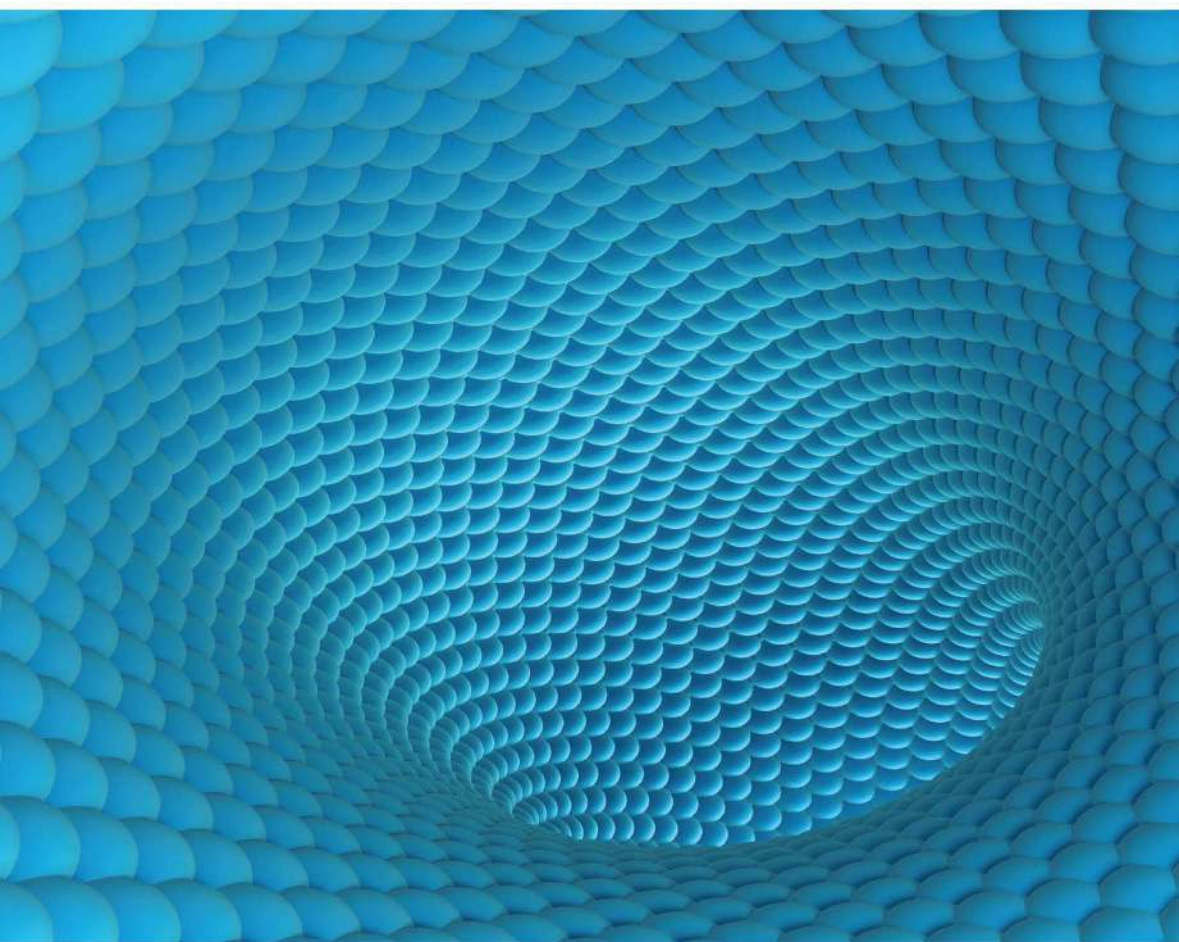


Plasticity

**Fundamentals
and Applications**
Second Edition



P.M. Dixit • U.S. Dixit



CRC Press
Taylor & Francis Group

Plasticity

This book covers the fundamentals of the elastic-plastic deformation including stress, strain, constitutive relations, fracture, anisotropy and contact problems along with a discussion of updated Lagrangian and Eulerian formulations. The second edition includes new material on thermal effects in plasticity and an introduction to crystal plasticity with review of all the chapters including more solved examples and a solutions manual.

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CRC Press

Taylor & Francis Group

Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business

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Second edition published 2025

by CRC Press

2385 NW Executive Center Drive, Suite 320, Boca Raton FL 33431

and by CRC Press

4 Park Square, Milton Park, Abingdon, Oxon, OX14 4RN

CRC Press is an imprint of Taylor & Francis Group, LLC

© 2025 P.M. Dixit and U.S. Dixit

First edition published by CRC Press 2015

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Library of Congress Cataloging-in-Publication Data

Names: Dixit, Prakash Mahadeo, author. | Dixit, Uday S., author.

Title: Plasticity : fundamentals and applications / P.M. Dixit and U.S. Dixit.

Description: Second edition. | Boca Raton : CRC Press, 2025. | Includes bibliographical references and index.

Subjects: LCSH: Plasticity. | Deformations (Mechanics)

Classification: LCC TA418.14 .D59 2025 (print) | LCC TA418.14 (ebook) |

DDC 620.1/1233—dc23/eng/20250224

LC record available at <https://lccn.loc.gov/2024048359>

LC ebook record available at <https://lccn.loc.gov/2024048360>

ISBN: 978-1-032-38399-6 (hbk)

ISBN: 978-1-032-38402-3 (pbk)

ISBN: 978-1-003-34487-2 (ebk)

DOI: 10.1201/9781003344872

Typeset in Times

by Apex CoVantage, LLC

To my wife, Rekha, and my daughter, Rashmi.

Prakash M. Dixit

To my teachers.

Uday S. Dixit



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Preface

This book was first published in 2015. The main objective of the book was to impart the knowledge of fundamentals of plasticity that is relevant for applying theory to practical problems. The book presented essential theory of classical plasticity and analytical solutions of some typical problems. Several universities in India and abroad used this book as a textbook and authors have been receiving requests for providing the solutions of the problems.

Considering the popularity of the book, it was decided to enhance its utility. Over the years, micromanufacturing has become popular in industry. Microforming and micromachining problems cannot be solved with classical macro-plasticity theory. Considering this, a chapter on Introduction to Crystal Plasticity has been added. Similarly, considering the popularity of thermal-based processes such as laser forming, a chapter on Thermal Considerations in Plasticity has been added. Apart from this, some other chapters have also been revised appropriately. For example, in Chapter 11, a section on ‘Effect of Lode angle on ductile fracture’ has been added. The number of solved and exercise problems has been increased.

The authors thank Dr. S.S. Gautam, Dr. Rajkumar Shufen and Prof. Anish Roy for contributing Chapters 11, 13 and 14 of the book, respectively. Several persons have helped in various ways in bringing out the second edition of the book. The assistance of Dr. Faladrum Sharma in proofreading and drawing some figures of Chapter 14 is acknowledged. Mr. Abhijeet Manoj Dhulekar has helped in improving the quality of some figures. Finally, support provided by CRC Press through Dr. Gagandeep Singh is gratefully acknowledged.

The solution manual of the book can be provided to instructors on demand. The focus of the book is not on explaining the numerical techniques for solving plasticity problems. Rather, it is on strengthening the basic fundamentals. We hope the students, teachers and practicing engineers will find the revised book more useful.

Prakash M. Dixit Uday S. Dixit



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1 Solid Mechanics and Its Applications

1.1 INTRODUCTION

In day-to-day life, any material is found in one of the three forms—solid, liquid and gas. A solid has a definite shape and volume. Although it is possible to change the shape of a solid, by application of force, in general, a solid offers significant resistance to the change of its shape. For most of the solids, even after the changed shape, volume remains fairly constant. The molecules in the solids are closely packed and cannot move freely, although they vibrate about their mean position.

Liquids generally have a constant volume, but they do not possess any specific shape. They adopt the shape of the container in which they are put. In liquids, the molecules remain in contact as they slide past each other. They can take normal compressive stress but not shear stress. Under the action of shear stress, a liquid starts flowing. The rate of shear strain depends on the shear stress applied. On the other hand, in solids, shear strain can be expressed as a function of shear stress up to a certain threshold shear stress.

Gases do not have any fixed volume and shape. The molecules in the gas keep moving with a large mean free path. Gas can easily adapt to the shape and size of the container in which it is kept.

Materials can be classified into solid, liquid or gas depending on their condition at room temperature and at atmospheric pressure. A material can undergo transformation from one form to another, depending on the pressure and temperature. In this book, the effect of forces on solids will be studied. Solid mechanics is the science of the effect of forces and motions on solids. Solid mechanics can be divided into two parts—rigid body mechanics and deformable body mechanics. In rigid body mechanics, it is assumed that the distance between two particles remains unchanged. There are two major divisions of rigid body mechanics—statics and dynamics. In statics, the focus is on studying the equilibrium of bodies. In dynamics, the motion of bodies is studied, which may be caused due to forces. Dynamics is further divided into kinematics and kinetics. Kinematics studies various aspects of the motion of bodies without considering the forces. For example, one may be interested to find the velocity and acceleration of a piston with respect to time in an internal combustion engine for a given crankshaft angular velocity. Kinematics can answer this. If one also wants to find the amount of force on the piston for causing a specified motion, kinetics has to be applied. Kinetics studies the effect of forces on motion. In deformable body mechanics, the statics and dynamics of bodies, in which the distance between the particles of bodies may change, are studied. The rigid body mechanics with two opposing motivations are studied. On one hand, it is of interest to know the situations that will not allow the body to deform significantly or permanently. On the

other hand, it is of interest to know the forces that will allow the body to deform for the purpose of shaping the material.

Most of the solids behave in the following manner: up to a certain amount of load, they undergo recoverable deformation. The recoverable deformation is called elastic deformation. After the elastic limit is crossed, they undergo a deformation of such type that a part of deformation cannot be recovered; the material gets permanently deformed. The permanent deformation is called plastic deformation. The major focus of this book is on studying plasticity, but invariably without causing elastic deformation, the plastic deformation cannot be achieved. Hence, for a thorough understanding of plasticity theory, elasticity theory must be understood. The book, therefore, discusses the important concepts of elasticity as well.

1.2 CONTINUUM HYPOTHESIS

Most of the time, in solid mechanics, the macroscopic view of the material is considered. A body consists of several particles, and each particle can be subdivided into molecules and atoms. The atom can be further subdivided into electrons, protons and neutrons as well as many other elementary particles. It is not feasible to solve an engineering problem by treating a body as a conglomeration of such discrete particles. The body is assumed to consist of a continuous distribution of matter. In other words, a body is treated as a continuum. Figure 1.1 illustrates this concept. Here, a schematic diagram of a cross section of actual material is shown. There are a number of molecules, which are shown here, arranged in a systematic manner. In many materials, the arrangement may not be so systematic. However, there is always some void between two molecules and a force of attraction works between two molecules. In an idealized material, this void is ignored. The typical cell dimension in a metal is of the order of 0.5 nm. A body-centered cubic unit cell has effectively two atoms, ions or molecules. Assume that the resolution of the measuring instrument is 0.5 μm . In that case, in a cube of 0.5- μm size, there will be 1×10^9 molecules, and it will definitely have some weight. The gap between the molecules will not be observed here. However, if the resolution of a measuring

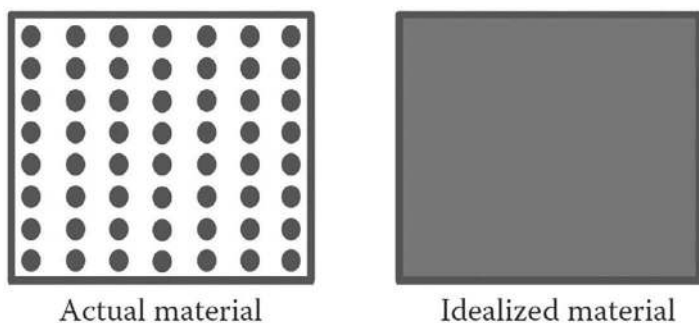


FIGURE 1.1 Concept of continuum.

instrument is 0.05 nm, then in a cube of 0.05 nm size, there is a possibility of having no molecule or having a fraction of a molecule. The material will not appear to be continuous, and the continuum hypothesis cannot be employed. In most of the engineering problems, resolution of the nanometer order is not required, and continuum theory is good enough.

The mass density ρ at a point P in a continuum is defined as the ratio of the mass element Δm to the volume element ΔV enclosing the point, in the limit when ΔV tends to 0. Thus,

$$\rho = \lim_{\Delta V \rightarrow 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV} \quad (1.1)$$

Note that, here, the assumption is that mass is the continuous function of the volume. In the left picture of Figure 1.1, mass is not a continuous function of volume. In the limit, one may end up in a void.

If a continuous body is deformed, then every particle of the body occupies a definite position in the deformed configuration. There exists a one-to-one correspondence between the particles of a continuum and the geometrical points of a region that the continuum occupies at any given instant of time. It can also be shown that during deformation, a boundary surface gets transformed to a boundary surface only. The particles that lie on a boundary surface of a material body in one configuration continue to remain on the boundary surface of the body in all configurations. In fact, a surface can deform only in the form of another surface. Similarly, all particles on a line remain on a line after deformation, although a straight line may become curved after deformation.

A dimensionless measure to assess the validity of the continuum hypothesis is the Knudsen number. The Knudsen number K_n is defined as

$$K_n = \frac{\lambda}{L}, \quad (1.2)$$

where λ is the molecular mean free path, and L is the characteristic length, for example, the cutting tool nose radius in a metal-cutting operation. If the Knudsen number is less than 0.01 then the continuum hypothesis is justified.

1.3 ELASTO-PLASTIC SOLIDS

A typical metal behaves in the following manner: when a load is applied, it undergoes elastic deformation. After the removal of the load, elastic deformation disappears. However, if the load exceeds a certain threshold, the metal deforms in such a manner that a part of its deformation permanently remains even after the removal of the load. During uni-axial loading, in the elastic region, the stress is proportional to strain. For most of the materials, the proportionality is linear, but it can be non-linear as well. Even for linearly elastic materials, there are some small regions before the yield point when the material behaves in a non-linear elastic way. Figure 1.2 shows the stress-strain diagram of a typical material in which there is no distinct yield point.

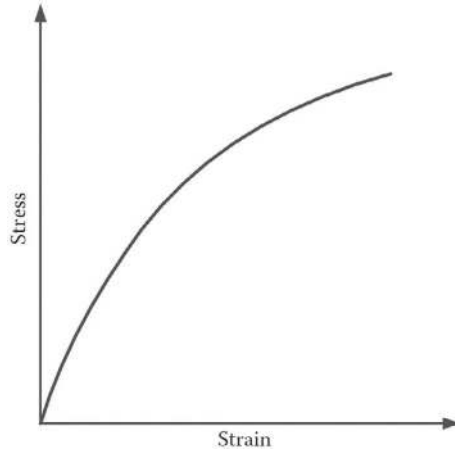


FIGURE 1.2 Stress-strain diagram of an elasto-plastic material.

One often makes several simplifying assumptions. In rigid perfectly plastic material, there is a threshold stress, called flow stress, until which no deformation takes place. After that, stress remains constant, but the strain keeps on increasing. When the load is removed, the stress becomes zero, but the permanent strain remains. In rigid-plastic materials, there is some strain-dependent hardening due to which the flow stress keeps increasing with further deformation. In visco-plastic materials, the dependency of strain rate on flow stress is considered. With increasing strain rate, the flow stress increases.

1.4 APPLICATIONS OF SOLID MECHANICS

The applications of solid mechanics can be broadly divided into two categories—(1) design of structure, machine, component or assembly; and (2) manufacture of a product. With the help of solid mechanics, one can calculate the deflection of a body under a specified force. One can also find the stresses coming out on the body and find if the body will not yield under a given loading. This information is very useful in design.

Figure 1.3 shows the photograph of a bridge. It contains trusses and beams. In this type of structure, no plastic deformation is expected. However, for the proper estimation of the factor of safety, one needs to apply plasticity theory. It is important to study the behavior of a structure after the structure has plastically deformed.

Figure 1.4 shows some examples of engineering products where solid mechanics finds extensive application. It shows a car, a CD driver, a hydraulic turbine and a robot. These are just a few examples. One can look around for innumerable products that are designed based on solid mechanics.

The second application of solid mechanics is in analyzing manufacturing processes. Two prominent manufacturing processes are metal-forming and machining processes. In metal-forming processes, the material is provided a desired shape by



FIGURE 1.3 A bridge.

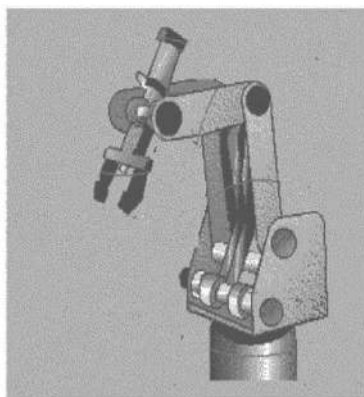
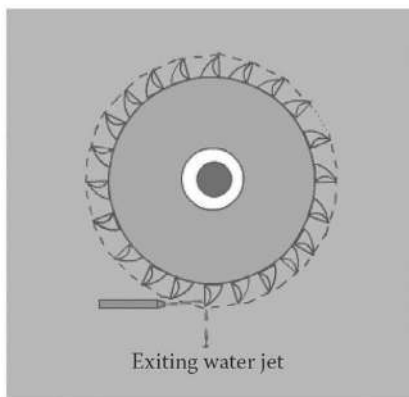


FIGURE 1.4 Some examples of engineering products.

plastic deformation. Figure 1.5 shows some examples of metal-forming processes. In sheet rolling, the thickness of the sheet is reduced by passing it in between two counter-rotating cylinders. In extrusion, the metal is pushed through a cavity to obtain a product of the desired cross-section. In closed-die forging, the metal is compressed

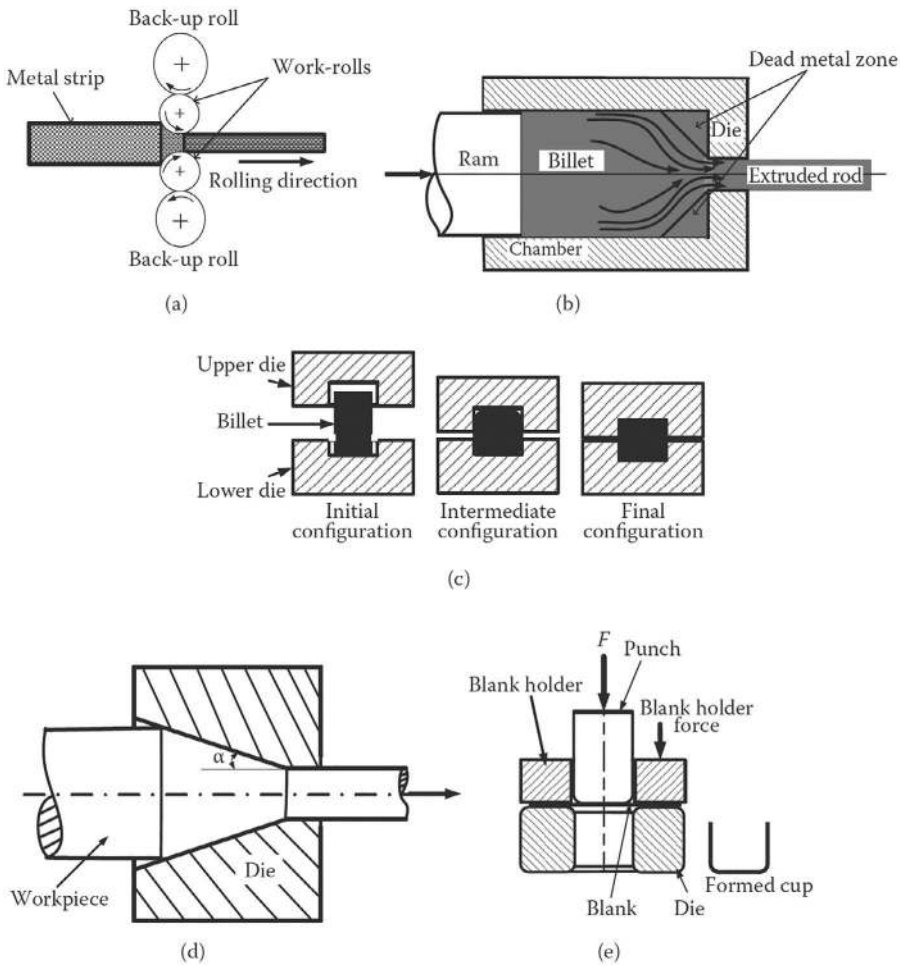


FIGURE 1.5 Examples of metal-forming processes: (a) rolling; (b) extrusion; (c) closed die forging; (d) wire drawing; (e) deep drawing.

between two die halves to obtain the desired shape. Wire drawing is the process of pulling the wire through a die to reduce its diameter. In the deep drawing process, a sheet is forced through the die by means of a punch to produce a cup. Without the proper application of plasticity theory, these processes cannot be analyzed. The forces required in the processes can be estimated with sufficient accuracy even with rigid-plastic assumption but the residual stresses in the product can be estimated only by elastic-plastic theory.

In machining, material is removed in the form of chips by applying a force on the material through a tool. Here, the work-piece material undergoes elastic-plastic deformation and material in particular zone fractures. The tool undergoes elastic deformation, but if the cutting forces exceed and/or the strength of the tool decreases because of excessive temperature, the tool may deform plastically or it may fracture.

The so-called non-traditional/non-conventional/advanced machining processes are different from conventional machining in the sense that there is no fixed tool that removes the material by physical contact with the workpiece. The non-traditional machining processes are classified based on the form of energy needed to remove the material. In mechanical processes, such as ultrasonic machining, abrasive jet machining, abrasive flow machining or water jet machining, mechanical energy in the form of kinetic energy or force times displacement is used to remove the material. Solid mechanics in general and plasticity theory in particular find application here. In electrical processes, such as electrochemical machining and electrochemical grinding, electrical energy is used to remove the material. In electrochemical machining, the material removal takes place atom by atom and is governed by Faraday's laws, and plasticity theory is not needed here. However, the design of the tool, considering the mechanical and thermal stresses coming on the tool, requires knowledge of plasticity theory. Thermal processes use thermal energy to melt or vaporize the material. Typical examples of such processes are electrical discharge machining, electron beam machining, laser beam machining and plasma arc machining. As the material removal is by melting and/or vaporization and not by fracture, solid mechanics does not find application. However, the effect of temperature on surrounding materials needs to be considered, and solid mechanics finds application here. Chemical etching processes, such as chemical milling and electropolishing, employ chemical energy to remove the material by chemical action. As the material removal is by chemical action, the modeling of the process does not require solid mechanics but in the design of equipment and tools, solid mechanics finds usual application.

1.5 SCOPE OF THIS TEXTBOOK

In this book, the basic fundamentals of solid mechanics of deformable bodies are explained, with an emphasis toward stresses generated in the material and its effect. More specifically, the book introduces the basic fundamentals of plasticity. Some application examples have been discussed.

Exercises

1. A 1-mm diameter and 2-mm length pin is compressed between two platens. Consider the distance between two atoms as 0.5 nm. By calculating the Knudsen number, justify the continuum hypothesis for this case.
2. What is the difference among kinematics, kinetics, dynamics and statics?
3. What are the two main applications of solid mechanics?
4. How do we differentiate solid, liquid and gas?
5. How does conventional machining differ from non-conventional machining?
6. What role does plasticity theory play in the modeling of metal-forming and machining processes?

2 Review of Algebra and Calculus of Vectors and Tensors

2.1 INTRODUCTION

There are a number of physical quantities that are completely specified by their magnitudes. For example, the mass of an object may be 10 kg. The temperature of a surface may be 50 °C. These quantities such as mass and temperature are called scalar quantities. The magnitude of a scalar quantity is independent of the coordinate system. Thus, in all coordinate systems, the mass of the object will be the same. Similarly, the temperature of a surface will be the same no matter what coordinate system is used. The scalars are invariant under the coordinate transformation.

Unlike scalars, there are a number of physical quantities that need the specification of direction apart from the magnitude. For example, the velocity of a particle has both magnitude and direction. Such quantities are called vectors. A vector can be graphically represented by an arrow. The length of the arrow represents the magnitude of the vector, and orientation in space represents the direction, pointing toward the arrowhead. A vector can be fully specified by its projections along the coordinate axes. Thus, in a two-dimensional Cartesian coordinate system, a vector can be specified by its x and y components, that is, its projections along x and y directions. In a three-dimensional space, a vector can be specified by three components. It is easy to understand a vector with the example of a position vector. The position vector of a point is the displacement needed for reaching that point from a reference point (say, origin). Thus, for point (x, y, z) , the position vector with respect to origin is written as $xe_1 + ye_2 + ze_3$, where the unit vectors e_1 , e_2 and e_3 are along the x , y and z directions, respectively. The components of a position vector will depend on the chosen coordinate system. It is possible to transform the components of a position vector from one system to another system by following a transformation rule. In fact, any physical quantity having three components in a three-dimensional space, whose components get transformed similarly to the components of a position vector, is a vector.

A scalar is also called a tensor of rank zero. A vector is also called a tensor of rank one. An example of a tensor of rank two is the stress at a point. In a three-dimensional space, it has nine components. It is completely specified if the direction and magnitude of force per unit area are determined on three mutually orthogonal planes. When the coordinate system is rotated, the components of a stress tensor get transformed following a transformation rule. In the same way, there are higher-order tensors. Unless indicated otherwise, the word ‘tensor’ shall be used to mean tensor of order (rank) two. In general, a tensor of rank r has the following:

1. It has n^r components in an n -dimensional space.
2. Its components transform in a particular fashion under the rotation of a coordinate system.

In this chapter, the algebra and calculus of vectors and tensors will be reviewed. Index notations will be used which help when writing lengthy equations in an abridged form. Relevant references for this chapter are Chandrashekaraiah and Debnath (1994), Arfken et al. (2005), Riley et al. (2006) and O'Neil (2007).

2.2 INDEX NOTATIONS

Suppose a vector has three components— a_1 , a_2 and a_3 . The components can be represented in the following form: a_i , where $i = 1, 2, 3$. It is known that in a three-dimensional space, a vector has three components. Hence, it is enough to write a_i to denote the components of the vector with the understanding that the index i varies from 1 to 3. Similarly, a_{ij} may represent the components of a tensor. In a three-dimensional space, it has nine components. Thus,

$$\begin{bmatrix} a_{ij} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \quad (2.1)$$

It is clear that in a three-dimensional space, the indices i and j vary from 1 to 3, and in a two-dimensional space, they vary from 1 to 2. This is called range convention.

For a three-dimensional space, the expression $a_i + b_i = 0$ means the following equations:

$$a_1 + b_1 = 0, \quad (2.2a)$$

$$a_2 + b_2 = 0, \quad (2.2b)$$

$$a_3 + b_3 = 0, \quad (2.2c)$$

The expression $a_i + b_{ij}$ is invalid as it is not possible to add a vector with a tensor. The expression $a_k + b_k$ also provides Equations 2.2a–2.2c. However, the expression $a_i + b_k$ is invalid as each term has different indices.

The expression $a_i b_k$ represents nine quantities, which can be represented in the form of a matrix:

$$\begin{bmatrix} a_i b_k \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}. \quad (2.3)$$

Let us consider the expression $a_i b_i$. Here, index i varies from 1 to 3 in a three-dimensional space. Obviously, unlike $a_i b_k$, the expression $a_i b_i$ cannot represent nine independent terms. It provides three terms $a_1 b_1$, $a_2 b_2$ and $a_3 b_3$. However, a convention of

adding the terms is adopted whenever there is a repeated index. Thus, since i is a repeated index in a term,

$$a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3. \quad (2.4)$$

The rule is that if an index in a term is repeated, then it means that the summation of the terms is obtained by assigning the values of the index over its range. This is called Einstein's summation convention. The expression $a_{ik} x_k = b_i$ means

$$a_{i1} x_1 + a_{i2} x_2 + a_{i3} x_3 = b_i. \quad (2.5)$$

Here, k is a repeated index in the term $a_{ik} x_k$. Three terms are obtained by assigning to k the values from 1 to 3. These three terms are added. In the process, index k disappears, whereas index i remains. Observe that the expression $a_{ij} x_j = b_i$ would also provide Equation 2.5. Thus, the repeated index may be replaced by any other index. For this reason, it is called a dummy index, whereas the non-repeated index is called a free index. Equation 2.5 implies the following three equations for three values of i :

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1, \quad (2.6a)$$

$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2, \quad (2.6b)$$

$$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3. \quad (2.6c)$$

Note that in an expression, each term should have the same free indices. Thus, the following are the valid expressions:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0, \quad (2.7)$$

$$\sigma_{ij} = C_{ijkl} \epsilon_{kl}, \quad (2.8)$$

$$p = \sigma_{ij} n_i n_j. \quad (2.9)$$

In Equation 2.7, j is the dummy index, and i is the free index. The free index i is present in each term. Varying the index from 1 to 3, three equations shall be obtained. In Equation 2.8, i and j are free indices that are present in each term, whereas k and l are dummy indices. Using summation convention,

$$\begin{aligned} C_{ijkl} \epsilon_{kl} &= C_{ij11} \epsilon_{11} + C_{ij12} \epsilon_{12} + C_{ij13} \epsilon_{13} \\ &+ C_{ij21} \epsilon_{21} + C_{ij22} \epsilon_{22} + C_{ij23} \epsilon_{23} \\ &+ C_{ij31} \epsilon_{31} + C_{ij32} \epsilon_{32} + C_{ij33} \epsilon_{33}. \end{aligned} \quad (2.10)$$

Here, k and l are varied from 1 to 3. Now, Equation 2.8 can be written as

$$\begin{aligned} \sigma_{ij} &= C_{ij11} \epsilon_{11} + C_{ij12} \epsilon_{12} + C_{ij13} \epsilon_{13} \\ &+ C_{ij21} \epsilon_{21} + C_{ij22} \epsilon_{22} + C_{ij23} \epsilon_{23} \\ &+ C_{ij31} \epsilon_{31} + C_{ij32} \epsilon_{32} + C_{ij33} \epsilon_{33}. \end{aligned} \quad (2.11)$$

The expression in Equation 2.11 represents nine equations that can be obtained by varying i and j from 1 to 3. In Equation 2.9, both i and j are dummy indices. There are no free indices in the expression. In expanded form, the equation is written as

$$\begin{aligned}
 p &= \sigma_{11}n_1n_1 + \sigma_{12}n_1n_2 + \sigma_{13}n_1n_3 + \sigma_{21}n_2n_1 + \sigma_{22}n_2n_2 \\
 &\quad + \sigma_{23}n_2n_3 + \sigma_{31}n_3n_1 + \sigma_{32}n_3n_2 + \sigma_{33}n_3n_3 \\
 &= \sigma_{11}n_1^2 + \sigma_{12}n_1n_2 + \sigma_{13}n_1n_3 + \sigma_{21}n_2n_1 + \sigma_{22}n_2^2 \\
 &\quad + \sigma_{23}n_2n_3 + \sigma_{31}n_3n_1 + \sigma_{32}n_3n_2 + \sigma_{33}n_3^2.
 \end{aligned} \tag{2.12}$$

The following expressions are invalid:

$$a_i b_j = c_i, \tag{2.13}$$

$$\sigma_{ij} + \varepsilon_{kl} = 0, \tag{2.14}$$

$$a_i b_i c_i = 0. \tag{2.15}$$

In Equation 2.13, the first term contains two free indices, i and j , whereas the second term contains only one free index i . Thus, all the terms are not having the same indices. Hence, the expression is invalid. In Equation 2.14, the first term has the free indices i and j , whereas the second term has indices k and l . Hence, it is an invalid expression. The valid expression will have the same free indices in each term. The following two expressions are the valid expressions:

$$\sigma_{ij} + \varepsilon_{ij} = 0; \quad \sigma_{kl} + \varepsilon_{kl} = 0. \tag{2.16}$$

In Equation 2.15, in the first term, index i occurs thrice. Hence, it is an invalid expression. In a term, an index can occur at most twice.

A comma (,) notation is also introduced here. The comma in the subscript indicates differentiation with respect to the coordinate. Thus,

$$a_{i,j} = \frac{\partial a_i}{\partial x_j}. \tag{2.17}$$

If ϕ is a scalar function of the coordinates, then

$$\phi_{,j} = \frac{\partial \phi}{\partial x_j}. \tag{2.18}$$

In a three-dimensional space, the index i can take the values 1, 2 and 3. Thus,

$$\{\phi_{,i}\} = \left\{ \begin{array}{c} \frac{\partial \phi}{\partial x_1} \\ \frac{\partial \phi}{\partial x_2} \\ \frac{\partial \phi}{\partial x_3} \end{array} \right\} = \text{gradient of } \phi. \tag{2.19}$$

Note that $\phi_{,i}$ indicates one component of the gradient vector.

Suppose that v_i indicates the component of a vector. The component $v_{i,j}$ denotes differentiation of v_i with respect to x_j . Thus,

$$v_{i,j} = \frac{\partial v_i}{\partial x_j}. \quad (2.20)$$

As i and j vary from 1 to 3, $v_{i,j}$ can take on nine values.

Example 2.1

Express $\sigma_{ij,j} + b_i = 0$ in an unabridged form.

SOLUTION

As per comma notation,

$$\sigma_{ij,j} = \frac{\partial \sigma_{ij}}{\partial x_j}. \quad (2.21)$$

In the previous expression, j occurs twice; hence, it is a dummy index. By summation convention,

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial \sigma_{i1}}{\partial x_1} + \frac{\partial \sigma_{i2}}{\partial x_2} + \frac{\partial \sigma_{i3}}{\partial x_3}. \quad (2.22)$$

Thus,

$$\sigma_{ij,j} + b_i = \frac{\partial \sigma_{i1}}{\partial x_1} + \frac{\partial \sigma_{i2}}{\partial x_2} + \frac{\partial \sigma_{i3}}{\partial x_3} + b_i = 0. \quad (2.23)$$

In Equation 2.23, i is a free index that can take on values 1, 2 and 3. Hence, the expression represents the following three equations:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + b_1 = 0, \quad (2.24a)$$

$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + b_2 = 0, \quad (2.24b)$$

$$\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + b_3 = 0. \quad (2.24c)$$

Example 2.2

Consider the following system of equations:

$$\begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}. \quad (2.25)$$

Express it in index notation.

SOLUTION

First, instead of x, y and z , 1, 2 and 3 are used. Thus, Equation 2.25 is written as

$$\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}. \quad (2.26)$$

The previous equation represents the following three equations:

$$t_1 = \sigma_{11}n_1 + \sigma_{12}n_2 + \sigma_{13}n_3, \quad (2.27a)$$

$$t_2 = \sigma_{21}n_1 + \sigma_{22}n_2 + \sigma_{23}n_3, \quad (2.27b)$$

$$t_3 = \sigma_{31}n_1 + \sigma_{32}n_2 + \sigma_{33}n_3. \quad (2.27c)$$

Using the free index i , these equations can be represented as

$$t_i = \sigma_{i1}n_1 + \sigma_{i2}n_2 + \sigma_{i3}n_3. \quad (2.28)$$

Using the dummy index j , the previous equation can be written as

$$t_i = \sigma_{ij}n_j. \quad (2.29)$$

Example 2.3

Given

$$[\sigma_{ij}] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

and

$$[\epsilon_{ij}] = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 6 & 7 & 8 \end{bmatrix},$$

calculate $\sigma_{ij}\epsilon_{ij}$.

SOLUTION

$$\begin{aligned} \sigma_{ij}\epsilon_{ij} &= \sigma_{11}\epsilon_{11} + \sigma_{12}\epsilon_{12} + \sigma_{13}\epsilon_{13} + \sigma_{21}\epsilon_{21} + \sigma_{22}\epsilon_{22} + \sigma_{23}\epsilon_{23} + \sigma_{31}\epsilon_{31} + \sigma_{32}\epsilon_{32} + \sigma_{33}\epsilon_{33} \\ &= 1 \times 1 + 2 \times 2 + 3 \times 4 + 4 \times 2 + 5 \times 3 + 6 \times 5 + 7 \times 6 + 8 \times 7 + 9 \times 8 \\ &= 240 \end{aligned}$$

2.3 KRONECKER DELTA AND LEVY-CIVITA SYMBOLS

Two important symbols are introduced in this section. The first one is a symbol introduced by a German mathematician named Leopold Kronecker (1823–1891) and is called the Kronecker delta (δ) symbol. It has two indices attached to δ as subscripts. The value of δ_{ij} is 1 if both indices have the same value and 0 if both indices are different. Thus,

$$\delta_{11} = \delta_{22} = \delta_{33} = 1, \quad \delta_{12} = \delta_{21} = \delta_{13} = \delta_{31} = \delta_{23} = \delta_{32} = 0. \quad (2.30)$$

Note that δ_{ij} represents the elements of an identity matrix, that is,

$$[\delta_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (2.31)$$

Observe that δ_{11} is 1, but δ_{ii} is not equal to 1. As i is a repeated dummy index in δ_{ii} ,

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3. \quad (2.32)$$

The Kronecker delta possesses the following important properties:

$$\begin{aligned} \text{(i)} \quad & a_i \delta_{ij} = a_j. \\ \text{(ii)} \quad & a_{ij} \delta_{jk} = a_{ik}. \\ \text{(iii)} \quad & \delta_{ij} \delta_{jk} = \delta_{ik}. \end{aligned} \quad (2.33)$$

To prove the first property, we proceed as follows. The expression on the left-hand side is $a_i \delta_{ij}$. Here, i is the dummy index, and j is the free index. Hence,

$$a_i \delta_{ij} = a_1 \delta_{1j} + a_2 \delta_{2j} + a_3 \delta_{3j}. \quad (2.34)$$

For $j = 1$, the right-hand-side expression becomes

$$a_1 \delta_{11} + a_2 \delta_{21} + a_3 \delta_{31} = a_1. \quad (2.35)$$

Similarly, for $j = 2$ and $j = 3$, it is a_2 and a_3 , respectively. Thus, the expression $a_i \delta_{ij}$ is a_1 for $j = 1$, a_2 for $j = 2$ and a_3 for $j = 3$. Therefore,

$$a_i \delta_{ij} = a_j. \quad (2.36)$$

To prove the second property, we proceed as follows:

$$a_{ij} \delta_{jk} = a_{i1} \delta_{1k} + a_{i2} \delta_{2k} + a_{i3} \delta_{3k}. \quad (2.37)$$

For $i = 1, k = 1$, the right-hand-side expression is equal to a_{11} . For $i = 1, k = 2$, it is equal to a_{12} and for $i = 2$ and $k = 2$, it is a_{22} . Proceeding this way, it can be seen that the right-hand-side expression is a_{ik} . Thus,

$$a_{ij}\delta_{jk} = a_{ik}. \quad (2.38)$$

Similarly, the third property can be proved.

Another important symbol is the Levi-Civita symbol, named after the Italian mathematician Tullio Levi-Civita (1873–1941). It is represented as ε_{ijk} . This symbol is also called the permutation symbol, the alternating symbol or the alternator. The value of ε_{ijk} is non-zero if i, j and k are different. If any two indices are equal, its value is 0. When i, j and k are different, ε_{ijk} is equal to 1, if the numerical values of i, j and k appear as in the sequence 12312. The value of ε_{ijk} is equal to -1 if the numerical values of i, j and k appear as in the sequence 32132. Thus,

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1; \quad \varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1. \quad (2.39)$$

The remaining 21 possible values of ε_{ijk} are 0.

The permutation symbol is very helpful in vector algebra. Consider three unit vectors $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ along the x, y and z directions, respectively. The cross-products of the unit vectors may be expressed as

$$\mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk} \mathbf{e}_k. \quad (2.40)$$

Equation 2.40 shows that if i and j are equal, the cross product is zero. Thus, the cross product of a unit vector with itself is 0. If i is 1 and j is 2, then

$$\mathbf{e}_1 \times \mathbf{e}_2 = \varepsilon_{12k} \mathbf{e}_k = \varepsilon_{121} \mathbf{e}_1 + \varepsilon_{122} \mathbf{e}_2 + \varepsilon_{123} \mathbf{e}_3 = \mathbf{e}_3. \quad (2.41)$$

Similarly, one can find the other cross-products. It is interesting to note that the values of nine different cross-products are given by just one expression of Equation 2.40.

Because the value of the alternating symbol is dependent on the order of the indices, the following relation holds good:

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} = -\varepsilon_{ikj} = -\varepsilon_{jik} = -\varepsilon_{kji}. \quad (2.42)$$

This means that when the two adjacent indices are interchanged, the value of the alternating symbols gets multiplied by -1 .

The following relation is called ε - δ identity or the permutation identity:

$$\varepsilon_{ijk} \varepsilon_{pqk} = \delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}. \quad (2.43)$$

It can be proved in the following way: the left-hand-side term will be non-zero only if i, j, p and q are different from k ; also, i should be different from j , and p should be different from k . This implies two possibilities: (1) i is equal to p , and j is equal to q ; and (2) i is equal to q , and j is equal to p . As k is a dummy index,

$$\varepsilon_{ijk}\varepsilon_{pqk} = \varepsilon_{ij1}\varepsilon_{pq1} + \varepsilon_{ij2}\varepsilon_{pq2} + \varepsilon_{ij3}\varepsilon_{pq3}. \quad (2.44)$$

In the right-hand side of Equation 2.44, out of the three terms, only one will be zero. If $i = p, j = q$, the value of that term will be one. If $i = q, j = p$, the value of the term will be negative one. Now, observing the expression on the right-hand side of ε - δ identity (Equation 2.43), it is seen that if $i = p, j = q$, the value of the expression will be one. If $i = q, j = p$, the value of the expression will be negative one. If $i = j, p = q$, the value of the expression will be zero. Thus, it is shown that the left-hand side of Equation 2.43 is the same as the right-hand side, and the identity is proved.

Example 2.4

Prove that

$$\begin{aligned} \varepsilon_{ijk}\varepsilon_{pqr} = & \delta_{ip}(\delta_{jq}\delta_{kr} - \delta_{jr}\delta_{kq}) + \delta_{iq}(\delta_{jr}\delta_{kp} - \delta_{jp}\delta_{kr}) \\ & + \delta_{ir}(\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}). \end{aligned} \quad (2.45)$$

SOLUTION

First, it shall be proved that

$$\varepsilon_{ijk}\varepsilon_{pqr}\det(a_{ij}) = \begin{vmatrix} a_{ip} & a_{iq} & a_{ir} \\ a_{jp} & a_{jq} & a_{jr} \\ a_{kp} & a_{kq} & a_{kr} \end{vmatrix}, \quad (2.46)$$

where $\det(a_{ij})$ denotes the determinant of matrix $[a_{ij}]$. It shall be proved by showing that the two sides of Equation 2.46 are equal. It is easy to see that if at least two of i, j, k or two of p, q, r are equal, then both sides are zero. If i, j, k are different from one another and p, q, r are also different from one another, then the following two cases may occur: (1) both i, j, k and p, q, r appear in the sequence 12312 or both appear in the sequence 32132. In this case, both sides of Equation 2.46 are equal to $\det(a_{ij})$. (2) Between i, j, k and p, q, r , one group appears in the sequence 12312 and the other in the sequence 32132. In this case, both sides of Equation 2.46 are equal to $-\det(a_{ij})$. This completes the proof for Equation 2.46.

Now, let $a_{ij} = \delta_{ij}$. From Equation 2.46,

$$\varepsilon_{ijk}\varepsilon_{pqr}\det(\delta_{ij}) = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix}. \quad (2.47)$$

Using the fact that $\det(\delta_{ij}) = 1$ and expanding the right-hand side of Equation 2.47, Equation 2.45 is obtained.

Example 2.5

Use Equation 2.45 to prove the ε - δ identity given by Equation 2.43.

SOLUTION

Replacing r with k in Equation 2.45,

$$\begin{aligned}\varepsilon_{ijk}\varepsilon_{pqk} &= \delta_{ip}(\delta_{jq}\delta_{kk} - \delta_{jk}\delta_{kq}) + \delta_{iq}(\delta_{jk}\delta_{kp} - \delta_{jp}\delta_{kk}) \\ &\quad + \delta_{ik}(\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp})\end{aligned}\quad (2.48)$$

Putting $\delta_{kk} = 3$ in Equation 2.48,

$$\varepsilon_{ijk}\varepsilon_{pqk} = \delta_{ip}(3\delta_{jq} - \delta_{jk}\delta_{kq}) + \delta_{iq}(\delta_{jk}\delta_{kp} - 3\delta_{jp}) + \delta_{ik}(\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}). \quad (2.49)$$

Using the third property of Equation 2.33,

$$\varepsilon_{ijk}\varepsilon_{pqk} = \delta_{ip}(3\delta_{jq} - \delta_{jq}) + \delta_{iq}(\delta_{jp} - 3\delta_{jp}) + (\delta_{jp}\delta_{iq} - \delta_{jq}\delta_{ip}). \quad (2.50)$$

Simplifying Equation 2.50, Equation 2.43 is obtained.

Example 2.6

Show that the determinant of a 3-by-3 matrix $[a_{ij}]$ can be expressed as

$$\det(a_{ij}) = \varepsilon_{pqr}a_{1p}a_{2q}a_{3r}. \quad (2.51)$$

SOLUTION

The determinant of a 3-by-3 matrix $[a_{ij}]$ is given by

$$\det(a_{ij}) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \quad (2.52)$$

In expanded form,

$$\begin{aligned}\det(a_{ij}) &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) \\ &\quad + a_{13}(a_{21}a_{32} - a_{22}a_{31}),\end{aligned}\quad (2.53)$$

or

$$\begin{aligned}\det(a_{ij}) &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} \\ &\quad - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.\end{aligned}\quad (2.54)$$

Observing the previous expression, it is seen that a typical term is $a_{1p}a_{2q}a_{3r}$. Further, when p, q, r appear in a sequence 12312, the term is positive; otherwise, it is negative. Therefore, the expression in Equation 2.54 can be written as $\varepsilon_{pqr}a_{1p}a_{2q}a_{3r}$, which is a summation of six terms with p, q, r as the dummy indices.

2.4 VECTORS

A vector is an entity that has magnitude and direction and can be added with another vector using the triangle law. The addition of vectors as per triangle law is done as follows: the vectors to be added are represented by straight arrows. The length of the arrow is equal to the length of the vector, and the direction is the same as that of the vector. The tail end of the second vector starts at the arrowhead of the first vector. These two vectors form the two sides of a triangle. The tail end of the first vector is joined with the arrowhead of the second vector to make the third side of the triangle. The length of the third side is equal to the magnitude of the resultant vector, which is directed from the tail end of the first vector to the arrowhead of the second vector. This is illustrated in Figure 2.1. In the figure, vector $\mathbf{a}$ is represented by arrow 1–2 and vector $\mathbf{b}$ by arrow 2–3. Then, the resultant vector $\mathbf{a} + \mathbf{b}$ is represented by arrow 1–3. The vectors also have a line of action, which is a hypothetical infinite straight line collinear with the vector. Thus, if two persons are moving side by side on a road with the same velocity, their velocities will have the same direction and magnitude but different lines of action. However, while adding two vectors, the line of action is not considered. Therefore, in Figure 2.1, vector $\mathbf{b}$ can also be represented by arrow 1–4. In that case, the tail ends of both vectors coincide at point 1. If 4 and 3 are joined, a parallelogram 1–2–3–4 is obtained. The resultant vector is still given by 1–3. Thus, a parallelogram law is obtained as an alternative to triangle law, which states:

If two vectors acting simultaneously at a point are represented both in magnitude and direction by the two adjacent sides of a parallelogram, then their resultant (i.e. the addition) is represented completely, both in magnitude and direction, by the diagonal of the parallelogram passing through the point.

All vectors should follow the parallelogram law for addition. If some physical quantity has direction and magnitude, but cannot be added as per the parallelogram law, it is not a vector. For example, finite rotations have direction (denoted by

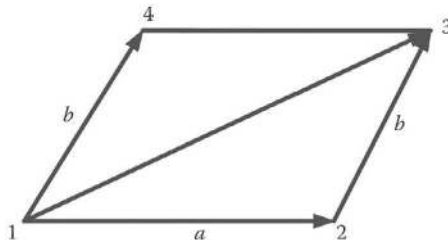


FIGURE 2.1 Addition of two vectors.

rotational axis) and magnitude but they do not follow the parallelogram law of addition. In fact, the addition of two finite rotations is not commutative. Hence, finite rotations are not vectors.

The position vector of a point can be considered as a basic vector quantity. The negative of a vector can also be defined as another vector with the same magnitude but opposite direction. Then, the subtraction of two vectors is basically the addition of one vector with the negative of another vector. It is appreciated that the addition (or subtraction) of two vectors will be a vector. The multiplication of a vector by a scalar means the multiplication of the magnitude of the vectors by the scalar, with the direction remaining unchanged. Considering the rules of addition, subtraction and multiplication of a vector by a scalar, it is easy to see that if the position of a particle can be represented by a vector, its displacement will also be a vector. This is because the displacement of the particle is the difference of its final and initial position vectors. Similarly, the velocity, which is the displacement divided by the scalar quantity time (notwithstanding that time tends to zero), is also a vector. In the same way, the acceleration is also a vector quantity. As the force can be obtained by the multiplication of a scalar mass with the acceleration, the force is also a vector and will have the same properties as the other vectors. Thus, the parallelogram law holds good for all vectors, that is, force, velocity, acceleration, etc. In the following subsections, the algebra and calculus of vectors are discussed.

2.4.1 NORM OF A VECTOR

In an n -dimensional Cartesian space, a vector can be represented by n components, which can be obtained by projecting the vector along the axes. Thus, in three dimensions, a vector $\mathbf{x}$ has three components x_1 , x_2 and x_3 . In this case,

$$\mathbf{x} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3. \quad (2.55)$$

By convention, the vector $\mathbf{x}$ can be expressed in the following ways:

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \{x_1 \quad x_2 \quad x_3\}^T, \quad (2.56)$$

where the superscript ‘T’ denotes transpose. Note that a triplet $\{x_1 \ x_2 \ x_3\}^T$ is a physical vector only if it follows the law of parallelogram. In mathematics, the vectors need not represent any physical quantity and may not have the concept of direction and magnitude. For example, suppose four experiments are carried out on a simple pendulum, each time changing its length and comparing the experimentally obtained time period with the theoretical one. Let us say the percentage errors in four cases are $-1, 3, 2, -2$. The errors in all the experiments can be represented by a vector $\mathbf{x} = \{-1 \ 3 \ 2 \ -2\}^T$. There is no concept of magnitude and direction of this error vector. However, one may be interested to have a single measure for all the errors. One way is to treat it like a physical vector, say, a position vector, and obtain its magnitude, which is given by

$$|x| = \sqrt{(-1)^2 + 3^2 + 2^2 + (-2)^2} = \sqrt{18}, \quad (2.57)$$

which is actually the root sum squared error. However, this measure may not be appropriate in all situations. Sometimes, one may be interested in just knowing the maximum value of the error, which is 3% in the present case. Both these measures, 18% root sum squared error and 3% maximum error, are a type of a norm of the vector $\mathbf{x}$.

The norm of a vector $\mathbf{x}$ denoted by $\|\mathbf{x}\|$ maps a vector into a real number and has the following properties:

1. Positive definiteness: $\|\mathbf{x}\| > 0$, if $\mathbf{x} \neq 0$.
2. Positive homogeneity: $\|a\mathbf{x}\| = |a| \|\mathbf{x}\|$, where a is a real number.
3. Triangle inequality: for any two vectors $\mathbf{x}$ and $\mathbf{y}$,

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

In property (2), if one puts $a = 0$, one gets $\|\mathbf{0}\| = 0$. The conventional magnitude of an n -dimensional vector is actually a norm. (It can be shown that it satisfies all the properties of the norm.) It is called the Euclidean norm. An L_p norm of a vector is defined by

$$\|\mathbf{x}\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p \right)^{1/p}. \quad (2.58)$$

Putting $p = 1$ in Equation 2.58,

$$\|\mathbf{x}\|_1 = |x_1| + |x_2| + \cdots + |x_n|. \quad (2.59)$$

Thus, the L_1 norm provides the sum of the absolute components of a vector. This norm is also called the taxicab norm. Putting $p = 2$ in Equation 2.58,

$$\|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}. \quad (2.60)$$

Thus, L_2 norm is the Euclidean norm. In this book, $\|\mathbf{x}\|$ (without any subscript) will mean $\|\mathbf{x}\|_2$. It can be easily shown that the L_∞ norm of a vector is equal to the magnitude of its largest component. L_∞ norm is also called the maximum norm. A semi-norm is a norm with the requirement of positive definiteness removed.

Example 2.7

Find out the L_1 , L_2 and L_∞ norm of a three-dimensional vector $\mathbf{x}$, whose components are 3, 4 and -12 .

SOLUTION

$$\|\mathbf{x}\|_1 = |3| + |4| + |-12| = 19.$$

$$\begin{aligned}\|\mathbf{x}\|_2 &= \sqrt{3^2 + 4^2 + (-12)^2} = 13. \\ \|\mathbf{x}\|_\infty &= \max(|3|, |4|, |-12|) = 12.\end{aligned}$$

Example 2.8

Derive the expression for the L_∞ norm of an n -dimensional vector.

SOLUTION

Let the i th component of an n -dimensional vector $\mathbf{x}$ be the largest in magnitude. As per Equation 2.58

$$\|\mathbf{x}\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_i|^p + \cdots + |x_n|^p \right)^{1/p}, \quad (2.61)$$

which can be written as

$$\|\mathbf{x}\|_p = |x_i| \left(\left| \frac{x_1}{x_i} \right|^p + \left| \frac{x_2}{x_i} \right|^p + \cdots + 1 + \cdots + \left| \frac{x_n}{x_i} \right|^p \right)^{1/p} \quad (2.62)$$

As p tends to infinity, all the terms inside the small bracket except '1' tend to 0. Hence,

$$\|\mathbf{x}\|_\infty = |x_i| = \text{the absolute value of the largest component.} \quad (2.63)$$

2.4.2 ADDITION OF VECTORS

Two vectors can be added according to parallelogram (or triangle law). It can easily be shown that for the two vectors $\mathbf{x}$ and $\mathbf{y}$ that are at an angle θ ,

$$\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\|\cos\theta, \quad (2.64)$$

where the norm used is the L_2 norm. It is already decided to omit the subscript 2 in the case of the L_2 norm. The resultant vector $\mathbf{x} + \mathbf{y}$ makes an angle α with vector $\mathbf{x}$, where

$$\tan\alpha = \frac{\|\mathbf{y}\|\sin\theta}{\|\mathbf{x}\| + \|\mathbf{y}\|\cos\theta}. \quad (2.65)$$

The addition of two vectors can also be carried out by adding their components. Thus, if

$$\mathbf{x} = \{x_1 x_2 \dots x_n\}^T \text{ and } \mathbf{y} = \{y_1 y_2 \dots y_n\}^T,$$

then

$$\mathbf{x} + \mathbf{y} = \{x_1 + y_1, x_2 + y_2 \dots x_n + y_n\}^T. \quad (2.66)$$

The addition defined in the previous manner satisfies the parallelogram law.

2.4.3 DOT PRODUCT

In index notation, the dot product of two vectors $\mathbf{x}$ and $\mathbf{y}$ is defined as

$$\mathbf{x} \cdot \mathbf{y} = x_i y_i. \quad (2.67)$$

Thus, for obtaining the dot product, each component of a vector is multiplied by the corresponding component of the other vector, and all such multiplications are added. Other names for the dot product are inner product and scalar product. It can be shown that in Euclidean space, the dot product may be obtained as

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos \theta, \quad (2.68)$$

where θ is the angle between two vectors. As $\cos \theta$ is always less than or equal to 1, the dot product of two vectors is always less than or equal to the products of their magnitudes.

In fact, through algebra, it can be shown that the magnitude of the dot product of two vectors is always less than or equal to the products of their magnitudes. This is called the Cauchy-Schwarz inequality. Two vectors are called orthogonal if and only if their dot product is 0.

2.4.4 CROSS PRODUCT

The cross product of two vectors is a vector. In index notation, the cross-product $\mathbf{z}$ of two vectors $\mathbf{x}$ and $\mathbf{y}$ is defined as

$$z_k = \varepsilon_{ijk} x_i y_j. \quad (2.69)$$

In expanded form,

$$z_1 = \varepsilon_{ij1} x_i y_j = \varepsilon_{231} x_2 y_3 + \varepsilon_{321} x_3 y_2 = x_2 y_3 - x_3 y_2, \quad (2.70a)$$

$$z_2 = \varepsilon_{ij2} x_i y_j = \varepsilon_{312} x_3 y_1 + \varepsilon_{132} x_1 y_3 = x_3 y_1 - x_1 y_3, \quad (2.70b)$$

$$z_3 = \varepsilon_{ij3} x_i y_j = \varepsilon_{123} x_1 y_2 + \varepsilon_{213} x_2 y_1 = x_1 y_2 - x_2 y_1. \quad (2.70c)$$

Thus,

$$\mathbf{z} = (x_2 y_3 - x_3 y_2) \mathbf{e}_1 + (x_3 y_1 - x_1 y_3) \mathbf{e}_2 + (x_1 y_2 - x_2 y_1) \mathbf{e}_3, \quad (2.71)$$

that can be expressed as

$$\mathbf{z} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}. \quad (2.72)$$

It can be shown that in a Euclidean space, the magnitude of the cross product of two vectors is given by

$$\|\mathbf{x} \times \mathbf{y}\| = \|\mathbf{x}\| \|\mathbf{y}\| \sin \theta, \quad (2.73)$$

where θ is the angle between the two vectors. The direction of the cross-product vector is perpendicular to $\mathbf{x}$ as well as $\mathbf{y}$. Thus, the cross-product vector is normal to the plane containing the vectors $\mathbf{x}$ and $\mathbf{y}$. It can be shown that the magnitude of the cross product of two vectors is equal to the area of a parallelogram having two vectors as its adjacent edges.

As the cross product of $\mathbf{x}$ and $\mathbf{y}$ is a vector denoted by $\mathbf{x} \times \mathbf{y}$, its dot product of a vector $\mathbf{z}$ will be a scalar. The product $(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z}$ is called a scalar triple product and is given by

$$(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}. \quad (2.74)$$

It can be shown that the magnitude of the scalar triple product of three vectors is equal to the volume of the parallelepiped whose adjacent edges with a common corner can be represented by three vectors. The product $(\mathbf{x} \times \mathbf{y}) \times \mathbf{z}$ is the vector triple product. It is non-commutative.

Example 2.9

Express the scalar triple product $(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z}$ in index notations and show that $\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}) = (\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z}$.

SOLUTION

Using the definition of dot and cross products,

$$(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z} = \varepsilon_{ijk} x_i y_j z_k. \quad (2.75)$$

Similarly,

$$\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}) = x_k \varepsilon_{ijk} y_i z_j. \quad (2.76a)$$

In Equation 2.76, i, j and k are dummy indices. Replacing i by j, j by k and k by i will keep the value of the expression the same. Hence,

$$\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}) = x_i \varepsilon_{jki} y_j z_k = \varepsilon_{jki} x_i y_j z_k. \quad (2.76b)$$

However, $\varepsilon_{ijk} = \varepsilon_{jki}$. Hence,

$$\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}) = \varepsilon_{ijk} x_i y_j z_k = (\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z}. \quad (2.77)$$

2.4.5 DERIVATIVE OF A VECTOR FUNCTION

Let a vector $\mathbf{x}$ be a function of some parameter t . Its components will also be the function of t . Representing the component by $x_i(t)$, its derivatives with respect to t can be found in the usual manner of finding the derivatives of a function. The derivative of the vector $\mathbf{x}$ is the vector addition of the derivative of its components. Thus, the i th component of the n th derivative of vector $\mathbf{x}$ is given by

$$\left(\frac{d^n \mathbf{x}}{dt^n} \right)_i = \frac{d^n x_i}{dt^n}. \quad (2.78)$$

For two vector fields $\mathbf{x}$ and $\mathbf{y}$ and a scalar field ϕ , the following identities hold good:

$$\frac{d}{dt}(\mathbf{x} + \mathbf{y}) = \frac{d\mathbf{x}}{dt} + \frac{d\mathbf{y}}{dt}, \quad (2.79)$$

$$\frac{d}{dt}(\phi \mathbf{x}) = \frac{d\phi}{dt} \mathbf{x} + \phi \frac{d\mathbf{x}}{dt}, \quad (2.80)$$

$$\frac{d}{dt}(\mathbf{x} \cdot \mathbf{y}) = \mathbf{x} \cdot \frac{d\mathbf{y}}{dt} + \frac{d\mathbf{x}}{dt} \cdot \mathbf{y}, \quad (2.81)$$

$$\frac{d}{dt}(\mathbf{x} \times \mathbf{y}) = \mathbf{x} \times \frac{d\mathbf{y}}{dt} + \frac{d\mathbf{x}}{dt} \times \mathbf{y}. \quad (2.82)$$

2.4.6 GRADIENT OF A SCALAR FIELD

If f is a scalar field, that is, f is a function of the coordinates, then $a_i = f_{,i}$ are the components of a vector. This vector is called the gradient of f and is denoted by $\text{grad } f$ or ∇f . The operator ∇ may be written as $\partial/\partial x_i$. It is known that for infinitesimal change in the values of the coordinates x , y and z , the infinitesimal change in the function is given by

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz. \quad (2.83)$$

In index notation, it is written as

$$df = f_{,i} dx_i \text{ or } df = (\nabla f)_i dx_i. \quad (2.84)$$

In the vector notation, it is written as

$$df = \nabla f \cdot d\mathbf{x}. \quad (2.85)$$

Now, suppose there is a surface given by $f(x, y, z) = 0$, and one moves by an infinitesimal distance on this surface; then $d\mathbf{x}$ will be tangential to the surface, and df will be zero. It then follows from Equation 2.85 that ∇f will be perpendicular to $d\mathbf{x}$. Thus, ∇f will be directed along normal to the surface.

If there is a unit vector (i.e. a vector of magnitude 1) $\mathbf{a}$, then $\nabla f \cdot \mathbf{a}$ will indicate the rate of change of the function with respect to length along $\mathbf{a}$, which is called the directional derivative of f along $\mathbf{a}$. To see it clearly consider that a particle moves infinitesimally along $\mathbf{a}$, and its displacement vector is given by $d\mathbf{a}$. Let the components of this displacement vector be da_x , da_y and da_z . Then, the infinitesimal change in the function is given by

$$df = \frac{\partial f}{\partial x} da_x + \frac{\partial f}{\partial y} da_y + \frac{\partial f}{\partial z} da_z. \quad (2.86)$$

The magnitude of the displacement vector, this is, its L_2 norm is

$$\|d\mathbf{a}\| = \sqrt{(da_x)^2 + (da_y)^2 + (da_z)^2}. \quad (2.87)$$

Therefore, the rate of change (with respect to length) in the function is

$$\frac{df}{\|d\mathbf{a}\|} = \frac{\partial f}{\partial x} \frac{da_x}{\|d\mathbf{a}\|} + \frac{\partial f}{\partial y} \frac{da_y}{\|d\mathbf{a}\|} + \frac{\partial f}{\partial z} \frac{da_z}{\|d\mathbf{a}\|}. \quad (2.88)$$

However, by definition of the unit vector

$$\mathbf{a} = \frac{da_x}{\|d\mathbf{a}\|} \mathbf{e}_1 + \frac{da_y}{\|d\mathbf{a}\|} \mathbf{e}_2 + \frac{da_z}{\|d\mathbf{a}\|} \mathbf{e}_3. \quad (2.89)$$

Thus, Equation 2.88 may be written as

$$\frac{df}{\|d\mathbf{a}\|} = \nabla f \cdot \mathbf{a}. \quad (2.90)$$

The notation $df/d\mathbf{a}$ will be used for the directional derivative along $\mathbf{a}$. Note that

$$\frac{df}{da} = \nabla f \cdot \mathbf{a} = \|\nabla f\| \|\mathbf{a}\| \cos \phi, \quad (2.91)$$

where ϕ is the angle between $\mathbf{a}$ and normal (along ∇f) to the surface $f=0$. It then follows that the normal derivative along ∇f is the maximum. Thus, ∇f is the direction of the maximum increase in the function value.

Example 2.10

Find out the rate of change of function (with respect to length) $f = x^2 - 2y$ along line $x - y = 1$ at point $(1, 0)$.

SOLUTION

The gradient of the function is

$$\nabla f = 2x\mathbf{e}_1 - 2\mathbf{e}_2. \quad (2.92)$$

The slope of the line $x - y = 1$ is given by 1, that is, $\tan \theta = 1$, where θ is the angle that a line makes with the x -axis. As $\cos \theta = 1/\sqrt{2}$ and $\sin \theta = 1/\sqrt{2}$, the unit vector parallel to this line is given by

$$\mathbf{a} = \frac{1}{\sqrt{2}}\mathbf{e}_1 + \frac{1}{\sqrt{2}}\mathbf{e}_2. \quad (2.93)$$

The directional derivative along this unit vector is given by

$$\frac{df}{da} = \nabla f \cdot \mathbf{a} = (2x\mathbf{e}_1 - 2\mathbf{e}_2) \cdot \left(\frac{1}{\sqrt{2}}\mathbf{e}_1 + \frac{1}{\sqrt{2}}\mathbf{e}_2 \right) = \sqrt{2}x - \sqrt{2}. \quad (2.94)$$

For point $(1, 0)$, substituting $x = 1$ and $y = 0$ in Equation 2.94, the directional derivative comes out to be 0. Hence, the rate of change of function is zero.

For confidence in the concept of directional derivative, let us solve this problem by another method. As it is of interest to know the rate of change along line $x - y = 1$, $y = x - 1$ is substituted in the function to get

$$f = x^2 - 2x + 2. \quad (2.95)$$

For this function, df/dx is zero at $x = 1$. Hence, on the line $x - y = 1$, the rate of change of function is 0 at point $(1, 0)$. Thus, its directional derivative is zero at that point.

Example 2.11

Given the function $f = x - y$, find the rate of change (with respect to length) of this function along line $x - y = 1$.

SOLUTION

This is a trivial problem. As one moves along the line $x - y = 1$, $f = 1$, that is, it is a constant function. Thus, its rate of change is 0. However, it shall be solved using a gradient.

The gradient of the function (in matrix notation) is $\{1 \ -1\}^T$. The unit vector along the line is $\left\{ \frac{1}{\sqrt{2}} \ \frac{1}{\sqrt{2}} \right\}^T$. The directional derivative of the function is given by the dot product of these two vectors, which comes out to be zero.

2.4.7 DIVERGENCE AND CURL OF A VECTOR FIELD

Let v_i be the component of a vector field. Then $v_{i,i}$ is called the divergence of the vector field. The divergence of the vector field is a scalar field. The divergence is denoted by 'div'. If $\mathbf{v}$ is a vector field, $\text{div } \mathbf{v}$ is basically $\nabla \cdot \mathbf{v}$. In the x, y, z Cartesian system, it is given by

$$\text{div } \mathbf{v} = \left(\frac{\partial}{\partial x}\mathbf{e}_1 + \frac{\partial}{\partial y}\mathbf{e}_1 + \frac{\partial}{\partial z}\mathbf{e}_1 \right) \cdot (v_1\mathbf{e}_1 + v_2\mathbf{e}_1 + v_3\mathbf{e}_1) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}. \quad (2.96a)$$

The divergence of a constant vector field is zero. The gradient of a scalar field f is a vector. Its divergence is given by

$$\begin{aligned}\operatorname{div} \operatorname{grad} f &= \left(\frac{\partial}{\partial x} \mathbf{e}_1 + \frac{\partial}{\partial y} \mathbf{e}_1 + \frac{\partial}{\partial z} \mathbf{e}_1 \right) \cdot \left(\frac{\partial}{\partial x} \mathbf{e}_1 + \frac{\partial}{\partial y} \mathbf{e}_1 + \frac{\partial}{\partial z} \mathbf{e}_1 \right) f \\ &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.\end{aligned}\quad (2.96b)$$

Note that by notation,

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.\quad (2.97)$$

Thus,

$$\operatorname{div} \operatorname{grad} f = \nabla^2 f.\quad (2.98)$$

To understand the physical meaning of divergence, consider the flow of water. Let P with coordinates (x, y, z) be a point in the domain, as shown in Figure 2.2. Construct a rectangular parallelepiped of sides dx , dy and dz around the point, P being at the centroid. Let (u, v, w) be the components of the velocity at point P , and the size of the cube is so small that the velocity components vary only linearly in the parallelepiped. The volume of water coming out of the cube in unit time is given by

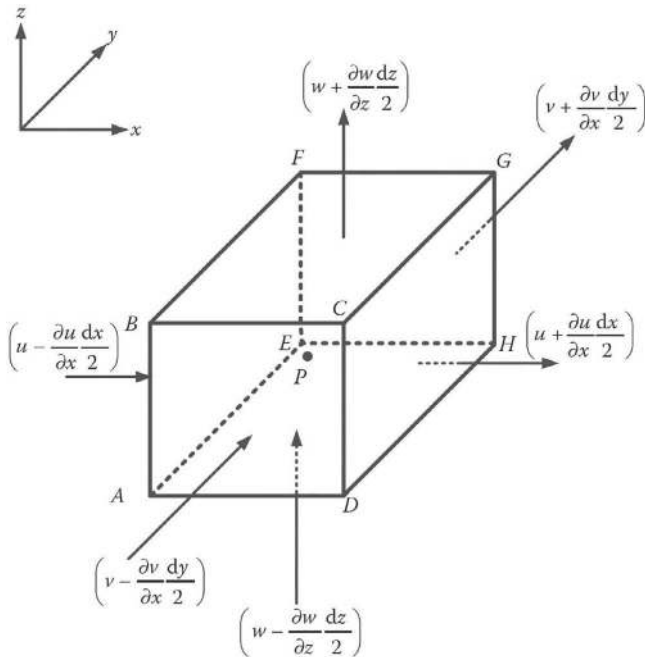


FIGURE 2.2 Flow of water through a rectangular parallelepiped.

$$dQ_{\text{out}} = \left(u + \frac{\partial u}{\partial x} \frac{dx}{2} \right) dydz + \left(v + \frac{\partial v}{\partial y} \frac{dy}{2} \right) dx dz + \left(w + \frac{\partial w}{\partial z} \frac{dz}{2} \right) dx dy. \quad (2.99)$$

The volume of water going into the cube in unit time is given by

$$dQ_{\text{in}} = \left(u - \frac{\partial u}{\partial x} \frac{dx}{2} \right) dy dz + \left(v - \frac{\partial v}{\partial y} \frac{dy}{2} \right) dx dz + \left(w - \frac{\partial w}{\partial z} \frac{dz}{2} \right) dx dy. \quad (2.100)$$

The net rate of outflow from the parallelepiped is given by

$$dQ_{\text{out}} - dQ_{\text{in}} = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dx dy dz. \quad (2.101)$$

The volume of the parallelepiped is $dx dy dz$. The rate of outflow per unit volume is called the divergence of the velocity field $\mathbf{v}$. Therefore, the divergence is given by

$$\text{div} \mathbf{v} = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right). \quad (2.102)$$

The curl of a vector field is a vector field. If $\mathbf{v}_i$ represents the component of a vector field, then the curl of the vector field $\mathbf{v}$ is denoted by $\text{curl } \mathbf{v}$ or $\nabla \times \mathbf{v}$ and its i th component is given by

$$(\nabla \times \mathbf{v})_i = \varepsilon_{imn} v_{n,m}. \quad (2.103)$$

It is obvious that the curl of a constant vector field is a zero vector. The curl can be expressed in the form of a determinant as follows:

$$\nabla \times \mathbf{v} = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}, \quad (2.104)$$

where the determinant is expanded in the usual way, but $\frac{\partial}{\partial y} v_3$ does not denote the product of two quantities; it is, in fact, the partial derivative of v_3 with respect to y . A similar interpretation should be made with the other term.

Let us find the value of $\text{curl grad } f$. The n th component of $\text{grad } f$ is given by $f_{,n}$. Therefore, from Equation 2.103, the i th component of $\text{curl grad } f$ is given by

$$(\text{curl grad } f)_i = \varepsilon_{imn} f_{,nm}. \quad (2.105)$$

In the previous expression, the dummy indices are interchanged to get

$$(\text{curl grad } f)_i = \varepsilon_{imn} f_{,mn}. \quad (2.106)$$

Note that

$$\varepsilon_{imn} = -\varepsilon_{imn} \text{ and } f_{,nm} = f_{,mn}. \quad (2.107)$$

Therefore, Equation 2.106 can be written as

$$(\text{curl } \nabla f)_i = -\varepsilon_{imn} f_{,nm}. \quad (2.108)$$

Comparing Equations 2.105 and 2.108, it is seen that

$$(\text{curl } \nabla f)_i = -(\text{curl } \nabla f)_i \text{ or } (\text{curl } \nabla f)_i = 0. \quad (2.109)$$

Hence, the curl of the gradient of a scalar field is zero.

Physically, the curl represents some sort of rotation. Let us consider a velocity field $x\mathbf{e}_1$. Here, all the particles are moving parallel to the x -axis, and velocities do not vary with y or z for a fixed x . The curl of this velocity field is zero. It indicates that there is no rotation. Now, suppose the velocity field is $y\mathbf{e}_1$. Here, also the velocities of all particles are along the x -direction. However, their magnitudes are changing along the y -direction; it is zero at the x -axis and increases linearly along the y -direction. Its curl is $-\mathbf{e}_3$. It can be easily visualized that this velocity field shows some clockwise rotation about the z -axis, which is indicated by its curl. In fact, it shall be shown in a later chapter that the curl of a velocity field is equal to twice the angular velocity.

2.4.8 GREEN'S THEOREM IN A PLANE

Consider a two-dimensional space and two functions $u_1(x, y)$ and $u_2(x, y)$. Let there be a closed curve C enclosing the area A in the two-dimensional space, as shown in Figure 2.3. In that case, Green's theorem states that

$$\oint_C (u_1 dx + u_2 dy) = \int_A \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right) dx dy \quad (2.110)$$

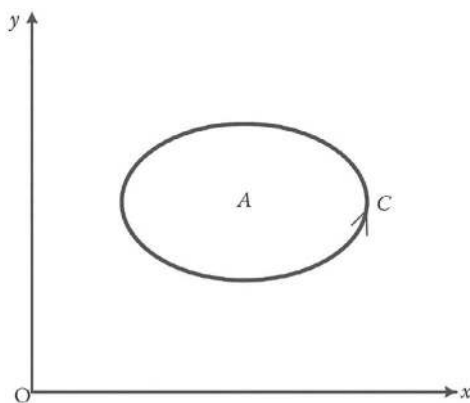


FIGURE 2.3 A simply connected closed curve with arrow indicating the direction of positive orientation.

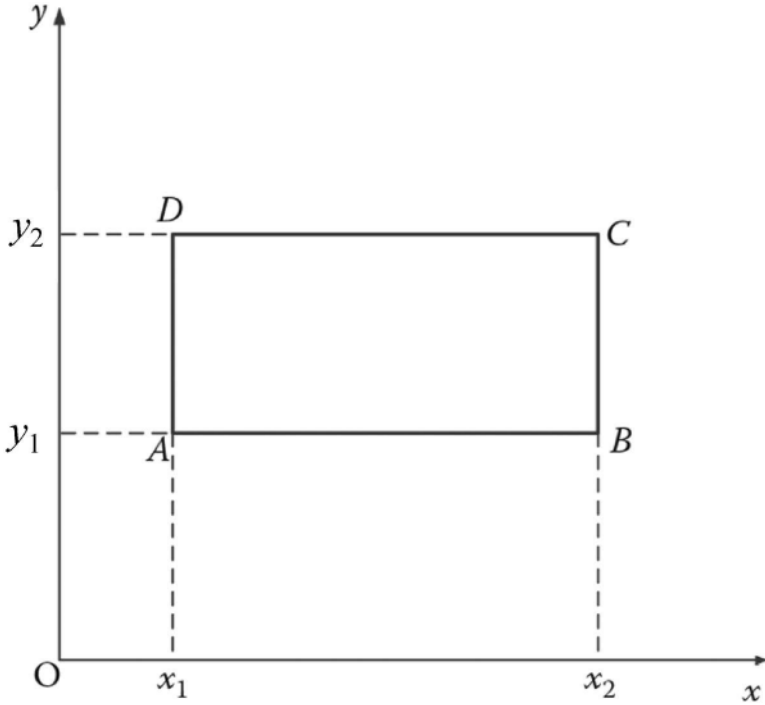


FIGURE 2.4 Rectangular region ABCD.

where ‘ $\oint$ ’ indicates the contour integration in the positive sense. While integrating, the curve $\mathcal{C}$ is moved on in such a manner that the enclosed region is always toward the left. It is also assumed that the partial derivatives of u_1 and u_2 exist.

An application of Green’s theorem is that the area integrals can be reduced to line integrals. Considering $u_1 = 0$ and $u_2 = x$, from Equation 2.110,

$$\oint_{\mathcal{C}} x dy = \int_A dx dy. \quad (2.111)$$

Equation 2.111 can be used to find the area enclosed by the curve. As a trivial example, consider a rectangular ABCD shown in Figure 2.4. Its area is given by $(x_2 - x_1)(y_2 - y_1)$. Let us find its area by contour integration (Equation 2.111) along ABCD. The contour integration is the sum of integrations along AB, BC, CD and DA. Along AB, the y coordinate does not change. Hence, dy is 0. Thus, the integral is zero along AB. On line BC, the x coordinate remains constant at x_2 , and the y coordinate changes from y_1 to y_2 . Hence, the line integral along BC is $x_2(y_2 - y_1)$. Similarly, the line integrals along CD and DA are 0 and $x_1(y_1 - y_2)$, respectively. Adding four line integrals (along AB, BC, CD and DA), the contour integral is obtained as $(x_2 - x_1)(y_2 - y_1)$, which is equal to the area of the region.

Example 2.12

Find out the area of the ellipse whose major axis is $2a$ and minor axis is $2b$.

SOLUTION

Consider the ellipse given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad (2.112)$$

whose major axis is $2a$ and $2b$. Its area can be obtained by evaluating the contour integration given in Equation 2.111. In the parametric form, Equation 2.112 may be written as

$$x = a \cos \theta, \quad y = b \sin \theta, \quad 0 < \theta < 2\pi. \quad (2.113)$$

Thus,

$$\oint_C x dy = \int_0^{2\pi} a \cos \theta b \cos \theta d\theta = \pi ab. \quad (2.114)$$

Hence, the area of the ellipse is πab .

2.4.9 DIVERGENCE THEOREM OF GAUSS

Let V be the volume of a three-dimensional region bounded by a closed regular (piecewise smooth) surface S ; then for a vector field $\mathbf{u}$ defined in V and S ,

$$\int_V \text{div} \mathbf{u} dV = \int_V \nabla \cdot \mathbf{u} dV = \int_S \mathbf{u} \cdot \mathbf{n} dS \quad (2.115)$$

where $\mathbf{n}$ is the unit outward normal to S . This is called the divergence theorem of Gauss (1777–1855). In index notation, the theorem is expressed as

$$\int_V u_{i,i} dV = \int_S u_i n_i dS. \quad (2.116)$$

There are many variants of the theorem but with some manipulation, one form can be derived from the other form. For example, consider the following statement of divergence theorem:

Let V be the volume of a three-dimensional region bounded by a closed regular surface S ; then for a scalar field f defined in V and S ,

$$\int_V \text{grad} f dV = \int_S f \mathbf{n} dS, \quad (2.117)$$

where $\mathbf{n}$ is the unit outward normal to S .

Equation 2.117 will be derived from Equation 2.116. In the expanded form, Equation 2.116 is written as

$$\int_V \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) dV = \int_S (u_x n_x + u_y n_y + u_z n_z) dS. \quad (2.118)$$

The previous equation is valid for any arbitrary $\mathbf{u}$. Consider the case in which u_y and u_z are zero. Then, from Equation 2.118,

$$\int_V \frac{\partial u_x}{\partial x} dV = \int_S u_x n_x dS. \quad (2.119)$$

As u_x is a function of x , y and z , it can be considered as a scalar field and be denoted by f . Thus,

$$\int_V \frac{\partial f}{\partial x} dV = \int_S f n_x dS. \quad (2.120)$$

Similarly,

$$\int_V \frac{\partial f}{\partial y} dV = \int_S f n_y dS \text{ and } \int_V \frac{\partial f}{\partial z} dV = \int_S f n_z dS. \quad (2.121)$$

From Equations 2.120 and 2.121,

$$\int_V \left(\frac{\partial f}{\partial x} \mathbf{e}_1 + \frac{\partial f}{\partial y} \mathbf{e}_2 + \frac{\partial f}{\partial z} \mathbf{e}_3 \right) dV = \int_S (f n_x \mathbf{e}_1 + f n_y \mathbf{e}_2 + f n_z \mathbf{e}_3) dS, \quad (2.122)$$

which is nothing but Equation 2.117 in expanded form.

Example 2.13

Using Gauss's divergence theorem, derive the continuity equation for a steady incompressible flow.

SOLUTION

Consider a region of volume V bounded by surface S . From an infinitesimally small segment of the surface, the flow coming out is $\mathbf{v} \cdot \mathbf{n} dS$, where $\mathbf{v}$ is the velocity vector and $\mathbf{n}$ is the outward normal to surface segment dS . The net outflow (actual outflow minus actual inflow) from the surface is given as

$$\text{net outflow} = \int_S \mathbf{v} \cdot \mathbf{n} dS. \quad (2.123)$$

As the flow is incompressible, the net outflow must be zero. Using this fact and the divergence theorem, the following is obtained:

$$\int_V \operatorname{div} \mathbf{v} dV = 0. \quad (2.124)$$

Now, region V can be made as small as one wishes, and Equation 2.124 holds good for every small region, implying that

$$\operatorname{div} \mathbf{v} = 0, \quad (2.125)$$

which is the continuity equation for incompressible steady flow. A little thinking will reveal that it is valid for incompressible non-steady flow as well.

2.4.10 INTEGRAL THEOREM OF STOKES

Let C be a simple closed curve in three-dimensional space and S be an open regular surface bounded by C ; then for a vector field $\mathbf{u}$ defined on S as well as C ,

$$\int_C \mathbf{u} \cdot \mathbf{t} ds = \int_S (\operatorname{curl} \mathbf{u}) \cdot \mathbf{n} dS \quad (2.126)$$

where $\mathbf{t}$ is the unit tangent to the curve and $\mathbf{n}$ is the unit normal to the surface, where the direction of $\mathbf{t}$ and $\mathbf{n}$ is chosen such that the curve is coherently oriented with respect to $\mathbf{n}$ (O'Neil, 2007). Even in a three-dimensional space, a curve can be represented by a single parameter. For example, in a Cartesian system, the three coordinates of the curve can be written as

$$x = x(t), y = y(t) \text{ and } z = z(t). \quad (2.127)$$

A curve is simple if the same point cannot be on the curve except for initial and terminal points. Initial and terminal points have to coincide for a 'closed curve'. Like a curve can be represented by one parameter, a surface S can be represented by two parameters u and v . Thus, the coordinates of the points on a surface are given as

$$x = x(u, v), y = y(u, v) \text{ and } z = z(u, v). \quad (2.128)$$

The normal to the surface is given by

$$\mathbf{N} = \frac{\partial(y, z)}{\partial(u, v)} \mathbf{e}_1 + \frac{\partial(z, x)}{\partial(u, v)} \mathbf{e}_2 + \frac{\partial(x, y)}{\partial(u, v)} \mathbf{e}_3, \quad (2.129)$$

where

$$\frac{\partial(y, z)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix}. \quad (2.130)$$

is called the Jacobian determinant of functions y and z . The unit normal $\mathbf{n}$ in Equation 2.126 is

$$\mathbf{n} = \frac{\mathbf{N}}{\|\mathbf{N}\|}. \quad (2.131)$$

If we stand on the curve C with our head on the tip of $\mathbf{n}$ and move on the curve in such a manner that the surface S is toward our left shoulders, then the curve C is called to be coherently oriented with respect to $\mathbf{n}$. Mathematically, for a coherently oriented curve with tangent $\mathbf{t}$ expressed as a function of parameter t , the following relation holds good:

$$(\mathbf{t} \times \dot{\mathbf{t}}) \cdot \mathbf{n} > 0, \quad (2.132)$$

where $\dot{\mathbf{t}}$ is the derivative with respect to t . Figure 2.5 shows a coherently oriented curve with respect to the normal to the surface. In the figure, tangent to the curve and normal to the surface are shown. Observe that as we walk on the curve with our head on the tip of the normal, the surface is toward our left.

Example 2.14

Find the unit normal to surface

$$x^2 + y^2 + z^2 = 9, \quad z \geq 0. \quad (2.133)$$

SOLUTION

This is an equation of a hemisphere of radius 3. Expressing z in terms of x and y , one gets

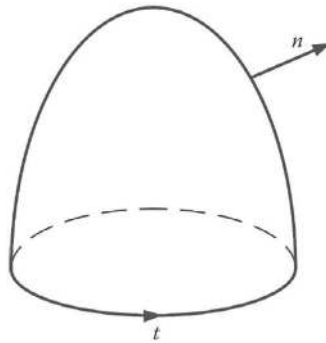


FIGURE 2.5 A coherently oriented curve with respect to the normal to the surface.

$$z = \sqrt{9 - x^2 - y^2}. \quad (2.134)$$

The parametric form of the surface is written as

$$x = u, y = v, z = \sqrt{9 - u^2 - v^2}. \quad (2.135)$$

Then,

$$\frac{\partial(y, z)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -\frac{x}{z} & -\frac{y}{z} \end{vmatrix} = \frac{x}{z}, \quad (2.136a)$$

$$\frac{\partial(z, x)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{x}{z} & -\frac{y}{z} \\ 1 & 0 \end{vmatrix} = \frac{y}{z}, \quad (2.136b)$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1. \quad (2.136c)$$

Hence, by Equation 2.129, a normal is given as

$$N = \frac{x}{z} \mathbf{e}_1 + \frac{y}{z} \mathbf{e}_2 + \mathbf{e}_3. \quad (2.137)$$

Its L_2 norm is given by

$$\|N\| = \sqrt{\frac{x^2}{z^2} + \frac{y^2}{z^2} + 1} = \frac{3}{z}. \quad (2.138)$$

By Equation 2.131, the unit normal is given by

$$\mathbf{n} = \frac{x}{3} \mathbf{e}_1 + \frac{y}{3} \mathbf{e}_2 + \frac{z}{3} \mathbf{e}_3. \quad (2.139)$$

Example 2.15

A particle is subjected to a force field $-y\mathbf{e}_1 + x\mathbf{e}_2$. It is constrained to move on the periphery of circle $x^2 + y^2 = 4$. Obtain the work done in one revolution. Solve this problem by carrying out first the contour integration and then the surface integration.

SOLUTION

The tangent to the circle $x^2 + y^2 = 4$ is given by

$$\mathbf{t} = -\frac{y}{2}\mathbf{e}_1 + \frac{x}{2}\mathbf{e}_2. \quad (2.140)$$

Hence, the work done is

$$\oint (-ye_1 + xe_2) \cdot \left(-\frac{y}{2}\mathbf{e}_1 + \frac{x}{2}\mathbf{e}_2\right) ds = \oint \left(\frac{x^2 + y^2}{2}\right) ds = 2(\text{perimeter}). \quad (2.141)$$

As the radius of the circle is 2, the perimeter is 4π . Hence, the work done is 8π .

Now, the curl of the force field is given by

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = 2\mathbf{e}_3 \quad (2.142)$$

Hence, by Stokes's theorem,

$$\text{work done} = \int_S 2\mathbf{e}_3 \cdot \mathbf{n} dS \quad (2.143)$$

Any surface that is enclosed by the circle can be chosen. A simple disk is taken. The normal to top surface of this disk is along $\mathbf{e}_3$. Hence,

$$\text{work done} = \int_S 2\mathbf{e}_3 \cdot \mathbf{e}_3 dS = 2(\text{top surface area of the disk}) = 8\pi. \quad (2.144)$$

Thus, both methods give the same work.

2.5 TRANSFORMATION RULES FOR VECTOR COMPONENTS UNDER THE ROTATION OF CARTESIAN COORDINATE SYSTEM

Consider two coordinate systems— x, y, z and x', y', z' —which are rotated with respect to each other. The unit vectors along x, y and z are $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$, and unit vectors along x', y' and z' are $\mathbf{e}'_1, \mathbf{e}'_2$ and $\mathbf{e}'_3$. In the x, y, z system, a vector $\mathbf{v}$ can be represented as

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 = v_p\mathbf{e}_p \quad (2.145)$$

The same vector is represented in the x', y', z' system as

$$\mathbf{v} = v'_1\mathbf{e}'_1 + v'_2\mathbf{e}'_2 + v'_3\mathbf{e}'_3 = v'_p\mathbf{e}'_p \quad (2.146)$$

Note that the vector remains the same in both systems. However, its components change. One wants to determine the component in one system as a function of components in other systems. For this purpose, one first writes

$$v_p\mathbf{e}_p = v'_q\mathbf{e}'_q. \quad (2.147)$$

Note that as p is a dummy index in Equation 2.146, it has been replaced by q . Taking the dot product of both sides with unit vector $\mathbf{e}_r$, one gets

$$v_p \mathbf{e}_p \cdot \mathbf{e}_r = v'_q \mathbf{e}'_q \cdot \mathbf{e}_r. \quad (2.148)$$

Now,

$$\mathbf{e}_p \cdot \mathbf{e}_r = \delta_{pr} \quad (2.149)$$

which means that the dot product of identical vectors is 1 and that of nonidentical vectors is 0. The dot product $\mathbf{e}'_q \cdot \mathbf{e}_r$ is equal to the cosine of the angle between the q th axis in the x', y', z' system and the r th axis in the x, y, z system, and it is denoted by α_{qr} . Hence, Equation 2.148 can be written as

$$v_p \delta_{pr} = v'_q \alpha_{qr}. \quad (2.150)$$

Using the substitution properties of Kronecker delta and the fact that scalar quantities commute, Equation 2.150 can be written as

$$v_r = \alpha_{qr} v'_q. \quad (2.151)$$

In the expanded form,

$$v_r = \alpha_{1r} v'_1 + \alpha_{2r} v'_2 + \alpha_{3r} v'_3. \quad (2.152)$$

The reader may be familiar with the previous expression, which states that the component of a vector along the r direction is equal to the sum of the projections of its three orthogonal components along the r direction. The set of direction cosines $[\alpha_{qr}]$, a total of nine components, is as follows in the expanded form:

$$[\alpha_{qr}] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}. \quad (2.153)$$

The reader may note that the rows of the previous matrix correspond to the axes in the x', y', z' system, and the columns correspond to the axes in the x, y, z system. The entries in the matrix correspond to the cosines of the angles between two lines. From three-dimensional coordinate geometry, recall that

$$\begin{aligned} \alpha_{11}^2 + \alpha_{12}^2 + \alpha_{13}^2 &= 1; \alpha_{21}^2 + \alpha_{22}^2 + \alpha_{23}^2 = 1; \alpha_{31}^2 + \alpha_{32}^2 + \alpha_{33}^2 = 1; \\ \alpha_{11}\alpha_{21} + \alpha_{12}\alpha_{22} + \alpha_{13}\alpha_{23} &= 0; \alpha_{31}\alpha_{21} + \alpha_{32}\alpha_{22} + \alpha_{33}\alpha_{23} = 0; \\ \alpha_{31}\alpha_{11} + \alpha_{32}\alpha_{12} + \alpha_{33}\alpha_{13} &= 0; \alpha_{11}^2 + \alpha_{21}^2 + \alpha_{31}^2 = 1; \alpha_{12}^2 + \alpha_{22}^2 + \alpha_{32}^2 = 1; \\ \alpha_{13}^2 + \alpha_{23}^2 + \alpha_{33}^2 &= 1; \alpha_{11}\alpha_{12} + \alpha_{21}\alpha_{22} + \alpha_{31}\alpha_{32} = 0; \\ \alpha_{12}\alpha_{13} + \alpha_{22}\alpha_{23} + \alpha_{32}\alpha_{33} &= 0; \alpha_{11}\alpha_{13} + \alpha_{21}\alpha_{23} + \alpha_{31}\alpha_{33} = 0. \end{aligned} \quad (2.154)$$

Thus, the matrix given by Equation 2.153 is an orthogonal matrix, that is,

$$\left[\alpha_{qr} \right]^T \left[\alpha_{qr} \right] = \left[\alpha_{qr} \right] \left[\alpha_{qr} \right]^T = [I]. \quad (2.155)$$

Thus,

$$\left[\alpha_{qr} \right]^{-1} = \left[\alpha_{qr} \right]^T. \quad (2.156)$$

Now, in the matrix form, Equation 2.151 is written as

$$\{v_r\} = \left[\alpha_{qr} \right]^T \{v'_q\}. \quad (2.157)$$

Therefore, in view of Equation 2.155,

$$\{v'_q\} = \left[\alpha_{qr} \right] \{v_r\} \quad (2.158)$$

The previous equation is written in index notation as

$$v'_q = \alpha_{qr} v_r. \quad (2.159)$$

In the previous equation, r is the dummy index and therefore,

$$v'_q = \alpha_{q1} v_1 + \alpha_{q2} v_2 + \alpha_{q3} v_3. \quad (2.160)$$

Thus, the component along the direction q in the primed (x' , y' , z') system is equal to the sum of the projections of the components of the unprimed system along the q direction. The reader may already be familiar with this statement.

Equations 2.151 and 2.159 form the part of the definition of the Cartesian vector, which is a tensor of order 1. A Cartesian tensor of order 1 is a quantity that consists of n components in an n -dimensional space and that follows the transformation rules given in Equations 2.151 and 2.159 under the rotation of the axis system.

Example 2.16

The components of a vector in the x , y , z system are given by $\{1 \ 2 \ 3\}^T$. The x' , y' , z' are obtained by rotating the original system about the z -axis through an angle of 30° in a counterclockwise way. Find the components of the vector in the x' , y' , z' system.

SOLUTION

Table 2.1 contains the angles between the axis of one system and the axis of the other system. With this, the matrix of direction cosine $[\alpha_{ij}]$ is written as

TABLE 2.1

**Angles between Axes of One System
with the Axes of the Other System**

	x	y	z
x'	30°	60°	90°
y'	120°	30°	90°
z'	90°	90°	0°

$$[\alpha_{ij}] = \begin{bmatrix} \cos 30^\circ & \cos 60^\circ & \cos 90^\circ \\ \cos 120^\circ & \cos 30^\circ & \cos 90^\circ \\ \cos 90^\circ & \cos 90^\circ & \cos 0^\circ \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (2.161)$$

Here, deliberately, the index i, j has been written in place of q, r to establish that one can name the free indexes in any way as long as it is consistent throughout the expression. Now, Equation 2.158 is $\{v'_i\} = [\alpha_{ij}]\{v_j\}$. Therefore,

$$\begin{Bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{Bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix} = \begin{Bmatrix} \frac{\sqrt{3}}{2} + 1 \\ -\frac{1}{2} + \sqrt{3} \\ 3 \end{Bmatrix} \quad (2.162)$$

Thus, the components of the vector in a rotated system have been obtained.

2.6 TENSORS

A scalar can be considered as a tensor of order zero. It remains invariant under the rotation of a coordinate system. A vector is a first-order tensor. Under the rotation of a coordinate system, it follows the transformation rule given by Equations 2.151 and 2.159. A second-order tensor has nine components, and they follow a certain transformation law that will be described in Section 2.6.1. The definition of tensor can be generalized to include higher-order tensors. Unless otherwise specified, a tensor will mean a tensor of order two.

2.6.1 TRANSFORMATION RULES FOR TENSOR COMPONENTS UNDER THE ROTATION OF CARTESIAN COORDINATE SYSTEM

Two vectors can be multiplied in a manner to yield a tensor of order two. It is called a tensor product or outer product, denoted by $\mathbf{u} \otimes \mathbf{v}$. In index notation, it is denoted as $u_i v_j$. It is clear that in a three-dimensional space, it has nine distinct components. Now,

$$u'_i v'_j = (\alpha_{ip} u_p) (\alpha_{jq} v_q) = \alpha_{ip} \alpha_{jq} u_p v_q \quad (2.163)$$

and

$$u_i v_j = (\alpha_{pi} u'_p) (\alpha_{qj} v'_q) = \alpha_{pi} \alpha_{qj} u'_p v'_q. \quad (2.164)$$

Any (physical) quantity that has n^2 components in an n -dimensional space and follows the transformation rule given by Equations 2.163 and 2.164 is called a tensor of order two. Thus, if a_{ij} and a'_{ij} are the components of tensor in the unprimed and primed Cartesian coordinate systems, respectively, then

$$a'_{ij} = \alpha_{ip} \alpha_{jq} a_{pq}; a_{ij} = \alpha_{pi} \alpha_{qj} a'_{pq}. \quad (2.165)$$

The transformation rule given in Equation 2.165 can be generalized. For example, for a tensor of the third order, the rule is

$$a'_{ijk} = \alpha_{ip} \alpha_{jq} \alpha_{kr} a_{pqr}; a_{ijk} = \alpha_{pi} \alpha_{qj} \alpha_{rk} a'_{pqr}. \quad (2.166)$$

Similarly, the transformation rule for higher-order tensors can be written.

Now, one wishes to write the transformation rules in the form of matrix multiplications. Given matrices $[a_{ij}]$ and $[b_{ij}]$, the matrix product in index notation is defined as

$$c_{ij} = a_{ip} b_{pj}. \quad (2.167)$$

The transpose of a matrix is obtained by interchanging the rows and columns of the matrix. The i - j th element of a matrix is the same as the j - i th element of a transpose matrix. With this knowledge, it is easy to see that the transformation rule given by Equation 2.165 can be written as

$$\mathbf{A}' = \boldsymbol{\alpha} \mathbf{A} \boldsymbol{\alpha}^T; \mathbf{A} = \boldsymbol{\alpha}^T \mathbf{A}' \boldsymbol{\alpha}. \quad (2.168)$$

It has been decided to use boldface letters for tensors.

Example 2.17

Stress at point is a tensor of rank two. Therefore, it is possible to use the transformation rule given by Equation 2.168 for finding the stress components along any axis system. Consider the case of plane stress with stress components σ_x , σ_y and τ_{xy} . Find the stress components along the rotated Cartesian system, where the axis x' makes an angle θ with axis x .

SOLUTION

The matrix of direction cosines is given by

$$\alpha = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad (2.169)$$

and the symmetric stress tensor in the x - y system is given as

$$\sigma = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix}. \quad (2.170)$$

Using Equation 2.168, the stress components in a new system are given by

$$\begin{aligned} \sigma' &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^T \\ &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \end{aligned} \quad (2.171)$$

that provide

$$\begin{aligned} \sigma' &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx}\cos\theta + \tau_{xy}\sin\theta & -\sigma_{xx}\sin\theta + \tau_{xy}\cos\theta \\ \tau_{xy}\cos\theta + \sigma_{yy}\sin\theta & -\tau_{xy}\sin\theta + \sigma_{yy}\cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta & -\sigma_{xx}\sin\theta\cos\theta + \sigma_{yy}\cos\theta\sin\theta \\ + 2\tau_{xy}\sin\theta\cos\theta & + \tau_{xy}(\cos^2\theta - \sin^2\theta) \\ -\sigma_{xx}\sin\theta\cos\theta & \sigma_{xx}\sin^2\theta \\ + \sigma_{yy}\cos\theta\sin\theta & + \sigma_{yy}\cos^2\theta \\ + \tau_{xy}(\cos^2\theta - \sin^2\theta) & -2\tau_{xy}\sin\theta\cos\theta \end{bmatrix} \end{aligned} \quad (2.172)$$

Thus,

$$\begin{aligned} \sigma'_{xx} &= \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta = \sigma_{xx}\frac{1+\cos 2\theta}{2} \\ &+ \sigma_{yy}\frac{1-\cos 2\theta}{2} + \tau_{xy}\sin 2\theta = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\ \sigma'_{yy} &= \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta = \sigma_{xx}\frac{1-\cos 2\theta}{2} \\ &+ \sigma_{yy}\frac{1+\cos 2\theta}{2} - \tau_{xy}\sin 2\theta = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \end{aligned} \quad (2.173)$$

and

$$\begin{aligned}\tau'_{xy} &= -\sigma_{xx} \sin \theta \cos \theta + \sigma_{yy} \cos \theta \sin \theta + \tau_{xy} (\cos^2 \theta - \sin^2 \theta) \\ &= -\frac{\sigma_{xx} - \sigma_{yy}}{2} \sin 2\theta + \tau_{xy} \cos 2\theta\end{aligned}\quad (2.174)$$

2.6.2 CONTRACTION AND QUOTIENT LAWS

If a force is represented as F_i in index notation and the displacement is represented by d_i , then the dot product $F_i d_i$, which is a scalar, will represent the work done. The tensor product of the force and displacement is denoted by $F_i d_j$, which is a tensor of rank two. Once j is changed to i , the rank of tensor reduces to zero. This is called contraction operation. Thus, $F_i d_i$ is the contraction of $F_i d_j$. Similarly, consider the tensor product of a tensor (of rank two) σ_{ij} and vector n_k , which gives $\sigma_{ij} n_k$. It can be shown that $\sigma_{ij} n_k$ is a tensor of order three. If k is replaced by j , the product $\sigma_{ij} n_j$ is the contraction of $\sigma_{ij} n_k$ and is of rank one. Thus, $\sigma_{ij} n_j$ is a vector that can be denoted by t_i . The relation $t_i = \sigma_{ij} n_j$ can be written in the matrix form as

$$\mathbf{t} = \boldsymbol{\sigma} \mathbf{n} \text{ or } \{t\} = [\boldsymbol{\sigma}] \{n\}, \quad (2.175)$$

whichever notation you like. Note that Equation 2.175 is the commonly known product of a matrix with a vector (a column matrix). It is observed that the pre-multiplication of a vector by the tensor provides another vector. The components of the vector $\mathbf{t}$ are the linear functions of the components of the vector $\mathbf{n}$. Thus, tensor (of rank two) can be called a linear map that assigns to each vector another vector. This is the alternative definition of tensor.

Now consider the reverse problem. If it is known that F_i are the components of a vector and $F_i d_i$ is a scalar, then one can conclude that d_i are the components of a vector. This is one type of quotient law. Quotient laws basically infer the nature of a quantity based on the outcome of its contracted product with a known quantity. Another quotient law can be written as follows:

If σ_{ij} are the components of the tensor (of rank two) and $\sigma_{ij} n_j$ are the components of the vectors, then n_j are the components of the vectors.

In a similar way, a number of quotient rules can be written.

Example 2.18

The components of a symmetric tensor $\mathbf{A}$ (symmetry implies $a_{ij} = a_{ji}$) are given by

$$[a] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 10 \\ 3 & 10 & 4 \end{bmatrix}. \quad (2.176)$$

The components of a skew-symmetric $\mathbf{B}$ (skew-symmetry implies $b_{ij} = -b_{ji}$) are given by

$$[b] = \begin{bmatrix} 0 & 8 & 11 \\ -8 & 0 & 10 \\ -11 & -10 & 0 \end{bmatrix}. \quad (2.177)$$

Find out the scalar product of the tensor $a_{ij}b_{ij}$.

SOLUTION

The scalar product $a_{ij}b_{ij}$ means multiplying each entry of $\mathbf{A}$ with the corresponding entry $\mathbf{B}$ and adding all nine products. Thus,

$$\begin{aligned} a_{ii}b_{ij} &= 1 \times 0 + 2 \times 8 + 3 \times 11 + 2 \times (-8) + 5 \times 0 \\ &\quad + 10 \times 10 + 3 \times (-11) + 10 \times (-10) + 4 \times 0 = 0 \end{aligned} \quad (2.178)$$

It can easily be proved that the scalar product of a symmetric tensor with a skew-symmetric tensor will be zero.

Example 2.19

From the quotient law, show that the mass moment of inertia is a tensor of rank two.

SOLUTION

The angular momentum ($\mathbf{L}$) is defined as

$$\mathbf{L} = \mathbf{I}\omega, \quad (2.179)$$

where $\mathbf{L}$ and the angular velocity ω are both vectors. In index notation, Equation 2.179 is written as

$$L_i = I_{ij}\omega_j. \quad (2.180)$$

From the direct relation, it is easy to prove that if $\mathbf{I}$ is a tensor, the right-hand side of Equation 2.180 will represent the components of the vector. Thus, the inverse relation is that if $\mathbf{L}$ and ω are vectors, $\mathbf{I}$ is a tensor (of rank two).

2.6.3 SOME IMPORTANT DEFINITIONS AND PROPERTIES OF TENSOR

The components of a tensor (of rank two) can be represented in the form of a matrix. Thus, the many properties and definitions are common between the matrix and tensors. However, the components of the matrix need not follow the transformation rule, and the matrix need not represent any physical quantity. If a matrix represents a physical quantity and its components follow the transformation rule, then it

is a tensor. Now, some properties of the tensor are briefly discussed. In many cases, details are left for the reader to carry out as exercise problems. Whenever a tensor is referred to without specifying its rank, it will be understood as a tensor of rank two.

1. *Zero tensor.* If all the components of a tensor are zero, the tensor is called zero tensor, denoted by $\mathbf{0}$. The vector product of this vector with any vector $\mathbf{v}$ will give a zero vector. Thus,

$$\mathbf{0}\mathbf{v}=\mathbf{0}. \quad (2.181)$$

Note that 0 in the left-hand side of the previous equation should be understood as a tensor having nine components in a three-dimensional space, whereas 0 in the right-hand side is a vector having three components in a three-dimensional space.

2. *Identity tensor.* The tensor $\mathbf{I}$ whose components are δ_{ij} is referred to as an identity tensor. For any vector $\mathbf{v}$,

$$\mathbf{I}\mathbf{v}=\mathbf{v}. \quad (2.182)$$

3. *Product of two tensors.* The ordinary product $\mathbf{C}$ of two tensors (of rank two) $\mathbf{A}$ and $\mathbf{B}$ is a tensor of rank two and is written as

$$\mathbf{C}=\mathbf{A}\mathbf{B}. \quad (2.183)$$

In index notation,

$$C_{ik}=A_{ij}B_{jk}. \quad (2.184)$$

Thus, it is a contracted product. One can have a scalar product defined by

$$c=A_{ij}B_{ij}, \quad (2.185)$$

or the non-contracted tensor product defined by

$$C_{ijkl}=A_{ij}B_{kl}. \quad (2.186)$$

Note that $\mathbf{A}$ and $\mathbf{B}$ commute during a scalar product. Some books denote the scalar product of $\mathbf{A}$ and $\mathbf{B}$ as $\mathbf{A}:\mathbf{B}$, where the double dot indicates contraction of two indices.

4. *Transpose of a tensor.* The transpose of a tensor is obtained by interchanging the indices of its components. Thus,

$$A_{ij}^T=A_{ji} \quad (2.187)$$

It can be shown that for all vectors $\mathbf{u}$ and $\mathbf{v}$,

$$\mathbf{A}\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{A}^T \mathbf{v}. \quad (2.188)$$

Also, for tensors $\mathbf{A}$ and $\mathbf{B}$,

$$(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T; (\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T. \quad (2.189)$$

5. *Additive decomposition of a tensor into a symmetric and a skew part.*

A tensor $\mathbf{A}$ can be decomposed into a symmetric $\mathbf{E}$ and a skewsymmetric $\mathbf{W}$ part, such that

$$\mathbf{A} = \mathbf{E} + \mathbf{W}, \quad (2.190)$$

where

$$\mathbf{E} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T); \mathbf{W} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T). \quad (2.191)$$

6. *Inverse of a tensor.* Given a tensor $\mathbf{A}$, if there exists a tensor $\mathbf{B}$ such that

$$\mathbf{AB} = \mathbf{I}, \quad (2.192)$$

then $\mathbf{B}$ is called the inverse of $\mathbf{A}$. If $\mathbf{B}$ is equal to $\mathbf{A}^T$, the tensor $\mathbf{A}$ is called an orthogonal tensor. An orthogonal tensor whose determinant is 1 is called the proper orthogonal tensor. If the determinant of $\mathbf{A}$ is 0, then the matrix corresponding to $\mathbf{A}$ is called a singular matrix, and the tensor is non-invertible.

The reader is already familiar with the procedure to determine the inverse of a matrix. In index notation, the i - j th component of the tensor $\mathbf{B}$ is given by

$$B_{ij} = \frac{1}{\det(a_{ij})} \left[\frac{1}{2} \varepsilon_{jpq} \varepsilon_{irs} A_{pr} A_{qs} \right]. \quad (2.193)$$

7. *Invariants of a tensor.* In a three-dimensional space, a tensor $\mathbf{A}$ has three principal invariants (that remain unchanged during coordinate transformation) as described here:

- a. First invariant $\mathbf{I}_A$: In index notation, it is written as A_{ii} , which is called trace of $\mathbf{A}$ and is denoted as $\text{tr}(\mathbf{A})$.
- b. Second invariant $\mathbf{II}_A$: It is given by

$$\mathbf{II}_A = \frac{1}{2} \left[(\text{tr} \mathbf{A})^2 - \text{tr} \mathbf{A}^2 \right] = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} \quad (2.194)$$

- c. Third invariant $\mathbf{III}_A$: The determinant of $\mathbf{A}$ is the third invariant.

8. *Positive-definite tensor.* A tensor $\mathbf{A}$ is a positive-definite tensor if $\mathbf{v} \cdot \mathbf{A}\mathbf{v} > 0$ for all non-zero vectors $\mathbf{v}$. It is called positive semi-definite if $\mathbf{v} \cdot \mathbf{A}\mathbf{v} \geq 0$.
9. *Negative-definite tensor.* A tensor $\mathbf{A}$ is a negative-definite tensor if $\mathbf{v} \cdot \mathbf{A}\mathbf{v} < 0$ for all non-zero vectors $\mathbf{v}$. It is called negative semi-definite if $\mathbf{v} \cdot \mathbf{A}\mathbf{v} \leq 0$.

2.6.4 EIGENVALUES OF A TENSOR

It is known that a tensor $\mathbf{A}$ carries out a linear transformation of a vector $\mathbf{x}$ by the relation

$$\mathbf{A}\mathbf{x} = \mathbf{b}. \quad (2.195)$$

It is possible that for some non-zero vector $\mathbf{x}$, the vector $\mathbf{b}$ may be parallel to $\mathbf{x}$, that is, $\mathbf{b} = \lambda \mathbf{x}$. Thus, for some scalar λ and some vector $\mathbf{x}$, the following relation holds good:

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}. \quad (2.196)$$

The previous equation is called an eigenvalue problem, λ is called the eigenvalue, and $\mathbf{x}$ is the eigenvector. Equation 2.196 may also be written as

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0. \quad (2.197)$$

Equation 2.197 implies that

$$\det(\mathbf{A} - \lambda\mathbf{I}) = |\mathbf{A} - \lambda\mathbf{I}| = 0. \quad (2.198)$$

The roots of Equation 2.198 will provide the eigenvalues. For a particular eigenvalue, Equation 2.197 may be used for finding the eigenvector. Note that eigenvectors are not unique. A normalized eigenvalue is the eigenvalue whose magnitude is one.

It shall be proved that eigenvalues of a Hermitian matrix are real, and eigenvectors corresponding to distinct eigenvalues of a real symmetric matrix are orthogonal to one another. First, let us define the Hermitian matrix. Given a matrix $\mathbf{A}$, the complex conjugate matrix $\mathbf{A}^*$ is formed by taking the complex conjugate of each element. The adjoint of $\mathbf{A}$ is formed by transposing $\mathbf{A}^*$. The matrix is called Hermitian (or self-adjoint) if $\mathbf{A} = (\mathbf{A}^*)^T$.

Let λ_i and λ_j be two eigenvalues of a Hermitian matrix. Then

$$\mathbf{A}\mathbf{x}_i = \lambda_i\mathbf{x}_i \quad (2.199a)$$

$$\mathbf{A}\mathbf{x}_j = \lambda_j\mathbf{x}_j \quad (2.199b)$$

Here, we are not using summation convention.

Pre-multiplying both sides of Equation 2.199a by $(\mathbf{x}_j^*)^T$ and Equation 2.199b by $(\mathbf{x}_i^*)^T$, one gets

$$(\mathbf{x}_j^*)^T \mathbf{A}\mathbf{x}_i = \lambda_i (\mathbf{x}_j^*)^T \mathbf{x}_i \quad (2.200a)$$

$$(\mathbf{x}_i^*)^T \mathbf{A}\mathbf{x}_j = \lambda_j (\mathbf{x}_i^*)^T \mathbf{x}_j. \quad (2.200b)$$

Taking the adjoint of Equation 2.200b,

$$\left(\mathbf{x}_j^*\right)^T \left(\mathbf{A}^*\right)^T \mathbf{x}_i = \lambda_j^* \left(\mathbf{x}_j^*\right)^T \mathbf{x}_i. \quad (2.201)$$

As $\mathbf{A}$ is Hermitian, the previous equation can be written as

$$\left(\mathbf{x}_j^*\right)^T \mathbf{A} \mathbf{x}_i = \lambda_j^* \left(\mathbf{x}_j^*\right)^T \mathbf{x}_i. \quad (2.202)$$

Comparing it with Equation 2.200a, one gets

$$\left(\lambda_j^* - \lambda_i\right) \left(\mathbf{x}_j^*\right)^T \mathbf{x}_i = 0. \quad (2.203)$$

In the previous equation, replacing j by i

$$\left(\lambda_i^* - \lambda_i\right) \left(\mathbf{x}_i^*\right)^T \mathbf{x}_i = 0 \quad (2.204)$$

As $\left(\mathbf{x}_i^*\right)^T \mathbf{x}_i$ is a positive number, $\lambda_i^* = \lambda_i$. This implies that λ_i is real. This result holds good for a real symmetric matrix as it is a special case of the Hermitian matrix.

Now, for a real symmetric matrix, if λ_i and λ_j are two distinct eigenvalues, then Equations 2.199a and 2.199b hold good. Pre-multiplying both sides of Equation 2.199a by $\left(\mathbf{x}_j\right)^T$ and Equation 2.199b by $\left(\mathbf{x}_i\right)^T$, one gets

$$\left(\mathbf{x}_j\right)^T \mathbf{A} \mathbf{x}_i = \lambda_i \left(\mathbf{x}_j\right)^T \mathbf{x}_i \quad (2.205a)$$

$$\left(\mathbf{x}_i\right)^T \mathbf{A} \mathbf{x}_j = \lambda_j \left(\mathbf{x}_i\right)^T \mathbf{x}_j. \quad (2.205b)$$

Taking the transpose of Equation 2.205b and using the fact that $\mathbf{A}$ is symmetric, one gets

$$\left(\mathbf{x}_j\right)^T \mathbf{A} \mathbf{x}_i = \lambda_j \left(\mathbf{x}_j\right)^T \mathbf{x}_i. \quad (2.206)$$

Subtracting Equations 2.205a and 2.206, one gets

$$\left(\lambda_i - \lambda_j\right) \left(\mathbf{x}_j\right)^T \mathbf{x}_i = 0. \quad (2.207)$$

As both the eigenvalues are distinct, the previous equation implies that

$$\left(\mathbf{x}_j\right)^T \mathbf{x}_i = 0. \quad (2.208)$$

Thus, the eigenvectors corresponding to distinct eigenvalues of a real symmetric matrix are orthogonal.

Example 2.20

A sheet is subjected to plane stress conditions. The symmetric stress components at a point are as follows:

$$\sigma_x = 10 \text{ MPa}, \sigma_y = 2 \text{ MPa}, \tau_{xy} = 3 \text{ MPa}.$$

Find out the principal stress and principal directions.

SOLUTION

In the form of matrix, stress components are represented as

$$[\sigma_{ij}] = \begin{bmatrix} 10 & 3 \\ 3 & 2 \end{bmatrix}. \quad (2.209)$$

The eigenvalues of the previous matrix will be the principal stresses and eigenvectors the principal direction. Let λ be the eigenvalue. Then

$$\begin{vmatrix} 10-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} = 0 \text{ or } \lambda^2 - 12\lambda + 11 = 0, \quad (2.210)$$

which gives $\lambda = 11$ and 1 MPa. Thus, the maximum principal stress is 11 MPa and the minimum principal stress is 1 MPa.

