.

Defining $\Omega^* = \{\omega \in \Omega : X(\omega) = Y(\omega)\}$, we have $\mathbf{P}(\Omega^*) = 1$, as is stated by the definition.

One can see it in another way: denoting $\Omega_0 = \{\omega \in \Omega : X(\omega) \neq Y(\omega)\}$, to say that $X = Y$ a.s. means $\mathbf{P}(\Omega_0) = 0$.

Our interest in the notion of almost sure equality will become evident again when the integral with respect to the probability measure $\mathbf{P}$ is defined.

*The framework of measure theory was established. By virtue of its historical development, it encompassed that of probability theory. The abstract framework of General topology had prepared Bernard for the dryness of the first theoretical elements:*

- *$\sigma$-algebras constitute the structure of sets that underlie the construction of the integral.*
- *Measurable functions, whose definitions rely on that of $\sigma$-algebras, are those whose integrals with respect to a measure can be constructed.*
- *Measures extend to a $\sigma$-algebra the notion of cardinality of a set or the length of an interval.*
- *A random variable is a special case of a measurable function in the probability framework.*
- *The law of a random variable measures the distribution of its values.*

*Aware that he is only just beginning to study Henri Brolle's book, Bernard is a little frustrated at feeling he is no closer to his target. He feels torn. On the one hand, he is inclined to let himself be convinced by this brilliant mathematician, his father's friend, who urges him to understand the concepts deeply. On the other hand, he absolutely needs to convince Ann, his boss at SofterWorld Consulting, that these in-depth mathematical studies are important for understanding the study assigned to him.*

*Fortunately, he realizes all the same that the stochastic framework will allow him to describe the variability of physical quantities, such as the velocity of a river and the concentration of rejected waste.*

*But how will these elements enable him to save his reputation within his own company and establish his position at the pharmaceutical plant, all the while minimizing the environmental risk? Convincing Ann will require truly concrete elements of modeling.*

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## Chapter 3

# Integration with Respect to a Measure I: Construction

*May 16th, in the early morning.*

*The young engineer had awakened from a sleep filled with integrals and measure spaces. He had spent his entire evening rereading the notes he had written down during his last meeting with Laurent. He is aware that he is entering a new world, by discovering measure theory, but he wonders about the concrete links with the integral of functions.*

*Naturally, for Bernard, the word "integral" is still attached to infinitesimal calculus. He discovered it in this last context, and it is not easy for him to brush aside an intuition he has matured over some many years. Indeed, this measure theory represents for him a kind of foreign country, whose culture and language are unknown to him.*

*He is in a hurry to meet his father's friend at the Centre Emile Borel. Today's session is supposed to be devoted to the construction of the integral with respect to a measure. How will the concepts of measurable functions and* $\sigma$*-algebras meet to form an integral?*

*Besides, he hopes to gain insights from his exchanges with Laurent that would convince his boss that a deep understanding of mathematical concepts will enhance his expertise in answering her complex questions on modeling.*

## 3.1 Integral with Respect to a Measure

**Bernard:** (smiling) At last! I am about to understand how these concepts are really necessary to integrate a function. How does it relate to an integral? Recall that we only want to measure the area under a curve...

**Laurent:** It is true that, for a continuous real-valued function $f$, the integral over $[0, 1]$ is simply a fraction of the area under $y = f(x)$. But remember what we discussed last weak about our interest in considering functions that are not necessarily continuous. For instance, what do you think about the particular case of the function $\mathbf{1}_{\mathbf{Q}}$?

**Bernard:** OK, you win! The curve is not drawable. I need to think about this. Let's go back to Henri Brolle's book.

### 3.1.1 *Simple functions*


The step functions play an important role in the construction of the Riemann integral. More than its use in defining objects, from a practical point of view, it provides an approximation of (Riemann) integrals by the Riemann sums.

In measure theory, *simple functions* constitute the starting point of the construction of the integral.

In the construction of the integral, we consider functions taking their values in $\mathbf{R}$ or $\overline{\mathbf{R}}_+ = \mathbf{R}_+ \cup \{+\infty\} = [0, \infty]$, equipped with the Borel $\sigma$-algebras $\mathcal{B}(\mathbf{R})$ and $\mathcal{B}\big([0, \infty]\big)$.

**Definition 3.1.** A measurable function $f$ from $(\Omega, \mathcal{A})$ to $\big(\mathbf{R}, \mathcal{B}(\mathbf{R})\big)$ (resp., to $\Big([0, \infty], \mathcal{B}\big([0, \infty]\big)\Big)$) is called a *simple function* (resp., a *nonnegative simple function*) if it only takes a finite number of values.

Any simple function different from the zero function (resp., any nonnegative simple function) can be uniquely expressed in the so-called *canonical form* $f = \sum_{k=1}^{N} \alpha_k \mathbf{1}_{A_k}$, where

- ⋆ $N \in \mathbf{N}^*$,
- ⋆ $(\alpha_k)_{1 \le k \le N}$ are in $\mathbf{R}^*$ (resp., in $[0, \mathbf{R}^*_+)$ such that $\alpha_1 < \cdots < \alpha_N$,
- ⋆ $A_1, \ldots, A_N$ are nonempty and pairwise disjoint in $\mathcal{A}$.

The set of simple functions is a vector space: $\mathcal{E}(\mathcal{A}) = \mathrm{Span}_{\mathbf{R}}(\mathbf{1}_A; A \in \mathcal{A})$.

The set of nonnegative simple functions is denoted by

$$\mathcal{E}_+(\mathcal{A}) = \left\{ \sum_{k=1}^{N} \alpha_k \mathbf{1}_{A_k};\ N \in \mathbf{N}^*,\ \alpha_1, \ldots, \alpha_N \in (0, \infty], \right.$$
$$\left. A_1, \ldots, A_N \in \mathcal{A} \right\}.$$

**Bernard:** Please, Laurent, how can we see why there cannot exist different ways to express a simple function?

**Laurent:** Be careful! It is not true in general. But it becomes true if we impose that $(\alpha_k)_{1 \le k \le N}$ are distinct values. We can even be precise in that the $(A_k)_{1 \le k \le N}$ are uniquely determined by $A_k = f^{-1}(\{\alpha_k\})$ for all $1 \le k \le N$.

**Bernard:** As usual, I need to try to prove it by myself in order to understand this point well.

First of all, we consider the $N$ distinct nonzero values taken by the function $f$, and we order them such as $\alpha_1 < \alpha_2 < \cdots < \alpha_N$.

Then, for each $k \in \{1, \ldots, N\}$, we set $A_k = f^{-1}\big(\{\alpha_k\}\big)$ and $g = \sum_{k=1}^{N} \alpha_k \mathbf{1}_{A_k}$. In addition, we set $\alpha_0 = 0$ and $A_0 = f^{-1}(\{0\}) = \Omega \setminus (A_1 \cup \cdots \cup A_N)$.

The $(\alpha_k)_{0 \le k \le N}$ are all the possible values for $f$ and each $x \in \Omega$ belongs to exactly one $A_k$, which leads to $f(x) = \alpha_k = g(x)$.

We can conclude that for all $x \in \Omega$, $f(x) = g(x)$. OK, I understand that the conditions on $(\alpha_k)_{1 \le k \le N}$ and $(A_k)_{1 \le k \le N}$ give the uniqueness of the expression $f = \sum_{k=1}^{N} \alpha_k \, \mathbf{1}_{A_k}$.

**Laurent:** Now, let us note something that will be useful when comparing this integral with the old Riemann integral. In the case of $\Omega = [a, b] \subset \mathbf{R}$, we can note that:

*step functions are a particular case of simple functions.*

Recall that a step function is a finite linear combination of indicator functions of intervals of $\mathbf{R}$. More precisely, $f\colon [a, b] \to \mathbf{R}$ is a step function if there exist an integer $n \ge 2$, a subdivision $a = a_0 < a_1 < \cdots < a_n = b$ of $[a, b]$ and real numbers $(\alpha_k)_{0 \le k \le n-1}$ and $(\beta_k)_{0 \le k \le n}$ such that

$$f = \sum_{k=0}^{n-1} \alpha_k \mathbf{1}_{(a_k, a_{k+1})} + \sum_{k=0}^{n} \beta_k \mathbf{1}_{\{a_k\}}.$$

This means, for all $x \in \mathbf{R}$,

$$f(x) = \begin{cases} \alpha_k & \text{if } a_k < x < a_{k+1}, \\ \beta_k & \text{if } x = a_k, \\ 0 & \text{otherwise.} \end{cases}$$

You can remark that a step function only takes a finite number of distinct values and that the intervals $\big((a_k, a_{k+1})\big)_{0 \le k \le n-1}$ and singletons $\big(\{a_k\}\big)_{0 \le k \le n}$ are Borel sets. Thus, any step function is a Borel simple function.

But since $(A_k)_{1 \le k \le N}$ in the expression of a simple function are not always intervals of $\mathbf{R}$, you can also remark that a simple function is not necessarily a step function.

**Bernard:** Something is clear to me: on the one hand, we discretize the domain of $x$ for step functions and, on the other hand, we look at the values taken by simple functions.

**Laurent:** Exactly. It is the major difference between Lebesgue and Riemann integrals. Before that, let us see how measurable functions can be approximated by simple functions.

The role of simple functions in the integration of measurable functions comes from the following result: any nonnegative measurable function can be approximated pointwise by simple functions.

**Theorem 3.2.** *For any measurable function* $f\colon (\Omega, \mathcal{A}) \to \big([0,\infty], \mathcal{B}([0,\infty])\big)$, *there exists a nondecreasing sequence of finite nonnegative simple functions* $(f_n)_{n\in\mathbf{N}^*}$ *which converges pointwise to* $f$.

**Proof.** For all $n \geq 1$ and for all $i \in \{0, 1, \ldots, n\,2^n - 1\}$, set

$$A_n = \{x : f(x) \geq n\}$$

$$B_{n,i} = \{x : i2^{-n} \leq f(x) < (i+1)2^{-n}\}$$

$$\text{and} \quad f_n = \sum_{i=0}^{n2^n-1} (i2^{-n})\mathbf{1}_{B_{n,i}} + n\mathbf{1}_{A_n}.$$

Since $f$ is measurable, for all $n$ and $i$, the sets $A_n$ and $B_{n,i}$ are in $\mathcal{A}$, and $f_n$ is a simple function.

Let us fix $x \in \Omega$.

$\star$ Either $x$ belongs to one $B_{n,i}$: $i\,2^{-n} \leq f(x) < (i+1)\,2^{-n}$.
In that case, either $x \in B_{n+1,2i}$ or $x \in B_{n+1,2i+1}$. This leads to

$$f_n(x) = i2^{-n} \leq f_{n+1}(x) = 2i2^{-(n+1)} \quad \text{or} \quad (2i+1)2^{-(n+1)}.$$

$\star$ Or $x$ belongs to one $A_n$.
In that case, $x \in B_{n+1,(n+1).2^{n+1}-1}$ or $x \in A_{n+1}$. This leads to

$$f_n(x) = n \leq f_{n+1}(x) = \big((n+1).2^{n+1} - 1\big)2^{-(n+1)} \quad \text{or} \quad n+1.$$

In each case, we have $f_{n+1}(x) \geq f_n(x)$.
Finally, note that for all $x \in \Omega$, $f_n(x)$ converges to $f(x)$ as $n \to \infty$. □

### 3.1.2 *Integration of nonnegative functions*

**Laurent:** The approximation of a finite nonnegative measurable function by simple functions is used to define the integral with respect to a measure starting from a definition for simple functions.

$\int f\, d\mu$ 515.BRO.m

We consider a measure space $(\Omega, \mathcal{A}, \mu)$, where $\mathcal{A}$ is a $\sigma$-algebra on $\Omega$ and $\mu$ a measure on $\mathcal{A}$.

The first step of the construction of the integral with respect to $\mu$ is the integration of nonnegative simple functions.

**Definition 3.3.** Let $f\colon (\Omega, \mathcal{A}) \to \big(\mathbf{R}_+, \mathcal{B}(\mathbf{R}_+)\big)$ be a simple function of the form $f = \sum_{i=1}^n \alpha_i \mathbf{1}_{A_i}$, where $(\alpha_i)_{1\le i\le n}$ are in $\mathbf{R}_+$ such that $0 < \alpha_1 < \cdots < \alpha_n < \infty$ and $(A_i)_{1\le i\le n}$ are nonempty pairwise disjoint elements of $\mathcal{A}$.

The integral of $f$ with respect to the measure $\mu$ on $\mathcal{A}$ is defined by

$$\mathcal{I}_\mu(f) = \sum_{i=1}^n \alpha_i \mu(A_i). \tag{3.1.1}$$

Moreover, the integral of the zero function is set to 0.

The integral is usually denoted by $\mu(f)$, $\int f\, d\mu$ or $\int_\Omega f(x)\mu(dx)$.

**Remark.** The unique writing of $f$ in the canonical form $f = \sum_{i=1}^n \alpha_i \mathbf{1}_{A_i}$, with $0 < \alpha_1 < \cdots < \alpha_n$ and $(A_i)_{1\le i\le n}$ nonempty and pairwise disjoint, leads to the well-defined quantity $\sum_{i=1}^n \alpha_i \mu(A_i)$.

However, it is not difficult to check that for any writing of the simple function (possibly zero) in the form $f = \sum_{j=1}^m \beta_j \mathbf{1}_{B_j}$, with the convention $0.\infty = 0$ when $\beta_j = 0$ and $\mu(B_j) = \infty$, the equality $\int f\, d\mu = \sum_{j=1}^m \beta_j \mu(B_j)$ still holds.

This remark is important in the proof of the following properties of the integral.

**Remark.** We could have defined the integral of a non-negative simple function possibly taking the infinite value, $f\colon (\Omega, \mathcal{A}) \to \big([0,\infty], \mathcal{B}([0,\infty])\big)$, thanks to Expression (3.1.1) with the additional convention $\infty.0 = 0$ if $\alpha_j = \infty$ and $\mu(A_j) = 0$.

**Laurent:** Let me check that you can prove that the equality still holds for any writing of the function $f$. There are two different cases which make the expression $f = \sum_{j=1}^{m} \beta_j \mathbf{1}_{B_j}$ not canonical in the sense of Definition 3.1.

**Bernard:** Yes, if I consider the definition of the canonical expression for a simple function, the two cases that appear to me are as follows:

- $(B_j)_{1\leq j\leq m}$ are pairwise disjoint, but $(\beta_j)_{1\leq j\leq m}$ are not distinct.
- $(\beta_j)_{1\leq j\leq m}$ are distinct, but $(B_j)_{1\leq j\leq m}$ are not pairwise disjoint.

**Laurent:** Absolutely, Bernard. And what happens in each case?

**Bernard:** In the first case, for $j \neq k$ such that $\beta_j = \beta_k$, I substitute the couple $(B_j, B_k)$ with $(B_j \cup B_k, \emptyset)$. Then, the corresponding contribution of $(j,k)$ in the sum becomes $\beta_j \mu(B_j \cup B_k) = \beta_j\, \mu(B_j) + \beta_k\, \mu(B_k)$.

Applying this substitution for each couple $(j,k)$ as above, we obtain the canonical form of $f$, and we observe that the sum $\sum_{j=1}^{m} \beta_j \mu(B_j)$ remains unchanged. Thus, it is a correct expression for the integral.

**Laurent:** Good. And in the second case?

**Bernard:** For $j \neq k$ such that $B_j \cap B_k \neq \emptyset$, we have

$$\beta_j \mathbf{1}_{B_j} + \beta_k \mathbf{1}_{B_k} = \beta_j \mathbf{1}_{B_j \setminus (B_j \cap B_k)} + \beta_k \mathbf{1}_{B_k \setminus (B_j \cap B_k)} + (\beta_j + \beta_k)\mathbf{1}_{B_j \cap B_k},$$

where $B_j \setminus (B_j \cap B_k), B_k \setminus (B_j \cap B_k), B_j \cap B_k$ are pairwise disjoint. And the contribution of $(j,k)$ to the integral is

$$\begin{aligned}&\beta_j \mu(B_j \setminus (B_j \cap B_k)) + \beta_k \mu(B_k \setminus (B_j \cap B_k)) + (\beta_j + \beta_k)\mu(B_j \cap B_k)\\ &\quad = \beta_j \mu(B_j) + \beta_k \mu(B_k).\end{aligned}$$

In the same way as in the first case, the sum $\sum_{j=1}^{m} \beta_j \mu(B_j)$ is a correct expression for the integral $s\int f\, d\mu$.

**Laurent:** Now, let us proceed with the properties of this integral of nonnegative simple functions.

**Proposition 3.4.** *Let $f$ and $g$ be two nonnegative simple functions from $(\Omega, \mathcal{A})$ to $\big(\mathbf{R}_+, \mathcal{B}(\mathbf{R}_+)\big)$ and $\mu$ be a measure over $(\Omega, \mathcal{A})$.*

(i) *For all* $\alpha$ *and* $\beta$ *in* $\mathbf{R}_+$, $\int(\alpha f + \beta g)\,d\mu = \alpha\int f\,d\mu + \beta\int g\,d\mu$.
(ii) *If* $f \leq g$, *then* $\int f\,d\mu \leq \int g\,d\mu$.

**Proof.** (i) Consider the expressions of the simple functions $f = \sum_{i=1}^{n} \alpha_i \mathbf{1}_{A_i}$, with $0 < \alpha_1 < \cdots < \alpha_n$ and $A_1, \cdots, A_n$ pairwise disjoint, and $g = \sum_{j=1}^{m} \beta_j \mathbf{1}_{B_j}$, with $0 < \beta_1 < \cdots < \beta_m$ and $B_1, \ldots, B_m$ pairwise disjoint.

In order to manipulate simultaneously $f$ and $g$, we need to express the two functions in a common partition of $\Omega$.

Let $A_0 = \Omega \setminus \left(\bigcup_{1\leq i\leq n} A_i\right)$, $B_0 = \Omega \setminus \left(\bigcup_{1\leq j\leq m} B_j\right)$ and $\alpha_0 = \beta_0 = 0$.

For each $i = 0, \ldots, n$, we write $A_i = \bigcup_{j=0}^{m}(A_i \cap B_j)$, and for each $j = 0, \ldots, m$, we write $B_j = \bigcup_{i=0}^{n}(A_i \cap B_j)$ (using the fact that $(A_i)_{0\leq i\leq n}$ and $(B_j)_{0\leq j\leq m}$ are two partitions of $\Omega$). Then, we have

$$f = \sum_{i=0}^{n} \alpha_i \mathbf{1}_{\bigcup_{j=0}^{m}(A_i\cap B_j)} = \sum_{i=0}^{n} \alpha_i \sum_{j=0}^{m} \mathbf{1}_{A_i\cap B_j} = \sum_{k=1}^{(n+1)(m+1)} \gamma_k \mathbf{1}_{C_k},$$

where $(C_k)_{1\leq k\leq (n+1)(m+1)}$ is an enumeration of $(A_i \cap B_j)_{(i,j)}$.

In the same way, we can write

$$g = \sum_{j=0}^{m} \beta_i \mathbf{1}_{\bigcup_{i=0}^{n}(A_i\cap B_j)} = \sum_{j=0}^{m} \beta_i \sum_{i=0}^{n} \mathbf{1}_{A_i\cap B_j} = \sum_{k=1}^{(n+1)(m+1)} \delta_k \mathbf{1}_{C_k}.$$

Hence,

$$\int f\,d\mu = \sum_{k=1}^{(n+1)(m+1)} \gamma_k \mu(C_k) \quad \text{and} \quad \int g\,d\mu = \sum_{k=1}^{(n+1)(m+1)} \delta_k \mu(C_k).$$

We also have

$$\int(\alpha f + \beta g)\,d\mu = \sum_{k=1}^{(n+1)(m+1)} (\alpha\gamma_k + \beta\delta_k)\mu(C_k),$$

and the result follows.

(ii) For this inequality, it suffices to remark that

$$\int g\,d\mu = \int f\,d\mu + \int (g-f)\,d\mu \geq \int f\,d\mu$$

using the fact that $\int (g-f)\,d\mu \geq 0$ due to the definition of the integral of a nonnegative simple function. $\square$

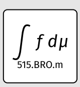
**Bernard:** I feel that this definition of the integral will allow me to be at peace with something that I have tried unsuccessfully to understand before. I was always told that the integral $\int$ of a discrete function could be written as a sum $\sum$.

I understand that by construction, it is obvious for a function that takes a finite number of values. The mystery dissipates...

**Laurent:** But the definition will be extended to much more than the integral of simple functions and will get further and further away from a discrete sum.

However, a common point we can rely on is the fact that an integral is a linear form on a well-chosen vector space. Of course, we are not there yet, since the set $\mathcal{E}_+$ of nonnegative simple functions is not a vector space.

**Bernard:** Of course, since if $f$ is in $\mathcal{E}_+$, $-f$ is not...




**Definition 3.5.** Let $f\colon (\Omega, \mathcal{A}) \to \big([0,\infty], \mathcal{B}([0,\infty])\big)$ be a nonnegative valued measurable function and $\mu$ be a measure over $(\Omega, \mathcal{A})$.

The integral of $f$ with respect to the measure $\mu$ is defined by

$$\int f\,d\mu = \sup_{\substack{h\in\mathcal{E}_+(\mathcal{A}),\\ h \text{ finite},\\ h\leq f}} \left(\int h\,d\mu\right).$$

The integral is also denoted by $\mathcal{I}_\mu(f)$, $\mu(f)$ or $\int_\Omega f(x)\,\mu(dx)$.

**Bernard:** I get anxious when definitions go by the dozen. Do these definitions give the same value in the end for a simple function?

**Laurent:** Yes, Bernard. If $f$ is a finite nonnegative simple function, for any finite nonnegative simple function $h \leq f$, we just saw that $\int h\, d\mu \leq \int f\, d\mu$.

Thus, $\int f\, d\mu$ in the sense of Definition 3.3 is equal to $\sup_{\substack{h \in \mathcal{E}_+(\mathcal{A}),\\ h \text{ finite},\\ h \leq f}} \int h\, d\mu$.

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**Proposition 3.6.** *Let $f$ and $g$ be two functions from $(\Omega, \mathcal{A})$ to $\big([0,\infty], \mathcal{B}([0,\infty])\big)$ and $\mu$ be a measure over $(\Omega, \mathcal{A})$.*

(i) *If $f \leq g$, then $\int f\, d\mu \leq \int g\, d\mu$.*
(ii) *If $\mu\big(\{x \in \Omega : f(x) > 0\}\big) = 0$, then $\int f\, d\mu = 0$.*

**Proof.** (i) Since $f \leq g$, we have

$$\int f\, d\mu = \sup_{\substack{h \in \mathcal{E}_+(\mathcal{A}),\\ h \text{ finite},\\ h \leq f}} \left(\int h\, d\mu\right) \leq \sup_{\substack{h \in \mathcal{E}_+(\mathcal{A}),\\ h \text{ finite},\\ h \leq g}} \left(\int h\, d\mu\right) = \int g\, d\mu.$$

(ii) $\star$ Assume first that $f$ is a nonnegative simple function: $f = \sum_{i=1}^{n} \alpha_i\, \mathbf{1}_{A_i}$, with $0 \leq \alpha_1 < \cdots < \alpha_n$ and $A_1, \ldots, A_n$ pairwise disjoint.

For all $i \in \{1, \ldots, n\}$ such that $\alpha_i > 0$, we have

$$\{x : f(x) = \alpha_i\} \subset \{x : f(x) > 0\},$$

and consequently,

$$\mu\big(\{x : f(x) > 0\}\big) = 0 \Rightarrow \mu(A_i) = \mu\big(\{x : f(x) = \alpha_i\}\big) = 0.$$

This leads to

$$\int f\, d\mu = \sum_{i=1}^{n} \alpha_i\, \mu(A_i) = 0.$$

$\star$ In the general case of a nonsimple function $f$, for all $h \in \mathcal{E}_+(\mathcal{A})$ such that $h \leq f$, if $\mu\left(\{x : f(x) > 0\}\right) = 0$, we have $\mu\big(\{x : h(x) > 0\}\big) = 0$, and consequently, $\int h\, d\mu = 0$. From the definition of the integral, we deduce that $\int f\, d\mu = 0$. $\square$

The linearity of the integral will be proved as a consequence of the following theorem, which is the first result about interchanging integral and limit.

**Laurent:** The issue of interchanging a limit and an integral is crucial in the theory of integration. There was a time, that you may have not known, when the subjects of the *concours* of the *Grandes Ecoles*, maybe not *CeePlus*, were only about showing the limit of the integral of complicated functions depending on a parameter.

To be more precise, the question can be expressed as follows:

> If $(f_n)_{n\in\mathbf{N}}$ is a sequence of functions that converges pointwise, do we have $\int(\lim f_n)\,d\mu = \lim\left(\int f_n\,d\mu\right)$?

**Bernard:** I already saw a monotone convergence theorem with Riemann's integral...

**Laurent:** (getting angry) Stop it, Bernard. You cannot say that! There is no general interchange theorem for Riemann's integral. The only reason why your *classes préparatoires*' teacher was able to state such an interchange theorem was that he was only interested in piecewise continuous functions over a segment.

For these functions, the Riemann integral coincides with the integral with respect to the Lebesgue measure. In fact, you did not study the real Riemann integral. I can tell you that it concerns a far larger class than the set of piecewise continuous functions.

And it is only in measure theory that such an interchanging theorem exists. It means you were tricked into thinking it was a real result of the Riemann integral. But, considering your math level, it must not have been too difficult to trick you...

**Bernard:** Don't get mad, Laurent. You must be right. As a matter of fact, I never saw the proof of such a result. I am therefore most curious about it.

**Theorem 3.7. (Monotone convergence theorem).** *Let $(f_n)_{n\in\mathbf{N}}$ be a nondecreasing sequence of measurable functions from $(\Omega,\mathcal{A})$ to $\big([0,\infty],\mathcal{B}([0,\infty])\big)$ and $\mu$ be a measure over $(\Omega,\mathcal{A})$. Let $f=\lim\uparrow f_n$, i.e. $\forall n\in\mathbf{N}$, $f_n\le f_{n+1}$ and $f_n\to f$.*

*Then,*

$$\int f\,d\mu = \lim\uparrow\int f_n\,d\mu.$$

**Proof.** $\star$ We first prove that $\int f\,d\mu \geq \lim\uparrow \int f_n\,d\mu$.

From Proposition 3.6, since $f_n \leq f_{n+1}$, we have $\int f_n\,d\mu \leq \int f_{n+1}\,d\mu$. Then, the sequence $\left(\int f_n\,d\mu\right)_{n\in\mathbf{N}}$ is nondecreasing.

The limit function $f$ is measurable and nonnegative, which implies that the integral $\int f\,d\mu$ is well defined. Since $f_n \leq f$ for all $n \in \mathbf{N}$, the sequence $\left(\int f_n\,d\mu\right)_{n\in\mathbf{N}}$ is bounded from above by $\int f\,d\mu$.

We deduce that $\left(\int f_n\,d\mu\right)_{n\in\mathbf{N}}$ admits a limit and $\int f\,d\mu \geq \lim\uparrow \int f_n\,d\mu$.

$\star$ In order to prove the converse inequality, we consider a (finite) function $h \in \mathcal{E}_+(\mathcal{A})$ such that $h \leq f$: $h = \sum_{i=1}^m \alpha_i\,\mathbf{1}_{A_i}$.

Let $a$ be any real number in $(0,1)$. Let us define $E_n = \{x \in \Omega : a\,h(x) \leq f_n(x)\}$.

Since $h$ and $f_n$ are measurable functions, we have $E_n \in \mathcal{A}$.

For all $n \in \mathbf{N}$, we have $f_n \geq a\,h\,\mathbf{1}_{E_n}$, and then

$$\int f_n\,d\mu \geq \int ah\mathbf{1}_{E_n}\,d\mu = a\sum_{i=1}^m \alpha_i\mu(A_i \cap E_n). \tag{3.1.2}$$

From the facts $f = \lim\uparrow f_n$, $h \leq f$ and $0 < a < 1$, we deduce that

$$\forall n \in \mathbf{N}, \quad E_n \subset E_{n+1} \quad \text{and} \quad \Omega = \bigcup_{n\in\mathbf{N}} E_n.$$

Then, for each $i = 1, \ldots, m$, $(A_i \cap E_n)$ grows to $A_i$ as $n \uparrow \infty$ and taking the limit $n \uparrow \infty$ in (3.1.2), we get

$$\lim\uparrow \int f_n\,d\mu \geq a\sum_{i=1}^m \alpha_i\mu(A_i) = a\int h\,d\mu.$$

Since this inequality holds for all $a \in (0,1)$, having $a$ tending to 1, we get

$$\forall h \in \mathcal{E}_+(\mathcal{A}) : h \leq f, \quad \lim\uparrow \int f_n\,d\mu \geq \int h\,d\mu.$$

This leads to

$$\lim\uparrow \int f_n\,d\mu \geq \int f\,d\mu.$$

$\square$

**Bernard:** What is this $a$? Why can't we take $a = 1$ at once?

**Laurent:** To make the proof correct, it is crucial that $\Omega = \bigcup_{n\in\mathbf{N}} E_n$, that is, for any $x \in \Omega$, there exists $n \in \mathbf{N}$ such that $x \in E_n$. Choosing $a \in (0, 1)$ allows for that.

**Bernard:** As usual, I need to check it by myself to understand this point. Sorry about that, Laurent. Let me start with $h \leq f$ and take $x \in \Omega$. Let me prove there exists $n \in \mathbf{N}$ such that $x \in E_n$.

If $h(x) = 0$, then $x \in E_n$ for all $n \in \mathbf{N}$.

Assume now that $h(x) > 0$. Since $h(x) \leq f(x)$, we can set $\varepsilon = f(x) - ah(x) > 0$.

By the definition of $f$, there exists $n \in \mathbf{N}$ such that $0 \leq f(x) - f_n(x) \leq \varepsilon/2$. We have $f_n(x) - ah(x) \geq \varepsilon/2 > 0$, and thus $x \in E_n$.

We can now claim that, thanks to $0 < a < 1$, $\Omega = \bigcup_{n\in\mathbf{N}} E_n$.

**Laurent:** Well done, Bernard. It's true that we understand arguments better when we try to prove them by ourselves. You are beginning to show the right attitude.

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**Proposition 3.8.** *Let $\mu$ be a measure over $(\Omega, \mathcal{A})$.*

(i) *Let $f, g$ be two measurable functions from $(\Omega, \mathcal{A})$ to $\big([0,\infty], \mathcal{B}([0,\infty])\big)$ and $a, b \in [0, \infty]$. We have*

$$\int (af + bg)\,d\mu = a \int f\,d\mu + b \int g\,d\mu.$$

(ii) *Let $(f_n)_{n\in\mathbf{N}}$ be measurable functions from $(\Omega, \mathcal{A})$ to $\big([0,\infty], \mathcal{B}([0,\infty])\big)$.*
*We have*

$$\int \Big(\sum_{n\in\mathbf{N}} f_n\Big)\,d\mu = \sum_{n\in\mathbf{N}} \Big(\int f_n\,d\mu\Big).$$

**Laurent:** You will realize that the additivity property of this integral comes from the monotone convergence theorem (Theorem 3.7). It is quite natural when you have its definition in mind.

**Bernard:** I assume that the starting point is the additivity property for the integral of simple functions. Can we have a look at the proof all the same?

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**Proof.** (i) Consider two sequences of $\mathbf{R}_+$-valued simple functions $(\varphi_n)_{n\in\mathbf{N}}$ and $(\psi_n)_{n\in\mathbf{N}}$ such that $\lim_{n\uparrow\infty}\uparrow\varphi_n = f$ and $\lim_{n\uparrow\infty}\uparrow\psi_n = g$.

We have $\lim_{n\uparrow\infty}\uparrow(a\varphi_n + b\psi_n) = af + bg$, and from the monotone convergence theorem,

$$\begin{aligned}\int (af+bg)\,d\mu &= \lim_{n\uparrow\infty}\uparrow \int (a\varphi_n + b\psi_n)\,d\mu \\ &= a\lim_{n\uparrow\infty}\uparrow\int \varphi_n\,d\mu + b\lim_{n\uparrow\infty}\uparrow\int \psi_n\,d\mu \\ &= a\int f\,d\mu + b\int g\,d\mu.\end{aligned}$$

(ii) Consider the $[0,\infty]$-valued measurable functions $F_n = \sum_{k=0}^{n} f_k$ for all $n \geq 0$. The nondecreasing sequence $(F_n)_{n\in\mathbf{N}}$ admits as a simple limit the $[0,\infty]$-valued measurable function $F = \sum_{n=0}^{\infty} f_n$. The monotone convergence theorem (Theorem 3.7) implies that $\int F\,d\mu = \lim_{n\uparrow\infty}\uparrow\int F_n\,d\mu$.

From (i) and a simple induction to extend the result to any finite sum of functions, for all $n \geq 0$, we have

$$\int F_n\,d\mu = \sum_{k=0}^{n}\int f_k\,d\mu.$$

This leads to

$$\int F\,d\mu = \lim_{n\uparrow\infty}\uparrow\sum_{k=0}^{n}\int f_k\,d\mu = \sum_{k=0}^{\infty}\int f_k\,d\mu.$$

□

**Laurent:** Thanks to this result, we can define the measure $f\mu$, where $f$ is a nonnegative measurable function and $\mu$ is a measure. We will see, especially in the context of probability, that it is a very important way to construct measures with particular properties when $\mu$ is the Lebesgue measure.

**Bernard:** Wait a minute, Laurent. You seem to be multiplying a function by a measure. It seems quite unnatural to me. At least, it deserves a precise definition. Let me read the book.

**Corollary 3.9.** *Let* $f\colon (\Omega, \mathcal{A}) \to \big([0,\infty], \mathcal{B}([0,\infty])\big)$ *be a measurable function and* $\mu$ *be a measure on* $\mathcal{A}$. *We define*

$$\forall A \in \mathcal{A}, \quad \nu(A) = \int \mathbf{1}_A f \, d\mu = \int_A f \, d\mu.$$

*Then,* $\nu$ *is a measure over* $(\Omega, \mathcal{A})$, *denoted by* $\nu = f\mu$.

*We say that* $\nu$ *is the measure of density* $f$ *with respect to the measure* $\mu$.

**Proof.** We note that $\nu(\emptyset) = 0$.

Moreover, if $(A_n)_{n\in\mathbf{N}}$ are pairwise disjoint elements of $\mathcal{A}$, we have

$$\nu\Big(\bigcup_{n\in\mathbf{N}} A_n\Big) = \int \Big(\sum_{n\in\mathbf{N}} \mathbf{1}_{A_n}\Big) f \, d\mu = \sum_{n\in\mathbf{N}} \int \mathbf{1}_{A_n} f \, d\mu = \sum_{n\in\mathbf{N}} \nu(A_n),$$

using Proposition 3.8(ii). □

**Remark.** If $\mu(A) = 0$, then $\nu(A) = \int \mathbf{1}_A f \, d\mu = 0$.

**Laurent:** You should realize that the $\sigma$-additivity of $f\mu$ is a direct consequence of the monotone convergence theorem applied to the integral with respect to $\mu$.

Let us examine the case of $\int f \, d\mu = 1$.

**Bernard:** In that case, we can see that $f\mu(\Omega) = \int f \, d\mu = 1$ and $f\mu$ is a probability measure.

A property is said to hold $\mu$-*almost everywhere* ($\mu$-a.e.) (or, simply, a.e.) if it is true outside a set of measure 0. Quite naturally, if $f, g$ are two measurable functions, $f = g$ $\mu$-a.e. means $\mu\big(\{x : f(x) \neq g(x)\}\big) = 0$.

**Laurent:** You will see that this property is important in the context of integration. And the starting point is the fact that two functions which are almost everywhere equal have the same integral.

You will also see that for a function that has a countable set of discontinuity points, these points are not seen by the integral with respect to any diffuse measure, as if the function was continuous.

**Proposition 3.10.** *Let $f\colon (\Omega, \mathcal{A}) \to \big([0,\infty], \mathcal{B}([0,\infty])\big)$ be a measurable function and $\mu$ be a measure over $(\Omega, \mathcal{A})$.*

(i) *For all $a > 0$, $\mu\big(\{x \in \Omega : f(x) \geq a\}\big) \leq \frac{1}{a}\int f\,d\mu$.*
(ii) *If $\int f\,d\mu < \infty$, then $f < \infty$ $\mu$-a.e.*
(iii) *$\int f\,d\mu = 0$ if and only if $f = 0$ $\mu$-a.e.*
(iv) *If $f = g$ $\mu$-a.e. then $\int f\,d\mu = \int g\,d\mu$.*

**Laurent:** Later on, we will see that Property 3.10(i) is often called the *Markov inequality* if $\mu$ is a probability measure.

**Bernard:** I have heard about it. But I remark that this property holds for any measure, not necessarily a probability measure.

**Laurent:** Absolutely. Pay close attention to the fact that in assertions (ii) and (iii), the assumptions on the integrals can only imply results that are valid almost everywhere (and not for any $x \in \Omega$). This is not at all surprising...

**Bernard:** For me, it is not totally obvious. The value of the integral depends on all the values of the function, no?

**Laurent:** No, Bernard. Let us study the proof of these properties before discussing this important point.

**Proof.** (i) Consider $A = \{x : f(x) \geq a\}$. By definition, we have $f \geq a\,\mathbf{1}_A$, so

$$\int f\,d\mu \geq \int a\mathbf{1}_A\,d\mu = a\,\mu(A).$$

(ii) Consider $A_\infty = \{x : f(x) = \infty\}$ and for all $n \geq 1$, $A_n = \{x : f(x) \geq n\}$.

Since $(A_n)_{n\geq 1}$ decreases to $A_\infty$, we have

$$\mu(A_\infty) = \mu\bigg(\bigcap_{n\geq 1} A_n\bigg) = \lim_{n\uparrow\infty} \downarrow \mu(A_n).$$

From (i), we have $\mu(A_n) \leq \frac{1}{n}\int f\,d\mu$ for all $n \geq 1$, and we deduce that $\mu(A_\infty) = 0$, which can be written as $f < \infty$ $\mu$-a.e.

(iii) $\star$ Assume first that $f = 0$ $\mu$-a.e. Consider $N = \{x : f(x) \neq 0\}$.

We have $\mu(N) = 0$, and consequently (it suffices to consider an increasing sequence of simple functions converging to $f$),

$$\int f \, d\mu = \int_N f \, d\mu = 0.$$

$\star$ Conversely, assume that $\int f \, d\mu = 0$. Consider $B_n = \{x : f(x) \geq 1/n\}$.

The increasing sequence $(B_n)_{n \geq 1}$ converges to $\bigcup_{n \geq 1} B_n = \{x : f(x) > 0\}$, and then

$$\mu(\{x : f(x) > 0\}) = \lim_{n \uparrow \infty} \uparrow \mu(B_n).$$

From (i), we have $\mu(B_n) \leq n \int f \, d\mu = 0$ for all $n \geq 1$. This leads to $\mu(\{x : f(x) > 0\}) = 0$, which can be written $f = 0$ $\mu$-a.e.

(iv) If $f = g$ $\mu$-a.e. then we have $\max(f,g) = \min(f,g)$ $\mu$-a.e. From this observation, applying (iii) to the nonnegative function $\max(f,g) - \min(f,g)$, which is equal to 0 almost everywhere, we deduce

$$\begin{aligned}\int \max(f,g) \, d\mu &= \int \min(f,g) \, d\mu + \int \underbrace{[\max(f,g) - \min(f,g)]}_{=0 \text{ a.e.}} \, d\mu \\ &= \int \min(f,g) \, d\mu.\end{aligned}$$

This equality for the nonnegative functions $f$ and $g$ which are equal almost everywhere can be applied to $f$ and $\max(f,g)$ (we can check that $f = \max(f,g)$ $\mu$-a.e.). This leads to

$$\int \max(f,g) \, d\mu = \int f \, d\mu.$$

By symmetry between $f$ and $g$, we also have $\int \max(f,g) \, d\mu = \int g \, d\mu$. □

**Laurent:** You need to know how to write each of these proofs at any moment. They are very basic for measure theory.

Moreover, note that for the point (iv), we could also use the fact that $\min(f,g) \leq f \leq \max(f,g)$ and $\min(f,g) \leq g \leq \max(f,g)$ to get

$$\int \min(f,g)\, d\mu \leq \int f\, d\mu \leq \int \max(f,g)\, d\mu,$$

$$\int \min(f,g)\, d\mu \leq \int g\, d\mu \leq \int \max(f,g)\, d\mu.$$

Since $\int \max(f,g)\, d\mu = \int \min(f,g)\, d\mu$, all these integrals are equal.

The concept of "almost everywhere" highlights the fact that **it is not necessary to know the value of a function at every point to determine the value of its integral** (point (iv)).

**Bernard:** What?

**Laurent:** Think of the construction of the integral, starting from the density of the simple functions. The values taken such that their preimages belong to a set of measure zero do not contribute to the integral. This is what is expressed by (iv).

For instance, assume that $f\colon \Omega \to \mathbf{R}_+$ is a measurable function.

For any $B \subset \mathbf{R}_+$ such that the subset $\Omega_0 = \{x \in \Omega : f(x) \in B\}$ satisfies $\Omega_0 \in \mathcal{A}$ and $\mu(\Omega_0) = 0$, we have $\displaystyle\int_\Omega f\, d\mu = \int_{\Omega\backslash\Omega_0} f\, d\mu$.

**Bernard:** Laurent, do you pretend the pointwise vision is not interesting any more?

**Laurent:** The infinitesimal vision demands to know more than the value of a function at each point, typically, for $C^1$, the existence of a derivative and its continuity at each point.

Infinitesimal calculus, as in the Riemann integral, demands these properties, which require in turn working in very small spaces as $C^k$ and very strong topological properties as the uniform convergence.

Honestly, defining a "good integral" only requires a global vision of the functions to integrate.

The real problems are not expressed in a pointwise form but in an integrated form (through an eye, a sensor, a filter, a measurement apparatus, etc.), i.e. in short, some continuous version of a weighted average. Physically, you do not have access to the value at a point, let alone all of the points. I guess that your boss, Ann, is sensitive to this modern vision of the application of mathematics...

The pointwise study probably comes from the beginning of infinitesimal calculus. Think of point mechanics, that is, the study of a virtual point that is the center of gravity, which is a mean point for some measure. It is at the heart of Newton's laws of motion.

**Bernard:** I have to admit that, with the piecewise continuous functions, some irregularities are already allowed, within a limited number of points. The engineer is only interested in averaged quantities, never in pointwise values, or sometimes in how frequently some values appear, which can be interpreted as a mean value.

Here, at last, are some things I can tell Ann to reassure her about my progress on the study.

**Laurent:** It is precisely the aim of measure theory: measuring the set where the equality of functions does not hold. When you say two functions are almost equal, it means there are not too many points such that the equality does not hold.

Mathematically, we write that the set of such points is of measure zero. In that sense, measure theory and everything that follows (e.g. partial differential equations, probabilities, and distributions) constitute a revolution with respect to 19th-century math.

Anyway, as we discussed earlier, a very comfortable point of view brought about by the integral with respect to a measure is the integration of the limit of a sequence of functions.

**Bernard:** This is what the monotone convergence theorem we just saw states.

**Laurent:** Yes, but it can be improved to what we will call the dominated convergence theorem. But before that, let us turn to an almost obvious consequence of the monotone convergence theorem that concerns the integration of the limit inferior of a sequence of nonnegative functions (which always exists, as you know it!).

**Theorem 3.11 (Fatou lemma).** *Let* $(f_n)_{n\in\mathbf{N}}$ *be a sequence of measurable functions from* $(\Omega, \mathcal{A})$ *to* $\big([0,\infty], \mathcal{B}([0,\infty])\big)$ *and* $\mu$ *be a measure over* $(\Omega, \mathcal{A})$. *We have*

$$\int (\liminf f_n)\, d\mu \leq \liminf \int f_n\, d\mu.$$

**Proof.** By definition, $\liminf f_n = \lim_{n\uparrow\infty} \uparrow \Big(\inf_{k\geq n} f_k\Big)$. The monotone convergence theorem implies that

$$\int \big(\liminf f_n\big)\, d\mu = \lim_{n\uparrow\infty} \int \Big(\inf_{k\geq n} f_k\Big)\, d\mu. \tag{3.1.3}$$

But, for all $p \geq n$, $\inf_{k \geq n} f_k \leq f_p$. This leads to

$$\forall p \geq n, \quad \int \Big( \inf_{k \geq n} f_k \Big) \, d\mu \leq \int f_p \, d\mu$$

and, consequently,

$$\int \Big( \inf_{k \geq n} f_k \Big) \, d\mu \leq \inf_{p \geq n} \int f_p \, d\mu.$$

Letting $n \uparrow \infty$, we get

$$\lim_{n \uparrow \infty} \int \Big( \inf_{k \geq n} f_k \Big) \, d\mu \leq \lim_{n \uparrow \infty} \inf_{p \geq n} \int f_p \, d\mu = \liminf \int f_n \, d\mu. \tag{3.1.4}$$

The result follows from (3.1.3) and (3.1.4). □

**Bernard:** You mentioned a dominated convergence theorem. Is there a link with the Fatou Lemma?

**Laurent:** First, do not underestimate the power of the Fatou Lemma. This result is easy to use since you only need the functions to be measurable and nonnegative, and the sequence does not even need to converge.

Secondly, the dominated convergence theorem concerns functions said to be integrable. It does not concern specifically nonnegative functions, but additional conditions are required. We now need to see what integrable means.

### 3.1.3 *Integrable functions*

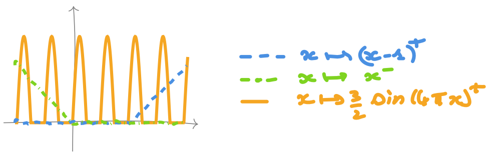
**Bernard:** If I understand well, starting from the integration of nonnegative functions we just saw, we will be able to define the integral of measurable functions that are not necessarily nonnegative.

I can imagine it relies on the decomposition of a function in nonnegative and nonpositive parts. If $f\colon \Omega \to \mathbf{R}$, we can define the functions $f^+$ and $f^-$ by

$$\forall x \in \Omega, \quad f(x) = f^+(x) - f^-(x),$$

where $f^+(x) = \max\big(f(x), 0\big)$ and $f^-(x) = -\min\big(f(x), 0\big)$.

The functions $f^+$ and $f^-$ are obviously nonnegative. They are also measurable if $f$ is measurable. Thus, we can apply the construction of the integral to $f^+$ and $f^-$.




**Laurent:** Of course. These two parts are involved in the definition of the integral. As you understood, the idea is to define the integral of $f$ as $\int f^+\,d\mu - \int f^-\,d\mu$. But how would you compute this quantity when the two integrals are infinite?

In order to ensure that the two integrals are finite, for any real-valued measurable function $f$, we consider the absolute value $|f|$, which is also measurable and nonnegative and thus can be integrated using the previous procedure.

This idea is developed in Lebesgue's early works.[1]

**Definition 3.12.** A measurable function $f\colon (\Omega, \mathcal{A}) \to \big(\mathbf{R}, \mathcal{B}(\mathbf{R})\big)$ is said to be *integrable with respect to the measure* $\mu$ over $(\Omega, \mathcal{A})$ if $\int |f|\,d\mu < \infty$.

We denote by $\mathcal{L}^1(\mu)$, or $\mathcal{L}^1(\Omega, \mathcal{A}, \mu)$ if we want to precise the measure space, the space of all the functions which are integrable with respect to the measure $\mu$.

In that case, we decompose $f = f^+ - f^-$, where $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$, and we define the integral of $f$ by

$$\int f\,d\mu = \int f^+\,d\mu - \int f^-\,d\mu.$$

The integral is well defined because the equality $|f| = f^+ + f^-$ implies that

$$\int f^+\,d\mu \leq \int |f|\,d\mu < \infty \quad \text{and} \quad \int f^-\,d\mu \leq \int |f|\,d\mu < \infty.$$

[1] See e.g. [Lebesgue (1910)].

**Bernard:** I can see the importance of noting that $|f| = f^+ + f^-$.

**Laurent:** Imposing $\int |f|\, d\mu < \infty$ is really the key point in guaranteeing that the integral of $\int f^+\, d\mu - \int f^-\, d\mu$ is well defined. If these two integrals were infinite, we would find it difficult to define $\infty - \infty$. You learned that as a teenager.

This issue does not appear when we consider nonnegative functions, whose integral may be infinite without any problem.

**Bernard:** What does this definition become for a nonnegative function $f$? Is a nonnegative measurable function necessarily integrable with respect to $\mu$?

**Laurent:** If you apply the strict definition, the answer is no. If $f$ is nonnegative and measurable, $\int f\, d\mu$ is always defined but can be infinite. However, since $|f| = f$ in this case, $f$ is said to be integrable if $\int f\, d\mu < \infty$.

Anyway, generally speaking, we can compare $\int f\, d\mu$ and $\int |f|\, d\mu$. Let us see that.

**Proposition 3.13.** *Let $\mu$ be a measure over $(\Omega, \mathcal{A})$ and $f$ be a function in $\mathcal{L}^1(\mu)$.*

*We have $\left|\int f\, d\mu\right| \leq \int |f|\, d\mu$.*

**Proof.** Thanks to the triangle inequality, we have

$$\left|\int f\, d\mu\right| = \left|\int f^+\, d\mu - \int f^-\, d\mu\right| \leq \left|\int f^+\, d\mu\right| + \left|\int f^-\, d\mu\right|.$$

Moreover, since $f^+$ and $f^-$ are nonnegative functions, we have

$$\left|\int f^+\, d\mu\right| + \left|\int f^-\, d\mu\right| = \int f^+\, d\mu + \int f^-\, d\mu = \int |f|\, d\mu.$$

□

**Laurent:** The space of integrable functions $\mathcal{L}^1(\mu)$ has an interesting algebraic structure which allows us to consider the integral of linear combinations of integrable functions. It is formalized in the following proposition.

$\int f\, d\mu$ 515.BRO.m

**Proposition 3.14.** *Let $\mu$ be a measure over $(\Omega, \mathcal{A})$. The set $\mathcal{L}^1(\mu)$ is a vector space, and the mapping $f \mapsto \int f\, d\mu$ is a linear form on $\mathcal{L}^1(\mu)$.*

**Proof.** $\star$ Let $f \in \mathcal{L}^1(\mu)$ and $a \in \mathbf{R}$.

We show that $af \in \mathcal{L}^1(\mu)$ and $\int af\, d\mu = a \int f\, d\mu$.

For the first point, it suffices to write $\int |af|\, d\mu = |a| \int |f|\, d\mu < \infty$.

For the second point, we distinguish the two disjoint cases $a \geq 0$ and $a < 0$.

For $a \geq 0$,

$$\begin{aligned}\int af\, d\mu &= \int (af)^+\, d\mu - \int (af)^-\, d\mu \\ &= a\left(\int f^+\, d\mu - \int f^-\, d\mu\right) = a \int f\, d\mu.\end{aligned}$$

And for $a < 0$,

$$\begin{aligned}\int af\, d\mu &= \int (af)^+\, d\mu - \int (af)^-\, d\mu \\ &= -a \int f^+\, d\mu + a \int f^-\, d\mu = a \int f\, d\mu.\end{aligned}$$

$\star$ Let $f, g \in \mathcal{L}^1(\mu)$.

We show that $f + g \in \mathcal{L}^1(\mu)$ and $\int (f+g)\, d\mu = \int f\, d\mu + \int g\, d\mu$.

The first point comes from the inequality $|f+g| \leq |f| + |g|$ and Proposition 3.6: $\int |f+g|\, d\mu \leq \int |f|\, d\mu + \int |g|\, d\mu < \infty$.

For the second point, we note that

$$f + g = (f+g)^+ - (f+g)^- = f^+ + g^+ - f^- - g^-,$$

which leads to

$$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+.$$

Using the additivity property of the integral of nonnegative functions (Proposition 3.8), we get

$$\begin{aligned}&\int (f+g)^+\, d\mu + \int f^-\, d\mu + \int g^-\, d\mu \\ &= \int (f+g)^-\, d\mu + \int f^+\, d\mu + \int g^+\, d\mu.\end{aligned}$$

All these integrals are finite (because $f, g, f+g \in \mathcal{L}^1(\mu)$), thus

$$\int (f+g)^+ \, d\mu - \int (f+g)^- \, d\mu$$
$$= \int f^+ \, d\mu - \int f^- \, d\mu + \int g^+ \, d\mu - \int g^- \, d\mu,$$

which is $\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu$. □

**Bernard:** I guess that, as in the nonnegative case, we can compare the integrals of $f$ and $g$ for real-valued functions when $f \leq g$ or $f = g$ $\mu$-a.e.

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**Proposition 3.15.** *Let $\mu$ be a measure over $(\Omega, \mathcal{A})$ and $f, g$ be functions in $\mathcal{L}^1(\mu)$.*

(i) *If $f \leq g$, then $\int f \, d\mu \leq \int g \, d\mu$.*
(ii) *If $f = g$ $\mu$-a.e., then $\int f \, d\mu = \int g \, d\mu$.*
(iii) *$f = 0$ $\mu$-a.e. if and only if $\int_A f \, d\mu = 0$ for all $A \in \mathcal{A}$.*

**Proof.** (i) The proof is exactly the same as in the case of nonnegative simple functions (Proposition 3.4(ii)).

(ii) If $f = g$ $\mu$-a.e. then we also have $f^+ = g^+$ $\mu$-a.e. and $f^- = g^-$ $\mu$-a.e.

Then, the conclusion follows from applying Proposition 3.10(iv) to the nonnegative functions $f^+, g^+$, on the one hand, and to $f^-, g^-$, on the other hand.

(iii) ⋆ If $f = 0$ $\mu$-a.e. then for any $A \in \mathcal{A}$, we have $f \, \mathbf{1}_A = 0$ $\mu$-a.e. and (ii) implies $\int f \, \mathbf{1}_A \, d\mu = 0$.

⋆ Conversely, assuming $\int_A f \, d\mu = 0$ for all $A \in \mathcal{A}$, we consider the two particular cases of $A^+ = f^{-1}\big([0, +\infty)\big)$ and $A^- = f^{-1}\big((-\infty, 0]\big)$, which are in $\mathcal{A}$ (since $f$ is measurable).

By the definition of $f^+$ and $f^-$, we have

$$\int_{A^+} f \, d\mu = \int f^+ \, d\mu \quad \text{and} \quad \int_{A^-} f \, d\mu = \int f^- \, d\mu.$$

Since $f^+$ and $f^-$ are nonnegative, the assumptions $\int f^+ \, d\mu = 0$ and $\int f^- \, d\mu = 0$ lead to $f^+ = 0$ $\mu$-a.e. and $f^- = 0$ $\mu$-a.e.

Using the decomposition $f = f^+ - f^-$, we conclude that $f = 0$ $\mu$-a.e. $\square$

**Remark.** Combining the two points (i) and (ii) of Proposition 3.15, if $f, g \in \mathcal{L}^1(\mu)$ are such that $f \leq g$ $\mu$-a.e. then $\int f \, d\mu \leq \int g \, d\mu$.

**Bernard:** It's going a little too fast for me. I need to check calmly that if $f = g$ $\mu$-a.e. then we also have $f^+ = g^+$ $\mu$-a.e. and $f^- = g^-$ $\mu$-a.e.

I remember that $f = g$ $\mu$-a.e. means $\mu(\{x \in \Omega : f(x) \neq g(x)\}) = 0$.

For the assertion on $f^+$ and $g^+$, I can write

$$\{x \in \Omega : f^+(x) \neq g^+(x)\} \subset \{x \in \Omega : f(x) \neq g(x)\}.$$

These two facts imply $\mu(\{x \in \Omega : f^+(x) \neq g^+(x)\}) = 0$, which means $f^+ = g^+$ $\mu$-a.e. All goes well.

In the same way, we have $f^- = g^-$ $\mu$-a.e.

Consequently, from Proposition 3.10(iv), we have $\int f^+ \, d\mu = \int g^+ \, d\mu$ and $\int f^- \, d\mu = \int g^- \, d\mu$.

We conclude that $\int f \, d\mu = \int g \, d\mu$.

**Laurent:** I see that you have understood the theoretical properties, Bernard. Let us write them when the measure is discrete. It is simpler to apprehend.

### 3.1.4 *Integration with respect to a discrete measure*

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Let us consider two particular cases of integration of a non-negative measurable function $f \colon (\Omega, \mathcal{A}) \to \big([0, \infty], \mathcal{B}(\mathbf{R})\big)$:

$\star$ *Integration with respect to the Dirac measure* $\mu = \delta_a$ on the $\sigma$-algebra $\mathcal{A}$, where $a \in \Omega$ and $\{a\} \in \mathcal{A}$.

We consider a nondecreasing sequence of simple functions $(\varphi_n)_{n \in \mathbf{N}}$ that converges pointwise to $f$ and satisfies the following:

for all $n \in \mathbf{N}$, $\varphi_n$ can be written as

$$\varphi_n = \sum_{k \leq k_n} \alpha_k \, \mathbf{1}_{A_k} + f(a) \, \mathbf{1}_{\{a\}},$$

where $(A_k)_{k \leq k_n}$ is a partition of $\Omega \setminus \{a\}$ with the elements of $\mathcal{A}$.

We have

$$\forall n \in \mathbf{N}, \quad \int_\Omega \varphi_n(x) \, \delta_a(dx) = f(a)$$

and, consequently,

$$\int_\Omega f(x) \, \delta_a(dx) = f(a).$$

$\star$ *Integration with respect to the counting measure* $\mu$ defined over the finite set $\Omega = \{a_k;\ 1 \leq k \leq n\}$ equipped with the $\sigma$-algebra $\mathcal{A} = \mathcal{P}(\Omega)$.

For all $A \subset \Omega$, we have $\mu(A) = \#(A) = \sum_{k=1}^n \mathbf{1}_A(a_k)$. Then, $\mu = \sum_{k=1}^n \delta_{a_k}$ and

$$\int_\Omega f(x)\mu(dx) = \sum_{k=1}^n \int_\Omega f(x)\delta_{a_k}(dx) = \sum_{k=1}^n f(a_k).$$

**Bernard:** Hold on. The first point goes too fast for me. Does such a sequence $(\varphi_n)_{n\in\mathbf{N}}$ always exist?

**Laurent:** It suffices to apply Theorem 3.2 to $g = f|_{\Omega\setminus\{a\}}$... This yields the existence of a nondecreasing sequence of simple functions $(g_n)_{n\in\mathbf{N}}$ defined on $\Omega \setminus \{a\}$ that converges pointwise to $g$. Then, you can consider the sequence $(\varphi_n)_{n\in\mathbf{N}}$, defined as, for all $x$ in $\Omega$,

$$\varphi_n(x) = g_n(x)\mathbf{1}_{\Omega\setminus\{a\}}(x) + f(a)\mathbf{1}_{\{a\}}(x).$$

Writing that, we have for all $n \in \mathbf{N}$

$$\int_\Omega \varphi_n(x)\delta_a(dx) = \underbrace{\int_\Omega g_n(x)\mathbf{1}_{\Omega\setminus\{a\}}(x)\delta_a(dx)}_{=0} + \underbrace{\int_\Omega f(a)\mathbf{1}_{\{a\}}(x)\delta_a(dx)}_{=f(a)}.$$

**Bernard:** Of course. We thus obtain a nondecreasing sequence of simple functions converging to $f$. By the monotone convergence theorem, we can claim that

$$\int_\Omega \varphi_n(x)\delta_a(dx) \underset{n\to\infty}{\longrightarrow} \int_\Omega f(x)\ \delta_a(dx).$$

We conclude that

$$\int_\Omega f(x)\delta_a(dx) = f(a).$$

**Laurent:** Exactly. And for the second point, you just need to understand that $\int f(x)(\mu+\nu)(dx) = \int f(x)\mu(dx) + \int f(x)\nu(dx)$.

**Bernard:** You are right to point this out, I can't remember if we had already mentioned it. But I can see it clearly enough to check that it is satisfied for simple functions such as $f = \sum \alpha_k \mathbf{1}_{A_k}$ (with the usual notations). Your equality reduces to

$$\sum \alpha_k(\mu+\nu)(A_k) = \sum \alpha_k \mu(A_k) + \sum \alpha_k \nu(A_k).$$

**Laurent:** If you understood this, you also understand that you only need to check it for $f = \mathbf{1}_A$. Anyway... The Dirac measure and the counting measure are particular cases of *discrete measures*. The integration with respect to these measures also has a simple expression.

**Definition 3.16.** A measure $\mu$ is said to be *discrete* if there exists a sequence $(a_n)_{n\in\mathbf{N}}$ (that can be finite) of elements of $\Omega$ such that $\mu\big(\{a_n;\ n\in\mathbf{N}\}^c\big) = 0$.

**Proposition 3.17.** *In the case where $\mathcal{A}$ contains all the singletons of $\Omega$, the measure $\mu$ is discrete if and only if it can be written as $\mu = \sum_{n=0}^{\infty} \alpha_n\, \delta_{a_n}$, where*

- ⋆ $(a_n)_{n\in\mathbf{N}}$ *is a sequence (that can be finite) in $\Omega$;*
- ⋆ $(\alpha_n)_{n\in\mathbf{N}}$ *is a sequence (that can be finite) in $\mathbf{R}_+$;*
- ⋆ $\delta_{a_n}$ *is the Dirac measure at $a_n$, for all $n\in\mathbf{N}$.*

**Proof.** $\star$ If $\mu = \sum_{n=0}^{\infty} \alpha_n \delta_{a_n}$, then $\mu\big(\{a_n;\ n \in \mathbf{N}\}^c\big) = 0$.

$\star$ Conversely, if $\mu\big(\{a_n;\ n \in \mathbf{N}\}^c\big) = 0$, where $(a_n)_{n\in\mathbf{N}}$ are distinct in $\Omega$, we remark that the two measures $\mu$ and $\sum \mu(\{a_n\})\delta_{a_n}$ coincide over $\{a_n;\ n \in \mathbf{N}\}$ and over $\{a_n;\ n \in \mathbf{N}\}^c$. Then, $\mu = \sum_{n=0}^{\infty} \mu(\{a_n\})\, \delta_{a_n}$. □

For any nonnegative measurable function $f\colon (\Omega, \mathcal{A}) \to \big([0,\infty], \mathcal{B}([0,\infty])\big)$ and any discrete measure $\mu = \sum_{n=0}^{\infty} \alpha_n \delta_{a_n}$, we have

$$\int_\Omega f(x)\mu(dx) = \sum_{n=0}^{\infty} \alpha_n \int_\Omega f(x)\delta_{a_n}(dx) = \sum_{n=0}^{\infty} \alpha_n f(a_n).$$

If $(a_n)_{n\in\mathbf{N}}$ are distinct, we have $\alpha_n = \mu(\{a_n\})$ for all $n \in \mathbf{N}$, and then

$$\int_\Omega f(x)\mu(dx) = \sum_{n=0}^{\infty} f(a_n)\mu\big(\{a_n\}\big). \tag{3.1.5}$$

**Bernard:** Using the decomposition $f = f^+ - f^-$, where $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$, it is true that, invoking linearity, (3.1.5) holds for any finite measurable function $f\colon \Omega \to \mathbf{R}$.

**Laurent:** Very good. And morally, discrete integration, that is $\mu$ or $f$ being discrete, reduces to a sum. And this expression is the same as the integral of a discrete function of the form $f = \sum_{n=0}^{\infty} \gamma_n \mathbf{1}_{\{a_n\}}$ with respect to any (not necessarily discrete) measure $\mu$:

$$\begin{aligned}\int_\Omega f(x)\mu(dx) &= \int \sum_{n=0}^{\infty} \gamma_n \mathbf{1}_{\{a_n\}}(x)\mu(dx) \\ &= \sum_{n=0}^{\infty} \gamma_n \mu\big(\{a_n\}\big) = \sum_{n=0}^{\infty} f(a_n)\mu\big(\{a_n\}\big).\end{aligned}$$

Let us see how all these translate in probability theory.

## 3.2 A Particular Case: Expectation of Random Variables

**Bernard:** Expectation... Well... Mathematically, I know the concept. I saw it in a variety of contexts.

**Laurent:** The problem is that people who do not master the concepts have a natural tendency to sum up a fluctuating physical quantity to a mean value. They may proceed to summing it up to a value and a standard deviation if they know a little more. Unfortunately, things are not that easy.

**Bernard:** But I heard in my engineer's curriculum that we know everything about the probability distribution if we know the mean value and the standard deviation!

**Laurent:** That is because people unfortunately think random variables are always Gaussian. This leads to Hurricane Katrina destroying New Orleans!


I am not an engineer, but I have understood one thing: the job of an engineer integrates performance and risk balance. Chasing performance generates a risk of catastrophe.

Be aware that your plant's environmental impact assessment is precisely a question of risk control. Your rigorous approach could impress your superiors and Ann in particular.

**Bernard:** Wait a minute, Laurent. You seem to know my boss Ann very well. How strange... Besides, how are performance and Katrina related?

**Laurent:** Managers usually obey the same principles. In the case of Ann, the French touch of her mathematical studies adds a particular flavor.

When building a dike to protect a city, if an engineer must choose between a 5 m and a 7 m high dike, the first idea is a question of performance: a 7 m high dike is a more efficient protection. However, the cost difference between the two options is in the order of a billion dollars...