Let us find the eigenvector corresponding to the eigenvalue of 11 MPa. By Equation 2.197

$$\begin{bmatrix} 10-11 & 3 \\ 3 & 2-11 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0, \quad (2.211)$$

which represent the following two equations:

$$-x_1 + 3x_2 = 0; 3x_1 - 9x_2 = 0. \quad (2.212)$$

The second equation is the scaled (by a factor of -3) version of the first equation. Thus, effectively, there is one equation that gives $x_1 = 3x_2$. Thus, multiple solutions are obtained. Taking $x_2 = \alpha$ (an arbitrary constant), x_1 is 3α . The normalized eigenvector components are

$$n_1 = \frac{3\alpha}{\sqrt{9\alpha^2 + \alpha^2}} = \frac{3}{\sqrt{10}}; n_2 = \frac{\alpha}{\sqrt{9\alpha^2 + \alpha^2}} = \frac{1}{\sqrt{10}}. \quad (2.213)$$

These are the direction cosines of the first eigenvector (with $n_3 = 0$) corresponding to the first principal direction.

For the principal stress of 1 MPa, Equation 2.197 gives

$$\begin{bmatrix} 10-1 & 3 \\ 3 & 2-1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0, \quad (2.214)$$

which represent the following two equations:

$$9x_1 + 3x_2 = 0; 3x_1 + x_2 = 0. \quad (2.215)$$

The first equation is the scaled (by a factor of 3) version of the second equation. Thus, effectively, there is one equation that gives $x_2 = -3x_1$. Thus, multiple solutions are obtained. Taking $x_1 = \alpha$ (an arbitrary constant), x_2 is -3α . The normalized eigenvector components are

$$n_1 = \frac{\alpha}{\sqrt{9\alpha^2 + \alpha^2}} = \frac{1}{\sqrt{10}}; n_2 = \frac{-3\alpha}{\sqrt{9\alpha^2 + \alpha^2}} = \frac{-3}{\sqrt{10}}. \quad (2.216)$$

It can be easily verified that both the eigenvectors are orthogonal to each other, that is, their dot product is zero.

Example 2.21

Find out the adjoint of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 2+i & 0 \\ 2-i & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad (2.217)$$

where $i \equiv \sqrt{-1}$. Also, find all the eigenvalues of $\mathbf{A}$.

SOLUTION

$$\mathbf{A}^* = \begin{bmatrix} 1 & 2-i & 0 \\ 2+i & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \quad (2.218)$$

The adjoint of $\mathbf{A}$ is given by

$$\text{adj}(\mathbf{A}) = (\mathbf{A}^*)^T = \begin{bmatrix} 1 & 2+i & 0 \\ 2-i & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \quad (2.219)$$

As the adjoint of the matrix is equal to itself, the matrix is Hermitian, and its eigenvalues will be real. If λ is the eigenvalue, then

$$\begin{vmatrix} 1-\lambda & 2+i & 0 \\ 2-i & 5-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0. \quad (2.220)$$

Expanding the determinant, one gets

$$(2-\lambda)(\lambda^2 - 6\lambda) = 0, \quad (2.221)$$

which gives $\lambda = 0, 2$ and 6 . Note that the sum of the eigenvalues is equal to the trace of the matrix $\mathbf{A}$.

2.6.5 POLAR DECOMPOSITION OF TENSORS

Every invertible tensor A can be decomposed into an orthogonal tensor Q and a positive-definite symmetric tensor U such that

$$A=QU. \quad (2.222)$$

Similarly, it can be decomposed into an orthogonal tensor Q and a positive-definite symmetric tensor V such that

$$A=VQ. \quad (2.223)$$

Starting from Equation 2.222,

$$A^T A = (QU)^T (QU) = U^T Q^T QU = U^T IU = U^T U = UU = U^2. \quad (2.224)$$

Similarly, starting from Equation 2.224, it can be shown that

$$AA^T = V^2. \quad (2.225)$$

From Equations 2.222 and 2.223,

$$VQ=QU. \quad (2.226)$$

Post-multiplying the previous equation by Q^T ,

$$V = QUQ^T. \quad (2.227)$$

For finding the U or V , one has to find the square root of a symmetric matrix. The square root of a symmetric matrix B of size n can be obtained as follows:

- Find out n orthonormal eigenvectors $b_1, b_2, \dots, b_n$ of B .
- Use the following transformation matrix:

$$\alpha = \begin{Bmatrix} b_1^T \\ b_2^T \\ \vdots \\ b_n^T \end{Bmatrix} \quad (2.228)$$

- Transform matrix B to B' using the following relation:

$$B' = \alpha B \alpha^T \quad (2.229)$$

- Observe that the matrix $\mathbf{B}'$ is diagonal with the i th elements on the leading diagonal as λ_i .
- The square root of $\mathbf{B}'$ denoted by $\sqrt{\mathbf{B}'}$ will be a diagonal matrix with the i th elements on the leading diagonal as $\sqrt{\lambda_i}$.
- The square root of $\mathbf{B}$ denoted by $\sqrt{\mathbf{B}}$ is obtained using the following transformation relation:

$$\sqrt{\mathbf{B}} = \boldsymbol{\alpha}^T \sqrt{\mathbf{B}'} \boldsymbol{\alpha}. \quad (2.230)$$

Once $\mathbf{U}$ or $\mathbf{V}$ of matrix $\mathbf{A}$ is obtained, $\mathbf{Q}$ can be obtained by post-multiplying or pre-multiplying $\mathbf{A}$.

Example 2.22

Carry out the polar decomposition of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & -4 \\ 2 & 0 & 0 \\ 0 & 4 & 6 \end{bmatrix}. \quad (2.231)$$

SOLUTION

$$\mathbf{U}^2 = \mathbf{A}^T \mathbf{A} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ -4 & 0 & 6 \end{bmatrix} \begin{bmatrix} 0 & 0 & -4 \\ 2 & 0 & 0 \\ 0 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 16 & 24 \\ 0 & 24 & 52 \end{bmatrix}. \quad (2.232)$$

The characteristic polynomial of $\mathbf{U}^2$ is

$$\begin{vmatrix} 4-\lambda & 0 & 0 \\ 0 & 16-\lambda & 24 \\ 0 & 24 & 52-\lambda \end{vmatrix} = 0, \quad (2.233)$$

or

$$(4-\lambda)(\lambda^2 - 68\lambda + 256) = 0, \quad (2.234)$$

which gives eigenvalues as

$$\lambda = 4, 4, 64. \quad (2.235)$$

The eigenvectors are denoted by $\{x_1 \ x_2 \ x_3\}^T$. The eigenvectors corresponding to $\lambda = 4$ are obtained by solving the following system of equations:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 12 & 24 \\ 0 & 24 & 48 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \quad (2.236)$$

The previous system effectively provides one equation:

$$x_2 + 2x_3 = 0 \quad (2.237)$$

Taking $x_3 = 1$, $x_2 = -2$, two eigenvectors are generated as

$$V_1 = \begin{Bmatrix} 1 \\ -2 \\ 1 \end{Bmatrix}, V_2 = \begin{Bmatrix} \alpha \\ -2 \\ 1 \end{Bmatrix}. \quad (2.238)$$

If both eigenvectors are orthogonal to each other, then

$$\alpha + 4 + 1 = 0 \text{ or } \alpha = -5. \quad (2.239)$$

Thus, the two orthogonal eigenvectors are

$$V_1 = \begin{Bmatrix} 1 \\ -2 \\ 1 \end{Bmatrix}, V_2 = \begin{Bmatrix} -5 \\ -2 \\ 1 \end{Bmatrix}. \quad (2.240)$$

The eigenvectors corresponding to $\lambda = 64$ are obtained by solving the following system of equations:

$$\begin{bmatrix} -60 & 0 & 0 \\ 0 & -48 & 24 \\ 0 & 24 & -12 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \quad (2.241)$$

which provides the following two linearly independent equations:

$$x_1 = 0; 2x_2 - x_3 = 0. \quad (2.242)$$

Taking $x_3 = 1$, one eigenvalue is obtained as

$$V_3 = \begin{Bmatrix} 0 \\ 1 \\ 2 \end{Bmatrix}. \quad (2.243)$$

It can be verified that V_3 is orthogonal to V_1 and V_2 . Dividing the eigenvectors by their L_2 norms, three orthonormal eigenvectors are obtained as

$$\mathbf{e}'_1 = \begin{Bmatrix} \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{Bmatrix}, \mathbf{e}'_2 = \begin{Bmatrix} \frac{-5}{\sqrt{30}} \\ \frac{-2}{\sqrt{30}} \\ \frac{1}{\sqrt{30}} \end{Bmatrix}, \mathbf{e}'_3 = \begin{Bmatrix} 0 \\ \frac{+1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{Bmatrix} \quad (2.244)$$

Hence, the matrix of direction cosines is given by

$$\alpha = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{-5}{\sqrt{30}} & \frac{-2}{\sqrt{30}} & \frac{1}{\sqrt{30}} \\ 0 & \frac{+1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}. \quad (2.245)$$

Transforming the matrix U^2 as per the previous matrix, one gets

$$U'^2 = \alpha U^2 \alpha^T = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 64 \end{bmatrix}. \quad (2.246)$$

Hence,

$$U' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}, \quad (2.247)$$

and

$$U = \alpha^T U' \alpha = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3.2 & 2.4 \\ 0 & 2.4 & 6.8 \end{bmatrix}. \quad (2.248)$$

$$Q = AU^{-1} = \begin{bmatrix} 0 & 0.6 & -0.8 \\ 1 & 0 & 0 \\ 0 & 0.8 & 0.6 \end{bmatrix}. \quad (2.249)$$

It can be verified that Q is orthogonal.

$$V = QUQ^T = \begin{bmatrix} 3.2 & 0 & -2.4 \\ 0 & 2 & 0 \\ -2.4 & 0 & 6.8 \end{bmatrix}. \quad (2.250)$$

$$\begin{aligned} A &= \begin{bmatrix} 0 & 0 & -4 \\ 2 & 0 & 0 \\ 0 & 4 & 6 \end{bmatrix} = QU = \begin{bmatrix} 0 & 0.6 & -0.8 \\ 1 & 0 & 0 \\ 0 & 0.8 & 0.6 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3.2 & 2.4 \\ 0 & 2.4 & 6.8 \end{bmatrix} \\ &= VQ = \begin{bmatrix} 3.2 & 0 & -2.4 \\ 0 & 2 & 0 \\ -2.4 & 0 & 6.8 \end{bmatrix} \begin{bmatrix} 0 & 0.6 & -0.8 \\ 1 & 0 & 0 \\ 0 & 0.8 & 0.6 \end{bmatrix}. \end{aligned} \quad (2.251)$$

2.6.6 TENSOR CALCULUS

Let g be a function of t whose values are scalars, vectors or tensors and whose domain is an open interval D of real numbers. The derivative $\dot{g}(t)$ of g at t , if it exists, is defined by

$$\dot{g}(t) = \frac{d}{dt} g(t) = \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} [g(t + \alpha) - g(t)]. \quad (2.252)$$

This implies that if the components of a tensor are represented in a matrix form, the derivative of the tensor can also be represented in a matrix form, in which each entry will be the derivative of individual entries. Here, it is assumed that axes do not change with time.

Example 2.23

Components of a tensor are given as

$$[a_{ij}] = \begin{bmatrix} 2t & t^2 & \sin t \\ t & 5 & e^t \\ t^3 & 7 & \ln(t) \end{bmatrix}. \quad (2.253)$$

Find the components of its derivatives with respect to time.

SOLUTION

By differentiating individual components, one gets

$$[\dot{a}_{ij}] = \begin{bmatrix} 2 & 2t & \cos t \\ 1 & 0 & e^t \\ 3t^2 & 0 & 1/t \end{bmatrix}. \quad (2.254)$$

It can very easily be proved the following relation exists for a tensor whose i - j th component is given by a_{ij} :

$$\frac{\partial a_{ij}}{\partial a_{pq}} = \delta_{ip} \delta_{jq}. \quad (2.255)$$

It states that if the component is partially differentiated with respect to itself, a value of one is obtained. If it is partially differentiated with respect to the other component, a value of zero is obtained.

Comma notation shall be adopted, which means

$$(\dots)_{,i} = \frac{\partial(\dots)}{\partial x_i}. \quad (2.256)$$

Now, note that

$$x_{i,j} = \frac{\partial x_i}{\partial x_j} = \delta_{ij}. \quad (2.257)$$

In the sequel, some special derivatives shall be described.

1. *Gradient.* The concept of gradient can be extended to vectors and tensors as well. If v_i are the components of the vector field, then $v_{i,j}$ represents the component of the gradient of the vector field. It can be shown that it is a tensor of rank two. Similarly, if a_{ij} are the components of a tensor field, then $a_{ij,k}$ represents the component of the gradient of the tensor and is a tensor of rank three.
2. *Divergence.* Let v_i be the component of a vector field. Then the gradient of the vector field is represented in index notation as $v_{i,j}$. Contracting it once, one gets $v_{i,i}$, which is called the divergence of the vector field. As the contraction operation reduces the rank of the tensor field by 2, $v_{i,i}$ is of rank zero and thus a scalar field. Similarly, if a_{ij} are the components of a tensor field, then $a_{ij,k}$ represents the component of the gradient of the tensor field. Applying the contraction operation, one gets $a_{ij,j}$, which represents the divergence of the tensor field. The divergence is denoted by 'div'. If a is a tensor field of rank two, $\text{div } a$ or $\nabla \cdot a$ is a vector field.
3. *Curl.* Let there be a tensor A with components A_{ij} . The curl of the tensor field A is defined as

$$\nabla \times A = \varepsilon_{ijk} A_{mj,i} e_k \otimes e_m, \quad (2.258)$$

or

$$(\nabla \times A)_{km} = \varepsilon_{ijk} A_{mj,i}. \quad (2.259)$$

4. *Laplacian.* The Laplacian of a scalar field f is $f_{,ii}$, of a vector field denoted by v_i is $v_{i,jj}$ and of a tensor field denoted by a_{ij} is $a_{ij,kk}$. The Laplacian operator is denoted by ∇^2 . Note that for the x, y, z coordinate system:

$$\nabla^2 f \equiv \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \equiv \text{div}(\text{grad } f). \quad (2.260)$$

2.6.7 DIVERGENCE THEOREM

Let V be the volume of a three-dimensional region bounded by a closed regular surface S . Then for a tensor field A defined in V and on S ,

$$\int_V \text{div } A dV = \int_S A n dS \quad (2.261)$$

where $\mathbf{n}$ is the unit outward normal to S . In index notation,

$$\int_V a_{ij,j} dV = \int_S a_{ij} n_j dS. \quad (2.262)$$

2.6.8 STOKES' THEOREM

Let C be a simple closed curve in a three-dimensional space and S be an open regular surface bounded by C . Then, for a vector field defined on S as well as C ,

$$\oint_C \mathbf{A} \cdot d\mathbf{s} = \int_S (\text{curl } \mathbf{A})^T \cdot \mathbf{n} dS, \quad (2.263)$$

where $\mathbf{t}$ is the unit vector tangent to C , which is assumed to be positively oriented relative to the unit normal $\mathbf{n}$ to S .

2.6.9 NORM OF A TENSOR

A tensor can be expressed in the form of a matrix. The norm of the matrix can be defined. However, the components of a matrix change with the reference frame. Thus, the norm of a tensor may change with the reference frame. Certain types of norms may remain invariant also with respect to rotation of the reference frame. In this section, the matrix norm shall be described. A matrix norm is a function that assigns to each $\mathbf{A} \in R^{n \times n}$ a real number $\|\mathbf{A}\|$ such that it has the following properties:

1. Positive definite property: $\|\mathbf{A}\| > 0$ if $\mathbf{A} \neq \mathbf{0}$ and $\|\mathbf{0}\| = 0$.
2. Absolute homogeneity: $\|\alpha \mathbf{A}\| = |\alpha| \|\mathbf{A}\|$, where α is a real number.
3. Triangle inequality: $\|\mathbf{A} + \mathbf{B}\| \leq \|\mathbf{A}\| + \|\mathbf{B}\|$.
4. Sub-multiplicativity: $\|\mathbf{A}\mathbf{B}\| \leq \|\mathbf{A}\| \|\mathbf{B}\|$.

It can be shown that Frobenius norm is defined by

$$\|\mathbf{A}\|_F = \left(\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2 \right)^{1/2} \quad (2.264)$$

is a matrix norm.

Suppose $\|\mathbf{A}\|_{\max} = \max_{1 \leq i, j \leq n} |a_{ij}|$ is defined. Will it be a matrix norm? Let us see if it satisfies property 4 of the matrix norm. Let an element of matrix $\mathbf{A}$ be denoted by a_{ij} , and that of $\mathbf{B}$ and $\mathbf{C}$ by b_{ij} and c_{ij} , respectively, where $\mathbf{C} = \mathbf{A}\mathbf{B}$.

$$c_{ij} = a_{ik} b_{kj}$$

Assume that all elements of $\mathbf{A}$ and $\mathbf{B}$ are positive, and a_{11} is the largest element of $\mathbf{A}$, and b_{11} is the largest element of $\mathbf{B}$. Then clearly,

$$c_{11} \geq a_{11} b_{11}.$$

Thus, property 4 is violated. Hence, it is not a matrix norm.

Induced (or operator) norm: Given a vector norm $\|\cdot\|_v$, the matrix norm induced by $\|\cdot\|_v$ is defined by

$$\|A\|_M = \max_{x \neq 0} \frac{\|Ax\|_v}{\|x\|_v} \quad (2.265)$$

The subscripts M and v shall be dropped henceforth.

Theorem 2.1

A vector norm and its induced matrix norm satisfy the inequality

$$\|Ax\| \leq \|A\| \|x\| \quad (2.266)$$

for all $A \in R^{n \times n}$ and $x \in R^n$.

Proof

If $x = 0$, the equality holds trivially. Otherwise,

$$\frac{\|Ax\|}{\|x\|} \leq \max_{x_1 \neq 0} \frac{Ax_1}{x_1} = \|A\|.$$

Hence, $\|Ax\| \leq \|A\| \|x\|$.

Theorem 2.2

The induced norm is a matrix norm.

Proof

If $A = 0$, for all vectors $\|Ax\| = 0$. Hence, $A = 0$. It is obvious from Equation 2.265 that, for other matrices, the operator norm will be positive. (One can always find a vector for which $Ax \neq 0$.) Hence, property 1 is satisfied.

Now for any real α , $\|\alpha(Ax)\| = |\alpha| \|Ax\|$, because of the property of the vector norm. Hence,

$$\|\alpha A\| = \max_{x \neq 0} \frac{\alpha(Ax)}{x} = \max_{x \neq 0} \frac{|\alpha Ax|}{x} = |\alpha| \max_{x \neq 0} \frac{Ax}{x} = |\alpha| \|A\|.$$

Thus, property 2 is satisfied.

Now,

$$\|A + B\| = \max_{x \neq 0} \frac{\|(A + B)x\|}{\|x\|} = \max_{x \neq 0} \frac{\|(Ax + Bx)\|}{\|x\|}.$$

Using the property of the vector norm,

$$\|A + B\| \leq \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} + \max_{x \neq 0} \frac{\|Bx\|}{\|x\|}.$$

Thus, $\|A+B\| \leq \|A\| + \|B\|$, and property 3 is satisfied.

Finally, in Equation 2.266, replace x by Bx and obtain

$$\|ABx\| \leq \|A\| \|Bx\| \leq \|A\| \|B\| \|x\|.$$

Now,

$$\|AB\| = \max_{x \neq 0} \frac{\|ABx\|}{\|x\|} \leq \|A\| \|B\|.$$

The norm induced by the p -norm is called the matrix p -norm. The matrix L_2 -norm is also called the spectral norm. Let us find an expression for the L_2 -norm.

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} \quad (2.267)$$

Now,

$$\frac{\|Ax\|_2^2}{\|x\|_2^2} = \frac{(Ax)^T (Ax)}{x^T x} = \frac{x^T A^T Ax}{x^T x}.$$

Let $B = A^T A$. Note that B is a symmetric matrix, as $B = B^T$. The following theorem shall be used:

For a symmetric matrix A , there exists (at least) n eigenvectors that are mutually orthogonal.

No summation convention will be used in the subsequent discussion.

Let the unit eigenvectors be denoted by b_i with λ_i as the corresponding eigenvalues. Thus,

$$Bb_i = \lambda_i b_i. \quad (2.268)$$

Pre-multiplying both sides by b_i^T , one gets

$$b_i^T Bb_i = \lambda_i b_i^T b_i \quad \text{or} \quad b_i^T A^T A b_i = \lambda_i b_i^T b_i \quad \text{or} \quad (Ab_i)^T (Ab_i) = \lambda_i b_i^T b_i$$

From the above, one gets

$$\lambda_i = \frac{(Ab_i)^T (Ab_i)}{b_i^T b_i}. \quad (2.269)$$

In the previous equation, both the numerator and the denominator are the dot products of a vector with itself and are positive. Hence, λ_i is positive, and matrix B is a positive-definite matrix.

The eigenvectors can be considered as the basis vectors. Thus,

$$\mathbf{x} = \sum_{i=1}^n y_i \mathbf{b}_i \quad (2.270)$$

Thus,

$$\mathbf{x}^T \mathbf{A}^T \mathbf{A} \mathbf{x} = \left(\sum_{i=1}^n y_i \mathbf{b}_i^T \right) \mathbf{B} \left(\sum_{i=1}^n y_i \mathbf{b}_i \right). \quad (2.271)$$

Considering the fact that unit eigenvectors are orthogonal and using Equation 2.268, the previous equation can be written as

$$\mathbf{x}^T \mathbf{A}^T \mathbf{A} \mathbf{x} = \sum_{i=1}^n \lambda_i y_i^2. \quad (2.272)$$

Similarly,

$$\mathbf{x}^T \mathbf{x} = \left(\sum_{i=1}^n y_i \mathbf{b}_i^T \right) \left(\sum_{i=1}^n y_i \mathbf{b}_i \right) = \sum_{i=1}^n y_i^2. \quad (2.273)$$

Hence,

$$\frac{\|\mathbf{A}\mathbf{x}\|_2^2}{\|\mathbf{x}\|_2^2} = \frac{\mathbf{x}^T \mathbf{A}^T \mathbf{A} \mathbf{x}}{\mathbf{x}^T \mathbf{x}} = \frac{\sum_{i=1}^n \lambda_i y_i^2}{\sum_{i=1}^n y_i^2} \quad (2.274)$$

Note that in the previous equations, if y_i is replaced by αy_i , the value remains unchanged. Hence, the case in which $\mathbf{x}$ is a unit vector can be considered, that is, $\sum_{i=1}^n y_i^2 = 1$. Thus, the square of L_2 norm will be the maximum value of $\sum_{i=1}^n \lambda_i y_i^2$ for $\sum_{i=1}^n y_i^2 = 1$. Without loss of generality, let the eigenvalues be arranged in the order of magnitude, such that λ_1 is the highest eigenvalue. Thus,

$$\sum_{i=1}^n \lambda_i y_i^2 = \lambda_1 - (\lambda_1 - \lambda_2) y_2^2 - (\lambda_1 - \lambda_3) y_3^2 - \dots - (\lambda_1 - \lambda_n) y_n^2. \quad (2.275)$$

This will be maximum if $y_2 = y_3 = \dots = y_n = 0$. Thus, $y_1 = 1$, and the maximum value is λ_1 . Thus, the L_2 norm of the matrix is equal to the square root of the maximum eigenvalue of $\mathbf{A}^T \mathbf{A}$.

Column-sum norm. The column-sum norm is given as

$$\|\mathbf{A}\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}|. \quad (2.276)$$

Proof

It is known that

$$\|A\|_1 = \max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} \quad (2.277)$$

Now,

$$\begin{aligned} \|Ax\|_1 &= |a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n| + |a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n| + \dots \\ &\quad + |a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n| \\ &\leq (|a_{11}| + |a_{21}| + \dots + |a_{n1}|)|x_1| + (|a_{12}| + |a_{22}| + \dots + |a_{n2}|)|x_2| + \dots \\ &\quad + (|a_{1n}| + |a_{2n}| + \dots + |a_{nn}|)|x_n| \end{aligned}$$

Let the m th column sum be the maximum. Then,

$$\begin{aligned} \|Ax\|_1 &= |a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n| + |a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n| \\ &\quad + |a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n| \\ &\leq (|a_{1m}| + |a_{2m}| + \dots + |a_{nm}|)(|x_1| + |x_2| + \dots + |x_n|) \end{aligned}$$

Thus,

$$\frac{\|Ax\|_1}{\|x\|_1} \leq (|a_{1m}| + |a_{2m}| + \dots + |a_{nm}|)$$

It is, however, noted that if one takes x as a vector whose m th element is one and other elements are zero,

$$\frac{\|Ax\|_1}{\|x\|_1} = (|a_{1m}| + |a_{2m}| + \dots + |a_{nm}|).$$

Thus,

$$\|A\|_1 = (|a_{1m}| + |a_{2m}| + \dots + |a_{nm}|)$$

and hence the proof.

Row-sum norm. The row-sum norm is given as

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|. \quad (2.278)$$

Proof

It is known that

$$\|A\|_{\infty} = \max_{x \neq 0} \frac{\|Ax\|_{\infty}}{\|x\|_{\infty}}$$

Now,

$$\begin{aligned} \|A\|_{\infty} &= \max \left(\begin{array}{c} |a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n|, |a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n|, \\ \dots |a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n| \end{array} \right) \\ &\leq |a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n| \\ &\leq (|a_{m1}| + |a_{m2}| + \dots + |a_{mn}|) x_{\max} \end{aligned}$$

Thus,

$$\frac{\|Ax\|_{\infty}}{\|x\|_{\infty}} \leq (|a_{m1}| + |a_{m2}| + \dots + |a_{mn}|) \quad (2.279)$$

However, if x is a vector having the elements equal to ± 1 (+ or – being decided by the sign of a_{mi}), the equality holds in Equation 2.279. Hence,

$$\|A\|_{\infty} = (|a_{m1}| + |a_{m2}| + \dots + |a_{mn}|)$$

and hence the proof.

2.7 TENSORS AND VECTORS IN CURVILINEAR COORDINATES

The Cartesian coordinate system is widely used. However, many times, it is convenient to work with a curvilinear coordinate system. In a general curvilinear coordinate system, at a point, coordinate axes need not be perpendicular to one another. In orthogonal curvilinear coordinates, coordinate axes are orthogonal to one another, although in space, the direction of the axes may keep changing. Two prominent orthogonal curvilinear coordinate systems are cylindrical and spherical coordinate systems. In this section, we shall concentrate on these two coordinate systems.

It is known that any point x, y, z is the intersection of three planes. It can also be the intersection of three surfaces, say, q_1, q_2 and q_3 . The direct relation between two coordinate systems is

$$x = x(q_1, q_2, q_3); y = y(q_1, q_2, q_3); z = z(q_1, q_2, q_3). \quad (2.280)$$

The inverse relations are given by

$$q_1 = q_1(x, y, z); q_2 = q_2(x, y, z); q_3 = q_3(x, y, z) \quad (2.281)$$

Considering unit vector $\hat{q}_i$ normal to surface $q_i = \text{constant}$ and in the direction of increasing q_i , a vector $\mathbf{v}$ can be expressed as follows:

$$\mathbf{v} = v_1 \hat{q}_1 + v_2 \hat{q}_2 + v_3 \hat{q}_3. \quad (2.282)$$

From the first principle, the following can be written:

$$dx = \frac{\partial x}{\partial q_1} dq_1 + \frac{\partial x}{\partial q_2} dq_2 + \frac{\partial x}{\partial q_3} dq_3 \quad (2.283)$$

and

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q_i} dq_i, \quad (2.284)$$

where the summation convention has been used.

Let the distance between two neighboring points be ds . Then

$$(ds)^2 = d\mathbf{r} \cdot d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q_i} \cdot \frac{\partial \mathbf{r}}{\partial q_j} dq_i dq_j = g_{ij} dq_i dq_j, \quad (2.285)$$

where g_{ij} are called the metric and form a second-rank symmetric tensor. From Equation 2.285,

$$g_{ij}(q_1, q_2, q_3) = \frac{\partial \mathbf{r}}{\partial q_i} \cdot \frac{\partial \mathbf{r}}{\partial q_j} = \frac{\partial x}{\partial q_i} \frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i} \frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i} \frac{\partial z}{\partial q_j}. \quad (2.286)$$

The vector $\partial \mathbf{r} / \partial q_i$ is the tangent to the curve $\mathbf{r}$ for $q_j = \text{constant}$, $j \neq i$. For an orthogonal coordinate system,

$$g_{ij} = 0, \quad i \neq j. \quad (2.287)$$

For orthogonal coordinates, let us define

$$h_1^2 = g_{11}, \quad h_2^2 = g_{22}, \quad h_3^2 = g_{33}. \quad (2.288)$$

Then

$$(ds)^2 = (h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2. \quad (2.289)$$

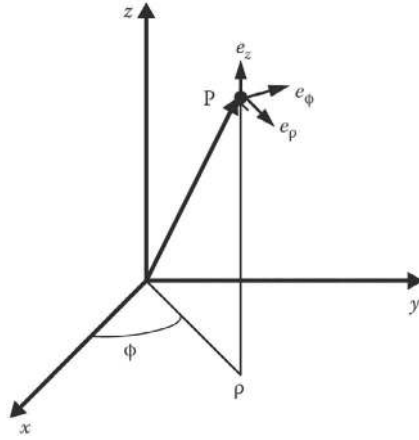
In the previous expression, the multipliers h_1 , h_2 and h_3 are called scale factors.

For Cartesian coordinates, $h_1 = h_2 = h_3 = 1$.

2.7.1 SCALE FACTORS FOR CYLINDRICAL AND SPHERICAL COORDINATES

For cylindrical polar coordinates (ρ, ϕ, z) (Figure 2.6),

$$x = \rho \cos \phi, \quad y = \rho \sin \phi, \quad z = z. \quad (2.290)$$

**FIGURE 2.6** Cylindrical polar coordinate system.

From Equation 2.286,

$$g_{11} = \left(\frac{\partial x}{\partial \rho} \right)^2 + \left(\frac{\partial y}{\partial \rho} \right)^2 + \left(\frac{\partial z}{\partial \rho} \right)^2 = (\cos \phi)^2 + (\sin \phi)^2 = 1. \quad (2.291)$$

Hence,

$$h_1 = \sqrt{g_{11}} = 1. \quad (2.292)$$

Similarly,

$$g_{22} = \left(\frac{\partial x}{\partial \phi} \right)^2 + \left(\frac{\partial y}{\partial \phi} \right)^2 + \left(\frac{\partial z}{\partial \phi} \right)^2 = (-\rho \sin \phi)^2 + (\rho \cos \phi)^2 = \rho^2 \quad (2.293)$$

providing

$$h_2 = \sqrt{g_{22}} = \rho \quad (2.294)$$

and

$$g_{33} = \left(\frac{\partial x}{\partial z} \right)^2 + \left(\frac{\partial y}{\partial z} \right)^2 + \left(\frac{\partial z}{\partial z} \right)^2 = 1, \quad (2.295)$$

providing

$$h_3 = \sqrt{g_{33}} = 1. \quad (2.296)$$

For spherical polar coordinates (r, θ, ϕ) (Figure 2.7),

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta. \quad (2.297)$$

Therefore,

$$\begin{aligned} g_{11} &= \left(\frac{\partial x}{\partial r} \right)^2 + \left(\frac{\partial y}{\partial r} \right)^2 + \left(\frac{\partial z}{\partial r} \right)^2 \\ &= (\sin \theta \cos \phi)^2 + (\sin \theta \sin \phi)^2 + (\cos \theta)^2 = 1, \end{aligned} \quad (2.298a)$$

$$\begin{aligned} g_{22} &= \left(\frac{\partial x}{\partial \theta} \right)^2 + \left(\frac{\partial y}{\partial \theta} \right)^2 + \left(\frac{\partial z}{\partial \theta} \right)^2 \\ &= (r \cos \theta \cos \phi)^2 + (r \cos \theta \sin \phi)^2 + (-r \sin \theta)^2 = r^2, \end{aligned} \quad (2.298b)$$

$$\begin{aligned} g_{33} &= \left(\frac{\partial x}{\partial \phi} \right)^2 + \left(\frac{\partial y}{\partial \phi} \right)^2 + \left(\frac{\partial z}{\partial \phi} \right)^2 \\ &= (-r \sin \theta \sin \phi)^2 + (r \sin \theta \cos \phi)^2 = r^2 \sin^2 \theta. \end{aligned} \quad (2.298c)$$

Hence, for spherical coordinates

$$h_1 = \sqrt{g_{11}} = 1, \quad h_2 = \sqrt{g_{22}} = r, \quad h_3 = \sqrt{g_{33}} = r \sin \theta. \quad (2.299)$$

2.7.2 GRADIENT OF A VECTOR

From calculus, it is known that

$$df = \frac{\partial f}{\partial q_1} dq_1 + \frac{\partial f}{\partial q_2} dq_2 + \frac{\partial f}{\partial q_3} dq_3. \quad (2.300)$$

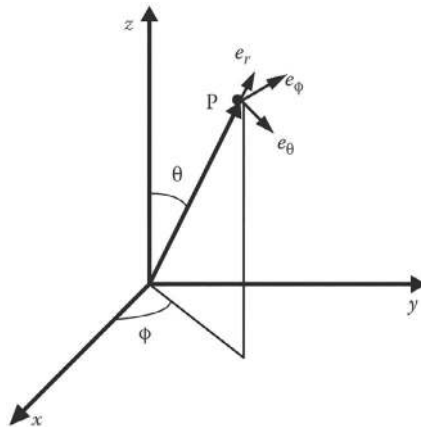


FIGURE 2.7 Spherical polar coordinate system.

Also, from the property of the gradient

$$df = \nabla f \cdot d\mathbf{r} \quad (2.301)$$

For curvilinear coordinates

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q_1} dq_1 + \frac{\partial \mathbf{r}}{\partial q_2} dq_2 + \frac{\partial \mathbf{r}}{\partial q_3} dq_3. \quad (2.302)$$

Now, from Equations 2.286 and 2.288

$$h_i = \left| \frac{\partial \mathbf{r}}{\partial q_i} \right|. \quad (2.303)$$

The unit vector along direction q_i is

$$\hat{\mathbf{q}}_i = \frac{\partial \mathbf{r} / \partial q_i}{|\partial \mathbf{r} / \partial q_i|} = \frac{\partial \mathbf{r} / \partial q_i}{h_i}. \quad (2.304)$$

Hence, Equation 2.302 is written as

$$d\mathbf{r} = h_1 \hat{\mathbf{q}}_1 dq_1 + h_2 \hat{\mathbf{q}}_2 dq_2 + h_3 \hat{\mathbf{q}}_3 dq_3. \quad (2.305)$$

Using Equations 2.300, 2.301 and 2.305

$$\frac{\partial f}{\partial q_1} dq_1 + \frac{\partial f}{\partial q_2} dq_2 + \frac{\partial f}{\partial q_3} dq_3 = \nabla f \cdot (h_1 \hat{\mathbf{q}}_1 dq_1 + h_2 \hat{\mathbf{q}}_2 dq_2 + h_3 \hat{\mathbf{q}}_3 dq_3). \quad (2.306)$$

By comparison

$$\nabla f = \frac{1}{h_1} \frac{\partial f}{\partial q_1} \hat{\mathbf{q}}_1 + \frac{1}{h_2} \frac{\partial f}{\partial q_2} \hat{\mathbf{q}}_2 + \frac{1}{h_3} \frac{\partial f}{\partial q_3} \hat{\mathbf{q}}_3, \quad (2.307)$$

which is the expression for the gradient in curvilinear coordinates. The gradient operator is given by

$$\nabla \equiv \frac{\hat{\mathbf{q}}_1}{h_1} \frac{\partial}{\partial q_1} + \frac{\hat{\mathbf{q}}_2}{h_2} \frac{\partial}{\partial q_2} + \frac{\hat{\mathbf{q}}_3}{h_3} \frac{\partial}{\partial q_3}. \quad (2.308)$$

2.7.3 DIVERGENCE OF A VECTOR

Before one derives an expression for divergence, three identities are considered that are needed in understanding the derivation for the divergence of a vector in curvilinear coordinates:

$$1. \quad \nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla \times \mathbf{a} - \mathbf{a} \cdot \nabla \times \mathbf{b} \quad (2.309)$$

Proof

In index notation, the i th component of $(\mathbf{a} \times \mathbf{b})$ is given by

$$(\mathbf{a} \times \mathbf{b})_i = \varepsilon_{ijk} a_j b_k.$$

Hence,

$$\begin{aligned} \text{LHS} &= \varepsilon_{ijk} (a_j b_k)_{,i} = \varepsilon_{ijk} a_j b_{k,i} + \varepsilon_{ijk} a_{j,i} b_k = b_k \varepsilon_{kij} a_{j,i} - a_j \varepsilon_{jik} b_{k,i} \\ &= b \cdot \nabla \times a - a \cdot \nabla \times b = \text{RHS} \end{aligned} \quad (2.310)$$

$$2. \quad \nabla q_1 = \frac{\hat{\mathbf{q}}_1}{h_1}, \quad \nabla q_2 = \frac{\hat{\mathbf{q}}_2}{h_2}, \quad \nabla q_3 = \frac{\hat{\mathbf{q}}_3}{h_3} \quad (2.311)$$

Proof

Using Equation 2.307 and putting $f = q_1$

$$\nabla q_1 = \frac{1}{h_1} \frac{\partial q_1}{\partial q_1} \hat{\mathbf{q}}_1 + \frac{1}{h_2} \frac{\partial q_1}{\partial q_2} \hat{\mathbf{q}}_2 + \frac{1}{h_3} \frac{\partial q_1}{\partial q_3} \hat{\mathbf{q}}_3 = \frac{\hat{\mathbf{q}}_1}{h_1}. \quad (2.312)$$

Similarly, other relations can be proved.

$$3. \quad \nabla \cdot \left(\frac{\hat{\mathbf{q}}_1}{h_2 h_3} \right) = 0 \quad (2.313)$$

Proof

$$\begin{aligned} \text{LHS} &= \nabla \cdot \left(\frac{\hat{\mathbf{q}}_1}{h_2 h_3} \right) = \nabla \cdot \left(\frac{\hat{\mathbf{q}}_2}{h_2} \times \frac{\hat{\mathbf{q}}_3}{h_3} \right) = \nabla \cdot (\nabla q_2 \times \nabla q_3) \\ &= \nabla q_3 \cdot \nabla \times \nabla q_2 - \nabla q_2 \cdot \nabla \times \nabla q_3 = 0 = \text{RHS} \end{aligned} \quad (2.314)$$

Similarly, we have

$$\nabla \cdot \left(\frac{\hat{\mathbf{q}}_2}{h_1 h_3} \right) = 0, \quad \nabla \cdot \left(\frac{\hat{\mathbf{q}}_3}{h_1 h_2} \right) = 0. \quad (2.315)$$

Now let $\mathbf{v}$ be a vector given by

$$\mathbf{v} = v_1 \hat{\mathbf{q}}_1 + v_2 \hat{\mathbf{q}}_2 + v_3 \hat{\mathbf{q}}_3. \quad (2.316)$$

The divergence of the vector is given by

$$\begin{aligned} \nabla \cdot \mathbf{v} &= \nabla \cdot (v_1 \hat{\mathbf{q}}_1 + v_2 \hat{\mathbf{q}}_2 + v_3 \hat{\mathbf{q}}_3) \\ &= \nabla \cdot \left(\frac{\hat{\mathbf{q}}_1}{h_2 h_3} h_2 h_3 v_1 + \frac{\hat{\mathbf{q}}_2}{h_1 h_3} h_1 h_3 v_2 + \frac{\hat{\mathbf{q}}_3}{h_1 h_2} h_1 h_2 v_3 \right). \end{aligned} \quad (2.317)$$

Making use of Equations 2.313 and 2.315 in Equation 2.317, one gets

$$\nabla \cdot \mathbf{v} = \frac{\hat{\mathbf{q}}_1}{h_2 h_3} \cdot \nabla (h_2 h_3 v_1) + \frac{\hat{\mathbf{q}}_2}{h_1 h_3} \cdot \nabla (h_1 h_3 v_2) + \frac{\hat{\mathbf{q}}_3}{h_1 h_2} \cdot \nabla (h_1 h_2 v_3). \quad (2.318)$$

Using Equation 2.308, the previous equation can be written as

$$\nabla \cdot \mathbf{v} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} (h_2 h_3 v_1) + \frac{\partial}{\partial q_2} (h_1 h_3 v_2) + \frac{\partial}{\partial q_3} (h_1 h_2 v_3) \right]. \quad (2.319)$$

2.7.4 LAPLACIAN OF A SCALAR

The Laplacian of a scalar field f is denoted by $\nabla^2 f$. The general expression for orthogonal curvilinear coordinates is given by

$$\begin{aligned} \nabla^2 f &= \nabla \cdot \nabla f = \nabla \cdot \left(\frac{1}{h_1} \frac{\partial f}{\partial q_1} \hat{\mathbf{q}}_1 + \frac{1}{h_2} \frac{\partial f}{\partial q_2} \hat{\mathbf{q}}_2 + \frac{1}{h_3} \frac{\partial f}{\partial q_3} \hat{\mathbf{q}}_3 \right) \\ &= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial f}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial f}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial f}{\partial q_3} \right) \right] \end{aligned} \quad (2.320)$$

Example 2.24

Convert the following steady-state heat-conduction equation from Cartesian to cylindrical polar coordinates:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{Q}}{k} = 0. \quad (2.321)$$

SOLUTION

The equation can be written as

$$\nabla^2 T + \frac{\dot{Q}}{k} = 0. \quad (2.322)$$

Now, Equation 2.320 is used to find the Laplacian. Here,

$$q_1 = r, \quad q_2 = \phi, \quad q_3 = z, \quad h_1 = 1, \quad h_2 = r, \quad h_3 = 1. \quad (2.323)$$

$$\begin{aligned} \nabla^2 T &= \frac{1}{r} \left[\frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{\partial}{\partial \phi} \left(\frac{1}{r} \frac{\partial T}{\partial \phi} \right) + \frac{\partial}{\partial z} \left(r \frac{\partial T}{\partial z} \right) \right] \\ &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}. \end{aligned} \quad (2.324)$$

Hence, the equation is given as

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{Q}}{k} = 0. \quad (2.325)$$

2.7.5 CURL OF A VECTOR

Given a vector $\mathbf{v} = v_1 \hat{\mathbf{q}}_1 + v_2 \hat{\mathbf{q}}_2 + v_3 \hat{\mathbf{q}}_3$, one finds the expression for its curl. First, consider the following identity:

$$\nabla \times (\phi \mathbf{u}) = \phi \nabla \times \mathbf{u} + \nabla \phi \times \mathbf{u}. \quad (2.326)$$

Proof

It can be shown that the i th component of the LHS is equal to that of the RHS.

$$\left[\nabla \times (\phi \mathbf{u}) \right]_i = \varepsilon_{ijk} (\phi u_k)_{,j} = \phi \varepsilon_{ijk} u_{k,j} + \varepsilon_{ijk} \phi_{,j} u_k = [\phi \nabla \times \mathbf{u}]_i + [\nabla \phi \times \mathbf{u}]_i \quad (2.327)$$

and hence the proof.

Also,

$$\nabla \times \left(\frac{q_1}{h_1} \right) = 0. \quad (2.328)$$

Proof

From Equation 2.311

$$\nabla \times \left(\frac{\hat{\mathbf{q}}_1}{h_1} \right) = \nabla \times (\nabla q_1) = \text{curl}(\text{grad } q_1) = 0. \quad (2.329)$$

Now, the curl of vector $\mathbf{v}$ is given by

$$\nabla \times (v_1 \hat{\mathbf{q}}_1 + v_2 \hat{\mathbf{q}}_2 + v_3 \hat{\mathbf{q}}_3) = \nabla \times (v_1 \hat{\mathbf{q}}_1) + \nabla \times (v_2 \hat{\mathbf{q}}_2) + \nabla \times (v_3 \hat{\mathbf{q}}_3). \quad (2.330)$$

Let us consider each term one by one.

$$\nabla \times (v_1 \hat{\mathbf{q}}_1) = \nabla \times \left[h_1 v_1 \frac{\hat{\mathbf{q}}_1}{h_1} \right]. \quad (2.331)$$

Now, using Equations 2.326 and 2.328, Equation 2.331 can be written as

$$\begin{aligned} \nabla \times \left[h_1 v_1 \frac{\hat{\mathbf{q}}_1}{h_1} \right] &= -\frac{\hat{\mathbf{q}}_1}{h_1} \times \nabla (h_1 v_1) \\ &= -\frac{\hat{\mathbf{q}}_1}{h_1} \times \left[\frac{1}{h_1} \frac{\partial (h_1 v_1)}{\partial q_1} \hat{\mathbf{q}}_1 + \frac{1}{h_2} \frac{\partial (h_1 v_1)}{\partial q_2} \hat{\mathbf{q}}_2 + \frac{1}{h_3} \frac{\partial (h_1 v_1)}{\partial q_3} \hat{\mathbf{q}}_3 \right] \\ &= -\frac{\hat{\mathbf{q}}_3}{h_1 h_2} \frac{\partial}{\partial q_2} (h_1 v_1) + \frac{\hat{\mathbf{q}}_2}{h_3 h_1} \frac{\partial}{\partial q_3} (h_1 v_1) \\ &= \frac{1}{h_1 h_2 h_3} \left[h_2 \hat{\mathbf{q}}_2 \frac{\partial}{\partial q_3} - h_3 \hat{\mathbf{q}}_3 \frac{\partial}{\partial q_2} \right] (h_1 v_1). \end{aligned} \quad (2.332)$$

Similarly,

$$\nabla \times \left[h_2 v_2 \frac{\hat{q}_2}{h_2} \right] = \frac{1}{h_1 h_2 h_3} \left[h_3 \hat{q}_3 \frac{\partial}{\partial q_1} - h_1 \hat{q}_1 \frac{\partial}{\partial q_3} \right] (h_2 v_2) \quad (2.333)$$

and

$$\nabla \times \left[h_3 v_3 \frac{\hat{q}_3}{h_3} \right] = \frac{1}{h_1 h_2 h_3} \left[h_1 \hat{q}_1 \frac{\partial}{\partial q_2} - h_2 \hat{q}_2 \frac{\partial}{\partial q_1} \right] (h_3 v_3). \quad (2.334)$$

Using Equations 2.332–2.334 and rearranging the terms, the curl of the vector is written as

$$\begin{aligned} \nabla \times \mathbf{v} = \frac{1}{h_1 h_2 h_3} & \left[h_1 \hat{q}_1 \left\{ \frac{\partial(h_3 v_3)}{\partial q_2} - \frac{\partial(h_2 v_2)}{\partial q_3} \right\} \right. \\ & \left. - h_2 \hat{q}_2 \left\{ \frac{\partial(h_3 v_3)}{\partial q_1} - \frac{\partial(h_1 v_1)}{\partial q_3} \right\} + h_3 \hat{q}_3 \left\{ \frac{\partial(h_2 v_2)}{\partial q_1} - \frac{\partial(h_1 v_1)}{\partial q_2} \right\} \right]. \end{aligned} \quad (2.335)$$

In the form of a determinant, it can be written as

$$\nabla \times \mathbf{v} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{q}_1 & h_2 \hat{q}_2 & h_3 \hat{q}_3 \\ \partial / \partial q_1 & \partial / \partial q_2 & \partial / \partial q_3 \\ h_1 v_1 & h_2 v_2 & h_3 v_3 \end{vmatrix}. \quad (2.336)$$

2.7.6 VOLUME OF AN INFINITESIMAL ELEMENT

As per Equation 2.305, the infinitesimal vector $d\mathbf{r}$ is given as the summation of three orthogonal vectors: $h_1 \hat{q}_1 dq_1$, $h_2 \hat{q}_2 dq_2$ and $h_3 \hat{q}_3 dq_3$. Now, the volume of the parallelepiped, which has diagonal $d\mathbf{r}$, is given by

$$dV = \left| (h_1 \hat{q}_1 dq_1) \cdot \{ (h_2 \hat{q}_2 dq_2) \times (h_3 \hat{q}_3 dq_3) \} \right| = h_1 h_2 h_3 dq_1 dq_2 dq_3. \quad (2.337)$$

If Equation 2.304 is used, then the volume of the parallelepiped becomes

$$dV = \left| \frac{\partial \mathbf{r}}{\partial q_1} \cdot \left(\frac{\partial \mathbf{r}}{\partial q_2} \times \frac{\partial \mathbf{r}}{\partial q_3} \right) \right| dq_1 dq_2 dq_3 \quad (2.338)$$

The previous equation simplifies to

$$dV = \begin{vmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} & \frac{\partial x}{\partial q_3} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} & \frac{\partial y}{\partial q_3} \\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} & \frac{\partial z}{\partial q_3} \end{vmatrix} dq_1 dq_2 dq_3 = \det(J) dq_1 dq_2 dq_3, \quad (2.339)$$

where J is called the Jacobian.

Exercises

1. For an incompressible fluid, the continuity equation in the Eulerian form is

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0,$$

where v_x , v_y and v_z are the components of the velocity field along the x , y and z directions, respectively. Express this equation in index notation.

2. The three-dimensional stress equilibrium equations are given by

$$\sigma_{ij,j} + \rho b_i = 0,$$

where σ is the stress tensor, and b is the body force per unit volume. Write down the stress equilibrium equations in unabridged notations.

3. Evaluate the following expressions:

- a. δ_{ii}
- b. $\varepsilon_{ijk} \varepsilon_{ijk}$
- c. $\varepsilon_{ijk} \varepsilon_{kji}$
- d. $\delta_{ij} \delta_{ik} \delta_{jk}$

4. Prove the following ε - δ identity:

$$\varepsilon_{ijk} \varepsilon_{pqk} = \delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}.$$

5. Prove that

$$\varepsilon_{ijk} \varepsilon_{pqr} = \delta_{ip} (\delta_{jq} \delta_{kr} - \delta_{jr} \delta_{kq}) + \delta_{iq} (\delta_{jr} \delta_{kp} - \delta_{jp} \delta_{kr}) + \delta_{ir} (\delta_{jp} \delta_{kq} - \delta_{jq} \delta_{kp}).$$

6. A second-order tensor is a linear transformation that maps vectors to vectors. For example, if $[\sigma]$ is the matrix containing stress components, $\{n\}$ is the vector of direction cosines to a plane and $\{t\}$ is the traction vector on the plane, then by Cauchy's relation

$$\{t\} = [\sigma] \{n\}.$$

Here, the stress tensor maps the direction cosine vector into the traction vector.

The transformation law for vectors is given by

$$v'_q = \alpha_{qr} v_r$$

Using the definition of tensor, find the transformation law for tensors.

7. If $\mathbf{a}$ and $\mathbf{b}$ are vectors with components a_i and b_i , respectively, then $a_i b_j$ are components of a second-order tensor, called the tensor product $\mathbf{a} \otimes \mathbf{b}$. Prove that

- a. If a_{ij} are components of a second-order tensor $\mathbf{A}$ and b_i are the components of a vector $\mathbf{b}$, then $a_{ij}b_k$ are the components of a third-order tensor (known as the tensor product of $\mathbf{A}$ and $\mathbf{b}$ in that order, denoted $\mathbf{A} \otimes \mathbf{b}$).
 - b. If a_{ij} and b_{ij} are the components of two second-order tensors $\mathbf{A}$ and $\mathbf{B}$, then $a_{ij}b_{kl}$ are the components of a fourth-order tensor (known as the tensor product of $\mathbf{A}$ and $\mathbf{B}$ in that order, denoted $\mathbf{A} \otimes \mathbf{B}$).
8. If a_{ij} and b_{ij} are the components of two second-order tensors $\mathbf{A}$ and $\mathbf{B}$ and c_i are the components of vector $\mathbf{c}$, then prove that
- a. $a_{ij}c_j$ are the components of a vector (known as the vector product of $\mathbf{A}$ and $\mathbf{c}$ in that order, denoted $\mathbf{Ac}$).
 - b. $a_{ik}b_{kj}$ are the components of a second-order tensor, called the product of $\mathbf{A}$ and $\mathbf{B}$ in that order denoted by $\mathbf{AB}$.
 - c. $a_{ij}b_{ij}$ is a scalar, called the scalar product of $\mathbf{A}$ and $\mathbf{B}$ and denoted by $\mathbf{A} \cdot \mathbf{B}$.
9. Prove the following quotient rules:
- a. Let a_i be an ordered triplet related to the x_i system. For an arbitrary vector with components b_i , if a_ib_i are scalar, then a_i are the components of a vector.
 - b. Let a_{ij} be a 3×3 matrix related to the x_i system. For an arbitrary vector with components b_i , if $a_{ij}b_j$ are components of a vector, then a_{ij} are the components of a tensor.
10. By using the transformation law for a vector, show that the vector product of $\mathbf{a} \times \mathbf{b}$ is also a vector.
11. Prove that if $\mathbf{A}$ is a second-order tensor, then it is a linear operator on vectors and its components are given by

$$a_{ij} = \mathbf{e}_i \cdot \mathbf{A} \mathbf{e}_j.$$

Also prove that, conversely, if $\mathbf{A}$ is a linear operator on vectors and a_{ij} are defined by the previous equation, then a_{ij} are components of a second-order tensor.

12. Show that every tensor with components a_{ij} can be represented in the form

$$\mathbf{A} = a_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$$

13. Prove that $(x_2, -x_1)$ are the components of a first-order Cartesian tensor in two dimensions, whereas (x_2, x_1) and (x_1^2, x_2^2) are not.
14. Let a_{ij} and b_{ij} be the components of two 3×3 matrices, respectively, and their scalar product is defined as $a_{ij}b_{ij}$. Prove that the scalar product of a symmetric and skew-symmetric matrix is zero.
15. Prove that the following are three invariants of a tensor $\mathbf{A}$ with components a_{ij} :
- a. $I_A = \text{tr} \mathbf{A} = a_{ii}$ (tr is a short form of trace).

$$\text{b. } I_A = \frac{1}{2} \left[(\text{tr} \mathbf{A})^2 - \text{tr}(\mathbf{A}^2) \right] = \frac{1}{2} [a_{ii}a_{kk} - a_{ik}a_{ki}].$$

$$\begin{aligned} \text{c. } III_A &= \frac{1}{6} \left[(\text{tr} \mathbf{A})^3 + 2\text{tr}(\mathbf{A}^3) - 3(\text{tr} \mathbf{A}^2)(\text{tr} \mathbf{A}) \right] \\ &= \frac{1}{6} [a_{ii}a_{ij}a_{kk} + 2a_{ik}a_{km}a_{mi} - 3a_{ik}a_{ki}a_{jj}]. \end{aligned}$$

Further, prove that the third invariant can also be written as

$$III_A = \det[a_{ij}] = \varepsilon_{ijk}a_{i1}a_{j2}a_{k3} = \varepsilon_{ijk}a_{1i}a_{2j}a_{3k}.$$

16. Let a_{ij} be the components of a tensor $\mathbf{A}$. If

$$a_{ij}^* = \frac{1}{2} \varepsilon_{ipq} \varepsilon_{jrs} a_{pr} a_{qs}.$$

Show that a_{ij}^* are the components of a tensor. If this tensor is denoted by $\mathbf{A}^*$, prove the following:

- $\mathbf{A}(\mathbf{A}^*)^T = (\mathbf{A}^*)^T \mathbf{A} = (\det \mathbf{A}) \mathbf{I}$.
- If $\mathbf{A}$ is invertible, then

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} (\mathbf{A}^*)^T$$

- For all vectors $\mathbf{a}, \mathbf{b}$

$$\mathbf{A}^* (\mathbf{a} \times \mathbf{b}) = \mathbf{A} \mathbf{a} \times \mathbf{A} \mathbf{b}.$$

- For all vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$,

$$(\mathbf{A}^*)^T \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot \mathbf{A} \mathbf{b} \times \mathbf{A} \mathbf{c}.$$

Here $\mathbf{A}^*$ is called the adjugate or cofactor of $\mathbf{A}$, and $(\mathbf{A}^*)^T$ is called the adjoint of $\mathbf{A}$.

- Obtain the expressions for three invariants of a deviatoric tensor of $\mathbf{A}$.
- Prove that a number λ is an eigenvalue of a tensor $\mathbf{A}$ if and only if it is a real root of the cubic equation

$$\lambda^3 - I_A \lambda^2 + II_A \lambda - III_A = 0.$$

- Prove that if $\mathbf{A}$ is a symmetric tensor, then all three roots of the characteristic equation of $\mathbf{A}$ are real, and therefore, $\mathbf{A}$ has exactly three (not necessarily distinct) eigenvalues.
- Prove that eigenvectors (principal directions) corresponding to two distinct eigenvalues (principal values) of a symmetric tensor are orthogonal.

21. Prove that a symmetric tensor has at least three mutually perpendicular principal directions.
22. Given a symmetric tensor $\mathbf{A}$, there exists at least one coordinate system with respect to which the matrix of $\mathbf{A}$ is diagonal.
23. Let $\mathbf{A}$ be a symmetric tensor with λ_i as eigenvalues and $\mathbf{v}_i$ as corresponding eigenvectors. Show that $\mathbf{A}$ can be represented as

$$\mathbf{A} = \sum_{k=1}^3 \lambda_k (\mathbf{v}_k \otimes \mathbf{v}_k)$$

This is known as the spectral representation of $\mathbf{A}$.

24. Every invertible tensor $\mathbf{A}$ can be represented in the form

$$\mathbf{A} = \mathbf{Q}\mathbf{U} = \mathbf{V}\mathbf{Q},$$

where $\mathbf{Q}$ is the orthogonal tensor and $\mathbf{U}$ and $\mathbf{V}$ are the positive-definite symmetric tensors, such that $\mathbf{U}^2 = \mathbf{A}^T\mathbf{A}$ and $\mathbf{V}^2 = \mathbf{A}\mathbf{A}^T$. Furthermore, the representations are unique.

25. If $\mathbf{Q}(t)$ is an orthogonal tensor, show that $\left(\frac{d\mathbf{Q}}{dt}\right)\mathbf{Q}^T$ is a skew tensor.
26. Prove that $x_{i,j} = \delta_{ij}$ and $x_{i,i} = 3$.
27. Express the divergence and curl of a vector and tensor field in index notation.
28. Prove the following divergence theorem for a tensor. Let V be the volume of a three-dimensional region bounded by a closed regular surface S . Then for a tensor field $\mathbf{A}$ defined in V and on S ,

$$\int_V \text{div} \mathbf{A} dV = \int_S \mathbf{A} n dS,$$

where n is the unit outward normal to S .

29. Prove the following Stoke's theorem for a tensor. Let C be a simple closed curve in a three-dimensional space and S be an open regular surface bounded by C . Then, for a vector field defined on S as well as C ,

$$\oint_C \mathbf{A} t ds = \int_S (\text{curl} \mathbf{A})^T n dS,$$

where t is the unit vector tangent to C , which is assumed to be positively oriented relative to the unit normal n to S .

ANSWERS

1. $v_{i,i} = 0$
2. $\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho b_x = 0$
 $\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + \rho b_y = 0$

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho b_z = 0$$

3. (a) 3; (b) 6; (c) -6; (d) 3

$$6. \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + \rho b_y = 0$$

3 Stress

3.1 INTRODUCTION

Whenever a body is subjected to external forces, internal forces are generated. Internal forces are the forces that one part of the body applies on the other part. Figure 3.1 shows a rod fixed at one end. It is pulled by a net force of F along the longitudinal direction applied at the other end. For convenience of understanding, it can be assumed that the force is uniformly distributed over the cross section of the rod. To obtain the internal force at a distance of x from the fixed end, the rod is hypothetically cut at this point. Consider the free body diagram of the right-hand side of the rod. This portion is subjected to the applied force F and an internal force that is applied by the left-side portion of the rod. As the entire body is in equilibrium, by Newton's first law of motion, the internal force on the right-side portion of the body must be equal to F directed toward the fixed end. The force is uniformly distributed over the cross section, but the figure shows a net force F passing through the centroid of the cross section. By Newton's third law, the right-hand-side portion exerts a force F on the left-hand-side portion of the rod. This force is directed toward the free end. Thus, the magnitude of the internal force at a distance of x from the fixed end is F . The magnitude of the internal force is independent of x in this case. Hence, the magnitude is the same at all the sections of the body. The direction of the force depends upon which portion of the rod is being considered.

Now, consider the same rod placed on a frictionless surface that is free to move, as shown in Figure 3.2. When the rod is subjected to a longitudinal force F at the right-hand side, it starts to accelerate. Assume that the acceleration is uniform everywhere. To obtain the internal force at a distance of x from the left-hand face, the rod is again hypothetically cut at this distance. Consider the right-hand-side portion. Let the internal force on it be F_i as shown. If the rod is homogeneous with length L and mass M , Newton's second law provides the following equation:

$$F - F_i = M \frac{(L-x)}{L} a. \quad (3.1)$$

However, $a = F/M$. Therefore, Equation 3.1 provides

$$F_i = F \frac{x}{L}. \quad (3.2)$$

Equation 3.2 shows that in this case, the internal force will vary linearly from zero at the left-hand side to F at the right-hand side.

The intensity of the internal force on a cross section can be defined as the force per unit area, which is called the stress. On a particular cross section, the internal

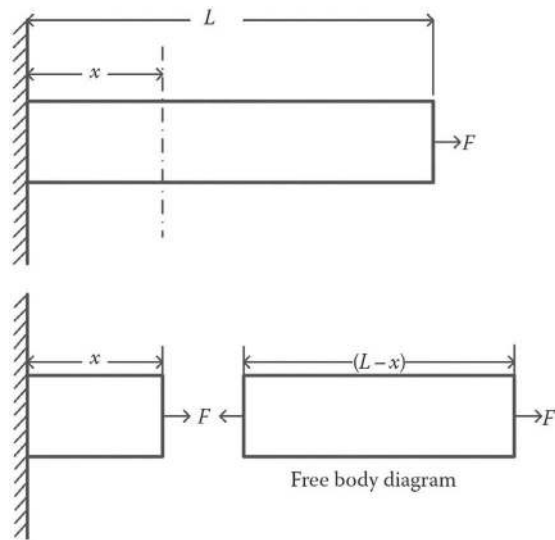


FIGURE 3.1 Internal force on an axially loaded rod.

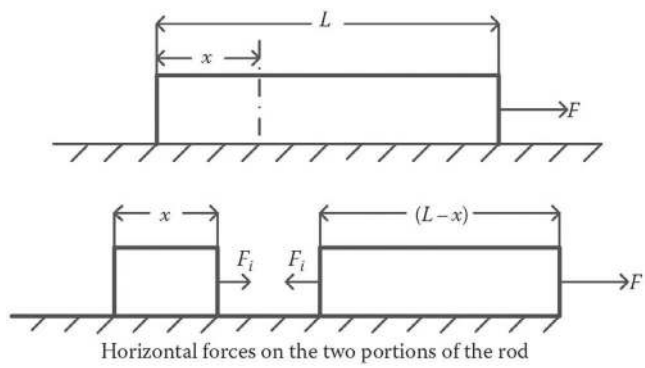


FIGURE 3.2 Internal forces on a moving rod.

force is a vector quantity; hence, its intensity is also a vector quantity. However, if the rod is cut at a plane inclined from the vertical, then the area over which the internal force acts changes. Thus, the intensity will depend on the plane under consideration. Such physical quantity is called a tensor of order two. In this chapter, stress will be discussed.

3.2 STRESS AT A POINT

Consider a body that is acted upon by the external and reaction forces. Let the body be hypothetically cut by a plane AA, as shown in Figure 3.3a. Figure 3.3b

shows the left part of the body. On the surface cut by the plane, the body is acted upon by the internal forces, which are due to the right portion of the body. These forces are distributed throughout the whole surface, although the intensity may not be constant over the surface. Take a small area ΔA surrounding a point P on the surface and let the total force on this area be $\Delta \mathbf{F}$. The ratio of $\Delta \mathbf{F}$ to ΔA in the limit of ΔA going to zero is called the stress vector or traction at point P. Thus,

$$\mathbf{t}_n = \lim_{\Delta A \rightarrow 0} \frac{\Delta \mathbf{F}}{\Delta A}, \quad (3.3)$$

where the subscript $\mathbf{n}$ indicates that the outward normal to the plane is $\mathbf{n}$. In general, the traction depends on the coordinates of point P and normal $\mathbf{n}$. Note that the normal is directed away from the material of the left part. There are an infinite number of possible planes passing through P, giving rise to an infinite number of traction vectors. Knowing the state of stress at a point means knowing the traction vector as a function of $\mathbf{n}$ at that point.

If normal to plane at the left portion is $\mathbf{n}$, it will be $-\mathbf{n}$ at the right portion. By Newton's third law, the force acting on the small area ΔA of the surface of the right body is $(-\Delta \mathbf{F})$. Hence, by definition,

$$\mathbf{t}_{(-n)} = \lim_{\Delta A \rightarrow 0} \frac{-\Delta \mathbf{F}}{\Delta A} = -\mathbf{t}_n \quad (3.4)$$

resolving the force $\Delta \mathbf{F}$ along the normal to the plane and in the plane. The normal component is ΔF_n , and the in-plane component is ΔF_s . The direct stress is defined as

$$\sigma_{nn} = \lim_{\Delta A \rightarrow 0} \frac{\Delta F_n}{\Delta A} \quad (3.5)$$

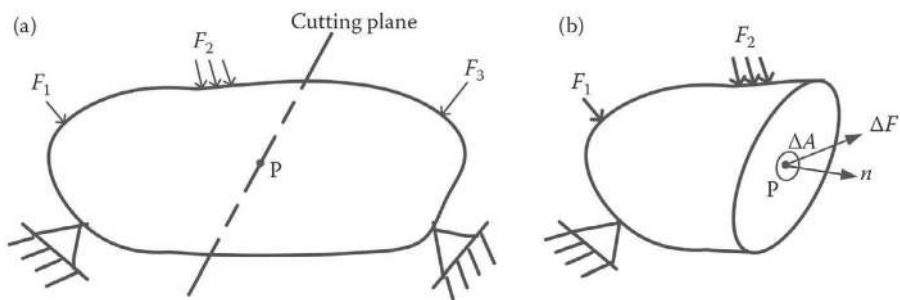


FIGURE 3.3 Stress vector at a point on a plane: (a) cutting plane passing through point P of the deformed configuration; (b) left side portion of the body (stress vector is along $\Delta \mathbf{F}$).

and the shear stress is defined as

$$\tau_{ns} = \lim_{\Delta A \rightarrow 0} \frac{\Delta F_s}{\Delta A}. \quad (3.6)$$

The direct stress is named tensile if the component of the force $\Delta \mathbf{F}$ along the normal to the plane is directed toward outward normal, that is, the directions of $\Delta \mathbf{F}_n$ and $\mathbf{n}$ are the same. If the resolved component of $\Delta \mathbf{F}$ along the normal is directed inward, that is, the directions of $\Delta \mathbf{F}_n$ and $\mathbf{n}$ are opposite, the stress is considered compressive. Usually, a tensile stress is taken as positive and a compressive stress as negative. Using Pythagoras theorem,

$$|\mathbf{t}_n|^2 = \sigma_{nn}^2 + \tau_{ns}^2. \quad (3.7)$$

If the traction vectors on three planes passing through a point are known, the traction vector on any plane passing through the point can be found. Consider that the traction vectors are known at planes with normals along x , y and z directions, respectively. Let us consider a point P and construct an infinitesimal parallelepiped of sides dx , dy and dz around it, as shown in Figure 3.4. The parallelepiped is made of six plane surfaces. On each surface, there will be traction vectors that can be resolved into three components in x , y and z directions. On the x plane, that is, the plane whose normal is along the positive x -axis, three components of the traction vector are σ_{xx} , σ_{xy} and σ_{xz} . These will be called stress components. The convention of calling a plane a positive plane is adopted if its normal is along the positive direction of axes. On a positive plane, a stress component is positive if it is directed toward the positive direction. Thus, σ_{xx} , σ_{xy} and σ_{xz} shown in the positive x -plane are positive. On a negative plane, that is, the plane whose normal is along the negative direction of the coordinate axis, the stress component directed toward a negative axis is considered positive. It is clear that σ_{xx} is a direct stress, and σ_{xy} and σ_{xz} are shear stresses. The material may be subjected to body forces and inertia forces, but as the volume of the parallelepiped tends to zero, the contribution of these forces also tends to zero. Assume that dx is zero, and only dy and dz remain, as shown in the figure. The volume of this element is zero, and the force balance provides that stress components on both sides of the planes must be equal. In a similar way, one can visualize that the components of the traction vector on opposite faces are equal in magnitude and opposite in direction as all sides shrink to zero. As there are six faces, the total number of components on these faces is 18. However, as the traction components on opposite faces are equal in magnitude, the tractions on six faces can be represented by nine components arranged in a matrix form:

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}. \quad (3.8)$$

These nine components represent the state of stress at point P.

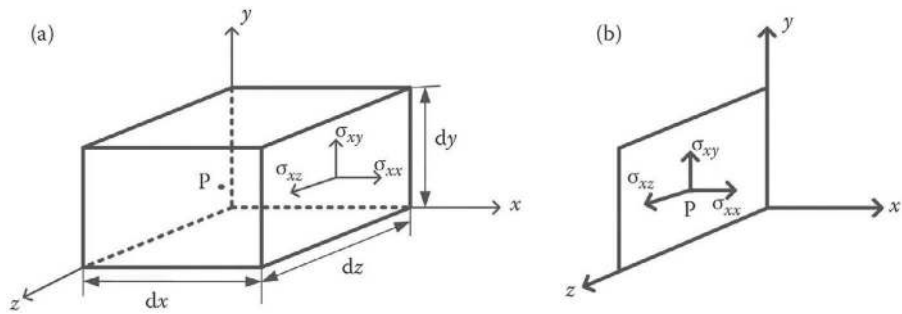


FIGURE 3.4 (a) A parallelepiped around point P (stress vector components shown on one face). (b) Parallelepiped turning into a plane as dx becomes zero; stress vector components on positive face are shown; the components' negative faces are equal and opposite to those on positive face.

Example 3.1

The traction vector at a point on a plane is $(3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}) \text{ N/m}^2$, where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors along x , y and z directions. If the direct stress on the plane is 5 N/m^2 , find the magnitude of shear stress.

SOLUTION

The magnitude of the traction vector is given by

$$|\mathbf{t}_n| = \sqrt{3^2 + 4^2 + 12^2} = 13 \text{ N/m}^2.$$

Using Equation 3.7,

$$\sigma_{nn}^2 + \tau_{ns}^2 = 13^2 = 169,$$

or

$$\tau_{ns}^2 = 169 - \sigma_{nn}^2 = 169 - 25 = 144.$$

Hence, the shear stress τ_{ns} is 12 N/m^2 .

3.3 SURFACE FORCES AND BODY FORCES

In continuum mechanics, a body consists of continuous volume enclosed by its bounding surfaces. When a body is cut (actually or hypothetically), each part of the body has its volume bounded by the surface. A body part is subjected to forces by the environment and another part of the body. Some type of force can only be applied on the surface of the body. It is not possible to apply these forces at a point in the interior

without breaking the body. When a workpiece is deformed by a hammer, the hammer strikes the surface only. Such types of forces are called surface forces. Surface forces are distributed over a finite area. One can obtain the intensity of the surface force at a point by dividing the force by the area on which it acts, in the limit of the area tending to zero. The intensity of the surface force is called surface traction. The total force on a surface S can be obtained by integrating the traction over the area. Thus,

$$\mathbf{F}_S = \int_S \mathbf{t}_n dS. \quad (3.9)$$

Note that $\mathbf{n}$ as well as $\mathbf{t}_n$ may keep changing from point to point on the surface.

A concentrated force acts on a zero area but can be represented with the help of a Dirac delta function as a distributed force. If $\mathbf{P}$ is a concentrated force acting at point (x_0, y_0, z_0) , its intensity is given by

$$\mathbf{t}_n = \mathbf{P} \delta_A(x - x_0, y - y_0, z - z_0), \quad (3.10)$$

where $\delta_A(x - x_0, y - y_0, z - z_0)$ is a Dirac delta function defined at (x_0, y_0, z_0) . The Dirac delta function has the following properties:

1. It is zero everywhere, except at the point (x_0, y_0, z_0) , at which it tends to infinity.
2. Its integration over an area enclosing the point (x_0, y_0, z_0) , is 1,

$$\int_A \delta_A(x - x_0, y - y_0, z - z_0) dA = 1, \quad (3.11)$$

provided that point (x_0, y_0, z_0) is contained in A . Here, $\delta_A(x - x_0, y - y_0, z - z_0)$ is a two-dimensional function, as on a surface, one coordinate can be expressed as a function of the other two. Similarly, one can define the one- and three-dimensional Dirac delta functions. From Equation 3.11, it is clear that in SI system, $\delta_A(x - x_0, y - y_0, z - z_0)$ has a unit of $1/\text{m}^2$.

Examples of surface forces are externally applied mechanical forces, fluid pressure exerted on the body and contact force between two bodies. The surface force may be resolved into two components: one normal to the surface and the other tangential to the surface. Normal force produces direct stress and tangential force the shear stress.

Body forces are the forces exerted on the interior points of a body by the environment. Their intensity is specified by force per unit volume or force per unit mass. If the body force per unit mass is denoted by $\mathbf{b}$, the body force per unit volume will be $\rho\mathbf{b}$. The total body force $\mathbf{F}_b$ in a volume V can be found by integrating $\rho\mathbf{b}$ over the volume.

$$\mathbf{F}_b = \int_V \rho\mathbf{b} dV. \quad (3.12)$$

The examples of body forces are gravitational force, electromagnetic force and inertia force. If there is a concentrated force P acting at point (x_0, y_0, z_0) , its intensity may be taken as $P\delta_V(x - x_0, y - y_0, z - z_0)$, where $\delta_V(x - x_0, y - y_0, z - z_0)$ is a three-dimensional Dirac delta function. It is zero except at (x_0, y_0, z_0) , where it tends to infinity. Also,

$$\int_V \delta_V(x - x_0, y - y_0, z - z_0) dV = 1 \quad (3.13)$$

provided that (x_0, y_0, z_0) is contained in V . Therefore,

$$\int_V P\delta_V(x - x_0, y - y_0, z - z_0) dV = P. \quad (3.14)$$

In SI system $\delta_V(x - x_0, y - y_0, z - z_0)$ has a unit of $1/\text{m}^3$.

3.4 MOMENTUM BALANCE LAWS

Let

$$\mathbf{a} = a_x \mathbf{e}_1 + a_y \mathbf{e}_2 + a_z \mathbf{e}_3 \quad (3.15)$$

be the acceleration vector of a particle designated by point (x, y, z) of the deformed configuration. As per Newton's second law (or Euler's first law for a particle), mass times acceleration is equal to the net force impressed on the particle, provided that the acceleration is measured in an inertial frame of reference. The inertial frame is a fixed reference frame or a frame translating with respect to a fixed reference frame with a constant velocity. The mass of a typical particle can be taken as ρdV , where ρ is the density at the point (x, y, z) , and dV is the infinitesimal volume. Due to the law of mass conservation, ρdV remains constant with respect to time. Integrating the net forces on all particles, one gets

$$\int_V \rho \mathbf{a} dV = \int_S \mathbf{t}_n dS + \int_V \rho \mathbf{b} dV \quad (3.16)$$

This is the balance of linear momentum in integral form. Note that internal forces cancel out one another as a result of Newton's third law.

Euler's second law, the balance of angular momentum, provides the following equation:

$$\int_V \mathbf{r} \times \rho \mathbf{a} dV = \int_S \mathbf{r} \times \mathbf{t}_n dS + \int_V \mathbf{r} \times \rho \mathbf{b} dV, \quad (3.17)$$

where $\mathbf{r}$ is the position vector of a typical point with respect to a fixed point in an inertial frame of reference. The accelerations are also measured with respect to the inertial frame of reference. Equation 3.17 remains valid if the position vector $\mathbf{r}$

and acceleration $\mathbf{a}$ are measured about the mass center or about any point accelerating toward the mass center.

Example 3.2

Show how Equation 3.17 is obtained from Euler's second law.

SOLUTION

Consider a point A, which is one of the following:

1. A fixed point in an inertial frame of reference
2. Center of mass
3. A point accelerating toward the mass center

Consider a particle of mass $\rho \, dV$ and velocity $\mathbf{v}$ with respect to point A. The angular momentum of the particle (or body of infinitesimal volume) about A is

$$\mathbf{dH}_A = \mathbf{r} \times \rho \mathbf{v} \, dV. \quad (3.18)$$

The angular momentum of the whole body about A is the integral of Equation 3.18. Thus,

$$\mathbf{H}_A = \int_V \mathbf{r} \times \rho \mathbf{v} \, dV. \quad (3.19)$$

Euler's second law states that the rate of change of angular momentum of the body about A is equal to the sum of moments of external forces about A. Therefore,

$$\frac{d\mathbf{H}_A}{dt} = \int_S \mathbf{r} \times \mathbf{t}_n \, dS + \int_V \mathbf{r} \times \rho \mathbf{b} \, dV. \quad (3.20)$$

Now,

$$\frac{d\mathbf{H}_A}{dt} = \frac{d\left(\int_V \mathbf{r} \times \rho \mathbf{v} \, dV\right)}{dt} = \frac{d\left(\int_V \mathbf{r} \times \mathbf{v} \rho \, dV\right)}{dt}. \quad (3.21)$$

As $\rho \, dV$ is constant with time, Equation 3.21 can be written as

$$\begin{aligned} \frac{d\mathbf{H}_A}{dt} &= \int_V \frac{d(\mathbf{r} \times \mathbf{v})}{dt} \rho \, dV = \int_V \frac{d\mathbf{r}}{dt} \times \mathbf{v} \rho \, dV + \int_V \mathbf{r} \times \frac{d\mathbf{v}}{dt} \rho \, dV \\ &= \int_V \mathbf{v} \times \mathbf{v} \rho \, dV + \int_V \mathbf{r} \times \frac{d\mathbf{v}}{dt} \rho \, dV = 0 + \int_V \mathbf{r} \times \frac{d\mathbf{v}}{dt} \rho \, dV \\ &= \int_V \mathbf{r} \times \rho \mathbf{a} \, dV. \end{aligned} \quad (3.22)$$

Substituting Equation 3.22 into Equation 3.20, Equation 3.17 is obtained.

3.5 THEOREM OF VIRTUAL WORK

A necessary and sufficient condition that the momentum balance laws be satisfied is that at any time

$$\int_S \mathbf{t}_n \mathbf{w} dS + \int_V \rho(\mathbf{b} - \mathbf{a}) \mathbf{w} dV = 0 \quad (3.23)$$

for every infinitesimal rigid displacement $\mathbf{w}$. For proving it, one can make use of Chasles's theorem. According to it, any displacement of a rigid body may be obtained from a single rotation about a selected point plus a translation of that point. Assume that infinitesimal rigid displacement $\mathbf{w}$ is composed of infinitesimal rotation $\delta\boldsymbol{\theta}$ about a fixed point plus infinitesimal translation $\delta\mathbf{u}$ of that point. (As the translation and rotation are infinitesimal, status quo is maintained.)

The infinitesimal rigid displacement $\mathbf{w}$ can now be written as

$$\mathbf{w} = \delta\mathbf{u} + \delta\boldsymbol{\theta} \times \mathbf{r}, \quad (3.24)$$

where $\mathbf{r}$ is the position vector of a point on the body with respect to a fixed point. Substituting Equation 3.24 in Equation 3.23 and rearranging,

$$\left(\int_S \mathbf{t}_n dS + \int_V \rho(\mathbf{b} - \mathbf{a}) dV \right) \delta\mathbf{u} + \int_S \mathbf{t}_n (\delta\boldsymbol{\theta} \times \mathbf{r}) dS + \int_V \rho(\mathbf{b} - \mathbf{a}) (\delta\boldsymbol{\theta} \times \mathbf{r}) dV = 0 \quad (3.25)$$

Now, using index notation,

$$\mathbf{t}_n (\delta\boldsymbol{\theta} \times \mathbf{r}) = (t_n)_i \varepsilon_{ijk} (\delta\theta)_j r_k = \varepsilon_{ijk} r_k (t_n)_i (\delta\theta)_j = (\mathbf{r} \times \mathbf{t}_n) \delta\theta. \quad (3.26)$$

Similarly,

$$\rho(\mathbf{b} - \mathbf{a}) (\delta\boldsymbol{\theta} \times \mathbf{r}) = (\mathbf{r} \times \rho(\mathbf{b} - \mathbf{a})) \delta\theta. \quad (3.27)$$

Hence, Equation 3.25 can be written as

$$\left(\int_S \mathbf{t}_n dS + \int_V \rho(\mathbf{b} - \mathbf{a}) dV \right) \delta\mathbf{u} + \left(\int_S (\mathbf{r} \times \mathbf{t}_n) dS + \int_V \mathbf{r} \times \rho(\mathbf{b} - \mathbf{a}) dV \right) \delta\boldsymbol{\theta} = 0 \quad (3.28)$$

As $\delta\mathbf{u}$ and $\delta\boldsymbol{\theta}$ are arbitrary, Equation 3.28 implies that

$$\int_S \mathbf{t}_n dS + \int_V \rho(\mathbf{b} - \mathbf{a}) dV = 0; \int_S (\mathbf{r} \times \mathbf{t}_n) dS + \int_V \mathbf{r} \times \rho(\mathbf{b} - \mathbf{a}) dV = 0 \quad (3.29)$$

which are the momentum balance equations (Equations 3.16 and 3.17). Thus, Equation 3.23 is a sufficient condition for the satisfaction of momentum balance laws. Conversely, if momentum balance laws are satisfied, then Equation 3.28 must hold good. In other words, Equation 3.23 must be satisfied. Thus, Equation 3.23 is a necessary condition.

It is to be noted that Equation 3.29 contains the balance of angular momentum with respect to a fixed point. It can be derived that the balance of angular momentum remains the same if instead of a fixed point, the mass center of the body or any point accelerating toward the mass center is taken.

3.6 CAUCHY'S THEOREM

A necessary and sufficient condition that the momentum balance laws be satisfied is that there exists a spatial tensor field $[\sigma]$ (called the Cauchy stress and denoted by σ) such that

1. For every unit vector $\mathbf{n}$,

$$\{t_n\} = [\sigma]^T \{n\}. \quad (3.30)$$

2. $[\sigma]$ is symmetric.
3. $[\sigma]$ satisfies the equation of motion

$$\text{div } \sigma + \rho \mathbf{b} = \rho \mathbf{a}. \quad (3.31)$$

Proof

1. Let us consider a tetrahedron as shown in Figure 3.5. The normal to the three faces of the tetrahedron are along $-\mathbf{e}_1$, $-\mathbf{e}_2$ and $-\mathbf{e}_3$ directions. Let us denote the corresponding areas of these faces as A_x , A_y and A_z and the area of the slanted face as A . The normal to the slanted face, $\mathbf{n}$, can be expressed as

$$\mathbf{n} = n_x \mathbf{e}_1 + n_y \mathbf{e}_2 + n_z \mathbf{e}_3, \quad (3.32)$$

where n_x , n_y and n_z are the direction cosines of $\mathbf{n}$. It can be easily shown that

$$n_x = \frac{A_x}{A}, n_y = \frac{A_y}{A}, n_z = \frac{A_z}{A}. \quad (3.33)$$

Here, it is shown using the divergence theorem. Recall that from the divergence theorem,

$$\int_S \mathbf{u} \mathbf{n} dS = \int_V \text{div } \mathbf{u} dV, \quad (3.34)$$

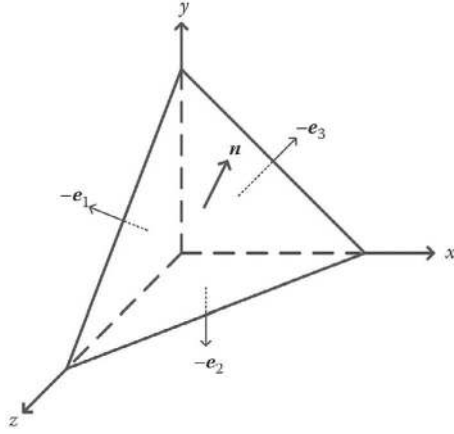


FIGURE 3.5 A tetrahedron.

where $\mathbf{u}$ is a vector field defined in volume V and its enclosing surface S . For the tetrahedron, the right-hand side of Equation 3.34 becomes 0 for a constant vector $\mathbf{u}$. Thus,

$$\mathbf{u}(n_x \mathbf{e}_1 + n_y \mathbf{e}_2 + n_z \mathbf{e}_3)A - \mathbf{u}e_1 A_x - \mathbf{u}e_2 A_y - \mathbf{u}e_3 A_z = 0. \quad (3.35)$$

Taking $\mathbf{u} = \mathbf{e}_1$, Equation 3.35 gives

$$n_x = \frac{A_x}{A}. \quad (3.36)$$

Similarly, taking $\mathbf{u} = \mathbf{e}_2$ in Equation 3.35, one gets

$$n_y = \frac{A_y}{A}, \quad (3.37)$$

and taking $\mathbf{u} = \mathbf{e}_3$ in Equation 3.35, one gets

$$n_z = \frac{A_z}{A}. \quad (3.38)$$

As the tetrahedron starts shrinking to a point, the surface integral dominates over the volume integral, because areas of the order of square of the side length and volume are the order of cube of the side length. Hence, Euler's first law given by Equation 3.16 becomes

$$\int_S \mathbf{t}_n dS = 0. \quad (3.39)$$

For the tetrahedron,

$$\mathbf{t}_n A + \mathbf{t}_{(-e_1)} A_x + \mathbf{t}_{(-e_2)} A_y + \mathbf{t}_{(-e_3)} A_z = 0, \quad (3.40)$$

which in view of Equation 3.4 becomes

$$\mathbf{t}_n A = \mathbf{t}_{e_1} A_x + \mathbf{t}_{e_2} A_y + \mathbf{t}_{e_3} A_z. \quad (3.41)$$

Dividing the whole expression by A and using Equations 3.36–3.38,

$$\mathbf{t}_n = \mathbf{t}_{e_1} n_x + \mathbf{t}_{e_2} n_y + \mathbf{t}_{e_3} n_z. \quad (3.42)$$

It is clear from Figure 3.4 and Equation 3.8 that

$$\mathbf{t}_{e_1} = \sigma_{xx} \mathbf{e}_1 + \sigma_{xy} \mathbf{e}_2 + \sigma_{xz} \mathbf{e}_3, \quad (3.43)$$

$$\mathbf{t}_{e_2} = \sigma_{yx} \mathbf{e}_1 + \sigma_{yy} \mathbf{e}_2 + \sigma_{yz} \mathbf{e}_3, \quad (3.44)$$

$$\mathbf{t}_{e_3} = \sigma_{zx} \mathbf{e}_1 + \sigma_{zy} \mathbf{e}_2 + \sigma_{zz} \mathbf{e}_3. \quad (3.45)$$

Substituting Equations 3.43–3.45 in Equation 3.42 and rearranging

$$\begin{aligned} \mathbf{t}_n = & (n_x \sigma_{xx} + n_y \sigma_{yx} + n_z \sigma_{zx}) \mathbf{e}_1 + (n_x \sigma_{xy} + n_y \sigma_{yy} + n_z \sigma_{zy}) \mathbf{e}_2 \\ & + (n_x \sigma_{xz} + n_y \sigma_{yz} + n_z \sigma_{zz}) \mathbf{e}_3 = (t_n)_x \mathbf{e}_1 + (t_n)_y \mathbf{e}_2 + (t_n)_z \mathbf{e}_3 \end{aligned} \quad (3.46)$$

In matrix form, Equation 3.46 is written as

$$\begin{Bmatrix} (t_n)_x \\ (t_n)_y \\ (t_n)_z \end{Bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} \quad (3.47)$$

or (considering Equation 3.8)

$$\{t_n\} = [\sigma]^T \{n\} \text{ or } \mathbf{t}_n = \sigma^T \mathbf{n}. \quad (3.48)$$

As $\{t_n\}$ and $\{n\}$ are vectors, quotient law suggests that $[\sigma]^T$ is a tensor and $[\sigma]$ being the transpose of the tensor is also a tensor. It is to be emphasized that Cauchy stress represents force intensity per unit area in deformed configuration.

- Figure 3.6 shows the forces acting on a parallelepiped. The moment about z -axis is given as

$$M_Z = (\sigma_{xy} - \sigma_{yx}) \Delta x \Delta y \Delta z \quad (3.49)$$

In equilibrium, this moment must vanish. Hence,

$$\sigma_{xy} = \sigma_{yx}. \quad (3.50)$$

Similarly, it can be shown that

$$\sigma_{xz} = \sigma_{zx} \text{ and } \sigma_{yz} = \sigma_{zy}. \quad (3.51)$$

Thus, the stress tensor is symmetric.

3. Substituting Equation 3.48 in Equation 3.16 and noting that the stress tensor is symmetric, one gets

$$\int_V \rho \mathbf{a} dV = \int_S \boldsymbol{\sigma} \mathbf{n} dS + \int_V \rho \mathbf{b} dV. \quad (3.52)$$

Applying the divergence theorem on the second term on the right-hand side,

$$\int_V \rho \mathbf{a} dV = \int_V \text{div}(\boldsymbol{\sigma}) dV + \int_V \rho \mathbf{b} dV, \quad (3.53)$$

or

$$\int_V (\text{div} \boldsymbol{\sigma} + \rho \mathbf{b} - \rho \mathbf{a}) dV = 0. \quad (3.54)$$

Note that the volume can be made as small as one wishes around a point, reducing the integral to function times volume. Hence,

$$\text{div} \boldsymbol{\sigma} + \rho \mathbf{b} - \rho \mathbf{a} = 0 \quad (3.55)$$

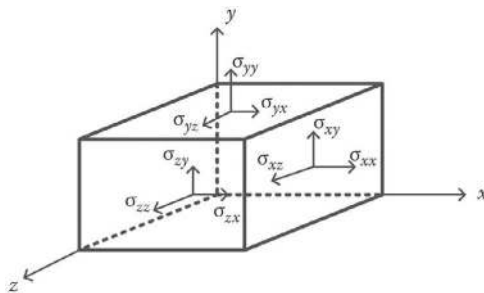


FIGURE 3.6 Stresses on a parallelepiped.

proving Equation 3.31. If the acceleration is zero, then

$$\operatorname{div} \sigma + \rho b = 0 \quad (3.56)$$

Equation 3.56 is called the equilibrium equation in vector form. It is to be noted that if the stress tensor is not symmetric, such as in the presence of body moments, the equilibrium equation will be expressed as

$$\operatorname{div} \sigma^T + \rho b = 0 \quad (3.57)$$

This amounts to three scalar equations. In orthogonal curvilinear coordinate system (q_1, q_2, q_3) , the scalar equations are expressed as (Boresi and Chong, 2000)

$$\begin{aligned} \frac{\partial}{\partial q_1} (h_2 h_3 \sigma_{q_1 q_1}) + \frac{\partial}{\partial q_2} (h_3 h_1 \sigma_{q_2 q_1}) + \frac{\partial}{\partial q_3} (h_1 h_2 \sigma_{q_3 q_1}) + h_3 \frac{\partial h_1}{\partial q_2} \sigma_{q_1 q_2} + h_2 \frac{\partial h_1}{\partial q_3} \sigma_{q_1 q_3} \\ - h_3 \frac{\partial h_2}{\partial q_1} \sigma_{q_2 q_2} - h_2 \frac{\partial h_3}{\partial q_1} \sigma_{q_3 q_3} + \rho h_1 h_2 h_3 b_{q_1} = 0 \end{aligned} \quad (3.58a)$$

$$\begin{aligned} \frac{\partial}{\partial q_1} (h_2 h_3 \sigma_{q_1 q_2}) + \frac{\partial}{\partial q_2} (h_3 h_1 \sigma_{q_2 q_2}) + \frac{\partial}{\partial q_3} (h_1 h_2 \sigma_{q_3 q_2}) + h_1 \frac{\partial h_2}{\partial q_3} \sigma_{q_2 q_3} + h_3 \frac{\partial h_2}{\partial q_1} \sigma_{q_1 q_2} \\ - h_1 \frac{\partial h_3}{\partial q_2} \sigma_{q_3 q_3} - h_3 \frac{\partial h_1}{\partial q_2} \sigma_{q_1 q_1} + \rho h_1 h_2 h_3 b_{q_2} = 0 \end{aligned} \quad (3.58b)$$

$$\begin{aligned} \frac{\partial}{\partial q_1} (h_2 h_3 \sigma_{q_1 q_3}) + \frac{\partial}{\partial q_2} (h_3 h_1 \sigma_{q_2 q_3}) + \frac{\partial}{\partial q_3} (h_1 h_2 \sigma_{q_3 q_3}) + h_2 \frac{\partial h_3}{\partial q_1} \sigma_{q_1 q_3} + h_1 \frac{\partial h_3}{\partial q_2} \sigma_{q_2 q_3} \\ - h_2 \frac{\partial h_1}{\partial q_3} \sigma_{q_1 q_1} - h_1 \frac{\partial h_2}{\partial q_3} \sigma_{q_2 q_2} + \rho h_1 h_2 h_3 b_{q_3} = 0 \end{aligned} \quad (3.58c)$$

where h_1 , h_2 and h_3 are scale factors as discussed in Chapter 2. For the Cartesian coordinate system

$$q_1 = x, q_2 = y, q_3 = z, h_1 = 1, h_2 = 1, h_3 = 1. \quad (3.59)$$

Substituting these in Equations 3.58a–3.58c,

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + \rho b_x = 0 \quad (3.60a)$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + \rho b_y = 0 \quad (3.60b)$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho b_z = 0. \quad (3.60c)$$

For the cylindrical coordinate system

$$q_1 = r, q_2 = \theta, q_3 = z, h_1 = 1, h_2 = r, h_3 = 1. \quad (3.61)$$

Substituting Equation 3.61 into Equation 3.58, the following equilibrium equations are obtained:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\partial \sigma_{zr}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \rho b_r = 0 \quad (3.62a)$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + \rho b_\theta = 0 \quad (3.62b)$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + \rho b_z = 0, \quad (3.62c)$$

where $(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{r\theta}, \sigma_{rz}, \sigma_{\theta z})$ represent the stress components defined relative to cylindrical coordinates (r, θ, z) .

For spherical coordinate systems,

$$q_1 = r, q_2 = \theta, q_3 = \phi, h_1 = 1, h_2 = r, h_3 = r \sin \theta. \quad (3.63)$$

Substituting Equation 3.56 into Equation 3.51, the following equilibrium equations are obtained:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{\phi r}}{\partial \phi} + \frac{1}{r} (2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \sigma_{r\theta} \cot \theta) + \rho b_r = 0 \quad (3.64a)$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{\phi\theta}}{\partial \phi} + \frac{1}{r} [(\sigma_{\theta\theta} - \sigma_{\phi\phi}) \cot \theta + 3\sigma_{r\theta}] + \rho b_\theta = 0 \quad (3.64b)$$

$$\frac{\partial \sigma_{r\phi}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{1}{r} (3\sigma_{r\phi} + 2\sigma_{\theta\phi} \cot \theta) + \rho b_\phi = 0 \quad (3.64c)$$

where $(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{\phi\phi}, \sigma_{r\theta}, \sigma_{\phi r}, \sigma_{\theta\phi})$ are defined relative to spherical coordinates (r, θ, ϕ) . One can convert equations of equilibrium to equations of motion by replacing body force per unit volume by $\rho(\mathbf{b} - \mathbf{a})$.

Example 3.3

Stress matrix at a point is given as

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \text{N/m}^2.$$

Find out the components of the traction vector on a plane whose normal is along the vector $(\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3)$.

SOLUTION

The direction cosines of the normal are given by

$$n_x = \frac{1}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{1}{\sqrt{14}}, n_y = \frac{2}{\sqrt{14}}, n_z = \frac{3}{\sqrt{14}}.$$

Using Cauchy's relation

$$\begin{Bmatrix} (t_n)_x \\ (t_n)_y \\ (t_n)_z \end{Bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{Bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{Bmatrix} = \begin{Bmatrix} 8/\sqrt{14} \\ 7/\sqrt{14} \\ 5/\sqrt{14} \end{Bmatrix} \text{ N/m}^2$$

Thus, the components are evaluated.

Example 3.4

Stress field matrix is given as

$$\begin{bmatrix} x^2 & x & 0 \\ x & y & 2z \\ 0 & 2z & z \end{bmatrix}.$$

Find out the body force field if the body is in equilibrium.

SOLUTION

The divergence of the stress field is given by

$$\left(\frac{\partial x^2}{\partial x} + \frac{\partial x}{\partial y} \right) \mathbf{e}_1 + \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial (2z)}{\partial z} \right) \mathbf{e}_2 + \left(\frac{\partial (2z)}{\partial y} + \frac{\partial z}{\partial z} \right) \mathbf{e}_3 = 2x\mathbf{e}_1 + 3\mathbf{e}_2 + \mathbf{e}_3$$

Using Equation 3.55 with $\mathbf{a} = 0$, one gets

$$\rho \mathbf{b} = -\text{div } \boldsymbol{\sigma} = -2x\mathbf{e}_1 - 3\mathbf{e}_2 - \mathbf{e}_3$$

which is the body force per unit volume field.

Example 3.5

Stress distribution in a long hollow cylinder subjected to axis-symmetric pressure is given by

$$\sigma_{rr} = A - \frac{B}{r^2}, \sigma_{\theta\theta} = A + \frac{B}{r^2}. \quad (3.65)$$

The previous equations are called Lamé's equations. Prove that this stress distribution satisfies the equilibrium equations.

SOLUTION

Substituting the expressions for radial and circumferential stresses in Equation 3.62a, one gets

$$\frac{2B}{r^3} + \frac{1}{r} \left(\frac{-B}{r^2} - \frac{B}{r^2} \right) = 0.$$

Similarly, it can be seen that Equations 3.62b and 3.62c are also satisfied.

3.7 TRANSFORMATION OF STRESS COMPONENTS

As the stress is a tensor, the transformation law for tensors can be applied to obtain the general formula for obtaining the components of a tensor in a coordinate system rotated with respect to the original system. Here, it is derived for the specific case of the stress. Cauchy's relation for the symmetric stress tensor is given as

$$\{t_n\} = [\sigma]\{n\}. \quad (3.66)$$

In the rotated (primed) system, the relation can be written as

$$\{t'_n\} = [\sigma']\{n'\}. \quad (3.67)$$

As $\{t_n\}$ and $\{n\}$ are vectors, they follow the transformation rule for vectors. Thus,

$$\{t'_n\} = [Q]\{t_n\}, \{n'\} = [Q]\{n\}. \quad (3.68)$$

Substituting Equation 3.68 in Equation 3.67,

$$[Q]\{t_n\} = [\sigma'][Q]\{n\}. \quad (3.69)$$

Using Equation 3.66,

$$[Q][\sigma]\{n\} = [\sigma'][Q]\{n\}. \quad (3.70)$$

This implies that

$$[\sigma'][Q] = [Q][\sigma]. \quad (3.71)$$

Post-multiplying both sides by the transpose of $[Q]$,

$$[\sigma'] [Q] [Q]^T = [Q] [\sigma] [Q]^T. \quad (3.72)$$

As $[Q]$ is an orthogonal matrix,

$$[Q] [Q]^T = [I], \quad (3.73)$$

where $[I]$ is an identity matrix with all diagonal terms equal to one and off-diagonal terms equal to zero. Thus, Equation 3.72 becomes

$$[\sigma'] = [Q] [\sigma] [Q]^T. \quad (3.74)$$

This is the transformation relation.

Example 3.6

Stress matrix at a point is given by

$$\begin{bmatrix} 13 & -15 & -17 \\ -15 & -15 & 20 \\ -17 & 20 & 22 \end{bmatrix} \text{ MPa}$$

in the coordinate system x - y - z . Find the stress matrix in the rotated coordinate system x' - y' - z' , where x' is along $3\mathbf{e}_1 + 4\mathbf{e}_3$, y' is along $\mathbf{e}_2$, and z' is along $-4\mathbf{e}_1 + 3\mathbf{e}_3$.

SOLUTION

The unit vectors along x' , y' and z' are obtained by dividing the given vectors by their magnitudes. They are $0.6\mathbf{e}_1 + 0.8\mathbf{e}_3$, $\mathbf{e}_2$ and $-0.8\mathbf{e}_1 + 0.6\mathbf{e}_3$, respectively. In other words,

$$\begin{Bmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{e}'_3 \end{Bmatrix} = \begin{bmatrix} 0.6 & 0 & 0.8 \\ 0 & 1 & 0 \\ -0.8 & 0 & 0.6 \end{bmatrix} \begin{Bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{Bmatrix}.$$

Hence,

$$[Q] = \begin{bmatrix} 0.6 & 0 & 0.8 \\ 0 & 1 & 0 \\ -0.8 & 0 & 0.6 \end{bmatrix}.$$

Now, using Equation 3.74, the stress matrix in the rotated coordinate is obtained as

$$\begin{bmatrix} 0.6 & 0 & 0.8 \\ 0 & 1 & 0 \\ -0.8 & 0 & 0.6 \end{bmatrix} \begin{bmatrix} 13 & -15 & -17 \\ -15 & -15 & 20 \\ -17 & 20 & 22 \end{bmatrix} \begin{bmatrix} 0.6 & 0 & -0.8 \\ 0 & 1 & 0 \\ 0.8 & 0 & 0.6 \end{bmatrix} = \begin{bmatrix} 2.44 & 7 & 9.08 \\ 7 & -15 & 24 \\ 9.08 & 24 & 32.56 \end{bmatrix} \text{ MPa}$$

3.8 STRESSES ON AN OBLIQUE PLANE

Consider a plane whose normal is denoted by $\{n\}$. The components of $\{n\}$ are called direction cosines and are denoted by n_i in index notations. In the expanded notations, they can be written as l, m and n , respectively. The traction vector on this plane is denoted by $\mathbf{t}_n$. Its dot product with $\mathbf{t}_n$ gives the normal stress σ_n on the oblique plane. Thus,

$$\sigma_n = \mathbf{t}_n \mathbf{n} = (t_n)_i n_i = \sigma_{ij} n_j n_i = \{n\}^T [\sigma] \{n\}. \quad (3.75)$$

If τ_n is the magnitude of shear stress on this plane, then

$$\tau_n^2 + \sigma_n^2 = |\mathbf{t}_n|^2. \quad (3.76)$$

The shear stress vector is given by

$$\tau_n = \mathbf{t}_n - \sigma_n (l\mathbf{e}_1 + m\mathbf{e}_2 + n\mathbf{e}_3). \quad (3.77)$$

Example 3.7

Stress matrix at a point is given by

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \text{ MPa.}$$

Find out the normal stress and shear stress along a plane whose normal is along $\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$.

Also, find the unit vector parallel to shear stress.

SOLUTION

The direction cosines of the normal are given as

$$l = \frac{1}{\sqrt{14}}, m = \frac{2}{\sqrt{14}}, n = \frac{3}{\sqrt{14}}$$

Using Equation 3.53 and employing matrix multiplications, the normal stress is given as

$$\sigma_n = \frac{1}{\sqrt{14}} \quad \frac{2}{\sqrt{14}} \quad \frac{3}{\sqrt{14}} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{Bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{Bmatrix} = 2.6429 \text{ MPa}$$

The traction vector on the plane is given by

$$[t_n] = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{Bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{Bmatrix} = \frac{1}{\sqrt{14}} \begin{Bmatrix} 8 \\ 7 \\ 5 \end{Bmatrix} \text{ MPa}$$

Using Equation 3.76,

$$\tau_n^2 = |t_n|^2 - \sigma_n^2 = \frac{1}{14} (8^2 + 7^2 + 5^2) - 2.6429^2 = 2.8722 (\text{MPa})^2$$

Hence, $\tau_n = 1.6948 \text{ MPa}$.

The shear stress vector is given by

$$\tau_n = t_n - \sigma_n (le_1 + me_2 + ne_3)$$

In matrix form,

$$\{\tau_n\} = \frac{1}{\sqrt{14}} \begin{Bmatrix} 8 \\ 7 \\ 5 \end{Bmatrix} - \frac{2.6429}{\sqrt{14}} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix} = \frac{1}{\sqrt{14}} \begin{Bmatrix} 5.3571 \\ 1.7142 \\ -2.9287 \end{Bmatrix}.$$

The unit vector along the shear stress is obtained by dividing the shear stress vector by its magnitude. Thus, the unit vector is

$$0.8448 e_1 + 0.2703 e_2 - 0.4618 e_3$$

Example 3.8

Stress matrix at a point is given by

$$\begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}.$$

Show that the normal stress on all planes is p .

SOLUTION

Consider a plane whose normal has direction cosines l , m and n . As per Equation 3.75, the normal stress is given as

$$\sigma_n = \begin{bmatrix} l & m & n \end{bmatrix} \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = p(l^2 + m^2 + n^2) = p.$$

Thus, it is seen that the normal stress is p and is independent of l , m and n .

3.9 PRINCIPAL STRESSES

An infinite number of planes can pass through a point. Assume that there is a plane on which the traction vector will be normal to the plane. In other words, there is no shear on that plane. If $\mathbf{n}$ is the normal on the plane and λ is the magnitude of the traction on that plane, then by Cauchy's relation,

$$\sigma \mathbf{n} = \lambda \mathbf{n} \text{ or } (\sigma - \lambda \mathbf{I}) \mathbf{n} = \mathbf{0}, \quad (3.78)$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{0}$ is the unit vector. This is clearly an eigenvalue problem with λ as the eigenvalue and $\mathbf{n}$ as the eigenvector. The eigenvalues and eigenvectors of σ are called principal stresses and principal directions, respectively. The three principal stresses are designated by σ_1 , σ_2 and σ_3 .

A necessary and sufficient condition for Equation 3.78 to have a non-trivial solution is

$$\det(\sigma - \lambda \mathbf{I}) = 0, \quad (3.79)$$

$$\begin{vmatrix} \sigma_{xx} - \lambda & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \lambda & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \lambda \end{vmatrix} = 0, \text{ or } \lambda^3 - I_\sigma \lambda^2 - II_\sigma \lambda - III_\sigma = 0, \quad (3.80)$$

where

$$I_\sigma = (\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = \sigma_{ii} \quad (3.81)$$

$$II_\sigma = -(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx}) + \sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2 = \frac{1}{2}(\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj}), \quad (3.82)$$

$$III_\sigma = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix} = \epsilon_{ijk} \sigma_{1i} \sigma_{2j} \sigma_{3k}. \quad (3.83)$$

It can be easily shown that the coefficients I_σ , II_σ and III_σ are invariant under the rotation of the coordinate system. Hence, no matter in what Cartesian coordinate system the stress is expressed at a point, the same cubic equation (Equation 3.80) is obtained. The coefficients I_σ , II_σ and III_σ are called the principal invariants of the stress σ .

It is known that the eigenvalues of a symmetric matrix are real. As the stress tensor is symmetric (in the absence of body moment), three real roots of Equation 3.80 can be obtained. These roots need not be distinct. If the roots are distinct, the corresponding eigenvectors (principal directions) will be orthogonal to one another. If the roots are not distinct, there will be more than one direction associated with the repeated root, and it is possible to choose three directions that will be perpendicular to one another.

We now show that one of the principal stresses is the maximum normal stress, and another one is the minimum normal stress. This can be done by solving the following optimization problem:

$$\text{Extremize } \sigma_n = \sigma_{ij}n_i n_j, \quad (3.84)$$

subject to

$$n_i n_i = 1. \quad (3.85)$$

As σ_n is the continuous function of n_i , the necessary condition for getting its extremum value is given by

$$\frac{\partial}{\partial n_k} [\sigma_{ij}n_i n_j - \lambda(n_i n_i - 1)] = 0, \quad (3.86)$$

where λ is the Lagrange multiplier. Equation 3.86 gives

$$\sigma_{ij}n_i \delta_{jk} + \sigma_{ij}n_j \delta_{ik} - 2\lambda n_i \delta_{ik} = 0. \quad (3.87)$$

Using the substitution property of Kronecker delta, Equation 3.87 is written as

$$\sigma_{ik}n_i + \sigma_{kj}n_j - 2\lambda n_k = 0. \quad (3.88)$$

In Equation 3.88, in the second expression, j is the dummy index. Replacing it by i and noting that the stress tensor is symmetric, one gets

$$2\sigma_{ik}n_i - 2\lambda n_k = 0 \text{ or } \sigma_{ki}n_i - \lambda n_k = 0. \quad (3.89)$$

Equation 3.89 is written in matrix form:

$$\sigma \mathbf{n} = \lambda \mathbf{n}, \quad (3.90)$$

which is the same as Equation 3.78. Hence, the maximum and minimum principal stresses are obtained on the planes whose normal is along principal directions.

Example 3.9

Stress matrix at a point is given by

$$\begin{bmatrix} 20 & -10 & 30 \\ -10 & 40 & 0 \\ 30 & 0 & -10 \end{bmatrix} \text{MPa.}$$

Find out the principal stresses and principal directions. Write down the stress matrix with reference to the principal direction. Show that the three principal invariants of the stress are the same for two stress matrices.

SOLUTION

Principal stresses are given by the eigenvalues of the stress matrix, and principal directions are the eigenvectors. The eigenvalues and eigenvectors can be determined by the method used in Chapter 2. However, here they are calculated using MATLAB®. The function EIG is used. Defining the given matrix as A,

$$\begin{aligned} [V, D] &= \text{eig}(A) \\ V &= \begin{bmatrix} -0.5324 & 0.6024 & -0.5947 \\ -0.0772 & 0.6650 & 0.7428 \\ 0.8430 & 0.4414 & -0.3076 \end{bmatrix} \\ D &= \begin{bmatrix} -28.9482 & 0 & 0 \\ 0 & 30.9421 & 0 \\ 0 & 0 & 48.0061 \end{bmatrix} \end{aligned}$$

Here, D is the diagonal matrix corresponding to principal stresses and column vectors of V provide corresponding eigenvectors. Thus, the principal stresses are -28.9482 , 30.9421 and 48.0061 , and the corresponding principal directions are

$$\begin{aligned} &-0.5324\mathbf{e}_1 - 0.0772\mathbf{e}_2 - 0.8430\mathbf{e}_3, 0.6024\mathbf{e}_1 + 0.6650\mathbf{e}_2 + 0.4414\mathbf{e}_3 \\ &\text{and } -0.5947\mathbf{e}_1 + 0.7428\mathbf{e}_2 - 0.3076\mathbf{e}_3 \end{aligned}$$

The stress matrix with reference to principal directions is

$$\begin{bmatrix} -28.9482 & 0 & 0 \\ 0 & 30.9421 & 0 \\ 0 & 0 & 48.0061 \end{bmatrix}.$$

This matrix is designated as **B**.

Now, the traces of **A** and **B** are given as

$$\text{tr}(\mathbf{A}) = 20 + 40 - 10 = 50, \text{tr}(\mathbf{B}) = -28.9482 + 30.9421 + 48.0061 = 50.$$

Thus, the first invariant $I_\sigma = 50$ is the same for **A** and **B**.

The second invariant for the matrix **B** is

$$\begin{aligned} II_\sigma &= -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \\ &= -(-28.9482 \times 30.9421 + 30.9421 \times 48.0061 - 28.9482 \times 48.0061) \\ &= 799.9987 \end{aligned}$$

For calculating the second invariant of matrix **A**, it can be written in the following convenient form:

$$II_\sigma = - \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} - \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} - \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix}.$$

Thus,

$$II_\sigma = - \begin{vmatrix} 20 & -10 \\ -10 & 40 \end{vmatrix} - \begin{vmatrix} 40 & 0 \\ 0 & -10 \end{vmatrix} - \begin{vmatrix} 20 & 30 \\ 30 & -10 \end{vmatrix} = -700 + 400 + 1100 = 800$$

There is a slight difference between the two invariants due to round-off errors.

The third invariant for matrix **B** is equal to the determinant of the matrix is the product of the principal stresses, which comes out to be $-42,999.93$. For matrix **A**, the DET function of MATLAB is used. The value of $\det(\mathbf{A})$ is obtained as $-43,000$. Third invariants differ slightly due to round-off errors.

3.10 MAXIMUM SHEAR STRESS

Suppose that the principal directions are chosen as coordinate axes. If σ_1 , σ_2 and σ_3 are the principal stresses at a point P and l , m and n are the direction cosines of normal to a plane passing through P, then Cauchy's relation provides the components of traction on that plane as

$$\begin{Bmatrix} (t_n)_x \\ (t_n)_y \\ (t_n)_z \end{Bmatrix} = \begin{Bmatrix} \sigma_1 l \\ \sigma_2 m \\ \sigma_3 n \end{Bmatrix}. \quad (3.91)$$

Direct (normal) stress on the plane is given by

$$\begin{aligned}\sigma_n &= \mathbf{t}_n \mathbf{n} = (\sigma_1 l \mathbf{e}_1 + \sigma_2 m \mathbf{e}_2 + \sigma_3 n \mathbf{e}_3) (l \mathbf{e}_1 + m \mathbf{e}_2 + n \mathbf{e}_3) \\ &= (\sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2)\end{aligned}\quad (3.92)$$

If the shear stress on that plane is τ , then

$$\tau^2 = |\mathbf{t}_n|^2 - \sigma_n^2 = \sigma_1^2 l^2 + \sigma_2^2 m^2 + \sigma_3^2 n^2 - (\sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2)^2. \quad (3.93)$$

The necessary conditions for maximizing or minimizing the magnitude of shear stress are given by

$$\frac{\partial \{ \tau^2 - \lambda (n_i n_i - 1) \}}{\partial n_k} = 0 \quad (3.94)$$

and

$$\frac{\partial \{ \tau^2 - \lambda (n_i n_i - 1) \}}{\partial \lambda} = 0, \quad (3.95)$$

where λ is a Lagrange multiplier. This leads to the following four equations:

$$\{ (\sigma_1 - 2\sigma_n) \sigma_1 - \lambda \} l = 0, \quad (3.96)$$

$$\{ (\sigma_2 - 2\sigma_n) \sigma_2 - \lambda \} m = 0, \quad (3.97)$$

$$\{ (\sigma_3 - 2\sigma_n) \sigma_3 - \lambda \} n = 0, \quad (3.98)$$

$$l^2 + m^2 + n^2 = 1. \quad (3.99)$$

These equations are satisfied by the following direction cosines:

1. $l = \pm 1, m = 0, n = 0$
2. $l = 0, m = \pm 1, n = 0$
3. $l = 0, m = 0, n = \pm 1$
4. $l = m = \pm 1/\sqrt{2}, n = 0$
5. $l = n = \pm 1/\sqrt{2}, m = 0$
6. $m = n = \pm 1/\sqrt{2}, l = 0$

Solutions (1), (2) and (3) provide $\tau^2 = 0$. Solutions (4), (5) and (6) provide the following values of τ^2 , respectively,

$$\tau^2 = \frac{1}{4}(\sigma_1 - \sigma_2)^2, \tau^2 = \frac{1}{4}(\sigma_1 - \sigma_3)^2, \tau^2 = \frac{1}{4}(\sigma_2 - \sigma_3)^2. \quad (3.100)$$

It is clear that the minimum value of τ^2 is 0. Thus, solutions (1), (2) and (3) provide global minima. It can be verified by showing the negative definiteness of the Hessian matrix that solutions (4), (5) and (6) provide the local maxima. Out of these, one corresponding to the difference of maximum and minimum principal stresses will provide the maximum shear stress. Thus,

$$\tau_{\max} = \frac{1}{2}(\sigma_{\max} - \sigma_{\min}) \quad (3.101)$$

and the plane of the maximum shear stress will bisect the maximum and minimum principal planes.

Example 3.10

The state of stress in a thin cylinder is given as

$$\sigma_{rr} = 0, \quad \sigma_{\theta\theta} = \frac{pr}{t}, \quad \sigma_{zz} = \frac{pr}{2t},$$

where p is the pressure, r is the radius of the cylinder, and t is the thickness. Find the maximum shear stress in the cylinder.

SOLUTION

The maximum shear stress is equal to half the difference between maximum and minimum shear stress. Thus,

$$\tau_{\max} = \frac{1}{2} \left(\frac{pr}{t} - 0 \right) = \frac{pr}{2t}.$$

3.11 OCTAHEDRAL STRESSES

A plane equally inclined to three orthogonal principal planes is called the octahedral plane. There are eight octahedral planes corresponding to $l = \pm 1/\sqrt{3}$, $m = \pm 1/\sqrt{3}$ and $n = \pm 1/\sqrt{3}$. From Equation 3.92, the direct stress acting on an octahedral plane is given by

$$\sigma_{\text{oct}} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}, \quad (3.102)$$

which is the mean of the principal stresses. As $(\sigma_1 + \sigma_2 + \sigma_3)$ is the first invariant and is equal to $(\sigma_x + \sigma_y + \sigma_z)$, the normal stress on the octahedral plane can be considered as the average of the trace of the stress tensor.

By Equation 3.93, the shear stress on an octahedral plane is found as

$$\tau_{\text{oct}} = \sqrt{\frac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{1}{9}(\sigma_1 + \sigma_2 + \sigma_3)^2}. \quad (3.103)$$

On simplification, one gets

$$\tau_{\text{oct}} = \frac{1}{3} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}. \quad (3.104)$$

Example 3.11

What is the value of the octahedral shear stress in the uni-axial tension test when the yielding impends?

SOLUTION

When yielding is about to start,

$$\sigma_1 = \sigma_y, \sigma_2 = 0, \sigma_3 = 0.$$

Using Equation 3.104,

$$\tau_{\text{oct}} = \frac{\sqrt{2}}{3} \sigma_y.$$

3.12 HYDROSTATIC AND DEVIATORIC STRESSES

A stress tensor can be decomposed as

$$\sigma_{ij} = \left(\frac{1}{3} \sigma_{kk} \right) \delta_{ij} + \sigma'_{ij}, \quad (3.105)$$

where σ' is the deviatoric part of the stress tensor. Putting $j = i$ in Equation 3.105,

$$\sigma_{ii} = \left(\frac{1}{3} \sigma_{kk} \right) \delta_{ii} + \sigma'_{ii}. \quad (3.106)$$

As $\delta_{ii} = 3$, one gets

$$\sigma_{ii} = \sigma_{kk} + \sigma'_{ii}. \quad (3.107)$$

As k is a dummy index, it can be replaced by i . Thus, one gets

$$\sigma'_{ii} = 0. \quad (3.108)$$

Thus, the trace of the deviatoric part of the stress tensor is zero. The other part of the stress tensor is called the hydrostatic part. In many books of metal forming, the negative of the mean stress is called pressure p . Thus,

$$p = -\frac{1}{3}\sigma_{kk}. \quad (3.109)$$

Then the hydrostatic part becomes $(-p)\delta_{ij}$. It can be easily shown that the hydrostatic part provides normal stress $(-p)$ on all planes. The shear stress on all the planes is zero due to the hydrostatic part.

Example 3.12

The stress matrix at a point is given as

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \text{N/m}^2.$$

Find out the components of the hydrostatic part of the stress tensor.

SOLUTION

The stress matrix for the hydrostatic part will have only diagonal components, each equal to one-third of the trace of the stress tensor. Thus,

$$\text{hydrostatic part} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{N/m}^2.$$

The deviatoric part is given as the difference between the stress matrix and its hydrostatics part. Thus,

$$\text{deviatoric part} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \text{N/m}^2.$$

3.13 MOHR'S CIRCLE

A graphical method of analyzing the state of stress at a point was published by Otto Mohr in 1882 (Mohr, 1882). By this method, one can easily calculate the normal and shear stress on any plane passing through the point. A plane is designated by its normal. The normal can make any angle between 0° and 360° from a reference line. However,

$$\sigma_n = \mathbf{t}_n \mathbf{n} = (-\mathbf{t}_n)(-\mathbf{n}) = (\mathbf{t}_{(-n)})(-\mathbf{n}) = \sigma_{(-n)}. \quad (3.110)$$

Thus, the normal stresses on a plane and on the plane making an angle of 180° to it are equal. Of course, the plane making an angle of 180° is just the other side of the plane. The same is true for shear stress. Therefore, normal and shear stresses on all planes with normal from 0° to 180° need to be known. In the following subsections, Mohr's circle for two- and three-dimensional cases is described. The treatment is different from Mohr's original paper.

3.13.1 TWO-DIMENSIONAL CASE

In the plane stress case, it is possible to describe the state of stress by the following matrix:

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}. \quad (3.111)$$

Denote two principal stresses by σ_1 and σ_2 , with the third principal stress being 0. Let us take the principal direction as reference axes. A plane making an angle θ with the principal plane corresponding to σ_1 has the following direction cosines:

$$l = \cos \theta, m = \sin \theta, n = 0. \quad (3.112)$$

Using Cauchy's relation,

$$(t_n)_x = \sigma_1 \cos \theta, (t_n)_y = \sigma_2 \sin \theta. \quad (3.113)$$

From Equation 3.76,

$$\tau_n^2 + \sigma_n^2 = |t_n|^2 = \sigma_1^2 \cos^2 \theta + \sigma_2^2 \sin^2 \theta. \quad (3.114)$$

Now,

$$\begin{aligned} \sigma_1^2 \cos^2 \theta + \sigma_2^2 \sin^2 \theta &= 2 \left(\sigma_1 \cos^2 \theta + \sigma_2 \sin^2 \theta \right) \left(\frac{\sigma_1 + \sigma_2}{2} \right) - \sigma_1 \sigma_2 \\ &= 2 \sigma_n \left(\frac{\sigma_1 + \sigma_2}{2} \right) - \sigma_1 \sigma_2. \end{aligned} \quad (3.115)$$

Thus, Equation 3.114 can be written as

$$\tau_n^2 + \sigma_n^2 = 2 \sigma_n \left(\frac{\sigma_1 + \sigma_2}{2} \right) - \sigma_1 \sigma_2. \quad (3.116)$$

If σ_n and τ_n are chosen as the coordinate axis, Equation 3.116 represents a circle. This circle is called Mohr's circle. Any point on the circle will represent direct and shear stress on a plane. With some algebraic manipulation, Equation 3.116 can be written as

$$\left(\sigma_n - \frac{\sigma_1 + \sigma_2}{2} \right)^2 + \tau_n^2 = \left(\frac{\sigma_1 - \sigma_2}{2} \right)^2. \quad (3.117)$$

Thus, the center of the Mohr circle is at $(\sigma_1 + \sigma_2)/2$ and its radius is equal to $(\sigma_1 - \sigma_2)/2$, that is, the maximum shear stress in $\sigma_1 - \sigma_2$ plane. We hasten to state that $(\sigma_1 - \sigma_2)/2$ will be the overall maximum (magnitude) shear stress only if σ_1 and σ_2 are of opposite sign, that is, one of them is the maximum principal stress, and the other is the minimum principal stress. Figure 3.7 shows a typical Mohr circle constructed by taking $(\sigma_1 + \sigma_2)/2$ as the center and $(\sigma_1 - \sigma_2)/2$ as the radius. The circle cuts the σ_n axis at two points representing the principal stresses.

The normal stress at a plane making an angle θ with the principal plane corresponding to σ_1 is given by

$$\sigma_n = \sigma_1 \cos^2 \theta + \sigma_2 \sin^2 \theta = \frac{(\sigma_1 + \sigma_2)}{2} + \frac{(\sigma_1 - \sigma_2)}{2} \cos 2\theta. \quad (3.118)$$

The shear stress component is obtained by resolving the traction components along the plane (see Figure 3.8). Thus,

$$\tau_n = -\sigma_1 \cos \theta \sin \theta + \sigma_2 \cos \theta \sin \theta = -\frac{(\sigma_1 - \sigma_2)}{2} \sin 2\theta. \quad (3.119)$$

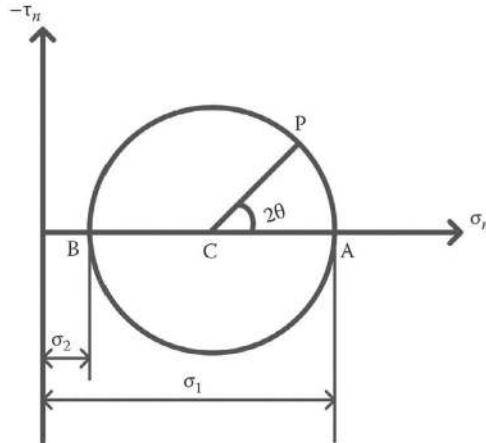


FIGURE 3.7 Two-Dimensional Mohr circle.

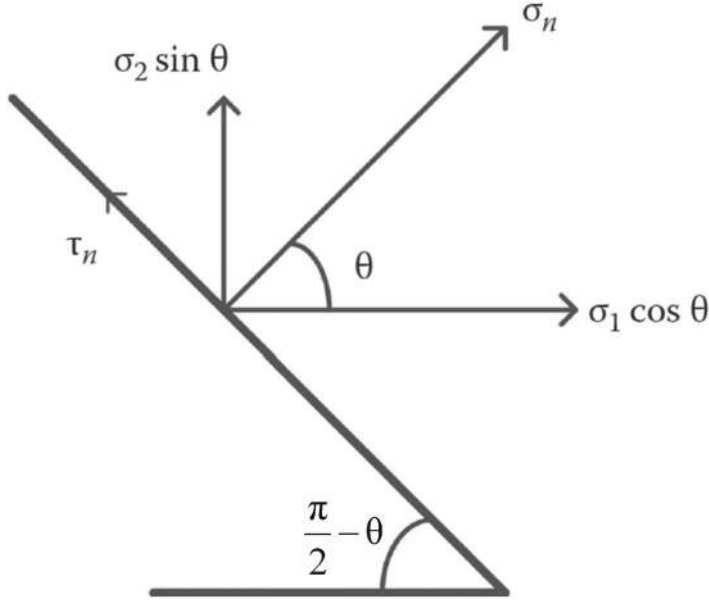


FIGURE 3.8 Resolution of traction components into normal and shear stress.

Now, if a point P at a radial location of 2θ (see Figure 3.7) is taken, its coordinates are clearly $(\sigma_n, -\tau_n)$. Thus, it is possible to find the direct and shear stress on any plane if the principal stress and principal directions are known.

If the principal stresses are not known but the state of stress given by Equation 3.111 is known, then Mohr's circle can be constructed. For this purpose, two end-points of a diameter are considered as $(\sigma_{xx}, -\sigma_{xy})$ and $(\sigma_{yy}, \sigma_{xy})$. These two points are 180° apart on the Mohr circle but represent the stresses on the planes, which are 90° apart. Now, Mohr's circle can be easily constructed, and principal stresses and principal directions can be interpreted from Mohr's circle. It is clear that the center point cuts the σ_n axis at $(\sigma_{xx} + \sigma_{yy})/2$, which is equal to $(\sigma_1 + \sigma_2)/2$.

Resolution of traction components into normal and shear stress.

Let us consider the situation in which the third principal stress is not zero but is along the z -direction. In that case,

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}. \quad (3.120)$$

Consider the plane making an angle θ with the principal plane corresponding to σ_1 (and perpendicular to z -plane) whose direction cosines are given by Equation 3.112. It can easily be verified that for this case, the expressions for σ_n and τ_n are the same as

in the plane stress case. Thus, Figure 3.7 Mohr's circle can also be used for the case in which the third principal plane is the z -plane.

Example 3.13

With respect to some coordinate system, the state of plane stress is given as

$$\begin{bmatrix} 6 & 3 \\ 3 & 14 \end{bmatrix} \text{MPa.}$$

Using the concept of Mohr's circle, find the maximum shear stress in the plane, principal stresses and the overall maximum shear stress.

SOLUTION

Two endpoints of a diameter of Mohr's circle are $(6, -3)$ and $(14, 3)$. The length of the diameter is

$$\sqrt{(6-14)^2 + (-3-3)^2} = 10 \text{ MPa.}$$

The maximum shear stress in the plane is equal to the radius of Mohr's circle. Hence, the maximum shear stress in the plane is 5 MPa.

The coordinates of the center of Mohr's circle are

$$\left(\frac{6+14}{2}, \frac{-3+3}{2} \right) = (10, 0).$$

Hence, the mean of the direct stresses is 10 MPa. The principal stresses in the plane are obtained by subtracting and adding maximum shear stress in the plane to this value. Thus, the principal stresses are 5 and 15 MPa. The third principal stress is 0, as it is in the plane state of stress. Overall, the maximum principal stress is $(15 - 0)/2 = 7.5$ MPa.

3.13.2 THREE-DIMENSIONAL CASE

Consider that at a point in a material, the principal stresses are σ_1 , σ_2 and σ_3 . A circle is drawn with $(\sigma_1, 0)$ and $(\sigma_2, 0)$ as the two endpoints of a diameter. Figure 3.9 shows such a circle. Any point on this circle represents the normal and shear stress on a plane, which is perpendicular to the principal direction corresponding to σ_3 . Similarly, a circle is drawn with $(\sigma_2, 0)$ and $(\sigma_3, 0)$ as the two endpoints of a diameter, and any point on this circle will represent the normal and shear stress on a plane, which is perpendicular to the principal direction corresponding to σ_1 . Finally, a circle is drawn with $(\sigma_1, 0)$ and $(\sigma_3, 0)$ as the two endpoints of a diameter, and any point on this circle will represent the normal and shear stress on a plane, which is perpendicular to the principal direction corresponding to σ_2 . These three circles represent

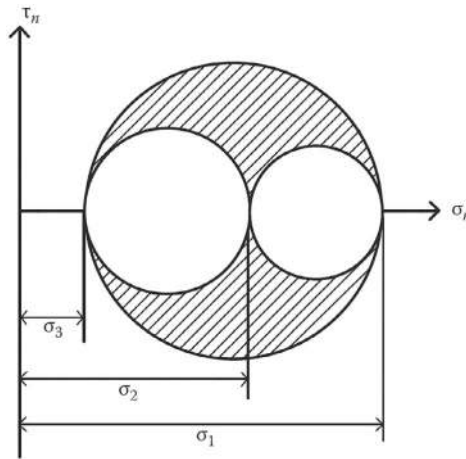


FIGURE 3.9 Mohr's three-dimensional representation of state of stress.

Mohr's representation of direct and shear stresses for a three-dimensional case. If C_1 , C_2 and C_3 represent sets of all points inside three circles, respectively, then the normal and shear stress at any plane will be presented by one of the points in C_1 , C_2 , C_3 (hatched area in Figure 3.9). For proof of this, one can refer to Chakrabarty (2006).

Exercises

1. In a uni-axial tensile test, the longitudinal stress is 100 MPa. On which planes will the direct stress be 50 MPa?
2. State of stress at a point is given as

$$\begin{bmatrix} 12 & -15 & -16 \\ -15 & -15 & 20 \\ -16 & 20 & 22 \end{bmatrix} \text{MPa.}$$

Find out the traction vector on a plane whose normal has direction cosines as

$$l = \frac{1}{\sqrt{2}}, \quad m = \frac{1}{\sqrt{2}}, \quad n = 0.$$

3. State of stress at a point for the x - y - z reference system is given as

$$\begin{bmatrix} 24 & 16 & 0 \\ 16 & 48 & 0 \\ 0 & 0 & 10 \end{bmatrix} \text{MPa.}$$

Find out the direction cosines of normal to a plane, on which the traction vector is along the positive x direction.

4. Find out the principal stresses and principal directions, when the state of stress at a point is expressed as

$$\begin{bmatrix} 24 & 16 & 0 \\ 16 & 48 & 0 \\ 0 & 0 & 10 \end{bmatrix} \text{MPa.}$$

5. Decompose the following stress matrix into hydrostatic and deviatoric stresses:

$$\begin{bmatrix} 24 & 16 & 0 \\ 16 & 48 & 0 \\ 0 & 0 & 10 \end{bmatrix} \text{MPa.}$$

6. In cylindrical polar coordinates, two stress components are given as

$$\sigma_{rr} = A + B \frac{b^2}{r^2} - \frac{(3 + \nu)}{8} \rho \omega^2 r^2$$

$$\sigma_{\theta\theta} = A - B \frac{b^2}{r^2} - \frac{(1 + 3\nu)}{8} \rho \omega^2 r^2$$

Other stress components are zero. Find the body force per unit volume.

7. A thick spherical shell subjected to uniform internal pressure has σ_{rr} , $\sigma_{\theta\theta}$ and $\sigma_{\phi\phi}$ as non-zero stress components, with other stress components being zero. Due to spherical symmetry, $\sigma_{\theta\theta} = \sigma_{\phi\phi}$. If

$$\sigma_{rr} = A + \frac{B}{r^3},$$

find out $\sigma_{\theta\theta}$.

8. In pure bending of curved bars, non-zero components of stresses are given in cylindrical polar coordinates as

$$\sigma_{rr} = -A \left(\frac{b^2}{r^2} - 1 \right) + B \ln \left(\frac{b}{r} \right)$$

$$\sigma_{\theta\theta} = A \left(\frac{b^2}{r^2} + 1 \right) - B \left\{ 1 - \ln \left(\frac{b}{r} \right) \right\}.$$

Other stress components are zero. Show that equilibrium equations are satisfied.

9. Components of the stress tensor σ with respect to the x, y, z coordinate system are given as

$$\begin{bmatrix} 2 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 4 \end{bmatrix} \text{ MPa.}$$

Find the components σ with respect to the rotated coordinate system x', y', z' . The unit vectors (e'_1, e'_2, e'_3) along the x', y', z' axes are given as

$$\begin{aligned} e'_1 &= 0.6e_1 + 0.8e_3 \\ e'_2 &= e_2 \\ e'_3 &= -0.8e_1 + 0.6e_3. \end{aligned}$$

10. Components of the stress tensor σ with respect to the x, y, z coordinate system are given as

$$\begin{bmatrix} 24 & 16 & 0 \\ 16 & 48 & 0 \\ 0 & 0 & 10 \end{bmatrix} \text{ MPa.}$$

Find the components σ with respect to the rotated coordinate system x', y', z' . The unit vectors (e'_1, e'_2, e'_3) along the x', y', z' axes are given as

$$\begin{aligned} e'_1 &= e_3 \\ e'_2 &= 0.8944 e_1 + 0.4472 e_2 \\ e'_3 &= 0.4472 e_1 + 0.8944 e_3. \end{aligned}$$

11. Principal stresses at a point are 9, 16 and 56 MPa. Find the normal and shear stresses on the octahedral plane.
12. The state of stress at a point is given by

$$\begin{aligned} \sigma_{xx} &= 10(1-x)^2 y, \sigma_{yy} = -\frac{10}{3}(12y - y^3 + 10), \\ \sigma_{xy} &= 10(1-x)(4-y^2), \sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0 \end{aligned}$$

Verify that the stress expressions satisfy the equations of equilibrium.

13. Principal stresses at a point are given as

$$\sigma_1 = 8 \text{ MPa}, \sigma_2 = 4 \text{ MPa}, \sigma_3 = -12 \text{ MPa.}$$

Find out the three principal invariants of the stress tensor.

14. Prove that the magnitude of shear stress on the octahedral plane is given by

$$|\tau_{\text{oct}}| = \left\{ \frac{2}{9} (I_{\sigma}^2 + 3II_{\sigma}^2) \right\}^{1/2}.$$

15. Stress field on a body in motion is given as

$$\begin{bmatrix} x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ MPa.}$$

There are no body forces. Find the acceleration of the body.

16. The vertices of a triangle are (3, 0, 0), (0, 4, 0) and (0, 0, 5) mm. If a uniformly distributed load of 1000 N is applied normally to the surface of the triangle, find the magnitude of the normal stress on the triangle.
17. The state of stress at a point is given as

$$\begin{bmatrix} 3 & 4 & 5 \\ 4 & 6 & 0 \\ 5 & 0 & 8 \end{bmatrix} \text{ MPa.}$$

At this point, find the traction vector on a plane that passes through (3, 0, 0), (0, 4, 0) and (0, 0, 12).

18. Direction cosines of two lines are given as $1/\sqrt{3}$, $1/\sqrt{3}$, $1/\sqrt{3}$ and 1, 0, 0. Find the direction cosines of planes that contain these lines. If the state of stress at a point is

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 10 & 3 \\ 0 & 3 & 20 \end{bmatrix} \text{ MPa,}$$

find the traction vectors on the planes containing the two lines.

19. The state of stress at a point is given as

$$\begin{bmatrix} 50 & 40 \\ 40 & -10 \end{bmatrix} \text{ MPa.}$$

Find out the equation of Mohr's circle in the form

$$(x-a)^2 + (y-b)^2 = r^2.$$

Find out the maximum shear stress and principal stresses.

20. Prove that for traction vector $\{t_x \ t_y \ t_z\}^T$ on any plane, the following relation holds good:

$$\frac{t_x^2}{\sigma_1^2} + \frac{t_y^2}{\sigma_2^2} + \frac{t_z^2}{\sigma_3^2} = 1,$$

where σ_1 , σ_2 and σ_3 are the principal stresses.

ANSWERS

1. Planes whose normal has the direction cosines l , m and n , where

$$l = \pm \frac{1}{\sqrt{2}}, m^2 + n^2 = \frac{1}{2}$$

$$2. \begin{Bmatrix} -3/\sqrt{2} \\ -30/\sqrt{2} \\ 4/\sqrt{2} \end{Bmatrix} \text{ MPa}$$

$$3. 0.9480, -0.3160, 0$$

$$4. \text{ Principal stresses: } 10 \text{ MPa, } 16 \text{ MPa, } 56 \text{ MPa. Directions } e_3, -0.8944e_1 + 0.4472e_2, 0.4472e_1 + 0.8944e_2$$

5. Hydrostatic part:

$$\begin{bmatrix} 30 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 30 \end{bmatrix} \text{ MPa}$$

Deviatoric part:

$$\begin{bmatrix} -6 & 16 & 0 \\ 16 & 18 & 0 \\ 0 & 0 & 12 \end{bmatrix} \text{ MPa}$$

$$6. \rho\omega^2 r$$

$$7. \sigma_{\theta\theta} = A - \frac{B}{2r^3}$$

$$9. \begin{bmatrix} 5.2 & 2.6 & 0.4 \\ 2.6 & 2 & -1.8 \\ 0.4 & -1.8 & 0.8 \end{bmatrix} \text{MPa}$$

$$10. \begin{bmatrix} 10 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 56 \end{bmatrix} \text{MPa}$$

$$11. 27 \text{ MPa}, 20.7 \text{ MPa}$$

$$13. 0, 112 \text{ MPa}, -144 \text{ MPa}$$

$$15. 1 \text{ m/s}^2$$

$$16. 72.12 \text{ MPa}$$

$$17. \begin{Bmatrix} 5.687 \\ 6.668 \\ 5.493 \end{Bmatrix} \text{MPa}$$

$$18. \begin{Bmatrix} 0 \\ 7/\sqrt{2} \\ -17/\sqrt{2} \end{Bmatrix} \text{MPa}$$

$$19. (\sigma_n - 20)^2 + \tau_n^2 = 2500, \text{ maximum shear stress} = 50 \text{ MPa, principal stresses} = 70 \text{ MPa}, -30 \text{ MPa}$$

4 Measures of Deformation and Rate of Deformation

4.1 INTRODUCTION

Strain is a relative measure of deformation. A relative measure of deformation is needed to estimate the stresses and the state of material. For example, a 1 mm change in the length of a 2 mm long steel rod is a large deformation and will cause permanent deformation, but a 1 mm change in the length of a 2 m long rod is a small deformation and is likely to keep the rod in an elastic state. The relative deformation can be expressed in a number of ways. Accordingly, there are different strain measures. If the deformation is very small, the longitudinal engineering strain can be expressed as the change in length divided by the original length. This definition would produce inconsistent results for a large deformation case. For example, if a rod is compressed to reduce its length by 50%, its longitudinal engineering strain will be -0.5 . If the same rod is stretched to bring it to its original state, its strain will be 1. The sum of these strains is not 0, which is inconsistent with physical understanding.

In this chapter, different strain measures are discussed ranging from small deformation to large deformation cases. In the plastic deformation, large deformation is encountered, and the stress depends on the history of deformation. Therefore, it is convenient to express deformation in the incremental or rate form.

4.2 DEFORMATION

When the forces are applied on a body, all or some particles of the body undergo displacement. It is possible that in spite of the displacement of particles, the distance between any two particles on the body remains unchanged. In that case, the angle between any pair of lines will also remain unchanged. This is called rigid body displacement and may consist of translation and rotation. In general, the distance between two particles can change, and the angle between any pair of lines may also change. This is called deformation. It is possible to have a deformation of the body in which only the distance between particles changes, but the angle between any pair of lines remains unchanged. For example, when a sphere is subjected to a hydrostatic state of compressive stress, it shrinks in size, but its shape remains the same. Consequently, the angle between any pair of lines remains unchanged. It is, however, impossible to have deformation in which the angle between any pairs of lines can change without causing the change of distance between any two particles.

In order to consider deformation at a point, it is observed if the points in the neighborhood of a point are changing their distances from that point. The neighborhood

of a point is defined as a set of points in the close vicinity of that point. A general displacement consists of the following three parts:

1. The point and its neighborhood may undergo translation. In that case, the neighborhood points do not move relative to the point.
2. There can be the rotation of the neighborhood of the point. In that case, neighborhood points undergo relative displacement, but it is not deformation as the distances between the points remain the same.
3. There can be displacement due to deformation of the neighborhood of the point. Here, there is a relative displacement. It is of interest to study deformation because the alteration due to changes of shape and size may affect the functionality of the body.

Figure 4.1 shows the undeformed and deformed configuration of a body. Due to deformation, point P_0 goes to P . Take two points Q_0 and S_0 very near to point P_0 , that is, in the neighborhood of P_0 , such that lines P_0Q_0 and P_0S_0 are orthogonal to each other. As a result of deformation, line P_0Q_0 changes to PQ and P_0S_0 changes to PS . The lengths of PQ and PS may be different from P_0Q_0 and P_0S_0 , and PQ and PS need not be orthogonal. In order to quantify deformation at a point, one needs to develop the following measures:

- A measure of change in linear dimension in every direction of that point
- A measure of change in angular dimension for every pair of directions at that point

These measures are called the strains, which will be discussed in subsequent sections.

It is clear that in the Cartesian system if the three components of displacement for each and every particle are known, one should be able to extract the information regarding deformation. Let us consider the point P_0 whose position vector in the initial configuration is given by

$$\mathbf{x}_0 = x_0\mathbf{e}_1 + y_0\mathbf{e}_2 + z_0\mathbf{e}_3 \quad (4.1)$$

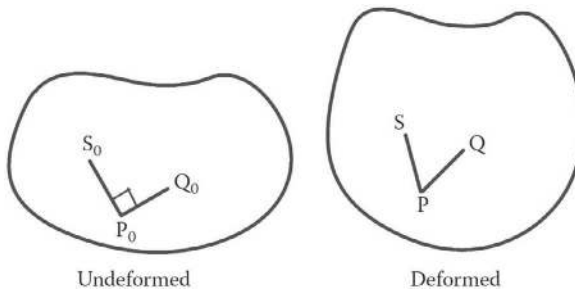


FIGURE 4.1 Undeformed and deformed configurations of a body.

and it undergoes a displacement

$$\mathbf{u} = u_x \mathbf{e}_1 + u_y \mathbf{e}_2 + u_z \mathbf{e}_3. \quad (4.2)$$

The gradient of the displacement vector can provide an idea about the relative displacement. The gradient of a vector is a tensor, and in an array form, it is represented as

$$[\nabla_0 \mathbf{u}] = \begin{bmatrix} \frac{\partial u_x}{\partial x_0} & \frac{\partial u_x}{\partial y_0} & \frac{\partial u_x}{\partial z_0} \\ \frac{\partial u_y}{\partial x_0} & \frac{\partial u_y}{\partial y_0} & \frac{\partial u_y}{\partial z_0} \\ \frac{\partial u_z}{\partial x_0} & \frac{\partial u_z}{\partial y_0} & \frac{\partial u_z}{\partial z_0} \end{bmatrix}. \quad (4.3)$$

Here ∇_0 indicates a gradient operator with respect to the coordinates in the initial configuration. It can be shown that the displacement gradient is a tensor. That means its components follow the rules of tensor transformation under the rotation of coordinate axes. In index notations, the deformation gradient tensor shall be indicated as $u_{i,j}$, where the semicolon (;) in the subscript denotes the derivative with respect to initial coordinates. In vector notation, it is denoted by $\nabla_0 \mathbf{u}$.

It is clear that the displacement gradient tensor is related to relative displacement. Hence, it should be able to provide the idea about strain as well as infinitesimal rotation. In the following subsections, it is shown that the symmetric part of the displacement gradient tensor provides a measure of small strain, and the antisymmetric part provides a measure of infinitesimal rotation.

4.2.1 LINEAR STRAIN TENSOR

Like every tensor, the displacement gradient tensor can be decomposed into symmetric and antisymmetric parts:

$$\nabla_0 \mathbf{u} = \frac{1}{2} \left\{ \nabla_0 \mathbf{u} + (\nabla_0 \mathbf{u})^T \right\} + \frac{1}{2} \left\{ \nabla_0 \mathbf{u} - (\nabla_0 \mathbf{u})^T \right\}. \quad (4.4)$$

Define strain tensor $\boldsymbol{\varepsilon}$ as the symmetric part of $\nabla_0 \mathbf{u}$, that is,

$$\boldsymbol{\varepsilon} = \frac{1}{2} \left\{ \nabla_0 \mathbf{u} + (\nabla_0 \mathbf{u})^T \right\}. \quad (4.5)$$

This can be considered an appropriate measure of deformation if it can measure the change in length of small lines emanating from all points under consideration and the change in angle for all pairs of orthogonal lines with intersections at that point. As shown in Example 4.1, for small strains,

$$\boldsymbol{\varepsilon}_n = \{n_0\}^T [\boldsymbol{\varepsilon}] \{n_0\} \quad (4.6)$$

and

$$\gamma_{n_1 n_2} = 2 \{n_{01}\}^T [\varepsilon] \{n_{02}\}, \quad (4.7)$$

where ε_n denotes the change in length per unit length along the direction $\mathbf{n}_0$ at point P_0 of the initial configuration, and $\gamma_{n_1 n_2}$ denotes the change in angle between two perpendicular directions $\mathbf{n}_{01}$ and $\mathbf{n}_{02}$ at P_0 . The change in angle in radians between two perpendicular directions is called shear strain. By convention, shear strain is positive if the angle between two orthogonal lines decreases, and shear strain is negative if the angle between two orthogonal lines increases. This convention ensures that a positive shear stress produces a positive strain and vice versa.

For the x, y, z coordinate system, Equation 4.5 provides

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{zx} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_x}{\partial x_0} & \frac{1}{2} \left(\frac{\partial u_x}{\partial y_0} + \frac{\partial u_y}{\partial x_0} \right) & \frac{1}{2} \left(\frac{\partial u_z}{\partial x_0} + \frac{\partial u_x}{\partial z_0} \right) \\ \frac{1}{2} \left(\frac{\partial u_x}{\partial y_0} + \frac{\partial u_y}{\partial x_0} \right) & \frac{\partial u_y}{\partial y_0} & \frac{1}{2} \left(\frac{\partial u_y}{\partial z_0} + \frac{\partial u_z}{\partial y_0} \right) \\ \frac{1}{2} \left(\frac{\partial u_z}{\partial x_0} + \frac{\partial u_x}{\partial z_0} \right) & \frac{1}{2} \left(\frac{\partial u_y}{\partial z_0} + \frac{\partial u_z}{\partial y_0} \right) & \frac{\partial u_z}{\partial z_0} \end{bmatrix} \quad (4.8)$$

The strain-displacement relation provided by Equation 4.8 is linear. By taking $\{n_0\}^T$ as $\{1 \ 0 \ 0\}^T$ in Equation 4.6, ε_n is obtained as ε_{xx} . Thus, ε_{xx} represents per unit change in length along the x -direction. Similarly, ε_{yy} and ε_{zz} represent per unit changes in length along the y - and z -directions, respectively. By taking $\{n_{01}\}^T$ as $\{1 \ 0 \ 0\}^T$ and $\{n_{02}\}^T$ as $\{0 \ 1 \ 0\}^T$ in Equation 4.7, the change in angle between x and y is obtained as $2\varepsilon_{xy}$. Similarly, the change in angle between the y - and z -directions is $2\varepsilon_{yz}$, and the change in angle between the x - and z -directions is $2\varepsilon_{zx}$.

Example 4.1

Prove Equations 4.6 and 4.7.

SOLUTION

The directional derivative of the x -component of displacement along a direction $\mathbf{n}_0$ is given by

$$\frac{\partial u_x}{\partial n_0} = \frac{\partial u_x}{\partial x_0} n_{x_0} + \frac{\partial u_x}{\partial y_0} n_{y_0} + \frac{\partial u_x}{\partial z_0} n_{z_0} = a. \quad (4.9)$$

The directional derivatives of y - and z -components are

$$\frac{\partial u_y}{\partial n_0} = \frac{\partial u_y}{\partial x_0} n_{x_0} + \frac{\partial u_y}{\partial y_0} n_{y_0} + \frac{\partial u_y}{\partial z_0} n_{z_0} = b \quad (4.10)$$

and

$$\frac{\partial u_z}{\partial n_0} = \frac{\partial u_z}{\partial x_0} n_{x_0} + \frac{\partial u_z}{\partial y_0} n_{y_0} + \frac{\partial u_z}{\partial z_0} n_{z_0} = c, \quad (4.11)$$

respectively. Assume a point P_0 at (x_0, y_0, z_0) and a close-by point Q_0 at $(x_0 + \Delta x_0, y_0 + \Delta y_0, z_0 + \Delta z_0)$ in the direction of $\mathbf{n}_0$. The distance between the two points P_0Q_0 is denoted by d . Thus,

$$d = \sqrt{(\Delta x_0)^2 + (\Delta y_0)^2 + (\Delta z_0)^2}. \quad (4.12)$$

After deformation, line P_0Q_0 becomes PQ . Then,

$$\text{coordinate of point } P = (x_0 + u_x, y_0 + u_y, z_0 + u_z) \quad (4.13)$$

and

$$\begin{aligned} &\text{coordinate of point } Q \\ &= (x_0 + \Delta x_0 + u_x + ad, y_0 + \Delta y_0 + u_y + bd, z_0 + \Delta z_0 + u_z + cd). \end{aligned} \quad (4.14)$$

In Equation 4.14, the assumption is that point Q_0 is very close to point P_0 . The distance between points PQ denoted by d_f is given by

$$d_f = \sqrt{(\Delta x_0 + ad)^2 + (\Delta y_0 + bd)^2 + (\Delta z_0 + cd)^2}. \quad (4.15)$$

Now,

$$\begin{aligned} &(\Delta x_0 + ad)^2 + (\Delta y_0 + bd)^2 + (\Delta z_0 + cd)^2 = (\Delta x_0)^2 + (\Delta y_0)^2 + (\Delta z_0)^2 \\ &\quad + 2d(a\Delta x_0 + b\Delta y_0 + c\Delta z_0) + d^2(a^2 + b^2 + c^2) \\ &= d^2(1 + a^2 + b^2 + c^2) + 2d(a\Delta x_0 + b\Delta y_0 + c\Delta z_0) \end{aligned} \quad (4.16)$$

Considering that the strain components are very small, such that $(a^2 + b^2 + c^2)$ is negligible compared to one,

$$(\Delta x_0 + ad)^2 + (\Delta y_0 + bd)^2 + (\Delta z_0 + cd)^2 = d^2 + 2d(a\Delta x_0 + b\Delta y_0 + c\Delta z_0) \quad (4.17)$$

Hence,

$$\begin{aligned} d_f &= \sqrt{d^2 + 2d(a\Delta x_0 + b\Delta y_0 + c\Delta z_0)} = d \left(1 + 2a \frac{\Delta x_0}{d} + 2b \frac{\Delta y_0}{d} + 2c \frac{\Delta z_0}{d} \right)^{1/2} \\ &= d \left(1 + 2an_{x_0} + 2bn_{y_0} + 2cn_{z_0} \right)^{1/2} \end{aligned} \quad (4.18)$$

where n_{x_0} , n_{y_0} and n_{z_0} are direction cosines of the line along n_0 .

As a , b and c are very small compared to one, the binomial theorem provides

$$d_f = d \left(1 + 2an_{x_0} + 2bn_{y_0} + 2cn_{z_0} \right)^{1/2} = d \left(1 + an_{x_0} + bn_{y_0} + cn_{z_0} \right) \quad (4.19)$$

Hence,

$$\begin{aligned} \varepsilon_n &= \frac{d_f - d}{d} = an_{x_0} + bn_{y_0} + cn_{z_0} = \begin{Bmatrix} n_{x_0} \\ n_{y_0} \\ n_{z_0} \end{Bmatrix}^T \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} \\ &= \begin{Bmatrix} n_{x_0} \\ n_{y_0} \\ n_{z_0} \end{Bmatrix}^T \begin{bmatrix} \frac{\partial u_x}{\partial x_0} & \frac{\partial u_x}{\partial y_0} & \frac{\partial u_x}{\partial z_0} \\ \frac{\partial u_y}{\partial x_0} & \frac{\partial u_y}{\partial y_0} & \frac{\partial u_y}{\partial z_0} \\ \frac{\partial u_z}{\partial x_0} & \frac{\partial u_z}{\partial y_0} & \frac{\partial u_z}{\partial z_0} \end{bmatrix} \begin{Bmatrix} n_{x_0} \\ n_{y_0} \\ n_{z_0} \end{Bmatrix} \end{aligned} \quad (4.20)$$

In index notation, Equation 4.20 is written as

$$\varepsilon_n = n_{0i} u_{i;j} n_{0j} = n_{0i} \left\{ \frac{1}{2} (u_{i;j} + u_{j;i}) + \frac{1}{2} (u_{i;j} - u_{j;i}) \right\} n_{0j}. \quad (4.21)$$

Note that by interchange of the index,

$$n_{0i} \frac{1}{2} (u_{i;j} - u_{j;i}) n_{0j} = n_{0j} \frac{1}{2} (u_{j;i} - u_{i;j}) n_{0i} = -n_{0i} \frac{1}{2} (u_{i;j} - u_{j;i}) n_{0j}. \quad (4.22)$$

Hence,

$$n_{0i} \frac{1}{2} (u_{i;j} - u_{j;i}) n_{0j} = 0. \quad (4.23)$$

Substituting Equation 4.23 in Equation 4.21, one gets

$$\varepsilon_n = n_{0i} \frac{1}{2} (u_{i;j} + u_{j;i}) n_{0j} = n_{0i} \varepsilon_{ij} n_{0j} \quad (4.24)$$

or in matrix notation

$$\varepsilon_n = \{n_0\}^T [\varepsilon] \{n_0\}, \quad (4.25)$$

which is exactly Equation 4.6.

For Equation 4.7, we proceed as follows. The vectors P_0Q_0 and PQ are given by

$$\begin{aligned} \overline{P_0Q_0} &= \Delta x_0 \mathbf{e}_1 + \Delta y_0 \mathbf{e}_2 + \Delta z_0 \mathbf{e}_3; \\ \overline{PQ} &= (\Delta x_0 + ad) \mathbf{e}_1 + (\Delta y_0 + bd) \mathbf{e}_2 + (\Delta z_0 + cd) \mathbf{e}_3 \end{aligned} \quad (4.26)$$

For the sake of convenience, denoting the component of these vectors with subscript 1,

$$\begin{aligned} \overline{P_0Q_0} &= \Delta x_{01} \mathbf{e}_1 + \Delta y_{01} \mathbf{e}_2 + \Delta z_{01} \mathbf{e}_3; \\ \overline{PQ} &= (\Delta x_{01} + a_1 d_1) \mathbf{e}_1 + (\Delta y_{01} + b_1 d_1) \mathbf{e}_2 + (\Delta z_{01} + c_1 d_1) \mathbf{e}_3 \end{aligned} \quad (4.27)$$

Let us consider a point S_0 , very near to point P_0 , such that P_0S_0 is perpendicular to P_0Q_0 . After deformation, P_0S_0 becomes PS . Similar to Equation 4.27, the vectors P_0S_0 and PS are denoted by

$$\begin{aligned} \overline{P_0S_0} &= \Delta x_{02} \mathbf{e}_1 + \Delta y_{02} \mathbf{e}_2 + \Delta z_{02} \mathbf{e}_3; \\ \overline{PS} &= (\Delta x_{02} + a_2 d_2) \mathbf{e}_1 + (\Delta y_{02} + b_2 d_2) \mathbf{e}_2 + (\Delta z_{02} + c_2 d_2) \mathbf{e}_3 \end{aligned} \quad (4.28)$$

As lines P_0Q_0 and P_0S_0 are orthogonal to each other, the dot product of the vectors formed by these lines is 0. Thus,

$$\Delta x_{01} \Delta x_{02} + \Delta y_{01} \Delta y_{02} + \Delta z_{01} \Delta z_{02} = 0. \quad (4.29)$$

If the angle between PQ and PS is θ , then

$$\begin{aligned} \overline{PQ} \cdot \overline{PS} &= d_1 d_2 \cos \theta = (\Delta x_{01} + a_1 d_1)(\Delta x_{02} + a_2 d_2) \\ &\quad + (\Delta y_{01} + b_1 d_1)(\Delta y_{02} + b_2 d_2) + (\Delta z_{01} + c_1 d_1)(\Delta z_{02} + c_2 d_2) \end{aligned} \quad (4.30)$$

Now,

$$(\Delta x_{01} + a_1 d_1)(\Delta x_{02} + a_2 d_2) = \Delta x_{01} \Delta x_{02} + \Delta x_{01} a_2 d_2 + \Delta x_{02} a_1 d_1 + a_1 d_1 a_2 d_2. \quad (4.31)$$

For small deformations, the last term of the right-hand side of the equation can be neglected. Thus,

$$(\Delta x_{01} + a_1 d_1)(\Delta x_{02} + a_2 d_2) = \Delta x_{01} \Delta x_{02} + \Delta x_{01} a_2 d_2 + \Delta x_{02} a_1 d_1. \quad (4.32)$$

Using the same approximation for y- and z-components, Equation 4.30 can be written as

$$\begin{aligned} d_1 d_2 \cos \theta = & \Delta x_{01} \Delta x_{02} + \Delta x_{01} a_2 d_2 + \Delta x_{02} a_1 d_1 + \Delta y_{01} \Delta y_{02} + \Delta y_{01} b_2 d_2 \\ & + \Delta y_{02} b_1 d_1 + \Delta z_{01} \Delta z_{02} + \Delta z_{01} c_2 d_2 + \Delta z_{02} c_1 d_1. \end{aligned} \quad (4.33)$$

Using Equation 4.29, Equation 4.33 can be written as

$$\cos \theta = \frac{\Delta x_{01} a_2 d_2 + \Delta x_{02} a_1 d_1 + \Delta y_{01} b_2 d_2 + \Delta y_{02} b_1 d_1 + \Delta z_{01} c_2 d_2 + \Delta z_{02} c_1 d_1}{d_1 d_2}. \quad (4.34)$$

Using the definition of direction cosines $n_{x_{01}} = \Delta x_{01}/d_1$, etc.,

$$\sin \left(\frac{\pi}{2} - \theta \right) = n_{x_{01}} a_2 + n_{x_{02}} a_1 + n_{y_{01}} b_2 + n_{y_{02}} b_1 + n_{z_{01}} c_2 + n_{z_{02}} c_1. \quad (4.35)$$

For a small change in the angle between lines $P_0 Q_0$ and $P_0 S_0$, Equation 4.35 can be written as

$$\frac{\pi}{2} - \theta = \begin{Bmatrix} n_{x_{01}} \\ n_{y_{01}} \\ n_{z_{01}} \end{Bmatrix}^T \begin{Bmatrix} a_2 \\ b_2 \\ c_2 \end{Bmatrix} + \begin{Bmatrix} n_{x_{02}} \\ n_{y_{02}} \\ n_{z_{02}} \end{Bmatrix}^T \begin{Bmatrix} a_1 \\ b_1 \\ c_1 \end{Bmatrix}. \quad (4.36)$$

From Equations 4.9–4.11, it is clear that

$$\begin{Bmatrix} a_1 \\ b_1 \\ c_1 \end{Bmatrix} = \begin{bmatrix} \frac{\partial u_x}{\partial x_0} & \frac{\partial u_x}{\partial y_0} & \frac{\partial u_x}{\partial z_0} \\ \frac{\partial u_y}{\partial x_0} & \frac{\partial u_y}{\partial y_0} & \frac{\partial u_y}{\partial z_0} \\ \frac{\partial u_z}{\partial x_0} & \frac{\partial u_z}{\partial y_0} & \frac{\partial u_z}{\partial z_0} \end{bmatrix} \begin{Bmatrix} n_{x_{01}} \\ n_{y_{01}} \\ n_{z_{01}} \end{Bmatrix}; \quad \begin{Bmatrix} a_2 \\ b_2 \\ c_2 \end{Bmatrix} = \begin{bmatrix} \frac{\partial u_x}{\partial x_0} & \frac{\partial u_x}{\partial y_0} & \frac{\partial u_x}{\partial z_0} \\ \frac{\partial u_y}{\partial x_0} & \frac{\partial u_y}{\partial y_0} & \frac{\partial u_y}{\partial z_0} \\ \frac{\partial u_z}{\partial x_0} & \frac{\partial u_z}{\partial y_0} & \frac{\partial u_z}{\partial z_0} \end{bmatrix} \begin{Bmatrix} n_{x_{02}} \\ n_{y_{02}} \\ n_{z_{02}} \end{Bmatrix}. \quad (4.37)$$

Using the definition of shear strain as the change in the angle between two perpendicular lines and Equation 4.37 with compact matrix notation for displacement gradient, Equation 4.36 is written as

$$\gamma_{n_1 n_2} = \begin{Bmatrix} n_{x_{01}} \\ n_{y_{01}} \\ n_{z_{01}} \end{Bmatrix}^T [\nabla_0 u] \begin{Bmatrix} n_{x_{02}} \\ n_{y_{02}} \\ n_{z_{02}} \end{Bmatrix} + \begin{Bmatrix} n_{x_{02}} \\ n_{y_{02}} \\ n_{z_{02}} \end{Bmatrix}^T [\nabla_0 u] \begin{Bmatrix} n_{x_{01}} \\ n_{y_{01}} \\ n_{z_{01}} \end{Bmatrix} \quad (4.38)$$

Taking the transpose of the second term on the right-hand side and using compact notation for the vectors of direction cosines, one gets

$$\gamma_{n_1 n_2} = \{n_{01}\}^T \left\{ [\nabla_0 u] + [\nabla_0 u]^T \right\} \{n_{02}\} = 2 \{n_{01}\}^T [\varepsilon] \{n_{02}\} \quad (4.39)$$

Thus, Equation 4.7 is proved.

4.2.2 INFINITESIMAL ROTATION TENSOR

As mentioned earlier, the displacement gradient can provide the idea about the relative displacement. It is also seen that the symmetric part of the displacement gradient provides strain. Thus, intuitively, the skew-symmetric part of the displacement gradient should provide infinitesimal rotation. The infinitesimal rotation tensor is denoted by ω . Then,

$$\omega = \frac{1}{2} \left\{ \nabla_0 u - (\nabla_0 u)^T \right\}. \quad (4.40)$$

In matrix notation, Equation 4.40 is written as

$$[\omega] = \frac{1}{2} \left\{ [\nabla_0 u] - (\nabla_0 u)^T \right\}. \quad (4.41)$$

In index notation,

$$\omega_{i,j} = \frac{1}{2} (u_{i,j} - u_{j,i}). \quad (4.42)$$

The diagonal components of ω are zero. The expression for the non-diagonal components of ω are as follows:

$$\begin{aligned} \omega_{zy} = -\omega_{yz} &= \frac{1}{2} \left(\frac{\partial u_z}{\partial y_0} - \frac{\partial u_y}{\partial z_0} \right), \\ \omega_{xz} = -\omega_{zx} &= \frac{1}{2} \left(\frac{\partial u_x}{\partial z_0} - \frac{\partial u_z}{\partial x_0} \right), \\ \omega_{yx} = -\omega_{xy} &= \frac{1}{2} \left(\frac{\partial u_y}{\partial x_0} - \frac{\partial u_x}{\partial y_0} \right). \end{aligned} \quad (4.43)$$

The components ω_{zy} , ω_{xz} and ω_{yx} represent the angle of rotation, respectively, about x -, y - and z -axes. The rotations are considered positive if they are counterclockwise and negative if clockwise.

To see that ω_{yx} represents the rotation about the z -axis, consider two orthogonal lines $P_0 Q_0$ and $P_0 S_0$ in the xy -plane (Figure 4.2). As a result of displacement, these lines change to PQ and PS . For small deformations, counterclockwise rotation

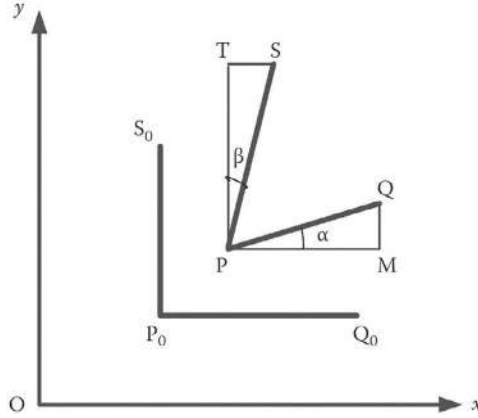


FIGURE 4.2 Deformation of two orthogonal lines.

of P_0Q_0 is given by $\partial u_y / \partial x_0 (\approx \alpha)$, and counterclockwise rotation of P_0S_0 is given by $-\partial u_x / \partial y_0 (\approx -\beta)$. Thus, the average rotation about the z -axis is

$$\omega_{yx} = -\omega_{xy} = \frac{1}{2} \left(\frac{\partial u_y}{\partial x_0} - \frac{\partial u_x}{\partial y_0} \right). \quad (4.44)$$

Similarly, other components can be proven.

Example 4.2

In a uni-axial tensile test of a bar, the displacement components are given as

$$u_x = 5 \times 10^{-3} x_0, u_y = -2 \times 10^{-3} y_0, u_z = -2 \times 10^{-3} z_0$$

Find out the strain field and infinitesimal rotation tensor.

SOLUTION

The displacement gradient tensor is given by

$$[\nabla_0 u] = \begin{bmatrix} 5 \times 10^{-3} & 0 & 0 \\ 0 & -2 \times 10^{-3} & 0 \\ 0 & 0 & -2 \times 10^{-3} \end{bmatrix}$$

This, itself, is symmetric. Hence, it is the strain field in matrix form, and the rotation tensor is zero.

Example 4.3

Given the displacement component as

$$u_x = 10^{-3} y_0, u_y = -10^{-3} x_0, u_z = 0$$

Find out the strain and infinitesimal rotation tensor.

SOLUTION

Here, the displacements are only in a two-dimensional space and they are not the function of the z -coordinate. Hence, the strain and rotation components associated with the z -direction are zero. The displacement gradient is given by

$$[\nabla_0 \mathbf{u}] = \begin{bmatrix} 0 & 10^{-3} \\ -10^{-3} & 0 \end{bmatrix}.$$

Its symmetric part gives strain. Thus,

$$[\nabla_0 \mathbf{u}] = \frac{1}{2} \left[\begin{bmatrix} 0 & 10^{-3} \\ -10^{-3} & 0 \end{bmatrix} + \begin{bmatrix} 0 & -10^{-3} \\ 10^{-3} & 0 \end{bmatrix} \right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, strain will be 0. As the symmetric part of the displacement gradient is 0, the displacement gradient is skew-symmetric.

4.3 DEFORMATION GRADIENT

In Section 4.2, a displacement gradient tensor is defined. The symmetric part of the displacement gradient tensor defines the infinitesimal strain, and the skew-symmetric part defines the infinitesimal rotation. For large deformation, the infinitesimal strain tensor provides physically unrealistic results. For example, consider the rigid body clockwise rotation θ about z_0 . The coordinates of the deformed and undeformed systems are related as follows:

$$\begin{aligned} x &= x_0 \cos \theta + y_0 \sin \theta \\ y &= -x_0 \sin \theta + y_0 \cos \theta \end{aligned} \quad (4.45)$$

The displacement components are

$$\begin{aligned} u_x &= x_0 \cos \theta + y_0 \sin \theta - x_0 \\ u_y &= -x_0 \sin \theta + y_0 \cos \theta - y_0 \\ u_z &= z_0 - z_0 = 0. \end{aligned} \quad (4.46)$$

The strain tensor is given by

$$\begin{aligned}
 [\varepsilon] &= \begin{bmatrix} \frac{\partial u_x}{\partial x_0} & \frac{1}{2} \left(\frac{\partial u_x}{\partial y_0} + \frac{\partial u_y}{\partial x_0} \right) & \frac{1}{2} \left(\frac{\partial u_z}{\partial x_0} + \frac{\partial u_x}{\partial z_0} \right) \\ \frac{1}{2} \left(\frac{\partial u_x}{\partial y_0} + \frac{\partial u_y}{\partial x_0} \right) & \frac{\partial u_y}{\partial y_0} & \frac{1}{2} \left(\frac{\partial u_y}{\partial z_0} + \frac{\partial u_z}{\partial y_0} \right) \\ \frac{1}{2} \left(\frac{\partial u_z}{\partial x_0} + \frac{\partial u_x}{\partial z_0} \right) & \frac{1}{2} \left(\frac{\partial u_y}{\partial z_0} + \frac{\partial u_z}{\partial y_0} \right) & \frac{\partial u_z}{\partial z_0} \end{bmatrix} \\
 &= \begin{bmatrix} \cos\theta - 1 & 0 & 0 \\ 0 & \cos\theta - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
 \end{aligned} \tag{4.47}$$

Unless the rotation is very small (making $\cos \theta$ almost equal to 1), this strain tensor is non-zero. Thus, rigid body rotation will provide non-zero strain, indicating some deformation. This is clearly wrong. Thus, some different strain measures need to be developed for a large deformation case.

A number of such strain measures have been developed, but almost all of them define a deformation gradient as a basis. Therefore, first, the concept of the deformation gradient will be explained.

Due to the continuum hypothesis, each particle in the undeformed configuration has a one-to-one mapping with the deformed configuration. Assume that a particle's position vector in the undeformed coordinate is $\mathbf{x}_0$, and its position at time t becomes $\mathbf{x}$. This fact is mathematically expressed as

$$\mathbf{x} = \mathbf{x}(\mathbf{x}_0, t) \tag{4.48}$$

The deformation gradient tensor $\mathbf{F}$ is defined as

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}. \tag{4.49}$$

In index notation, it is written as

$$F_{ij} = x_{i,j} \tag{4.50}$$

and in matrix form

$$[\mathbf{F}] = \begin{bmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} & \frac{\partial x}{\partial z_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} & \frac{\partial y}{\partial z_0} \\ \frac{\partial z}{\partial x_0} & \frac{\partial z}{\partial y_0} & \frac{\partial z}{\partial z_0} \end{bmatrix}. \tag{4.51}$$

By the chain rule of differentiation,

$$d\mathbf{x} = \mathbf{F} d\mathbf{x}_0 \quad (4.52)$$

Thus, the deformation gradient tensor operates on an infinitesimal line element in an undeformed configuration to provide the corresponding line element in a deformed configuration. In principle, it is possible to get back the undeformed configuration by further reverse deformation. Equation 4.52 provides

$$d\mathbf{x}_0 = \mathbf{F}^{-1} d\mathbf{x}. \quad (4.53)$$

In order for $\mathbf{F}$ to be invertible, the matrix of $\mathbf{F}$ must be non-singular. Therefore, a necessary condition is

$$\det(\mathbf{F}) \neq 0. \quad (4.54)$$

If there is no deformation, $\mathbf{F}$ must be the unit tensor and its determinant is equal to 1. As deformation is continuous, the determinant of the deformation gradient cannot be negative, because it cannot reach negative from a positive value of one without crossing zero. Hence,

$$\det(\mathbf{F}) > 0. \quad (4.55)$$

Assume that a parallelepiped of sides $d\mathbf{x}_{01}$, $d\mathbf{x}_{02}$ and $d\mathbf{x}_{03}$ has volume dV_0 . After deformation, its sides become $d\mathbf{x}_1$, $d\mathbf{x}_2$ and $d\mathbf{x}_3$ and volume dV . Then,

$$dV_0 = d\mathbf{x}_{03} \cdot (d\mathbf{x}_{01} \times d\mathbf{x}_{02}) = \varepsilon_{ijk} dx_{03_i} dx_{01_j} dx_{02_k} \quad (4.56)$$

and

$$dV = d\mathbf{x}_3 \cdot (d\mathbf{x}_1 \times d\mathbf{x}_2) = \varepsilon_{ijk} dx_{3_i} dx_{1_j} dx_{2_k}. \quad (4.57)$$

Also,

$$dx_{1_j} = F_{jl} dx_{01_l}, dx_{2_k} = F_{km} dx_{02_m}, dx_{3_i} = F_{in} dx_{03_n}. \quad (4.58)$$

Substituting Equation 4.58 in Equation 4.57,

$$dV = \varepsilon_{ijk} dx_{3_i} dx_{1_j} dx_{2_k} = \varepsilon_{ijk} F_{in} F_{jl} F_{km} dx_{03_n} dx_{01_l} dx_{02_m}. \quad (4.59)$$

It can be easily shown that

$$\varepsilon_{ijk} F_{in} F_{jl} F_{km} = \varepsilon_{nlm} \det(\mathbf{F}) \quad (4.60)$$

Hence, Equation 4.59 can be written as

$$dV = \det(\mathbf{F}) \varepsilon_{nlm} dx_{03_n} dx_{01_l} dx_{02_m} \quad (4.61)$$

As n, l and m are dummy indices, replacing them by i, j and k ,

$$dV = \det(\mathbf{F}) \varepsilon_{ijk} dx_{03_i} dx_{01_j} dx_{02_k} . \quad (4.62)$$

Substituting Equation 4.56 in Equation 4.62, one gets

$$dV = \det(\mathbf{F}) dV_0. \quad (4.63)$$

Thus, the determinant of the deformation gradient at a point is the ratio of infinitesimal volumes enclosing the point in the deformed and undeformed configurations. As the ratio cannot be zero or negative, the determinant of the deformation gradient is strictly positive. Henceforth, $\det(\mathbf{F})$ shall be denoted by J .

Now consider two non-collinear infinitesimal vectors $\mathbf{dx}_{01}$ and $\mathbf{dx}_{02}$ in undeformed configuration, which become $\mathbf{dx}_1$ and $\mathbf{dx}_2$ after deformation. It is to be noted that $\mathbf{dx}_1$ and $\mathbf{dx}_2$ will also be non-collinear because if they become collinear, then the two particles in the undeformed configuration will occupy the same position in deformed configuration and mapping will not be one-to-one.

Now,

$$\mathbf{dx}_1 \times \mathbf{dx}_2 = (dS) \mathbf{n}, \quad (4.64)$$

where dS is the area of the infinitesimal parallelogram formed by dx_1 and dx_2 , and $\mathbf{n}$ is the normal to that area. In index notation,

$$(dS) n_i = \varepsilon_{ijk} dx_{1_j} dx_{2_k} . \quad (4.65)$$

As

$$dx_{1_j} = F_{jl} dx_{01_l}, dx_{2_k} = F_{km} dx_{02_m} , \quad (4.66)$$

Equation 4.65 becomes

$$(dS) n_i = \varepsilon_{ijk} F_{jl} F_{km} dx_{01_l} dx_{02_m} . \quad (4.67)$$

Multiplying Equation 4.67 by F_{ip} for making use of Equation 4.60,

$$(dS) F_{ip} n_i = \varepsilon_{ijk} F_{ip} F_{jl} F_{km} dx_{01_l} dx_{02_m} = J \varepsilon_{plm} dx_{01_l} dx_{02_m} . \quad (4.68)$$

The previous equation can also be written as

$$(dS) F_{pi}^T n_i = J \varepsilon_{plm} dx_{01_l} dx_{02_m} , \quad (4.69)$$

which in tensor notation becomes

$$(dS) \mathbf{F}^T \mathbf{n} = J d\mathbf{x}_{01} \times d\mathbf{x}_{02} = J (dS_0) \mathbf{n}_0. \quad (4.70)$$

As $\mathbf{F}^T \mathbf{n}$ is in the direction of unit vector $\mathbf{n}_0$, it cannot be zero. This implies that the transpose of the deformation gradient (and therefore the deformation gradient) itself is non-singular and invertible.

Example 4.4

Someone claims that a material is deformed in such a manner that coordinates after deformation are given as

$$x = x_0, y = x_0, z = x_0,$$

where x_0 is the initial x -coordinate. Justify that this type of deformation is not possible.

SOLUTION

For the given deformation field, the deformation gradient is

$$[F] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Here, $\det(F) = 0$. Hence, the deformation field is not possible.

Physically, the given deformation field shows that all particles whose x -coordinate is the same will occupy the same position after deformation. Even a line will change to the point, which is not possible from the continuum hypothesis.

Example 4.5

Consider the following deformation field:

$$x = x_0 + y_0, y = x_0 - y_0, z = x_0 + y_0 - z_0.$$

Find out the determinant of the deformation gradient.

SOLUTION

The deformation gradient is given by

$$[F] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}.$$

Hence,

$$\det(\mathbf{F}) = 1(1-0) - 1(-1-0) = 2.$$

4.4 GREEN STRAIN TENSOR

It is clear that if the deformation gradient tensor is orthogonal, it just causes the rotation of an infinitesimal arc without any stretching. In general, the deformation gradient tensor is not orthogonal, but it can be multiplicatively decomposed into an orthogonal and symmetric tensor using the polar decomposition theorem. Thus,

$$\mathbf{F} = \mathbf{Q}\mathbf{U}, \quad (4.71)$$

where $\mathbf{Q}$ is the orthogonal tensor called the rotation tensor and $\mathbf{U}$ is called the right stretch tensor. The tensor $\mathbf{U}$ is symmetric and positive definite and represents the deformation of the neighborhood of the point. It can be easily seen that

$$\mathbf{U}^2 = \mathbf{F}^T \mathbf{F}. \quad (4.72)$$

Now, assume a unit vector $\mathbf{n}_0$ in the $\mathbf{u}_n$ deformed configuration, which becomes $\eta\mathbf{n}$ after deformation, where $\mathbf{n}$ is the unit vector and η is the ratio of lengths of infinitesimal arcs in deformed and undeformed configurations. Hence,

$$\eta\mathbf{n} = \mathbf{F}\mathbf{n}_0. \quad (4.73)$$

Therefore,

$$\eta^2 = \mathbf{F}\mathbf{n}_0 \cdot \mathbf{F}\mathbf{n}_0 = \mathbf{n}_0 \cdot (\mathbf{F}^T \mathbf{F}) \mathbf{n}_0 = \mathbf{n}_0 \cdot \mathbf{C}\mathbf{n}_0, \quad (4.74)$$

where $\mathbf{C}$ is called the Green deformation tensor. It was introduced by George Green in 1841. It is also called the right Cauchy-Green deformation tensor.

The normal strain at a point in some direction is usually defined as the relative elongation of an infinitesimal line in that direction. If the length of the line in the deformed configuration is ds and the corresponding length in the undeformed configuration is given by ds_0 , then the normal strain in the direction of ds_0 of the undeformed configuration is

$$e = \frac{ds - ds_0}{ds_0} = \eta - 1. \quad (4.75)$$

Then,

$$\begin{aligned} 1 + e &= (\eta^2)^{1/2} = (\mathbf{n}_0 \cdot \mathbf{C}\mathbf{n}_0)^{1/2} = \{1 + \mathbf{n}_0 \cdot (\mathbf{C} - \mathbf{I})\mathbf{n}_0\}^{1/2} \\ &= \{1 + \mathbf{n}_0 \cdot 2\mathbf{G}\mathbf{n}_0\}^{1/2}, \end{aligned} \quad (4.76)$$

where the symmetric tensor $\mathbf{G}$ is called the Green strain tensor. It was introduced by George Green in 1841 and by B. Saint-Venant in 1844. Hence, it is also known as the Green-Saint-Venant strain tensor. Some people also call it the Green-Lagrange tensor. The tensor is zero for the rigid body transformation. Thus, it can give an idea about the deformation. Rearranging Equation 4.76,

$$e = \{1 + 2\mathbf{n}_0 \cdot \mathbf{G}\mathbf{n}_0\}^{1/2} - 1. \quad (4.77)$$

In index notation,

$$e = \left\{1 + 2G_{ij}n_{0i}n_{0j}\right\}^{1/2} - 1. \quad (4.78)$$

From this,

$$(1 + e)^2 = 1 + 2e + e^2 = 1 + 2G_{ij}n_{0i}n_{0j}. \quad (4.79)$$

If normal strain, e , is small such that e^2 is negligible, then

$$e \approx G_{ij}n_{0i}n_{0j}. \quad (4.80)$$

For the x -axis, three components of $\mathbf{n}_0$ are 1, 0 and 0, respectively. Thus, to a first-order approximation, the component G_{xx} represents the normal strain along the x -direction. Similarly, G_{yy} and G_{zz} represent the normal strains along the y - and z -directions, respectively, to a first-order approximation.

From Equation 4.78,

$$1 + 2G_{ij}n_{0i}n_{0j} = (1 + e)^2 = \eta^2 \quad (4.81)$$

or

$$G_{ij}n_{0i}n_{0j} = \frac{\eta^2 - 1}{2} = \frac{(\mathrm{d}s)^2 - (\mathrm{d}s_0)^2}{2(\mathrm{d}s_0)^2}. \quad (4.82)$$

Thus, G_{xx} exactly represents half the change in square length per unit square length along the direction, which was initially along the x -direction.

Let us see if the Green strain tensor can estimate the change in angle between two lines. For this purpose, consider two unit vectors $\mathbf{n}_0$ and $\mathbf{n}_{01}$ in the undeformed configuration. If the angle between them is θ_0 , then

$$\cos\theta_0 = \mathbf{n}_0 \cdot \mathbf{n}_{01}. \quad (4.83)$$

After deformation, the direction of these vectors is represented by unit vectors $\mathbf{n}$ and $\mathbf{n}_1$ and the angle between them is given by

$$\cos\theta = \mathbf{n} \cdot \mathbf{n}_1. \quad (4.84)$$

If the ratio of deformed to undeformed length is η for the line initially along $\mathbf{n}_0$ and it is η_1 for the line initially along $\mathbf{n}_{01}$, then by using Equation 4.73,

$$\eta\eta_1\cos\theta = \mathbf{F}\mathbf{n}_0 \cdot \mathbf{F}\mathbf{n}_{01} = \mathbf{n}_0 \cdot \mathbf{F}^T \mathbf{F}\mathbf{n}_{01} = \mathbf{n}_0 \cdot \mathbf{C}\mathbf{n}_{01} \quad (4.85)$$

Equation 4.76 provides $\mathbf{C}$ equal to $(2\mathbf{G} + \mathbf{I})$, and thus,

$$\eta\eta_1\cos\theta = \mathbf{n}_0 \cdot (2\mathbf{G} + \mathbf{I})\mathbf{n}_{01} = \mathbf{n}_0 \cdot 2\mathbf{G}\mathbf{n}_{01} + \cos\theta_0 \quad (4.86)$$

Equation 4.86 is a useful equation that can calculate the angle between two lines in the deformed configuration passing through a point given the initial directions of the lines and the Green strain tensor at that point.

If two lines are orthogonal to each other in the undeformed configuration, that is, $\theta_0 = \pi/2$ and shear strain is defined as the reduction in the angle between the lines, that is,

$$\gamma = \frac{\pi}{2} - \theta, \quad (4.87)$$

then Equation 4.86 provides

$$\eta\eta_1\sin\gamma = 2\mathbf{n}_0 \cdot \mathbf{G}\mathbf{n}_{01} \quad (4.88)$$

or in index notation,

$$\eta\eta_1\sin\gamma = (1+e)(1+e_1)\sin\gamma = 2G_{ij}n_{0i}n_{01j} \quad (4.89)$$

If γ , e and e_1 are small, then

$$\gamma \approx 2G_{ij}n_{0i}n_{01j} \quad (4.90)$$

Let us express the Green strain tensor in terms of the displacement gradient. From Equation 4.74, $\mathbf{C}$ gets defined as $\mathbf{F}^T\mathbf{F}$ and from Equation 4.76, $\mathbf{G}$ gets defined as $(\mathbf{C} - \mathbf{I})/2$. Hence,

$$\begin{aligned} \mathbf{G} &= \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) = \frac{1}{2}\left\{(\mathbf{I} + \nabla_0\mathbf{u})^T(\mathbf{I} + \nabla_0\mathbf{u}) - \mathbf{I}\right\} \\ &= \frac{1}{2}\left\{\nabla_0\mathbf{u} + (\nabla_0\mathbf{u})^T + (\nabla_0\mathbf{u})^T\nabla_0\mathbf{u}\right\}. \end{aligned} \quad (4.91)$$

In terms of the index notation, Equation 4.91 can be written as

$$G_{ij} = \frac{1}{2}(u_{i;j} + u_{j;i} + u_{k,i}u_{k;j}). \quad (4.92)$$

Example 4.6

Consider the following deformation field:

$$x = x_0 + y_0, y = x_0 - y_0, z = x_0 + y_0 - z_0.$$

Find the angle between two lines with direction cosines as $\{1, 0, 0\}$ and $\{1/\sqrt{2}, 1/\sqrt{2}, 0\}$, which are initially at 45° .

SOLUTION

The deformation gradient is given by

$$[F] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

The Green deformation tensor is given by

$$[C] = [F]^T [F] = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix}.$$

Calculating the ratio of lengths of infinitesimal arcs in deformed and undeformed configurations in the direction $\{1, 0, 0\}$,

$$\eta^2 = \mathbf{n}_0 \cdot \mathbf{C} \mathbf{n}_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = 3.$$

Thus, $\eta = \sqrt{3}$. Calculating the ratio of lengths of infinitesimal arcs in deformed and undeformed configurations in the direction $\{1/\sqrt{2}, 1/\sqrt{2}, 0\}$,

$$\eta_1^2 = \mathbf{n}_0 \cdot \mathbf{C} \mathbf{n}_0 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{Bmatrix} = 4,$$

thus, $\eta_1 = 2$.

Using Equation 4.85

$$\eta \eta_1 \cos \theta = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{Bmatrix} = 4/\sqrt{2}.$$

Putting the value of η and η_{II} ,

$$\cos\theta = \sqrt{\frac{2}{3}},$$

or $\theta = 35.26^\circ$.

4.5 ALMANZI STRAIN TENSOR

In the previous section, the strain was expressed in terms of the coordinates of the undeformed configuration. In this section, strain is expressed as a function of the coordinates of the deformed configuration.

From Equation 4.73,

$$\mathbf{n}_0 = \eta \mathbf{F}^{-1} \mathbf{n}. \quad (4.93)$$

Therefore,

$$\frac{1}{\eta^2} = \mathbf{F}^{-1} \mathbf{n} \cdot \mathbf{F}^{-1} \mathbf{n} = \mathbf{n} \cdot (\mathbf{F}^{-1})^T \mathbf{F}^{-1} \mathbf{n} = \mathbf{n} \cdot (\mathbf{F} \mathbf{F}^T)^{-1} \mathbf{n}. \quad (4.94)$$

Now, define

$$\mathbf{B} = \mathbf{F} \mathbf{F}^T. \quad (4.95)$$

Hence,

$$\frac{1}{\eta^2} = \mathbf{n} \mathbf{B}^{-1} \mathbf{n} \quad (4.96)$$

The tensor $\mathbf{B}^{-1}$ was introduced by Cauchy in 1827 and is known as the Cauchy deformation tensor. The tensor $\mathbf{B}$ was introduced by Finger in 1894 and is known as the Finger deformation tensor. The tensor $\mathbf{B}$ is also called the left Cauchy-Green deformation tensor.

It is known that the deformation gradient tensor can be multiplicatively decomposed into an orthogonal and symmetric tensor using the polar decomposition theorem to get the following form:

$$\mathbf{F} = \mathbf{V} \mathbf{Q}, \quad (4.97)$$

where $\mathbf{Q}$ is the orthogonal tensor called the rotation tensor, and $\mathbf{V}$ is called the left stretch tensor. The tensor $\mathbf{V}$ is symmetric and positive definite and represents deformation of the neighborhood of the point. It can be easily seen that

$$\mathbf{V}^2 = \mathbf{F} \mathbf{F}^T = \mathbf{B}. \quad (4.98)$$

Defining e by Equation 4.75 and using Equation 4.96,

$$\begin{aligned} 1 + e &= \left(\eta^2 \right)^{1/2} = \left(\mathbf{n} \cdot \mathbf{B}^{-1} \mathbf{n} \right)^{-1/2} \\ &= \left\{ 1 - \mathbf{n} \cdot \left(\mathbf{I} - \mathbf{B}^{-1} \right) \mathbf{n} \right\}^{-1/2} \end{aligned} \quad (4.99)$$

Defining

$$\mathbf{A} = \frac{1}{2} \left(\mathbf{I} - \mathbf{B}^{-1} \right), \quad (4.100)$$

Equation 4.99 can be written as

$$e = \left(\eta^2 \right)^{1/2} - 1 = \left\{ 1 - 2\mathbf{n} \cdot \mathbf{A} \mathbf{n} \right\}^{-1/2} - 1 \quad (4.101)$$

The tensor $\mathbf{A}$ was introduced by E. Almansi in 1911 and by G. Hamel in 1912 and is known as the Almansi strain tensor.

In index notation,

$$e = \left\{ 1 - 2A_{ij}n_i n_j \right\}^{-1/2} - 1. \quad (4.102)$$

From this,

$$(1 + e)^{-2} = 1 - 2A_{ij}n_i n_j \quad (4.103)$$

If normal strain, e , is small such that e^2 and higher degree terms are negligible, then

$$e \approx A_{ij}n_i n_j \quad (4.104)$$

Thus, to a first-order approximation, the component A_{xx} represents the normal strain along the x -direction in the deformed configuration. Similarly, A_{yy} and A_{zz} represent the normal strains along the y - and z -directions in the deformed configuration, respectively, to a first-order approximation.

From Equation 4.102,

$$1 - 2A_{ij}n_i n_j = (1 + e)^{-2} = \frac{1}{\eta^2} = \frac{(ds_0)^2}{(ds)^2} \quad (4.105)$$

or

$$A_{ij}n_i n_j = \frac{(ds)^2 - (ds_0)^2}{2(ds)^2}. \quad (4.106)$$

Thus, A_{xx} exactly represents half the change in square length per unit final square length along the direction, which is finally along the x -direction.

Let us see if the Almansi strain tensor can estimate the change in angle between two lines. For this purpose, consider two unit vectors $\mathbf{n}_0$ and $\mathbf{n}_{01}$ in the undeformed configuration. If the angle between them is θ_0 , then

$$\cos\theta_0 = \mathbf{n}_0 \mathbf{n}_{01}. \quad (4.107)$$

After deformation, the direction of these vectors is represented by unit vectors $\mathbf{n}$ and $\mathbf{n}_1$ and the angle between them is given by

$$\cos\theta = \mathbf{n} \mathbf{n}_1. \quad (4.108)$$

If the ratio of deformed to undeformed length is η for the line initially along $\mathbf{n}_0$ and it is η_1 for the line initially along $\mathbf{n}_{01}$, then by using Equation 4.93,

$$\frac{1}{\eta\eta_1} \cos\theta_0 = \mathbf{F}^{-1} \mathbf{n} \mathbf{F}^{-1} \mathbf{n}_1 = \mathbf{n} \left(\mathbf{F} \mathbf{F}^T \right)^{-1} \mathbf{n}_1 = \mathbf{n} \mathbf{B}^{-1} \mathbf{n}_1 \quad (4.109)$$

Equation 4.100 provides $\mathbf{B}^{-1}$ equal to $(\mathbf{I} - 2\mathbf{A})$ and thus,

$$\frac{1}{\eta\eta_1} \cos\theta_0 = \mathbf{n} (\mathbf{I} - 2\mathbf{A}) \mathbf{n}_1 = \cos\theta - 2\mathbf{n} \mathbf{A} \mathbf{n}_1 \quad (4.110)$$

Equation 4.110 is a useful equation that can calculate the angle between two lines in the undeformed configuration passing through a point given the final directions of the lines and the Almansi strain tensor at that point.

If two lines are orthogonal to each other in the deformed configuration, that is, $\theta = \pi/2$ and shear strain is defined as the reduction in the angle between the lines from undeformed to deformed configuration, that is,

$$\gamma = \theta_0 - \frac{\pi}{2}, \quad (4.111)$$

then Equation 4.110 provides

$$\frac{1}{\eta\eta_1} \sin\gamma = 2\mathbf{n} \mathbf{A} \mathbf{n}_1 \quad (4.112)$$

or in index notation,

$$\sin\gamma = 2(1+e)(1+e_1) A_{ij} n_i n_{1j} \quad (4.113)$$

For small γ , e and e_1 ,

$$\gamma \approx 2A_{ij} n_i n_{1j} \quad (4.114)$$

Example 4.7

Consider the following deformation field:

$$x = x_0 + y_0, y = x_0 - y_0, z = x_0 + y_0 - z_0.$$

Find out the Almansi strain tensor.

SOLUTION

The deformation gradient is given by

$$[F] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}.$$

Finger deformation tensor is obtained as

$$[B] = [F][F]^T = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix}.$$

In matrix form, the Almansi strain tensor is given by

$$[A] = \frac{1}{2}([I] - [B]^{-1}) = \begin{bmatrix} -0.25 & 0 & 0.5 \\ 0 & 0.25 & 0 \\ 0.5 & 0 & 0 \end{bmatrix}.$$

4.6 LOGARITHMIC STRAIN TENSOR

As noted earlier, the deformation gradient tensor can be written as the multiplication of an orthogonal tensor $\mathbf{Q}$ and a symmetric tensor $\mathbf{U}$ with the help of the polar decomposition theorem. As $\mathbf{U}$ is symmetric, three orthogonal directions can be found such that if these directions are taken as coordinate axes, $\mathbf{U}$ will have only diagonal components. These orthogonal directions are called eigenvectors or principal directions, and the diagonal components of $\mathbf{U}$ are called eigenvalues or principal values of $\mathbf{U}$. Principal value λ_i of $\mathbf{U}$ represents the ratio of the deformed length to the initial length of a small line element along e_i . Then the logarithm of λ_i can be used as a measure of the normal strain. Thus, with respect to the coordinate system of e_i , the components of the logarithmic strain tensor are defined as

$$\varepsilon_{ij}^L = \begin{cases} \ln \lambda_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad (4.115)$$

For small deformation, εL gets reduced to infinitesimal strain tensor ε .

As the logarithmic strain is a tensor quantity, its component in any coordinate system can be found by the transformation rule for tensors.

Example 4.8

Consider the following deformation field:

$$x = x_0 + y_0, y = x_0 - y_0, z = x_0 + y_0 - z_0.$$

Find out the logarithmic strain tensor.

SOLUTION

The deformation gradient tensor is given by

$$[F] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}.$$

$$[U^2] = [F]^T [F] = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix}.$$

The eigenvalues of U^2 are 0.4384, 2.0 and 4.5616. Hence, the eigenvalues of U are 0.6621, 1.4142 and 2.1358. Thus, the logarithmic strain tensor is given by

$$[\varepsilon^L] = \begin{bmatrix} \ln(0.6621) & 0 & 0 \\ 0 & \ln(1.4142) & 0 \\ 0 & 0 & \ln(2.1358) \end{bmatrix}$$

$$= \begin{bmatrix} -0.4123 & 0 & 0 \\ 0 & 0.3466 & 0 \\ 0 & 0 & 0.7588 \end{bmatrix}.$$

This strain system is defined with respect to the axis system made of the eigenvectors of U (or the same as that of U^2). The eigenvectors are as follows:

$$\left\{ \begin{bmatrix} -0.2610 \\ -0.2610 \\ -0.9294 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 0.7071 \\ -0.7071 \\ 0 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 0.6572 \\ 0.6572 \\ -0.3690 \end{bmatrix} \right\}.$$

4.7 STRAIN-DISPLACEMENT RELATION IN CURVILINEAR COORDINATE

Consider the displacement vector field $\mathbf{u}$ that has three components u_x , u_y and u_z in the Cartesian coordinate system. In the orthogonal curvilinear coordinate system (q_1, q_2, q_3) , its components are u_{q_1} , u_{q_2} , u_{q_3} . The gradient of the vector field is defined as (Brower, 2009)

$$[\nabla_0 \mathbf{u}] = \begin{bmatrix} \frac{1}{h_1} \left(\frac{\partial u_{q_1}}{\partial q_1} + \frac{\partial h_1}{\partial q_2} \frac{u_{q_2}}{h_2} + \frac{\partial h_1}{\partial q_3} \frac{u_{q_3}}{h_3} \right) & \frac{1}{h_2} \left(\frac{\partial u_{q_1}}{\partial q_2} - \frac{\partial h_2}{\partial q_1} \frac{u_{q_2}}{h_1} \right) \\ \frac{1}{h_1} \left(\frac{\partial u_{q_2}}{\partial q_1} - \frac{\partial h_1}{\partial q_2} \frac{u_{q_1}}{h_2} \right) & \frac{1}{h_2} \left(\frac{\partial u_{q_2}}{\partial q_2} + \frac{\partial h_2}{\partial q_1} \frac{u_{q_1}}{h_1} + \frac{\partial h_2}{\partial q_3} \frac{u_{q_3}}{h_3} \right) \\ \frac{1}{h_1} \left(\frac{\partial u_{q_3}}{\partial q_1} - \frac{\partial h_1}{\partial q_3} \frac{u_{q_1}}{h_3} \right) & \frac{1}{h_2} \left(\frac{\partial u_{q_3}}{\partial q_2} - \frac{\partial h_2}{\partial q_3} \frac{u_{q_2}}{h_3} \right) \\ \frac{1}{h_3} \left(\frac{\partial u_{q_1}}{\partial q_3} - \frac{\partial h_3}{\partial q_1} \frac{u_{q_1}}{h_1} \right) & \\ \frac{1}{h_3} \left(\frac{\partial u_{q_2}}{\partial q_3} - \frac{\partial h_3}{\partial q_2} \frac{u_{q_2}}{h_2} \right) & \\ \frac{1}{h_3} \left(\frac{\partial u_{q_3}}{\partial q_3} + \frac{\partial h_3}{\partial q_1} \frac{u_{q_1}}{h_1} + \frac{\partial h_3}{\partial q_2} \frac{u_{q_2}}{h_2} \right) & \end{bmatrix} \quad (4.116)$$

where h_1 , h_2 and h_3 are scale factors, as discussed in Chapter 2. For the Cartesian coordinate system,

$$q_1 = x_0, q_2 = y_0, q_3 = z_0, h_1 = 1, h_2 = 1, h_3 = 1. \quad (4.117)$$

Substituting these in Equation 4.116,

$$[\nabla_0 \mathbf{u}] = \begin{bmatrix} \frac{\partial u_x}{\partial x_0} & \frac{\partial u_x}{\partial y_0} & \frac{\partial u_x}{\partial z_0} \\ \frac{\partial u_y}{\partial x_0} & \frac{\partial u_y}{\partial y_0} & \frac{\partial u_y}{\partial z_0} \\ \frac{\partial u_z}{\partial x_0} & \frac{\partial u_z}{\partial y_0} & \frac{\partial u_z}{\partial z_0} \end{bmatrix} \quad (4.118)$$

For a cylindrical coordinate system,

$$q_1 = r_0, q_2 = \theta_0, q_3 = z_0, h_1 = 1, h_2 = r_0, h_3 = 1 \quad (4.119)$$

$$[\nabla_0 u] = \begin{bmatrix} \frac{\partial u_r}{\partial r_0} & \frac{1}{r_0} \left(\frac{\partial u_r}{\partial, 0} - u_r \right) & \frac{\partial u_r}{\partial z_0} \\ \frac{\partial u_\theta}{\partial r_0} & \frac{1}{r_0} \left(\frac{\partial u_\theta}{\partial \theta_0} + u_r \right) & \frac{\partial u_\theta}{\partial z_0} \\ \frac{\partial u_z}{\partial r_0} & \frac{1}{r_0} \frac{\partial u_z}{\partial \theta_0} & \frac{\partial u_z}{\partial z_0} \end{bmatrix} \quad (4.120)$$

From Equation 4.5, strain components in the cylindrical coordinate system are given by

$$\begin{aligned} \varepsilon_{rr} &= \frac{\partial u_r}{\partial r_0}, \varepsilon_{\theta\theta} = \frac{1}{r_0} \frac{\partial u_\theta}{\partial \theta_0} + \frac{u_r}{r_0}, \varepsilon_{zz} = \frac{\partial u_z}{\partial z_0}, \varepsilon_{r\theta} = \varepsilon_{\theta r} = \frac{1}{2} \left(\frac{1}{r_0} \frac{\partial u_r}{\partial \theta_0} + \frac{\partial u_\theta}{\partial r_0} - \frac{u_\theta}{r_0} \right), \\ \varepsilon_{rz} &= \varepsilon_{zr} = \frac{1}{2} \left(\frac{\partial u_r}{\partial z_0} + \frac{\partial u_z}{\partial r_0} \right), \varepsilon_{\theta z} = \varepsilon_{z\theta} = \frac{1}{2} \left(\frac{\partial u_\theta}{\partial z_0} + \frac{1}{r_0} \frac{\partial u_z}{\partial, 0} \right). \end{aligned} \quad (4.121)$$

For spherical coordinate systems,

$$q_1 = r_0, q_2 = \theta_0, q_3 = \phi_0, h_1 = 1, h_2 = r_0, h_3 = r_0 \sin \theta_0. \quad (4.122)$$

Hence,

$$[\nabla_0 u] = \begin{bmatrix} \frac{\partial u_r}{\partial r_0} & \frac{1}{r_0} \left(\frac{\partial u_r}{\partial \theta_0} - u_\theta \right) & \frac{1}{r_0 \sin \theta} \left(\frac{\partial u_r}{\partial \phi_0} - \sin \theta u_\phi \right) \\ \frac{\partial u_\theta}{\partial r_0} & \frac{1}{r_0} \left(\frac{\partial u_\theta}{\partial \theta_0} + u_r \right) & \frac{1}{r_0 \sin \theta} \left(\frac{\partial u_\theta}{\partial \phi_0} - \cos \theta u_\phi \right) \\ \frac{\partial u_\phi}{\partial r_0} & \frac{1}{r_0} \frac{\partial u_\phi}{\partial \theta} & \frac{1}{r_0 \sin \theta} \left(\frac{\partial u_\phi}{\partial \phi_0} + \sin \theta u_r + \cos \theta u_\theta \right) \end{bmatrix} \quad (4.123)$$

From Equation 4.5, strain components in the spherical coordinate system are given by

$$\begin{aligned} \varepsilon_{rr} &= \frac{\partial u_r}{\partial r_0}, \varepsilon_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta_0} + \frac{u_r}{r_0}, \varepsilon_{\phi\phi} = \frac{1}{r_0 \sin \theta} \frac{\partial u_\phi}{\partial \phi_0} + \frac{u_r}{r_0} + \frac{\cot \theta}{r_0} u_\theta \\ \varepsilon_{r\theta} &= \varepsilon_{\theta r} = \frac{1}{2} \left(\frac{\partial u_r}{\partial \theta_0} + \frac{\partial u_\theta}{\partial r_0} - \frac{u_\theta}{r_0} \right), \varepsilon_{r\phi} = \varepsilon_{\phi r} = \frac{1}{2} \left(\frac{1}{r_0 \sin \theta} \frac{\partial u_r}{\partial \phi_0} + \frac{\partial u_\phi}{\partial r_0} - \frac{u_\phi}{r_0} \right) \\ \varepsilon_{\theta\phi} &= \varepsilon_{\phi\theta} = \frac{1}{2} \left(\frac{1}{r_0 \sin \theta} \frac{\partial u_\theta}{\partial \phi_0} + \frac{1}{r_0} \frac{\partial u_\phi}{\partial \theta} - \cot \theta \frac{u_\phi}{r} \right). \end{aligned} \quad (4.124)$$

Displacement gradient $\mathbf{F}$ is related with the deformation gradient in the following manner:

$$\mathbf{F} = \nabla_0 \mathbf{u} + \mathbf{I} \quad (4.125)$$

Once the deformation gradient is known in any coordinate system, various strain measures in that system can be obtained.

Example 4.9

The displacement components in a cylindrical polar coordinate system are given as

$$\begin{aligned} u_r &= -\frac{2p}{\pi E} \cos\theta \ln r - \frac{(1-\nu)p}{\pi E} \theta \sin\theta + A \sin\theta + B \cos\theta, \\ u_\theta &= \frac{2\nu p}{\pi E} \sin\theta + \frac{2p}{\pi E} \ln r \sin\theta - \frac{(1-\nu)p}{\pi E} \theta \cos\theta + \frac{(1-\nu)p}{\pi E} \sin\theta \\ &\quad + A \cos\theta - B \sin\theta + Cr \\ u_z &= 0. \end{aligned}$$

Find out the infinitesimal strain components.

SOLUTION

$$\begin{aligned} \varepsilon_{rr} &= \frac{\partial u_r}{\partial r} = -\frac{2p}{\pi E} \frac{\cos\theta}{r} \\ \varepsilon_{\theta\theta} &= \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} = \frac{2\nu p}{\pi E} \frac{\cos\theta}{r} \\ \varepsilon_{r\theta} &= \varepsilon_{\theta r} = \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) = 0 \end{aligned}$$

Other components are easily seen to be zero.

Example 4.10

The displacement components in spherical polar coordinates are given as follows:

$$u_r = \frac{p}{E} \frac{(1-2\nu)r + (1+\nu)(b^3/2r^2)}{b^3/a^3 - 1}, u_\theta = 0, u_\phi = 0.$$

Find out the infinitesimal strain components.

SOLUTION

$$\begin{aligned}\varepsilon_{rr} &= \frac{\partial u_r}{\partial r} = \frac{p}{E} \frac{(1-2\nu) - (1+\nu)(b^3/r^3)}{b^3/a^3 - 1}, \\ \varepsilon_{\theta\theta} &= \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} = \frac{p}{E} \frac{(1-2\nu) + (1+\nu)(b^3/2r^3)}{b^3/a^3 - 1}, \\ \varepsilon_{\phi\phi} &= \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{\cot \theta}{r} u_\theta = \frac{p}{E} \frac{(1-2\nu) + (1+\nu)(b^3/2r^3)}{b^3/a^3 - 1}\end{aligned}$$

Other components are zero.

4.8 TRANSFORMATION OF STRAIN COMPONENTS

As the strain is a tensor, the transformation law for tensors can be applied to obtain the general formula for obtaining the components of a tensor in a coordinated system rotated with respect to the original system. Thus, the strain tensor ε becomes ε' in rotated coordinates as per the following relation:

$$[\varepsilon'] = [Q][\varepsilon][Q]^T, \quad (4.126)$$

where $[Q]$ is the matrix of direction cosines.

Example 4.11

Components of the linear strain tensor with respect to the (x, y, z) coordinate system are given as

$$[\varepsilon] = 10^{-5} \begin{bmatrix} -2 & 3 & -1 \\ 3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Find the components of the strain tensor with respect to the rotated coordinate system (x', y', z') .

The unit vectors $(\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3')$ along (x', y', z') are given as

$$\begin{aligned}\mathbf{e}_1' &= 0.6\mathbf{e}_1 + 0.8\mathbf{e}_3, \\ \mathbf{e}_2' &= \mathbf{e}_2, \\ \mathbf{e}_3' &= -0.8\mathbf{e}_1 + 0.6\mathbf{e}_3,\end{aligned}$$

SOLUTION

The transformation matrix is given as

$$[Q] = \begin{bmatrix} 0.6 & 0 & 0.8 \\ 0 & 1 & 0 \\ -0.8 & 0 & 0.6 \end{bmatrix}.$$

Using Equation 4.126, the strain components in the rotated coordinate system are given as

$$\begin{aligned} [\varepsilon'] &= \begin{bmatrix} 0.6 & 0 & 0.8 \\ 0 & 1 & 0 \\ -0.8 & 0 & 0.6 \end{bmatrix} 10^{-5} \begin{bmatrix} -2 & 3 & -1 \\ 3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.6 & 0 & -0.8 \\ 0 & 1 & 0 \\ 0.8 & 0 & 0.6 \end{bmatrix} \\ &= 10^{-5} \begin{bmatrix} -1.04 & 1.8 & 1.72 \\ 1.8 & 1 & -2.4 \\ 1.72 & -2.4 & 0.04 \end{bmatrix} \end{aligned}$$

4.9 PRINCIPAL STRAINS

For any real symmetric tensor, there surely exists an orthogonal coordinate system in which the tensor can be expressed as a matrix consisting of only the diagonal matrix. As strain is a tensor, three principal directions of strain can be found such that shear strains associated with these directions are zero. The strains along these directions are called principal strains. These directions remain orthogonal in the deformed configuration. Principal strains are basically the eigenvalues of the strain tensor, and principal directions are the corresponding eigenvectors.

For a linear strain tensor, the principal strains are determined as the roots of the following equation:

$$\lambda^3 - I_\varepsilon \lambda^2 - II_\varepsilon \lambda - III_\varepsilon = 0, \quad (4.127)$$

where

$$I_\varepsilon = \varepsilon_{ii}, \quad (4.128)$$

$$II_\varepsilon = \frac{1}{2} (\varepsilon_{ij} \varepsilon_{ij} - \varepsilon_{ii} \varepsilon_{jj}), \quad (4.129)$$

$$III_\varepsilon = \varepsilon_{ijk} \varepsilon_{1i} \varepsilon_{2j} \varepsilon_{3k}. \quad (4.130)$$

Here, I_ε , II_ε and III_ε are the three principal invariants of the linear strain tensor. One of the three principal strains provides the maximum normal strain, and one of them provides the minimum normal strain.

4.10 MAXIMUM SHEAR STRAIN

Assume that the maximum principal strain is ε_1 , and the minimum principal strain is ε_2 . The corresponding unit principal directions are $\mathbf{e}_1$ and $\mathbf{e}_2$. The maximum shear strain is given by

$$|\gamma_{n_1 n_2}|_{\max} = |\varepsilon_1 - \varepsilon_2|. \quad (4.131)$$

Further, the directions of the maximum shear strain are given by

$$\mathbf{n}_{01} = \pm \frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2), \mathbf{n}_{02} = \pm \frac{1}{\sqrt{2}}(\mathbf{e}_1 - \mathbf{e}_2)$$

Example 4.12

Components of the linear strain tensor with respect to the (x, y, z) coordinate system are given as

$$[\varepsilon] = 10^{-5} \begin{bmatrix} -2 & 3 & -1 \\ 3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Find out the principal strains and the maximum shear strain.

SOLUTION

The eigenvalues of the strain matrix are given as -4×10^{-5} , 1×10^{-5} , and 3×10^{-5} . These are the principal strains. The maximum shear strain is the difference between maximum and minimum principal strains. Thus, it is equal to 7×10^{-5} .

4.11 OCTAHEDRAL STRAIN

A plane equally inclined to three orthogonal principal planes is called the octahedral plane. There are eight octahedral planes corresponding to $l = \pm 1/\sqrt{3}$, $m = \pm 1/\sqrt{3}$ and $n = \pm 1/\sqrt{3}$. Like octahedral stresses are found, octahedral strain can also be found. The normal strain acting on an octahedral plane is given by

$$\varepsilon_{oct} = \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3}{3}, \quad (4.132)$$

which is the mean of the principal strains. As $(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$ is the first invariant and is equal to $(\varepsilon_x + \varepsilon_y + \varepsilon_z)$, the normal stress on the octahedral plane can be considered as the average of the trace of the strain tensor.

The maximum shear strain with respect to some line in the octahedral plane and the normal to plane is found as

$$\gamma_{oct} = \frac{2}{3} \sqrt{(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2}. \quad (4.133)$$

It is possible to find the direction cosine of the line in the octahedral plane that provides the maximum shear strain with respect to normal to the plane.

Example 4.13

Derive Equation 4.133.

SOLUTION

Considering that the direction cosines of the normal to octahedral plane are $1/\sqrt{3}$, $1/\sqrt{3}$ and $1/\sqrt{3}$. Let us consider a line in the octahedral plane whose direction cosines are l , m and n . As this line is perpendicular to the normal to the octahedral plane,

$$l/\sqrt{3} + m/\sqrt{3} + n/\sqrt{3} = 0 \text{ or } l + m + n = 0. \quad (4.134)$$

Also,

$$l^2 + m^2 + n^2 = 1 \quad (4.135)$$

The shear strain between normal to octahedral plane and the line having direction cosines l , m and n is

$$\gamma = 2[l \quad m \quad n] \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} \begin{Bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{Bmatrix} = \frac{2}{\sqrt{3}}(l\varepsilon_1 + m\varepsilon_2 + n\varepsilon_3) \quad (4.136)$$

We want to maximize the shear strain subject to the constraints of Equations 4.134 and 4.135. The Lagrange function for this optimization problem is

$$L = \frac{2}{\sqrt{3}}(l\varepsilon_1 + m\varepsilon_2 + n\varepsilon_3) + \lambda_1(l + m + n) + \lambda_2(l^2 + m^2 + n^2 - 1) \quad (4.137)$$

The necessary conditions for the extremum of the function are

$$\frac{\partial L}{\partial l} = \frac{2}{\sqrt{3}}\varepsilon_1 + \lambda_1 + 2\lambda_2 l = 0, \quad (4.138)$$

$$\frac{\partial L}{\partial m} = \frac{2}{\sqrt{3}}\varepsilon_2 + \lambda_1 + 2\lambda_2 m = 0, \quad (4.139)$$

$$\frac{\partial L}{\partial n} = \frac{2}{\sqrt{3}}\varepsilon_3 + \lambda_1 + 2\lambda_2 n = 0, \quad (4.140)$$

$$\frac{\partial L}{\partial \lambda_1} = l + m + n = 0, \quad (4.141)$$

$$\frac{\partial L}{\partial \lambda_2} = l^2 + m^2 + n^2 - 1 = 0. \quad (4.142)$$

These five equations can be solved for five unknowns l, m, n, λ_1 and λ_2 . Here, they are solved by shortcuts. Adding Equations 4.138–4.140,

$$\frac{2}{\sqrt{3}}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + 3\lambda_1 + 2\lambda_2(l + m + n) = 0. \quad (4.143)$$

In view of Equation 4.141, Equation 4.143 can be written as

$$\frac{2}{\sqrt{3}}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + 3\lambda_1 = 0 \text{ or } \lambda_1 = -\frac{2}{3\sqrt{3}}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3). \quad (4.144)$$

Equation 4.138 provides

$$l = -\frac{\left(\lambda_1 + \frac{2}{\sqrt{3}}\varepsilon_1\right)}{2\lambda_2} = \frac{(\varepsilon_2 + \varepsilon_3 - 2\varepsilon_1)}{3\sqrt{3}\lambda_2}, \quad (4.145)$$

similarly, Equations 4.139 and 4.140 provide

$$m = \frac{(\varepsilon_1 + \varepsilon_3 - 2\varepsilon_2)}{3\sqrt{3}\lambda_2}, \quad (4.146)$$

$$n = \frac{(\varepsilon_1 + \varepsilon_2 - 2\varepsilon_3)}{3\sqrt{3}\lambda_2}. \quad (4.147)$$

Substituting Equations 4.145–4.147 into Equation 4.142, one gets

$$\lambda_2 = 3\left\{(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2\right\}^{1/2}. \quad (4.148)$$

Now, substituting Equations 4.145–4.148 into Equation 4.136, one gets

$$\gamma = -\frac{2}{3}\sqrt{(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2} \quad (4.149)$$

It is also noted that if (l, m, n) is one solution, the other solution is $(-l, -m, -n)$, which provides

$$\gamma = \frac{2}{3}\sqrt{(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2}. \quad (4.150)$$

Equation 4.150 thus provides the maximum shear strain and is known as the octahedral shear strain.

4.12 VOLUMETRIC STRAIN

Suppose the initial volume is dV_0 and the final volume is dV . Then the volumetric strain is defined as

$$G_v = \frac{1}{2} \left\{ \left(\frac{dV}{dV_0} \right)^2 - 1 \right\} \quad (4.151)$$

If the principal strains for the Green-Lagrange strain tensor are G_1 , G_2 and G_3 , then

$$\left(\frac{ds_1}{ds_{10}} \right)^2 = 1 + 2G_1. \quad (4.152)$$

A similar relation is for other principal strains. Hence,

$$\left(\frac{dV}{dV_0} \right)^2 = \left(\frac{ds_1 ds_2 ds_3}{ds_{10} ds_{20} ds_{30}} \right)^2 = (1 + 2G_1)(1 + 2G_2)(1 + 2G_3). \quad (4.153)$$

Hence, the volumetric strain is obtained as

$$G_v = I_G + 2II_G + 4III_G, \quad (4.154)$$

where I_G , II_G and III_G are principal invariants. The first principal invariant contains only linear terms, and the second and third invariants contain non-linear terms. For small strains, the second and third invariants become vanishingly small. Hence, for small strains, the infinitesimal volumetric strain is given as

$$\varepsilon_v \approx \frac{dV - dV_0}{dV_0} = I_\varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \quad (4.155)$$

4.13 MEAN AND DEVIATORIC STRAIN

The concept of mean and deviatoric strain is explained for linear strain, and the same can be extended to other types of strain. A strain tensor can be decomposed as

$$\varepsilon_{ij} = \left(\frac{1}{3} \varepsilon_{kk} \right) \delta_{ij} + \varepsilon'_{ij} \quad (4.156)$$

where ε' is the deviatoric part of the strain tensor. Putting $j = i$ in Equation 4.156,

$$\varepsilon_{ii} = \left(\frac{1}{3} \varepsilon_{kk} \right) \delta_{ii} + \varepsilon'_{ii}. \quad (4.157)$$

As $\delta_{ii} = 3$, one gets

$$\varepsilon_{ii} = \varepsilon_{kk} + \varepsilon'_{ii} \quad (4.158)$$

As k is a dummy index, it can be replaced by i . Thus, one gets

$$\varepsilon'_{ii} = 0. \quad (4.159)$$

Thus, the trace of the deviatoric part of the strain tensor is zero. The other part of the strain tensor is called the mean strain. Thus, the mean strain is equal to one-third of the trace of the strain tensor.

4.14 MOHR'S CIRCLE FOR STRAIN

Similar to Mohr's circle for stress, Mohr's circle for strain can be constructed. Figure 4.3 shows a typical Mohr's circle. The abscissa represents the normal strain, and the ordinate represents half of the engineering shear strain. Circle cuts the abscissa at points A and B, whose x -coordinates are the principal strains. The diameter of the circle is equal to the maximum engineering shear strain. If the shear strain is positive, the point representing the x -axis is plotted at a distance $\gamma_\theta/2$ below the ε_θ -axis and the point representing the y -axis is plotted at a distance $\gamma_\theta/2$ above the ε_θ -axis. If the shear strain is negative, the point representing the x -axis is plotted at a distance $\gamma_\theta/2$ above the ε_θ -axis and the point representing the y -axis is plotted at a distance $\gamma_\theta/2$ below the ε_θ -axis. As in the case of Mohr's stress circle, the relative angular position in the physical system is doubled in Mohr's circle, but the orientation remains the same.

In the case of the plane strain, the strain components are ε_{xx} , ε_{yy} and γ_{xy} . Usually, the normal strains in three directions are measured with the help of a strain rosette. With the knowledge of three normal strains at different orientations, the principal strains and shear strains can be calculated.

Example 4.14

A 45° strain rosette consists of three strain gages that measure the normal strain in three directions, in which the angle between orientation one and

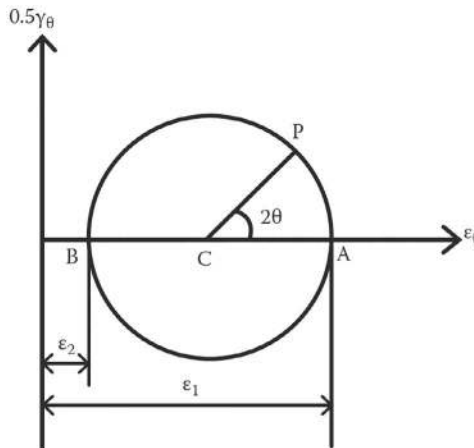


FIGURE 4.3 A two-dimensional Mohr circle for strain.

orientation two is 45° and the angle between orientation two and orientation three is 45° . Assume that the strain of an angle ϕ from the maximum principal strain direction is 1×10^{-3} , the strain at angle $(45^\circ + \phi)$ from the maximum principal strain direction is 2×10^{-3} , and the strain from an angle of $(90^\circ + \phi)$ from the maximum principal strain direction is 4×10^{-3} . These are the readings of three strain gages of the 45° strain rosette. Find the magnitude of principal strains in the plane of the rosette.

SOLUTION

Let the principal strain be ε_1 and ε_2 . From Figure 4.3, it is clear that the normal strain at the orientation θ is given by

$$\varepsilon_\theta = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) + \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos 2\theta.$$

Using this formula, for three orientations,

$$1 \times 10^{-3} = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) + \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos 2\phi \quad (4.160)$$

$$\begin{aligned} 2 \times 10^{-3} &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2) + \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos 2(45^\circ + \phi) \\ &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2) - \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\sin 2\phi \end{aligned} \quad (4.161)$$

$$\begin{aligned} 4 \times 10^{-3} &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2) + \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos 2(90^\circ + \phi) \\ &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2) - \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos 2\phi \end{aligned} \quad (4.162)$$

From Equations 4.160 and 4.162,

$$(\varepsilon_1 + \varepsilon_2) = 5 \times 10^{-3} \quad (4.163)$$

Substituting Equation 4.163 in Equations 4.160 and 4.161,

$$(\varepsilon_1 - \varepsilon_2)\cos 2\phi = -3 \times 10^{-3} \quad (4.164)$$

and

$$(\varepsilon_1 - \varepsilon_2)\sin 2\phi = 1 \times 10^{-3} \quad (4.165)$$

Squaring Equations 4.164 and 4.165 and adding,

$$(\varepsilon_1 - \varepsilon_2)^2 = 10 \times 10^{-6}. \quad (4.166)$$

It is already assumed that ε_1 is greater than ε_2 ; hence, taking the positive square root of Equation 4.166,

$$(\varepsilon_1 - \varepsilon_2) = 3.1622 \times 10^{-3}. \quad (4.167)$$

Equations 4.163 and 4.166 provide

$$\varepsilon_1 = 4.0811 \times 10^{-3}, \varepsilon_2 = 0.9189 \times 10^{-3}. \quad (4.168)$$

The direction of principal strains can also be found by calculating the angle ϕ .

4.15 INCREMENTAL STRAIN TENSOR

4.15.1 INTRODUCTION

In Sections 4.4–4.6, three measures of large (i.e. finite) deformation have been discussed, namely, the Green strain, the Almansi strain and the logarithmic strain. The plastic deformation is certainly large. So, one might think that one of these strain measures would be an appropriate measure to use in the elasto-plastic stress-strain relation. However, in elasto-plastic behavior, the stress does not depend uniquely on the current value of strain. Instead, it depends on the history of deformation, that is, it depends on whether there is any unloading and the stress level at which the unloading takes place. Further, the behavior in unloading is elastic. (This point will be discussed in more detail in Section 5.3.1.) So, the convenient way to describe this behavior is to express it either in the incremental form or in the rate form. In this section, some measures of incremental deformation are discussed. The rate measure of deformation is discussed in the next section. First, the incremental linear strain tensor is discussed.

4.15.2 INCREMENTAL LINEAR STRAIN TENSOR

Figure 4.4 shows three configurations in a beam-bending problem: (1) *initial configuration*; (2) deformed configuration at current time t (called the *current configuration*); and (3) deformed configuration at time $t + dt$, where dt is the time increment (called the *incremental configuration*). A material particle occupies position P_0 in the initial configuration, position P in the current configuration and position P' in the incremental configuration. Let

$$\mathbf{x} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \quad (4.169)$$

be the current position vector of the particle (i.e. the position vector of point P) and

$$d\mathbf{u} = du_x\hat{\mathbf{i}} + du_y\hat{\mathbf{j}} + du_z\hat{\mathbf{k}} \quad (4.170)$$

be the incremental displacement vector (of point P) in time dt . Here, $(\hat{i}, \hat{j}, \hat{k})$ are the unit vectors along the (x, y, z) axes.

It is seen in Section 4.2 that the gradient of a vector is a tensor. Therefore, the quantity $\nabla(\mathbf{du})$, which is a gradient of the vector $\mathbf{du}$ with respect to the current position vector $\mathbf{x}$, is a tensor. Since the gradient is not with respect to the initial configuration, the operator ∇ here does not have the subscript zero like the gradient symbol in Section 4.2. The components of $\nabla(\mathbf{du})$ with respect to the (x, y, z) coordinate system are given by

$$[\nabla(\mathbf{du})] = \begin{bmatrix} \frac{\partial(du_x)}{\partial x} & \frac{\partial(du_x)}{\partial y} & \frac{\partial(du_x)}{\partial z} \\ \frac{\partial(du_y)}{\partial x} & \frac{\partial(du_y)}{\partial y} & \frac{\partial(du_y)}{\partial z} \\ \frac{\partial(du_z)}{\partial x} & \frac{\partial(du_z)}{\partial y} & \frac{\partial(du_z)}{\partial z} \end{bmatrix}. \quad (4.171)$$

It is shown in Section 4.2.1 that if $\mathbf{u}$ is the displacement from the initial configuration to a deformed configuration, then the symmetric part of the gradient of $\mathbf{u}$ (with respect to the position vector in the initial configuration) can be chosen as a measure of that deformation, provided that the deformation is small. Now, *assume that the incremental deformation from the current configuration is small.* (Note that, mathematically, this assumption means the components of the tensor $\nabla(\mathbf{du})$ are

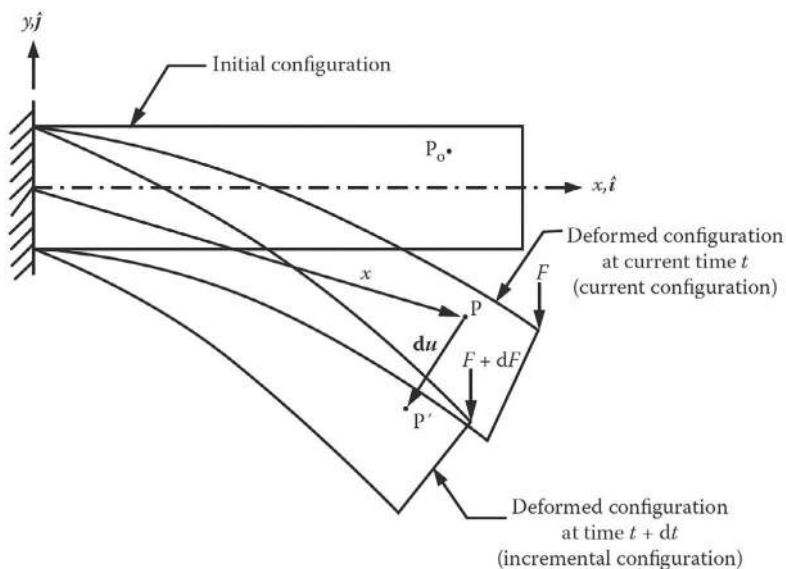


FIGURE 4.4 Incremental deformation in the time interval dt in beam bending.

small compared to one.) Since $\mathbf{du}$ is the incremental displacement from the current configuration, the symmetric part of the tensor $\nabla(\mathbf{du})$ can be selected as a measure of the incremental deformation. Thus, our measure of incremental deformation is the tensor $\mathbf{d}\boldsymbol{\varepsilon}$, which is the symmetric part of $\nabla(\mathbf{du})$. It is called the *incremental linear strain tensor*. The relation between $\mathbf{d}\boldsymbol{\varepsilon}$ and $\mathbf{du}$ can be written, in tensor notation, as

$$\mathbf{d}\boldsymbol{\varepsilon} = \frac{1}{2} \left(\nabla(\mathbf{du}) + (\nabla(\mathbf{du}))^T \right), \quad (4.172)$$

and in index notation as

$$d\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial(du_i)}{\partial x_j} + \frac{\partial(du_j)}{\partial x_i} \right) = \frac{1}{2} (du_{i,j} + du_{j,i}), \quad (4.173)$$

where it is understood that the comma indicates the differentiation with respect to the current coordinates. Let the components of $d\varepsilon$ with respect to the (x, y, z) coordinate system be

$$[d\varepsilon] = \begin{bmatrix} d\varepsilon_{xx} & d\varepsilon_{xy} & d\varepsilon_{zx} \\ d\varepsilon_{xy} & d\varepsilon_{yy} & d\varepsilon_{yz} \\ d\varepsilon_{zx} & d\varepsilon_{yz} & d\varepsilon_{zz} \end{bmatrix}. \quad (4.174)$$

Then using Equation 4.171, the component form of Equation 4.172 can be written as

$$\begin{aligned} d\varepsilon_{xx} &= \frac{\partial(du_x)}{\partial x}, d\varepsilon_{yy} = \frac{\partial(du_y)}{\partial y}, d\varepsilon_{zz} = \frac{\partial(du_z)}{\partial z}, \\ d\varepsilon_{xy} &= \frac{1}{2} \left(\frac{\partial(du_x)}{\partial y} + \frac{\partial(du_y)}{\partial x} \right), \\ d\varepsilon_{yz} &= \frac{1}{2} \left(\frac{\partial(du_y)}{\partial z} + \frac{\partial(du_z)}{\partial y} \right), \\ d\varepsilon_{zx} &= \frac{1}{2} \left(\frac{\partial(du_z)}{\partial x} + \frac{\partial(du_x)}{\partial z} \right). \end{aligned} \quad (4.175)$$

Equation (4.175) represents the *incremental strain-displacement relations*.

To find the incremental displacements, incremental strains and incremental stresses in a deformable body, one needs to solve three sets of incremental equations in the current configuration. Such a formulation is called the *updated Lagrangian formulation*, which is described in more detail later. The previous equation is *one set of governing equations* for this formulation when the incremental deformation is small.

The physical interpretation of the components of $\mathbf{d}\boldsymbol{\varepsilon}$ is similar to that of the components of the linear strain tensor. The component $d\varepsilon_{xx}$ represents the *change in current length per unit current length* along the direction, which is currently along the x -direction. The components $d\varepsilon_{yy}$ and $d\varepsilon_{zz}$ have a similar interpretation. The component $d\varepsilon_{xy}$ represents *half the change in angle* between the directions, which are currently along the x - and y -directions. The components $d\varepsilon_{yz}$ and $d\varepsilon_{zx}$ have similar interpretations. The sign convention for the components of $\mathbf{d}\boldsymbol{\varepsilon}$ is the same as that of the components of the linear strain tensor.

Just like the other strain tensors of Chapter 4, the tensor $\mathbf{d}\boldsymbol{\varepsilon}$ has the principal values, principal directions, principal invariants and the hydrostatic and deviatoric parts. They are defined similarly. The incremental volumetric strain $d\varepsilon_v$, when the incremental deformation is small, is defined by an equation similar to Equation 4.155:

$$d\varepsilon_v = d\varepsilon_{ii}. \quad (4.176)$$

4.15.3 INCREMENTAL INFINITESIMAL ROTATION TENSOR

Define the tensor $\mathbf{d}\boldsymbol{\omega}$ as the antisymmetric part of $\nabla(\mathbf{d}\mathbf{u})$:

$$\mathbf{d}\boldsymbol{\omega} = \frac{1}{2} \left[\nabla(\mathbf{d}\mathbf{u}) - \{ \nabla(\mathbf{d}\mathbf{u}) \}^T \right]. \quad (4.177)$$

It can be shown that the tensor $\mathbf{d}\boldsymbol{\omega}$ represents the incremental rotation of a neighborhood of the particle in time dt , if the rotation is small (i.e. if the components of the tensor $\nabla(\mathbf{d}\mathbf{u})$ are small compared to one). It is called the *incremental infinitesimal rotation tensor*. In index notation, Equation 4.177 can be written as

$$d\omega_{ij} = \frac{1}{2} \left(\frac{\partial (du_i)}{\partial x_j} - \frac{\partial (du_j)}{\partial x_i} \right) = \frac{1}{2} (du_{i,j} - du_{j,i}), \quad (4.178)$$

where it is understood that the comma indicates the differentiation with respect to the current coordinates. Let the components of $\mathbf{d}\boldsymbol{\omega}$ with respect to the (x, y, z) coordinate system be

$$[\mathbf{d}\boldsymbol{\omega}] = \begin{bmatrix} d\omega_{xx} & d\omega_{xy} & d\omega_{xz} \\ d\omega_{yx} & d\omega_{yy} & d\omega_{yz} \\ d\omega_{zx} & d\omega_{zy} & d\omega_{zz} \end{bmatrix}. \quad (4.179)$$

Then using Equation 4.171, the component form of Equation 4.177 can be written as

$$\begin{aligned} d\omega_{xx} &= d\omega_{yy} = d\omega_{zz} = 0, \\ d\omega_{zy} &= -d\omega_{yz} = \frac{1}{2} \left(\frac{\partial (du_z)}{\partial y} - \frac{\partial (du_y)}{\partial z} \right), \end{aligned}$$

$$\begin{aligned}
d\omega_{xz} &= -d\omega_{zx} = \frac{1}{2} \left(\frac{\partial(du_x)}{\partial z} - \frac{\partial(du_z)}{\partial x} \right), \\
d\omega_{yx} &= -d\omega_{xy} = \frac{1}{2} \left(\frac{\partial(du_y)}{\partial x} - \frac{\partial(du_x)}{\partial y} \right).
\end{aligned} \tag{4.180}$$

The components $d\omega_{zy}$, $d\omega_{xz}$ and $d\omega_{yx}$ represent the angle of incremental rotation, respectively, about x -, y - and z -axes. Their sign convention is similar to that for the components of the infinitesimal rotation tensor.