The public figures are certainly not accurate, but if you can imagine that the average height of the waves is 3 m, then one can think that, economically, the best choice is a 5 m high dike.

But it was said at the time of Hurricane Katrina that, if the dike had been 7 m high, the city would not have been destroyed...

**Bernard:** OK, Laurent. As an engineer myself, I feel a bit stung by what you say. What could have been done, knowing that the mean value was known?

**Laurent:** Yes, the mean value is important. And we will detail its properties, but more than that, the important thing is the distribution of the heights of waves during the preceding hurricanes.

Essentially, the height of the dike should have been chosen with respect to the probability that the height of a wave could be larger than 5 m.

**Bernard:** Ah, yes. I understand better what my really experienced (and old) colleagues mean when they assert that "the design should be based on the extreme values and not the mean value".

**Laurent:** Exactly, Bernard! And in order to do that, they need to know the distribution of the probability of random variables. And we will see that the expectation (or mean value) of a random variable is really an integral with respect to its law (or distribution).

To summarize, we are facing a situation where a poor or too imprecise appreciation of the risk can lead to destruction and death. You could develop this story, transposing it to the case of your pharmaceutical plant's waste, in your next discussion with Ann.

### 3.2.1 *Expectation and transfer theorem*

$\mathbf{E}[\varphi(X)]$
519.FRA.f

Naturally, the general framework of the theory of integration with respect to a measure applies to the particular case of probability measures (it was the starting point of probability theory from Kolmogorov's axiomatic, see [Kolmogorov (1933)]).

As a measurable function from $(\Omega, \mathcal{F})$ to $(E, \mathcal{E})$, a random variable on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ can be integrated with respect to the probability measure $\mathbf{P}$ to define the *expectation*.

**Definition 3.18.** In the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, let $X$ be a real random variable integrable with respect to the probability measure $\mathbf{P}$, i.e. such that $\int_\Omega |X(\omega)| \; \mathbf{P}(d\omega) < \infty$ (which is denoted by $X \in \mathcal{L}^1(\mathbf{P})$).

The random variable $X$ is said to have a first-order moment, and the quantity

$$\mathbf{E}[X] = \int X \, d\mathbf{P} = \int_\Omega X(\omega) \, \mathbf{P}(d\omega)$$

is called *the expectation* (or *expected value* or *mean value*) of $X$.

By the construction of the integral with respect to the measure $\mathbf{P}$, the expectation is characterized by the relation $\mathbf{E}[\mathbf{1}_A] = \mathbf{P}(A)$ for all $A \in \mathcal{F}$.

**Bernard:** Finally, there is no real theory of probability. Actually, it reduces to being a part of measure theory.

**Laurent:** You are wrong, Bernard. Probability is a theory in itself, which certainly has its roots in measure theory, but it has several specific subjects. For instance, the question of *independence of random variables.*

Another point is the fact that in probability, we study random variables through the distribution of its values, and consequently, we work in the space of values rather than in $\Omega$.

$\mathbf{E}[\varphi(X)]$ 519.FRA.f

It is often more convenient to express the expectation $\mathbf{E}[X]$ of a random variable as an integral in the space of values $(E, \mathcal{E})$ with respect to its probability law $P_X$ (instead of $\mathbf{P}$). In a way, the *transfer theorem* makes it possible to forget the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ in favor of the probability space $(E, \mathcal{E}, P_X)$.

**Laurent:** Recall that the integral of a nonnegative measurable function is more or less defined though its approximation by simple functions. It means that it relies on the values taken by the function, weighted by the measure of the preimages of these values.

**Bernard:** From what you say, I understand that the integral should be expressed in terms of the values taken by the function. But the point which is unclear to me is the measure that we need to consider.

**Laurent:** That is precisely the point. The measure on the domain $\Omega$ can be transported on the codomain $\mathbf{R}$ in the case of a real-valued function. We obtain the transfer theorem, which is THE starting theorem of probability theory.

$\mathbf{E}[\varphi(X)]$
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**Theorem 3.19 (Transfer theorem).** *In the probability space* $(\Omega, \mathcal{F}, \mathbf{P})$, *let* $X$ *be a random variable with values in* $(E, \mathcal{E})$.

*Then, the law of* $X$ *is the only probability measure* $P_X$ *on* $(E, \mathcal{E})$ *such that*

$$\int_E h(x) P_X(dx) = \int_\Omega h\big(X(\omega)\big) \mathbf{P}(d\omega) \tag{3.2.1}$$

*for all bounded measurable functions* $h\colon (E, \mathcal{E}) \to \big(\mathbf{R}, \mathcal{B}(\mathbf{R})\big)$.

*In addition, for any measurable function* $h\colon (E, \mathcal{E}) \to \big(\mathbf{R}, \mathcal{B}(\mathbf{R})\big)$, *we have*

$$h(X) \in \mathcal{L}^1(\mathbf{P}) \Longleftrightarrow h \in \mathcal{L}^1(P_X).$$

*If one of these equivalent conditions is satisfied, Expression* (3.2.1) *holds.*

**Laurent:** By the way, Bernard, we denote usually by $h(X)$ the random variable $h \circ X\colon (\Omega, \mathcal{F}) \to \big(\mathbf{R}, \mathcal{B}(\mathbf{R})\big)$. According to this notation, we can write

$$\mathbf{E}[h(X)] = \int_\Omega h\big(X(\omega)\big)\, \mathbf{P}(d\omega),$$

and Theorem 3.19 tells that the measure $P_X$ on $(E, \mathcal{E})$ is characterized by

$$\int_E h(x)\, P_X(dx) = \mathbf{E}[h(X)]$$

as soon as $h(X) \in \mathcal{L}^1(\mathbf{P})$.

E[$\varphi(X)$] 519.FRA.f

**Proof.** $\star$ Assume that $\mu$ is some probability measure on $(E, \mathcal{E})$ such that

$\mathbf{E}[h(X)] = \int_E h(x)\,\mu(dx)$ for any function $h\colon (E, \mathcal{E}) \to \big(\mathbf{R}, \mathcal{B}(\mathbf{R})\big)$ that is measurable and bounded.

The particular case of $h = \mathbf{1}_A$ for $A \in \mathcal{E}$ gives

$$\mathbf{P}[X^{-1}(A)] = \mathbf{E}\big[\mathbf{1}_{X^{-1}(A)}\big] = \mathbf{E}\big[\mathbf{1}_A(X)\big] = \mu(A).$$

This shows that $\mu$ is equal to $P_X$, the law of $X$, as in Definition 2.28.
$\star$ Conversely, let us show that (3.2.1) holds for any bounded measurable function $h\colon (E, \mathcal{E}) \to \big(\mathbf{R}, \mathcal{B}(\mathbf{R})\big)$. This fact is a direct consequence of the construction of the integral:

— We first remark that $\mathbf{E}[\mathbf{1}_A(X)] = \mathbf{P}[X^{-1}(A)] = P_X(A) = \int_E \mathbf{1}_A(x) P_X(dx)$ for all $A \in \mathcal{E}$.

By linearity, this implies that (3.2.1) holds for any simple function $h$.

— For any nonnegative measurable function $h\colon (E, \mathcal{E}) \to \big([0,\infty], \mathcal{B}([0,\infty])\big)$, we consider a nondecreasing sequence $(\varphi_n)_{n\in\mathbf{N}}$ of nonnegative simple functions that converges pointwise to $h$ (see Theorem 3.2), i.e. such that for all $x \in E$ and $n \in \mathbf{N}$, $\varphi_{n+1}(x) \geq \varphi_n(x)$ and $\varphi_n(x) \to h(x)$ as $n \to \infty$.

From $\mathbf{E}[\varphi_n(X)] = \int_E \varphi_n(x)\,P_X(dx)$, for all $n \in \mathbf{N}$, the monotone convergence theorem (Theorem 3.7) implies that $\mathbf{E}[h(X)] = \int_E h(x)\,P_X(dx)$.

— For any measurable function $h\colon (E, \mathcal{E}) \to \big(\mathbf{R}, \mathcal{B}(\mathbf{R})\big)$, we can apply the previous point to the nonnegative measurable function $|h|$, and we can claim that $\mathbf{E}[|h(X)|] = \int_E |h(x)|\,P_X(dx)$.

This implies the equivalence $h(X) \in \mathcal{L}^1(\mathbf{P}) \iff h \in \mathcal{L}^1(P_X)$.

When $h \in \mathcal{L}^1(P_X)$, i.e. $\int_E |h(x)|\,P_X(dx) < +\infty$, we decompose $h = h^+ - h^-$ into $h^+ = \max(h, 0)$ and $h^- = -\min(h, 0)$.

We can claim that $\int_E h^+(x)\,P_X(dx) < +\infty$ and $\int_E h^-(x)\,P_X(dx) < +\infty$, and then,

$$\begin{aligned}\mathbf{E}\big[h(X)\big] &= \mathbf{E}\big[h^+(X)\big] - \mathbf{E}\big[h^-(X)\big] \\ &= \int_E h^+(x) P_X(dx) - \int_E h^-(x) P_X(dx) = \int_E h(x) P_X(dx).\end{aligned}$$

In the specific case where $h$ is a bounded measurable function, we have $h \in \mathcal{L}^1(P_X)$ and (3.2.1) holds. □

**Laurent:** Note that the inequality $|X| \leq 1 + |X|^2$ holds (to convince yourself, you can consider the cases $|X| \geq 1$ and $|X| < 1$).

Then, we can deduce that, if $\mathbf{E}[|X|^2] < +\infty$, we have $\mathbf{E}[|X|] < +\infty$.

Consequently, in that case, we can define the *variance* of $X$ by

$$\mathrm{Var}(X) = \mathbf{E}\left[X^2\right] - \left(\mathbf{E}[X]\right)^2 = \mathbf{E}\left[(X - \mathbf{E}[X])^2\right].$$

**Bernard:** I'm a bit confused by the inequality $|X| \leq 1 + |X|^2$. Thinking about it, I realize it's true... even if not sharp. And I admit it is sufficient.

**Laurent:** Well. Let us turn to a clever extension of the definition of convexity.

E[φ(X)] 519.FRA.f

**Theorem 3.20 (Jensen inequality).** *In the probability space* $(\Omega, \mathcal{F}, \mathbf{P})$, *let* $X$ *be a real random variable and* $\varphi : \mathbf{R} \to \mathbf{R}$ *be a convex map.*

*If* $X$ *and* $\varphi(X)$ *lie in* $L^1(\Omega, \mathcal{F}, \mathbf{P})$, *then*

$$\varphi(\mathbf{E}[X]) \leq \mathbf{E}[\varphi(X)].$$

**Proof.** The map $\varphi$ being convex, it admits a right-derivative $\varphi'_d$, so that

$$\forall x, y \in \mathbf{R}, \quad \varphi'_d(y)(x - y) \leq \varphi(x) - \varphi(y).$$

Taking $x = X$ and $y = \mathbf{E}[X]$, we get

$$\varphi'_d(\mathbf{E}[X])(X - \mathbf{E}[X]) \leq \varphi(X) - \varphi(\mathbf{E}[X]).$$

Since $X$ and $\varphi(X)$ lie in $L^1(\Omega, \mathcal{F}, \mathbf{P})$, we can take the expectation and

$$\begin{aligned}\varphi'_d(\mathbf{E}[X])\mathbf{E}(X - \mathbf{E}[X]) &\leq \mathbf{E}[\varphi(X)] - \mathbf{E}[\varphi(\mathbf{E}[X])] \\ &= \mathbf{E}[\varphi(X)] - \varphi(\mathbf{E}[X])\mathbf{E}[\mathbf{1}].\end{aligned}$$

We conclude that $0 \leq \mathbf{E}[\varphi(X)] - \varphi(\mathbf{E}[X])$. □

**Bernard:** Why do you call it an extension of convexity?

**Laurent:** Here is a tip for you, Bernard: take $\Omega = \{0, 1\}$, $\mathbf{P} = \dfrac{\delta_0 + \delta_1}{2}$, and remember what convex means...

### 3.2.2 *Expectation of discrete random variables*

**Bernard:** I remember from my *classes préparatoires*, before entering engineering school, that we talked about discrete random variables $X\colon \Omega \to E$ when $\Omega$ was discrete.

But I thought at the time that, since we almost never considered $\Omega$, this definition could be extended to the case where we only assume that the space of values $E$ is discrete (with $\Omega$ being discrete or not).

In particular, the fact that $\Omega$ is discrete implies that the set of values $X(\Omega) \subset E$ taken by $X$ is discrete, right?

**Laurent:** Definitely, Bernard. I believe I understand what was disturbing you. Rest assured, everything becomes clear when we consider the notion of discrete measure. Indeed, this notion allows us to cover the two cases you mention: either if $\Omega$ is a discrete space or if the codomain $E$ is discrete, the law of $X$ is necessarily a discrete measure. And this is the definition that is usually taken for a discrete variable.

E[φ(X)] 519.FRA.f

**Definition 3.21.** A random variable $X$ defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with values in $(E, \mathcal{E})$ is said to be *discrete* if its law $P_X$ is a discrete probability measure.

**Example.** ⋆ When the space of values $E$ is finite or countable and $\mathcal{E} = \mathcal{P}(E)$, the distribution of every random variable $X\colon (\Omega, \mathcal{F}) \to (E, \mathcal{E})$ is a discrete probability measure (see Section 3.1.4).

Then, in this case, every random variable is discrete.

If $E \subset \mathbf{R}$ and $X \in \mathcal{L}^1(\mathbf{P})$, according to Section 3.1.4 and the transfer theorem, we have

$$\mathbf{E}[X] = \int_\Omega X(\omega)\,\mathbf{P}(d\omega) = \int_E x\,P_X(dx) = \sum_{x_k \in E} x_k\,P_X(\{x_k\}).$$

Denoting $p_k = P_X(\{x_k\}) = \mathbf{P}(X = x_k)$, for any measurable function $h\colon E \to \mathbf{R}$ such that $\sum_k |h(x_k)|\, p_k < \infty$, we have

$$\mathbf{E}[h(X)] = \int_E h(x) P_X(dx) = \sum_k \int_E h(x) p_k \delta_{x_k}(dx) = \sum_k h(x_k) p_k.$$

$\star$ An even more special case is when $\Omega$ is finite or countable and $\mathcal{F} = \mathcal{P}(\Omega)$.

This implies that the law $P_X = \mathbf{P} \circ X^{-1}$ of any random variable $X\colon (\Omega, \mathcal{F}) \to (E, \mathcal{E})$ is a discrete probability measure.

In that case, if $E \subset \mathbf{R}$ and $X \in \mathcal{L}^1(\mathbf{P})$, (3.1.5) and the transfer theorem imply that

$$\mathbf{E}[X] = \int_\Omega X(\omega)\mathbf{P}(d\omega) = \sum_{\omega\in\Omega} X(\omega)\mathbf{P}(\{\omega\}) = \sum_{x_i \in X(\Omega)} x_i\, P_X(\{x_i\}).$$

**Laurent:** I admit that these examples do not really illustrate the notion of discrete variable or measure. But I am quite sure that you already met some of them. Let us review the famous ones.

### 3.2.3 *Examples of discrete probability distributions*

$\mathbf{E}[\varphi(X)]$
519.FRA.f

**Uniform discrete law**

A random variable $X$ is said to have a *uniform distribution* over the finite set $\{1, 2, \ldots, n\}$ if it takes its values in this set and if

$$\forall k \in \{1, \ldots, n\}, \quad \mathbf{P}(X = k) = \frac{1}{n}.$$

This distribution represents what is commonly referred to as the *equiprobability* in finite spaces.

Then, we have $\mathbf{E}[X] = \frac{n+1}{2}$ and $\mathrm{Var}(X) = \frac{n^2-1}{12}$.

**Bernoulli distribution**
A random variable $X$ is said to have a *Bernoulli distribution* with parameter $p \in (0, 1)$ if it takes the two values 0 and 1 with probability

$$\mathbf{P}(X = 1) = p \quad \text{and} \quad \mathbf{P}(X = 0) = 1 - p.$$

This distribution is used to describe the result of a random experiment which can either succeed or fail (pass/fail experiment).

Then, we have $\mathbf{E}[X] = p$ and $\mathrm{Var}(X) = p(1 - p)$.

**Binomial distribution**
A random variable $N$ is said to have a *binomial distribution* $B(n, p)$, where $n \in \mathbf{N}^*$ and $p \in (0, 1)$, if it takes values in $\{0, 1, \ldots, n\}$ and

$$\forall k \in \{0, 1, \ldots, n\}, \quad \mathbf{P}(N = k) = \binom{n}{k} \, p^k (1 - p)^{n-k}.$$

The binomial law represents the distribution of the number of successes of a random pass/fail experiment with probability $p$, repeated $n$ times *independently.*

Note that the random variable $N$ can be written as a sum of $n$ *independent* Bernoulli's variables $(X_1, \ldots, X_n)$, where each variable $X_i$ represents the success or failure of the $i$th experiment, so

$$N = X_1 + X_2 + \cdots + X_n.$$

We deduce that $\mathbf{E}[N] = np$ and $\mathrm{Var}(N) = np(1 - p)$.

**Geometric distribution**
A random variable $N$ is said to have a *geometric distribution* with parameter $p \in (0, 1)$ if

$$\forall n \in \mathbf{N}, \quad \mathbf{P}(N = n) = p^n (1 - p).$$

The geometric law represents the distribution of the number of consecutive successes of a random pass/fail experiment with probability of success $p$, repeated *independently.*

Then, we have $\mathbf{E}[N] = \frac{p}{1-p}$ and $\mathrm{Var}(N) = \frac{p}{(1-p)^2}$.

**Poisson distribution**

A random variable $X$ is said to have a *Poisson distribution* with parameter $\lambda > 0$ if

$$\forall n \in \mathbf{N}, \quad \mathbf{P}(X = n) = \frac{\lambda^n}{n!} e^{-\lambda}.$$

Then, we have $\mathbf{E}[X] = \lambda$ and $\mathrm{Var}(X) = \lambda$.

*The structure of measure theory was finally taking shape. The integral with respect to a measure was built. Bernard finds it very nice that applying the definition to discrete functions leads to an expression through discrete sums, which is also the case when considering a discrete probability measure. So, the disturbance he had always been feeling during his engineering courses, when he had to switch from the integral symbol $\int$ to the symbol $\sum$ depending on the situation should now vanish.*

*However, the fact that the integral does not rely in any way on its infinitesimal properties is a revolution to him.*

- *The integral is defined for measurable functions from a measure space $(\Omega, \mathcal{A})$ to $\big(\mathbf{R}, \mathcal{B}(\mathbf{R})\big)$. It is elaborated upon with respect to a measure $\mu$ that is defined on the $\sigma$-algebra $\mathcal{A}$ of the domain $\Omega$.*
- *The starting point of the construction is the definition of simple function.*

  *For $h = \sum_{i=1}^n \alpha_i \mathbf{1}_{A_i}$, where $(\alpha_i)_{1\leq i\leq n}$ belong to $\mathbf{R}$ and $(A_i)_{1\leq i \leq n}$ are in $\mathcal{A}$, we define $\int h\, d\mu = \sum_{i=1}^n \alpha_i\, \mu(A_i)$.*
- *This definition is extended to all the measurable nonnegative functions $f$ through $\int f\, d\mu = \sup\limits_{\substack{h\in\mathcal{E}_+(\mathcal{A}),\\ h \text{ finite},\, h\leq f}} \left( \int h\, d\mu \right)$.*
- *A measurable function $f : (\Omega, \mathcal{A}) \to \big(\mathbf{R}, \mathcal{B}(\mathbf{R})\big)$ is integrable with respect to the measure $\mu$ on $(\Omega, \mathcal{A})$ if $\int |f|\, d\mu < \infty$.*

  *In that case, we decompose $f = f^+ - f^-$, where $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$, and then we define $\int f\, d\mu = \int f^+\, d\mu - \int f^-\, d\mu$.*
- *The monotone convergence property of measures, which follows from $\sigma$-additivity, implies naturally the monotone convergence theorem for the integral.*

- *The beginning of the theory of probability fits naturally in the framework of measure theory. In particular, the expectation of a random variable can be expressed as an integral with respect to its law.*

*About this last point, if the law of a random variable is a discrete measure, we find again the expression that Bernard had learned during his time at* classes préparatoires*.*

*Besides, thanks to this subject of probability, he is happy to go back to work with things to tell his boss Ann. The discussion with Laurent has made him realize that, through the management of the environmental risk, rigorous probability should play a role in the study he had been assigned to. He is in a haste to understand how mastering theoretical concepts can allow him to give relevant answers in the real world.*

*However, the young engineer is intrigued. He wonders how a theoretical mathematician could be sensitive to elements that would convince Ann. Why has Laurent demanded that he tell no one about their working sessions?*

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## Chapter 4

# Integration with Respect to a Measure II: Convergence and Functional Spaces

*May 23rd, in the early morning.*

*Laurent is meditating while having his usual café-croissant. A big step has been taken with the son of his climbing companion, a week ago, but he has to tread cautiously. His experience has shown him how the impatience to get results and the harshness of mathematical rigor could lead young people away from studying theoretical concepts. Ah, the unfortunate opposition, so widely and unduly spread, between theory and application...*

*Until now, Bernard seems to him inclined to really understand the mathematical ideas. But he is inevitably torn between his renewed scientific curiosity and the necessity of satisfying his bosses at SofterWorld Consulting.*

*Laurent's idea is to maintain the young engineer's curiosity through the exploration of the mathematical foundations of modeling, all the while ensuring a sense of fulfillment.*

*He is well aware that Ann would do anything to drive Bernard away from theoretical considerations, which she now deems sterile. So, Bernard has to ensure that he reveals the arguments showing that the industrial waste produced by this plant will be all the more controlled if the mathematical formalization of the problem is mastered.*

*For this week's session, the goal is to get the definition of integral closer to the question of convergence, which is so dear to approximation-loving engineers. Measure theory will look more adaptable than Riemann's theory in studying the integral of the limit of a sequence of functions. In particular, it allows us to define functional spaces with pleasant topological properties, which is important for applications.*

*Laurent remembers that Bernard is particularly sensitive to the special case of probability, which he has not understood well during his studies.*

## 4.1 $L^p$ Spaces

**Bernard:** Laurent, I understood that this construction of an integral is more general than what I know of the Riemann integral. Still, I wish to know the price of this generality...

**Laurent:** Fine. Let us talk about this now. The first thing to understand is that functions which are *almost* equal have the same integral. We already saw this result in Propositions 3.10 and 3.15 along the construction. Recall that, thanks to measure theory, the "almost" word is very precisely defined.

Anyway, let us talk about nonnegative functions whose integrals vanish...

**Bernard:** For the integral I know about, if the integral of a given nonnegative function vanishes, then I know that the function vanishes everywhere.

**Laurent:** Careful, Bernard. If the function is not continuous, this is obviously false! You are right in the special context of the Riemann integral for a continuous function. But, you know that we now consider functions which are not necessarily continuous... For instance, think about the indicator function of the rationals $\mathbf{1}_{\mathbf{Q}}$.

**Bernard:** Wait. My old Riemann integral also integrates functions which are less than continuous! If a nonnegative function vanishes everywhere except on a countable set of points, its Riemann integral vanishes.

**Laurent:** You have obviously mastered the fine properties of the Riemann integral. You are an expert! You really deserve to be named after Riemann...

An analogous property of the integral we just defined will depend on the measure.

**Bernard:** Of course, I forgot that this new integral depends on the properties of the underlying measure! In particular, I understood that if $f$ is integrable with respect to a measure $\mu$, one cannot say whether or not $f$ is integrable with respect to another measure $\nu$.

**Laurent:** Listen to me, Bernard. Assume that $f$ is a nonzero nonnegative real function and $N$ belongs to the $\sigma$-algebra $\mathcal{A}$, with $\mu(N) = 0$. The construction of our integral leads to $\int f\, d\mu = \int f(1 - \mathbf{1}_N)\, d\mu$. It implies that the integral of the nonnegative function $\mathbf{1}_N f$ vanishes, even if the function is nonzero.

**Bernard:** Ah. But, in your example, what can be said of the link between $f$ and $\mathbf{1}_N f$?

**Laurent:** That is the point. I was about to ask you this precise question...

**Bernard:** If I understood correctly, if I have $g = f(1 - \mathbf{1}_N)$, the measure of the subset $\{x \in \Omega : f(x) \neq g(x)\}$ is 0 (this subset is in $\mathcal{A}$ and included in $N$).

**Laurent:** Excellent! You have understood what the *almost everywhere* equality is! We already saw it last week. Recall that the two functions $f$ and $g$ from $\Omega$ to $\mathbf{R}$ are equal $\mu$-almost everywhere ($f = g$ $\mu$-almost everywhere, or $\mu$-a.e. for short) if $\mu\big(\{x \in \Omega : f(x) \neq g(x)\}\big) = 0$.

And Proposition 3.10(iii) stated that if $f = g$ $\mu$-a.e. we have $\int f\, d\mu = \int g\, d\mu$.

**Bernard:** Then, we could say that functions that are equal $\mu$-a.e. are *equivalent* for the integral, right?

**Laurent:** Precisely. Now, let us read the $L^p$ chapter...

### 4.1.1 *Definition of $L^p$ spaces*

In the measure space $(\Omega, \mathcal{A}, \mu)$, for all $p \geq 1$, we define

$$\mathcal{L}^p(\mu) = \left\{ f : \Omega \to \mathbf{R} \text{ measurable} : \int |f|^p \, d\mu < \infty \right\}$$

$$\text{and} \quad \mathcal{L}^\infty(\mu) = \{ f : \Omega \to \mathbf{R} \text{ measurable} : \exists\, C > 0, |f| \leq C \quad \mu - \text{a.e.}\}.$$

For all $p \in [1, \infty]$, we define an *equivalence relation* on $\mathcal{L}^p(\mu)$ by

$$f \sim g \text{ if and only if } f = g \ \mu - \text{a.e.}$$

Remember that an equivalence relation $\sim$ on a set $E$ is a binary relation that satisfies the following three conditions:

- ⋆ *reflexive*: $\forall x \in E$, $x \sim x$;
- ⋆ *symmetric*: $\forall x, y \in E$, if $x \sim y$ then $y \sim x$;
- ⋆ *transitive*: $\forall x, y, z \in E$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

**Example.** — The relation on real numbers "*$x$ and $y$ are congruent, modulo $\pi$*" is an equivalence relation.
— The relation on convergent real sequences "*$(x_n)_{n \in \mathbf{N}}$ and $(y_n)_{n \in \mathbf{N}}$ have the same limit*" is an equivalence relation.

**Definition 4.1.** For $p \in [1, \infty]$, we define the *quotient set* $L^p(\mu) = \mathcal{L}^p(\mu)/\sim$, composed of all the *equivalence classes* of $\sim$, i.e. the subsets of the form

$$C(f) = \{ g \in \mathcal{L}^p(\mu) : f = g \quad \mu - \text{a.e.}\}.$$

We observe that $f \sim g$ if and only if $C(f) = C(g)$.

In practice, we identify $C(f)$ and $f$, which means that:
*In $L^p(\mu)$, we identify functions which are equal* $\mu$-a.e.

**Bernard:** Do we really need to use these really abstract notions to compute the surface under a curve???

**Laurent:** Fear not! You already encountered this notion of equivalence relation, which we can illustrate with the computation of angles. You know it quite well.

In a plane, a geometric angle $\overset{\curvearrowright}{\theta}$ should not be mixed up with its measure $\theta$. This was the first time you were acquainted with equivalence relations:

> Two geometric angles are the same if and only if their measures are equal up to an integer times $2\pi$.
>
> Formally, we have $\overset{\curvearrowright}{\theta} = \overset{\curvearrowright}{\theta'}$ if and only if $(\theta - \theta')/(2\pi) \in \mathbf{Z}$.
>
> We define the equivalence relation $\mathcal{R}$ on the measures of angles:

$$x\mathcal{R}y \iff (x-y)/(2\pi) \in \mathbf{Z},$$

$$(\text{which is usually denoted} \quad x \equiv y \pmod{2\pi})$$

To identify the measures of a geometric angle means to work in the quotient space $\mathbf{R}/(2\pi\mathbf{Z}) = \mathbf{R}/\mathcal{R}$. And, for the sake of simplicity, we even drop the $\equiv$ symbol to write $x = y \pmod{2\pi}$.

**Bernard:** I remember that! This example shows how this formalism can simply express an obvious reality.

**Laurent:** We apply what we just did for angles to function $f$ such that $|f|^p$ is integrable. Equivalence relations for functions allow us to reduce integrable functions to the values of their integrals.

**Bernard:** What? Some very different functions have the same integral!!!

**Laurent:** More precisely, I mean: we identify functions that are such that they have the same value of integral on any element of the $\sigma$-algebra.

Then, according to Proposition 3.15(iii), $f = g$ in $L^p$ means $|f|^p$ and $|g|^p$ are integrable and $f = g$ a.e.

**Bernard:** My impression is that if we are not careful when manipulating objects in $L^p$, we can easily be mistaken. For instance, what is the meaning of $f(x)$ if $f$ is in $L^1$?

**Laurent:** You pinpointed the most difficult point. The elements of $L^1$ are classes of functions that are identified thanks to the equivalence relation but are not necessarily equal, and consequently, writing "for $f \in L^1$, for all $x$, $f(x) = \exp(-x^2)$" has no meaning.

**Bernard:** I am lost. How do you manipulate objects then?

**Laurent:** Recall that we are interested in $f$ as a whole, not pointwise, so morally it does not matter. To manipulate objects correctly, we should write:

"Let $f \in L^1$ and the integrable function $g\colon x \mapsto \exp(-x^2)$. We have $f = g$ in $L^1$."

A common abuse of notations allows us to write this statement without quantifying $x$,

$$f(x) = \exp(-x^2) \text{ in } L^1,$$

or, even worse,

$$\text{for all } x, \quad f(x) = \exp(-x^2) \text{ in } L^1.$$

In any case, equality in $L^1$ means

$$\int_\Omega \left| f(x) - \exp(-x^2) \right| \mu(dx) = 0.$$

Even if it does not seem rigorous, this abuse is very useful (and widely used) in order to simplify notations.

Let us go on with the properties of $L^p$ spaces.

For $f\colon \Omega \to \mathbf{R}$ measurable, for all $p \in [1, \infty[$, we consider

$$\|f\|_{L^p} = \left( \int |f|^p \, d\mu \right)^{1/p} \quad \text{with the convention } \infty^{1/p}$$

$$= \infty \quad \text{and} \quad \|f\|_{L^\infty} = \inf\{C \in [0, \infty] : |f|$$

$$\leq C \quad \mu - \text{a.e.}\}.$$

We now show that $\|\cdot\|_{L^p}$ and $\|\cdot\|_{L^\infty}$ are, respectively, norms on the vector spaces $L^p(\mu)$ and $L^\infty(\mu)$.

**Theorem 4.2 (Hölder inequality).** *Let $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $f, g$ are two measurable functions from $(\Omega, \mathcal{A})$ to $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$.*

*Then, we have $\int |fg| \, d\mu \leq \|f\|_{L^p} \, \|g\|_{L^q}$.*

Particularly, if $f \in L^p(\mu)$ and $g \in L^q(\mu)$, then $fg \in L^1(\mu)$.

**Proof.** $\star$ If $\|f\|_{L^p} = 0$, which implies $f = 0$ $\mu$-a.e. or if $\|f\|_{L^p} = \infty$, the Hölder inequality obviously holds.

$\star$ If $p = 1$ (and consequently $q = \infty$), by the definition of $\|g\|_{L^\infty}$, we can write

$$\int |fg|\, d\mu \leq \|g\|_{L^\infty} \int |f|\, d\mu,$$

and the Hölder inequality follows.

The case of $p = \infty$ follows from the case of $p = 1$ by the permutation of $f$ and $g$.

$\star$ We assume now that $1 < p < \infty$ and $\|f\|_{L^p}, \|g\|_{L^p} \in (0, \infty)$.

— For $0 < \alpha < 1$, the inequality $x^\alpha - \alpha x \leq 1 - \alpha$ holds for all $x \in \mathbf{R}_+$ (it suffices to study the function $x \mapsto x^\alpha - \alpha x$).

— For $0 < \alpha < 1$, the inequality $u^\alpha v^{1-\alpha} \leq \alpha u + (1 - \alpha)v$ holds for all $u, v \in \mathbf{R}_+$. If $v \neq 0$, this inequality follows from the previous one by setting $x = u/v$. The case of $v = 0$ is obvious.

— The Hölder inequality follows from the integration of the previous inequality applied to $\alpha = 1/p$, $u = (|f(x)|/\|f\|_{L^p})^p$ and $v = (|g(x)|/\|g\|_{L^q})^q$.

□

**Bernard:** I encountered the Holder inequality before, especially for discrete sums and the Riemann integral. But I note that the book aims at showing that the function $f \mapsto \|f\|_{L^p}$ is a norm as the notation suggests. The separation and homogeneity axioms are obvious. The triangle inequality $\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$ is not so clear...

**Laurent:** Precisely, we will see that this triangle inequality, which is called the Minkowski inequality in our context, is a direct consequence of the Hölder inequality. It suffices to write

$$|f+g|^p = |f+g|\,|f+g|^{p-1} \leq |f|\,|f+g|^{p-1} + |g|\,|f+g|^{p-1}.$$

$\int f\,d\mu$ 515.BRO.m

**Theorem 4.3 (Minkowski inequality).** *Let $p \in [1, \infty]$. For any $f, g \in L^p(\mu)$, we have $f + g \in L^p(\mu)$ and $\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$.*

**Proof.** Consider $f, g \in L^p(\mu)$.

- ⋆ Assume first that $1 < p < \infty$. The triangle inequality implies $|f + g| \leq |f| + |g|$ and then $|f + g|^p \leq (|f| + |g|)^p \leq 2^p(|f|^p + |g|^p)$ (the last inequality holds when $|f(x)| \leq |g(x)|$ and also when $|g(x)| \leq |f(x)|$). We deduce that $f + g \in L^p(\mu)$.
- ⋆ To prove the upper bound for $\|f + g\|_{L^p}$, we write

$$|f + g|^p = |f + g|\,|f + g|^{p-1} \leq |f|\,|f + g|^{p-1} + |g|\,|f + g|^{p-1}$$

and

$$\int |f + g|^p \,d\mu \leq \int |f|\,|f + g|^{p-1} \,d\mu + \int |g|\,|f + g|^{p-1} \,d\mu.$$

Using the Hölder inequality applied to each term of the second member, denoting by $q$ the conjugate exponent of $p$, we get

$$\begin{aligned}\int |f + g|^p \,d\mu &\leq \|f\|_{L^p} \left(\int |f + g|^{q(p-1)} \,d\mu\right)^{1/q} \\ &\quad + \|g\|_{L^p} \left(\int |f + g|^{q(p-1)} \,d\mu\right)^{1/q} \\ &\leq \|f\|_{L^p} \left(\int |f + g|^p \,d\mu\right)^{1-1/p} \\ &\quad + \|g\|_{L^p} \left(\int |f + g|^p \,d\mu\right)^{1-1/p},\end{aligned}$$

since $q = \frac{p}{p-1}$.

We can assume $\int |f+g|^p \,d\mu \neq 0$ (if $\int |f+g|^p \,d\mu = 0$, the inequality to be proved is obviously satisfied). The previous inequality then becomes

$$\left(\int |f + g|^p \,d\mu\right)^{1/p} \leq \|f\|_{L^p} + \|g\|_{L^p}.$$

⋆ In the case of $p = 1$, the inequality to be proved comes directly from the integration of $|f + g| \leq |f| + |g|$.

⋆ In the case of $p = \infty$, we can write $|f| \leq \|f\|_{L^\infty}$ a.e. and $|g| \leq \|g\|_{L^\infty}$. We deduce that $|f + g| \leq |f| + |g| \leq \|f\|_{L^\infty} + \|g\|_{L^\infty}$ a.e. Then, $\|f + g\|_{L^\infty} \leq \|f\|_{L^\infty} + \|g\|_{L^\infty}$. □

**Bernard:** I saw it once in some exercise! But it was in the context of finite sums:

$$\left(\sum_{i=1}^{n} |y_i + z_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |z_i|^p\right)^{1/p}.$$

**Laurent:** What you say is just a special case where $\mu$ is a discrete measure $\mu = \sum_{i=1}^{n} \delta_{x_i}$.

**Bernard:** Let me check that by myself, please. In that case, with this discrete measure, we compute $\int |f(x)|^p \, \mu(dx) = \int \sum_{i=1}^{n} |f(x)|^p \, \delta_{x_i}(dx) = \sum_{i=1}^{n} |f(x_i)|^p$. Then, the Minkowski inequality reduces to

$$\left(\sum_{i=1}^{n} |f(x_i) + g(x_i)|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |f(x_i)|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |g(x_i)|^p\right)^{1/p},$$

which is the inequality I saw during my *classes préparatoires*, with $y_i = f(x_i)$ and $z_i = g(x_i)$.

Thanks to the Minkowski inequality, we are now able to show that $L^p$ is a complete normed vector space.

**Theorem 4.4 (Riesz–Fischer).** *For all $p \in [1, \infty]$, the mapping $f \mapsto \|f\|_{L^p}$ is a norm on $L^p(\mu)$.*

*Moreover, the space $L^p(\mu)$ equipped with the norm $\|\cdot\|_{L^p}$ is a Banach space,* i.e. *a complete normed vector space.*

**Proof.** $\star$ **$L^p(\mu)$ is a vector space for $p \in [1, \infty]$**

$$a \in \mathbf{R}, f \in L^p(\mu) \Rightarrow af \in L^p(\mu)$$

$$f, g \in L^p(\mu) \Rightarrow f + g \in L^p(\mu),$$

where the second implication comes from the Minkowski inequality.

We now check that $\|\cdot\|_{L^p}$ is a norm. The triangle inequality is precisely given by the Minkowski inequality. For all $a \in \mathbf{R}$ and $f \in L^p(\mu)$, we have $\|af\|_{L^p} = |a| \, \|f\|_{L^p}$.

The last condition to check is that $\|f\|_{L^p} = 0 \Rightarrow f = 0$.

Assume that for some $f \in L^p(\mu)$, we have $\|f\|_{L^p} = 0$. We deduce that $|f|^p = 0$ a.e. and thus $f = 0$ a.e. Since in the quotient space $L^p(\mu)$ functions that are equal almost everywhere are the same, we can deduce that $f = 0$.

$\star$ **For $1 < p < \infty$, $L^p(\mu)$ is complete for the topology of the $L^p$ norm.**

Let $(f_n)_{n\in\mathbf{N}}$ be a Cauchy sequence in $L^p(\mu)$. Let us prove that it converges in $L^p(\mu)$.

Let $(k_n)_{n\in\mathbf{N}}$ be an increasing sequence such that $\forall n \geq 1$, $\|f_{k_{n+1}} - f_{k_n}\|_{L^p} \leq 2^{-n}$.

— We will first prove that the sequence $(g_n)_{n\geq 1}$, defined by $g_n = f_{k_n}$ for all $n \geq 1$, converges almost everywhere. The monotone convergence theorem implies that

$$\int \left( \sum_{n=1}^{\infty} |g_{n+1} - g_n| \right)^p d\mu = \lim_{N\to\infty} \int \left( \sum_{n=1}^{N} |g_{n+1} - g_n| \right)^p d\mu.$$

Using the Minkowski inequality, we get

$$\int \left( \sum_{n=1}^{\infty} |g_{n+1} - g_n| \right)^p d\mu$$

$$\leq \lim_{N\to\infty} \left( \sum_{n=1}^{N} \|g_{n+1} - g_n\|_{L^p} \right)^p = \left( \sum_{n=1}^{\infty} \|g_{n+1} - g_n\|_{L^p} \right)^p < \infty.$$

Then, we can claim that $\sum_{n=1}^{\infty} |g_{n+1} - g_n| < \infty$ a.e. and set $h = g_1 + \sum_{n=1}^{\infty} (g_{n+1} - g_n)$.

The function $h$ is well defined on the set where the series converges. And we give to $h$ the value 0 out of this set. The function $h$ is then measurable.

— By the definition of $h$, we have $g_N \to h$ a.e., but we need to prove the convergence for the $L^p$ norm.

Considering the fact that $|h| = \liminf_{N\to\infty} |g_N|$ a.e. the Fatou Lemma implies

$$\int |h|^p \, d\mu \le \liminf_{N\to\infty} \int |g_N|^p \, d\mu \le \sup_N \int |g_N|^p \, d\mu$$
$$= \sup_N (\|g_N\|_{L^p})^p < \infty,$$

since $(g_n)_{n\ge1}$ is bounded in $L^p(\mu)$, as a Cauchy sequence in $L^p(\mu)$.

We deduce that $\int |h|^p \, d\mu < \infty$ and $h \in L^p(\mu)$.

Finally,

$$\int |h - g_n|^p \, d\mu \le \liminf_{N\to\infty} \int |g_N - g_n|^p \, d\mu = \liminf_{N\to\infty} (\|g_N - g_n\|_{L^p})^p.$$

For $N > n$, using the triangle inequality for the $L^p$ norm, we get

$$\|g_N - g_n\|_{L^p} \le \|g_N - g_{N-1}\|_{L^p} + \cdots + \|g_{n+1} - g_n\|_{L^p}$$
$$\le 2^{-n} \sum_{k\ge0} 2^{-k} = 2^{-n+1}.$$

This yields $\left(\|h - g_n\|_{L^p}\right)^p \le 2^{p(-n+1)}$ for all $n \ge 1$, which implies that $g_n$ converges to $h$ in $L^p(\mu)$, as $n \to \infty$.

— From the inequality $\|f_m - h\|_{L^p} \le \|f_m - g_n\|_{L^p} + \|g_n - h\|_{L^p}$, we deduce that $f_m$ converges to $h$ in $L^p(\mu)$ as $m \to \infty$.

We conclude that any Cauchy sequence in $L^p(\mu)$ converges in $L^p(\mu)$. Then, the normed vector space $L^p(\mu)$ is complete.

⋆ For $p = \infty$, the proof can be adapted and is even simpler. □

**Laurent:** Could you detail rigorously the last argument for completeness in the case of $p < \infty$?

**Bernard:** You're right, Laurent, I really need to check it by myself. Consider $\varepsilon > 0$.

- I recall that $(f_n)_{n\in\mathbf{N}}$ is a Cauchy sequence in $L^p$. It means that there exists $N$ such that for all $m, m' \geq N$, $\|f_m - f_{m'}\|_{L^p} \leq \varepsilon/2$.
- And, since $(g_n)_{n\in\mathbf{N}}$ converges to $h$ in $L^p$, there also exists $N' \in \mathbf{N}$, which can be chosen to be greater than $N$, such that for all $n \geq N'$, $\|g_n - h\|_{L^p} \leq \varepsilon/2$.

Finally, the inequality

$$|f_m - h| \leq |f_m - g_n| + |g_n - h| = |f_m - f_{k_n}| + |g_n - h|$$

is valid for all $m \geq N$ and $n \geq N'$ (which implies that $k_n \geq n \geq N' \geq N$).

Thanks to the Minkowski inequality, I can conclude that

$$\begin{aligned}\forall n \geq N, \quad \|f_m - h\|_{L^p} &\leq \|f_m - f_{k_n}\|_{L^p} + \|g_n - h\|_{L^p} \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

As noted in the proof of Theorem 4.4, we have shown that any Cauchy sequence $(f_n)_{n\in\mathbf{N}}$ in $L^p(\mu)$ admits a subsequence $(f_{k_n})_{n\in\mathbf{N}}$ that converges $\mu$-almost everywhere. We can retain the following statement of this result.

**Corollary 4.5.** *Any sequence of functions that converges in $L^p(\mu)$ admits a subsequence that converges $\mu$-almost everywhere.*

This result obviously holds for sequences of random variables. A specific proof, based on the Borel–Cantelli lemma, is usually given in this case.

**Laurent:** Often, when one discovers this result, particularly in France where the first notion of convergence that is taught is the pointwise convergence, one always tries to link:

- the convergence $f_n \to f$ in $L^p$, that is,

$$\forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbf{N} : \quad n \geq N_\varepsilon \Longrightarrow \int_\Omega |f_n(x) - f(x)|^p \,\mu(dx) < \varepsilon,$$

- to the convergence $f_n(x) \to f(x)$ for every $x \in \Omega$, that is,

$$\forall x \in \Omega,\ \forall \varepsilon > 0,\ \exists N_{x,\varepsilon} \in \mathbf{N} : \quad n \geq N_{x,\varepsilon} \Longrightarrow |f_n(x) - f(x)| < \varepsilon.$$

As we explained earlier, the point in seeing a function $f$ as an element of a functional space is to not having to consider $f$ pointwise.

**Bernard:** I confess that, every time we encountered a new notion of convergence, I tried to refer to the pointwise convergence... I admit that it is a disease I caught in my previous studies. And I'm aware that I need to heal because, without special hypotheses, there is no implication from one convergence to another... especially to the pointwise convergence.

**Laurent:** Indeed, we need something as strong as the dominated convergence theorem to go from a pointwise convergence to a $L^p$ convergence.

Conversely, the only general result is that the $L^p$ convergence implies the existence of a subsequence converging pointwise. There is nothing more.

### 4.1.2 $L^2$ *Hilbert spaces*

**Laurent:** Among the $L^p$ spaces, $L^2$ has a specific structure since it can be equipped with an inner product.

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For any measurable function $f\colon (\Omega, \mathcal{A}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$, we can write

$$f \in L^2(\mu) \Longleftrightarrow \|f\|_{L^2} = \left( \int |f|^2 \, d\mu \right)^{1/2} < \infty.$$

The mapping

$$\begin{aligned} L^2(\mu) \times L^2(\mu) &\to \quad \mathbf{R} \\ (f, g) \qquad &\mapsto \langle f, g \rangle = \int fg \, d\mu \end{aligned}$$

is an inner product, for which $\| \cdot \|_{L^2}$ is the associated norm.

According to the Riesz–Fischer theorem (Theorem 4.4), the space $L^2(\mu)$ equipped with the inner product $\langle \cdot, \cdot \rangle$ is a Hilbert space.

**Remark.** In general, $L^2(\mu) \not\subset L^1(\mu)$ except if $\mu(\Omega) < \infty$ (which is the case if $\mu$ is a probability measure).

The particular case of the Hölder inequality, when $p = 2$, reduces to the Cauchy–Schwarz inequality seen in the framework of Hilbert spaces (Theorem 1.41 in Section 1.5) and that does not need to be further proved.

**Theorem 4.6 (Cauchy–Schwarz inequality).** *Let $f$ and $g$ be two functions in $L^2(\mu)$. We have*

$$\int |fg| \, d\mu \leq \left( \int |f|^2 \, d\mu \right)^{1/2} \left( \int |g|^2 \, d\mu \right)^{1/2}.$$

**Corollary 4.7.** *Let $f, g \in L^2(\mu)$. Then, the product $fg$ belongs to $L^1(\mu)$.*

## 4.2 Sequences of Integrals

**Laurent:** You noted that the point is to be able to exchange a limit and an integral. Consider a sequence of measurable real-valued functions $(f_n)_{n\in\mathbf{N}}$ that converges to a function $f$ in some sense (pointwise or almost everywhere, for instance). What can you say about the convergence of the sequence of $\int f_n \, d\mu$ with respect to the integrability of $f$?

To go further, can one say that $\int f_n \, d\mu \to \int f \, d\mu$?

**Bernard:** To answer these questions, we need to be at least convinced that $\lim f_n$ is integrable... Are there conditions that ensure that?

**Laurent:** Some exist, surely. You should know that if the measurable functions $(f_n)_{n\in\mathbf{N}}$ are nonnegative and the sequence $(f_n)_{n\in\mathbf{N}}$ is nondecreasing, then $\int (\lim f_n)\, d\mu$ is defined and

$$\lim_{n\to\infty} \left( \int f_n \, d\mu \right) = \int \left( \lim_{n\to\infty} f_n \right) d\mu.$$

**Bernard:** Ah! I recognize the monotone convergence theorem. What about integrable functions? What can we say?

**Laurent:** We saw that the monotone convergence theorem leads to the Fatou Lemma, which concerns nonnegative functions, and thanks to the Fatou Lemma, we will be able to prove the dominated convergence theorem.

More generally, as well as being interested in the convergence of $\int f_n(x)\, \mu(dx)$, we can be interested in studying the functions $u \mapsto \int f(u,x)\, \mu(dx)$, which is an integral depending on a parameter. In particular, their continuity, differentiability, and so on. This will help the proving properties of the convolution of functions.

### 4.2.1 *Dominated convergence theorem*

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**Theorem 4.8 (Dominated convergence).** *Let $(f_n)_{n\in\mathbf{N}}$ be real-valued functions in $\mathcal{L}^1(\mu)$. Assume that the following two conditions are satisfied:*

(i) *There exists a measurable function $f\colon (\Omega, \mathcal{A}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ such that $f_n(x)$ converges to $f(x)$ $\mu$-a.e. as $n \to \infty$.*
(ii) *There exists a measurable function $g\colon (\Omega, \mathcal{A}) \to (\mathbf{R}_+, \mathcal{B}(\mathbf{R}_+))$ such that $\int g\, d\mu < \infty$ and for all $n \in \mathbf{N}$, $|f_n| \le g$ $\mu$-a.e.*

*Then, $f$ is in $\mathcal{L}^1(\mu)$ and $\lim_{n\to\infty} \int f_n\, d\mu = \int f\, d\mu$.*

*Moreover, the sequence $(f_n)_{n\in\mathbf{N}}$ converges in $L^1$ to $f$,* i.e. $\lim_{n\to\infty} \int |f_n - f|\, d\mu = 0$.

**Bernard:** I note that Condition (ii) implies the functions $(f_n)_{n\in\mathbf{N}}$ are integrable.

**Laurent:** You are right. But you have to admit that the goal in a book is not to give a minimalist statement. Here, the $\mathcal{L}^1$ assumption for $(f_n)_n \in \mathbf{N}$ seems superfluous when Condition (ii) holds. But stating it makes the theorem clearer for the reader who is just discovering the concepts.

**Proof.** $\star$ Assume first that the functions $(f_n)_{n\in\mathbf{N}}$ satisfy the following stronger conditions:

(i′) For all $x \in \Omega$, $\lim_{n\to\infty} f_n(x) = f(x)$.
(ii′) For all $x \in \Omega$ and for all $n \in \mathbf{N}$, $|f_n(x)| \leq g(x)$.

Letting $n$ go to $\infty$ in (ii′), we get that for all $x \in \Omega$, $|f(x)| \leq g(x)$. Consequently, $f \in \mathcal{L}^1(\mu)$.

We have $\forall n \in \mathbf{N}$, $|f_n - g| \leq 2g$ and, $\forall x \in \Omega$, $|f_n(x) - f(x)| \to 0$ as $n \to \infty$.

By the Fatou Lemma (Theorem 3.11),

$$\liminf \int (2g - |f_n - f|)\, d\mu \geq \int \underbrace{\liminf (2g - |f_n - f|)}_{2g}\, d\mu.$$

Hence, recall that $\liminf(-u_n) = -\limsup u_n$ for any real sequence $(u_n)_{n\in\mathbf{N}}$,

$$2\int g\, d\mu - \limsup \int |f_n - f|\, d\mu \geq 2\int g\, d\mu,$$

and then, $\limsup \int |f_n - f|\, d\mu \leq 0$.

We deduce that $\liminf \int |f_n - f|\, d\mu = \limsup \int |f_n - f|\, d\mu = 0$, which implies $\lim_{n\to\infty} \int |f_n - f|\, d\mu = 0$.

By the inequality $|\int f_n\, d\mu - \int f\, d\mu| \leq \int |f_n - f|\, d\mu$, we obtain the convergence $\lim_{n\to\infty} \int f_n\, d\mu = \int f\, d\mu$.

$\star$ If the strong conditions (i′) and (ii′) are not satisfied, but only conditions (i) and (ii) are satisfied, we consider

$$A = \left\{ x \in \Omega : \lim_{n\to\infty} f_n(x) = f(x) \text{ and } \forall n \in \mathbf{N}, |f_n(x)| \leq g(x) \right\}.$$

By assumption, we have $\mu(\Omega \setminus A) = 0$.

We define the functions $(\tilde{f}_n)_{n\in\mathbf{N}}$ and $\tilde{f}$ by

$$\forall x \in \Omega, \forall n \in \mathbf{N}, \quad \tilde{f}_n(x) = f_n(x)\ \mathbf{1}_A(x) \text{ and } \tilde{f}(x) = f(x)\ \mathbf{1}_A(x).$$

The functions $(\tilde{f}_n)_{n\in\mathbf{N}}$ and $\tilde{f}$ satisfy conditions (i′) and (ii′), and consequently the results hold for these functions.

Since $f = \tilde{f}$ a.e. and for all $n \in \mathbf{N}$, $f_n = \tilde{f}_n$ a.e. we have $\int f\,d\mu = \int \tilde{f}\,d\mu$, for all $n \in \mathbf{N}$, $\int f_n\,d\mu = \int \tilde{f}_n\,d\mu$ and $\int |f_n - f|\,d\mu = \int |\tilde{f}_n - \tilde{f}|\,d\mu$.

The result follows. □

**Laurent:** Can you think of a $L^p$ version of this theorem? Can you change $\mathcal{L}^1$ into $\mathcal{L}^p$?

**Bernard:** To be sure, I would need to prove that under the following two conditions,

(*i*) *there exists a measurable function* $f\colon (\Omega, \mathcal{A}) \to \big(\mathbf{R}, \mathcal{B}(\mathbf{R})\big)$ *such that* $f_n(x)$ *converges to* $f(x)$ $\mu$-a.e. *as* $n \to \infty$;

(*ii*) *there exists a measurable function* $g\colon (\Omega, \mathcal{A}) \to (\mathbf{R}_+, \mathcal{B}(\mathbf{R}_+))$ *such that* $\int g^p\,d\mu < \infty$ *and for all* $n \in \mathbf{N}$, $|f_n| \le g$ $\mu$-a.e.

we can conclude that *the sequence* $(f_n)_{n\in\mathbf{N}}$ *converges in* $L^p$ *to* $f$, i.e. $f$ *is in* $\mathcal{L}^p(\mu)$ *and* $\lim_{n\to\infty} \int |f_n - f|^p\,d\mu = 0$.

**Laurent:** Exactly. How would you start? Do we start the proof all over again?

**Bernard:** I understand you are trying to drag me into the mentality of mathematicians... Let's not do it all over again. We could use the result we just proved in $\mathcal{L}^1$.

Under these two conditions, we set $h_n = |f_n - f|^p$, and we apply the dominated convergence theorem to $(h_n)_{n\in\mathbf{N}}$.

**Laurent:** Wow! You are almost reasoning like a mathematician. Congratulations, Bernard. You think before you start to compute.

Let us see if you are always able to do that. Consider the measure $\mu = \sum_{m\ge1} \delta_m$ and the functions $f_n\colon x \mapsto \frac{1}{x}\,\mathbf{1}_{[1,n]}(x)$. For all $n \ge 1$, $f_n$ is bounded by 1 and integrable since $\int f_n\,d\mu = \sum_{m=1}^{n} \frac{1}{m} \le \log(n+1)$.

The sequence $(f_n)_{n\ge1}$ converges to $f\colon x \mapsto \frac{1}{x}\,\mathbf{1}_{[1,+\infty)}$. And $\int f\,d\mu = \sum_{n\ge1} \frac{1}{n} = \infty$.

So, $f$ is not in $\mathcal{L}^1(\mu)$.

**Bernard:** It means there is a problem with the dominated convergence theorem, then! I am under the impression that this sequence satisfies all the conditions.

**Laurent:** It does not seem so! Check more carefully, Bernard.

The sequence $(|f_n|)_{n\geq 1}$ is not bounded by an integrable function since 1, or, more precisely, the function $x \mapsto 1$, is not $\mu$-integrable.

In conclusion, it is crucial to be able to bound $(|f_n|)_{n\geq 1}$ by an integrable function.

### 4.2.2 *Application: Integrals depending on a parameter*

**Laurent:** As we said earlier, a natural extension is the study of functions defined as an integral with a parameter. It amounts to replacing $(f_n)_{n\in\mathbf{N}}$ with $f\colon (u,x) \mapsto f(u,x)$, that is, the dependence on $n \in \mathbf{N}$ with the dependence on $u \in U$.

We consider the integral of functions which depend on a parameter $u$ in a metric space $(U,d)$, $f\colon (U,\Omega) \to \mathbf{R}$, $(u,x) \mapsto f(u,x)$. For instance, the space $(U,d)$ can be $\mathbf{R}$ endowed with its usual distance.

**Theorem 4.9 (Continuity at a point).** *Let $u_0 \in U$. Assume that the following three conditions hold*:

(i) *For all $u \in U$, the function $x \mapsto f(u,x)$ is measurable.*
(ii) *For $\mu$-almost every $x \in \Omega$, the function $u \mapsto f(u,x)$ is continuous at $u_0$.*
(iii) *There exists a function $g \in \mathcal{L}^1(\mu)$ with nonnegative values such that, for all $u \in U$, $|f(u,x)| \leq g(x)$ for $\mu$-almost every $x$.*

*Then, the function $u \mapsto F(u) = \int_\Omega f(u,x)\ \mu(dx)$ is well defined at every point $u \in U$ and continuous at $u_0$.*

**Proof.** Condition (iii) implies that the function $x \mapsto f(u,x)$ is in $\mathcal{L}^1(\mu)$. Then, $F$ is well defined in $U$.

Let $(v_n)_{n\in\mathbf{N}}$ be any sequence in $U$ that converges to $u_0$.

The continuity property of the function $u \mapsto f(u, x)$ at $u_0$ for $\mu$-almost every $x$ (Condition (ii)) implies that $f(v_n, x)$ converges to $f(u_0, x)$ as $n \to \infty$, for $\mu$-almost every $x$.

Finally, the dominated convergence theorem (Theorem 4.8) implies that

$$\lim_{n\to\infty} \int_\Omega f(v_n, x)\ \mu(dx) = \int_\Omega f(u_0, x)\ \mu(dx),$$

and the continuity of $F$ at $u_0$ follows by sequential characterization. □

**Remark.** From the previous proof, Condition (ii) of continuity at $u_0$ for almost every $x$ can be substituted with the following: for almost every $x$, $\lim_{u\to u_0} f(u, x) = \ell(u_0, x)$.

In that case, the conclusion becomes:

- ⋆ from Condition (i), the function $x \mapsto \ell(u_0, x)$ is measurable;
- ⋆ the integrals $\int_\Omega \ell(u_0, x)\ \mu(dx)$ and $F(u)$ are well defined at every $u \in U$ and

$$\lim_{u\to u_0} \underbrace{\int_\Omega f(u, x)\ \mu(dx)}_{F(u)} = \int_\Omega \underbrace{\lim_{u\to u_0} f(u, x)}_{\ell(u_0,x)}\ \mu(dx).$$

**Bernard:** If I understand well the hypotheses of this theorem, the quantity $\ell(u_0, x)$ is only defined for $x$ belonging to the complementary of some set $\Omega_0$ with $\mu(\Omega_0) = 0$. So, why can't we consider the integral $\int_\Omega \ell(u_0, x)\ \mu(dx)$?

**Laurent:** How disappointing you can be sometimes... Were you sleeping through the construction of the integral? We can extend the function $x \mapsto \ell(u_0, x)$ to all $x \in \Omega$ by assigning an arbitrary value on $\Omega_0$.

By the construction of the integral, the given value on $\Omega_0$ does not count in the value of $\int_\Omega \ell(u_0, x)\ \mu(dx)$ since $\mu(\Omega_0) = 0$.

You have to admit that the convergence condition on the function $u \mapsto f(u, x)$ at $u_0$ is not required for all $x \in \Omega$.

**Bernard:** It seems incredible that everything falls into place with this integral.

**Laurent:** As in all well-built theories! Do not forget what I already told you: measure theory is a cornerstone of mathematical analysis. It was a revolution!

**Example.** (From Le Gall (2022))

(1) Let $\mu$ be a diffuse measure on $\big(\mathbf{R}, \mathcal{B}(\mathbf{R})\big)$ and $\varphi \colon \mathbf{R} \to \mathbf{R}$ be an integrable function. Consider

$$\forall u \in \mathbf{R}, \quad F(u) = \int_{\mathbf{R}} \mathbf{1}_{(-\infty,u]}(x)\ \varphi(x)\ \mu(dx).$$

Then, Theorem 4.9 implies that the function $F \colon u \mapsto F(u)$ is continuous on $\mathbf{R}$.

(2) In the measure space $(\mathbf{R}, \mathcal{B}(\mathbf{R}), \mu)$, let $\varphi \colon \mathbf{R} \to \mathbf{R}$ be an integrable function and $h \colon \mathbf{R} \to \mathbf{R}$ be a continuous and bounded function.
Then, by Theorem 4.9, the function $h * \varphi$ defined by

$$\forall u \in \mathbf{R}, \quad h * \varphi(u) = \int_{\mathbf{R}} h(u-x)\ \varphi(x)\ \mu(dx)$$

is continuous on $\mathbf{R}$.

**Laurent:** Can you prove that Theorem 4.9 applies to Example (1) above?

**Bernard:** Let me see... Define $f \colon (u,x) \mapsto \mathbf{1}_{(-\infty,u]}(x)\,\varphi(x)$, and let $u_0 \in \mathbf{R}$.

Condition (i): $f$ is obviously measurable as the product of measurable functions.

Condition (ii): Let $x \in \mathbf{R}$. Note that

$$\forall u > u_0, \quad f(u,x) - f(u_0,x) = \mathbf{1}_{(u_0,u]}(x)\,\varphi(x),$$
$$\forall u < u_0, \quad f(u,x) - f(u_0,x) = \mathbf{1}_{(u,u_0]}(x)\,\varphi(x).$$

Consider the case $x < u_0$. Then, for $u > u_0$, $f(u,x) - f(u_0,x) = 0$ and for $u \in (x,u_0]$, $f(u,x) - f(u_0,x) = 0$ since $x \notin (u,u_0]$.

In the same way, if $x > u_0$, then for $u < x$, $f(u,x) - f(u_0,x) = 0$.

Finally, if $x = u_0$, for $u > u_0$, $f(u,x) - f(u_0,x) = 0$ and for $u < u_0$, $f(u,x) - f(u_0,x) = \varphi(u_0)$.

Thus, $u \mapsto f(u,x)$ is continuous at $u_0$ for all $x \neq u_0$ (consequently, a.e. since $\mu(\{u_0\}) = 0$).

Condition (iii): for all $x, u \in \mathbf{R}$, $|f(u,x)| \leq |\varphi(x)|$ and $\varphi$ is integrable.

Conclusion: the theorem applies!

**Laurent:** You see that writing the proof allows you to understand something else: without any assumption on the measure $\mu$, the function $F$ is continuous at each point $u_0$ such that $\mu(\{u_0\}) = 0$.

And what about Example (2) on the previous page?

**Bernard:** Define $f\colon (u,x) \mapsto h(u-x)\,\varphi(x)$. Conditions (i) and (ii) are obviously fulfilled. And for almost all $x \in \mathbf{R}$ and all $u \in \mathbf{R}$, $|f(u,x)| \leq \|h\|_\infty |\varphi(x)|$ and $\varphi$ is integrable, so Condition (iii) is also fulfilled.

But... it makes me think of a convolution.

**Laurent:** Yes. But wait. We will treat this notion later on.

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For the following result, we assume that the set $U$ of parameters is an open interval denoted by $I \subset \mathbf{R}$.

**Theorem 4.10 (Differentiation under the integral symbol).** *Let $u_0 \in I$. Assume that the following three conditions hold:*

(i) *For all $u \in I$, the function $x \mapsto f(u,x)$ is in $\mathcal{L}^1(\mu)$.*

(ii) *For $\mu$-almost every $x \in \Omega$, the function $u \mapsto f(u,x)$ is differentiable at $u_0$. Its derivative is denoted by $\frac{\partial f}{\partial u}(u_0,x)$.*

(iii) *There exists a nonnegative function $g \in \mathcal{L}^1(\mu)$ such that, for all $u \in I$, $|f(u,x) - f(u_0,x)| \leq g(x)\,|u-u_0|$ for $\mu$-almost every $x$.*

*Then, the function $u \mapsto F(u) = \int_\Omega f(u,x)\,\mu(dx)$ is differentiable at $u_0$. Its derivative at $u_0$ is given by $F'(u_0) = \int_\Omega \frac{\partial f}{\partial u}(u_0,x)\,\mu(dx)$.*

**Proof.** Let $(v_n)_{n\in\mathbf{N}}$ be a sequence in $I \setminus \{u_0\}$, such that $\lim_{n\to\infty} v_n = u_0$, and consider $\varphi_n(x) = \frac{f(v_n,x)-f(u_0,x)}{v_n-u_0}$.

From Condition (ii), for $\mu$-almost every $x$, $\lim_{n\to\infty} \varphi_n(x) = \frac{\partial f}{\partial u}(u_0, x)$.