4.16 MATERIAL AND LOCAL TIME DERIVATIVE

Before discussing the rate of deformation, the material time derivative and local time derivative shall be discussed. Assume that there is a variable f , which is expressed as a function of initial coordinates and time. Thus,

$$f = f(\mathbf{x}_0, t). \tag{4.181}$$

The coordinate $\mathbf{x}_0$ designates a particle. Thus, the function f provides the value of the variable for each particle at any time. The rate of change of the variable with respect to time can be obtained for each particle. It is defined as

$$\frac{Df}{Dt} = \left. \frac{\partial f(\mathbf{x}_0, t)}{\partial t} \right|_{\mathbf{x}_0}. \tag{4.182}$$

Here, the derivative with respect to time is taken keeping $\mathbf{x}_0$ as fixed. This is called the material time derivative of the variable. For example, if the velocity field is given by

$$\mathbf{v} = \mathbf{v}(\mathbf{x}_0, t), \tag{4.183}$$

then the material time derivative of the velocity is given by

$$\frac{D\mathbf{v}}{Dt} = \left. \frac{\partial \mathbf{v}(\mathbf{x}_0, t)}{\partial t} \right|_{\mathbf{x}_0}. \tag{4.184}$$

Equation 4.184 provides the acceleration of each particle as a function of the time.

Now, let us consider that the value of a variable is known as a function of spatial coordinates and time, that is,

$$\phi = \phi(\mathbf{x}, t). \tag{4.185}$$

The rate of change of the variable with time can be observed at any particular location. This is called the local time derivative and is expressed mathematically as

$$\frac{\partial \phi}{\partial t} = \left. \frac{\partial \phi(\mathbf{x}, t)}{\partial t} \right|_{\mathbf{x}}. \quad (4.186)$$

Here, the derivative with respect to time is taken keeping $\mathbf{x}$ as fixed. Thus, if the velocity field is given by

$$\mathbf{v} = \mathbf{v}(\mathbf{x}, t), \quad (4.187)$$

then the local time derivative of the velocity is given as

$$\frac{\partial \mathbf{v}}{\partial t} = \left. \frac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial t} \right|_{\mathbf{x}}. \quad (4.188)$$

Note that Equation 4.188 does not tell the acceleration of a particle as a function of time. It just tells how the velocities of particles passing through a point $\mathbf{x}$ are changing. If there is no change in the velocities of the particles passing through any arbitrary point $\mathbf{x}$, the motion is said to be in a steady state. A steady-state motion does not imply that the acceleration of the particles is zero. It just means that the velocities of all the particles passing through the arbitrary spatial point remain the same at all times. However, any individual particle may undergo a change in the velocity. For example, when there is a steady flow of water in a nozzle, the velocity of a particle keeps increasing as it moves forward, and thus, it experiences acceleration. However, all particles passing through a point will experience the same velocity, although they pass through that point at different times.

The material derivative can also be expressed in spatial form. Referring to Equation 4.185, the material time derivative can be found using the chain rule. Thus, for any scalar function of spatial coordinates and time

$$\frac{D\phi}{Dt} = \left. \frac{\partial \phi}{\partial t} \right|_{\mathbf{x}_0} = \left. \frac{\partial \phi}{\partial t} \right|_{\mathbf{x}} + \left. \frac{\partial \phi(\mathbf{x}, t)}{\partial x_i} \right|_t \left. \frac{\partial x_i}{\partial t} \right|_{\mathbf{x}_0}, \quad (4.189)$$

or

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + v_i \phi_{,i} = \frac{\partial \phi}{\partial t} + (\mathbf{v} \cdot \nabla) \phi. \quad (4.190)$$

Here, $(\mathbf{v} \cdot \nabla)\phi$ called the convective rate of ϕ .

Example 4.15

At a point $\mathbf{x}$ in the current configuration of a fluid, the temperature is given by

$$T = t(x + y^2 + 2z).$$

The velocity field of the fluid has the following components:

$$v_x = xt, v_y = yt, v_z = zt.$$

Find out the material rate of change of T .

SOLUTION

The material rate of change of T is given by Equation 4.190, with ϕ replaced by T . Thus,

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z}.$$

Now,

$$\frac{\partial T}{\partial t} = (x + y^2 + 2z), \frac{\partial T}{\partial x} = t, \frac{\partial T}{\partial y} = 2yt, \frac{\partial T}{\partial z} = 2t.$$

Hence,

$$\frac{DT}{Dt} = (x + y^2 + 2z) + xt^2 + 2y^2t^2 + 2zt^2.$$

4.17 RATE OF DEFORMATION TENSOR

Let $\mathbf{x}$ be the position vector of a point P , that is,

$$\mathbf{x} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3. \quad (4.191)$$

Let $\mathbf{v}$ be the velocity at point $\mathbf{x}$, that is,

$$\mathbf{v} = v_x\mathbf{e}_1 + v_y\mathbf{e}_2 + v_z\mathbf{e}_3. \quad (4.192)$$

The spatial gradient of velocity, $\nabla\mathbf{v}$, is called the velocity gradient tensor. Its components in the Cartesian system are given by

$$[\nabla\mathbf{v}] = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial z} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} \\ \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} \end{bmatrix}. \quad (4.193)$$

In many books on continuum, the velocity gradient tensor is denoted by $\mathbf{L}$. The symmetric part of the velocity gradient tensor is called the strain rate tensor or rate of deformation tensor. Thus,

$$\dot{\boldsymbol{\varepsilon}} = \frac{1}{2} \left\{ \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right\}. \quad (4.194)$$

For Cartesian coordinates, using index notation,

$$\dot{\varepsilon}_{ij} = \frac{1}{2} (v_{i,j} + v_{j,i}). \quad (4.195)$$

In many books of continuum mechanics, the strain rate tensor is called the rate of deformation tensor and is denoted by $\mathbf{D}$. This is done to avoid confusion considering that this tensor is not the time derivative of the linear strain tensor.

The material time derivative of the linear strain is given by

$$\begin{aligned} \frac{D\boldsymbol{\varepsilon}}{Dt} &= \frac{1}{2} \frac{D}{Dt} \left\{ \nabla^0 \mathbf{u} + (\nabla^0 \mathbf{u})^T \right\} \\ &= \frac{1}{2} \left\{ \nabla^0 \frac{D\mathbf{u}}{Dt} + \left(\nabla^0 \frac{D\mathbf{u}}{Dt} \right)^T \right\} \\ &= \frac{1}{2} \left\{ \nabla^0 \mathbf{v} + (\nabla^0 \mathbf{v})^T \right\}. \end{aligned} \quad (4.196)$$

For small deformation, the derivative with initial coordinates is the same as the derivative with current coordinates. Hence, for small deformation,

$$\frac{D\boldsymbol{\varepsilon}}{Dt} \simeq \frac{1}{2} \left\{ \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right\} = \dot{\boldsymbol{\varepsilon}}. \quad (4.197)$$

It can be easily shown that the rate of deformation tensor defined by Equation 4.194 is sufficient for finding out the rate of change of the length of an infinitesimal line element passing through a point. Also, it can find the rate of change of the angle between two lines passing through the point. To see it, consider that the infinitesimal line element is $d\mathbf{x}$ in the direction of unit vector $\mathbf{n}$ and the length of the line element is ds . Thus,

$$d\mathbf{x} = (ds) \mathbf{n}. \quad (4.198)$$

Taking the material time derivative of the previous equation with respect to time,

$$\frac{D}{Dt} (d\mathbf{x}) = (ds) \frac{D\mathbf{n}}{Dt} + \mathbf{n} \frac{D}{Dt} (ds). \quad (4.199)$$

Now,

$$\frac{D}{Dt} (d\mathbf{x}) = \frac{D}{Dt} (\mathbf{F} d\mathbf{x}_0) = \frac{D\mathbf{F}}{Dt} d\mathbf{x}_0. \quad (4.200)$$

It can be shown (see Example 4.16) that

$$\frac{D\mathbf{F}}{Dt} = (\nabla \mathbf{v})\mathbf{F}. \quad (4.201)$$

Hence,

$$\frac{D}{Dt}(\mathbf{dx}) = (\nabla \mathbf{v})\mathbf{F} \mathbf{dx}_0 = (\nabla \mathbf{v})\mathbf{dx} = (ds)(\nabla \mathbf{v})\mathbf{n}. \quad (4.202)$$

Thus, Equation 4.199 can be written as

$$(ds)(\nabla \mathbf{v})\mathbf{n} = (ds)\frac{D\mathbf{n}}{Dt} + \mathbf{n}\frac{D}{Dt}(ds). \quad (4.203)$$

Taking the dot product of each term in the previous equation with $\mathbf{n}$,

$$(ds)\mathbf{n} \cdot (\nabla \mathbf{v})\mathbf{n} = (ds)\mathbf{n} \cdot \frac{D\mathbf{n}}{Dt} + \mathbf{n} \cdot \mathbf{n} \frac{D}{Dt}(ds). \quad (4.204)$$

Observe that

$$\frac{D(\mathbf{n} \cdot \mathbf{n})}{Dt} = \frac{D(1)}{Dt} = 0 = 2\mathbf{n} \cdot \frac{D\mathbf{n}}{Dt}. \quad (4.205)$$

Thus, Equation 4.204 becomes

$$\frac{1}{(ds)} \frac{D}{Dt}(ds) = \mathbf{n} \cdot (\nabla \mathbf{v})\mathbf{n} = \mathbf{n} \cdot \frac{1}{2} \left\{ (\nabla \mathbf{v}) + (\nabla \mathbf{v})^T \right\} \mathbf{n} = \mathbf{n} \cdot \dot{\boldsymbol{\varepsilon}} \mathbf{n}. \quad (4.206)$$

Equation 4.206 can be used to find the rate of stretching in any direction. For the x -direction, the component of vector $\mathbf{n}$ is given by $\{1 \ 0 \ 0\}^T$. Hence, the rate of stretching (per unit rate of change of length) in the x -direction is given by $\dot{\varepsilon}_{xx}$. Similarly, $\dot{\varepsilon}_{yy}$ and $\dot{\varepsilon}_{zz}$ represent the rate of stretching in the y - and z -directions, respectively.

Now, consider two unit vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ in the current configuration. If the angle between them is θ , then

$$\cos\theta = \mathbf{n}_1 \cdot \mathbf{n}_2. \quad (4.207)$$

Taking the material time derivative,

$$-\sin\theta \frac{D\theta}{Dt} = \mathbf{n}_1 \cdot \frac{D\mathbf{n}_2}{Dt} + \mathbf{n}_2 \cdot \frac{D\mathbf{n}_1}{Dt}. \quad (4.208)$$

From Equation 4.203,

$$\frac{D\mathbf{n}}{Dt} = (\nabla \mathbf{v})\mathbf{n} - \frac{1}{(ds)} \frac{D}{Dt}(ds)\mathbf{n}. \quad (4.209)$$

Due to Equation 4.206, Equation 4.209 can be written as

$$\frac{D\mathbf{n}}{Dt} = (\nabla \mathbf{v})\mathbf{n} - (\mathbf{n} \cdot \dot{\mathbf{e}}\mathbf{n})\mathbf{n}. \quad (4.210)$$

Hence,

$$\frac{D\mathbf{n}_1}{Dt} = (\nabla \mathbf{v})\mathbf{n}_1 - (\mathbf{n}_1 \cdot \dot{\mathbf{e}}\mathbf{n}_1)\mathbf{n}_1; \quad \frac{D\mathbf{n}_2}{Dt} = (\nabla \mathbf{v})\mathbf{n}_2 - (\mathbf{n}_2 \cdot \dot{\mathbf{e}}\mathbf{n}_2)\mathbf{n}_2. \quad (4.211)$$

Therefore,

$$\begin{aligned} \mathbf{n}_1 \cdot \frac{D\mathbf{n}_2}{Dt} + \mathbf{n}_2 \cdot \frac{D\mathbf{n}_1}{Dt} &= \mathbf{n}_1 \cdot (\nabla \mathbf{v})\mathbf{n}_2 + \mathbf{n}_2 \cdot (\nabla \mathbf{v})\mathbf{n}_1 \\ &\quad - (\mathbf{n}_1 \cdot \mathbf{n}_2)(\mathbf{n}_1 \cdot \dot{\mathbf{e}}\mathbf{n}_1 + \mathbf{n}_2 \cdot \dot{\mathbf{e}}\mathbf{n}_2) \end{aligned} \quad (4.212)$$

It is easily seen that

$$\mathbf{n}_2 \cdot (\nabla \mathbf{v})\mathbf{n}_1 = \mathbf{n}_1 \cdot (\nabla \mathbf{v})^T \mathbf{n}_2 \quad (4.213)$$

Thus,

$$\mathbf{n}_1 \cdot \frac{D\mathbf{n}_2}{Dt} + \mathbf{n}_2 \cdot \frac{D\mathbf{n}_1}{Dt} = 2\mathbf{n}_1 \cdot \dot{\mathbf{e}}\mathbf{n}_2 - (\mathbf{n}_1 \cdot \mathbf{n}_2)(\mathbf{n}_1 \cdot \dot{\mathbf{e}}\mathbf{n}_1 + \mathbf{n}_2 \cdot \dot{\mathbf{e}}\mathbf{n}_2). \quad (4.214)$$

Substituting Equation 4.214 in Equation 4.208,

$$\sin\theta \frac{D\theta}{Dt} = -2\mathbf{n}_1 \cdot \dot{\mathbf{e}}\mathbf{n}_2 + (\mathbf{n}_1 \cdot \mathbf{n}_2)(\mathbf{n}_1 \cdot \dot{\mathbf{e}}\mathbf{n}_1 + \mathbf{n}_2 \cdot \dot{\mathbf{e}}\mathbf{n}_2) \quad (4.215)$$

For $\theta = \pi/2$, Equation 4.215 becomes

$$\mathbf{n}_1 \cdot \dot{\mathbf{e}}\mathbf{n}_2 = -\frac{1}{2} \frac{D\theta}{Dt}. \quad (4.216)$$

The term on the right-hand side may be called the shearing rate. In many textbooks, the term on the right-hand side is considered as half the shearing rate. Notwithstanding the convention adopted by different textbooks, the rate of deformation tensor can exactly find the shearing rate.

Example 4.16

Prove that

$$\frac{D\mathbf{F}}{Dt} = (\nabla \mathbf{v})\mathbf{F}$$

SOLUTION

Index notation shall be used to prove it.

$$\begin{aligned}\frac{D}{Dt}(x_{i;j}) &= \frac{\partial}{\partial t} \left(\frac{\partial x_i}{\partial x_{0j}} \right) \bigg|_{x_{0k}} = \frac{\partial}{\partial x_{0j}} \left(\frac{\partial x_i}{\partial t} \right) \bigg|_{x_{0k}} \\ &= \frac{\partial}{\partial x_{0j}} \left(\frac{Dx_i}{Dt} \right) = v_{i;j} = v_{i,k} x_{k;j}\end{aligned}$$

Therefore,

$$\frac{D\mathbf{F}}{Dt} = (\nabla \mathbf{v}) \mathbf{F}.$$

4.18 SPIN TENSOR

The unsymmetrical part of the velocity gradient

$$\dot{\omega} = \frac{1}{2} \left\{ \nabla \mathbf{v} - (\nabla \mathbf{v})^T \right\} \quad (4.217)$$

is called spin or vorticity tensor. In index notation, it is written as

$$\dot{\omega}_{i,j} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right) = \frac{1}{2} (v_{i,j} - v_{j,i}). \quad (4.218)$$

In expanded notation, its components are given as

$$\begin{aligned}\dot{\omega}_{xx} &= \dot{\omega}_{yy} = \dot{\omega}_{zz} = 0, \\ \dot{\omega}_{zy} &= -\dot{\omega}_{yz} = \frac{1}{2} \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right), \\ \dot{\omega}_{xz} &= -\dot{\omega}_{zx} = \frac{1}{2} \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right), \\ \dot{\omega}_{yx} &= -\dot{\omega}_{xy} = \frac{1}{2} \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right).\end{aligned} \quad (4.219)$$

Following the method similar to Section 4.2.2, it can be shown that $\dot{\omega}_{zy}$, $\dot{\omega}_{xz}$ and $\dot{\omega}_{yx}$ represent the components of average angular velocity, respectively, about the x -, y - and z -axes. It can also be seen that

$$\dot{\omega}_{zy} \mathbf{e}_1 + \dot{\omega}_{xz} \mathbf{e}_2 + \dot{\omega}_{yx} \mathbf{e}_3 = \frac{1}{2} \text{curl} \mathbf{v}. \quad (4.220)$$

4.19 ON RELATION BETWEEN INCREMENTAL STRAIN AND STRAIN RATE TENSORS

Note that the symbol $\mathbf{du}$ in Section 4.15.2 does not represent *the relative displacement* of a particle with respect to its neighboring particle. Instead, it is an *infinitesimal change in the displacement of a single particle in time dt* (Figure 4.4). Then, the velocity $\mathbf{v}$ of the particle would be $\mathbf{du}/dt$. In view of this relation between $\mathbf{du}$ and $\mathbf{v}$, a comparison of Equations 4.170 and 4.194 gives the following relation between the incremental linear strain tensor $\mathbf{d}\boldsymbol{\varepsilon}$ and the strain rate tensor $\dot{\boldsymbol{\varepsilon}}$:

$$\mathbf{d}\boldsymbol{\varepsilon} = \dot{\boldsymbol{\varepsilon}} dt, \quad (4.221)$$

or in index notation

$$d\varepsilon_{ij} = \dot{\varepsilon}_{ij} dt. \quad (4.222)$$

Further, one gets a similar relation between the incremental infinitesimal rotation tensor $d\boldsymbol{\omega}$ and the spin tensor $\dot{\boldsymbol{\omega}}$:

$$d\boldsymbol{\omega} = \dot{\boldsymbol{\omega}} dt \quad (4.223)$$

or

$$d\omega_{ij} = \dot{\omega}_{ij} dt. \quad (4.224)$$

Equation 4.222 is used to find what happens when the strain rate tensor $\dot{\boldsymbol{\varepsilon}}$ is integrated. For this purpose, a very simple deformation is considered. The deformation in the control volume of Figure 4.5 is such that the particle paths are straight lines parallel to the x -axis. It means that the particles do not rotate, and therefore both $\dot{\boldsymbol{\omega}}$ and $d\boldsymbol{\omega}$ are 0 at every point of the control volume. Consider a typical particle path where the length of the particle is ℓ_0 when it enters the control volume (i.e. at time $t = t_0$) and ℓ_f when it leaves the control volume (i.e. at time $t = t_f$).

Further, let the length at time t be ℓ and the length increment in time dt be $d\ell$. Then, from Equation 4.222 and the definition of $d\varepsilon_{xx}$, one gets

$$\dot{\varepsilon}_{xx} dt = d\varepsilon_{xx} = \frac{d\ell}{\ell}. \quad (4.225)$$

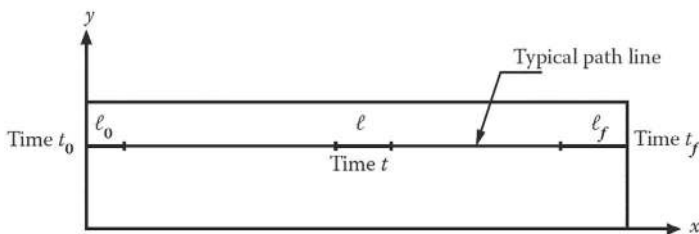


FIGURE 4.5 Simple deformation in which particle path lines are parallel to the x -axis.

Integration of $\dot{\epsilon}_{xx}$ from $t = t_0$ to $t = t_f$ gives

$$\int_{t_0}^{t_f} \dot{\epsilon}_{xx} dt = \int_{t_0}^{t_f} d\epsilon_{xx} = \int_{\ell_0}^{\ell_f} \frac{d\ell}{\ell} = \ln \frac{\ell_f}{\ell_0} = \epsilon_f \neq \epsilon_{xx}. \quad (4.226)$$

Here, ϵ_f is the value of the one-dimensional logarithmic strain ϵ at $t = t_f$.

Thus, when a normal strain rate component is integrated, the *logarithmic strain* is obtained in that direction and *not* the corresponding component of the linear strain tensor. This result is based on the assumption that there is *no rotation* of particles. When the rotation of particles is present, no physically meaningful quantity emerges from the integration of a strain rate component (Malvern, 1969). Generalizing this result to a three-dimensional case, it can be stated that the integration of the strain rate tensor $\dot{\epsilon}$ gives the logarithmic strain tensor when there is no rotation of the particles. When the rotation of the particles is present, the integration of $\dot{\epsilon}$ does not give any physically meaningful tensor.

4.20 COMPATIBILITY CONDITIONS

In a symmetric strain tensor field, there are six functions of strain components. These are integrated to find the three displacement components. It is not possible to assign arbitrary functions to the strain components because some combinations may provide discontinuous values of displacement components. Thus, strain components need to satisfy certain compatibility conditions.

Consider the case of an infinitesimal strain tensor. We have

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x}, \epsilon_{yy} = \frac{\partial u_y}{\partial y}, \epsilon_{xy} = \frac{1}{2} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right). \quad (4.227)$$

Differentiating the first strain component twice with respect to x , the second component twice with respect to y and the third component first with respect to y and then with respect to x , one gets

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} - 2 \frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} = 0. \quad (4.228)$$

This is one compatibility condition. Similarly, the other compatibility conditions can be obtained:

$$\frac{\partial^2 \epsilon_{yy}}{\partial z^2} + \frac{\partial^2 \epsilon_{zz}}{\partial y^2} - 2 \frac{\partial^2 \epsilon_{yz}}{\partial y \partial z} = 0, \quad (4.229)$$

$$\frac{\partial^2 \epsilon_{zz}}{\partial x^2} + \frac{\partial^2 \epsilon_{xx}}{\partial z^2} - 2 \frac{\partial^2 \epsilon_{zx}}{\partial z \partial x} = 0, \quad (4.230)$$

$$\frac{\partial^2 \epsilon_{xx}}{\partial y \partial z} - \frac{\partial}{\partial x} \left[-\frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z} \right] = 0, \quad (4.231)$$

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} - \frac{\partial}{\partial y} \left[\frac{\partial \varepsilon_{yz}}{\partial x} - \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} \right] = 0, \quad (4.232)$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} - \frac{\partial}{\partial z} \left[\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} - \frac{\partial \varepsilon_{xy}}{\partial z} \right] = 0. \quad (4.233)$$

It can be shown that out of the six compatibility conditions (Equations 4.228–4.233), only three are independent.

It can be shown that these conditions are sufficient only for simply connected regions. One needs additional compatibility conditions for multiple connected regions. For other strain measures, similar types of compatibility conditions can be derived.

Exercises

1. Consider the case of small deformation. The displacement at point (x_0, y_0, z_0) is given by

$$u_x = 1 \times 10^{-3} x_0, u_y = -0.3 \times 10^{-3} y_0, u_z = -0.3 \times 10^{-3} z_0.$$

Find the displacement gradient $\nabla_0 \mathbf{u}$. If the displacement gradient is calculated based on current coordinates, how much deviation will occur? Essentially, you have to calculate the norm of $(\nabla \mathbf{u} - \nabla_0)$ and compare it with the norm of $\nabla_0 \mathbf{u}$.

2. In a certain deformation, the displacement components are found to be

$$u_x = 3x_0^2 + y_0, \quad u_y = 2y_0^2 + z_0, \quad u_z = 4z_0^2 + x_0.$$

Find out the linear strain field. What is the linear strain tensor at the origin? What is the normal strain at the origin in a direction equally inclined to all the axes?

3. Consider the following relation between current and initial coordinates:

$$x = x_0^2, y = y_0^2, z = z_0^2,$$

where the domain of the coordinates is the first quadrant, that is, all the coordinates are positive. Find the deformation gradient in matrix form, $[F]$, and the determinant of the deformation gradient. Find $[F]^{-1}$ first by inverting $[F]$ and then by expressing the initial coordinates as a function of current coordinates and calculating $x_{0i,j}$.

4. Deformation defined by an equation of the form

$$x_i = a_{ij} x_{0j} + c_i,$$

where a_{ij} and c_i are functions of time, is called homogeneous deformation. Express this relation for homogeneous deformation in expanded form (three

scalar equations). Obtain the deformation gradient, and based on it, explain why this deformation is called homogeneous deformation. Find the condition that the homogeneous deformation represents rigid body transformation.

5. Show that the following measure may be adopted to represent the relative deviation of linear strain from Green strain:

$$\% \text{deviation} = \frac{(\nabla_0 \mathbf{u})^T (\nabla_0 \mathbf{u})_I}{(\nabla_0 \mathbf{u}) + (\nabla_0 \mathbf{u})^T + (\nabla_0 \mathbf{u})^T (\nabla_0 \mathbf{u})_I} \times 100.$$

Use this measure to find the percentage deviation at the origin for the deformation of Exercise Problem 4.3.

6. Prove that

$$\mathbf{B} = \mathbf{I} + \nabla_0 \mathbf{u} + \nabla_0 \mathbf{u}^T + (\nabla_0 \mathbf{u})(\nabla_0 \mathbf{u})^T.$$

7. Prove that

$$\nabla_0 \mathbf{u} = (\nabla \mathbf{u}) \mathbf{F}.$$

8. The displacement components in cylindrical polar coordinates are given as

$$u_r = -r(Az + B), \quad u_z = A\left(\frac{1}{2}r^2 + z^2\right) + 2Bz, \quad u_\theta = 0.$$

Find out the linear strain components.

9. Which of the following displacement components (in cylindrical polar coordinates) are not possible from a continuum point of view? Provide proper justification.

$$(A) u_\theta = 10 \quad (B) u_\theta = \theta \sin \theta \quad (C) u_\theta = \theta \cos \theta$$

The range of θ is from 0 to 2π .

10. Given

$$[\mathbf{F}] = \begin{bmatrix} 1 & 0.1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find out the logarithmic strain tensor.

11. Consider a strain tensor in the Cartesian system

$$[\varepsilon] = \begin{bmatrix} 0.01 & 0.003 & 0 \\ 0.003 & 0.002 & 0 \\ 0 & 0 & 0.006 \end{bmatrix}.$$

Transform this to a coordinate system, whose axes are

$$e'_1 = 0.3162e_1 - 0.9487e_2,$$

$$e'_2 = e_3,$$

$$e'_3 = -0.9487e'_1 - 0.3162e'_2.$$

12. a. Prove that the right stretch tensor $\mathbf{U}$ and left stretch tensor $\mathbf{V}$ are related by the following relation:

$$\mathbf{U} = \mathbf{Q}^T \mathbf{V} \mathbf{Q}.$$

- b. Prove that $\mathbf{U}$ and $\mathbf{V}$ have the same eigenvalues.
 c. Prove that the eigenvalues of $\mathbf{U}$ represent principal stretches.
 13. For small displacement gradients given by

$$[\nabla u] = 10^{-3} \begin{bmatrix} 4 & 1 & 4 \\ -1 & -4 & 0 \\ 0 & 2 & 6 \end{bmatrix},$$

Find out the octahedral normal and shear strain.

14. Show that the volumetric strain is given by

$$G_V = \frac{1}{2} \{(J+1)(J-1)\}.$$

Hence, find the value of J for isochoric (volume-preserving) deformation.
 Can volumetric strain be negative?

15. Find out the volumetric strain for the deformation field given by

$$x = x_0 + y_0,$$

$$y = x_0 - 2y_0,$$

$$z = x_0 + y_0 - z_0.$$

16. The velocity components in spatial coordinates are given by

$$v_x = \frac{x}{(1+t)}, v_y = \frac{y}{(1+t)}, v_z = \frac{3z}{(1+t)}.$$

Find out the acceleration in spatial and material form.

17. For the following velocity field, show that the motion is isochoric, that is, $\text{tr } \mathbf{D} = 0$:

$$v_x = -\frac{y}{x^2 + y^2}, v_y = \frac{x}{x^2 + y^2}, v_z = 0.$$

18. A deformation gradient is given by

$$[F] = \begin{bmatrix} 1/3 & 1/3 & 0 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix}.$$

Is this deformation field isochoric? Is the trace of linear displacement tensor zero in this case?

19. Prove the following theorem of Kirchhoff:

If the displacement vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ correspond to the same infinitesimal strain tensor, then $\mathbf{u}_1 - \mathbf{u}_2$ is a rigid displacement.

20. Show that in index notation, the compatibility conditions (Equation 4.228–4.233) can be written as

$$\varepsilon_{ij,kk} + \varepsilon_{kk,ij} - \varepsilon_{ik,kj} - \varepsilon_{jk,ki}.$$

ANSWERS

$$1. [\nabla_0 \mathbf{u}] = 10^{-3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.3 & 0 \\ 0 & 0 & -0.3 \end{bmatrix}$$

$$\frac{\nabla_0 \mathbf{u} - \nabla \mathbf{u}_1}{\nabla_0 \mathbf{u}_1} \times 100 = 0.1\%$$

$$2. [\varepsilon] = \begin{bmatrix} 6x_0 & 1/2 & 1/2 \\ 1/2 & 4y_0 & 1/2 \\ 1/2 & 1/2 & 8z_0 \end{bmatrix}$$

At $x_0 = 0$,

$$[\varepsilon] = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

Normal strain in the direction equally inclined to the axis is one.

$$3. [F] = \begin{bmatrix} 2x_0 & 0 & 0 \\ 0 & 2y_0 & 0 \\ 0 & 0 & 2z_0 \end{bmatrix}$$

$$[F]^{-1} = \begin{bmatrix} \frac{1}{2x_0} & 0 & 0 \\ 0 & \frac{1}{2y_0} & 0 \\ 0 & 0 & \frac{1}{2z_0} \end{bmatrix}$$

$$4. \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_0 \\ y_0 \\ z_0 \end{Bmatrix} + \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix}$$

$$[F] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = [a_{i,j}]$$

For rigid body transformation, $[a_{ij}]$ is orthogonal.

5. 33.33%

8. $\varepsilon_{rr} = -(Az + B), \varepsilon_{\theta\theta} = -(Az + B), \varepsilon_{zz} = 2(Az + B)$

Other components are 0.

9. A and C, because the values are different at $\theta = 0$ and $\theta = 2\pi$.

10. $\begin{bmatrix} 0.6863 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1.3187 \end{bmatrix}$

11. $\begin{bmatrix} 0.001 & 0 & 0 \\ 0 & 0.006 & 0 \\ 0 & 0 & 0.011 \end{bmatrix}$

13. $2 \times 10^{-3}, 9.3808 \times 10^{-3}$

14. $J = 1$, volumetric deformation can be negative.

15. 4

16. $a_x = 0, a_y = 0, a_z = \frac{6z}{(1+t)^2} = 6z_0(1+t)$

18. Yes, No

5 Incremental and Rate Type of Elastic-Plastic Constitutive Relations for Isotropic Materials, Objective Incremental Stress and Stress Rate Measures

5.1 INTRODUCTION

To determine the displacements, strains and stresses in a deformable body, one needs to solve the set of the following three governing equations: (1) strain-displacement or kinematic relation; (2) stress-strain or constitutive relation and (3) equation of motion. The equation of motion is related to the balance of forces and moments acting on the body and constitutes one of the postulates of mechanics. It must be satisfied by every rigid or deformable body. It has been described in Section 3.6. The strain-displacement relation describes the geometric changes (or the deformation) that a deformable body undergoes due to external forces acting on it. Depending on whether the deformation is infinitesimal (i.e. small) or finite (i.e. large), this relation has different forms. Further, the case of finite deformation can also be analyzed either incrementally or using the rate form of the measure of deformation. Chapter 4 describes all these cases: linear strain tensor for infinitesimal deformation; Green, Almansi and logarithmic strain tensors for finite deformation; incremental strain tensor for infinitesimal incremental deformation; and the rate of deformation tensor for the rate form.

The stress-strain relation is a type of constitutive relation. The constitutive relations describe various responses (like mechanical, thermal, electrical, etc.) of a material and hence are different for different materials. These relations are based on experimental observations. In this book, only pure mechanical response is considered, that is, the mechanical response like deformation caused by mechanical stimuli like forces and moments. At a point, such a response is usually expressed as a relation between the stress and an appropriate measure of deformation (i.e. strain) or the rate of deformation. There are three basic mechanical responses: (1) elastic, (2) plastic and (3) viscous. However, sometimes a material exhibits a combination of these basic

responses like elasto-plastic or visco-elastic. Further, the same material may respond differently over different ranges of deformation, like the metals, which are elastic at small deformation but become elasto-plastic at large deformation. Therefore, it is quite difficult to express the complete mechanical response of a material over the whole range of deformation by a single constitutive relation. As the title of the book suggests, in this chapter, expressing the elasto-plastic response of materials (at large deformation) in a mathematical form is of interest. However, it is instructive to first study the elastic response of materials at small deformation. In elastic response, the stress depends only on the current value of strain. Further, this relation is one-to-one. It means if the external forces acting on the body are removed (i.e. if the stress is reduced to zero value), the strain will also reduce to the zero value, thereby bringing the body to the original undeformed configuration.

5.2 ELASTIC STRESS-STRAIN RELATIONS FOR SMALL DEFORMATION

For the case of small deformation, as stated earlier, the appropriate measure of deformation is the linear strain tensor ϵ . Therefore, for small deformation, the constitutive equation becomes a relation between σ and ϵ . Since the constitutive equations are based on experimental observations, the experimental observations about the relation between σ and ϵ in the simplest experiment, that is, the *tension test*, are first studied.

5.2.1 ONE-DIMENSIONAL EXPERIMENTAL OBSERVATIONS

In tension tests, a rod of uniform cross section is stretched by an (axial) tensile force F_x , as shown in Figure 5.1. The geometry of the rod and the loading suggest that the state of stress is one-dimensional (1-D) and homogeneous in the region away from the ends. Then, the only non-zero stress component is σ_{xx} and it is constant. Further, the state of strain can also be assumed to be homogeneous in the region away from the ends. Additionally, the observed deformation suggests that the shear strain components are zero. Thus, there are three non-zero strain components, namely, ϵ_{xx} , ϵ_{yy} and ϵ_{zz} , and all are constant.

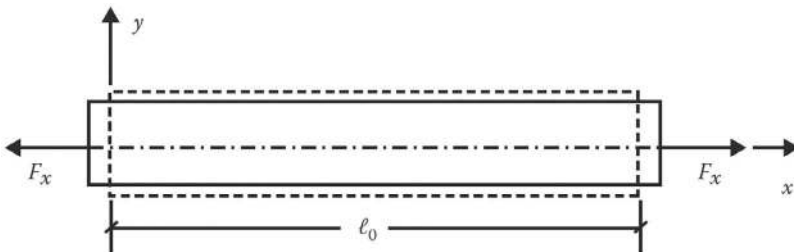


FIGURE 5.1 Rod stretched by axial tensile forces. The dashed lines indicate the undeformed configuration.

For the rod of Figure 5.1, the following is defined:

$$\sigma_0 = \frac{F_x}{A_0}, \quad (5.1)$$

$$e = \frac{\Delta \ell}{\ell_0}, \quad (5.2)$$

where A_0 is the initial area of the cross section of the rod, ℓ_0 is the initial length of the rod, and $\Delta \ell$ is the change in length corresponding to the (axial) tensile force F_x . Since this is a case of small deformation, the area A_0 does not change much. Therefore, σ_0 is almost equal to the σ_{xx} component of the stress tensor. However, this is not true for the case of large deformation. Therefore, σ_0 is called the *engineering* or *nominal stress*. Again, for the case of small deformation (i.e. when the change in length $\Delta \ell$ is small), e is almost equal to $\partial u / \partial x$ and thus approximately represents the ε_{xx} component of the *linear strain* tensor. However, for the case of large deformation, ε_{xx} or $\partial u / \partial x$ is not equal to e . Therefore, e is called the *engineering strain*.

Figure 5.2 shows the graph of σ_0 versus e up to fracture for a typical metal (i.e. mild steel). The figure shows that the variation of σ_0 with respect to e is linear when the deformation is small. As stated earlier, for the case of small deformation, σ_0 reduces to σ_{xx} and e to ε_{xx} . Thus, for small deformation, σ_{xx} varies linearly with ε_{xx} . It should be noted that the stress-strain relations need not be linear for all elastic materials. There are exceptions like rubber (an elastic material), for which the stress-strain relations are non-linear.

5.2.2 GENERALIZED (I.E., THREE-DIMENSIONAL) STRESS-STRAIN RELATIONS

For the case of small deformation, the 1-D experimental observation of Figure 5.2 is generalized by assuming that each stress component depends linearly on all the components of the linear strain tensor. Thus,

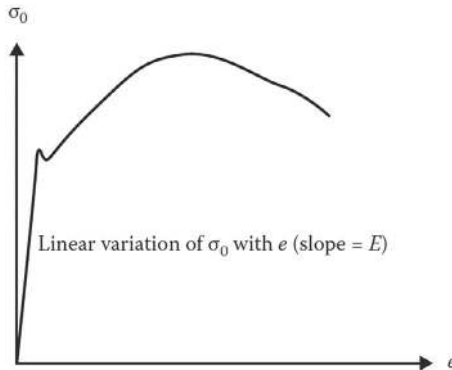


FIGURE 5.2 Variation of engineering stress with engineering strain for a ductile material in tension test.

$$\begin{aligned}
\sigma_{xx} &= C_{xxxx}\varepsilon_{xx} + C_{xxxy}\varepsilon_{xy} + C_{xxxz}\varepsilon_{xz} + C_{xxyx}\varepsilon_{yx} + \dots + C_{xxzz}\varepsilon_{zz}, \\
\sigma_{xy} &= C_{xyxx}\varepsilon_{xx} + C_{xyxy}\varepsilon_{xy} + C_{xyxz}\varepsilon_{xz} + C_{xyyx}\varepsilon_{yx} + \dots + C_{xyzx}\varepsilon_{zx}, \\
&\text{-----}, \\
&\text{-----}, \\
&\text{-----}, \\
&\text{-----}, \\
&\text{-----}, \\
&\text{-----}, \\
\sigma_{zz} &= C_{zzxx}\varepsilon_{xx} + C_{zzxy}\varepsilon_{xy} + C_{zzxz}\varepsilon_{xz} + C_{zzyx}\varepsilon_{yx} + \dots + C_{zzzz}\varepsilon_{zz}.
\end{aligned} \tag{5.3}$$

The stress-strain relations given by Equation 5.3 have 81 constants called the material or elastic constants. These constants represent the elastic behavior of the material for the case of small deformation. These constants are determined from experiments.

Using index notation, Equation 5.3 can be written as

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}. \tag{5.4}$$

Here, C_{ijkl} are the components of a *fourth-order tensor* C , called the *elasticity tensor*. In three dimensions, a fourth-order tensor has $3^4 = 81$ components.

One can reduce the number of elastic constants by using the symmetry of the stress and strain tensors:

$$\sigma_{ij} = \sigma_{ji}, \quad \varepsilon_{kl} = \varepsilon_{lk}. \tag{5.5}$$

Then, the components C_{ijkl} satisfy the following symmetry relations:

$$C_{ijkl} = C_{jikl}, \quad C_{ijkl} = C_{ijlk}. \tag{5.6}$$

Because of these symmetry relations, the number of independent components of the tensor C reduces from 81 to 36. Further reduction is possible if the material possesses certain symmetry.

5.2.3 STRESS-STRAIN RELATIONS FOR ISOTROPIC MATERIALS

Many elastic materials possess the symmetry called isotropy, that is, their response is the same in every direction. Mathematically, it means that the elastic constants in the stress-strain relations do not change under any arbitrary rotation of the coordinate system. Because of this restriction, the number of independent elastic constants reduces to only two. (Actually, because of this restriction, 24 out of the 36 constants become zero, and the remaining 12 can be expressed in terms of only two constants). These independent elastic constants are denoted by λ and μ and are called Lamé's constants. Then, for isotropic materials, the stress-strain relations become

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}. \quad (5.7)$$

In tensor notation, they can be written as

$$\boldsymbol{\sigma} = \lambda (\text{tr} \boldsymbol{\varepsilon}) \mathbf{1} + 2\mu \boldsymbol{\varepsilon}. \quad (5.8)$$

The component forms of the previous two equations become

$$\begin{aligned} \sigma_{xx} &= \lambda (\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu \varepsilon_{xx}, \\ \sigma_{yy} &= \lambda (\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu \varepsilon_{yy}, \\ \sigma_{zz} &= \lambda (\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu \varepsilon_{zz}, \\ \sigma_{xy} &= 2\mu \varepsilon_{xy}, \\ \sigma_{yz} &= 2\mu \varepsilon_{yz}, \\ \sigma_{zx} &= 2\mu \varepsilon_{zx}. \end{aligned} \quad (5.9)$$

Sometimes, expressions are needed for the components of $\boldsymbol{\varepsilon}$ in terms of the stress components. They are obtained by inverting Equation 5.9:

$$\begin{aligned} \varepsilon_{xx} &= \frac{1}{E} \left[-\nu (\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) + (1 + \nu) \sigma_{xx} \right], \\ \varepsilon_{yy} &= \frac{1}{E} \left[-\nu (\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) + (1 + \nu) \sigma_{yy} \right], \\ \varepsilon_{zz} &= \frac{1}{E} \left[-\nu (\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) + (1 + \nu) \sigma_{zz} \right], \\ \varepsilon_{xy} &= \frac{(1 + \nu)}{E} \sigma_{xy}, \\ \varepsilon_{yz} &= \frac{(1 + \nu)}{E} \sigma_{yz}, \\ \varepsilon_{zx} &= \frac{(1 + \nu)}{E} \sigma_{zx}, \end{aligned} \quad (5.10)$$

where

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad \nu = \frac{\lambda}{2(\lambda + \mu)}. \quad (5.11)$$

In index and tensor notations, Equation 5.10 can be expressed as

$$\varepsilon_{ij} = \frac{1}{E} \left[-\nu \sigma_{kk} \delta_{ij} + (1 + \nu) \sigma_{ij} \right], \quad (5.12)$$

$$\varepsilon = \frac{1}{E} \left[-\nu (\text{tr} \sigma) 1 + (1 + \nu) \sigma \right]. \quad (5.13)$$

It can be shown that the slope of the straight portion of the stress-strain curve in the tension test (Figure 5.2) is equal to E :

$$E = \frac{\sigma_{xx}}{\varepsilon_{xx}}. \quad (5.14)$$

It is called the *Young's modulus*. Further, it can be shown that the ratio of the transverse normal strain to the axial normal strain in tension test is equal to $-\nu$:

$$\nu = -\frac{\varepsilon_{yy}}{\varepsilon_{xx}} = -\frac{\varepsilon_{zz}}{\varepsilon_{xx}}. \quad (5.15)$$

The symbol ν indicates *Poisson's ratio*. Equations 5.9 and 5.10 are called the *generalized Hooke's law*. When the two equations given by Equation 5.11 for λ and μ are solved, one gets

$$\lambda = \frac{Ev}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}. \quad (5.16)$$

There is a third form of the stress-strain relations (only for isotropic materials). One gets this form by substituting the decompositions of stress and strain tensors (Equations 3.105 and 4.156) in the stress-strain relations (Equation 5.7) and equating the hydrostatic and deviatoric parts on each side:

$$\left(\frac{1}{3} \sigma_{kk} \right) = 3K \left(\frac{1}{3} \varepsilon_{kk} \right), \quad (5.17)$$

$$\sigma'_{ij} = 2\mu \varepsilon'_{ij}, \quad (5.18)$$

where

$$K = \frac{(3\lambda + 2\mu)}{3}. \quad (5.19)$$

In tensor notation, the third form can be expressed as

$$\left(\frac{1}{3} \text{tr} \sigma \right) = 3K \left(\frac{1}{3} \text{tr} \varepsilon \right). \quad (5.20)$$

$$\sigma' = 2\mu \varepsilon'. \quad (5.21)$$

For small deformation, the volumetric strain is given by $\text{tr} \varepsilon = \varepsilon_{ll}$ (Equation 4.155). Thus, for small deformation, K is the ratio of the hydrostatic part of stress to the volumetric strain:

$$K = \frac{(1/3)\sigma_{kk}}{\varepsilon_{ll}} = \frac{(1/3)\text{tr}\sigma}{\text{tr}\varepsilon}. \quad (5.22)$$

It is called the *bulk modulus*. By taking the trace of Equation 5.13 and using Equation 5.22 for K , one gets

$$K = \frac{E}{3(1-2\nu)}. \quad (5.23)$$

If the decompositions of stress and strain tensors (Equations 3.105 and 4.156) in the *inverse* stress-strain relations are substituted (Equation 5.10), the hydrostatic and deviatoric parts on each side are equated, and expressions 5.16 and 5.23 for μ and K are used, one gets the same equations (Equations 5.18 and 5.19).

Equation 5.17 or 5.20 shows that, when the deformation is small, the elastic constant K relates the hydrostatic parts of stress and strain tensors. Further, it implies that, in isotropic materials, the hydrostatic part of the stress tensor causes only the change in volume (without a change in shape). Similarly, Equation 5.18 or 5.21 shows that the elastic constant μ relates the deviatoric parts of stress and strain tensors. Therefore, it is called the *shear modulus*. Further, it implies that, in isotropic materials, the change in shape (without change in volume) is caused only by the deviatoric part of the stress tensor.

Equation 5.17 is a scalar equation and, because of the symmetry of σ'_{ij} and ε'_{ij} , the tensor equation 5.18 represents six scalar equations. Hence, it appears that this form has seven scalar relations. However, because of the constraints $\sigma'_{kk} = 0$ (Equation 3.108) and $\varepsilon'_{kk} = 0$ (Equation 4.159), only five out of the six equations from the set Equation 5.18 are independent.

The constants E and ν are measured from the *tension test* using relations 5.14 and 5.15. Equation 5.23 shows that for compressible materials (finite K), ν has to be less than $(1/2)$. For incompressible materials ($K \rightarrow \infty$), ν must be $1/2$. The constant μ is measured from the *torsion test*. In the torsion test, the slope of the torque versus the twist curve is equal to μJ where J is the torsion constant of the shaft cross section. The constant K is measured from the *dilatation test* as the slope of the pressure versus the volumetric strain diagram. The constant λ cannot be measured experimentally but needs to be calculated either from E and ν using Equation 5.16 or from μ and K using Equation 5.19. Since the number of independent elastic constants is only two, one needs to measure any two elastic constants and then the other three can be obtained from appropriate relations. Using the sign conventions for stress and strain components described in Sections 3.2 and 4.2.1, experimental observations in real materials show that the signs of E , ν , μ , λ and K are all positive.

5.3 EXPERIMENTAL OBSERVATIONS ON ELASTIC-PLASTIC BEHAVIOR

The material behavior that causes the plastic deformation differs from the elastic behavior. In plastic deformation, the body does not return to the original undeformed

configuration after the removal of the external forces. This behavior is expressed by the stress-strain relations that are not one-to-one like the elastic stress-strain relations. Further, it means that the material behaviors in loading and unloading are different, whereas the behavior in unloading is elastic. Therefore, the stress-strain relations developed for the elastic behavior in Section 5.2 need to be modified. In this section, it is planned to develop the *stress-strain relations for elasto-plastic behavior*.

Besides the elasto-plastic stress-strain relations, four more things are needed for the analysis of plastic deformation. The first thing is that a typical metal behaves elastically for small deformation and then behaves elasto-plastically as the deformation increases. Therefore, a criterion is needed that decides when the elastic behavior ends and the elasto-plastic behavior begins. This criterion is called the yield criterion, expressed as a scalar function of the stress components. Therefore, the *(initial) yield criterion* of materials needs to be developed. Secondly, for continued (or subsequent) plastic deformation beyond initial yielding, additional stress is required. It means that the (initial) yield criterion changes with the plastic deformation in subsequent yielding. This phenomenon is called the *hardening*, and a *criterion for subsequent yielding* needs to be developed for its modeling. Thirdly, when a combination of the stress components decreases, the material again behaves elastically. This is called the unloading phenomenon. An *unloading criterion* is needed for modeling this behavior. Fourthly, the constitutive equation for plastic behavior is usually expressed either in the rate form or in the increment form. The stress tensors appearing in these constitutive equations have to be objective, that is, they have to be invariant under a change of reference frame. The Cauchy stress tensor (introduced in Chapter 3) is objective, but its rate or increment is not objective. Therefore, the *objective stress rate and objective incremental stress measures* need to be developed. Constitutive equation for large elastic deformation is also sometimes expressed in the rate or incremental form. In that case, the stress measures appearing in these constitutive equations also have to be objective.

Before the equations necessary for describing the plastic behavior are developed, the 1-D experimental observations on plasticity are studied based on tension tests. These observations provide valuable insight in the phenomenon of plasticity. Further, they provide a useful basis for the development of three-dimensional yield criterion, hardening relations, unloading criterion, etc.

5.3.1 1-D EXPERIMENTAL OBSERVATIONS ON PLASTICITY

Again, consider a rod of uniform cross-section stretched by an axial tensile force F_x , as shown in Figure 5.1. As before, let A_0 be its initial area of cross section, ℓ_0 be its initial length and $\Delta\ell$ be the change in length corresponding to F_x . The *engineering* (or *nominal*) stress σ_0 and the *engineering strain* e have been defined by Equations 5.1 and 5.2. Further, the variation of σ_0 with e has been plotted in Figure 5.2. It is observed that, after certain deformation, the value of F_x , and along with that, the value of σ_0 decreases. However, the *true stress* never decreases with the deformation. Further, it is observed that the value of e at fracture, for most metals, is more than 0.5. Thus, the plastic deformation is usually quite large. Therefore, it is more appropriate to use a measure of deformation that can represent large deformation.

One such measure is the *logarithmic strain*. The diagram of the true stress versus the logarithmic strain is more useful for studying the phenomenon of plasticity. Therefore, such a diagram is constructed.

In a 1-D case, the logarithmic strain (denoted by ε) is defined as

$$\varepsilon = \ln \frac{\ell}{\ell_0}, \quad (5.24)$$

where ℓ is the current length (i.e. the length in the deformed configuration). It is also called the *natural strain*. Since,

$$\ell = \ell_0 + \Delta\ell, \quad (5.25)$$

the combination of Equations 5.2, 5.24 and 5.25 gives the following relationship between ε and e :

$$\varepsilon = \ln(1 + e). \quad (5.26)$$

When the deformation is small (i.e. when $e < 0.05$), ε reduces to e . The expression for true stress σ is given by

$$\sigma = \frac{F_x}{A}, \quad (5.27)$$

where A is the current area of the cross section. It is observed that the volume remains constant during plastic deformation. This condition gives the following relation between A and A_0 :

$$A = \frac{\ell_0}{\ell} A_0. \quad (5.28)$$

Substitution of Equations 5.1, 5.2, 5.25 and 5.28 in Equation 5.27 gives the following relationship between σ and σ_0 :

$$\sigma = (1 + e) \sigma_0. \quad (5.29)$$

Again, when the deformation is small (i.e. when $e < 0.05$), σ reduces to σ_0 . Using Equations 5.26 and 5.29, the graph of σ_0 versus e (Figure 5.2) is converted into the variation of σ with ε , which is shown in Figure 5.3. From this figure, the following observations can be made about the plasticity.

5.3.1.1 Elastic Region

The stress at point Y (end of the straight part of Figure 5.3) is denoted as σ_y . If the rod is stressed up to any level less than σ_y (say, up to point A), then it returns to the original undeformed configuration after unloading. Therefore, the straight part OY corresponds to elastic behavior.

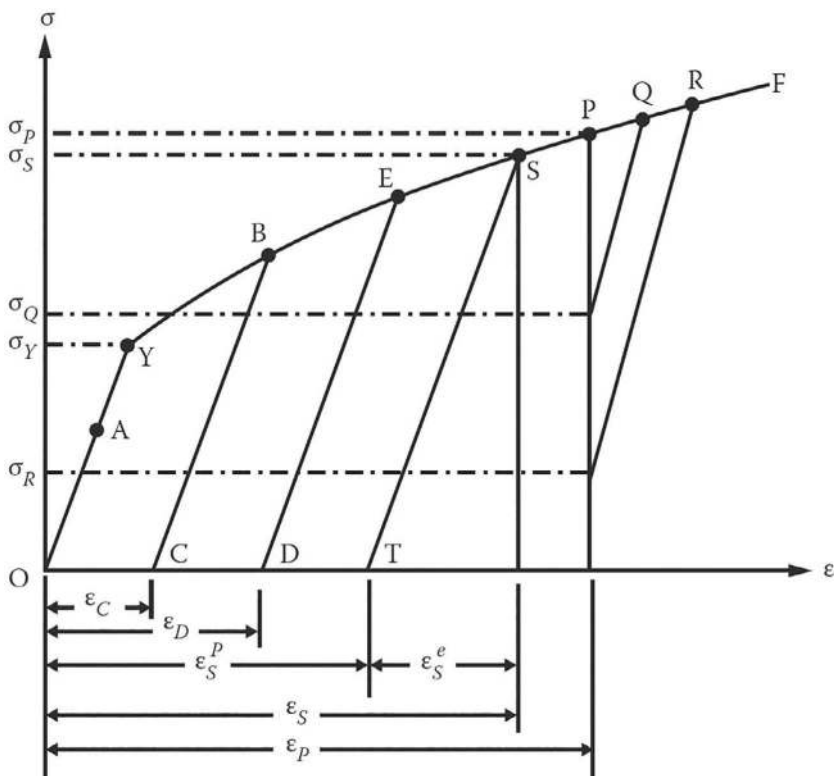


FIGURE 5.3 Variation of true stress with logarithmic strain in tension test.

5.3.1.2 Yield Stress

It is observed that when the stress reaches the value σ_Y (Figure 5.3), the material yields, that is, it starts flowing suddenly leading to large deformation. The value σ_Y is called the *yield stress*. It marks the transition from elastic to plastic behavior. Thus, in 1-D state of stress, *initial yielding* criterion can be stated as

$$\sigma - \sigma_Y = 0 \quad (5.30)$$

The generalization of this initial yield condition to a three-dimensional state of stress is discussed in Section 5.4.

In some materials, the actual stress-strain curve (Johnson and Mellor, 1972; Khan and Huang, 1995) differs from the curve of Figure 5.3 in the neighborhood of σ_Y in certain respects. The first difference is that the end of the elastic behavior does not coincide with the end of the straight part of the curve. The second difference is the drop in the stress after initial yielding leading to upper and lower yield points. However, for the ease of mathematical modeling, these finer aspects of yielding are

neglected, and the existence of a sharp yield point at the end of the straight part is assumed. For materials like aluminum for which there is no sharp yield point, the yield stress is defined as the stress corresponding to 0.2% permanent strain. The concept of permanent strain is defined later.

It is observed that the value of σ_y increases with the strain rate but decreases with temperature.

5.3.1.3 Plastic Region

The curved part of Figure 5.3 beyond point Y corresponds to the plastic behavior. Some of the characteristics of plastic behavior are as follows.

If the rod is stressed beyond σ_y up to point B and if the loading continues, then the stress-strain curve follows the path BF leading to fracture at point F. Therefore, part YF is called the loading path. However, if one unloads from point B to zero stress level, then the stress-strain curve follows the straight path BC (parallel to OY) leaving a permanent strain ϵ_c in the rod, called the plastic strain. Thus, the rod does not attain the initial undeformed configuration on unloading. If the rod with an initial plastic strain ϵ_D is loaded, then the stress-strain curve first follows a straight path from point D to point E and then it follows the curved path EF. It means the rod behaves elastically up to point E and yields at the stress level corresponding to point E, which is greater than σ_y . Thus, a rod with some initial plastic strain yields at a higher stress level than an undeformed rod. This is called *subsequent or continued yielding*. The condition for this yielding is developed later.

The stress-strain relationship corresponding to plastic behavior is not *one-to-one*. This can be observed by determining the value of stress corresponding to the strain value of ϵ_p . The stress is equal to σ_p if we are on the loading path. However, it is equal to σ_Q if the rod is loaded up to point Q and then it is unloaded to the strain level ϵ_p . Further, it will be equal to σ_R if the rod is loaded up to point R and then it is unloaded up to the strain level ϵ_p . Thus, the stress corresponding to the strain level ϵ_p is not unique but depends on the *history* of deformation. Further, there is one type of stress-strain relationship corresponding to the loading path and a different one for the unloading path. Generalization to three-dimensional plastic stress-strain relations is discussed in Section 5.6.

To avoid the mathematical complexity in the analysis of plastic behavior, plastic stress-strain relations are sometimes simplified by idealizing the actual stress-strain behavior. First simplification arises by neglecting the elastic strain as it is small compared to the plastic strain in metals. Then, the stress-strain curve of Figure 5.3 starts from point Y and has only the plastic part YF. Such a material is called *rigid-plastic*. Otherwise, it is called *elastic-plastic*. In the second simplification, the part YF is assumed straight. Such a material is called *linearly hardening*. (The phenomenon of hardening is discussed in Strain Hardening.) In the third simplification, the part YF is assumed to be straight as well as parallel to the strain axis. Such a material is called *ideal or perfectly plastic*. Various combinations of these simplifications lead to the following four idealizations: (1) rigid perfectly plastic material, (2) rigid-plastic material with linear hardening, (3) elastic-perfectly plastic material and (4) elastic-plastic material with linear hardening.

5.3.1.4 Strain Hardening

Figure 5.3 shows that the stress increases with strain beyond point Y. It means, that beyond initial yielding, the stress required to cause subsequent yielding (or continued material flow) increases with the strain. This phenomenon is called *strain hardening*. The yield stress in subsequent yielding depends on the plastic part of deformation. To develop a mathematical expression for subsequent yielding, a graph of the variation of stress with the plastic part of strain is constructed from Figure 5.3. First, the plastic part of strain corresponding to σ_Y (i.e. the stress at point S of Figure 5.3) is determined by unloading from point S to the zero stress level (i.e. to point T). Then, OT is the plastic part of strain (denoted by ε^p) corresponding to σ_Y , and the remaining is the elastic part (ε^e) of strain. Similarly, the plastic part of strain corresponding to all values of stress greater than σ_Y is determined. The graph of σ versus ε^p is shown in Figure 5.4.

From Figure 5.4, the criterion for subsequent (or continued) yielding for a 1-D state of stress can be expressed as

$$\sigma - h(\varepsilon^p) = 0, \quad (5.31)$$

where the function h is called the hardening function. For the zero value of ε^p , the function h reduces to the constant value σ_Y . Thus, for $\varepsilon^p = 0$, the criterion for subsequent yielding (Equation 5.31) reduces to the criterion for initial yielding (Equation 5.30). The generalization of Equation 5.31 to a three-dimensional case is discussed in Section 5.5.

Several forms of function h are available in the literature. Some commonly used forms are given in the following (Chakrabarty, 1987; Hill, 1950). The original expressions for these functions are in terms of the total strain. Here, they have been appropriately modified to express them in terms of the plastic part of strain. Further, the symbols for the material constants also have been changed.

1. Ludwik's expression:

$$\sigma = \sigma_Y + K(\varepsilon^p)^n. \quad (5.32)$$

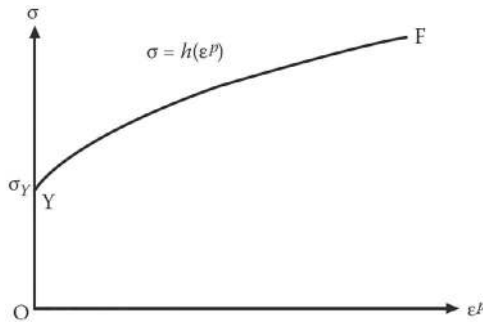


FIGURE 5.4 Variation of true stress with the plastic part of logarithmic strain in tension test.

This expression does not give a good fit at large strains as the experimental stress-strain curves of most metals have a constant slope at large strain.

2. Swift's expression:

$$\sigma = \sigma_Y \left[1 + K \varepsilon^p \right]^n. \quad (5.33)$$

This expression gives a better fit of experimental stress-strain curves at large strains than Ludwik's expression.

3. Voce's expression:

$$\sigma = \sigma_Y + K \left[1 - e^{-(n\varepsilon^p)} \right]. \quad (5.34)$$

This expression gives a good fit of experimental stress-strain curves at moderate values of strain.

Table 5.1 provides a number of other expressions that are commonly used in metal forming.

In all the previous expressions, K and n are the material constants called the *hardening parameters*, which are determined by fitting the previous equations with the experimental curves of true stress versus the plastic part of logarithmic strain (Figure 5.4). In all the previous equations, σ reduces to σ_Y (i.e., to the initial yield stress) when ε^p is 0 (i.e., at initial yielding). When n is equal to 1, Equations 5.32 and 5.33 represent a *linear hardening* curve.

5.3.1.5 Temperature Softening

It is observed that, beyond initial yielding, the stress required to cause subsequent yielding decreases with temperature rise. This phenomenon is called the *temperature softening*. In this case, the function h of Equation 5.31 also depends on temperature. This effect needs to be included in the plastic stress-strain relations in thermo-elasto-plastic problems.

5.3.1.6 Viscoplasticity

It is observed that, beyond initial yielding, the stress required to cause subsequent yielding increases with the strain rate (or the rate of deformation). This phenomenon is called *viscoplasticity*. This increase in the stress is due to the *viscous* resistance of the material to further yielding. In this case, the function h of Equation 5.31 also depends on some measure of the rate of deformation. This effect needs to be included in the plastic stress-strain relations while analyzing the impact problems of elasto-plastic materials.

5.3.1.7 Isochoric Deformation

As stated earlier, the volume remains constant during plastic deformation. Thus, the plastic deformation is *isochoric*. This imposes a constraint on plastic deformation.

5.3.1.8 Large Deformation

As stated earlier, the deformation in the plastic region is quite large. As a result, the linear or infinitesimal strain tensor ϵ cannot be used as a measure of deformation. One has to look for some other measure of deformation to represent the plastic deformation. Non-linear, incremental and rate measures of deformation have been discussed in Chapter 4.

TABLE 5.1
Typical Empirical Models Showing the Dependency of Flow Stress

S. No.	Empirical model	Comment
1.	$\sigma = K\epsilon^n$	Hollomon law; deviates at low strain; for high strain, ϵ may be treated total as well as plastic strain rate, but for low strain, it refers to plastic strain
2.	$\sigma = \sigma_y + K\epsilon^n$	Ludwik law; ϵ is plastic strain; does not give good fit over a wide range
3.	$\sigma = \sigma_y(1 + \epsilon/b)^n$ $\sigma = C(m + \epsilon)^n$	Swift's generalized power law; suitable for a wide range; ϵ should be plastic strain when elastic and plastic strains are of comparable magnitude
4.	$\sigma = \sigma_y + K[1 - m\epsilon^{-m}]$	Voce law; ϵ should be plastic strain when elastic and plastic strains are of comparable magnitude
5.	$\epsilon = \frac{\sigma}{E} \left\{ 1 + \alpha \left(\frac{\sigma}{\sigma_0} \right)^{m-1} \right\}$	Ramburg-Osgood equation; considers elasticity; ϵ is total strain
6.	$\sigma = \sigma_y \tanh \left(\frac{E\epsilon}{\sigma_y} \right)$	Pragar's law for ideally plastic material; ϵ is total strain
7.	$Z = \dot{\epsilon} \exp \left(\frac{Q}{RT} \right) = A \{ \sinh(\alpha \sigma) \}^n$	Relation considering strain rate and temperature; Z is the Hollomon parameter or temperature-corrected (plastic) strain rate
8.	$\sigma = k\epsilon^n \dot{\epsilon}^m \exp \left(\frac{\beta}{T} \right)$	Relation considering strain, strain rate, and temperature; m is (plastic) strain-rate sensitivity
9.	$\sigma = \left(A + B\epsilon^n \right) \left(1 + C \ln \frac{\dot{\epsilon}}{\dot{\epsilon}_0} \right) \left\{ 1 - \left(\frac{T - T_0}{T_{melt} - T_0} \right)^m \right\}$	Johnson-Cook model; widely used in machining; strain and strain rate are usually taken plastic
10.	$\sigma = \sigma_0 \epsilon^n \left(\frac{\dot{\epsilon}}{\dot{\epsilon}_0} \right)^m \left(\frac{T}{T_0} \right)^{-r}$	Power law

Source: Reprinted from *Materials & Design*, Vol. 32, Dixit, U.S., Joshi, S. N., and Davim, J. P., Incorporation of material behavior in modeling of metal forming and machining processes: A review, pp. 3655–3670. Copyright 2011, with permission from Elsevier.

5.3.1.9 Hysteresis

If the rod loaded up to point B (Figure 5.5) is unloaded up to zero stress level, then it follows the straight path BC leaving a plastic strain ϵ_C in the rod. On further loading, the initial straight path CD, which the stress-strain curve follows, has a slightly different slope than the unloading path BC. This phenomenon is called the *hysteresis* and the loop BCD is called the hysteresis loop.

The actual hysteresis loop is much smaller than what is shown in Figure 5.5, and therefore, its effect on the plastic stress-strain relations can be neglected. Thus, it is assumed that the slopes of both the straight-line parts BC as well as CD are identical and are equal to Young's modulus.

5.3.1.10 Bauschinger Effect

In a compression test, the numerical value of the *yield stress in compression* is observed to be exactly equal to σ_y , the *yield stress in tension*. However, this numerical equality of yield stress in tension and compression does not hold in reversed loading after the yielding.

If the rod is first loaded in *tension* up to point B (Figure 5.6), then unloaded to the zero stress level (i.e. to point C) leaving a plastic strain ϵ_C in the rod and then loaded in *compression*, it follows the path CD, where the new yield stress σ_D (in compression) is smaller in magnitude than the stress σ_B (the yield stress in tension corresponding to the initial strain of ϵ_C). This phenomenon is called the *Bauschinger effect*. This lowering of the yield stress in reversed loading is caused by the residual stresses (at the *microscopic scale*) left in the rod after unloading. The Bauschinger effect can be removed after mild annealing. The Bauschinger effect will be neglected in the analysis and it is assumed that the yield stress in tension and compression are numerically equal.

5.3.1.11 Effect of Hydrostatic Stress on Yielding

It is observed that the yield stress is unaffected by the hydrostatic part of the stress tensor (Hill, 1950). Thus, yielding is essentially caused by the deviatoric part of the stress tensor. This observation is needed in developing the initial yield criteria of Section 5.4.

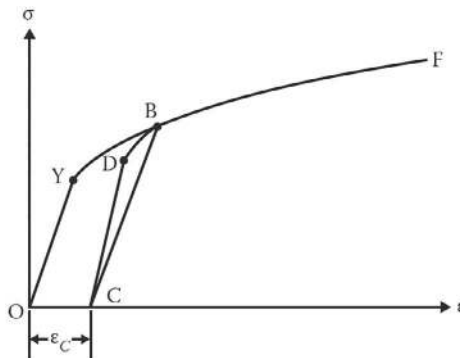


FIGURE 5.5 Hysteresis loop.

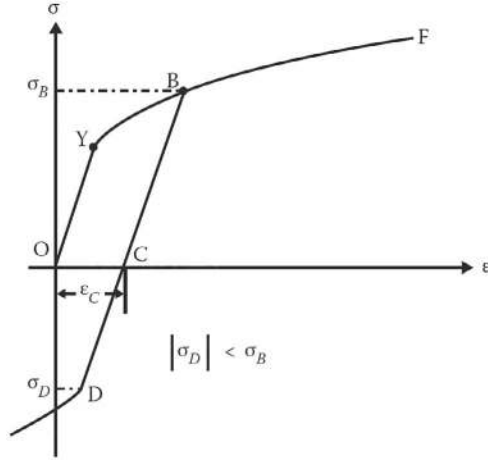


FIGURE 5.6 Bauschinger effect.

5.3.1.12 Anisotropy

The microstructure of metals is crystalline in nature, and the crystallographic directions are randomly oriented in an annealed metal. This means that in an annealed metal, there are no preferred directions and thus, it is isotropic at the macroscopic level. However, when it is subjected to cold-forming processes like drawing, extrusion, rolling, etc., the crystallographic directions gradually rotate toward a common axis, thus creating a preferred direction. Thus, after cold forming, the metal becomes anisotropic. When this metal is subjected to further plastic deformation, the yield criteria and the plastic stress-strain relations used for the analysis of this problem should incorporate anisotropy.

5.4 CRITERIA FOR INITIAL YIELDING OF ISOTROPIC MATERIALS

Yield criterion is a law that defines the limit of elastic behavior. For a 1-D state of stress (i.e. when only one stress component is non-zero), Equation 5.30 represents the initial yield criterion. Its generalization to a three-dimensional state of stress involves all the stress components:

$$f(\sigma_{ij}) = 0. \quad (5.35)$$

The function f is called the *yield function*. Next, the form of f is simplified based on some experimental observations and the property of isotropy.

One of the experimental observations is that the yielding depends only on the deviatoric part of the stress tensor. Then, in Equation 5.35, σ_{ij} should be replaced by σ'_{ij} , the components of the deviatoric part of σ . Then, the yield criterion becomes

$$f(\sigma'_{ij}) = 0. \quad (5.36)$$

For isotropic materials, the yield function does not change with a change in the coordinate system. It implies that the function f should be an invariant of σ' , and thus should be a function of the three principal invariants J_1 , J_2 and J_3 are defined by equations similar to Equations 3.81–3.83. Since the first invariant ($J_1 = \text{tr}\sigma'$) is 0 (Equation 3.108), the yield criterion becomes

$$f(J_2, J_3) = 0. \quad (5.37)$$

At the initial yielding, the yield stress in compression is observed to be numerically equal to the yield stress in tension. A generalization of this observation to the three-dimensional case implies that the value of f does not get affected if σ'_{ij} in Equation 5.37 is replaced by $-\sigma'_{ij}$. This means f becomes an even function of σ'_{ij} . Since J_2 is an even function of σ'_{ij} (Equation similar to Equation 3.82), and J_3 is an odd function of σ'_{ij} (Equation similar to Equation 3.83), f can be any function of J_2 but should be an *even* function of J_3 .

The specific dependence of f on J_2 and J_3 is decided by making a hypothesis and testing it against the experimental results. Two such hypotheses are discussed here: (1) von Mises yield criterion and (2) the Tresca yield criterion.

5.4.1 VON MISES YIELD CRITERION

In 1913, von Mises proposed a criterion in which f is assumed to be independent of J_3 and a linear function of J_2 . Thus, the von Mises yield criterion (henceforth simply called the Mises criterion) can be stated as

$$f(J_2, J_3) \equiv J_2 - k = 0. \quad (5.38)$$

Huber anticipated this criterion in 1904. Hencky in 1924 and Nadai in 1933 provided its physical interpretation. Hencky and Nadai observed that J_2 can be related, respectively, to the distortion strain energy density (work done per unit volume by the deviatoric part of the stress which results in the change of shape but not change in volume) and the octahedral shear stress τ_{oct} (Equation 3.104). Thus, as per Hencky's interpretation, Equation 5.38 states that whenever the distortion strain energy density reaches a critical value, the yielding occurs. Similarly, Nadai's interpretation of Equation 5.38 is that the yielding occurs whenever the octahedral shear stress reaches a critical value.

The material constant k is determined from the tension test (Section 5.3.1). In the tension test, the matrix of the stress components with respect to the (x, y, z) coordinate system is given by

$$[\sigma] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (5.39)$$

where σ is given by Equation 5.27. The previous equation gives $\text{tr}\sigma = \sigma$. Then, using Equation 3.106, the matrix of the deviatoric part can be written as

$$[\sigma'] = \begin{bmatrix} \frac{2}{3}\sigma & 0 & 0 \\ 0 & -\frac{1}{3}\sigma & 0 \\ 0 & 0 & -\frac{1}{3}\sigma \end{bmatrix}. \quad (5.40)$$

Then, J_2 is calculated as (Equation similar to Equations 3.82 and 3.108)

$$J_2 = \frac{1}{2} \sigma'_{ij} \sigma'_{ij} = \frac{1}{3} \sigma^2. \quad (5.41)$$

At the initial yielding, σ is equal to σ_Y , and therefore, J_2 becomes $(1/3)\sigma_Y^2$. Substitution of this value of J_2 in Equation 5.38 gives $k = (1/3)\sigma_Y^2$. Then, the Mises criterion becomes

$$f(J_2, J_3) \equiv J_2 - \frac{1}{3} \sigma_Y^2 = 0. \quad (5.42)$$

The Mises criterion is also expressed in two alternate forms. The first alternate form involves the following invariant of σ' :

$$\sigma_{eq} = (3J_2)^{1/2} = \left(\frac{3}{2} \sigma'_{ij} \sigma'_{ij} \right)^{1/2}. \quad (5.43)$$

Equation 5.40 shows that, in tension tests, σ_{eq} is equal to σ . Therefore, σ_{eq} is called the *equivalent or effective or generalized stress*. Since, at the initial yielding, σ is equal to σ_Y , the Mises criterion in terms of σ_{eq} takes the form

$$f(\sigma_{eq}) \equiv \sigma_{eq} - \sigma_Y = 0. \quad (5.44)$$

The second alternate form is in terms of the principal stresses σ_i . In the coordinate system of the principal directions, the matrix of σ becomes

$$[\sigma] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}. \quad (5.45)$$

Then, the matrix of the deviatoric part σ' can be expressed as

$$[\sigma'] = \begin{bmatrix} \sigma_1 - \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) & 0 & 0 \\ 0 & \sigma_2 - \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) & 0 \\ 0 & 0 & \sigma_3 - \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \end{bmatrix}. \quad (5.46)$$

Then, J_2 becomes

$$J_2 = \frac{1}{2} \sigma'_{ij} \sigma'_{ij} = \frac{1}{2} \left\{ \left[\sigma_1 - \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \right]^2 + \left[\sigma_2 - \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \right]^2 + \left[\sigma_3 - \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \right]^2 \right\} = \frac{1}{6} \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]. \quad (5.47)$$

Substitution of this value of J_2 in Equation 5.42 leads to the following form for the Mises criterion:

$$f(\sigma_1, \sigma_2, \sigma_3) \equiv \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] - 2\sigma_Y^2 = 0. \quad (5.48)$$

One can express the yield criterion in terms of the principal stresses only for isotropic materials. This is because the principal stresses, being the roots of an equation involving the principal invariants (Equation 3.80), are invariants of the stress tensor.

Example 5.1

Express the Mises criterion in terms of nine components of stress with respect to the x, y, z coordinate system.

SOLUTION

From Equation 5.47,

$$\begin{aligned} J_2 &= \frac{1}{2} \sigma'_{ij} \sigma'_{ij} = \frac{1}{2} \left(\sigma_{ij} - \frac{\sigma_{kk}}{3} \delta_{ij} \right) \left(\sigma_{ij} - \frac{\sigma_{kk}}{3} \delta_{ij} \right) \\ &= \frac{1}{2} \left(\sigma_{ij} \sigma_{ij} - \frac{\sigma_{kk}^2}{9} \delta_{ij} \delta_{ij} \right). \end{aligned}$$

But $\delta_{ij} \delta_{ij} = 3$. Hence,

$$\begin{aligned} J_2 &= \frac{1}{2} \sigma'_{ij} \sigma'_{ij} = \frac{1}{2} \left(\sigma_{ij} \sigma_{ij} - \frac{\sigma_{kk}^2}{3} \right) \\ &= \frac{1}{6} (3\sigma_{ij} \sigma_{ij} - \sigma_{kk}^2). \end{aligned}$$

Writing in expanded notations,

$$J_2 = \frac{1}{6} \left\{ 3(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 + \sigma_{xy}^2 + \sigma_{yx}^2 + \sigma_{xz}^2 + \sigma_{zx}^2 + \sigma_{yz}^2 + \sigma_{zy}^2) - (\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 + 2\sigma_{xx}\sigma_{yy} + 2\sigma_{yy}\sigma_{zz} + 2\sigma_{zz}\sigma_{xx}) \right\}$$

$$= \frac{1}{6} \left\{ (\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{xx} - \sigma_{zz})^2 \right\} \\ + \frac{1}{2} \left\{ \sigma_{xy}^2 + \sigma_{yx}^2 + \sigma_{yz}^2 + \sigma_{zy}^2 + \sigma_{zx}^2 + \sigma_{xz}^2 \right\}.$$

Hence, by Equation 5.43, the Mises criterion at yielding is

$$\frac{1}{2} \left\{ (\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{xx} - \sigma_{zz})^2 \right\} \\ + \frac{3}{2} \left\{ \sigma_{xy}^2 + \sigma_{yx}^2 + \sigma_{yz}^2 + \sigma_{zy}^2 + \sigma_{zx}^2 + \sigma_{xz}^2 \right\} - \sigma_Y^2 = 0.$$

Example 5.2

The matrix of stress tensor σ at a point, with respect to the (x, y, z) coordinate system, is given by

$$[\sigma] = \begin{bmatrix} 50 & 120 & 0 \\ 120 & 50 & 0 \\ 0 & 0 & -100 \end{bmatrix} \text{ MPa}.$$

Check whether yielding occurs at this point using the Mises criterion: (1) in terms of J_2 , (2) in terms of σ_{eq} , (3) in terms of principle stresses and (4) in terms of the component of stresses in the (x, y, z) coordinate system. The yield stress of the material is 270 MPa.

SOLUTION

Here, $\sigma_{kk} = 0$. Hence, $\sigma' = \sigma$.

1. The Mises criterion in terms of J_2 is

$$J_2 - \frac{1}{3} \sigma_Y^2 = 0.$$

Here,

$$J_2 = \frac{1}{2} \sigma'_{ij} \sigma'_{ij} = \frac{1}{2} (50^2 + 50^2 + 100^2 + 2 \times 120^2) = 21,900 (\text{MPa})^2.$$

Hence,

$$21,900 - \frac{1}{3}(270)^2 = -2400 < 0,$$

indicating that yielding does not occur.

2. From Equation 5.43,

$$\sigma_{eq} = (3J_2)^{1/2} = (3 \times 21,900)^{1/2} = 256.32 \text{ MPa}.$$

The yield criterion in terms of σ_{eq} is

$$\sigma_{eq} - \sigma_Y = 0.$$

Here,

$$\sigma_{eq} - \sigma_Y = 256.32 - 270 = -13.68 < 0,$$

indicating that yielding does not occur.

3. Principal stresses are the eigenvalues of σ , which can be found by solving

$$\begin{vmatrix} 50 - \lambda & 120 & 0 \\ 120 & 50 - \lambda & 0 \\ 0 & 0 & -100 - \lambda \end{vmatrix} = 0.$$

Three roots of the previous equation are the principal stresses given here:

$$\sigma_1 = 170 \text{ MPa}, \sigma_2 = -70 \text{ MPa}, \sigma_3 = -100 \text{ MPa}.$$

The Mises yield criterion in terms of the principal stresses is

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 - 2\sigma_Y^2 = 0.$$

Here,

$$\begin{aligned} & (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 - 2\sigma_Y^2 \\ &= (170 + 70)^2 + (-70 + 100)^2 + (-100 - 170)^2 - 2 \times 270^2 \\ &= -14,400 < 0, \end{aligned}$$

indicating that the yielding does not occur.

4. For symmetric stress tensor, the Mises yield criterion in terms of the components of stresses is given as

$$(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{xx} - \sigma_{zz})^2 + 6(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{xz}^2) - 2\sigma_Y^2 = 0.$$

Here,

$$(50 - 50)^2 + (50 + 100)^2 + (50 + 100)^2 + 6(120^2) - 2 \times 270^2 = -14,400 < 0,$$

indicating that yielding does not occur.

5.4.2 TRESCA YIELD CRITERION

The Tresca yield criterion (henceforth simply called the Tresca criterion) was proposed in 1864 on the basis of experimental observations on the extrusion of metals through dies of various shapes. Thus, it was not developed from Equation 5.37. This criterion states that yielding occurs whenever the maximum shear stress at that point reaches the critical value. When the principal stresses are not ordered, the maximum shear stress at a point is given by (Equation 3.100)

$$|\sigma_s|_{\max} = \frac{1}{2} \max [|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|]. \quad (5.49)$$

If the critical value of the maximum shear stress (i.e. its value at yielding) is k_1 , then the Tresca criterion can be expressed as

$$f(\sigma_1, \sigma_2, \sigma_3) \equiv \left[(\sigma_1 - \sigma_2)^2 - 4k_1^2 \right] \left[(\sigma_2 - \sigma_3)^2 - 4k_1^2 \right] \left[(\sigma_3 - \sigma_1)^2 - 4k_1^2 \right] = 0. \quad (5.50)$$

As before, the value of k_1 is evaluated from the tension test (Section 5.3.1). The principal stresses corresponding to the stress matrix in the tension test (Equation 5.39) are

$$\sigma_1 = \sigma, \sigma_2 = \sigma, \sigma_3 = 0. \quad (5.51)$$

where σ , given by Equation 5.27, has the value σ_Y at the initial yielding. Substitution of Equation 5.51 in Equation 5.50 gives $k_1 = \sigma_Y/2$. Then, the Tresca criterion (Equation 5.50) takes the form

$$f(\sigma_1, \sigma_2, \sigma_3) \equiv \left[(\sigma_1 - \sigma_2)^2 - \sigma_Y^2 \right] \left[(\sigma_2 - \sigma_3)^2 - \sigma_Y^2 \right] \left[(\sigma_3 - \sigma_1)^2 - \sigma_Y^2 \right] = 0. \quad (5.52)$$

In terms of the invariants J_2 and J_3 , Equation 5.52 becomes (Chakrabarty, 1987)

$$f(J_2, J_3) \equiv 4 \left(J_2 - \frac{\sigma_Y^2}{4} \right) (J_2 - \sigma_Y^2)^2 - 27J_3^2 = 0. \quad (5.53)$$

This form is not very convenient for application purposes.

Example 5.3

The matrix of stress tensor σ at a point, with respect to the (x, y, z) coordinate system, is given by

$$[\sigma] = \begin{bmatrix} 150 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & -110 \end{bmatrix} \text{ MPa.}$$

Use the Tresca criterion given in the form of Equation 5.53 and verify that the material starts yielding if the yield stress is 260 MPa.

SOLUTION

In this case,

$$\sigma_{kk} = \frac{150 + 50 - 110}{3} = 30 \text{ MPa.}$$

Subtracting this value from each diagonal element, one gets

$$[\sigma'] = \begin{bmatrix} 120 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & -140 \end{bmatrix} \text{ MPa.}$$

Now,

$$J_2 = \frac{1}{2} \sigma'_{ij} \sigma'_{ij} = \frac{1}{2} (120^2 + 20^2 + 140^2) = 17,200 (\text{MPa})^2.$$

Alternatively, the sum of the determinants of three 2×2 matrices extracted from the deviatoric stress matrix can be taken and can be multiplied by -1 to calculate

$$\begin{aligned} J_2 &= - \left| \begin{array}{cc} 120 & 0 \\ 0 & 20 \end{array} \right| - \left| \begin{array}{cc} 20 & 0 \\ 0 & -140 \end{array} \right| - \left| \begin{array}{cc} 120 & 0 \\ 0 & -140 \end{array} \right| \\ &= -2400 + 2800 + 16,800 = 17,200 (\text{MPa})^2. \end{aligned}$$

The third invariant J_3 is equal to the determinant of the stress matrix, which is $-33,600 \text{ (MPa)}^3$.

Now,

$$\begin{aligned} & 4 \times \left(17,200 - \frac{260^2}{4} \right) \left(17,200 - 260^2 \right)^2 - 27 \times 336,000^2 \\ &= 3.048192 \times 10^{12} - 3.048192 \times 10^{12} = 0. \end{aligned}$$

Hence, the given state of stress causes yielding.

5.4.3 GEOMETRIC REPRESENTATION OF YIELD CRITERIA

Equations 5.48 and 5.52 imply that the Mises and Tresca criteria represent geometrical surfaces in a three-dimensional stress space of $(\sigma_1, \sigma_2, \sigma_3)$, called the principal stress space. The *Mises surface* is a right circular cylinder of radius $(\sqrt{2/3})\sigma_Y$, whereas the *Tresca surface* is a right (regular) hexagonal prism completely inscribed in the Mises cylinder, as shown in Figure 5.7 (Johnson and Mellor, 1972). The axis of the cylinder and prism is along the line $\sigma_1 = \sigma_2 = \sigma_3$. When the state of stress is purely hydrostatic, all the principal stresses are equal. Therefore, this line is called the *hydrostatic line*. Further, this axis is perpendicular to the plane $\sigma_1 + \sigma_2 + \sigma_3 = 0$. Since $\text{tr}\sigma$ is zero in a purely deviatoric state of stress, this plane is called the *deviatoric* or π plane.

Figure 5.7 can be used to find out graphically when the state of stress at a material particle will reach the yield level. For this purpose, first express the state of stress at the material particle in terms of the principal stresses $(\sigma_1, \sigma_2, \sigma_3)$. Then, locate the point $(\sigma_1, \sigma_2, \sigma_3)$ in the principal stress space. Denote this point by Q . (If the stress level at the material particle is elastic, point Q will be inside the yield surfaces.) Then, the vector $\overline{OQ}$ represents the state of stress at the material particle. Further, the component OG , along the hydrostatic line, represents the hydrostatic part of the stress, whereas the component OR , along the deviatoric plane, represents the deviatoric part. Now, let there be an increase in the stress level at the material particle such that only the hydrostatic component OG increases. Then, the point Q can never reach the yield surfaces, and hence there can never be any yielding. On the other hand, if the increase in the stress level is such that the deviatoric component OR or both the components increase sufficiently, then the point Q reaches the yield surfaces. When that happens, there will be yielding at the material particle. Note that the Tresca prism is completely inside the Mises cylinder except at the six edges. Therefore, if point Q reaches any one of these six edges, then the yielding occurs according to both the Tresca and Mises criteria. Otherwise, point Q will reach the Tresca prism first indicating yielding according to the Tresca criterion.

When the state of stress at a particle has zero hydrostatic part, the geometrical representation of the yield criteria reduces to curves: the intersections of the yield surfaces with the deviatoric plane. These curves are called *yield loci on the deviatoric plane*. The Mises yield locus is a circle of radius $(\sqrt{2/3})\sigma_Y$, whereas

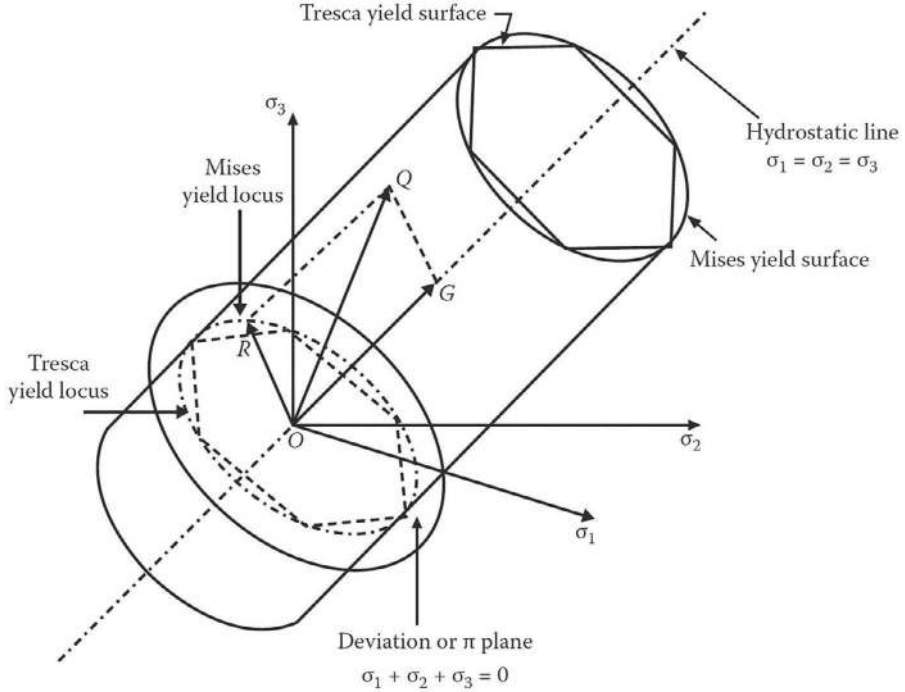


FIGURE 5.7 Geometric representation of the yield criteria in the principal stress space.

the Tresca yield locus is a regular hexagon, completely inscribed in the Mises circle (Figure 5.8).

If the state of stress at a particle is of the *plane stress* type (i.e. if one of the three principal stresses is zero at the particle), the geometrical representation of the yield criteria reduces to different curves. If it is assumed that the principal stresses are not ordered and σ_3 is zero, then the equations of these curves, as obtained from Equations 5.48 and 5.52, become

$$f(\sigma_1, \sigma_2) \equiv \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 - \sigma_Y^2 = 0, \text{ (Mises criterion)}, \quad (5.54)$$

$$f(\sigma_1, \sigma_2) \equiv [(\sigma_1 - \sigma_2)^2 - \sigma_Y^2][\sigma_1^2 - \sigma_Y^2][\sigma_2^2 - \sigma_Y^2] = 0. \text{ (Tresca criterion)}. \quad (5.55)$$

In this case, the Mises criterion (Equation 5.54) represents an ellipse in the two-dimensional stress plane of (σ_1, σ_2) , whereas the Tresca criterion (Equation 5.55) represents a hexagon (but not regular), completely inscribed in the Mises ellipse. Both these curves (Figure 5.9) are the intersections of the yield surfaces with the plane $\sigma_3 = 0$.

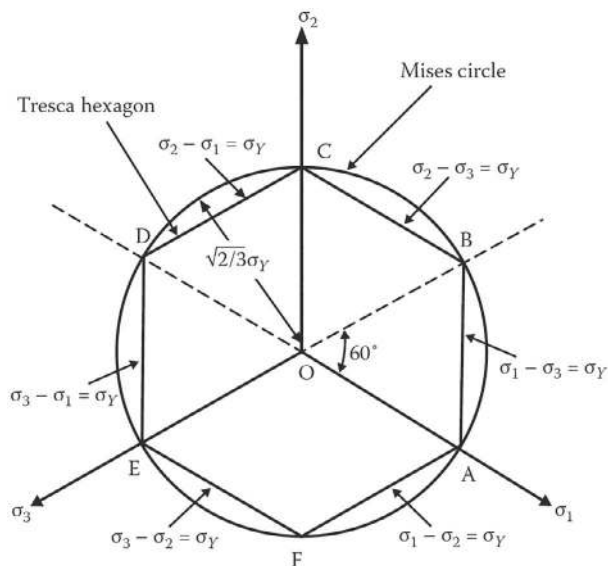


FIGURE 5.8 Loci of the Mises and Tresca yield surfaces on the deviatoric plane. The axes (σ_1 , σ_2 , σ_3 , respectively) are not in the deviatoric plane.

5.4.4 CONVEXITY OF YIELD SURFACES

It is observed that the regions bounded by the yield surfaces of both the Mises and Tresca criteria (Figure 5.7) are *convex*. A region is defined as convex if a straight-line segment joining any two points of the region lies completely inside the region. Note that the regions bounded by the yield loci of Figures 5.8 and 5.9 are also convex. While developing the anisotropic yield criteria, one of the requirements is that their geometric representations must lead to convex yield surfaces.

5.4.5 EXPERIMENTAL VALIDATION

Lode (1925) and Taylor and Quinney (1931) performed experiments on thin-walled tubes to validate the Mises and Tresca criteria with experimental results on yielding. In Lode's experiments, the loadings were axial force and internal pressure, whereas in the experiments of Taylor and Quinney, the loadings were axial force and torque.

Both experiments indicate that the Mises criterion is in better agreement with experimental results (Dixit and Dixit, 2008). Therefore, normally, the Mises criterion is used in the analysis of plastic deformation. The experiments further indicate that the Tresca criterion is conservative as far as prediction of yielding is concerned. Therefore, it is preferred in the design of structures and machine elements where the objective is to avoid yielding. The Tresca yield surface is not smooth like the Mises yield surface (Figure 5.7). Normal to the Tresca yield surface does not exist along the six edges of the hexagonal prism. This creates difficulties in applying the elasto-plastic stress-strain relations along the edges as these relations depend on the

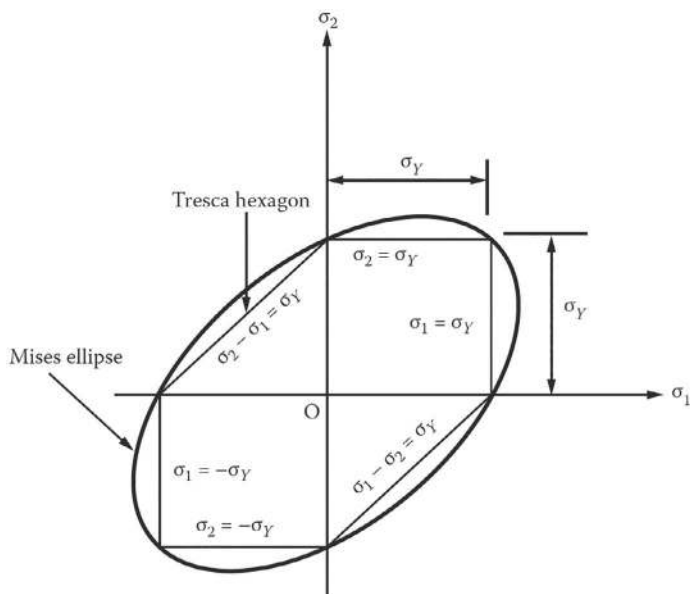


FIGURE 5.9 Loci of the Mises and Tresca yield surfaces on the plane $\sigma_3 = 0$.

normal. This is another reason why only the Mises criterion is used in the analysis of plastic deformation.

5.5 MODELING OF ISOTROPIC HARDENING OR CRITERION FOR SUBSEQUENT ISOTROPIC YIELDING

Figure 5.4 represents the 1-D experimental result on subsequent yielding in the form of variation of yield stress with plastic deformation. Equation 5.31 represents its mathematical form, with some specific expressions of hardening function h being given by Equations 5.32–5.34. The approach adopted for the generalization of the 1-D initial yield criterion to the three-dimensional state of stress is not convenient for the generalization of Equation 5.31. Therefore, a different approach is followed: we look at the graphical form of the initial yield criteria for obtaining more insight into the phenomenon of subsequent yielding. Since yielding depends only on the deviatoric part of stress, it is convenient to consider a two-dimensional curve called the yield locus (Figure 5.8), which is the intersection of the yield surface with the deviatoric plane. It is expected that, in general, both the size and shape of the initial yield locus would change with the complete history of plastic deformation since the last annealing. Further, due to the Bauschinger effect, the center of the yield locus would move away from the origin.

To simplify the development of the generalized criterion for subsequent yielding, it is assumed that the material remains *isotropic during hardening*. This assumption of isotropy automatically excludes the Bauschinger effect. (It means that the center

of the yield locus remains at the origin.) Further, the assumption of isotropy implies the following:

- During subsequent yielding, the shape of the yield locus does not change, only its size changes
- The change in size depends on the invariants of a tensor describing the history of plastic deformation since the last annealing

As a first step in this development, the evaluation of the invariants needs to be discussed. As stated earlier, the convenient way of describing the elasto-plastic behavior is to use an incremental measure of deformation like the incremental linear strain tensor. However, in most elasto-plastic problems, the deformation is accompanied by large rotation. As a result, integration of the incremental linear strain tensor up to the current time does not give a physically meaningful measure of the history of deformation. Therefore, the invariants of the incremental linear strain tensor are first evaluated, and then they are integrated, along the path of deformation, to get scalar measures of the history of deformation. Since the invariants are coordinate-independent functions, the presence of rotation does not affect their values during the integration process.

Note that the incremental linear strain tensor $d\epsilon_{ij}$ (developed in Section 4.15) contains both the elastic and plastic parts. However, the change in the size of the yield locus depends only on the plastic part of the history of deformation. Therefore, the plastic part of $d\epsilon_{ij}$ needs to be separated first. Figure 5.3 shows that, for the 1-D case, the elastic and plastic parts are additive, not just for the incremental strain, but for the total strain also. Therefore, for a three-dimensional case, it is assumed that the elastic and plastic parts of $d\epsilon_{ij}$ are additive, at least for the case of small incremental deformation. Thus, it is assumed that

$$d\epsilon_{ij} = d\epsilon_{ij}^e + d\epsilon_{ij}^p, \quad (5.56)$$

where $d\epsilon_{ij}^e$ and $d\epsilon_{ij}^p$ are, respectively, the elastic and plastic parts of $d\epsilon_{ij}$. The principal invariants $I_{d\epsilon^p}$, $II_{d\epsilon^p}$ and $III_{d\epsilon^p}$ of the tensor $d\epsilon_{ij}^p$ are now defined by the equations similar to Equations 4.128–4.130:

$$I_{d\epsilon^p} = d\epsilon_{ii}^p, \quad (5.57)$$

$$II_{d\epsilon^p} = \frac{1}{2} \left(d\epsilon_{ij}^p d\epsilon_{ij}^p - d\epsilon_{ii}^p d\epsilon_{jj}^p \right), \quad (5.58)$$

$$III_{d\epsilon^p} = \epsilon_{ijk} d\epsilon_{ii}^p d\epsilon_{jj}^p d\epsilon_{kk}^p. \quad (5.59)$$

The incremental volumetric strain corresponding to $d\epsilon_{ij}^p$ is defined by

$$d\epsilon_v^p = d\epsilon_{ii}^p. \quad (5.60)$$

No change in volume during plastic deformation implies that $d\varepsilon_{ii}^p$ is zero. Then, Equations 5.57 and 5.58 for the first two invariants get modified as

$$I_{d\varepsilon^p} = 0, \quad (5.61)$$

$$II_{d\varepsilon^p} = \frac{1}{2} d\varepsilon_{ij}^p d\varepsilon_{ij}^p. \quad (5.62)$$

Since $I_{d\varepsilon^p}$ is zero, the size of the yield locus (for isotropic hardening) should depend only on the integrals of $I_{d\varepsilon^p}$ and $III_{d\varepsilon^p}$ along the path of deformation.

Regarding the dependence of the size of the yield locus on the integrals of $I_{d\varepsilon^p}$ and $III_{d\varepsilon^p}$, there are two hypotheses: (1) *strain-hardening hypothesis* and (2) *work-hardening hypothesis*. Both of these are considered one by one. Regarding the choice of the initial yield locus, the Mises yield locus is chosen because it has been decided to use the Mises criterion for initial yielding.

5.5.1 STRAIN-HARDENING HYPOTHESIS FOR MISES MATERIAL

In this hypothesis, the size of the yield locus is assumed to be independent of $III_{d\varepsilon^p}$. Further, it is assumed to depend not on the integral of $I_{d\varepsilon^p}$ but on the following related invariant:

$$d\varepsilon_{eq}^p = \left(\frac{4}{3} II_{d\varepsilon^p} \right)^{1/2} = \left(\frac{2}{3} d\varepsilon_{ij}^p d\varepsilon_{ij}^p \right)^{1/2}. \quad (5.63)$$

The choice of this invariant makes it convenient to evaluate the hardening function of the three-dimensional criterion from the results of the tension test. It is so because, in the tension test, this invariant is equal to the plastic part of the incremental axial strain. This is shown in the following. First, as stated in Section 5.2.1, the shear strain components are zero in the tension test:

$$d\varepsilon_{xy}^p = d\varepsilon_{yz}^p = d\varepsilon_{zx}^p = 0. \quad (5.64)$$

Further, the condition of zero volume change in plastic deformation, along with Equation 5.60, leads to

$$d\varepsilon_{xx}^p + d\varepsilon_{yy}^p + d\varepsilon_{zz}^p = 0. \quad (5.65)$$

Additionally, the transverse symmetry of the rod implies

$$d\varepsilon_{yy}^p = d\varepsilon_{zz}^p. \quad (5.66)$$

Substitution of Equations 5.64–5.66 in expression 5.63 reduces $d\varepsilon_{eq}^p$, in the tension test, to $d\varepsilon_{xx}^p$. Therefore, $d\varepsilon_{eq}^p$ is called the *equivalent plastic strain increment*. It is also called the *effective or generalized plastic strain increment*. Integration of $d\varepsilon_{eq}^p$ along the path of deformation gives

$$\varepsilon_{eq}^p = \int d\varepsilon_{eq}^p. \quad (5.67)$$

The quantity ε_{eq}^p is called the *equivalent* (or *effective* or *generalized*) *plastic strain*.

Now, the strain-hardening hypothesis is applied to the Mises yield locus on the deviatoric plane. The radius of the Mises yield locus in initial yielding is $(\sqrt{2/3})\sigma_Y$, which is equal to $(\sqrt{2/3})\sigma_{eq}$, because of Equation 5.44. σ_{eq} is chosen as a measure of the size of the Mises yield locus. Then, the strain-hardening hypothesis becomes

$$\sigma_{eq} = H(\varepsilon_{eq}^p), \quad (5.68)$$

where H is called the three-dimensional hardening function, which depends on the material. This equation indicates that hardening depends, not just on the difference between the final and initial deformations of the particle but also on the sum of all infinitesimal plastic increments until the final deformation since the last annealing.

The function H is evaluated from the tension test. In Section 5.4.2, it was shown that the equivalent stress σ_{eq} reduces to the axial stress σ in the tension test. Next, it was shown that the equivalent plastic strain ε_{eq}^p , in the tension test, reduces to ε_p (the plastic part of the logarithmic strain). For this, we proceed as follows. Earlier in this section, it was shown that the equivalent plastic strain increment $d\varepsilon_{eq}^p$ reduces to $d\varepsilon_{xx}^p$ (plastic part of the incremental axial strain) in the tension test. Further, in Section 4.19 it was shown that in the absence of rotation, the integral of $d\varepsilon_{xx}$ becomes equal to the logarithmic strain ε in the x -direction (Equation 4.226). Equality of the integral of $d\varepsilon_{xx}$ and the logarithmic strain ε remains valid in the tension test also as there is no rotation in this test. Since the elastic ($d\varepsilon_{xx}^e$) and plastic ($d\varepsilon_{xx}^p$) parts of $d\varepsilon_{xx}$ are additive in the tension test (Figure 5.3), the integral of $d\varepsilon_{xx}^p$ becomes equal to ε^p (the plastic part of ε). Thus, in the tension test, one gets

$$\varepsilon_{eq}^p = \int d\varepsilon_{eq}^p = \int d\varepsilon_{xx}^p = \varepsilon^p. \quad (5.69)$$

This means that the three-dimensional hardening function H reduces to a function between σ and ε^p in the tension test. In other words, for isotropic hardening, the function H is the same as the 1-D hardening function h describing the stress-strain curve of Figure 5.4. As stated earlier, this curve is commonly represented by Equations 5.32–5.34.

5.5.2 WORK-HARDENING HYPOTHESIS FOR MISES MATERIAL

In the work-hardening hypothesis, the size of the yield locus is assumed to depend on the total plastic work done (per unit volume) to achieve the present plastic

deformation since the last annealing. This quantity is the integral of the incremental plastic work (per unit volume) along the path of deformation:

$$W^p = \int dW^p = \int \sigma_{ij} d\varepsilon_{ij}^p. \quad (5.70)$$

Since work is a coordinate-independent quantity, $dW^p = \sigma_{ij} d\varepsilon_{ij}^p$ becomes an invariant of the tensor $d\varepsilon_{ij}^p$. Now, the work-hardening hypothesis is applied to the Mises material. Since σ_{eq} is a measure of the size of the Mises yield locus, the work-hardening hypothesis can be expressed as

$$\sigma_{eq} = F(W^p), \quad (5.71)$$

where the hardening function F depends on the material.

As before, the function F is evaluated from the tension test. As stated earlier, in the tension test, the equivalent stress σ_{eq} reduces to the only non-zero stress, that is, the axial stress σ . Therefore, the expression for W^p in the tension test becomes

$$W_{\text{tension-test}}^p = \int \sigma d\varepsilon^p. \quad (5.72)$$

Thus, for the work-hardening hypothesis, the function F reduces to a function between σ and $W_{\text{tension-test}}^p$ in the tension test. To construct this function, $W_{\text{tension-test}}^p$ is first found corresponding to all values of σ greater than σ_Y from Figure 5.4. Then, a graph of σ versus $W_{\text{tension-test}}^p$ is plotted. Finally, a mathematical equation similar to Equations 5.32–5.34 is fitted through this graph.

It can be shown that the work-hardening hypothesis and the strain-hardening hypothesis are equivalent for the Mises material.

5.5.3 CRITERION FOR SUBSEQUENT YIELDING FOR MISES MATERIAL BASED ON STRAIN-HARDENING HYPOTHESIS

In this section, the criterion for subsequent yielding for the material that yields according to the Mises yield criterion is developed. It is assumed that the hardening is governed by the strain-hardening hypothesis. Further, it is assumed that the 1-D hardening function h can be approximated by the power law given by Equation 5.32. Then, the *three-dimensional hardening function* H becomes

$$\sigma_{eq} = H(\varepsilon_{eq}^p) \equiv \sigma_Y + K(\varepsilon_{eq}^p)^n. \quad (5.73)$$

Then, using Equations 5.42, 5.44 and 5.73, the *criterion for subsequent yielding* for the Mises material can be expressed as

$$f(J_2, J_3; \varepsilon_{eq}^p) \equiv J_2 - \frac{1}{3} H^2(\varepsilon_{eq}^p) = 0. \quad (5.74)$$

The function f is called the *yield function*. For subsequent yielding, the yield function, besides being a function of the invariants J_2 and J_3 , also depends on the equivalent plastic strain (called the hardening parameter). For initial yielding, the value of H reduces to σ_Y , and thus, Equation 5.74 reduces to Equation 5.42.

Example 5.4

The matrices of the stress tensor σ at points A, B and C, with respect to the (x, y, z) coordinate system, are

$$[\sigma]_A = \begin{bmatrix} 80 & 60 & 0 \\ 60 & -40 & 0 \\ 0 & 0 & 20 \end{bmatrix} \text{ MPa,}$$

$$[\sigma]_B = \begin{bmatrix} 100 & 70 & 0 \\ 70 & -40 & 0 \\ 0 & 0 & 30 \end{bmatrix} \text{ MPa,}$$

$$[\sigma]_C = \begin{bmatrix} 120 & 80 & 0 \\ 80 & -40 & 0 \\ 0 & 0 & 40 \end{bmatrix} \text{ MPa,}$$

1. Find the equivalent stress at points A, B and C.
2. The values of the equivalent plastic strain ϵ_{eq}^p at points A, B and C are 0, 0.1 and 0.2, respectively. The true stress-logarithmic plastic strain curve of the material in the tension test is to be approximated by Ludwik's relation:

$$\sigma_{eq} = \sigma_Y + K(\epsilon_{eq}^p)^n.$$

Find the material constants σ_Y , K and n using the previous data.

SOLUTION

The expression for equivalent stress is

$$\sigma_{eq} = \sqrt{\frac{3}{2} \sigma'_{ij} \sigma'_{ij}}.$$

Hence, first, the deviatoric parts of stress tensors are calculated as follows:

$$\begin{aligned}
 [\sigma']_A &= \begin{bmatrix} 80 & 60 & 0 \\ 60 & -40 & 0 \\ 0 & 0 & 20 \end{bmatrix} - \frac{(80 - 40 + 20)}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 60 & 60 & 0 \\ 60 & -60 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ MPa,}
 \end{aligned}$$

$$\begin{aligned}
 [\sigma']_B &= \begin{bmatrix} 100 & 70 & 0 \\ 70 & -40 & 0 \\ 0 & 0 & 30 \end{bmatrix} - \frac{(100 - 40 + 30)}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 70 & 70 & 0 \\ 70 & -70 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ MPa,}
 \end{aligned}$$

$$\begin{aligned}
 [\sigma']_C &= \begin{bmatrix} 120 & 80 & 0 \\ 80 & -40 & 0 \\ 0 & 0 & 40 \end{bmatrix} - \frac{(120 - 40 + 40)}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 80 & 80 & 0 \\ 80 & -80 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ MPa.}
 \end{aligned}$$

1. The equivalent stress at points A, B and C is now calculated as follows:

$$\sigma_{eq}|_A = \sqrt{\frac{3}{2}(60^2 + 60^2 + 60^2 + 60^2)} = 146.97 \text{ MPa,}$$

$$\sigma_{eq}|_B = \sqrt{\frac{3}{2}(70^2 + 70^2 + 70^2 + 70^2)} = 171.46 \text{ MPa,}$$

$$\sigma_{eq}|_C = \sqrt{\frac{3}{2}(80^2 + 80^2 + 80^2 + 80^2)} = 195.96 \text{ MPa.}$$

2. Given the equivalent plastic strain at points A, B and C as

$$\varepsilon_{eq}^p \Big|_A = 0, \quad \varepsilon_{eq}^p \Big|_B = 0.1, \quad \varepsilon_{eq}^p \Big|_C = 0.2.$$

Ludwik's relation is given by

$$\sigma_{eq} = \sigma_Y + K \left(\varepsilon_{eq}^p \right)^n.$$

Putting the values of equivalent stress and equivalent strain at point A,

$$146.97 = \sigma_Y + K (0)^n.$$

Hence,

$$\sigma_Y = 146.97 \text{ MPa}.$$

Putting the values of equivalent stress and equivalent strain at points B and C,

$$171.46 = 146.97 + K (0.1)^n, \quad 195.96 = 146.97 + K (0.2)^n,$$

or,

$$K (0.1)^n = 24.49, \quad K (0.2)^n = 48.99.$$

From the previous two equations,

$$2^n = \left(\frac{48.99}{24.49} \right) = 1.9967.$$

This gives $n = 0.9976$. Now,

$$K (0.1)^{0.9976} = 24.49.$$

This gives $K = 243.55 \text{ MPa}$. Considering that the precision in experimental evaluation of flow stress is approximately $\pm 5\%$,

$$\sigma_y = 147 \text{ MPa}, \quad n = 0.1, \quad K = 244 \text{ MPa}.$$

5.5.4 EXPERIMENTAL VALIDATION OF ISOTROPIC HARDENING

The few experimental studies on the change of the yield locus/surface due to hardening indicate that the yield locus/surface does distort or translate due to hardening (Naghdi et al., 1958; Philips and Das, 1985). Thus, the assumption of isotropic hardening is not supported by experiments except in the case of *proportional loading* (Lubahn and Felgar, 1961). Modeling of the translation of the yield locus/surface with hardening, called kinematic hardening, is discussed in Prager (1955) and Ziegler (1959). (It is important if the process involves too many unloading steps.) However, there has been no attempt to model the distortion of the yield locus/surface with hardening due to the unavailability of the required experimental data.

5.6 ELASTIC-PLASTIC STRESS-STRAIN AND STRESS-STRAIN RATE RELATIONS FOR ISOTROPIC MATERIALS

In Section 5.2, 1-D experimental observations are generalized to develop the three-dimensional stress-strain relations for linearly elastic material. However, such an approach is not feasible for developing the three-dimensional elasto-plastic stress-strain relations. The earliest attempts to develop these relations were based on Saint-Venant's proposal: 'the principal directions of the plastic part of the incremental linear strain tensor coincide with the principal directions of the stress tensor (Hill, 1950)'. Based on this proposal, Levy and Mises proposed three-dimensional stress-strain relations for rigid perfectly plastic material, whereas Prandtl and Reuss developed them for elastic-perfectly plastic material. However, these relations are not applicable to hardening materials.

In developing the three-dimensional stress-strain relations for hardening materials, two approaches have emerged: (1) an approach based on *Drucker's postulate for stable plastic material* (Malvern, 1969) and (2) an approach based on the *postulate of plastic potential*. In this book, the first approach is followed. For discussion on the first approach, the reader should refer to Hill's (1950) book.

5.6.1 DRUCKER'S POSTULATE FOR STABLE PLASTIC MATERIAL

In the tension test, the graph of true stress versus the logarithmic strain for a typical ductile material (Figure 5.3) has been reproduced in Figure 5.10. Now, consider a rod in an initial equilibrium state with σ^* as the true stress. Let an external agency (distinct from the agency that caused the existing state of stress) slowly apply a set of self-equilibrating forces on this rod to change the stress level to σ and then slowly remove them to bring the stress level back to the equilibrium level of σ^* . In this case, the original configuration may or may not be restored after the removal of the forces by the external agency. Such a cycle is called the *closed stress cycle* or *closed loading-unloading path* and is denoted by C_σ . For the material of Figure 5.10, it can be shown that

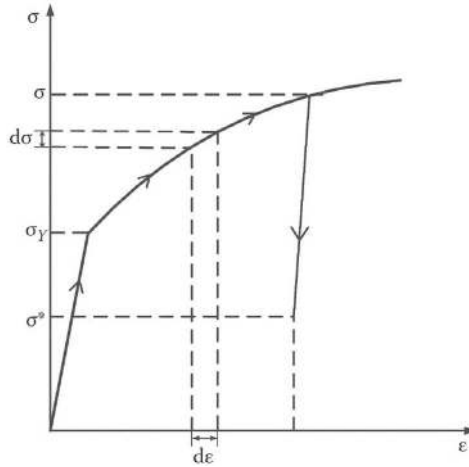


FIGURE 5.10 Graph of true stress versus logarithmic strain in tension test (hardening material).

$$\int_{C_\sigma} (\sigma - \sigma^*) d\epsilon \geq 0, \quad (\text{zero if } \sigma \text{ is less than } \sigma_Y), \quad (5.75)$$

and

$$d\sigma d\epsilon > 0 \quad (5.76)$$

However, if one has a material whose behavior in the tension test is as shown in Figure 5.11, then the following is obtained:

$$\int_{C_\sigma} (\sigma - \sigma^*) d\epsilon < 0, \quad (\text{If } \sigma^* \text{ is close to } \sigma_1, \text{ the maximum stress}), \quad (5.77)$$

$$d\sigma d\epsilon < 0, \quad (\text{If } d\sigma \text{ is after } \sigma_1, \text{ the maximum stress}). \quad (5.78)$$

Drucker calls the material of Figure 5.10 a *stable plastic material* whereas that of Figure 5.11 an *unstable plastic material*.

As a generalization of the hardening behavior in the tension test, Drucker made the following postulate for a stable plastic material in the three-dimensional state of stress. In a *stable plastic material*

- The total work performed by the external energy during the *closed stress cycle* is non-negative (it is zero only if the deformation is purely elastic).
- The incremental work done by the external agency during the application of additional stresses is positive.

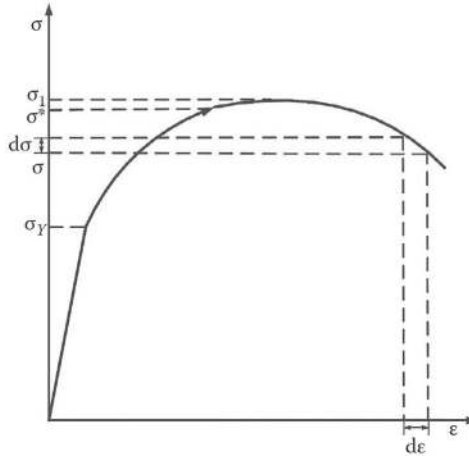


FIGURE 5.11 Graph of true stress versus logarithmic strain in tension test (softening material).