Finally, Condition (iii) allows us to apply the dominated convergence theorem (Theorem 4.8) and to conclude that

$$\lim_{n\to\infty} \frac{F(v_n) - F(u_0)}{v_n - u_0} = \lim_{n\to\infty} \int_\Omega \varphi_n(x)\,\mu(dx) = \int_\Omega \frac{\partial f}{\partial u}(u_0, x)\,\mu(dx),$$

which proves the result by the sequential characterization of the limit. □

**Remark.** Other results can be stated about the differentiation of functions defined by an integral. For instance, the following Conditions (ii′) and (iii′) (see e.g. Lang (1993b, Lemma 2.2, p. 226)) are stronger than Conditions (ii) and (iii) of Theorem 4.10:

(ii′) For $\mu$-almost every $x$, the function $u \mapsto f(u, x)$ is differentiable over $I$.

(iii′) There exists a function $g \in \mathcal{L}^1(\mu)$ such that, for $\mu$-almost every $x$,

$$\forall u \in I, \quad \left|\frac{\partial f}{\partial u}(u, x)\right| \leq g(x).$$

If Conditions (i), (ii′) and (iii′) hold, then the function $F$ is differentiable over $I$.

**Bernard:** I guess it is the same as for continuity, right? From Condition (ii), the quantity $\frac{\partial f}{\partial u}(u_0, x)$ is only defined for $x$ belonging to the complementary of some set $\Omega_0$ with $\mu(\Omega_0) = 0$.

It can be extended for all $x \in \Omega$ by assigning an arbitrary value on $\Omega_0$.

Since $\mu(\Omega_0) = 0$, the quantity $\int_\Omega \frac{\partial f}{\partial u}(u_0, x)\,\mu(dx)$ can be considered.

**Laurent:** And do you understand the remark well?

**Bernard:** Sure. This is some kind of calculus stuff that I know pretty well.

Assume that $f$ satisfies (ii′) and (iii′) over $I \times \Omega$. Then, for each $u_0 \in I$ fixed, the finite increment theorem implies that, for almost all $x \in \Omega$,

$$\forall u \in I, \quad |f(u,x) - f(u_0,x)| \le g(x)\,|u - u_0|.$$

Then, Theorem 4.10 implies that $F$ is differentiable at $u_0$.

**Laurent:** Good. Let us turn now to the implications in probability theory. Let us return to the book of André François.

## 4.3 Functional Spaces of Random Variables

### 4.3.1 *Convergence properties coming from the general theory of integration*

$\mathbf{E}[\varphi(X)]$ 519.FRA.f

The expectation has the general convergence properties of the integral with respect to a measure: the monotone convergence theorem, the Fatou Lemma and the dominated convergence theorem.

**Theorem 4.11.** *Let $(X_n)_{n\in\mathbf{N}}$ be a sequence of real random variables on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$.*

- ⋆ *Monotone convergence: If $X_n \ge 0$ for all $n \in \mathbf{N}$ and if $(X_n)_{n\in\mathbf{N}}$ is nondecreasing and converges to $X$ a.s., then $\lim_{n\to\infty} \mathbf{E}[X_n] = \mathbf{E}[X]$.*
- ⋆ *Fatou Lemma: If $X_n \ge 0$ for all $n \in \mathbf{N}$, then $\mathbf{E}[\liminf_{n\to\infty} X_n] \le \liminf_{n\to\infty} \mathbf{E}[X_n]$.*
- ⋆ *Dominated convergence: If $\lim_{n\to\infty} X_n = X$ a.s. and if there exists $Z \in \mathcal{L}^1(\mathbf{P})$ such that $|X_n| \le Z$ for all $n \in \mathbf{N}$, then $\lim_{n\to\infty} \mathbf{E}[X_n] = \mathbf{E}[X]$.*

**Proposition 4.12 (Markov inequality).** *Let $X$ be a real random variable in $\mathcal{L}^1(\mathbf{P})$. For all real $a > 0$, we have*

$$\mathbf{P}(|X| \ge a) \le \frac{\mathbf{E}[|X|]}{a}.$$

**Bernard:** I recognize properties that we discussed for the integral with respect to any measure. Right now, I can see no specificity of the probability theory with respect to measure theory...

**Laurent:** Indeed. The specificity of the probability theory arises mainly with the concept of independence and the fact that one studies the behavior of a random variable means one studies its law. But not so fast.

### 4.3.2 *The nth moment of a random variable for $n \in [1, +\infty)$*

**Laurent:** Recall that after defining the integrable functions with respect to a measure $\mu$, we defined and studied the $L^p(\mu)$ spaces. Regarding the probability measures, one usually calls the *nth-order moment of a random variable* $X$ the value of $\mathbf{E}[X^n]$, when $X \in L^n(\mathbf{P})$. Again, nothing new here.

$\mathbf{E}[\varphi(X)]$ 519.FRA.f

**Definition 4.13.** A real random variable $X$ on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is said to have an *nth moment* if $\int |X|^n \, d\mathbf{P} < \infty$.

We note that $X \in L^n(\mathbf{P})$.

**Laurent:** Please note that it should be the definition of $\mathcal{L}^n(\mathbf{P})$, and not $L^n(\mathbf{P})$, that should be the quotient space of $\mathcal{L}^n(\mathbf{P})$ with respect to the equivalence relation $\sim$, defined as follows: $X \sim Y$ if and only if $X = Y$ almost surely.

But, in probability theory, we are used to considering only the abusive notation $L^n(\mathbf{P})$ instead of $\mathcal{L}^n(\mathbf{P})$.

$\mathbf{E}[\varphi(X)]$ 519.FRA.f

The fact that a probability measure is a finite measure implies the decreasing inclusion of spaces $L^n(\mathbf{P})$, $n \geq 1$. We know that this result is false for measures that are not finite (e.g. the Lebesgue measure).

**Proposition 4.14.** *Let $p, q \in \mathbf{R}$ such that $1 \leq p < q$.*

*Then, for any real random variable $X$ on $(\Omega, \mathcal{F}, \mathbf{P})$, we have*

$$\int |X|^q \, d\mathbf{P} < \infty \Rightarrow \int |X|^p \, d\mathbf{P} < \infty.$$

*This can be written as $L^q(\mathbf{P}) \subset L^p(\mathbf{P})$.*

**Proof.** Without loss of generality, we can assume that $p = 1$ and $q > 1$ (if $1 < p < q$, it suffices to consider $Y = |X|^p$ instead of $X$ and remark that $Y^{q/p} = |X|^q$).

The result is a direct consequence of the Hölder inequality:

$$\int |X| \, d\mathbf{P} \leq \left( \int |X|^q \, d\mathbf{P} \right)^{1/q} \underbrace{\left( \int 1 \, d\mathbf{P} \right)^{1-1/q}}_{=1} .$$

This implies that if $\int |X|^q \, d\mathbf{P} < \infty$, then we have $\int |X| \, d\mathbf{P} < \infty$. □

Note that to prove the inclusion $L^q(\mathbf{P}) \subset L^p(\mathbf{P})$, we could have simply noted that $|X|^p \leq 1 + |X|^q$ when $0 < p < q$. This implies that if $\int |X|^q \, d\mathbf{P} < \infty$, then we have $\int |X|^p \, d\mathbf{P} < \infty$.

**Bernard:** There is a thing I never understood rigorously: how does knowing moments of several orders of a random variable $X$ bring more elements on the distribution of $X$?

For instance, in some engineering courses, it seemed that the randomness of a variable was completely known through the two first moments, meaning the average and the standard deviation...

**Laurent:** STOP! We will answer this question precisely later on, more precisely when we study the Gaussian vectors.

But get out of your head at once the belief that the law of a random variable sums up to knowing the moments of orders 1 and 2! This is really false, Bernard.

$\mathbf{E}[\varphi(X)]$ 519.FRA.f

A direct consequence of Proposition 4.14 is that $L^2(\mathbf{P}) \subset L^1(\mathbf{P})$. This allows us to define the *variance* of a random variable in $L^2(\mathbf{P})$.

**Definition 4.15.** For any real random variable $X$ in $L^2(\mathbf{P})$, $\mathbf{E}[X]$ and $\mathbf{E}[X^2]$ are well defined and finite. In that case, the *variance of* $X$ is the quantity

$$\mathrm{Var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \mathbf{E}\left[(X - \mathbf{E}[X])^2\right] .$$

The quantity $\sqrt{\text{Var}(X)}$ is called the *standard deviation of* $X$, often denoted by $\sigma$.

The variance of a random variable therefore expresses its quadratic deviation from its mean. The following proposition shows how it measures the concentration of the values of $X$ around $\mathbf{E}[X]$.

**Proposition 4.16.** *Let $X$ be a random variable in $L^2(\mathbf{P})$. Then, the following properties hold:*

(i) *For all $a, b \in \mathbf{R}$,* $\text{Var}(aX + b) = a^2 \text{ Var}(X)$.
(ii) *For all real $a > 0$, we have*

$$\mathbf{P}(|X - \mathbf{E}[X]| \geq a) \leq \frac{\text{Var}(X)}{a^2} \quad \textit{called the Chebyshev inequality.}$$

(iii) *$X$ is constant almost surely if and only if* $\text{Var}(X) = 0$.

**Proof.** (i) Since $X^2$ is in $L^1(\mathbf{P})$, $(aX + b)^2$ is also in $L^1(\mathbf{P})$.
Then, we have

$$\begin{aligned}
\text{Var}(aX + b) &= \mathbf{E}\left[(aX + b - \mathbf{E}[aX + b])^2\right] \\
&= \mathbf{E}\left[(aX + b - a\,\mathbf{E}[X] - b)^2\right] \\
&= \mathbf{E}\left[a^2(X - \mathbf{E}[X])^2\right] = a^2 \text{ Var}(X).
\end{aligned}$$

(ii) The inequality comes from

$$\mathbf{P}(|X - \mathbf{E}[X]| \geq a) = \mathbf{P}(|X - \mathbf{E}[X]|^2 \geq a^2),$$

and the Markov inequality for the random variable $(X - \mathbf{E}[X])^2$.

(iii) If $\text{Var}(X) = 0$, then $\mathbf{E}\left[(X - \mathbf{E}[X])^2\right] = 0$.
We deduce that $\mathbf{P}(X - \mathbf{E}[X] \neq 0) = 0$ and then $\mathbf{P}(X - \mathbf{E}[X] = 0) = 1$.
The converse implication is obvious. □

**Bernard:** The last point is treated a bit too fast for me. I would prefer splitting clearly the two implications: if $X$ is almost surely constant, I can see that $X = \mathbf{E}[X]$ almost surely and consequently $\mathbf{E}[(X - \mathbf{E}[X])^2] = 0$. This proves the direct implication.

**Laurent:** Very good. This is indeed the easy implication.

**Bernard:** Conversely, if we assume $\mathrm{Var}(X) = 0$, how can we deduce that $X$ is constant almost surely?

**Laurent:** This a classical reasoning when manipulating measures. It suffices to write

$$\mathbf{P}(|X - \mathbf{E}[X]| \neq 0) = \mathbf{P}\left(\bigcup_{n\geq 1}\left\{|X - \mathbf{E}[X]| \geq \frac{1}{n}\right\}\right).$$

**Bernard:** This is very clever... Now, I can see. We use the Chebyshev inequality:

$$\mathbf{P}(|X - \mathbf{E}[X]| \neq 0) \leq \sum_{n\geq 1} \mathbf{P}\left(|X - \mathbf{E}[X]| \geq \frac{1}{n}\right) \leq \sum_{n\geq 1} n^2 \,\mathrm{Var}(X).$$

We deduce that $\mathbf{P}\big(|X - \mathbf{E}[X]| \neq 0\big) = 0$, that is, $\mathbf{P}\big(X = \mathbf{E}[X]\big) = 1$.

**Laurent:** All these concepts are at the basis of probability theory. The real question now is how to construct a probability on $\mathbf{R}$. This question will arise at the same time as the construction of the Lebesgue measure.

*The framework of measure theory is now well posed. Laurent is reviewing the topics studied during these last sessions:*

- *the rigorous construction of the integral with respect to a measure,*
- *the particular cases of discrete measures and of probability measures,*
- *the Banach spaces $L^p(\mu)$, as well as the Hilbert space $L^2(\mu)$,*
- *the convergence of the sequences of integrals $(\int f_n \, d\mu)_{n\in\mathbf{N}}$.*

*Laurent is feeling delighted that Bernard has found renewed thirst for deeply understanding the mathematical objects. But he is also aware that the young engineer has to discuss regularly his thoughts with his professional circle.*

*Laurent knows perfectly well that to convince Ann, it is crucial to show quickly the interest in theoretical developments toward mastering concrete applications. She has followed a path to a world where scientific value is measured in terms of usefulness for the mercantile world and in the quickness of getting results.*

*Bernard's knowledge is now sufficient on the subject of abstract integration. They have to quickly get closer to the real framework in which the modeling of industrial waste of his pharmaceutical plant lies. They have to go over the case of discrete probability measures. How does one define a measure on* $\mathbf{R}$*? Would the corresponding integral be an extension to Riemann's? Could it really overlook infinitesimal considerations about functions? The study of these questions is still to come...*

Chapter 5

# Integration on R or $\mathrm{R}^d$: The Lebesgue Measure, Cumulative Distribution Functions and Densities

*May 30th, in the early morning.*

*Laurent planned to attack the definition of the Lebesgue measure on* **R**. *But he is hesitant to go into the details of the theory behind the construction. It will depend on the reaction of the young engineer.*

*Ann's specter is wandering in his head. He is especially wary of the emanations of Papadiamantis... her mentor.*

*Depending on the depths of concepts, they will be able – or not – to detail the comparison between the Lebesgue and Riemann integrals, particularly the relation between integration and differentiation of functions.*

*In any case, he knows that the young engineer will be thrilled to enter probability theory, beyond the discrete case.*

## 5.1 The Borel–Lebesgue Measure on the Borel $\sigma$-Algebra

**Bernard:** To sum up what I have understood of measure theory up to now, when we want to integrate functions from $\Omega$ to $\mathbf{R}$:

- we define $\sigma$-algebras: $\mathcal{A}$ on $\Omega$ and $\mathcal{B}$ on $\mathbf{R}$,
- we consider functions that are measurable from $(\Omega, \mathcal{A})$ to $(\mathbf{R}, \mathcal{B})$,
- we define a measure $\mu$ on $\mathcal{A}$,

- we define the integrals of measurable nonnegative functions from $\Omega$ to $\mathbf{R}$ (based on simple functions),
- we define the space $\mathcal{L}^p(\mu)$ and then $L^p(\mu)$.

**Laurent:** Nice summary, Bernard. You have all the steps.

**Bernard:** All this brings us back to the question of the measure. I think I understand well what a $\sigma$-algebra is, but a measure seems more complicated to apprehend. However our discussion on the probabilities on discrete spaces made me understand that defining a measure on a countable (finite or infinite) space is simple. This comes from the $\sigma$-additivity property of the measures.

But, in our "real cases", where $\Omega = \mathbf{R}$ or $\mathbf{R}^d$, I can't even imagine how to define a measure.

**Laurent:** You are pointing at the (only) really technical aspect of measure theory. I believe you have reached the level of understanding to tackle this problem. We will see it when studying the *Lebesgue measure* and its consequences for probability theory.

$\int f\,d\mu$ 515.BRO.m

The goal of this chapter is to define a measure on $\mathbf{R}$ or $\mathbf{R}^d$, endowed with the Borel $\sigma$-algebra $\mathcal{B}(\mathbf{R})$ or $\mathcal{B}(\mathbf{R}^d)$, which extends the notion of length in $\mathbf{R}$ or volume in $\mathbf{R}^d$. Here, we focus on the case of $\mathbf{R}$, but the case of $\mathbf{R}^d$ is completely similar.

For all subsets $A$ of $\mathbf{R}$, we define

$$\lambda^*(A) = \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i);\ A \subset \bigcup_{i \in \mathbf{N}^*} (a_i, b_i) \right\}, \tag{5.1.1}$$

where the infimum is taken over all countable covers of $A$ by intervals $\big((a_i, b_i)\big)_{i \in \mathbf{N}^*}$, with $a_i, b_i \in \mathbf{R}$, $a_i \leq b_i$, for all $i \geq 1$.

In the particular cases of bounded intervals, we remark that for all $a, b \in \mathbf{R}$ such that $a \leq b$,

$$\lambda^*\big((a,b)\big) = \lambda^*\big((a,b]\big) = \lambda^*\big([a,b)\big) = \lambda^*\big([a,b]\big) = b - a. \tag{5.1.2}$$

Indeed:

$\star$ The fact that $\lambda^*\big((a,b)\big) = b - a$ comes directly from the definition of $\lambda^*$.

⋆ We observe that $\lambda^*$ is nondecreasing and, consequently,

$$\lambda^*\big((a,b)\big) \leq \lambda^*\big((a,b]\big) \leq \lambda^*\big([a,b]\big) \quad \text{and}$$
$$\lambda^*\big((a,b)\big) \leq \lambda^*\big([a,b)\big) \leq \lambda^*\big([a,b]\big).$$

⋆ Then, it suffices to prove that $\lambda^*\big([a,b]\big) = b-a$ to conclude that (5.1.2) holds.

For all $n \geq 1$, we have $[a,b] \subset (a-1/n, b+1/n)$, and then

$$b-a \leq \lambda^*\big([a,b]\big) \leq \lambda^*\big((a-1/n, b+1/n)\big) = b-a+2/n.$$

The result follows by taking $n \to \infty$.

For unbounded intervals, for all $a, b \in \mathbf{R}$, we have

$$\lambda^*\big((-\infty,b)\big) = \lambda^*\big((-\infty,b]\big) = \lambda^*\big((a,+\infty)\big) = \lambda^*\big([a,+\infty)\big) = +\infty. \tag{5.1.3}$$

⋆ The fact that $\lambda^*\big((-\infty,b)\big) = \lambda^*\big((a,+\infty)\big) = +\infty$ comes directly from the definition of $\lambda^*$.

⋆ The cases of $(-\infty,b]$ and $[a,+\infty)$ are treated in the same way as the bounded case.

**Bernard:** Formula (5.1.1) seems to give a general process to define a measure.

**Laurent:** You are going too fast, Bernard. The so-defined $\lambda^*$ has the flavor of a measure, but it is not a measure. In particular, it is not $\sigma$-additive: in general,

$$\lambda^*\Bigg(\bigcup_{n\in\mathbf{N}} A_n\Bigg) \neq \sum_{n\in\mathbf{N}} \lambda^*(A_n)$$

even if the $(A_n)_{n\in\mathbf{N}}$ are pairwise-disjoint.

Actually, we can see easily that the left-hand term is less than the right-hand term.

Did you not note another point? We do not need a $\sigma$-algebra to define it.

**Bernard:** Ah! Right. I confess I hadn't noted this point. You are right, I wasn't rigorous enough. But if it's not a measure, what is it?

And if it's almost a measure, can we define a measure starting from it?

**Laurent:** $\lambda^*$ is a so-called *outer-measure*, and it so happens that the heart of the Caratheodory theory lies in studying the outer-measures and considering restrictions that are indeed measures on some $\sigma$-algebras. I suggest you leave this by the side at first and admit that, starting from $\lambda^*$, we obtain, restricting it to $\mathcal{B}(\mathbf{R})$, the Borel–Lebesgue measure.

From the mapping $\lambda^*$ defined on all subsets of $\mathbf{R}$, Caratheodory's general theory of *outer-measures* allows us to construct the *Borel–Lebesgue measure* defined on the Borel $\sigma$-algebra (see Halmos (1974); Munroe (1953); Rogers (1998) for instance).

Without proof, we can retain the following definition.

**Definition 5.1 (Borel–Lebesgue measure).** The restriction of $\lambda^*$ to the Borel sets is a measure on the Borel $\sigma$-algebra $\mathcal{B}(\mathbf{R})$. This so-called *Borel–Lebesgue measure* is usually denoted by $\underline{\lambda}$.

Since the intervals $(a, b)$, $(a, b]$, $[a, b)$ and $[a, b]$ are in $\mathcal{B}(\mathbf{R})$ for all $a, b \in \mathbf{R}$ such that $a \le b$, (5.1.2) implies that

$$\underline{\lambda}\big((a,b)\big) = \underline{\lambda}\big((a,b]\big) = \underline{\lambda}\big([a,b)\big) = \underline{\lambda}\big([a,b]\big) = b - a.$$

In the same way, (5.1.3) implies that

$$\underline{\lambda}\big((-\infty,b)\big) = \underline{\lambda}\big((-\infty,b]\big) = \underline{\lambda}\big((a,+\infty)\big) = \underline{\lambda}\big([a,+\infty)\big) = +\infty.$$

These relations show that the Lebesgue measure extends to $\mathcal{B}(\mathbf{R})$ the notion of length in $\mathbf{R}$.

Actually, Caratheodory's construction proves that these relations characterize the Borel–Lebesgue measure. We retain the following.

**Proposition 5.2.** *The Borel–Lebesgue measure is the only measure* $\underline{\lambda}$ *on the Borel* $\sigma$*-algebra* $\mathcal{B}(\mathbf{R})$ *such that* $\underline{\lambda}\big((a,b)\big) = b - a$ *for all* $a, b \in \mathbf{R}$ *such that* $a \le b$.

**Bernard:** The particular cases of unbounded intervals suggest that, in Expression (5.1.1) defining $\lambda^*$, some $a_i$ can be equal to $-\infty$ and some $b_i$ can be equal to $+\infty$. Is it correct?

**Laurent:** As written in this book, it is not allowed, but one could. In that case, the covers considered in Expression (5.1.1) would possibly contain intervals of the form $(a_i, +\infty)$ or $(-\infty, b_i)$.

For instance, it could happen when $A$ is an unbounded subset of $\mathbf{R}$. In that case, the relative quantity $b_i - a_i$ would be substituted with $+\infty$. But I repeat that it was not the choice of the author of this book.

**Bernard:** It doesn't seem important. Since the definition of $\lambda^*$ relies on an infimum, my impression is that the specific cover involving an unbounded interval doesn't impact the quantity $\lambda^*(A)$.

**Laurent:** You are right. We can also remark that we do not need to consider unbounded intervals in the covers. Since these intervals can be written as

$$(-\infty, b_i) = \bigcup_{n\in\mathbf{N}^*} (b_i - n, b_i) \quad \text{and} \quad (a_i, +\infty) = \bigcup_{n\in\mathbf{N}^*} (a_i, a_i + n),$$

we could restrict the $a_i$'s and $b_i$'s to be (finite) real numbers in Expression (5.1.1) defining $\lambda^*$.

Let us talk about the case of $\mathbf{R}^d$, which is said to be similar to $\mathbf{R}$.

Expression (5.1.1) for $\lambda^*(A)$ can also be considered for any subset $A$ of $\mathbf{R}^d$. In that case, $a_i$ and $b_i$ are in $\mathbf{R}^d$ for all $i \in \mathbf{N}^*$. Writing $a_i = \left(a_i^{(1)}, \dots, a_i^{(d)}\right)$ and $b_i = \left(b_i^{(1)}, \dots, b_i^{(d)}\right)$, we have $(a_i, b_i) = \left(a_i^{(1)}, b_i^{(1)}\right) \times \cdots \times \left(a_i^{(d)}, b_i^{(d)}\right)$. With this notation, the definition of $\lambda^*(A)$ for $A \subset \mathbf{R}^d$ is the same as in the case of $A \subset \mathbf{R}$.

**Definition 5.3 (Borel–Lebesgue measure in $\mathbf{R}^d$).** The restriction of $\lambda^*$ to Borel sets is a measure on the Borel $\sigma$-algebra $\mathcal{B}(\mathbf{R}^d)$. The so-called *Borel–Lebesgue measure* is usually denoted by $\underline{\lambda}^{(d)}$.

For any bounded interval $(a, b)$ of $\mathbf{R}^d$, we have

$$\underline{\lambda}^{(d)}\big((a,b)\big) = \left(b^{(1)} - a^{(1)}\right) \dots \left(b^{(d)} - a^{(d)}\right).$$

This relation shows that the Borel–Lebesgue measure of $\mathbf{R}^d$ extends the notion of volume (or area when $d = 2$) in $\mathbf{R}^d$. Once again, Caratheodory's construction proves that this relation characterizes the Borel–Lebesgue measure of $\mathbf{R}^d$. We retain the following.

**Proposition 5.4.** *The Borel–Lebesgue measure of* $\mathbf{R}^d$ *is the only measure* $\underline{\lambda}^{(d)}$ *on the Borel* $\sigma$*-algebra* $\mathcal{B}(\mathbf{R}^d)$ *such that* $\underline{\lambda}^{(d)}\big((a,b)\big) = \big(b^{(1)} - a^{(1)}\big)\ldots\big(b^{(d)} - a^{(d)}\big)$ *for all* $a = (a^{(1)}, \ldots, a^{(d)})$ *and* $b = (b^{(1)}, \ldots, b^{(d)})$ *in* $\mathbf{R}^d$ *such that* $a \leq b$.

**Bernard:** I am curious about the detailed construction of the Borel–Lebesgue measure. In particular, I wonder how the mapping $\lambda^*$ and the Borel $\sigma$-algebra are connected.

**Laurent:** This is precisely the key point of the definition of the measure $\underline{\lambda}$, starting from the outer-measure $\lambda^*$. I understand your eagerness, and if you are interested, we will see all the details later. Before that, let me say that you do not need it to understand the Lebesgue integral. Just accept that the Borel–Lebesgue measure exists on the Borel $\sigma$-algebra.

**Bernard:** OK OK. I'll try to be patient. Let's proceed to the famous Lebesgue integral.

**Laurent:** Before that, we need to extend a little bit the Borel $\sigma$-algebra and the Borel–Lebesgue measure. The so-called *Lebesgue* $\sigma$*-algebra* and *Lebesgue measure* are considered in order to define an integral with good properties.

## 5.2 The Lebesgue $\sigma$-Algebra and the Lebesgue Measure

### 5.2.1 *Completion of a measure space*

The Lebesgue integral is defined for functions which are measurable with respect to a $\sigma$-algebra that is larger than the Borel $\sigma$-algebra. It allows, in particular, to consider the class of Riemann-integrable functions on a segment $[a, b]$ of $\mathbf{R}$.

**Definition 5.5 (Completion of a $\sigma$-Algebra).** Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. The class of *$\mu$-negligible sets* of $\Omega$, or *negligible sets* if there is no confusion, is

$$\mathcal{N}_\mu = \{A \in \mathcal{P}(\Omega) : \exists B \in \mathcal{A} \text{ such that } A \subset B \text{ and } \mu(B) = 0\}.$$

The completion of the $\sigma$-algebra $\mathcal{A}$ with respect to $\mu$ is the $\sigma$-algebra $\overline{\mathcal{A}} = \sigma(\mathcal{A} \cup \mathcal{N}_\mu)$.

**Bernard:** So, the completion of a $\sigma$-algebra is simply adding the zero-measure sets.

**Laurent:** You need to be more precise, Bernard. If I strictly consider what you said, there would be nothing to do since all the sets you can measure would be already in the $\sigma$-algebra...

You need to understand that the negligible sets are not necessarily in the $\sigma$-algebra $\mathcal{A}$. And by definition, when you say "zero-measure sets", you consider subsets that are already in $\mathcal{A}$ since you consider their measures.

The point is that you want to add to the $\sigma$-algebra all the subsets of its zero-measure elements, which are precisely the negligible sets. And it is not all. The $\sigma$-algebra $\overline{\mathcal{A}}$ does not reduce to the addition of these subsets to $\mathcal{A}$. Actually, it consists in adding all the subsets between two elements, $B$ and $B'$, of $\mathcal{A}$ such that $B \subset B'$ and $\mu(B' \setminus B) = 0$.

**Bernard:** Oh! Slow down, please. I need to read this carefully in the book.

**Proposition 5.6.** *The completion of the $\sigma$-algebra $\mathcal{A}$ with respect to the measure $\mu$ is the $\sigma$-algebra*

$$\overline{\mathcal{A}} = \{A \in \mathcal{P}(\Omega) : \exists B, B' \in \mathcal{A} \text{ such that } B \subset A \subset B' \text{ and } \mu(B' \setminus B) = 0\}.$$

**Proof.** Let us note that

$$\mathcal{B} = \{A \in \mathcal{P}(\Omega) : \exists B, B' \in \mathcal{A} \text{ such that } B \subset A \subset B' \\ \text{and } \mu(B' \setminus B) = 0\}.$$

We need to prove that $\overline{\mathcal{A}} = \mathcal{B}$.

⋆ We first remark that $\mathcal{A} \cup \mathcal{N}_\mu \subset \mathcal{B}$. Any $A$ in $\mathcal{N}_\mu$ belongs to $\mathcal{B}$: it suffices to take $B = \emptyset$ in the expression of $\mathcal{B}$. And obviously, any $B$ in $\mathcal{A}$ belongs to $\mathcal{B}$.

⋆ Then, we check that $\mathcal{B}$ is a $\sigma$-algebra.

Of course, $\emptyset \in \mathcal{B}$, and if $A \in \mathcal{B}$, then $\Omega \setminus A \in \mathcal{B}$.

Consider $(A_n)_{n\in\mathbf{N}}$ in $\mathcal{B}$. By definition, there exists $(B_n)_{n\in\mathbf{N}}$ and $(B'_n)_{n\in\mathbf{N}}$ in $\mathcal{A}$ such that $B_n \subset A_n \subset B'_n$ and $\mu(B'_n \setminus B_n) = 0$ for all $n \in \mathbf{N}$.

Then, we have $\bigcup_{n\in\mathbf{N}} B_n \subset \bigcup_{n\in\mathbf{N}} A_n \subset \bigcup_{n\in\mathbf{N}} B'_n$ and

$$\mu\left(\bigcup_{n\in\mathbf{N}} B'_n \Big\backslash \bigcup_{n\in\mathbf{N}} B_n\right) \leq \sum_{n\in\mathbf{N}} \mu\left(B'_n \Big\backslash \bigcup_{n\in\mathbf{N}} B_n\right) \\ \leq \sum_{n\in\mathbf{N}} \mu(B'_n \setminus B_n) = 0.$$

Then, $\bigcup_{n\in\mathbf{N}} A_n$ belongs to $\mathcal{B}$.

⋆ Let $A$ be in $\mathcal{B}$ and $B, B'$ in $\mathcal{A}$ such that $B \subset A \subset B'$ and $\mu(B' \setminus B) = 0$.

We have $A \setminus B \in \mathcal{N}_\mu$ since $A \setminus B \subset B' \setminus B$ and $\mu(B' \setminus B) = 0$.

We can write $A = B \cup (A \setminus B) \in \sigma(\mathcal{A} \cup \mathcal{N}_\mu) = \overline{\mathcal{A}}$. Then, $\mathcal{B} \subset \overline{\mathcal{A}}$.

Since $\overline{\mathcal{A}}$ is the smallest $\sigma$-algebra containing $\mathcal{A} \cup \mathcal{N}_\mu$ by definition, we can conclude that $\mathcal{B} = \overline{\mathcal{A}}$. □

**Bernard:** If I understand well: the measure $\mu$ can be easily extended to the subsets $A$ between $B$ and $B'$ in the proof.

∫ f dμ 515.BRO.m The description of the completed $\sigma$-algebra with respect to the measure $\mu$ allows us to extend the measure naturally.

**Proposition 5.7.** *There exists a unique measure* $\overline{\mu}$ *over* $\left(\Omega, \overline{\mathcal{A}}\right)$ *which extends the measure* $\mu$ *defined on* $(\Omega, \mathcal{A})$.

**Proof.** $\star$ We define the function $\overline{\mu}\colon \overline{\mathcal{A}} \to \mathbf{R}_+$ by $\overline{\mu}(A) = \mu(A)$ for $A \in \mathcal{A}$ and $\overline{\mu}(A) = \mu(B)$ for $B \subset A \subset B'$, with $B, B' \in \mathcal{A}$ and $\mu(B' \setminus B) = 0$.

The $\sigma$-additivity of $\overline{\mu}$ follows from the $\sigma$-additivity of $\mu$.

$\star$ To prove the uniqueness of the extension, let us consider two measures $\nu_1$ and $\nu_2$ on $\overline{\mathcal{A}}$ that extend the measure $\mu$.

- We have $\nu_1(A) = \nu_2(A) = \mu(A)$ for any $A \in \mathcal{A}$.
- For any $A \subset \Omega$ such that $B \subset A \subset B'$ with $B, B' \in \mathcal{A}$ and $\mu(B' \setminus B) = 0$, we have $\mu(B) = \nu_1(B) \subset \nu_1(A) \subset \nu_1(B') = \mu(B')$. Then, $\nu_1(A) = \mu(B)$.
  In the same way, we have $\nu_2(A) = \mu(B)$.

We conclude that $\nu_1(A) = \nu_2(A)$ for all $A \in \overline{\mathcal{A}}$. □

**Definition 5.8.** The measure space $\left(\Omega, \overline{\mathcal{A}}, \overline{\mu}\right)$ is called the *completion of* $(\Omega, \mathcal{A}, \mu)$.

We say that $\left(\Omega, \overline{\mathcal{A}}, \overline{\mu}\right)$ is a *complete measure space* since every $\mu$-negligible set is measurable (and its measure is equal to zero).

**Bernard:** The definition is clear to me now. But I wonder which functions are measurable with respect to this completed $\sigma$-algebra.

**Laurent:** Oh! It is not difficult at all! You simply have to bring the definitions together. A function $f\colon (\Omega, \overline{\mathcal{A}}) \to (E, \mathcal{E})$ is measurable if for all $B \in \mathcal{E}$, $f^{-1}(B) \in \overline{\mathcal{A}}$.

It follows that, by the definition of $\overline{\mathcal{A}}$, a function $f\colon (\Omega, \overline{\mathcal{A}}) \to (E, \mathcal{E})$ is measurable if for all $B \in \mathcal{E}$, there exists $C, C' \in \mathcal{A}$ such that $C \subset f^{-1}(B) \subset C'$ and $\mu(C' \setminus C) = 0$.

Moreover, we can view a $\overline{\mathcal{A}}$-measurable function as an *almost* $\mathcal{A}$*-measurable function.*

**Proposition 5.9.** *Let* $f\colon \Omega \to E$ *be a function. Assume that there exists a measurable function* $\varphi\colon (\Omega, \mathcal{A}) \to (E, \mathcal{E})$

*such that*

$$\{x \in \Omega : f(x) \neq \varphi(x)\} \in \mathcal{N}_\mu,$$

*where* $\mathcal{N}_\mu$ *denotes the class of negligible sets of* $\Omega$ *(with respect to the measure* $\mu$*).*

*Then, the function* $f$ *is measurable with respect to the completed* $\sigma$*-algebra* $\overline{\mathcal{A}} = \sigma(\mathcal{A} \cup \mathcal{N}_\mu)$*.*

**Proof.** Denote $A = \{x \in \Omega : f(x) \neq \varphi(x)\}$. Let $B \in \mathcal{E}$. We have

$$f^{-1}(B) = \Big(\varphi^{-1}(B) \cap (\Omega \setminus A)\Big) \cup \Big(f^{-1}(B) \cap A\Big),$$

where

- ⋆ $\varphi^{-1}(B) \in \mathcal{A}$ and $A \in \mathcal{N}_\mu$ imply $\varphi^{-1}(B) \cap (\Omega \setminus A) \in \sigma(\mathcal{A} \cup \mathcal{N}_\mu)$;
- ⋆ $f^{-1}(B) \cap A \subset A$ implies $f^{-1}(B) \cap A \in \mathcal{N}_\mu$.

Then, we can conclude that $f^{-1}(B) \in \sigma(\mathcal{A} \cup \mathcal{N}_\mu) = \overline{\mathcal{A}}$ and $f$ is $\overline{\mathcal{A}}$-measurable. □

**Bernard:** OK. I got it. If I haven't lost my train of thought, we are studying the Lebesgue integral. So, I guess we must see what happens if $\mathcal{A}$ is the Borel $\sigma$-algebra.

### 5.2.2 *Lebesgue measure on the Lebesgue σ-algebra*

**Definition 5.10.** The completion of the Borel $\sigma$-algebra is called the *Lebesgue* $\sigma$*-algebra*.

The extension of the Borel–Lebesgue measure $\underline{\lambda}$ to the Lebesgue $\sigma$-algebra $\overline{\mathcal{B}(\mathbf{R})}$ is called *the Lebesgue measure* and is denoted by $\lambda$.

The Lebesgue measure on $\overline{\mathcal{B}(\mathbf{R})}$ can also be defined as the restriction of $\lambda^*$ to $\overline{\mathcal{B}(\mathbf{R})}$. We can admit the following result.

**Proposition 5.11.** *The Lebesgue measure* $\lambda$ *is the only measure on* $\overline{\mathcal{B}(\mathbf{R})}$ *such that* $\lambda\big((a,b)\big) = b - a$ *for all* $a, b \in \mathbf{R}$ *such that* $a \leq b$*.*

**Remark.** Note that Definition 5.10 and Proposition 5.11 are easily extended to the Lebesgue measure $\lambda^{(d)}$ on the measurable space $\big(\mathbf{R}^d, \overline{\mathcal{B}(\mathbf{R}^d)}\big)$.

**Laurent:** I have to say that the Lebesgue $\sigma$-algebra can be introduced in a different way from the completion of the Borel $\sigma$-algebra.

As we already discussed, the Lebesgue measure is defined as the restriction of the outer-measure $\lambda^*$ to some class of subsets. Actually, we are used to calling *Lebesgue $\sigma$-algebra* this class of subsets on which $\lambda^*$ is a measure.

And it can be proved that this $\sigma$-algebra is precisely the completion of the Borel $\sigma$-algebra.

**Bernard:** But what is truly the interest of this Lebesgue $\sigma$-algebra? In my understanding, the Borel $\sigma$-algebra is sufficient to consider an integral on **R**.

**Laurent:** You are absolutely right, Bernard. But this slight extension of the Borel $\sigma$-algebra and of the Borel–Lebesgue measure allow us to consider Riemann-integrable functions (which are not necessarily Borel functions) and to compare the Lebesgue integral with the Riemann integral.

And the price is not too high. Some interesting regularity properties on the measure $\lambda$ can be derived, which allows us to obtain approximations of functions in $L^p(\lambda)$ using continuous functions.

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**Proposition 5.12.** *The Lebesgue measure over* **R** *is* regular *in the sense that for all* $A \in \overline{\mathcal{B}(\mathbf{R})}$,

$$\begin{aligned}\lambda(A) &= \inf\{\lambda(U) : U \text{ open and } A \subset U\}\\ &= \sup\{\lambda(F) : F \text{ compact and } F \subset A\}.\end{aligned}$$

**Bernard:** If I understand well, since open sets and compact sets in **R** belong to the Borel $\sigma$-algebra $\mathcal{B}(\mathbf{R})$, we could write $\underline{\lambda}(U)$ and $\underline{\lambda}(F)$, instead of $\lambda(U)$ and $\lambda(F)$.

Moreover, if this result is true for any $A \in \overline{\mathcal{B}(\mathbf{R})}$, it is also true for any $A \in \mathcal{B}(\mathbf{R})$. Then, for all $A \in \mathcal{B}(\mathbf{R})$, $\underline{\lambda}(A) = \lambda(A)$, and finally

$$\begin{aligned}\underline{\lambda}(A) &= \inf\{\underline{\lambda}(U) : U \text{ open and } A \subset U\} \\ &= \sup\{\underline{\lambda}(F) : F \text{ compact and } F \subset A\},\end{aligned}$$

which means that the Borel–Lebesgue measure $\underline{\lambda} = \lambda|_{\mathcal{B}}$ is also regular.

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**Proof.** We first remark that, since $\lambda$ is nondecreasing, for all $A \in \overline{\mathcal{B}(\mathbf{R})}$, we have

$$\begin{aligned}\sup\{\lambda(F) &: F \text{ compact and } F \subset A\} \leq \lambda(A) \\ &\leq \inf\{\lambda(U) : U \text{ open and } A \subset U\}.\end{aligned}$$

To complete the proof, we need to show that the converse inequalities hold.

$\star$ Proof of $\lambda(A) \geq \inf\{\lambda(U) : U \text{ open and } A \subset U\}$.

If $\lambda(A) = \infty$, the inequality $\lambda(A) \geq \inf\{\lambda(U) : U \text{ open and } A \subset U\}$ is obvious.

Now, we assume that $\lambda(A) < \infty$.

By the definition of $\lambda$, we have $\lambda(A) = \lambda^*(A)$, and by the definition of $\lambda^*$: for all $\varepsilon > 0$, there exists an open cover of $A$, $A \subset \bigcup_i (a_i, b_i)$, such that

$$\sum_i (b_i - a_i) \leq \lambda(A) + \varepsilon.$$

Setting $U = \bigcup_i (a_i, b_i)$, we have

$$\lambda(U) \leq \sum_i (b_i - a_i) \leq \lambda(A) + \varepsilon.$$

To summarize, for all $\varepsilon > 0$, we proved the existence of an open set $U$ such that $A \subset U$ and $\lambda(U) \leq \lambda(A) + \varepsilon$.

We conclude that $\lambda(A) \geq \inf\{\lambda(U) : U \text{ open and } A \subset U\}$.

$\star$ Proof of $\lambda(A) \leq \sup\{\lambda(F) : F \text{ compact and } F \subset A\}$.

– First, assume that $A \subset C$, for some compact subset $C$ of $\mathbf{R}$.

Applying the previous point to $C \setminus A$, for all $\varepsilon > 0$, there exists an open set $U$ such that $C \setminus A \subset U$ and $\lambda(U) \leq \lambda(C \setminus A) + \varepsilon$.

Since $A \subset C$, we can write $C = A \cup (C \setminus A)$. Then, $C \setminus U \subset A$ since

$$C \setminus U = C \cap (\mathbf{R} \setminus U) = [A \cap (\mathbf{R} \setminus U)] \cup \underbrace{[(C \setminus A) \cap (\mathbf{R} \setminus U)]}_{=\emptyset} \subset A.$$

This implies that the compact set $C \setminus U$ satisfies

$$\lambda(C \setminus U) \geq \lambda(C) - \lambda(U) \geq \lambda(C) - \lambda(C \setminus A) - \varepsilon = \lambda(A) - \varepsilon.$$

To summarize, for all $\varepsilon > 0$, we proved the existence of a compact set $F = C \setminus U$ such that $F \subset A$ and $\lambda(A) - \varepsilon \leq \lambda(F)$.

We conclude that $\lambda(A) \leq \sup\{\lambda(F) : U \text{ compact and } F \subset A\}$.

– Second, if $A$ is not included in some compact set, we apply the result to $A \cap [-n, n]$ for all $n \in \mathbf{N}$, and we write $\lambda(A) = \lim_{n\to\infty} \lambda(A \cap [-n, n])$. □

The regularity of the Lebesgue measure allows us to obtain the first result of the density of $L^p(\lambda)$ (and $L^p(\lambda^{(d)})$), which is fundamental.

### 5.2.3 $L^p(\lambda^{(d)})$ *spaces and continuity*

**Bernard:** It's weird: the integral with respect to the Lebesgue measure exists to go further than integrating continuous functions, right? So, I don't see how continuity could be a relevant property for the $L^p$ spaces and finally for integrability.

**Laurent:** This is a very smart question, Bernard. Really. Even if the Lebesgue integral is designed to integrate a larger set of functions than the continuous functions, nevertheless, the continuous functions play a very important role in the characterization of the $L^p$ spaces.

**Bernard:** I don't understand. What you say isn't precise enough for me.

**Laurent:** Right. I mean that, if a function $f$ belongs to $L^p(\lambda)$, then $f$ can be approximated by a sequence $(\varphi_n)_{n\in\mathbf{N}}$ of continuous functions on $\mathbf{R}$. More precisely, $\|f - \varphi_n\|_{L^p}$ converges to 0 as $n$ goes to $\infty$.

**Bernard:** What an impressive result! I must admit I have no idea how to begin the proof...

**Laurent:** At first sight, it may appear so. But, looking into it, you can just go back to the definition of the integral.

So, the first step is to prove the result for simple functions and then to approximate the simple functions by compactly supported continuous functions. We will now see how these results follow on from each other.

**Lemma 5.13.** *For any $p \geq 1$ or $p = \infty$, the set of simple functions is dense in $L^p(\lambda^{(d)})$.*

**Proof.** Let $f$ be in $\mathcal{L}^p\big(\lambda^{(d)}\big)$.

By the decomposition $f = f^+ - f^-$, where $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$, we can restrict ourselves to the case where $f$ only takes nonnegative values.

According to Theorem 3.2, there exists an increasing sequence of nonnegative $\overline{\mathcal{B}(\mathbf{R}^d)}$-measurable simple functions $(f_n)_{n\in\mathbf{N}}$ that converges pointwise to $f$.

We distinguish the two cases $p \geq 1$ and $p = \infty$.

⋆ For $p \in [1, +\infty)$, we have the following:
  - $|f_n - f|^p$ converges to 0 almost everywhere (actually, for all $x \in \mathbf{R}^d$);
  - for all $n \in \mathbf{N}$, $|f_n - f|^p \leq 2^p f^p$, with $f^p \in \mathcal{L}^1\big(\lambda^{(d)}\big)$.

  The dominated convergence theorem implies that

$$\int |f_n - f|^p \, d\lambda^{(d)} \longrightarrow 0, \quad \text{as } n \to \infty,$$

  which states that $(f_n)_{n\in\mathbf{N}}$ converges to $f$ in $L^p\big(\lambda^{(d)}\big)$.

⋆ For $p = \infty$, since $0 \leq f(x) < \infty$ for almost every $x$, we can claim that, for $n$ sufficiently large, $|f_n(x) - f(x)| \leq 2^{-n}$ a.e. (see the definition of $f_n$ in the proof of Theorem 3.2).

We conclude that $\|f_n - f\|_{L^\infty} \leq 2^{-n}$ and. consequently, $(f_n)_{n\in\mathbf{N}}$ converges to $f$ in $L^\infty\big(\lambda^{(d)}\big)$.

In passing, it shows that $f_n \in L^p\big(\lambda^{(d)}\big)$ for all $n \in \mathbf{N}$. □

**Remark.** In the same way, we can claim that the set of $\mathcal{B}(\mathbf{R}^d)$-measurable simple functions is dense in $L^p\big(\underline{\lambda}^{(d)}\big)$.

**Laurent:** As you can see, this result is not attached to $\mathbf{R}^d$ and the Lebesgue measure. It remains true for any measure space $(\Omega, \mathcal{A}, \mu)$.

**Bernard:** I understand that the simple functions are dense in $L^p\big(\lambda^{(d)}\big)$. Finally, this is not a surprise since the integral is defined starting from simple functions. Now, how do the continuous functions come into play?

**Laurent:** To come to this, we need to use the density of the simple functions in $L^p$ and observe that each simple function can be approximated by a continuous function.

The key point is really that, considering two closed disjoint subsets $A$ and $B$ of $\mathbf{R}^d$, there exists a continuous function $f$ on $\mathbf{R}^d$ such that $f|_A = 1$ and $f|_B = 0$.

Let us have a precise look at that.

**Notation.** For any open set $U$ of $\mathbf{R}^d$, $C_c(U)$ denotes the space of continuous functions $f\colon U \to \mathbf{R}$ such that there exists a compact set $K \subset U$ with $f(x) = 0$ for all $x \in U \setminus K$. Such a function $f$ is said to be *continuous with compact support.*

**Lemma 5.14.** *Let $V$ be an open set of $\mathbf{R}^d$ and $K$ a compact set such that $K \subset V$.*

*Then, there exists a function $f$ in $C_c(\mathbf{R}^d)$ such that $\mathbf{1}_K \leq f \leq \mathbf{1}_V$,* i.e. *$f = 1$ on $K$ and $f = 0$ on $\mathbf{R}^d \setminus V$.*

In the context of more general spaces than $\mathbf{R}^d$, namely *locally compact Hausdorff spaces*, some authors call this result *the Urysohn lemma* (e.g. Rudin (1987, Lemma 2.12, p. 39)). Others reserve the name Urysohn lemma when the function $f$ is only assumed to be continuous (e.g. Lang (1993a, Theorem 4.2, p. 40)).

In $\mathbf{R}^d$ (and in any metric space), the use of the metric makes the proof easier.

**Proof.** $\star$ Consider two disjoint closed subsets $A$ and $B$ of $\mathbf{R}^d$.

Denoting by $d(x, A)$ (resp., $d(x, B)$) the distance from $x \in \mathbf{R}^d$ to $A$ (resp., $B$), the function $f\colon \mathbf{R}^d \to [0, 1]$, defined by

$$\forall x \in \mathbf{R}^d, \quad f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)},$$

is continuous and satisfies the following assertions:

$$\forall x \in A,\ f(x) = 0 \quad \text{and} \quad \forall x \in B,\ f(x) = 1.$$

Since $K \subset V$, where $K$ is a compact (and thus closed) subset and $V$ is an open subset of $\mathbf{R}^d$, the subsets $A = \mathbf{R}^d \setminus V$ and $B = K$ are two disjoint closed subsets of $\mathbf{R}^d$. Then, we just showed that there exists a continuous function $f\colon \mathbf{R}^d \to [0, 1]$ such that $f(x) = 0$ for all $x \in \mathbf{R}^d \setminus V$ and $f(x) = 1$ for all $x \in K$.

Without any additional assumptions on $K$ and $V$, we cannot claim that the previously defined function $f$ has compact support.

$\star$ We consider an open subset $W$ whose closure $\overline{W}$ is compact such that

$$K \subset W \subset \overline{W} \subset V.$$

Let us check that such a subset $W$ exists. Since $V$ is open, for any $x \in K \subset V$, there exists $\delta_x > 0$ such that $B(x, \delta_x) \subset V$. By the compactness of $K$, from the covering $K \subset \bigcup_{x \in K} B(x, \delta_x/2)$, we can extract a finite subcover $K \subset \bigcup_{1 \le i \le n} B(x_i, \delta_{x_i}/2)$, where $x_1, \dots, x_n \in K$.
Denoting $W = \bigcup_{1 \le i \le n} B(x_i, \delta_{x_i}/2)$, we have

$$\overline{W} = \bigcup_{1 \le i \le n} \overline{B}(x_i, \delta_{x_i}/2) \subset \bigcup_{1 \le i \le n} B(x_i, \delta_{x_i}) \subset V.$$

Then, $\overline{W}$ is compact and $K \subset W \subset \overline{W} \subset V$.

⋆ The subsets $A = \overline{W} \setminus W = \overline{W} \cap (\mathbf{R}^d \setminus W)$ and $B = K$ are closed and disjoint. The restriction to $\overline{W}$ of the function $f$ defined above satisfies the following:

– For all $x \in K$, $f(x) = 1$.
– For all $x \in \overline{W} \setminus W$, $f(x) = 0$.

We then extend $f$ to $\mathbf{R}^d \setminus \overline{W}$ by giving the value 0.

Then, this function $f$ defined on $\mathbf{R}^d$ has compact support (since its support is included in $\overline{W}$). Moreover, $f(x) = 1$ for all $x \in K$ and $f(x) = 0$ for all $x \in \mathbf{R}^d \setminus W$ (and, consequently, for all $x \in \mathbf{R}^d \setminus V \subset \mathbf{R}^d \setminus W$). □

**Bernard:** Ah! I can see the big picture. For $f \in L^p$ and $\varepsilon > 0$:

- one can find a simple function $\psi$ such that $\|f - \psi\|_{L^p} < \varepsilon$,
- one can find a continuous function $\varphi$ such that $\|\psi - \varphi\|_{L^p} < \varepsilon$.

The proof should look like something containing these two ingredients.

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**Theorem 5.15.** *For any* $p \in [1, +\infty)$, *the space* $C_c(\mathbf{R}^d)$ *is dense in* $L^p\big(\lambda^{(d)}\big)$.

**Proof.** Following Lemma 5.13, it suffices to prove that for any $A \in \overline{\mathcal{B}(\mathbf{R}^d)}$ such that $\lambda^{(d)}(A) < +\infty$ and for any $\varepsilon > 0$, there exists a function $f$ in $C_c(\mathbf{R}^d)$ such that $\|f - \mathbf{1}_A\|_{L^p} \leq \varepsilon$.

We fix $A \in \overline{\mathcal{B}(\mathbf{R}^d)}$ such that $\mathbf{1}_A \in L^p\big(\lambda^{(d)}\big)$ and $\varepsilon > 0$.

Since $\lambda^{(d)}(A) < \infty$, from Proposition 5.12 (which remains valid for the Lebesgue measure $\lambda^{(d)}$ in $\mathbf{R}^d$), there exist a compact $K$ and an open set $V$ of $\mathbf{R}^d$ such that $K \subset A \subset V$ and $\lambda^{(d)}(V) < \lambda^{(d)}(K) + \varepsilon$.

According to Lemma 5.14, there exists $f \in C_c(\mathbf{R}^d)$ such that $\mathbf{1}_K \leq f \leq \mathbf{1}_V$, i.e. $f(x) = 1$ for all $x \in K$ and $f(x) = 0$ for all $x \in \mathbf{R}^d \setminus V$.

For $p \in [1, +\infty)$, the inequality $\lambda^{(d)}(V) < \lambda^{(d)}(K) + \varepsilon$ implies

$$\int_{\mathbf{R}^d} |f(x) - \mathbf{1}_K(x)|^p \, \lambda^{(d)}(dx) < \varepsilon.$$

Then, by the triangle inequality of the norm $\|\cdot\|_{L^p}$, we get

$$\|f - \mathbf{1}_A\|_{L^p} \le \|f - \mathbf{1}_K\|_{L^p} + \|\mathbf{1}_K - \mathbf{1}_A\|_{L^p} \le \varepsilon^{1/p} + \varepsilon^{1/p}.$$

We conclude that any function in $L^p(\lambda^{(d)})$ can be approximated up to $\varepsilon$ by a function in $C_c(\mathbf{R}^d)$, in the sense of the norm $\|\cdot\|_{L^p}$. □

**Bernard:** I'm still not at ease with Lebesgue's $\sigma$-algebra. So, I wonder whether this result is still true with Borel's $\sigma$-algebra...

Oh! How stupid I can be! Since $L^p(\underline{\lambda}^{(d)}) \subset L^p(\lambda^{(d)})$, any $f$ in $L^p(\underline{\lambda}^{(d)})$ is the limit of a sequence $(\varphi_n)_{n\in\mathbf{N}}$ in $C_c(\mathbf{R}^d)$ for the $L^p$ norm.

Then, the result holds for the Borel $\sigma$-algebra.

**Laurent:** I am delighted to see that you acquired automatisms in measure theory. You are getting intuitive, Bernard...

**Bernard:** Stop! Wait a minute, Laurent! I don't know if I missed something: what happens for $p = \infty$? This case is not covered by Theorem 5.15.

**Laurent:** You are right. This book does not consider the case of $L^\infty(\lambda^{(d)})$. But I can tell you that the space $C_c(\mathbf{R}^d)$ is not dense in $L^\infty(\lambda^{(d)})$.

If I remember well, this question is treated in Rudin (1987, p. 70). Actually, the closure of $C_c(\mathbf{R}^d)$ in $L^\infty(\lambda^{(d)})$ is not the whole space $L^\infty(\lambda^{(d)})$. This closure is the space of continuous functions which *vanish at infinity*, usually denoted by $C_0(\mathbf{R}^d)$.

> *A continuous function* $f\colon \mathbf{R}^d \to \mathbf{R}$ *is said to vanish at infinity if for any* $\varepsilon > 0$, *there exists a compact* $K \subset \mathbf{R}^d$ *such that* $|f(x)| < \varepsilon$ *for all* $x \in \mathbf{R}^d \setminus K$.

## 5.3 Caratheodory Theory and the Lebesgue Measure

**Bernard:** Could you do me a favor, Laurent? I feel a little frustrated. I realized that it was not necessary to understand the details of the construction of Lebesgue's measure, but I would like us to spend some time on the matter.

**Laurent:** Very good. Let's discuss a little about the Caratheodory theory, at least in the particular case of Lebesgue's measure. This is well explained in Henri Brolle's book. As we mentioned before, it is all about the study of $\lambda^*$, defined by (5.1.1). But we must be careful not to get too lost in formalism...

### 5.3.1 *Detailed construction of the Lebesgue measure*

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**Theorem 5.16.** *The function* $\lambda^* : \mathcal{P}(\mathbf{R}) \to [0, \infty]$ *defined on the subsets of* $\mathbf{R}$ *by*

$$\forall A \subset \mathbf{R}, \quad \lambda^*(A) = \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i);\ A \subset \bigcup_{i \in \mathbf{N}^*} (a_i, b_i) \right\},$$

*where the infimum is taken over all countable covers of* $A$ *by finite intervals* $\big((a_i, b_i)\big)_{i \in \mathbf{N}^*}$ *with* $a_i \leq b_i$ *for all* $i \geq 1$, *is an* outer-measure, *in the sense that*:

(i) *for all* $A \subset \mathbf{R}$, $\lambda^*(A) \in [0, \infty]$;
(ii) $\lambda^*(\emptyset) = 0$;
(iii) *for any subsets* $A, B$ *of* $\mathbf{R}$ *such that* $A \subset B$, $\lambda^*(A) \leq \lambda^*(B)$;
(iv) *for any subsets* $(A_i)_{i \in \mathbf{N}^*}$ *of* $\mathbf{R}$, $\lambda^*\left(\bigcup_{i \in \mathbf{N}^*} A_i\right) \leq \sum_{i=1}^{\infty} \lambda^*(A_i)$.

**Proof.** We check the four required conditions in order that the expression for $\lambda^*(A)$ defines an outer-measure.

(i) For all $A \subset \mathbf{R}$, we can see directly that $\lambda^*(A)$ is nonnegative.
(ii) For any $a \in \mathbf{R}$, $\emptyset$ can be covered by the interval $(a, a)$. Since $\lambda^*(\emptyset)$ is defined as an infimum over all the covers of $\emptyset$, this implies $\lambda^*(\emptyset) \leq 0$.
Then, $\lambda^*(\emptyset) = 0$.
(iii) Consider $A \subset B \subset \mathbf{R}$. For any countable cover of $B$ by finite intervals $\big((a_i, b_i)\big)_{i \in \mathbf{N}^*}$ with $a_i \leq b_i$ for all $i \geq 1$, we have $A \subset B \subset \bigcup_{i \in \mathbf{N}^*} (a_i, b_i)$ and, consequently, $\lambda^*(A) \leq \sum_{i=1}^{\infty} (b_i - a_i)$. Taking the infimum over all countable covers of $B$ by finite intervals, we get $\lambda^*(A) \leq \lambda^*(B)$.
(iv) Let $(A_i)_{i \in \mathbf{N}^*}$ be subsets of $\mathbf{R}$. We assume that $\sum_{i=1}^{\infty} \lambda^*(A_i) < +\infty$ (if not the result is obvious).

We fix $\varepsilon > 0$. For any $i \in \mathbf{N}^*$, we consider a countable cover $\big(a_j^{(i)}, b_j^{(i)}\big)_{j\in\mathbf{N}^*}$ of $A_i$ by finite intervals such that

$$\sum_{j=1}^{\infty} \big(b_j^{(i)} - a_j^{(i)}\big) \leq \lambda^*(A_i) + \varepsilon\, 2^{-i}.$$

Since $\mathbf{N}^* \times \mathbf{N}^*$ is countable, we can order the intervals $\big(a_j^{(i)}, b_j^{(i)}\big)_{i,j\in\mathbf{N}^*}$ as $(c_k, d_k)_{k\in\mathbf{N}^*}$ and note that $(c_k, d_k)_{k\in\mathbf{N}^*}$ is a countable cover of $\bigcup_{i\in\mathbf{N}^*} A_i$ by finite intervals. Then,

$$\begin{aligned}\lambda^*\Big(\bigcup_{i\in\mathbf{N}^*} A_i\Big) &\leq \sum_{k=1}^{\infty}(d_k - c_k) = \sum_{i,j} \big(b_j^{(i)} - a_j^{(i)}\big)\\ &\leq \sum_{i=1}^{\infty} \big(\lambda^*(A_i) + \varepsilon\, 2^{-i}\big) \leq \sum_{i=1}^{\infty} \lambda^*(A_i) + \varepsilon.\end{aligned}$$

Since this inequality holds for any $\varepsilon > 0$, we can conclude that

$$\lambda^*\Big(\bigcup_{i\in\mathbf{N}^*} A_i\Big) \leq \sum_{i=1}^{\infty} \lambda^*(A_i).$$

□

**Remark.** Theorem 5.16 can be generalized to any mapping $\mu\colon \mathcal{P}(\Omega) \to [0, \infty]$, defined on the subsets of any set $\Omega$ by

$$\forall A \subset \Omega, \quad \mu(A) = \inf\Big\{\sum_{i=1}^{\infty} \tau(C_i);\ A \subset \bigcup_{i\in\mathbf{N}^*} C_i\Big\},$$

where ⋆ $\tau\colon \mathcal{C} \to [0, \infty]$ is a mapping defined on some collection $\mathcal{C} \subset \mathcal{P}(\Omega)$ such that $\emptyset \in \mathcal{C}$ and $\tau(\emptyset) = 0$,

⋆ The infimum is taken over all countable covers of $A$ by $\big(C_i\big)_{i\in\mathbf{N}^*}$ in $\mathcal{C}$.

In that case, the mapping $\mu$ is an outer-measure.

**Bernard:** I realize that this construction is very general. The remark highlights the fact that there is no difficulty in defining an outer-measure.

**Laurent:** Definitely. The question is now: how to define a measure starting from an outer-measure?

**Bernard:** If you allow me, Laurent, the question is rather: how to define the Lebesgue measure $\lambda$ on the $\sigma$-algebra $\overline{\mathcal{B}(\mathbf{R})}$ (or simply the Borel–Lebesgue measure $\underline{\lambda}$ on the $\sigma$-algebra $\mathcal{B}(\mathbf{R})$) from the outer-measure $\lambda^*$ defined on the subsets of $\mathbf{R}$?

**Laurent:** The key to the mystery is the definition of the $\sigma$-algebra of the $\lambda^*$-measurable sets. Let us move on.

**Definition 5.17 ($\lambda^*$-measurable sets).** A subset $E$ of $\mathbf{R}$ is said to be $\lambda^*$-*measurable* if, for all subsets $A, B$ of $\mathbf{R}$ such that $A \subset E$ and $B \subset \mathbf{R} \setminus E$ (such $A$ and $B$ are said *separated by* $E$), we have $\lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B)$.

The collection of $\lambda^*$-measurable sets is denoted by $\mathcal{M}_{\lambda^*}$.

The following statement is equivalent: the subset $E$ of $\mathbf{R}$ is said to be $\lambda^*$-measurable if

$$\forall A \subset \mathbf{R}, \quad \lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \setminus E). \tag{5.3.1}$$

**Remark.** In the same way, we can define the $\mu$-measurable sets, for any outer-measure $\mu$ defined on $\Omega$.

**Bernard:** Since $\lambda^*$ is an outer-measure, we already know that for any subsets $A, B$ of $\mathbf{R}$, $\lambda^*(A \cup B) \leq \lambda^*(A) + \lambda^*(B)$.

Then, if we want to prove that $E$ is $\lambda^*$-measurable, it suffices to check that $\lambda^*(A \cup B) \geq \lambda^*(A) + \lambda^*(B)$ for any $A \subset E$ and $B \subset \mathbf{R} \setminus E$.

As $\lambda^*$ is nondecreasing, I remark that this inequality automatically holds as soon as either $\lambda^*(A) = \infty$ or $\lambda^*(B) = \infty$. Consequently, it suffices to consider the cases $\lambda^*(A) < \infty$ and $\lambda^*(B) < \infty$.

**Laurent:** Now, it remains to prove that the $\lambda^*$-measurable sets form a $\sigma$-algebra and that the restriction of $\lambda^*$ to the $\lambda^*$-measurable sets is a measure.

The following properties of the $\lambda^*$-measurable sets show that they constitute a $\sigma$-algebra.

**Theorem 5.18.** *The $\lambda^*$-measurable sets satisfy the following properties*:

(i) *If $\lambda^*(N) = 0$, where $N \subset \mathbf{R}$, then $N$ is $\lambda^*$-measurable.*
(ii) *If $E$ is a $\lambda^*$-measurable subset of $\mathbf{R}$, then $\mathbf{R} \backslash E$ is $\lambda^*$-measurable.*
(iii) *If $E_1, E_2$ are two $\lambda^*$-measurable subsets of $\mathbf{R}$, then $E_1 \cup E_2$ is $\lambda^*$-measurable.*
(iv) *If $(E_n)_{n \in \mathbf{N}}$ are pairwise disjoint $\lambda^*$-measurable subsets of $\mathbf{R}$, then*

$$\bigcup_{n \in \mathbf{N}} E_n \text{ is } \lambda^*\text{-measurable and } \lambda^*\left(\bigcup_{n \in \mathbf{N}} E_n\right) = \sum_{n=0}^{\infty} \lambda^*(E_n).$$

**Proof.** (i) Assume that $\lambda^*(N) = 0$ and $A \subset N$, $B \subset \mathbf{R} \setminus N$.

We have $\lambda^*(B) \leq \lambda^*(A \cup B) \leq \lambda^*(A) + \lambda^*(B)$. And since $\lambda^*$ is nondecreasing, we have $0 \leq \lambda^*(A) \leq \lambda^*(N) = 0$.

This leads to $\lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B)$. Then, $N$ is $\lambda^*$-measurable.

(ii) By symmetry, the $\lambda^*$-measurability property is stable with respect to complementation.

(iii) Let $E_1, E_2$ be two $\lambda^*$-measurable sets.

Let $A$ and $B$ be two subsets of $\mathbf{R}$ such that $A \subset E_1 \cup E_2$ and $B \subset \mathbf{R} \setminus (E_1 \cup E_2)$.

Since $B \cap E_1 = \emptyset$, we have

$$A \cup B = [A \cap E_1] \cup [(A \cup B) \cap (\mathbf{R} \setminus E_1)].$$

The two sets $A \cap E_1$ and $(A \cup B) \cap (\mathbf{R} \setminus E_1)$ are separated by $E_1$, then

$$\lambda^*(A \cup B) = \lambda^*(A \cap E_1) + \lambda^*\big((A \cup B) \cap (\mathbf{R} \setminus E_1)\big).$$

We have $(A \cup B) \cap (\mathbf{R} \setminus E_1) = \big(A \cap (\mathbf{R} \setminus E_1)\big) \cup B$ ,and the two sets $A \cap (\mathbf{R} \setminus E_1)$ and $B$ are separated by $E_2$. Then,

$$\lambda^*\big((A \cup B) \cap (\mathbf{R} \setminus E_1)\big) = \lambda^*\big(A \cap (\mathbf{R} \setminus E_1)\big) + \lambda^*(B).$$

From the $\lambda^*$-measurability of $E_1$, we deduce that

$$\begin{aligned}\lambda^*(A\cup B) &= \lambda^*(A\cap E_1) + \lambda^*\big(A\cap(\mathbf{R}\setminus E_1)\big) + \lambda^*(B)\\ &= \lambda^*(A)+\lambda^*(B).\end{aligned}$$

Then, $E_1\cup E_2$ is $\lambda^*$-measurable.

(iv) Let $(E_n)_{n\in\mathbf{N}}$ be pairwise disjoint $\lambda^*$-measurable sets.
$\star$ We start by proving that $E=\bigcup_{n\in\mathbf{N}} E_n$ is $\lambda^*$-measurable.
Let $A$ and $B$ be two sets such that $A\subset E$ and $B\subset \mathbf{R}\setminus E$.
For all $n\in\mathbf{N}$, we have

$$\begin{aligned}\lambda^*(A\cup B) &= \lambda^*\big((A\cap E)\cup B\big)\\ &\ge \lambda^*\Big(\big(A\cap\bigcup_{i\le n}E_i\big)\cup B\Big) = \lambda^*\Big(A\cap \bigcup_{i\le n}E_i\Big) + \lambda^*(B),\end{aligned}$$

since $\bigcup_{i\le n}E_i$ is $\lambda^*$-measurable (by (iii), as a finite union) and $B\subset \mathbf{R}\setminus\bigcup_{i\le n}E_i$.

The set $E_n$ is $\lambda^*$-measurable and $(E_i)_{i\le n}$ are pairwise disjoint sets, then

$$\begin{aligned}\lambda^*\Big(A\cap\bigcup_{i\le n}E_i\Big) &= \lambda^*\Big(\big(A\cap\bigcup_{i\le n-1}E_i\big)\cup(A\cap E_n)\Big)\\ &= \lambda^*\Big(A\cap\bigcup_{i\le n-1}E_i\Big)+\lambda^*(A\cap E_n).\end{aligned}$$

By an iteration of the argument, we get

$$\lambda^*\Big(A\cap\bigcup_{i\le n}E_i\Big) = \lambda^*(A\cap E_1)+\cdots+\lambda^*(A\cap E_n).$$

We deduce that

$$\forall n\in\mathbf{N}^*,\quad \lambda^*(A\cup B)\ge \sum_{i=1}^{n}\lambda^*(A\cap E_i)+\lambda^*(B)$$

and, letting $n$ go to infinity and using Theorem 5.16(iv),

$$\lambda^*(A \cup B) \geq \sum_{i=1}^{\infty} \lambda^*(A \cap E_i) + \lambda^*(B) \geq \lambda^*(A \cap E) + \lambda^*(B). \quad (5.3.2)$$

Since $A \cap E = A$, we deduce that $\lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B)$ and $E$ is $\lambda^*$-measurable.

$\star$ Taking $A = E$ and $B = \emptyset$ in (5.3.2), we get $\lambda^*\left(\bigcup_{n\in\mathbf{N}^*} E_n\right) = \sum_{n=1}^{\infty} \lambda^*(E_n)$. □

**Bernard:** As I remember, all these properties imply that the collection $\mathcal{M}_{\lambda^*}$ is a $\sigma$-algebra. And the restriction of $\lambda^*$ to $\mathcal{M}_{\lambda^*}$ should be a measure. One thing puzzles me though. We don't seem to have used the definition of $\lambda^*$ in this proof.

**Laurent:** Exactly. According to the following remark in this book, this fact can be stated for any outer-measure.

According to Lemma 2.2 and Theorem 5.18, we can state the following.

**Corollary 5.19.** *The collection $\mathcal{M}_{\lambda^*}$ of the $\lambda^*$-measurable subsets of* $\mathbf{R}$ *is a $\sigma$-algebra. Moreover, the restriction of $\lambda^*$ to $\mathcal{M}_{\lambda^*}$ is a measure.*

**Remark.** The proof of Theorem 5.18 can be easily adapted to any outer-measure $\mu$ on a set $\Omega$ in the place of $\lambda^*$ on $\mathbf{R}$. We can state the following: *for any outer-measure $\mu$ on any set $\Omega$, the collection $\mathcal{M}_\mu$ of the $\mu$-measurable subsets of $\Omega$ is a $\sigma$-algebra and the restriction of $\mu$ to $\mathcal{M}_\mu$ is a measure on the space $(\Omega, \mathcal{M}_\mu)$.*

**Bernard:** The story of the Lebesgue measure seems finished now, isn't it? The restriction of $\lambda^*$ to $\mathcal{M}_{\lambda^*}$ is certainly the Lebesgue measure.

**Laurent:** Wait a minute, Bernard. We first need to check that $\mathcal{M}_{\lambda^*}$ contains the Lebesgue $\sigma$-algebra. And it is not sufficient...

**Bernard:** I feel like $\mathcal{M}_{\lambda^*}$ contains all the intervals $(a, b)$ with $a, b \in \mathbf{R}$, but thinking about it calmly, I'm not sure I can justify it rigorously.

**Laurent:** That is precisely the point. Let us study it.

**Proposition 5.20.** *The following inclusion holds:* $\mathcal{B}(\mathbf{R}) \subset \mathcal{M}_{\lambda^*}$.

**Proof.** $\star$ First, we prove that the interval $(a, +\infty)$ is $\lambda^*$-measurable for any $a \in \mathbf{R}$. Then, we aim to prove that for any subset $A$ of $\mathbf{R}$, we have

$$\lambda^*(A) = \lambda^*\big(A \cap (a, +\infty)\big) + \lambda^*\big(A \setminus (a, +\infty)\big).$$

Let $A \subset \mathbf{R}$ and $\varepsilon > 0$. By the definition of $\lambda^*(A)$, there exist finite intervals $\big((a_i, b_i)\big)_{i \in \mathbf{N}^*}$ such that $A \subset \bigcup_{i \in \mathbf{N}^*} (a_i, b_i)$ and $\sum_{i=1}^{\infty} (b_i - a_i) \leq \lambda^*(A) + \varepsilon$.

For each $i \in \mathbf{N}^*$, we have

$$\ell\big((a_i, b_i) \cap (a, +\infty)\big) + \ell\big((a_i, b_i) \setminus (a, +\infty)\big) = \ell\big((a_i, b_i)\big) = b_i - a_i,$$

where $\ell\big((c, d)\big) = d - c$ denotes the length of the finite interval $(c, d)$.

Since $\big((a_i, b_i) \cap (a, +\infty)\big)_{i \in \mathbf{N}^*}$ (resp., $\big((a_i, b_i) \setminus (a, +\infty)\big)_{i \in \mathbf{N}^*}$) covers $A \cap (a, +\infty)$ (resp., $A \setminus (a, +\infty)$) by finite intervals, this leads to

$$\begin{aligned}
&\lambda^*\big(A \cap (a, +\infty)\big) + \lambda^*\big(A \setminus (a, +\infty)\big) \\
&\qquad \leq \sum_{i=1}^{\infty} \ell\big((a_i, b_i) \cap (a, +\infty)\big) + \sum_{i=1}^{\infty} \ell\big((a_i, b_i) \setminus (a, +\infty)\big) \\
&\qquad = \sum_{i=1}^{\infty} (b_i - a_i).
\end{aligned}$$

Finally, for all $\varepsilon > 0$,

$$\lambda^*\big(A \cap (a, +\infty)\big) + \lambda^*\big(A \setminus (a, +\infty)\big) \leq \lambda^*(A) + \varepsilon.$$

Then, we can claim that $\lambda^*\big(A \cap (a, +\infty)\big) + \lambda^*\big(A \setminus (a, +\infty)\big) \leq \lambda^*(A)$ for all $A \subset \mathbf{R}$, and thus $(a, +\infty)$ is $\lambda^*$-measurable.

⋆ According to Lemma 2.7, the $\sigma$-algebra $\mathcal{B}(\mathbf{R})$ is generated by the intervals $(a, +\infty)$, where $a \in \mathbf{R}$. Then, the fact that $\mathcal{M}_{\lambda^*}$ is a $\sigma$-algebra which contains all the intervals $(a, +\infty)$ implies the inclusion $\mathcal{B}(\mathbf{R}) \subset \mathcal{M}_{\lambda^*}$.

⋆ Since $\mathcal{B}(\mathbf{R}) \subset \mathcal{M}_{\lambda^*}$, Theorem 5.18(iv) implies that the restriction of the outer-measure $\lambda^*$ to the $\sigma$-algebra $\mathcal{B}(\mathbf{R})$ is a measure, which is denoted by $\lambda^*|_{\mathcal{B}(\mathbf{R})}$.

By definition, any $\lambda^*|_{\mathcal{B}(\mathbf{R})}$-negligible set $N$ is a subset of $\mathbf{R}$ that is included in some $A \in \mathcal{B}(\mathbf{R})$ such that $\lambda^*(A) = 0$. Then, we have $\lambda^*(N) = 0$, and from Theorem 5.18(i), $N$ is $\lambda^*$-measurable.

We conclude that $\mathcal{M}_{\lambda^*}$ is a $\sigma$-algebra that contains the Borel $\sigma$-algebra $\mathcal{B}(\mathbf{R})$ and the collection $\mathcal{N}$ of the negligible sets. This implies that $\mathcal{M}_{\lambda^*}$ contains the Lebesgue $\sigma$-algebra $\overline{\mathcal{B}(\mathbf{R})}$. □

**Remark.** Actually, the equality $\mathcal{M}_{\lambda^*} = \overline{\mathcal{B}(\mathbf{R})}$ holds (see for instance Le Gall (2022, Proposition 3.7, p. 49)).

**Bernard:** You taught me to be careful, Laurent, and I try not to conclude too quickly. If I gather the results: we know that the restriction of $\lambda^*$ to the Lebesgue $\sigma$-algebra is a measure $\lambda$ that verifies $\lambda\big((a, b)\big) = b - a$ for any finite interval $(a, b)$, which thus extends the notion of length.

Before claiming that the measure defined in this way is precisely the Lebesgue measure, I wonder if there can be other measures on $\overline{\mathcal{B}(\mathbf{R})}$ that satisfy this (without being defined from $\lambda^*$ as we just saw).

**Laurent:** Excellent question, my dear Bernard! The answer to this problem of characterizing measures requires some additional material, notably the notion of $\pi$-systems and monotone classes.

### 5.3.2 *Characterization of the Lebesgue measure*

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Characterizing a measure on the Lebesgue $\sigma$-algebra $\overline{\mathcal{B}(\mathbf{R})}$ (or even on the Borel $\sigma$-algebra $\mathcal{B}(\mathbf{R})$) by its values on the finite intervals of $\mathbf{R}$ is part of a more general issue:

*Assume that a $\sigma$-algebra $\mathcal{A}$ of $\Omega$ is generated by a collection $\mathcal{C}$ of subsets of $\Omega$. Are there conditions on $\mathcal{C}$ ensuring that a measure on $\mathcal{A} = \sigma(\mathcal{C})$ is characterized by its values on $\mathcal{C}$?*

We will see that one of the conditions on $\mathcal{C}$ is the stability by finite intersections. The following definitions clarify the stability properties that are relevant to answering the question.

**Definition 5.21 ($\pi$-system).** A collection $\mathcal{C}$ of sets is called a *$\pi$-system* if it is closed under finite intersections. This can be stated as follows: for all $A, B \in \mathcal{C}$, $A \cap B \in \mathcal{C}$.

**Bernard:** My first feeling is that, regarding stability properties, the concept of $\sigma$-algebra is very rich, and unsurprisingly, any $\sigma$-algebra is a $\pi$-system.

However, I imagine that the goal here is to reduce the stability conditions that define a $\sigma$-algebra, in order to obtain collections that are simpler to handle than a $\sigma$-algebra while remaining rich enough to characterize the $\sigma$-algebra they generate.

**Laurent:** Wonderful! This is precisely the point. Could you illustrate it with the Borel $\sigma$-algebra of $\mathbf{R}$?

**Bernard:** Well, following the problem we are tackling and the definition of $\lambda^*$, my first idea is the collection of finite open intervals $\{(a, b);\ a, b \in \mathbf{R} \text{ with } a \leq b\}$. I think I don't need to explain that $(a, b) \cap (c, d)$ is not a finite open interval, in general.

I am also thinking of the collections $\{(-\infty, a);\ a \in \mathbf{R}\}$ or $\{(-\infty, a];\ a \in \mathbf{R}\}$ or $\{[a, +\infty);\ a \in \mathbf{R}\}$ or $\{(a, +\infty);\ a \in \mathbf{R}\}$.

**Laurent:** Definitely, all these collections are $\pi$-systems. And remember that each of these collections generates the Borel $\sigma$-algebra $\mathcal{B}(\mathbf{R})$. They are obviously simpler to handle than $\mathcal{B}(\mathbf{R})$.

Let us now consider the second structure of collection of sets.

**Definition 5.22.** A collection $\mathcal{C}$ of sets is said to be:

$\star$ *closed under nondecreasing limits* if for all sets $(A_n)_{n \in \mathbf{N}}$ in $\mathcal{C}$ such that $A_n \subset A_{n+1}$ for all $n \in \mathbf{N}$, the set $\bigcup_{n \in \mathbf{N}} A_n$ belongs to $\mathcal{C}$.

⋆ *closed under differences* if for all sets $A, B$ in $\mathcal{C}$ such that $B \subset A$, the set $A \setminus B$ belongs to $\mathcal{C}$.

**Definition 5.23 (Monotone class).** A collection $\mathcal{C}$ of subsets of $\Omega$ is called a *monotone class* (or a $\lambda$-*system* in the terminology of Dynkin) if the following three conditions hold:

(i) $\Omega$ belongs to $\mathcal{C}$;
(ii) $\mathcal{C}$ is closed under nondecreasing limits;
(iii) $\mathcal{C}$ is closed under differences.

**Bernard:** I wonder if some of the previous examples of $\pi$-systems of $\mathbf{R}$ are also monotone classes. For instance, I consider the collection $\{(-\infty, a];\ a \in \mathbf{R}\}$.

For $a < b$, the set $(-\infty, b] \setminus (-\infty, a] = (a, b]$ is not in the collection. Then, the collection is not a monotone class.

**Laurent:** Definitely. I confirm that none of your examples are monotone classes.

The following lemma should clarify the difference between monotone classes and $\sigma$-algebras.

**Lemma 5.24.** *Let $\mathcal{C}$ be a monotone class. If the class $\mathcal{C}$ is also a $\pi$-system, then $\mathcal{C}$ is a $\sigma$-algebra.*

**Proof.** Since $\Omega \in \mathcal{C}$ and $\mathcal{C}$ is closed under differences, we can claim that $\mathcal{C}$ is closed under complementation and that $\emptyset \in \mathcal{C}$.

To prove that $\mathcal{C}$ is a $\sigma$-algebra, it remains to prove that $\mathcal{C}$ is closed under countable unions.

Let $(A_n)_{n \in \mathbf{N}}$ be in $\mathcal{C}$.

For all $n \in \mathbf{N}$, the set $B_n = \bigcup_{k \leq n} A_k$ belongs to $\mathcal{C}$ (since $\Omega \setminus B_n = \bigcap_{k \leq n} (\Omega \setminus A_k)$ belongs to $\mathcal{C}$).

The sequence $(B_n)_{n \in \mathbf{N}}$ is nondecreasing, then $\bigcup_{n \in \mathbf{N}} A_n = \bigcup_{n \in \mathbf{N}} B_n$ belongs to $\mathcal{C}$. □

**Bernard:** If I understand well, the sole property of closure under finite intersections makes the difference between monotone classes and $\sigma$-algebras.

But I wonder if there is any real point in weakening the conditions of a $\sigma$-algebra to get those of a monotone class.

**Laurent:** You will discover it very quickly in concrete situations. In particular, when we characterize a probability measure by its values on a generating $\pi$-system.

Before we do that, let us look at the result that underlies everything involving monotone classes.

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**Theorem 5.25 (Monotone class lemma).** *If $\mathcal{C}$ is a $\pi$-system, then the smallest monotone class that contains $\mathcal{C}$ is $\mathcal{D} = \sigma(\mathcal{C})$.*

**Proof.** $\star$ The intersection of any family of monotone classes is a monotone class: if $(\mathcal{C}_t)_{t\in\mathcal{T}}$ are monotone classes, then $\bigcap_{t\in\mathcal{T}} \mathcal{C}_t$ is a monotone class.

The intersection $\mathcal{D}$ of all the monotone classes that contain $\mathcal{C}$ is the smallest one. The $\sigma$-algebra $\sigma(\mathcal{C})$ is obviously a monotone class which contains $\mathcal{C}$. Then, we have $\mathcal{D} \subset \sigma(\mathcal{C})$.

It only remains to prove that $\mathcal{D}$ is a $\sigma$-algebra to conclude.

$\star$ According to Lemma 5.24, it suffices to prove that $\mathcal{D}$ is closed under finite intersections.

In order to do that, for all $B \subset \Omega$, we define $\mathcal{D}_B = \{A \in \mathcal{D} : A \cap B \in \mathcal{D}\}$, and we prove that $\mathcal{D}_B = \mathcal{D}$ for all $B \in \mathcal{D}$.

$\star$ For any $B \subset \Omega$, we prove that the collection $\mathcal{D}_B$ is closed under nondecreasing limits and under differences.

Let $(A_n)_{n\in\mathbf{N}}$ be a nondecreasing sequence of sets in $\mathcal{D}_B$. We have

$$\left(\bigcup_{n\in\mathbf{N}} A_n\right) \cap B = \bigcup_{n\in\mathbf{N}} (A_n \cap B),$$

where $(A_n \cap B)_{n\in\mathbf{N}}$ is a nondecreasing sequence, and for all $n \in \mathbf{N}$, $A_n \cap B$ belongs to the monotone class $\mathcal{D}$. Then, $\bigcup_{n\in\mathbf{N}}(A_n \cap B)$ belongs to $\mathcal{D}$. This leads to $\bigcup_{n\in\mathbf{N}} A_n \in \mathcal{D}_B$.

Let $A_1, A_2$ in $\mathcal{D}_B$ with $A_1 \subset A_2$. We have

$$(A_2 \setminus A_1) \cap B = (A_2 \cap B) \setminus (A_1 \cap B),$$

where $A_1 \cap B$ and $A_2 \cap B$ are in the monotone class $\mathcal{D}$ and $A_2 \cap B \subset A_1 \cap B$. Then, $(A_2 \cap B) \setminus (A_1 \cap B)$ belongs to $\mathcal{D}$. This leads to $(A_2 \setminus A_1) \cap B \in \mathcal{D}$.

When $B \in \mathcal{D}$, we also have $\Omega \in \mathcal{D}_B$, and we can conclude that $\mathcal{D}_B$ is a monotone class.

$\star$ We now prove that for $B \in \mathcal{C}$, $\mathcal{D}_B = \mathcal{D}$.

For all $C \in \mathcal{C}$, $B \cap C$ belongs to the $\pi$-system $\mathcal{C} \subset \mathcal{D}$.

Since $C \in \mathcal{D}$, we can claim that $C \in \mathcal{D}_B$.

The monotone class $\mathcal{D}_B$ contains $\mathcal{C}$ and is included in $\mathcal{D}$. Then, $\mathcal{D}_B = \mathcal{D}$.

$\star$ Let us prove now that for $B \in \mathcal{D}$, $\mathcal{D}_B = \mathcal{D}$.

For all $C \in \mathcal{C}$, we have $B \in \mathcal{D}_C$ because $\mathcal{D}_C = \mathcal{D}$. Then, $B \cap C$ belongs to $\mathcal{D}$, and thus $C \in \mathcal{D}_B$.

As previously, the monotone class $\mathcal{D}_B$ contains $\mathcal{C}$ and is included in $\mathcal{D}$. Then, $\mathcal{D}_B = \mathcal{D}$.

$\star$ The fact that $\mathcal{D}_B = \mathcal{D}$ for all $B \in \mathcal{D}$ implies that $\mathcal{D}$ is closed under finite intersections. By Lemma 5.24, the fact that the monotone class $\mathcal{D}$ is a $\pi$-system implies that $\mathcal{D}$ is a $\sigma$-algebra. □

**Laurent:** This result shows that a monotone class is not far from being a $\sigma$-algebra. It provides a useful way to determine the $\sigma$-algebra generated by a collection of sets.

We have reached the point where we can see how the values on a $\pi$-system characterize the values of a measure on the $\sigma$-algebra generated by this $\pi$-system.

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**Theorem 5.26.** *Let $\mu$ and $\nu$ be two probability measures on the measurable space $(\Omega, \mathcal{A})$. Assume that $\mu$ and $\nu$ coincide on a $\pi$-system $\mathcal{C} \subset \mathcal{A}$ such that $\sigma(\mathcal{C}) = \mathcal{A}$.*

*Then, $\mu = \nu$.*

The result can be also stated as follows: *If $\mu(C) = \nu(C)$ for all $C$ in some $\pi$-system $\mathcal{C}$, then $\mu(A) = \nu(A)$ for all $A$ in $\mathcal{A}$, as $\mathcal{A} = \sigma(\mathcal{C})$.*

**Proof.** Let us define $\mathcal{B} = \{A \in \mathcal{A} : \mu(A) = \nu(A)\}$. Our goal is to prove that $\mathcal{B} = \mathcal{A}$.

First, we show that $\mathcal{B}$ in a monotone class:

$\star$ Since $\mu(\Omega) = \nu(\Omega) = 1$, we have $\Omega \in \mathcal{B}$.

$\star$ We show that $\mathcal{B}$ is closed under nondecreasing limits.

Let $(A_n)_{n\in\mathbf{N}}$ be elements of $\mathcal{B}$ such that $A_n \subset A_{n+1}$ for all $n \in \mathbf{N}$ and set $A = \bigcup_{n\in\mathbf{N}} A_n$.

The monotone convergence theorem for measures implies that $\mu(A) = \lim_{n\to\infty} \mu(A_n)$ and $\nu(A) = \lim_{n\to\infty} \nu(A_n)$. By the definition of $\mathcal{B}$, we have $\mu(A_n) = \nu(A_n)$ for all $n \in \mathbf{N}$.

This leads to $\mu(A) = \nu(A)$ and $A \in \mathcal{B}$.

$\star$ We show that $\mathcal{B}$ is closed under differences.

Let $A, B$ be elements of $\mathcal{B}$ such that $B \subset A$. We have $\mu(A) = \nu(A)$ and $\mu(B) = \nu(B)$. This implies $\mu(A \setminus B) = \mu(A) - \mu(B) = \nu(A) - \nu(B) = \nu(A \setminus B)$ and then $A \setminus B \in \mathcal{B}$.

Finally, $\mathcal{B}$ is a monotone class that contains $\mathcal{C}$.

Then, Theorem 5.25 implies $\sigma(\mathcal{C}) \subset \mathcal{B} \subset \mathcal{A}$. We conclude that $\mathcal{B} = \mathcal{A}$. □

**Bernard:** Laurent, the Lebesgue measure isn't a probability measure, right? Then, this characterization result cannot be applied to it.

**Laurent:** You are right. But this result can still be used thanks to the fact that the Lebesgue measure is $\sigma$-finite. The picture is almost complete.

**Theorem 5.27.** *The restriction of the outer-measure* $\lambda^*$ *to the Lebesgue* $\sigma$*-algebra* $\overline{\mathcal{B}(\mathbf{R})}$ *is the Lebesgue measure* $\lambda$.

*The Lebesgue measure is the only measure on* $\overline{\mathcal{B}(\mathbf{R})}$ *such that*

$$\forall a, b \in \mathbf{R}, \quad \lambda\big((a,b)\big) = b - a.$$

**Proof.** Let us denote by $\mu$ the restriction of the outer-measure $\lambda^*$ to $\overline{\mathcal{B}(\mathbf{R})}$. As a consequence of Theorem 5.18 and Proposition 5.20, we have $\mu(N) = 0$ for all negligible sets $N$ and $\mu\big((a,b)\big) = b - a$ for all $a, b \in \mathbf{R}$ with $a \le b$.

We have to prove that there is only one measure on $\overline{\mathcal{B}(\mathbf{R})}$ which satisfies this.

Assume that $\nu$ is another measure which satisfies these equalities.

We remark that $\mathbf{R} = \bigcup_{n\in\mathbf{N}^*}(-n,n)$, and for all $n \in \mathbf{N}^*$, we denote by $\mu_n$ and $\nu_n$ the measures defined by

$$\forall A \in \overline{\mathcal{B}(\mathbf{R})}, \quad \mu_n(A) = \mu\big(A \cap (-n,n)\big) \quad \text{and} \quad \nu_n(A) = \nu\big(A \cap (-n,n)\big).$$

For each $n \in \mathbf{N}^*$, $\mu_n/n$ and $\nu_n/n$ are probability measures that coincide on all finite open intervals and on negligible sets. The collection of finite open intervals and negligible sets is a $\pi$-system that generates the $\sigma$-algebra $\overline{\mathcal{B}(\mathbf{R})}$.

Then, Theorem 5.26 implies $\mu = \nu$. □

**Laurent:** Finally, Theorem 5.27 closes the definition and the construction of the Lebesgue measure, does it not?

**Bernard:** It provides a justification for Definition 5.10 concerning the Lebesgue measure and its characterization on intervals $(a, b)$ (Proposition 5.11). And since $\mathcal{B}(\mathbf{R}) \subset \overline{\mathcal{B}(\mathbf{R})}$, it also justifies the corresponding Definition 5.1 and Proposition 5.2 concerning the Borel–Lebesgue measure $\underline{\lambda}$ on $\mathcal{B}(\mathbf{R})$.

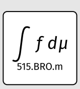
Obviously, I'm not going to tell my boss Ann about these developments... They seem far removed from my pharmaceutical plant.

## 5.4 Lebesgue Integral

**Laurent:** You had better... Now that the Lebesgue measure over $\mathbf{R}^d$ is completely clear to you, we can discuss the Lebesgue integral.

**Bernard:** And to compare it with the integral that I know, it is more or less the Riemann integral.

**Laurent:** One step at a time, but we will get there. And it will make the need for the Lebesgue $\sigma$-algebra clear, that is, with respect to the Borel one.




**Definition 5.28 (Lebesgue integral).** The integral of measurable functions from $\mathbf{R}^d$ to $\mathbf{R}$, endowed with their Lebesgue $\sigma$-algebra, with respect to the Lebesgue measure is called *the Lebesgue integral.*

The integral of $f\colon \mathbf{R}^d \to \mathbf{R}$ is denoted by either

$$\int_{\mathbf{R}^d} f \, d\lambda^{(d)} \quad \text{or} \quad \int_{\mathbf{R}^d} f(x)\, \lambda^{(d)}(dx) \quad \text{or}$$
$$\times \int_{\mathbf{R}^d} f(x_1, \ldots, x_d)\, \lambda^{(d)}(dx_1, \ldots, dx_d).$$

In the specific case of the function $f\colon \mathbf{R} \to \mathbf{R}$ $(d = 1)$, the integral is denoted by $\int_{\mathbf{R}} f \, d\lambda$ or $\int_{\mathbf{R}} f(x)\, \lambda(dx)$.

It is sometimes interesting to consider the integration of functions over subsets of $\mathbf{R}^d$. For instance, if $f\colon \mathbf{R}^d \to \mathbf{R}$ is an integrable function and $E$ is a subset of $\mathbf{R}^d$, we denote

$$\int_E f \, d\lambda^{(d)} = \int_{\mathbf{R}^d} (f \mathbf{1}_E)\, d\lambda^{(d)} = \int_{\mathbf{R}^d} f(x) \mathbf{1}_E(x) \lambda^{(d)}(dx).$$

The integral over $E$ is well defined since $|f\,\mathbf{1}_E| \leq |f|$ implies that the function $(f\,\mathbf{1}_E)$ is integrable.

The following definition considers functions which may not be integrable but are integrable over compact subsets of $\mathbf{R}$.

**Definition 5.29.** A function $f\colon \mathbf{R}^d \to \mathbf{R}$ is said to be *locally integrable* if the function $(f\,\mathbf{1}_K)$ is integrable for any compact $K \subset \mathbf{R}^d$.

The set of locally integrable functions is denoted by $\mathcal{L}^1_{\text{loc}}\big(\lambda^{(d)}\big)$ or $\mathcal{L}^1_{\text{loc}}$, and we have $\mathcal{L}^1 \subset \mathcal{L}^1_{\text{loc}}$.

**Bernard:** As I said the other day, since the very first time I heard about integrating a function, the operation appeared to be linked both to its *primitive function* and to the *area under the curve*. What happens in the case of the Lebesgue integral?

**Laurent:** My answer is always the same, Bernard. When you talk of the area under a curve, I guess you unconsciously imagine the frontier of the area to be continuous. Our framework goes beyond the case of continuous functions. We cannot sum up the Lebesgue integral to computing an area.

On the other subject, as far as the primitive functions are concerned, there exist results for the Lebesgue integral that are less simple than what you know for your integral because the function is not continuous.

### 5.4.1 *Fundamental theorem of calculus*

**Laurent:** In the Riemann setting, if $f\colon [a,b] \to \mathbf{R}$ is a $C^1$ function, you certainly know that

$$f(b) - f(a) = \int_a^b f'(t)\,dt,$$

and if $g\colon [a,b] \to \mathbf{R}$ is a continuous function, the function $G$ defined by

$$\forall x \in [a,b], \quad G(x) = \int_a^x g(t)\,dt$$

is differentiable in $(a,b)$ and for all $x \in (a,b)$, $G'(x) = g(x)$.

**Bernard:** I know. And these results forged my intuition that integrating and differentiating a function are the inverse operations of each other. But I have a feeling these results are no longer true for the Lebesgue integral.

**Laurent:** There are several ways to tackle this link between differentiation and integration with respect to the Lebesgue measure. In any case, the very first step is the following fact:

> *Any nondecreasing function is differentiable almost everywhere.*

**Bernard:** Oh! Very interesting, Laurent. Since I learned that any differentiable function $f\colon \mathbf{R} \to \mathbf{R}$ whose derivative satisfies $f'(x) \geq 0$, for all $x \in (a,b)$, is nondecreasing on $(a,b)$, I have been wondering if some kind of converse is true.

All I know is that any nondecreasing function on a compact subset has a finite number of discontinuities.

If I understand well what you said, we cannot say that a nondecreasing function is differentiable, but we can say that it is differentiable almost everywhere.

Wow. How can this result be proved?

**Laurent:** The key argument is due to Vitali.

There are several ways to prove that a monotone function $\mathbf{R} \to \mathbf{R}$ is differentiable almost everywhere. For instance, three different approaches are considered in Kolmogorov and Fomin (1975, Theorem 6, p. 321), Royden (1968, Theorem 2, p. 96) and Tao (2011, Theorem 1.6.25, p. 129). We choose here to present the second one, which relies on the concept of Vitali covering.

For $A \subset \mathbf{R}$, we say that a collection $\mathcal{I}$ of intervals covers $A$ *in the sense of Vitali* if *for all* $x \in A$ *and all* $\varepsilon > 0$, *there exists an interval* $I \in \mathcal{I}$ *such that*

$$x \in I \quad \text{and} \quad \lambda(I) < \varepsilon.$$

**Theorem 5.30 (Vitali lemma, (Vitali, 1908)).** *Let $A$ be a subset of* $\mathbf{R}$ *with $\lambda^*(A) < +\infty$ and $\mathcal{I}$ a collection of intervals that covers $A$ in the sense of Vitali.*

*Then, for all* $\varepsilon > 0$, *there exists a finite collection $\{I_1, \ldots, I_n\}$ of pairwise disjoint intervals in $\mathcal{I}$ such that*

$$\lambda^*\left(A \Big\backslash \bigcup_{k=1}^{n} I_k\right) < \varepsilon.$$

**Proof.** Without loss of generality, we can assume that all the intervals in $\mathcal{I}$ are closed. If not, we can replace each interval $I_k$ with its closure $\overline{I_k}$ and remark that $\lambda^*\left[\left(\bigcup_{k=1}^n \overline{I_k}\right) \setminus \left(\bigcup_{k=1}^n I_k\right)\right] = 0$.

By the definition of $\lambda^*$, there exists an open subset $U$ of $\mathbf{R}$ (actually a countable union of finite open intervals) such that $A \subset U$ and $\lambda^*(U) < +\infty$.

Without loss of generality, we can assume that $I \subset U$, for all $I \in \mathcal{I}$. If not, we can use the fact that $\mathcal{I}$ is a Vitali covering and $U$ is open to replace the interval with a smaller one.

$\star$ We now construct a sequence $(I_n)_{n\in\mathbf{N}^*}$ in $\mathcal{I}$ by induction in the following way:

- We choose any $I_1 \in \mathcal{I}$.
- If $I_1, \dots, I_n$ are chosen, we define

$$k_n = \sup_{\substack{I\in\mathcal{I}:\\ I\cap(I_1\cup\cdots\cup I_n)=\emptyset}} \lambda^*(I).$$

We know that $k_n < +\infty$ because $I \subset U$ for all $I \in \mathcal{I}$ and $\lambda^*(U) < +\infty$.
If $A \subset \bigcup_{k=1}^n I_k$, we set $I_{n+1} = \emptyset$. Otherwise, there exists $I_{n+1} \in \mathcal{I}$ disjoint from $I_1 \cup \cdots \cup I_n$ such that $\lambda^*(I_{n+1}) > k_n/2$ (by the definition of $k_n$).

$\star$ Since the intervals $(I_n)_{n\in\mathbf{N}^*}$ in $\mathcal{I}$ are pairwise disjoint and all included in $U$, we have

$$\sum_{n=1}^{+\infty} \lambda^*(I_n) = \lambda^*\left(\bigcup_{n\in\mathbf{N}^*} I_n\right) \leq \lambda^*(U) < +\infty.$$

Then, considering $\varepsilon > 0$, there exists $n \in \mathbf{N}^*$ such that $\sum_{k=n+1}^{+\infty} \lambda^*(I_k) < \frac{\varepsilon}{5}$.

$\star$ We now consider the set $D = A \setminus \bigcup_{k=1}^n I_k$. We aim to prove that $\lambda^*(D) < \varepsilon$.

Let $x \in D$. The closed set $\bigcup_{k=1}^n I_k$ does not contain $x$.

Since $\mathcal{I}$ covers $A$ in the sense of Vitali, there exists an interval $I \in \mathcal{I}$ such that $x \in I$ and $\lambda^*(I)$ is small enough to ensure that $I \cap \left(\bigcup_{k=1}^n I_k\right) = \emptyset$. The interval $I$ is now fixed.

For any $m \in \mathbf{N}^*$, if $I \cap \left(\bigcup_{k=1}^m I_k\right) = \emptyset$, the definition of $k_m$ and $I_{m+1}$ implies $\lambda^*(I) \leq k_m < 2\,\lambda^*(I_{m+1})$.

The fact that $\lambda^*(I_{m+1})$ tends to 0 as $m \to \infty$ implies the existence of a first integer $m$ such that the previous inequality fails and, consequently, $I \cap I_m \neq \emptyset$.

We have obviously $m > n$ and $\lambda^*(I) \leq k_{m-1} < 2\lambda^*(I_m)$.

Since $x \in I$ and $I \cap I_m \neq \emptyset$, we can claim that the distance from $x$ to the midpoint of $I_m$ is smaller than $\lambda^*(I) + \lambda^*(I_m)/2 \leq 5\lambda^*(I_m)/2$.

Consider now, for each $k \in \mathbf{N}^*$, the interval $J_k$ defined by the following: $J_k$ has the same midpoint as $I_k$ and $\lambda^*(J_k) = 5\,\lambda^*(I_k)$.

We can claim that $D \subset \bigcup_{k=n+1}^{\infty} J_k$ and, consequently,

$$\lambda^*(D) \leq \sum_{k=n+1}^{\infty} \lambda^*(J_k) = 5 \sum_{k=n+1}^{\infty} \lambda^*(I_k) < \varepsilon.$$

□

**Remark.** Theorem 5.30 is stated as Vitali did in Vitali (1908) (see also the discussion in Lebesgue (1910, pp. 365 and 391)). However, the result can also be stated in the following way:

> *There exists a finite or countable collection of pairwise disjoint intervals* $(I_n)_{n\geq 1}$ *in* $\mathcal{I}$ *such that*
>
> $$\lambda^*\left(A \setminus \bigcup_{n\geq 1} I_n\right) = 0.$$

**Bernard:** I understand that this new concept is a technical one, directed toward proving the almost differentiability of nondecreasing functions. However, I really need to form an intuition about it.

**Laurent:** You are right, Bernard. In order to do that, we can consider two examples. For any subset $A$ of $\mathbf{R}$, the two collections of intervals $\{[x-h, x];\ x \in A, h > 0\}$ and $\{[y, y+h];\ y \in A, h > 0\}$ are Vitali coverings of $A$.

Moreover, I can say that you only need these examples to understand the proof of the following theorem.

**Theorem 5.31.** *Let* $f\colon [a,b] \to \mathbf{R}$ *be a nondecreasing function.*

*Then,* $f$ *is differentiable almost everywhere and the derivative* $f'$ *satisfies*

$$0 \leq f' < +\infty \quad \textit{almost everywhere.}$$

**Proof.** We consider the Dini derivatives:

$$\overline{L}_+(x) = \limsup_{\substack{h\to 0\\ h>0}} \frac{f(x+h)-f(x)}{h},$$

$$\overline{L}_-(x) = \limsup_{\substack{h\to 0\\ h>0}} \frac{f(x)-f(x-h)}{h},$$

$$\underline{L}_+(x) = \liminf_{\substack{h\to 0\\ h>0}} \frac{f(x+h)-f(x)}{h},$$

$$\underline{L}_-(x) = \liminf_{\substack{h\to 0\\ h>0}} \frac{f(x)-f(x-h)}{h}.$$

The idea of the proof is to show that the set of $x$ for which these quantities are not equal has Lebesgue measure zero.

Here, we only consider the set $E = \big\{x \in (a,b) : \overline{L}_+(x) > \underline{L}_-(x)\big\}$. The other combinations can be treated the same way.

In order to prove that $\lambda(E) = 0$, we write

$$E = \bigcup_{u,v\in \mathbf{Q}} E_{u,v} \quad \text{with}$$

$$E_{u,v} = \big\{x \in (a,b) : \overline{L}_+(x) > u > v > \underline{L}_-(x)\big\},$$

and we aim to prove that $\lambda(E_{u,v}) = 0$ for all $u, v \in \mathbf{Q}$.

We fix $u, v \in \mathbf{Q}$ such that $u > v$. By the definition of $\lambda$ (actually of $\lambda^*$), for any $\varepsilon > 0$, there exists an open set $U$ such that $E_{u,v} \subset U$ and $\lambda(U) < \lambda(E_{u,v}) + \varepsilon$.

$\star$ For any $x \in E_{u,v}$, since $U$ is open and $v > \underline{L}_-(x)$, for all $\eta > 0$, there exists $0 < \xi \le \eta$ such that

$$[x-\xi, x] \subset U \quad \text{and} \quad f(x) - f(x-\xi) < v\,\xi.$$

Those intervals $[x-\xi, x]$ cover $E_{u,v}$ in the sense of Vitali. Hence, Theorem 5.30 implies the existence of a finite collection $\big(I_k\big)_{1\le k\le n}$ of pairwise disjoint intervals $I_k = [x_k - \xi_k, x_k]$ $(k = 1, \ldots, n)$ such that

$$\lambda\big(E_{u,v} \setminus A\big) < \varepsilon \quad \text{where} \quad A = \bigcup_{k=1}^{n} I_k.$$

By summing over these disjoint intervals, we get

$$\sum_{k=1}^{n} \big[f(x_k) - f(x_k - \xi_k)\big] < v \sum_{k=1}^{n} \xi_k \leq v\,\lambda(U) < v\,\big(\lambda(E_{u,v}) + \varepsilon\big).$$

⋆ In the same way, for any $y \in A$ and for all $\eta > 0$, there exists $0 < \kappa \leq \eta$ and $k \in \{1, \ldots, n\}$ such that

$$[y, y + \kappa] \subset I_k \quad \text{and} \quad f(y + \kappa) - f(y) > u\,\kappa.$$

Those intervals $[y, y + \kappa]$ cover $A$ in the sense of Vitali. Hence, Theorem 5.30 implies the existence of a finite collection $\big(J_k\big)_{1 \leq k \leq m}$ of pairwise disjoint intervals $J_k = [y_k, y_k + \kappa_k]$ $(k = 1, \ldots, m)$ such that

$$\lambda\big(A \setminus B\big) < \varepsilon \quad \text{where } B = \bigcup_{k=1}^{m} J_k.$$

By summing over these disjoint intervals, we get

$$\begin{aligned}\sum_{k=1}^{m} \big[f(y_k + \kappa_k) - f(y_k)\big] &> u \sum_{k=1}^{m} \kappa_k = u\lambda(B) > u\big(\lambda(A) - \varepsilon\big) \\ &> u\big(\lambda(E_{u,v}) - 2\varepsilon\big).\end{aligned}$$

⋆ For each $I_l = [x_l - \xi_l, x_l]$, by summing over the intervals $J_k = [y_k, y_k + \kappa_k] \subset I_l$ and using the nondecreasingness of $f$, we get

$$\sum_{k : J_k \subset I_l} \big[f(y_k + \kappa_k) - f(y_k)\big] \leq f(x_l) - f(x_l - \xi_l).$$

Finally, we get

$$u\left(\lambda(E_{u,v})-2\varepsilon\right) < \sum_{k=1}^{m}\left[f(y_k+\kappa_k)-f(y_k)\right]$$
$$\leq \sum_{l=1}^{n}\left[f(x_l)-f(x_l-\xi_l)\right] < v\big(\lambda(E_{u,v})+\varepsilon\big)$$

and, thus, taking $\varepsilon \to 0$, $u\lambda(E_{u,v}) \leq v\,\lambda(E_{u,v})$.

Since $u > v$ and $\lambda(E_{u,v}) \leq b-a < +\infty$, we conclude that $\lambda(E_{u,v}) = 0$.

⋆ We proved that there exists a Borel set $\Lambda$ such that $\lambda(\Lambda) = 0$ and the limit

$$L(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

is defined for all $x \in [a,b] \setminus \Lambda$. We extend the definition of $L(x)$ to every $x \in [a,b]$ with $L(x) = 0$ if $x \in \Lambda$.

It remains to prove that $L(x)$ is finite almost everywhere.

For any $n \in \mathbf{N}^*$, we consider

$$\forall x \in (a,b), \quad g_n(x) = \frac{f(x+1/n)-f(x)}{1/n}.$$

In this expression, we set $f(x+1/n) = f(b)$ when $x+1/n > b$.

The function $L$ is measurable and $g_n(x) \to L(x)$ for almost every $x$. Since $f$ is nondecreasing, we have $g_n(x) \geq 0$ for all $x$, and we can apply the Fatou lemma:

$$\int_{[a,b]} L\,d\lambda \leq \liminf_{n\to\infty}\int_{[a,b]} g_n\,d\lambda$$
$$= \liminf_{n\to\infty}\left(n\int_{[b,b+1/n]} f\,d\lambda + n\int_{[a,a+1/n]} f\,d\lambda\right)$$
$$\leq f(b)-f(a) < +\infty.$$

We can conclude that $L$ is finite almost everywhere and the result follows. □

**Bernard:** What an impressive proof where Vitali's coverings show their crucial role! And what an interesting result! I guess that we cannot say more about a nondecreasing function.

**Laurent:** If you hope to claim that any nondecreasing function is differentiable at any point, I can tell you that this is false. The Cantor function, also called Devil's staircase, constitutes a famous counterexample.

**Bernard:** What a shame! I still enjoy the result almost everywhere.

**Laurent:** We are now able to study the first fundamental theorem of calculus for the Lebesgue integral.

The Lebesgue integral considers functions that are not necessarily continuous, and the following two results show that integrating and differentiating cannot be considered generally as inverse operators.

**Theorem 5.32 (Primitive function of an integrable function).** *Let* $f\colon \mathbf{R} \to \mathbf{R}$ *be a locally integrable function and* $a \in \mathbf{R}$.

$$\text{For all } u \in \mathbf{R}, \text{ we consider } F(u) = \begin{cases} \int_{[a,u]} f\, d\lambda & \text{for } u \geq a, \\ -\int_{[u,a]} f\, d\lambda & \text{for } u < a. \end{cases}$$

*The function* $F\colon \mathbf{R} \to \mathbf{R}$ *is continuous on* **R** *and differentiable almost everywhere. We have* $F' = f$ *almost everywhere.*

**Proof.** $\star$ We only prove the continuity for $u \in [a, +\infty)$ (the case of $u \in (-\infty, a)$ follows the same reasoning). The result can be proved using the dominated convergence theorem (Theorem 4.8). For the sake of brevity, we simply apply Theorem 4.9, which gives the continuity of integrals depending on a parameter.

Let us fix $u_0$ in an open interval $(\alpha, \beta) \subset [a, +\infty)$ and check the conditions of Theorem 4.9 for the function $u \mapsto F(u) = \int_{\mathbf{R}} f(u,x)\lambda(dx)$, with $f(u,x) = f(x)\mathbf{1}_{[a,u]}(x)$:

(i) For all $u \in (\alpha, \beta)$, the function $x \mapsto f(u,x)$ is measurable.
(ii) The function $u \mapsto f(u,x)$ is continuous at $u_0$, for all $x \in \mathbf{R} \setminus \{a, u_0\}$ (so almost everywhere).
(iii) For all $u \in (\alpha, \beta)$, $|f(u,x)| \leq |f(x)|\, \mathbf{1}_{(\alpha,\beta)}(x) = g(x)$ and $g \in \mathcal{L}^1$.

Hence, we can conclude that $F$ is continuous at $u_0$.

For $u_0 = a$, the dominated convergence theorem applies.

⋆ We now prove the differentiability of $F$ in the case of a nonnegative function $f$ (the general case follows by the decomposition $f = f^+ - f^-$, where $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$).

The function $F$ is nondecreasing, and thus Theorem 5.31 implies that $F$ is differentiable almost everywhere, and we denote by $F'$ its derivative (extended by 0, where $F$ is not differentiable).

⋆ We first assume that the function $f$ is bounded on $\mathbf{R}$ by a constant $K > 0$.

For any $n \in \mathbf{N}^*$, we consider

$$\forall x \in \mathbf{R}, \quad G_n(x) = \frac{F(x + 1/n) - F(x)}{1/n}.$$

We have the following:

(i) for all $n \in \mathbf{N}^*$, the function $G_n$ is measurable (because $F$ is continuous);
(ii) for almost every $x \in \mathbf{R}$, $G_n(x)$ tends to $F'(x)$ as $n$ goes to infinity;
(iii) for all $n \in \mathbf{N}^*$ and all $x \in \mathbf{R}$,

$$|G_n(x)| = G_n(x) = n \int_{(x, x+1/n]} f \, d\lambda \leq K.$$

Thus, the dominated convergence theorem implies that for any $\alpha \leq \beta$,

$$\int_{[\alpha,\beta]} G_n \, d\lambda \longrightarrow \int_{[\alpha,\beta]} F' \, d\lambda \quad \text{as } n \to \infty.$$

This leads to

$$n \int_{[\beta,\beta+1/n]} F \, d\lambda - n \int_{[\alpha,\alpha+1/n]} F \, d\lambda \longrightarrow \int_{[\alpha,\beta]} F' \, d\lambda \quad \text{as } n \to \infty.$$

Using the continuity of $F$, we conclude that

$$\forall \alpha \leq \beta, \quad F(\beta) - F(\alpha) = \int_{[\alpha,\beta]} F' \, d\lambda$$

and, by the definition of $F$,

$$\forall \alpha \leq \beta, \quad \int_{[\alpha,\beta]} f \, d\lambda = \int_{[\alpha,\beta]} F' \, d\lambda.$$

Using a monotone class argument ($\mathcal{B}(\mathbf{R})$ is generated by the intervals $[\alpha, \beta]$, with $\alpha \leq \beta$), we conclude that for all $A \in \mathcal{B}(\mathbf{R})$, $\int_A f \, d\lambda = \int_A F' \, d\lambda$, which implies that $F' = f$ almost everywhere.

$\star$ For an unbounded function $f$, we consider the function $f_n = \min(f, n)$ for any $n \in \mathbf{N}^*$, which is bounded by $n$, and the corresponding function $F_n$ (defined from $f_n$ as $F$ is defined from $f$).

The previous point proves that, for any $n$, $F_n$ is differentiable almost everywhere and $F_n' = f_n$ almost everywhere.

For all $x \geq a$, the relation $f(x) = f_n(x) + \big(f(x) - f_n(x)\big)$ implies that

$$F(x) = F_n(x) + G_n(x), \quad \text{where } G_n(x) = \int_{[a,x]} (f - f_n) \, d\lambda.$$

Since $f - f_n$ is nonnegative, the function $G_n$ is nondecreasing. Hence, its difference quotients are nonnegative and $F'(x) \geq F_n'(x) = f_n(x)$.

By taking the limit $n \to \infty$, we get $F'(x) \geq f(x)$. The case of $x < a$ can be treated in the same way, and this inequality holds for all $x \in \mathbf{R}$.

Integrating on any interval $[\alpha, \beta]$, we get

$$\int_{[\alpha,\beta]} F' \, d\lambda \geq \int_{[\alpha,\beta]} f \, d\lambda = F(\beta) - F(\alpha).$$

In the proof of Theorem 5.31, we showed that $\int_{[\alpha,\beta]} F' \, d\lambda \leq F(\beta) - F(\alpha)$.

The result follows as in the previous point. □

It is important to note that if the function $f$ is not continuous, the function $F\colon u \mapsto \int_{[a,u]} f\,d\lambda$ is not differentiable at every $x \in \mathbf{R}$. And consequently, its derivative is not defined at every $x \in \mathbf{R}$. The differentiability part of Theorem 5.32 only stands almost everywhere.

**Bernard:** I've had the feeling for some time that you place the difference between the Riemann and Lebesgue integrals mostly on the integration of noncontinuous functions.

And, if I understand well, the previous theorem tells us that integration and differentiation are not inverse operations in the Lebesgue sense.

However, wasn't it already the case for Riemann?

**Laurent:** You are right. Even for the Riemann integral (the real one, I mean), considering noncontinuous functions, integration and differentiation are no longer inverse operators.

Anyway, let us examine what happen when we integrate a derivative. We do not often see the general result discussed in books.

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The following result is a particular case of the characterization of the functions $f$ such that Expression (5.4.1) holds. Actually, Royden (1968, Theorem 13 and Corollary 14, pp. 106–107) prove that a function $f$ satisfies (5.4.1) if and only if $f$ is *absolutely continuous* (which covers the case of differentiable functions).

The key argument is the use of Theorem 5.32 with the locally integrable function $f'$ and the fact that if $g$ is an absolutely continuous function such that $g' = 0$ almost everywhere (note also that any absolutely continuous function is differentiable almost everywhere), then $g$ is constant (see Royden (1968, Lemma 12, p. 105)).

**Theorem 5.33 (Integration of a derivative).** *Let $f\colon \mathbf{R} \to \mathbf{R}$ be a function satisfying the two following conditions:*

- ⋆ *$f$ is differentiable at every point of $\mathbf{R}$.*
- ⋆ *The derivative $f'$ is a locally integrable function.*

*Then, for all* $a, b \in \mathbf{R}$ *such that* $a \leq b$,

$$f(b) - f(a) = \int_{[a,b]} f' \, d\lambda. \tag{5.4.1}$$

We omit the proof, which can be found in Rudin (1987, Theorem 7.21, p. 149).

**Remark.** In addition to the discussion that precedes the theorem, a natural question is: can we weaken the assumption of Theorem 5.33 to functions that are only differentiable almost everywhere? We refer to the discussion in Rudin (1987, p. 144).

The *Cantor function* (also called *Devil's staircase*) is a famous example of THE function $f\colon [0,1] \to \mathbf{R}$ such that $f(0) = 0$, $f(1) = 1$, $f$ nondecreasing and $f' = 0$ almost everywhere. For such a function, the left-hand part of (5.4.1) is equal to 1 and the right-hand part is equal to 0.

**Bernard:** I'm a little sad that I haven't read the proof, or at least parts of it.

**Laurent:** Do not worry, Bernard. It is very technical but not very interesting, or even very instructive. I once put this proof in my course exam, and the students had many difficulties...

In many books on measure theory (for instance Lang's[1] or Tao's[2]), the two previous results are limited to the case of regulated functions instead of locally integrable. We will see later that, in this case, the integral coincides with the Riemann integral.

You can capture the spirit in the case of a differentiable function $f$ such that $f'$ is continuous (or simply bounded). In this case, the result is a simple consequence of the dominated convergence theorem.

**Bernard:** Let me try. I should be able to handle the case where $f'$ is continuous.

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[1]See Lang (1993a).
[2]See Tao (2011).

For $a \leq b$ and $n \geq 1$, I consider $(a_k)_{0 \leq k \leq n}$ in $[a, b]$ such that

$$a_0 = a, \quad a_n = b \text{ and } a_{k+1} - a_k = 1/n \quad \forall k = 0, \ldots, n-1,$$

and I define the sequence of functions $(g_n)_{n \in \mathbf{N}^*}$ by

$$\forall x \in [a, b], \quad g_n(x) = \sum_{k=0}^{n-1} \frac{f(a_{k+1}) - f(a_k)}{1/n} \, \mathbf{1}_{[a_k, a_{k+1})}(x).$$

Now, by the mean value theorem (also called the finite-increments formula), for all $k = 0, \ldots, n-1$, there exists $\xi_k \in (a_k, b_k)$ such that $f(a_{k+1}) - f(a_k) = f'(\xi_k)/n$.

From the relation

$$\forall x \in [a, b], \quad g_n(x) = \sum_{k=0}^{n-1} f'(\xi_k) \, \mathbf{1}_{[a_k, a_{k+1})}(x),$$

I deduce that for all $x \in [a, b]$, $g_n(x)$ converges to $f'(x)$ as $n$ goes to infinity.

Moreover, since $f'$ is continuous, we can consider $M = \sup_{x \in [a,b]} |f'(x)|$.

Then, since $|g_n(x)| \leq M$ for all $x \in [a, b]$, the dominated convergence theorem allows us to claim that

$$\int_{[a,b]} \lim_{n \to \infty} g_n \, d\lambda = \lim_{n \to \infty} \int_{[a,b]} g_n \, d\lambda.$$

Finally, I deduce that

$$\int_{[a,b]} f' \, d\lambda = \lim_{n \to \infty} \sum_{k=0}^{n-1} \big[ f(a_{k+1}) - f(a_k) \big] = f(b) - f(a).$$

**Laurent:** Very good, Bernard. I remark that you did not really use the assumption of continuity of $f'$. You only use the fact that $f'$ is bounded on $[a, b]$.

**Bernard:** Oh you're right! I proved the result in the case of $f'$ bounded on $[a, b]$, but possibly not continuous.

Anyway. You said that the integral of regulated functions coincides with the Riemann integral. I guess it is now time to see how this Lebesgue integral generalizes my old Riemann integral.

### 5.4.2 *Lebesgue integral vs. Riemann integral*

**Bernard:** Before tackling the comparison between the Riemann and Lebesgue integrals, we should probably agree on their definitions.

**Laurent:** Excellent idea. I can already tell you that if you wish to talk about the measurability of Riemann integrable functions, the necessity of the Lebesgue $\sigma$-algebra appears.

**Short refresher on the Riemann integral**
Let $[a, b]$ be a fixed compact interval of $\mathbf{R}$ (with $a < b$).

A function $h$ from $[a, b]$ to $\mathbf{R}$ is a *step function* if there exist a finite subdivision $a = x_0 < x_1 < \cdots < x_N = b$ and real numbers $(\alpha_k)_{1\le k\le N}$ and $(\beta_k)_{0\le k\le N}$, with $N \ge 1$, such that

$$h = \sum_{k=1}^{N} \alpha_k \mathbf{1}_{(x_{k-1},x_k)} + \sum_{k=0}^{N} \beta_k \mathbf{1}_{\{x_k\}}.$$

This means for all $x \in [a, b]$,

$$h(x) = \begin{cases} \alpha_k & \text{if } x_{k-1} < x < x_k, \\ \beta_k & \text{if } x = x_k, \\ 0 & \text{otherwise.} \end{cases}$$

The set of step functions on $[a, b]$ is denoted by $Step_{[a,b]}$ or $Step$.

We remark that any step function is a simple function for the Borel $\sigma$-algebra, which can be written as $Step_{[a,b]} \subset \mathcal{E}\big(\mathcal{B}([a, b])\big)$.

We define $I_R(h) = \sum_{k=1}^{N} \alpha_k (x_k - x_{k-1})$, and we remark that

$$I(h) = \sum_{k=1}^{N} \alpha_k \lambda\big((x_{k-1}, x_k)\big) = \int_{[a,b]} h \, d\lambda.$$

**Definition 5.34.** A (bounded) function $f : [a, b] \to \mathbf{R}$ is said to be *Riemann-integrable* if

$$\sup_{\substack{h\in Step \\ h\le f}} I_R(h) = \inf_{\substack{h\in Step \\ h\ge f}} I_R(h).$$

In that case, the common value is denoted by $I_R(f) = \int_a^b f(x)\, dx$.

**Laurent:** As you can see, from this definition, the class of Riemann-integrable functions is not reduced to continuous functions (and is a little bit larger than the class of regulated functions).

**Proposition 5.35.** *Let $f\colon [a,b] \to \mathbf{R}$ be a Riemann-integrable function.*

*Then, $f$ is measurable with respect to the completed $\sigma$-algebra $\overline{\mathcal{B}([a,b])}$, integrable with respect to the Lebesgue measure and*

$$\int_a^b f(x)\,dx = \int_{[a,b]} f\,d\lambda.$$

**Proof.** Since $f$ is Riemann-integrable, we can find a sequence of step functions $(h_n)_{n\in\mathbf{N}}$ on $[a,b]$ such that for all $n \in \mathbf{N}$, $f \le h_n$ and $I_R(h_n) \downarrow I_R(f)$ as $n \uparrow \infty$.

We can assume that the sequence $(h_n)_{n\in\mathbf{N}}$ is nonincreasing. If not, we substitute $h_n$ with $\inf(h_0, h_1, \ldots, h_n)$ for all $n \in \mathbf{N}$.

Then, we set $h_\infty = \lim h_n \ge f$.

In the same way, we can find a nondecreasing sequence of step functions $(\tilde{h}_n)_{n\in\mathbf{N}}$ on $[a,b]$ such that for all $n \in \mathbf{N}$, $\tilde{h}_n \le f$ and $I_R(\tilde{h}_n) \uparrow I_R(f)$. We set $\tilde{h}_\infty = \lim \tilde{h}_n \le f$.

The functions $h_\infty$ and $\tilde{h}_\infty$ are Borel functions (and bounded).

Thanks to the dominated convergence theorem,

$$\int_{[a,b]} h_\infty\,d\lambda = \lim \int_{[a,b]} h_n\,d\lambda = \lim I_R(h_n) = I_R(f)$$

and

$$\int_{[a,b]} \tilde{h}_\infty\,d\lambda = \lim \int_{[a,b]} \tilde{h}_n\,d\lambda = \lim I_R(\tilde{h}_n) = I_R(f).$$

Then, $\int_{[a,b]} (h_\infty - \tilde{h}_\infty)\,d\lambda = 0$.

Since $h_\infty \ge \tilde{h}_\infty$, we deduce that $h_\infty = \tilde{h}_\infty$ almost everywhere.

From the inequalities $\tilde{h}_\infty \le f \le h_\infty$, we conclude that $f$ coincides almost everywhere with a Borel function. Then, thanks to Proposition 5.9, $f$ is measurable with respect to the completed $\sigma$-algebra $\overline{\mathcal{B}([a,b])}$.

From $f = h_\infty$ a.e. we conclude that $\int_{[a,b]} f\,d\lambda = \int_{[a,b]} h_\infty\,d\lambda = I_R(f)$. □

**Bernard:** If I understand well, starting with the definition of a Riemann-integrable function, there is no element that allows us to say that it is a Borel function. Consequently, its Riemann integral cannot be compared to its integral with respect to the Borel–Lebesgue measure. This last integral is not necessarily defined.

**Laurent:** However, what we can say is that this Riemann-integrable function is necessarily measurable with respect to the completed $\sigma$-algebra. We observe that the completion of the Borel $\sigma$-algebra allows us to obtain the equality of the Riemann integral and the Lebesgue integral... And this comparison holds only if the considered $\sigma$-algebra is the Lebesgue $\sigma$-algebra.

### Improper Riemann integral

**Definition 5.36.** Let $a, b \in \overline{\mathbf{R}}$, $a < b$. A function $f\colon (a,b) \to \mathbf{R}$ is said to be *locally Riemann-integrable* if for all $c, d \in (a,b)$, $c < d$, the Riemann integral $\int_c^d f(t)\,dt$ is well defined.

**Remark.** The same definition stands for $f\colon [a,b) \to \mathbf{R}$ if $a \in \mathbf{R}$.

**Definition 5.37.** For $a \in \mathbf{R}$ and $b \in \mathbf{R} \cup \{+\infty\}$, let $f\colon [a,b) \longrightarrow \mathbf{R}$ be a locally Riemann-integrable function. We define $\int_a^b f(x)\,dx = \lim_{\substack{X \to b \\ X < b}} \int_a^X f(x)\,dx$. This object is called the *improper Riemann integral* of $f$ over $[a,b)$.

Moreover, if $\int_a^b |f(x)|\,dx$ is defined, the integral is said to be *absolutely convergent*. Otherwise, it is said to be *semi-convergent*.

The following result extends Proposition 5.35 and compares the improper Riemann integral on $[a,b)$ with the corresponding Lebesgue integral.

**Theorem 5.38.** *A function* $f\colon [a,b) \to \mathbf{R}$ *locally Riemann-integrable is integrable with respect to the Lebesgue measure if and only if it is Riemann absolutely convergent.*

*In this case, we have* $\int_a^b f(x)\,dx = \int_{[a,b)} f\,d\lambda$.

**Remark.** We deduce from Theorem 5.38 that the functions whose Riemann integral is semi-convergent are not integrable with respect to the Lebesgue measure.

For example, since the Riemann integral $\int_0^{+\infty} \frac{\sin x}{x}\, dx$ is semi-convergent, the function $x \mapsto \frac{\sin x}{x}$ is not integrable with respect to the Lebesgue measure.

**Laurent:** We saw the framework of the integral with respect to any measure and to the Lebesgue measure in particular. We are now ready for the theory of probability.

## 5.5 Cumulative Distribution Functions and Densities

### 5.5.1 *Construction of a probability measure*

**Laurent:** In the framework of probability, it is crucial to know how a probability measure is precisely defined, and what it means. A probability measure is not merely any measure that is used for integration. The measure is attached to a certain reality.

**Bernard:** I'm aware of this, Laurent. Using different probability measures to estimate the probability of an event can lead to very different results. If I understood well, this choice of probability measure is at the heart of understanding the paradoxes at the beginning of probability calculus.

**Laurent:** That is indeed the point. Let us start at the beginning of the theory, which takes its roots in the Caratheodory theory. We already saw this theory for the Lebesgue measure, so we will just give an overview.

Let us take the book of probability. Maybe we can skip the part about $\pi$-systems.

**Bernard:** Yes, I remember. A $\pi$-system is a collection $\mathcal{C}$ of subsets of $\Omega$ which is closed under finite intersections: for all $A, B \in \mathcal{C}$, $A \cap B \in \mathcal{C}$. And if $\mathcal{C}$ is a $\pi$-system such that $\sigma(\mathcal{C}) = \mathcal{F}$, a probability measure is characterized by its values on $\mathcal{C}$.

The notion of $\sigma$-algebra can be difficult to understand when defining a probability measure. It is often convenient to use the notions of Boolean algebra and $\pi$-systems, which are easier to handle.

**Definition 5.39.** A collection $\mathcal{F}_0$ of subsets of $\Omega$ is called *a Boolean algebra on* $\Omega$ if:

(i) $\Omega \in \mathcal{F}_0$;
(ii) $F \in \mathcal{F}_0 \Rightarrow \Omega \setminus F \in \mathcal{F}_0$;
(iii) $F, G \in \mathcal{F}_0 \Rightarrow F \cup G \in \mathcal{F}_0$.