To derive the consequences of the previous postulate, let σ_{ij}^* be the initial equilibrium state in a body and σ_{ij} be the stress level after the application of self-equilibrating forces by the external agency. As before, let C_σ denote the *closed stress cycle*. Then, the first statement implies that

$$\int_{C_\sigma} (\sigma_{ij} - \sigma_{ij}^*) d\varepsilon_{ij} \geq 0, \quad (\text{zero only for purely elastic deformation}), \quad (5.79)$$

Earlier, an additive decomposition of the incremental linear strain tensor $d\varepsilon_{ij}$ is assumed (Equation 5.56). Further, if $d\varepsilon_{ij}$ in the inequality 5.79 is purely elastic, then it becomes an equation with zero right side. Therefore, when the decomposition $d\varepsilon_{ij} = d\varepsilon_{ij}^e + d\varepsilon_{ij}^p$ is substituted in the previous inequality, it implies

$$\int_{C_\sigma} (\sigma_{ij} - \sigma_{ij}^*) d\varepsilon_{ij}^p > 0. \quad (5.80)$$

The path C_σ in the previous inequality is such that the integrand satisfies the following inequality:

$$(\sigma_{ij} - \sigma_{ij}^*) d\varepsilon_{ij}^p > 0. \quad (5.81)$$

The second statement, along with the decomposition $d\varepsilon_{ij} = d\varepsilon_{ij}^e + d\varepsilon_{ij}^p$, implies that

$$d\sigma_{ij} d\varepsilon_{ij}^p > 0. \quad (5.82)$$

Now, consider a combined nine-dimensional space of σ_{ij} and $d\epsilon_{ij}^p$ (Figure 5.12). The yield criterion $f(\sigma_{ij}, \epsilon_{eq}^p) = 0$ represents a surface in this space, called the yield surface. (In Section 5.4, this has been seen for the Mises and Tresca criteria for initial yielding in terms of the principal stresses.) It is assumed that σ_{ij}^* represents a state of stress before initial yielding, and therefore, it lies inside the initial yield surface. Further, it is assumed that σ_{ij} represents a state of stress after initial yielding, and therefore, it lies on one of the (subsequent) yield surfaces. The tensors σ_{ij} , σ_{ij}^* and $d\epsilon_{ij}^p$ can be represented as geometrical vectors in this space. Further, since the yield surface expands with an increase in the plastic strain, the geometrical representation of $d\epsilon_{ij}^p$ must be pointing outward from the (current) yield surface. Therefore, it can be shown that the inequality 5.81 implies that the yield surface $f(\sigma_{ij}, \epsilon_{eq}^p) = 0$ is *convex* (see Figure 5.12).

Next, consider the inequality 5.82. The vectorial representation of $d\sigma_{ij}$ must point outward from the (current) yield surface if it is a loading process from the current stress state. (It becomes unloading if it points inward.) The vectorial representation of $d\sigma_{ij}$ can be decomposed as

$$d\sigma_{ij} = d\sigma_{ij}^t + d\sigma_{ij}^n. \quad (5.83)$$

where $d\sigma_{ij}^t$ represents the vectorial representation tangent to the current yield surface, and $d\sigma_{ij}^n$ represents the vectorial representation normal to it. The tangential part ($d\sigma_{ij}^t$) denotes neutral loading, and therefore, it causes only the elastic deformation. Thus,

$$d\sigma_{ij}^t d\epsilon_{ij}^p = 0. \quad (5.84)$$

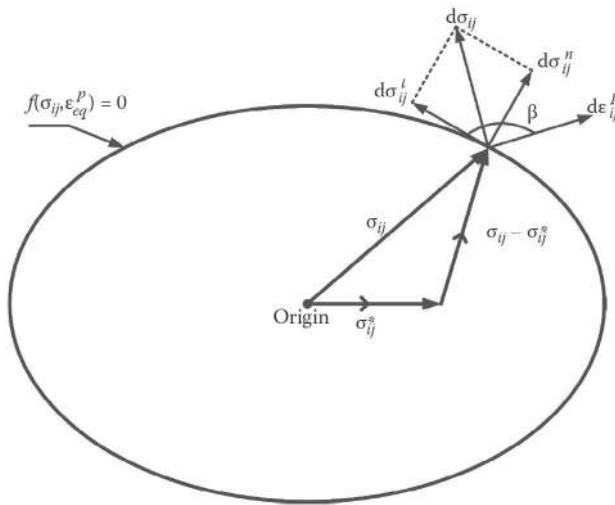


FIGURE 5.12 Graphical representation of yield surface in a nine-dimensional stress space of σ_{ij} .

The previous equation can be expressed as

$$d\sigma_{ij}^t \varepsilon_{ij}^p \cos \beta = 0 \quad (5.85)$$

where $\|(\cdot)\|$ represents the norm of the tensor (Equation 2.264), and β is the ‘angle’ between the vectorial representations of $d\sigma_{ij}^t$ and $d\varepsilon_{ij}^p$.

For the given non-zero $d\varepsilon_{ij}^p$, the previous equation must be true for every direction in the tangential plane to the yield surface (i.e. for an infinite number of non-zero values of $d\sigma_{ij}^t$). Then, the only solution of Equation 5.85 is

$$\cos \beta = 0 \quad \text{or} \quad \beta = \pi / 2 \quad (5.86)$$

The previous equation implies that the vectorial representations of $d\varepsilon_{ij}^p$ must be normal to the (current) yield surface. This is called the ‘normality rule’.

Since the vector $\partial f / \partial \sigma_{ij}$ is also normal to the (current) yield surface, the vectorial representations of $d\varepsilon_{ij}^p$ and $\partial f / \partial \sigma_{ij}$ must be parallel. It means, in the tensorial form, $d\varepsilon_{ij}^p$ must be a scalar multiple of $\partial f / \partial \sigma_{ij}$:

$$d\varepsilon_{ij}^p = d\lambda \frac{\partial f}{\partial \sigma_{ij}}. \quad (5.87)$$

The scalar $d\lambda$ must be positive since both $d\varepsilon_{ij}^p$ and $\partial f / \partial \sigma_{ij}$ point outward from the yield surface.

5.6.2 ASSOCIATED FLOW RULE

Equation 5.87 is called the *associated flow rule*. The yield function f depends on the stress tensor σ_{ij} through the invariants J_2 and J_3 of its deviatoric part. Using the chain rule, the associated flow rule (Equation 5.87) can be written as

$$d\varepsilon_{ij}^p = d\lambda \left(\frac{\partial f}{\partial J_2} \frac{\partial J_2}{\partial \sigma_{ij}} + \frac{\partial f}{\partial J_3} \frac{\partial J_3}{\partial \sigma_{ij}} \right). \quad (5.88)$$

The derivatives of the invariants J_2 and J_3 with respect to σ_{ij} are given by Dixit and Dixit (2008):

$$\frac{\partial J_2}{\partial \sigma_{ij}} = \sigma'_{ij}, \quad (5.89)$$

$$\frac{\partial J_3}{\partial \sigma_{ij}} = p_{ij}, \quad (5.90)$$

where $\mathbf{p}$ is the deviatoric part of $\boldsymbol{\sigma}'^2$ (i.e. square of $\boldsymbol{\sigma}'$). Substitution of Equations 5.89 and 5.90 in Equation 5.88 gives the following expression for the associated flow rule:

$$d\varepsilon_{ij}^p = d\lambda \left(\frac{\partial f}{\partial J_2} \sigma'_{ij} + \frac{\partial f}{\partial J_3} p_{ij} \right), \quad (5.91)$$

For a specific yield function (for example, the Mises or the Tresca yield function), the derivatives of f can be evaluated with respect to J_2 and J_3 and then the expression 5.91 of the associated flow rule can be simplified. However, before that, an important consequence of Equation 5.91 will be discussed. This is done in the next paragraph.

The consequence of Equation 5.91 is that the principal directions of the plastic part $d\varepsilon^p$ of the incremental linear strain tensor are the same as those of the stress tensor σ . To show this, the following two results are used:

Result 1: The principal directions of the deviatoric part of a tensor are the same as those of the tensor itself. This can be shown by combining the equation governing the eigenvalues and eigenvectors of a tensor (Equation 3.80) with the one that defines the deviatoric part of a tensor (Equation 3.105).

Result 2: The principal directions of the square of a tensor are the same as those of the tensor itself. This again can be shown using Equation 3.80.

Let $\hat{e}_i$ be the principal directions of the stress tensor σ . Then, Result 1 implies that $\hat{e}_i$ would be the principal directions of σ' also. Further, using Result 2, it can be shown that the principal directions of σ'^2 also would be $\hat{e}_i$. Again, Result 1 would imply that the principal directions of p (i.e. the deviatoric part of σ'^2) also would be $\hat{e}_i$. As per Equation 5.91, the tensor $d\varepsilon^p$ is a linear combination of the tensors σ' and p . Since the principal directions of σ' and p are identical and equal to $\hat{e}_i$ the principal directions of $d\varepsilon^p$ also would be equal to $\hat{e}_i$. Thus, one gets the result that the principal directions of $d\varepsilon^p$ coincide with those of σ . This was proposed by Saint-Venant in 1870–1871.

Next, a graphical representation of the associated flow rule is considered (Equation 5.87). For that purpose, $\hat{e}_i$ is used as the coordinate axes to represent the matrices of σ and $d\varepsilon^p$. In this coordinate system, these matrices become

$$[\sigma] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}, \quad [d\varepsilon^p] = \begin{bmatrix} d\varepsilon_1^p & 0 & 0 \\ 0 & d\varepsilon_2^p & 0 \\ 0 & 0 & d\varepsilon_3^p \end{bmatrix} \quad (5.92)$$

where σ_i and $d\varepsilon_i^p$ are the principal values of the tensors σ and $d\varepsilon^p$, respectively. Next, the yield function f is expressed in terms of σ_i and the hardening parameter $\varepsilon_{eq}^p : f(\sigma_1, \sigma_2, \sigma_3; \varepsilon_{eq}^p)$. Then, the yield criterion $f=0$ can be represented as a surface (called the yield surface) in the three-dimensional principal stress space of $(\sigma_1, \sigma_2, \sigma_3)$. (In Section 5.4, this has been seen for the Mises and Tresca criteria for initial yielding.)

Then, the associated flow rule (Equation 5.87), in terms of the components with respect to $\hat{e}_i$ as the coordinate axes become

$$\{d\varepsilon^p\} = d\lambda \{\nabla f\} \quad (5.93)$$

where

$$\{d\varepsilon^p\}^T \equiv \{d\varepsilon_1^p, d\varepsilon_2^p, d\varepsilon_3^p\}, \quad (5.94)$$

$$\{\nabla f\}^T = \left\{ \frac{\partial f}{\partial \sigma_1}, \frac{\partial f}{\partial \sigma_2}, \frac{\partial f}{\partial \sigma_3} \right\}. \quad (5.95)$$

Note that the geometrical representation of the gradient array $\{\nabla f\}$ is normal to the yield surface $f = 0$. Further, it has been seen earlier that the geometrical representation of the array $\{d\varepsilon\}$ is also perpendicular to the yield surface (i.e. the ‘normality rule’). Figure 5.13 shows the graphical representation of the normality rule for the Mises and Tresca yield functions on the deviatoric plane in the three-dimensional principal stress space of $(\sigma_1, \sigma_2, \sigma_3)$.

The matrix of σ'_i in the coordinate system of $\hat{e}_i$ becomes

$$[\sigma'] = \begin{bmatrix} \sigma'_1 & 0 & 0 \\ 0 & \sigma'_2 & 0 \\ 0 & 0 & \sigma'_3 \end{bmatrix}. \quad (5.96)$$

where σ'_i are its principal values. Then, the geometrical vector corresponding to the array

$$\{\sigma'\}^T \equiv \{\sigma'_1, \sigma'_2, \sigma'_3\} \quad (5.97)$$

lies on the deviatoric plane (see Section 5.4.3). Note that, in Figure 5.13, the vector $\{d\varepsilon^p\}$ is parallel to the vector $\{\sigma'\}$ for the Mises material but not for the Tresca material. It means that the tensor $d\varepsilon_{ij}^p$ is a scalar multiple of σ'_{ij} for the Mises material but not for the Tresca material. This is consistent with Equation 5.91, where the derivative $\partial f / \partial J_3$ is zero for the Mises material but not for the Tresca material. There is another difference between the Mises and Tresca yield criteria as far as the evaluation of $d\varepsilon_{ij}^p$ is concerned. For the Mises surface the normal is defined at every point of the yield surface. However, for the Tresca surface, it is not defined along the six edges of the prism (i.e. at the corners of the yield loci of Figure 5.13). Therefore, the vector $\{d\varepsilon^p\}$ is not uniquely determined at the edges of the Tresca surface. This is one of the reasons for not using the Tresca yield criterion in our analysis.

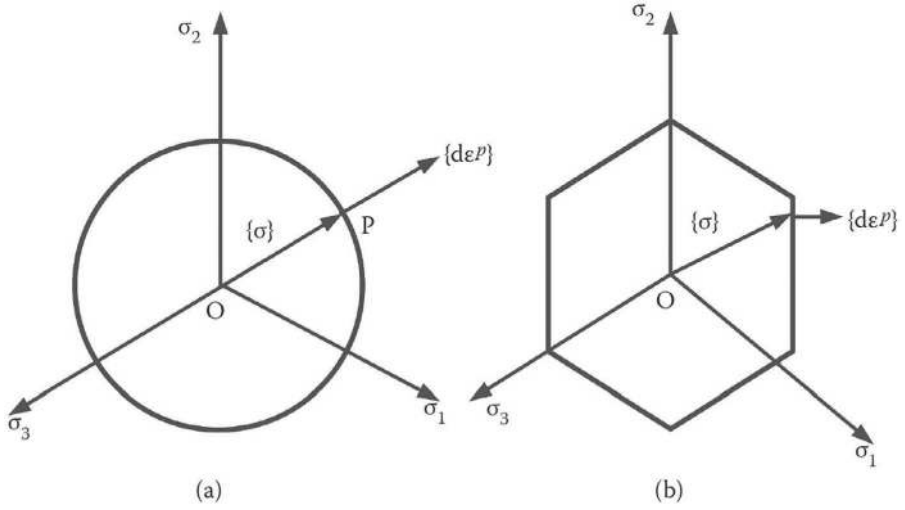


FIGURE 5.13 Graphical representation of ‘normality rule’ for Mises and Tresca yield functions: (a) Mises yield locus on the deviatoric plane; (b) Tresca yield locus on the deviatoric plane.

Next, the procedure for determining $d\lambda$ for the materials that obey the strain-hardening hypothesis is discussed. In this case, the yield function can be expressed as

$$f(\sigma_{ij}; \varepsilon_{eq}^p) = 0. \quad (5.98)$$

Setting the differential of f to zero, one gets

$$df \equiv \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial f}{\partial \varepsilon_{eq}^p} d\varepsilon_{eq}^p = 0. \quad (5.99)$$

This is called the consistency relation. It means that the state $(\sigma_{ij} + d\sigma_{ij}, \varepsilon_{eq}^p + d\varepsilon_{eq}^p)$ also lies on the yield surface. Combining the definition of $d\varepsilon_{eq}^p$ (Equation 5.63) and the associated flow rule (Equation 5.87), one gets

$$d\varepsilon_{eq}^p = \left(\frac{2}{3} d\varepsilon_{ij}^p d\varepsilon_{ij}^p \right)^{1/2} = \sqrt{\frac{2}{3}} \left(\frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} \right)^{1/2} d\lambda. \quad (5.100)$$

Substituting the previous expression in Equation 5.99, one gets the following expression for $d\lambda$:

$$d\lambda = -\sqrt{\frac{3}{2}} \left(\frac{\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij}}{\frac{\partial f}{\partial \varepsilon_{eq}^p} \left(\frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} \right)^{1/2}} \right). \quad (5.101)$$

To evaluate $d\lambda$ for the Mises material, the Mises yield function for the hardening material is differentiated (Equation 5.74) with respect to σ_{ij} and ε_{eq}^p and $\partial J_2 / \partial \sigma_{ij} = \sigma'_{ij}$ is noted (Equation 5.89). Then, one gets

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial J_2}{\partial \sigma_{ij}} = \sigma'_{ij}, \quad (5.102)$$

$$\frac{\partial f}{\partial \varepsilon_{eq}^p} = -\frac{1}{3} (2HH'). \quad (5.103)$$

Here, H' is the derivative of the hardening function (Equation 5.68) with respect to the equivalent plastic strain. Substitution of the previous expressions in Equation 5.101 along with the equality $H = \sigma_{eq}$ and the expression $\sigma'_{ij} \sigma'_{ij} = (2/3) \sigma_{eq}^2$ (Equation 5.43) leads to

$$d\lambda = \frac{9}{4} \frac{\sigma'_{ij} d\sigma_{ij}}{H' \sigma_{eq}^2}. \quad (5.104)$$

Now, the associated flow rule for the Mises material is obtained. The Mises yield function (Equation 5.74) is linear in J_2 and independent of J_3 . Therefore, the associated flow rule (Equation 5.91) becomes

$$d\varepsilon_{ij}^p = d\lambda \sigma'_{ij}. \quad (5.105)$$

Thus, as stated earlier, the tensor $d\varepsilon_{ij}^p$ is a scalar multiple of σ'_{ij} for the Mises material. This is also expressed graphically in Figure 5.13a. Substituting the expression for $d\lambda$ (Equation 5.104) and changing the dummy indices from i and j to k and l , one gets the following expression for the associated flow rule of the Mises material:

$$d\varepsilon_{ij}^p = \frac{9}{4} \frac{\sigma'_{ij} \sigma'_{kl}}{H' \sigma_{eq}^2} d\sigma_{kl}. \quad (5.106)$$

Note that, in the previous expression, the differential is *not of the deviatoric part but of the whole stress tensor*. This form of the associated flow rule is convenient for the updated Lagrangian formulation but not for the Eulerian formulation.

For the Eulerian formulation Equation 5.106 needs to be in a different form. Instead of obtaining this form from Equation 5.106, we proceed differently. The $d\lambda$ for the Mises material is found by a simpler procedure starting from Equation 5.105. By squaring all the scalar equations of the set (Equation 5.105) and using the definitions of the equivalent stress σ_{eq} (Equation 5.43) and equivalent plastic strain increment $d\varepsilon_{eq}^p$ (Equation 5.63), one gets

$$d\lambda = \frac{3}{2} \frac{d\varepsilon_{eq}^p}{\sigma_{eq}}. \quad (5.107)$$

Then, substituting this value of $d\lambda$ in Equation 5.105, one gets

$$d\varepsilon_{ij}^p = \frac{3}{2} \frac{d\varepsilon_{eq}^p}{\sigma_{eq}} \sigma'_{ij}. \quad (5.108)$$

This is the form of the associated flow rule that is convenient for the Eulerian formulation. Derivation of this equation from Equation 5.106 is given in Dixit and Dixit (2008).

Starting from Equations 5.106 and 5.108, the following two constitutive relations for the Mises material are now derived: (1) elastic-plastic incremental stress-strain relation for the updated Lagrangian formulation and (2) elastic-plastic stress-strain rate relation for the Eulerian formulation.

5.6.3 ELASTIC-PLASTIC INCREMENTAL STRESS-STRAIN RELATION FOR THE MISES MATERIAL

The associated flow rule (Equation 5.106) is a constitutive relationship between the incremental stress tensor $d\sigma_{kl}$ and only the plastic part of incremental linear strain tensor $d\varepsilon_{ij}$. However, a relationship is needed between $d\sigma_{kl}$ and the whole of $d\varepsilon_{ij}$. To develop this relationship, first, it is needed to relate $d\sigma_{kl}$ with the elastic part of $d\varepsilon_{ij}$ and then to combine this relationship with Equation 5.106.

To relate $d\sigma_{kl}$ with $d\varepsilon_{ij}^e$, recall that it has been decided to neglect the hysteresis in the tension test. Then, the slope of the unloading path becomes equal to the slope of the elastic path (line OY of Figure 5.3). This means that, in a three-dimensional elastic-plastic state of deformation, the relation between $d\sigma_{kl}$ and $d\varepsilon_{ij}^e$ is the same as the constitutive equation of the linearly elastic material (Equation 5.7 or 5.12 or 5.17 and 5.18). To relate $d\varepsilon_{ij}^e$ with $d\sigma_{kl}$, the inverse stress-strain relationship is used (Equation 5.12):

$$d\varepsilon_{ij}^e = \frac{1}{E} \left[-\nu d\sigma_{kk} \delta_{ij} + (1 + \nu) d\sigma_{ij} \right]. \quad (5.109)$$

Using the identity Equation 2.38, the previous equation can be rewritten as

$$d\varepsilon_{ij}^e = \frac{1}{E} \left[-\nu \delta_{kl} \delta_{ij} + (1 + \nu) \delta_{ik} \delta_{jl} \right] d\sigma_{kl}. \quad (5.110)$$

Next, the elastic and plastic parts of the constitutive equation are combined. For this, as before, it is assumed that the elastic and plastic parts of the incremental linear strain tensor are additive (Equation 5.56). By adding Equations 5.109 and 5.110, one obtains

$$\begin{aligned} d\epsilon_{ij} &= d\epsilon_{ij}^e + d\epsilon_{ij}^p, \\ &= \left\{ \frac{1}{E} \left[-\nu \delta_{kj} \delta_{ij} + (1 + \nu) \delta_{ik} \delta_{jl} \right] + \frac{9}{4} \frac{\sigma'_{ij} \sigma'_{kl}}{H' \sigma_{eq}^2} \right\} d\sigma_{kl}. \end{aligned} \quad (5.111)$$

Inverting this relationship (Chakrabarty, 1987), one gets

$$d\sigma_{ij} = C_{ijkl}^{EP} d\epsilon_{kl}, \quad (5.112)$$

where the fourth-order elastic-plastic constitutive tensor C_{ijkl}^{EP} is given by

$$C_{ijkl}^{EP} = 2\mu \left[\frac{\nu}{1-2\nu} \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} - \frac{9\mu}{2} \frac{\sigma'_{ij} \sigma'_{kl}}{(H' + 3\mu) \sigma_{eq}^2} \right]. \quad (5.113)$$

Note that, the tensor C_{ijkl}^{EP} depends on (1) the elastic constants μ and ν , (2) the hardening curve through $\sigma_{eq} = H(\epsilon_{eq}^p)$ and its slope H' and (3) the current stress through σ' . The incremental stress in Equation 5.112 has to be *objective*. The definition of objectivity and an objective incremental stress tensor, namely, the increment of the second Piola-Kirchhoff stress tensor, is discussed in Section 5.7.

5.6.4 ELASTIC-PLASTIC STRESS-STRAIN RATE RELATION FOR THE MISES MATERIAL

In Eulerian formulation, the strain rate tensor $\dot{\epsilon}_{ij}$ is the measure of deformation. Therefore, the constitutive relation must be expressed in terms of $\dot{\epsilon}_{ij}$. To develop this relationship, $\dot{\epsilon}_{ij}$ is first decomposed into elastic and plastic parts. Similar to the decomposition of $d\epsilon_{ij}$ (Equation 5.56), the decomposition of $\dot{\epsilon}_{ij}$ into elastic and plastic parts is assumed to be additive:

$$\dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^e + \dot{\epsilon}_{ij}^p. \quad (5.114)$$

Then, the plastic constitutive relation, that is, the associated flow rule (Equation 5.108), is expressed in terms of $\dot{\epsilon}_{ij}^p$. Next, the elastic constitutive equation is obtained as a relation between the stress rate and $\dot{\epsilon}_{ij}^e$. Finally, the two constitutive relations are added.

In order to obtain the associated flow rule in terms of $\dot{\epsilon}_{ij}^p$, $d\epsilon_{ij}^p$ is first related with $\dot{\epsilon}_{ij}^p$. This is done by substituting the decompositions of $d\epsilon_{ij}$ and $\dot{\epsilon}_{ij}$ (Equations 5.56

and 5.114) in the equation relating these two quantities (Equation 4.222) and equating the plastic parts of both sides. This gives us

$$d\epsilon_{ij}^p = \dot{\epsilon}_{ij}^p dt. \quad (5.115)$$

Next, the equivalent plastic strain increment $d\epsilon_{eq}^p$ is expressed in terms of $\dot{\epsilon}_{ij}^p$. For this purpose, an invariant of $\dot{\epsilon}_{ij}^p$ is defined (called the *equivalent, effective or generalized plastic strain rate*):

$$\dot{\epsilon}_{eq}^p = \left(\frac{2}{3} \dot{\epsilon}_{ij}^p \dot{\epsilon}_{ij}^p \right)^{1/2}. \quad (5.116)$$

(It can be shown that this invariant reduces to the plastic part of the axial strain rate in the tension test.) Then, Equation 5.115 is substituted in the expression for $d\epsilon_{eq}^p$ (Equation 5.63) and the previous definition of $\dot{\epsilon}_{eq}^p$ (Equation 5.116) is used to get

$$d\epsilon_{eq}^p = \dot{\epsilon}_{eq}^p dt. \quad (5.117)$$

Before we proceed further, it is necessary to express the equivalent plastic strain ϵ_{eq}^p (the argument of the hardening function H) in terms of $\dot{\epsilon}_{eq}^p$ by substituting the previous expression in Equation 5.67:

$$\epsilon_{eq}^p = \int \dot{\epsilon}_{eq}^p dt. \quad (5.118)$$

(Here, the integration is to be carried along the path line of the material particle.) Finally, the expressions for $d\epsilon_{ij}^p$ and $d\epsilon_{eq}^p$ (Equations 5.115 and 5.117) are substituted in Equation 5.108 and the factor dt is canceled from both sides to obtain the associated flow rule in terms of $\dot{\epsilon}_{ij}^p$:

$$\dot{\epsilon}_{ij}^p = \frac{3}{2} \frac{\dot{\epsilon}_{eq}^p}{\sigma_{eq}} \sigma'_{ij}. \quad (5.119)$$

For the rate form of the constitutive relation, the relationship in terms of the hydrostatic and deviatoric parts is convenient. Since the trace of σ' is zero (Equation 3.108), by taking the trace of the previous equation, one gets the hydrostatic part of $\dot{\epsilon}_{eq}^p$ as 0:

$$\dot{\epsilon}_{kk}^p = 0. \quad (5.120)$$

Therefore, the whole of $\dot{\epsilon}_{ij}^p$ is equal to its deviatoric part. Thus,

$$\dot{\epsilon}_{ij}^p = \frac{3}{2} \frac{\dot{\epsilon}_{eq}^p}{\sigma_{eq}} \sigma'_{ij}. \quad (5.121)$$

Similar to Equation 4.176, $\dot{\varepsilon}_{kk}^p$ represents the volumetric strain rate corresponding to the plastic part of the strain rate tensor. Thus, Equation 5.120 is consistent with the observation that, in plastic deformation, the volume change is zero.

Next, the rate form of the elastic constitutive equation is obtained in terms of $\dot{\varepsilon}_{ij}^e$. As before, it is assumed that $d\varepsilon_{ij}^e$ is related to $d\sigma_{ij}$ by the constitutive equation of a linearly elastic material, that is, by any one form given by Equation 5.7 or 5.12 or 5.17 and 5.18. As stated previously, the form in terms of the hydrostatic and deviatoric parts is convenient here. Therefore, Equations 5.17 and 5.18 are used. Writing these equations in incremental form, replacing $d\varepsilon_{ij}$ by $d\varepsilon_{ij}^e$, and interchanging the sides, one gets

$$d\varepsilon_{kk}^e = \frac{d\sigma_{kk}}{3K}, \quad (5.122)$$

$$d\varepsilon_{ij}^{'e} = \frac{d\sigma'_{ij}}{2\mu}. \quad (5.123)$$

Now, the strain increments are expressed in terms of the strain rates (Equation 4.222) and the stress increments in terms of the stress rates, and the factor dt is canceled from both sides to get

$$\dot{\varepsilon}_{kk}^e = \frac{\dot{\sigma}_{kk}}{3K}, \quad (5.124)$$

$$\dot{\varepsilon}_{ij}^{'e} = \frac{\dot{\sigma}'_{ij}}{2\mu}. \quad (5.125)$$

Here, $\dot{\sigma}_{kk}$ is the time rate of σ_{kk} , and $\dot{\sigma}'_{ij}$ is the time rate of σ'_{ij} .

Finally, the elastic (Equations 5.124 and 5.125) and plastic (Equations 5.120 and 5.121) constitutive relations are added to obtain

$$\dot{\varepsilon}_{kk} = \dot{\varepsilon}_{kk}^e + \dot{\varepsilon}_{kk}^p = \frac{\dot{\sigma}_{kk}}{3K} + 0 = \frac{\dot{\sigma}_{kk}}{3K}, \quad (5.126)$$

$$\dot{\varepsilon}'_{ij} = \dot{\varepsilon}'_{ij}^e + \dot{\varepsilon}'_{ij}^p = \frac{\dot{\sigma}'_{ij}}{2\mu} + \frac{3\dot{\varepsilon}_{eq}^p}{2\sigma'_{eq}} \sigma'_{ij}. \quad (5.127)$$

Note that, in the previous constitutive relation, $\dot{\varepsilon}_{ij}$ depends on (1) the elastic material constants K and μ , (2) the material hardening curve through $\sigma'_{eq} = H\left(\int \dot{\varepsilon}_{eq}^p dt\right)$, (3)

the current stress through σ' and (4) the current stress rate through $\dot{\sigma}$. The stress rates in Equations 5.126 and 5.127 have to be *objective*. The definition of objectivity and an objective stress rate tensor, namely, the Jaumann stress rate tensor, are discussed in Section 5.6.5.

Now, two special cases of the constitutive relation (Equations 5.126 and 5.127) are derived. In the first case, hardening is neglected. Then, the size of the initial yield surface remains constant. As a result, the equivalent stress σ_{eq} remains constant at the value of σ_Y , as per Equation 5.73. Substitution of $\sigma_{eq} = \sigma_Y$ in Equations 5.126 and 5.127 gives the constitutive relation for non-hardening materials called the *Prandtl-Reuss equations*. They were proposed by Prandtl in 1924 for plane problems and by Reuss in 1930 for general cases based on Saint-Venant's proposal.

In the second case, the elastic deformation is neglected. Equations 5.126 and 5.127 contain both the stress and stress rate, making the resulting problem difficult to solve. To avoid this difficulty, $\dot{\epsilon}_{ij}^e$ is neglected compared to $\dot{\epsilon}_{ij}^p$, thus assuming the material to be a *rigid-plastic material*. Since the elastic deformation in metals is normally very small, this assumption is justified. When $\dot{\epsilon}_{ij}^e$ is neglected, Equation 5.126 becomes meaningless and Equation 5.127 reduces to

$$\dot{\epsilon}_{ij} \cong \frac{3\dot{\sigma}_{eq}}{2\sigma_{eq}} \sigma'_{ij}. \quad (5.128)$$

If this material is also non-hardening, the equivalent stress σ_{eq} (in Equation 5.128) becomes equal to σ_Y . This resulting equation, for *rigid-plastic non-hardening materials*, is called the *Levy-Mises equation*. It was proposed independently by Levy in 1871 and Mises in 1913 based on Saint-Venant's proposal.

Equation 5.128 is usually written in the following form:

$$\sigma'_{ij} = \frac{2\sigma_{eq}}{3\dot{\epsilon}_{eq}} \dot{\epsilon}_{ij}. \quad (5.129)$$

This equation implies that, in *rigid-plastic materials*, only the deviatoric part of stress can be determined from the plastic deformation, and the hydrostatic part is constitutively indeterminate. In *rigid-plastic materials*, the (plastic) deformation cannot produce any change in volume. Therefore, the hydrostatic part, which arises as a reaction to this incompressibility constraint, is determined from the following condition:

$$\dot{\epsilon}_{kk} = 0. \quad (5.130)$$

5.6.5 VISCOPLASTICITY AND TEMPERATURE SOFTENING

If the material also possesses viscoplasticity besides strain hardening, then similar to the case of strain hardening, it is assumed that

- During subsequent yielding, the shape and the center of the yield locus remain unchanged
- Only the size of the yield locus depends on $\dot{\epsilon}_{ij}^p$ (i.e. the plastic part of strain rate tensor) through its invariant $\dot{\epsilon}_{eq}^p$

As stated earlier, σ_{eq} is a measure of the size of the yield locus for the Mises material. Therefore, for the material possessing both strain hardening and viscoplasticity, σ_{eq} becomes a function of ε_{eq}^p and $\dot{\varepsilon}_{eq}^p$:

$$\sigma_{eq} = H\left(\varepsilon_{eq}^p, \dot{\varepsilon}_{eq}^p\right). \quad (5.131)$$

Similar to the dependence of function H on ε_{eq}^p , the dependence on $\dot{\varepsilon}_{eq}^p$ is also determined from the tension test. In the tension test, σ_{eq} reduces to the axial stress and $\dot{\varepsilon}_{eq}^p$ reduces to the plastic part of the axial strain rate. Thus, the dependence of H on $\dot{\varepsilon}_{eq}^p$ is found from the graph of axial stress versus the plastic part of the axial strain rate.

If the material also possesses temperature softening besides strain hardening, then it is assumed that, during subsequent yielding, the shape and the center of the yield locus remain unchanged. Further, the size of the yield locus depends on the temperature T . Therefore, for the material possessing both strain hardening and temperature softening, σ_{eq} becomes a function of ε_{eq}^p and T :

$$\sigma_{eq} = H\left(\varepsilon_{eq}^p, T\right). \quad (5.132)$$

Again, the dependence of H on T is determined from the graph of axial stress versus temperature in the tension test.

If the material possesses strain hardening, viscoplasticity, as well as temperature softening, then σ_{eq} becomes a function of ε_{eq}^p , $\dot{\varepsilon}_{eq}^p$ and T :

$$\sigma_{eq} = H\left(\varepsilon_{eq}^p, \dot{\varepsilon}_{eq}^p, T\right). \quad (5.133)$$

For the materials possessing viscoplasticity or/and temperature softening besides strain hardening, the equivalent stress (σ_{eq}) in the constitutive equations (Equations 5.112, 5.113 and 5.126, 5.127) should be evaluated from Equations 5.131 or 5.132 or 5.133 depending on the case. Further, similar to Equation 5.73, the function H in these expressions (Equations 5.131–5.133) can be approximated as a power law in $\left(\varepsilon_{eq}^p, \dot{\varepsilon}_{eq}^p\right)$, $\left(\varepsilon_{eq}^p, T\right)$ or $\left(\varepsilon_{eq}^p, \dot{\varepsilon}_{eq}^p, T\right)$. Additionally, H' in Equation 5.113 means $\partial H / \partial \varepsilon_{eq}^p$.

5.7 OBJECTIVE INCREMENTAL STRESS AND OBJECTIVE STRESS RATE TENSORS

The stress rate tensor in Equations 5.126 and 5.127 and the incremental stress tensor in Equation 5.112 have to be *objective* tensors. It means that these tensors have to be frame-invariant or invariant under a change of the reference frame.

The essence of the concept of frame invariance is the *postulate* that the direction of the stress vector (at every point on every plane) should be invariant under a change of frame. It leads to the following mathematical relation:

$$\boldsymbol{\sigma}^* = \boldsymbol{Q}(t) \boldsymbol{\sigma} \boldsymbol{Q}^T(t) \quad (5.134)$$

where $\boldsymbol{\sigma}^*$ and $\boldsymbol{\sigma}$ are the representations of the Cauchy stress tensor with respect to fixed and moving frames, respectively, and $\boldsymbol{Q}(t)$ is the angular velocity of the moving frame. This derivation is given in Dixit and Dixit (2008). Any tensor quantity that satisfies the previous relation is called an objective tensor. Thus, the Cauchy stress tensor is objective, but its rate

$$\dot{\boldsymbol{\sigma}}^* = \dot{\boldsymbol{Q}}(t) \boldsymbol{\sigma} \boldsymbol{Q}^T(t) + \boldsymbol{Q}(t) \dot{\boldsymbol{\sigma}} \boldsymbol{Q}^T(t) + \boldsymbol{Q}(t) \boldsymbol{\sigma} \dot{\boldsymbol{Q}}^T(t) \quad (5.135)$$

is not objective. Thus, its increment is also not objective. Therefore, the Cauchy stress increment cannot be used in the incremental constitutive relation (Equation 5.112) or the Cauchy stress rate in the rate constitutive equation (Equations 5.126 and 5.127).

Another way to look at the objective stress rate or objective incremental stress tensors is as follows. Note that, in general, a material particle gets both deformed as well as rotated in a typical time increment. Thus, at any time, it has both the rate of deformation as well as the rate of rotation. Suppose that, in the current time increment Δt (or at the current time t), the particle does not deform (or has no rate of deformation), but only rotates (or has only the rate of rotation), then $d\boldsymbol{\varepsilon}$ (or $\dot{\boldsymbol{\varepsilon}}$) will be zero but not $d\boldsymbol{\omega}$ (or $\dot{\boldsymbol{\omega}}$). In this case, the objective incremental stress tensor (or the objective stress rate) is expected to be a zero tensor as it is frame invariant. Thus, in a simpler term, an incremental stress tensor is considered objective if it reduces to a zero tensor in the event of the increment being a pure rotation. Similarly, a stress rate tensor is considered objective if it reduces to a zero tensor in the event of the rate of deformation being zero.

Quite a few objective incremental stress and objective stress rate tensors have been proposed. Only one objective incremental stress tensor and one objective stress rate tensor are presented here, which are commonly employed in the literature and which are simpler to use. Further, their objectivity will be proved using the simpler definition.

Before that, the relation between the Cauchy stress tensors at time t and $t + \Delta t$ shall be obtained when the increment consists of pure rotation. This is done in Section 5.7.1.

For this section, the following notation is followed. To denote the time, the left subscript will be used for incremental quantities and the left superscript for all other quantities. In Section 4.15, only a small increment was considered and therefore, the symbol $d(\cdot)$ was used to denote an incremental quantity. In this section, an incremental quantity at time t would be denoted by the symbol $\Delta(\cdot)$ to emphasize the fact that the increment size may not be small.

5.7.1 RELATION BETWEEN CAUCHY STRESS TENSORS WHEN THE INCREMENT IS PURE ROTATION

Consider the two deformed configurations of a body at times t and $t + \Delta t$ (Figure 5.14).

Similar to Equation 4.49, the incremental deformation gradient tensor $\Delta \boldsymbol{F}$ at a point is defined as

$${}_t\Delta\mathbf{F} = \frac{\partial {}^{t+\Delta t}\mathbf{x}}{\partial {}^t\mathbf{x}} = \frac{\partial ({}^t\mathbf{x} + {}_t\Delta\mathbf{u})}{\partial {}^t\mathbf{x}} = \mathbf{1} + \nabla({}_t\Delta\mathbf{u}) \quad (5.136)$$

where ${}^{t+\Delta t}\mathbf{x}$ is the position vector of the particle at time $t + \Delta t$, ${}^t\mathbf{x}$ is the position vector at time t , ${}_t\Delta\mathbf{u}$ is the incremental displacement vector during the time increment Δt , $\mathbf{1}$ is the unit tensor, and the symbol ∇ denotes the gradient with respect to ${}^t\mathbf{x}$ (Figure 5.14). Further, similar to Equation 5.71, the tensor ${}_t\Delta\mathbf{F}$ can be decomposed as

$${}_t\Delta\mathbf{F} = ({}_t\Delta\mathbf{R})({}_t\Delta\mathbf{U}). \quad (5.137)$$

Here, ${}_t\Delta\mathbf{R}$ is an orthogonal tensor representing the incremental rotation of the particle during the time increment dt , whereas ${}_t\Delta\mathbf{U}$ is a positive-definite symmetric tensor representing the incremental stretching during the same increment.

Now, assume that the increment at a particle (which occupies the position ${}^tP$ at time t) is such that there is only rotation and no deformation during the time increment Δt (Figure 5.14). In this situation, there is no incremental deformation or stretching at point ${}^tP$, and therefore, the tensor ${}_t\Delta\mathbf{U}$ at point ${}^tP$ reduces to the unit tensor $\mathbf{1}$. Thus, when there is no incremental deformation, the incremental deformation gradient tensor consists of only the incremental rotation:

$${}_t\Delta\mathbf{F} = {}_t\Delta\mathbf{R}. \quad (5.138)$$

Figure 5.14a shows the stress vector ${}^t\mathbf{t}$ at point ${}^tP$ on a plane with (unit outward) normal ${}^t\mathbf{n}$, whereas Figure 5.14b shows the stress vector ${}^{t+\Delta t}\mathbf{t}$ at the same particle on the same plane but at time $t + \Delta t$. (The right subscript on the stress vector denoting the normal has been omitted here.) At time $t + \Delta t$, the normal to the plane is denoted by ${}^{t+\Delta t}\mathbf{n}$. Since the increment at point ${}^tP$ consists of only pure rotation ${}_t\Delta\mathbf{R}$, these four vectors are related by the following relations:

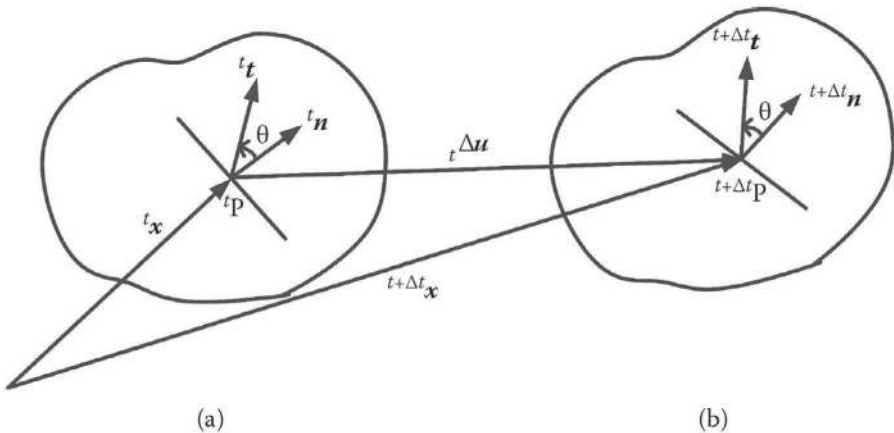


FIGURE 5.14 Deformed configurations: (a) at time t ; (b) at time $t + \Delta t$.

$${}^{t+\Delta t}\mathbf{t} = ({}_t\Delta\mathbf{R})({}^t\mathbf{t}), \quad (5.139)$$

$${}^{t+\Delta t}\mathbf{n} = ({}_t\Delta\mathbf{R})({}^t\mathbf{n}). \quad (5.140)$$

The relations between the Cauchy stress tensors and the stress vectors at times t and $t + \Delta t$ is given by Cauchy's relation:

$${}^t\mathbf{t} = ({}^t\boldsymbol{\sigma})({}^t\mathbf{n}), \quad (5.141)$$

$${}^{t+\Delta t}\mathbf{t} = ({}^{t+\Delta t}\boldsymbol{\sigma})({}^{t+\Delta t}\mathbf{n}). \quad (5.142)$$

Substituting Equations 5.139 and 5.140 in Equation 5.142, one gets

$$({}_t\Delta\mathbf{R})({}^t\mathbf{t}) = ({}^{t+\Delta t}\boldsymbol{\sigma})({}_t\Delta\mathbf{R})({}^t\mathbf{n}). \quad (5.143)$$

Further, the substitution of Equation 5.141 in Equation 5.143 leads to

$$({}_t\Delta\mathbf{R})({}^t\boldsymbol{\sigma})({}^t\mathbf{n}) = ({}^{t+\Delta t}\boldsymbol{\sigma})({}_t\Delta\mathbf{R})({}^t\mathbf{n}). \quad (5.144)$$

Since this equation is true for every surface (with normal ${}^t\mathbf{n}$) passing through point ${}^tP$, one gets

$$({}_t\Delta\mathbf{R})({}^t\boldsymbol{\sigma}) = ({}^{t+\Delta t}\boldsymbol{\sigma})({}_t\Delta\mathbf{R}) \quad (5.145)$$

Since $({}_t\Delta\mathbf{R})$ is an orthogonal tensor, it satisfies the following relation:

$$\begin{aligned} (\Delta\mathbf{R})(\Delta\mathbf{R})^T &= (\Delta\mathbf{R})^T(\Delta\mathbf{R}) = \mathbf{1}, \\ \Delta\mathbf{R}^{-1} &= \Delta\mathbf{R}^T, \quad \Delta\mathbf{R}^{-T} = \Delta\mathbf{R}. \end{aligned} \quad (5.146)$$

Then, Equation 5.145 becomes

$${}^{t+\Delta t}\boldsymbol{\sigma} = ({}_t\Delta\mathbf{R})({}^t\boldsymbol{\sigma})({}_t\Delta\mathbf{R})^T. \quad (5.147)$$

Thus, when the increment consists of pure rotation, the Cauchy stress tensors at times t and $t + \Delta t$ are related by the previous relation containing the incremental rotation. This relation would be used to prove the objectivity of incremental tensors.

5.7.2 PIOLA-KIRCHOFF STRESS TENSORS

The first Piola-Kirchhoff stress tensor is the three-dimensional generalization of the idea of 1-D 'engineering stress', introduced in Section 5.2.1. The development of the

first Piola-Kirchoff stress tensor is started by defining a *pseudo stress vector* ${}^0\mathbf{t}$ such that it satisfies the following relation:

$${}^0\mathbf{t}({}_0dS) = {}^t\mathbf{t}({}_tdS). \quad (5.148)$$

Here, ${}_0dS$ and ${}_tdS$ are the elemental areas in the undeformed and deformed configurations at time t , respectively. Thus, ${}^0\mathbf{t}$ is the *force acting on the deformed element at time t , but it is per unit undeformed area*. (Here, the notation of Chapter 4 has been appropriately modified to make it consistent with the notation of this section.) In terms of the Cauchy stress tensor at time t , the previous relation can be written as

$${}^0\mathbf{t}({}_0dS) = ({}^t\boldsymbol{\sigma})({}^t\mathbf{n})({}_tdS) \quad (5.149)$$

Modifying the relation between ${}_0dS$ and ${}_tdS$ (Equation 4.70) by introducing the subscripts/superscripts denoting the time, one gets

$$({}_tdS)^t \mathbf{n} = {}^tJ({}_0dS)({}^t\mathbf{F}^{-T})({}^0\mathbf{n}) \quad (5.150)$$

where

$${}^tJ = \det {}^t\mathbf{F}, \quad (5.151)$$

${}^t\mathbf{F}$ is the deformation gradient tensor at time t (defined by Equation 4.49) and ${}^0\mathbf{n}$ is the (unit outward) normal to the area ${}_0dS$. Elimination of ${}^t\mathbf{n}({}_tdS)$ from Equations 5.149 and 5.150 leads to

$${}^0\mathbf{t}({}_0dS) = {}^tJ({}^t\boldsymbol{\sigma})({}^t\mathbf{F}^{-T})({}^0\mathbf{n})({}_0dS). \quad (5.152)$$

Now, a tensor ${}^t_0\mathbf{P}$ is defined by the relation

$${}^t_0\mathbf{P}({}^0\mathbf{n}) = {}^0\mathbf{t} \quad (5.153)$$

Eliminating ${}^0\mathbf{t}$ from Equations 5.152 and 5.153 and canceling the factor ${}^0\mathbf{n}$, one gets the following expression for the tensor ${}^t_0\mathbf{P}$:

$${}^t_0\mathbf{P} = {}^tJ({}^t\boldsymbol{\sigma})({}^t\mathbf{F}^{-T}). \quad (5.154)$$

The tensor ${}^t_0\mathbf{P}$ is called the first Piola-Kirchoff stress tensor. It is not symmetric. Further, it appears in the equation of motion of the Lagrangian formulation for large deformation problems. Since this tensor involves quantities both from the deformed configuration at time t and the undeformed configuration (at time 0), its symbol contains both the times: 0 and t . A similar thing appears in the symbol for the second Piola-Kirchoff stress tensor.

To get a symmetric tensor from Equation 5.154, the right-hand side is pre-multiplied by ${}^t\mathbf{F}^{-1}$. The resulting tensor ${}^t_0\mathbf{S}$, called the second Piola-Kirchhoff stress tensor, is given by

$${}^t_0\mathbf{S} = {}^tJ \left({}^t\mathbf{F}^{-1} \right) \left({}^t\boldsymbol{\sigma} \right) \left({}^t\mathbf{F}^{-T} \right). \quad (5.155)$$

This tensor appears in the integral form of the updated Lagrangian formulation. It does not seem to have any physical interpretation. Another way to define this tensor is as follows. Consider the stress power for an elemental volume ${}_t dV$ at time t :

$${}_t dP = \left({}^t\sigma_{ij} \right) \left({}^tD_{ij} \right) ({}_t dV). \quad (5.156)$$

where ${}^tD_{ij}$ is the rate of deformation tensor at time t defined by Equation 4.174 where it is denoted by ${}^t\dot{\epsilon}_{ij}$. Using Equation 4.63 for ${}_t dV$, the stress power for an elemental volume ${}_0 dV$ in the undeformed configuration becomes

$${}_t dP = {}^tJ \left({}^t\sigma_{ij} \right) \left({}^tD_{ij} \right) ({}_0 dV) \quad (5.157)$$

where, as before, ${}^tJ$ is given by Equation 5.151. The second Piola-Kirchhoff stress tensor is defined as the tensor that is energy conjugate to the Green (or Green-Lagrange) tensor ${}^t\mathbf{G}$ defined in Section 4.4. The energy is to be taken per unit of undeformed volume. Then,

$${}^tJ \left({}^t\sigma_{kl} \right) \left({}^tD_{kl} \right) = \left({}^t_0S_{ij} \right) \left(\dot{G}_{ij} \right) \quad (5.158)$$

where $\dot{G}$ is the time derivative of ${}^t\mathbf{G}$. Using the definitions of ${}^t\mathbf{G}$ (Equation 4.91) and ${}^t\mathbf{D}$ (Equation 4.173, where it is denoted by ${}^t\dot{\epsilon}$) and the expression for the time derivative of ${}^t\mathbf{F}$ (Equation 4.80), one gets the following relation between $\dot{G}$ and $\dot{D}$:

$$\dot{G}_{ij} = \left({}^tF_{ik}^T \right) \left({}^tD_{kl} \right) \left({}^tF_{lj} \right). \quad (5.159)$$

Eliminating $\dot{G}$ from Equations 5.158 and 5.159, one gets

$${}^tJ \left({}^t\sigma_{kl} \right) \left({}^tD_{kl} \right) = \left({}^tF_{ki} \right) \left({}^t_0S_{ij} \right) \left({}^tF_{jl}^T \right) \left({}^tD_{kl} \right). \quad (5.160)$$

Since this relation is true for every ${}^t\mathbf{D}$, one gets the following expression for ${}^t_0\mathbf{S}$:

$${}^t_0S_{ij} = {}^tJ \left({}^tF_{ik}^{-1} \right) \left({}^t\sigma_{kl} \right) \left({}^tF_{lj}^{-T} \right) \quad (5.161)$$

which is the index form of Equation 5.155.

Note that the expression 5.151 for ${}^tJ$ can be expressed as the ratio of ${}^0\rho$ (the density in the undeformed configuration at time t) and ${}^t\rho$ (the density in the deformed configuration):

$${}^tJ = {}^0\rho / {}^t\rho \quad (5.162)$$

Substituting this expression in Equations 5.154 and 5.155, one gets the following alternate expressions for the first and second Piola-Kirchoff stress tensors:

$${}^t_0\mathbf{P} = \left({}^0\rho / {}^t\rho\right) \left({}^t\boldsymbol{\sigma}\right) \left({}^t\mathbf{F}^{-T}\right), \quad (5.163)$$

$${}^t_0\mathbf{S} = \left({}^0\rho / {}^t\rho\right) \left({}^t\mathbf{F}^{-1}\right) \left({}^t\boldsymbol{\sigma}\right) \left({}^t\mathbf{F}^{-T}\right). \quad (5.164)$$

The previous definitions of the first and second Piola-Kirchoff stress tensors correspond to the deformation from the undeformed configuration at time 0 to the deformed configuration at time t . During the incremental analysis, similar tensors involving the deformation from time t to time $t + \Delta t$ are needed. In this situation, the previous definitions of these tensors get modified as

$${}^{t+\Delta t}_t\mathbf{P} = \left({}^t\rho / {}^{t+\Delta t}\rho\right) \left({}^{t+\Delta t}_t\boldsymbol{\sigma}\right) \left({}_t\Delta\mathbf{F}^{-T}\right), \quad (5.165)$$

$${}^{t+\Delta t}_t\mathbf{S} = \left({}^t\rho / {}^{t+\Delta t}\rho\right) \left({}_t\Delta\mathbf{F}^{-1}\right) \left({}^{t+\Delta t}_t\boldsymbol{\sigma}\right) \left({}_t\Delta\mathbf{F}^{-T}\right). \quad (5.166)$$

5.7.3 INCREMENT OF SECOND PIOLA-KIRCHOFF STRESS TENSOR (OBJECTIVE INCREMENTAL STRESS TENSOR)

The second Piola-Kirchoff stress tensor can be decomposed as

$${}^{t+\Delta t}_t\mathbf{S} = {}^t_t\mathbf{S} + {}_t\Delta\mathbf{S} \quad (5.167)$$

where ${}_t\Delta\mathbf{S}$ is its increment. Note that when the left superscript $(t + \Delta t)$ becomes equal to the left subscript (t) , the incremental deformation gradient tensor ${}_t\Delta\mathbf{F}$, as per Equation 5.136, reduces to the unit tensor $\mathbf{1}$. Further, the densities ${}^{t+\Delta t}\rho$ and ${}^t\rho$ become identical. Then, ${}^t_t\mathbf{S}$ becomes

$${}^t_t\mathbf{S} = \left({}^t\rho / {}^t\rho\right) \left(\mathbf{1}^{-1}\right) \left({}^t\boldsymbol{\sigma}\right) \left(\mathbf{1}^{-T}\right) = {}^t\boldsymbol{\sigma}. \quad (5.168)$$

Substitution of Equation 5.168 in Equation 5.167 leads to the following expression for the increment of the second Piola-Kirchoff stress tensor:

$${}_t\Delta\mathbf{S} = {}^{t+\Delta t}_t\mathbf{S} - {}^t_t\boldsymbol{\sigma}. \quad (5.169)$$

Now, to prove that ${}_t\Delta\mathbf{S}$ is an incremental objective stress tensor, it would be shown that it reduces to a zero tensor if the increment consists of pure rotation.

Note that, when an increment consists of pure rotation, the incremental deformation gradient tensor ${}_{\Delta} \mathbf{F}$ reduces to the incremental rotation tensor ${}_{\Delta} \mathbf{R}$ (Equation 5.138). Further, since ${}_{\Delta} \mathbf{R}$ is an orthogonal tensor, it satisfies relation 5.146. Additionally, the density ${}^{t+\Delta t} \rho$ becomes equal to ${}^t \rho$, as there is no incremental deformation. In this situation, expression 5.166 for the second Piola-Kirchoff stress tensor becomes

$$\begin{aligned} {}^{t+\Delta t} {}_t \mathbf{S} &= \left({}^t \rho / {}^{t+\Delta t} \rho \right) \left({}_t \Delta \mathbf{F}^{-1} \right) \left({}^{t+\Delta t} \boldsymbol{\sigma} \right) \left({}_t \Delta \mathbf{F}^{-T} \right) \\ &= \left({}^t \rho / {}^t \rho \right) \left({}_t \Delta \mathbf{R}^{-1} \right) \left({}^{t+\Delta t} \boldsymbol{\sigma} \right) \left({}_t \Delta \mathbf{R}^{-T} \right) \\ &= \left({}_t \Delta \mathbf{R}^T \right) \left({}^{t+\Delta t} \boldsymbol{\sigma} \right) \left({}_t \Delta \mathbf{R} \right). \end{aligned} \quad (5.170)$$

Earlier, it has been shown that the Cauchy stress tensor at time $t + \Delta t$ reduces to Equation 5.147 when the increment consists of pure rotation. Substituting Equation 5.147 in the previous equation and using the orthogonality property (Equation 5.146), the previous expression for ${}^{t+\Delta t} {}_t \mathbf{S}$ reduces to

$$\begin{aligned} {}^{t+\Delta t} {}_t \mathbf{S} &= \left({}_t \Delta \mathbf{R}^T \right) \left({}^{t+\Delta t} \boldsymbol{\sigma} \right) \left({}_t \Delta \mathbf{R} \right) \\ &= \left({}_t \Delta \mathbf{R}^T \right) \left(\left({}_t \Delta \mathbf{R} \right) \left({}^t \boldsymbol{\sigma} \right) \left({}_t \Delta \mathbf{R} \right)^T \right) \left({}_t \Delta \mathbf{R} \right) \\ &= \left(\left({}_t \Delta \mathbf{R}^T \right) \left({}_t \Delta \mathbf{R} \right) \right) \left({}^t \boldsymbol{\sigma} \right) \left(\left({}_t \Delta \mathbf{R} \right)^T \left({}_t \Delta \mathbf{R} \right) \right) \\ &= \left({}^t \boldsymbol{\sigma} \right). \end{aligned} \quad (5.171)$$

Thus, when an increment consists of pure rotation, the second Piola-Kirchoff stress tensor reduces to the Cauchy stress tensor at time t .

Substitution of Equation 5.171 in Equation 5.169 implies that, in the event of the increment being a pure rotation, the increment of the second Piola-Kirchoff stress tensor reduces to a zero tensor:

$${}_t \Delta \mathbf{S} = {}^t \boldsymbol{\sigma} - {}^t \boldsymbol{\sigma} = \mathbf{0}. \quad (5.172)$$

Thus, the tensor ${}_{\Delta} \mathbf{S}$ is an objective incremental stress tensor. This tensor is one of the commonly used objective incremental stress tensors in the incremental elastoplastic constitutive Equation 5.112.

5.7.4 RELATION BETWEEN FINITE AND INFINITESIMAL INCREMENTAL ROTATION TENSORS FOR SMALL INCREMENT

From Equation 5.137, the following expression is obtained for the incremental rotation tensor:

$${}_t\Delta \mathbf{R} = ({}_t\Delta \mathbf{F})({}_t\Delta \mathbf{U})^{-1}. \quad (5.173)$$

The following relation between ${}_t\Delta \mathbf{U}$ and ${}_t\Delta \mathbf{F}$ is also obtained from Equation 5.137. Similar to Equation 4.72, one gets

$$({}_t\Delta \mathbf{U})^2 = ({}_t\Delta \mathbf{F})^T ({}_t\Delta \mathbf{F}) \quad (5.174)$$

Now, it is assumed that the increment size is very small, that is, the incremental deformation and rotation are very small. Mathematically, it means that the components of the tensor $\nabla({}_t\Delta \mathbf{U})$ are very small compared to one. Then, substituting expression 5.136 for ${}_t\Delta \mathbf{F}$ and neglecting the second-order terms, one obtains

$$\begin{aligned} ({}_t\Delta \mathbf{U})^2 &= (\mathbf{1} + \nabla({}_t\Delta \mathbf{u}))^T (\mathbf{1} + \nabla({}_t\Delta \mathbf{u})) \\ &\simeq \mathbf{1} + \nabla({}_t\Delta \mathbf{u}) + (\nabla({}_t\Delta \mathbf{u}))^T. \end{aligned} \quad (5.175)$$

Taking the square root, using the binomial expansion, and neglecting higher-order terms, one gets

$$\begin{aligned} {}_t\Delta \mathbf{U} &\simeq \mathbf{1} + \left(\nabla({}_t\Delta \mathbf{u}) + (\nabla({}_t\Delta \mathbf{u}))^T \right)^{1/2} \\ &\simeq \mathbf{1} + \frac{1}{2} \left(\nabla({}_t\Delta \mathbf{u}) + (\nabla({}_t\Delta \mathbf{u}))^T \right). \end{aligned} \quad (5.176)$$

Substituting expression 5.136 for ${}_t\Delta \mathbf{F}$ and the previous expression for ${}_t\Delta \mathbf{U}$, using the binomial expansion for the inverse, and neglecting the higher-order terms, expression 5.173 for ${}_t\Delta \mathbf{R}$ becomes

$$\begin{aligned} {}_t\Delta \mathbf{R} &= ({}_t\Delta \mathbf{F})({}_t\Delta \mathbf{U})^{-1} \\ &= (\mathbf{1} + \nabla({}_t\Delta \mathbf{u})) \left(\mathbf{1} + \frac{1}{2} \left(\nabla({}_t\Delta \mathbf{u}) + (\nabla({}_t\Delta \mathbf{u}))^T \right) \right)^{-1} \\ &\simeq (\mathbf{1} + \nabla({}_t\Delta \mathbf{u})) \left(\mathbf{1} - \frac{1}{2} \left(\nabla({}_t\Delta \mathbf{u}) + (\nabla({}_t\Delta \mathbf{u}))^T \right) \right) \\ &\simeq \mathbf{1} + \nabla({}_t\Delta \mathbf{u}) - \frac{1}{2} \left(\nabla({}_t\Delta \mathbf{u}) + (\nabla({}_t\Delta \mathbf{u}))^T \right) \\ &\simeq \mathbf{1} + \frac{1}{2} \left(\nabla({}_t\Delta \mathbf{u}) - (\nabla({}_t\Delta \mathbf{u}))^T \right). \end{aligned} \quad (5.177)$$

As per Equation 4.177 of Section 4.15.3, the term $1/2 (\nabla({}_t\Delta \mathbf{u}) - (\nabla({}_t\Delta \mathbf{u}))^T)$ represents the incremental infinitesimal rotation tensor ${}_t\Delta \omega$. Thus, when the incremental rotation is small, the (finite) incremental rotation tensor ${}_t\Delta \mathbf{R}$ reduces to

$${}_t\Delta\mathbf{R} \approx \mathbf{1} + {}_t\mathbf{d}\boldsymbol{\omega}. \quad (5.178)$$

5.7.5 JAUMANN STRESS TENSOR (OBJECTIVE STRESS RATE TENSOR)

When the increment consists of pure rotation, the relation between the Cauchy stress tensors at times t and $t + \Delta t$ is given by Equation 5.147. Now, it is assumed that the increment size is very small. Then, the (finite) incremental rotation tensor ${}_t\Delta\mathbf{R}$ reduces to the expression in Equation 5.178 involving the incremental infinitesimal rotation tensor ${}_t\mathbf{d}\boldsymbol{\omega}$. Substituting this expression (Equation 5.178) in Equation 5.147, changing the symbol of the incremental time from Δt to dt , and neglecting the second-order term, one gets

$$\begin{aligned} {}^{t+dt}\boldsymbol{\sigma} &= ({}_t\Delta\mathbf{R})({}^t\boldsymbol{\sigma})({}_t\Delta\mathbf{R})^T \\ &= (\mathbf{1} + {}_t\mathbf{d}\boldsymbol{\omega})({}^t\boldsymbol{\sigma})(\mathbf{1} + {}_t\mathbf{d}\boldsymbol{\omega})^T \\ &\approx {}^t\boldsymbol{\sigma} + ({}_t\mathbf{d}\boldsymbol{\omega})({}^t\boldsymbol{\sigma}) + ({}^t\boldsymbol{\sigma})({}_t\mathbf{d}\boldsymbol{\omega})^T. \end{aligned} \quad (5.179)$$

Now, using Equation 5.179, the rate of Cauchy stress tensor only due to rotation can be obtained as

$$\begin{aligned} ({}^t\dot{\boldsymbol{\sigma}}_{rot})dt &= {}^{t+dt}\boldsymbol{\sigma} - {}^t\boldsymbol{\sigma} \\ &= \left({}^t\boldsymbol{\sigma} + ({}_t\mathbf{d}\boldsymbol{\omega})({}^t\boldsymbol{\sigma}) + ({}^t\boldsymbol{\sigma})({}_t\mathbf{d}\boldsymbol{\omega})^T \right) - {}^t\boldsymbol{\sigma} \\ &= ({}_t\mathbf{d}\boldsymbol{\omega})({}^t\boldsymbol{\sigma}) + ({}^t\boldsymbol{\sigma})({}_t\mathbf{d}\boldsymbol{\omega})^T. \end{aligned} \quad (5.180)$$

Using the relation between the incremental infinitesimal rotation tensor ${}_t\mathbf{d}\boldsymbol{\omega}$ and the spin tensor ${}^t\dot{\boldsymbol{\omega}}$ given by Equation 4.223 of Section 4.19 and canceling the factor dt , one gets

$${}^t\dot{\boldsymbol{\sigma}}_{rot} = ({}^t\dot{\boldsymbol{\omega}})({}^t\boldsymbol{\sigma}) + ({}^t\boldsymbol{\sigma})({}^t\dot{\boldsymbol{\omega}})^T. \quad (5.181)$$

Now, it is assumed that the rate of Cauchy stress tensor only due to deformation can be expressed as the difference between the total rate (${}^t\dot{\boldsymbol{\sigma}}$) and the rate only due to rotation (${}^t\dot{\boldsymbol{\sigma}}_{rot}$). This rate is denoted by the symbol ${}^t\dot{\boldsymbol{\sigma}}^0$. Using Equation 5.181, this rate can be expressed as

$${}^t\dot{\boldsymbol{\sigma}}^0 = {}^t\dot{\boldsymbol{\sigma}} - ({}^t\dot{\boldsymbol{\omega}})({}^t\boldsymbol{\sigma}) - ({}^t\boldsymbol{\sigma})({}^t\dot{\boldsymbol{\omega}})^T. \quad (5.182)$$

This is called the *Jaumann stress rate*. It is shown in Dixit and Dixit (2008) that this stress rate is objective. This stress rate is one of the commonly used objective stress rate tensors in the rate constitutive Equations 5.126 and 5.127.

The product of the Jaumann stress rate and the time increment dt is called the Jaumann stress increment. It is related to the Cauchy stress increment ($d\sigma$) and incremental infinitesimal rotation tensor ($d\omega$) by the following relation:

$$\begin{aligned} d\sigma &= \overset{0}{\sigma} dt \\ &= {}^t\dot{\sigma}dt - \left({}^t\dot{\omega}dt\right)\left({}^t\sigma\right) - \left({}^t\sigma\right)\left({}^t\dot{\omega}dt\right)^T \\ &= d\sigma - \left({}_td\omega\right)\left({}^t\sigma\right) - \left({}^t\sigma\right)\left({}_td\omega\right)^T \end{aligned} \quad (5.183)$$

The Jaumann stress increment is also used as an objective incremental stress tensor in the incremental elasto-plastic constitutive Equation 5.112.

5.8 UNLOADING CRITERION

The true stress-logarithmic strain curve (Figure 5.3) of the tension test shows that, in 1-D state of stress, unloading (at a point on the curve YF) occurs if the stress decreases, that is, if $d\sigma < 0$. However, this is not true for the compression test, where σ itself is negative, and therefore, unloading occurs if $d\sigma > 0$. Thus, the unloading criterion, which is valid for both tension and compression tests, can be stated as

$$\sigma d\sigma < 0. \quad (5.184)$$

To extend this criterion to a three-dimensional state of stress, one turns to a graphical representation of the yield criteria. In the 1-D state of stress, the graphical representation of the yield criterion is just a pair of two points $\sigma = \pm h(\epsilon^p)$ on the σ -axis. Equation 5.184 means that, during the unloading, the direction of the stress increment $d\sigma$ is inward from the yield points (i.e. toward the origin of the σ -axis). In a three-dimensional state of stress, the graphical representation of the yield criterion is a surface in the stress space of the principal stresses ($\sigma_1, \sigma_2, \sigma_3$). However, for convenience, its locus is considered on the deviatoric plane.

Let $\{d\sigma\}$ be the array of the principal values of the stress increment $d\sigma_{ij}$. Then, it can be represented as a geometrical vector in the stress space. It is expected that, during unloading, the vector $\{d\sigma\}$ will be pointing inward from the yield locus, as shown in Figure 5.15. Then, it will make an obtuse angle with the vector $\{\nabla f\}$, which is normal to the yield locus. This condition can be stated as

$$\{\nabla f\}^T \{d\sigma\} < 0. \quad (5.185)$$

In index notation, this condition can be expressed as (Chakrabarty, 1987)

$$\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} < 0. \quad (5.186)$$

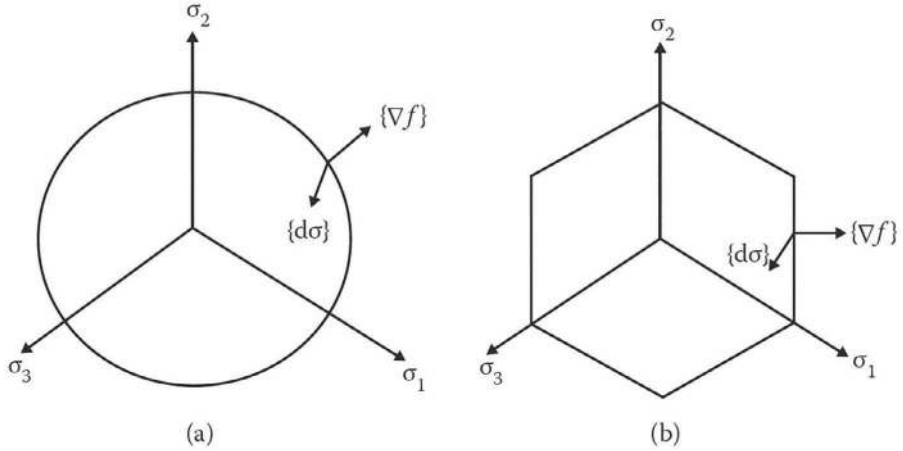


FIGURE 5.15 Unloading criteria. The vector $\{d\sigma\}$ points inward from the yield locus and makes an obtuse angle with the outward normal: (a) Mises yield locus on the deviatoric plane; (b) Tresca yield locus on the deviatoric plane.

For the Mises material, the derivative of the yield function f with respect to σ_{ij} is given by Equation 5.102. Using this result, the unloading criterion (Equation 5.186) becomes

$$\sigma'_{ij} d\sigma_{ij} < 0. \quad (5.187)$$

Exercises

1. The Young's modulus of elasticity of steel is 210 GPa, and the Poisson's ratio is 0.29. Find its shear modulus and bulk modulus. If the yield strength of the material is 350 MPa, find the percentage change in the volume of the specimen in a uni-axial tensile test, just before yielding.
2. A metal is strained up to 0.1 by applying a stress of 400 MPa in a uni-axial tensile test. After that, the stress is released. Find the permanent strain left if Young's modulus of elasticity is 200 GPa. The ultimate strength of the metal is more than 400 MPa.
3. A material follows the Johnson-Cook model for certain ranges of temperature, strain and strain rate. The parameters of the Johnson-Cook model are as follows:

$$A = 598 \text{ MPa}, \quad B = 768 \text{ MPa}, \quad n = 0.2, \quad C = 0.01, \\ \dot{\epsilon}_0 = 1 \times 10^{-3} \text{ s}^{-1}, \quad m = 0.8, \quad T_{melt} = 1768 \text{ K}, \quad T_0 = 303 \text{ K}.$$

Find out the flow stress of the material for the strain of 0.2, strain rate of 18 s^{-1} and temperature of 500°C .

4. The yield strength of a material is 250 MPa. Find out if the material can sustain the following state of stress with respect to the (x, y, z) coordinate system without yielding by (1) the Mises criterion and (2) Tresca criterion:

$$[\sigma] = \begin{bmatrix} 50 & 120 & 0 \\ 120 & 50 & 0 \\ 0 & 0 & -100 \end{bmatrix} \text{MPa.}$$

5. The matrix of the stress tensor σ at a point, with respect to the (x, y, z) coordinate system, is

$$[\sigma] = \begin{bmatrix} 240 & 0 & 180 \\ 0 & 60 & 0 \\ 180 & 0 & -120 \end{bmatrix} \text{MPa.}$$

- Find the matrix of the deviatoric part σ' at that point.
- Using the Mises yield function as the plastic potential and the associated flow rule,

$$d\varepsilon_{ij}^p = d\lambda \sigma'_{ij}$$

find the matrix of the tensor $d\varepsilon^p$ (i.e. the plastic part of the incremental linear strain tensor) in terms of $d\lambda$.

6. The matrix of the stress tensor σ at a point, with respect to the (x, y, z) coordinate system, is

$$[\sigma] = \begin{bmatrix} 70 & 0 & 0 \\ 0 & 70 & 0 \\ 0 & 0 & 140 \end{bmatrix} \text{MPa.}$$

- Find the matrices of σ' , σ'^2 and p . Note that p is the deviatoric part of σ'^2 .
- This is a case of subsequent yielding. Therefore, the Tresca yield criteria become

$$f(J_2, J_3, \varepsilon_{eq}^p) \equiv 4 \left(J_2 - \frac{\sigma_{eq}^2}{4} \right) (J_2 - \sigma_{eq}^2) - 27 J_3^2 = 0.$$

Find the derivatives $\frac{\partial f}{\partial J_2}$ and $\frac{\partial f}{\partial J_3}$.

- When the Tresca yield function is used as the plastic potential, the associated flow rule becomes

$$d\varepsilon_{ij}^p = d\lambda \frac{\partial f}{\partial \sigma_{ij}} = d\lambda \left(\frac{\partial f}{\partial J_2} \sigma'_{ij} + \frac{\partial f}{\partial J_3} p_{ij} \right).$$

Find the matrix of the tensor $\mathbf{d}\boldsymbol{\varepsilon}^p$ in terms of $d\lambda$.

7. *Plane deformation* of a square block (which is infinitely long in the z -direction) is shown in the figure. The block behaves *elastically*. The velocity components are $v_x = 2y$, $v_y = 0$, $v_z = 0$. Therefore, the components of the strain rate tensor and the spin tensor are

$$\begin{aligned} \dot{\varepsilon}_{xx} = 0, \dot{\varepsilon}_{yy} = 0, \dot{\varepsilon}_{xy} = 1/s, \quad \dot{\varepsilon}_{ij} = 0 \text{ for other } i \text{ and } j; \\ \dot{\omega}_{xy} = 1/s, \dot{\omega}_{yz} = 0, \dot{\omega}_{zx} = 0, \quad \dot{\omega}_{ij} = 0 \text{ for other } i \text{ and } j. \end{aligned}$$

- a. Find the component of the Jaumann stress rate tensor using the rate form of the elastic stress-strain relations:

$$\dot{\varepsilon}_{kk} = \frac{\sigma_{kk}}{3K}, \quad \varepsilon'_{ij} = \frac{\sigma'_{ij}}{2\mu}.$$

- b. Using the results of part (a), find the Cauchy stress rate tensor:

$$\dot{\underline{\underline{\sigma}}} = \underline{\underline{\dot{\sigma}}} + \left(\underline{\underline{\dot{\omega}}} \underline{\underline{\sigma}} + \underline{\underline{\sigma}} \underline{\underline{\dot{\omega}}}^T \right)$$

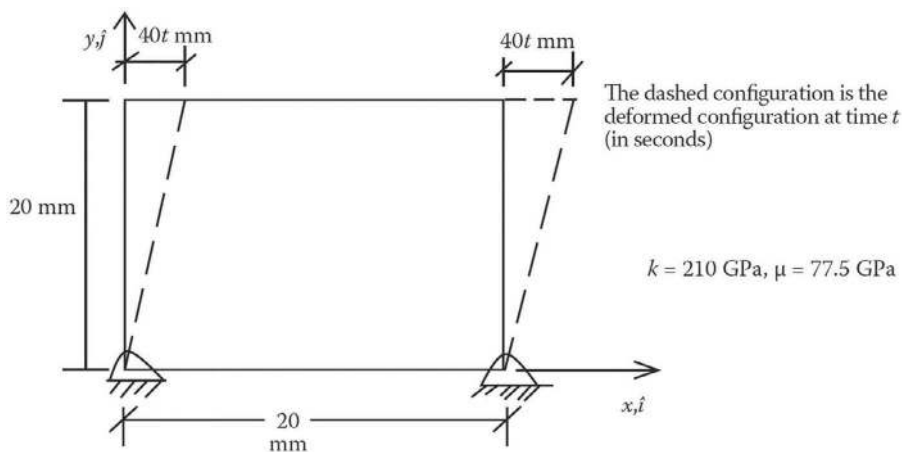


FIGURE 5.0E7

ANSWERS

1. Shear modulus: 81.39 GPa and bulk modulus: 166.67 GPa. percentage change in the volume: 0.07%.
2. 0.098
3. 757.21 MPa
4. The material will not yield as per the Mises criterion but will yield as per the Tresca criterion.
- 5.

$$\text{a. } \begin{bmatrix} 180 & 0 & 180 \\ 0 & 0 & 0 \\ 180 & 0 & -180 \end{bmatrix} \text{ MPa} \quad \text{b. } d\lambda \begin{bmatrix} 180 & 0 & 180 \\ 0 & 0 & 0 \\ 180 & 0 & -180 \end{bmatrix}$$

6.

$$\text{a. } [\sigma'] = \begin{bmatrix} -23.33 & 0 & 0 \\ 0 & -23.33 & 0 \\ 0 & 0 & 46.67 \end{bmatrix} \text{ MPa},$$

$$[\sigma']^2 = \begin{bmatrix} 544.28 & 0 & 0 \\ 0 & 544.28 & 0 \\ 0 & 0 & 2178.08 \end{bmatrix} (\text{MPa})^2,$$

$$[p] = \begin{bmatrix} -544.6 & 0 & 0 \\ 0 & -544.6 & 0 \\ 0 & 0 & 1089.2 \end{bmatrix} (\text{MPa})^2$$

$$\text{b. } \frac{\partial f}{\partial J_2} = 4(J_2 - \sigma_{eq}^2) \left(3J_2 - \frac{3}{2} \sigma_{eq}^2 \right), \quad \frac{\partial f}{\partial J_3} = -54J_3$$

$$\text{c. } [d\epsilon^p] = d\lambda \begin{bmatrix} -653672486.5 & 0 & 0 \\ 0 & -653672486.5 & 0 \\ 0 & 0 & 1307304957 \end{bmatrix}.$$

7.

$$\text{a. } \begin{bmatrix} 0 & 155000 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ MPa/s} \quad \text{b. } \begin{bmatrix} \sigma_{yx} + \sigma_{xy} & \sigma_{yy} + 155000 & \sigma_{yz} \\ \sigma_{yy} & 0 & 0 \\ \sigma_{zy} & 0 & 0 \end{bmatrix} \text{ MPa/s}$$

6 Eulerian and Updated Lagrangian Formulations

6.1 INTRODUCTION

In this chapter, all three of the governing equations developed in Chapters 3–5 are collected to develop the following two formulations for elasto-plastic problems: updated Lagrangian formulation and Eulerian formulation. In the updated Lagrangian formulation, the final deformed configuration is analyzed in several increments using the governing equations in the incremental form. The domain, the deformation and the stresses are updated at the end of each increment. This formulation can also be used for steady-state problems like rolling, drawing, extrusion, etc. However, for such problems, it is possible to identify a fixed region in the space (called the control volume) where the deformation gets concentrated. Thus, the flow-type Eulerian formulation is convenient for such problems where only the control volume containing the deformation is analyzed. In this formulation, the velocity is treated as a primary unknown, and therefore, the strain rate tensor (or the rate of deformation tensor) is employed as the measure of deformation, and the constitutive equation is expressed in rate form.

Since some of the governing equations are differential equations in space and time variables, the boundary and initial conditions are needed. They are also discussed in this section. One of the governing equations, namely, the equation of motion (Equation 3.55), is in terms of the acceleration vector $\mathbf{a}$. Since the primary variable is the velocity vector in the Eulerian formulation, $\mathbf{a}$ is needed to be expressed in terms of the velocity vector. Further, for the updated Lagrangian formulation, the equation of motion (Equation 3.55) is needed to be put in the incremental form. These things are discussed first before developing the two formulations. For further details, one can refer Dixit and Dixit (2008).

6.2 EQUATION OF MOTION IN TERMS OF VELOCITY DERIVATIVES

As stated earlier, in Eulerian formulation the primary variable of the problem is the velocity vector $\mathbf{v}$. The acceleration vector $\mathbf{a}$ is the time rate of the velocity vector. Note that the velocity vector $\mathbf{v}$ of a particle, besides depending explicitly on time t , also depends implicitly on t through its position vector $\mathbf{x}$. Therefore, using the chain rule, the acceleration vector $\mathbf{a}$ can be expressed as

$$a_i \equiv \frac{dv_i}{dt} = \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_j} \frac{dx_j}{dt} = \frac{\partial v_i}{\partial t} + v_{i,j} v_j \quad (6.1)$$

Here, the last equality follows from the definition of the velocity vector $\mathbf{v}$: that it is the rate of change of the position vector $\mathbf{x}$ of the particle. The comma in the second term of the last equality indicates the derivative with respect to the components of the position vector $\mathbf{x}$.

In tensor notation, the previous equation can be written as

$$\mathbf{a} \equiv \frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\nabla \mathbf{v})\mathbf{v} \quad (6.2)$$

where the velocity gradient tensor $\nabla \mathbf{v}$ has been defined in Section 4.17. The derivative $d\mathbf{v}/dt$ is called the *material time derivative* of the velocity vector. The physical interpretation of the two parts of $d\mathbf{v}/dt$ is as follows. The first part consists of the partial derivative of $\mathbf{v}$ with respect to time. It represents the change in the velocity vector of the point of the control volume that the particle occupies at time t . It is called the *unsteady* term of the material time derivative. Since the particle continues to change its position with time, the second part consists of the partial derivative of $\mathbf{v}$ with respect to $\mathbf{x}$. It represents the change in the velocity vector due to the change in its position. This term is called the *convective* term of the material time derivative. Equation 6.2 shows that the acceleration vector $\mathbf{a}$ is a non-linear function of the velocity vector $\mathbf{v}$. Because of this, in Eulerian formulation, the equation of motion becomes a *non-linear* equation.

Substituting the expression for the acceleration vector (Equation 6.2), the equation of motion (Equation 3.55) now becomes

$$\rho \left(\frac{\partial v_i}{\partial t} + v_{i,j} v_j \right) = \rho b_i + \sigma_{ij,j} \quad (6.3)$$

For a *steady process*, the first part of the acceleration vector, namely, $\partial \mathbf{v}_i / \partial t$ is zero.

For a *rigid-plastic material*, it is convenient to decompose the last term of the equation of motion into the hydrostatic and deviatoric parts. In metal forming and machining literature, the negative of the hydrostatic part is often called *pressure* and is denoted by the letter p :

$$p = -\frac{1}{3} \sigma_{kk} \quad (6.4)$$

Note that, unlike in fluids, the hydrostatic part in solids is sometimes tensile (i.e. p is sometimes negative). However, whenever the hydrostatic part becomes tensile at a point, there is a likelihood of material separation at that point. This aspect will be dealt with in Chapter 11 when the theories of fracture are discussed. Substituting Equation 6.4 in the expression for decomposition of the stress tensor (Equation 3.105), one gets

$$\sigma_{ij} = -p \delta_{ij} + \sigma'_{ij} \quad (6.5)$$

Now, the divergence of the first term is evaluated on the right side. Using the product rule and the identity 2.36 and noting that δ is a constant, one obtains

$$\left(p\delta_{ij}\right)_{,j} = p_{,j}\delta_{ij} + p\delta_{ij,j} = p_{,i} + 0 = p_{,i} \quad (6.6)$$

Taking the divergence of each side of Equation 6.5 and using Equation 6.6, one obtains

$$\sigma_{ij,j} = -p_{,i} + \sigma'_{ij,j} \quad (6.7)$$

Substituting Equation 6.7, the equation of motion, Equation 6.3 now becomes

$$\rho \left(\frac{\partial v_i}{\partial t} + v_{i,j}v_j \right) = \rho b_i - p_{,i} + \sigma'_{ij,j} \quad (6.8)$$

6.3 INCREMENTAL EQUATION OF MOTION

For the updated Lagrangian formulation, the equation of motion needs to be expressed in an incremental form. This can be done as follows: let a_i and σ_{ij} be the components of the acceleration vector and the stress tensor, respectively, at a particle at time t . Then, a_i and σ_{ij} will satisfy the equation of motion given by Equation 3.55 in the deformed configuration at time t (called the *current configuration*). Let da_i and $d\sigma_{ij}$ be the increments in the acceleration vector and the stress tensor at the particle during the time increment dt . Then, $a_i + da_i$ and $\sigma_{ij} + d\sigma_{ij}$ will satisfy the equation of motion in the deformed configuration at time $t + dt$. This equation will be similar to Equation 3.55, except that the derivatives will be now with respect to the position vector of the particle at time $t + dt$. However, the deformed configuration at time $t + dt$ is not known, and therefore, the position vector of the particle at time $t + dt$ is also unknown. Note that while developing a measure of incremental deformation, *it is assumed that the incremental deformation during the time interval dt is small*. It means that the deformed configuration at time t does not change much geometrically during the time interval dt . Therefore, the derivative with respect to the position vector (of a particle) at time $t + dt$ will be approximately equal to the derivative with respect to the position vector at time t . Then, the approximate equation of motion at time $t + dt$ will be

$$\rho(a_i + da_i) = \rho(b_i + db_i) + (\sigma_{ij} + d\sigma_{ij})_{,j} \quad (6.9)$$

where db_i is the body force increment (per unit mass) in the time interval dt , and the comma denotes the derivative with respect to the components of the position vector at time t . Subtracting the index form of the transposed Equation 3.55 from the previous equation, one gets the following form of the incremental equation of motion:

$$\rho da_i = \rho db_i + d\sigma_{ij,j} \quad (6.10)$$

Hill (1950) has derived the incremental *equilibrium* equation, taking into account the change in the position vector during the time interval dt . One can extend his derivation to obtain the incremental *equation of motion*. The incremental equation of motion is not really convenient for the finite-element formulation of the problem. Therefore, Hill's incremental equilibrium equation or the corresponding incremental equations of motion are not presented here.

6.4 EULERIAN FORMULATION

As stated earlier, the Eulerian formulation is convenient for the analysis of steady-state problems like rolling, drawing, extrusion, etc. In this formulation, a region fixed in space where the deformation is concentrated (called the *control volume*) is chosen as the domain for the analysis. Further, the velocity vector v_i is treated as the primary unknown and the strain rate tensor $\dot{\epsilon}_{ij}$ is employed as the measure of deformation. The constitutive equation is expressed in rate form.

In any elasto-plastic problem, there is always some temperature change due to the dissipation of mechanical energy into heat. If it is assumed that the temperature change is small, then the problem can be approximated as *isothermal*. For isothermal problems, the velocity field v_i , the strain rate field $\dot{\epsilon}_{ij}$ and the stress field σ_{ij} in the control volume is governed by the following three equations.

6.4.1 GOVERNING EQUATIONS (ELASTO-PLASTIC MATERIAL)

1. *Strain rate-velocity relations* (Equation 4.173), six scalar equations

$$\dot{\epsilon}_{ij} = \frac{1}{2} (v_{i,j} + v_{j,i}) \quad (6.11)$$

2. *Elasto-plastic stress-strain rate relations* (Equations 5.67, 5.73, 5.126 and 5.127), six scalar equations.

In the plastic zone

$$\dot{\epsilon}_{kk} = \frac{\sigma_{kk}^{\circ}}{3K}, \quad \dot{\epsilon}'_{ij} = \frac{1}{2\mu} \sigma'_{ij} + \frac{3\dot{\epsilon}_{eq}^p}{2\sigma_{eq}} \sigma'_{ij} \quad (6.12a)$$

where

$$\sigma_{eq} = \sigma_Y + K \left(\epsilon_{eq}^p \right)^n, \quad \epsilon_{eq}^p = \int \dot{\epsilon}_{eq}^p dt \quad (6.12b)$$

In the elastic zone

$$\dot{\epsilon}_{kk} = \frac{\sigma_{kk}^{\circ}}{3K}, \quad \dot{\epsilon}'_{ij} = \frac{1}{2\mu} \sigma'_{ij} \quad (6.12c)$$

Here, the superscript $\circ$ denotes that it is the *Jaumann stress rate*. The Jaumann stress rate is related to the Cauchy stress rate through spin tensor by Equation 5.182. The spin tensor is given by Equation (4.218). Thus,

$$\dot{\sigma}_{kk}^{\circ} = \dot{\sigma}_{kk} - \left(\dot{\omega}_{kl} \sigma_{lk} + \sigma_{kl} \dot{\omega}_{lk}^T \right) \quad (6.12d)$$

and

$$\dot{\sigma}_{ij}^{\circ} = \dot{\sigma}_{ij}' - \left(\dot{\omega}_{il} \sigma_{lj}' + \sigma_{il}' \dot{\omega}_{lj}^T \right) \quad (6.12e)$$

where

$$\dot{\omega}_{ij} = \frac{1}{2} \left(v_{i,j} - v_{j,i} \right). \quad (6.12f)$$

Note that the time derivative of the Cauchy stress in Equations 6.12d and 6.12e is the material time derivative.

3. *Equations of motion* (Equation 6.3), three scalar equations

$$\rho \left(\frac{\partial v_i}{\partial t} + v_{i,j} v_j \right) = \rho b_i + \sigma_{ij,j} \quad (6.13)$$

The equation of conservation of mass (also called the continuity equation) is needed if the density is treated as unknown. However, for isothermal processes, the change in density is very small, and ρ can be treated as a constant. Thus, we have 15 scalar equations for 15 unknowns: (1) three velocity components v_i ; (2) six strain rate components $\dot{\epsilon}_{ij}$ and (3) six stress components σ_{ij} . To solve these equations for the given material, the material properties have to be supplied: (1) the density ρ ; (2) the elastic properties K and μ and (3) the yield stress σ_y and the hardening parameters K and n . Further, the body force b_i (per unit mass) also has to be specified.

The governing Equations 6.12 and 6.13 are non-linear differential equations and therefore need to be solved by an iterative scheme.

6.4.2 GOVERNING EQUATIONS (RIGID-PLASTIC MATERIAL)

The stress-strain rate relations (Equations 6.12a–6.12c) involve the stress rate terms. Because of this, sometimes, the iterative scheme does not converge. In that case, one can still obtain a reasonably accurate solution (in the plastic deformation zone) by simplifying these equations by neglecting the elastic part of the deformation. This amounts to assuming the material to be *rigid-plastic*.

The governing Equation 6.11 is applicable for the rigid-plastic material. Even though the equation of motion (Equation 6.13) is also applicable, it is convenient to replace it with

an alternate expression (Equation 6.8), which involves the separation of the stress tensor into the hydrostatic and deviatoric parts. The constitutive relation (Equation 6.12), however, needs to be replaced by the one for the rigid-plastic material (Equation 5.129). For *rigid-plastic* materials, the hydrostatic part of stress is constitutively indeterminate. Therefore, an additional equation in the form of the incompressibility constraint is needed (Equation 5.130), since the hydrostatic part arises as a reaction to this constraint. Thus, for rigid-plastic materials, there is an *additional governing equation*. The four governing equations of the Eulerian formulation are as follows.

1. *Strain rate-velocity relations* (Equation 4.173), six scalar equations

$$\dot{\varepsilon}_{ij} = \frac{1}{2} (v_{i,j} + v_{j,i}) \quad (6.14)$$

2. *Rigid-plastic stress-strain rate relations* (Equations 5.67, 5.73 and 5.129), six scalar equations

$$\sigma'_{ij} = \frac{2\sigma_{eq}}{3\dot{\varepsilon}_{eq}^p} \dot{\varepsilon}_{ij} \quad (6.15a)$$

where

$$\sigma_{eq} = \sigma_Y + K \left(\varepsilon_{eq}^p \right)^n, \quad \varepsilon_{eq}^p = \int \dot{\varepsilon}_{eq}^p dt \quad (6.15b)$$

Note that, now, the constitutive equation *does not* contain the stress rate terms.

3. *Equations of motion* (Equation 6.8), three scalar equations

$$\rho \left(\frac{\partial v_i}{\partial t} + v_{i,j} v_j \right) = \rho b_i - p_{,i} + \sigma'_{ij,j} \quad (6.16)$$

4. *Incompressibility constraint* (Equation 5.130), one scalar equation

$$\dot{\varepsilon}_{kk} = 0 \quad (6.17)$$

In isothermal and isochoric (i.e. no volume change) process, the density ρ can still depend on the hydrostatic part of stress. However, as before, the change in density is assumed to be small, and ρ is treated as a constant. Thus, now, there are 16 scalar equations for 16 unknowns: (1) three velocity components v_i ; (2) six strain rate components $\dot{\varepsilon}_{ij}$; (3) six deviatoric stress components σ'_{ij} and (4) one hydrostatic stress component p . Since the elastic deformation has been neglected, the elastic material properties are not needed to solve these equations.

These governing equations are also non-linear and, therefore, need to be solved by an iterative scheme. However, they are easier to solve than the governing equations of the elastic-plastic materials.

All these 16 equations are differential equations in spatial variables x_j and time t . Therefore, boundary and initial conditions are required for solving these equations.

6.4.3 BOUNDARY CONDITIONS

The typical boundary conditions are as follows. Let the boundary of the domain be denoted by S .

1. On a part of the boundary (S_v), a velocity vector v is specified. Thus,

$$v_i = v_i^* \text{ on } S_v \quad (6.18)$$

where v_i^* represents the specified value. This is called the *kinematic* or *velocity* boundary condition.

2. On the remaining part of the boundary (S_t), a stress vector $\mathbf{t}_n = \boldsymbol{\sigma} \hat{\mathbf{n}}$ is specified. Thus,

$$(t_n)_i \equiv \sigma_{ij} n_j = (t_n)_i^* \text{ on } S_t \quad (6.19)$$

where $(t_n)_i^*$ represents the specified value. This is called the *stress* or *traction* boundary condition. Note that the parts S_v and S_t have to be disjoint. Further, their union has to be equal to the total boundary S . Thus,

$$S_v \cap S_t = \phi, S = S_v \cup S_t \quad (6.20)$$

The individual parts S_v and S_t may consist of several disjoint segments.

In practice, the boundary conditions differ from those specified by Equations 6.18 and 6.19. Sometimes at a point, all the three components of the velocity vector v or the stress vector $\mathbf{t}_n$ may not be known. Instead, only one velocity component and two stress components or two velocity components and one stress component may be known. Such boundary conditions are called *mixed* boundary conditions. Further, on a boundary inclined to coordinate axes or on a boundary where friction is present, individual components of v or $\mathbf{t}_n$ may not be known. Instead, a combination of their components is known. For the *rigid-plastic material*, the boundary conditions are the same.

6.4.4 INITIAL CONDITIONS

Note that the governing Equations 6.12d, 6.12e and 6.13 involve the *first* (partial) time derivative of the velocity vector as well as of the hydrostatic and deviatoric parts of the stress tensor. Therefore, the initial values of v_i , σ_{kk} and σ'_{ij} are needed to be specified at every point of the control volume. Thus, the initial conditions are

$$v_i = v_i^\circ, \sigma_{kk} = \sigma_{kk}^\circ, \sigma'_{ij} = \sigma'_{ij}^\circ \text{ at } t = t_0 \quad (6.21)$$

where $\overset{\circ}{v}_i$, $\overset{\circ}{\sigma}_{kk}$ and $\overset{\circ}{\sigma}'_{ij}$ are the specified values at the initial time t_0 . For a *steady* process, the partial time derivative (i.e. the unsteady part of the material time derivative) of the velocity vector as well as of the stress tensor is zero. Therefore, the initial conditions are *not* needed.

For the *rigid-plastic material*, the constitutive equation does not contain the time derivative of the stress tensor. Therefore, only one initial condition is needed, namely, on the velocity vector.

6.5 EXAMPLE OF EULERIAN FORMULATION: A WIRE DRAWING PROBLEM

A possible control volume for the *wire drawing* problem is shown in Figure 6.1. Since the geometry and loading are *axisymmetric*, only a typical r - z plane of the wire is considered. While choosing the control volume, the boundaries AB and EF are placed sufficiently away from the die interface CD to simplify the boundary conditions on these boundaries by taking advantage of the uniform velocity fields existing there. The figure also shows the possible *plastic boundaries*. These boundaries are not known a priori but have to be determined as a part of the solution.

Since the problem is axisymmetric, it is convenient to use the cylindrical polar coordinates (r, θ, z) . Then, the non-zero components of the velocity vector, the strain rate tensor and the stress tensor become

$$\{v\} = \begin{Bmatrix} v_r \\ 0 \\ v_z \end{Bmatrix}, [\dot{\epsilon}] = \begin{bmatrix} \dot{\epsilon}_{rr} & 0 & \dot{\epsilon}_{zr} \\ 0 & \dot{\epsilon}_{\theta\theta} & 0 \\ \dot{\epsilon}_{zr} & 0 & \dot{\epsilon}_{zz} \end{bmatrix}, [\sigma] = \begin{bmatrix} \sigma_{rr} & 0 & \sigma_{zr} \\ 0 & \sigma_{\theta\theta} & 0 \\ \sigma_{zr} & 0 & \sigma_{zz} \end{bmatrix}. \quad (6.22)$$

Further, all these non-zero components become independent of the coordinate θ . Now, the boundary conditions are written for the problem of Figure 6.1. Because of axisymmetry, there are only two boundary conditions on each boundary and not three.

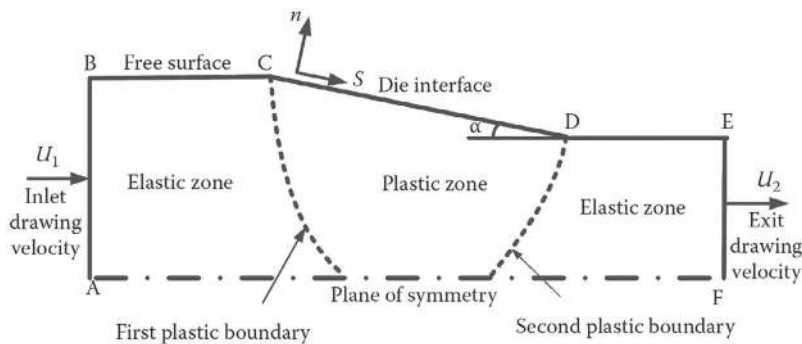


FIGURE 6.1 Domain for the Eulerian formulation of wire drawing—it is a control volume consisting of a typical r - z plane.

6.5.1 INLET AND EXIT BOUNDARIES AB AND EF

As stated earlier, the boundaries AB and EF are chosen sufficiently away from the die interface CD. Therefore, it can be assumed that, at these boundaries, the velocity vector has only a z -component and is uniform over the whole cross section of the wire. Let U_2 be the specified *drawing velocity*. Then, the boundary conditions at the exit boundary EF become

$$v_r = 0, v_z = U_2 \quad (6.23)$$

Let U_1 be the velocity (along the z -axis) at the inlet boundary AB. It can be expressed in terms of U_2 and the *fractional reduction* r_d using the conservation of mass equation. Since the density is treated as a constant, the conservation of mass implies

$$U_1 A_1 = U_2 A_2 \quad (6.24)$$

where A_1 and A_2 are the areas of cross section of the wire at the inlet and exit boundaries, respectively. The definition of the fractional reduction r_d is

$$r_d = 1 - \frac{A_2}{A_1} \quad (6.25)$$

Eliminating A_2/A_1 from Equations 6.24 and 6.25, one gets

$$U_1 = U_2 (1 - r_d) \quad (6.26)$$

Now, the boundary condition at the inlet boundary AB can be written as

$$v_r = 0, v_z = U_2 (1 - r_d) \quad (6.27)$$

6.5.2 STRESS-FREE BOUNDARIES BC AND DE

The boundary BC is a stress-free surface. It is assumed that the *die does not have the land portion*. Then, the whole of the boundary DE is also a stress-free surface. On the stress-free surfaces, the stress vector is zero at every point. Therefore, the boundary conditions at the boundaries BC and DE can be expressed as

$$t_r = 0, t_z = 0 \quad (6.28)$$

where t_r and t_z are the components of the stress vector $\mathbf{t}_n$ in cylindrical polar coordinates. Sometimes an alternate set of boundary conditions is used on these boundaries. This set is as follows. Since the direction of the velocity vector at the boundaries BC and DE is always along the z -axis, the boundary condition of Equation 6.28 may be modified to specify v_r to be zero instead of t_r being zero. The modified boundary condition is expected to give a more accurate velocity field.

6.5.3 PLANE OF SYMMETRY AF

On the plane of symmetry, the normal component of the velocity vector and the shear components of the stress vector are zero at every point. Therefore, the boundary conditions at the boundary AF can be written as

$$v_r = 0, t_z = 0. \quad (6.29)$$

Note that this is a mixed type of boundary condition.

6.5.4 DIE INTERFACE CD

Let n be the direction normal to the die interface and s be the direction along the interface. At the interface, there cannot be any material flow along the normal direction n . Therefore, the component of the velocity vector along the direction n must be zero. Using the die semi-angle α , the normal component of the velocity vector can be expressed as

$$v_n = v_r \cos \alpha + v_z \sin \alpha \quad (6.30)$$

It is assumed that the frictional (or shear) stress exerted by the die in the s -direction is assumed to be governed by *Coulomb's law*:

$$|t_s| = f |t_n| \quad (6.31)$$

where f is the coefficient of friction, and t_s and t_n are the components of the stress vector $\mathbf{t}_n$ along the directions s and n , respectively. Since the material flow at the interface is in the positive s -direction, the frictional stress will be in the opposite direction, that is, in the negative s -direction. Further, the normal stress exerted by the die is always compressive, that is, in the negative n -direction. Therefore, both t_s and t_n are negative. Then, Equation 6.31 becomes

$$t_s = f t_n \quad (6.32)$$

Using the die semi-angle α , t_s and t_n can be expressed in terms of t_r and t_z :

$$t_s = -t_r \sin \alpha + t_z \cos \alpha, t_n = t_r \cos \alpha + t_z \sin \alpha \quad (6.33)$$

Eliminating t_s and t_n from Equations 6.32 and 6.33, Coulomb's law can be written as

$$-(\sin \alpha + f \cos \alpha) t_r + (\cos \alpha - f \sin \alpha) t_z = 0 \quad (6.34)$$

Now, the boundary conditions at the boundary CD become

$$\begin{aligned} v_r \cos \alpha + v_z \sin \alpha &= 0 \\ -(\sin \alpha + f \cos \alpha) t_r + (\cos \alpha - f \sin \alpha) t_z &= 0 \end{aligned} \quad (6.35)$$

These boundary conditions involve combinations of v_r and v_z and t_r and t_z .

The shear stress at the die interface is subject to a constraint that it cannot exceed its maximum value. The maximum value for the Mises material can be found as follows. The maximum value of the shear component of the stress vector at a point *with respect to the orientation of the plane* on which it acts is given by (Equation 3.101)

$$\tau = \frac{(\sigma_1 - \sigma_3)}{2} \quad (6.36)$$

where σ_1 and σ_3 are, respectively, the largest and smallest principal stresses at the point. Note that the values of σ_1 and σ_3 change from point to point. However, they have to satisfy the Mises yield criterion during the plastic deformation. Using Equations 5.44, 5.48 and 5.68, the Mises criterion for subsequent yielding, in terms of the principal stresses, can be written as

$$\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] - 2\sigma_{eq}^2 = 0 \quad (6.37)$$

$$\sigma_{eq} = H(\varepsilon_{eq}^p)$$

Maximizing ε of Equation 6.36 with respect to σ_1 and σ_3 subject to the constraint of Equation 6.37, one gets

$$\tau_{\max} = \frac{\sigma_{eq}}{\sqrt{3}}, \sigma_{eq} = H(\varepsilon_{eq}^p) \quad (6.38)$$

Thus, the maximum value of the shear stress on the die interface cannot exceed $\sigma_{eq} / \sqrt{3}$ for the Mises material.

6.5.5 LOCATION OF PLASTIC BOUNDARIES

The first plastic boundary corresponds to the process of initial yielding at the material particles. The second plastic boundary corresponds to the process of unloading at these particles. In Eulerian formulation, the velocity, strain rate and stress at every point of the control volume are determined in a single step. Thus, it is not an incremental procedure. Therefore, there is no scope for carrying out the elastic analysis first, then applying the initial yield criterion, then carrying out the plastic analysis and finally checking for the unloading.

In elasto-plastic analysis, the plastic boundaries are determined iteratively starting with an initial choice of the plastic boundaries. Usually, the initial choice corresponds to the straight lines parallel to the r -axis passing through points C and D. In each iteration, elastic constitutive equations are applied in the inlet and exit regions, and elasto-plastic constitutive relations are applied in the plastic zone. At the end of the iteration, the stress field of the solution is used to correct the plastic boundaries using the initial yielding and unloading criterion.

However, in rigid-plastic analysis, it is not possible to locate the plastic boundaries iteratively on the basis of initial yielding and unloading criteria. Instead, the whole control volume is treated as the plastic zone, and only the rigid-plastic

constitutive relations are used. After the analysis, it is observed that the strain rates are very small in the inlet and exit regions compared to their values in the middle region. Therefore, the inlet and exit regions can be interpreted as rigid. A sufficiently small value of the equivalent plastic strain rate $\dot{\epsilon}_{eq}^p$ is used as a *cut-off* to demarcate the plastic zone from the rest of the control volume. The cutoff can be a small percent of the maximum value of the equivalent plastic strain rate over the control volume. Accuracy of the plastic boundaries depends on the cutoff value.

Another drawback of the rigid-plastic analysis is that the stresses in the inlet and exit regions are highly inaccurate since they are determined by the rigid-plastic constitutive relations. The value of the equivalent stress σ_{eq} in the inlet region is equal to σ_y , whereas in the exit region, it is $H(\epsilon_{eq}^p)$. Thus, one does not get any reasonable estimates of the residual stresses. However, accuracy of the deformation and stress fields in the plastic region is quite good. Further, estimate of the power required to carry out the process is also reasonably accurate.

6.6 UPDATED LAGRANGIAN FORMULATION

In this formulation, the final deformed configuration is analyzed through several increments using the incremental form of the governing equations. After a certain number of increments, the current (or present) configuration is obtained, that is the deformed configuration at the current time t . The deformation and stress fields at time t are already known through the analysis of earlier increments up to the previous increment. For the analysis of the current increment, the *current configuration as the reference configuration* is considered.

In this formulation, the incremental displacement vector $d\mathbf{u}$ is treated as the primary unknown. To simplify the analysis, it is assumed that the increment size is small. Then, the incremental linear strain tensor $d\epsilon$ can be used as the measure of incremental deformation. In this formulation, no simplification of any kind is achieved if the elastic deformation is neglected. Therefore, the elasto-plastic constitutive relation is used. Let du_i , $d\epsilon_{ij}$ and $d\sigma_{ij}$ be the components of the incremental displacement vector, the incremental linear strain tensor and the incremental stress tensor, respectively. Then, for an *isothermal* process, the increments du_i , $d\epsilon_{ij}$ and $d\sigma_{ij}$ are governed by the following three governing equations.

6.6.1 GOVERNING EQUATIONS

1. *Incremental strain-displacement relations* (Equation 4.173), six scalar equations:

$$d\epsilon_{ij} = \frac{1}{2}(du_{i,j} + du_{j,i}) \quad (6.39)$$

2. *Incremental elastic-plastic stress-strain relations* (Equations 5.11, 5.73, 5.112 and 5.113), six scalar equations.

After yielding:

$$d \overset{\circ}{\sigma}_{ij} = C_{ijkl}^{EP} d\epsilon_{kl} \quad (6.40a)$$

where

$$C_{ijkl}^{EP} = 2\mu \left[\frac{\nu}{1-2\nu} \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} - \frac{9\mu}{2} \frac{\sigma'_{ij} \sigma'_{kl}}{(H' + 3\mu) \sigma_{eq}^2} \right] \quad (6.40b)$$

$$\sigma_{eq} = \sigma_Y + K \left(\epsilon_{eq}^p \right)^n \quad (6.40c)$$

$$\nu = \frac{\lambda}{2(\lambda + \mu)} \quad (6.40d)$$

Before yielding and after unloading:

$$d \overset{\circ}{\sigma}_{ij} = C_{ijkl}^E d\epsilon_{kl}, \quad (6.40e)$$

where

$$C_{ijkl}^E = \lambda \delta_{kl} \delta_{ij} + 2\mu \delta_{ik} \delta_{jl}. \quad (6.40f)$$

Here, the superscript $\circ$ on the stress increment in Equations 6.40a and 6.40e means that it is the *Jaumann stress increment* given by Equation 5.183, where the incremental infinitesimal rotation tensor $d\omega$ is given by Equation (4.177). Thus, we have

$$d \overset{\circ}{\sigma}_{ij} = d\sigma_{ij} - \left(d\omega_{il} \sigma_{lj} + \sigma_{il} d\omega_{lj}^T \right) \quad (6.40g)$$

$$\omega_{ij} = \frac{1}{2} \left(du_{i,j} - du_{j,i} \right) \quad (6.40h)$$

3. *Incremental equations of motion* (Equation 6.10), three scalar equations

$$\rho da_i = \rho db_i + d\sigma_{ij,j} \quad (6.41)$$

As decided earlier, ρ is treated as a constant. Therefore, the equation of conservation of mass is not needed. Thus, there are 15 scalar equations for 15 unknowns: (1) three incremental displacement components du_i ; (2) six incremental linear strain components $d\epsilon_{ij}$ and (3) six incremental stress components σ_{ij} . To solve these equations for the given material, the material properties have to be supplied: (1) density ρ ; (2) elastic properties λ and μ

and (3) the yield stress σ_y and the hardening parameters K and n . Further, the incremental body force db_i (per unit mass) also has to be specified.

Equations 6.39, 6.40h and 6.41 are differential equations in spatial variables x_j and time t . Therefore, the boundary and initial conditions are required for solving these equations.

6.6.2 BOUNDARY CONDITIONS

The typical boundary conditions are as follows. As before, the boundary of the domain is denoted by S .

1. On a part of the boundary (S_u), an incremental displacement vector $d\mathbf{u}$ is specified. Thus,

$$du_i = du_i^* \text{ on } S_u \quad (6.42)$$

where du_i^* represents the specified value. This is called the *kinematic* or *displacement* boundary condition.

2. On the remaining part of the boundary (S_t), an incremental stress vector $d\mathbf{t}_n = d\sigma\hat{\mathbf{n}}$ is specified. Thus,

$$(dt_n)_i \equiv d\sigma_{ij}n_j = (dt_n)_i^* \text{ on } S_t \quad (6.43)$$

where $(dt_n)_i^*$ represents the specified value. This is called the *stress* or *traction* boundary condition.

Note that parts S_u and S_t have to satisfy the relations similar to the one given by Equation 6.20. Further, each of S_u and S_t may consist of several disjoint segments. In practice, the boundary conditions differ from those specified by Equations 6.42 and 6.43. Sometimes there are *mixed* boundary conditions. Further, on a boundary inclined to coordinate axes or on a boundary where friction is present, a combination of the components of $d\mathbf{u}$ or $d\mathbf{t}_n$ is known.

6.6.2.1 Initial Conditions

The governing Equation 6.41 contains the incremental acceleration vector, which involves two time derivatives of the incremental displacement vector. Therefore, the values of the incremental displacement vector du_i and the incremental velocity vector dv_i is needed to be specified at the beginning of the increment (i.e. at the current time t) at every point of the current configuration. Thus, the initial conditions are

$$du_i = du_i^t, dv_i = dv_i^t, \text{ at time } t \quad (6.44)$$

where du_i^t and dv_i^t are the specified values at time t . For a *quasi-static* problem, the incremental acceleration term can be neglected from Equation 6.41. Then, the initial conditions are not needed.

6.6.2.2 Updating Scheme

After solving the incremental equations (Equations 6.39–6.41) along with the boundary and initial conditions, one gets the incremental displacement vector du_i , the incremental linear strain tensor $d\epsilon_{ij}$, and the incremental stress tensor $d\sigma_{ij}$. Using du_i , the geometry is updated to get the deformed configuration at time $t + dt$. Further, by adding du_i and $d\sigma_{ij}$ to u_i and σ_{ij} , one gets the displacement vector and the stress tensor at time $t + dt$. This completes the analysis of the current increment. After this, we go to the next increment. For this increment, the deformed configuration at time $t + dt$ is used as the reference configuration. Therefore, the incremental equations (Equations 6.39–6.41) are solved over the deformed configuration at time $t + dt$ along with the boundary and initial conditions for this configuration. This process is continued until the desired deformation is achieved.

6.7 EXAMPLE OF UPDATED LAGRANGIAN FORMULATION: FORGING OF A CYLINDRICAL BLOCK

The domain for the problem of forging a cylindrical block is shown in Figure 6.2. It is assumed that *both the platens move with the same velocity but in the opposite direction*. Therefore, only half the block needs to be considered for the analysis. Further, the geometry and loading are *axisymmetric*. Therefore, only a typical r - z plane of half the block is considered.

Since the problem is axisymmetric, it is convenient to use the cylindrical polar coordinates (r, θ, z) . Then, the non-zero components of the incremental displacement vector, the incremental linear strain tensor and the incremental stress tensor become

$$\{du\} = \begin{Bmatrix} du_r \\ 0 \\ du_z \end{Bmatrix}, [d\epsilon] = \begin{bmatrix} d\epsilon_{rr} & 0 & d\epsilon_{rz} \\ 0 & d\epsilon_{\theta\theta} & 0 \\ d\epsilon_{rz} & 0 & d\epsilon_{zz} \end{bmatrix}, [d\sigma] = \begin{bmatrix} d\sigma_{rr} & 0 & d\sigma_{rz} \\ 0 & d\sigma_{\theta\theta} & 0 \\ d\sigma_{rz} & 0 & d\sigma_{zz} \end{bmatrix} \quad (6.45)$$

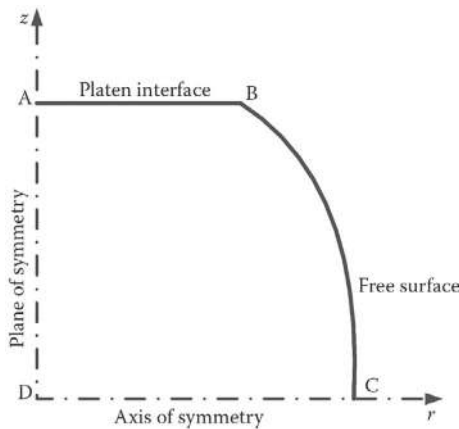


FIGURE 6.2 Domain for the updated Lagrangian formulation of forging of a cylindrical block. It is a typical r - z plane of half the deformed block at time t .

Further, all these non-zero components become independent of the coordinate θ . Now, the boundary conditions are written for the problem of Figure 6.2. Because of axisymmetry, there are only two boundary conditions on each boundary and not three.

6.7.1 STRESS-FREE BOUNDARY BC

The boundary BC is a stress-free surface. On the stress-free surface, the incremental stress vector is zero at every point. Therefore, the boundary conditions at the boundary BC can be expressed as

$$dt_r = 0, dt_z = 0 \quad (6.46)$$

where dt_r and dt_z are the components of the incremental stress vector $d\mathbf{t}_n$ in cylindrical polar coordinates.

6.7.2 PLANE OF SYMMETRY DC

On the plane of symmetry, the normal component of the incremental displacement vector and the shear components of the incremental stress vector are zero at every point. Therefore, the boundary conditions at the boundary DC can be written as

$$dt_r = 0, du_z = 0 \quad (6.47)$$

where du_z is the z -component of the incremental displacement vector. Note that this is a mixed type of boundary condition.

6.7.3 PLANE OF SYMMETRY AD

On the plane of symmetry, the normal component of the incremental displacement vector and the shear components of the incremental stress vector are zero at every point. Therefore, the boundary conditions at the boundary DC can be written as

$$dt_r = 0, dt_z = 0 \quad (6.48)$$

Note that this is a mixed type of boundary condition.

6.7.4 PLATEN INTERFACE AB

At the interface, the z -component of the incremental displacement vector must be equal to the incremental platen displacement.