The following fundamental theorem shows that, to define a probability measure, it is sufficient to know its values on a Boolean algebra. In the following section, we will see that it allows us to characterize a measure on **R** by its distribution function.

**Theorem 5.40 (Caratheodory extension theorem).** *Let $\mathcal{F}_0$ be a Boolean algebra on $\Omega$ and $\mathcal{F} = \sigma(\mathcal{F}_0)$.*

⋆ *If $\mu_0 \colon \mathcal{F}_0 \to [0, +\infty]$ is a $\sigma$-additive function, then there exists a measure $\mu$ on $(\Omega, \mathcal{F})$ such that $\mu = \mu_0$ on $\mathcal{F}_0$.*
⋆ *If $\mu_0(\Omega) < \infty$, then this extension is unique.*

**Bernard:** Boolean algebra... Yet another new concept!!! Is it really necessary? If I understand well, a Boolean algebra is a particular case of $\pi$-system, right?

**Laurent:** Yes, Bernard. The concept of Boolean algebra allows us to concentrate the Caratheodory theory in Theorem 5.40. However, it is not strictly necessary. Recall how we built the Lebesgue measure $\lambda$ from the outer measure $\lambda^*$ and how we characterized the measure by its values on finite open intervals.

**Bernard:** I guess we could make the same construction for any probability measure, without the use of Theorem 5.40.

**Laurent:** Definitely. Let's see what happens for **R**.

### 5.5.2 *Probability measure on* $\mathbf{R}$

In the case where the probability space is $\big(\mathbf{R}, \mathcal{B}(\mathbf{R}), \mathbf{P}\big)$, the notion of the distribution function is more easily manipulated than the probability measure itself, all the while fully characterizing it.

**Definition 5.41.** In the probability space $\big(\mathbf{R}, \mathcal{B}(\mathbf{R}), \mathbf{P}\big)$, where $\mathcal{B}(\mathbf{R})$ is the Borel $\sigma$-algebra, the *cumulative distribution function* (or simply *distribution function*) of the probability measure $\mathbf{P}$ is the function $F\colon \mathbf{R} \to [0,1]$, defined by $F(x) = \mathbf{P}\big((-\infty, x]\big)$ for all $x \in \mathbf{R}$.

**Bernard:** How can a measure, which seems a complicated object, be reduced to a mere function? I had finally adjusted to all your theoretical jargon, and now you tell me simple functions are sufficient...

**Laurent:** Calm down, Bernard. Measures are the ideal objects to construct an integral, this is for fact. What we say here is that, in the case of $\mathbf{R}$, a Borel measure is characterized by a function.

But pay attention. This does not mean they are identified. Maybe you encountered people who think this is true, but, sadly, they have it all wrong.

**Theorem 5.42.** *The probability measure of a probability space* $(\mathbf{R}, \mathcal{B}(\mathbf{R}), \mathbf{P})$ *is characterized by its cumulative distribution function.*

**Proof.** We have to show that if $\mathbf{P}$ and $\mathbf{Q}$ are two probability measures on the measurable space $\big(\mathbf{R}, \mathcal{B}(\mathbf{R})\big)$ such that $\mathbf{P}\big((-\infty, a]\big) = \mathbf{Q}\big((-\infty, a]\big)$ for all $a \in \mathbf{R}$, then $\mathbf{P} = \mathbf{Q}$.

In order to do that, we remark that the collection $\pi(\mathbf{R}) = \{(-\infty, a];\ a \in \mathbf{R}\}$ is a $\pi$-system. Since the probability measures $\mathbf{P}$ and $\mathbf{Q}$ coincide on the $\pi$-system $\pi(\mathbf{R})$, they coincide on the generated $\sigma$-algebra $\sigma(\pi(\mathbf{R}))$.

Since the Borel $\sigma$-algebra $\mathcal{B}(\mathbf{R})$ is generated by the intervals $(-\infty, a]$ $(a \in \mathbf{R})$ (Lemma 2.7), the result follows. $\square$

**Bernard:** I understand that, if I have two probability measures $\mathbf{P}$ and $\mathbf{Q}$ on $\mathcal{B}(\mathbf{R})$ with the same distribution function, then $\mathbf{P}(A) = \mathbf{Q}(A)$ for all $A \in \mathcal{B}(\mathbf{R})$.

More formally,

$$\Big[\mathbf{P}\big((-\infty, x]\big) = \mathbf{Q}\big((-\infty, x]\big),\ \forall x \in \mathbf{R}\Big] \\ \Longrightarrow \Big[\mathbf{P}\big(A\big) = \mathbf{Q}(A),\ \forall A \in \mathcal{B}(\mathbf{R})\Big].$$

I think I understand the power of the distribution function and, more generally, the power of the $\pi$-systems.

One thing remains unclear to me: what is a distribution function? More precisely, what are the conditions for a function $F\colon \mathbf{R} \to [0,1]$ to be the distribution function of some probability measure?

**Laurent:** Here you go, Bernard. These conditions are given by the following theorem.

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**Theorem 5.43.** *A function $F\colon \mathbf{R} \to [0,1]$ is the cumulative distribution function of a probability measure on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ if and only if all three following conditions hold:*

(i) *$F$ is nondecreasing;*
(ii) *$F$ is right-continuous;*
(iii) $\lim_{x \to -\infty} F(x) = 0$ *and* $\lim_{x \to +\infty} F(x) = 1$.

**Proof.** $\star$ Assume that $F$ is the distribution function of a probability measure $\mathbf{P}$ on $\big(\mathbf{R}, \mathcal{B}(\mathbf{R})\big)$.

(i) For all $x, y \in \mathbf{R}$ such that $x < y$, we have $(-\infty, x] \subset (-\infty, y]$, and then

$$F(x) = \mathbf{P}\big((-\infty, x]\big) \leq \mathbf{P}\big((-\infty, y]\big) = F(y),$$

which proves that $F$ is nondecreasing.

(ii) By the sequential characterization of the right continuity, it suffices to prove that, for any sequence $(x_n)_{n \in \mathbf{N}}$ in $\mathbf{R}$ with $x_n \to_{x_n \leq x} x$, $F(x_n)$ converges to $F(x)$ as $n$ goes to infinity.

Since $F$ is nondecreasing, it suffices to consider a nonincreasing sequence $(x_n)_{n\in\mathbf{N}}$ that converges to $x$.

The sequence of intervals $\big((-\infty, x_n]\big)_{n\in\mathbf{N}}$ is nonincreasing and converges to $\bigcap_{n\in\mathbf{N}}(-\infty, x_n] = (-\infty, x]$.

Thanks to the monotone convergence property of measures, we get

$$\lim_{n\to\infty} \mathbf{P}\big((-\infty, x_n]\big) = \mathbf{P}\left(\bigcap_{n\in\mathbf{N}}(-\infty, x_n]\right) = \mathbf{P}\big((-\infty, x]\big).$$

(iii) As previously, we consider a nonincreasing sequence $(y_n)_{n\in\mathbf{N}}$ that tends to $-\infty$ and a nondecreasing sequence $(z_n)_{n\in\mathbf{N}}$ that tends to $+\infty$.

The sequences of intervals $\big((-\infty, y_n]\big)_{n\in\mathbf{N}}$ and $\big((-\infty, z_n]\big)_{n\in\mathbf{N}}$ are, respectively, nonincreasing and nondecreasing.

Then, by the monotone convergence property of measures, we get

$$\lim_{n\to\infty} \mathbf{P}\big((-\infty, y_n]\big) = \mathbf{P}\left(\bigcap_{n\in\mathbf{N}}(-\infty, y_n]\right) = \mathbf{P}(\emptyset) = 0$$

and

$$\lim_{n\to\infty} \mathbf{P}\big((-\infty, z_n]\big) = \mathbf{P}\left(\bigcup_{n\in\mathbf{N}}(-\infty, z_n]\right) = \mathbf{P}(\Omega) = 1.$$

By the sequential characterization of limits (for a nondecreasing function $F$), we can conclude that $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to+\infty} F(x) = 1$.

⋆ The converse part of the theorem relies on the Caratheodory theory. The idea is to build a probability measure $\mathbf{P}$ on $\big(\mathbf{R}, \mathcal{B}(\mathbf{R})\big)$ such that

$$\forall a, b \in \mathbf{R} \text{ with } a \leq b, \quad \mathbf{P}\big((a, b]\big) = F(b) - F(a). \tag{5.5.1}$$

In order to do that, we consider the collection $\mathcal{A}(\mathbf{R})$ of all finite unions of pairwise disjoint intervals on the form:

⋆ $(a, b]$, with $a \in \mathbf{R} \cup \{-\infty\}$, $b \in \mathbf{R}$ and $a \leq b$,
⋆ or $(a, +\infty)$, with $a \in \mathbf{R} \cup \{-\infty\}$.

The collection $\mathcal{A}(\mathbf{R})$ is a Boolean algebra that generates the Borel $\sigma$-algebra $\mathcal{B}(\mathbf{R})$. Then, we can use Theorem 5.40 to define a probability measure **P** such that (5.5.1) holds.

We do not give all the details here, but the starting point is to define a function $\mu_0\colon \mathcal{A}(\mathbf{R}) \to [0,1]$ as follows: for all $A \in \mathcal{A}(\mathbf{R})$ with $A = \bigcup_{1\le k\le n}(a_k, b_k] \cup \bigcup_{1\le l\le m}(c_l, +\infty)$, where all the intervals are pairwise disjoint,

$$\mu_0(A) = \sum_{k=1}^{n} \big[F(b_k) - F(a_k)\big] + \sum_{l=1}^{m} \big[1 - F(c_l)\big],$$

with the convention $F(-\infty) = 0$.

We first remark that this definition implies that for all $a, b \in \mathbf{R}$ such that $a \le b$, $\mu_0\big((a,b]\big) = F(b) - F(a)$.

Then, according to Theorem 5.40, it simply remains to prove that $\mu_0$ is $\sigma$-additive to obtain an extension **P** to $\mathcal{B}(\mathbf{R})$ whose distribution function is $F$. We omit this part of the proof. □

**Bernard:** Since I don't really understand all the details hidden behind Theorem 5.40, I don't like very much the existence part of this proof. I mean the existence of a probability **P** such that (5.5.1) holds.

**Laurent:** Oh, Bernard! You do not even understand that you do know all the details of Theorem 5.40. Actually, this theorem is a summary of Caratheodory's theory, which you have seen in detail during the construction of the Lebesgue measure.

**Bernard:** Really? What you say is very surprising because I was just about to suggest defining the measure **P** in the same way as we had defined $\lambda$.

My idea was to mimic the construction of the Lebesgue measure, which starts from the notion of the length of an interval $\ell\big((a,b)\big) = b - a$, then to define the outer-measure $\lambda^*$ using covers of any subset of **R** by finite open intervals and, finally, to restrict it to the $\lambda^*$-measurable sets that contains the Lebesgue $\sigma$-algebra.

Instead of $\ell\big((a,b)\big)$, we can use $F(b) - F(a)$ to define an outer-measure $\mathbf{P}^*$, and we carry out the same procedure to obtain a probability measure $\mathbf{P}$ on $\mathcal{B}(\mathbf{R})$ such that (5.5.1) holds.

**Laurent:** Great, Bernard! Really! Following the remarks after Theorems 5.16 and 5.18, let me summarize the construction in the following array:

| | Lebesgue measure $\lambda$ | Probability measure $\mathbf{P}$ |
|---|---|---|
| Characterization | $\forall a \leq b,\ \lambda\big((a,b)\big) = b - a$ | $\forall a \leq b,\ \mathbf{P}\big((a,b]\big) = F(b) - F(a)$ |
| Outer-measure | $\lambda^*(A) = \inf\left\{ \sum_{i=1}^{\infty}(b_i - a_i)\right\}$ where the infimum is taken on all covers $A \subset \bigcup_{i\in\mathbf{N}^*}(a_i, b_i)$ with $a_i, b_i \in \mathbf{R}$ and $a_i \leq b_i$ for all $i$ | $\mathbf{P}^*(A) = \inf\left\{ \sum_{i=1}^{\infty}\big(F(b_i) - F(a_i)\big)\right\}$ where the infimum is taken on all covers $A \subset \bigcup_{i\in\mathbf{N}^*}(a_i, b_i]$ with $a_i, b_i \in \mathbf{R}$ and $a_i \leq b_i$ for all $i$ |
| $\sigma$-algebra | $\overline{\mathcal{B}(\mathbf{R})} \subset \mathcal{M}_{\lambda^*}$ | $\mathcal{B}(\mathbf{R}) \subset \mathcal{M}_{\mathbf{P}^*}$ |
| Measure | $\lambda = \lambda^*\vert_{\overline{\mathcal{B}(\mathbf{R})}}$ | $\mathbf{P} = \mathbf{P}^*\vert_{\mathcal{B}(\mathbf{R})}$ |

**Bernard:** If I understand well the construction of the probability $\mathbf{P}$ from the function $F$, the Caratheodory theory is the key to sum up a finite Borel measure to a nondecreasing function with the correct limits.

**Laurent:** You are absolutely right. But, do not be too quick about the regularity. There is no reason why a distribution function should be *a priori* continuous. Let us see that with the following result.

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According to Theorem 5.43, a distribution function is not necessarily continuous. In fact, its discontinuity points are the *atoms* of the probability measure it characterizes.

**Proposition 5.44.** *Let $F$ be the cumulative distribution function of a probability measure $\mathbf{P}$ over $\big(\mathbf{R}, \mathcal{B}(\mathbf{R})\big)$. For all $x \in \mathbf{R}$, we have*

$$\mathbf{P}\big(\{x\}\big) = F(x) - F(x-), \quad \textit{where } F(x-) = \lim_{\substack{u\to x\\ u<x}} F(u).$$

**Proof.** It suffices to write $(-\infty, x] = (-\infty, x) \cup \{x\}$ and to use the additivity property of measures: $\mathbf{P}\big((-\infty,x]\big) = \mathbf{P}\big((-\infty,x)\big) + \mathbf{P}\big(\{x\}\big)$.

By remarking that $\mathbf{P}\big((-\infty,x)\big) = F(x-)$, the result follows. $\square$

We have seen previously that the distribution function is more easily manipulated than the probability measure it represents. It is often convenient to use this characterization to describe the law of a random variable.

**Laurent:** Bernard, to check that you understood how easier the manipulation of a distribution function is, can you apply it to the variable $Y = X^2$, when the law of $X$ is known?

**Bernard:** OK. I consider $F_X$ the cumulative distribution function of $X$, which determines its law as we just saw. And I compute the cumulative distribution function of $Y$ by writing

$$\begin{aligned}\forall y \geq 0, \quad F_Y(y) &= \mathbf{P}(X^2 \leq y) = \mathbf{P}(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= \mathbf{P}(X \leq \sqrt{y}) - \mathbf{P}(X < -\sqrt{y}) \\ &= F_X(\sqrt{y}) - F_X\big((-\sqrt{y})-\big).\end{aligned}$$

The law of $Y$ is then completely determined. In the special case where $F_X$ is continuous (the law of $X$ has no atom), we have

$$F_Y(y) = \begin{cases} F_X(\sqrt{y}) - F_X(-\sqrt{y}) & \text{if } y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the law of $Y$ has no atom either.

**Laurent:** Very good. Now, let us look at a very special case of the cumulative distribution function using the Lebesgue integral. But remember that this is a particular case.

### 5.5.3 *Particular case: Probability on* **R** *defined by a density*

E[φ(X)]
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**Definition 5.45.** If the probability measure $\mathbf{P} = f\,\lambda$, where $\lambda$ is the Lebesgue measure of $\mathbf{R}$ and $f\colon \mathbf{R} \to \mathbf{R}$ is a nonnegative Borel function such that $\int_{\mathbf{R}} f(x)\,\lambda(dx) = 1$, the probability measure $\mathbf{P}$ is said to admit $f$ as a *density of probability* (or simply *density*).

In this case, we have

$$\forall A \in \mathcal{B}(\mathbf{R}), \quad \mathbf{P}(A) = \int_A f(x)\lambda(dx). \tag{5.5.2}$$

With the vocabulary of measure theory, a probability measure $\mathbf{P}$ that admits a density $f$ is said to be *absolutely continuous* with respect to the Lebesgue measure $\lambda$ and $f$ is the *Radon–Nikodym derivative* of $\mathbf{P}$ with respect to $\lambda$.

**Bernard:** I remember that from Corollary 3.9 that the product of a measure by a nonnegative measurable function defines another measure. Here, we have the particular case where $\int_{\mathbf{R}} f(x)\lambda(dx) = 1$ implies that $f\lambda$ is a probability measure.

After the cumulative distribution function, here is another function that characterizes a probability measure. There are many such functions, finally.

**Laurent:** That is correct. But you need to be very careful, Bernard. Not every probability measure admits a density. It is really a particular case.

$\mathbf{E}[\varphi(X)]$
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Theorem 5.42 allows us to characterize the density of a probability measure through its link with the cumulative distribution function.

**Theorem 5.46.** *Let $\mathbf{P}$ be a probability measure over $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ and $f\colon \mathbf{R} \to \mathbf{R}$ be a nonnegative function, integrable over $\mathbf{R}$ with respect to the Lebesgue measure $\lambda$. The two following assertions are equivalent*:

(i) *The function $f$ is a density of the probability measure $\mathbf{P}$.*
(ii) *The cumulative distribution function of $\mathbf{P}$ is*

$$F\colon x \mapsto F(x) = \int_{(-\infty,x]} f(t)\,\lambda(dt). \tag{5.5.3}$$

**Proof.** $\star$ (i)$\Rightarrow$(ii): If $f$ is the density of $\mathbf{P}$, we can apply (5.5.2) to any $A = (-\infty, x]$ with $x \in \mathbf{R}$,

$$\forall x \in \mathbf{R}, \quad \mathbf{P}\big((-\infty, x]\big) = \int_{(-\infty,x]} f(x)\lambda(dx).$$

$\star$ (ii)$\Rightarrow$(i): If (5.5.3) holds, the dominated convergence theorem implies that, for any sequence $(x_n)_{n\in\mathbf{N}}$ that tends to $+\infty$ as $n$ goes to $\infty$, we have

$$\int_{\mathbf{R}} f(x)\lambda(dx) = \lim_{n\to\infty} \int_{\mathbf{R}} f(x)\mathbf{1}_{(-\infty,x_n]}\lambda(dx) = \lim_{n\to\infty} F(x_n) = 1.$$

Then, according to Definition 5.45, we can consider the probability measure $\mathbf{Q} = f.\lambda$ defined by

$$\forall A \in \mathcal{B}(\mathbf{R}), \quad \mathbf{Q}(A) = \int_A f(t)\,\lambda(dt).$$

We observe that the cumulative distribution function of $\mathbf{Q}$ is $F$.

Then, by Theorem 5.42, we deduce that $\mathbf{Q} = \mathbf{P}$, and (5.5.2) holds. Consequently, the function $f$ is a density of $\mathbf{P}$. □

**Bernard:** If I understand well the proof and, more specifically the logical link between (5.5.2) and (5.5.3), for any nonnegative Borel function $f$ such that $\|f\|_{L^1} = 1$,

$$\begin{bmatrix} \forall x \in \mathbf{R}, \\ \mathbf{P}\big((-\infty, x)\big) = \displaystyle\int_{]-\infty,x]} f(t)\lambda(dt) \end{bmatrix} \Longleftrightarrow \begin{bmatrix} \forall A \in \mathcal{B}(\mathbf{R}), \\ \mathbf{P}(A) = \displaystyle\int_A f(x)\lambda(dx) \end{bmatrix}.$$

It looks like a miracle, doesn't it?

**Laurent:** The miracle has a name: it is called the Caratheodory theory. You do not seem to realize how powerful it is. I do not know how many times I have told you this. It looks like you do not hear me clearly. Would you be deaf, Bernard?

**Bernard:** Another point. When the probability measure $\mathbf{P}$ admits a density, one can certainly say more than the three properties of Theorem 5.43 about the distribution $F$.

I think I have my own theorem.

**Bernard's theorem** *Let* $\mathbf{P}$ *be a probability measure over* $\big(\mathbf{R}, \mathcal{B}(\mathbf{R})\big)$. *If* $\mathbf{P}$ *admits a density of probability, then its cumulative distribution function is continuous.*

**Proof.** According to (5.5.2), we have

$$\forall a \in \mathbf{R}, \quad \mathbf{P}\big(\{a\}\big) = \int_{\{a\}} f(x)\, \lambda(dx).$$

Since the Lebesgue measure has no atoms, i.e. $\lambda\big(\{x\}\big) = 0$ for all $x \in \mathbf{R}$, we can claim that $\mathbf{P}\big(\{a\}\big) = 0$.

Then, according to Proposition 5.44, the cumulative distribution function $F$ of $\mathbf{P}$ satisfies $F(a) = F(a-)$.

Finally, $F$ is continuous. □

**Laurent:** *What? You cannot be serious*, Bernard. This is true but you cannot call this a theorem. Everybody knows that any function defined as the integral of some $L^1_{loc}(\lambda)$ function is continuous!!!

And we can even say a little more. Recall Theorem 5.32. Expression (5.5.3) implies that $F$ is differentiable almost everywhere and $F' = f$ $\lambda$-a.e.

Besides, what about the uniqueness of the density, if there exists one?

**Bernard:** I think you just gave me the key, Laurent. If I assume that $f_1$, $f_2$ are two densities of a probability measure $\mathbf{P}$ whose cumulative distribution function is denoted by $F$, as you said, Theorem 5.32 implies $F' = f_1 = f_2$ $\lambda$-a.e.

Then, I can conclude that the density of a cumulative distribution function is uniquely determined up to a $\lambda$-negligible set.

**Laurent:** So, Bernard. Now, I hope you can understand that the Dirac measure does not have a density.

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**Remark.** Let us consider the Dirac probability measure $\delta_a$ at $a \in \mathbf{R}$ on $\mathcal{B}(\mathbf{R})$:

$$\forall A \in \mathcal{B}(\mathbf{R}), \quad \delta_a(A) = \begin{cases} 1 & \text{if } a \in A, \\ 0 & \text{otherwise.} \end{cases}$$

The cumulative distribution function $F$ of $\delta_a$ is given by

$$\forall x \in \mathbf{R}, \quad F(x) = \begin{cases} 0 & \text{if } x < a, \\ 1 & \text{if } x \geq a. \end{cases}$$

Consequently, the probability measure $\delta_a$ does **not** have a density.

**Bernard:** Ah, it shows that at least discrete probability measures do not have densities. I guess they are not the only ones. I really need to keep that in mind.

Now that we have defined a probability measure from its density, when it exists, we should also be able to express the expectation from the density, right?

I guess it should look like writing "$P_X(dx) = f(x)\lambda(dx)$" in the integral that defines $\mathbf{E}[X]$. But it needs to be explained rigorously.

**Laurent:** Definitely, Bernard. We will arrive at a reformulation of the transfer theorem in the particular case of real random variables whose law admits a density.

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In the case of a real-valued random variable with a density, i.e. when its law is a probability measure over $\big(\mathbf{R}, \mathcal{B}(\mathbf{R})\big)$ that admits a density, the transfer theorem (Theorem 3.19) can be stated using the density as an integral with respect to the Lebesgue measure.

**Theorem 5.47.** *Let $X$ be a real-valued random variable whose distribution $P_X$ admits a density $f_X$. And let $h\colon \mathbf{R} \to \mathbf{R}$ be a Borel function such that $\int_{\mathbf{R}} |h(x)| f_X(x)\lambda(dx) < \infty$.*

*Then, the random variable $h(X)$ belongs to $\mathcal{L}^1(\mathbf{P})$ and*

$$\mathbf{E}[h(X)] = \int_{\mathbf{R}} h(x) f_X(x)\lambda(dx).$$

**Proof.** It suffices to rewrite the proof of the transfer theorem (Theorem 3.19) in the particular case of $P_X = f_X.\lambda$. □

Thanks to Theorem 5.47, the expectation of a real-valued random variable $X$ which admits a density can be obtained: if

$\int_{\mathbf{R}} |x|\, f_X(x)\, \lambda(dx) < \infty$, then $X$ belongs to $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbf{P})$ and

$$\mathbf{E}[X] = \int_{\mathbf{R}} x\, f_X(x)\, \lambda(dx).$$

**Laurent:** Do you not see, Bernard, that the statement of Theorem 5.47 can be strengthened?

**Bernard:** I was just thinking that. If I gather all the elements, I have the following statement:

> **Theorem.** *Let $X$ be a real-valued random variable. The two following assertions are equivalent:*
>
> (i) *The law of $X$ admits a density $f_X$.*
>
> (ii) *For any Borel function $h\colon \mathbf{R} \to \mathbf{R}$ with $\int_{\mathbf{R}} |h(x)|\, f_X(x)\lambda(dx) < \infty$, we have*
>
> $$h(X) \in \mathcal{L}^1(\mathbf{P}) \quad \textit{and} \quad \mathbf{E}[h(X)] = \int_{\mathbf{R}} h(x) f_X(x)\lambda(dx).$$

This gives a particular form of the transfer theorem, as a characterization of the law of a random variable.

### 5.5.4 *Probability measure on* $\mathbf{R}^d$

**Laurent:** We have just seen the zero degree of probability, in which it appears as a particular case of measure theory. In reality, probability theory is a theory in its own right, which does not sum up as a subset of measure theory.

If one has a question about stochastic modeling, stochastic processes or random functions such as Brownian motion, a lot more than measure theory is needed.

An additional step would be to consider random vectors $X = (X_1, \ldots, X_d)$ that would allow us to study the distributions of their components with respect to each other. For instance, we aim at describing the distribution of $X_1$ with respect to the distributions of the random variables $X_2, \ldots, X_d$.

**Bernard:** It's still very mysterious to me. I suppose it will get clearer as we go on, as is always the case with math. Anyway, I can see us getting closer to my problem, which isn't in one dimension.

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**Definition 5.48.** We consider the probability space $\big(\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d), \mathbf{P}\big)$, where $\mathcal{B}(\mathbf{R}^d)$ is the Borel $\sigma$-algebra of $\mathbf{R}^d$.

The *cumulative distribution function* of $\mathbf{P}$ is the function $F\colon \mathbf{R}^d \to [0, 1]$ such that

$$\forall (x_1, \cdots, x_d) \in \mathbf{R}^d, \quad F(x_1, \ldots, x_d) = \mathbf{P}\Big(\prod_{k=1}^{d} (-\infty, x_k]\Big).$$

As in the case of $\mathbf{R}$, the distribution function determines the probability measure it represents. However, mainly because the order of $\mathbf{R}^d$ is only partial, handling the distribution function in $\mathbf{R}^d$ is less easy than in $\mathbf{R}$.

**Theorem 5.49.** *The probability measure* $\mathbf{P}$ *of the probability space* $\big(\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d), \mathbf{P}\big)$ *is characterized by its cumulative distribution function* $F$.

**Proof.** The proof is similar to that of Theorem 5.42. As a result:

⋆ the $\sigma$-algebra $\mathcal{B}(\mathbf{R}^d)$ is generated by the quadrants of the form $\prod_{k=1}^{d} (-\infty, x_k]$;

⋆ the collection of the quadrants of the form $\prod_{k=1}^{d} (-\infty, x_k]$ is a $\pi$-system. □

**Bernard:** I can see it's the same as in $\mathbf{R}$, as far as Caratheodory is concerned. See, it finally stuck in my head! I can even imagine that a distribution function in $\mathbf{R}^d$ has the same properties as in $\mathbf{R}$, nondecreasing and with the good limits, even if I understand that the nondecreasing property is less important than in $\mathbf{R}$.

Maybe we can consider, as in $\mathbf{R}$, the case where the distribution function is written as an integral with respect to the Lebesgue measure of $\mathbf{R}^d$:

$$\text{for } x \in \mathbf{R}^d,$$
$$F(x) = \int_{(-\infty, x_1) \times \ldots \times (-\infty, x_d)} f(u_1, \ldots, u_d) \lambda^{(d)}(du_1, \ldots, du_d).$$

**Laurent:** Indeed, the notion of density is a tool for calculating probabilities in $\mathbf{R}^d$, as conveniently as in $\mathbf{R}$.

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**Definition 5.50.** Let $f$ be a nonnegative Borel function on $\mathbf{R}^d$ such that $\int_{\mathbf{R}^d} f \; d\lambda^{(d)} = 1$. The probability measure $\mathbf{P}$ over the space $\left(\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d)\right)$ is said to admit the function $f$ as a density of probability if $\mathbf{P} = f.\lambda^{(d)}$, where $\lambda^{(d)}$ denotes the Lebesgue measure of $\mathbf{R}^d$, i.e. for all $A \in \mathcal{B}(\mathbf{R}^d)$,

$$\mathbf{P}(A) = \int_A f(x)\lambda^{(d)}(dx) = \int_{\mathbf{R}^d} f(x_1, \cdots, x_d) \times \mathbf{1}_A(x_1, \ldots, x_d)\lambda^{(d)}(dx_1, \ldots, dx_d). \tag{5.5.4}$$

From Expression (5.5.4), the cumulative distribution function is given by
for all $(x_1, \cdots, x_d) \in \mathbf{R}^d$,

$$F(x_1, \ldots, x_d) = \int_{(-\infty, x_1]\times\cdots\times]-\infty, x_d]} f(u_1, \ldots, u_d)\lambda^{(d)}(du_1, \cdots, du_d).$$

Moreover, this relation characterizes the existence of $f$ as a density of $\mathbf{P}$.

Theorem 5.47 also extends to a $\mathbf{R}^d$-valued random variable: for a Borel function $h \colon \mathbf{R}^d \to \mathbf{R}$ such that the random variable $h(X)$ is integrable, we have

$$\mathbf{E}[h(X)] = \int_{\mathbf{R}^d} h(x)f(x)\lambda^{(d)}(dx).$$

**Bernard:** I can see that we applied to $\mathbf{R}^d$ what we had seen in $\mathbf{R}$, changing random variables into random vectors. But I have no idea how what we saw can link the components of the vector...

**Laurent:** You are right. But, for that, you need to understand really well the construction of the integral in product spaces. It will allow us to speak of the independence of random variables, which is the very beginning of probability theory.

### 5.5.5 *Example of probability laws on* R

**Laurent:** Now that we have seen all this, you know everything about real-valued random variables.

**Bernard:** Wait a minute. Even if I'm French and used to abstract theoretical lectures, I have shown my limits... Now, I would like to get a picture of a law.

**Laurent:** Let us see toy examples, then. Sigh.

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**Constant random variable**
A random variable $X$ is said to be *constant* if there exists $c \in \mathbf{R}$ such that

$$\mathbf{P}(X = c) = 1.$$

The law $P_X$ is the *Dirac distribution* $\delta_c$. Then, we have $\mathbf{E}[X] = c$ and $\mathrm{Var}(X) = 0$.

**Uniform law**
A random variable $X$ is said to have a *uniform distribution* over the interval $[a, b] \subset \mathbf{R}$ if its law has the probability density $f\colon \mathbf{R} \to \mathbf{R}_+$, defined by

$$\forall x \in \mathbf{R}, \quad f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } x \in [a, b]; \\ 0 & \text{otherwise.} \end{cases}$$

**Exponential law**
A random variable $X$ is said to have an *exponential distribution* $\mathcal{E}(\alpha)$ ($\alpha > 0$) if its law has the probability density $f\colon \mathbf{R} \to \mathbf{R}_+$, defined by

$$\forall x \in \mathbf{R}, \quad f(x) = \begin{cases} \alpha e^{-\alpha x} & \text{if } x \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have $\mathbf{E}[X] = \frac{1}{\alpha}$ and $\mathrm{Var}(X) = \frac{1}{\alpha^2}$.

**Gamma law**
A random variable $X$ is said to have a *gamma distribution* $\gamma(p, \alpha)$ ($p > 0$ and $\alpha > 0$) if its law has the probability density $f\colon \mathbf{R} \to \mathbf{R}_+$,

defined by

$$\forall x \in \mathbf{R}, \quad f(x) = \begin{cases} \dfrac{\alpha}{\Gamma(p)}\,(\alpha x)^{p-1} e^{-\alpha x} & \text{if } x \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have $\mathbf{E}[X] = \frac{p}{\alpha}$ and $\mathrm{Var}(X) = \frac{p}{\alpha^2}$.

**Normal (or Gaussian) law**

A random variable $X$ is said to have a *normal distribution* $\mathcal{N}(m, \sigma^2)$ (with $(m, \sigma) \in \mathbf{R} \times \mathbf{R}_+$) if its law has the probability density $f\colon \mathbf{R} \to \mathbf{R}_+$, defined by

$$\forall x \in \mathbf{R}, \quad f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right].$$

Then, we have $\mathbf{E}[X] = m$ and $\mathrm{Var}(X) = \sigma^2$.

**Laurent:** I suggest you show me what you understood in the case of the exponential law, whose computations are simple.

**Bernard:** OK, let's see. Let's check at first that $f\colon x \mapsto \alpha e^{-\alpha x}\mathbf{1}_{x \geq 0}$ is a density of probability. It is clear that it is piecewise continuous, so it is a Borel function and nonnegative. It remains to prove that its integral over $\mathbf{R}$ is 1:

$$\int_{\mathbf{R}} f\, d\lambda = \int_{[0,+\infty)} \alpha\, e^{-\alpha x}\, \lambda(dx) = \left[-e^{-\alpha x}\right]_0^{+\infty} = 1.$$

Just for fun, I will compute the cumulative distribution function:

$$F\colon x \mapsto \begin{cases} 0 \text{if } x < 0 \\ \displaystyle\int_{(-\infty,x]} f\, d\lambda = \int_{[0,x]} \alpha e^{-\alpha x} \lambda(dx) = 1 - e^{-\alpha x} \text{ otherwise.} \end{cases}$$

Now, I will compute the expectation and the variance of $X$ of density $f$:

$$\mathbf{E}[X] = \int_{\mathbf{R}} X\, d\mathbf{P} = \int_{\mathbf{R}} x\, P_X(dx) = \int_{\mathbf{R}} x\, f(x)\, \lambda(dx)$$

$$= \int_{[0,+\infty)} x\,\alpha\,e^{-\alpha x}\,\lambda(dx) = \left[-x\,e^{-\alpha x}\right]_0^{+\infty} + \int_{[0,+\infty)} e^{-\alpha x}\,\lambda(dx) = \frac{1}{\alpha}.$$

and

$$\begin{aligned}\mathrm{Var}(X) &= \int_{\mathbf{R}} X^2\,d\mathbf{P} - (\mathbf{E}[X])^2 = \int_{\mathbf{R}} x^2\,f(x)\lambda(dx) - \frac{1}{\alpha^2} \\ &= \int_{[0,+\infty)} x^2\,\alpha\,e^{-\alpha x}\,\lambda(dx) - \frac{1}{\alpha^2} \\ &= \left[-x^2\,e^{-\alpha x}\right]_0^{+\infty} + 2\int_{[0,+\infty)} x\,e^{-\alpha x}\,\lambda(dx) - \frac{1}{\alpha^2} = \frac{1}{\alpha^2}.\end{aligned}$$

**Laurent:** The computations for the other laws are similar. It is not useful to do them here. The important point is to remember that the density characterized the law, and consequently it allows us to easily compute the first two moments of $X$, which are the expectation and the variance.

But be careful, young Bernard, you must absolutely not think that the law of $X$ can be deduced from knowing the expectation and the variance. It is a really common mistake. The only thing we can say is that the law is completely determined by its cumulative distribution function and, here, by its density since it exists.

Note also that the function $F$ is continuous. Thus, for all $x \in \mathbf{R}$, $\mathbf{P}(X = x) = 0$, i.e. the law $P_X$ has no atom.

*This chapter got us very far from Manolis Papadiamantis' or Harry Bannan's spheres...*

*Laurent is a bit anxious about his student. This week, they spent a lot of time on very fine details. Except perhaps for the question of probabilities, they haven't really talked about engineering issues.*

*Fortunately, he is convinced that a deep understanding of the construction of measures, tackled in the particular case of the Lebesgue measure, will enable them to be precise and efficient in their forthcoming discussions. In particular, when talking about independence between components of a random vector, they will be able to draw on some mastery of integration over a product space.*

*As for Bernard, he is really pleased with the session. The theoretical concepts of $\sigma$-algebra and measure become very concrete when considering the Borel $\sigma$-algebra on* $\mathbf{R}$ *and the Borel–Lebesgue measure, which precede the definition of the Lebesgue measure and the associated integral. He finally understands how the Lebesgue integral can be seen as a generalization of the Riemann integral. In any case, it makes it possible to integrate a wider class of functions.*

*Bernard retains the following elements:*

- *The Borel–Lebesgue measure on* $\mathbf{R}$ *extends to the Borel sets the notion of length of an interval of* $\mathbf{R}$.
- *It is necessary to complete the $\sigma$-algebra with negligible sets..., essentially to compare the Riemann and Lebesgue integrals.*
- *The set $C_c(\mathbf{R})$ of compactly supported functions is dense in $L^p(\lambda)$, for all $p \in [1, +\infty)$.*
- *As the Lebesgue integral concerns functions that are not necessarily continuous, the integration is not the inverse operation of the derivation.*
- *The Lebesgue and Riemann integrals coincide when considering Riemann-integrable functions.*
- *The constructions of these two integrals differ in their point of view. The Riemann integral is based on an approximation by stepped functions and, therefore, relies on a subdivision of the (compact) starting set of the function to be integrated. In contrast, the Lebesgue integral is based on an approximation by stepped functions and, therefore, relies on a subdivision of the (arrival) set of values taken by the function to be integrated.*
- *The construction of the Lebesgue measure naturally transposes to probability measures on* $\mathbf{R}$ *or* $\mathbf{R}^d$.
- *Any probability measure on* $\mathbf{R}$ *or* $\mathbf{R}^d$ *is characterized by its distribution function.*
- *Some probability measures have a density... but not all.*

*Bernard cannot wait for the next session, when he will finally understand the notion of independence between random variables. He feels he will be one step closer to modeling his pharmaceutical plant problem.*

## Chapter 6

# Measures on Product Spaces

*June 6th, in the early morning.*

*Laurent's gaze is immersed in the froth of his cappuccino. His thoughts wander to the Caratheodory theory, particularly around the discussions during the previous session. He knows he has to provide concrete, effective weapons to help Bernard. The latter has to convince Ann of his ability to study the environmental impact of a pharmaceutical plant. Given her personal background, Ann can only be convinced by theoretical considerations if they lead to solid, effective practical answers.*

*He plans to focus the session on the integration of functions defined on a product of measurable spaces, of the type* $E \times F$ *in which the case of* $\mathbf{R}^d$ *naturally fits. The question naturally pertains to understanding how measures on* $E$ *and* $F$ *can be used to define a measure on* $E \times F$.

*Beyond a real understanding of the Fubini theorem for multivariate integration, he is keen to show the regularizing effect of the convolution product and its consequences on the results of approximating* $L^p$ *functions by smooth functions.*

*This session is intended to open up the possibility of considering multi-dimensional quantities in modeling, whose evolution can be described by partial differential equations. On the treatment of randomness, it will enable the study of the dependence or independence between the components of a random vector.*

**Laurent:** Bernard, the problem that you explained to me at the beginning of our discussions, about the positioning of a plant, is obviously multivariate. The three space variables $x, y$, and $z$ naturally appear, but so does the time variable $t$. Thus, the functions that we are going to consider will belong to the space $\mathbf{R}_+ \times \mathbf{R}^3$.

You will see that integrating functions defined on product spaces provides one more striking illustration of the beauty of measure theory.

Remember that, in the framework of the Riemann theory, integrating a function of two variables, say $f\colon \mathbf{R}^2 \to \mathbf{R}$, means integrating with respect to one variable and then with respect to the other:

$$\int \left( \int f(x, y)\ dx \right) dy.$$

Unfortunately, such a process presents several difficulties...

**Bernard:** I never understood that precisely. Why the result would not depend on the chosen order of integration has always been a mystery to me.

**Laurent:** We will see that Lebesgue's construction provides many answers on the subject.

The theory of integration presented in the previous chapters concerns real-valued functions defined on measure spaces. Following the specific case of $\mathbf{R}^d$, we may consider functions on some product space $E \times F$, as soon as it has the structure of a measure space.

- ⋆ The first question is to define such a structure of the measure space $(E \times F, \mathcal{C}, \eta)$, where $(E, \mathcal{A}, \mu)$ and $(F, \mathcal{B}, \nu)$ are measure spaces.
- ⋆ Then, considering a function $f\colon E \times F \to \mathbf{R}_+$, is there a link between the integral $\int_{E\times F} f\, d\eta$ and the two integrals $\int_E f(x, y)\mu(dx)$ and $\int_F f(x, y)\nu(dy)$?

**Bernard:** In the framework of the Riemann integral, I retained that functions were considered sufficiently regular for the *Fubini theorem* to justify that

$$\int \left( \int f(x, y)\, dx \right) dy = \int \left( \int f(x, y)\, dy \right) dx.$$

**Laurent:** Recall that there exist functions that do not satisfy this equality. For instance, consider the function $f\colon [0,1]^2 \to \mathbf{R}$, defined by $f(0,0) = 0$ and $f(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$, for all $(x,y) \in [0,1]^2 \setminus \{(0,0)\}$.

We can see that $f$ is measurable. In order to check that, for all $n \geq 1$, we consider

$$\forall x, y \in [0,1], \quad f_n(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2 + \frac{1}{n}}.$$

For all $n \geq 1$, the function $f_n$ is measurable (because it is continuous) and thus the pointwise limit $f = \lim_{n\to\infty} f_n$ is also measurable.

For $y \neq 0$, integration by parts leads to

$$\begin{aligned}\int_0^1 f(x,y)\,dx &= \int_0^1 \frac{-1}{x^2+y^2}\,dx + \int_0^1 \frac{2x^2}{(x^2+y^2)^2}\,dx \\ &= -\int_0^1 \frac{1}{x^2+y^2}\,dx + \left[\frac{-1}{x^2+y^2}x\right]_0^1 + \int_0^1 \frac{1}{x^2+y^2}\,dx.\end{aligned}$$

Then,

$$\int_0^1 f(x,y)\,dx = \frac{-1}{1+y^2},$$

and finally

$$\int_0^1 \left(\int_0^1 f(x,y)\,dx\right) dy = -\frac{\pi}{4}.$$

By symmetry, we get

$$\int_0^1 \left(\int_0^1 f(x,y)\,dy\right) dx = \frac{\pi}{4}.$$

We conclude that the order of integration is important for the function $f$. The Fubini theorem does not seem to be valid in this case. Actually, we will see that the problem comes from the fact that *the function $f$ is not integrable on* $[0,1]^2$.

## 6.1 $\sigma$-Algebra of a Product Space and Product Measure

**Bernard:** If I understand well what you said; measure theory cleans up this question of integration on a product space. Since this is measure theory, I guess the story starts with the definition of a $\sigma$-algebra on $E \times F$, right?

Well, I'll go for it. I suppose we start from two measurable spaces $(E, \mathcal{A})$ and $(F, \mathcal{B})$. The natural idea is to consider the collection of subsets $A \times B \subset E \times F$, with $A \in \mathcal{A}$ and $B \in \mathcal{B}$, where, as usual, $A \times B = \big\{(x, y) \in E \times F : x \in A, y \in B\big\}$.

**Laurent:** It is a good starting point. But the collection $\{A \times B; A \in \mathcal{A}, B \in \mathcal{B}\}$ is not a $\sigma$-algebra...

For instance, if $\mathcal{A}$ and $\mathcal{B}$ are the Borel $\sigma$-algebras of $\mathbf{R}$, the union of the two subsets $[0,1] \times [0,1]$ and $[1,2] \times [1,2]$ does not have the form $A \times B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

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**Definition 6.1 (Product $\sigma$-algebra).** Let $(E, \mathcal{A})$ and $(F, \mathcal{B})$ be two measurable spaces.

The product space $E \times F$ can be endowed with the *product $\sigma$-algebra*, denoted by $\mathcal{A} \otimes \mathcal{B}$ and defined as

$$\mathcal{A} \otimes \mathcal{B} = \sigma(\mathcal{A} \times \mathcal{B}) = \sigma(A \times B; A \in \mathcal{A}, B \in \mathcal{B}).$$

The elements of $\mathcal{A} \otimes \mathcal{B}$ of the form $A \times B$, with $A \in \mathcal{A}$ and $B \in \mathcal{B}$, are called *measurable rectangles*.

**Remark.** Every subset $A \times B$, with $A \in \mathcal{A}$ and $B \in \mathcal{B}$, belongs to $\mathcal{A} \otimes \mathcal{B}$.

But since $\mathcal{A} \otimes \mathcal{B} \neq \mathcal{A} \times \mathcal{B}$, there are elements of $\mathcal{A} \otimes \mathcal{B}$ which cannot be written in the form $A \times B$, with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

This definition can be extended to any finite number of measure spaces $(E_1, \mathcal{A}_1), \ldots, (E_n, \mathcal{A}_n)$: the collection

$$\begin{aligned}\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n &= \sigma(\mathcal{A}_1 \times \cdots \times \mathcal{A}_n) \\ &= \sigma(A_1 \times \cdots \times A_n; A_k \in \mathcal{A}_k,\ \forall k = 1, \ldots, n)\end{aligned}$$

is called *the product $\sigma$-algebra* on the space $E_1 \times \cdots \times E_n$.

**Example.** We already know that $\mathcal{B}\big(\mathbf{R}^d\big)$ is the Borel $\sigma$-algebra of $\mathbf{R}^d$. Now, we can state that the product space $\mathbf{R}^d = \mathbf{R} \times \cdots \times \mathbf{R}$ can be equipped with the product $\sigma$-algebra $\mathcal{B}(\mathbf{R}) \otimes \cdots \otimes \mathcal{B}(\mathbf{R})$.

Actually, since these two $\sigma$-algebras are generated by the product of open intervals $(a_1, b_1) \times \cdots \times (a_d, b_d)$, where $(a_1, \ldots, a_d)$ and $(b_1, \ldots, b_d)$ are in $\mathbf{R}^d$, we have $\mathcal{B}(\mathbf{R}^d) = \mathcal{B}(\mathbf{R}) \otimes \cdots \otimes \mathcal{B}(\mathbf{R})$.

**Bernard:** This example goes a little too fast for me. Could we discuss a little more slowly the fact that the two $\sigma$-algebras $\mathcal{B}(\mathbf{R}) \otimes \cdots \otimes \mathcal{B}(\mathbf{R})$ and $\mathcal{B}(\mathbf{R}^d)$ are equal?

**Laurent:** Sure, Bernard. First, we remark that $\mathcal{B}(\mathbf{R}) \otimes \cdots \otimes \mathcal{B}(\mathbf{R})$ is the smallest $\sigma$-algebra that contains all the subsets $U_1 \times \cdots \times U_d$, where $U_1, \ldots, U_d$ are open subsets of $\mathbf{R}$.

**Bernard:** No no no. For me, this is not obvious at all, Laurent! By definition, $\mathcal{B}(\mathbf{R}) \otimes \cdots \otimes \mathcal{B}(\mathbf{R})$ is generated by the subsets $A_1 \times \cdots \times A_d$, where $A_1, \ldots, A_d$ are in $\mathcal{B}(\mathbf{R})$. But what you say is not clear, even if I guess this relies on the fact that $\mathcal{B}(\mathbf{R})$ is generated by the open subsets of $\mathbf{R}$.

**Laurent:** OK, Bernard. Let us look at the details in the case of $d = 2$.

We consider the $\sigma$-algebra $\mathcal{T}$ generated by the products $U \times V$ of the open subsets $U, V$ of $\mathbf{R}$. We want to check that $\mathcal{T} = \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R})$.

- For all the open subsets $U, V$ of $\mathbf{R}$ (which implies $U, V \in \mathcal{B}(\mathbf{R})$), the set $U \times V$ is in $\mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R})$. This leads to $\mathcal{T} \subset \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R})$ by the definition of $\mathcal{T}$.

The reverse inclusion relies on the following two points:

- We first fix any open subset $U$ of $\mathbf{R}$ and consider $\mathcal{A}_1 = \{B \in \mathcal{B}(\mathbf{R}) : U \times B \in \mathcal{T}\}$.
  We remark that $\mathcal{A}_1$ is a $\sigma$-algebra (which is not difficult to check) that contains the open subsets of $\mathbf{R}$ (by definition of $\mathcal{T}$). Thus, $\mathcal{A}_1 = \mathcal{B}(\mathbf{R})$.
  We can claim that for any open subset $U$ of $\mathbf{R}$ and any $B \in \mathcal{B}(\mathbf{R})$, $U \times B \in \mathcal{T}$.
- We fix any $B \in \mathcal{B}(\mathbf{R})$ and consider $\mathcal{A}_2 = \{A \in \mathcal{B}(\mathbf{R}) : A \times B \in \mathcal{T}\}$. We remark that $\mathcal{A}_2$ is a $\sigma$-algebra (as $\mathcal{A}_1$) which contains the open subsets of $\mathbf{R}$ (by the previous point). Thus, $\mathcal{A}_2 = \mathcal{B}(\mathbf{R})$.

We can claim that for any $A, B \in \mathcal{B}(\mathbf{R})$, $A \times B \in \mathcal{T}$.
This implies $\mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R}) \subset \mathcal{T}$.

**Bernard:** OK, I understand now that $\mathcal{B}(\mathbf{R}) \otimes \cdots \otimes \mathcal{B}(\mathbf{R})$ is generated by the subsets $U_1 \times \cdots \times U_d$, where $U_1, \ldots, U_d$ are open subsets of $\mathbf{R}$. But what about $\mathcal{B}\big(\mathbf{R}^d\big)$?

**Laurent:** Remember that we already saw that $\mathcal{B}(\mathbf{R})$ is generated by the open sets $(a, b)$, where $a, b \in \mathbf{R}$. We used the fact that any open subset of $\mathbf{R}$ can be written as a countable union of the open intervals of $\mathbf{R}$.

In the same way, considering the $\sigma$-algebra $\mathcal{T}$ generated by the product of open intervals $(a_1, b_1) \times \cdots \times (a_d, b_d)$, where $a_1, \ldots, a_d, b_1, \ldots, b_d$ are in $\mathbf{R}$, we have $\mathcal{B}\big(\mathbf{R}^d\big) = \mathcal{T}$ (it suffices to write any open subset of $\mathbf{R}^d$ as a countable union of subsets of type $(a_1, b_1) \times \cdots \times (a_d, b_d)$).

We can now conclude that $\mathcal{B}\big(\mathbf{R}^d\big) = \mathcal{T} = \mathcal{B}(\mathbf{R}) \otimes \cdots \otimes \mathcal{B}(\mathbf{R})$.

**Bernard:** I accept this last quick argument. All of this was not so trivial, was it?

Now, if I remember well the scheme of integration, we must deal with measurability questions with respect to product $\sigma$-algebras.

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Following the general construction of the integral presented in Chapter 3, the question is now to clarify the concept of measurability of functions with respect to the product $\sigma$-algebras. After that, we will be able to build a measure on $\mathcal{A} \otimes \mathcal{B}$ based on the measures on $\mathcal{A}$ and $\mathcal{B}$.

In the following, we use the usual dedicated notations (see for instance Halmos (1974, p. 141) and Rudin (1987, p. 161)).

Let $(E, \mathcal{A})$ and $(F, \mathcal{B})$ be two measure spaces. For $C \subset E \times F$, we define the following:

⋆ *x-sections*: $\forall x \in E$, $C_x = \{y \in F : (x, y) \in C\}$;
⋆ *y-sections*: $\forall y \in E$, $C^y = \{x \in E : (x, y) \in C\}$.

**Theorem 6.2.** *For any $C \in \mathcal{A} \otimes \mathcal{B}$, we have*

$$\forall x \in E, \quad C_x \in \mathcal{B} \quad \text{and} \quad \forall y \in F, \quad C^y \in \mathcal{A}.$$

**Proof.** Let $x \in E$. Consider the collection $\mathcal{C} = \{C \in \mathcal{A} \otimes \mathcal{B} : C_x \in \mathcal{B}\}$, which is clearly a $\sigma$-algebra.

For $C = A \times B$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we have $C_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A. \end{cases}$

Then, $C_x \in \mathcal{B}$ and thus $C \in \mathcal{C}$. This yields $\mathcal{A} \times \mathcal{B} \subset \mathcal{C}$.

Since $\mathcal{C}$ is a $\sigma$-algebra which contains $\mathcal{A} \times \mathcal{B}$, we have $\mathcal{A} \otimes \mathcal{B} = \sigma(\mathcal{A} \times \mathcal{B}) \subset \mathcal{C}$.

Finally, $\mathcal{C} = \mathcal{A} \otimes \mathcal{B}$, which proves the first assertion.

The case of $C^y$, with $y \in F$, is treated in the same way. □

**Bernard:** I guess that these sections will provide a rigorous way to connect the integral of a function $(x, y) \mapsto f(x, y)$ to the integrals with respect to $x$ and with respect to $y$.

**Laurent:** That is true, but you are going too fast, Bernard.

Recall the construction of the integral. Before talking about integration, we first need to talk about measurability with respect to $\mathcal{A} \otimes \mathcal{B}$ and then to define a measure.

Thanks to Theorem 6.2, we are able to connect the measurability of the function of two variables, $f : (x, y) \mapsto f(x, y)$, with the measurability with respect to each variable $x$ and $y$.

For any function $f$ defined on $E \times F$, we define

⋆ $\forall x \in E,\ f_x : y \mapsto f(x, y)$;
⋆ $\forall y \in F,\ f^y : x \mapsto f(x, y)$.

**Theorem 6.3.** *For any measurable function $f : (E \times F, \mathcal{A} \otimes \mathcal{B}) \to (G, \mathcal{G})$, we have*

$$\forall x \in E, f_x \textit{ is } \mathcal{B} - \textit{measurable} \quad \textit{and} \quad \forall y \in F, f^y \textit{ is } \mathcal{A} - \textit{measurable.}$$

**Proof.** Let $x \in E$. For all $D \in \mathcal{G}$,

$$\begin{aligned} f_x^{-1}(D) &= \{y \in F : f(x, y) \in D\} = \{y \in F : (x, y) \in f^{-1}(D)\} \\ &= \left(f^{-1}(D)\right)_x \in \mathcal{B}, \end{aligned}$$

since $f^{-1}(D) \in \mathcal{A} \otimes \mathcal{B}$, using Theorem 6.2.

The case of $f^y$, with $y \in F$, is treated in the same way. □

**Bernard:** Now that we've talked about measurability, we can finally talk about measures. Phew.

**Laurent:** Definitely. But remember that the construction of measures is a tricky aspect of the Caratheodory theory. We observed this when defining the Lebesgue measure. Brolle's book only gives the main ideas in its proof, and I think this is enough for us. But if you need more details, you can find them in Rudin's book.[1]

The following result provides a definition/characterization of the *product measure* defined on $\mathcal{A} \otimes \mathcal{B}$, from the measure $\mu$ on $\mathcal{A}$ and the measure $\nu$ on $\mathcal{B}$.

**Theorem 6.4.** *Let $\mu$ be a $\sigma$-finite measure over $(E, \mathcal{A})$ and $\nu$ be a $\sigma$-finite measure over $(F, \mathcal{B})$.*

(i) *There exists a unique measure $m$ over $(E \times F, \mathcal{A} \otimes \mathcal{B})$ such that*

$$\forall A \in \mathcal{A}, \forall B \in \mathcal{B}, \quad m(A \times B) = \mu(A)\nu(B), \tag{6.1.1}$$

*with the convention $0.\infty = 0$.*
*The measure $m$ is $\sigma$-finite and denoted by $m = \mu \otimes \nu$.*

(ii) *For all $C \in \mathcal{A} \otimes \mathcal{B}$,*

$$\mu \otimes \nu(C) = \int_E \nu(C_x)\mu(dx) = \int_F \mu(C^y)\nu(dy). \tag{6.1.2}$$

**Proof.** $\star$ In order to prove the uniqueness, we use the fact that the measurable rectangles $A \times B$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$ constitute a $\pi$-system.

We assume that (6.1.1) holds, and we consider two increasing sequences $(A_n)_{n \in \mathbf{N}}$ in $\mathcal{A}$ and $(B_n)_{n \in \mathbf{N}}$ in $\mathcal{B}$ such that $E = \bigcup_{n \in \mathbf{N}} A_n$, $F = \bigcup_{n \in \mathbf{N}} B_n$ and for all $n \in \mathbf{N}$, $\mu(A_n) < \infty$ and $\nu(B_n) < \infty$.

Setting $C_n = A_n \times B_n$ for all $n \in \mathbf{N}$, we have $E \times F = \bigcup_{n \in \mathbf{N}} C_n$. We observe that:

– for all $n \in \mathbf{N}$, we have $m(C_n) = \mu(A_n)\nu(B_n) < \infty$.

[1] See Rudin (1987, Theorem 8.6, p. 163, and Definition 8.7).

– thanks to the Caratheodory theory, the finite measure $m_n(\cdot) = m(\cdot \cap C_n)$ is characterized by its values on the $\pi$-system $\{A \times B; A \in \mathcal{A}, B \in \mathcal{B}\}$.

By increasing limits, we conclude that $m$ is characterized by its values on rectangles $A \times B$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$. This proves the uniqueness.

$\star$ In order to prove the existence, we prove that the first equality of (6.1.2) gives a $\sigma$-finite measure which satisfies Condition (i).

For $C \in \mathcal{A} \otimes \mathcal{B}$, we set $m(C) = \int_E \nu(C_x)\mu(dx)$.

To this end, we need to justify that $x \mapsto \nu(C_x)$ is $\mathcal{A}$-measurable. The argument relies on the fact that, when $\nu$ is finite, $\mathcal{G} = \{C \in \mathcal{A} \otimes \mathcal{B} : x \mapsto \nu(C_x)$ is $\mathcal{A}$ – measurable$\}$ is a monotone class which contains the rectangles $A \times B$, with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then, the monotone class lemma (Theorem 5.25) implies $\mathcal{G} = \mathcal{A} \otimes \mathcal{B}$.

We apply this result to the finite measure $\nu_n(\cdot) = \nu(\cdot \cap B_n)$, where $(B_n)_{n \in \mathbf{N}}$ is defined as above.

We can now check that $m$ is a measure and that $m(A \times B) = \mu(A)\nu(B)$, for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. This proves the existence of the product measure.

$\star$ In order to prove the second equality of (6.1.2), we prove the equality when $C$ is a rectangle.

For $C \in \mathcal{A} \otimes \mathcal{B}$, we set $m'(C) = \int_F \mu(C^y)\nu(dy)$.

Using the same arguments as for $m$, we can claim that $m'$ is a $\sigma$-finite measure which satisfies (6.1.1).

Because of the uniqueness, we deduce that $m = m'$. □

**Remark.** The assumption of $\sigma$-finiteness of the measures $\mu$ and $\nu$ is essential.

For example, if $E = F = \mathbf{R}$ and $\mathcal{A} = \mathcal{B} = \mathcal{B}(\mathbf{R})$, take the Lebesgue measure for $\mu$ and a counting measure for $\nu$.

For $C = \{(x, x); x \in \mathbf{R}\}$, we have

$$\infty = \int_E \nu(C_x)\mu(dx) \neq \int_F \mu(C^y)\nu(dy) = 0.$$

**Laurent:** Bernard, you can see that the product measure $\mu \otimes \nu$ is characterized by its values on rectangles. Morally, this fact holds because the $\sigma$-algebra $\mathcal{A} \otimes \mathcal{B}$ is generated by rectangles. Do you remember that we have already encountered something similar?

**Bernard:** Let me think. The Lebesgue measure of $\mathbf{R}$ is characterized by $\lambda([a, b]) = b - a$ and the Borel $\sigma$-algebra $\mathcal{B}(\mathbf{R})$ is generated by intervals $[a, b]$, with $a, b \in \mathbf{R}$. It is the same story.

Without entering into all the details as you said Laurent, is it really obvious that $\mathcal{G} = \{C \in \mathcal{A} \otimes \mathcal{B} : x \mapsto \nu(C_x) \text{ is } \mathcal{A} - \text{measurable}\}$ is a $\sigma$-algebra? And why does $\nu$ need to be finite at this stage?

**Laurent:** This is not clear from the simple observation of the definition of $\mathcal{G}$. But from its definition, we can claim directly that $\mathcal{G}$ is a monotone class:

– The closure under nondecreasing limits comes from the monotone convergence theorem.
– The closure under differences comes from the measure of the difference, for which $\nu$ needs to be finite.

Then, since $\mathcal{G}$ contains the measurable rectangles and since the measurable rectangles form a $\pi$-system, the monotone class lemma implies that $\mathcal{G}$ is the $\sigma$-algebra generated by the measurable rectangles. This is $\mathcal{A} \otimes \mathcal{B}$.

**Bernard:** Oh yes. I've got it. All the rest is clear enough to me. Now, what about integration with respect to a product measure? And what about the so-called Fubini theorem?

## 6.2 Multiple Integrals, Fubini Theorems and Convolution

**Laurent:** Bernard, just to check that you have understood the construction of the integral, could you recall the different steps?

**Bernard:** OK, I'll go ahead.

The construction of $\mathcal{I}_{\mu\otimes\nu}(f) = \int_{E\times F} f(x)\, \mu \otimes \nu(dx)$, for a measurable function $f\colon (E\times F, \mathcal{A}\otimes\mathcal{B}) \to \big(\mathbf{R}, \mathcal{B}(\mathbf{R})\big)$ and the product measure $\mu \otimes \nu$ on $\mathcal{A} \otimes \mathcal{B}$, relies on the following steps:

- For $f = \sum_{i=1}^{n} \alpha_i \, \mathbf{1}_{A_i}$, where $A_i \in \mathcal{A} \otimes \mathcal{B}$ and $\alpha_i \in \mathbf{R}_+$ for all $1 \leq i \leq n$,

$$\mathcal{I}_{\mu\otimes\nu}(f) = \int_{E\times F} f(x)\, \mu \otimes \nu(dx) = \sum_{i=1}^{n} \alpha_i\, \mu \otimes \nu(A_i).$$

- For all $f\colon (E \times F, \mathcal{A} \otimes \mathcal{B}) \to \big(\mathbf{R}, \mathcal{B}(\mathbf{R})\big)$ measurable and nonnegative,

$$\mathcal{I}_{\mu\otimes\nu}(f) = \sup \left\{\mathcal{I}_{\mu\otimes\nu}(\varphi);\ \varphi \text{ simple function such that } \varphi \leq f\right\}.$$

- For all $f\colon (E \times F, \mathcal{A} \otimes \mathcal{B}) \to \big(\mathbf{R}, \mathcal{B}(\mathbf{R})\big)$ measurable,

$$f \in \mathcal{L}^1(E \times F, \mu \otimes \nu) \Longleftrightarrow \mathcal{I}_{\mu\otimes\nu}(|f|) < +\infty.$$

  We set $\mathcal{I}_{\mu\otimes\nu}(f) = \mathcal{I}_{\mu\otimes\nu}(f^+) - \mathcal{I}_{\mu\otimes\nu}(f^-)$, where $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$.

**Laurent:** Good. We can now move on to the specific properties of this integral. Do you remember the basics of differential calculus?

### 6.2.1 *Chain rule*

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Let us consider the special case of the Lebesgue measure $\lambda^{(d)} = \lambda \otimes \cdots \otimes \lambda$ on $\mathbf{R}^d$.

The Jacobian determinant of the function of $d$ variables $\Phi\colon \mathbf{R}^d \to \mathbf{R}^d$ is denoted by $J\Phi(x) = \det\left[\left(\frac{\partial \Phi_j}{\partial x_i}(x)\right)_{i,j}\right]$, where $\Phi(x) = (\Phi_1(x), \ldots, \Phi_d(x))$ with $x \in \mathbf{R}^d$.

**Theorem 6.5.** *Let $U$ and $V$ be open subsets of $\mathbf{R}^d$. Let $\Phi$ be a $C^1$-diffeomorphism from $V$ onto $U$ of the Jacobian determinant $J\Phi$. Let $f\colon U \to \mathbf{R}$ be a Borel function.*

*Then, $f$ is integrable on $U$ if and only if $f \circ \Phi.|J\Phi|$ is integrable on $V$.*

*In that case, we have $\int_U f(y)\lambda^{(d)}(dy) = \int_V f \circ \Phi(x).|J\Phi(x)|\, \lambda^{(d)}(dx)$.*

**Bernard:** Fortunately, I remember the concepts of Jacobian matrix and determinant. But the fact that $\lambda^{(d)} = \lambda \otimes \cdots \otimes \lambda$ is not obvious to me.

- In Chapter 5, we defined the Lebesgue measure $\lambda$ over $\big(\mathbf{R}, \mathcal{B}(\mathbf{R})\big)$ and the Lebesgue measure $\lambda^{(d)}$ over $\big(\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d)\big)$.
- Here, we define a product $\sigma$-algebra $\mathcal{B}(\mathbf{R}) \otimes \cdots \otimes \mathcal{B}(\mathbf{R})$ on $\mathbf{R}^d$ and then a product measure $\lambda \otimes \cdots \otimes \lambda$ over $\big(\mathbf{R}^d, \mathcal{B}(\mathbf{R}) \otimes \cdots \otimes \mathcal{B}(\mathbf{R})\big)$.