As far as the other boundary condition is concerned, the following is observed. Nearer to point A (center of the platen), the block material *sticks* to the platen since $t_r + dt_r$ (i.e. the frictional or shear stress exerted by the platen in the r -direction) satisfies the inequality $|t_r + dt_r| < f|t_z + dt_z|$. Here, f is the coefficient of friction and $t_z + dt_z$ is the normal stress exerted by the platen in the z -direction. On the other hand, nearer

the free edge (point B), the block material *slips* relative to the platen in the outward direction and thus $t_r + dt_r$ and $t_z + dt_z$ are governed by *Coulomb's law*:

$$|t_r + dt_r| = f |t_z + dt_z|. \quad (6.49)$$

Note that the friction boundary condition has to be in terms of the total stress vector $\mathbf{t}_n + d\mathbf{t}_n$ at time $t + dt$ and not in terms of the incremental stress vector $d\mathbf{t}_n$. The previous boundary condition can be simplified by using the following observation. The material flow at the interface is in the positive r -direction. Therefore, the frictional stress will be in the opposite direction, that is, in the negative r -direction. Further, the normal stress exerted by the platen is always compressive, that is, in the negative z -direction. Thus, both $t_r + dt_r$ and $t_z + dt_z$ are negative. Then, Equation 6.49 becomes

$$t_r + dt_r = f(t_z + dt_z). \quad (6.50)$$

When the block material sticks to the platen, the r -component of the incremental displacement vector must be zero.

Thus, the boundary conditions at the boundary AB become

$$\begin{aligned} t_r + dt_r - f(t_z + dt_z) &= 0, \quad (\text{for slipping}), \\ du_r &= 0, \quad (\text{for sticking}) \\ du_z &= du_z^* \end{aligned} \quad (6.51)$$

where du_z^* is the prescribed *incremental displacement of the platen*. Note that this boundary condition involves a combination of $t_r + dt_r$ and $t_z + dt_z$. Here also, the shear stress at the platen interface is subject to a constraint that it cannot exceed its maximum value.

Exercises

1. Plane strain rolling of a strip is shown in Figure 6.0E1. In this figure, the directions n and s vary continuously from C to D. Further, G is a neutral point, that is, the point at which the frictional stress changes its direction.

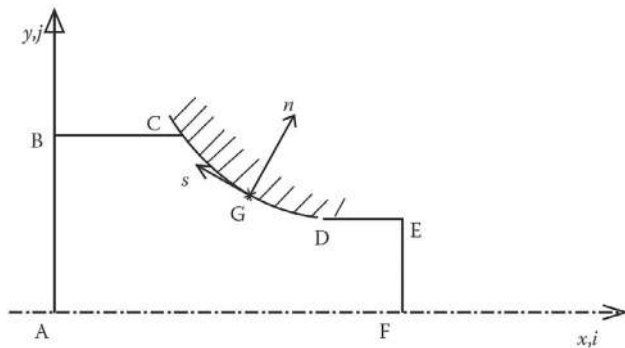


FIGURE 6.0E1 The half side view of a plane strain rolling.

In this process, the velocity vector, the strain rate tensor and the stress tensor have only the following non-zero components:

$$\{v\} = \begin{Bmatrix} v_x \\ v_y \end{Bmatrix}, [\dot{\epsilon}] = \begin{bmatrix} \dot{\epsilon}_{xx} & \dot{\epsilon}_{xy} & 0 \\ \dot{\epsilon}_{xy} & \dot{\epsilon}_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}, [\sigma] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix}.$$

As a result, the stress vector also has only two non-zero components:

$$\begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ 0 \end{Bmatrix} = \begin{Bmatrix} \sigma_{xx}n_x + \sigma_{xy}n_y \\ \sigma_{xy}n_x + \sigma_{yy}n_y \\ 0 \end{Bmatrix}.$$

Write down two boundary conditions on each of the six boundaries: AB, BC, CD, DE, EF and FA. The boundary conditions could be (1) velocity b.c., that is, (v_x, v_y) specified; or (2) stress b.c., that is, (t_x, t_y) specified or (3) mixed b.c., that is, (v_x, t_y) or (t_x, v_y) specified.

2. A cantilever beam is bent by applying a uniformly distributed stress at the top surface. Figure 6.0E2 shows the different configurations at time t . During the time t to $t + dt$, the applied uniformly distributed stress is dt_y in the y -direction. Write down the boundary conditions for all of the six faces.

ANSWERS

1. On AB, EF, BC and DE: $t_x=0, t_y=0$ (because they are free surfaces),
On BC and DE: $t_x=0, t_y=0$ (because they are free surfaces).
On CD: $v_n = v_x \sin \alpha + v_y \cos \alpha = 0$.

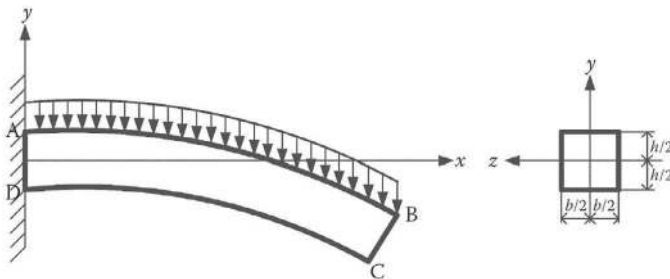


FIGURE 6.0E2 A cantilever beam loaded by a uniformly distributed load.

$$t_x (\cos \alpha + f \sin \alpha) = t_y (\sin \alpha - f \cos \alpha) \text{ Before neutral point.}$$

$$t_x (\cos \alpha - f \sin \alpha) = t_y (\sin \alpha + f \cos \alpha) \text{ After neutral point.}$$

On AF: $v_y = 0$, because of symmetry and $t_x = 0$, again because of symmetry.

2. a. At the support: zero displacement (and rotation).
- b. At the surface on which load is acting: $t_y =$ given load per unit area, other components of traction are zero.
- c. At other surfaces all components of traction are zero.

7 Calculus of Variations and Extremum Principles

7.1 INTRODUCTION

Consider a large-sized vertical wooden board mounted on a wall. A small-sized block, such that it can be treated as a particle, has to slide down from a point P with coordinates (x_1, y_1) to another point Q with coordinates (x_2, y_2) under the influence of gravity. Figure 7.1 shows points P and Q joined by an arbitrary curve. In order to slide the small block from point P to Q, one can carve a curved or straight slot joining points P and Q. The block will move within the slot. Let us assume that there is no friction between the block and the slot, and just as the block is treated as a particle, the slot is treated as a line. The question is what type of slot should be made so that the block starting from rest from point P reaches point Q in the least possible time.

Let us formulate this problem. The time to travel from point P to point Q is given by

$$t_{PQ} = \int_P^Q \frac{ds}{v}, \quad (7.1)$$

where s is the arc length, and v is the speed. The speed at any point can be found by conservation of energy. Let the velocity at coordinate y be v . Then by conservation of energy principle,

$$\frac{1}{2}mv^2 = mg(y_1 - y), \quad (7.2)$$

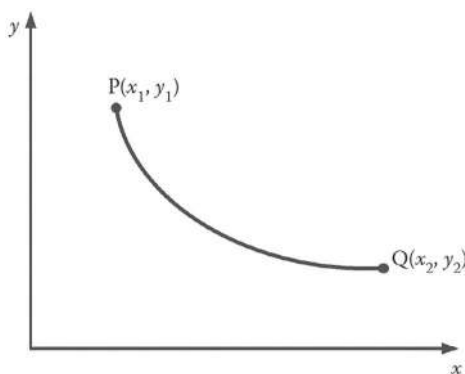


FIGURE 7.1 Arbitrary path between two points P and Q in a vertical plane.

where m is the mass of the particle, and g is the gravitational acceleration. This provides

$$v = \sqrt{2g(y_1 - y)}. \quad (7.3)$$

Also,

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \quad (7.4)$$

Denoting dy/dx by y' and substituting Equations 7.3 and 7.4 in Equation 7.1,

$$t_{PQ} = \int_{x_1}^{x_2} \frac{\sqrt{1 + y'^2}}{\sqrt{2g(y_1 - y)}} dx. \quad (7.5)$$

Now the problem is to find y (and hence y') that minimizes t_{PQ} . This problem is called the famous brachistochrone problem, although the statement of the original problem was not exactly the same as the present one. The brachistochrone is a Greek word for 'shortest time'. This problem was posed by Johann Bernoulli in 1696. It is said that Newton solved this problem in a day. This was the beginning of the calculus of variations.

Calculus of variations provides a way to convert the problem of maximization/minimization of an integral such as in Equation 7.5 into a differential equation with appropriate boundary conditions. The procedure involves assuming y_0 as the solution and giving a small variation δy to y from the true solution y_0 . If y_0 indeed gives the extremum (maximum or minimum) value of the integral, the vanishingly small variation δy should not cause any variation to the integral, that is, $\delta t_{PQ} = 0$. This is the basic concept of calculus of variations and the reason for the word 'variations' in the name of this branch of calculus.

Calculus of variations can accomplish the following tasks:

1. It can convert the problem of extremization (maximization or minimization) of an integral containing one or more unknown functions and/or their derivatives into corresponding one or more than one differential equation along with appropriate boundary conditions.
2. It can convert the problem of solving a differential equation with boundary conditions to corresponding extremization of an integral form. The integral form such as that of Equation 7.5 is called the variational form. However, variational form does not exist for all differential equations.

As an example from the strength of materials, consider the problem of deflection of an Euler-Bernoulli beam. The differential equation of the problem is

$$EI_{zz} \frac{d^4 w}{dz^4} = q, \quad (7.6)$$

where EI_{zz} is the flexural rigidity of the beam, w is the transverse deflection of the beam, which is a function of longitudinal coordinate x , and q is the load intensity (transverse load per unit length) function. For a built-in beam, the boundary conditions are

$$w = 0 \text{ and } w' = 0 \text{ at both ends.} \quad (7.7)$$

With the help of the calculus of variations, one can obtain the following variational form:

$$I = \int_0^l \left(\frac{1}{2} EI_{zz} \left(\frac{d^2 w}{dx^2} \right)^2 - qw \right) dx, \quad (7.8)$$

where l is the length of the beam. The variational form given by Equation 7.8 needs to be minimized to obtain the exact deflection of the beam. Thus, the technique of calculus of variations can get the form given by Equation 7.8 from Equations 7.6 and 7.7. Alternatively, the form given by Equation 7.8 can be converted to the differential form given by Equation 7.6.

In this chapter, a brief discussion of the calculus of variations is provided. The discussion starts by defining the functional. Then, techniques to convert a variational form to a differential form and vice versa have been described. Finally, some extremum principles are discussed. For further details, refer to Bathe (1982), Reddy (1993), Cook et al. (1989), Riley et al. (1998) and Arfken and Weber (2005).

7.2 FUNCTIONAL

The discussion in this chapter is confined to a real domain only. In the real domain, a function maps one real variable or more than one real variable to a real dependent variable. For example,

$$w = x^2 + y^2 + 2z^3 \quad (7.9)$$

is a function of x , y and z . Here, x , y and z are called independent variables, and w is called a dependent variable.

A functional maps one function or more than one function to a real dependent variable. Hence, a functional can be called a function of functions, whose arguments are functions. Thus,

$$I(y) = \int_0^4 y dx \quad (7.10)$$

is a functional. Here, I is a dependent variable, and y is an independent function. It can be easily seen that

$$I(x) = \int_0^4 x dx = 8, \quad (7.11)$$

$$I(x^2) = \int_0^4 x^2 dx = \frac{64}{3}, \quad (7.12)$$

and

$$I(\cos x) = \int_0^4 \cos x dx = \sin 4. \quad (7.13)$$

Thus, functions x , x^2 and $\cos x$ are mapped to 8, 64 and $\sin 4$, respectively. Other examples of functionals include the following:

1.

$$I(u) = \int_0^l \left\{ \frac{1}{2} EA \left(\frac{du}{dx} \right)^2 + 5u \right\} dx, \quad (7.14)$$

where u is the independent function (of x).

2.

$$I(T) = \int_0^7 \int_0^5 \left\{ \left(\frac{\partial T}{\partial x} \right)^2 + \left(\frac{\partial T}{\partial y} \right)^2 \right\} dx dy, \quad (7.15)$$

where T is the independent function (of x and y).

3.

$$I(u, v) = \int_0^1 \int_0^3 \left\{ \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 - 2uv \right\} dx dy, \quad (7.16)$$

where u and v are the independent functions (of x and y).

4.

$$I(w) = \frac{1}{2} \int_0^{10} EI \left(\frac{d^2 w}{dx^2} \right)^2 dx - \frac{P}{2} \int_0^{10} \left(\frac{dw}{dx} \right)^2 dx + \frac{1}{2} k w_{10}^2, \quad (7.17)$$

where w is an independent function of x , w_{10} is the function value at $x = 10$, k and P are constants, and EI is a known function of x .

5.

$$I(w) = \int_0^4 \int_0^3 \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2 \left(\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - 2 \frac{\partial^2 w}{\partial x \partial y} \right) - 5w \right\} dx dy, \quad (7.18)$$

where w is the independent function (of x and y).

Consider the following functional of y :

$$I(y) = \int_a^b F(x, y, y') dx, \quad (7.19)$$

where $F(x, y$ and $y')$ is an expression containing the variable x , function y and the first derivative of function y . An example of this form is Equation 7.14. Note that $F(x, y$ and $y')$ is not a functional, because for a given y , it does not provide a real value. However, the expression of F at some specified x is a functional. Now, denote the expression containing y , its first two derivatives and independent variable x by $F(x, y, y', y'')$. The expression of F at a certain specified point x is a functional, and a definite integral with this functional as an integrand is also a functional. Thus, for constant limits a and b ,

$$I(y) = \int_a^b F(x, y, y', y'') dx, \quad (7.20)$$

is a functional with argument y . An example of this type of functional is given by Equation 7.17. There, the last term is outside the definite integral but can be brought within it by using the Dirac delta function. Thus,

$$\frac{1}{2}kw_{10}^2 = \int_0^{10} \frac{1}{2}kw^2 \delta(x-10) dx. \quad (7.21)$$

A functional $I(y)$ is said to be linear in y if and only if it satisfies the following relation for any scalars α and β and any independent functions y and z :

$$I(\alpha y + \beta z) = \alpha I(y) + \beta I(z). \quad (7.22)$$

Thus, the functional given by Equation 7.14 is not linear because

$$\begin{aligned} I(\alpha y + \beta z) &= \int_0^l \left\{ \frac{1}{2} EA \left(\frac{d(\alpha y + \beta z)}{dx} \right)^2 + 5(\alpha y + \beta z) \right\} dx \\ &\neq \alpha \int_0^l \left\{ \frac{1}{2} EA \left(\frac{dy}{dx} \right)^2 + 5y \right\} dx + \beta \int_0^l \left\{ \frac{1}{2} EA \left(\frac{dz}{dx} \right)^2 + 5z \right\} dx. \end{aligned} \quad (7.23)$$

The following functional is a linear functional:

$$I(y) = \int_0^{10} 5xy dx \quad (7.24)$$

because

$$I(\alpha y + \beta z) = \int_0^{10} 5x(\alpha y + \beta z) dx = \alpha \int_0^{10} 5xy dx + \beta \int_0^{10} 5xz dx. \quad (7.25)$$

A functional $B(y, z)$ is said to be bilinear if it is linear in each of its arguments y and z , that is, for any scalars α and β ,

$$B(\alpha y_1 + \beta y_2, z) = \alpha B(y_1, z) + \beta B(y_2, z), \quad (7.26)$$

$$B(y, \alpha z_1 + \beta z_2) = \alpha B(y, z_1) + \beta B(y, z_2). \quad (7.27)$$

An example of a bilinear functional is

$$I(u, v) = \int_0^{10} 5x \frac{du}{dx} \frac{dv}{dx} dx. \quad (7.28)$$

Here,

$$\begin{aligned} I(\alpha y_1 + \beta y_2, z) &= \alpha \int_0^{10} 5x \frac{d(\alpha y_1 + \beta y_2)}{dx} \frac{dz}{dx} dx \\ &= \alpha \int_0^{10} 5x \frac{dy_1}{dx} \frac{dz}{dx} dx + \beta \int_0^{10} 5x \frac{dy_2}{dx} \frac{dz}{dx} dx = \alpha I(y_1, z) + \beta I(y_2, z) \end{aligned} \quad (7.29)$$

and

$$\begin{aligned} I(y, \alpha z_1 + \beta z_2) &= \int_0^{10} 5x \frac{dy}{dx} \frac{d(\alpha z_1 + \beta z_2)}{dx} dx \\ &= \alpha \int_0^{10} 5x \frac{dy}{dx} \frac{dz_1}{dx} dx + \beta \int_0^{10} 5x \frac{dy}{dx} \frac{dz_2}{dx} dx = \alpha I(y, z_1) + \beta I(y, z_2). \end{aligned} \quad (7.30)$$

The functional is said to be symmetric if

$$B(y, z) = B(z, y). \quad (7.31)$$

The functional given by Equation 7.28 is symmetric.

Example 7.1

Identify each of the following expressions if they are functional or not:

1.

$$EI \frac{d^2 y}{dx^2} + qx$$

2.

$$\int_0^x y dx$$

3.

$$\left(EI \frac{d^2 y}{dx^2} + qx \right)_{x=0}$$

4.

$$\int_0^{10} y dx$$

5.

$$\int_0^7 \frac{dy}{dx} \frac{dz}{dx} dx$$

6.

$$\int_0^3 \int_0^7 z(x, y) dx dy$$

SOLUTION

Expression 1 is not a functional, as for a given y , the expression becomes a function and not a real number. Expression 2 is also not a functional, as for a given y , this expression also becomes a function of x . Expression 3 is a functional, as for a given y , this expression provides a real number. Expression 4 is also a functional, as it maps each y into a real number. In Expression 5, two functions y and z are mapped into a real number; hence, it is a functional. Expression 6 maps a function of two variables into a real number; hence, it is a functional.

7.3 EXTREMIZATION OF A FUNCTIONAL

In this section, the technique of converting a maximization or minimization problem in the form of differential equation(s) will be studied. Only the necessary conditions will be derived. By the application of the sufficiency condition, one can know if the solution of a differential equation maximizes, minimizes or neither maximizes nor minimizes the variational form. Akin to the point of inflection or saddle point in the

multivariable optimization problem, one may get a situation in which the functional is neither maximized nor minimized. However, many a time, some physical reasoning can tell whether the functional is minimized or maximized. For example, in the brachistochrone problem, it is obvious that there is only a shortest path between two points because a zigzag path can be constructed to delay the time beyond any prescribed time limit. Thus, the maximum time is unbounded. The minimum time is not unbounded. In any case, the minimum time has to be greater than zero.

7.3.1 FUNCTIONAL CONTAINING THE FORM $F(x, y, y')$

Consider the following functional:

$$I = \int_a^b F(x, y, y') dx, \quad (7.32)$$

where a and b are fixed real numbers. Note that F is a function of x , y and y' . All of them are independent. Of course, for a given y , y' is fixed, but it is not the other way around. Hence, both y and y' can be treated as independent. It is also clear that functional I is the function of y and y' .

Assume that function $y_0(x)$ extremizes the functional I . A small perturbation to $y_0(x)$ can be provided by adding a scalar multiple of an arbitrary function $\eta(x)$. Thus, in the vicinity of $y_0(x)$, one can write

$$y = y_0(x) + \varepsilon \eta(x). \quad (7.33)$$

It is assumed that the arbitrary function $\eta(x)$ is sufficiently differentiable, and ε is a small real number.

Substituting Equation 7.33 into Equation 7.32, one gets I as a function of ε . Thus,

$$I(\varepsilon) = \int_a^b F(x, y_0 + \varepsilon \eta, y'_0 + \varepsilon \eta') dx. \quad (7.34)$$

Applying Taylor's series expansion up to the second-order derivative terms for fixed x ,

$$\begin{aligned} F(x, y_0 + \varepsilon \eta, y'_0 + \varepsilon \eta') &= F(x, y_0, y'_0) + \left(\varepsilon \eta \frac{\partial F}{\partial y} + \varepsilon \eta' \frac{\partial F}{\partial y'} \right)_{(y=y_0, y'=y'_0)} \\ &+ \left(\frac{\varepsilon^2 \eta^2}{2} \frac{\partial^2 F}{\partial y^2} + \varepsilon^2 \eta \eta' \frac{\partial^2 F}{\partial y \partial y'} + \frac{\varepsilon^2 \eta'^2}{2} \frac{\partial^2 F}{\partial y'^2} \right)_{(y=y_0, y'=y'_0)} + \dots \end{aligned} \quad (7.35)$$

Note that the derivatives are evaluated at y_0 and y'_0 . It has been indicated in Equation 7.35, but in subsequent equations, this indication will be omitted for the sake of

brevity. As I is a function of ε , the necessary condition for its being the maximum or the minimum is that its first derivative with respect to ε vanishes. Applying Leibniz's rule to Equation 7.34,

$$\frac{dI}{d\varepsilon} = \int_a^b \frac{dF(x, y_0 + \varepsilon\eta, y'_0 + \varepsilon\eta')}{d\varepsilon} dx. \quad (7.36)$$

Now, from Equation 7.35,

$$\frac{dF}{d\varepsilon} = \left(\eta \frac{\partial F}{\partial y} + \eta' \frac{\partial F}{\partial y'} \right) + \left(\varepsilon \eta^2 \frac{\partial^2 F}{\partial y^2} + 2\varepsilon \eta \eta' \frac{\partial^2 F}{\partial y \partial y'} + \varepsilon \eta'^2 \frac{\partial^2 F}{\partial y'^2} \right) + \varepsilon(\text{other terms}). \quad (7.37)$$

It is further reminded that the partial derivatives in Equation 7.37 are evaluated at y_0 and y'_0 . Also, the other terms contain ε as well. Now, if y_0 extremizes the function, then the first derivative of I with respect to ε should vanish at $\varepsilon = 0$. Thus,

$$\left. \frac{dI}{d\varepsilon} \right|_{\varepsilon=0} = \int_a^b \left(\eta \frac{\partial F}{\partial y} + \eta' \frac{\partial F}{\partial y'} \right) dx = 0. \quad (7.38)$$

Integrating the second term by parts, taking η' as a first function, one gets

$$\int_a^b \left\{ \frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \right\} \eta dx + \left(\frac{\partial F}{\partial y'} \eta \right) \Big|_b - \left(\frac{\partial F}{\partial y'} \eta \right) \Big|_a = 0. \quad (7.39)$$

As η is arbitrary, it can always be chosen from the set of functions that become 0 at the $x = a$ and $x = b$. Thus,

$$\int_a^b \left\{ \frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \right\} \eta dx = 0. \quad (7.40)$$

Now one makes use of the following lemma:

If $f(x)$ is continuous in a closed interval $[a, b]$, and if

$$\int_a^b f(x) \eta(x) dx = 0 \quad (7.41)$$

for every continuous function $\eta(x)$ defined on a closed interval $[a, b]$ such that $\eta(a) = \eta(b) = 0$, then $f(x) = 0$ for all x in $[a, b]$.

Proof: Suppose that the function $f(x)$ is non-zero, say, negative, at some point in $[a, b]$. Then $f(x)$ is also negative in some interval $[x_1, x_2]$ contained in $[a, b]$, because a

continuous function cannot be abruptly negative just at one point. Now choose a continuous function $\eta(x)$ that is negative in the open interval (x_1, x_2) and zero elsewhere. For example, one such function is

$$\eta(x) = \begin{cases} (x_1 - x)(x_2 - x) & \text{for } x \in (x_1, x_2) \\ 0 & \text{Otherwise} \end{cases}. \quad (7.42)$$

$$\int_a^b f(x)\eta(x)dx = \int_{x_1}^{x_2} f(x)(x_1 - x)(x_2 - x)dx > 0 \quad (7.43)$$

and cannot be zero. Therefore, if Equation 7.41 holds good, then $f(x)$ should be zero everywhere in $[a, b]$.

In view of the lemma, Equation 7.40 provides

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0. \quad (7.44)$$

Equation 7.44 is called the Euler equation or Euler-Lagrange equation and is, in general, of second order.

Because of the Euler-Lagrange equation, Equation 7.39 provides

$$\left(\frac{\partial F}{\partial y'} \eta \right) \Big|_b - \left(\frac{\partial F}{\partial y'} \eta \right) \Big|_a = 0. \quad (7.45)$$

$\eta(a) = 0$ can always be chosen as η is arbitrary (unless of course y is prescribed at point $x = a$, in that case η has to be zero); then

$$\left(\frac{\partial F}{\partial y'} \eta \right) \Big|_b = 0. \quad (7.46)$$

This gives us

$$\text{At } x = b, \text{ either } \frac{\partial F}{\partial y'} = 0 \text{ or } \eta = 0. \quad (7.47)$$

Note that while arriving at Equation 7.47, it is not implied that $\eta(a)$ must be equal to 0. Being arbitrary, it can be zero or non-zero, and it has been chosen to be 0 to arrive at the boundary condition given by Equation 7.47. If it is not arbitrary, it has to be taken as zero for a prescribed y . Similarly, choosing $\eta(b) = 0$, one gets

$$\text{At } x = \alpha, \text{ either } \frac{\partial F}{\partial y'} = 0 \text{ or } \eta = 0. \quad (7.48)$$

Thus, at both of the boundaries, either of the following conditions has to be satisfied:

$$\frac{\partial F}{\partial y'} = 0 \quad (7.49)$$

or condition

$$\eta = 0. \quad (7.50)$$

The boundary condition given by Equation 7.50 is called the geometric or essential boundary condition. This boundary condition means that y must be prescribed. (The variation of y should be zero.) It is essential to have at least one boundary condition of this type for getting the unique function y as a result of the solution. The boundary condition given by Equation 7.49 is called the natural boundary condition. It is also called the force boundary condition for mechanics problems.

Example 7.2

Consider the following differential equation:

$$I = \int_0^{10} \frac{1}{2} \left(\frac{dy}{dx} \right)^2 dx - 5y(5), \quad (7.51)$$

where $y(5)$ is the value of function y at $x = 5$. Use the Euler-Lagrange equation to find the differential equation that minimizes I .

SOLUTION

Equation 7.51 can be written as

$$\int_0^{10} \left\{ \frac{1}{2} y'^2 - 5y\delta(x-5) \right\} dx, \quad (7.52)$$

where $\delta(x-5)$ is the Dirac delta function defined at $x = 5$. Equation 7.52 is in the form of Equation 7.32. Here,

$$F = \frac{1}{2} y'^2 - 5y\delta(x-5). \quad (7.53)$$

The Euler-Lagrange equation is given by

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = -5\delta(x-5) - \frac{d}{dx} (y') = 0. \quad (7.54)$$

Thus, the desired differential equation is

$$y'' + 5\delta(x-5) = 0. \quad (7.55)$$

How does one know that this differential equation minimizes and not maximizes? Well, one can observe Equation 7.51 and see that it is possible to choose a high value of y' and make I as large as one wishes. Thus, the functional is unbounded for maximization.

Example 7.3

Prove that the length of the curve joining (x_1, y_1) and (x_2, y_2) will be the minimum if it is a straight line.

SOLUTION

An infinitesimal length on the curve is given by

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \quad (7.56)$$

The total length L is given by

$$\int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \quad (7.57)$$

Here,

$$F = \sqrt{1 + y'^2}. \quad (7.58)$$

The Euler-Lagrange equation provides

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = - \frac{d}{dx} \left(\frac{2y'}{2\sqrt{1+y'^2}} \right) = 0, \quad (7.59)$$

or

$$\frac{y'}{\sqrt{1+y'^2}} = C, \quad (7.60)$$

where C is a constant. Thus,

$$y' = \sqrt{\frac{C}{1-C}} = m. \quad (7.61)$$

This provides

$$y = mx + c, \quad (7.62)$$

which is the equation of a straight line.

Again, Equation 7.62 must provide a minimum, because the problem is unbounded for maximization. It is possible to make as long a path as one wishes between two points by increasing the ordinate of the middle point of the path.

7.3.2 ALTERNATE FORM OF EULER-LAGRANGE EQUATION

A somewhat simpler form of the Euler-Lagrange equation is given as

$$\frac{\partial F}{\partial x} - \frac{d}{dx} \left(F - y' \frac{\partial F}{\partial y'} \right) = 0. \quad (7.63)$$

This form is particularly useful when F does not contain x explicitly. This form was discovered in 1868 by Beltrami and is known as the Beltrami identity. The derivation of this form is presented.

Equation 7.63 can be expanded as

$$\frac{\partial F}{\partial x} - \frac{dF}{dx} + \frac{d}{dx} \left(y' \frac{\partial F}{\partial y'} \right) = 0. \quad (7.64)$$

However,

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} + \frac{\partial F}{\partial y'} \frac{dy'}{dx}. \quad (7.65)$$

Substituting Equation 7.65 into Equation 7.64 and simplifying, one gets

$$\frac{d}{dx} \left(y' \frac{\partial F}{\partial y'} \right) = \frac{\partial F}{\partial y} \frac{dy}{dx} + \frac{\partial F}{\partial y'} \frac{dy'}{dx}. \quad (7.66)$$

Further expanding the left-hand-side term, one gets

$$\frac{\partial F}{\partial y'} \frac{dy'}{dx} + y' \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = \frac{\partial F}{\partial y} \frac{dy}{dx} + \frac{\partial F}{\partial y'} \frac{dy'}{dx}, \quad (7.67)$$

or

$$y' \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = \frac{\partial F}{\partial y} y'. \quad (7.68)$$

Dividing Equation 7.68 by y' (assuming that y is not a constant function) and rearranging

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0, \quad (7.69)$$

which is nothing but the Euler-Lagrange equation.

Example 7.4

Find out the solution of the famous brachistochrone problem, that is, find the function y that minimizes time t_{PQ} in Equation 7.5.

SOLUTION

For the sake of convenience, Y is substituted for $(y_1 - y)$ in Equation 7.5. Thus, Equation 7.5 becomes

$$t_{PQ} = \int_{x_1}^{x_2} \frac{\sqrt{1+Y'^2}}{\sqrt{2gY}} dx. \quad (7.70)$$

Here,

$$F = \frac{\sqrt{1+Y'^2}}{\sqrt{2gY}} \quad (7.71)$$

and F does not explicitly depend on x . Hence, from Equation 7.63,

$$\frac{d}{dx} \left(F - Y' \frac{\partial F}{\partial Y'} \right) = 0 \text{ or } \left(F - Y' \frac{\partial F}{\partial Y'} \right) = c, \quad (7.72)$$

where c is a constant. Substituting the value of F from Equation 7.71,

$$\left(\frac{\sqrt{1+Y'^2}}{\sqrt{2gY}} - \frac{Y'^2}{\sqrt{2gY}\sqrt{1+Y'^2}} \right) = c. \quad (7.73)$$

After simplification, one gets

$$(1+Y'^2)Y = \frac{1}{2gc^2} = k^2. \quad (7.74)$$

From Equation 7.74,

$$\frac{dY}{dx} = \sqrt{\frac{k^2 - Y}{Y}} \text{ or } dx = \sqrt{\frac{Y}{k^2 - Y}} dY. \quad (7.75)$$

Integrating both sides,

$$x = \int \sqrt{\frac{Y}{k^2 - Y}} dY. \quad (7.76a)$$

Substituting $Y = k^2 \sin^2 \theta$,

$$x = \int \frac{k \sin \theta}{\sqrt{k^2 - k^2 \sin^2 \theta}} k^2 2 \sin \theta \cos \theta d\theta = k^2 \left(\theta - \frac{\sin(2\theta)}{2} \right). \quad (7.76b)$$

Thus, the parametric equation of the curve providing the least time is given as

$$x = \frac{k^2}{2} (2\theta - \sin 2\theta), Y = \frac{k^2}{2} (1 - \cos 2\theta). \quad (7.77)$$

7.3.3 FUNCTIONAL CONTAINING THE FORM $F = (x, y_1, y_1', y_2, y_2', \dots, y_n, y_n')$

Consider the case in which a functional I is a function of n functions and their derivatives with respect to x . Thus,

$$F \equiv F(x, y_1, y_1', y_2, y_2', \dots, y_n, y_n') \equiv F(x, y_i, y_i', i = 1 \text{ to } n) \quad (7.78)$$

and

$$I = \int_a^b F dx. \quad (7.79)$$

For the extremization of the function, one can follow the same methodology as discussed in Section 7.3.2. Let $y_{i0}, y_{20}, \dots, y_{n0}$ extremize the functional. All n functions can be perturbed and written as

$$y_i = y_{i0} + \varepsilon_i \eta_i(x), i = 1 \text{ to } n, \quad (7.80)$$

where $\eta_i(x)$ are perturbation functions, and ε_i are infinitesimally small real numbers. Substituting Equation 7.80 in Equation 7.79, one gets

$$I = \int_a^b F(x, y_{i0} + \varepsilon_i \eta_i, y'_{i0} + \varepsilon_i \eta'_i, i = 1 \text{ to } n) dx. \quad (7.81)$$

As y_{i0} are not the variable functions but fixed functions at which I becomes extremum and η_i are the arbitrary chosen fixed functions, I in Equation 7.81 is a function of ε_i . The necessary condition for it to be extremum is

$$\frac{\partial I}{\partial \varepsilon_1} = \frac{\partial I}{\partial \varepsilon_2} = \dots = \frac{\partial I}{\partial \varepsilon_n} = 0. \quad (7.82)$$

The Taylor series expansion of the function F in Equation 7.81 is

$$F(x, y_{i0} + \varepsilon_i \eta_i, y'_{i0} + \varepsilon_i \eta'_i \quad i = 1 \text{ to } n) = F(x, y_{i0}, y'_{i0}, i = 1 \text{ to } n) + \left(\frac{\partial F}{\partial y_1} \varepsilon_1 \eta_1 + \frac{\partial F}{\partial y'_1} \varepsilon_1 \eta'_1 \right) + \left(\frac{\partial F}{\partial y_2} \varepsilon_2 \eta_2 + \frac{\partial F}{\partial y'_2} \varepsilon_2 \eta'_2 \right) + \dots + \left(\frac{\partial F}{\partial y_n} \varepsilon_n \eta_n + \frac{\partial F}{\partial y'_n} \varepsilon_n \eta'_n \right) + \text{higher order terms.} \quad (7.83)$$

Note that all the derivatives are evaluated at y_{i0} and their derivatives with respect to x , but they are not written for the sake of brevity. Also, all the higher-order terms contain at least a factor of form $\varepsilon_i \varepsilon_j$. Now, if y_{i0} really extremizes the functions, Equation 7.82 should be satisfied when ε_i is equal to 0 for $i = 1$ to n . This gives us the following n equations:

$$\begin{aligned} \left. \frac{\partial I}{\partial \varepsilon_1} \right|_{\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_n = 0} &= \int_a^b \left(\frac{\partial F}{\partial y_1} \eta_1 + \frac{\partial F}{\partial y'_1} \eta'_1 \right) dx = 0, \\ \left. \frac{\partial I}{\partial \varepsilon_2} \right|_{\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_n = 0} &= \int_a^b \left(\frac{\partial F}{\partial y_2} \eta_2 + \frac{\partial F}{\partial y'_1} \eta'_2 \right) dx = 0, \\ &\dots \dots \dots \\ \left. \frac{\partial I}{\partial \varepsilon_n} \right|_{\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_n = 0} &= \int_a^b \left(\frac{\partial F}{\partial y_n} \eta_n + \frac{\partial F}{\partial y'_n} \eta'_n \right) dx = 0. \end{aligned} \quad (7.84)$$

Any typical equation in Equation 7.84 is similar to Equation 7.38, and its second term can be integrated by part like in Section 7.3.1. The rest of the procedure is similar to that already discussed in Section 7.3.1. Employing that procedure, one gets the following set of n ordinary differential equations:

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'_i} \right) = 0, \quad i = 1 \text{ to } n. \quad (7.85)$$

One also gets the following $2n$ boundary conditions:

$$\frac{\partial F}{\partial y'_i} \eta_i = 0 \text{ at } x = a \text{ and } x = b, \text{ for } i = 1 \text{ to } n. \quad (7.86)$$

This means at each boundary, either the function y_i should be satisfied (an essential boundary condition), or $\frac{\partial F}{\partial y'_i}$ should be equal to 0.

7.3.4 FUNCTIONAL CONTAINING THE FUNCTION OF n INDEPENDENT VARIABLES

Let us consider the functional containing the function and its derivative as in Section 7.2.1 but consider y as a function of n independent variables. The function will be in the form of a definite multiple integral. Thus,

$$I = \int \dots \int_R F \left(y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \dots, \frac{\partial y}{\partial x_n}, x_1, x_2, \dots, x_n \right) dx_1, dx_2, \dots, dx_n, \quad (7.87)$$

where R is the fixed region in an n -dimensional space. As usual, we consider that y_0 extremizes I . A general function in the vicinity of y_0 can be written as

$$y = y_0 + \varepsilon \eta(x_1, x_2, \dots, x_n), \quad (7.88)$$

where ε is a small number, and η is an arbitrary function of n variables satisfying essential (geometric) boundary conditions. For the ease of representation, the convention of attaching a suffix x_i with a function shall be used to indicate the first derivative of the function with respect to x_i . Substituting Equation 7.88 in F and carrying out the Taylor series expansion, one gets

$$\begin{aligned} F(y, y_{x_1}, y_{x_2}, \dots, y_{x_n}, x_1, x_2, \dots, x_n) &= F(y_0, y_{0x_1}, y_{0x_2}, \dots, y_{0x_n}, x_1, x_2, \dots, x_n) \\ &+ \left(\frac{\partial F}{\partial y} \varepsilon \eta + \frac{\partial F}{\partial y_{x_1}} \varepsilon \eta_{x_1} + \dots + \frac{\partial F}{\partial y_{x_n}} \varepsilon \eta_{x_n} \right) \\ &+ \varepsilon^2 (\text{higher order terms}). \end{aligned} \quad (7.89)$$

When Equation 7.89 is substituted in Equation 7.87, I is obtained as a function of ε . The necessary condition that function y_0 actually extremizes the functional I is given by

$$\left. \frac{dI}{d\varepsilon} \right|_{\varepsilon=0} = \int \dots \int_R \left(\frac{\partial F}{\partial y} \eta + \frac{\partial F}{\partial y_{x_1}} \eta_{x_1} + \dots + \frac{\partial F}{\partial y_{x_n}} \eta_{x_n} \right) dx_1, dx_2, \dots, dx_n = 0 \quad (7.90)$$

Integration by part needs to be carried out so that the integrand becomes η times some function. This can be done in the following way. Note that

$$\frac{\partial F}{\partial y_{x_i}} \eta_{x_i} = \frac{\partial}{\partial x_i} \left(\frac{\partial F}{\partial y_{x_i}} \eta \right) - \frac{\partial}{\partial x_i} \left(\frac{\partial F}{\partial y_{x_i}} \right) \eta. \quad (7.91)$$

Using Equation 7.91, Equation 7.90 can be written as

$$\begin{aligned}
& \int \dots \int_R \left\{ \frac{\partial F}{\partial y} - \frac{\partial}{\partial x_1} \left(\frac{\partial F}{\partial y_{x_1}} \right) - \frac{\partial}{\partial x_2} \left(\frac{\partial F}{\partial y_{x_2}} \right) - \dots - \frac{\partial}{\partial x_n} \left(\frac{\partial F}{\partial y_{x_n}} \right) \right\} \eta \, dx_1, dx_2, \dots, dx_n \\
& + \int \dots \int_R \left\{ \frac{\partial}{\partial x_1} \left(\frac{\partial F}{\partial y_{x_1}} \eta \right) + \frac{\partial}{\partial x_2} \left(\frac{\partial F}{\partial y_{x_2}} \eta \right) + \dots + \frac{\partial}{\partial x_n} \left(\frac{\partial F}{\partial y_{x_n}} \eta \right) \right\} dx_1, dx_2, \dots, dx_n = 0.
\end{aligned} \tag{7.92}$$

The second integral can be converted to an $(n-1)$ -dimensional surface integral using the generalized Green's theorem or divergence theorem in an n -dimensional space.

Let the $(n-1)$ -dimensional surface enclosing the region be represented by Γ and the direction cosines of the normal to the surface be represented by $(n_1, n_2, \dots, n_n)$. (Index 1, 2, ..., n can be written as x, y, z , etc.)

In that case, the second term of Equation 7.92 is given by

$$\int \dots \int_R \left\{ \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{\partial F}{\partial y_{x_i}} \eta \right) \right\} dx_1, dx_2, \dots, dx_n = \oint_{\Gamma} \sum_{i=1}^n \left(\frac{\partial F}{\partial y_{x_i}} n_i \right) \eta \, d\Gamma, \tag{7.93}$$

where $d\Gamma$ is the infinitesimal area on the surface Γ . Substituting Equation 7.93 in Equation 7.92 and writing in the compact way, one gets

$$\int \dots \int_R \left\{ \frac{\partial F}{\partial y} - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{\partial F}{\partial y_{x_i}} \right) \right\} \eta \, dx_1, dx_2, \dots, dx_n + \oint_{\Gamma} \sum_{i=1}^n \left(\frac{\partial F}{\partial y_{x_i}} n_i \right) \eta \, d\Gamma = 0. \tag{7.94}$$

Now, one can choose η such that it vanishes on the boundary of the region R , that is, on surface Γ . Thus, in Equation 7.94, the first integral can be made equal to 0 by the specific choice of η . However, as η is still arbitrary except for being zero on Γ , it implies that

$$\frac{\partial F}{\partial y} - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{\partial F}{\partial y_{x_i}} \right) = 0. \tag{7.95}$$

This is the required differential equation. The boundary conditions are either η is zero on the boundary or

$$\oint_{\Gamma} \sum_{i=1}^n \left(\frac{\partial F}{\partial y_{x_i}} n_i \right) d\Gamma = 0. \tag{7.96}$$

Equation 7.96 is the natural boundary condition.

Example 7.5

The variational form of the two-dimensional steady-state heat conduction problem without heat generation is

$$\text{Minimize } I = \frac{1}{2} \int_A \left\{ \left(\frac{\partial T}{\partial x} \right)^2 + \left(\frac{\partial T}{\partial y} \right)^2 \right\} dx dy. \quad (7.97)$$

Find out the corresponding differential equation and form of boundary conditions.

SOLUTION

Here,

$$F = \frac{1}{2} (T_x^2 + T_y^2). \quad (7.98)$$

Using Equation 7.95,

$$\frac{\partial F}{\partial T} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial T_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial T_y} \right) = 0 - \frac{\partial^2 T}{\partial x^2} - \frac{\partial^2 T}{\partial y^2} = 0. \quad (7.99)$$

Hence, the differential equation is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0. \quad (7.100)$$

Boundary conditions should be either in the form of prescribed temperature or in the form

$$\frac{\partial T}{\partial x} n_1 + \frac{\partial T}{\partial y} n_2 = 0. \quad (7.101)$$

It is usual to write n_1 as n_x and n_2 as n_y .

7.3.5 FUNCTIONAL DEPENDENT ON THE FUNCTIONS AND ITS DERIVATIVES UP TO ORDER N

Consider the functional of the following form:

$$I = \int_a^b F(x, y, y', y'') dx. \quad (7.102)$$

This expression is dependent on the function y and its first- and second-order derivatives. Making use of Equation 7.33 that expresses y in terms of supposedly extremum y_0 and a perturbed portion and carrying out the Taylor series expansion, F can be written as

$$F(x, y, y', y'') = F(x, y_0, y'_0, y''_0) + \left(\frac{\partial F}{\partial y} \varepsilon \eta + \frac{\partial F}{\partial y'} \varepsilon \eta' + \frac{\partial F}{\partial y''} \varepsilon \eta'' \right) + \varepsilon^2 (\text{higher order terms}). \quad (7.103)$$

Hence,

$$\frac{dI}{d\varepsilon} = \int_a^b \frac{dF}{d\varepsilon} dx = \int_a^b \left\{ \left(\frac{\partial F}{\partial y} \eta + \frac{\partial F}{\partial y'} \eta' + \frac{\partial F}{\partial y''} \eta'' \right) + 2\varepsilon (\text{higher order terms}) \right\} dx. \quad (7.104)$$

It is emphasized that the derivatives in Equation 7.104 have been taken at y_0 and its first and second derivatives. The condition for the extremum is

$$\left. \frac{dI}{d\varepsilon} \right|_{\varepsilon=0} = \int_a^b \left(\frac{\partial F}{\partial y} \eta + \frac{\partial F}{\partial y'} \eta' + \frac{\partial F}{\partial y''} \eta'' \right) dx = 0. \quad (7.105)$$

Now, the integration by parts is carried out to have only η and a function multiplied to it in the integral. Note that the second term of the integrand is integrated by parts once, whereas the third term is integrated by parts twice. As a result, the following expression is obtained:

$$\int_a^b \left\{ \frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial y''} \right) \right\} \eta dx + \left\{ \frac{\partial F}{\partial y'} - \frac{d}{dx} \left(\frac{\partial F}{\partial y''} \right) \right\} \eta \Big|_a^b + \left(\frac{\partial F}{\partial y''} \right) \eta' \Big|_a^b = 0. \quad (7.106)$$

Choosing η in a manner such that the function η and its first derivative vanish at the boundary, one gets

$$\int_a^b \left\{ \frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial y''} \right) \right\} \eta dx = 0, \quad (7.107)$$

where η is arbitrary. Hence, the required differential equation is

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial y''} \right) = 0. \quad (7.108)$$

Its solution will provide function y_0 . Now, as on the boundary η and η' are independent, at $x = a$ and $x = b$, the following conditions should be satisfied:

$$\left\{ \frac{\partial F}{\partial y'} - \frac{d}{dx} \left(\frac{\partial F}{\partial y''} \right) \right\} \eta = 0 \quad \text{and} \quad \left(\frac{\partial F}{\partial y''} \right) \eta' = 0. \quad (7.109)$$

Thus, at each end, there are two sets of boundary conditions:

1.

$$\left\{ \frac{\partial F}{\partial y'} - \frac{d}{dx} \left(\frac{\partial F}{\partial y''} \right) \right\} = 0 \quad \text{or } y \text{ prescribed.} \quad (7.110a)$$

2.

$$\left(\frac{\partial F}{\partial y''} \right) = 0 \quad \text{or } y' \text{ prescribed.} \quad (7.110b)$$

Prescription of y and y' is called the essential or geometric boundary condition, and the other boundary conditions are called the natural boundary conditions. Note that for the fourth-order differential equation, one needs a minimum of two essential boundary conditions, out of which at least one should be the prescribed y .

Generalizing this approach, it can be seen that if F contains the function variable y and its derivative up to the n th order, the Euler differential equation is given by

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial y''} \right) - \dots + (-1)^n \frac{d^n}{dx^n} \left(\frac{\partial F}{\partial y^{(n)}} \right) = 0, \quad (7.111)$$

where $y^{(n)}$ indicates the n th derivative of y . There will be $2n$ boundary conditions (n essential and n natural similar to Equation 7.110).

Example 7.6

The total potential energy of a simply supported beam of length l is given by

$$\Pi = \int_0^l \frac{1}{2} \left\{ EI \left(\frac{d^2 w}{dx^2} \right)^2 - qw \right\} dx, \quad (7.112)$$

where w is the transverse deflection, and EI is called the flexural rigidity of the beam. Find the governing differential equation and boundary conditions of the problem.

SOLUTION

For finding the governing differential equation, the total potential energy has to be minimized. Here,

$$F = \frac{1}{2}EIw'' - qw. \quad (7.113)$$

From Equation 7.108,

$$-q + \frac{d^2}{dx^2}(EIw'') = 0 \quad \text{or} \quad \frac{d^2}{dx^2}\left(EI \frac{d^2w}{dx^2}\right) = q. \quad (7.114)$$

For a simply supported beam, deflection at both ends is 0. Hence, $w = 0$ at $x = 0$ and $x = l$. The slope need not be zero. Hence, a natural boundary condition has to be used from Equation 7.110b, that is, $EIw'' = 0$ at both ends. In other words, the bending moment at both ends is zero, which is indeed the case.

7.4 SOLUTION OF EXTREMIZATION PROBLEMS USING δ OPERATOR

In Section 7.3, Euler's equations for different cases were derived. In the following, the concept of a variational operator is introduced. All the equations derived in Section 7.3 can be obtained by making the first variation of the functional as zero. It is strongly suggested that students should use a variational operator and solve problems by this technique rather than memorizing Euler's equations.

7.4.1 VARIATIONAL OPERATOR

In Section 7.3 the methodology of perturbing the function from the assumed extremum function $y_0(x)$ is followed. The perturbed function is $y_0(x) + \epsilon\eta(x)$. The perturbation $\epsilon\eta(x)$ is called variation δy , provided that ϵ is infinitesimally small. The operator δ is called the variational operator. In many ways, the variational operator δ is similar to the differential operator d and has similar mathematical properties. However, both are conceptually different. The differential of a function, dy , is a first-order approximation to the change in function along a particular curve. For example, let a function be $y(x)$. The rate of change of a function at a point is given by dy/dx . For an infinitesimal change in x , the change in y will be $(dy/dx)dx$, that is, dy . Here, d is a differential operator. Note that any numerical value cannot be assigned to dy . All that can be said is that it is an infinitesimal change along the curve $y(x)$. However, for a smooth curve, it can be said that, at maxima or minima, dy has to be zero. In contrast to the differential, the variation δy is a first-order approximation to the change from curve to curve. Thus, $\delta y = \epsilon\eta(x)$ is basically a function with arbitrary $\eta(x)$ and infinitesimally small ϵ . By nature, dy is a real number, and δy is a function of x . If F is a function of (x, y, z) , then dF is a real number (although it is infinitesimally small) while δF is a function of (x, y, z) .

The variation of a functional I can also be defined. It is the variation in I when a very small perturbation is provided to the function. By nature, δI is an infinitesimal

real number, because I is a real number for known values of independent functions. Consider the following functional:

$$I = \int_a^b F\left(x, y, \frac{dy}{dx}\right) dx. \quad (7.115)$$

Suppose that this functional is evaluated at y_0 and then at $y_0 + \varepsilon \eta(x)$, it can be appreciated that the infinitesimal change in I will be $(dI/d\varepsilon)\varepsilon$. It can be denoted by δI calling it as the first variation of I . It was discussed in Section 7.3 that if $y_0(x)$ extremizes the functional I , the following necessary condition must be satisfied:

$$\left. \frac{dI}{d\varepsilon} \right|_{\varepsilon=0} = 0. \quad (7.116)$$

Multiplying it by ε , the following can be written:

$$\left. \frac{dI}{d\varepsilon} \right|_{\varepsilon=0} \varepsilon = \delta I = 0, \quad \text{at } y = y_0. \quad (7.117)$$

This is consistent with the reasoning that at the extremum of a functional, δI should be zero, just as dy is zero for the extremum of a function. The imposition of $\delta I = 0$ at $y = y_0$ gives us a differential equation with y_0 as a dependent variable. However as before, subscript '0' is omitted, and the differential equation is obtained in y itself. Thus, for getting the differential equation, just $\delta I = 0$ shall be imposed for getting a differential equation in y (or differential equations in $y_1, y_2, \dots, y_n$ if I is a function of $y_1, y_2, \dots, y_n$).

7.4.2 PROPERTIES OF VARIATIONAL OPERATOR

It is not difficult to see that the variational operator behaves in a similar way as a differential operator. If u and v are two functions and α and β are known constant functions, then

$$\delta(\alpha u \pm \beta v) = \alpha \delta u \pm \beta \delta v, \quad (7.118)$$

$$\delta(uv) = v \delta u + u \delta v, \quad (7.119)$$

$$\delta\left(\frac{u}{v}\right) = \frac{v \delta u - u \delta v}{v^2}. \quad (7.120)$$

In addition, differentiation and variation jointly have the following properties: Property I: Differentiation and variation commute, that is,

$$\frac{d}{dx}(\delta y) = \delta\left(\frac{dy}{dx}\right). \quad (7.121)$$

Proof: If the perturbed function is $y + \varepsilon\eta$, the corresponding perturbed first-derivative function is $y' + \varepsilon\eta'$. Thus,

$$\delta\left(\frac{dy}{dx}\right) \equiv \delta y' = \varepsilon\eta'. \quad (7.122)$$

Now,

$$\frac{d}{dx}(\delta y) = \frac{d}{dx}(\varepsilon\eta) = \varepsilon \frac{d\eta}{dx} \equiv \varepsilon\eta' \quad (7.123)$$

Comparing Equations 7.122 and 7.123, one gets Equation 7.121.

Property II: Integration and variation commute, that is,

$$\delta \int_a^b y dx = \int_a^b \delta y dx. \quad (7.124)$$

Proof: By definition,

$$\begin{aligned} \delta \int_a^b y dx &= \int_a^b (y + \varepsilon\eta) dx - \int_a^b y dx \\ &= \int_a^b y dx + \int_a^b \varepsilon\eta dx - \int_a^b y dx. \\ &= \int_a^b \varepsilon\eta dx = \int_a^b \delta y dx \end{aligned} \quad (7.125)$$

and hence the proof.

The chain rule for differentiation is also applicable for variation. For example, consider an expression $F(x, y, y', y'')$, where y is a function of x . For an infinitesimal ε ,

$$\delta F(x, y, y', y'') = F(x, y + \varepsilon\eta, y' + \varepsilon\eta', y'' + \varepsilon\eta'') - F(x, y, y', y''). \quad (7.126)$$

By Taylor's series expansion of the first term on the right-hand side, one gets

$$\delta F = \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' + \frac{\partial F}{\partial y''} \delta y'' + (\varepsilon^2). \quad (7.127)$$

As a first-order approximation, $O(\varepsilon^2)$ is zero. Hence,

$$\delta F = \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' + \frac{\partial F}{\partial y''} \delta y''. \quad (7.128)$$

Similarly, if F is a function of $x, y, y', y'', \dots, y_{(n)}$, then

$$\delta F = \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' + \frac{\partial F}{\partial y''} \delta y'' + \dots + \frac{\partial F}{\partial y^{(n)}} \delta y^{(n)}. \quad (7.129)$$

Example 7.7

Given

$$F = x^2 \left(\frac{d^2 y}{dx^2} \right)^3 + 5y \left(\frac{dy}{dx} \right)^2 + 10y^2, \quad (7.130)$$

find out δF in terms of variations in y and its derivatives.

SOLUTION

It is easy to write the expression for δF , considering that the variational operator behaves like the differential operator. However, coefficient x^2 of the first term has to be treated as constant as there is no variation of x^2 . Thus, making use of Equations 7.118 and 7.119, the following can be written:

$$\delta F = 3x^2 \left(\frac{d^2 y}{dx^2} \right)^2 \delta \left(\frac{d^2 y}{dx^2} \right) + 10y \left(\frac{dy}{dx} \right) \delta \left(\frac{dy}{dx} \right) + 5 \left(\frac{dy}{dx} \right)^2 \delta y + 20y \delta y. \quad (7.131)$$

7.4.3 CONVERTING VARIATIONAL FORM TO DIFFERENTIAL EQUATION

For converting a variational form I into corresponding differential equations, the following steps can be followed:

- Step 1: Impose the condition $\delta I = 0$.
- Step 2: Take the variational sign inside the integral and, using the properties of the variational operator, convert I into the form of summation of terms containing $\delta y, \delta y', \delta y'',$ etc., inside the integral.
- Step 3: Integrate $\delta y', \delta y'',$ etc., by parts to have all the terms inside the integral with δy alone. Due to this, some boundary terms will be generated.
- Step 4: Now, the integrand associated with δy is made equal to 0 to get the differential equation. The boundary terms associated with $\delta y, \delta y', \delta y'',$ etc., are individually made zero to get the set of boundary conditions.

For example, consider the variational form given by Equation 7.32. We write

$$\delta I = \int_a^b \delta F(x, y, y') dx = \int_a^b \left(\frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right) dx = 0 \quad (7.132)$$

Integrating the second term of the integrand by parts, one gets

$$\int_a^b \left(\frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right) dx = \int_a^b \left(\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \right) \delta y dx + \frac{\partial F}{\partial y'} \delta y \Big|_a^b. \quad (7.133)$$

As δy is arbitrary,

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0, \quad (7.134)$$

which is the required differential equation. At $x = a$ and $x = b$,

$$\frac{\partial F}{\partial y'} \delta y = 0, \quad (7.135)$$

implying that at both the boundaries, either y is prescribed or $\partial F / \partial y' = 0$.

Example 7.8

A cantilever beam of flexural rigidity EI is supported transversely on a spring of stiffness k at the other end. An axial compressive load P is applied at the spring-supported end. The total potential energy of the beam is given by

$$\Pi = \frac{1}{2} \int_0^L EI \left(\frac{d^2 w}{dx^2} \right)^2 dx - \frac{P}{2} \int_0^L \left(\frac{dw}{dx} \right)^2 dx + \frac{1}{2} k w_L^2, \quad (7.136)$$

where w_L is the value of transverse deflection w at $x = L$. Find the governing differential equation and boundary conditions of the problem.

SOLUTION

At equilibrium, the total potential energy is minimized. Therefore, the first variation of $\Pi = 0$. Thus,

$$\delta \Pi = \frac{1}{2} \int_0^L EI \delta \left(\frac{d^2 w}{dx^2} \right)^2 dx - \frac{P}{2} \int_0^L \delta \left(\frac{dw}{dx} \right)^2 dx + \frac{1}{2} k \delta w_L^2 = 0. \quad (7.137)$$

Applying the properties of the variational operator, one gets

$$\int_0^L EI \left(\frac{d^2 w}{dx^2} \right) \delta \left(\frac{d^2 w}{dx^2} \right) dx - P \int_0^L \left(\frac{dw}{dx} \right) \delta \left(\frac{dw}{dx} \right) dx + k w_L \delta w_L = 0.$$

Integrating by parts once,

$$\begin{aligned}
 & EI \left(\frac{d^2 w}{dx^2} \right) \delta \left(\frac{dw}{dx} \right) \Big|_0^L - \int_0^L \frac{d}{dx} \left\{ EI \left(\frac{d^2 w}{dx^2} \right) \right\} \delta \left(\frac{dw}{dx} \right) dx - P \left(\frac{dw}{dx} \right) \delta w \Big|_0^L \\
 & + P \int_0^L \left(\frac{d^2 w}{dx^2} \right) \delta w dx + k w_L \delta w_L = 0.
 \end{aligned} \tag{7.138}$$

Integrating the second term of Equation 7.138 again by parts,

$$\begin{aligned}
 & EI \left(\frac{d^2 w}{dx^2} \right) \delta \left(\frac{dw}{dx} \right) \Big|_0^L - \frac{d}{dx} \left\{ EI \left(\frac{d^2 w}{dx^2} \right) \right\} \delta w \Big|_0^L + \int_0^L \frac{d^2}{dx^2} \left\{ EI \left(\frac{d^2 w}{dx^2} \right) \right\} \delta w dx \\
 & - P \left(\frac{dw}{dx} \right) \delta w \Big|_0^L + P \int_0^L \left(\frac{d^2 w}{dx^2} \right) \delta w dx + k w_L \delta w_L = 0.
 \end{aligned} \tag{7.139}$$

As δw is arbitrary, it can be chosen in such a manner as to make boundary terms zero. With this, one gets

$$\int_0^L \left[\frac{d^2}{dx^2} \left\{ EI \left(\frac{d^2 w}{dx^2} \right) \right\} + P \left(\frac{d^2 w}{dx^2} \right) \right] \delta w dx = 0. \tag{7.140}$$

The previous equation implies that

$$\frac{d^2}{dx^2} \left\{ EI \left(\frac{d^2 w}{dx^2} \right) \right\} + P \left(\frac{d^2 w}{dx^2} \right) = 0, \tag{7.141}$$

which is the required differential equation. For the boundary condition at $x = 0$, collecting all terms corresponding to $x = 0$ in Equation 7.139:

$$EI \left(\frac{d^2 w}{dx^2} \right) \delta \left(\frac{dw}{dx} \right) \Big|_0 - \left[\frac{d}{dx} \left\{ EI \left(\frac{d^2 w}{dx^2} \right) \right\} + P \left(\frac{dw}{dx} \right) \right] \delta w \Big|_0 = 0. \tag{7.142}$$

As $\delta(dw/dx)$ and δw are arbitrary independent quantities, at $x = 0$,

$$\begin{aligned}
 & EI \left(\frac{d^2 w}{dx^2} \right) \delta \left(\frac{dw}{dx} \right) \Big|_0 = 0, \\
 & \left[\frac{d}{dx} \left\{ EI \left(\frac{d^2 w}{dx^2} \right) \right\} + P \left(\frac{dw}{dx} \right) \right] \delta w \Big|_0 = 0.
 \end{aligned} \tag{7.143}$$

Thus, at $x = 0$, either slope and deflection should be prescribed or the terms associated with them should be zero. In this case, as the cantilever beam is fixed, slopes and deflections are zero.

In a similar manner, collecting the terms corresponding to $x = L$, one gets

$$\begin{aligned} EI \left(\frac{d^2 w}{dx^2} \right) \delta \left(\frac{dw}{dx} \right) \Big|_L &= 0, \\ \left[\frac{d}{dx} \left\{ EI \left(\frac{d^2 w}{dx^2} \right) \right\} + P \left(\frac{dw}{dx} \right) - kw \right] \delta w \Big|_L &= 0. \end{aligned} \quad (7.144)$$

As the slope and deflections are not prescribed at this end, we have the following boundary conditions:

$$EI \left(\frac{d^2 w}{dx^2} \right) \Big|_L = 0, - \left[\frac{d}{dx} \left\{ EI \left(\frac{d^2 w}{dx^2} \right) \right\} + P \left(\frac{dw}{dx} \right) \right] \Big|_L = -kw_L \quad (7.145)$$

The first boundary condition states that the bending moment is zero, whereas the second states that the shear force is equal to $-kw_L$.

7.5 OBTAINING VARIATIONAL FORM FROM A DIFFERENTIAL EQUATION

Consider a differential equation of the form

$$A(\phi) - f = 0, \quad (7.146)$$

where A is the differential operator. With this differential equation, appropriate boundary conditions should also be provided. To convert it into a variation form, the reverse process of the process used in Section 7.4 is followed. Thus, first, we write

$$\int_D \{A(\phi) - f\} \delta \phi \, dD = 0, \quad (7.147)$$

where D is the domain of the differential equation. Afterward, integration by parts is applied to get the expression in the following form:

$$\delta \left[\int_D \{A^*(\phi) - f\phi\} \, dD + \text{boundary terms} \right] = 0. \quad (7.148)$$

Note that boundary terms can also be written inside the integral by using the Dirac delta function. From Equation 7.148, it is concluded that the variational form is given by

$$I(\phi) = \int_D \{A^*(\phi) - f\phi\} dD + \text{boundary terms.} \quad (7.149)$$

In the process, orders of derivatives in the expression get reduced. For two or more coupled equations a similar inverse procedure can be employed. As an example, consider two coupled differential equations in dependent variables ϕ_1 and ϕ_2 :

$$A_1(\phi_1, \phi_2) = 0, \quad A_2(\phi_1, \phi_2) = 0. \quad (7.150)$$

To convert it to a variational form, the following can be written:

$$\int_D [\{A_1(\phi_1, \phi_2)\} \delta\phi_1 + \{A_2(\phi_1, \phi_2)\} \delta\phi_2] dD = 0. \quad (7.151)$$

Now, integration by parts is carried out to get a form of $\delta I(\phi_1, \phi_2) = 0$.

Example 7.9

Consider the following differential equation for an axially loaded rod of Young's modulus of elasticity E , cross-sectional area A , and length L :

$$\frac{d}{dx} \left(EA \frac{du}{dx} \right) + q = 0, \quad (7.152)$$

where q is the load per unit length. One end of the rod is fixed and the other end is supported to the wall by a spring of stiffness k (Figure 7.2). Hence, the boundary conditions are

$$\begin{aligned} \text{at } x = 0, u &= 0, \\ \text{at } x = l, EA \frac{du}{dx} &= -ku. \end{aligned} \quad (7.153)$$

Obtain the variational form of this problem.

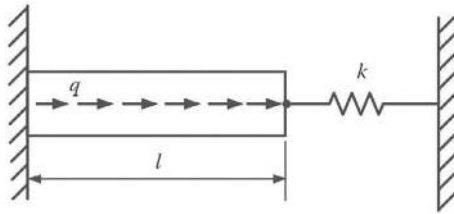


FIGURE 7.2 A rod fixed at one end and supported by spring on the other end.

SOLUTION

Let

$$\delta I = \int_0^l \left\{ \frac{d}{dx} \left(EA \frac{du}{dx} \right) + q \right\} \delta u \, dx = 0 \quad (7.154)$$

Integrating Equation 7.154 by parts, one gets

$$\left(EA \frac{du}{dx} \delta u \right) \Big|_0^l - \int_0^l EA \frac{du}{dx} \frac{d}{dx} (\delta u) \, dx + \int_0^l q \delta u \, dx = 0. \quad (7.155)$$

Putting the boundary conditions, $\delta u = 0$ at $x = 0$ and $EA(du/dx) = -ku_l$ at $x = l$ and using the properties of the variational operator, Equation 7.155 is written as

$$\int_0^l \delta \left[-\frac{1}{2} EA \left(\frac{du}{dx} \right)^2 \right] dx + \int_0^l \delta (qu) \, dx - ku_l \delta u_l = 0. \quad (7.156)$$

For convenience, the expression in Equation 7.156 is multiplied by (-1) , the variational operator is taken outside the integral sign, and $u_l \delta u_l$ is written as $\delta(u_l^2/2)$ to get

$$\delta \left\{ \int_0^l \frac{1}{2} EA \left(\frac{du}{dx} \right)^2 dx + \frac{1}{2} ku_l^2 - \int_0^l qu \, dx \right\} = 0. \quad (7.157)$$

Hence,

$$I = \int_0^l \frac{1}{2} EA \left(\frac{du}{dx} \right)^2 dx + \frac{1}{2} ku_l^2 - \int_0^l qu \, dx. \quad (7.158)$$

In the previous expression, the first two terms represent the strain energy, and the last term represents the work potential.

Example 7.10

The two-dimensional heat conduction equation in an isotropic medium of thermal conductivity k with per unit volume heat generation rate as $\dot{Q}(x, y)$ is given by

$$k \frac{\partial^2 T}{\partial x^2} + k \frac{\partial^2 T}{\partial y^2} + \dot{Q}(x, y) = 0. \quad (7.159)$$

Assume that at boundary Γ_T , the temperature is prescribed, and at the remaining boundary Γ_q , the normal heat flux is q^* . Obtain the variational form of the problem.

SOLUTION

Let

$$\delta I = \int_A \left(k \frac{\partial^2 T}{\partial x^2} + k \frac{\partial^2 T}{\partial y^2} + \dot{Q}(x, y) \right) \delta T \, dx \, dy = 0. \quad (7.160)$$

Here, the divergence theorem can be applied for a two-dimensional region to reduce the order of derivatives, which in a way is integration by parts. Thus,

$$\begin{aligned} \int_A \left(k \frac{\partial^2 T}{\partial x^2} + k \frac{\partial^2 T}{\partial y^2} \right) \delta T \, dx \, dy &= \int_A k \nabla^2 T \delta T \, dA \\ &= \int_A k \nabla \cdot (\nabla T \delta T) \, dA - \int_A k \nabla T \cdot (\nabla \delta T) \, dA \\ &= \int_{\Gamma} k (\nabla T \cdot \mathbf{n}) \delta T \, d\Gamma - \int_A k \left(\frac{\partial T}{\partial x} \frac{\partial \delta T}{\partial x} + \frac{\partial T}{\partial y} \frac{\partial \delta T}{\partial y} \right) dA \end{aligned} \quad (7.161)$$

Fourier's law of conduction for homogeneous and isotropic material provides

$$\mathbf{q} = -k \nabla T, \quad (7.162)$$

where $\mathbf{q}$ is the heat flux vector. The normal heat flux is given by

$$q_n = \mathbf{q} \cdot \mathbf{n} = -k \nabla T \cdot \mathbf{n}. \quad (7.163)$$

Substituting Equation 7.163 in Equation 7.161,

$$\begin{aligned} \int_A \left(k \frac{\partial^2 T}{\partial x^2} + k \frac{\partial^2 T}{\partial y^2} \right) \delta T \, dx \, dy &= - \int_{\Gamma} k q_n \delta T \, d\Gamma - \int_A k \left(\frac{\partial T}{\partial x} \frac{\partial \delta T}{\partial x} + \frac{\partial T}{\partial y} \frac{\partial \delta T}{\partial y} \right) dA \\ &= - \int_{\Gamma_T} k q_n \delta T \, d\Gamma - \int_{\Gamma_q} k q_n \delta T \, d\Gamma - \int_A k \left(\frac{\partial T}{\partial x} \frac{\partial \delta T}{\partial x} + \frac{\partial T}{\partial y} \frac{\partial \delta T}{\partial y} \right) dA. \end{aligned} \quad (7.164)$$

In view of the prescribed boundary conditions, the first term in Equation 7.164 becomes zero, and in the second term, q_n is written as q^* . Thus,

$$\begin{aligned}
& \int_A \left(k \frac{\partial^2 T}{\partial x^2} + k \frac{\partial^2 T}{\partial y^2} \right) \delta T \, dx \, dy \\
&= - \int_{\Gamma_q} k q^* \delta T \, d\Gamma - \int_A k \left(\frac{\partial T}{\partial x} \frac{\partial \delta T}{\partial x} + \frac{\partial T}{\partial y} \frac{\partial \delta T}{\partial y} \right) dA.
\end{aligned} \tag{7.165}$$

Now, using the properties of the variational operator,

$$\begin{aligned}
& \int_A \left(k \frac{\partial^2 T}{\partial x^2} + k \frac{\partial^2 T}{\partial y^2} \right) \delta T \, dx \, dy \\
&= -\delta \int_{\Gamma_q} k q^* T \, d\Gamma - \delta \int_A \frac{k}{2} \left\{ \left(\frac{\partial T}{\partial x} \right)^2 + \left(\frac{\partial T}{\partial y} \right)^2 \right\} dA.
\end{aligned} \tag{7.166}$$

Therefore, Equation 7.160 can be written as

$$\delta I = -\delta \int_{\Gamma_q} k q^* T \, d\Gamma - \delta \int_A \frac{k}{2} \left\{ \left(\frac{\partial T}{\partial x} \right)^2 + \left(\frac{\partial T}{\partial y} \right)^2 \right\} dA + \int_A \dot{Q}(x, y) \delta T \, dA = 0. \tag{7.167}$$

or

$$\delta I = -\delta \left[\int_{\Gamma_q} k q^* T \, d\Gamma + \int_A \frac{k}{2} \left\{ \left(\frac{\partial T}{\partial x} \right)^2 + \left(\frac{\partial T}{\partial y} \right)^2 \right\} dA - \int_A \dot{Q}(x, y) T \, dA \right] = 0. \tag{7.168}$$

Hence,

$$I = \int_A \frac{k}{2} \left\{ \left(\frac{\partial T}{\partial x} \right)^2 + \left(\frac{\partial T}{\partial y} \right)^2 \right\} dA - \int_A \dot{Q}(x, y) T \, dA + \int_{\Gamma_q} k q^* T \, d\Gamma. \tag{7.169}$$

Note that the entire expression can also be multiplied by -1 .

Example 7.11

Convert the stress equilibrium equations into a variational form. Assume that the material remains in the elastic zone.

SOLUTION

The stress equilibrium equations are given by

$$\sigma_{ij,j} + \rho b_i = 0. \tag{7.170}$$

where index i varies from one to three for a three-dimensional case. Thus, Equation 7.170 represents three scalar equations corresponding to force balance in the x , y , and z directions. To convert these equations to a variational form, the equations are multiplied by δu_x , δu_y , and δu_z , respectively, they are added, and the resulting expression is integrated over the domain. Thus, using the index notations,

$$\delta I = \int_V (\sigma_{ij,j} + \rho b_i) \delta u_i dV = 0. \quad (7.171)$$

Now, the first term can be integrated by parts to reduce the order of derivative on stress tensor σ . In this process, the order of derivative on δu_i will increase, but the overall order will be one instead of two. Concentrating on the integration of the first term,

$$\begin{aligned} \int_V \sigma_{ij,j} \delta u_i dV &= \int_V (\sigma_{ij} \delta u_i)_{,j} dV - \int_V \sigma_{ij} \delta u_{i,j} dV \\ &= \int_S \sigma_{ij} \delta u_i n_j dS - \int_V \sigma_{ij} \delta u_{i,j} dV \quad (\text{using divergence theorem}). \end{aligned} \quad (7.172)$$

In the previous expression, S is the surface enclosing the volume. The second integrand in Equation 7.172 can be simplified as follows:

$$\sigma_{ij} u_{i,j} = \frac{\sigma_{ij}}{2} (u_{i,j} + u_{j,i}) + \frac{\sigma_{ij}}{2} (u_{i,j} - u_{j,i}) = \sigma_{ij} \varepsilon_{ij} + \sigma_{ij} \omega_{ij}. \quad (7.173)$$

In Equation 7.173, the second term is the scalar product of a symmetric and skew-symmetric tensor; hence, it is zero. Therefore, Equation 7.172 can be written as

$$\int_V \sigma_{ij,j} \delta u_i dV = \int_S \sigma_{ij} \delta u_i n_j dS - \int_V \sigma_{ij} \delta \varepsilon_{ij} dV \quad (7.174)$$

Using Cauchy's law ($\sigma_{ij} n_j = t_i$), Equation 7.174 is written as

$$\int_V \sigma_{ij,j} \delta u_i dV = \int_S t_i \delta u_i dS - \int_V \sigma_{ij} \delta \varepsilon_{ij} dV. \quad (7.175)$$

For the elastic range, the following can be written:

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl}. \quad (7.176)$$

where C_{ijkl} are the components of a fourth-order tensor $\mathbf{C}$, which is called the elasticity tensor. Hence, Equation 7.174 is written as

$$\int_V \sigma_{ij,j} \delta u_i dV = \int_S t_i \delta u_i dS - \int_V C_{ijkl} \epsilon_{kl} \delta \epsilon_{ij} dV. \quad (7.177)$$

Using Equation 7.177, Equation 7.171 can be written as

$$\int_S t_i \delta u_i dS - \int_V C_{ijkl} \epsilon_{kl} \delta \epsilon_{ij} dV + \int_V \rho b_i \delta u_i dV = 0. \quad (7.178)$$

Based on Equation 7.119, it can be easily verified that

$$C_{ijkl} \epsilon_{kl} \delta \epsilon_{ij} = \delta \left(\frac{1}{2} \epsilon_{ij} C_{ijkl} \epsilon_{kl} \right). \quad (7.179)$$

Thus, after multiplication with (-1) , the expression in Equation 7.178 becomes

$$\begin{aligned} \delta I &= - \int_S t_i \delta u_i dS + \int_V \delta \left(\frac{1}{2} \epsilon_{ij} C_{ijkl} \epsilon_{kl} \right) dV - \int_V \rho b_i \delta u_i dV \\ &= \delta \left[- \int_S t_i \delta u_i dS + \int_V \left(\frac{1}{2} \epsilon_{ij} C_{ijkl} \epsilon_{kl} \right) dV - \int_V \rho b_i \delta u_i dV \right] = 0. \end{aligned} \quad (7.180)$$

Hence,

$$I = \int_V \left(\frac{1}{2} \epsilon_{ij} C_{ijkl} \epsilon_{kl} \right) dV - \int_S t_i \delta u_i dS - \int_V \rho b_i \delta u_i dV. \quad (7.181)$$

Note that in writing the variational form of Equation 7.181, the first term has been made representing the strain energy as positive.

7.6 PRINCIPLE OF VIRTUAL WORK

The principle of the virtual work for a particle states that if a particle is in equilibrium, the total work of all forces acting on the particle for any virtual displacement vanishes. Virtual displacements are small displacements compatible with boundary conditions. They are considered so small that there is no change in the forces acting on the particle. Denoting the resultant force by $\Sigma \mathbf{F}$ and the virtual displacement vector by $\delta \mathbf{u}$, the work done on the particle is written as

$$\delta W = \Sigma \mathbf{F} \cdot \delta \mathbf{u} = 0. \quad (7.182)$$

The virtual work is zero because the resultant force is zero and the dot product of any vector with a zero vector is zero. It is, thus, clear that $\delta \mathbf{u}$ need not be displacement; it

can be any arbitrary vector. However, it is customary to treat $\delta \mathbf{u}$ as a virtual (not real) displacement. Equation 7.182 also suggests the sufficiency condition of the equilibrium of a particle. If $\delta \mathbf{u}$ is arbitrary, the only possibility is to have $\Sigma \mathbf{F} = 0$ for identity to be satisfied.

The principle of virtual work can easily be extended to a rigid body as it is composed of particles. For a rigid body in equilibrium, the total work of all external forces and external moments acting on the body in a virtual displacement field vanishes. The virtual displacement functions in the x , y and z directions can be denoted by δu_x , δu_y and δu_z , respectively. These can be considered as the variations of displacement functions.

For a non-rigid body, the principle of the virtual work can be written as follows:

‘When we subject a loaded body in equilibrium to small compatible virtual displacements without violating the essential boundary conditions, the total internal virtual work is equal to the total external virtual work’. (Note that sign convention for work may differ from book to book, and in some books, the total internal virtual work is taken equal to the negative of the total external virtual work.)

Consider Equation 7.178 of Example 7.11. This equation can be written as

$$\int_V \sigma_{ij} \delta \varepsilon_{ij} dV = \int_S t_i \delta u_i dS + \int_V \rho b_i \delta u_i dV. \quad (7.183)$$

Here, the right-hand side expression is the internal virtual work, which is stored as the virtual strain energy for a conservative system, and the left-hand side represents the external virtual work.

The virtual work can also be expressed in rate form. Suppose that v_i is any continuous velocity field consistent with the boundary conditions, and rate of deformation is computed based on this velocity field. Then, the principle of the virtual work can be expressed as

$$\int_V \sigma_{ij} \dot{\varepsilon}_{ij} dV = \int_S t_i v_i dS + \int_V \rho b_i v_i dV. \quad (7.184)$$

7.7 PRINCIPLE OF MINIMUM POTENTIAL ENERGY

For a conservative system, there is a popular principle called the principle of minimum potential energy. It can be expressed as follows:

Among all possible configurations of a conservative system satisfying internal compatibility and essential boundary conditions, those that keep the body in stable equilibrium make the potential energy minimum with respect to small admissible variations of displacement. The admissible variations are those that preserve the geometric constraints of the system.

The principle of minimum potential energy is applicable even if the material behavior is non-linear. This principle can be derived from the virtual work principle. From

the principle of virtual work, the total external virtual work is equal to the total internal virtual work, that is,

$$\delta W_e = \delta W_i. \quad (7.185)$$

Now, for a conservative system, the total internal virtual work is stored as the virtual strain energy δU . Hence,

$$\delta(U - W_e) = \delta \Pi = 0, \quad (7.186)$$

where Π is called the total potential energy. Note that $(-W_e)$ is called the work potential due to the external load. It is different from the actual work done by the system. For example, if a spring of stiffness k is pulled by a force F to deflect it by an amount x the work done will be $Fx/2$, assuming that the force F increases from 0 to F in a quasi-static manner, whereas the work potential due to F will be considered as $(-Fx)$ because it is assumed that the force remains constant during virtual displacement. The strain energy of the spring will be taken as $(kx^2/2)$.

Variational forms of a number of differential equations have already been obtained in Section 7.5. In a generalized way, all of them represent the total potential energy corresponding to the respective differential equation. In fact, the extremization of potential energy provides a differential equation corresponding to equilibrium. However, the configuration for which the potential energy will be the maximum will be unstable because any slight disturbance will decrease the potential energy and increase the kinetic energy. Hence, for stable equilibrium, the potential energy must be the minimum. Corresponding to neutral equilibrium, there is neither maxima nor minima, but the first variation of potential energy is zero.

7.8 SOLUTION OF VARIATIONAL PROBLEMS BY RITZ METHOD

A powerful approximate method to solve variational problems has been developed by Swiss physicist Walter Ritz (1878–1909). The method first approximates a dependent variable as a linear combination of a number of functions. The approximate form must satisfy the essential boundary conditions, but the natural boundary conditions need not be satisfied because they are already contained in the variational form. For convergence, it is desirable that functions used in the approximation should be from a complete set of functions. A complete set of functions means that the error between any dependent variable function and approximating function may be reduced as much as needed by choosing the sufficient number of terms from the complete set. For example, a beam deflection can be approximated as a linear combination of functions $1, x, x^2$, etc., not omitting any lower-degree term. The approximating function is substituted in the variational form resulting in a function of the coefficients of the approximating function. Now, the variational form is extremized by treating the coefficient as design variables. The optimization procedure provides the values of the design variables and thus the approximate solution.

In most of the physical problems, the variational form is of the following form:

$$I(u) = \frac{1}{2} B(u, u) - l(u), \quad (7.187)$$

where $B(u, u)$ is a bilinear symmetric functional and $l(u)$ is a linear functional. An n -term approximation of u may be written as follows:

$$u_n = \phi_0 + \sum_{i=1}^n c_i \phi_i(x), \quad (7.188)$$

where ϕ_0 is the lowest possible degree function to take into account the non-homogenous boundary conditions and ϕ_i are the functions fulfilling the requirement of completeness. That means that by choosing an appropriate n , L_2 -norm $\|u - u_n\|_2$ can be made as small as desired. Substituting Equation 7.188 in Equation 7.187, one gets

$$I = \frac{1}{2} B \left(\phi_0 + \sum_{i=1}^n c_i \phi_i(x), \phi_0 + \sum_{i=1}^n c_i \phi_i(x) \right) - l \left(\phi_0 + \sum_{i=1}^n c_i \phi_i(x) \right). \quad (7.189)$$

The necessary condition for the extremization of I is

$$\frac{\partial I}{\partial c_j} = 0, \quad j = 1, \dots, n. \quad (7.190)$$

This provides

$$\sum_{i=1}^n B(\phi_j, \phi_i) c_i = l(\phi_j) - B(\phi_j, \phi_0), \quad j = 1, 2, \dots, n. \quad (7.191)$$

Equation 7.191 is a system of n simultaneous equations.

Some books name the Ritz method as the Rayleigh-Ritz method. However, Leissa (2005) has studied the related works of Lord Rayleigh and W. Ritz and concluded that Rayleigh's method is quite different from Ritz's method. In many problems of vibration, Rayleigh used only one approximating function (not a series of approximating functions as used in Ritz's method), and in some places, he also used a linear combination of functions, but the method to obtain the solution is different from Ritz's method.

Example 7.12

Find the deflection function for a cantilever beam of flexural rigidity EI loaded by a transverse load P at its free end by using the Ritz method.

SOLUTION

The total potential energy of the beam is given by

$$\Pi = \int_0^l \frac{1}{2} EI \left(\frac{d^2 w}{dx^2} \right)^2 dx - P w_l, \quad (7.192)$$

where w_l is the deflection of the free end. Assume that

$$w = a + bx + cx^2 dx^3. \quad (7.193)$$

Imposing the essential boundary conditions, at $x = 0$, both deflection and slopes zero, one gets $a = b = 0$. Now, substituting $w = cx^2 + dx^3$ in Equation 7.192,

$$\Pi = \int_0^l \frac{1}{2} EI (2c + 6dx)^2 dx - P (cl^2 + dl^3). \quad (7.194)$$

The expression in Equation 7.194 is a function of c and d . The necessary condition for minimization is that its partial derivatives with respect to c and d must vanish. Thus,

$$\begin{aligned} \int_0^l 2EI (2c + 6dx) dx - Pl^2 &= 0, \\ \int_0^l 6x EI (2c + 6dx) dx - Pl^3 &= 0. \end{aligned} \quad (7.195)$$

After simplification, one gets

$$\begin{aligned} 4EIc + 6EI dl - Pl &= 0, \\ 6EIc + 12EI dl - Pl &= 0. \end{aligned} \quad (7.196)$$

Solving them, one gets

$$c = \frac{Pl}{2EI}, \quad d = -\frac{P}{6EI}. \quad (7.197)$$

Hence,

$$w = \frac{Px^2}{6EI} (3l - x). \quad (7.198)$$

Exercises

1. Among the following expressions, which are functional?
 - a.

$$\int_0^{21} (x^2 + 2 \cos x + 5 \sin x e^{2x}) dx.$$

b.

$$\int_0^{21} (x^2 + e^{2x} + y) dx, \text{ where } y \text{ is a function of } x.$$

c.

$$\int_0^4 (y^2 + 5y'^2 + xy) dx, \text{ where } y \text{ is a function of } x \text{ and } y' \text{ is the first derivative of } y$$

d.

$$\int_0^x y dx$$

e.

$$\int_6^{14} \int_0^7 xy dx dy, \text{ where } x \text{ and } y \text{ are both independent variables.}$$

f.

$$\int_{t_1}^{t_2} \int_0^l \left\{ \frac{1}{2} \rho A \left(\frac{\partial u}{\partial t} \right)^2 - \frac{1}{2} EA \left(\frac{\partial u}{\partial x} \right)^2 \right\} dx dt, u \text{ is a function of } x \text{ and } t.$$

2. Consider the following variational form:

$$\int_0^{15} (y'^2 - 5y) dx.$$

It is also known that at $x = 0$, $y = 7$. The variational form has to be minimized. Find the Euler-Lagrange equation and boundary conditions.

3. Find out the Euler-Lagrange equation and possible boundary conditions that minimize the following functional:

$$I = \int_0^{10} \left(y'^2 + \frac{y^2}{64} + 2xy \right) dx.$$

4. Find out the Euler-Lagrange equation with possible boundary conditions that minimize the following functional:

$$I = \int_0^7 \left(y'^2 + \frac{1}{y^2} \right) dx.$$

5. Consider the functional

$$\Pi = \int_0^l \frac{1}{2} EA \left(\frac{du}{dx} \right)^2 dx - \int_0^l q u dx.$$

Find out the Euler-Lagrange equation and the corresponding boundary conditions.

6. Assume that y_1 and y_2 are two unknown functions of x , which are 0 at $x = 0$ and 5. Find y_1 and y_2 for minimizing the following functional:

$$I = \int_0^5 \left\{ \frac{1}{2} \left(\frac{dy_1}{dx} \right)^2 + \frac{1}{2} \left(\frac{dy_2}{dx} \right)^2 - 5y_1 - 5y_2 \right\} dx.$$

7. A thin stretched membrane is fixed at its periphery. Let T be the uniform tension per unit length in the membrane, w be the vertical displacement (transverse to the x - y plane), and $q(x, y)$ be the vertical load per unit area. The total potential energy of the membrane is

$$I = \int_A \frac{1}{2} \left\{ \left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 - 2qw \right\} dA.$$

Find out the governing differential equation for this problem.

8. Obtain the governing differential equation and boundary conditions to minimize the following functional:

$$I = \int_0^{10} (y''^2 + y'^2 - y^2) dx.$$

9. The governing equations for the free vibrations of a rod of length l can be obtained from Hamilton's principle. According to this principle, the motion of the particles of the rod from time t_1 to t_2 is such that the following line integral has a stationary value for the actual path of motion:

$$H = \int_{t_1}^{t_2} \int_0^l \left\{ \frac{1}{2} \rho A \left(\frac{\partial u}{\partial t} \right)^2 - \frac{1}{2} EA \left(\frac{\partial u}{\partial x} \right)^2 \right\} dx dt.$$

where A is the area of the cross section of the rod, ρ is the density, E is Young's modulus of elasticity and u is the displacement of a particle. The area of the cross section, the density and Young's modulus of elasticity may be functions of x , while u is a function of x and t . It is assumed that the position of particles is prescribed at time t_1 and t_2 . Obtain the governing differential equation for the free vibration of the rod.

10. Consider the following differential equation:

$$\frac{d^2 y}{dx^2} - y - x = 0$$

with $y = 0$ at $x = 0$ and $x = 1$. Obtain the corresponding variational form. If the boundary conditions are changed to $y = 0$ at $x = 0$ and $dy/dx = 0$ at $x = 1$, how will the variational form change?

11. The total potential energy of a simply supported beam of length l loaded by a concentrated force P at the middle is

$$\Pi = \int_0^l \frac{1}{2} EI \left(\frac{d^2 v}{dx^2} \right)^2 dx - Pv(x = l/2),$$

where EI is the modulus of flexural rigidity, and v is the transverse deflection of the beam. Assuming $v = A \sin(\pi x/l)$, find the maximum deflection of the beam by the Ritz method. Assume EI to be constant.

12. The principle of the virtual work provides two important theorems for rigid-plastic material—lower bound and upper bound theorems. In the lower-bound theorem, a statically admissible stress field is assumed, which satisfies the equilibrium equations and the boundary conditions without violating the yield criterion. Let $(\sigma_{ij}, \dot{\epsilon}_{ij})$ refer to the actual stress and the associated strain rate corresponding to any yield point state. If σ_{ij}^* is any other statically admissible state of stress, then show by the principle of virtual work,

$$\int_S (t_j - t_j^*) v_j dS = \int_V (\sigma_{ij} - \sigma_{ij}^*) \dot{\epsilon}_{ij} dV + \int_{S_D} (k - \tau^*) [v] dS_D,$$

where t_j denotes the component of actual traction, v_j is the actual velocity component, k is the yield shear strength, τ^* is the assumed shear stress along the velocity discontinuity surface and $[v]$ is the magnitude of velocity discontinuity across the velocity discontinuity surface S_D . In general, the asterisk denotes the assumed values. Further, show by using maximum work inequality for rigid-plastic material that

$$\int_S (t_j - t_j^*) v_j dS \geq 0.$$

If the tractions are specified on S_F and velocities are prescribed on S_v , then

$$\int_{S_v} t_j v_j dS_v \geq \left(\int_{S_v} t_j^* v_j dS_v = \int_{S_v} n_i \sigma_{ij}^* v_j dS_v \right).$$

The previous inequality is the lower bound theorem: the rate of work done by the actual surface tractions on S_b is greater than or equal to that done by the surface tractions in any statically admissible stress field.

In a similar way, prove the upper bound theorem, which states that the rate of work done by the unknown surface tractions on S_b is less than or equal to the rate of internal energy dissipated in any kinematically admissible velocity field. The kinematically admissible velocity field satisfies the incompressibility condition and velocity boundary conditions.

ANSWERS

1. b , c and f

2. $y'' + 2.5 = 0$, $0 < x < 15$,

with boundary conditions: (i) $y = 7$ at $x=0$, (ii) $y' = 0$ at $x=15$.

3. $y'' - \frac{y}{64} - x = 0$, $0 < x < 10$,

with either $y' = 0$ or y prescribed at $x=0$ and 10 .

4. $y'' + \frac{1}{y^3} = 0$, $0 < x < 7$.

At each boundary (i.e., at $x = 0$ and at $x = 7$), either $y' = 0$ or y prescribed.

5.

$$\frac{d}{dx} \left(EA \frac{du}{dx} \right) + q = 0.$$

At each boundary (i.e., at $x = 0$ and at $x = 1$), either the $(du/dx) = 0$ or u prescribed.

6.

$$y_1 = 12.5x - 2.5x^2 \text{ and } y_1 = 12.5x - 2.5x^2.$$

7.

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + q = 0.$$

8.

$$\frac{d^4 y}{dx^4} - \frac{d^2 y}{dx^2} - y = 0.$$

There are two sets of boundary conditions:

- a. $\frac{dy}{dx} - \frac{d^3y}{dx^3} = 0$ or y prescribed.
- b. $y'' = 0$ or y' prescribed.

For the fourth-order differential equation, at least two essential boundary conditions should be prescribed, out of which one must be prescribed value of y . Also, from each set one boundary condition should be chosen from each of the two sets for each end.

9.

$$\rho A \frac{\partial^2 u}{\partial t^2} = EA \frac{\partial^2 u}{\partial x^2}.$$

10.

$$I = \int_0^1 \left\{ \frac{1}{2} \left(\frac{dy}{dx} \right)^2 + \frac{1}{2} y^2 + xy \right\} dx.$$

The form will remain unchanged after changing the boundary conditions here.

11.

$$\frac{2Pl^3}{\pi^4 EI}.$$

8 Two-Dimensional and Axisymmetric Elasto-Plastic Problems

8.1 INTRODUCTION

In this chapter, solutions are presented to some two-dimensional and axisymmetric problems. Ideally, the two formulations described in Chapter 6 should be used, namely, the Eulerian and the updated Lagrangian formulations. However, even for simple problems, the use of the updated Lagrangian formulation requires a lot of computation. Therefore, only the Eulerian formulation will be used, with the current configuration as the control volume. (The only exception to this is the one-dimensional [1-D] problem of Section 8.2 where the total Lagrangian formulation is used.) The three governing equations need to be solved: (1) equilibrium equation, (2) stress-strain (or strain rate) relation and (3) strain-displacement (or strain rate-velocity) relation. A simplified approach shall be used to solve these problems. Especially, if the material is assumed to be perfectly plastic, then it becomes possible to find the stress field using only the equilibrium equation and the yield criterion. After that, the strain and displacement fields can be obtained by using the other two governing equations. Before some two-dimensional (plane stress/strain) or axisymmetric problems are solved, it is instructive to start with a 1-D problem. Therefore, first, the problem of symmetric beam bending of a perfectly plastic material shall be discussed.

8.2 SYMMETRIC BEAM BENDING OF A PERFECTLY PLASTIC MATERIAL (1-D PROBLEM)

For symmetric bending, the beam cross section needs to have an axis of symmetry. For concreteness, the cross section is assumed to be a rectangle. The x -axis is taken along the undeformed beam axis and the y - and z -axes are within the cross section. Initial geometric dimensions of the beam are l_0 as the length, h_0 as the height of the cross section (along the y -axis) and b_0 as the width of the cross section (along the z -axis). Further, the beam material is assumed to be perfectly plastic with σ_y as the yield stress. Regarding the states of stress and deformation, the following assumptions are made:

- All the stress components except σ_{xx} are zero. Thus, the state of stress is 1-D. Further, σ_{xx} is denoted as σ
- All the shear strain components are zero. Further, the normal strain component along the x -axis would be denoted as ϵ

8.2.1 PURE BENDING

An undeformed beam with end moments M (about the z -axis) along with the geometric dimensions and coordinate axes is shown in Figure 8.1. Before starting the analysis of plastic deformation, the results of the elastic analysis are reviewed based on the elementary theory of bending.

8.2.1.1 Elastic Analysis

The governing equations are given in the following.

- *Strain-displacement relation.* Assuming that the plane cross sections remain plane after bending, the following expression for the change in length per unit length for the fibers along the x -axis is obtained:

$$\varepsilon \equiv \varepsilon_{xx} = -\frac{y}{\rho}, \quad (8.1)$$

where

$$\rho = \frac{\left[1 + \left(\frac{dv}{dx} \right)^2 \right]^{3/2}}{\frac{d^2v}{dx^2}} \quad (8.2)$$

is the radius of curvature, and v is the transverse displacement (called the deflection) of the beam, assumed to be independent of y and z . Since ρ depends non-linearly on the displacement v , this relation remains valid even for the case of *large deformation*. However, the elastic deformation is small, and therefore, it is assumed that the cross-sectional dimensions up to yielding remain as h_0 and b_0 .

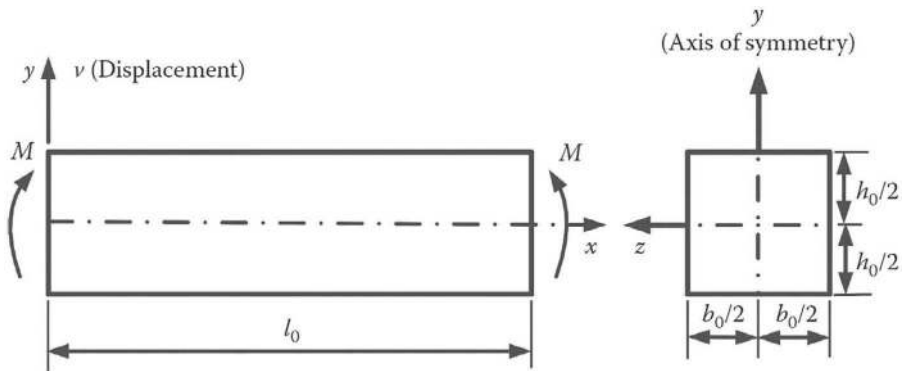


FIGURE 8.1 Undeformed beam of rectangular cross section subjected to end moments.

- *Stress-strain relation:* Since all the stress components except $\sigma_{xx} \equiv \sigma$ are zero, the only non-zero elastic stress-strain relation, for isotropic and linearly elastic material, becomes

$$\sigma = E\varepsilon. \quad (8.3)$$

- *Equilibrium equation.* For slender members like beams, the equilibrium equation is not written in a differential form. Instead, it is written in terms of the stress resultants, like bending moments, which represent certain integrals of the stress components over the cross section. For the case of pure bending, the bending moment (about the z -axis) on every cross section is the same and equal to the applied end moment M . Further, there are no other stress resultants on any cross section other than the bending moment (about the z -axis). The applied moment M is related to the non-zero stress component $\sigma_{xx} \equiv \sigma$ by the following relation:

$$M = - \int_{A_0} \sigma y \, dA_0, \quad (8.4)$$

where A_0 is the undeformed area of the cross section.

The solution of Equations 8.1, 8.3 and 8.4 leads to the following expressions for the stress and strain components:

$$\sigma = - \frac{My}{I}, \quad (8.5)$$

$$\varepsilon = - \frac{My}{EI}, \quad (8.6)$$

where

$$M = \frac{EI}{\rho} \quad (8.7)$$

and

$$I = \int_{A_0} y^2 \, dA_0 \quad (8.8)$$

is the moment of inertia of the cross section about the z -axis. For a rectangular cross section shown in Figure 8.1, the initial value of I is

$$I = \frac{b_0 h_0^3}{12}. \quad (8.9)$$

As stated earlier, in pure bending, the bending moment does not vary with x . As a result, the stress and strain distributions on every cross section remain the same. Further, they vary linearly with y as represented by Equations 8.5 and 8.6. The stress distribution on a typical cross section is shown in Figure 8.2.

Relation 8.7 is called the *moment-curvature* relation. In beam bending problems, ρ is taken as the measure of deformation, and thus, Equation 8.7 represents the relation between the applied generalized force (i.e., bending moment) and the resulting deformation. In the elastic case, E is a constant and thus the moment-curvature relation is linear. However, during the plastic deformation, as shown in Section 8.2.1.2, the relation becomes non-linear.

8.2.1.2 Plastic Analysis

In this problem, the Eulerian formulation is not employed. Since there is no unloading, instead of using the updated Lagrangian formulation, the total Lagrangian formulation is used. The governing equations of this formulation are given in the following.

- As stated earlier, the *strain-displacement relation* (Equation 8.1) remains valid even for large deformation. Therefore, for plastic analysis, the same relation is used
- The *equilibrium equation* (Equation 8.4), of course, remains valid for the case of plastic deformation except that, now, the integration should be over the deformed area of the cross section. It is assumed that the deformed area is approximately rectangular with h as the height and b as the weight
- However, it is needed to change the stress-strain relation as the relation 8.3 is for the elastic behavior. For a perfectly plastic material and 1-D state of stress, this relation is the same as the yield criterion (Equation 5.30):

$$\sigma = \sigma_Y. \quad (8.10)$$

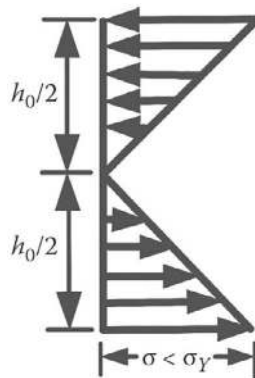


FIGURE 8.2 Stress distribution on a typical cross section (elastic deformation).

Figure 8.2 shows that the maximum values of the stress occur at the top and bottom surfaces. (i.e., $y = \pm h_0/2$). Therefore, as the applied moment M is increased, yielding occurs first at these surfaces. Let us denote the maximum positive strain ($y = h_0/2$) and bending moment at yielding by ε_Y and M_Y , respectively. Then, using, Equations 8.3, 8.5 and 8.9, they are given by

$$\varepsilon_Y = \sigma_Y / E, \quad (8.11)$$

$$M_Y = -\frac{\sigma_Y (b_0 h_0^3 / 12)}{-h_0 / 2} = \frac{b_0 h_0^2}{6} \sigma_Y. \quad (8.12)$$

Thus, at $M = M_Y$, the plastic zone consists of only the top and bottom surfaces.

As the applied moment is further increased, the stress in the top and bottom fibers cannot increase beyond σ_Y as the material is perfectly plastic. Therefore, the fibers below $y = h_0/2$ and above $y = -h_0/2$ start yielding, and thus, the plastic zone starts growing toward the center surface $y = 0$. During the plastic deformation, h and b are taken to be the current cross-sectional dimensions. The stress distributions on a typical cross section corresponding to $M = M_Y$ and $M > M_Y$ are shown in Figure 8.3a and b, respectively.

In Figure 8.3b, the plastic zone is at the distance of $y = \pm y_Y$ from the center. The relation between the applied moment M and the quantity y_Y can be obtained as follows. From Figure 8.3b, one gets

$$\sigma = -\frac{y}{y_Y} \sigma_Y, \quad 0 \leq y \leq y_Y; \quad \sigma = -\sigma_Y, \quad y_Y \leq y \leq h/2. \quad (8.13)$$

Since $dA = bdy$ for a rectangular cross section, the limits of integration in Equation 8.4 become $y = -h/2$ and $y = h/2$. Further, the stress distribution is antisymmetric about $y = 0$. Then, Equation 8.4 becomes

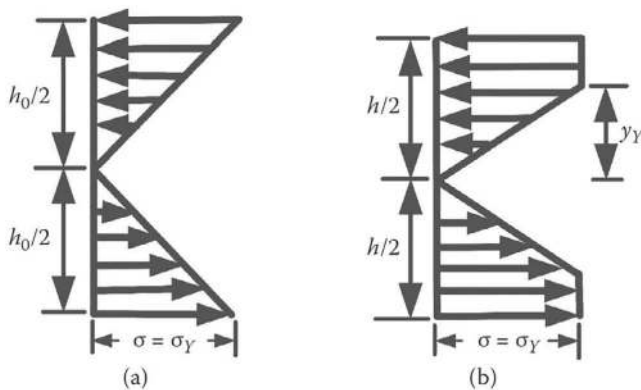


FIGURE 8.3 Stress distributions on a typical cross section (elasto-plastic deformation): (a) $M = M_Y$; (b) $M > M_Y$.

$$M = -\int_A \sigma y dA = -\int_{-h/2}^{h/2} \sigma y b dy = -2 \int_0^{h/2} \sigma b y dy. \quad (8.14)$$

Substituting Equation 8.13 in the previous relation, one gets

$$\begin{aligned} M &= -2b \left[\int_0^{y_Y} -\frac{y}{y_Y} \sigma_Y y dy + \int_{y_Y}^{h/2} -\sigma_Y y dy \right] \\ &= \frac{bh^2}{12} \sigma_Y \left[3 - \left(\frac{y_Y}{h/2} \right)^2 \right]. \end{aligned} \quad (8.15)$$

As the applied moment is increased further, in the limiting case, the yielding takes place in all the fibers, and the whole beam becomes plastic. In this case, the stress in the top half becomes $-\sigma_Y$, whereas in the bottom half, it becomes σ_Y . Further, the value of y_Y becomes zero, and therefore, the expression for the external moment (denoted by M_L) becomes

$$M_L = \frac{bh^2}{4} \sigma_Y. \quad (8.16)$$

Note that, for a perfectly plastic material, the applied moment cannot be increased beyond this value. Therefore, this value of the applied moment is called the *limit/collapse/fully plastic moment*. The stress distribution corresponding to the limit moment is shown in Figure 8.4.

Next, the moment-curvature relation corresponding to plastic deformation, that is, for $M \geq M_Y$, is derived. First, the expression is obtained for y_Y in terms of ρ . At $M =$

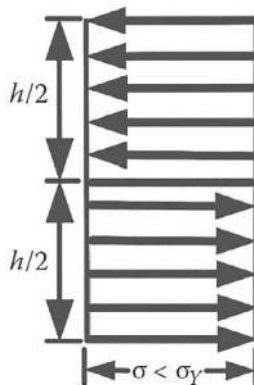


FIGURE 8.4 Stress distributions on a typical cross section (elasto-plastic deformation) at the limiting moment M_L .

M_Y (i.e., when the yielding first starts at the top and bottom surfaces), let ρ_Y be the radius of curvature. Then, it can be related to M_Y by Equation 8.7:

$$\frac{1}{\rho_Y} = \frac{M_Y}{EI}. \quad (8.17)$$

Further, it can be related to the corresponding maximum positive strain ε_Y (at $y = -h/2$) by Equation 8.1:

$$\varepsilon_Y = \frac{h/2}{\rho_Y}. \quad (8.18)$$

When $M > M_Y$, again Equation 8.1 can be used to relate the radius of curvature ρ to the corresponding maximum positive strain ε_Y (at $y = -y_Y$):

$$\varepsilon_Y = \frac{y_Y}{\rho}. \quad (8.19)$$

Eliminating ε_Y from Equations 8.18 and 8.19, one gets

$$\frac{y_Y}{(h/2)} = \frac{(1/\rho_Y)}{(1/\rho)}. \quad (8.20)$$

Now, the elimination of y_Y from Equations 8.15 and 8.20 gives the moment-curvature relation for $M \geq M_Y$:

$$M = \frac{bh^2}{12} \sigma_Y \left[3 - \left(\frac{1/\rho_Y}{1/\rho} \right)^2 \right]. \quad (8.21)$$

Equations 8.16 and 8.21 show that, at the limit moment M_L , the curvature ($1/\rho$) becomes infinite. The graph of the moment-curvature relations (Equation 8.7 for $M \leq M_Y$ and Equation 8.21 for $M \geq M_Y$) is shown in Figure 8.5. Note that, in the elastic region ($0 \leq M \leq M_Y$), the graph is a straight line, while in the plastic region ($M_Y \leq M \leq M_L$), it is a curve. The value of M_L is attained only asymptotically. It means the applied moment M tends to M_L , the curvature ($1/\rho$) of the beam tends to infinity.

8.2.2 BENDING IN THE PRESENCE OF SHEAR FORCE

In this section, one skips the elastic analysis and straightaway proceeds to the plastic analysis. The *strain-displacement relation* Equation 8.1, even though valid for plastic deformation, has been derived assuming that the plane cross sections remain plane after bending. This assumption is strictly valid only for the case of pure bending, that is when there is no shear stress. However, in slender members like beams, the shear stress created by the applied transverse forces is smaller than the bending stresses by an order of the magnitude. Therefore, it is assumed that the strain-displacement relation Equation 8.1 remains *approximately valid* in the presence of shear force.

When the shear force is present, the bending moment varies with the x -coordinate. In Section 8.2.1, certain expressions are derived for the bending moment (Equations 8.12, 8.15 and 8.16), the stress distribution (Equation 8.13) and the moment-curvature relation (Equation 8.21) at and after the yielding for the case of pure bending (i.e., when the bending moment did not vary with x). In the present problem also, these relations can be applied but only at those cross sections at which the yielding is occurring or has already occurred.

Figure 8.6 shows an undeformed beam of the same geometry (b_0 , h_0 and $l_0 = 2a_0$) and the same material (σ_Y) as the last section. However, now, it is simply supported

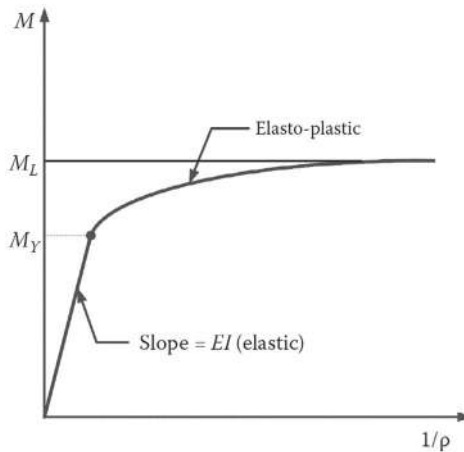


FIGURE 8.5 Graph of the moment-curvature relation for elastic and elasto-plastic deformation.

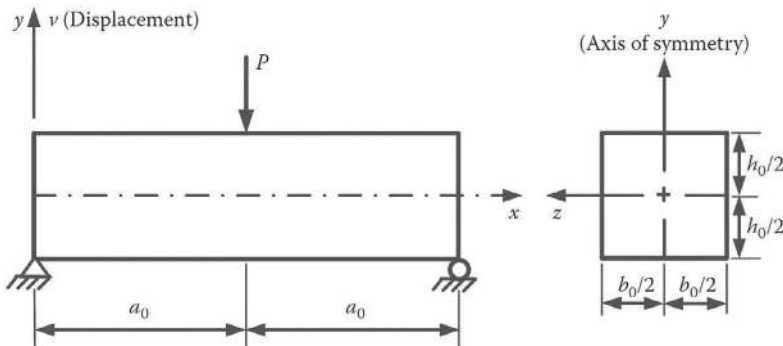


FIGURE 8.6 Undeformed simply supported beam of rectangular cross section subjected to downward transverse point force at the center.

at the two ends and is acted on by a downward transverse point force P at the center (i.e., at $x = a_0$). The expression for the bending moment M is given by

$$M = \frac{Px}{2}, \quad 0 \leq x \leq a_0; \quad (8.22)$$

$$M = \frac{P(-x + 2a_0)}{2}, \quad a_0 \leq x \leq 2a_0.$$

Figure 8.7 shows the bending moment diagram.

As per Figure 8.7, the maximum bending moment occurs at $x = a_0$. Therefore, when the applied transverse force is increased, the yielding first occurs at this cross section according to Equation 8.5. Further, it occurs at $y = \pm h_0/2$. Thus, in the present problem, the yielding first occurs at the lines $y = \pm h_0/2$ of the top and bottom surfaces, unlike in the problem of pure bending, where it first occurs at the entire top and bottom surfaces. At the yielding, the bending moment at this cross section reaches the value M_Y (Equation 8.12). Then, using Equations 8.12 and 8.22 with $M = M_Y$ and $x = a_0$, the value of P at yielding (denoted by P_Y) is given by

$$P_Y = \frac{2M_Y}{a_0} = \frac{2(b_0 h_0^2 \sigma_Y / 6)}{a_0} = \frac{b_0 h_0^2}{3a_0} \sigma_Y. \quad (8.23)$$

As the applied transverse force is increased further, the stress at the lines ($x = a_0$, $y = \pm h_0/2$) cannot increase beyond σ_Y as the material is perfectly plastic. Therefore, the lines at the cross section $x = a_0$ below $y = h_0/2$ and above $y = -h_0/2$ start yielding. Further, the lines in the top and bottom surfaces on either side of the line $x = a_0$ also start yielding. Thus, the plastic zone starts growing toward the center surface $y = 0$ as well as toward the left and right from the lines ($x = a_0$, $y = \pm h_0/2$). During the plastic deformation, a is taken to be the half-length and h and b to be the cross-sectional dimensions. Figure 8.8a and b show the bending moment diagrams corresponding to the applied transverse forces $P = P_Y$ and $P > P_Y$. Figure 8.9a and b shows the corresponding plastic zones. (In Figure 8.9a and b the undeformed domain has been used.)

The equation of the plastic boundary corresponding to $P > P_Y$ can be obtained as follows. From the bending moment diagram of Figure 8.8b (for $P > P_Y$), the bending

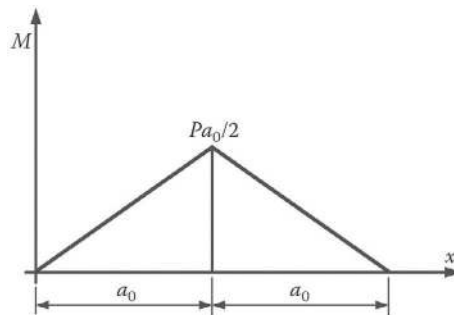


FIGURE 8.7 Bending moment diagram for elastic deformation.

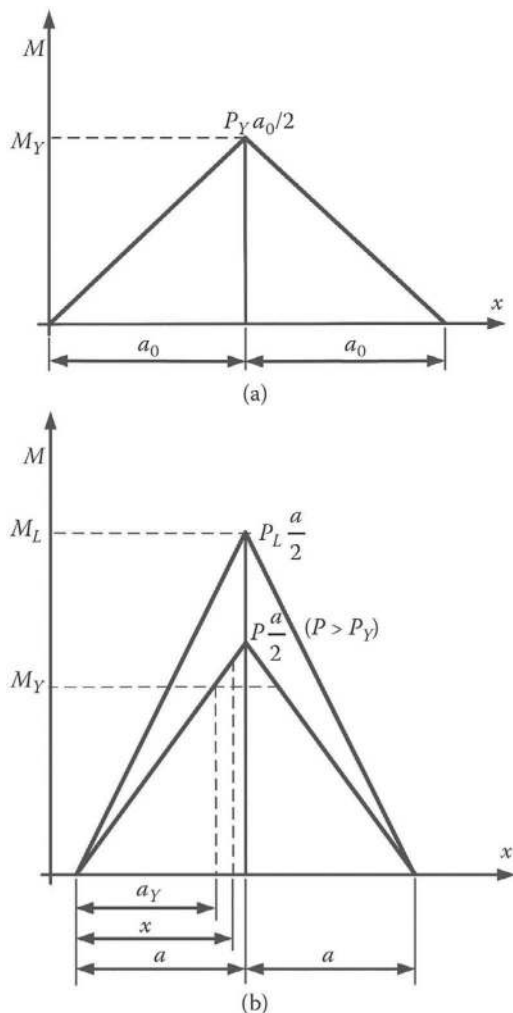


FIGURE 8.8 Bending moment diagrams for elasto-plastic deformation: (a) $P = P_Y$; (b) $P > P_Y$.

moment at any cross section x can be expressed in terms of the distance a_Y at which the plastic zone begins, that is, the cross section at which the bending moment reaches the value M_Y . From the similarity of the triangles, this relation is given by

$$\frac{M}{M_Y} = \frac{x}{a_Y}. \quad (8.24)$$

For a perfectly plastic material, the expression for $M (> M_Y)$ is given by Equation 8.15, in which y_Y is the distance of the plastic boundary from the center surface $y = 0$ at any x (see Figure 8.9b). Approximating $(bh^2/6)$ in Equation 8.15 as M_Y (similar to Equation 8.12) and eliminating (M/M_Y) from Equations 8.15 and 8.24 gives the following equation of the plastic boundary:

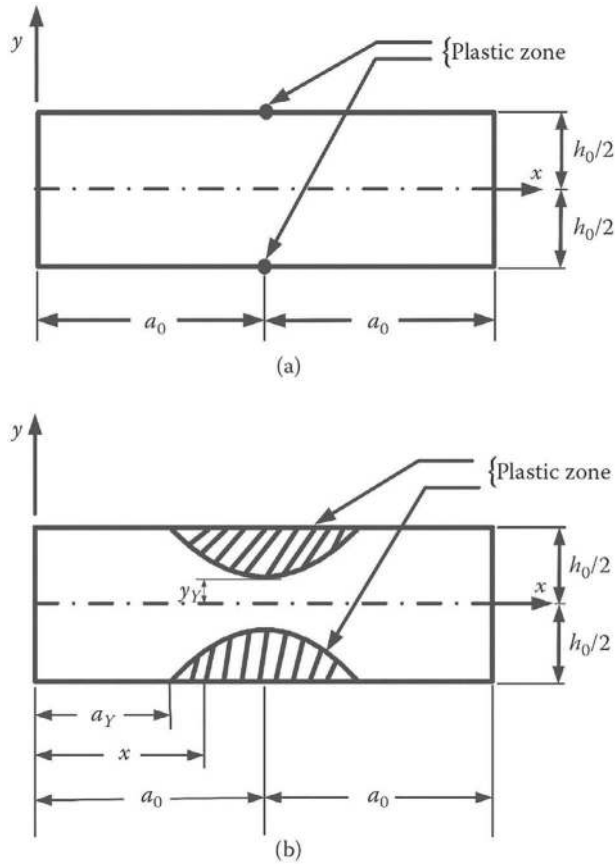


FIGURE 8.9 Plastic zones (drawn on undeformed configuration): (a) $P = P_Y$; (b) $P > P_Y$.

$$\frac{x}{a_Y} = \frac{1}{2} \left[3 - \left(\frac{y_Y}{h/2} \right)^2 \right]. \quad (8.25)$$

Note that this relation is valid strictly for $x > a_Y$.

As the applied transverse force is further increased, in the limiting case, the yielding takes place at all the lines of the cross section $x = a$ and the whole cross section becomes plastic. Thus, at this cross section, the stress in the top half becomes $-\sigma_Y$, while in the bottom half, it becomes σ_Y . Then, the value of the bending moment M at this cross section becomes equal to the limit moment M_L . The corresponding value of the applied transverse force (denoted by P_L) is obtained from Equations 8.16 and 8.22 with $M = M_L$ and $x = a$:

$$P_L = \frac{2M_L}{a} = \frac{2}{a} \left(\frac{bh^2}{4} \sigma_Y \right) = \frac{bh^2}{2a} \sigma_Y. \quad (8.26)$$

Note that, for a perfectly plastic material, the applied transverse force cannot be increased beyond this value. Therefore, this value of the force is called the *limit/collapse force*. The corresponding bending moment diagram is also shown in Figure 8.8b. Further, the corresponding plastic zone is shown in Figure 8.10. (In this diagram, the undeformed domain has been used.) The plastic boundary in this figure is also described by Equation 8.25.

Note that, when the bending moment at the cross section $x = a$ becomes equal to M_L , the curvature ($1/\rho$) at this cross section becomes infinite as per the moment-curvature relation (Equation 8.21 and Figure 8.5). As per Equation 8.2, the expression for the curvature is

$$\frac{1}{\rho} = \frac{\frac{d}{dx} \left(\frac{dv}{dx} \right)}{\left[1 + \left(\frac{dv}{dx} \right)^2 \right]^{3/2}}. \quad (8.27)$$

So, when ($1/\rho$) becomes infinite at $x = a$, the derivative of (dv/dx) also becomes infinite. It means that the slope (dv/dx) of the beam becomes discontinuous at $x = a$. The cross section that has become completely plastic and the discontinuous slope is called the *plastic hinge*. Thus, in this case, the cross section at $x = a$ becomes a plastic hinge.

8.3 HOLE EXPANSION IN AN INFINITE PLATE (PLANE STRESS AND AXISYMMETRIC PROBLEM)

For this problem, it is advantageous to use the cylindrical polar coordinate system (r, θ, z). Even though the plate is infinite, it is convenient to consider the outer boundary of the plate at $r = b_0$ with $b_0 \rightarrow \infty$. The plate is shown in Figure 8.11 where a_0 is the initial radius of the hole, h_0 is the initial thickness, and p is the pressure acting on the plate at the inner boundary $r = a_0$. The plate material is assumed to yield according to the Mises criterion with σ_y as the yield stress.

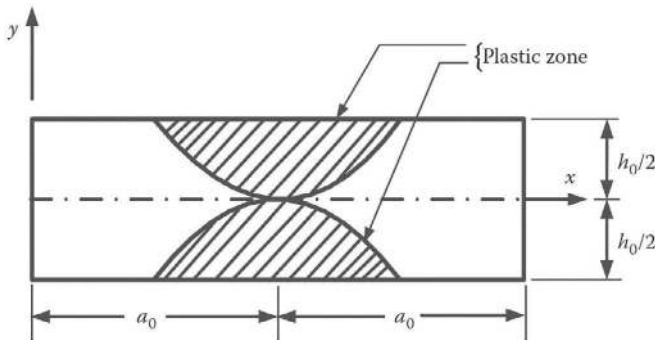


FIGURE 8.10 Plastic zones (drawn on undeformed configuration) at $P = P_L$.

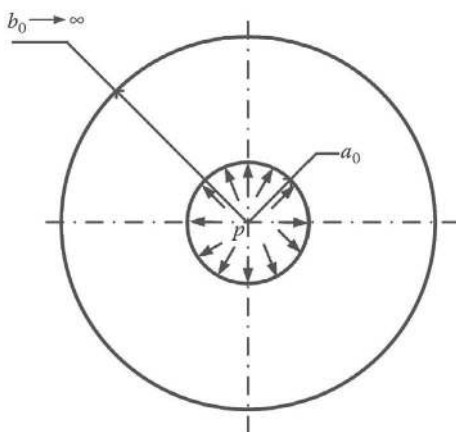


FIGURE 8.11 An infinite plate with a hole subjected to pressure at the inner boundary. Initial thickness is h_0 .

With respect to the cylindrical polar coordinate system, the stress matrix becomes

$$[\sigma] = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{zr} \\ \sigma_{r\theta} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{zr} & \sigma_{\theta z} & \sigma_{zz} \end{bmatrix} \quad (8.28)$$

However, this is a plane stress problem in the r - θ plane. Therefore, the following stress components are zero:

$$\sigma_{zr} = \sigma_{\theta z} = \sigma_{zz} = 0. \quad (8.29)$$

Further, the remaining stress components become independent of the z -coordinate. Additionally, the problem is axisymmetric. Therefore, one more stress component is zero:

$$\sigma_{r\theta} = 0 \quad (8.30)$$

and the remaining stress components become independent of the θ -coordinate. Then, the matrix of the non-zero stress components become

$$[\sigma] = \begin{bmatrix} \sigma_{rr} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (8.31)$$

The boundary conditions of the problem are

$$\sigma_{rr} = -p, \text{ at } r = a_0; \quad (8.32)$$

$$\sigma_{rr} \rightarrow 0, \text{ at } r = b_0. \quad (8.33)$$

The solution of the elastic analysis, assuming the body force to be negligible, is (Timoshenko and Goodier, 1970)

$$\sigma_{rr} = -\frac{pa_0^2}{r^2}, \quad \sigma_{\theta\theta} = \frac{pa_0^2}{r^2}. \quad (8.34)$$

8.3.1 INITIAL YIELDING

Note that, at every point, the principal stresses are

$$\sigma_1 = \sigma_{rr}, \quad \sigma_2 = \sigma_{\theta\theta}, \quad \sigma_3 = 0. \quad (8.35)$$

(One of the principal stresses is zero because it is a plane stress problem.) Then, the Mises criterion for the plane stress problem (Equation 5.54) becomes

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 - \sigma_Y^2 = 0. \quad (8.36)$$

Substituting the expressions for σ_1 and σ_2 from Equation 8.35 and those of σ_{rr} and $\sigma_{\theta\theta}$ from Equation 8.34, Equation 8.36 becomes

$$\left(-\frac{pa_0^2}{r^2}\right)^2 - \left(-\frac{pa_0^2}{r^2}\right)\left(\frac{pa_0^2}{r^2}\right) + \left(\frac{pa_0^2}{r^2}\right)^2 - \sigma_Y^2 = 0$$

or

$$\frac{p}{(\sigma_Y / \sqrt{3})} = \left(\frac{r}{a_0}\right)^2. \quad (8.37)$$

The positive sign is chosen in the square root as p and σ_Y are positive quantities. Equation 8.37 shows that:

- Since $r \geq a_0$, yielding cannot take place until the applied pressure p reaches the value $\sigma_Y / \sqrt{3}$
- At $p = \sigma_Y / \sqrt{3}$, the yielding first occurs at $r = a_0$. Thus, at the beginning, the plastic boundary is at $r = a_0$
- As the applied pressure p is increased further, the plastic boundary starts moving radially outward

Figure 8.12 shows the plastic boundary when $p = \sigma_Y / \sqrt{3}$.

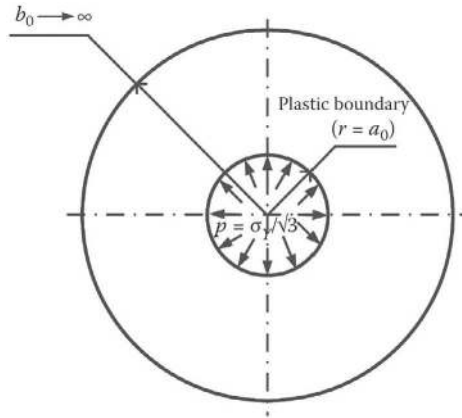


FIGURE 8.12 Plastic boundary at the initial yielding when the pressure is $\sigma_Y/\sqrt{3}$.

In Section 8.3.2, the elasto-plastic solution is first discussed by assuming the plate to be perfectly plastic. In the subsequent solution, the elasto-plastic solution corresponding to a hardening case is presented. For the case of a perfectly plastic material, it is possible to solve for the stresses using only the equilibrium equation along with the yield criterion, that is, without using the stress-strain and strain-displacement relations. On the other hand, for a hardening material, all of the three governing equations have to be solved simultaneously.

8.3.2 ELASTO-PLASTIC ANALYSIS FOR A PERFECTLY PLASTIC MATERIAL

The Eulerian formulation is used with the current configuration as the control volume. Figure 8.13 shows the state of deformation in the plate when p is greater than $\sigma_Y/\sqrt{3}$. In the current deformed configuration, the radius of the hole is a . The region between $r = a$ and $r = c$ is plastic. The plastic boundary is at $r = c$ and the region beyond $r = c$ is elastic. The parameter c has to be found as a part of the solution.

8.3.2.1 Stresses in the Elastic Region

To find the stresses in the elastic region, the advantage of the analogy between the elastic regions of Figures 8.12 and 8.13 is taken. In Figure 8.12, the plastic boundary is at $r = a_0$ and the pressure acting on the boundary is $\sigma_Y/\sqrt{3}$. The corresponding expressions for the elastic stresses are given by Equation 8.34 with $p = \sigma_Y/\sqrt{3}$. In Figure 8.13, the plastic boundary is at $r = c$, but the pressure acting on the boundary is the same: $p = \sigma_Y/\sqrt{3}$. Therefore, the expressions for the elastic stresses in Figure 8.13 would be the same as given by Equation 8.34 with p replaced by $\sigma_Y/\sqrt{3}$ and a_0 replaced by c :

$$\sigma_{rr} = -\frac{\sigma_Y}{\sqrt{3}} \frac{c^2}{r^2}, \quad \sigma_{\theta\theta} = \frac{\sigma_Y}{\sqrt{3}} \frac{c^2}{r^2}. \quad (8.38)$$

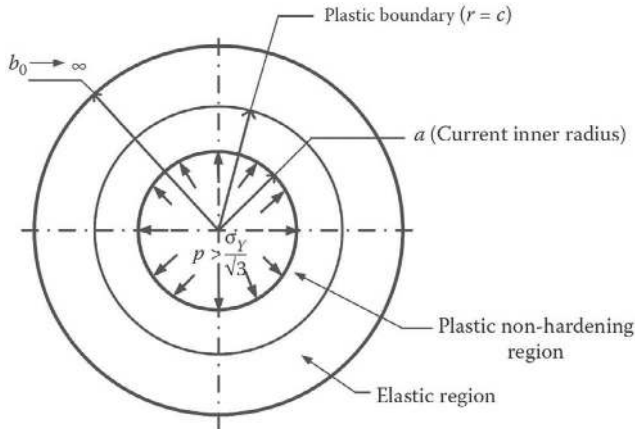


FIGURE 8.13 Plastic boundary when the pressure exceeds $\sigma_Y/\sqrt{3}$.

8.3.2.2 Stresses in the Plastic Region

There are two non-zero stress components ($\sigma_{rr}, \sigma_{\theta\theta}$) and there are two governing equations in terms of these two stress components: (1) equilibrium equation; and (2) yield criterion. Thus, it is possible to find these two stress components first and then find the non-zero strain and displacement components using the stress-strain and strain-displacement equations. The two *governing equations* for ($\sigma_{rr}, \sigma_{\theta\theta}$) are given in the following.

- *Equilibrium equation.* Since the problem is both axisymmetric and plane stress, the equilibrium equation in the θ - and z -directions are identically satisfied. It is assumed that the body force is negligible. Then, the equilibrium equation in the r -direction becomes

$$\frac{d\sigma_{rr}}{dr} = \frac{\sigma_{\theta\theta} - \sigma_{rr}}{r}. \quad (8.39)$$

- *Yield criterion.* As stated earlier, the material yields according to the Mises criterion with σ_Y as the yield stress. In this section, the material is assumed to be perfectly plastic. Then, σ_Y does not change with the plastic deformation but remains constant. Further, the problem is a plane stress problem with σ_{rr} and $\sigma_{\theta\theta}$ as the non-zero principal stresses. Then, the Mises yield criterion (Equations 8.35 and 8.36) becomes

$$\sigma_{rr}^2 - \sigma_{rr}\sigma_{\theta\theta} + \sigma_{\theta\theta}^2 - \sigma_Y^2 = 0. \quad (8.40)$$

This is a non-linear algebraic equation.

Boundary condition: since, Equation 8.39 is a first-order ordinary differential equation, only one boundary condition is needed. Equation 8.32 provides this

boundary condition with a_0 replaced by the current hole radius a . The other boundary condition (Equation 8.33) has been used while finding the elastic stresses.

Condition for determination of c : the continuity of the stress component σ_{rr} at the plastic boundary ($r = c$) is used to determine the parameter c .

Solution: out of the two governing equations, one is a non-linear algebraic equation. The standard solution technique for such problems is to introduce a new variable that would satisfy the algebraic equation identically. This technique also reduces the number of dependent variables from two to one. For the present problem, this variable (denoted by ϕ) is defined as

$$\sigma_{rr} = \frac{2\sigma_Y}{\sqrt{3}} \sin\left(\frac{\pi}{6} + \phi\right), \sigma_{\theta\theta} = \frac{2\sigma_Y}{\sqrt{3}} \sin\left(\frac{\pi}{6} - \phi\right). \quad (8.41)$$

Note that these expressions satisfy the yield criterion (Equation 8.40) identically. Further, the continuity of σ_{rr} at $r = c$, along with Equations 8.38 and 8.41, implies that

$$2 \sin\left(\frac{\pi}{6} + \phi\right) = 1, \quad (8.42)$$

which leads to

$$\phi = 0, \quad \text{at } r = c. \quad (8.43)$$

To find the equation satisfied by ϕ , expressions 8.41 are substituted into the equilibrium Equation 8.39. Upon simplification, one gets the following non-linear differential equation:

$$\frac{d\phi}{dr} = -\frac{2}{(\sqrt{3} - \tan\phi)r}. \quad (8.44)$$

The continuity of σ_{rr} is used at $r = c$ (Equation 8.43) as the boundary condition for this differential equation. The boundary condition Equation 8.32 would be used later for the determination of c .

The separation of the variable technique is used to solve this problem. Equation 8.44 can be rearranged as

$$(\sqrt{3} - \tan\phi)d\phi = -2\frac{dr}{r}. \quad (8.45)$$

Integration of this equation and simplification leads to

$$\exp(\sqrt{3}\phi)\cos\phi = \frac{c_1}{r^2}. \quad (8.46)$$

The constant c_1 is determined from the boundary condition 8.43. Its value is

$$c_1 = c^2. \quad (8.47)$$

Then, the solution to the differential Equation 8.44 and the boundary condition Equation 8.43 is

$$\exp(\sqrt{3}\phi)\cos\phi = \left(\frac{c}{r}\right)^2. \quad (8.48)$$

For the given applied pressure p , the radius c of the plastic boundary is obtained using the boundary condition Equation 8.32 with a_0 replaced by a . Let α be the value of ϕ at $r = a$. Then, Equation 8.48 gives

$$\exp(\sqrt{3}\alpha)\cos\alpha = \left(\frac{c}{a}\right)^2 \quad (8.49)$$

Further, substitution of the expression 8.41 for σ_{rr} in the boundary condition Equation 8.32 with $\phi = \alpha$ at $r = a$ leads to

$$-\frac{2\sigma_Y}{\sqrt{2}}\sin\left(\frac{\pi}{6} + \alpha\right) = -p. \quad (8.50)$$

Elimination of α from Equations 8.49 and 8.50 gives the expression for c in terms of a , p and σ_Y : $c = c(a, p, \sigma_Y)$.

Once $c = c(a, p, \sigma_Y)$ is known, the expressions for the stress components (σ_{rr} , $\sigma_{\theta\theta}$) as functions of r , corresponding to the applied pressure p and the material (σ_Y) and geometric (a) parameters, are obtained as follows.

- For a chosen value of the radial coordinate r , the value of the variable ϕ is obtained from Equation 8.48
- For this value of ϕ , the stress components (σ_{rr} , $\sigma_{\theta\theta}$) are obtained using the expression 8.41

Note that as per Equation 8.50, the maximum value of the pressure that can be applied on a perfectly plastic plate is

$$p_{\max} = \frac{2\sigma_Y}{\sqrt{3}}. \quad (8.51)$$

This happens when $\alpha = \pi/3$. The corresponding value of c is given by Equation 8.49:

$$c = a \left[\exp(\sqrt{3}\alpha)\cos\alpha \right]^{1/2} \Big|_{\alpha=\pi/3} = (1.751)a. \quad (8.52)$$

This means, that in a perfectly plastic plate, the plastic boundary can move only up to the distance of $(1.751)a$ where a is the current hole radius. However, the experimental observations on real plates are different. A real plate can withstand the pressure above $2\sigma_Y/\sqrt{3}$ and the plastic boundary can move beyond $(1.751)a$. The experimental observations on a real plate when the applied pressure exceeds $2\sigma_Y/\sqrt{3}$ are as follows:

- The plate thickens near the inner boundary up to $r = \rho$ due to *hardening*.
- Beyond $r = \rho$, the *non-hardening* solution of this section remains valid.
- Beyond $r = (1.751)\rho$, the elastic solution given by Equation 8.38 remains valid.

This is shown in Figure 8.14.

8.3.3 ELASTO-PLASTIC ANALYSIS FOR A HARDENING MATERIAL

In this section, the plastic analysis of the plate is carried out when the applied pressure p exceeds $2\sigma_Y/\sqrt{3}$, assuming it to be a hardening material. The Eulerian formulation is used with the current configuration as the control volume. The governing equations are as follows.

Governing Equation:

- *Yield criterion.* Equation 8.40 represents the Mises yield criterion for a perfectly plastic material. For a hardening material, the yield stress σ_Y in this equation gets replaced by the equivalent stress σ_{eq} , which depends on the equivalent plastic strain ε_{eq}^p through the hardening function (H) of the material (Equation 5.68). Thus, the yield criterion becomes

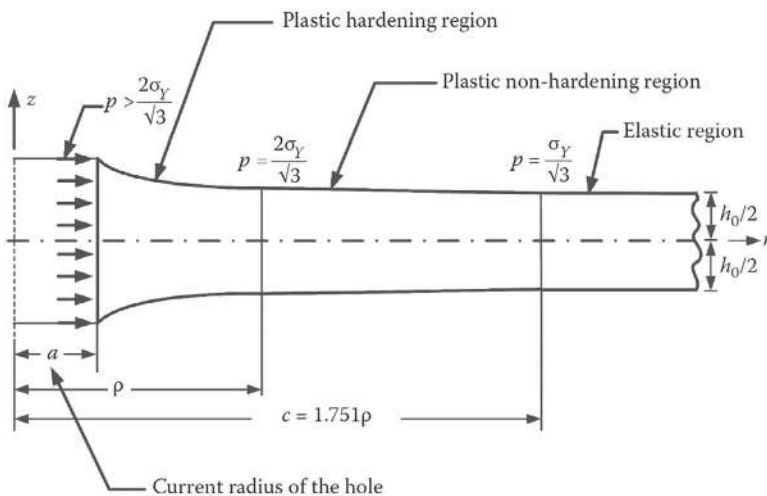


FIGURE 8.14 Deformation of the plate when the pressure exceeds $2\sigma_Y/\sqrt{3}$.

$$\sigma_{rr}^2 - \sigma_{rr}\sigma_{\theta\theta} + \sigma_{\theta\theta}^2 = \sigma_{eq}^2. \quad (8.53)$$

- *Equilibrium equation.* The free diagram of a typical element is shown in Figure 8.15. As stated earlier, the only non-zero stress components are $(\sigma_{rr}, \sigma_{\theta\theta})$. Due to hardening, the thickness h of the plate varies. However, because of axisymmetry, σ_{rr} , $\sigma_{\theta\theta}$ and h do not vary with θ . The equilibrium of the forces in the r -direction gives the following equation:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr}}{h} \frac{\partial h}{\partial r} = \frac{\sigma_{\theta\theta} - \sigma_{rr}}{r}. \quad (8.54)$$

Here, all the quantities also depend on the time t besides the radial coordinate r . Therefore, the derivative with respect to r is denoted as a partial derivative

- *Stress-velocity relations.* In order to avoid considering the strain rates as unknowns, strain rates are eliminated from the stress-strain rate and strain rate-velocity relations to obtain the stress-velocity relations

Stress-strain rate relations. In metals, the elastic strain rate is much smaller than the plastic strain rate. Therefore, the stress-strain rate relations, after neglecting the elastic strain rate, become (Equation 5.128)

$$\dot{\epsilon}_{ij} = \frac{3\dot{\epsilon}_{eq}}{2\sigma_{eq}} \sigma'_{ij}. \quad (8.55)$$

Since the strain rates are not being considered as unknowns, we like to eliminate $\dot{\epsilon}_{eq}$ from the previous relation using the hardening relation of the plate material.

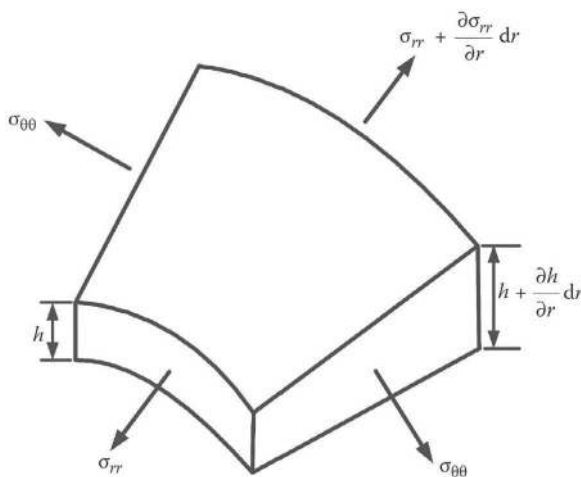


FIGURE 8.15 Free body diagram of a small element in the plastic hardening region.

Since the elastic deformation has been neglected, the argument of the hardening function becomes ε_{eq} rather than ε_{eq}^p . Thus,

$$\sigma_{eq} = H(\varepsilon_{eq}). \quad (8.56)$$

Then,

$$\dot{\varepsilon}_{eq} = \frac{\dot{\sigma}_{eq}}{H'}. \quad (8.57)$$

The following hardening relation is assumed (Voce's hardening relation, see Equation 5.34):

$$\sigma_{eq} \equiv H(\varepsilon_{eq}) = \sigma_Y + K[1 - \exp(-n\varepsilon_{eq})] \quad (8.58)$$

where K and n are the hardening parameters. From Equation 8.58, one gets

$$H' = Kn \exp(-n\varepsilon_{eq}). \quad (8.59)$$

Elimination of $\exp(-n\varepsilon_{eq})$ from Equations 8.58 and 8.59 leads to the following expression for H' :

$$H' = n[(\sigma_Y + K) - \sigma_{eq}]. \quad (8.60)$$

After eliminating $\dot{\varepsilon}_{eq}$ and H' from Equations 8.55, 8.57 and 8.60, the stress-strain rate relation becomes

$$\dot{\varepsilon}_{ij} = \frac{3\sigma_{eq}}{2n\sigma_{eq}[(\sigma_Y + K) - \sigma_{eq}]} \sigma'_{ij}. \quad (8.61)$$

Now, the right-hand side of this equation is independent of $\dot{\varepsilon}_{eq}$. It does, however, depend on σ_{eq} , but it would be considered as an independent variable.

From Equation 8.31, one gets the following expression for the matrix of the deviator stress:

$$[\sigma'] = \begin{bmatrix} \frac{2\sigma_{rr} - \sigma_{\theta\theta}}{3} & 0 & 0 \\ 0 & \frac{2\sigma_{\theta\theta} - \sigma_{rr}}{3} & 0 \\ 0 & 0 & -\frac{\sigma_{rr} - \sigma_{\theta\theta}}{3} \end{bmatrix}. \quad (8.62)$$

Then, the expressions for the three non-zero strain rates become

$$\begin{aligned}\dot{\epsilon}_{rr} &= \frac{\sigma_{eq} (2\sigma_{rr} - \sigma_{\theta\theta})}{2n\sigma_{eq} [(\sigma_Y + K) - \sigma_{eq}]'} \\ \dot{\epsilon}_{\theta\theta} &= \frac{\dot{\sigma}_{eq} (2\sigma_{\theta\theta} + \sigma_{rr})}{2n\sigma_{eq} [(\sigma_Y + K) - \sigma_{eq}]'} \\ \dot{\epsilon}_{zz} &= -\frac{\dot{\sigma}_{eq} (\sigma_{rr} + \sigma_{\theta\theta})}{2n\sigma_{eq} [(\sigma_Y + K) - \sigma_{eq}]'}.\end{aligned}\quad (8.63)$$

Strain rate-velocity relations. Since the problem is both plane stress and axisymmetric, v_θ and v_r depend only on r and t . Further, $v_\theta = 0$. Then, the two non-zero strain rate components are

$$\dot{\epsilon}_{rr} = \frac{\partial v_r}{\partial r}, \quad \dot{\epsilon}_{\theta\theta} = \frac{v_r}{r}. \quad (8.64)$$

In a plane stress problem, $\dot{\epsilon}_{zz}$ is non-zero but approximately constant across the plate thickness. Therefore, instead of relating it to the z derivative of v_z , it is approximated as the ratio of the thickness rate to the current thickness:

$$\dot{\epsilon}_{zz} = \frac{\dot{h}}{h}. \quad (8.65)$$

Hence, h is being used as an independent variable instead of v_z . Note that the variable h also appears in the equilibrium Equation 8.54.

Stress-velocity relations. In Eulerian formulation, the time rate of a function $f(r, t)$ is defined as (Equation 6.1)

$$\dot{f} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial r} \frac{dr}{dt} = \frac{\partial f}{\partial t} + v_r \frac{\partial f}{\partial r}.$$

Therefore,

$$\dot{\sigma}_{eq} = \frac{\partial \sigma_{eq}}{\partial t} + v_r \frac{\partial \sigma_{eq}}{\partial r}, \quad (8.66)$$

$$\dot{h} = \frac{\partial h}{\partial t} + v_r \frac{\partial h}{\partial r}. \quad (8.67)$$

Now, $\dot{\epsilon}_{rr}$, $\dot{\epsilon}_{\theta\theta}$ and $\dot{\epsilon}_{zz}$ are eliminated from Equations 8.63–8.65 and substituted for $\dot{\sigma}_{eq}$ and $\dot{h}$ from Equations 8.66 and 8.67. Then, one gets

$$\frac{\partial v_r}{\partial r} = \frac{(2\sigma_{rr} - \sigma_{\theta\theta})}{2n\sigma_{eq}[(\sigma_Y + K) - \sigma_{eq}]} \left(\frac{\partial \sigma_{eq}}{\partial t} + v_r \frac{\partial \sigma_{eq}}{\partial r} \right), \quad (8.68)$$

$$\frac{v_r}{r} = \frac{(2\sigma_{\theta\theta} - \sigma_{rr})}{2n\sigma_{eq}[(\sigma_Y + K) - \sigma_{eq}]} \left(\frac{\partial \sigma_{eq}}{\partial t} + v_r \frac{\partial \sigma_{eq}}{\partial r} \right), \quad (8.69)$$

$$\frac{1}{h} \left(\frac{\partial h}{\partial t} + v_r \frac{\partial h}{\partial r} \right) = - \frac{(\sigma_{rr} + \sigma_{\theta\theta})}{2n\sigma_{eq}[(\sigma_Y + K) + \sigma_{eq}]} \left(\frac{\partial \sigma_{eq}}{\partial t} + v_r \frac{\partial \sigma_{eq}}{\partial r} \right). \quad (8.70)$$

Equations 8.53, 8.54 and 8.68–8.70 are the five governing equations of this problem for five unknown variables: two stress components σ_{rr} and $\sigma_{\theta\theta}$, the equivalent stress σ_{eq} , the radial velocity component v_r and the plate thickness h . Except for Equation 8.53, which is a non-linear algebraic equation, the remaining four are non-linear partial differential equations in the independent variables r and t . All the equations are coupled. These equations can be simplified as follows.

Simplification of the Governing Equations:

- *First simplification.* In a finite plate, the stress waves (elastic and/or plastic) created at the inner boundary travel in the radially outward direction and get reflected at the outward boundary and then again at the inner boundary. This process of reflection continues. As a result, the solution at a fixed r changes continuously with t . In an infinite plate, the stress waves keep traveling in the radially outward direction without any reflection as the outer boundary is at infinity. Therefore, in an infinite plate, the dependence of the solution on r and t is related. As a result, it is possible to express this dependence on a single parameter ξ :

$$\xi = \frac{r}{t}. \quad (8.71)$$

For a function $f = f(r, t) \equiv f(\xi)$, the r and t derivatives can be expressed using Equation 8.71 as

$$\begin{aligned} \frac{\partial f}{\partial r} &= \frac{df}{d\xi} \frac{\partial \xi}{\partial r} = \frac{1}{t} \frac{df}{d\xi}, \\ \frac{\partial f}{\partial t} &= \frac{df}{d\xi} \frac{\partial \xi}{\partial t} = -\frac{r}{t^2} \frac{df}{d\xi} = -\frac{\xi}{t} \frac{df}{d\xi}. \end{aligned} \quad (8.72)$$

Using the relations 8.72, the governing Equations 8.54 and 8.68–8.70 can be expressed as the (non-linear) ordinary differential equations in the independent variable ξ :

$$\begin{aligned}
 \frac{d\sigma_{rr}}{d\xi} + \frac{\sigma_{rr}}{h} \frac{dh}{d\xi} &= \frac{\sigma_{\theta\theta} - \sigma_{rr}}{\xi}, \\
 \frac{dv_r}{d\xi} &= \frac{(v_r - \xi)(2\sigma_{rr} - \sigma_{\theta\theta})}{2n\sigma_{eq}[(\sigma_Y + K) - \sigma_{eq}]} \frac{d\sigma_{eq}}{d\xi}, \\
 \frac{v_r}{\xi} &= \frac{(v_r - \xi)(2\sigma_{\theta\theta} - \sigma_{rr})}{2n\sigma_{eq}[(\sigma_Y + K) - \sigma_{eq}]} \frac{d\sigma_{eq}}{d\xi}, \\
 \frac{1}{h} \frac{dh}{d\xi} &= -\frac{(\sigma_{rr} + \sigma_{\theta\theta})}{2n\sigma_{eq}[(\sigma_Y + K) - \sigma_{eq}]} \frac{d\sigma_{eq}}{d\xi}.
 \end{aligned} \tag{8.73}$$

Second simplification. The technique used for the case of perfectly plastic material can also be used here. In this technique, a new variable is introduced that satisfies the yield criterion (Equation 8.53) identically. This technique also reduces the number of dependent variables by one. Similar to expression 8.41, the new variable ϕ is defined by the equation

$$\sigma_{rr} = -\frac{2\sigma_{eq}}{\sqrt{3}} \sin\left(\frac{\pi}{6} + \phi\right), \sigma_{\theta\theta} = \frac{2\sigma_{eq}}{\sqrt{3}} \sin\left(\frac{\pi}{6} - \phi\right). \tag{8.74}$$

Then, the yield criterion (Equation 8.53) is satisfied identically. The remaining four governing equations (Equation 8.73) become

$$\begin{aligned}
 \left(\frac{1}{\sigma_{eq}} \frac{d\sigma_{eq}}{d\xi} + \frac{1}{h} \frac{dh}{d\xi} \right) \tan\left(\frac{\pi}{6} + \phi\right) + \frac{d\phi}{d\xi} &= -\frac{2}{\xi} \frac{1}{(\sqrt{3} - \tan\phi)}, \\
 \frac{dv_r}{d\xi} &= -\frac{(v_r - \xi)}{2n[(\sigma_Y + K) - \sigma_{eq}]} \left[\cos\phi(\sqrt{3} + \tan\phi) \right] \frac{d\sigma_{eq}}{d\xi}, \\
 \frac{v_r}{\xi} &= \frac{(v_r - \xi)}{2n[(\sigma_Y + K) - \sigma_{eq}]} \left[\cos\phi(\sqrt{3} - \tan\phi) \right] \frac{d\sigma_{eq}}{d\xi}, \\
 \frac{1}{h} \frac{dh}{d\xi} &= -\frac{\sin\phi}{n[(\sigma_Y + K) - \sigma_{eq}]} \frac{d\sigma_{eq}}{d\xi}.
 \end{aligned} \tag{8.75}$$

- *Third simplification.* A partial nondimensionalization of the governing Equation 8.75 is carried out by introducing the following two nondimensional variables:

$$\eta = \frac{h}{h_0}, \quad s = \frac{\sigma_{eq}}{\sigma_Y + K}. \quad (8.76)$$

- Note that h_0 is the initial thickness of the plate. The variable ϕ is already nondimensional, whereas the dimension of the fourth variable u is that of the velocity. Using expression 8.76, Equation 8.75 becomes

$$\left(\frac{1}{s} \frac{ds}{d\xi} + \frac{1}{\eta} \frac{d\eta}{d\xi} \right) \tan \left(\frac{\pi}{6} + \phi \right) + \frac{d\phi}{d\xi} = -\frac{2}{\xi} \frac{1}{(\sqrt{3} - \tan \phi)}, \quad (8.77)$$

$$\frac{dv_r}{d\xi} = -\frac{(v_r - \xi)}{2n(1-s)} \left[\cos \phi (\sqrt{3} + \tan \phi) \right] \frac{ds}{d\xi}, \quad (8.78)$$

$$\frac{v_r}{\xi} = \frac{(v_r - \xi)}{2n(1-s)} \left[\cos \phi (\sqrt{3} - \tan \phi) \right] \frac{ds}{d\xi}, \quad (8.79)$$

$$\frac{1}{\eta} \frac{d\eta}{d\xi} = \frac{\sin \phi}{n(1-s)} \frac{ds}{d\xi}. \quad (8.80)$$

- Fourth simplification. In this simplification, certain equations are combined to put them in a form that would be convenient for numerical integration. Equation 8.79 is rearranged first:

$$\frac{ds}{d\xi} = -\frac{2n(1-s)}{(\xi - v_r)} \frac{\sec \phi}{(\sqrt{3} - \tan \phi)} \frac{v_r}{\xi}. \quad (8.81)$$

Then, $ds/d\xi$ is eliminated from Equations 8.78 and 8.79. This leads to

$$\frac{dv_r}{d\xi} = -\frac{(\sqrt{3} + \tan \phi)}{(\sqrt{3} - \tan \phi)} \frac{v_r}{\xi}. \quad (8.82)$$

Next, $ds/d\xi$ is eliminated again from Equations 8.79 and 8.80 to get

$$\frac{d\eta}{d\xi} = -\frac{\eta}{(\xi - v_r)} \frac{2 \tan \phi}{(\sqrt{3} - \tan \phi)} \frac{v_r}{\xi}. \quad (8.83)$$

Finally, $ds/d\xi$ and $d\eta/d\xi$ are eliminated from Equation 8.77 using Equations 8.79 and 8.83. This leads to

$$\frac{d\phi}{d\xi} = \frac{\left\{ (v_r / \xi) \left[\sqrt{3} \sec \phi + (n(1-s)/s) (1 + \sqrt{3} \tan \phi) \right] \sec((\pi/6) + \phi) - 2 \right\}}{(\xi - v_r)(\sqrt{3} - \tan \phi)}. \quad (8.84)$$

Equations 8.81–8.84 are four coupled non-linear ordinary differential equations (in independent variable ξ) in four unknowns: s , v_r , η and ϕ . They can be solved by an appropriate numerical scheme. They need to be solved simultaneously as they are coupled.

Boundary conditions. First, one needs to choose an appropriate timescale. Note that, as the applied pressure increases beyond $2\sigma_Y/\sqrt{3}$, the radius ρ of the interface (between the hardening and perfectly plastic regions as shown in Figure 8.14) starts increasing. Thus, both the time t and the radius ρ increase simultaneously. Assuming a linear relationship between them and choosing the proportionality constant to be one for convenience, the following can be written:

$$t \propto \rho, \text{ (proportionality constant} = 1\text{)}. \quad (8.85)$$

Then,

$$\xi = \frac{r}{t} = \frac{r}{\rho}. \quad (8.86)$$

From Figure 8.14, the following conditions are presented at the interface $r = \rho$ or $\xi = 1$.

$$h = h_0, \sigma_{eq} = \sigma_Y, \sigma_{rr} = -\frac{2\sigma_Y}{\sqrt{3}} \text{ at } \xi = 1. \quad (8.87)$$

Since the solution is being considered only after p exceeds $2\sigma_Y/\sqrt{3}$, the radial velocity v_r can be zero at the interface (i.e., at $\xi = 1$). Using the previous information and the assumptions and Equations 8.74 and 8.76, one gets the following boundary conditions:

Boundary condition at $\xi = 1$:

$$s = \frac{\sigma_{eq}}{\sigma_Y + K} = \frac{\sigma_Y}{\sigma_Y + K} = \frac{1}{1 + K/\sigma_Y}, \quad (8.88)$$

$$v_r = 0, \quad (8.89)$$

$$\eta = \frac{h}{h_0} = \frac{h_0}{h_0} = 1, \quad (8.90)$$

$$\sin\left(\frac{\pi}{6} + \phi\right) = -\frac{\sigma_{rr}}{(2\sigma_{eq}/\sqrt{3})} = -\frac{(-2\sigma_{eq}/\sqrt{3})}{(2\sigma_{eq}/\sqrt{3})} = 1 \Rightarrow \phi = \frac{\pi}{3}. \quad (8.91)$$

Numerical solution. The four coupled non-linear ordinary differential Equations 8.81–8.84 can be solved along with the boundary conditions for Equations 8.88–8.91 by standard numerical techniques. The variations of σ_{rr}/σ_y , $\sigma_{\theta\theta}/\sigma_y$ and h/h_0 with respect to the nondimensional radial coordinate r/c are given in the book by Chakrabarty (2010b) for the following values of material parameters:

$$K = 1.5\sigma_y, n = 9. \quad (8.92)$$

8.4 ANALYSIS OF PLASTIC DEFORMATION IN THE FLANGE OF CIRCULAR CUP DURING DEEP DRAWING PROCESS (PLANE STRESS AND AXISYMMETRIC PROBLEM)

In a circular cup deep drawing process, a circular sheet is drawn into a hollow cylindrical cup with the help of an annular die, a cylindrical punch and an annular blank holder. The setup of the process is shown in Figure 8.16. The process consists of the following:

- Radial drawing of the outer annulus of the sheet, called the flange. During the drawing, the flange thickens to some extent. Further, if the pressure applied by the blank holder is not enough, then the flange can wrinkle
- Bending and stretching of the sheet at the die radius and punch radius
- Stretching in the wall of the cup. The greatest thinning occurs in the wall near the punch radius

There is friction at the interfaces with the punch, the die and the blank holder. Further, there could be a loss of the contact of the sheet at the interfaces with the die and the blank holder. This happens during the wrinkling of the flange. An analytical

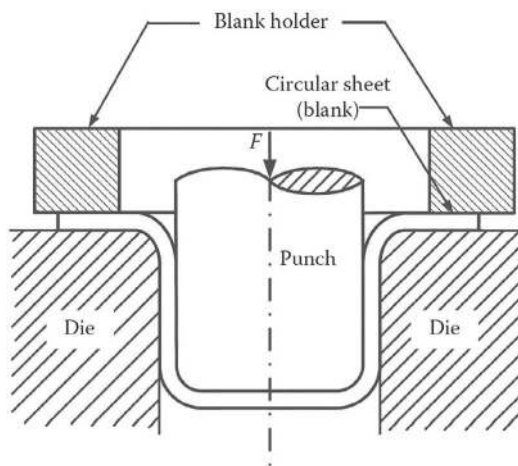


FIGURE 8.16 Setup of circular cup deep drawing process.

solution of the process involving all of the three aspects as well as the friction and the loss of contact at the interfaces is not possible.

However, it is possible to analyze the plastic deformation in the flange under certain assumptions. The analysis of flange deformation is useful as a significant part of the work done by the punch force F is spent in this part of the process. The following assumptions are made to make the problem both the plane stress as well as axisymmetric:

- The flange thickening is negligible
- Lubrication at the interfaces with the die and the blank holder is such that the friction at these interfaces is negligible
- The pressure applied by the blank holder is enough to prevent the wrinkling of the flange
- Further, the stresses in the plane of the flange are greater than the pressure applied by the die or the blank holder by an order of the magnitude. Thus, $\sigma_{zz} \simeq 0$

Then, with respect to the cylindrical polar coordinates (r, θ, z) , the stress matrix becomes

$$[\sigma] = \begin{bmatrix} \sigma_{rr} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (8.93)$$

with only $(\sigma_{rr}, \sigma_{\theta\theta})$ as the non-zero stress components.

It is further assumed that

- The plate material is perfectly plastic and yields according to the Mises criterion with σ_y as the yield stress
- The punch force F is large enough to make the whole flange plastic

The Eulerian formulation is used with the current configuration as the control volume. As observed in the problem of Section 8.3 (i.e., in the problem of hole expansion in an infinite plate), it is possible to determine only the stresses without simultaneously determining the strain rates or the velocities in a plane stress/axisymmetric problem if the material is perfectly plastic. Therefore, only the stresses shall be determined in Section 8.4.1. In the problem of Section 8.3, the strain rates and velocities were determined only in the hardening region as they were not significant in the non-hardening region. In the present problem, even though the material is assumed to be perfectly plastic, the strains (and not the strain rates) shall be determined in subsequent sections as they are significant.

8.4.1 DETERMINATION OF STRESSES

The domain of the problem is shown in Figure 8.17. In this figure, b denotes the current radius of the sheet (that keeps reducing), whereas a represents the smallest

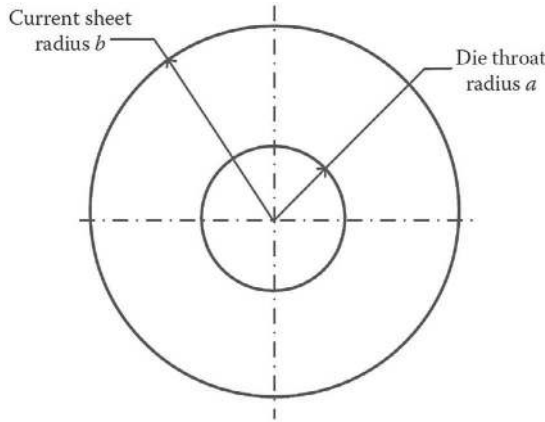


FIGURE 8.17 Domain for the plastic analysis of flange.

radius up to which the sheet is in contact with the die. This is called the die throat radius. The two *governing equations* for $(\sigma_{rr}, \sigma_{\theta\theta})$ are given as follows.

- *Equilibrium equations.* Since the problem is both axisymmetric and plane stress, the equilibrium equations in the θ - and z -directions are satisfied identically. It is assumed that the body force is negligible. Then, the equilibrium equation in the r -direction becomes

$$\frac{\partial \sigma_{rr}}{\partial r} = \frac{\sigma_{\theta\theta} - \sigma_{rr}}{r}. \quad (8.94)$$

Here, the stress components also depend on the time t besides the radial coordinate r . Therefore, the derivative with respect to r is denoted as a partial derivative

- *Yield criterion.* As stated earlier, the material is perfectly plastic and yields according to the Mises criterion. Then, the yield stress σ_Y does not change with the plastic deformation but remains constant. Further, the problem is a plane stress problem with $(\sigma_{rr}, \sigma_{\theta\theta})$ as the non-zero principal stresses. Then, the Mises yield criterion is given by Equation 8.40:

$$\sigma_{rr}^2 - \sigma_{rr}\sigma_{\theta\theta} + \sigma_{\theta\theta}^2 - \sigma_Y^2 = 0. \quad (8.95)$$

This is a non-linear algebraic equation.

Boundary condition. Since Equation 8.94 is a first-order ordinary differential equation, only one boundary condition is needed. Note that the outer edge of the sheet ($r = b$) is a free surface. Then, the boundary condition becomes

$$\sigma_{rr} = 0, \text{ at } r = b. \quad (8.96)$$

Solution: Out of the two governing equations, one is a non-linear algebraic equation. As stated in Section 8.3, the standard solution technique used for such problems is to introduce a new variable that would satisfy the algebraic equation identically. This technique also reduces the number of dependent variables from two to one. Unlike the problem of Section 8.3, in this problem, σ_{rr} is positive, whereas $\sigma_{\theta\theta}$ is negative. Therefore, for the present problem, the new variable (denoted by ϕ) is defined slightly differently than the problem of Section 8.3 (i.e., differently than Equation 8.41):

$$\sigma_{rr} = \frac{2\sigma_y}{\sqrt{3}} \sin\phi, \quad \sigma_{\theta\theta} = -\frac{2\sigma_y}{\sqrt{3}} \cos\left(\frac{\pi}{6} + \phi\right). \quad (8.97)$$

These expressions satisfy the yield criterion (Equation 8.95) identically.

To find the equation satisfied by ϕ , expressions 8.97 are substituted into the equilibrium Equation 8.94. Upon simplification, one gets the following non-linear differential equation:

$$\frac{\partial\phi}{\partial r} = -\frac{(\sqrt{3} + \tan\phi)}{2r}. \quad (8.98)$$

Equations 8.96 and 8.97 imply that at $r = b$:

$$\frac{2\sigma_y}{\sqrt{3}} \sin\phi = 0. \quad (8.99)$$

which leads to the following boundary condition for the new variable ϕ :

$$\phi = 0, \text{ at } r = b. \quad (8.100)$$

The separation of the variable technique is used to solve this problem. Equation 8.98 can be rearranged as

$$\frac{d\phi}{(\sqrt{3} + \tan\phi)} = -\frac{dr}{2r} \quad (8.101)$$

Integration of this equation and simplification leads to

$$r^2 = \frac{\sqrt{3}}{2} c_1 \exp(-\sqrt{3}\phi) \sec\left(\frac{\pi}{6} - \phi\right). \quad (8.102)$$

The constant c_1 is determined from the boundary condition Equation 8.100. Its value is

$$c_1 = b^2. \quad (8.103)$$

Then, the solution to the differential Equation 8.98 and the boundary condition Equation 8.100 is

$$r^2 = \frac{\sqrt{3}}{2} b^2 \exp(-\sqrt{3}\phi) \sec\left(\frac{\pi}{6} - \phi\right). \quad (8.104)$$

Thus, the stress components (σ_{rr} , $\sigma_{\theta\theta}$) are given by expression 8.97 where ϕ depends on r through the relation Equation 8.104.

8.4.2 DETERMINATION OF STRAINS

As in the problem of Section 8.3, the elastic strain rate can be neglected, being much smaller than the plastic strain rate in metals. Further, the equivalent stress σ_{eq} becomes equal to σ_Y in a perfectly plastic material. Then, the stress-strain rate relations become (Equation 5.128)

$$\dot{\epsilon}_{ij} = \frac{3\dot{\epsilon}_{eq}}{2\sigma_Y} \sigma'_{ij}. \quad (8.105)$$

where the deviator stress tensor σ'_{ij} and the equivalent stress σ_{eq} can be obtained from the expressions of the stress components (σ_{rr} , $\sigma_{\theta\theta}$) of Section 8.4.1.

Note that, in a perfectly plastic material, the strain rates can be determined uniquely only up to an arbitrary multiplicative constant. To verify this fact, let $\dot{\epsilon}_{ij}^{(1)}$ be a solution of Equation 8.105 corresponding to the given stress field. Then, $\dot{\epsilon}_{ij}^{(1)}$ satisfies the following equation:

$$\dot{\epsilon}_{ij}^{(1)} = \frac{3\dot{\epsilon}_{eq}^{(1)}}{2\sigma_Y} \sigma'_{ij} \quad (8.106)$$

where $\dot{\epsilon}_{eq}^{(1)}$ is the equivalent strain corresponding to $\dot{\epsilon}_{ij}^{(1)}$ obtained from Equation 5.116. Now, consider

$$\dot{\epsilon}_{ij}^{(2)} = -\dot{\epsilon}_{ij}^{(1)} \quad (8.107)$$

where α is any arbitrary constant. Then, from Equation 5.116, one gets

$$\dot{\epsilon}_{eq}^{(2)} = -\dot{\epsilon}_{eq}^{(1)} \quad (8.108)$$

It can be easily seen that $\dot{\epsilon}_{ij}^{(2)}$ also satisfies Equation 8.105, no matter what α is. This shows that, in a perfectly plastic material, the strain rates can be determined uniquely only up to an arbitrary multiplicative constant. However, this constant can be determined from the kinematic boundary conditions of the problem.

From Equation 8.105, the explicit expressions can be obtained for the three non-zero strain rates $\dot{\epsilon}_{rr}$, $\dot{\epsilon}_{\theta\theta}$ and $\dot{\epsilon}_{zz}$ (up to an arbitrary multiplicative constant). Since the motion of the material particles in the flange region does not involve any rotation, one can integrate these individual strain rate expressions to obtain the total strains in r , θ (hoop) and z (thickness) directions. However, this approach is not pursued as both the determination of the strain rate components and their integration are not easy. Therefore, an alternate procedure is followed.

8.4.2.1 Determination of Logarithmic Hoop Strain

Two configurations need to be considered for the determination of strain. Figure 8.18 shows two configurations of the flange: (1) the initial configuration at $t = 0$; and (2) the current configuration at time t .

Consider a circular fiber at a radial location r_0 in the initial configuration, which is at the location r in the current configuration. Then, the logarithmic hoop strain is given by

$$\epsilon_{\theta\theta} = \ln \left(\frac{2\pi r}{2\pi r_0} \right) = \ln \left(\frac{r}{r_0} \right) \quad (8.109)$$

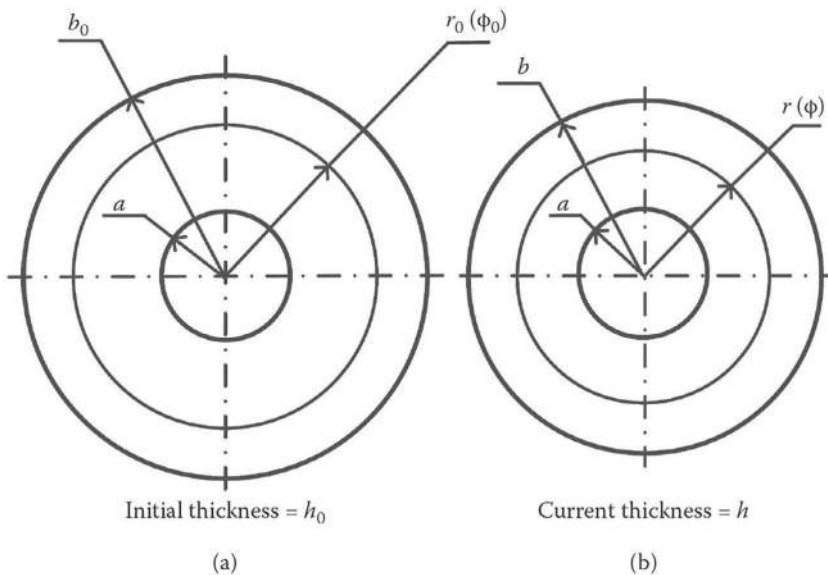


FIGURE 8.18 Two configurations of the flange for the determination of strains: (a) initial configuration at $t = 0$; (b) current configuration at time t .

where r can be obtained from

$$r = r_0 + \int_{r_0}^r dr = r_0 + \int_{r_0}^r \frac{dr}{dt} dt = r_0 + \int_{r_0}^r v_r dt. \quad (8.110)$$

However, as stated earlier, it is not easy to obtain the explicit expression for $\dot{\epsilon}_{rr} = \partial v_r / \partial r$. Further, since the stresses depend only implicitly on time t , it is not possible to find v_r as an explicit function of t . Therefore, an alternative approach is followed.

A fiber is identified in the initial configuration by the variable ϕ_0 by relating it to its location r_0 by a relation similar to Equation 8.104:

$$r_0^2 = \frac{\sqrt{3}}{2} b_0^2 \exp(-\sqrt{3}\phi_0) \sec\left(\frac{\pi}{6} - \phi_0\right), \quad (8.111)$$

where b_0 is the initial radius of the sheet. Of course, in the current configuration, the current value of the variable ϕ is related to the current location r by Equation 8.104. Then, the expression for the hoop strain, in terms of ϕ_0 and ϕ , becomes

$$-\epsilon_{\theta\theta} = \ln\left(\frac{r_0}{r}\right) = \ln\left\{\frac{b_0}{b} \left(\frac{\sec(\pi/6 - \phi_0)}{\sec(\pi/6 - \phi)}\right)^{0.5} \exp\left(-\frac{\sqrt{3}}{2}(\phi_0 - \phi)\right)\right\}. \quad (8.112)$$

In order to calculate the hoop strain in a fiber at location r_0 , and thus identified by ϕ_0 in the initial configuration, its current value ϕ needs to be found. The procedure to obtain ϕ corresponding to ϕ_0 is explained in the next few paragraphs.

Since the outer radius b of the sheet varies with time t , the rate of change of ϕ is obtained by differentiating Equation 8.104 with respect to b :

$$\frac{d\phi}{db} = -\frac{2[(1/r)dr/db - (1/b)]}{\tan(\pi/6 - \phi) + \sqrt{3}}. \quad (8.113)$$

In order to facilitate the integration, dr/db is expressed in terms of the radial velocity v_r . For that purpose, a suitable implicit timescale needs to be chosen. Since b decreases with t , t is considered proportional to $b_0 - b$. Choosing the proportionality constant as one, the implicit timescale becomes

$$t = b_0 - b. \quad (8.114)$$

Then, the expression for dr/db in terms of v_r becomes

$$\frac{dr}{db} = -\frac{dr}{dt} = -v_r. \quad (8.115)$$

Using the previous expression, Equation 8.113 is expressed in terms of v_r :

$$\frac{d\phi}{db} = \frac{2[(v_r/r) + (1/b)]}{[\tan(\pi/6 - \phi) + \sqrt{3}]} \quad (8.116)$$

Before this equation is integrated, v_r needs to be expressed in terms of ϕ .

Use of the stress-strain rate and the strain rate-velocity relations gives the following differential equation for v_r (Equation 8.184 of Appendix A):

$$\frac{\partial v_r}{\partial \phi} = \frac{(\sqrt{3}\tan\phi + 1)}{(\tan\phi + \sqrt{3})} v_r. \quad (8.117)$$

Now, a boundary condition is needed for v_r at the outer surface (i.e., at $r = b$ or $\phi = 0$). Since v_r is in the negative r -direction, a suitable negative value for it can be assumed at the outer boundary. It is assumed to be -1 . Thus, the boundary condition becomes

$$v_r = -1, \text{ at } \phi = 0. \quad (8.118)$$

The solution of the differential Equation 8.117 along with the boundary condition Equation 8.118 is

$$v_r^2 = \frac{\sqrt{3}}{2} \exp(\sqrt{3}\phi) \sec\left(\frac{\pi}{6} - \phi\right). \quad (8.119)$$

Before this expression is substituted in Equation 8.116, it is simplified using Equation 8.104. Eliminating $(\sqrt{3}/2)\sec(\pi/6 - \phi)$ from Equations 8.104 and 8.119 and taking the square root of the resulting expression, one gets

$$v_r = -\frac{r}{b} \exp(\sqrt{3}\phi). \quad (8.120)$$

(Here, the negative sign is chosen for the square root because v_r is negative.)

Substitution of the previous expression for v_r in Equation 8.116 and further simplification leads to

$$\frac{[\tan(\pi/6 - \phi) + \sqrt{3}]d\phi}{[\exp(\sqrt{3}\phi) - 1]} = -\frac{2db}{b}. \quad (8.121)$$

Integrating the previous equation from the initial to the current configuration, one gets

$$\int_{\phi_0}^{\phi} \frac{[\tan(\pi/6 - \phi) + \sqrt{3}]}{[\exp(\sqrt{3}\phi) - 1]} d\phi = - \int_{b_0}^b \frac{2db}{b} = 2\ln\left(\frac{b_0}{b}\right). \quad (8.122)$$

Given the initial radius (b_0) of the sheet and the location of the fiber in the initial configuration (ϕ_0), its current location (ϕ) in the configuration defined by the current radius (b) is found from the previous expression where the integration has to be done numerically.

Once the current location of the fiber (ϕ) is determined from the previous equation, the logarithmic hoop strain in the current configuration is found from expression 8.112.

8.4.2.2 Determination of Logarithmic Thickness Strain

The differential equation for the current plate thickness h (Equation 8.191 of Appendix B) is

$$\frac{dh}{h} = \frac{(\sqrt{3}\tan\phi - 1)}{2} \frac{dr}{r}. \quad (8.123)$$

On the right-hand side, there are two independent variables related to each other: r as well as ϕ . Therefore, the right-hand side is rearranged into two terms to make each term a function of a single independent variable. The rearrangement is

$$\frac{dh}{h} = -\frac{2dr}{r} + \frac{dr}{r} \frac{(\sqrt{3}\tan\phi + 3)}{2}. \quad (8.124)$$

In order to eliminate dr/r from the first term of the right-hand side, Equations 8.115, 8.120 and 8.121 are used. First, the elimination of v_r from Equations 8.115 and 8.120 gives

$$\frac{dr}{r} = \frac{db}{b} \exp(\sqrt{3}\phi). \quad (8.125)$$

Further, the elimination of db/b from Equations 8.121 and 8.125 leads to

$$\frac{dr}{r} = -\frac{1}{2} \frac{[\tan(\pi/6 - \phi) + \sqrt{3}] \exp(\sqrt{3}\phi)}{[\exp(\sqrt{3}\phi) - 1]} d\phi. \quad (8.126)$$

Simple trigonometric manipulation gives

$$\tan(\pi/6 - \phi) + \sqrt{3} = \frac{4\sqrt{3}}{(\sqrt{3}\tan\phi + 3)}. \quad (8.127)$$

Eliminating $\tan(\pi/6 - \phi) + \sqrt{3}$ from Equations 8.126 and 8.127, one gets

$$\begin{aligned} \frac{dr}{r} &= -\frac{1}{2} \frac{\left[\exp(\sqrt{3}\phi) \right]}{\left[\exp(\sqrt{3}\phi) - 1 \right]} \frac{4\sqrt{3}}{(\sqrt{3}\tan\phi + 3)} d\phi \\ &= -\frac{(2\sqrt{3}) \left[\exp(\sqrt{3}\phi) \right] d\phi}{\left[\exp(\sqrt{3}\phi) - 1 \right] (\sqrt{3}\tan\phi + 3)}. \end{aligned} \quad (8.128)$$

Substitution of this expression in the second term on the right-hand side of Equation 8.124 leads to

$$\frac{dh}{h} = -\frac{2dr}{r} - \frac{(\sqrt{3}) \left[\exp(\sqrt{3}\phi) \right] d\phi}{\left[\exp(\sqrt{3}\phi) - 1 \right]}. \quad (8.129)$$

The boundary condition for this differential equation is that the thickness of the fiber in the initial configuration (i.e., at $r = r_0$ or $\phi = \phi_0$) is h_0 . Thus,

$$h = h_0, \quad \text{at } r = r_0, \quad \text{or } \phi = \phi_0. \quad (8.130)$$

The solution of the differential Equation 8.129 along with the boundary condition is

$$\ln\left(\frac{h}{h_0}\right) = \ln\left(\frac{r_0}{r}\right)^2 + \ln\left(\frac{\exp(\sqrt{3}\phi_0) - 1}{\exp(\sqrt{3}\phi) - 1}\right). \quad (8.131)$$

Using Equations 8.104 and 8.111, the ratio $(r_0/r)^2$ can be expressed in terms of $(b_0/b)^2$, ϕ_0 and ϕ :

$$\left(\frac{r_0}{r}\right)^2 = \left(\frac{b_0}{b}\right)^2 \frac{\exp(-\sqrt{3}\phi_0) \sec(\pi/6 - \phi_0)}{\exp(-\sqrt{3}\phi) \sec(\pi/6 - \phi)}. \quad (8.132)$$

Elimination of $(r_0/r)^2$ from Equations 8.131 and 8.132 gives the following expression for the logarithmic thickness strain:

$$\ln\left(\frac{h}{h_0}\right) = \ln\left(\frac{b_0}{b}\right)^2 + \ln\left(\frac{1 - \exp(-\sqrt{3}\phi_0)}{1 - \exp(-\sqrt{3}\phi)}\right) + \ln\left(\frac{\sec(\pi/6 - \phi_0)}{\sec(\pi/6 - \phi)}\right). \quad (8.133)$$

In Equation 8.133, the current location (ϕ) of the fiber in the configuration defined by the current radius (b) of the sheet corresponding to its initial location (ϕ_0) in the

configuration defined by the initial radius (b_0) is obtained from expression 8.122 by numerical integration. The variation of logarithmic thickness strain with nondimensional radial coordinate r/b_0 is given in the book by Chakrabarty (2010b) for different values of b_0/a ratios.

8.5 NECKING OF A CYLINDRICAL ROD

In the tension test, initially, the state of stress in the rod is 1-D (i.e., uni-axial) both before as well as after yielding. However, at some level of plastic deformation, the inherent defects in the rod change the state of stress from 1-D to 3-D (i.e., triaxial), thereby initiating necking (Figure 8.19). The analysis of necking in cylindrical rods was first carried out by Bridgman (1944). The solution presented in this section is essentially due to Bridgman.

The problem of the necking of a cylindrical rod is axisymmetric. Therefore, with respect to the cylindrical polar coordinates (r, θ, z), the stress matrix becomes

$$[\sigma] = \begin{bmatrix} \sigma_{rr} & 0 & \sigma_{rz} \\ 0 & \sigma_{\theta\theta} & 0 \\ \sigma_{rz} & 0 & \sigma_{zz} \end{bmatrix} \quad (8.134)$$

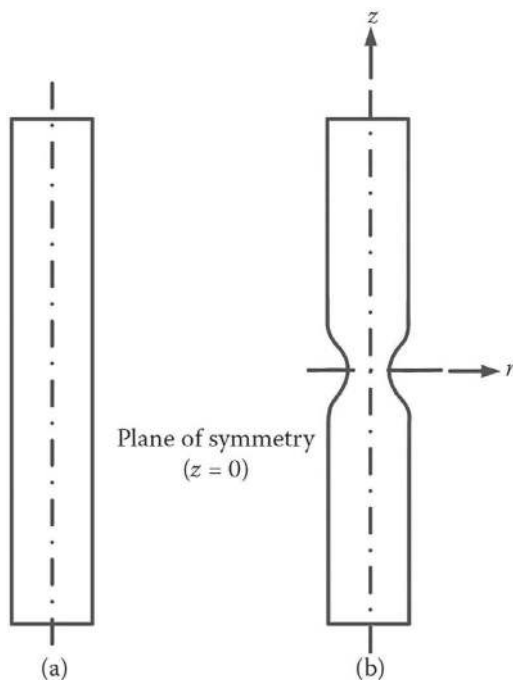


FIGURE 8.19 States of stress in tension test of rod before and after necking: (a) uni-axial state (only σ_{zz}) before necking; (b) triaxial state (σ_{rr} , $\sigma_{\theta\theta}$, σ_{zz}) after necking.

with $(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{rz})$ as the non-zero stress components. To simplify the solution procedure further, the following assumptions are made.

- The material is assumed to be perfectly plastic. Further, it is assumed to yield according to the Mises criterion with σ_y as the yield stress
- Only the plane of symmetry is analyzed (i.e., the plane $z = 0$) as shown in Figure 8.19b. Since necking starts only after the whole rod has yielded, it is understood that the whole domain is plastic. It is assumed that, in the plane of symmetry, σ_{rz} is zero. Thus,

$$\sigma_{rz} \Big|_{z=0} = 0. \quad (8.135)$$

- Then, the number of non-zero stress components reduces to three
- As done in the problems of Sections 8.3 and 8.4, the stress-strain rate relations are simplified by assuming that the elastic strain rate is negligible compared to the plastic strain rate. Then, using the stress-strain relation of Equation 8.105, the expressions for the strain rate components $\dot{\epsilon}_{rr}$ and $\dot{\epsilon}_{\theta\theta}$ become

$$\dot{\epsilon}_{rr} = \frac{3\dot{\epsilon}_{eq}}{2\sigma_y} \left(\sigma_{rr} - \frac{1}{3} \text{tr}[\sigma] \right), \quad \dot{\epsilon}_{\theta\theta} = \frac{3\dot{\epsilon}_{eq}}{2\sigma_y} \left(\sigma_{\theta\theta} - \frac{1}{3} \text{tr}[\sigma] \right). \quad (8.136)$$

- In the problems of Sections 8.3 and 8.4, the stress components σ_{rr} and $\sigma_{\theta\theta}$ were of opposite signs and different magnitudes. In the present problem, both these stress components are of the same sign (i.e., compressive). It is possible to reduce the number of non-zero stress components to two by assuming that σ_{rr} and $\sigma_{\theta\theta}$ are equal in magnitude. This happens if it is assumed that the radial velocity v_r is proportional to r :

$$v_r = \alpha r, \quad (\alpha \text{ constant}) \quad (8.137)$$

Since the circumferential velocity v_θ is zero in axisymmetric problems, the strain rate-velocity relations, using Equation 8.137, become

$$\dot{\epsilon}_{rr} = \frac{\partial v_r}{\partial r} = \alpha, \quad \dot{\epsilon}_{\theta\theta} = \frac{v_r}{r} = \alpha. \quad (8.138)$$

Now, it is easy to see that the stress-strain rate (Equation 8.136) and the strain rate-velocity (Equation 8.138) relations imply

$$\sigma_{rr} = \sigma_{\theta\theta}. \quad (8.139)$$

8.5.1 ANALYSIS IN THE PLANE OF SYMMETRY ($z = 0$)

Equations 8.134, 8.135 and 8.139 show that, in the plane of symmetry, the number of independent non-zero stress components is only two: (1) $\sigma_{rr} = \sigma_{\theta\theta}$ and (2) σ_{zz} .

Therefore, the two governing equations (the equilibrium equation in the r -direction and the yield criterion) are sufficient to find the two unknown stress components. These governing equations are as follows.

Governing equations for $(\sigma_{rr} = \sigma_{\theta\theta}, \sigma_{zz})$:

- *Equilibrium equation.* In an axisymmetric problem, the equilibrium equation in the θ -direction is satisfied identically. It is assumed that the body force is negligible. Then, the equilibrium equation in the z -direction, in view of Equation 8.135, reduces to $\partial\sigma_{zz}/\partial z$ being 0 in the plane of symmetry. This relation is not required in the present problem. Finally, the equilibrium equation in the r -direction, in view of Equation 8.139, becomes

$$\left(\frac{\partial\sigma_{rr}}{\partial r} + \frac{\partial\sigma_{rz}}{\partial z} \right) \Big|_{z=0} = 0. \quad (8.140)$$

- Here, the stress components, besides being dependent on the radial coordinate r , also depend on the axial coordinate z . Therefore, the derivatives are denoted as partial derivatives.
- *Yield criterion.* As stated earlier, the material is perfectly plastic and yields according to the Mises criterion. Then, the yield stress σ_Y does not change with the plastic deformation but remains constant. Further, in view of Equation 8.135, the stress components $(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz})$ are the principal stresses in the present problem. Then, the Mises yield criterion (Equation 5.48) with $\sigma_1 = \sigma_{rr}$, $\sigma_2 = \sigma_{\theta\theta}$ and $\sigma_3 = \sigma_{zz}$ becomes

$$\left[(\sigma_{rr} - \sigma_{\theta\theta})^2 + (\sigma_{\theta\theta} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{rr})^2 \right] - 2\sigma_Y^2 = 0. \quad (8.141)$$

Substituting the equality $\sigma_{rr} = \sigma_{\theta\theta}$ (Equation 8.139) in the previous relation and taking the square root, one gets

$$(\sigma_{zz} - \sigma_{rr}) = \sigma_Y. \quad (8.142)$$

Here, the positive sign is chosen for the square root as $\sigma_{zz} > 0$ and $\sigma_{rr} < 0$.

Elimination of σ_{rr} from the two governing Equations 8.140 and 8.142 leads to the following differential equation for σ_{zz} as a function of r in the plane of symmetry:

$$\left(\frac{\partial\sigma_{zz}}{\partial r} + \frac{\partial\sigma_{rz}}{\partial z} \right) \Big|_{z=0} = 0. \quad (8.143)$$

However, this equation contains an additional unknown $(\partial\sigma_{rz}/\partial z) \Big|_{z=0}$ that needs

to be expressed in terms of some known and/or measurable quantities. Bridgman (1944) expressed it in terms of σ_Y and the two geometric parameters: (1) the smallest radius of the rod in the necked region (denoted by a); and (2) the radius of curvature of the necked arc, in the r - z plane, at the smallest cross section (denoted by R). The

procedure to do this is explained in Section 8.5.1.1. However, once it is done, the boundary condition for the resulting differential equation is needed.

Boundary condition for σ_{zz} :

In the plane of symmetry, the boundary $r = a$ is a part of a free surface. Therefore, the stress component σ_{rr} is zero along this boundary:

$$(\sigma_{rr} = 0) \Big|_{z=0}, \quad \text{at } r = a. \quad (8.144)$$

However, the boundary condition is needed in terms of σ_{zz} . The previous boundary condition can be expressed in terms of σ_{zz} using the simplified yield criterion (Equation 8.142). Elimination of σ_{rr} from Equations 8.141 and 8.142 leads to

$$(\sigma_{zz} = \sigma_Y) \Big|_{z=0}, \quad \text{at } r = a. \quad (8.145)$$

8.5.1.1 Simplification of Differential Equation

The simplification of the differential equation, that is, expressing the unknown $(\partial\sigma_{rz}/\partial z) \Big|_{z=0}$ in terms of some known geometric parameters, is done in two steps.

In the first step, the unknown $(\partial\sigma_{rz}/\partial z) \Big|_{z=0}$ is expressed in terms of the radius of curvature of the trajectory of the first principal stress (denoted by ρ). In the second step, the radius of curvature ρ is expressed in terms of the geometric parameters a and R , which were defined earlier.

Expressing $(\partial\sigma_{rz}/\partial z) \Big|_{z=0}$ in terms of ρ :

Mohr's circle for the stress state in the r - z plane, at a point very close to the plane of symmetry, is shown in Figure 8.20. Here, ψ is the clockwise inclination of the first principal axis with the z -axis. Then, from this figure, one gets the following expression for the stress component σ_{rz} :

$$\sigma_{rz} = \frac{(\sigma_1 - \sigma_2)}{2} \sin(2\psi). \quad (8.146)$$

For very small values of ψ , the principal stresses in the r - z plane can be approximated as

$$\sigma_1 = \sigma_{zz'} \quad \sigma_2 = \sigma_{rr} \quad (8.147)$$

Further, for very small values of ψ ,

$$\sin(2\psi) \approx 2\psi. \quad (8.148)$$

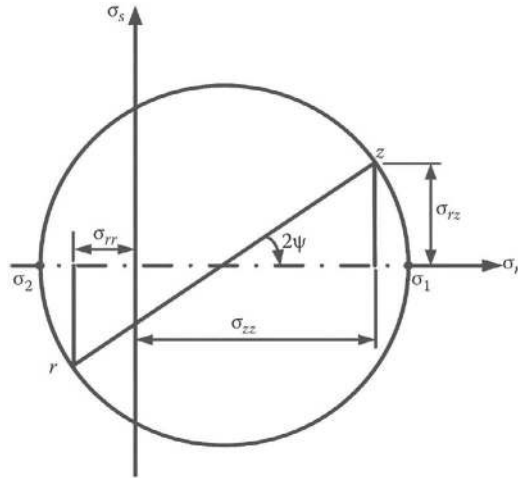


FIGURE 8.20 Mohr's circle for the state of stress in the r - z plane at a point close to the plane of symmetry.

Substituting the previous approximations (Equations 8.147 and 8.148) and the simplified yield condition (Equation 8.142) in Equation 8.146, for very small values of ψ , one gets

$$\sigma_{rz} = \sigma_Y \psi. \quad (8.149)$$

Then, for very small values of ψ (i.e., in the vicinity of the plane of symmetry $z=0$), the partial differentiation of the previous equation with respect to z leads to

$$\frac{\partial \sigma_{rz}}{\partial z} = \sigma_Y \frac{\partial \psi}{\partial z}. \quad (8.150)$$

Figure 8.21 shows the trajectory of the first principal axis in the r - z plane in the vicinity of the plane of symmetry. The trajectory is symmetric about the plane of symmetry. Further, at the plane of symmetry, its tangent is along the z -axis since the z -axis is the first principal direction at the plane of symmetry. At a point close to the plane of symmetry, ψ (i.e., the clockwise inclination of the first principal axis with the z -axis) at that point is shown. Let ρ be the radius of curvature of the trajectory of the first principal axis at the plane of symmetry. Then, from the definition of the radius of curvature, one gets

$$\left. \frac{\partial \psi}{\partial z} \right|_{z=0} \approx \frac{1}{\rho}. \quad (8.151)$$

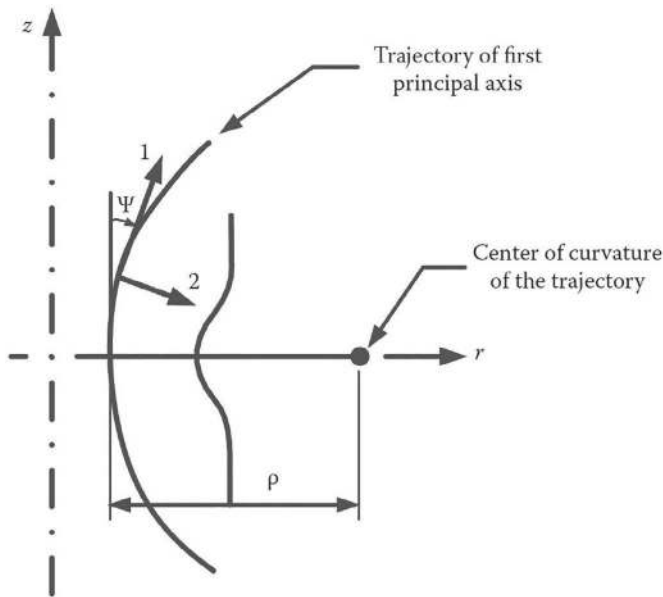


FIGURE 8.21 Trajectory of the first principal axis in the r - z plane in the vicinity of the plane of symmetry.

Evaluating Equation 8.150 at $z = 0$ and then substituting for $(\partial\psi / \partial z) |_{z=0}$, one gets

$$\frac{\partial\sigma_{rz}}{\partial z} \Big|_{z=0} = \sigma_Y \frac{\partial\psi}{\partial z} \Big|_{z=0} = \frac{\sigma_Y}{\rho}. \quad (8.152)$$

Expressing ρ in Terms of a and R :

The trajectories of the first and second principal axes are always orthogonal. They are denoted as trajectory-1 and trajectory-2, respectively. It is assumed that, in the vicinity of the plane of symmetry, they are circular (Bridgman, 1944). Figure 8.22 shows two of the trajectory-1 in the vicinity of the plane of symmetry and the intersecting trajectory-2 at points Q and S.

In Figure 8.22, O is the origin of the coordinate axes. Further, the trajectory-1 passing through point S denotes the necking arc. It intersects the r -axis at point T. Thus, the distance OT is a (i.e., the smallest radius of the rod in the necked region). The radius of curvature of the necked arc at point T is denoted as R . Then the distance TD is R where D is the center of curvature. Similarly, the trajectory-1 passing through point Q intersects the r -axis at point P. The radius of curvature of this trajectory is denoted at point P as ρ . Then the distance PC is ρ where C is the center of curvature. The tangent to the trajectory-1 at point Q intersects the z -axis at point B. Thus, point B becomes the center of curvature of trajectory-2 at point Q. Further, trajectory-2 intersects the r -axis at point N.

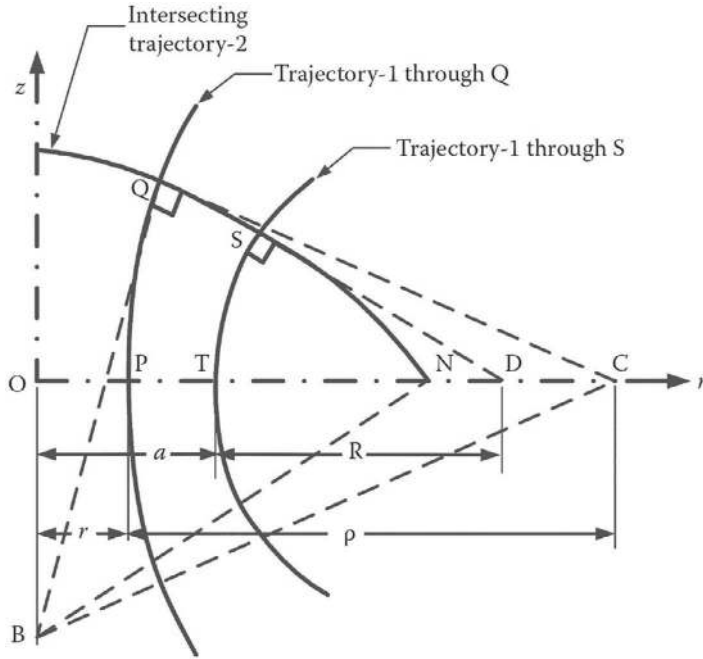


FIGURE 8.22 Two trajectories of the first principal axis and an intersecting trajectory of the second principal axes (in r - z plane) in the vicinity of the plane of symmetry.

From the geometry of Figure 8.22, ρ can be expressed as

$$\rho^2 \equiv (QC)^2 = (BC)^2 - (BQ)^2. \quad (8.153)$$

Since the points Q and N lie on the trajectory-2 (assumed circular), using the Pythagoras theorem, the following can be written:

$$(BQ)^2 \equiv (BN)^2 = (OB)^2 + (ON)^2. \quad (8.154)$$

Note that the distance OC is the sum of the radial coordinate r of point P and the radius of the curvature ρ of the trajectory-1 (through Q) at point P. Then, the following can be written using the Pythagoras theorem:

$$(BC)^2 = (OB)^2 + (OC)^2 = (OB)^2 + (r + \rho)^2. \quad (8.155)$$

Substituting expressions 8.154 and 8.155 for BQ and BC in expression 8.153 for ρ , and canceling the term $(OB)^2$, one gets the following expression for the distance ON:

$$\rho^2 = (r + \rho)^2 - (ON)^2 \Rightarrow (ON)^2 = r^2 + 2r\rho. \quad (8.156)$$

Note that for the trajectory-1 through S, the distance OD is the sum of the radial coordinate a of point T and its radius of the curvature R at point T. Similar analysis for the trajectory-1 through S gives the following expression for the distance ON:

$$(\text{ON})^2 = a^2 + 2aR. \quad (8.157)$$

Elimination of $(\text{ON})^2$ from Equations 8.156 and 8.157 gives the following expression for ρ in terms of r , a and R :

$$\rho = \frac{a^2 + 2aR - r^2}{2r}. \quad (8.158)$$

Substitution of this expression in Equation 8.152 gives the following expression for $(\partial\sigma_{rz}/\partial z)|_{z=0}$ in terms of σ_Y , the geometric parameters a and R and the radial coordinate r :

$$\frac{\partial\sigma_{rz}}{\partial z}|_{z=0} = \frac{\sigma_Y}{\rho} = \frac{2r\sigma_Y}{a^2 + 2aR - r^2}. \quad (8.159)$$

Elimination of $(\partial\sigma_{rz}/\partial z)|_{z=0}$ from Equations 8.143 and 8.159 gives the following modified differential equation for σ_{zz} with respect to r :

$$\frac{\partial\sigma_{zz}}{\partial r}|_{z=0} = \frac{-2r\sigma_Y}{a^2 + 2aR - r^2}. \quad (8.160)$$

This differential equation is to be solved along with the boundary condition 8.145.

8.5.1.2 Solution of the Modified Differential Equation

To solve the differential Equation 8.160 in the plane of symmetry ($z = 0$), it is rearranged as

$$d\sigma_{zz} = \sigma_Y \frac{-2r dr}{a^2 + 2aR - r^2}. \quad (8.161)$$

By integrating it, one gets

$$\sigma_{zz} = \sigma_Y \ln(a^2 + 2aR - r^2) + \ln C, \quad (8.162)$$

where C is a constant. The use of the boundary condition $\sigma_{zz} = \sigma_Y$ at $r = a$ (Equation 8.145) gives the following expression for the constant C :

$$\ln C = \sigma_Y - \sigma_Y \ln(2aR). \quad (8.163)$$

Elimination of the constant $\ln C$ from Equations 8.162 and 8.163 and a certain simplification lead to the following expression for σ_{zz} in the plane of symmetry:

$$\sigma_{zz} = \sigma_Y + \sigma_Y \ln \left(\frac{a^2 + 2aR - r^2}{2aR} \right). \quad (8.164)$$

Using the simplified yield condition of Equation 8.142 and the previous expression for σ_{zz} , one gets the following expression for σ_{rr} in the plane of symmetry:

$$\sigma_{rr} \equiv \sigma_{zz} - \sigma_Y = \sigma_Y \ln \left(\frac{a^2 + 2aR - r^2}{2aR} \right). \quad (8.165)$$

Since $\sigma_{\theta\theta}$ is equal to σ_{rr} (Equation 8.139), the expression for it in the plane of symmetry is the same:

$$\sigma_{\theta\theta} \equiv \sigma_{rr} = \sigma_Y \ln \left(\frac{a^2 + 2aR - r^2}{2aR} \right). \quad (8.166)$$

Before the necking, only σ_{zz} is non-zero and is uniform over the cross section (i.e., independent of r). Since the material is assumed perfectly plastic, its value is equal to the yield stress σ_Y . However, after the necking, besides σ_{zz} , the other normal stress components σ_{rr} and $\sigma_{\theta\theta}$ also exist in the plane of symmetry. (On either side of the plane of symmetry, the shear stress σ_{rz} also exists.) Further, they are not uniform over the cross section but vary with the radial coordinate r (in the plane of symmetry) as per Equations 8.164–8.166. Equation 8.164 shows that the value of σ_{zz} is equal to σ_Y at the outer surface ($r = a$). However, it starts increasing as we move toward the center. Its maximum value occurs at the center of the cross section ($r = 0$). The maximum value is $\sigma_{zz} = \sigma_Y + \sigma_Y \ln((a^2 + 2aR)/2aR)$, which is greater than σ_Y . Even though the material is assumed perfectly plastic, in a multiaxial state of stress, individual stress components can be greater than σ_Y . This is because, in a multiaxial state of stress, a combination of the stress components (like in Equation 8.142) becomes equal to σ_Y rather than an individual component. The combination is decided by the yield criterion.

Exercises

1. A stepped bar, shown in Figure 8.E1, is fixed at both ends and is subjected to an axial force P at the point of discontinuity in the area of the cross section. The yield stress of the bar material is σ_Y . When the bar behaves *elastically*, the stresses are given by

$$(\sigma_{xx})_a = \frac{1}{5} \frac{P}{A}, \quad (\sigma_{xx})_b = -\frac{2}{5} \frac{P}{A}$$

where A is the area of cross section of part a (i.e., the left part).

- a. Which part of the bar yields first and why? Let P_1 be the value of the axial force at *first yielding*. Show that

$$P_1 = \frac{5}{2} \sigma_Y A.$$

- b. Assume that the bar material is *perfectly plastic*. Let P_2 be the value of the axial force when the other part also yields. Using the equilibrium equation and the yield criterion, show that

$$P_2 = 3\sigma_Y A.$$

- c. Now, assume that the bar material is linearly *hardening*:

$\sigma_{eq} = \sigma_Y + K \varepsilon_{eq}^p$, ($K = E/4$), (K = hardening parameter, E = Young's modulus).

This means that, in the 1-D state of stress, the incremental plastic stress-strain relation is given by

$$d\varepsilon_{xx}^p = \frac{1}{K} d\sigma_{xx}.$$

- i. Using the incremental equilibrium equation, the incremental stress-strain (elastic or elasto-plastic) relation, and the incremental compatibility condition (i.e., the incremental change in length of part a is equal to the magnitude of the incremental change in length of part b), show that, for $P > P_1$, the incremental stresses are given by

$$(d\sigma_{xx})_a = \frac{5}{9} \frac{dP}{A} \text{ (elastic)}, (d\sigma_{xx})_b = -\frac{2}{9} \frac{dP}{A} \text{ (elasto-plastic)}.$$

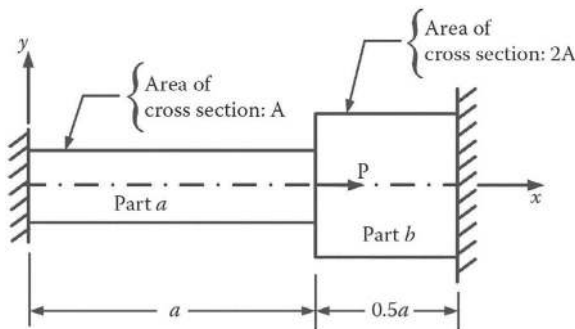


FIGURE 8.0E1

- ii. Using the results of part (i) and the yield criterion, show that the value of the axial force when the other part also yields is given by

$$P_2 = \frac{34}{10} \sigma_Y A.$$

2. A circular rod of radius a is subjected to an axial force F and the torque T , as shown in Figure 8.E2. The stress matrix in the rod, in cylindrical polar coordinates (r, θ, z) , is

$$[\sigma] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \sigma_{\theta z} \\ 0 & \sigma_{\theta z} & \sigma_{zz} \end{bmatrix}. \quad (8.167)$$

During the elastic behavior, the non-zero stresses are given by

$$\sigma_{zz} = \frac{F}{\pi a^2}, \quad \sigma_{\theta z} = \frac{2Tr}{\pi a^4}. \quad (8.168)$$

a. *Yielding*

- Find the three principal stresses in terms of σ_{zz} and $\sigma_{\theta z}$.
- Show that the Mises criterion for this problem, in terms of the principal stresses, reduces to

$$\sigma_{zz}^2 + 3\sigma_{\theta z}^2 = \sigma_Y^2, \quad (8.169a)$$

and

$$F^2 + 12 \left(\frac{T}{a} \right)^2 \left(\frac{r}{a} \right)^2 = (\pi a^2 \sigma_Y)^2. \quad (8.169b)$$

where σ_Y is the initial yield stress of the material.

- Show that the yielding first begins at $r = a$.

b. *Plastic analysis*

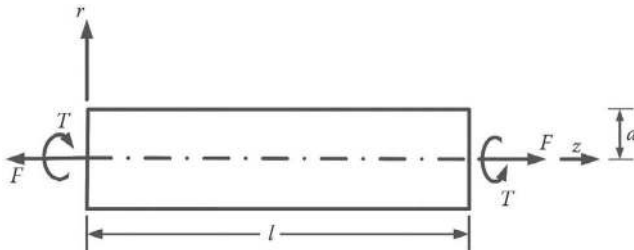


FIGURE 8.0E2

- As F and T are increased, the plastic boundary moves inward. Let $r = c < a$ be the plastic boundary at time t
- In the plastic region, σ_{zz} is expected to be a function of r , whereas $\sigma_{\theta z}$ is expected to be a different function of r than given by Equation 8.168. Since both σ_{zz} and $\sigma_{\theta z}$ are expected to be independent of θ and z , the equilibrium equations are identically satisfied
- Assume the following expressions for the components of the incremental displacement vector:

$$du_r = -\frac{r}{2l}dl, \quad du_\theta = \frac{rz}{l}d\phi, \quad du_z = \frac{z}{l}dl \quad (8.170)$$

where dl and $d\phi$ are the incremental changes in length and twist

- The matrix of the incremental linear strain tensor in the rod is

$$[d\varepsilon] = \begin{bmatrix} d\varepsilon_{rr} & 0 & 0 \\ 0 & d\varepsilon_{\theta\theta} & d\varepsilon_{\theta z} \\ 0 & d\varepsilon_{\theta z} & d\varepsilon_{zz} \end{bmatrix}.$$

- Using Equation 8.170 and the incremental strain-displacement relations, show that

$$d\varepsilon_{rr} = -\frac{1}{2} \frac{dl}{l}, \quad d\varepsilon_{\theta\theta} = -\frac{1}{2} \frac{dl}{l}, \quad d\varepsilon_{zz} = \frac{dl}{l}, \quad d\varepsilon_{\theta z} = \frac{r d\phi}{2l}. \quad (8.171)$$

- From the $[\sigma]$ matrix of Equation 8.167, find the matrix of the deviatoric part $[\sigma']$ in terms of σ_{zz} and $\sigma_{\theta z}$.
- Assume that the material is perfectly plastic and neglect the elastic deformation. Then, the constitutive behavior is governed by the Levy-Mises equation. Using Equation 8.171 and the expressions for σ'_{ij} from part ii, write down the Levy-Mises equation for non-zero components in terms of dl , l , $d\phi$, r , $d\varepsilon_{eq}$, σ_Y , σ_{zz} and $\sigma_{\theta z}$ to show that

$$\frac{dl}{l} = \frac{d\varepsilon_{eq}}{\sigma_Y} \sigma_{zz}, \quad \frac{d\phi}{dl} = \frac{3}{r} \frac{\sigma_{\theta z}}{\sigma_{zz}}. \quad (8.172)$$

- Assume that the ratio $d\phi/dl$ is held constant during the plastic deformation. Then, using the yield criterion (Equation 8.169a) and the second condition of Equation 8.172, show that the expressions for the stress components σ_{zz} and $\sigma_{\theta z}$ in the plastic region are

$$\sigma_{zz} = \frac{\sigma_Y}{\sqrt{1 + (3\beta^2)r^2}}, \quad \sigma_{\theta z} = \frac{(\beta\sigma_Y)r}{\sqrt{1 + (3\beta^2)r^2}}, \quad \beta = \frac{1}{3} \frac{d\phi}{dl} \text{ (constant)}.$$



FIGURE 8.0E3

3. Figure 8.E3 shows a uniform and homogeneous shaft of the circular cross section subjected to pure torsion under the action of the torque T . Make the following assumptions:

- The strain-displacement relation $\epsilon_{\theta z} = r(d\phi/dz)$ holds even after yielding where $d\phi/dz$ is the twist per unit length

$$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{zz} = \sigma_{r\theta} = \sigma_{zr} = 0$$

- The shaft material is perfectly plastic (with yield stress $= \sigma_Y$) and obeys the Mises initial yield criterion
- The equilibrium equation is given by

$$T = \int_A r \sigma_{\theta z} dA.$$

- a. Show that, after initial yielding, the torque-twist relationship is given by

$$T = \frac{3}{4} T_Y \left[1 - \frac{1}{4} \left(\frac{\phi_Y}{\phi} \right)^3 \right],$$

where T_Y is the torque at which the initial yielding starts and ϕ_Y is the twist corresponding to T_Y .

- b. Show that the fully plastic (or limit or collapse) torque is given by $T_L = (4/3)T_Y$.
- c. Sketch the variations of $\sigma_{\theta z}$ with r for the following four cases: (i) $T < T_Y$; (ii) $T = T_Y$; (iii) $T_Y < T < T_L$; and (iv) $T = T_L$.
4. Figure 8.E4a shows a cantilever beam of length l , height $2h$, and unit thickness, subjected to a uniformly distributed (downward) force q . The beam material is elastic-perfectly plastic with σ_Y as the yield stress. The bending moment diagram is shown in Figure 8.E4b.
- a. Show that yielding first occurs at the top and bottom corners of the fixed end at the value $q = q_Y$ where

$$q_Y = \frac{4}{3} \left(\frac{h}{l} \right)^2 \sigma_Y$$

- b. Show that the force corresponding to the plastic zone (extending up to $x = a$) shown in Figure 8.E4c is

$$q = \frac{4}{3} \left(\frac{h}{l-a} \right)^2 \sigma_Y. \quad (8.173)$$

- c. Using the stress distribution over the cross section at x (where the height of the elastic core is c), show that the bending moment at this section is

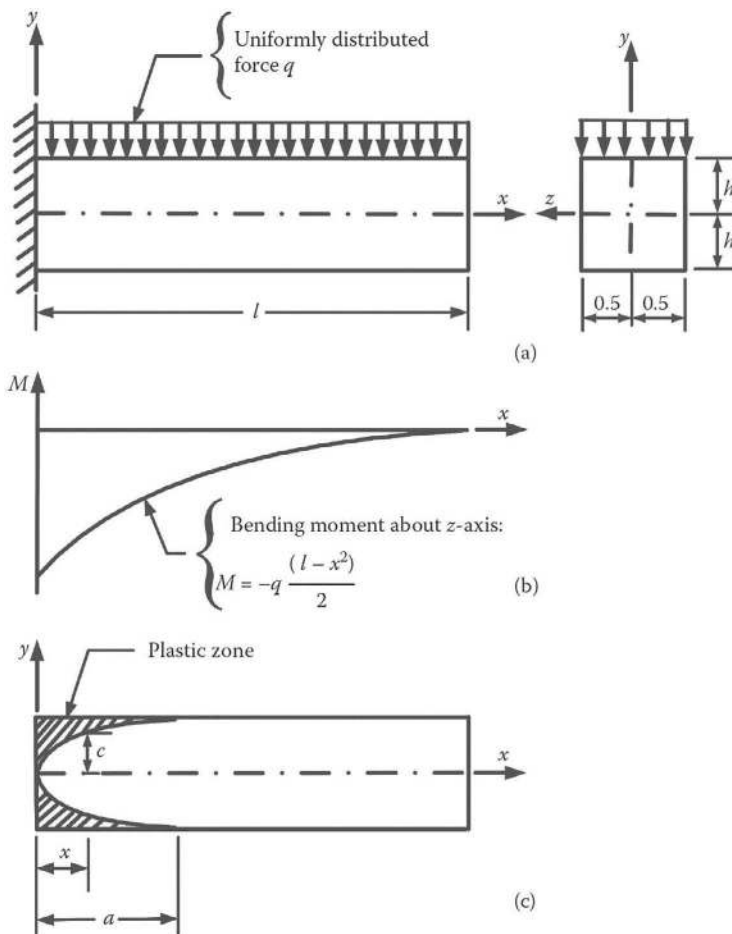


FIGURE 8.0E4

$$M = -\sigma_Y \left(\frac{3h^2 - c^2}{3} \right).$$

- d. Equating the expression for M of part (c) with the expression for the bending moment diagram of Figure 8.E4b, and eliminating the corresponding force q using the expression 8.173, show that (for given a), the dependence of c on x is given by

$$c = h \left[3 - \frac{2(l-x)^2}{(l-a)^2} \right]^{0.5}.$$

5. Figure 8.E5 shows a thick-walled cylinder (with a and b as the internal and external radii, respectively) subjected to internal pressure p and in the state of both *axisymmetry* and *plane strain*. It is assumed that the *Poisson's ratio* (ν) is 0.5. Further, the cylinder material is assumed to be *perfectly plastic* (with σ_Y as the yield stress) and yield according to the *Mises* criterion. When the cylinder behaves elastically, the stresses are

$$\begin{aligned} \sigma_{rr} &= \frac{a^2}{b^2 - a^2} \left(1 - \frac{b^2}{r^2} \right) p, & \sigma_{\theta\theta} &= \frac{a^2}{b^2 - a^2} \left(1 + \frac{b^2}{r^2} \right) p \\ \sigma_{zz} &= \frac{a^2}{b^2 - a^2} p = \frac{\sigma_{rr} + \sigma_{\theta\theta}}{2}, & \sigma_{ij} &= 0 \text{ (for } i \neq j \text{)}. \end{aligned} \quad (8.174)$$

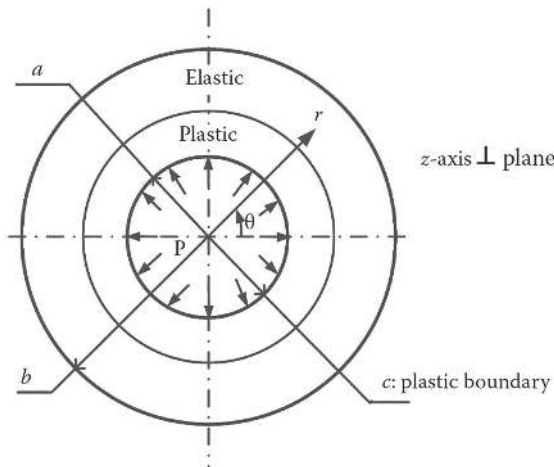


FIGURE 8.0E5

- a. Show that yielding first occurs at $r = a$ at the value $p = p_Y$ where

$$p_Y = \frac{1}{\sqrt{3}} \frac{b^2 - a^2}{b^2} \sigma_Y.$$

- b. For $p > p_Y$, using the equilibrium equation

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

and the yield criterion, show that the stresses in the *plastic* region (i.e., for $r \leq c$) are

$$\sigma_{rr} = -p + \frac{2}{\sqrt{3}} \sigma_Y \ln(r/a), \quad \sigma_{\theta\theta} = -p + \frac{2}{\sqrt{3}} \sigma_Y [\ln(r/a) + 1]$$

Note: The internal pressure p , in terms of p_c (i.e., the value of $-\sigma_{rr}$ at the *plastic boundary* $r = c$), becomes

$$p = p_c + \frac{2}{\sqrt{3}} \sigma_Y \ln(c/a). \quad (8.175)$$

- c. Neglecting the elastic deformation, using the stress-strain rate relation in the z -direction and the plane strain condition, show that σ_{zz} in the plastic region is

$$\sigma_{zz} = -p + \frac{1}{\sqrt{3}} \sigma_Y [\ln(r/a) + 1].$$

- d. *Note:* Expressions for the stresses in the *elastic* region (i.e., $r \geq c$), using Equation 8.174, become

$$\sigma_{rr} = \frac{c^2}{b^2 - c^2} \left(1 - \frac{b^2}{r^2} \right) p_c, \quad \sigma_{\theta\theta} = \frac{c^2}{b^2 - c^2} \left(1 + \frac{b^2}{r^2} \right) p_c, \quad \sigma_{zz} = \frac{c^2}{b^2 - c^2} p_c = \frac{\sigma_{rr} + \sigma_{\theta\theta}}{2}.$$

Using the yield criterion at $r = c$, find the expression for p_c in terms of b , c and σ_Y .

- e. Using expression 8.175 for p and the expression for p_c from part (d), show that the *limit* pressure (i.e., the pressure when the whole cylinder becomes plastic) is

$$p_L = \frac{2}{\sqrt{3}} \sigma_Y \ln(b/a).$$

6. For the problem of hole expansion in an infinite plate (Section 8.3), choose the following values for the geometric and material parameters:

$$\sigma_Y = 250 \text{ MPa}, K = 500 \text{ MPa}, n = 3, a_0 = 100 \text{ mm}, h_0 = 1 \text{ mm}.$$

- Solve the four coupled non-linear ordinary differential Equations 8.81–8.84 numerically along with the boundary conditions 8.88–8.91 to obtain s , v_r , η and ϕ . Then, find the equivalent stress σ_{eq} from s using Equation 8.76. Find the non-dimensional stress components σ_{rr}/σ_Y and $\sigma_{\theta\theta}/\sigma_Y$ from ϕ using Equation 8.74.
 - Using the results of part (a), plot the variations of the non-dimensional stress components σ_{rr}/σ_Y and $\sigma_{\theta\theta}/\sigma_Y$ and the non-dimensional thickness $\eta = h/h_0$ (Equation 8.76) with respect to the radial coordinate r for the plastic hardening region $a \leq r \leq \rho$ using the relation $r = \rho\xi$ (Equation 8.86). Choose the current hole radius as $a = 105 \text{ mm}$ and the radius of the hardening-nonhardening interface as $\rho = 120 \text{ mm}$.
 - Using expressions 8.41, plot the variations of the non-dimensional stress components σ_{rr}/σ_Y and $\sigma_{\theta\theta}/\sigma_Y$ with respect to the radial coordinate r for the plastic non-hardening region $\rho \leq r \leq c = 1.751\rho$ using relation 8.48 between r and ϕ .
 - Using expressions 8.38, plot the variations of the non-dimensional stress components σ_{rr}/σ_Y and $\sigma_{\theta\theta}/\sigma_Y$ with respect to the radial coordinate r for the elastic region $r \geq c = 1.751\rho$.
7. For the problem of flange analysis in circular cup deep drawing (Section 8.4), find the variations of logarithmic hoop strain and logarithmic thickness strain using the following procedure:
- The initial radius of the sheet is $b_0 = 260 \text{ mm}$ and the throat radius is $a = 100 \text{ mm}$. In the initial configuration, choose nine values of r_0 lying equidistant between the interval (a, b_0) . For each of these nine values of r_0 , find the corresponding value of ϕ_0 from Equation 8.111:

$$r_0^2 = \frac{\sqrt{3}}{2} b_0^2 \exp(-\sqrt{3}\phi_0) \sec\left(\frac{\pi}{6} - \phi_0\right)$$

- Let the radius of the blank in the current configuration be $b = 200 \text{ mm}$. Corresponding to nine values of ϕ_0 in the initial configuration (of part [a]), find the corresponding values of ϕ in the current configuration by evaluating numerically the following integral (Equation 8.122):

$$\int_{\phi_0}^{\phi} \frac{[\tan(\pi/6 - \phi) + \sqrt{3}] d\phi}{[\exp(\sqrt{3}\phi) - 1]} = 2 \ln\left(\frac{b_0}{b}\right).$$

- c. Corresponding to nine values of ϕ of part (b), find the corresponding values of the current radial coordinate r from Equation 8.104:

$$r^2 = \frac{\sqrt{3}}{2} b^2 \exp(-\sqrt{3}\phi) \sec\left(\frac{\pi}{6} - \phi\right).$$

- d. Find the variation of the logarithmic hoop strain $-\varepsilon_{\theta\theta}$ (given by Equation 8.112) with respect to r in the current configuration ($b = 200$) using the nine values of ϕ_0 , ϕ and r from parts (a), (b) and (c).
- e. Find the variation of the logarithmic thickness strain $\ln(h/h_0)$ (given by Equation 8.133) with respect to r in the current configuration ($b = 200$) using the nine values of ϕ_0 , ϕ and r from parts (a), (b) and (c).
8. In the problem of the necking of a cylindrical rod (Section 8.5), using the expression for the axial stress σ_{zz} (Equation 8.164), show that the average value of σ_{zz} over the cross section is

$$(\sigma_{zz})_{av} = \sigma_Y \left[\left(1 + \frac{2R}{a}\right) \ln\left(1 + \frac{a}{2R}\right) \right].$$

APPENDIX A: DIFFERENTIAL EQUATION FOR RADIAL VELOCITY

Using Equation 8.93 for the stress matrix, the matrix for the deviatoric stress becomes

$$[\sigma'] = \begin{bmatrix} \frac{2\sigma_{rr} - \sigma_{\theta\theta}}{3} & 0 & 0 \\ 0 & \frac{2\sigma_{\theta\theta} - \sigma_{rr}}{3} & 0 \\ 0 & 0 & -\frac{\sigma_{rr} + \sigma_{\theta\theta}}{3} \end{bmatrix}. \quad (8.176)$$

From the stress-strain relations in Equation 8.105, the $\dot{\varepsilon}_{rr}$ and $\dot{\varepsilon}_{\theta\theta}$ components of the strain rate tensor can be written as follows:

$$\dot{\varepsilon}_{rr} = \frac{3\dot{\varepsilon}_{eq}}{2\sigma_Y} \sigma'_{rr}, \quad \dot{\varepsilon}_{\theta\theta} = \frac{3\dot{\varepsilon}_{eq}}{2\sigma_Y} \sigma'_{\theta\theta}. \quad (8.177)$$

Substituting the expressions for σ'_{rr} and $\sigma'_{\theta\theta}$ from Equation 8.176, the previous expressions become

$$\dot{\varepsilon}_{rr} = \frac{3\dot{\varepsilon}_{eq}}{2\sigma_Y} \left(\frac{2\sigma_{rr} - \sigma_{\theta\theta}}{3} \right), \quad \dot{\varepsilon}_{\theta\theta} = \frac{3\dot{\varepsilon}_{eq}}{2\sigma_Y} \left(\frac{2\sigma_{\theta\theta} - \sigma_{rr}}{3} \right). \quad (8.178)$$

The strain rate-velocity relations corresponding to the previous strain rate components are

$$\dot{\epsilon}_{rr} = \frac{\partial v_r}{\partial r}, \quad \dot{\epsilon}_{\theta\theta} = \frac{v_r}{r}. \quad (8.179)$$

Elimination of $\dot{\epsilon}_{rr}$, $\dot{\epsilon}_{\theta\theta}$ and $\dot{\epsilon}_{eq}/\sigma_Y$ from Equations 8.178 and 8.179 leads to

$$\frac{\partial v_r}{\partial r} = \frac{(2\sigma_{rr} - \sigma_{\theta\theta})}{(2\sigma_{\theta\theta} - \sigma_{rr})} \frac{v_r}{r}. \quad (8.180)$$

Using expressions 8.97 for σ_{rr} and $\sigma_{\theta\theta}$ in terms of the new independent variable ϕ , one gets

$$\frac{(2\sigma_{rr} - \sigma_{\theta\theta})}{(2\sigma_{\theta\theta} - \sigma_{rr})} = \frac{2\sin\phi - (-\cos(\pi/6 + \phi))}{2(-\cos(\pi/6 + \phi)) - \sin\phi} = -\frac{(\sqrt{3}\tan\phi + 1)}{2}. \quad (8.181)$$

Substitution of the previous expression in Equation 8.180 leads to the following differential equation for v_r with respect to r :

$$\frac{\partial v_r}{\partial r} = -\frac{(\sqrt{3}\tan\phi + 1)}{2} \frac{v_r}{r}. \quad (8.182)$$

However, the differential equation with respect to ϕ is needed. For that, the chain rule is used:

$$\frac{\partial v_r}{\partial \phi} = \frac{\partial v_r}{\partial r} \frac{\partial r}{\partial \phi}. \quad (8.183)$$

Note that the expression for $\partial r/\partial \phi$ is readily available as the stress equilibrium Equation 8.98 in terms of ϕ . Substituting for $\partial v_r/\partial r$ and $\partial r/\partial \phi$ from Equations 8.182 and 8.98 in Equation 8.183, one gets the required differential equation with respect to ϕ :

$$\frac{\partial v_r}{\partial \phi} = \frac{\partial v_r}{\partial r} \frac{\partial r}{\partial \phi} = \left[\frac{(\sqrt{3}\tan\phi + 1)}{2} \frac{v_r}{r} \right] \left[-\frac{2r}{(\sqrt{3} + \tan\phi)} \right] = \frac{(\sqrt{3}\tan\phi + 1)}{(\tan\phi + \sqrt{3})} v_r. \quad (8.184)$$

APPENDIX B: DIFFERENTIAL EQUATION FOR CURRENT PLATE THICKNESS

From the stress-strain relations 8.105, the $\dot{\epsilon}_{\theta\theta}$ and $\dot{\epsilon}_{zz}$ components of the strain rate tensor can be written as follows:

$$\dot{\epsilon}_{\theta\theta} = \frac{3\dot{\epsilon}_{eq}}{2\sigma_Y} \sigma'_{\theta\theta} \quad \dot{\epsilon}_{zz} = \frac{3\dot{\epsilon}_{eq}}{2\sigma_Y} \sigma'_{zz}. \quad (8.185)$$

Substituting the expressions for $\sigma'_{\theta\theta}$ and σ'_{zz} from Equation 8.176, the previous expressions become

$$\dot{\epsilon}_{\theta\theta} = \frac{3\dot{\epsilon}_{eq}}{2\sigma_Y} \left(\frac{2\sigma_{\theta\theta} - \sigma_{rr}}{3} \right) \cdot \dot{\epsilon}_{zz} = -\frac{3\dot{\epsilon}_{eq}}{2\sigma_Y} \left(\frac{\sigma_{rr} + \sigma_{\theta\theta}}{3} \right). \quad (8.186)$$

As done in the problem of Section 8.3 (i.e., in the problem of hole expansion in an infinite plate), the constant $\dot{\epsilon}_{zz}$ across the plate thickness can be approximated as the ratio of the thickness rate to the current thickness, instead of relating it to the z derivative of v_z :

$$\dot{\epsilon}_{zz} = \frac{\dot{h}}{h} = \frac{1}{h} \frac{dh}{dt}. \quad (8.187)$$

The strain rate-velocity relation for $\dot{\epsilon}_{\theta\theta}$ component (Equation 8.179) can be written as

$$\dot{\epsilon}_{\theta\theta} = \frac{v_r}{r} = \frac{1}{r} \frac{dr}{dt}. \quad (8.188)$$

Elimination of $\dot{\epsilon}_{\theta\theta}$, $\dot{\epsilon}_{zz}$ and $\dot{\epsilon}_{eq}/\sigma_Y$ from Equations 8.186–8.188 leads to

$$\frac{dh}{h} = -\frac{(\sigma_{rr} + \sigma_{\theta\theta})}{(2\sigma_{\theta\theta} - \sigma_{rr})} \frac{dr}{r}. \quad (8.189)$$

Using expressions 8.97 for σ_{rr} and $\sigma_{\theta\theta}$ in terms of the new independent variable ϕ , one gets

$$\frac{(\sigma_{rr} + \sigma_{\theta\theta})}{(2\sigma_{\theta\theta} - \sigma_{rr})} = \frac{\sin\phi + (-\cos(\pi/6 + \phi))}{2(-\cos(\pi/6 + \phi)) - \sin\phi} = -\frac{(\sqrt{3}\tan\phi - 1)}{2}. \quad (8.190)$$

Substitution of the previous expression in Equation 8.189 leads to the following differential equation for h :

$$\frac{dh}{h} = \frac{(\sqrt{3}\tan\phi - 1)}{2} \frac{dr}{r}. \quad (8.191)$$

9 Contact Mechanics

9.1 INTRODUCTION

Contact mechanics is concerned with the determination of stresses and deformation when two bodies are in contact. There are several examples in which a force is applied by a body to another body by physical contact, some of which are given as follows:

1. Contact between a wheel and a rail. Here, both the contacting bodies are considered deformable.
2. Contact between a tire and a road. Here, the tire can be considered deformable and the road can be considered rigid.
3. Contact between two gear teeth. Here, both the teeth need to be considered deformable.
4. Contact between a cam and a follower. Here, the cam and the follower are considered deformable bodies.
5. Indentation of a material by a punch. Usually, the punch is considered rigid.
6. Bearings. In rolling element bearings, there is a contact between rolling elements and inner/outer races of bearings. In journal bearing, there can be a metal-to-metal contact between shaft and bearing surfaces in dry conditions. In the presence of lubrication, either the shaft is fully supported on the lubricant or there is a partial contact in the boundary lubrication situation.
7. Several metal-forming processes, like forging, deep drawing and rolling.

The study of contact mechanics is useful for getting information about material behavior based on hardness tests, for understanding the tribology (friction, lubrication and wear) and for modeling of many manufacturing processes.

There are two types of contacting surfaces: (1) conforming and (2) non-conforming. Conforming surfaces have some contact area even in the absence of any force. For example, a rectangular cross section punch on a flat surface and a collar fitted on a shaft are examples of conforming surfaces. In non-conforming surfaces, a load needs to be applied to get a finite area. For example, a rolling cylinder on a flat floor surface will be only in line contact if there is no load. When some load is applied, the floor surface and the surface of the cylinder contact over a finite area. During the contact, bodies may undergo elastic or plastic deformation. Even if the deformation is elastic, the problem is non-linear because the contact area changes with force.

Mathematical analysis of contact problems started with the seminal work of Hertz (1881). Hertz assumed that the geometry of general curved surfaces in the vicinity of contact can be described by quadratic terms only. Geometric compatibility condition leads to an elliptic contact area. Neglecting friction and assuming that the bodies in contact deform as though they were elastic half-spaces, Hertz deduced that the

contact pressure distribution has to be elliptic for producing an elliptic contact area. Hertz did not consider friction, adhesion and van der Waals forces.

Improvement over the Hertzian theory was provided by Johnson et al. (1971) with the Johnson, Kendall, Roberts (JKR) theory. Consideration of adhesive force becomes necessary when the loads are low and contacting surfaces are smooth and clean. The JKR model accounts for van der Waals forces within the contact zone. Attraction due to van der Waals force weakens the force of elastic repulsion, and sometimes contact can be sustained even in the presence of a slight tensile load. Derjaguin, Muller, Toporov (DMT) theory (1975) considers van der Waals interaction outside the elastic contact, which gives rise to an additional load. Tabor (1977) has analyzed the validity of these models under different conditions. In this chapter, we will concentrate on the contact between two solids under the presence of high load, and therefore, van der Waals forces will not be considered.

9.2 HERTZ THEORY

The pioneering work on contact mechanics was carried out by Hertz (1881). The formulae derived by the Hertz theory are still used in practice. In this section, a brief review of the Hertz theory is presented. All the basic concepts are explained, but some derivations are omitted for the sake of brevity. The books by Johnson (1985) and Timoshenko and Goodier (1970) are valuable references to the Hertz theory.

Hertz employed the following assumptions:

1. The radii of curvature of the contacting bodies are large compared with contact length. Hence, each surface can be considered as an elastic half-space. (An elastic half-space can be thought of as a cube, whose one face is defined, but the other five faces are at infinity. The load is usually applied on the defined face.)
2. The dimension of each contacting body is large when compared with the contact zone. Thus, it is assumed that the contact stresses are dependent only on the local geometry of the contacting bodies.
3. The contacting bodies are perfectly smooth, and there is no friction.
4. Bodies are homogeneous, isotropic and linearly elastic. Elastic limits of the material are not exceeded.
5. There is no effect of van der Waals forces.

9.2.1 GEOMETRY OF UNSTRESSED SURFACE IN THE REGION OF CONTACT

Consider two contacting non-conforming surfaces that meet at point O, as shown in Figure 9.1. The case is considered in which the load is applied along the common normal. It is assumed that the z -coordinate of a point of a body is a quadratic function of x and y , where the z -coordinate is measured from the tangent plane. For Body 1, the z -coordinate is positive in the upward direction, and for Body 2, it is positive in the downward direction in Figure 9.1. For body 1,

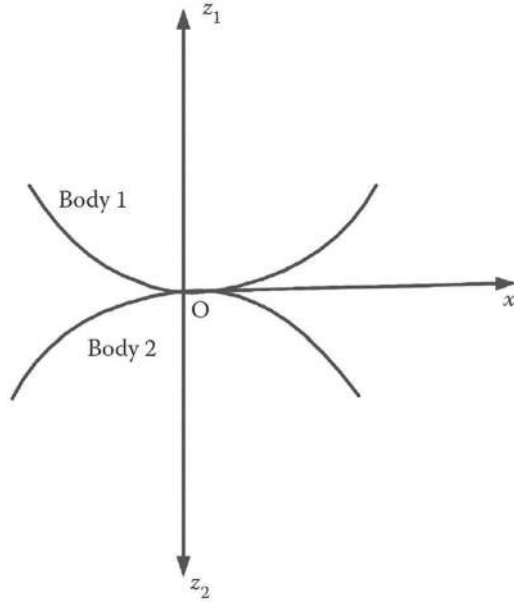


FIGURE 9.1 Contact between two non-conforming surfaces.

$$z_1 = A_1 x^2 + A_2 xy + A_3 y^2. \quad (9.1)$$

With proper choice of coordinate axes, the coefficient of xy can be made 0. To see it, consider that the coordinate system X - Y is obtained by rotating the x - y system by an angle θ , as shown in Figure 9.2. For this case,

$$x = X \cos \theta - Y \sin \theta; \quad y = X \sin \theta + Y \cos \theta. \quad (9.2)$$

Substituting Equation 9.2 in Equation 9.1,

$$\begin{aligned} z_1 = & A_1 (X \cos \theta - Y \sin \theta)^2 + A_2 (X \cos \theta - Y \sin \theta)(X \sin \theta + Y \cos \theta) \\ & + A_3 (X \sin \theta + Y \cos \theta)^2 \end{aligned} \quad (9.3)$$

In Equation 9.3, the coefficient of XY is 0 if

$$-A_1 \sin 2\theta + A_3 \sin 2\theta + A_2 (\cos^2 \theta - \sin^2 \theta) = 0, \quad (9.4)$$

that is,

$$\tan 2\theta = \frac{A_2}{(A_1 - A_3)}. \quad (9.5)$$

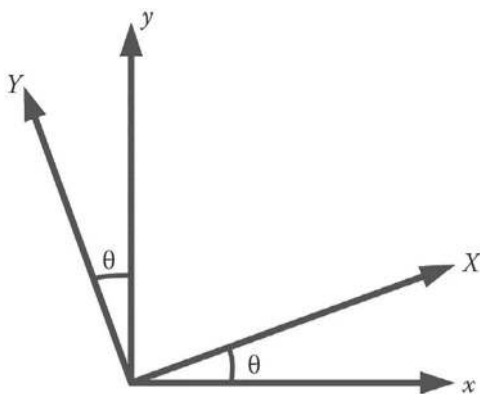


FIGURE 9.2 Original (x - y) and rotated (X - Y) coordinate systems.