I understand that $\mathcal{B}(\mathbf{R}) \otimes \cdots \otimes \mathcal{B}(\mathbf{R}) = \mathcal{B}\big(\mathbf{R}^d\big)$, and I remember that it wasn't so clear to me. And the fact that $\lambda^{(d)} = \lambda \otimes \cdots \otimes \lambda$, as claimed in Brolle's book, is mysterious.

**Laurent:** Bernard, you have done the hardest part. It suffices to check that the Lebesgue measure $\lambda^{(d)}$ satisfies condition (6.1.1) of the characterization of the product measure $\lambda \otimes \cdots \otimes \lambda$. Remember that this condition was the definition/characterization of $\lambda^{(d)}$ (see Proposition 5.4 and the remark after Proposition 5.11). This leads to $\lambda^{(d)} = \lambda \otimes \cdots \otimes \lambda$.

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The proof requires a very good mastery of the Radon–Nikodym–Lebesgue theorem. Only a few underlying ideas are given here.

We refer to Rudin (1987, pp. 153–154) for a detailed proof of Theorem 6.5.

**Idea of proof.** It suffices to prove the equality for any Borel function $f\colon U \to [0, \infty]$.

The proof relies on the two following steps:

(1) For all $E$ in the Lebesgue $\sigma$-algebra of $\mathbf{R}^d$, we can prove that

$$\lambda^{(d)}\big(\Phi(E \cap V)\big) = \int_V \mathbf{1}_E |J\Phi|\, d\lambda^{(d)},$$

using the Radon–Nikodym derivative.

(2) For all $A$ in the Lebesgue $\sigma$-algebra of $\mathbf{R}^d$, we can prove that

$$\int_U \mathbf{1}_A \, d\lambda^{(d)} = \int_V (\mathbf{1}_A \circ \Phi) |J\Phi|\, d\lambda^{(d)},$$

thanks to step (1) applied to $E = \Phi^{-1}(A)$ with the help of the characterization of the Lebesgue $\sigma$-algebra.

We deduce that the equality holds for simple functions and then for any Borel function $f\colon U \to [0,\infty]$. □

**Example.** Consider the integral $\int_D e^{x^2+y^2} \lambda^{(2)}(dx,dy)$, where $D$ is the unit disk of $\mathbf{R}^2$ centered at the origin: $D = \{(x,y) \in \mathbf{R}^2 : x^2+y^2 \le 1\}$.

Define the change of variables $\Phi\colon \mathbf{R}^2 \to \mathbf{R}^2$ by

$$(r,\theta) \mapsto (x = r\cos\theta, y = r\sin\theta).$$

Since the Jacobian determinant $J\Phi(r,\theta)$ equals $r$, we deduce from Theorem 6.5 that

$$\int_D e^{x^2+y^2} \lambda^{(2)}(dx,dy) = \int_{\{(0\le r\le 1; 0\le\theta\le 2\pi)\}} e^{r^2} r \lambda^{(2)}(dr,d\theta).$$

In the following paragraph, we will see that this integral can be computed easily thanks to the Fubini–Tonelli theorem.

**Bernard:** Oh, great! I understand at last what was said in my physics course: $dx\,dy = r\,dr\,d\theta$ — the famous conversion between Cartesian and polar coordinates.

### 6.2.2 *Fubini theorems*

The connection between (1) the integral of a function of two variables and (2) the successive integrations with respect to each variable is the object of two theorems, where the name of Fubini appears.

The first one concerns the integration of nonnegative functions.

**Theorem 6.6 (Fubini–Tonelli).** *Let $\mu$ be a $\sigma$-finite measure over $(E,\mathcal{A})$ and $\nu$ be a $\sigma$-finite measure over $(F,\mathcal{B})$.*

*Let $f\colon (E\times F, \mathcal{A}\otimes\mathcal{B}) \to \big([0,\infty], \mathcal{B}([0,\infty])\big)$ be a measurable function.*

(i) *The function* $x \mapsto \int f(x,y)\nu(dy)$ *is* $\mathcal{A}$*-measurable, and the function* $y \mapsto \int f(x,y)\mu(dx)$ *is* $\mathcal{B}$*-measurable.*
(ii) *The following equalities hold*:

$$\int_{E\times F} f(x,y)\mu\otimes\nu(dx,dy) = \int_E \left(\int_F f(x,y)\nu(dy)\right)\mu(dx) = \int_F \left(\int_E f(x,y)\mu(dx)\right)\nu(dy).$$

**Proof.** (i) For $C \in \mathcal{A}\otimes\mathcal{B}$, we consider $f = \mathbf{1}_C$.

As in the proof of Theorem 6.4, we can claim that the function

$$x \mapsto \int \underbrace{\mathbf{1}_C(x,y)}_{\mathbf{1}_{C_x}(y)}\nu(dy) = \nu(C_x) \quad \text{is } \mathcal{A} - \text{measurable}$$

and the function

$$y \mapsto \int \underbrace{\mathbf{1}_C(x,y)}_{\mathbf{1}_{C^y}(x)}\mu(dy) = \mu(C^y) \quad \text{is } \mathcal{B} - \text{measurable.}$$

By linearity, we deduce that (i) holds for any nonnegative simple function and, by increasing limits, for any nonnegative measurable function.

(ii) For $f = \mathbf{1}_C$, where $C \in \mathcal{A}\otimes\mathcal{B}$, the equality to be proved is expressed as

$$\mu\otimes\nu(C) = \int_E \nu(C_x)\mu(dx) = \int_F \mu(C^y)\nu(dy),$$

which is exactly the characterization of $\mu\otimes\nu$.

By linearity, we deduce that the equality holds for any nonnegative simple function. By increasing limits, the equality holds for any nonnegative measurable function. □

**Bernard:** Wonderful. I admire the power of this theory of integration, in which everything seems to fit together in a natural way. This is another obscure point that is becoming clear to me.

I now understand why we can integrate a function of two variables, successively integrating with respect to each of the variables.

**Laurent:** I understand your enthusiasm, but do not go so fast. We have only treated here the nonnegative functions. We still have to talk about integrable functions, and this is the object of the second Fubini theorem.

In the second theorem, we extend to integrable functions with respect to the product measure the results obtained for nonnegative functions.

**Theorem 6.7 (Fubini–Lebesgue).** *Let $\mu$ be a $\sigma$-finite measure over $(E, \mathcal{A})$ and $\nu$ be a $\sigma$-finite measure over $(F, \mathcal{B})$.*

*Let $f\colon (E \times F, \mathcal{A} \otimes \mathcal{B}) \to \big(\mathbf{R}, \mathcal{B}(\mathbf{R})\big)$ be a function integrable with respect to $\mu \otimes \nu$* (i.e. $f \in \mathcal{L}^1(\mu \otimes \nu)$)*. Then, we have the following:*

(i) *For $\mu$-almost every $x$, the function $y \mapsto f(x, y)$ is in $\mathcal{L}^1(\nu)$*
*For $\nu$-almost every $y$, the function $x \mapsto f(x, y)$ is in $\mathcal{L}^1(\mu)$.*

(ii) *The functions $x \mapsto \int f(x, y)\nu(dy)$ and $y \mapsto \int f(x, y)\mu(dx)$ are well defined (except on a set of zero measure) and are, respectively, in $\mathcal{L}^1(\mu)$ and $\mathcal{L}^1(\nu)$.*

(iii) *The following equalities hold:*

$$\int_{E\times F} f(x, y)\, \mu \otimes \nu(dx, dy) = \int_E \left( \int_F f(x, y)\nu(dy) \right) \mu(dx)$$
$$= \int_F \left( \int_E f(x, y)\mu(dx) \right) \nu(dy).$$

**Bernard:** I guess it is always the same story to obtain results for integrable functions starting from results for nonnegative functions. It certainly suffices to decompose real-valued functions as $f = f^+ - f^-$, where $f^+ = \max(f, 0)$ is the positive part of $f$ and $f^- = -\min(f, 0)$ is its negative part, and apply Theorem 6.6.

**Laurent:** I see that you have acquired the right reflexes. But before that, we need to consider $|f| = f^+ + f^-$.

**Proof.** (i) We apply Theorem 6.6 to $|f|$, which is measurable and nonnegative:

$$\int_E \left( \int_F |f(x,y)|\nu(dy) \right) \mu(dx) = \int_{E\times F} |f| \, d(\mu \otimes \nu) < +\infty.$$

We deduce that

$$\int_F |f(x,y)|\nu(dy) < +\infty \quad \mu - \text{a.e.}$$

In the same way,

$$\int_E |f(x,y)|\mu(dx) < +\infty \quad \nu - \text{a.e.}$$

(ii) We decompose $f = f^+ - f^-$, with $f^+ = \max(f,0)$ and $f^- = -\min(f,0)$.

For $\mu$-almost every $x$, we have

$$\int_F f(x,y)\nu(dy) = \int_F f^+(x,y)\,\nu(dy) - \int_F f^-(x,y)\nu(dy).$$

Thus, $x \mapsto \int_F f(x,y)\nu(dy)$ is well defined, except on a set of zero measure.

Moreover,

$$\int_E \left| \int_F f(x,y)\nu(dy) \right| \mu(dx) \le \int_E \left( \int_F |f(x,y)|\nu(dy) \right) \mu(dx)$$
$$= \int_{E\times F} |f| \, d(\mu \otimes \nu).$$

Hence, $x \mapsto \int_F f(x,y)\nu(dy)$ is in $\mathcal{L}^1(\mu)$.

The function $y \mapsto \int_E f(x,y)\mu(dx)$ is treated in the same way.

(iii) By Theorem 6.6, the equality to be proved holds for the nonnegative measurable functions $f^+$ and $f^-$. Then, by additivity, we deduce that the equality holds for $f = f^+ - f^-$. □

**Example (*Continued*).** From the Fubini–Tonelli Theorem 6.6 (the function is nonnegative), we have

$$\int_{\{(0\leq r\leq 1;0\leq\theta\leq 2\pi)\}} e^{r^2} r\lambda^{(2)}(dr, d\theta) = \int_{[0,1]}\left(\int_{[0,2\pi)} e^{r^2} r\lambda(d\theta)\right)\lambda(dr).$$

Then, using the primitive function of $r \mapsto r\ e^{r^2}$,

$$\int_D e^{x^2+y^2}\lambda^{(2)}(dx, dy) = 2\pi\int_{[0,1]} r\,e^{r^2}\lambda(dr) = 2\pi\left[\frac{1}{2}\,e^{r^2}\right]_0^1 = \pi(e-1).$$

**Laurent:** An important consequence of the Fubini–Lebesgue theorem is the famous integration-by-parts result that you certainly used for the Riemann integral. But I warn you: the integration by parts needs integration conditions to hold...

**Theorem 6.8 (Integration by parts).** *For $\alpha, \beta \in \mathbf{R}$ with $\alpha < \beta$ and $f, g$ two functions in $L^1(\lambda|_{[\alpha,\beta]})$, we consider*

$$\forall x \in [\alpha, \beta], \quad F(x) = \int_{[\alpha,x]} f(t)\lambda(dt) \quad \text{and}$$

$$G(x) = \int_{[\alpha,x]} g(t)\lambda(dt).$$

*Then, for all $a, b \in (\alpha, \beta)$ with $a < b$, denoting $[GF]_a^b = G(b)F(b) - G(a)F(a)$,*

$$\int_{[a,b]} G(t)f(t)\lambda(dt) = [GF]_a^b - \int_{[a,b]} g(t)F(t)\lambda(dt). \tag{6.2.1}$$

**Proof.** $\star$ Using the Fubini–Tonelli theorem, we have

$$\int_{[\alpha,\beta]}\int_{[\alpha,\beta]} |f(t)g(u)|\lambda(dt)\lambda(du) = \|f\|_{L^1}\|g\|_{L^1} < +\infty.$$

Hence, the function $(t,u) \mapsto |f(t)g(u)|$ is integrable on $[\alpha,\beta]^2$ $\left(\in \mathcal{L}^1\big(\lambda^{(2)}|_{[\alpha,\beta]^2}\big)\right)$.

$\star$ For any $(u,t)$ in $[\alpha,\beta]^2$, we can claim that

$$\mathbf{1}_{[\alpha,t]}(u)\mathbf{1}_{[\alpha,x]}(t) = \mathbf{1}_{[\alpha,x]}(u)\big(\mathbf{1}_{[\alpha,x]}(t) - \mathbf{1}_{[\alpha,u)}(t)\big). \tag{6.2.2}$$

It suffices to check that the two functions are equal to 1 if and only if $\alpha \leq u \leq t \leq x$.

$\star$ For any $x$ in $[\alpha,\beta]$, we have

$$\int_{[\alpha,x]} G(t)f(t)\lambda(dt) = \int_{[\alpha,x]} \left(\int_{[\alpha,t]} g(u)\lambda(du)\right) f(t)\lambda(dt),$$

which can be rewritten as

$$\int_{[\alpha,x]} G(t)f(t)\lambda(dt) = \int_{[\alpha,\beta]} \left(\int_{[\alpha,\beta]} g(u)\mathbf{1}_{[\alpha,t]}(u)\lambda(du)\right) \times\ f(t)\mathbf{1}_{[\alpha,x]}(t)\lambda(dt).$$

Thanks to (6.2.2), we can write

$$\int_{[\alpha,x]} G(t)f(t)\lambda(dt) = \int_{[\alpha,\beta]}\int_{[\alpha,\beta]} f(t)g(u) \times \mathbf{1}_{[\alpha,x]}(u)\big(\mathbf{1}_{[\alpha,x]}(t) - \mathbf{1}_{[\alpha,u)}(t)\big)\lambda(du)\lambda(dt).$$

We can claim that the three functions

$$(t,u) \mapsto \left|\mathbf{1}_{[\alpha,t]}(u)\mathbf{1}_{[\alpha,x]}(t)f(t)g(u)\right|,$$
$$(t,u) \mapsto \left|\mathbf{1}_{[\alpha,x]}(u)\mathbf{1}_{[\alpha,x]}(t)f(t)g(u)\right|$$
$$\text{and} \quad (t,u) \mapsto \left|\mathbf{1}_{[\alpha,x]}(u)\mathbf{1}_{[\alpha,u)}(t)f(t)g(u)\right|$$

are integrable since they are bounded by the integrable function $(t,u) \mapsto |f(t)g(u)|$.

Hence, we have

$$\int_{[\alpha,x]} G(t)f(t)\lambda(dt)$$

$$= \int_{[\alpha,x]} \int_{[\alpha,x]} f(t)g(u)\lambda(du)\lambda(dt)$$

$$- \int_{[\alpha,\beta]} \int_{[\alpha,\beta]} \mathbf{1}_{[\alpha,x]}(u)g(u)\mathbf{1}_{[\alpha,u]}(t)f(t)\lambda(du)\lambda(dt).$$

Using the Fubini–Lebesgue theorem, this leads to

$$\int_{[\alpha,x]} G(t)f(t)\lambda(dt) = G(x)F(x) - \int_{[\alpha,x]} g(u)F(u)\lambda(du).$$

$\star$ Let $a, b \in (\alpha, \beta)$, with $a < b$. Using $[a,b] = [\alpha,b] \setminus [\alpha,a)$, by subtracting the previous expression for $x = a$ to the expression for $x = b$, we get (6.2.1). $\square$

**Laurent:** You certainly noted that the word *derivative* is not mentioned in Theorem 6.8, contrary to the result you certainly learned in the context of the Riemann integral. However, remember that the fundamental theorem of calculus allows us to say that the functions $f$ and $g$ are the *almost derivatives* of $F$ and $G$.

**Bernard:** I see. Is it because $F$ and $G$ are not differentiable at any point, and $f$ and $g$ are not necessarily continuous that the integrability conditions are so important?

**Laurent:** Not only, Bernard.

For instance, the formula $\int_{[0,1]} uv'\,d\lambda = [uv] - \int_{[0,1]} u'v\,d\lambda$ does not hold for the functions $x \mapsto u(x) = x^2 \sin(1/x^3)$ (extended at 0 by 0) and $x \mapsto v'(x) = 1$ on $[0,1]$.

**Bernard:** I see. There are integration problems in 0 in the two integrals. I am now convinced that the integration condition is crucial in this theorem.

### 6.2.3 *Convolution and regularizing sequences*

**Laurent:** Another important consequence of the Fubini Theorems is the study of the convolution of two functions. As an engineer, Bernard, I guess that you have encountered this notion many times during your studies and afterward.

**Bernard:** I can confirm. The convolution between the functions $f$ and $g$ was defined by a formula like $x \mapsto \int f(x-y)\, g(y)\, dy$. Since we need to be mathematically rigorous here, I guess that the integrability questions are important again. But I have to confess that mathematical rigor was almost never mentioned during my engineering studies.

**Laurent:** How revolting! Without mathematical rigor, you can say anything. And that is why you came to me, Bernard... Let me remind you that your less-than-rigorous engineering education has not prepared you for the advanced mathematics you need to study your pharmaceutical plant's waste problem.

For instance, did you have any idea that the convolution allows to approximate $L^p(\lambda)$ by infinitely differentiable functions with compact support?

**Bernard:** Please calm down, Laurent. I am now convinced of the importance of rigor.

If I remember correctly, I have mostly encountered convolution when it was about optics or more generally about physical quantities observed through a sensor.

#### 6.2.3.1 *Convolution of functions*

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**Definition 6.9 (Convolution of two functions).** Let $f$ and $g$ be two measurable functions on $\mathbf{R}^d$ and denote by $\lambda^{(d)}$ the Lebesgue measure on $\mathbf{R}^d$.

If $\int_{\mathbf{R}^d} |f(x-y)g(y)|\lambda^{(d)}(dy) < \infty$, we define the *convolution of $f$ and $g$* as the function denoted by $f * g$ such that

$$\forall x \in \mathbf{R}^d, \quad f * g(x) = \int_{\mathbf{R}^d} f(x-y)g(y)\lambda^{(d)}(dy).$$

In the following, we study conditions ensuring $f * g$ is well defined and integrable.

**Laurent:** Do you think that we can consider the convolution as a commutative product of functions? More precisely, does the equality $f * g = g * f$ hold?

**Bernard:** I need to be rigorous... The first question is: if $f * g(x)$ is well defined, can we say that $g * f(x)$ is well defined?

The question can be reformulated as: if $\int_{\mathbf{R}^d} |f(x - y) g(y)| \lambda^{(d)}(dy) < \infty$, can we say that $\int_{\mathbf{R}^d} |g(x-y)f(y)|\lambda^{(d)}(dy) < \infty$?

The answer seems to come from the change of variable theorem, setting $z = x - y$. OK, let me be rigorous. Considering the change of variable $\Phi(y) = x - y$, the function $\Phi$ is a $C^1$-diffeomorphism of $\mathbf{R}^d$ such that $|J\Phi(y)| = 1$ for all $y \in \mathbf{R}^d$. Then, from Theorem 6.5, we can conclude that

$$\int_{\mathbf{R}^d} |f(x-y)g(y)|\lambda^{(d)}(dy) < \infty \Leftrightarrow \int_{\mathbf{R}^d} |g(x-y)f(y)|\lambda^{(d)}(dy) < \infty,$$

and that $f * g(x) = g * f(x)$.

But is this equality sufficient to claim that the convolution is a product of functions? Is it possible to give a space of functions where the convolution is well defined?

**Laurent:** Actually, the following result shows that the convolution is a product in $L^1\big(\lambda^{(d)}\big)$. You will see that it is merely a question of interchanging integrations with respect to $x$ and $y$.

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**Theorem 6.10.** *For $f, g \in L^1\big(\lambda^{(d)}\big)$, $f * g(x)$ is well defined* $\lambda^{(d)}$-a.e.

*Moreover, $f * g \in L^1\big(\lambda^{(d)}\big)$ and $\|f * g\|_{L^1} \leq \|f\|_{L^1}\|g\|_{L^1}$.*

**Proof.** $\star$ From the Fubini–Tonelli theorem (Theorem 6.6),

$$\int_{\mathbf{R}^d}\left(\int_{\mathbf{R}^d} |f(x-y)|\,|g(y)|\lambda^{(d)}(dy)\right)\lambda^{(d)}(dx)$$
$$= \int_{\mathbf{R}^d}\left(\int_{\mathbf{R}^d} |f(x-y)|\,|g(y)|\lambda^{(d)}(dx)\right)\lambda^{(d)}(dy)$$

$$= \int_{\mathbf{R}^d} |g(y)| \left( \int_{\mathbf{R}^d} |f(x-y)| \lambda^{(d)}(dx) \right) \lambda^{(d)}(dy)$$

$$= \int_{\mathbf{R}^d} |g(y)| \lambda^{(d)}(dy) \int_{\mathbf{R}^d} |f(x)| \lambda^{(d)}(dx) < +\infty.$$

Hence, $\int_{\mathbf{R}^d} |f(x-y)|\, |g(y)| \lambda^{(d)}(dy) < +\infty$ for $\lambda^{(d)}$-almost every $x$. Finally, $f * g(x)$ is well defined for $\lambda^{(d)}$-almost every $x \in \mathbf{R}^d$.

$\star$ In order to consider the integrability of $f * g$, we arbitrarily extend the definition of the function to the whole $\mathbf{R}^d$ (e.g. we can give the value 0 where $f * g$ is not defined).

We conclude with

$$\int_{\mathbf{R}^d} |f * g(x)| \lambda^{(d)}(dx) \leq \underbrace{\int_{\mathbf{R}^d} \left( \int_{\mathbf{R}^d} |f(x-y)|\, |g(y)| \lambda^{(d)}(dy) \right) \lambda^{(d)}(dx)}_{= \|f\|_{L^1} \|g\|_{L^1} < +\infty}.$$

$\square$

**Laurent:** With a little extra technique, this result can be easily improved, for $f \in L^1\big(\lambda^{(d)}\big)$ and $g \in L^p\big(\lambda^{(d)}\big)$ with $p \in [1, \infty]$, using the Hölder inequality.

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**Theorem 6.11.** *Let $f \in L^1\big(\lambda^{(d)}\big)$ and $g \in L^p\big(\lambda^{(d)}\big)$, with $p \in [1, \infty]$.*

*Then, for $\lambda^{(d)}$-almost every $x \in \mathbf{R}^d$, $f * g(x)$ is well defined. Moreover, $f * g \in L^p\big(\lambda^{(d)}\big)$ and $\|f * g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}$.*

**Proof.** $\star$ The case $p = 1$ has already been treated in Theorem 6.10.

$\star$ The case $p = \infty$ corresponds to a function $g$ bounded almost everywhere. In that case, the function $f * g$ is obviously well defined and bounded by $\|f\|_{L^1} \|g\|_{L^\infty}$.

$\star$ For the case of $1 < p < \infty$, we consider $q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, writing $|f(x-y)| = |f(x-y)|^{1/p} |f(x-y)|^{1/q}$, the

Hölder inequality leads to

$$\int_{\mathbf{R}^d} |f(x-y)|^{1/p}\,|g(y)|\,|f(x-y)|^{1/q}\lambda^{(d)}(dy)$$
$$\leq \left(\int_{\mathbf{R}^d} |f(x-y)|\,|g(y)|^p\lambda^{(d)}(dy)\right)^{1/p}$$
$$\times \left(\int_{\mathbf{R}^d} |f(x-y)|\lambda^{(d)}(dy)\right)^{1/q},$$

Since $f$ and $|g|^p$ are in $L^1(\lambda^{(d)})$, $\int_{\mathbf{R}^d} |f(x-y)|\,|g(y)|^p\lambda^{(d)}(dy) < \infty$ for $\lambda^{(d)}$-almost every $x \in \mathbf{R}^d$ by Theorem 6.10.

Hence, we can claim that $f * g(x)$ is well defined for $\lambda^{(d)}$-almost every $x \in \mathbf{R}^d$.

Moreover, integrating with respect to $x$ the inequality

$$|f * g(x)|^p \leq \left(\int_{\mathbf{R}^d} |f(x-y)|\,|g(y)|^p\lambda^{(d)}(dy)\right)\left(\|f\|_{L^1}\right)^{p/q},$$

and using the Fubini–Tonelli theorem to switch the integrations with respect to $x$ and $y$, we get $\left(\|f * g\|_{L^p}\right)^p \leq \left(\|g\|_{L^p}\right)^p \|f\|_{L^1}\left(\|f\|_{L^1}\right)^{p/q}$.

The result follows. □

**Bernard:** OK Laurent, we have defined a kind of commutative product of functions in $L^1(\lambda^{(d)})$. But when we use the word "product", we expect certain properties to hold. In particular, we expect that $(f + a\,g) * h = f * h + a\,g * h$ for all $f, g, h \in L^1(\lambda^{(d)})$ and $a \in \mathbf{R}$.

This point seems to be obvious from the linearity of the integral.

**Laurent:** You are right, Bernard. Moreover, a product usually has an identity element, i.e. a function $h$ such that $f * h = h * f$ for all $f \in L^1(\lambda^{(d)})$.

Rather than answering the question of the existence of such an identity element, we will see that we can approximate it.

**Bernard:** Are there some important applications of convolution? You mentioned that it allows to approximate $L^p(\lambda)$ by infinitely differentiable functions...

**Laurent:** There are indeed many applications, for instance when we want to know the law of the sum of two *independent* random variables. But we first look at the regularizing role of convolution.

Remember that the space $C_c\big(\mathbf{R}^d\big)$ of continuous functions $\mathbf{R}^d \to \mathbf{R}$ with compact support is dense in $L^p\big(\lambda^{(d)}\big)$ (Theorem 5.15). The convolution by smooth functions allows us to improve this result: the space $C_c^\infty\big(\mathbf{R}^d\big)$ of infinitely differentiable functions with compact support (which is obviously included in $C_c\big(\mathbf{R}^d\big)$) is dense in $L^p\big(\lambda^{(d)}\big)$.

Before studying this result, let us study the regularity of the convolution of a function in $L^1\big(\lambda^{(d)}\big)$ with a function in $C_c^\infty\big(\mathbf{R}^d\big)$.

**Theorem 6.12.** *Let $f \in L^p\big(\lambda^{(d)}\big)$ and $\varphi\colon \mathbf{R}^d \to \mathbf{R}$ be an infinitely differentiable function with compact support* (i.e. $\varphi \in C_c^\infty(\mathbf{R}^d)$).

*Then, $f * \varphi(x)$ is defined for all $x \in \mathbf{R}^d$, $f * \varphi \in C^\infty\big(\mathbf{R}^d\big)$ and for all $n \geq 1$, $D^n(f * \varphi) = f * D^n(\varphi)$, where $D^n$ denotes any differentiation operator of order $n$.*

**Proof.** For $f \in L^p\big(\lambda^{(d)}\big)$ and $\varphi \in C_c^\infty(\mathbf{R}^d)$, by the Hölder inequality and the fact that $\varphi$ has a compact support, the expression $f * \varphi(x) = \int_{\mathbf{R}^d} f(x-y)\varphi(y)\lambda^{(d)}(dy)$ is well defined for all $x \in \mathbf{R}^d$.

The relation $D^n(f * \varphi) = f * D^n(\varphi)$ for all $n \geq 1$ is a direct consequence, by induction, of the remark following Theorem 4.10 about the differentiation under the integral symbol.

This proves that $f * \varphi \in C^\infty\big(\mathbf{R}^d\big)$. □

6.2.3.2 *Approximation of the identity*

According to Theorem 6.10, the convolution is a commutative operation in $L^1\big(\lambda^{(d)}\big)$. The identity element of this operation would be a function $h$ in $L^1\big(\lambda^{(d)}\big)$ such that

$$\forall f \in L^1\big(\lambda^{(d)}\big), \quad f * h = h * f = f.$$

Without answering the question of the existence of such a function $h$, we define an *approximation of the identity* as a family of functions $(\eta_\varepsilon)_{\varepsilon>0}$ such that

$$\forall f \in L^1\big(\lambda^{(d)}\big), \quad \eta_\varepsilon * f \to f \quad \text{as } \varepsilon \to 0.$$

Several authors present different concepts, which are related to the approximation of the identity: *Dirac sequences*, *approximations of the Dirac measure*, *regularizing sequences* and *mollifiers* (see for instance Lang (1993b, p. 227), Le Gall (2022, p. 97) and Rudin (1991, p. 173)).

**Definition 6.13 (Dirac sequence).** A sequence $(\varphi_n)_{n\in\mathbf{N}}$ of continuous functions $\mathbf{R}^d \to \mathbf{R}$ is *a Dirac sequence on* $\mathbf{R}^d$ if the following three conditions hold:

(i) For all $n \in \mathbf{N}$ and for all $x \in \mathbf{R}^d$, $\varphi_n(x) \geq 0$.
(ii) For all $n \in \mathbf{N}$, $\int_{\mathbf{R}^d} \varphi_n(x)\lambda^{(d)}(dx) = 1$.
(iii) For all $\delta > 0$, $\lim_{n\to\infty} \big( \int_{\{|x|>\delta\}} \varphi_n(x)\lambda^{(d)}(dx)\big) = 0$.

A Dirac sequence can also be called an *approximation of the Dirac measure*.

When the third point is substituted with the stronger condition that:

(iii′) for all $n \in \mathbf{N}$, the support of $\varphi_n$ is included in a ball $B(0, \delta_n)$ and $\delta_n$ converges to 0 as $n \to \infty$,

the sequence $(\varphi_n)_{n\in\mathbf{N}}$ is called a *Dirac sequence with shrinking support*.

Such a sequence can be defined from a function $\varphi\colon \mathbf{R}^d \to \mathbf{R}$ that is continuous or infinitely differentiable, with compact support, and such that $\int_{\mathbf{R}^d} \varphi(x)\lambda^{(d)}(dx) = 1$.

For all $n \in \mathbf{N}$, we set $\varphi_n(x) = n^d\, \varphi(nx)$ for all $x \in \mathbf{R}^d$.

A sequence built in this way is called a *regularizing sequence*.

**Example.** Starting from the function $x \mapsto \varphi(x) = c \exp\left(-\frac{1}{1-|x|^2}\right) \mathbf{1}_{\{|x|<1\}}$, where $c$ is such that $\int_{\mathbf{R}^d} \varphi(x)\lambda^{(d)}(dx) = 1$, we obtain a regularizing sequence $(\varphi_n)_{n\in\mathbf{N}}$ of functions in $C_c^\infty\big(\mathbf{R}^d\big)$.

**Bernard:** Can we check precisely if such a regularizing sequence provides an approximation of the identity?

**Laurent:** Of course, Bernard. It is not obvious at all. We need to check if for any $f \in L^1\big(\lambda^{(d)}\big)$, we have $\varphi_n * f \to f$ as $n \to \infty$.

Well... obviously, there exist several statements for the approximation of the identity, depending on the considered type of convergence, the assumptions on the considered class of functions and the precise conditions on the Dirac sequence.

I remember some precise statements by Lang and Le Gall[2] for the uniform convergence, but I know that Henri Brolle follows the same ideas in his book *Measure Theory*.

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**Theorem 6.14.** *Let $(\varphi_n)_{n\in\mathbf{N}}$ be a Dirac sequence on $\mathbf{R}^d$. Considering a continuous and bounded function $f\colon \mathbf{R}^d \to \mathbf{R}$, the sequence of functions $\big(\varphi_n * f\big)_{n\in\mathbf{N}}$ converges to $f$ uniformly on any compact subset of $\mathbf{R}^d$.*

**Proof.** Since $\varphi_n$ is in $L^1\big(\lambda^{(d)}\big)$ and $f$ bounded, $\varphi_n * f(x)$ is well defined for any $x \in \mathbf{R}^d$.

We fix a compact subset $K$ of $\mathbf{R}^d$ and $\varepsilon > 0$. For any $n \in \mathbf{N}$, $x \in K$ and $\delta > 0$, we decompose

$$\begin{aligned}\varphi_n * f(x) - f(x) &= \int_{|y|\le\delta} \big(f(x-y) - f(x)\big)\varphi_n(y)\lambda^{(d)}(dy) \\ &\quad + \int_{|y|>\delta} \big(f(x-y) - f(x)\big)\varphi_n(y)\lambda^{(d)}(dy).\end{aligned}$$

[2]See Lang (1993b, Theorems 3.1, p. 228) and Le Gall (2022, Proposition 5.8, p. 98).

Since $f$ is uniformly continuous on any compact, for any compact subset $K_1$ of $\mathbf{R}^d$, there exists $0 < \delta < 1$ such that

$$\forall x, z \in K_1 : |z - x| \leq \delta, \quad |f(z) - f(x)| \leq \varepsilon.$$

Choosing the compact $K_1 = \overline{\bigcup_{u \in K} \overline{B}(u, 1)}$ (where $\overline{B}(u, 1)$ is the closed ball centered in $u$ and of radius 1) yields

$$\forall x \in K,\ \forall y \in \mathbf{R}^d : |y| \leq \delta, \quad |f(x - y) - f(x)| \leq \varepsilon.$$

Consequently, by Condition (ii) of Definition 6.13, we have

$$\left| \int_{|y| \leq \delta} \big(f(x-y) - f(x)\big) \varphi_n(y) \lambda^{(d)}(dy) \right| \leq \varepsilon \int_{|y| \leq \delta} \varphi_n(y) \lambda^{(d)}(dy) \leq \varepsilon.$$

Furthermore, using the boundedness of $f$, we have

$$\left| \int_{|y| > \delta} \big(f(x-y) - f(x)\big) \varphi_n(y) \lambda^{(d)}(dy) \right|$$
$$\leq 2\|f\|_\infty \int_{|y| > \delta} \varphi_n(y) \lambda^{(d)}(dy).$$

Using Condition (iii) of Definition 6.13, we conclude that there exists $n_0$ such that

$$\forall n \geq n_0, \forall x \in K, \quad |\varphi_n * f(x) - f(x)| \leq 2\varepsilon.$$

□

**Laurent:** For the convergence in $L^1$ (more generally in $L^p$), Henri Brolle chooses the same assumptions as Le Gall.[3] But I remember different statements by Lang[4] for a larger class of functions $f$ but for a Dirac sequence with shrinking support.

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[3] See Le Gall (2022, Proposition 5.8, p. 98).

[4] See Lang (1993b, Theorem 3.2 and Corollary 3.4, pp. 234–235)

**Theorem 6.15.** *Let $(\varphi_n)_{n\in\mathbf{N}}$ be a Dirac sequence on $\mathbf{R}^d$. Considering $f \in L^p(\lambda^{(d)})$ with $p \in [1,\infty)$, the sequence $(\varphi_n * f)_{n\in\mathbf{N}}$ converges to $f$ in $L^p$.*

**Proof.** According to Theorem 6.11, since $\varphi_n \in L^1(\lambda^{(d)})$ and $f \in L^p(\lambda^{(d)})$, $\varphi_n * f$ is well defined $\lambda^{(d)}$-a.e. and belongs to $L^p(\lambda^{(d)})$.

$\star$ For $\varepsilon > 0$, according to Theorem 5.15, there exists a continuous function $g$ with compact support (i.e. $g \in C_c(\mathbf{R}^d)$) such that $\|f - g\|_{L^p} < \varepsilon$.

Since the function $z \mapsto |z|^p$ is convex and $y \mapsto \varphi_n(x-y)$ is a probability density, the Jensen inequality implies, for any $n \in \mathbf{N}$,

$$\begin{aligned}
&\int_{\mathbf{R}^d} |\varphi_n * f(x) - \varphi_n * g(x)|^p \lambda^{(d)}(dx) \\
&\quad = \int_{\mathbf{R}^d} \left| \int_{\mathbf{R}^d} \varphi_n(x-y)\big(f(y) - g(y)\big)\lambda^{(d)}(dy) \right|^p \lambda^{(d)}(dx) \\
&\quad \leq \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} \varphi_n(x-y)\left|f(y) - g(y)\right|^p \lambda^{(d)}(dy)\lambda^{(d)}(dx).
\end{aligned}$$

Then, by the Fubini–Tonelli theorem, for any $n \in \mathbf{N}$,

$$\begin{aligned}
&\int_{\mathbf{R}^d} |\varphi_n * f(x) - \varphi_n * g(x)|^p \lambda^{(d)}(dx) \\
&\quad \leq \int_{\mathbf{R}^d} \left|f(y) - g(y)\right|^p \left( \int_{\mathbf{R}^d} \varphi_n(x-y)\lambda^{(d)}(dx) \right) \lambda^{(d)}(dy) \\
&\quad = \int_{\mathbf{R}^d} \left|f(y) - g(y)\right|^p \lambda^{(d)}(dy).
\end{aligned}$$

This leads to $\|\varphi_n * f - \varphi_n * g\|_{L^p} < \varepsilon$ for any $n \in \mathbf{N}$.

$\star$ Then, we study $\|\varphi_n * g - g\|_{L^p}$ for any $n \in \mathbf{N}$ using the previous arguments:

$$\begin{aligned}
&\int_{\mathbf{R}^d} |\varphi_n * g(x) - g(x)|^p \lambda^{(d)}(dx) \\
&\quad = \int_{\mathbf{R}^d} \left| \int_{\mathbf{R}^d} \varphi_n(y)\big(g(x-y) - g(x)\big)\lambda^{(d)}(dy) \right|^p \lambda^{(d)}(dx)
\end{aligned}$$

$$\leq \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \varphi_n(y) \big| g(x-y) - g(x) \big|^p \lambda^{(d)}(dy) \lambda^{(d)}(dx)$$
$$\leq \int_{\mathbf{R}^d} \varphi_n(y) \left( \int_{\mathbf{R}^d} \big| g(x-y) - g(x) \big|^p \lambda^{(d)}(dx) \right) \lambda^{(d)}(dy).$$

⋆ We claim that the function $h\colon y \mapsto \int_{\mathbf{R}^d} \big| g(x-y) - g(x) \big|^p \lambda^{(d)}(dx)$ is bounded and that $h(y) \to 0$ as $y \to 0$.

The boundedness comes from the Minkowski inequality applied to $\big(\|g_y - g\|_{L^p}\big)^p$, with $g_y\colon x \mapsto g(x-y)$, and the fact that $g$ has compact support.

Finally, $h(y) \to 0$ as $y \to 0$ follows from the dominated convergence theorem.

⋆ It remains to prove that $\int_{\mathbf{R}^d} \varphi_n(y) h(y) \lambda^{(d)}(dy)$ converges to 0 as $n$ goes to $\infty$.

There exists $\delta > 0$ such that $|y| < \delta \Rightarrow |h(y)| < \varepsilon^p$. We decompose

$$\int_{\mathbf{R}^d} \varphi_n(y) h(y) \lambda^{(d)}(dy) = \int_{|y| \leq \delta} \varphi_n(y) h(y) \lambda^{(d)}(dy) + \int_{|y| > \delta} \varphi_n(y) h(y) \lambda^{(d)}(dy).$$

The first term can be bounded by Condition (ii) of Definition 6.13

$$\left| \int_{|y| \leq \delta} \varphi_n(y) h(y) \lambda^{(d)}(dy) \right| \leq \varepsilon^p \int_{|y| \leq \delta} \varphi_n(y) \lambda^{(d)}(dy) \leq \varepsilon^p.$$

For the second term, we have

$$\left| \int_{|y| > \delta} \varphi_n(y) h(y) \lambda^{(d)}(dy) \right| \leq \|h\|_\infty \int_{|y| > \delta} \varphi_n(y) \lambda^{(d)}(dy).$$

But Condition (iii) implies the existence of $n_0 \in \mathbf{N}$ such that for all $n \geq n_0$, $\int_{|y| > \delta} \varphi_n(y) \lambda^{(d)}(dy) < \varepsilon^p / \|h\|_\infty$.

Finally, we can conclude that for all $n \geq n_0$, $\|\varphi_n * g - g\|_{L^p} < 2\,\varepsilon$ and, consequently, by the triangle inequality,

$$\|\varphi_n * f - f\|_{L^p} \leq \|\varphi_n * f - \varphi_n * g\|_{L^p} + \|\varphi_n * g - g\|_{L^p} + \|f - g\|_{L^p} < 4\varepsilon.$$

□

**Bernard:** Finally, is there any concrete interest to approximate a function $f$ in $L^p\big(\lambda^{(d)}\big)$ by the functions $(\varphi_n * f)_{n\in\mathbf{N}}$?

**Laurent:** It depends on the properties of the functions $(\varphi_n)_{n\in\mathbf{N}}$. Morally, the functions $(\varphi_n * f)_{n\in\mathbf{N}}$ inherit the properties of $(\varphi_n)_{n\in\mathbf{N}}$.

More precisely, if the functions $(\varphi_n)_{n\in\mathbf{N}}$ are infinitely differentiable, the functions $(\varphi_n * f)_{n\in\mathbf{N}}$ are also infinitely differentiable and approximate $f$ (in the $L^p$ sense).

**Bernard:** Great! I remember that any function $f$ in $L^p$ can be approximated by a function $g$ in $C_c(\mathbf{R}^d)$. By the way, this fact has been used in the previous proof. Maybe we could use a convolution of this $g$ by a function $\varphi_n$, which is infinitely differentiable, to obtain an approximation of $f$ by $\varphi_n * g$ which is in $C_c^\infty(\mathbf{R}^d)$?

**Laurent:** Excellent, Bernard. This is precisely the following result.

**Theorem 6.16.** *For any open subset $\mathcal{U}$ of $\mathbf{R}^d$, the set $C_c^\infty(\mathcal{U})$ is dense in $L^p\big(\lambda^{(d)}|_{\mathcal{U}}\big)$ for $p \in [1, +\infty)$.*

**Proof.** ⋆ We consider $f \in L^p\big(\lambda^{(d)}\big)$ and the regularizing sequence $(\varphi_n)_{n\in\mathbf{N}^*}$ built upon the function $\varphi \in C_c^\infty\big(\mathbf{R}^d\big)$, defined as $x \mapsto \varphi(x) = c\, \exp\left(-\frac{1}{1-|x|^2}\right)\mathbf{1}_{\{|x|<1\}}$, where $c$ is such that $\int_{\mathbf{R}^d} \varphi(x)\lambda^{(d)}(dx) = 1$.

For $\varepsilon > 0$, Theorem 5.15 implies there exists $g \in C_c\big(\mathbf{R}^d\big)$ such that $\|f - g\|_{L^p} < \varepsilon$.

By Theorem 6.15, the sequence $(\varphi_n * g)_{n\in\mathbf{N}^*}$ converges to $g$ in $L^p\big(\lambda^{(d)}\big)$.

Then, there exists $n_0 \in \mathbf{N}^*$ such that, for all $n \geq n_0$, $\|f - \varphi_n * g\|_{L^p} < 2\varepsilon$.

Since $\varphi_n * g$ is infinitely differentiable, by Theorem 6.12, the result is proved for $\mathcal{U} = \mathbf{R}^d$.

⋆ For $\mathcal{U} \subsetneq \mathbf{R}^d$, we remark that the proof of Theorem 5.15 can be easily adapted for $\mathcal{U}$ and $\lambda^{(d)}|_{\mathcal{U}}$ instead of $\mathbf{R}^d$ and $\lambda^{(d)}$, respectively.

Consider $f \in L^p\big(\lambda^{(d)}|_{\mathcal{U}}\big)$ and the regularizing sequence $(\varphi_n)_{n\in\mathbf{N}^*}$ defined above.

Let $\varepsilon > 0$ and $g \in C_c(\mathcal{U})$ such that $\|f - g\|_{L^p} < \varepsilon$.

Note that $\text{supp}(\varphi_n * g) \subset \text{supp}(\varphi_n) + \text{supp}(g)$. Since $\text{supp}(\varphi_n) \subset B(0, 1/n)$, there exists $n_1$ such that for all $n \geq n_1$, $\text{supp}(\varphi_n * g) \subset \text{supp}(g) + B(0, 1/n) \subset \mathcal{U}$.

We conclude as in the case of $\mathcal{U} = \mathbf{R}^d$. □

## 6.3 Probability on $\mathbf{R}^d$ and Independence of Random Variables

**Laurent:** Thanks to the notions of product $\sigma$-algebra and product measure we just saw, we are now able to study random vectors $X = (X_1, \dots, X_d)$, where $(X_k)_{1 \leq k \leq d}$ are real-valued random variables.

**Bernard:** I don't understand. We were already able to consider random vectors, as soon as we had defined a random variable. It suffices to study random variables with values in $\big(\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d)\big)$.

**Laurent:** Of course, we can. But our goal here is to see how $\mathbf{R}^d$ as a product space can inherit knowledge of the real-valued case, for instance, cumulative distribution function, expectation and covariance matrix.

But the more important point is to study the *dependence* between the components $X_k$ of a random vector $X = (X_1, \dots, X_d)$. I mean, how the distribution of the values of the variable $X_1$ acts on those of the variables $X_2, \dots, X_d$.

Let us go back to *Foundations of Probability*.

### 6.3.1 *Random vectors*

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In the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, a random vector of size $d$ is a random variable, $X = (X_1, \dots, X_d)$, with values in $\big(\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d)\big)$. Then, its law $P_X$ is the probability measure on the space $\big(\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d)\big)$ defined by

$$\forall A \in \mathcal{B}(\mathbf{R}^d), \quad P_X(A) = \mathbf{P}\big(X^{-1}(A)\big) = \mathbf{P}(X \in A).$$

The cumulative distribution function of $X$ (or $P_X$) is the function $F_X\colon \mathbf{R}^d \to [0,1]$, defined by

$$\forall x_1,\dots,x_d \in \mathbf{R}, \quad F_X(x_1,\dots,x_d) = P_X\big((-\infty,x_1]\times\cdots\times(-\infty,x_d]\big) = \mathbf{P}(X_1 \le x_1,\dots,X_N \le x_d).$$

**Laurent:** Bernard, do you remember the reason why, in the real case, a cumulative distribution function $F\colon x \mapsto \mathbf{P}\big((-\infty,x]\big)$ characterizes the probability $\mathbf{P}$?

**Bernard:** Yes. It was a part of the Caratheodory theory we saw in the *Measure Theory* book. The collection $\Pi(\mathbf{R}) = \{(-\infty,x];\ x \in \mathbf{R}\}$ of subsets of $\mathbf{R}$ forms a $\pi$-system such that $\sigma\big(\Pi(\mathbf{R})\big) = \mathcal{B}(\mathbf{R})$. Then, any probability measure on $\mathcal{B}(\mathbf{R})$ is characterized by its values on $\Pi(\mathbf{R})$.

As I say this, I think the same thing must be true when $\mathbf{R}$ is substituted with $\mathbf{R}^d$. The collection $\Pi(\mathbf{R}^d) = \{(-\infty,x_1]\times\cdots\times(-\infty,x_d];\ x_1,\dots,x_d \in \mathbf{R}\}$ of subsets of $\mathbf{R}^d$ certainly forms a $\pi$-system such that $\sigma\big(\Pi(\mathbf{R}^d)\big) = \mathcal{B}(\mathbf{R}^d)$.

**Laurent:** You are right. And consequently you can claim that the cumulative distribution function $F_X$ characterizes the law $P_X$ of the random vector $X$.

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When the cumulative distribution function of $X$ can be written as an integral with respect to the Lebesgue measure of $\mathbf{R}^d$, i.e. when for all $x_1,\dots,x_d \in \mathbf{R}$,

$$F_X(x_1,\dots,x_d) = \int_{(-\infty,x_1]\times\cdots\times(-\infty,x_d]} f_X(u_1,\dots,u_d)\lambda^{(d)} \times (du_1,\dots,du_d),$$

where:

- ⋆ $\lambda^{(d)}$ is the Lebesgue measure of $\mathbf{R}^d$ ($\lambda^{(d)} = \lambda\otimes\cdots\otimes\lambda$),

- $\star$ $f_X \colon \mathbf{R}^d \to \mathbf{R}_+$ is a Borel function that is integrable with respect to $\lambda^{(d)}$,
- $\star$ $\int_{\mathbf{R}^d} f_X \, d\lambda^{(d)} = 1$,

the random vector $X$ (or the law of $X$) is said to have the density $f_X$.

In that case,

$$\forall A \in \mathcal{B}(\mathbf{R}^d), \quad \mathbf{P}(X \in A) = \int_A f_X \, d\lambda^{(d)}$$

and, more generally, for any Borel function $h \colon \mathbf{R}^d \to \mathbf{R}$ such that

$$\int_{\mathbf{R}^d} |h(x_1, \ldots, x_d)| f_X(x_1, \ldots, x_d) \lambda^{(d)}(dx_1, \ldots, dx_d) < \infty,$$

the real random variable $h(X_1, \ldots, X_d)$ is in $L^1(\mathbf{P})$ and

$$\mathbf{E}\big[h(X_1, \ldots, X_d)\big] = \int_{\mathbf{R}^d} h(x_1, \ldots, x_d) f_X(x_1, \ldots, x_d) \lambda^{(d)} \times (dx_1, \ldots, dx_d).$$

**Laurent:** Note that even if this special case is very important, not all random vectors have densities. Nothing new here. It is the same as in one dimension.

6.3.1.1 *Integration and moments*

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**Definition 6.17.** Let $X = (X_1, \ldots, X_d)$ be a random vector of $(\Omega, \mathcal{F}, \mathbf{P})$. If the real-valued random variables $X_1, \ldots, X_d$ are in $L^1(\mathbf{P})$, then the vector

$$\mathbf{E}[X] = \big(\mathbf{E}[X_1], \ldots, \mathbf{E}[X_d]\big)$$

of $\mathbf{R}^d$ is called *the expectation* or *mean of* $X$.

**Definition 6.18.** In the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, let $X$ and $Y$ be two real-valued random variables in $L^2(\mathbf{P})$. The *covariance of* $X$ *and* $Y$ is the finite real number

$$\operatorname{Cov}(X, Y) = \mathbf{E}\big[(X - \mathbf{E}[X])(Y - \mathbf{E}[Y])\big] = \mathbf{E}[XY] - \mathbf{E}[X]\,\mathbf{E}[Y].$$

**Justification.** Since $L^2(\mathbf{P}) \subset L^1(\mathbf{P})$, the quantities $\mathbf{E}[X]$ and $\mathbf{E}[Y]$ are well defined. The integrability of $XY$ comes from the Cauchy–Schwarz inequality

$$\mathbf{E}[|XY|] \leq \big(\mathbf{E}[X^2]\big)^{1/2} \big(\mathbf{E}[Y^2]\big)^{1/2} < \infty,$$

since $X$ and $Y$ are in $L^2(\mathbf{P})$. □

As a direct consequence of the linearity of the expectation $\mathbf{E}$, we can state the following property of the covariance function.

**Proposition 6.19.** *The function* $\operatorname{Cov}\colon L^2(\mathbf{P}) \times L^2(\mathbf{P}) \to \mathbf{R}$ *is bilinear.*

**Bernard:** The covariance of two random variables has the same flavor as the variance of a variable in $L^2(\mathbf{P})$. More precisely, if I take $Y = X$ in the definition, I obtain $\operatorname{Cov}(X, X) = \mathbf{E}\big[(X - \mathbf{E}[X])^2\big] = \operatorname{Var}(X)$.

**Laurent:** That is true. You studied a little bit of bilinear algebra in your *classes préparatoires*, right? If we go back to this framework, we can say that $X \mapsto \operatorname{Var}(X)$ is the quadratic form associated with the bilinear form $(X, Y) \mapsto \operatorname{Cov}(X, Y)$. Then, the following result is well known.

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**Proposition 6.20.** *In the probability space* $(\Omega, \mathcal{F}, \mathbf{P})$, *let* $X$ *and* $Y$ *be two real-valued random variables in* $L^2(\mathbf{P})$. *The following statements hold:*

(i) $\operatorname{Cov}(X, X) = \operatorname{Var}(X)$.

(ii) $\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$.
(iii) $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X, Y)$.

**Proof.** Assertions (i) and (ii) are obvious.

In order to prove the third point, we write

$$\begin{aligned}\mathrm{Var}(X+Y) &= \mathbf{E}\left[(X+Y-\mathbf{E}[X+Y])^2\right] \\ &= \mathbf{E}[((X-\mathbf{E}[X]) + (Y-\mathbf{E}[Y]))^2] \\ &= \mathbf{E}[(X-\mathbf{E}[X])^2 + (Y-\mathbf{E}[Y])^2 \\ &\quad + 2(X-\mathbf{E}[X])(Y-\mathbf{E}[Y])] \\ &= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y).\end{aligned}$$

□

More generally, for any real-valued random variables $X_1, \ldots, X_n$ in $L^2(\mathbf{P})$, we have

$$\mathrm{Var}(X_1 + \cdots + X_n) = \sum_{1 \le i \le n} \mathrm{Var}(X_i) + 2 \sum_{1 \le i < j \le n} \mathrm{Cov}(X_i, X_j).$$

**Definition 6.21.** In the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, let $X$ and $Y$ be two real-valued random variables in $L^2(\mathbf{P})$. When $\mathrm{Var}(X) \neq 0$ and $\mathrm{Var}(Y) \neq 0$, the quantity

$$\rho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sigma(X)\sigma(Y)},$$

where $\sigma(X) = \sqrt{\mathrm{Var}(X)}$ and $\sigma(Y) = \sqrt{\mathrm{Var}(Y)}$ are the standard deviations of $X$ and $Y$, respectively, is called *the correlation coefficient between $X$ and $Y$*.

Thanks to the Cauchy–Schwarz inequality, the correlation coefficient satisfies $|\rho| \le 1$.

**Laurent:** The correlation coefficient naturally appears in regression problems in statistics. However, in probability theory, the covariance is usually preferred.

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**Definition 6.22.** In the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, let $X = (X_1, \ldots, X_d)$ be a random vector such that $X_k \in L^2(\mathbf{P})$ for all $k = 1, \ldots, d$.

The *covariance matrix* of $X$ is the matrix $\Sigma = [c_{ij}]_{1 \leq i,j \leq d}$ defined as

$$\forall i, j = 1, \ldots, d, \quad c_{ij} = \operatorname{Cov}(X_i, X_j).$$

**Proposition 6.23.** *A covariance matrix $\Sigma = [c_{ij}]_{1 \leq i,j \leq d}$ is symmetric and positive (semi-definite), i.e.*

- ⋆ *for all $i, j = 1, \ldots, d$, we have $c_{ij} = c_{ji}$;*
- ⋆ *for all vectors $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{R}^d$, $\alpha \Sigma \alpha^t = \sum_{1 \leq i,j \leq d} \alpha_i c_{ij} \alpha_j \geq 0$.*

**Proof.** For all $i, j = 1, \ldots, d$, we have $c_{ij} = \operatorname{Cov}(X_i, X_j) = \operatorname{Cov}(X_j, X_i) = c_{ji}$. And,

$$\sum_{1 \leq i,j \leq d} \alpha_i \, c_{ij} \, \alpha_j = \sum_{1 \leq i,j \leq d} \operatorname{Cov}(\alpha_i X_i, \alpha_j X_j)$$

$$= \operatorname{Cov}\left( \sum_{i=1}^{d} \alpha_i X_i, \sum_{j=1}^{d} \alpha_j X_j \right) = \operatorname{Var}\left( \sum_{j=1}^{d} \alpha_j X_j \right) \geq 0.$$

□

**Laurent:** You will understand that these properties of symmetry and positiveness are characteristic of a covariance matrix. We will see later that any symmetric and positive matrix is the covariance matrix of a Gaussian random vector.

### 6.3.1.2 *Chain rule*

**Bernard:** I understand that the fact that a random vector having a density is a particular case and not the general case at all. But I guess this case is important. Do such random vectors have special properties?

**Laurent:** Since the distribution of the vector $X$ is totally determined by its density, there are lots of things to say. Maybe the first one is to observe that the vector $h(X)$, where $h \colon \mathbf{R}^d \to \mathbf{R}^d$ is a suitable transformation, also admits a density. This fact simply comes from the change of variables formula (Theorem 6.5). And this is the only thing to keep in mind, certainly not the precise result... In all the situations, we will use a specific change of variables.

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Let $X$ be a random vector with values in an open subset $\Delta \subset \mathbf{R}^d$. We are interested in the random vector $h(X)$ when $h\colon \Delta \to D \subset \mathbf{R}^d$ is a $C^1$-diffeomorphism, i.e. if $h$ is bijective and $h$ and $h^{-1}$ are continuously differentiable.

Remember that the Jacobian determinant of $h$ at point $x \in \Delta$ is

$$Jh(x) = \det \left[ \left( \frac{\partial h_i}{\partial x_j}(x) \right)_{1 \le i,j \le d} \right],$$

where $h(x) = \big(h_1(x_1, \dots, x_d), \dots, h_d(x_1, \dots, x_d)\big)$.

**Theorem 6.24.** *Let $X = (X_1, \dots, X_d)$ be a random vector with values in $\Delta \subset \mathbf{R}^d$, whose distribution has the density $f_X$.*

*If $h\colon \Delta \to D \subset \mathbf{R}^d$ is a $C^1$-diffeomorphism, then the random vector $Y = h(X)$ has the density $f_Y$, defined by*

$$\forall y \in \mathbf{R}^d, \quad f_Y(y) = \frac{f_X(h^{-1}(y))}{|Jh(h^{-1}(y))|} \mathbf{1}_D(y),$$

*where $Jh$ is the Jacobian determinant of $h$.*

**Proof.** Let $B \in \mathcal{B}(\mathbf{R}^d)$ and $A = h^{-1}(B \cap D)$. Using the change of variable formula, we get

$$\mathbf{P}(X \in A) = \int_A f_X(x) \lambda^{(d)}(dx) = \int_{B \cap D} f_X(h^{-1}(y)) \times |\det Jh^{-1}(y)| \lambda^{(d)}(dy).$$

Since $\mathbf{P}(X \in A) = \mathbf{P}(Y \in B \cap D)$, the relation

$$\mathbf{P}(Y \in B \cap D) = \int_{B \cap D} f_X(h^{-1}(y)) |Jh^{-1}(y)| \lambda^{(d)}(dy) \tag{6.3.1}$$

shows that the law of $Y$ is absolutely continuous with respect to the Lebesgue measure and that the probability density of $Y$ is given by

$$\forall y \in \mathbf{R}^N, \quad f_Y(y) = \frac{f_X(h^{-1}(y))}{|Jh(h^{-1}(y))|} \mathbf{1}_D(y).$$

□

### 6.3.1.3 *Marginal probability distributions*

**Laurent:** Another interesting fact to note is that the distribution of a random vector governs the distribution of all its subvectors. In particular, from the law of a vector $X = (X_1, \ldots, X_d)$, we can derive the law of each variable $X_k$ $(1 \leq k \leq d)$.

However, note that we cannot deduce the law of the vector from the simple knowledge of the laws of $X_k$ $(1 \leq k \leq d)$. The concept of dependence between the variables $(1 \leq k \leq d)$ appears here...

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**Definition 6.25.** Let $X = (X_1, \ldots, X_d)$ be a random vector. The distribution of any subvector $(X_{i_1}, \ldots, X_{i_k})$ is called the *marginal probability distribution* of the probability measure $P_X$.

In particular, the distributions of the real-valued random variables $X_k$ $(1 \leq k \leq d)$ are marginal distributions of $P_X$.

**Proposition 6.26.** *Let $X = (X_1, \ldots, X_d)$ be a random vector of the cumulative distribution function $F_X$ given by*

$$\forall x_1, \ldots, x_d \in \mathbf{R}, \quad F_X(x_1, \ldots, x_d) = \mathbf{P}(X_1 \leq x_1, \ldots, X_d \leq x_d).$$

*For all $1 \leq k < d$, the cumulative distribution function of $(X_1, \ldots, X_k)$ is defined by*

$$\begin{aligned} F_{(X_1,\ldots,X_k)}(x_1, \ldots, x_k) &= \lim_{x_{k+1},\ldots,x_d \to +\infty} F_X(x_1, \ldots, x_d) \\ &= F_X(x_1, \ldots, x_k, +\infty, \ldots, +\infty). \end{aligned}$$

In particular, the (marginal) distribution of $X_k$ $(1 \leq k \leq d)$ is determined by its cumulative distribution function $F_{X_k}$:

$$x_k \mapsto F_{X_k}(x_k) = F_X(+\infty, \ldots, +\infty, x_k, +\infty, \ldots, +\infty).$$

**Proof.** Let us consider a vector $(x_1, \ldots, x_k)$ and $\left(x_{k+1}^{(n)}\right)_{n \in \mathbf{N}}, \ldots, \left(x_d^{(n)}\right)_{n \in \mathbf{N}}$ be some nondecreasing sequences that converge to $+\infty$.

Let us set

$$E_n = \left\{X_1 \leq x_1, \ldots, X_k \leq x_k, X_{k+1} \leq x_{k+1}^{(n)}, \ldots, X_d \leq x_d^{(n)}\right\}.$$

For all $n \in \mathbf{N}$, we have $E_n \subset E_{n+1}$ and

$$\lim_{n\to\infty} E_n = \bigcup_{n=0}^{\infty} E_n = \{X_1 \le x_1, \dots, X_k \le x_k\}.$$

The monotone convergence property for a measure implies that

$$\lim_{n\to\infty} \mathbf{P}(E_n) = \mathbf{P}\left(X_1 \le x_1, \dots, X_k \le x_k\right),$$

that is,

$$\lim_{n\to\infty} F_X\left(x_1, \dots, x_k, x_{(k+1)}^{(n)}, \dots, x_d^{(n)}\right) = F_{(X_1,\dots,X_k)}(x_1, \dots, x_k).$$

□

**Laurent:** This result is quite obvious. If you think about it, it just states that for all $x_1, \dots, x_k \in \mathbf{R}$,

$$\begin{aligned}&\mathbf{P}(X_1 \le x_1, \dots, X_k \le x_k)\\ &\quad = \mathbf{P}(X_1 \le x_1, \dots, X_k \le x_k, X_{k+1} \le +\infty, \dots, X_d \le +\infty).\end{aligned}$$

As you know, the left-hand side of the equality characterizes the distribution of the subvector $(X_1, \dots, X_k)$ $(k < d)$, and it is obtained thanks to the distribution of the vector $(X_1, \dots, X_d)$.

**Bernard:** Yes, as you write it, the result looks obvious, but it's good to see a rigorous proof. I understand that the result doesn't only stand for subvectors of the first $k$ components.

**Laurent:** Definitely, Bernard. It is convenient to write it like that. But it is true for any subvector $(X_{l_1}, \dots, X_{l_k})$ of $(X_1, \dots, X_d)$, where $1 \le l_1 < \cdots < l_k \le d$.

E[φ(X)]
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In the case of random vectors with a probability density, the following holds.

**Proposition 6.27.** *Let $X = (X_1, \dots, X_d)$ be a random vector whose distribution has a density $f_X$. For $k < d$, the random* *vector* $(X_1, \ldots, X_k)$ *admits the density* $f_{(X_1,\ldots,X_k)}$, *defined by the following: for all* $x_1, \ldots, x_k \in \mathbf{R}$,

$$f_{(X_1,\ldots,X_k)}(x_1, \ldots, x_k) = \int_{\mathbf{R}^{d-k}} f_X(x_1, \ldots, x_d)\ \lambda^{(d-k)} \times (dx_{k+1}, \ldots, dx_d).$$

**Proof.** By the definition of the density $f_X$, the cumulative distribution function $F_X$ is given as follows: for all $x_1, \ldots, x_d \in \mathbf{R}$,

$$F_X(x_1, \ldots, x_d) = \int_{(-\infty,x_1]\times\cdots\times(-\infty,x_d]} f_X(u_1, \ldots, u_d)\lambda^{(d)} \times (du_1, \ldots, du_d).$$

Then, Proposition 6.26 implies

$$F_{(X_1,\ldots,X_k)}(x_1, \ldots, x_k) = \int_{(-\infty,x_1]\times\cdots\times(-\infty,x_k]\times\mathbf{R}\times\cdots\times\mathbf{R}} \times f_X(u_1, \ldots, u_d)\lambda^{(d)}(du_1, \ldots, du_d).$$

Thanks to the Fubini–Lebesgue theorem, we can write

$$\begin{aligned} &F_{(X_1,\ldots,X_k)}(x_1, \ldots, x_k) \\ &\quad = \int_{(-\infty,x_1]\times\cdots\times(-\infty,x_k]} \left( \int_{\mathbf{R}^{d-k}} f_X(u_1, \ldots, u_d) \right. \\ &\qquad \left. \times\ \lambda^{(d-k)}(du_{k+1}, \ldots, du_d) \right) \lambda^{(k)}(du_1, \ldots, du_k). \end{aligned}$$

This equality shows that the distribution of $(X_1, \ldots, X_k)$ has a density given by the following:
for all $x_1, \ldots, x_k \in \mathbf{R}$,

$$f_{(X_1,\ldots,X_k)}(x_1, \ldots, x_k) = \int_{\mathbf{R}^{d-k}} f_X(u_1, \ldots, u_d)\lambda^{(d-k)}(du_{k+1}, \ldots, du_d).$$

□

**Laurent:** Once again, Proposition 6.27 could be stated for any subvector $(X_{l_1}, \dots, X_{l_k})$ of $(X_1, \dots, X_d)$, where $1 \leq l_1 < \cdots < l_k \leq d$. Could you apply it to the particular case of a two-component random vector $(X, Y)$?

**Bernard:** Sure. Let me assume that $(X, Y)$ admits a probability density $f$. According to Proposition 6.27, the random variables $X$ and $Y$ also admit densities $f_X$ and $f_Y$, defined by the following: for all $x, y \in \mathbf{R}$,

$$f_X(x) = \int_{\mathbf{R}} f(x, v)\lambda(dv) \quad \text{and} \quad f_Y(y) = \int_{\mathbf{R}} f(u, y)\lambda(du).$$

**Laurent:** We have just arrived to the point where the dependence between the distributions of $X$ and $Y$ is questioned. This is where probability theory really begins.

### 6.3.2 *Independence of random variables*

#### 6.3.2.1 *From σ-algebras to random variables*

$\mathbf{E}[\varphi(X)]$ 519.FRA.f

The elementary definition of independence between events of a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is naturally extended to sub-$\sigma$-algebras of $\mathcal{F}$.

**Definition 6.28.** In the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, the sub-$\sigma$-algebras $(\mathcal{F}_i)_{i \in I}$ of $\mathcal{F}$ are said to be *independent* if for any finite subset $J$ of $I$ and any collection of events $(A_i)_{i \in J}$ such that $A_i \in \mathcal{F}_i$ for all $i \in J$,

$$\mathbf{P}\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} \mathbf{P}(A_i).$$

By remembering that a random variable $X\colon (\Omega, \mathcal{F}) \to (E, \mathcal{E})$ generates a sub-$\sigma$-algebra $X^{-1}(\mathcal{E})$ of $\mathcal{F}$, we can define the independence between random variables.

**Definition 6.29.** Let $(X_i)_{i \in I}$ be a family of random variables, with values in the spaces $(E_i, \mathcal{E}_i)$. The variables $X_i$ are said to be *independent* if the generated $\sigma$-algebras $\sigma(X_i) = X_i^{-1}(\mathcal{E}_i)$ are independent.

For any family $(A_i)_{i\in I}$ such as $A_i \in \mathcal{E}_i$ for all $i \in I$, we have $X_i^{-1}(A_i) \in \sigma(X_i)$, and by the definition of the notation "$\mathbf{P}(X \in A)$", the independence between the random variables $X_i$ $(i \in I)$ reads as follows:
for all finite subsets $J$ of $I$,

$$\mathbf{P}(X_i \in A_i; \forall i \in J) = \mathbf{P}\left(\bigcap_{i\in J} X_i^{-1}(A_i)\right) = \prod_{i\in J} \mathbf{P}\big(X_i^{-1}(A_i)\big)$$
$$= \prod_{i\in J} \mathbf{P}\big(X_i \in A_i\big).$$

**Bernard:** I had a hard time getting an intuition about the $\sigma$-algebra-based definition. Fortunately, this comment clarifies the concept of independence.

Two random variables $X$ and $Y$ with values in $(E, \mathcal{E})$ are independent if for all $A$ and $B$ in $\mathcal{E}$, we have $\mathbf{P}(X \in A, Y \in B) = \mathbf{P}(X \in A)\,\mathbf{P}(Y \in B)$.

**Laurent:** This is true, but I call your attention to the fact that the definition does not require that the variables take their values in the same space $(E, \mathcal{E})$. The two variables $X\colon (\Omega, \mathcal{F}) \to (E, \mathcal{A})$ and $Y\colon (\Omega, \mathcal{F}) \to (F, \mathcal{B})$ are independent if

$$\forall A \in \mathcal{A}, \forall B \in \mathcal{B}, \quad \mathbf{P}(X \in A, Y \in B) = \mathbf{P}(X \in A)\,\mathbf{P}(Y \in B).$$

There is an important point to understand about independence between random variables. Dependence does not mean causality. Some time ago, some of my colleagues used to give a striking example to keep this in mind. They used to say that a statistical study conducted by a prestigious US university found a correlation between the number of mental illnesses in the country and the number of television sets per household...

**Bernard:** What? Could it be serious?

**Laurent:** In front of all the laughter, they pointed out that while they did not dispute the seriousness of this conclusion, there was no way to conclude that watching television could cause mental illness. Other factors could obviously be involved (for example, the way we eat or the impact on the brain of the waves emitted by all electronic devices, why not?)

Dependence (and correlation) between $X$ and $Y$ does not mean at all that $X$ implies $Y$, or even the vice versa. It only means that the distributions of $X$ and $Y$ are related mathematically. We will see later that independence between $X$ and $Y$ is simply a relation between the laws $P_X$, $P_Y$ and $P_{(X,Y)}$. Let us move on.

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Since independence concerns the distributions of random variables, this property is naturally expressed via the transfer theorem (through the definition of expectation). For the sake of clarity, we only consider the case of two random variables in the following.

**Theorem 6.30.** *Let $X$ and $Y$ be two random variables with values in the measurable spaces $(E, \mathcal{A})$ and $(F, \mathcal{B})$, respectively. $X$ and $Y$ are independent if the following equivalent conditions hold:*

(i) *For any $A \in \mathcal{A}$ and $B \in \mathcal{B}$,*

$$\mathbf{P}(X \in A, Y \in B) = \mathbf{P}(X \in A)\,\mathbf{P}(Y \in B). \tag{6.3.2}$$

(ii) *For any bounded measurable functions $f\colon E \to \mathbf{R}$ and $g\colon F \to \mathbf{R}$,*

$$\mathbf{E}\big[f(X)\,g(Y)\big] = \mathbf{E}\big[f(X)\big]\,\mathbf{E}\big[g(Y)\big]. \tag{6.3.3}$$

**Proof.** ⋆ Statement (i) is a reformulation of Definition 6.29 of the independence between the two random variables $X$ and $Y$.

⋆ In order to prove the implication (i)⇒(ii), it suffices to remark that for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, the equality $\mathbf{P}\big(X^{-1}(A) \cap Y^{-1}(B)\big) = \mathbf{P}\big(X^{-1}(A)\big)\,\mathbf{P}\big(Y^{-1}(B)\big)$ can be rewritten as $\mathbf{E}\big[\mathbf{1}_{X^{-1}(A)}\,\mathbf{1}_{Y^{-1}(B)}\big] = \mathbf{E}\big[\mathbf{1}_{X^{-1}(A)}\big]\,\mathbf{E}\big[\mathbf{1}_{Y^{-1}(B)}\big]$ and, finally,

$$\mathbf{E}\big[\mathbf{1}_A(X)\,\mathbf{1}_B(Y)\big] = \mathbf{E}\big[\mathbf{1}_A(X)\big]\,\mathbf{E}\big[\mathbf{1}_B(Y)\big].$$

By linearity, Expression (6.3.3) is satisfied for all simple functions $f$ and $g$. Then, by increasing limits, it is satisfied for all nonnegative measurable functions $f$ and $g$. Finally, by decomposing $f = f^+ - f^-$

and $g = g^+ - g^-$, it is satisfied for all bounded measurable functions $f$ and $g$.

⋆ In order to prove to converse implication (ii)⇒(i), it suffices to consider the particular case of $f = \mathbf{1}_A$ and $g = \mathbf{1}_B$. □

**Remark.** In particular, considering the functions $\mathbf{1}_{\{x\}}$ and $\mathbf{1}_{\{y\}}$ in (6.3.3), the discrete random variables $X$ and $Y$ with values in $E$ and $F$, respectively, are independent if and only if

$$\forall (x,y) \in E \times F, \quad \mathbf{P}(X = x, Y = y) = \mathbf{P}(X = x)\,\mathbf{P}(Y = y).$$

**Laurent:** As a consequence of Theorem 6.30, we can claim that if $X$ and $Y$ are independent random variables, then $\varphi(X)$ and $\psi(Y)$ are also independent, for any (not necessarily bounded) measurable functions $\varphi\colon E \to \mathbf{R}$ and $\psi\colon F \to \mathbf{R}$.

**Bernard:** Yes, this is clear thanks to this theorem. I can't wait to see the independence between $X$ and $Y$ being formalized in terms of the laws $P_X$, $P_Y$ and $P_{(X,Y)}$.

### 6.3.2.2 *Independence and product measures*

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We recall that the product of two $\sigma$-algebras $\mathcal{A}$ and $\mathcal{B}$ is the $\sigma$-algebra generated by the measurable rectangles $A \times B$, with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. The product $\sigma$-algebra is denoted by $\mathcal{A} \otimes \mathcal{B} = \sigma(\mathcal{A} \times \mathcal{B})$.

**Proposition 6.31.** *Let $X$ and $Y$ be two random variables with values in $(E, \mathcal{A})$ and $(F, \mathcal{B})$, respectively, which can be also stated as follows: let $(X,Y)$ be a random variable with values in $E \times F$, equipped with the product $\sigma$-algebra $\mathcal{A} \otimes \mathcal{B}$.*

*$X$ and $Y$ are independent if and only if the law $P_{(X,Y)}$ of the vector $(X,Y)$ satisfies $P_{(X,Y)} = P_X \otimes P_Y$, where $P_X$ is the law of $X$ and $P_Y$ is the law of $Y$.*

**Proof.** From the definition, $X$ and $Y$ are independent if for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$,

$$\mathbf{P}\big((X,Y) \in A \times B\big) = \mathbf{P}(X \in A)\mathbf{P}(Y \in B),$$

that is,

$$P_{(X,Y)}(A \times B) = P_X(A)\, P_Y(B).$$

By the definition of the product measure, we have $P_X \otimes P_Y(A \times B) = P_X(A)\, P_Y(B)$.

Consequently, we can claim that $X$ and $Y$ are independent if and only if for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$,

$$P_{(X,Y)}(A \times B) = P_X \otimes P_Y(A \times B).$$

Since the $\sigma$-algebra $\mathcal{A} \otimes \mathcal{B}$ is generated by the $\pi$-system $\mathcal{A} \times \mathcal{B}$, we conclude that $X$ and $Y$ are independent if and only if $P_{(X,Y)} = P_X \otimes P_Y$. □

**Bernard:** OK for general random variables. But when the variables $X$ and $Y$ are real-valued, their laws are characterized by their cumulative distribution functions. I guess that the independence of $X$ and $Y$ can be expressed in those terms.

**Laurent:** You are right, Bernard. And I can tell you that when $(X,Y)$ admits a density of probability, the independence can be expressed in terms of the densities of $X$, $Y$ and $(X,Y)$.

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**Theorem 6.32.** *The real-valued random variables $X$ and $Y$ are independent if and only if the cumulative distribution functions $F_X$ of $X$, $F_Y$ of $Y$ and $F_{(X,Y)}$ of $(X,Y)$ satisfy*

$$\forall (x,y) \in \mathbf{R}^2, \quad F_{(X,Y)}(x,y) = F_X(x)F_Y(y). \tag{6.3.4}$$

**Proof.** ⋆ The direct part follows directly from Theorem 6.30 in the particular case of $A = (-\infty, x]$ and $B = (-\infty, y]$ for any $x, y \in \mathbf{R}$:

$$\mathbf{P}(X \le x, Y \le y) = \mathbf{P}(X \le x)\mathbf{P}(Y \le y).$$

$\star$ Conversely, let us assume that (6.3.4) holds.

We fix $y \in \mathbf{R}$, and we consider the functions $\mu\colon \mathcal{B}(\mathbf{R}) \to [0,1]$ and $\nu\colon \mathcal{B}(\mathbf{R}) \to [0,1]$, defined by

$$\begin{aligned} \forall A \in \mathcal{B}(\mathbf{R}), \quad \mu(A) &= \mathbf{P}(X \in A, Y \le y) \\ \nu(A) &= \mathbf{P}(X \in A)\mathbf{P}(Y \le y). \end{aligned}$$

$\mu$ and $\nu$ are finite measures over $\big(\mathbf{R}, \mathcal{B}(\mathbf{R})\big)$ that coincide on the sets $(-\infty, x]$, $x \in \mathbf{R}$. Actually, from (6.3.4), for all $x \in \mathbf{R}$, we have

$$\begin{aligned} \mu\big((-\infty, x]\big) &= \mathbf{P}(X \le x, Y \le y) = \mathbf{P}(X \le x)\mathbf{P}(Y \le y) \\ &= \nu\big((-\infty, x]\big). \end{aligned}$$

Since the sets $(-\infty, x]$, $x \in \mathbf{R}$, form a $\pi$-system that generates the Borel $\sigma$-algebra $\mathcal{B}(\mathbf{R})$, we deduce that $\mu = \nu$, that is,

$$\forall A \in \mathcal{B}(\mathbf{R}), \quad \mathbf{P}(X \in A, Y \le y) = \mathbf{P}(X \in A)\mathbf{P}(Y \le y).$$

Then, we fix $A \in \mathcal{B}(\mathbf{R})$ and consider the functions $\tilde{\mu}\colon \mathcal{B}(\mathbf{R}) \to [0,1]$ and $\tilde{\nu}\colon \mathcal{B}(\mathbf{R}) \to [0,1]$, defined by

$$\begin{aligned} \forall B \in \mathcal{B}(\mathbf{R}), \quad \tilde{\mu}(B) &= \mathbf{P}(X \in A, Y \in B) \\ \tilde{\nu}(B) &= \mathbf{P}(X \in A)\ \mathbf{P}(Y \in B). \end{aligned}$$

$\tilde{\mu}$ and $\tilde{\nu}$ are finite measures over $\big(\mathbf{R}, \mathcal{B}(\mathbf{R})\big)$, that coincide on the sets $(-\infty, y]$, $y \in \mathbf{R}$. Since the sets $(-\infty, y]$, $y \in \mathbf{R}$, form a $\pi$-system that generates the Borel $\sigma$-algebra $\mathcal{B}(\mathbf{R})$, we deduce that $\tilde{\mu} = \tilde{\nu}$, that is,

$$\forall A, B \in \mathcal{B}(\mathbf{R}), \quad \mathbf{P}(X \in A, Y \in B) = \mathbf{P}(X \in A)\mathbf{P}(Y \in B).$$

This proves the independence between $X$ and $Y$. □

**Bernard:** What a clear use of Caratheodory's theory! Very elegant.

Now, I guess we can more or less differentiate this equality on the cumulative distribution functions to obtain a similar one for densities when they exist.

**Laurent:** Morally, you are right. But be careful, Bernard, the cumulative distribution functions are not differentiable in general and the rigorous link between these two functions is the fundamental theorem of calculus (Theorem 5.32).

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Following Theorem 6.32, the independence property can also be expressed in terms of density when the laws admit densities. We only state this unsurprising result since density and cumulative distribution function are linked by the fundamental theorem of calculus. We refer to Jacod and Protter (2012) for a detailed proof.

**Theorem 6.33.** *Let $(X, Y)$ be a random vector whose distribution admits a density with respect to the Lebesgue measure.*

*The random variables $X$ and $Y$ are independent if and only if the probability densities $f_X$ of $X$, $f_Y$ of $Y$ and $f_{(X,Y)}$ of $(X, Y)$ satisfy almost everywhere*

$$\forall (x, y) \in \mathbf{R}^2, \quad f_{(X,Y)}(x, y) = f_X(x) f_Y(y).$$

**Laurent:** Let me check if you have understood correctly how the independence property is an important concept when several random variables that are involved.

For example, consider two *independent and identically distributed* random variables (usually noted *i.i.d.*) $X$ and $Y$ that have the normal law $\mathcal{N}(0.1)$.

What can you say about the distribution of the variable $Z = \frac{X}{Y}$?

**Bernard:** My first idea is the use of the transfer theorem. The law of $Z$ is the pushforward measure of $P_{(X,Y)}$, the law of $(X, Y)$, by the function $(x, y) \mapsto \frac{x}{y}$:

$$\forall A \in \mathcal{B}(\mathbf{R}), \quad P_Z(A) = \int_{\mathbf{R}^2} \mathbf{1}_A\left(\frac{x}{y}\right) P_{(X,Y)}(dx, dy).$$

**Laurent:** And what can you say about $P_{(X,Y)}$ when $X$ and $Y$ are independent?

**Bernard:** The independence of $X$ and $Y$ can be written as $P_{(X,Y)} = P_X \otimes P_Y$. Then, for all $A \in \mathcal{B}(\mathbf{R})$,

$$\begin{aligned} P_Z(A) &= \int_{\mathbf{R}^2} \mathbf{1}_A\left(\frac{x}{y}\right) P_X \otimes P_Y(dx, dy) \\ &= \int_{\mathbf{R}^2} \mathbf{1}_A\left(\frac{x}{y}\right) f(x) f(y) \lambda^{(2)}(dx, dy), \end{aligned}$$

where $f\colon x \mapsto \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$ is the density of the law $\mathcal{N}(0,1)$.

**Laurent:** Go on. You have almost proved that the law of $Z$ admits a density.

**Bernard:** Don't pressure me, please. I can apply the change of variables

$$\begin{aligned} \Phi\colon \mathbf{R}^* \times \mathbf{R}^* &\to \mathbf{R}^* \times \mathbf{R}^* \\ (x, y) &\mapsto \left(x, t = \frac{x}{y}\right). \end{aligned}$$

The function $\Phi$ is a $C^1$-diffeomorphism of the Jacobian determinant

$$J\Phi(x,y) = \begin{vmatrix} 1 & \dfrac{1}{y} \\ 0 & -\dfrac{x}{y^2} \end{vmatrix} = -\frac{x}{y^2} \quad \Rightarrow \quad J\Phi^{-1}(x,t) = -\frac{x}{t^2}.$$

Then, I have

$$\forall A \in \mathcal{B}(\mathbf{R}), \quad P_Z(A) = \int_{\mathbf{R}^2} \mathbf{1}_A(t) f(x) f\left(\frac{x}{t}\right) \frac{|x|}{t^2} \lambda^{(2)}(dx, dt).$$

Now, I think I'm almost done. I set, for all $t \in \mathbf{R}^*$,

$$\begin{aligned} k(t) &= \int_{\mathbf{R}} f(x)\, f\left(\frac{x}{t}\right) \frac{|x|}{t^2} \lambda(dx) = \frac{1}{2\pi t^2} \int_{\mathbf{R}} |x| \\ &\quad \times \exp\left[-\frac{x^2}{2}\left(1 + \frac{1}{t^2}\right)\right] \lambda(dx) \\ &= \frac{1}{\pi t^2} \left[ -\frac{\exp\left[-\frac{x^2}{2}\left(1 + \frac{1}{t^2}\right)\right]}{1 + \frac{1}{t^2}} \right]_0^{+\infty} = \frac{1}{\pi(1+t^2)}, \end{aligned}$$

and I write $P_Z(A) = \int_{\mathbf{R}} \mathbf{1}_A(t)\, k(t)\lambda(dt)$ for all $A \in \mathcal{B}(\mathbf{R})$, using the Fubini–Lebesgue theorem.

**Laurent:** Great. You proved that the law of the random variable $Z$ has the density $k$. It is called the *Cauchy law.*

We can remark that the functions $x \mapsto \frac{x^k}{\pi(1+x^2)}$ are not integrable for $k \geq 1$. The Cauchy variables are not in $L^1(\mathbf{P})$, and thus they do not have any moment.

### 6.3.2.3 *Independence and correlation*

**Laurent:** Unfortunately, engineers are used to confusing independence and noncorrelation. We will see that the independence of random variables implies that they are not correlated. But the converse is false.

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**Proposition 6.34.** *In the probability space* $(\Omega, \mathcal{F}, \mathbf{P})$, *let* $X$ *and* $Y$ *be two independent random variables.*

(i) *If* $X$ *and* $Y$ *are in* $L^1(\mathbf{P})$, *we have*

$$\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y].$$

(ii) *If* $X$ *and* $Y$ *are in* $L^2(\mathbf{P})$, *then they are not correlated*, i.e. $\mathrm{Cov}(X, Y) = 0$.
*Consequently*,

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y).$$

**Proof.** (i) This point is a direct consequence of Theorem 6.30.

(ii) If $X$ and $Y$ are in $L^2(\mathbf{P})$, we can compute

$$\mathrm{Cov}(X, Y) = \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y] = 0,$$

which leads to

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X, Y) = \mathrm{Var}(X) + \mathrm{Var}(Y).$$

□

**Bernard:** Laurent, the identity function is not bounded... so it seems strange to pretend that we use the equality $\mathbf{E}\big[f(X)g(Y)\big] =$

$\mathbf{E}\big[f(X)\big]\mathbf{E}\big[g(Y)\big]$, which holds for the bounded measurable functions $f$ and $g$, to claim that $\mathbf{E}[XY] = \mathbf{E}[X]\,\mathbf{E}[Y]$...

**Laurent:** You are right. But Expression (6.3.3) does not only hold for bounded measurable functions $f$ and $g$. Actually, it holds as soon as $f(X)$ and $g(Y)$ are integrable. This is merely a question of construction of the integrals $\mathbf{E}\big[f(X)\big]$ and $\mathbf{E}\big[g(Y)\big]$.

There is another important point to mention. Some students are used to reducing the independence of the random variables $X$ and $Y$ to the equality $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$. And Theorem 6.30 precisely tells us that this equality is not sufficient to claim the independence of $X$ and $Y$. The role of the functions $f$ and $g$ is crucial in the characterization of the independence.

Moreover, the relation $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$ only tells us that $\mathrm{Cov}(X, Y) = 0$, which does not necessarily imply the independence of $X$ and $Y$. There are several classical examples of variables $X$ and $Y$ such that $\mathrm{Cov}(X, Y) = 0$ and $X$ and $Y$ are not independent. For instance, consider a discrete random variable $X$ such that

$$\mathbf{P}(X = 0) = \mathbf{P}(X = -1) = \mathbf{P}(X = 1) = 1/3,$$

and the random variable $Y = \mathbf{1}_{\{X=0\}}$.

You can see easily that $X$ and $Y$ are not independent (it suffices to remark that $\mathbf{P}(X = 0, Y = 0) \neq \mathbf{P}(X = 0)\mathbf{P}(Y = 0)$).

However, since $\mathbf{E}[X] = 0$ (by symmetry) and $XY = 0$ (by the definition of $Y$), we have

$$\mathrm{Cov}(X, Y) = \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y] = 0.$$

The variables $X$ and $Y$ are not independent, but they are not correlated.

**Bernard:** Oh, I understand that independence is an important property in the study of a set of random variables. I can't wait to see how this shows up in my engineering problem.

*Many things had just become clear in Bernard's mind.*

- *First, the framework of measure theory naturally applies to the products of measure spaces.*

- *The definition of the product measure on the product $\sigma$-algebra leads naturally to Fubini's Theorems, a result that had always seemed rather mysterious to him in the context of the Riemann integral.*
- *Thanks to the regularizing effect of the convolution product, the space $C_c^\infty$ of infinitely differentiable functions with compact support is dense in the space $L^p$.*
- *Independence between two random variables $X$ and $Y$ is defined very naturally from the law of the pair $(X, Y)$ and the product measure $P_X \otimes P_Y$. More precisely, $X$ and $Y$ are independent if and only if $P_{(X,Y)} = P_X \otimes P_Y$.*
- *The independence of variables $X$ and $Y$ is equivalent to the equality*

$$\mathbf{E}\big[f(X)g(Y)\big] = \mathbf{E}\big[f(X)\big]\mathbf{E}\big[g(Y)\big]$$

  *for all bounded measurable functions $f$ and $g$.*

*The young engineer was still struck by the fact that the independence of real random variables $X$ and $Y$ could not be summed up as $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$. The latter equality, which only expresses the noncorrelation of $X$ and $Y$, seemed a weaker notion than independence. He really needed to break the bad intuitions he had formed during his studies.*

*During this session, Bernard felt that he had acquired the knowledge he needed to model the discharges from his pharmaceutical plant. Not only would the quantities he had to consider be functions of position in space and time, but he would also need to be able to express the independence or nonindependence of the random variables that would represent the uncertainties in the model parameters.*

*He should start scoring points with Ann...*

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# Chapter 7

# Generalized Functions

*June 13rd, in the early morning.*

*Still in his favorite café, Laurent is morose. He has had some bad news about his health recently. Nevertheless, Bernard is about to arrive, and it is time to focus on the task he has set for himself.*

*He starts leafing through the book* Functional Analysis, *which he wrote a few years earlier. This time, he is going to have to introduce generalized functions from a slightly different angle. Or at least make the introduction of this concept quite natural for the engineer. He smiles as he recalls the state of mind he was in when writing this book. He was filled with the desire to present a complete mathematical theory.*

*Now, he has to return to the basics. He recalls that it was rather simple considerations that led to the development of this new theory. In fact, it is now part of many engineering courses, at least in France. They would just have to avoid the trap of going into too much technical detail.*

*Laurent is aware that Papadiamantis has no interest in teaching distributions. His mathematical background is very weak, and he is content with riding the wave of data science and artificial intelligence. Unfortunately, Papadiamantis had considerable influence on Ann's change of heart when she was his student.*

*The idea behind this theory was to stop thinking of functions in a pointwise manner, assigning values to real numbers; instead, they should be considered as functions, assigning values to areas of* $\mathbf{R}$*, such as open intervals. He will easily remind Bernard that the* *idealized vision of a real number does not really have a material reality. From his studies, Bernard should certainly be aware that when a scientist or engineer makes a physical measurement, they are unable to concentrate it on an (infinitesimal) point of* $\mathbf{R}$*. Measurements are carried out on physical points, each of which has a certain thickness.*

*He will explain to Bernard how the Lebesgue integral allows us to associate with any integrable function* $f$ *a generalized function* $\mathcal{T}_f$*, which is extremely regular in a new sense.*

*His ideas are clear for today's session. He still rereads the introduction to his book before his student arrives.*

*Bernard, meanwhile, was seriously questioned by Ann the day before. He is in a state of high stress...*

**Laurent:** Now, Bernard. We saw many times that considering the functions from a nonlocal point of view has opened many doors. My book will take us through the illuminating concepts of generalized functions, which in turn allow for the definition of new functional Hilbert spaces with more regularity: the Sobolev spaces. They are essential to the modern study of partial differential equations (PDEs).

{f} 515.COR.f

In this chapter, we will highlight the one-dimensional case, in particular in the definition of the convergence of a sequence of test-functions, in the notion of differentiation and for the definition and regularity of Sobolev spaces.

In the one-dimensional case, we will work in the context of an open interval $\mathcal{I} \subset \mathbf{R}$. In the multi-dimensional case, we will work with an open subset $\mathcal{U} \subset \mathbf{R}^d$.

To lighten the reading, when nothing is mentioned, it means that the definitions and results are valid both in one or several dimensions: they are stated then for an open subset $\mathcal{U}$, but we may only give the proofs of theorems in one dimension.

Note that the reference measure, especially as the $L^p$ spaces are mentioned, is the Lebesgue measure.

## 7.1 Distributions on the Space of Compactly Supported Smooth Functions

**Laurent:** The purpose of the *generalized functions*, also called *distributions*, is to obtain regularity properties such as derivability on some objects by using infinitely differentiable functions as dual functions.

**Bernard:** I'm sorry, Laurent. I understand every word of the sentence, one by one, but the sentence as a whole has no meaning for me.

**Laurent:** You are not making much of an effort today, Bernard. You are late. If you do not want to work to understand, I have more interesting things to do. My life will not last much longer, I want to make the most of it.

**Bernard:** Sorry. I am being very impatient, I know. But I really want to understand and I really need you. Ann is breathing down on my neck these days.

**Laurent:** Hmm... Consider a sequence of functions $(f_n)_{n\in\mathbf{N}}$, mapping $\mathbf{R}$ to $\mathbf{R}$, all differentiable, and admitting a limit $f_\infty$. What can you say about the differentiability of $f_\infty$?

**Bernard:** Let me see. If I remember my infinitesimal calculus well, it depends on the type of convergence that you consider.

**Laurent:** Indeed, to be able to say that the derivative of $f_\infty$ is the limit of $f'_n$, many regularity conditions on $(f'_n)_{n\in\mathbf{N}}$ are required. The idea of the distributions is to benefit from the regularity of *test-functions...*

**Bernard:** What do you mean by "test-functions"?

**Laurent:** As we will see soon, the functions will be identified as linear forms on vector spaces of infinitely differentiable, so-called *smooth*, functions.

### 7.1.1 *Definitions*

#### 7.1.1.1 *Test-functions*

**Notation 7.1.** We denote by $\mathcal{D}(\mathcal{U})$ the space $C_c^\infty(\mathcal{U})$ of all infinitely differentiable functions with compact support in $\mathcal{U}$:

$$\mathcal{D}(\mathcal{U}) = \{f \in C^\infty(\mathcal{U}) : \quad \text{supp}(f) \text{ bounded}\}.$$

**Bernard:** Please stop. We already have a notation for the space of compactly supported functions. Why is it interesting to have a new notation?

**Laurent:** The difference in notation relies on the difference between the associated topologies. What topology would you recommend?

**Bernard:** The sup-norm $f \mapsto \sup_{x\in\mathcal{U}} |f(x)|$ is well defined over $C_c^\infty(\mathcal{U})$; am I wrong?

**Laurent:** No, you are not. But the $C_c^\infty(\mathcal{U})$ space equipped with the sup-norm is actually not interesting because it is not complete.

**Bernard:** So, if I understand well, you mean the notation $\mathcal{D}(\mathcal{U})$ is attached to a particular topology?

**Laurent:** Exactly. But we will see only part of it, just enough for our purpose.

... we equip $\mathcal{D}(\mathcal{U})$ with the following notion of convergence.

**Definition 7.2.** **(Multi-dimensional case).** Let $(\phi_n)_{n\in\mathbf{N}}$ be a sequence of functions of $\mathcal{D}(\mathcal{U})$ and $\phi \in \mathcal{D}(\mathcal{U})$. We say that $\phi_n$ *converges to* $\phi$ *in* $\mathcal{D}(\mathcal{U})$ if:

(i) the supports of the limit $\phi$ and of all the functions $(\phi_n)$ are included in a fixed compact, that is,

$$\text{there exists a compact subset } K \text{ of } \mathcal{U} : \ \forall n \in \mathbf{N},$$
$$\text{supp}(\phi_n) \subset K, \ \text{supp}(\phi) \subset K;$$

(ii) $\forall k \in \mathbf{N}$ and $\alpha \in \mathbf{N}^d$, $D^\alpha \phi_n \longrightarrow D^\alpha \phi$ uniformly in $\mathcal{U}$:

$$\forall k \in \mathbf{N}, \forall \varepsilon > 0, \exists N \in \mathbf{N} : \ \forall n \in \mathbf{N},\ n \geq N,$$
$$\|D^\alpha \phi_n - D^\alpha \phi\|_\infty \leq \varepsilon,$$

where, for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{N}^d$, we denote

$$D^\alpha \psi = \partial^{|\alpha|}_{\alpha_1 \ldots \alpha_d} \psi = \frac{\partial^{|\alpha|}\psi}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}},$$

with $|\alpha| = \sum_{k=1}^d \alpha_k$.

Such functions are often called *test-functions*.

**Remark.** It is usual to extend $\phi \in \mathcal{D}(\mathcal{U})$ to $\mathbf{R}^d$ with the value 0 outside $\mathcal{U}$. The extension belongs naturally to $\mathcal{D}(\mathbf{R}^d)$.

**Notation 7.3.** We then write $\phi_n \to \phi$ in $\mathcal{D}(\mathcal{U})$ or $\phi_n \underset{\mathcal{D}(\mathcal{U})}{\longrightarrow} \phi$.

**Laurent:** Do not be afraid of this definition, which is a bit complicated because it is stated in the multi-dimensional case. The main sense can be easily understood in the case of an open interval.

{f} 515.COR.f

**Definition 7.4. (One-dimensional case).** Let $(\phi_n)_{n\in\mathbf{N}}$ be a sequence of functions of $\mathcal{D}(\mathcal{I})$ and $\phi \in \mathcal{D}(\mathcal{I})$. We say that $\phi_n$ *converges to* $\phi$ *in* $\mathcal{D}(\mathcal{I})$ if:

(i) the supports of the limit $\phi$ and of all functions $(\phi_n)$ are included in a fixed compact, that is, there exist $a, b \in \mathbf{R}$, $a < b$, such that $[a, b] \subset \mathcal{I}$ and

$$\forall n \in \mathbf{N},\ \operatorname{supp}(\phi_n) \subset [a, b], \text{ and } \operatorname{supp}(\phi) \subset [a, b];$$

(ii) $\forall k \in \mathbf{N}$, $\phi_n^{(k)} \to \phi^{(k)}$ uniformly in $\mathcal{I}$:

$$\forall k \in \mathbf{N}, \forall \varepsilon > 0, \exists N \in \mathbf{N} : \ \forall n \in \mathbf{N},\ n \geq N,$$
$$\|\phi_n^{(k)} - \phi^{(k)}\|_\infty \leq \varepsilon,$$

where $\psi^{(k)}$ denotes the $k$th derivative of $\psi$.

**Laurent:** The problem of $\mathcal{D}(\mathcal{U})$ is that it cannot be equipped with a metric that induces this topology...[1]

#### 7.1.1.2 *Distributions*

**Bernard:** You were talking about linear forms as the definition for your "generalized" functions. Since you insisted on the specific topology of $\mathcal{D}(\mathcal{U})$, I imagine that it plays its role now.

**Laurent:** Indeed. The main goal of the generalized functions is to help solve analytical problems. Then, the topology on $\mathcal{D}(\mathcal{U})$ is essential. Actually, generalized functions are defined as continuous linear forms.

**Bernard:** You talk about continuity, but you have not defined the topology, only the convergence of sequences.

**Laurent:** Exactly. This very special topology I do not want to detail has a crucial property: it is equivalent for a linear form to be continuous or *sequentially continuous*.

{f} 515.COR.f

**Definition 7.5.** If $T\colon \mathcal{D}(\mathcal{U}) \to \mathbf{R}$ is linear, $T$ is a *distribution* (or, equivalently, a *generalized function*) if it is *sequentially continuous*:

$$\forall(\phi_n), \phi \in \mathcal{D}(\mathcal{U}) \text{ such that } \phi_n \underset{\mathcal{D}(\mathcal{U})}{\longrightarrow} \phi, \qquad T(\phi_n) \to T(\phi). \quad (7.1.1)$$

The *space of distributions* $\mathcal{D}'(\mathcal{U})$ is the set of these linear forms.

Usually, applying a distribution $T \in \mathcal{D}'(\mathcal{U})$ to a function $\phi \in \mathcal{D}(\mathcal{U})$ is denoted by $\langle T, \phi\rangle$ $(= T(\phi))$. This notation is called a *duality bracket*.

**Bernard:** If I understand well, the convergence $T(\phi_n) \to T(\phi)$ is in $\mathbf{R}$, with its usual topology.

**Laurent:** What else could that be? The quantities $T(\phi_n)$ are real. The topology of $\mathcal{D}(\mathcal{U})$ only appears in $\phi_n \to \phi$. This is a major point of duality, as we saw it previously. It allows us to identify properties

[1]For further reference, see Bony (2001); Schwartz (1966); Gelfand and Šilov (1964).

of an object through a transfer to dual objects – here, test-functions – via a duality bracket. We already saw the concept of topological dual space, if you remember.

**Bernard:** Yes! I recall that a separable Hilbert space $\mathcal{H}$ can be identified to its topological dual space $\mathcal{H}'$, which is the set of continuous linear forms.

**Bernard:** Can we consider a sequence of distributions and study its convergence?

**Laurent:** This question is interesting, and we can address it immediately. There is a very weak notion of convergence. Actually it is the pointwise convergence on $\mathcal{D}(\mathcal{U})$.

{f} 515.COR.f

**Definition 7.6.** Let $(T_n)_{n\in\mathbf{N}}$ be a sequence of elements of $\mathcal{D}'(\mathcal{U})$, and let $T \in \mathcal{D}'(\mathcal{U})$. We say that $T_n$ tends to $T$ *in the sense of distributions* in $\mathcal{D}'(\mathcal{U})$ if

$$\forall \phi \in \mathcal{D}(\mathcal{U}), \quad \lim_{n\to+\infty} \langle T_n, \phi\rangle = \langle T, \phi\rangle .$$

### 7.1.2 *Some particular cases*

**Bernard:** You sometimes called distributions "generalized functions". I don't understand how their definition extends the concept of function.

**Laurent:** Ah. This is a very good question. Actually, many functions can be seen as distributions, which are called *regular distributions*.

#### 7.1.2.1 *Regular distributions*

{f} 515.COR.f

**Definition 7.7.** Let $f$ be locally integrable on $\mathcal{U}$, that is, $f \in L^1_{loc}\big(\lambda^{(d)}|_{\mathcal{U}}\big)$.

We define the linear form $\mathcal{T}_f \colon \mathcal{D}(\mathcal{U}) \to \mathbf{R}, \phi \longmapsto \int_{\mathcal{U}} f\phi \, d\lambda^{(d)}$.

The linear map $\mathcal{T}_f$ is a distribution, which is called a *regular distribution.*

**Justification.** The linearity is checked at once.

Moreover, for any sequence $(\phi_n)_{n\in\mathbf{N}}$ converging to $\phi$ in $\mathcal{D}(\mathcal{U})$ such that there exists a compact $K$ containing $\bigcup_{n\in\mathbf{N}} \operatorname{supp}(\phi_n)$, we get for all $n \in \mathbf{N}$,

$$|\langle \mathcal{T}_f, \phi_n - \phi \rangle| = \left| \int_{\mathcal{U}} f(\phi_n - \phi)\, d\lambda^{(d)} \right| \leq \|f \mathbf{1}_K\|_{L^1} \, \|\phi_n - \phi\|_\infty.$$

This proves the sequential continuity of $\mathcal{T}_f$. □

**Laurent:** We can now see how some functions can be identified as distributions.

{f} 515.COR.f

**Theorem 7.8.** *The map $\mathcal{T}\colon L^1_{loc}(\lambda|_{\mathcal{U}}) \to \mathcal{D}'(\mathcal{U})$ is one-to-one, and we identify $L^1_{loc}(\lambda^{(d)}|_{\mathcal{U}})$ as a vector subspace of $\mathcal{D}'(\mathcal{U})$.*

*We will denote $\mathcal{T}(f) = \mathcal{T}_f$ for $f \in L^1_{loc}$.*

**Notation 7.9.** When there is no ambiguity, we may therefore identify $\mathcal{T}_f$ and $f$ for functions belonging to $L^1_{loc}(\lambda^{(d)}|_{\mathcal{U}})$.

**Bernard:** The duality bracket can thus be seen as a kind of inner product on $L^2(\lambda^{(d)}|_{\mathcal{U}})$.

**Laurent:** Definitely, but for the fact that $\phi$ does not range the whole $L^2(\lambda^{(d)}|_{\mathcal{U}})$ space.

**Bernard:** Can we choose another measure than the Lebesgue measure in that particular definition of distribution?

**Laurent:** Of course, we can. But in that case, this distribution would not be called a regular distribution. For instance, we can consider the case of the integration of $\phi \in \mathcal{D}(\mathcal{U})$ with respect to the Dirac measure: for $a \in \mathcal{U}$, the map $\mathcal{D}(\mathcal{U}) \to \mathbf{R}$, defined by $\phi \mapsto \int_{\mathcal{U}} \phi(u)\delta_a(du)$, is a distribution...

{f} 515.COR.f

**Proof.** Consider $f \in L^1\big(\lambda^{(d)}|_{\mathcal{U}}\big)$ such that $\mathcal{T}_f = 0$.

Let us show that $f$ vanishes almost everywhere on any compact by constructing adapted test-functions and applying $\mathcal{T}$ to them.

Let $K$ be a compact subset of $\mathbf{R}^d$ such that $K \subset \mathcal{U}$.

Using the fact that $|f| = f\,\mathrm{sgn}(f)$, we will now prove that $|f|\,\mathbf{1}_K = 0$.

⋆ Define $s_K = \mathbf{1}_K\,\mathrm{sgn}(f)$, and for $n \geq 1$, $s_n = \mathbf{1}_K\, f/(|f| + 1/n)$. They are all bounded by 1 a.e.

Note that $f$ is finite a.e. since $f \in L^1\big(\lambda^{(d)}|_{\mathcal{U}}\big)$. So, for any $x$ such that $f(x)$ is finite,

$$\frac{f(x)}{|f(x)| + 1/n} - \mathrm{sgn}(f(x)) = \frac{f(x) - \mathrm{sgn}(f(x))\,(|f(x)| + 1/n)}{|f(x)| + 1/n}$$

$$= -1/n\,\frac{\mathrm{sgn}(f(x))}{|f(x)| + 1/n} \underset{n\to\infty}{\longrightarrow} 0.$$

So, $s_n \to s_K$ a.e. and $s_K$ is measurable and belongs to $L^1\big(\lambda^{(d)}|_{\mathcal{U}}\big)$.

⋆ Consider a Dirac sequence with the shrinking support $(\rho_n)_{n\geq 1}$ (see Definition 6.13). Recall that $\mathrm{supp}(\rho_n) \subset B_n$, where $B_n = \{y \in \mathbf{R}^d : \|y\|_\infty \leq 1/n\}$.

Note that $\mathrm{supp}\big(\rho_n * s_K\big) \subset K + B_n = \{x + y : x \in K \text{ and } y \in B_n\}$, and since $\mathcal{U}$ is an open set of $\mathbf{R}^d$ and $K$ is a compact, there exists $\delta > 0$ such that $K + B(0,\delta) \subset \mathcal{U}$.

So, there exists $N \geq 1$ such that, for all $n \in \mathbf{N}$, $n \geq N$, $\mathrm{supp}\big(\rho_n * s_K\big) \subset \mathcal{U}$ and $\rho_n * s_K$ belongs to $C_c^\infty(\mathcal{U})$.

For any $n \in \mathbf{N}$, with $n \geq N$, $\rho_n * s_K$ being in $C_c^\infty(\mathcal{U})$, we have

$$\langle \mathcal{T}_f, (\rho_n * s_K)\rangle = 0 = \int_{\mathcal{U}} f\big(\rho_n * s_K\big)\,d\lambda^{(d)}. \tag{7.1.2}$$

⋆ Since for all $n \geq 1$, $\|\rho_n * s_K\|_{L^\infty} \leq \|\rho_n\|_{L^1}\,\|s_K\|_{L^\infty} = 1$ thanks to Theorem 6.11 and to Definition 6.13(ii), $\big(\|\rho_n * s_K\|_{L^\infty}\big)_n$ is uniformly bounded by 1.

So, for all $n \geq 1$, $|f(\rho_n * s_K)| \leq |f|\,\|\rho_n * s_K\|_{L^\infty} \leq |f|$ a.e. and $f(\rho_n * s_K)$ belongs to $L^1\big(\lambda^{(d)}|_{\mathcal{U}}\big)$.

Also, Theorem 6.15 implies that $\|\rho_n * s_K - s_K\|_{L^1}$ tends to 0 so, up to a subsequence, $\rho_n * s_K \to s_K$ a.e. thanks to Corollary 4.5, and $f(\rho_n * s_K) \to f\, s_K = |f|\,\mathbf{1}_K$ a.e.

Applying the dominated convergence theorem yields $\|f(\rho_n * s_K) - |f|\mathbf{1}_K\|_{L^1} \to 0$.

From (7.1.2), we infer that

$$0 = \int_{\mathcal{U}} f(\rho_n * s_K)\, d\lambda^{(d)} \to \int_{\mathcal{U}} |f|\,\mathbf{1}_K\, d\lambda^{(d)}.$$

So, for all compact $K$ included in $\mathcal{U}$, $|f|\mathbf{1}_K = 0$. This concludes the proof. □

7.1.2.2 *The Dirac distribution*

**Bernard:** I recall that $\int_{\mathcal{U}} \phi(u)\delta_a(du) = \phi(a)$. However the sequential continuity is less obvious to me.

{f} 515.COR.f

**Definition 7.10.** The *Dirac distribution* at a point $a \in \mathcal{U}$, denoted by $\delta_a$, is the distribution

$$\forall \phi \in \mathcal{U}, \quad \delta_a(\phi) = \phi(a).$$

**Justification.** The linearity of $\delta_a$ is straightforward.

Moreover, we have, for all $\phi \in \mathcal{D}(\mathcal{U})$, $|\delta_a(\phi)| \le \|\phi\|_\infty$.

Let $(\phi_n)_{n\in\mathbf{N}}$ be a sequence in $\mathcal{D}(\mathcal{U})$ converging to $\phi \in \mathcal{D}(\mathcal{U})$. The convergence in $\mathcal{D}(\mathcal{U})$ implies that $\|\phi_n - \phi\|_\infty \to 0$. Then,

$$|\delta_a(\phi_n) - \delta_a(\phi)| = |\delta_a(\phi_n - \phi)| = |\phi_n(a) - \phi(a)| \le \|\phi_n - \phi\|_\infty.$$

So, $|\delta_a(\phi_n) - \delta_a(\phi)| \to 0$, and we conclude that $\delta_a$ is a distribution. □

**Bernard:** The notation $\delta_a$ stands, at the same time, for the Dirac measure and the Dirac distribution. In my understanding, they are not the same object.

**Laurent:** True. But we just saw that they are similar. This confusion of notations is justified by the equality $\langle \delta_a, \phi \rangle = \int_{\mathcal{U}} \phi(u)\delta_a(du)$.

One very important point is that the Dirac distribution is not regular, at least for the Lebesgue measure. It cannot be identified as a function in $L^1_{loc}(\lambda^d)$. But it can be approximated by regular distributions.

{f} 515.COR.f

**Theorem 7.11.** *Assume* $0 \in \mathcal{U}$ *and let* $(\rho_n)_{n\geq 1}$ *be a Dirac sequence with shrinking support. Then,* $\mathcal{T}_{\rho_n}$ *converges to* $\delta_0$*.*

**Proof.** Let $\phi \in \mathcal{D}(\mathcal{U})$. Let us show that $\langle \mathcal{T}_{\rho_n}, \phi\rangle = \int_{\mathcal{U}} \rho_n \phi \, d\lambda^{(d)} \to 0$ as $n \to \infty$. Let $\varepsilon > 0$. Since $\phi$ is continuous, there exists $\delta > 0$ such that for all $x \in B(0,\delta)$, $|\phi(x) - \phi(0)| < \varepsilon$. There also exists $N \geq 1$ such that for all $n \geq N$, the support of $\rho_n$ is a subset of $B(0,\delta)$.

Let $n \in \mathbf{N}$ and $n \geq N$. Then,

$$\left|\int_{\mathcal{U}} \rho_n \phi \, d\lambda^{(d)} - \phi(0)\right| = \left|\int_{B(0,\delta)} \rho_n(\phi - \phi(0)) \, d\lambda^{(d)}\right| \leq \int_{B(0,\delta)} \rho_n \, |\phi - \phi(0)| \, d\lambda^{(d)} \leq \varepsilon.$$

This concludes the proof. □

**Bernard:** So, the Dirac distribution can be approximated by a Dirac sequence. This name was well chosen... But the theorem does not prove that it is not regular. I heard about the Dirac function a hundred times in my physics and signal processing courses!

**Laurent:** It was false. However, it is used because of the theorem. Now, prove that it is not regular. The last proof can help you.

**Bernard:** Let me see. I assume that there exists $f \in L^1_{loc}(\lambda^d)$ such that $\forall \phi \in \mathcal{D}(\mathcal{U})$, $\int_{\mathbf{R}^d} f \phi d\lambda^{(d)} = \phi(0)$. If I choose $\phi$ such that $\phi(0) = 1$, $1 = \int_{\mathbf{R}^d} f(x)\phi(nx)\lambda^{(d)}(dx) = n^{-d} \int_{\mathbf{R}^d} f(x/n)\phi(x)\lambda^{(d)}(dx)$.

This last sequence goes to 0 thanks to the dominated convergence theorem!

**Laurent:** Very good, Bernard. I am truly happy to see you reasoning like this. Let us see one last special subspace of $\mathcal{D}'(\mathcal{U})$.

#### 7.1.2.3 $L^2\left(\lambda^{(d)}|_{\mathcal{U}}\right)$ *functions as distributions*

**Laurent:** As we discussed earlier, having a Hilbert structure to work on is very practical, and we will be using it extensively to solve PDEs. Let us study $L^2(\lambda^{(d)})$.

{f} 515.COR.f

**Proposition 7.12.** *The vector space $L^2(\lambda^{(d)})$ can be identified as a subspace of $\mathcal{D}'(\mathcal{U})$. In particular, for all $f \in L^2(\lambda^{(d)})$ and for all $\phi \in \mathcal{D}(\mathcal{U})$, $\langle \mathcal{T}_f, \phi \rangle = \langle f, \phi \rangle_{L^2}$.*

**Justification.** It suffices to remark that $L^2(\lambda^{(d)})$ is a subspace of $L^1_{loc}(\lambda^{(d)})$. The map $f \mapsto \mathcal{T}_f$ is thus one-to-one from $L^2(\lambda^{(d)})$ to $\mathcal{D}'(\mathcal{U})$. □

### 7.1.3 *Operations on distributions*

**Bernard:** I understand the concept of generalized functions. But can we make operations on them, as can be done with functions?

**Laurent:** If you mean derivation for instance, the smoothness of the test-functions provides deeper results for distributions than for functions. Mathematically, this is the main point of this new concept.

**Bernard:** Primarily, I was thinking of products. Since they are linear forms, one cannot multiply two distributions, right?

**Laurent:** Good thinking, Bernard. This is true. Nonetheless, there are other types of products that can be operated. Let us see.

#### 7.1.3.1 *Multiplying by a smooth function*

**Laurent:** The first natural way to define distributions starting from a fixed one is to multiply it by functions.

{f} 515.COR.f

**Definition 7.13.** Let $T$ be a distribution, i.e. $T \in \mathcal{D}'(\mathcal{U})$, and $f$ be a function in $C^\infty(\mathcal{U})$.

Then, the linear map $f\,T \colon \phi \in \mathcal{D}(\mathcal{U}) \mapsto \langle T, f\phi \rangle$ is a distribution.

**Justification.** The linearity is straightforward.

Let $(\phi_n)_{n\in\mathbf{N}}$ be a sequence in $\mathcal{D}(\mathcal{U})$ converging to $\phi \in \mathcal{D}(\mathcal{U})$. Note that, thanks to the general Leibniz formula (differentiation of a product), it is easy to show that $(f\phi_n)_{n\in\mathbf{N}}$ converges to $f\phi$ in $\mathcal{D}(\mathcal{U})$.

We have

$$\langle f\,T, \phi_n \rangle = \langle T, f\phi_n \rangle \to \langle T, f\phi \rangle = \langle f\,T, \phi \rangle .$$

We conclude that $f\,T$ is sequentially continuous and, thus, a distribution. □

**Remark.** Note that this operation is possible due to the stability of $\mathcal{D}(\mathcal{U})$ when multiplied by a $C^\infty$ function.

**Bernard:** I now begin to understand the importance of reasoning via the duality.

**Laurent:** The best is yet to come. But first, I would like you to solve the equation $xT = 0$ in $\mathcal{D}'(\mathbf{R})$. Of course, there is an abuse of notation: $xT$ stands for the distribution $fT$, where $f : x \mapsto x$.

**Bernard:** Let me see. We are looking for distributions $T$ such that $xT$ is the zero distribution... My guess is $\delta_0$. And more generally $c\delta_0$, with $c \in \mathbf{R}$. And they obviously solve the problem since $\langle x\delta_0, \phi\rangle = \langle \delta_0, x\phi\rangle = 0\phi(0) = 0$.

**Laurent:** Good guess. Now to the mathematical question: are there others?

**Bernard:** Reasoning with distributions requires reasoning with duality.
• I take a test-function $\psi \in \mathcal{D}(\mathbf{R})$. Then, $\langle xT, \psi\rangle = \langle T, x\psi\rangle$... To get the complete definition of $T : \phi \mapsto \langle T, \phi\rangle$, I need to express each $\phi \in \mathcal{D}(\mathbf{R})$ in the form $x\psi$ with $\psi \in \mathcal{D}(\mathbf{R})$. To do that, I need to single out the value of $\phi$ at $x = 0$. But the function $x \mapsto (\phi(x) - \phi(0))/x$, although in $C^\infty(\mathbf{R})$, does not have a compact support, so it's not in $\mathcal{D}(\mathbf{R})$...
• So, I choose a function $\theta \in \mathcal{D}(\mathbf{R})$ such that $\theta(0) = 1$. And $\phi = \phi(0)\theta + \phi - \phi(0)\theta$, noting that $\phi - \phi(0)\theta$ vanishes at $x = 0$. I see that $(\phi - \phi(0)\theta)/x$ belongs to $\mathcal{D}(\mathbf{R})$, and I can write it as

$$\frac{\phi(x) - \phi(0)\theta}{x} = \psi(x) = \begin{cases} \dfrac{\phi(x) - \phi(0)\theta(x)}{x} & \text{if } x \neq 0, \\ \phi'(0) - \phi(0)\theta'(0) & \text{if } x = 0. \end{cases}$$

• Going back to the equation, $0 = \langle T, x\psi\rangle = \langle T, \phi - \phi(0)\theta\rangle$, and I conclude that $\langle T, \phi\rangle = \phi(0)\langle T, \theta\rangle = \langle T, \theta\rangle \langle \delta_0, \phi\rangle$.

So, the only $T$ such that $xT = 0$ are of the form $c\delta_0$.

**Laurent:** Very good, Bernard. You have made real progress.

#### 7.1.3.2 *Differentiating a distribution*

**Laurent:** You will now understand the point of taking infinitely differentiable functions as test-functions.

**Bernard:** I was just thinking that the differentiability had been of no use until now...

**Laurent:** Be patient. We now tackle the issue of differentiating distributions.

{f} 515.COR.f

We can define the notion of derivative for any distribution. Then, for this definition, the distributions are infinitely differentiable.

**Definition 7.14.** Let $T$ be a distribution over $\mathcal{U} \subset \mathbf{R}^d$. For all $i \in \{1, \dots, d\}$, the functional

$$S_i\colon \mathcal{D}(\mathcal{U}) \to \mathbf{R}, \quad \phi \mapsto \langle S_i, \phi\rangle = -\langle T, \partial_{x_i}\phi\rangle$$

is a distribution.

The distributions $S_i$ are called the *derivatives of order* 1 of $T$. Using the notation $S_i = \partial_{x_i} T$, we have

$$\forall \phi \in \mathcal{D}(\mathcal{U}), \quad \langle \partial_{x_i} T, \phi\rangle = -\langle T, \partial_{x_i}\phi\rangle .$$

**Justification.** Fix $i$ to be in $\{1, \dots, d\}$. We first note that $\partial_{x_i} T$ is linear (since $T$ is linear).

Let $(\phi_n)_{n\in\mathbf{N}}$ be a sequence in $\mathcal{D}(\mathcal{U})$ converging to $\phi \in \mathcal{D}(\mathcal{U})$.

Note that $(\partial_{x_i}\phi_n)_{n\in\mathbf{N}}$ converges to $\partial_{x_i}\phi \in \mathcal{D}(\mathcal{U})$. By duality, $\langle \partial_{x_i} T, \phi_n\rangle = -\langle T, \partial_{x_i}\phi_n\rangle = -\langle T, \partial_{x_i}\phi_n\rangle \underset{n\to\infty}{\longrightarrow} -\langle T, \partial_{x_i}\phi\rangle = \langle \partial_{x_i} T, \phi\rangle$ .

We conclude that $\partial_{x_i} T$ is a distribution. □

**Definition 7.15.** For all $\alpha \in \mathbf{N}^d$, the distribution $D^\alpha T$ is defined as

$$\forall \phi \in \mathcal{D}(\mathcal{U}), \quad \langle D^\alpha T, \phi\rangle = (-1)^{|\alpha|} \langle T, D^\alpha \phi\rangle .$$

**Bernard:** I think I begin to understand why distributions are interesting. Contrary to usual functions, which can be nondifferentiable, any generalized function is differentiable at any order.

**Laurent:** You are perfectly right. And this important point results from the fact that the test-functions ($\phi \in \mathcal{D}(\mathcal{U})$) are extremely regular. And the distributions inherit this regularity. This is the beauty of duality.

But be careful: the differentiability of distributions is not an infinitesimal notion! This is why this definition is often called *derivation in the sense of distributions*.

**Bernard:** This is confusing. You defined the derivation of distributions. So, the definition applies to distributions.

**Laurent:** But we sometimes derive functions in the sense of distributions. In that case, we use the identification of $L^1_{loc}$ functions for regular distributions. The derivative of $f \in L^1_{loc}$ in the sense of distributions is the derivative distribution $\mathcal{T}'_f$.

Let us see how all these are written in one dimension for an interval.

{f} 515.COR.f

**Let us consider the particular case of one dimension.**

**Definition 7.16.** Let $T$ be a distribution over $\mathcal{I}$ an open interval of $\mathbf{R}$.

The functional

$$S\colon \mathcal{D}(\mathcal{I}) \to \mathbf{R}, \quad \phi \mapsto -\langle T, \phi' \rangle$$

is a distribution, called the *first derivative* of $T$ and denoted by $T' = S$.

For all $k \in \mathbf{N}$, the distribution $T^{(k)}$ is defined as

$$\forall \phi \in \mathcal{D}(\mathcal{I}), \quad \langle T^{(k)}, \phi \rangle = (-1)^k \langle T, \phi^{(k)} \rangle.$$

**Theorem 7.17.** *Assume that $T' = 0$ in $\mathcal{D}'(\mathcal{I})$.*

*Then, $T$ is a regular distribution, and there exists $C \in \mathbf{R}$ such that $T = \mathcal{T}_{x \mapsto C}$.*

**Proof.** Let $\theta \in \mathcal{D}(\mathcal{I})$ such that $\int_{\mathcal{I}} \theta \, d\lambda = 1$. Let $\phi \in \mathcal{D}(\mathcal{I})$.

Define $\psi : x \mapsto \int_{\mathcal{I}} \left(\phi \, d\lambda - \theta \int_{\mathcal{I}} \phi\right) \mathbf{1}_{[\min(\text{supp}(\phi) \cup \text{supp}(\theta)), x]} \, d\lambda$. Note that $\psi \in \mathcal{D}(\mathcal{I})$.

Then, $\langle T, \psi \rangle = 0 = -\langle T, \psi' \rangle = -\langle T, \phi \rangle + \int_{\mathcal{I}} \phi \, d\lambda \, \langle T, \theta \rangle$.

So, $T$ is regular and, by identification, is equal to a constant. □

**Notation 7.18.** Let $f \in L^1_{loc}\big(\lambda|_{\mathcal{I}}\big)$. In the case where the derivative of $f$ in the sense of distributions, $(\mathcal{T}_f)'$, is regular, i.e. if there exists $g \in L^1_{loc}\big(\lambda|_{\mathcal{I}}\big)$ such that $(\mathcal{T}_f)' = \mathcal{T}_g$, then we note that $f' = g$. It is a very common abuse of notation.

**Bernard:** Yet, it seems dangerous to use the same notation for different notions.

**Laurent:** Come on, Bernard. You are right, you need to be careful. But beware of the fact that you already did it for the integral symbol. You have improved; I believe you can manage. Let us see how to handle it in a practical case.

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**Example.** Let $\mathcal{I} = \mathbf{R}$. Consider the Heaviside function $H_0 = \mathbf{1}_{\mathbf{R}^+}$.

It is locally integrable with respect to the Lebesgue measure, so $\mathcal{T}_{H_0}$ is a distribution.

$H_0$ is not continuous at 0, so it is not differentiable at 0 (in the infinitesimal sense)...

However, in the sense of distributions, we can write that, for all $\phi \in \mathcal{D}(\mathbf{R})$,

$$\big\langle (\mathcal{T}_{H_0})', \phi \big\rangle = -\big\langle \mathcal{T}_{H_0}, \phi' \big\rangle = -\int_{\mathbf{R}} \mathbf{1}_{x \geq 0} \, \phi' \, d\lambda$$

$$= -\int_{\mathbf{R}_+} \phi'(x) \, \lambda(dx) = \phi(0) = \langle \delta_0, \phi \rangle \, .$$

So, the derivative of $H_0$ in the sense of distributions is the Dirac mass at 0.

**Bernard:** I understand better and better that these new notions replace beneficially the derivative and antiderivative that come from infinitesimal calculus. However, I wonder if these notions, the old and the new, are linked.

**Laurent:** Let us have a look at the Fundamental theorem of calculus. Recall that if $f$ is locally integrable with respect to the Lebesgue measure, any primitive function $F$ in the sense of Theorem 5.32 is differentiable (in the infinitesimal sense) almost everywhere and $F' = f$ a.e. In the formalism of distributions, here is how it is written.

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**Theorem 7.19.** *Let $f \in L^1_{loc}(\lambda)$, $a \in \mathbf{R}$ and $F$ be defined as in Theorem 5.32. Then, $(\mathcal{T}_F)' = \mathcal{T}_f$.*

**Proof.** We first note that the function $F$ being continuous, it is locally integrable, Then, the regular distribution $\mathcal{T}_F$ is well defined. It remains to obtain its derivative.

Let $\phi \in \mathcal{D}(\mathcal{I})$. Let $\alpha, \beta \in \mathbf{R}$, $\alpha < \beta$ be such that $\text{supp}(\phi) \subset [\alpha, \beta]$. Assume that $a < \alpha$.

Then, from the Fubini–Lebesgue theorem,

$$\begin{aligned}\big\langle (\mathcal{T}_F)', \phi \big\rangle &= -\int_{(\alpha,\beta)} F\phi' \, d\lambda = -\int_{(\alpha,\beta)} \phi'(x) \\ &\quad \times \left[\int_{(\alpha,\beta)} \mathbf{1}_{(a,x)}(y) f(y)\, \lambda(dy)\right] \lambda(dx) \\ &= -\int_{(\alpha,\beta)} \phi'(x) \int_{(\alpha,\beta)} \mathbf{1}_{(\alpha,x)}(y) f(y)\, \lambda(dy)\, \lambda(dx) \\ &= -\int_{(\alpha,\beta)^2} \phi'(x) \mathbf{1}_{(a,x)}(y) f(y)\, \lambda^{(2)}(dx, dy).\end{aligned}$$

As $\mathbf{1}_{(\alpha,\beta)}(x)\mathbf{1}_{(\alpha,x)}(y) = \mathbf{1}_{\{\alpha<x<y<\beta\}}(x,y) = \mathbf{1}_{(y,\beta)}(x)\mathbf{1}_{(\alpha,\beta)}(y)$, we get

$$\big\langle (\mathcal{T}_F)', \phi \big\rangle = -\int_{(\alpha,\beta)^2} \phi'(x) \mathbf{1}_{(y,\beta)}(x) f(y)\, \lambda^{(2)}(dx, dy)$$

$$= -\int_{(\alpha,\beta)} f(y)(\underbrace{\phi(\beta)}_{=0} - \phi(y))\,\lambda(dy)$$

$$= \int_{(\alpha,\beta)} f(y)\phi(y)\,\lambda(dy) = \langle \mathcal{T}_f, \phi \rangle\,.$$

The result follows.

The cases $\alpha < a < \beta$ and $\beta < a$ are treated in the same way. □

**Laurent:** When you consider locally integrable functions as belonging to a larger world, i.e. the distributions' one, then basic analytical questions can be answered.

**Bernard:** Then, if I sum up, in the sense of distributions, any locally integrable function is differentiable, even if it is not differentiable in the classical sense.

But in the specific case where $f$ admits a derivative $f'$ in the infinitesimal sense, are the two different notions of derivative connected?

**Laurent:** When $f$ is a differentiable function in the infinitesimal sense and the derivative $f'$ is locally integrable, the regular distribution $\mathcal{T}_{f'}$ is well defined, as is $\mathcal{T}_f$ since $f$ is continuous. And we can claim that $\mathcal{T}_{f'} = \big(\mathcal{T}_f\big)'$.

**Bernard:** This follows from Theorem 5.33, right? We have $f(x) - f(c) = \int_{\mathcal{I}} \mathbf{1}_{(c,x)} f'\, d\lambda$. And we get $\mathcal{T}_{f'} = \big(\mathcal{T}_{f+f(c)}\big)' = \big(\mathcal{T}_f + \mathcal{T}_{f(c)}\big)' = \big(\mathcal{T}_f\big)'$ from Theorem 7.19. It makes me wonder... Is it enough that $f$ is differentiable almost everywhere to consider the distribution $\mathcal{T}_{f'}$?

**Laurent:** Definitely. But in that case, we cannot say that $f$ is continuous and Theorem 5.33 cannot be applied. For instance, let us study the case where $f$ is piecewise $C^1$.

**Theorem 7.20 (Jump formula in dimension 1).** *Let $\mathcal{I} = (a, b)$ with $a < b$. Let $a = a_0 < a_1 < \ldots < a_k < a_{k+1} = b$ be points of $\mathcal{I}$ ($k \geq 0$).*

*Assume that the function $f\colon \mathcal{I} \to \mathbf{R}$ satisfies: for all $i = 0, \dots, k$, $f$ is differentiable on $(a_i, a_{i+1})$ and $f'$ is continuous on $(a_i, a_{i+1})$ and can be extended by continuity on $[a_i, a_{i+1}]$ (such a function $f$ is said to be* piecewise continuously differentiable*).*

*Then, we have*

$$(\mathcal{T}_f)' = \mathcal{T}_{f'} + \sum_{i=1}^{k} \big(f(a_i^+) - f(a_i^-)\big)\delta_{a_i},$$

*where $f(a_i^+) = \lim_{\substack{x\to a_i \\ x>a_i}} f(x)$ and $f(a_i^-) = \lim_{\substack{x\to a_i \\ x<a_i}} f(x)$.*

**Proof.** The functions $f$ and $f'$ are piecewise continuous, so they are locally integrable.

For $\phi \in \mathcal{D}(\mathcal{I})$, integration by parts gives

$$\begin{aligned}
\big\langle (\mathcal{T}_f)', \phi \big\rangle &= -\int_{(a,b)} f(x)\phi'(x)\,\lambda(dx) \\
&= -\sum_{i=0}^{k} \int_{(a_i,a_{i+1})} f(x)\phi'(x)\,\lambda(dx) \\
&= -\sum_{i=0}^{k} \left( \big[f(x)\phi(x)\big]_{a_i^+}^{a_{i+1}^-} - \int_{(a_i,a_{i+1})} f'(x)\phi(x)\,\lambda(dx) \right).
\end{aligned}$$

So,

$$\begin{aligned}
\big\langle (\mathcal{T}_f)', \phi \big\rangle &= f(a_0^+)\underbrace{\phi(a_0^+)}_{=0} - f(a_1^-)\phi(a_1^-) \\
&\quad + \sum_{i=1}^{k-1} \big(f(a_i^+)\phi(a_i^+) - f(a_{i+1}^-)\phi(a_{i-1}^-)\big) \\
&\quad + f(a_k^-)\phi(a_k^-) - f(a_{k+1}^-)\underbrace{\phi(a_{k+1}^-)}_{=0} \\
&\quad + \int_{(a,b)} f'(x)\phi(x)\,\lambda(dx).
\end{aligned}$$

Since $\phi$ is continuous, for all $i \in \{1, \dots, k\}$, we have $\phi(a_i^-) = \phi(a_i^+) = \phi(a_i)$. So,

$$\langle (\mathcal{T}_f)', \phi \rangle = \sum_{i=1}^{k} (f(a_i^+) - f(a_i^-)\phi(a_i) + \int_{(a,b)} f'\phi d\lambda,$$

and the result follows. □

## 7.2 Sobolev Spaces in One Dimension

### 7.2.1 *Definition of $H^k(\lambda|_{\mathcal{I}})$, $k \geq 1$*

**Bernard:** In my opinion, it's too complicated to have to go through the distribution $\mathcal{T}_f$ in order to deal with the derivative of $f$.

**Laurent:** Do not worry, Bernard. Until now, we have not stated any property of the derivative $f'$ of $f$ in the sense of distributions. Particularly, it is interesting to see it as an element of some functional space and to be able to approximate it by a sequence of more classical functions.

**Bernard:** It's not clear to me.

**Laurent:** Let us discover together the Sobolev spaces.

{f} 515.COR.f

The Sobolev spaces are the measure theory counterpart of the spaces of continuously differentiable functions. In the following, the regular distribution $\mathcal{T}_f$ is identified with the locally integrable function $f$. In the same way, when $(\mathcal{T}_f)'$ is a regular distribution, it is identified with $f'$.

**Bernard:** In my experience, the abuses of notation are traps until I become completely familiar with the objects.

**Laurent:** You are wise, and we are going to be cautious. At first, be aware that the derivative of a regular distribution is not necessarily a regular distribution.

{f} 515.COR.f

**Definition 7.21.** The *Sobolev space of order 1* over an open interval $\mathcal{I}$ of $\mathbf{R}$ is defined as

$$H^1(\lambda|_{\mathcal{I}}) = \{v \in L^2(\lambda|_{\mathcal{I}}) : v' \in L^2(\lambda|_{\mathcal{I}})\},$$

where $\lambda|_{\mathcal{I}}$ denotes the Lebesgue measure on $\mathbf{R}$ restricted to $\mathcal{I}$ and $v'$ is the locally integrable function identified with the regular distribution $(\mathcal{T}_v)'$.

**Definition 7.22.** The bilinear form on $H^1(\lambda|_{\mathcal{I}})$

$$\langle v, w\rangle_{H^1} : (v, w) \mapsto \langle v, w\rangle_{L^2} + \langle v', w'\rangle_{L^2}$$

defines an inner product on $H^1(\lambda|_{\mathcal{I}})$.

The induced norm

$$v \mapsto \|v\|_{H^1} = \sqrt{\|v\|_{L^2}^2 + \|v'\|_{L^2}^2}$$

satisfies, for all $v \in H^1(\lambda|_{\mathcal{I}})$,

$$\|v\|_{L^2} \leq \|v\|_{H^1} \quad \text{and} \quad \|v'\|_{L^2} \leq \|v\|_{H^1}. \tag{7.2.1}$$

**Justification.** We note that $\langle ., .\rangle_{H^1}$ is a symmetric bilinear form as the sum of two inner products.

Moreover, $\langle v, v\rangle_{H^1} = 0$ implies that $\|v'\|_{L^2} = 0$, so $(\mathcal{T}_v)' = 0$ and $v$ is a constant in $L^2$. Since $\|v\|_{L^2} = 0$, we conclude that $v = 0$. ☐

**Bernard:** Once we've equipped a space with an inner product, shouldn't we hope that it is a Hilbert space in order to benefit from all its nice properties for free?

**Laurent:** Definitely. Your guess is right.

{f} 515.COR.f

**Theorem 7.23.** *The Sobolev space $H^1(\lambda|_{\mathcal{I}})$ equipped with $\|.\|_{H^1}$ is complete.*

*Consequently, $H^1(\lambda|_{\mathcal{I}})$ equipped with $\langle ., .\rangle_{H^1}$ is a Hilbert space.*

**Proof.** Let $(v_n)_{n\in\mathbf{N}}$ be a Cauchy sequence of the elements of $H^1\big(\lambda|_{\mathcal{I}}\big)$.

According to Inequalities (7.2.1), $(v_n)_{n\in\mathbf{N}}$ and $(v_n')_{n\in\mathbf{N}}$ are both Cauchy sequences of $L^2\big(\lambda|_{\mathcal{I}}\big)$. Since $L^2\big(\lambda|_{\mathcal{I}}\big)$ is complete, there exist $v, w \in L^2\big(\lambda|_{\mathcal{I}}\big)$ such that $v_n \to v$ and $v_n' \to w$ in $L^2\big(\lambda|_{\mathcal{I}}\big)$.

We thus need to show that $(\mathcal{T}_v)' = \mathcal{T}_w$, i.e. $v' = w$, which leads to $v \in H^1\big(\lambda|_{\mathcal{I}}\big)$.

Let $\phi \in \mathcal{D}(\mathcal{I})$. We have

$$\big\langle (\mathcal{T}_{v_n})', \phi \big\rangle = -\int_{\mathcal{I}} v_n \phi' \, d\lambda \to -\int_{\mathcal{I}} v\phi' \, d\lambda = \big\langle (\mathcal{T}_v)', \phi \big\rangle \qquad \text{as } n \to \infty,$$

since $\left|\int_{\mathcal{I}} (v_n - v)\phi'\right| \le \|v_n - v\|_{L^2} \|\phi'\|_{L^2}$ for all $n$, thanks to the Cauchy–Schwarz inequality.

But we also have

$$\begin{aligned} \big\langle (\mathcal{T}_{v_n})', \phi \big\rangle = \big\langle \mathcal{T}_{(v_n)'}, \phi \big\rangle &= \int_{\mathcal{I}} v_n' \phi \, d\lambda \to \int_{\mathcal{I}} w\phi \, d\lambda \\ &= \langle \mathcal{T}_w, \phi \rangle \quad \text{as } n \to \infty. \end{aligned}$$

By uniqueness of the limit, we deduce that $v' = w$, so $(v_n)_{n\in\mathbf{N}}$ converges to $v$ in $H^1\big(\lambda|_{\mathcal{I}}\big)$. We conclude that $H^1\big(\lambda|_{\mathcal{I}}\big)$ is complete. □

**Bernard:** I wonder if it makes sense to consider functions whose derivatives, in the sense of distributions, are in $L^p$ instead of $L^2$.

**Laurent:** Indeed, some people denote the space $H^1$ by $W^{1,2}$ in order to specify the fact that the derivatives of order 1 are in $L^2$. More generally, we can define the space $W^{1,p}$, for $p \ge 1$, as the space of functions in $L^p$ whose derivative of order 1 is in $L^p$. But the Hilbert structure is lost if $p$ is not 2.

**Bernard:** In the same way as we defined spaces $C^k$ of functions with continuous derivatives of order $m \le k$, is it interesting to consider that functions whose derivative of order $k$ are in $L^p$?

**Laurent:** Yes again. The Sobolev space $H^k = W^{k,2}$ is defined as the space of functions whose derivatives of order $m = 0, \ldots, k$ are in $L^2$.

{f} 515.COR.f

**Definition 7.24.** For $k \in \mathbf{N}$, we define the *Sobolev space of order* $k$ over an open interval $\mathcal{I}$ of $\mathbf{R}$ as

$$H^k(\lambda|_{\mathcal{I}}) = \left\{u \in L^2(\lambda|_{\mathcal{I}}) : u' \in H^{k-1}(\lambda|_{\mathcal{I}})\right\}$$
$$= \{u \in L^2(\lambda|_{\mathcal{I}}) : u^{(m)} \in L^2(\lambda|_{\mathcal{I}}), \quad \forall m = 0, \ldots, k\},$$

where $\lambda|_{\mathcal{I}}$ denotes the Lebesgue measure on $\mathbf{R}$ restricted to $\mathcal{I}$.

**Remark.** The space $H^k(\lambda|_{\mathcal{I}})$ is a Hilbert space when equipped with the inner product

$$\langle v, w\rangle_{H^k} : (u, v) \mapsto \sum_{0 \le m \le k} \left\langle u^{(m)}, v^{(m)}\right\rangle_{L^2}.$$

**Bernard:** I'm not sure I understand the point of defining this inner product, instead of simply the $L^2$ one, since $H^k$ is a subspace of $L^2$. In my understanding, it would benefit from the Hilbert properties of $L^2$. Particularly, what is the point of considering all the derivatives up to $k$?

**Laurent:** You could have asked the question for $H^1$. The issue is the same. It is a good question, but it shows that you are forgetting an important point of topology. You have mastered the finite-dimensional vector spaces. As you now know, the functional spaces are infinite-dimensional. And in order to get some useful properties, such as projections, the spaces have to be complete for the topology we consider.

**Bernard:** I know. So, what?

**Laurent:** Unfortunately, the spaces $H^k(\lambda|_{\mathcal{I}})$ are not complete for the $L^2$ topology. For instance, $C^\infty([0, 1])$ is a subset of $H^k(\lambda|_{(0,1)})$ and $C^\infty([0, 1])$ is dense in $L^2(\lambda|_{(0,1)})$ (for the $L^2$-norm). So, $H^k(\lambda|_{(0,1)})$ is dense in $L^2(\lambda|_{(0,1)})$ (for the $L^2$-norm). We can conclude that $H^k(\lambda|_{(0,1)})$ is not a closed subset of $L^2(\lambda|_{(0,1)})$, and consequently it is not complete.

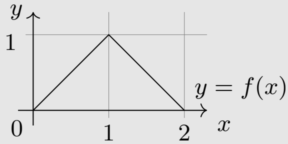
**Bernard:** Oh! I could have thought of that by myself. So, the space $H^k(\lambda|_{(0,1)})$ is not equal to $L^2(\lambda|_{(0,1)})$!

**Laurent:** Indeed. For instance, the Heaviside function is not in $H^1$ since its derivative is the Dirac distribution. You seem hesitant. Let us consider an example.

{f} 515.COR.f

**Example.** Let $\mathcal{I} = (0,2)$. Consider the following "hat"-function:

$$f : x \mapsto \begin{cases} x & \text{if } x \in (0,1], \\ 2-x & \text{if } x \in [1,2). \end{cases}$$




We will see later that $f$ plays a very important role in the approximation by finite elements.

The function $f$ has no derivative on $(0,2)$ in the infinitesimal sense, but it is continuous over $[0,2]$. This implies $f \in L^2(\lambda|_{(0,2)})$. Thus, its derivative in the sense of distributions can be defined. Let us compute it.

Let $\phi \in \mathcal{D}\big((0,2)\big)$.

$$\begin{aligned} \big\langle (\mathcal{T}_f)', \phi \big\rangle &= -\big\langle \mathcal{T}_f, \phi' \big\rangle = -(f, \phi')_{L^2} = -\int_{(0,2)} f\phi' \, d\lambda \\ &= -\int_{(0,1)} x\phi'(x)\,\lambda(dx) - \int_{(1,2)} (2-x)\phi'(x)\,\lambda(dx) \\ &= -[x\phi(x)]_0^1 + \int_{(0,1)} \phi\, d\lambda - [(2-x)\phi(x)]_1^2 - \int_{(1,2)} \phi\, d\lambda \\ &= -\phi(1) + \phi(1) - \int_{(0,2)} (\mathbf{1}_{(1,2)} - \mathbf{1}_{(0,1)})\phi\, d\lambda \\ &= \int_{(0,2)} (-\mathbf{1}_{(1,2)} + \mathbf{1}_{(0,1)})\phi\, d\lambda. \end{aligned}$$

We deduce that

$$(\mathcal{T}_f)' = -\mathbf{1}_{(1,2)} + \mathbf{1}_{(0,1)},$$

and $(\mathcal{T}_f)'$ is a regular distribution.
As $f' \in L^2\big(\lambda|_{(0,2)}\big)$, we conclude that $f \in H^1\big(\lambda|_{(0,2)}\big)$.

Moreover, we can claim that the function $f$ is not in $H^2\big(\lambda|_{(0,2)}\big)$. More precisely, we have $(\mathcal{T}_f)'' = -2\delta_1$.

**Bernard:** It seems that this computation can be avoided by using directly the jump formula stated in Theorem 7.20.

**Laurent:** Good! I am happy to see that you are warming up to the subject.

### 7.2.2 *Regularity of the Sobolev spaces and density*

**Laurent:** Let us now study what the functions in the Sobolev spaces look like.

**Bernard:** Oh, that question seems very strange to me. In my understanding, the local properties are given by the regularity, which is an infinitesimal issue, and the Sobolev spaces only rely on integration properties. So, regularity doesn't seem to fit here.

**Laurent:** You are absolutely right. However, as well as there are some relations between the Lebesgue and Riemann integrals, there are some infinitesimal properties satisfied by functions in the Sobolev spaces... at least in one dimension.

{f} 515.COR.f

**Theorem 7.25.** *Let $u$ be an element of $H^1\big(\lambda|_{\mathcal{I}}\big)$, where $\lambda|_{\mathcal{I}}$ denotes the Lebesgue measure on $\mathbf{R}$ restricted to the open interval $\mathcal{I}$.*

*There exists a function $\tilde{u}$ continuous on $\overline{\mathcal{I}}$ in the class $u$,* i.e. *identifying the functions and their equivalence classes in $H^1\big(\lambda|_{\mathcal{I}}\big)$, the equality $\tilde{u} = u$ holds.*

*Hereinafter, an element $u$ of $H^1\big(\lambda|_{\mathcal{I}}\big)$ is identified as a continuous representative function, that is, with an abuse of notation, $u$ is supposed to be continuous.*

*Moreover, if $\mathcal{I} = (a, b)$, $a, b \in \mathbf{R}$, $a < b$, then the two following assertions hold:*

(i) *There exists a constant $c > 0$, depending only on $b - a$, such that*

$$\forall u \in H^1\big(\lambda|_{(a,b)}\big), \quad \|u\|_\infty \leq c\,\|u\|_{H^1}. \tag{7.2.2}$$

(ii) *From any bounded sequence of* $H^1(\lambda|_{(a,b)})$*, one can extract a subsequence that converges to a function in* $C([a,b])$ *for the topology of* $\|.\|_\infty$.

**Laurent:** You must remember here that $H^1$ is defined from a quotient space. Consequently, elements of $H^1$ are not functions but classes of functions which are equal almost everywhere.

**Bernard:** I understand why this statement is so complicated. When we read $u \in H^1$, I understand that $u$ is not a function and that there exists a continuous function $\tilde{u}$ in the class $u$.

Finally, with the abuse of notation $\tilde{u} = u$ (instead of $\tilde{u} \in u$), we can write $u(x) = \tilde{u}(x)$ almost everywhere and $\tilde{u}' = u'$ in the sense of distributions.

**Laurent:** Wow! Students seldom reach as clear a vision of this very difficult point. As you said, we are used to manipulate these classes of functions as if they were simply functions. And this is very confusing.

{f} 515.COR.f

**Remark.** The canonical injection from $H^1(\lambda|_{\mathcal{I}})$ to $C(\overline{\mathcal{I}})$ is a *continuous* linear map (i) and even a *compact operator* (ii).

**Proof.** Let $u$ be a function in $H^1(\lambda|_{\mathcal{I}})$. By definition, $u'$ belongs to $L^2(\lambda|_{\mathcal{I}})$, so $u'$ is locally integrable. Thanks to Theorem 5.32, any function $U$ such that

$$\forall (x,y) \in \mathcal{I}^2,\ x > y \quad U(x) - U(y) = \int_{[y,x]} u' \, d\lambda$$

is continuous everywhere on $\mathcal{I}$.

By extending $u'$ by the value 0 on $\mathbf{R} \setminus \mathcal{I}$, the previous formula implies that $U$ is continuous on $\overline{\mathcal{I}}$. Moreover, $U$ is differentiable almost everywhere and $U'(x) = u'(x)$ for almost every $x \in \mathcal{I}$.

Then, thanks to Theorem 7.19 and by definition of the derivation in the sense of distributions, $(\mathcal{T}_U)' = \mathcal{T}_{u'} = (\mathcal{T}_u)'$. So, $(\mathcal{T}_u - \mathcal{T}_U)' = 0$ and, consequently, Theorem 7.17 implies that there exists $C \in \mathbf{R}$ such that $u - U = C$ in $L^1_{loc}(\lambda|_{\mathcal{I}})$.

Since $U$ is continuous thanks to the Fundamental theorem of calculus, the result is proved by taking $\tilde{u} = U + C$.

Assume now that $\mathcal{I} = (a, b)$, with $a < b$.

(i) Let $x, y \in \overline{\mathcal{I}}$ with $x > y$. Then, considering $u$ as a continuous function, a representative of the class $u$ in $H^1\big(\lambda|_{\mathcal{I}}\big)$, we can write $u(x) = u(y) + \int_{[y,x]} u'\, d\lambda$.

The Cauchy–Schwarz inequality yields $|u(x)| \leq |u(y)| + \sqrt{|y - x|}\ \|u'\|_{L^2}$.

We note that this inequality also holds when $x \leq y$.

Then, we can claim that $|u(x)|^2 \leq 2\big(|u(y)|^2 + |y - x|\ \|u'\|^2_{L^2}\big)$.

Integrating over $(a, b)$ with respect to the variable $y$ yields

$$(b - a)\|u\|^2_\infty \leq 2\big(\|u\|^2_{L^2} + (b - a)^2\ \|u'\|^2_{L^2}\big).$$

Consequently,

$$\begin{aligned}\|u\|^2_\infty &\leq \frac{2}{b - a}\|u\|^2_{L^2} + 2(b - a)\ \|u'\|^2_{L^2} \\ &\leq 2 \max\big((b - a)^{-1}, (b - a)\big)\ \|u\|^2_{H^1}.\end{aligned}$$

(ii) The proof is a direct consequence of the Arzelà–Ascoli theorem (e.g. see Rudin (1987, Chapter 11 (Harmonic functions))).

According to this result, the closure of $B = \{u \in H^1\big(\lambda|_{(a,b)}\big) : \|u\|_{H^1} \leq 1\}$ in $C([a, b])$, for the topology of $\|.\|_\infty$, is compact if it is:

- $\star$ *bounded as a subset of* $C([a, b])$ (for the norm $\|.\|_\infty$);
- $\star$ *equicontinuous*, that is, for every $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\forall u \in B, \forall x, y \in \mathcal{I}, \quad |x - y| < \eta \Rightarrow |u(x) - u(y)| < \varepsilon.$$

Let us check that these two conditions hold.

Thanks to (7.2.2), for every $u \in B$, $\|u\|_\infty \leq c$, so $B$ is bounded in $C([a, b])$. The first point is proved.

Let $x, y \in [a, b]$, with $x > y$. As before, we can write $u(x) - u(y) = \int_{[y,x]} u' \, d\lambda$. As $\|u\|_{H^1} \le 1$, the Cauchy–Schwarz inequality yields

$$|u(x) - u(y)| \le \sqrt{|y - x|} \; \|u'\|_{L^2} \le \sqrt{|y - x|} \; \|u\|_{H^1} \le \sqrt{|y - x|}.$$

By noting that this inequality also holds when $x \le y$, we conclude that $B$ is equicontinuous and, finally, its closure is compact.

Let $(u_n)_{n \in \mathbf{N}}$ be a sequence in $H^1\big(\lambda|_{(a,b)}\big)$ bounded by $M > 0$.

The sequence $(u_n/M)_{n \in \mathbf{N}}$ is in $B$. Since the closure of $B$ is compact in $\big(C([a, b]), \|.\|_\infty\big)$ which is metric, we conclude that there exists a subsequence $(u_{n_k})_{k \in \mathbf{N}}$ that converges to some $u$ in $C([a, b])$ (for the topology of $\|.\|_\infty$). □

**Bernard:** Arf! This proof is very technical. Let me gather all the details of the topology chapter again. The proof oscillates between the different norms $\|.\|_{L^2}$, $\|.\|_{H^1}$ and $\|.\|_\infty$.

**Laurent:** I admit it. Many mistakes can be made when using the wrong norms. For instance, $B$, at the end of the proof, is compact for the topology of $\|.\|_\infty$. But we cannot conclude that the limit of the converging subsequence is in $H^1$.

**Bernard:** I understand that we need to compare these norms. And I realize that the Cauchy–Schwarz inequality is the key argument when $\|.\|_{L^2}$ is involved.

{f} 515.COR.f

**Remark.** ⋆ From the proof, we can deduce that the functions in $H^1\big(\lambda|_{\mathcal{I}}\big)$ can be continuously extended to $\overline{\mathcal{I}}$. They are more regular than continuous functions but less regular than $C^1$ functions since they are not necessarily differentiable at every point.
⋆ Iterating on the order, we can observe that $H^k\big(\lambda|_{\mathcal{I}}\big) \subset C^{k-1}(\overline{\mathcal{I}})$ for all $k \ge 1$.
⋆ In dimension $d \ge 2$, this regularity result is false.

**Laurent:** As in the case of the $L^p$ spaces, there are some density results for the space $H^1$. Once again, the smooth functions with compact support play a crucial role in approximating the functions in $H^1$.

{f} 515.COR.f

**Theorem 7.26 (Density of smooth functions in $H^1(\lambda|_{\mathcal{I}})$).**
*The space $C^\infty([a,b])$ is dense in $H^1\big(\lambda|_{(a,b)}\big)$.*

**Proof.** Let $u \in H^1\big(\lambda|_{(a,b)}\big)$. Let $\varepsilon > 0$.

We need to show that there exists $V \in C^\infty\big([a,b]\big)$ such that $\|u - V\|_{H^1} < \varepsilon$.

Since $u'$ belongs to $L^2\big(\lambda|_{(a,b)}\big)$, according to Theorem 6.16, there exists $v \in C_c^\infty\big((a,b)\big)$ such that $\|u' - v\|_{L^2} < \varepsilon$.

Define the function $V$ in $C^\infty\big([a,b]\big)$ by $V \colon x \mapsto u(a) + \int_{(a,x)} v\, d\lambda$.

For $x \in [a,b]$, we have

$$\big(u(x) - V(x)\big)^2 = \left| \int_{(a,x)} (u' - v) d\lambda \right|^2 \leq (b-a)\, \|u' - v\|_{L^2}^2 \leq (b-a)\, \varepsilon^2.$$

Then, integrating over $[a,b]$, we get

$$\|u - V\|_{H^1}^2 = \|u - V\|_{L^2}^2 + \|u' - v\|_{L^2}^2 \leq (b-a)\, \varepsilon^2 + \varepsilon^2,$$

and the result follows. □

**Remark.** The following results also hold:

⋆ The space $C_c^\infty(\mathbf{R})$ is dense in $H^1(\lambda)$.

⋆ For any $a \in \mathbf{R}$, the space $C_c^\infty\big([a,+\infty)\big)$ is dense in $H^1\big(\lambda|_{(a,+\infty)}\big)$.

Their proofs can be found in Allaire (2007).

**Laurent:** You must have noted that the compact support of functions is mentioned in the remark and not in the theorem. That simply because $C_c^\infty\big([a,b]\big) = C^\infty\big([a,b]\big)$ since $[a,b]$ is compact.

But there is a more subtle point...

**Bernard:** I guess that it's about topology since it's a question of density.

**Laurent:** Definitely. The result is that any function in $H^1$ can be approximated by a sequence of functions in $C_c^\infty$. But this approximation must be understood in the sense of the $H^1$ norm.

**Bernard:** Between this result and the one expressing that the functions in $C_c^\infty$ approximate the functions in $L^p$, it looks like the smooth functions approximate everything. Does it mean that all those spaces $L^p$ and $H^1$ are roughly the same?

**Laurent:** You pointed out the major mistake that could be made at this stage. The two results – $C_c^\infty$ *is dense in* $L^p$ and $C_c^\infty$ *is dense in* $H^1$ – are fundamentally different.

The first assertion states that any function $f$ in $L^p$ can be approximated, in the sense of the $\|.\|_{L^p}$-norm, by a sequence of functions $(\varphi_n)_{n\in\mathbf{N}}$ in $C_c^\infty$: $\|f-\varphi_n\|_{L^p} \to 0$ as $n \to \infty$. Meanwhile, the second one states that any function $g$ of $H^1$ can be approximated, in the sense of the $\|.\|_{H^1}$-norm, by a sequence of functions $(\psi_n)_{n\in\mathbf{N}}$ in $C_c^\infty$: $\|g - \psi_n\|_{H^1} \to 0$ as $n \to \infty$.

But we know that the two norms $\|.\|_{L^p}$ and $\|.\|_{H^1}$ are not equivalent.

**Bernard:** Oh I see. Then, the convergence $\|f - \varphi_n\|_{L^p} \to 0$ does not imply that $\|f - \varphi_n\|_{H^1} \to 0$.

**Laurent:** And vice versa. Moreover, the second expression does not make any sense if $f$ is not in $H^1$.

### 7.2.3 *Trace theorem and integration by parts*

**Laurent:** Remember that integrating by parts is not an obvious issue for the Lebesgue integral. However the result holds and is a consequence of the Fubini–Lebesgue theorem. In the present setting, the considered derivative stands in the sense of distributions.

**Bernard:** Yes, but we saw that when the function is differentiable in the infinitesimal sense, it is also differentiable in the sense of distributions, and the two notions of derivative coincide.

**Laurent:** Not exactly. The derivative in the infinitesimal sense needs to be locally integrable. In this case, the two derivatives coincide as functions in $L^1$ or as distributions.

One of the benefits of the $H^1$ space is to avoid this kind of concern. Then, integrating by parts the functions in $H^1\big(\lambda|_{(a,b)}\big)$ can then be seen as more natural.

{f} 515.COR.f

**Definition 7.27.** For $u \in H^1(\lambda|_{\mathcal{I}})$, we call the *trace of* $u$ the restriction $u|_{\partial\mathcal{I}}$, where $\partial\mathcal{I}$ is the boundary of $\mathcal{I}$. Moreover the linear operator $\gamma_0 : u \mapsto u|_{\partial\mathcal{I}}$ is called the *trace operator.*

On a bounded interval $\mathcal{I} = (a, b)$, $a, b \in \mathbf{R}$, the functions in $H^1(\lambda|_{(a,b)})$ are identified as functions in $C([a, b])$: we can then define their values at the extreme points $a$ and $b$.

More precisely, thanks to Theorem 7.25, we can state the following corollary.

**Corollary 7.28 (Trace theorem).** *Let $\mathcal{I} = (a, b)$, with $a < b$. There exists a constant $c$ depending only on $b - a$ such that*

$$\forall u \in H^1(\lambda|_{(a,b)}), \quad \max(|u(a)|, |u(b)|) \leq c\,\|u\|_{H^1}.$$

*Then, the trace operator $\gamma_0$ is a continuous linear operator from $H^1(\lambda|_{(a,b)})$ to $\mathbf{R}^2$.*

**Laurent:** Let me check something. In Corollary 7.28, the trace operator $\gamma_0$ is a mapping from the infinite-dimensional space $H^1$ to the two-dimensional space $\mathbf{R}^2$. In order to prove the continuity of $\gamma_0$, can you tell me what topologies are considered upon both spaces?

**Bernard:** Let me think. Both are normed vector spaces. For $H^1$, we consider the norm $\|.\|_{H^1}$ as previously defined. Now, for $\mathbf{R}^2$, we have several possible choices. Reading the conclusions of the corollary, I would rather consider the infinity norm: $(x, y) \mapsto \|(x, y)\|_{\mathbf{R}^2} = \max(|x|, |y|)$.

**Laurent:** Excellent! And how is the continuity criterion for $\gamma_0$ written?

**Bernard:** Since $\gamma_0$ is a mapping from $H^1$ to $\mathbf{R}^2$ that is obviously linear, it is continuous as soon as there exists a constant $C > 0$ such that

$$\forall u \in H^1, \quad \|\gamma_0(u)\|_{\mathbf{R}^2} \leq C\,\|u\|_{H^1}.$$

If I rewrite this criterion as $\gamma_0(u)$ explicitly, I obtain

$$\forall u \in H^1, \quad \|(u(a), u(b))\|_{\mathbf{R}^2} \leq C\,\|u\|_{H^1},$$

which is precisely the conclusion of the corollary.

**Laurent:** Perfect. Now that you have mastered the trace theorem in one dimension, let us use it to integrate by parts.

{f} 515.COR.f

Thanks to the definition and existence of the trace operator in dimension $d = 1$, we can now state two extended versions of the "classical" integration by parts as follows.

**Theorem 7.29 (Integration by parts I).** *Let* $u, v \in H^1(\lambda|_{(a,b)})$, *where* $a, b \in \mathbf{R}$, $a < b$, *and* $\lambda|_{(a,b)}$ *denotes the Lebesgue measure on* $\mathbf{R}$ *restricted to* $(a, b)$. *Then,*

$$\int_{(a,b)} uv'\, d\lambda = -\int_{(a,b)} vu'\, d\lambda + \big[u(b)v(b) - u(a)v(a)\big].$$

**Proof.** This result can be seen as a consequence of Theorem 6.8 using the fact that $u$ and $v$ are primitive functions (in the sense of the Lebesgue Integration) of their derivatives in the sense of distributions (see the proof of Theorem 7.25).

However, here is a direct proof using a density argument.

Thanks to Theorem 7.25, $uv$ is continuous on $[a, b]$. Moreover, the Cauchy–Schwarz inequality implies that $uv, uv'$ and $u'v$ belong to $L^1\big(\lambda|_{(a,b)}\big)$.

Since $C_c^\infty\big([a, b]\big)$ is dense in $H^1\big(\lambda|_{(a,b)}\big)$ (see Theorem 7.26), there exist two sequences $(u_n)_{n\in\mathbf{N}}$ and $(v_n)_{n\in\mathbf{N}}$ of elements of $C_c^\infty\big([a, b]\big)$ converging, respectively, to $u$ and $v$ in $H^1\big(\lambda|_{(a,b)}\big)$.

For all $n \in \mathbf{N}$, since $u_n v_n$ is in $C_c^\infty\big([a, b]\big)$, we have

$$(u_n v_n)(b) - (u_n v_n)(a) = \int_{(a,b)} (u_n v_n)'\, d\lambda = \int_{(a,b)} (u_n v_n' + u_n' v_n)\, d\lambda. \tag{7.2.3}$$

Since $u_n \to u$ and $v_n \to v$ in $H^1\big(\lambda|_{(a,b)}\big)$, we know that $u_n \to u$, $v_n \to v$, $u_n' \to u'$ and $v_n' \to v'$ in $L^2\big(\lambda|_{(a,b)}\big)$. These convergences imply that $\|u_n\|_{L^2} \to \|u\|_{L^2}$, $\|v_n\|_{L^2} \to \|v\|_{L^2}$, $\|u_n'\|_{L^2} \to \|u'\|_{L^2}$ and $\|v_n'\|_{L^2} \to \|v'\|_{L^2}$, and all these converging sequences are bounded. Thus, there exists a constant $C > 0$ such that for all $n \in \mathbf{N}$, $\max\big(\|u_n\|_{L^2}, \|v_n\|_{L^2}, \|u_n'\|_{L^2}, \|v_n'\|_{L^2}\big) \le C$.

Then, $\int_{(a,b)} u_n v_n' \, d\lambda$ converges to $\int_{(a,b)} uv' \, d\lambda$, since

$$\left| \int_{(a,b)} (u_n v_n' - uv') \, d\lambda \right| \leq \left| \int_{(a,b)} (u_n - u) v_n' \, d\lambda \right| + \left| \int_{(a,b)} u(v_n' - v') \, d\lambda \right|$$
$$\leq C \, \|u_n - u\|_{L^2} + \|u\|_{L^2} \, \|v_n' - v'\|_{L^2},$$

by the Cauchy–Schwarz inequality.

In the same way, $\int_{(a,b)} u_n' v_n \, d\lambda$ converges to $\int_{(a,b)} u'v \, d\lambda$.

Besides, since $\|u_n - u\|_\infty \leq c \, \|u_n - u\|_{H^1}$ and $\|v_n - v\|_\infty \leq c \, \|v_n - v\|_{H^1}$, for any $x \in [a, b]$, we have $u_n(x) \to u(x)$ and $v_n(x) \to v(x)$.

This implies $(u_n v_n)(b) - (u_n v_n)(a) \to u(b)v(b) - u(a)v(a)$.

Taking the limit in (7.2.3), the result follows. □

**Bernard:** I don't understand very well how this result improves Theorem 6.8.

**Laurent:** There are several points:

- Firstly, the definition of the $H^1$ space gathers conditions allowing a formula of the same kind as in the $C^1$ case.
- Secondly, there is no derivative in the statement of Theorem 6.8.

The fact that $f$ is the derivative of $F \colon x \mapsto \int_{[0,x]} f \, d\lambda$ in the sense of distributions is another result.

Remember that we already talked at length about this precise issue.

Once again, Theorem 7.29 gathers these two different results.

**Bernard:** OK. I think I see more or less. But I need to read my notes again.

**Laurent:** We can complete Theorem 7.29 using the following result. Formally, replacing $u$ with $u'$, we get an equality which is very useful in the context of PDEs.

{f} 515.COR.f

**Corollary 7.30 (Integration by parts II).** *Let* $u \in H^2(\lambda|_{(a,b)})$ *and* $v \in H^1(\lambda|_{(a,b)})$. *Then,*

$$\int_{(a,b)} u'v'\, d\lambda + \int_{(a,b)} u''v\, d\lambda = [u'(b)v(b) - u'(a)v(a)].$$

**Bernard:** Wait! What do $u'(a)$ and $u'(b)$ mean? $u$ is an equivalence class, right?

**Laurent:** Remember that $H^2(\lambda|_{(a,b)}) \subset C^1([a,b])$, with the usual abuse of notation. Then, $u(x)$ and $u'(x)$ are well defined for $u \in H^2$.

### 7.2.4 $H_0^1(\lambda|_{(a,b)})$ *and the Poincaré inequality*

**Laurent:** The question of the values of $u$ at the boundary of $(a,b)$, that is at $a$ and $b$, is essential to solve some PDE problems in the good spaces. Now, the goal is to define a space that has the same properties as $H^1$ but whose elements vanish at the boundary.

But things are not that easy...

{f} 515.COR.f

Let $(a,b)$ be a bounded interval of $\mathbf{R}$. We know that $C_c^\infty((a,b))$ is dense in $L^2(\lambda|_{(a,b)})$ (see Theorem 5.15), but $C_c^\infty((a,b))$ is *not* dense in $H^1(\lambda|_{(a,b)})$, as shown by the following example.

**Example.** The constant function $f\colon x \in (0,1) \mapsto 1$ belongs to $H^1((0,1))$.

Thanks to Theorem 7.25, $f$ can be extended to $[0,1]$ such that $f(0) = f(1) = 1$.

Then, if $C_c^\infty((0,1))$ were dense in $H^1(\lambda|_{(0,1)})$, for all $\varepsilon \in (0,1)$, there would exist $\phi \in C_c^\infty((0,1))$ such that $1 = |f(0)| = |\phi(0) - f(0)| \leq \|\phi - f\|_\infty \leq c\,\|\phi - f\|_{H^1} \leq \varepsilon$, which is impossible.

**Bernard:** I realize that all of this relies on the fact that the norm $\|.\|_\infty$ is bounded by the norm $\|.\|_{H^1}$ up to a multiplicative constant.

**Laurent:** Definitely. Since such an inequality does not hold between the norms $\|.\|_\infty$ and $\|.\|_{L^2}$, we cannot build a similar counterexample.

Keep in mind that $C_c^\infty\big((a,b)\big)$ is dense in $L^2\big(\lambda|_{(a,b)}\big)$.

Now, we will determine the set of functions that can be approximated by functions in $C_c^\infty\big((a,b)\big)$ with the $H^1$-norm.

{f} 515.COR.f

**Definition 7.31.** Let $a, b \in \mathbf{R}$, $a < b$. We define $H_0^1\big(\lambda|_{(a,b)}\big)$ as the closure of $C_c^\infty\big((a,b)\big)$ for the $H^1$ norm, i.e. all the limits of sequences $(\varphi_n)_{n\in\mathbf{N}}$ in $C_c^\infty\big((a,b)\big)$ that converge for the $H^1$-norm.

The previous example shows that the sequences of functions in $C_c^\infty\big((a,b)\big)$ that converge in $H^1\big(\lambda|_{(a,b)}\big)$ do not reach all the elements of $H^1\big(\lambda|_{(a,b)}\big)$. Then, $H_0^1\big(\lambda|_{(a,b)}\big)$ cannot be equal to $H^1\big(\lambda|_{(a,b)}\big)$.

**Proposition 7.32.**

(i) *The space* $H_0^1\big(\lambda|_{(a,b)}\big)$ *is a proper vector subspace of* $H^1\big(\lambda|_{(a,b)}\big)$.

(ii) *The space* $H_0^1\big(\lambda|_{(a,b)}\big)$ *equipped with the inner product of* $H^1$ *is a Hilbert space.*

(iii) *The space* $H_0^1\big(\lambda|_{(a,b)}\big)$ *can be expressed, using the trace operator, as the relation* $H_0^1\big(\lambda|_{(a,b)}\big) = \gamma_0^{-1}\big((0,0)\big)$.

**Proof.** (i) This point follows from the previous example.

(ii) By definition, $H_0^1\big(\lambda|_{(a,b)}\big)$ is a closed vector subspace of $H^1\big(\lambda|_{(a,b)}\big)$, which is complete. So, $H_0^1\big(\lambda|_{(a,b)}\big)$ is complete for the $H^1$-norm and is thus a Hilbert space.

(iii) $\star$ Let us prove that $H_0^1\big(\lambda|_{(a,b)}\big) \subset \gamma_0^{-1}\big((0,0)\big)$.

Let $u \in H_0^1\big(\lambda|_{(a,b)}\big)$. By the definition of $H_0^1\big(\lambda|_{(a,b)}\big)$, there exists a sequence $(u_n)_{n\in\mathbf{N}}$ of functions in $C_c^\infty\big((a,b)\big)$ such that $\|u_n - u\|_{H^1} \to 0$.

Thanks to Theorem 7.25 and recalling that $u_n(a) = 0$,

$$|u(a)| = |u(a) - u_n(a)| \le \|u - u_n\|_\infty \le c\,\|u - u_n\|_{H^1} \to 0.$$

In the same way, $u(b) = 0$. We conclude that $\gamma_0(u) = \big(u(a), u(b)\big) = (0,0)$.

$\star$ To prove that $\gamma_0^{-1}\big((0,0)\big) \subset H_0^1\big(\lambda|_{(a,b)}\big)$, we use a characterization of the closure of vector subspaces in normed vector spaces (e.g. see Rudin (1987, Theorem 5.19)): if $x_0$ does not belong to the closure of a subspace $M$ of a normed vector space $X$, then there exists a continuous linear form $\Phi$ on $X$ such that $\Phi(x_0) = 1$ and $\Phi(m) = 0$ for all $m \in M$.

Let us assume that there exists $u_0 \in \gamma_0^{-1}((0,0))$ such that $u_0$ does not belong to $H_0^1\big(\lambda|_{(a,b)}\big)$, which is the closure of $C_c^\infty\big((a,b)\big)$.

According to the previous characterization, there exists a continuous linear form $\Phi$ on $H^1\big(\lambda|_{(a,b)}\big)$ such that $\Phi(u_0) = 1$ and $\Phi(\varphi) = 0$ for all $\varphi \in C_c^\infty\big((a,b)\big)$.

Remember that the Riesz–Fréchet theorem applies to the Hilbert space $H^1\big(\lambda|_{(a,b)}\big)$: there exists $u_\Phi \in H^1\big(\lambda|_{(a,b)}\big)$ such that for all $w \in H^1\big(\lambda|_{(a,b)}\big)$, $\Phi(w) = (u_\Phi, w)_{H^1}$.

Then, $(u_\Phi, u_0)_{H^1} = 1$ and $(u_\Phi, \varphi)_{H^1} = 0$ for all $\varphi \in C_c^\infty\big((a,b)\big)$.

Recall that for all $v \in H^1\big(\lambda|_{(a,b)}\big)$, $(u_\Phi, v)_{H^1} = \int_{(a,b)}(u'_\Phi v' + u_\Phi v)\, d\lambda$.

This implies that for all $\varphi \in C_c^\infty\big((a,b)\big)$, $0 = \int_{(a,b)}(u'_\Phi \varphi' + u_\Phi \varphi)\, d\lambda = \langle -u''_\Phi + u_\Phi, \varphi\rangle$ and, consequently, $u''_\Phi = u_\Phi$ in the sense of distributions.

Since $u_\Phi$ belongs to $H^1\big(\lambda|_{(a,b)}\big) \subset L^2\big(\lambda|_{(a,b)}\big)$, $u''_\Phi$ belongs to $L^2\big(\lambda|_{(a,b)}\big)$ and $u_\Phi$ belongs to $H^2\big(\lambda|_{(a,b)}\big)$.

Then, we can claim that $u''_\Phi - u_\Phi = 0$ in $L^2\big(\lambda|_{(a,b)}\big)$.

Since $u_0 \in H^1\big(\lambda|_{(a,b)}\big)$ and $u_\Phi \in H^2\big(\lambda|_{(a,b)}\big)$, we can apply our integration by parts II (Corollary 7.30) to $u_\Phi$ and $u_0$:

$$
\begin{aligned}
(u_0, u_\Phi)_{H^1} &= \int_{(a,b)} (u'_\Phi u'_0 + u_\Phi u_0)\, d\lambda \\
&= \int_{(a,b)} (-u''_\Phi + u_\Phi) u_0\, d\lambda + \big[u'_\Phi(b)u_0(b) - u'_\Phi(a)u_0(a)\big]\,.
\end{aligned}
$$

Knowing that $u''_\Phi - u = 0$ in $L^2\big(\lambda|_{(a,b)}\big)$ and $u_0 \in \gamma_0^{-1}\big((0,0)\big)$, that is, $u_0(a) = u_0(b) = 0$, we find that $(u_0, u_\Phi)_{H^1} = 0$.

Recalling that $(u_0, u_\Phi)_{H^1} = 1$ yields the contradiction.

We can conclude that $\gamma_0^{-1}\big((0,0)\big) \subset H_0^1\big(\lambda|_{(a,b)}\big)$. $\square$

**Laurent:** According to this result, we observe that the $H^1$-norm is a norm on $H_0^1$. But there exists a simpler inner product on $H_0^1$, whose associated norm is equivalent to the $H^1$-norm.

**Bernard:** Why didn't we equip the whole $H^1$ with this magic inner product then?

**Laurent:** Simply because it is an inner product on $H_0^1$ but not on $H^1$. The question of the value on the boundary is crucial here.

Consider the bilinear form $\Gamma : (u,v) \mapsto \int_{(a,b)} u'v'\,d\lambda$ on $H^1\big(\lambda|_{(a,b)}\big)$. Thanks to the Cauchy–Schwarz inequality, this bilinear form is continuous. But it is not an inner product since if $u$ in $H^1$ is such that $\Gamma(u,u) = 0$, then $u' = 0$ (in $L^2$).

Then, $u$ is constant but not necessarily zero.

**Bernard:** And what is the difference with $H_0^1$?

**Laurent:** If you add the fact that $u = 0$ on the boundary, then $\Gamma(u,u) = 0$ implies $u = 0$.

Moreover, the triangle inequality $\big(\Gamma(u+v, u+v)\big)^{1/2} \leq \big(\Gamma(u,u)\big)^{1/2} + \big(\Gamma(v,v)\big)^{1/2}$ comes from the Minkowski inequality.

The condition $\big(\Gamma(\alpha u, \alpha u)\big)^{1/2} = |\alpha| \big(\Gamma(u,u)\big)^{1/2}$ is obvious.

We can conclude that $\Gamma$ is an inner product on $H_0^1$.

The following inequality, due to Henri Poincaré, proves that the norm associated with $\Gamma$ is equivalent to the $H^1$-norm on $H_0^1$.

{f} 515.COR.f

**Theorem 7.33 (Poincaré inequality).** *Let $\lambda|_{(a,b)}$ be the restriction of the Lebesgue measure on $\mathbf{R}$ to the interval $(a,b)$.*

*Then, the following inequality holds*:

$$\forall v \in H_0^1\big(\lambda|_{(a,b)}\big), \quad \|v\|_{L^2} \leq (b-a)\,\|v'\|_{L^2}.$$

**Proof.** Let $v \in H_0^1\big(\lambda|_{(a,b)}\big)$.

For any $x \in [a,b]$, since $v(x) = \int_{(a,x)} v'\,d\lambda$, the Cauchy–Schwarz inequality implies that

$$\big|v(x)\big| = \left|\int_{(a,x)} v'\,d\lambda\right| \leq (b-a)^{1/2}\,\|v'\|_{L^2}.$$

Consequently, $\|v\|_{L^2} \leq (b-a)\,\|v'\|_{L^2}$, which proves the result. □

This result allows us to define the following norm on $H_0^1$.

**Definition 7.34.** The semi-norm $v \mapsto \|v'\|_{L^2}$, defined on $H^1\big(\lambda|_{(a,b)}\big)$, is a norm on $H_0^1\big(\lambda|_{(a,b)}\big)$, which is denoted by

$$\|v\|_{H_0^1} : v \mapsto \|v'\|_{L^2}.$$

Moreover, we have $\forall v \in H_0^1\big(\lambda|_{(a,b)}\big)$, $\|v\|_{H_0^1} \leq \|v\|_{H^1} \leq \big(1 + (b - a)^2\big)^{1/2} \|v\|_{H_0^1}$.

Then, on $H_0^1\big(\lambda|_{(a,b)}\big)$, the norm $\|.\|_{H_0^1}$ is equivalent to the norm $\|.\|_{H^1}$.

**Proposition 7.35.** *Consider the bilinear form $\langle .,. \rangle_{H_0^1}$ on $H_0^1\big(\lambda|_{(a,b)}\big)$*

$$\langle .,. \rangle_{H_0^1} : (u,v) \mapsto (u,v)_{H_0^1} = \int_{(a,b)} u'v'\,d\lambda.$$

*The space $H_0^1\big(\lambda|_{(a,b)}\big)$ equipped with the inner product $\langle .,. \rangle_{H_0^1}$ is a Hilbert space.*

**Bernard:** Very well. But the modeling problems in engineering are not usually in one dimension.

Can all these results be extended to larger dimension?

**Laurent:** More or less. However, the proofs of some of them are much more complicated because it is very difficult to define a measure on the boundary. You will see that they are seldom written in books.

## 7.3 Sobolev Spaces in the Multi-Dimensional Case

**Laurent:** Let us have a look at the multi-dimensional situation. As I just told you, we are going to investigate the adaptation of the one-dimensional results in that case, without proving them.

{f} 515.COR.f

Here is a review of the main multi-dimensional results. As they are significantly harder to prove than in one dimension, their proofs are beyond the scope of this book.

**Notation 7.36.** In the sequel, the notation $\mathcal{U}$ stands for an open subset of $\mathbf{R}^d$. The notation $\lambda^{(d)}$ stands for the Lebesgue measure of $\mathbf{R}^d$, and $\lambda^{(d)}|_{\mathcal{U}}$ denotes its restriction to $\mathcal{U}$.

If $\mathcal{U}$ is bounded in at least one direction, its boundary is denoted by $\partial\mathcal{U}$.

In this case, we will sometimes assume that $\mathcal{U}$ is a smooth open set, that is, locally, its boundary $\partial\mathcal{U}$ is the backward image of the interval $(-1, 0]$ by a smooth function, at least of class $C^1$.

Then, the boundary of the open set is said to be *smooth of class* $C$.[1] Additionally, when the open set is bounded, it is said to be a *smooth open bounded set of class* $C^1$.

The normal vector to the boundary at point $x$ is denoted by $n(x)$.

We will assume that there exists a measure $\mu$ on $\mathcal{U}$ whose support is $\partial\mathcal{U}$ and which is compatible with $\lambda^{(d)}$, e.g. the natural measure whose support is the edge of a rectangle, a sphere, and so on.

### 7.3.1 *Definition of* $H^k(\lambda^{(d)}|_{\mathcal{U}})$, $k \geq 1$

**Laurent:** Do you remember the nabla operator? You may have encountered it in differential calculus or in fluid mechanics or electromagnetics.

**Bernard:** Of course. The nabla operator, denoted by the $\nabla$ symbol, is used as a vector representation of the gradient: $\nabla u = (\partial_{x_1} u, \ldots, \partial_{x_d} u)$, where $\partial_{x_i} u$ denotes the derivative of the function $u$ along the direction $x_i$.

**Laurent:** Yes, specifying that, here, the notion of derivative is taken in the sense of distributions.

{f} 515.COR.f

**Definition 7.37.** In higher dimensions, the *first-order Sobolev space* is defined as

$$\begin{aligned} H^1(\lambda^{(d)}|_{\mathcal{U}}) &= \{v \in L^2(\lambda^{(d)}|_{\mathcal{U}}) \ : \ \forall i \in \{1,\dots,d\}, \\ &\qquad \partial_{x_i} v \in L^2(\lambda^{(d)}|_{\mathcal{U}})\} \\ &= \{v \in L^2(\lambda^{(d)}|_{\mathcal{U}}) \ : \ \nabla v \in L^2(\lambda^{(d)}|_{\mathcal{U}})\}, \end{aligned}$$

and is equipped with the inner product

$$\begin{aligned} \langle v, w\rangle_{H^1} : (v,w) &\mapsto \langle v, w\rangle_{L^2} + \langle \nabla v, \nabla w\rangle_{L^2} \\ &= \langle v, w\rangle_{L^2} + \sum_{i=1}^{d} \langle \partial_{x_i} v, \partial_{x_i} w\rangle_{L^2} \\ &= \int_{\mathcal{U}} vw \, d\lambda^{(d)} + \sum_{i=1}^{d} \int_{\mathcal{U}} \partial_{x_i} v \, \partial_{x_i} w \, d\lambda^{(d)}. \end{aligned}$$

The associated norm in $H^1(\lambda^{(d)}|_{\mathcal{U}})$,

$$\|v\|_{H^1} : v \mapsto \left(\|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2\right)^{1/2} = \left(\|v\|_{L^2}^2 + \sum_{i=1}^{d} \|\partial_{x_i} v\|_{L^2}^2\right)^{1/2},$$

satisfies, for all $v \in H^1(\lambda^{(d)}|_{\mathcal{U}})$ and all $i \in \{1,\dots,d\}$,

$$\|v\|_{L^2} \le \|v\|_{H^1} \quad \text{and} \quad \|\partial_{x_i} v\|_{L^2} \le \|\nabla v\|_{L^2} \le \|v\|_{H^1}.$$

**Bernard:** Wait a minute. We must be careful with the notation $\|\nabla u\|_{L^2}$ since $\nabla u$ is a $d$-dimensional vector.

**Laurent:** It is a natural abuse of notation. For any finite-dimensional vector $X$, we write $X \in L^p$ when each component $X_i \in L^p$.

**Bernard:** And if I understand well, in the specific case of $L^2$, the inner product is defined as $\langle X, Y\rangle_{L^2} = \sum_i \langle X_i, Y_i\rangle_{L^2}$.

Besides, I hope the space $L^2$ with this inner product is a Hilbert space.

**Laurent:** Of course. And we can see that it is also true for $H^1$.

{f} 515.COR.f

**Theorem 7.38.** *The Sobolev space* $H^1\big(\lambda^{(d)}|_{\mathcal{U}}\big)$ *equipped with* $\|.\|_{H^1}$ *is complete.*

*Consequently,* $H^1\big(\lambda^{(d)}|_{\mathcal{U}}\big)$ *equipped with* $\langle .,.\rangle_{H^1}$ *is a Hilbert space.*

**Definition 7.39.** For $k \in \mathbf{N}$, we define the Sobolev space

$$H^k\big(\lambda^{(d)}|_{\mathcal{U}}\big) = \{v \in L^2\big(\lambda^{(d)}|_{\mathcal{U}}\big) \;:\; \forall \alpha \in \mathbf{N}^d,\ |\alpha| \le k,\\ D^\alpha v \in L^2\big(\lambda^{(d)}|_{\mathcal{U}}\big)\}.$$

**Remark.** The Sobolev space $H^k\big(\lambda^{(d)}|_{\mathcal{U}}\big)$ equipped with the inner product

$$\langle v, w\rangle_{H^k} : (u,v) \mapsto \sum_{\alpha \in \mathbf{N}^d, |\alpha| \le k} \int_{\mathcal{U}} \langle D^\alpha u, D^\alpha v\rangle_{L^2},$$

is a Hilbert space.

### 7.3.2 *Regularity of the Sobolev spaces and density*

**Laurent:** We must be careful when extending the one-dimensional results. For instance, the multi-dimensional counterpart of Theorem 7.25 is weaker.

Although it is not really necessary for your immediate understanding, if you want to see the proofs of the results in several dimensions, I suggest you read Brézis and Allaire[2] in complement of this *Functional Analysis* by Laurent Corps.

{f} 515.COR.f

**Theorem 7.40 (Rellich theorem).** *Let* $\mathcal{U}$ *be a regular bounded open subset of* $\mathbf{R}^d$ *of class* $C^1$ *and* $\lambda^{(d)}|_{\mathcal{U}}$ *be the Lebesgue measure on* $\mathbf{R}^d$ *restricted to* $\mathcal{U}$. *Then:*

[2]See [Brezis (1987)] and [Allaire (2007)].

⋆ *if* $d \leq 2$, $H^1(\lambda^{(d)}|_{\mathcal{U}}) \subset L^q(\lambda^{(d)}|_{\mathcal{U}})$, $\forall q \in [1, +\infty)$,
⋆ *if* $d > 2$, $H^1(\lambda^{(d)}|_{\mathcal{U}}) \subset L^q(\lambda^{(d)}|_{\mathcal{U}})$, $\forall q \in \left[1, \frac{2d}{d-2}\right)$,

*with compact injections.*

**Remark.** ⋆ If the dimension $d$ is larger than 1, then, generally, a function belonging to $H^1(\lambda^{(d)}|_{\mathcal{U}})$ has no continuous representative.

Here is a counterexample, given by Allaire (see [Allaire (2007, Ex. 4.3.2)]).

Consider $\mathcal{U} = B(0,1)$ in $\mathbf{R}^2$ and $v\colon (x,y) \mapsto \left|\ln\left((x^2 + y^2)^{1/2}/2\right)\right|^\alpha$, for $\alpha \in (0, 1/2)$. Then, $v \in H^1(\lambda|_{B(0,1)})$ and $v \notin C\left(\overline{B(0,1)}\right)$.

⋆ If $\mathcal{U}$ is bounded of class $C^1$, we can show that there is a continuous injection from $H^k(\lambda^{(d)}|_{\mathcal{U}})$ to $C(\overline{\mathcal{U}})$ if $k > d/2$ (see Allaire (2007)).

**Theorem 7.41.** *Assume that $\mathcal{U}$ is a bounded regular open set of class $C^1$, or $\mathcal{U} = \mathbf{R}^d$, or $\mathcal{U}$ is a half-space of $\mathbf{R}^d$, for instance $\mathcal{U} = \{x \in \mathbf{R}^d : x_d > 0\}$.*

*Then, $C_c^\infty(\overline{\mathcal{U}})$ is dense in $H^1(\lambda^{(d)}|_{\mathcal{U}})$.*

### 7.3.3 *Trace theorem and integration by parts*

**Bernard:** If I understand well that there will be a problem: we need the trace, and we just saw that the functions on $H^1(\lambda^{(d)}|_{\mathcal{U}})$ are not necessarily continuous on $\overline{\mathcal{U}}$.

**Laurent:** That is the reason why we first define the trace on $C(\overline{\mathcal{U}})$ and then extend it to $H^1(\lambda^{(d)}|_{\mathcal{U}})$ by density.

{f}
515.COR.f

**Definition 7.42.** Let $\mathcal{U}$ be a regular open subset of class $C^1$. Assume that $u$ is a continuous function on $\overline{\mathcal{U}}$.

The restriction $u|_{\partial\mathcal{U}}$ to the boundary is called the *trace of* $u$, and the linear operator $\gamma_0 : u \mapsto u|_{\partial\mathcal{U}}$ is called the *trace operator.*

Recall that $\lambda^{(d)}|_{\mathcal{U}}$ denotes the restriction to $\mathcal{U}$ of the Lebesgue measure on $\mathbf{R}^d$ and $\mu$ denotes the measure on $\partial\mathcal{U}$.

**Proposition 7.43.** *The trace operator*

$$\gamma_0 \colon H^1\big(\lambda^{(d)}|_{\mathcal{U}}\big) \cap C\big(\overline{\mathcal{U}}\big) \longrightarrow L^2(\mu|_{\partial\mathcal{U}})$$
$$v \longmapsto v|_{\partial\mathcal{U}},$$

*is continuous for the norms* $\|.\|_{H^1}$ *and* $\|.\|_{L^2(\mu|_{\partial\mathcal{U}})}$.

Consequently, there exists a constant $C$ depending only on $\mathcal{U}$ such that

$$\forall v \in H^1\big(\lambda^{(d)}|_{\mathcal{U}}\big) \cap C\big(\overline{\mathcal{U}}\big), \quad \|v\|_{L^2(\mu)} \leq C\,\|v\|_{H^1}.$$

The trace operator can be extended to $H^1\big(\lambda^{(d)}|_{\mathcal{U}}\big)$ by the density of $H^1\big(\lambda^{(d)}|_{\mathcal{U}}\big) \cap C\big(\overline{\mathcal{U}}\big)$ in $H^1\big(\lambda^{(d)}|_{\mathcal{U}}\big)$ and continuity thanks to Theorem 1.39.

**Theorem 7.44.** *The trace operator is a continuous linear map from* $H^1\big(\lambda^{(d)}|_{\mathcal{U}}\big)$ *to* $L^2(\mu)$.

*By extension, for* $u \in H^1\big(\lambda^{(d)}|_{\mathcal{U}}\big)$, $\gamma_0(u) = u|_{\partial\mathcal{U}}$ *is also called the* trace of $u$.

**Remark.** The trace operator $\gamma_0$ is not surjective: the image of $H^1\big(\lambda^{(d)}|_{\mathcal{U}}\big)$ by $\gamma_0$ is not equal to $L^2(\mu)$.

We denote $H^{1/2}(\mu) = \gamma_0\Big(H^1\big(\lambda^{(d)}|_{\mathcal{U}}\big)\Big)$.

**Laurent:** We have a multi-dimensional counterpart of the extended integration by parts that we saw earlier. Recall that the derivatives must be understood in the sense of distributions.

The novelty in this precise context is the appearance of the outer normal $n$ of $\partial\mathcal{U}$.

{f} 515.COR.f

**Theorem 7.45 (Integration by parts I).** *Let* $u, v \in H^1\big(\lambda^{(d)}|_{\mathcal{U}}\big)$. *Then,*

$$\forall i \in \{1, \dots, d\}, \quad \int_{\mathcal{U}} u\, \partial_{x_i} v\, d\lambda^{(d)} = -\int_{\mathcal{U}} v\, \partial_{x_i} u\, d\lambda^{(d)} + \int_{\partial\mathcal{U}} u\, v\, n_i\, d\mu.$$

**Theorem 7.46 (Integration by parts II).** *Let* $u \in H^2\big(\lambda^{(d)}|_{\mathcal{U}}\big)$ *and* $v \in H^1\big(\lambda^{(d)}|_{\mathcal{U}}\big)$. *Then,*

$$\int_{\mathcal{U}} \langle \nabla u, \nabla v \rangle \, d\lambda^{(d)} + \int_{\mathcal{U}} \Delta u \, v \, d\lambda^{(d)} = \int_{\partial \mathcal{U}} v \, \langle \nabla u, n \rangle \, d\mu.$$

### 7.3.4 $H_0^1\big(\lambda^{(d)}|_{\mathcal{U}}\big)$ *and the Poincaré inequality*

**Laurent:** As in one dimension, we still need the space of $H^1$-functions of several variables that vanish at the boundary. As we will see, the definition in the multi-dimensional case is exactly the same as previously.

{f} 515.COR.f

**Definition 7.47.** Let $\mathcal{U}$ be an open subset of $R^d$. We define $H_0^1\big(\lambda^{(d)}|_{\mathcal{U}}\big)$ as the closure of $C_c^\infty(\mathcal{U})$ in $H^1\big(\lambda^{(d)}|_{\mathcal{U}}\big)$ for the norm $\|.\|_{H^1}$.

**Proposition 7.48.**

(i) $H_0^1\big(\lambda^{(d)}|_{\mathcal{U}}\big)$ *is a proper subspace of* $H^1\big(\lambda^{(d)}|_{\mathcal{U}}\big)$:
$H_0^1\big(\lambda^{(d)}|_{\mathcal{U}}\big) \subsetneq H^1\big(\lambda^{(d)}|_{\mathcal{U}}\big)$.

(ii) $H_0^1\big(\lambda^{(d)}|_{\mathcal{U}}\big)$ *equipped with the* $H^1$ *inner product is a Hilbert space.*

(iii) *If furthermore* $\mathcal{U}$ *is a regular open bounded subset of class* $C^1$, *then* $H_0^1\big(\lambda^{(d)}|_{\mathcal{U}}\big) = \gamma_0^{-1}(\{0\})$.

**Bernard:** I note that the regularity of $\mathcal{U}$ is needed to link $H_0^1$ to the trace operator. It's no surprise to me since regularity was needed to define the trace operator on $H^1$.

**Laurent:** Good. But no regularity assumption is necessary for the Poincaré inequality.

{f} 515.COR.f

**Theorem 7.49 (Poincaré inequality).** *If $\mathcal{U}$ is a bounded open set in at least one direction of $\mathbf{R}^d$, then there exists a constant $C$ depending only on $\lambda^{(d)}|_{\mathcal{U}}$ such that*

$$\forall v \in H_0^1\big(\lambda^{(d)}|_{\mathcal{U}}\big), \quad \|v\|_{L^2} \leq C\,\|\nabla v\|_{L^2}.$$

**Bernard:** I realize that the Poincaré inequality keeps the same form in this context. I imagine that the result about the semi-norm being a norm on $H_0^1$ remains true.

Let's see.

{f} 515.COR.f

**Definition 7.50.** If $\mathcal{U}$ is bounded in at least one direction, the semi-norm $v \mapsto \|\nabla v\|_{L^2}$ defined on $H^1$ is a norm on $H_0^1\big(\lambda^{(d)}|_{\mathcal{U}}\big)$, which is denoted by

$$\|v\|_{H_0^1} : v \mapsto \|\nabla v\|_{L^2} = \left(\sum_{k=1}^{d} \|\partial_{x_i} v\|_{L^2}^2\right)^{1/2}.$$

Moreover, we have

$$\forall v \in H_0^1\big(\lambda^{(d)}|_{\mathcal{U}}\big), \quad \|v\|_{H_0^1} \leq \|v\|_{H^1} \leq \big(1 + C^2\big)^{1/2}\,\|v\|_{H_0^1}.$$

Then, on $H_0^1\big(\lambda^{(d)}|_{\mathcal{U}}\big)$, the norm $\|.\|_{H^1}$ is equivalent to the norm $\|.\|_{H_0^1}$.

**Proposition 7.51.** *Consider the bilinear form on $H_0^1\big(\lambda^{(d)}|_{\mathcal{U}}\big)$*

$$\langle v, w\rangle_{H_0^1} : (v, w) \mapsto \int_{\mathcal{U}} \langle \nabla v, \nabla w\rangle\, d\lambda^{(d)}.$$

*The space $H_0^1\big(\lambda^{(d)}|_{\mathcal{U}}\big)$ equipped with the inner product $\langle ., .\rangle_{H_0^1}$ is a Hilbert space.*

*Bernard is delighted to have learned so much from his father's friend. He feels that he's finally mastered the essential concepts to mathematically model the pharmaceutical plant's waste.*

*He is aware, however, that to fully understand the modeling Laurent presented at their first meeting, he is still missing many things.*

*In particular, Laurent has not yet told him about the equations that link the quantities. And the descriptions of the dependence of random quantities is a mystery to him.*

*The world of modeling has yet to be discovered...*

*Laurent and Bernard agree to focus their future discussions in this direction.*

**To be continued in Part II...**

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# Notation

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