Thus, it is possible to write

$$z_1 = AX^2 + BY^2. \quad (9.6)$$

In the X - z_1 plane,

$$z_1 = AX^2. \quad (9.7)$$

Assuming that the projection of contacting Body 1 in the vicinity of contact on the X - z_1 plane is part of a circle (Figure 9.3),

$$X^2 = z_1 (2R_1 - z_1) \approx 2R_1 z_1, \quad (9.8)$$

as z_1 is very small in comparison to R_1 . The radius R_1 is called the first principal radius of curvature. Hence,

$$A = \frac{1}{2R_1}. \quad (9.9)$$

Similarly,

$$B = \frac{1}{2R'_1}, \quad (9.10)$$

where R'_1 is called the second principal radius of curvature (in the Y - z_1 plane). Thus,

$$z_1 = \frac{X^2}{2R_1} + \frac{Y^2}{2R'_1}. \quad (9.11)$$

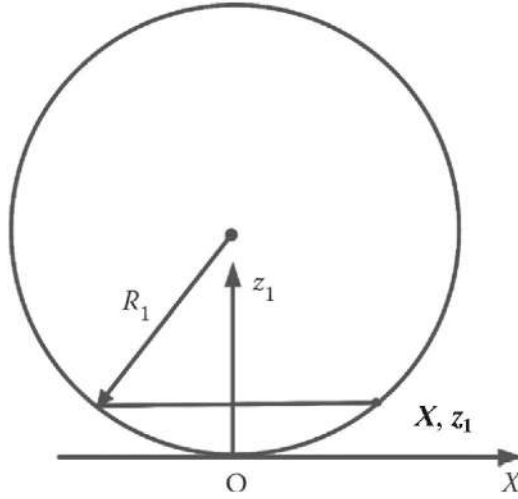


FIGURE 9.3 Projection of contacting bodies in the X - z_1 plane.

In practice, the planes containing the principal radii of curvature for two bodies may be different. Hence, it is appropriate to write

$$z_1 = \frac{x_1^2}{2R_1} + \frac{y_1^2}{2R'_1}, z_2 = \frac{x_2^2}{2R_2} + \frac{y_2^2}{2R'_2} \quad (9.12)$$

for two bodies, respectively.

For finding the deflection during contact between two bodies, it is better to adopt common coordinate axes, say, X - Y , for the two bodies as shown in Figure 9.4. Let x_1 make angle β_1 with X and x_2 make angle β_2 with X , such that

$$\beta_1 + \beta_2 = \alpha, \quad (9.13)$$

where α is the angle between x_1 and x_2 (coordinate axes of two bodies). The transformation of coordinates is given by the following equations:

$$\begin{aligned} x_1 &= X \cos \beta_1 - Y \sin \beta_1, y_1 = X \sin \beta_1 + Y \cos \beta_1; \\ x_2 &= X \cos \beta_2 + Y \sin \beta_2, y_2 = -X \sin \beta_2 + Y \cos \beta_2. \end{aligned} \quad (9.14)$$

Substituting Equation 9.14 in Equation 9.12,

$$\begin{aligned} z_1 + z_2 &= \frac{(X \cos \beta_1 - Y \sin \beta_1)^2}{2R_1} + \frac{(X \sin \beta_1 + Y \cos \beta_1)^2}{2R'_1} \\ &\quad + \frac{(X \cos \beta_2 + Y \sin \beta_2)^2}{2R_2} + \frac{(-X \sin \beta_2 + Y \cos \beta_2)^2}{2R'_2}. \end{aligned} \quad (9.15)$$

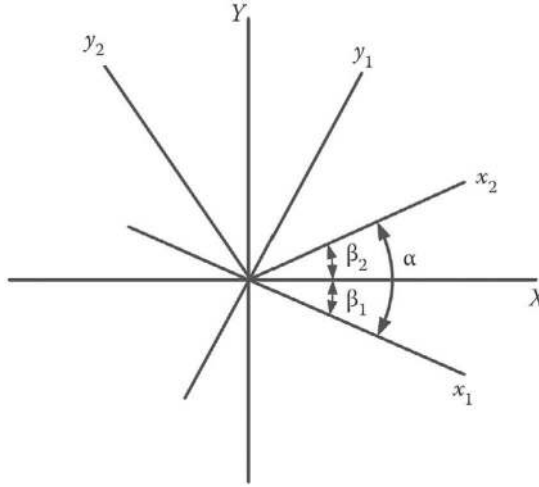


FIGURE 9.4 Common coordinate axes X - Y and principal axis for two bodies.

The previous expression has the term of XY , but its coefficient can be made 0 by appropriate selection of coordinate axes X - Y . The necessary condition for that is

$$-\left(\frac{1}{2R_1} - \frac{1}{2R'_1}\right)\sin 2\beta_1 + \left(\frac{1}{2R_2} - \frac{1}{2R'_2}\right)\sin 2\beta_2 = 0. \quad (9.16)$$

Equations 9.13 and 9.16 together decide β_1 and β_2 . Now, performing some simple algebraic manipulation, Equation 9.15 can be written as

$$z_1 + z_2 = AX^2 + BY^2, \quad (9.17)$$

where

$$A + B = \frac{1}{2R_1} + \frac{1}{2R'_1} + \frac{1}{2R_2} + \frac{1}{2R'_2}, \quad (9.18)$$

$$\begin{aligned} (A - B)^2 &= \left(\frac{1}{2R_1} - \frac{1}{2R'_1}\right)^2 + \left(\frac{1}{2R_2} - \frac{1}{2R'_2}\right)^2 \\ &\quad + 2\left(\frac{1}{2R_1} - \frac{1}{2R'_1}\right)\left(\frac{1}{2R_2} - \frac{1}{2R'_2}\right)\cos 2\alpha \end{aligned} \quad (9.19)$$

Example 9.1

Two cylinders of radii R_1 and R_2 are in contact with their axes perpendicular to each other. The distance between the surfaces of both cylinders can be expressed as

$$h = Ax^2 + By^2.$$

Express A and B in terms of R_1 and R_2 and specify the orientation of axes x and y .

SOLUTION

As per the notations used in this section,

$$h = z_1 + z_2 = Ax^2 + By^2,$$

as per Equation 9.17. In the x_1 - z_1 plane, the radius of curvature of the first cylinder is R_1 . The curvature in y_1 - z_1 plane is $R'_1 = \infty$. Similarly, $R'_2 = \infty$. From Equation 9.18

$$A + B = \frac{1}{2R_1} + \frac{1}{2R_2}. \quad (i)$$

From Equation 9.19 (putting $\alpha = 90^\circ$)

$$(A - B)^2 = \frac{1}{4R_1^2} + \frac{1}{4R_2^2} - \frac{2}{4R_1R_2} = \left(\frac{1}{2R_1} - \frac{1}{2R_2} \right)^2,$$

or

$$A - B = \frac{1}{2R_1} - \frac{1}{2R_2}. \quad (ii)$$

Solving (i) and (ii),

$$A = \frac{1}{2R_1} \text{ and } B = \frac{1}{2R_2}.$$

Using Equation 9.16

$$-\frac{1}{2R_1} \sin 2\beta_1 + \frac{1}{2R_2} \sin 2\beta_2 = 0. \quad (iii)$$

From Equation 9.13

$$\beta_2 = \frac{\pi}{2} - \beta_1.$$

Putting this in (iii), $\beta_1 = 0$ is obtained. Hence, $\beta_2 = \pi/2$. Thus, the x -axis is perpendicular to the axis of cylinder 1, and the y -axis is along the axis of the cylinder. Can you find other possible orientations?

9.2.2 BOUSSINESQ SOLUTION

To determine the stresses and deformations in the contact problems, it is helpful to understand the Boussinesq solution, which provides the expressions for stress and deformations in a semi-infinite body subjected to a concentrated load on the surface. Assume that a force P is acting at plane $z = 0$ of an elastic half-space at origin (Figure 9.5). The solution to this problem was provided by J. Boussinesq in 1885. For a detailed derivation of the solution, one can refer to the book of Timoshenko and Goodier (1970). Here, some important results are reproduced. The stress components are expressed in the r, θ, z coordinate system instead of the x, y, z system. It is obvious that the stress distribution is symmetric about the z -axis. We have the following non-zero stress components:

$$\begin{aligned}\sigma_r &= \frac{P}{2\pi} \left[(1-2\nu) \left\{ \frac{1}{r^2} - \frac{z}{r^2} (r^2 + z^2)^{-\frac{1}{2}} \right\} - 3r^2 z (r^2 + z^2)^{-\frac{5}{2}} \right], \\ \sigma_\theta &= \frac{P}{2\pi} (1-2\nu) \left\{ -\frac{1}{r^2} + \frac{z}{r^2} (r^2 + z^2)^{-1/2} + z (r^2 + z^2)^{-3/2} \right\}, \\ \sigma_z &= -\frac{3P}{2\pi} z^3 (r^2 + z^2)^{-\frac{5}{2}}, \\ \tau_{rz} &= -\frac{3P}{2\pi} r z^2 (r^2 + z^2)^{-5/2}.\end{aligned}\tag{9.20}$$

The displacement components are given as

$$\begin{aligned}u_r &= \frac{P(1-2\nu)(1+\nu)}{2\pi E r} \left\{ z (r^2 + z^2)^{-1/2} - 1 + \frac{1}{(1-2\nu)} r^2 z (r^2 + z^2)^{-3/2} \right\}, \\ u_z &= \frac{P}{2\pi E} \left[(1+\nu) z^2 (r^2 + z^2)^{-3/2} + 2(1-\nu^2) (r^2 + z^2)^{-1/2} \right].\end{aligned}\tag{9.21}$$

At the bounding surface $z = 0$, the displacement components are given as

$$\begin{aligned}(u_r)_{z=0} &= \frac{-P(1-2\nu)(1+\nu)}{2\pi E r}, \\ (u_z)_{z=0} &= \frac{P(1-\nu^2)}{\pi E r}.\end{aligned}\tag{9.22}$$

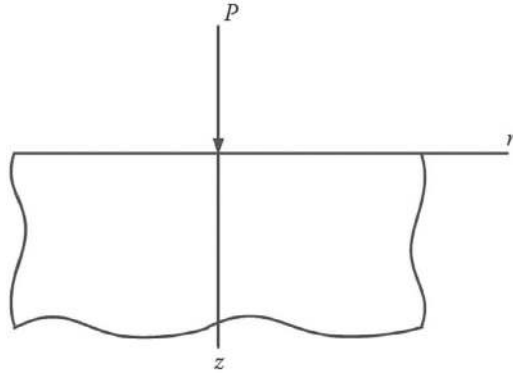


FIGURE 9.5 Concentrated force P on the boundary of a semi-infinite body.

Example 9.2

On a semi-infinite body of steel, a force P is applied as shown in Figure 9.5. What load will initiate the yielding at location (1 mm, 1 mm) if the yield stress is 300 MPa? Poisson's ratio for steel may be taken as 0.3.

SOLUTION

Substituting $r = 1$ mm, $z = 1$ mm and $\nu = 0.3$ in Equation 9.20,

$$\begin{aligned}\sigma_r &= -0.06578P, \sigma_\theta = 3.8549 \times 10^{-3}P, \\ \sigma_z &= -0.0844P, \tau_{rz} = -0.0844P,\end{aligned}$$

where P is in newtons and stress components are in megapascals. In the matrix form, the stress components are written as

$$[\sigma] = P \begin{bmatrix} -0.06578 & 0 & -0.0844 \\ 0 & 3.85 \times 10^{-3} & 0 \\ -0.0844 & 0 & -0.0844 \end{bmatrix}.$$

Now,

$$\frac{1}{3} \text{tr} \sigma = -0.04878P.$$

The deviatoric stress is obtained by subtracting this value from the diagonal components. Thus,

$$[\sigma'] = P \begin{bmatrix} -0.017 & 0 & -0.0844 \\ 0 & 0.05263 & 0 \\ -0.0844 & 0 & -0.0356 \end{bmatrix}.$$

The scalar product of the deviatoric stress tensor with itself is obtained by adding all the squared elements of the deviatoric stress matrix. Thus,

$$\sigma' : \sigma' = \sigma'_{ij} \sigma'_{ij} = 0.01853 P^2.$$

As per the Mises criterion, at yielding

$$\frac{3}{2} \sigma' : \sigma' = 0.01853 P^2 = \sigma_Y^2 = 300^2.$$

Hence,

$$P = 1800 \text{ N} = 1.8 \text{ kN}.$$

For applying the Tresca criterion, one needs to determine the principal stress, which is the eigenvalues of $[\sigma]$. Here, one gets the principal stresses in decreasing order as

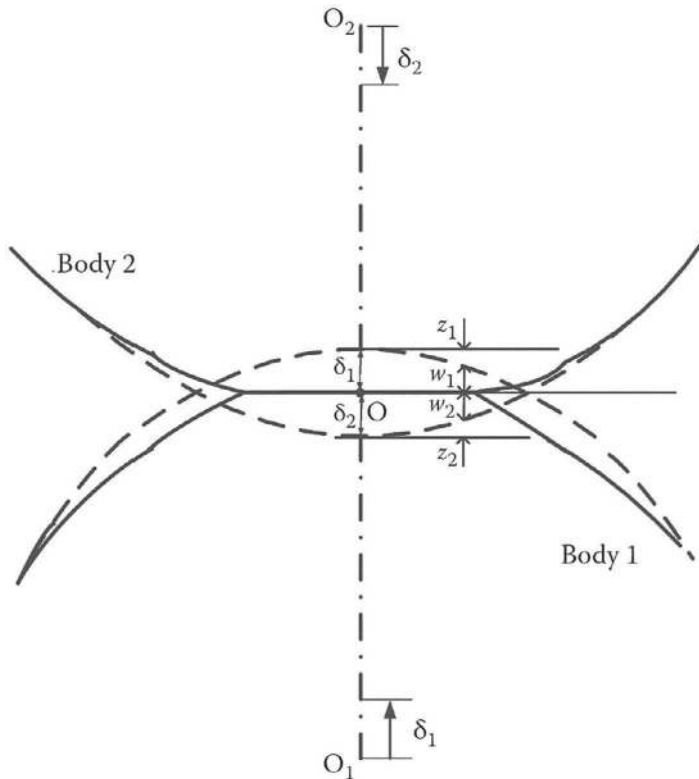


FIGURE 9.6 Contact between two non-conforming bodies.

$$\sigma_1 = 0.0098P, \sigma_2 = 0.0038P, \sigma_3 = -0.16P.$$

By the Tresca criterion,

$$\sigma_3 - \sigma_1 = 0.1698P = \sigma_Y = 300.$$

This gives

$$P = 1767 \text{ N} = 1.77 \text{ kN}.$$

9.2.3 PRESSURE AND DEFLECTIONS IN THE CONTACT REGION

Figure 9.6 shows two non-conforming bodies pressed against each other. The undeformed shapes of the bodies are shown by dotted lines. The mutual approach of the bodies toward each other is δ . The z -coordinates of both the bodies are measured from the tangent plane in undeformed configuration. Let w_1 denote the deformation of a point on Body 1, whose z -coordinate is z_1 and w_2 and z_2 are the corresponding quantities for Body 2. Note that z_1 and z_2 coordinates pertain to undeformed configuration. Then, in the contact region,

$$(z_1 + w_1) + (z_2 + w_2) = \delta. \quad (9.23)$$

In other words, it means that the bodies will approach each other to cover for the already existing gap ($z_1 + z_2$) and the local deformation. Outside the area of contact,

$$(z_1 + w_1) + (z_2 + w_2) > \delta \quad (9.24)$$

Assuming

$$z_1 + z_2 = Ax^2 + By^2, \quad (9.25)$$

in the contact region,

$$Ax^2 + By^2 + (w_1 + w_2) = \delta. \quad (9.26)$$

Assuming the surface to be plane, the deformation at a point (x, y) owing to the pressure p applied in a small neighborhood is given by

$$(u_z)_{z=0} = \frac{(1-\nu^2)}{\pi E} \frac{p(x', y')}{r} dx' dy'. \quad (9.27)$$

By the principle of superposition, the total vertical deformation as a function of (x, y) is given as

$$(u_z)_{z=0} = \frac{(1-\nu^2)}{\pi E} \iint \frac{p(x', y')}{r} dx' dy', \quad (9.28)$$

where A_c denotes the contact region. As both the bodies are subjected to the same contact pressure by Newton's third law, Equation 9.28 provides

$$w_1 = \frac{(1-\nu_1^2)}{\pi E_1} \iint_{A_c} \frac{p(x', y')}{r} dx' dy', \quad (9.29)$$

$$w_2 = \frac{(1-\nu_2^2)}{\pi E_2} \iint_{A_c} \frac{p(x', y')}{r} dx' dy'. \quad (9.30)$$

Substituting Equations 9.29 and 9.30 in Equation 9.26,

$$\left\{ \frac{(1-\nu_1^2)}{\pi E_1} + \frac{(1-\nu_2^2)}{\pi E_2} \right\} \iint_{A_c} \frac{p(x', y')}{r} dx' dy' = \delta - Ax^2 - By^2. \quad (9.31)$$

Based on the observation, Hertz assumed that the contact region is elliptical. It can be shown (Johnson, 1985) that Equation 9.31 is satisfied if

$$p = p_0 \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right)^{1/2} \quad (9.32)$$

over the area of contact bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad (9.33)$$

If the total contact force is P , it can be shown that

$$p = \frac{3P}{2\pi ab} \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right)^{1/2}. \quad (9.34)$$

Substituting Equation 9.34 in Equation 9.31 and performing some mathematical manipulation (Puttock and Thwaite, 1969), one gets

$$\delta = \frac{3}{4} P \left\{ \frac{(1-\nu_1^2)}{\pi E_1} + \frac{(1-\nu_2^2)}{\pi E_2} \right\} \int_0^\infty \frac{d\psi}{\left\{ (a^2 + \psi)(b^2 + \psi)\psi \right\}^{1/2}}, \quad (9.35)$$

$$A = \frac{3}{4}P \left\{ \frac{(1-\nu_1^2)}{\pi E_1} + \frac{(1-\nu_2^2)}{\pi E_2} \right\} \int_0^\infty \frac{d\psi}{(a^2 + \psi) \left\{ (a^2 + \psi)(b^2 + \psi)\psi \right\}^{1/2}}, \quad (9.36)$$

$$B = \frac{3}{4}P \left\{ \frac{(1-\nu_1^2)}{\pi E_1} + \frac{(1-\nu_2^2)}{\pi E_2} \right\} \int_0^\infty \frac{d\psi}{(b^2 + \psi) \left\{ (a^2 + \psi)(b^2 + \psi)\psi \right\}^{1/2}}. \quad (9.37)$$

Constants A and B are dependent on the geometry of the contacting bodies. Using the values of A and B , a and b can be obtained from Equations 9.36 and 9.37. Afterward, δ can be obtained from Equation 9.35. Now, some specific cases shall be solved.

9.2.4 TWO SPHERES IN CONTACT

For the two spheres of radii R_1 and R_2 in contact,

$$z_1 = \frac{x^2}{2R_1} + \frac{y^2}{2R_1}, \quad z_2 = \frac{x^2}{2R_2} + \frac{y^2}{2R_2}. \quad (9.38)$$

In this case,

$$z_1 + z_2 = x^2 \left(\frac{1}{2R_1} + \frac{1}{2R_2} \right) + y^2 \left(\frac{1}{2R_1} + \frac{1}{2R_2} \right). \quad (9.39)$$

Hence,

$$A = B = \left(\frac{1}{2R_1} + \frac{1}{2R_2} \right). \quad (9.40)$$

From Equation 9.36 or 9.37,

$$A = B = \frac{3}{4}P \left\{ \frac{(1-\nu_1^2)}{\pi E_1} + \frac{(1-\nu_2^2)}{\pi E_2} \right\} \int_0^\infty \frac{d\psi}{(a^2 + \psi)^2 \psi^{1/2}}. \quad (9.41)$$

It is obvious that due to axisymmetry, the area of the contact is a circle with $a = b$. The previous integral can be solved by substituting $\psi = a^2 \tan^2 \theta$ to obtain the following expression:

$$A = B = \left(\frac{1}{2R_1} + \frac{1}{2R_2} \right) = \frac{3\pi}{8a^3}P \left\{ \frac{(1-\nu_1^2)}{\pi E_1} + \frac{(1-\nu_2^2)}{\pi E_2} \right\}. \quad (9.42)$$

Hence,

$$a^3 = \frac{3\pi}{8} P \frac{\left\{ \frac{(1-\nu_1^2)}{\pi E_1} + \frac{(1-\nu_2^2)}{\pi E_2} \right\}}{\left(\frac{1}{2R_1} + \frac{1}{2R_2} \right)}. \quad (9.43)$$

Let us define the relative curvature as

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \quad (9.44)$$

and equivalent Young's modulus of elasticity E^* by the following relation

$$\frac{1}{E^*} = \frac{(1-\nu_1^2)}{E_1} + \frac{(1-\nu_2^2)}{E_2}. \quad (9.45)$$

From Equation 9.43,

$$a^3 = \frac{3PR}{4E^*}. \quad (9.46)$$

Solving Equation 9.35 for the specific case,

$$\delta = \frac{a^2}{R} = \left(\frac{9P^2}{16RE^{*2}} \right)^{1/3}. \quad (9.47)$$

Comparing Equations 9.32 and 9.34, it is observed that

$$p_0 = \frac{3P}{2\pi ab} = \frac{3P}{2\pi a^2} = \left(\frac{6PE^{*2}}{\pi^3 R^2} \right)^{1/3}. \quad (9.48)$$

Equations 9.46–9.48 can also be used when one sphere is in contact with an internal sphere. If the sphere of radius R_2 makes contact with the sphere of radius R_1 on the inner surface, then R in the equations can be taken as

$$\frac{1}{R} = \frac{1}{R_2} - \frac{1}{R_1}. \quad (9.49)$$

If a sphere of radius R_1 is in contact with a plane, then R_2 can be taken as infinity.

Example 9.3

Two spheres are in contact. They are subjected to load P and their mutual approach is δ . If the load is increased to $2P$, what will be their mutual approach?

SOLUTION

From Equation 9.47

$$\delta \propto P^{2/3}.$$

Hence, the mutual approach with $2P$ load is

$$(2P / P)^{2/3} \delta = 1.587\delta \cong 1.6\delta.$$

Thus, doubling the load increases the mutual approach by a factor of 1.6. This factor of 1.6 is not dependent on the respective radii and elastic constants of the spheres.

Example 9.4

A steel spherical ball of radius 5 mm is in contact with a flat surface of steel. Find the displacement of the center of the ball when a load of 800 N is applied. If the surface is not flat but has a 6 mm radius of curvature, what will be the displacement of the center of the ball when the same load is applied? Take Young's modulus of elasticity as 210 GPa and Poisson's ratio as 0.3.

SOLUTION

Case 1, sphere in contact with flat surface:

Given $P = 800$ N, $R = 5$ mm. Using Equation 9.45

$$\frac{1}{E^*} = \frac{(1-0.3^2)}{210} + \frac{(1-0.3^2)}{210} = \frac{0.91}{105},$$

and hence $E^* = 115$ GPa.

From Equation 9.47

$$\delta = \left(\frac{9 \times 800^2}{16 \times 5 \times 10^{-3} \times 115^2 \times 10^{18}} \right)^{1/3} = 17.6 \times 10^{-6} \text{ m} = 17.6 \text{ } \mu\text{m}$$

Case 2, sphere in contact with curved surface:

Using Equation 9.49

$$\frac{1}{R} = \frac{1}{5} - \frac{1}{6} = \frac{1}{30},$$

or $R = 30$ mm. Hence, from Equation 9.47

$$\delta = \left(\frac{9 \times 800^2}{16 \times 30 \times 10^{-3} \times 115^2 \times 10^{18}} \right)^{1/3} = 9.7 \times 10^{-6} \text{ m} = 9.7 \text{ } \mu\text{m}.$$

9.2.5 TWO CYLINDERS IN CONTACT ALONG A LINE PARALLEL TO THEIR AXES

As already discussed, the pressure distribution over the elliptical contact area is given by Equation 9.34. In the case of the contact between two cylinders, the semi-major axis a is infinity. The integrated pressure across the minor axis of the ellipse in the plane $x = 0$ is

$$\bar{P} = \frac{3P}{2\pi ab} \int_{-b}^b \left(1 - \frac{y^2}{b^2} \right)^{1/2} dy = \frac{3P}{4a}. \quad (9.50)$$

Here, both P and a tend to infinity, but the per unit length load (P/a) remains finite. Now, the pressure distribution as a function of y is given by

$$p(y) = \frac{3P}{2\pi ab} \left(1 - \frac{y^2}{b^2} \right)^{1/2} = \frac{2\bar{P}}{\pi b} \left(1 - \frac{y^2}{b^2} \right)^{1/2}. \quad (9.51)$$

The z -coordinates of two cylinders are given as

$$z_1 = B_1 y^2, z_2 = B_2 y^2. \quad (9.52)$$

The cylinders are initially in contact over a line of length $2a$. For long cylinders, Equation 9.31 becomes

$$\left\{ \frac{(1 - \nu_1^2)}{\pi E_1} + \frac{(1 - \nu_2^2)}{\pi E_2} \right\} \iint_{A_c} \frac{p(x', y')}{r} dx' dy' = \delta - B y^2 = \delta - \frac{1}{2R} y^2, \quad (9.53)$$

where

$$r^2 = (y - y')^2 + x'^2. \quad (9.54)$$

Substituting the expression of p from Equation 9.51 and considering a to be much larger than b , the integral in Equation 9.53 can be evaluated. After simplification, one gets

$$b^2 = 2 \left\{ \frac{(1-\nu_1^2)}{\pi E_1} + \frac{(1-\nu_2^2)}{\pi E_2} \right\} \frac{\bar{P}}{B}. \quad (9.55)$$

If the two cylinders have radii R_1 and R_2 , then

$$b^2 = 4 \frac{R_1 R_2}{R_1 + R_2} \left\{ \frac{(1-\nu_1^2)}{\pi E_1} + \frac{(1-\nu_2^2)}{\pi E_2} \right\} \bar{P}. \quad (9.56)$$

If a cylinder of radius R_1 makes contact with a flat surface, then R_2 is taken as infinity and

$$b^2 = 4 R_1 \left\{ \frac{(1-\nu_1^2)}{\pi E_1} + \frac{(1-\nu_2^2)}{\pi E_2} \right\} \bar{P}. \quad (9.57)$$

It is to be noted that the expression for δ will indicate infinite displacement. In this case, the mutual approach of the bodies has to be calculated by taking into consideration the shape of the roll and the type of loading on each body.

Example 9.5

A cylinder of radius R_1 makes contact with the rigid concave surface of radius R_2 by an external force P per unit length in the axial plane. Find the length of contact.

SOLUTION

Using Equation 9.56, but replacing R_2 by $(-R_1)$,

$$b^2 = 4 \frac{R_1 R_2}{-R_1 + R_2} \left\{ \frac{(1-\nu_1^2)}{\pi E_1} + \frac{(1-\nu_2^2)}{\pi E_2} \right\} \bar{P}.$$

Here, E_2 tends to infinity. Hence,

$$b^2 = 4 \frac{R_1 R_2}{-R_1 + R_2} \frac{(1-\nu_1^2)}{\pi E_1} \bar{P}.$$

The contact length will be $2b$.

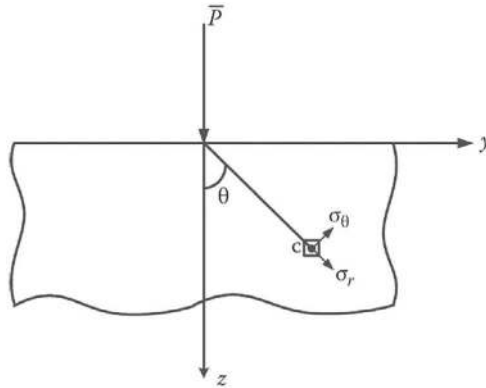


FIGURE 9.7 Front view of the line loading on an elastic half-space.

9.2.6 ALTERNATE DERIVATION FOR THE CONTACT BETWEEN TWO CYLINDERS

The contact problem between two cylinders can be solved by a different approach. Consider an elastic half-space, on the surface of which a line load per unit length P is applied (Figure 9.7). The loading gives rise to a simple radial compressive distribution of stress directed toward O , the point of application of the load. The stresses are described by the equations:

$$\sigma_r = -\frac{2\bar{P}}{\pi r} \cos\theta, \quad \sigma_\theta = \tau_{r\theta} = 0. \quad (9.58)$$

In the y - z plane,

$$\begin{aligned} \sigma_y &= \sigma_r \sin^2\theta = -\frac{2\bar{P}}{\pi} \frac{y^2 z}{(y^2 + z^2)^2}, \\ \sigma_z &= \sigma_r \cos^2\theta = -\frac{2\bar{P}}{\pi} \frac{z^3}{(y^2 + z^2)^2}, \\ \tau_{yz} &= \sigma_r \sin\theta \cos\theta = -\frac{2\bar{P}}{\pi} \frac{yz^2}{(y^2 + z^2)^2}. \end{aligned} \quad (9.59)$$

It can be shown that surface displacements are given by

$$\begin{aligned} w_r \left(\text{at } \theta = \pm \frac{\pi}{2} \right) &= -\frac{(1-2\nu)(1+\nu)\bar{P}}{2E}, \\ w_\theta \left(\text{at } \theta = +\frac{\pi}{2} \right) &= -w_\theta \left(\text{at } \theta = -\frac{\pi}{2} \right) = -\frac{(1-\nu^2)2\bar{P}}{\pi E} \ln(r_0/r), \end{aligned} \quad (9.60)$$

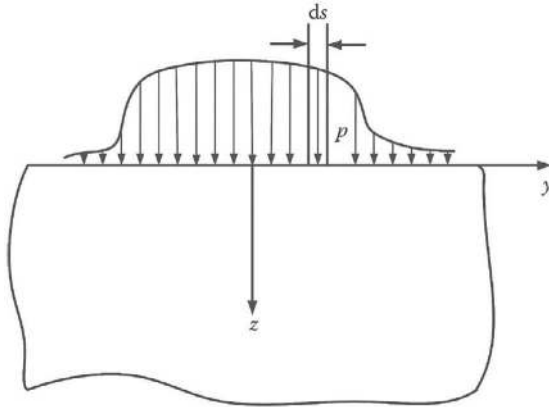


FIGURE 9.8 Distributed load (loading same across every plane parallel to the y - z plane).

where r_0 is the arbitrary datum at which the vertical displacement is 0.

Equations 9.59 and 9.60 can be used for deriving the expressions for stresses and deflections under distributed normal loads. Consider Figure 9.8 in which pressure p is applied at the surface. The stresses at point (y, z) are given by

$$\begin{aligned}\sigma_y &= -\frac{2z}{\pi} \int_{-\infty}^{\infty} \frac{p(s)(y-s)^2 ds}{\left\{(y-s)^2 + z^2\right\}^2}, \\ \sigma_z &= -\frac{2z^3}{\pi} \int_{-\infty}^{\infty} \frac{p(s) ds}{\left\{(y-s)^2 + z^2\right\}^2}, \\ \tau_{yz} &= -\frac{2z^2}{\pi} \int_{-\infty}^{\infty} \frac{p(s)(y-s) ds}{\left\{(y-s)^2 + z^2\right\}^2}.\end{aligned}\tag{9.61}$$

The displacement at the surface is given by

$$\begin{aligned}w_y &= -\frac{(1-2\nu)(1+\nu)}{2E} \left\{ \int_{-\infty}^y p(s) ds - \int_y^{\infty} p(s) ds \right\}, \\ w_z &= \frac{2(1-\nu^2)}{\pi E} \left[\int_{-\infty}^y p(s) \ln \frac{r_0}{|y-s|} ds + \int_y^{\infty} p(s) \ln \frac{r_0}{|-y+s|} ds \right].\end{aligned}\tag{9.62}$$

Now, substitute the pressure distribution given by Equation 9.51 into Equation 9.62. Writing $\eta = |y/b|$, so that within contact zone $\eta < 1$, the vertical displacement w_z of the surface is given by

$$w_z = \frac{2\bar{P}(1-\nu^2)}{\pi E} (\eta^2 + C), \quad (9.63)$$

while outside the contact area

$$w_z = \frac{2\bar{P}(1-\nu^2)}{\pi E} \left\{ \ln \left(\eta + \sqrt{\eta^2 - 1} \right) + \frac{1}{2 \left(\eta + \sqrt{\eta^2 - 1} \right)} + C + \frac{1}{2} \right\}. \quad (9.64)$$

The constant C can be chosen to make the displacement 0 at some outside contact point.

The comparison of Equations 9.53 and 9.63 provides

$$b^2 = \frac{4\bar{P}R}{\pi E^*}, \quad (9.65)$$

which is the same result as in Equation 9.55.

9.2.7 STRESSES IN CONTACT PROBLEM

Once the contact pressure is obtained, the stresses can be calculated by the expressions provided in Sections 9.2.2 to 9.2.6. In the case of contact between two spheres, the contact area is a circle with radius a . The pressure distribution is given by

$$p = p_0 \sqrt{1 - \frac{r^2}{a^2}}. \quad (9.66)$$

Equation 9.20 may be used for obtaining the stresses. On a typical small element of the area $r d\theta dr$, the distributed force can be treated as concentrated force $p r d\theta dr$. Then, the Boussinesq solution provides the stresses due to this force (see Equation 9.20). The overall stresses can be obtained by integrating the expression for stresses thus obtained over the entire contact area.

On the surface ($z = 0$) within the contact area, the stress components are given by

$$\sigma_r = \frac{(1-2\nu)}{3} \frac{a^2 p_0}{r^2} \left\{ 1 - \left(1 - \frac{r^2}{a^2} \right)^{3/2} \right\} - p_0 \left\{ 1 - \frac{r^2}{a^2} \right\}^{1/2},$$

$$\sigma_{\theta} = \frac{-(1-2\nu)}{3} \frac{a^2 p_0}{r^2} \left\{ 1 - \left(1 - \frac{r^2}{a^2} \right)^{3/2} \right\} - 2\nu p_0 \left\{ 1 - \frac{r^2}{a^2} \right\}^{1/2}, \quad (9.67)$$

$$\sigma_z = -2\nu p_0 \left\{ 1 - \frac{r^2}{a^2} \right\}^{1/2}.$$

Outside the contact area,

$$\sigma_r = -\sigma_{\theta} = \frac{(1-2\nu)}{3} \frac{a^2 p_0}{r^2}, \sigma_z = 0. \quad (9.68)$$

It can be seen that the greatest value of tensile stress is $(1-2\nu)p_0/3$ and occurs at the edge of the contact area.

Within the bulk material, the stresses along the z -axis are given by

$$\sigma_r = \sigma_{\theta} = -(1+\nu) p_0 \left\{ 1 - \frac{z}{a} \tan^{-1} \frac{a}{z} \right\} + \frac{1}{2} p_0 \left\{ 1 + \frac{z^2}{a^2} \right\}^{-1}, \quad (9.69)$$

$$\sigma_z = -p_0 \left\{ 1 + \frac{z^2}{a^2} \right\}^{-1}.$$

In a similar manner, the stresses in the case of contact between two cylinders can be found. Here, in Equation 9.61, the elliptic pressure distribution can be substituted. The stresses along the z -axis are given as

$$\sigma_y = -p_0 \left\{ \left(1 + \frac{2z^2}{b^2} \right) \left(1 + \frac{z^2}{b^2} \right)^{-1/2} - 2 \frac{z}{b} \right\} \quad (9.70)$$

$$\sigma_z = -p_0 \left\{ 1 + \frac{z^2}{b^2} \right\}^{-\frac{1}{2}},$$

where the cylinders are in contact from $y = -b$ to $y = +b$.

The maximum shear stress τ is given by

$$\tau = p_0 \left\{ \frac{z}{b} - \frac{z^2}{b^2} \left(1 + \frac{z^2}{b^2} \right)^{-1/2} \right\}. \quad (9.71)$$

This expression has a maximum value of $0.3p_0$ below the surface at the point where $z = 0.78 b$.

Example 9.6

One steel sphere of radius 10 cm is in contact with a rigid floor. How much total load needs to be applied on the sphere for yielding to start at the edge of the contact area? The shear yield stress of the material is 200 MPa.

SOLUTION

The Tresca criterion is used. By Equation 9.68 and taking Poisson's ratio as 0.3, the maximum shear stress at the edge ($r = a$)

$$\frac{2(1-2\nu)p_0}{3} = \frac{1.2p_0}{3} = 400,$$

or $p_0 = 1000$ MPa. Let the contact radius be a . The total load is given by

$$P = \int_0^a p_0 \sqrt{1 - \frac{r^2}{a^2}} 2\pi r dr = \frac{2}{3} p_0 \pi a^2$$

Hence,

$$a = \left(\frac{3P}{2\pi p_0} \right)^{1/2}.$$

From Equation 9.46

$$a^3 = \frac{3PR}{4E^*},$$

or

$$\left(\frac{3P}{2\pi p_0} \right)^{3/2} = \frac{3PR}{4E^*}$$

Hence,

$$P = \frac{\pi^3 R^2 p_0^3}{6E^{*2}}.$$

This equation could have been obtained directly from Equation 9.46.

In the present case, $R = 0.1$ m, $p_0 = 1000 \times 10^6$ Pa and $E^* = 210 \times 10^9$ Pa. Hence, $P = 1170$ N.

9.3 ELASTIC-PLASTIC INDENTATION

In the case of the contact between a cylinder and a plane surface, the maximum shear stress occurs at a depth of $0.78b$ from the surface and its magnitude is $0.3p_0$. From Equation 9.51

$$p_0 = \frac{2\bar{P}}{\pi b}. \quad (9.72)$$

If there is no strain hardening, the maximum shear stress is equal to shear yield stress k . Hence,

$$k = 0.3p_0 = 0.19 \frac{\bar{P}}{b}. \quad (9.73)$$

From Equation 9.73, the load to cause yielding can be estimated if the total contact length $2b$ is known. For the Tresca criterion, $k = Y/2$ and for the Mises criterion, $k = Y/\sqrt{3}$. With the use of Equation 9.65, it can be shown that the critical load per unit length for the indentation of the cylinder on the flat surface is given by

$$\bar{P}_Y = 35.27 \frac{k^2 R}{E^*}. \quad (9.74)$$

Note that the previous equation is also valid for the contact between two cylinders when R is taken as the equivalent radius obtained using Equation 9.44.

When rigid flat punch indents on the flat surface, then Equation 9.74 cannot be employed. This is the case of the contact between two conforming surfaces. Figure 9.9 shows a rigid smooth flat punch kept on an elastic half-space. When the load W is applied, the elastic half-space keeps deforming. It can be shown that the pressure distribution given by

$$p(x) = \frac{p_0}{\sqrt{1 - \frac{y^2}{b^2}}} \quad (9.75)$$

produces uniform displacement over the contact length $2b$. This type of pressure distribution produces the maximum pressure at the center of the punch and infinite pressure at the two edges. In practice, the edges can never be perfectly sharp, and the pressure will be finite. If the length of the punch in the x -direction is L , then

$$\frac{W}{L} = \bar{P} = \int_{-b}^b \frac{p_0}{\sqrt{1 - \frac{y^2}{b^2}}} dy = \pi p_0 b, \quad (9.76)$$

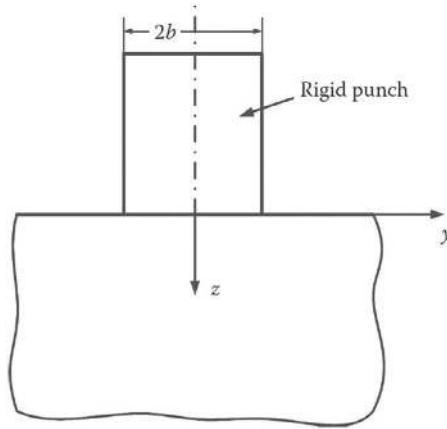


FIGURE 9.9 A rigid punch on an elastic half-space.

The load required can be estimated to cause initial yielding. However, most of the time, our interest is to find the load when the yielding has reached the surface of the elastic half-space and a significant zone around the punch has reached the plastic state. In the plastic zone, the elastic strains can be neglected in comparison to the plastic strain. In Sections 9.3.1, 9.3.2 and 9.3.3, three methods of finding the approximate indentation pressure are described.

Example 9.7

A rigid cylinder is supported on a soft metal base. It is desired that the base material should not deform due to the weight of the block. The factor of safety for this requirement comes out to be 1.5. It is desired to increase the factor of safety to 3. By what factor should the yield strength of base material be changed?

SOLUTION

For doubling the safety factor, the load-carrying capacity should be doubled. From Equation 9.74,

$$\bar{P}_Y \propto k^2 \text{ or } k \propto (\bar{P}_Y)^{1/2}.$$

Hence,

$$\frac{k_{\text{new}}}{k_{\text{old}}} = \left(\frac{\text{new factor of safety}}{\text{old factor of safety}} \right)^{1/2} = \sqrt{2} = 1.41$$

Thus, the yield strength increases by a factor of 1.41.

9.3.1 SOLUTION OF FLAT PLATE INDENTATION PROBLEM BY UPPER BOUND METHOD

In the upper bound method, a kinematically admissible velocity field, which satisfies the continuity equation and velocity boundary conditions, is assumed. The total power obtained based on the assumed velocity field provides an upper bound on the actual power. The velocity field can also be assumed to consist of some rigid blocks that may have tangential velocity discontinuities with the neighboring blocks. There is no velocity discontinuity in the normal direction on the surface of the block. The power consumed internally in the deformation regions can be calculated. If the strain rate is continuous, the power per unit volume is given by the scalar product of the stress and the strain-rate tensor. It is to be noted that the actual stress tensor is not known. The stress tensor is calculated from the assumed strain-rate tensor using the constitutive relation for the material. At the surface of the velocity discontinuity, the power per unit area is equal to the shear yield stress of the material multiplied by the magnitude of the velocity discontinuity. The total internal power is taken as the sum of the powers in the continuous strain-rate zone and the velocity discontinuity zone. If it is considered that the flow field consists of the movement of some rigid blocks, there is no power dissipation in the blocks, but one has to consider the power dissipation between the blocks due to velocity discontinuity. The internal power consumed plus the frictional power is equated to the external power for evaluating the external forces.

Let us try to solve the indentation problem by the upper bound theorem. Let a flat rigid frictionless punch be indenting on a half-space made of some material, as shown in Figure 9.10. Once the punch starts moving downward, some material beneath the punch moves downward, and the material adjacent to the punch moves upward. There is a piling-up of the material near the edges of the punch. A possible deformation field is shown in Figure 9.10. ACB, BCD and CDE are three equal isosceles triangles. The angle BAC is equal to θ . The punch is considered perfectly smooth.

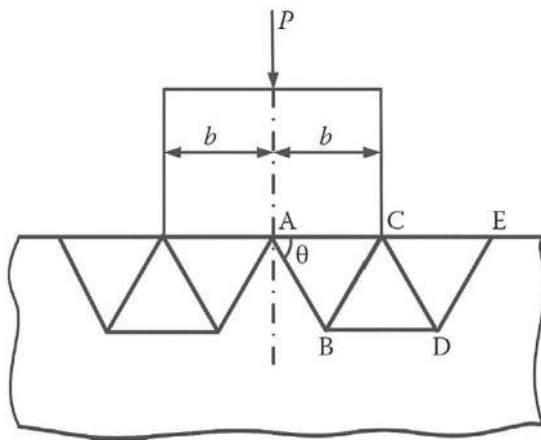


FIGURE 9.10 Possible deformation field for indentation problem, as the movement of blocks.

Let the punch move downward with unit velocity. Consider $\triangle ACB$. This triangle slides down as a rigid body parallel to line AB. Let its velocity be V_1 . The vertical component of this velocity should be equal to the punch velocity to avoid any velocity discontinuity normal to the line interface. Hence,

$$V_1 \sin \theta = 1. \quad (9.77)$$

Thus, $V_1 = \csc \theta$. Now, consider $\triangle BCD$. This triangle moves horizontally, parallel to line BD. Let its velocity be V_2 . Now, the component of velocity of $\triangle ACB$ normal to interface BC is equal to the component of velocity of $\triangle BCD$ normal to BC. Hence,

$$\csc \theta \cos \left(-\frac{\pi}{2} + 2\theta \right) = V_2 \sin \theta. \quad (9.78)$$

Hence, $V_2 = 2 \cot \theta$. Now, consider $\triangle CDE$. This triangle moves as a rigid body parallel to DE with velocity V_3 . Across the line CD, the normal velocities of $\triangle BCD$ and $\triangle CDE$ should match. Hence,

$$2 \cot \theta \sin \theta = V_3 \sin (\pi - 2\theta). \quad (9.79)$$

Thus, $V_3 = \csc \theta$.

As all these blocks are rigid, there is no internal power dissipation inside them. There is power dissipation due to tangential velocity discontinuities along AB, BC, BD, CD and DE.

9.3.1.1 Power Dissipation along AB

The tangential velocity discontinuity along AB is $\csc \theta$. The length of AB is $(b \sec \theta)/2$. Hence, power dissipation along AB is $k \csc \theta (b \sec \theta)/2$, where k is the shear yield strength of the material. This simplifies to $ka \csc 2\theta$.

9.3.1.2 Power Dissipation along BC

The component of velocity of $\triangle ABC$ along BC is $\csc \theta \sin(-\pi/2 + 2\theta) = -\csc \theta \cos 2\theta$, directed from C to B. The component of velocity of $\triangle BCD$ along BC is $2 \cot \theta \cos \theta$ directed from B to C. Hence, the magnitude of relative sliding velocity between the two triangles is

$$V_{BC} = 2 \cot \theta \cos \theta - \csc \theta \cos 2\theta = \csc \theta \quad (9.80)$$

The length of BC is $(b \sec \theta)/2$. Hence, power dissipation along BC is $k \csc \theta (b \sec \theta)/2$, where k is the shear yield strength of the material. This simplifies to $kb \csc 2\theta$.

9.3.1.3 Power Dissipation along BD

The tangential velocity discontinuity along BD is $2 \cot \theta$. The length of BD is a . Hence, the power dissipation along BD is $2kbc \cot \theta$.

9.3.1.4 Power Dissipation along CD

The component of velocity of $\triangle BCD$ along CD is $2\cot\theta\cos\theta$, directed from C to D. The component of velocity of $\triangle CDE$ along CD is $\operatorname{cosec}\theta\cos(\pi-2\theta)$ from D to C. Hence, the magnitude of the relative sliding velocity between the two triangles is

$$V_{CD} = 2\cot\theta\cos\theta - \operatorname{cosec}\theta\cos 2\theta = \operatorname{cosec}\theta \quad (9.81)$$

The length of BC is $(b\sec\theta)/2$. Hence, power dissipation along CD is $k\operatorname{cosec}\theta(b\sec\theta)/2$. This simplifies to $kb\operatorname{cosec}2\theta$.

9.3.1.5 Power Dissipation along DE

The tangential velocity discontinuity along DE is $\operatorname{cosec}\theta$. The length of BC is $(b\sec\theta)/2$. Hence, power dissipation along BC is $k\operatorname{cosec}\theta(b\sec\theta)/2$, where k is the shear yield strength of the material. This simplifies to $kb\operatorname{cosec}2\theta$.

The total power dissipation is given by

$$\begin{aligned} P_t &= kb\operatorname{cosec}2\theta + kb\operatorname{cosec}2\theta + 2kb\cot\theta + kb\operatorname{cosec}2\theta + kb\operatorname{cosec}2\theta \\ &= 2kb(2\operatorname{cosec}2\theta + \cot\theta). \end{aligned} \quad (9.82)$$

This power dissipation calculation is for the right portion of the line of symmetry. The power dissipation on the left portion of the line of symmetry is exactly the same. The power supplied by the indenting punch is $p_{av}2b$, where p_{av} is the average pressure on the punch. Equating power supplied to power dissipation

$$2p_{av}b = 4kb(2\operatorname{cosec}2\theta + \cot\theta). \quad (9.83)$$

Thus,

$$p_{av} = 2k(2\operatorname{cosec}2\theta + \cot\theta). \quad (9.84)$$

The average pressure obtained is a function of the assumed angle θ . Differentiating average pressure with respect to θ and equating it to zero,

$$\frac{dp_{av}}{d\theta} = 2k(-4\operatorname{cosec}2\theta\cot 2\theta - \operatorname{cosec}^2\theta) = 0. \quad (9.85)$$

This provides $\tan\theta = \sqrt{2}$. Thus, the minimum value of average pressure is obtained at $\theta = \tan^{-1}(\sqrt{2})$. That it is the minimum value and not the maximum value that can be verified by taking the double derivative of average pressure with respect to θ and noting that at minimum, its value is positive. Alternatively, it is observed that at $\theta = 0$, p_{av} becomes infinite. Hence, the maximum value occurs at $\theta = 0$, and $\theta = \tan^{-1}(\sqrt{2})$ actually provides the minimum value of the average punch pressure. This is still expected to be greater than the actual average pressure as it is an upper-bound

solution, but among various possible solutions for different values of θ , it is the closest to the actual solution. Hence, Equation 9.84 provides the closest value of average pressure as $5.66k$. The yield shear stress k is $Y/2$ for the Tresca criterion and $Y/\sqrt{3}$ for the Mises criterion. Hence, the average pressure is equal to $2.83Y$ for the Tresca criterion and $3.27Y$ for the Mises criterion.

Example 9.8

A rigid punch of width 5 mm and length 50 mm is used to deform a material of yield strength 80 MPa. How much load should be applied on the punch as per the upper bound method?

SOLUTION

As per the Mises criterion, the average pressure is $3.27Y = 261.6$ MPa.

Total load = average pressure $\times$ cross-sectional area = $261.6 \times 10^6 \times 5 \times 50 \times 10^{-6} = 65400$ N = 65.4 kN.

9.3.2 SOLUTION OF FLAT PLATE INDENTATION BY SLIP LINE FIELD METHOD

Slip line field is a method of modeling plane-strain rigid-plastic metal-forming problems; although there have been attempts to apply the method to strain-hardening materials and for axisymmetric problems as well. The method is based on the fact that through each point in the plane of plastic flow, one can consider a pair of orthogonal curves along which the shear stress has its maximum value. These curves are called slip lines or shear lines, and material particles are assumed to flow along these lines. Figure 9.11 shows a set of orthogonal slip lines. These are designated as α and β slip lines. By convention, the algebraically greatest principal stress direction bisects the angle between α and β directions taken as a right-handed pair of curvilinear axes. The following two Hencky's equations for slip lines are presented:

$$p + 2k\phi = \text{constant along an } \alpha \text{ line,} \quad (9.86)$$

$$p - 2k\phi = \text{constant along a } \beta \text{ line.} \quad (9.87)$$

where ϕ is the counterclockwise angle of α line with the x -axis, and p is the pressure (negative of hydrostatic stress).

Figure 9.12 shows the proposed slip line field for the indentation of a semi-infinite medium by a flat punch. Punch is considered perfectly smooth; hence, the shear stress on the punch face is zero. Thus, one principal stress is along the punch face. The other principal stress is along the punch travel and is expected to be compressive. It is clear that slip lines that are the lines of maximum shear stress must meet the punch face at 45° . Similarly, the slip lines meet at a free surface (which is a principal plane) at 45° . In the triangular regions, the stresses are constant. These regions are joined by centered fan regions.

In the region ABD, one principal stress (on plane AB) is zero. Hence, the other principal stress has to be $(-2k)$. The Mohr circle is shown in Figure 9.13. The hydrostatic

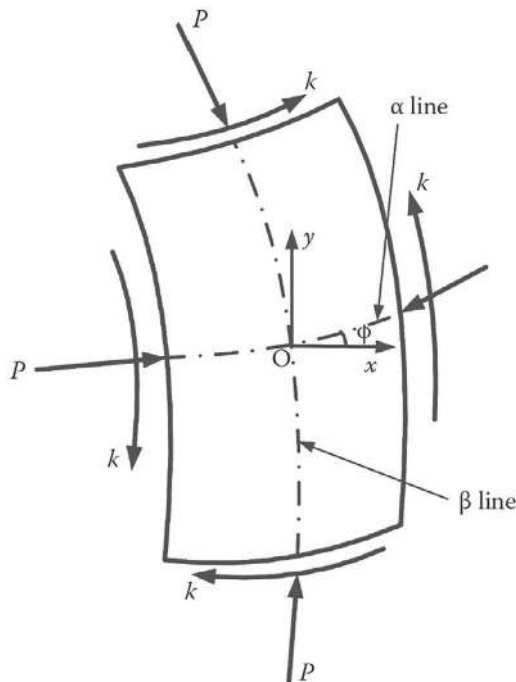


FIGURE 9.11 A set of α and β slip lines.

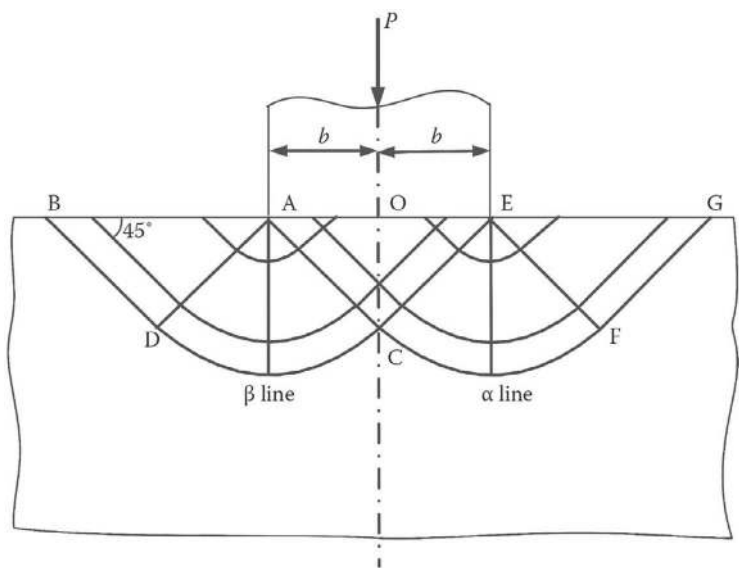


FIGURE 9.12 Slip line field for the indentation by a flat punch.

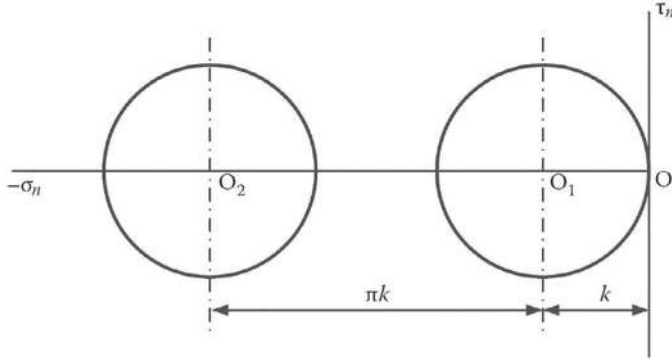


FIGURE 9.13 Mohr circles for the slip line field of indentation by a flat punch.

stress, which happens to be the mean of the two principal stresses in the plane strain case, is $(-k)$. Now, by Hencky's equation, the mean stress is $(-k - k\pi)$. A Mohr circle with this value as the center is constructed. It is seen that the greatest (magnitude-wise) principal stress is $(-2k - k\pi)$. Hence, the uniform pressure on the punch is given by

$$p_p = 2k \left(1 + \frac{\pi}{2} \right). \quad (9.88)$$

If the contact length of the punch is $2b$, the load per unit width of the punch is given by

$$\bar{P} = 2bp_p = 4kb \left(1 + \frac{\pi}{2} \right) \quad (9.89)$$

Example 9.9

A rigid punch of width 5 mm and length 50 mm is used to deform a material of yield strength 80 MPa. How much load should be applied on the punch as per the slip line method?

SOLUTION

As per Equation 9.89, the load per unit width of the punch is given by

$$\bar{P} = 2k \times \text{contact length} \times \left(1 + \frac{\pi}{2} \right) = 2.57Y \times \text{contact length}.$$

Hence,

$$P = 2.57Y \times \text{contact area} = 2.57 \times 80 \times 10^6 \times 5 \times 50 \times 10^{-6} = 51400 \text{ N} = 51.4 \text{ kN}$$

Compare this with the solution of Example 9.8.

9.3.3 SOLUTION OF FLAT PLATE INDENTATION BY NUMERICAL METHODS

When the contact geometry cannot be described adequately by the quadratic terms, it becomes necessary to use numerical methods. The contact area can be assumed, and the displacement compatibility can be applied. Then, the equilibrium equations are solved considering the suitable constitutive relations. If the normal stresses in some regions come out to be tensile, the region is excluded from the contact area in the subsequent iteration. If the shear stress exceeds the product of the static coefficient of friction and the normal stress, it is understood that there is a slip. In the subsequent iterations, the slip condition is introduced there. During slip, the shear stress will be the product of the kinetic coefficient of friction and the normal stress.

It is common to use the finite-element method for solving the contact problem. The finite-element method discretizes the domain by a number of elements. Elements contain the internal nodes and boundary nodes are shared by more than one element. When the deformation is large, there is a possibility that a pair of nodes, initially in contact, may undergo a large relative displacement. If a node-to-node interface model, as shown in Figure 9.14a, is used, remeshing becomes essential for carrying out the subsequent analysis. To avoid frequent remeshing, a node-to-segment model, as shown in Figure 9.14b, can be used. However, the contact conditions acquire a complex form when expanded in terms of the nodal variables. To impose these, an additional set of finite-element equations involving the contact stiffness needs to be developed using the principle of the virtual work of the contact forces.

Johnson (1985) has described a numerical method in which a number of nodes are placed in the contact area. The pressure acting on the contact region may be approximated by piecewise linear pressure distributions, in which unknowns are the nodal values of pressure. For example, in a two-dimensional case, at each node, a triangular pressure distribution can be assumed in which the peak pressure will be at the node, and the other two vertices of the triangle will be at zero pressure. If the spacing between the nodes is appropriate, there will be overlapping of the pressure distributions. If the gap between two bodies is defined by $h(x, y)$, then in the contact region,

$$w_1 + w_2 + h(x, y) - \delta = 0 \quad (9.90)$$

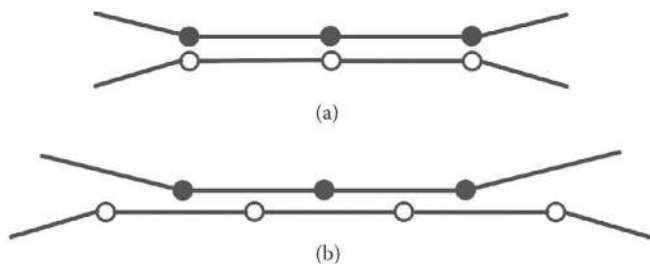


FIGURE 9.14 Two interface models: (a) node-to-node; (b) node-to-segment.

where δ is the prescribed approach of the two bodies, and w_1 and w_2 can be obtained by superposing the displacements due to all nodal pressure distributions. Displacement due to a particular pressure distribution may be obtained by the theory of loading on elastic half-space. Thus, one can get the following n equations:

$$\sum_{j=1}^{j=n} c_{ij} p_j = (h_i - \delta), i = 1, n, \quad (9.91)$$

where c_{ij} is the influence coefficient indicating the displacement at node i due to unit pressure at node j , and h_i is the gap between the bodies at node i .

If δ is prescribed, the nodal values of pressure may be obtained by solving the previous n simultaneous linear equations. If instead of δ , the total load P is prescribed, one gets an addition equation:

$$P = A \sum_{i=1}^n p_i, \quad (9.92)$$

where A is a constant depending upon the form and size of the pressure element. With the set of equations given in Equations 9.91 and 9.92, the unknown nodal pressures and approach distance can be found. If some pressure value comes out to be negative, the corresponding region is excluded from the contact zone in the next iteration. Iterations are continued until the pressure everywhere is positive in the contact zone.

9.4 CAVITY MODEL

For solving the indentation model, a cavity model has been described in the book of Johnson (1985). The same is presented here in a somewhat expanded form and by including the effect of strain hardening as carried out by Rao (2000). Rao (2000) has also carried out some experiments on the indentation of three materials by a spherical indenter and compared the experimental results by upper bound, slip line field and cavity model methods. These results are reproduced here and show that a cavity model with strain hardening provides a good match with the experimental results.

In the cavity model, the contact surface of the indenter is encased in a hemispherical core of radius a , as shown in Figure 9.15. In the case of a spherical indenter, the radius of the hemisphere is equal to the radius of the contact circle at the surface of the material. Let r , θ and ϕ be the spherical coordinates. Here, θ is the angle made by the radius vector with the vertical z -axis, and ϕ is the angle measured around that axis. By virtue of spherical symmetry $\sigma_\theta = \sigma_\phi$, stress is a function of r only and the following equilibrium equation is satisfied:

$$\frac{\partial \sigma_r}{\partial r} = \frac{2}{r} (\sigma_\theta - \sigma_r). \quad (9.93)$$

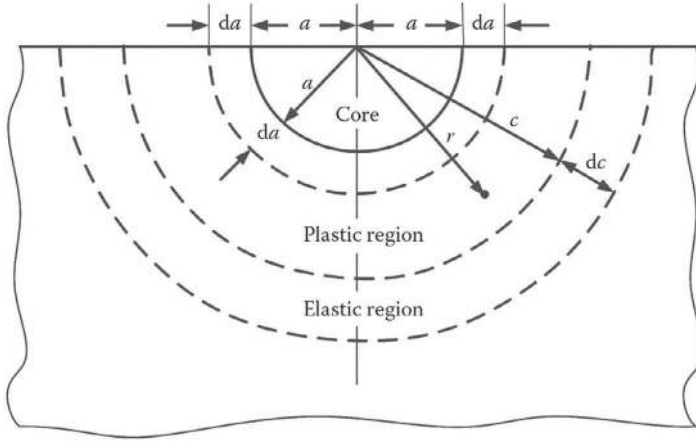


FIGURE 9.15 A cavity model.

For sufficiently small values of the pressure, the deformation is purely elastic. Subsequently, elastic-plastic deformation commences.

If the radial distance is denoted by u , then the stress-strain relation for the elastic portion may be written as

$$\varepsilon_r = \frac{\partial u_r}{\partial r} = \frac{1}{E}(\sigma_r - 2\nu\sigma_\theta), \quad (9.94)$$

$$\varepsilon_\theta = \varepsilon_\phi = \frac{u_r}{r} = \frac{1}{E}\{(1-\nu)\sigma_\theta - \nu\sigma_r\}. \quad (9.95)$$

By eliminating u_r from the previous two equations

$$r \frac{\partial}{\partial r} \{(1-\nu)\sigma_\theta - \nu\sigma_r\} = (\sigma_r - \sigma_\theta)(1+\nu). \quad (9.96)$$

Substituting $(\sigma_r - \sigma_\theta)$ value from Equation 9.93 in the previous expression,

$$\frac{\partial}{\partial r}(\sigma_r + 2\sigma_\theta) = 0 \quad (9.97)$$

This is the equation of strain compatibility expressed in terms of stresses.

From Equation 9.97,

$$\sigma_r + 2\sigma_\theta = C, \quad (9.98)$$

where C is a constant. The equilibrium Equation 9.93 can be written as

$$\frac{d\sigma_r}{C - 3\sigma_r} = \frac{dr}{r} \quad (9.99)$$

Integrating both sides, one gets

$$\sigma_r = A + \frac{B}{r^3}, \quad (9.100)$$

where A is $C/3$, and B is the other constant.

By applying boundary conditions $\sigma_r = 0$ at $r = b$ and $\sigma_r = -p$ at $r = a$,

$$\sigma_r = \frac{-p\left(\frac{b^3}{r^3} - 1\right)}{\left(\frac{b^3}{a^3} - 1\right)} \text{ and } \sigma_\theta = \sigma_\phi = \frac{p\left(\frac{b^3}{2r^3} + 1\right)}{\left(\frac{b^3}{a^3} - 1\right)} \quad (9.101)$$

Thus, σ_r is compressive, and σ_θ is tensile throughout the material. The stress-strain relation corresponding to ε_θ in Equation 9.95 then gives

$$u(r) = \frac{p}{E\left(\frac{b^3}{a^3} - 1\right)} \left[(1 - 2\nu)r + \frac{b^3}{2r^2}(1 + \nu) \right] \quad (9.102)$$

If the internal pressure is increased to a critical value p_e , plastic yielding begins at the radius where the yield criterion is first satisfied. Since, $\sigma_\theta = \sigma_\phi$, the Tresca and Mises criteria both reduce to

$$\sigma_\theta - \sigma_r = H, \quad (9.103)$$

where H is the flow stress. At the elastic-plastic boundary, when there is no strain hardening, flow stress is equal to σ_Y . Therefore, from Equations 9.101 and 9.103,

$$\sigma_Y = \frac{3}{2} \frac{b^3}{r^3} \frac{p_e}{\left(\frac{b^3}{a^3} - 1\right)}. \quad (9.104)$$

This gives the pressure needed to cause initial yielding at $r = a$ as

$$p_e|_{\text{initial yielding}} = \frac{2\sigma_Y}{3} \left(1 - \frac{a^3}{b^3} \right). \quad (9.105)$$

From Equations 9.102 and 9.105, the radial displacement of the internal surface at this stage can be written as

$$u(a) = \frac{\sigma_y a}{3E} \left[(1+\nu) + 2(1-2\nu) \frac{a^3}{b^3} \right] \quad (9.106)$$

With further increase in the internal pressure, the plastic zone spreads outward, and the elastic/plastic boundary denoted by c is a spherical surface at each stage, as shown in Figure 9.15. In the elastic region, one gets

$$\sigma_r = \frac{-2\sigma_y c^3}{3b^3} \left(\frac{b^3}{r^3} - 1 \right) \text{ and } \sigma_\theta = \frac{2\sigma_y c^3}{3b^3} \left(\frac{b^3}{2r^3} + 1 \right); \quad c \leq r \leq b \quad (9.107)$$

The radial displacement in the elastic region is found as

$$u(r) = \frac{2\sigma_y c^3}{3Eb^3} \left[(1-2\nu)r + \frac{b^3}{2r^2} (1+\nu) \right]; \quad c \leq r \leq b \quad (9.108)$$

The flow stress after strain hardening can be taken as $\sigma_y (1 + K \varepsilon_{eq}^n)^n$ where σ_y is the flow stress at zero strain, K and n are the hardening coefficients, and ε_{eq}^p is the equivalent plastic strain. The equivalent strain is defined as

$$\varepsilon_{eq} = \int \sqrt{\frac{2}{3} d\varepsilon_{ij}^p d\varepsilon_{ij}^p} = \int \sqrt{\frac{2}{3} (d\varepsilon_\theta^{p^2} + d\varepsilon_r^{p^2} + d\varepsilon_\phi^{p^2})}. \quad (9.109)$$

Considering $d\varepsilon_r^p = -2 d\varepsilon_\phi^p = -2 d\varepsilon_\theta^p$, Equation 9.109 provides

$$\varepsilon_{eq} = \int 2 d\varepsilon_\theta^p = 2\varepsilon_\theta^p. \quad (9.110)$$

The equilibrium equation for the plastic zone is

$$\frac{d\sigma_r}{dr} = \frac{2}{r} \sigma_y (1 + K \varepsilon_{eq}^n)^n \quad (9.111)$$

The elastic-plastic boundary is at $r = c$. Now, for any radius r in the plastic zone,

$$\int_r^c \frac{d\sigma_r}{dr} dr = \int_r^c \frac{2}{r} \sigma_y (1 + K \varepsilon_{eq}^n)^n dr \quad (9.112)$$

or

$$\sigma_r(c) - \sigma_r(r) = \int_r^c \frac{2}{r} \sigma_y \left(1 + K \varepsilon_{eq}^p\right)^n dr. \quad (9.113)$$

From Equation 9.107 and assuming that $b \rightarrow \infty$,

$$\sigma_r(c) = \frac{-2\sigma_y}{3} \quad (9.114)$$

Thus, from Equation 9.113

$$\sigma_r(r) = -\frac{2\sigma_y}{3} - \int_r^c \frac{2}{r} \sigma_y \left(1 + K \varepsilon_{eq}^p\right)^n dr. \quad (9.115)$$

Using Equation 9.103 and the expression for flow stress in terms of equivalent strain,

$$\sigma_\theta = \sigma_y \left(1 + K \varepsilon_{eq}^p\right)^n - \frac{2\sigma_y}{3} - \int_r^c \frac{2}{r} \sigma_y \left(1 + K \varepsilon_{eq}^p\right)^n dr. \quad (9.116)$$

At $r = a$, $\sigma_r = -p$, therefore,

$$p = \frac{2\sigma_y}{3} + \int_r^c \frac{2}{r} \sigma_y \left(1 + K \varepsilon_{eq}^p\right)^n dr. \quad (9.117)$$

The integral on the right-hand side of Equation 9.117 can be evaluated numerically provided that the elastic-plastic boundary radius c and the plastic strain in the plastic zone are known. In the beginning, these are unknown. Hence, to start with, strain hardening can be neglected, and a plastic strain field can be obtained. Now, a revised estimate for the pressure and strain field is made. The procedure continues iteratively until convergence is obtained, that is, strain values in two subsequent iterations do not differ appreciably. In Sections 9.4.1 and 9.4.2, the procedures for determining the elastic-plastic boundary and plastic strains are discussed.

9.4.1 DETERMINATION OF ELASTIC-PLASTIC BOUNDARY RADIUS

For small strains,

$$\varepsilon_r + \varepsilon_\theta + \varepsilon_\phi = \left(\varepsilon_r^e + \varepsilon_\theta^e + \varepsilon_\phi^e\right) + \left(\varepsilon_r^p + \varepsilon_\theta^p + \varepsilon_\phi^p\right). \quad (9.118)$$

Due to volume constancy of the plastic deformation,

$$\varepsilon_r + \varepsilon_\theta + \varepsilon_\phi = (\varepsilon_r^e + \varepsilon_\theta^e + \varepsilon_\phi^e) \quad (9.119)$$

On the left-hand side of Equation 9.119, using the strain-displacement relation, and on the right-hand side, the elastic stress-strain relation

$$\frac{\partial u}{\partial r} + \frac{2u}{r} = \frac{(1-2\nu)}{E} (\sigma_r + 2\sigma_\theta) \quad (9.120)$$

Equation 9.120 is valid in the elastic as well as the plastic zone. Substituting Equations 9.115 and 9.116 in it,

$$\frac{\partial u}{\partial r} + \frac{2u}{r} = \frac{(1-2\nu)}{E} \left\{ 2\sigma_y \left(1 + K\varepsilon_{eq}^p \right)^n - 2\sigma_y - \int_r^c \frac{6}{r} \sigma_y \left(1 + K\varepsilon_{eq}^p \right)^n dr \right\} \quad (9.121)$$

Multiplying by r^2 on both sides of the previous expression results in

$$\begin{aligned} \frac{\partial}{\partial r} (r^2 u) &= \frac{(1-2\nu)}{E} \left\{ 2\sigma_y r^2 \left(1 + K\varepsilon_{eq}^p \right)^n - 2\sigma_y r^2 \right. \\ &\quad \left. - \int_r^c \frac{6}{r} \sigma_y r^2 \left(1 + K\varepsilon_{eq}^p \right)^n dr \right\}. \end{aligned} \quad (9.122)$$

For $b \rightarrow \infty$, Equation 9.108 gives

$$u(c) = \frac{2\sigma_y (1+\nu)c}{3E} \quad (9.123)$$

By integrating Equation 9.122 from r to c and using $u(c)$ from Equation 9.123, for the plastic zone,

$$u(r) = \frac{\sigma_y (1+\nu)c^3}{3Er^2} + \frac{(1-2\nu)}{Er^2} \left[\int_r^c (2\sigma_y - 2H) r^2 dr - \int_r^c r^2 \int_r^c \frac{6}{r} H dr dr \right], \quad (9.124)$$

where H is equal to $\sigma_y \left(1 + K\varepsilon_{eq}^p \right)^n$.

Therefore,

$$u(a) = \frac{\sigma_y (1+\nu)c^3}{3Ea^2} + \frac{(1-2\nu)}{Ea^2} \left[\int_a^c (2\sigma_y - 2H) a^2 dr - \int_a^c r^2 \int_a^c \frac{6}{r} H dr dr \right] \quad (9.125)$$

A computer code is developed to find $u(r)$, $u(a)$, and $\frac{du(a)}{dc}$. The condition of similarity of strain field, that is, $dc/da = c/a$, can be used. For a conical indenter, the conservation of the volume of the core requires

$$2\pi a^2 du(a) = \pi a^2 \tan \beta da, \quad (9.126)$$

where $(\pi - 2\beta)$ is the included angle of the cone. For a spherical indenter, $\tan \beta$ can be taken as a/R . Equation 9.126 provides $du(a)/da$, whereas $du(a)/dc$ is calculated numerically. Hence, dc/da can be calculated and equated to c/a . This provides c . Once c is known, the core pressure is calculated from Equation 9.117.

Since the stress in the material immediately below an indenter is not purely hydrostatic and if p denotes the hydrostatic component, then the normal stress can be written as

$$\sigma_z = -\left(p + \frac{2H}{3}\right) \quad (9.127)$$

and the radial stress is

$$\sigma_r = -\left(p - \frac{H}{3}\right) \quad (9.128)$$

Therefore, a best estimation to the indentation pressure can be obtained from Equation 9.127, and the corresponding indentation load P can be calculated.

9.4.2 DETERMINATION OF PLASTIC STRAIN

In the case of strain hardening, still, the total hoop strain is given by u/r , and from Equation 9.124,

$$\begin{aligned} \text{total hoop strain} &= \frac{u(r)}{r} = \frac{\sigma_y(1+\nu)c^3}{3Er^3} + \frac{(1-2\nu)}{Er^3} \\ &\times \left[\int_r^c (2\sigma_y - 2H)r^2 dr + \int_r^c r^2 \int_r^c \frac{6}{r} H dr dr \right] \end{aligned} \quad (9.129)$$

By substituting σ_r from Equation 9.115 and σ_θ value from Equation 9.116 in the expression of the elastic hoop strain given by Equation 9.95,

$$\varepsilon_\theta^e = \varepsilon_\phi^e = \frac{1}{E} \left[H(1-\nu) - \frac{2\sigma_y}{3} - \int_r^c \frac{(2-4\nu)}{r} H dr \right] \quad (9.130)$$

By subtracting Equation 9.130 from Equation 9.129, the expression for plastic hoop strain is given as

$$\epsilon_{\theta}^p = \frac{u(r)}{r} - \frac{1}{E} \left[H(1-\nu) - \frac{2\sigma_y}{3}(1-2\nu) - \int_r^c \frac{(2-4\nu)}{r} Hdr \right]. \tag{9.131}$$

9.4.3 TYPICAL RESULTS

Rao (2000) carried out experiments on the indentation of copper aluminum and mild steel by a spherical indenter. He compared the experimental results with the cavity model, the upper bound theory and the slip line theory. Theories provide the pressure for a given indentation diameter. The total load is obtained by integrating the pressure over the projected area. The projected area is $\pi d^2/4$, where d is the indentation diameter. Note that the upper bound and the slip line theory of plane strain problems have been used. This itself is an approximation. Based on the compression test, yield strengths and hardening parameters of copper, aluminum and steel are estimated. It is observed that experimental results match very well with the cavity model in all the cases (Figures 9.16 to 9.18).

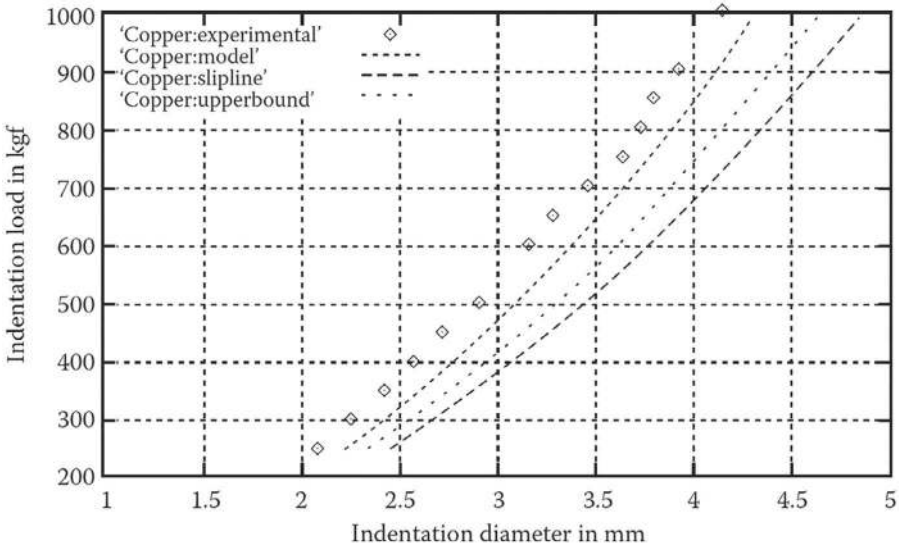


FIGURE 9.16 Comparison of experimental results with cavity model, slip line field theory and upper bound theory for copper. (From Rao, P.M.S., *Determination of material parameters through the study of indentation of the material by a sphere*, M.Tech. Thesis, IIT Guwahati, 2000.)

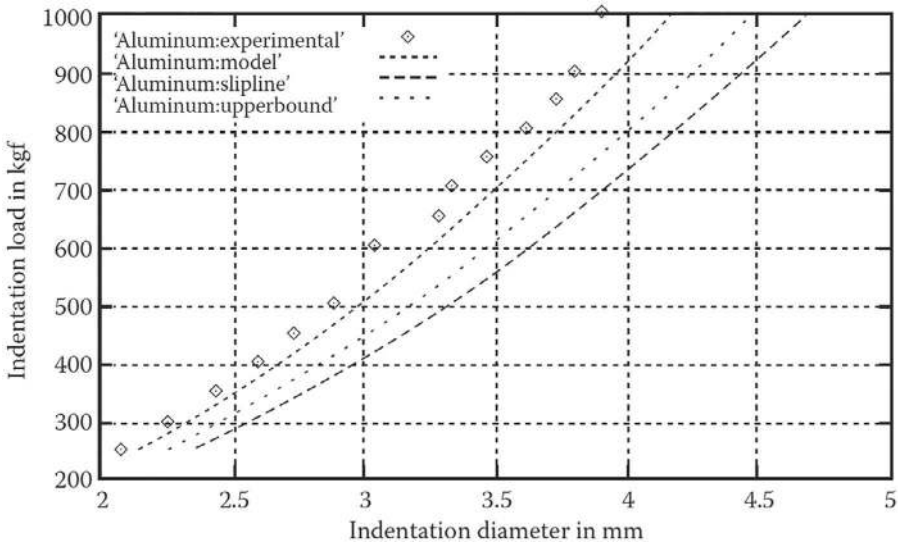


FIGURE 9.17 Comparison of experimental results with cavity model, slip line field theory and upper bound theory for aluminum. (From Rao, P.M.S., *Determination of material parameters through the study of indentation of the material by a sphere*, M.Tech. Thesis, IIT Guwahati, 2000.)

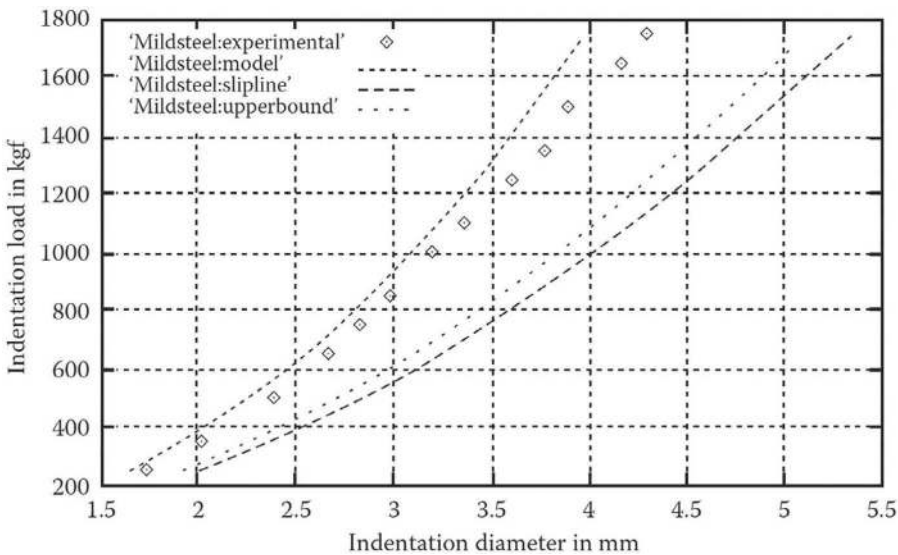


FIGURE 9.18 Comparison of experimental results with cavity model, slip line field theory and upper bound theory for mild steel. (From Rao, P.M.S., *Determination of material parameters through the study of indentation of the material by a sphere*, M.Tech. Thesis, IIT Guwahati, 2000.)

9.5 SLIDING OF ELASTIC-PLASTIC SOLIDS

Two bodies are said to be in sliding contact when there is a relative peripheral velocity of their surfaces at their point of contact. If two bodies slide without friction, the contact stresses due to pressure between two bodies are unaffected by the sliding motion. In practice, there is always some kinetic friction between the surfaces. A number of analytical methods have been proposed for the case of frictional contact. In this section, the basic principles of sliding contact shall only be discussed, and the mathematical details shall be left. Interested readers can refer to other specialized books, for example, Johnson (1985).

Whenever the two bodies are in sliding motion, there is a tangential traction due to friction. Usually, the magnitude of tangential traction is much smaller than the normal traction, and it is helpful to assume that there is no coupling between the two tractions. In that case, it is possible to superimpose the results of normal and tangential tractions for finding the stresses and deformations. The tangential tractions acting at each interface are equal in magnitude and opposite in direction. Mathematically,

$$q_1(x, y) = -q_2(x, y). \quad (9.132)$$

If the two bodies are made of the same material, these tractions will produce the same amount of normal displacement in opposite directions. Hence, they do not cause any change in the distribution of the normal pressure. However, if the material of the two bodies is different and the magnitude of the tangential traction is large, the normal pressure distribution will be affected by the tangential traction.

The sliding bodies usually follow Amonton's law, according to which

$$\frac{|q(x, y)|}{p(x, y)} = \frac{|Q|}{P} = \mu_k, \quad (9.133)$$

where Q is the total tangential load, P is the total normal load, and μ_k is the kinetic coefficient of friction. If the bodies are not sliding, but sliding is incipient, then μ_k is replaced by the static coefficient of friction μ_s . A tangential force whose magnitude is less than the force of limiting friction does not give rise to sliding motion. Even if there is no sliding as a whole, there may be some regions in the contact zone where the slip takes place. At other regions, there is sticking. Relative tangential displacements are constant within the stick region. All surface points within a stick region undergo the same tangential displacement. The statement is true even if the materials of the two bodies are different, but the overall relative displacements are then divided unequally between the two bodies.

Within the stick region, the following relation will hold good:

$$|q(x, y)| \leq \mu_s |p(x, y)|. \quad (9.134)$$

In the region of slip,

$$|q(x, y)| = \mu_k |p(x, y)|. \quad (9.135)$$

In addition, the direction of friction traction, q , must oppose the direction of slip.

Division of the contact area into stick and slip regions is not known *a priori* and must be found by trial and error. First, the entire contact area can be considered the stick region. The slip is likely to occur in those regions where the tangential traction exceeds its limiting value. For solving the problem analytically, the traction distribution can be assumed to have the same form as the pressure distribution.

9.6 ROLLING CONTACT

Rolling is defined as a relative angular motion between two bodies in contact about an axis parallel to their common tangent plane. Along with rolling, the bodies may slide, which means the contact surfaces of the two bodies have relative motion with respect to each other. In addition, the bodies may spin also. Spin is the rotation about their common normal. When sliding and spin are absent, it is called pure rolling.

Pure rolling can be divided into two parts—free rolling and tractive rolling. In free rolling, the tangential force at the contact zone is zero. This case is no different from the contact of two stationary bodies. In tractive rolling, some tangential force is present. In the contact zone, some portions may slip and some may stick. A difference between the tangential strains in the two bodies in the stick areas leads to a small apparent slip, which is commonly called creep.

Suppose a cylinder is rolling on a perfectly rigid surface. If the contact surface gets stretched, the effective radius of the cylinder will increase and the cylinder will cover more distance in one revolution. The fractional increase in the distance covered due to stretching is called the creep ratio. In this case, the creep ratio is positive, but it can also be negative if, instead of stretching, there is compression.

Consider the rolling of two cylinders in contact. Let the common normal be along the z -axis. A typical point on the interface goes from x to x' as per the following relation:

$$x' = x + w_x(x, t), \quad (9.136)$$

where w_x is the displacement of the point. The velocity at any instant will be given by

$$U = \frac{dx'}{dt} = \frac{dx}{dt} + \frac{\partial w_x}{\partial x} \frac{dx}{dt} + \frac{\partial w_x}{\partial t}. \quad (9.137)$$

Now,

$$\frac{dx}{dt} = V + \delta V, \quad (9.138)$$

where V is the common velocity called the rolling speed, and δV is called the creep speed. Thus,

$$U = V + \delta V + V \frac{\partial w_x}{\partial x} + \frac{\partial w_x}{\partial t}. \quad (9.139)$$

For steady-state rolling, the last term in expression 9.139 will be zero, giving

$$U = V + \delta V + V \frac{\partial w_x}{\partial x}. \quad (9.140)$$

The velocity of microslip is given by the difference of velocities between the velocities of two bodies. Thus,

$$\dot{s} = U_1 - U_2 = (\delta V_1 - \delta V_2) + V \left\{ \frac{\partial w_{x1}}{\partial x} - \frac{\partial w_{x2}}{\partial x} \right\}, \quad (9.141)$$

or

$$\frac{\dot{s}}{V} = \frac{(\delta V_1 - \delta V_2)}{V} + \left\{ \frac{\partial w_{x1}}{\partial x} - \frac{\partial w_{x2}}{\partial x} \right\} = \xi + \left\{ \frac{\partial w_{x1}}{\partial x} - \frac{\partial w_{x2}}{\partial x} \right\}, \quad (9.142)$$

where ξ is the creep ratio between the two surfaces. A similar expression for slip is obtained in the y -direction.

In a stick region,

$$\dot{s}_x = \dot{s}_y = 0 \quad (9.143)$$

and

$$|q(x, y)| \leq \mu p(x, y). \quad (9.144)$$

In a slip region,

$$|q(x, y)| = \mu p(x, y), \quad (9.145a)$$

and

$$\frac{q(x, y)}{|q(x, y)|} = -\frac{\dot{s}(x, y)}{|\dot{s}(x, y)|}. \quad (9.145b)$$

Equation 9.145b states that the direction of q must oppose the slip velocity.

These are the basic equations of the rolling. Analysis for the specific cases can be carried out. As the stick and slip zones are not known *a priori*, the analysis has to be done in an iterative manner.

In practical rolling, even if the material is elastic, during one cycle of loading and unloading, the material loses some energy due to hysteresis. The power imparted by the cylinder during loading is

$$\dot{W} = \omega \int_0^a P(x) x dx \quad (9.146)$$

where a is the contact length.

Considering the Hertizian pressure distribution,

$$\dot{W} = \frac{2}{3\pi} Pa\omega. \quad (9.147)$$

If the torque required to overcome the hysteresis loss is T , then

$$T\omega = -\dot{W} = \frac{2}{3\pi} \alpha Pa\omega, \quad (9.148)$$

where α is called the hysteresis loss factor. Usually, its value is less than 1%.

Analogous to the case of sliding friction, the rolling resistance can be expressed as

$$F = \mu_r P, \quad (9.149)$$

where μ_r is the coefficient of rolling resistance. If R is the radius of the roll,

$$FR = T = \frac{2}{3\pi} \alpha Pa, \quad (9.150)$$

This gives

$$F = \mu_r P = \frac{2}{3\pi} \frac{\alpha Pa}{R}, \quad (9.151)$$

or

$$\mu_r = \frac{2}{3\pi} \frac{\alpha a}{R}. \quad (9.152)$$

In the case of a cylinder, the loading contact length a is proportional to the square root of PR . Hence,

$$F \propto P^{3/2} R^{-1/2}. \quad (9.153)$$

It shows that a bigger wheel will produce less rolling resistance. This simple analysis is due to Tabor. It assumes that α is a material constant. This assumption may be true for rubber-like materials but not for metals. For metals, the hysteresis loss factor increases with a/R . For the rolling sphere, Tabor's model gives

$$F \propto P^{4/3} R^{-2/3}. \quad (9.154)$$

Hysteresis is one cause of energy dissipation. Microslip, plastic deformation and surface roughness are also responsible for increasing the rolling resistance.

9.7 PRINCIPLE OF VIRTUAL WORK AND DISCRETIZATION OF CONTACT PROBLEMS

Consider two bodies, Body 1 and Body 2, in contact (as shown in Figure 9.19). For each body, let S_u be the surface on which the displacement or Dirichlet boundary condition is specified and S_f be the surface on which the traction or Neumann boundary conditions are specified. Let S^1 be the surface area of Body 1 that can come in contact with Body 2 and S^2 be the surface area of Body 2 that can come in contact with Body 1. S^1 and S^2 make a contact pair. Let tS_c be the actual contact area common to both S^1 and S^2 at time t . The virtual work principle provides

$$\begin{aligned} & \left\{ \int_{0_v}^t \mathbf{S} \cdot \delta_0^t \boldsymbol{\varepsilon} d^0 V \right\}_{\text{body 1}} + \left\{ \int_{0_v}^t \mathbf{S} \cdot \delta_0^t \boldsymbol{\varepsilon} d^0 V \right\}_{\text{body 2}} \\ &= \left\{ \int_{t_v}^t \mathbf{f}^B \cdot \delta \mathbf{u} d^t V + \int_{t_s f}^t \mathbf{f}^s \cdot \delta \mathbf{u} d^t S \right\}_{\text{body 1}} \\ &+ \left\{ \int_{t_v}^t \mathbf{f}^B \cdot \delta \mathbf{u} d^t V + \int_{t_s f}^t \mathbf{f}^s \cdot \delta \mathbf{u} d^t S \right\}_{\text{body 2}} \\ &+ \int_{t_{S_c}}^t \mathbf{f}^{21} \cdot \delta (\mathbf{u}^2 - \mathbf{u}^1) d^t S. \end{aligned} \quad (9.155)$$

The last term in the previous equation is called the contact integral. The left-hand side expresses the internal virtual work evaluated based on the second Piola-Kirchhoff stress $\mathbf{S}$ (reference configuration at time $t = 0$) and Green-Lagrange strain tensor $\boldsymbol{\varepsilon}$. The symbol ${}^t\mathbf{f}^B$ denotes body force per unit volume and ${}^t\mathbf{f}^s$ denotes surface traction at time t . The symbol ${}^t\mathbf{f}^{21}$ indicates traction on Body 2 at the contact surface.

The boundary conditions are as follows:

1. The gap is non-negative. Let $\mathbf{x}^2$ be the position vector of a contact point on the second body and $\mathbf{x}^1$ the position vector on the first body. Let $\mathbf{n}$ be the

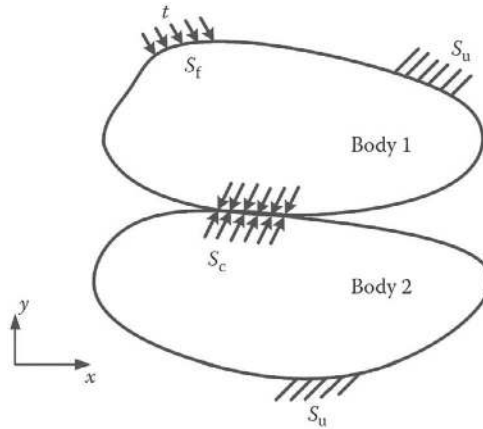


FIGURE 9.19 Contact of two bodies.

normal vector on the surface of the second body directed outward to the body. Then, the gap is given by

$$g = (\mathbf{x}^1 - \mathbf{x}^2) \cdot \mathbf{n} \quad (9.156)$$

2. There can be only compression between two bodies. If the dot product of the traction vector on Body 1 and the normal vector on Body 2 is λ , then

$$\lambda \geq 0. \quad (9.157)$$

3. If there is gap, there is no compressive stress, and if there is compressive stress on the body, there cannot be any gap between two bodies. This condition is expressed as

$$g\lambda = 0. \quad (9.158)$$

4. The Coulomb friction boundary condition is given by

$$|t_s| \leq \mu |t_n|, \quad (9.159)$$

where t_s and t_n are the tangential and normal components of traction, respectively. In case the equality holds in Equation 9.159, then there is slipping, and the tangential traction opposes the relative motion. In the case of an inequality condition, there is sticking, and there is no relative motion.

The contact problem is usually solved by finite element method (FEM). In all FEM formulations, there is a difficulty of identifying the contact zone. There are

many algorithms for that. One of them is the master-slave (node-to-segment) algorithm. One of the contacting surfaces is considered to be the master surface, and the other surface is called the slave surface. The master surface is represented by the boundary cells and the slave surface by the boundary nodes. The algorithm utilizes *a priori* information about the region of possible contact and identifies contact pairs by the penetration of the slave nodes through the master's boundary cells. Penetrations can be detected by several algorithms. A slave node has a normal projection on the closest master segment. For details, refer to Goudreau and Hallquist (1982) and Yastrebov (2013).

Exercises

1. Two steel spheres of diameter D_1 and D_2 are pressed together by a load P . Prove that their mutual approach is given by

$$\delta = 2 \times 10^{-5} P^{2/3} \left(\frac{1}{D_1} + \frac{1}{D_2} \right)^{1/3},$$

where P is in gf (1 gf = 9.81×10^{-3} N) and δ , D_1 , and D_2 are in mm.

2. A steel sphere of diameter D is compressed by load P between two parallel platens of steel. Prove that the total compression is given by

$$\delta = 2 \times 10^{-5} P^{2/3} \left(\frac{1}{D} \right)^{1/3},$$

where P is in gf (1 gf = 9.81×10^{-3} N) and δ and D are in mm.

3. Show that if two equal cylinders are kept one previous the other with their axes perpendicular to each other, they can be analyzed as if a sphere of the same radius is kept on a flat surface.
4. Prove that when a sphere of radius R indents a flat workpiece, the normal load P_Y to initiate yield in the flat workpiece is given by

$$P_Y = \frac{\pi^3 R^2}{6E^2} (1.6\sigma_Y)^3,$$

where

$$\frac{1}{E} = \frac{1-\nu_s^2}{E_s^2} + \frac{1-\nu_f^2}{E_f^2},$$

where E_s and E_f are the Young's moduli of elasticity of the sphere and workpiece, respectively, and ν_s and ν_f are the corresponding Poisson's ratios.

5. Solve the flat plate indentation problem by an upper bound similar to that done in Section 9.3.1, but for this, take $\theta = 45^\circ$.
6. Use the slip line method to solve the indentation by a smooth indenter problem and show that the Mises yield criterion provides approximately $3\sigma_Y$ pressure on punch.
7. During the rolling process, the roll gets flattened due to the rolling load. Using contact analysis, show that the deformed roll radius is given by

$$R' = R \left(1 + \frac{P}{C\delta} \right),$$

where R is the roll radius, P is the rolling load per unit width of the strip, δ is the draft and is equal to change in thickness, and c is given by

$$C = \frac{\pi E_r}{16(1 - \nu_r^2)'} ,$$

where E_r is the Young's modulus of elasticity, and ν_r is the Poisson's ratio of roll.

8. Consider a spherical cavity in an infinite medium. The cavity is subjected to an internal pressure p . If the radius of the elastic-plastic interface is c , show that

$$\frac{c}{a} = \left[\frac{E}{3(1 - \nu)\sigma_Y} \right]^{1/3} .$$

10 Dynamic Elasto-Plastic Problems

10.1 INTRODUCTION

Solutions to some dynamic elasto-plastic problems (mostly one-dimensional) are presented in this chapter to bring out the effect of the dynamic or inertia terms. Ideally, the two formulations described in Chapter 6 should be used, namely, the Eulerian and updated Lagrangian formulations. However, for the problems described in this chapter, the total Lagrangian formulation in the steps of loading and unloading is convenient. In each step, the three governing equations need to be solved: (1) equation of motion, (2) stress-strain relation and (3) strain-displacement relation. A simplified approach will be used to solve these problems. First, the problem of longitudinal stress wave propagation in a rod shall be discussed. This problem was first solved independently by von Karman (1942) and Taylor (1942).

10.2 LONGITUDINAL STRESS WAVE PROPAGATION IN A ROD (1-D PROBLEM)

Figure 10.1 shows a homogeneous elasto-plastic rod of uniform cross section (of area A) and length l . The coordinate axis x is along the axis of the rod, and its origin is at the left end. The situations in which the rod can undergo dynamic or time-dependent deformation are as follows. The rod may be fixed at the left end and subjected to time-dependent axial force or velocity at the right end. The rod may be free at both ends and traveling with a uniform velocity and may impact against a rigid or deformable support.

Consider the case in which the left end is fixed, and the right end is subjected to an axial force. If the problem is static, then it is assumed that the applied axial force is increased very slowly to the desired value to keep the accelerations at a negligible level. In this case, at each instant, there is an instantaneous reaction (in the opposite direction but of equal magnitude) at the left end to maintain the equilibrium of the rod. However, in dynamic problems, the force at the right end is applied suddenly. In that case, a small region at the right end is acted on by an unbalanced force. This unbalanced force creates some acceleration in that region leading to the development of non-homogeneous velocity and displacement fields. Due to this non-homogeneous displacement field, the corresponding strain and stress fields are created. With time, these acceleration, velocity, displacement, strain and stress fields propagate to the fixed left end and are then reflected from its end and start traveling to the right end. Then, they get reflected at the right end and again propagate toward the fixed left end. This whole phenomenon is called

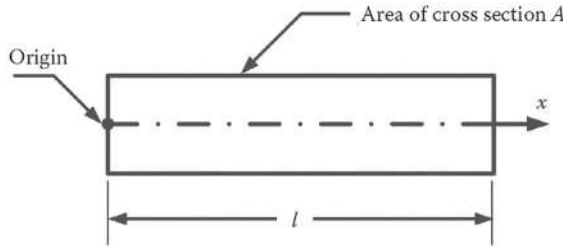


FIGURE 10.1 Homogeneous elasto-plastic rod of uniform cross section.

the ‘stress wave’ propagation. Since the direction of the wave propagation is along the (or in the opposite) direction of the particle velocity, these waves are called the ‘longitudinal’ waves.

It is assumed that the deformation is small. Then, the small strain $\varepsilon \equiv \varepsilon_{xx}$ can be used as the measure of axial deformation. The relation between ε and the axial displacement u is given by the following *strain-displacement relation*:

$$\varepsilon = \frac{\partial u}{\partial x}. \quad (10.1)$$

In this problem, it is assumed that the only non-zero stress is $\sigma \equiv \sigma_{xx}$. Applying Newton’s second law to a small axial element of length dx at a distance x from the left end gives the following *equation of motion*:

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x}, \quad (10.2)$$

where t is the time. Note that, when the only non-zero stress is σ_{xx} , the first equation of the set represented by Equation 3.55 reduces to the previous equation. The *stress-strain relation* can be expressed as follows:

$$\sigma = \sigma(\varepsilon). \quad (10.3)$$

The three equations (strain-displacement relation, equation of motion and stress-strain relation) are combined. Using the chain rule and the strain-displacement relation Equation 10.1, one gets

$$\frac{\partial \sigma}{\partial x} = \frac{d\sigma}{d\varepsilon} \frac{\partial \varepsilon}{\partial x} = \frac{d\sigma}{d\varepsilon} \frac{\partial^2 u}{\partial x^2}. \quad (10.4)$$

Substitution of the previous relation in Equation 10.2 leads to

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\rho} \frac{d\sigma}{d\varepsilon} \frac{\partial^2 u}{\partial x^2}. \quad (10.5)$$

Now, the quantity $c = c(\epsilon)$ is defined as

$$c^2 = \frac{1}{\rho} \frac{d\sigma}{d\epsilon}. \quad (10.6)$$

With this definition of c , Equation 10.5 becomes

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \quad (10.7)$$

This is a one-dimensional wave equation with c as the speed of the wave propagation. In common engineering materials, the slope of the stress-strain curve decreases with strain. Thus, as per Equation 10.6, the wave speed c decreases with the strain.

10.2.1 METHOD OF CHARACTERISTICS

Equation 10.7 is a second-order hyperbolic equation. Therefore, it has two characteristics or characteristic curves in the x - t plane. The characteristics can be found as follows. The velocity of a particle is given by

$$v = \frac{\partial u}{\partial t} \quad (10.8)$$

Using Equations 10.1 and 10.8, the wave Equation 10.7 can be expressed as

$$\frac{\partial v}{\partial t} = c^2 \frac{\partial \epsilon}{\partial x}. \quad (10.9)$$

Further, it is easy to establish the following relation between the velocity v and the strain ϵ :

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial t} \right) = \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial \epsilon}{\partial t}. \quad (10.10)$$

Along any curve in the x - t plane, the variations in ϵ and v can be expressed as

$$d\epsilon = \frac{\partial \epsilon}{\partial t} dt + \frac{\partial \epsilon}{\partial x} dx. \quad (10.11)$$

$$dv = \frac{\partial v}{\partial t} dt + \frac{\partial v}{\partial x} dx. \quad (10.12)$$

Using the relations in Equations 10.9 and 10.10, the second relation also can be expressed in terms of the derivatives $\partial \epsilon / \partial t$ and $\partial \epsilon / \partial x$:

$$dv = c^2 \frac{\partial \epsilon}{\partial x} dt + \frac{\partial \epsilon}{\partial t} dx = \frac{\partial \epsilon}{\partial t} dx + \frac{\partial \epsilon}{\partial x} (c^2 dt). \quad (10.13)$$

Then, Equations 10.11–10.13 can be expressed in the array form as follows:

$$\begin{Bmatrix} d\varepsilon \\ dv \end{Bmatrix} = \begin{bmatrix} dt & dx \\ dx & c^2 dt \end{bmatrix} \begin{Bmatrix} \partial\varepsilon / \partial t \\ \partial\varepsilon / \partial x \end{Bmatrix}. \quad (10.14)$$

Equation 10.14 shows that the derivatives $\partial\varepsilon/\partial t$ and $\partial\varepsilon/\partial x$ can be uniquely determined from the given values of $d\varepsilon$ and dv along any curve in the x - t plane if the determinant of the coefficient matrix does not vanish. The curves C in the x - t plane along which the determinant of the coefficient matrix vanishes are called the characteristic curves of the wave Equation 10.7. These curves are given by the following equation:

$$\det \begin{bmatrix} dt & dx \\ dx & c^2 dt \end{bmatrix} = 0 \Rightarrow \frac{dx}{dt} = \pm c. \quad (10.15)$$

Since c is a function of ε , the characteristic lines are not straight lines but curves. However, for a linearly elastic behavior, c is a constant giving rise to straight-line characteristics. Across the characteristics the derivatives $\partial\varepsilon/\partial t$ and $\partial\varepsilon/\partial x$ are discontinuous. The relation between the differentials of v and ε along the characteristics can be obtained from Equation 10.13:

$$dv = \frac{\partial\varepsilon}{\partial t} dx + \frac{\partial\varepsilon}{\partial x} (c^2 dt) = \frac{\partial\varepsilon}{\partial t} (\pm c dt) + \frac{\partial\varepsilon}{\partial x} c (\pm dx) = \pm c \left(\frac{\partial\varepsilon}{\partial t} dt + \frac{\partial\varepsilon}{\partial x} dx \right) = \pm c d\varepsilon \quad (10.16)$$

Substituting Equation 10.6 in the previous equation, the relation between the differentials of dv and $d\sigma$ can be obtained as follows:

$$dv = \pm c d\varepsilon = \pm c (d\sigma / \rho c^2) = \pm d\sigma / \rho c. \quad (10.17)$$

In the method of characteristics, the solution (i.e., the velocity and stress fields) is constructed graphically by first constructing the characteristics ($dx/dt = \pm c$) in the x - t plane and then finding the variations of the velocity and stress with x and t using Equation 10.17. This method has been used in solving the problem of Section 10.2.5.

10.2.2 CONDITIONS AT THE SURFACES OF DISCONTINUITY IN WAVE PROPAGATION

When the applied velocity or stress is discontinuous in time (discontinuity at $t = 0$ means that they do not increase from the value zero but are applied abruptly), the characteristics originating from the point of disturbance become lines of discontinuity for v and ε . In this case, the jump conditions across the characteristics are needed to obtain the solution. These jump conditions can be derived as follows.

The differential of the displacement is given by

$$du \equiv \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dx = v dt + \varepsilon dx. \quad (10.18)$$

Since the displacement is continuous across the characteristics, that is $du = 0$ across the characteristics, one of the jump conditions becomes

$$[v] = c_s [\varepsilon] = 0, \quad (10.19)$$

where $[v]$ and $[\varepsilon]$ denote, respectively, the jumps in the velocity and strain, and c_s represents the speed of the propagation of the discontinuity. The differential form of the momentum equation for the element of length $dx = c_s dt$ across the characteristics is

$$\rho v dx + \sigma dt = 0. \quad (10.20)$$

Then, the second jump condition becomes

$$\rho c_s [v] + [\sigma] = 0. \quad (10.21)$$

Elimination of $[v]$ from Equations 10.19 and 10.21 leads to the following relation between the jumps of σ and ε :

$$\rho c_s^2 [\varepsilon] = [\sigma]. \quad (10.22)$$

10.2.3 ELASTIC SOLUTION OF 1-D WAVE EQUATION

When the force applied at the right end is small, the wave propagation takes place elastically. In this case, the stress-strain relation in Equation 10.3 becomes

$$\sigma = E\varepsilon, \quad (10.23)$$

where E is the Young's modulus. Then, expression 10.6 for c becomes

$$c^2 = \frac{E}{\rho} \equiv c_e^2. \quad (10.24)$$

The strain-displacement relation (Equation 10.1) and the equation of motion (Equation 10.2) remain the same. Then, the wave Equation 10.7 becomes

$$\frac{\partial^2 u}{\partial t^2} = c_e^2 \frac{\partial^2 u}{\partial x^2}, \quad (10.25)$$

and the corresponding characteristics are given by Equation 10.15 with c replaced by c_e . In the corresponding relations between dv , $d\varepsilon$ and $d\sigma$ (Equations 10.16 and 10.17) along the characteristics also, c should be replaced by c_e .

A general solution of this equation (Kreyszig, 2010), due to D'Alembert, is

$$u = f(x - c_e t) + g(x + c_e t), \quad (10.26)$$

where f and g are arbitrary functions. The first part $f(x - c_e t)$ represents the wave propagating in the positive x -direction from the point of disturbance, whereas the second part $g(x + c_e t)$ is the wave traveling in the negative x -direction. Using the strain-displacement relation (Equation 10.1) and the stress-strain relation (Equation 10.23), the stress field corresponding to the displacement field of Equation 10.26 becomes

$$\sigma \equiv E \frac{\partial u}{\partial x} = E \left[f'(x - c_e t) + g'(x + c_e t) \right], \quad (10.27)$$

where the prime denotes the derivative with respect to the argument. In an infinite rod ($-\infty < x < \infty$), the boundary conditions at the infinity are that the displacement and stress must be bounded. In that case, the functions f and g can be determined from the non-zero initial conditions on the displacement and velocity. For a finite rod with zero initial displacement and velocity, it is possible to determine f and g from the boundary conditions, but the procedure is cumbersome. Therefore, other techniques like the separation of variables (with Fourier expansion of the boundary values) or the Laplace transform are used to obtain the solution. The method of characteristics also can be used.

The D'Alembert's solution can be used to study the reflection of waves at the free and fixed boundaries. Consider a wave generated at the right end $x = l$ and traveling in the negative x -direction toward the left end:

$$u = g(x + c_e t). \quad (10.28)$$

The corresponding stress is given by

$$\sigma = E g'(x + c_e t). \quad (10.29)$$

After reflection at the left end $x = 0$, there would be a wave of the same amplitude but traveling in the positive x -direction. The expression for this wave would be

$$u = g(x - c_e t), \quad (10.30)$$

with the corresponding stress being

$$\sigma = E g'(x - c_e t). \quad (10.31)$$

If the left end $x = 0$ is fixed, the sum of the displacements of the incident and reflected waves should be 0. Then, Equations 10.28 and 10.30 along with the condition $x = 0$ imply

$$u \equiv g(+c_e t) + g(-c_e t) = 0 \Rightarrow g(+c_e t) = -g(-c_e t) \Rightarrow g'(+c_e t) = +g'(-c_e t) \quad (10.32)$$

Using the previous relation, the sum of the stresses due to the incident and reflected waves (Equations 10.29 and 10.31) at the end $x = 0$ becomes

$$\sigma = Eg'(+c_e t) + Eg'(-c_e t) = E[g'(+c_e t) + g'(-c_e t)] = 2Eg'(+c_e t). \quad (10.33)$$

Thus, the stress at the fixed end gets doubled after the reflection. Therefore, an elastic wave can be reflected as a plastic wave. On the other hand, if the left end $x = 0$ is free, then the sum of the stresses of the incident and reflected waves should be zero. Then, Equations 10.29 and 10.31 along with the condition $x = 0$ imply

$$\sigma = Eg'(+c_e t) + Eg'(-c_e t) = 0 \Rightarrow g'(+c_e t) = -g'(-c_e t) \Rightarrow g(+c_e t) = +g(-c_e t). \quad (10.34)$$

Using the previous relation, the sum of the displacements due to the incident and reflected waves (Equations 10.28 and 10.30) at the end $x = 0$ becomes

$$u = g(+c_e t) + g(-c_e t) = 2g(+c_e t) \quad (10.35)$$

Thus, the displacement at the free end gets doubled after the reflection. As a result, the velocity at the free end also gets doubled after the reflection. Further, to maintain the surface stress-free, a compression wave at the free surface needs to be reflected as a tension wave and vice versa.

10.2.4 1-D WAVE EQUATION FOR UNLOADING

When the applied force at the right end is increased sufficiently, the maximum stress in the rod exceeds the yield stress and the wave propagation becomes plastic. Even in impact problems where a sudden velocity is applied at the right end, the reflection at the fixed left end results in a doubling of the stress. Thus, in these problems also, eventually the wave propagation becomes plastic.

During the plastic wave propagation, unloading at a cross section may occur if the stress decreases. Suppose, at a cross section, the stress and strain have reached the values σ^* and ε^* , respectively, and after that, the stress starts decreasing. Then, the stress-strain relation, during unloading, becomes

$$\sigma - \sigma^* = E(\varepsilon - \varepsilon^*). \quad (10.36)$$

The strain-displacement relation (Equation 10.1) and the equation of motion (Equation 10.2) remain the same. Then, the wave Equation 10.25 becomes

$$\frac{\partial^2 u}{\partial t^2} = c_e^2 \frac{\partial^2 u}{\partial x^2} + \frac{d}{dx} \left(\frac{\sigma^*}{\rho} - c_e^2 \varepsilon^* \right) \quad (10.37)$$

The corresponding characteristics are the same as those for the elastic waves:

$$\frac{dx}{dt} = \pm c_e. \quad (10.38)$$

The corresponding relations between dv , $d\varepsilon$ and $d\sigma$ along the characteristics are also the same as those for the elastic waves:

$$dv = \pm c_e d\varepsilon, \quad \rho c_e dv = \pm d\sigma \quad (10.39)$$

10.2.5 PLASTIC SOLUTION OF 1-D WAVE EQUATION IN ROD IMPACTED AGAINST RIGID SUPPORT

Consider the homogeneous elasto-plastic rod of uniform cross section (of area A) and length l shown in Figure 10.1. The rod is stationary and is impacted at the left end ($x = 0$) at time $t = 0$ by a rigid support moving with the velocity U . Then the boundary and initial conditions are given as follows.

Boundary conditions:

$$v = U, \quad \text{at } x = 0, \quad t > 0; \quad (10.40)$$

$$\sigma = 0, \quad \text{at } x = l, \quad t > 0. \quad (10.41)$$

Initial conditions:

$$v = \sigma = 0, \quad \text{at } t = 0; \quad 0 < x < l, \quad (10.42)$$

Let us assume that the material is linearly hardening. Then, the stress-strain relation becomes

$$\begin{aligned} \sigma &= E\varepsilon, \quad \text{for } \sigma \leq \sigma_Y; \\ \sigma &= E_p\varepsilon, \quad \text{for } \sigma > \sigma_Y, \end{aligned} \quad (10.43)$$

where σ_Y is the yield stress and E_p is the slope of the stress-strain curve in the plastic range. In this case, the wave speed of plastic waves also becomes constant, and its value is given by

$$c_p^2 = \frac{E_p}{\rho}. \quad (10.44)$$

Since both the elastic and plastic wave speeds are constant, all the characteristics become straight lines.

Since the disturbance is provided at the origin ($x = 0$), the characteristics of the positive slope originate at this point. Further, for a sufficiently large value of U , both the elastic and plastic waves get generated simultaneously. Figure 10.2 shows a partial characteristic field of this problem. In this figure, OA is the elastic characteristic corresponding to the wave speed c_e , whereas OB is the plastic characteristic with the wave speed c_p . Since $c_e > c_p$ and the t -axis is the ordinate, the slope of the

The stress σ_2 above the characteristic OA (i.e., in region 2) is compressive and its value is σ_Y . Thus, the stress jumps across this characteristic is $-\sigma_Y$. Let v_2 and ε_2 be the velocity and strain above the characteristic OA (i.e., in region 2). The value of c_s along this characteristic is c_e . Then, using this value and the jump conditions 10.21 and 10.22, v_2 and ε_2 can be calculated as

$$\rho c_e (v_2 - 0) + (-\sigma_Y - 0) = 0 \Rightarrow v_2 = \sigma_Y / \rho c_e, \quad (10.46)$$

$$\rho c_e^2 (\varepsilon_2 - 0) - (-\sigma_Y - 0) = 0 \Rightarrow \varepsilon_2 = -\sigma_Y / \rho c_e^2 = -\sigma_Y / E. \quad (10.47)$$

Thus, the velocity, strain and stress fields in region 2 are

$$v_2 = \sigma_Y / \rho c_e, \varepsilon_2 = -\sigma_Y / E, \sigma_2 = -\sigma_Y, \text{ in region 2.} \quad (10.48)$$

The velocity, strain and stress across the characteristic OB are also discontinuous. To the left of the characteristic OB (i.e., in region 3), the velocity v_3 is U . Thus, the velocity jump across this characteristic is $(U - \sigma_Y / \rho c_e)$. Let ε_3 and σ_3 be the strain and stress to the left of the characteristic OB (i.e., in region 3). The value of c_s along this characteristic is c_p . Then, using this value and the jump conditions of Equations 10.19 and 10.21, ε_3 and σ_3 can be calculated as

$$\begin{aligned} (U - \sigma_Y / \rho c_e) + c_p (\varepsilon_3 - (-\sigma_Y / E)) &= 0 \\ \Rightarrow \varepsilon_3 &= (-\sigma_Y / E) - (U - \sigma_Y / \rho c_e) / c_p, \end{aligned} \quad (10.49)$$

$$\rho c_p (U - (\sigma_Y / \rho c_e)) + (\sigma_3 - (-\sigma_Y)) = 0 \Rightarrow \sigma_3 = -\sigma_Y - \rho c_p (U - \sigma_Y / \rho c_e). \quad (10.50)$$

Thus, the velocity, strain and stress fields in region 3 are

$$v_2 = U, \varepsilon_2 = -\sigma_Y / E - (U - \sigma_Y / \rho c_e) / c_p, \sigma_2 = -\sigma_Y - \rho c_p (U - \sigma_Y / \rho c_e). \quad (10.51)$$

Equation 10.51 shows that the bar would develop plasticity only when

$$U > \sigma_Y / \rho c_e. \quad (10.52)$$

The velocity, strain and stress across the unloading elastic characteristic AB are also discontinuous. Again, the velocity v_4 , strain ε_4 and stress σ_4 above the characteristic AB (i.e., in region 4) become zero:

$$v_4 = \varepsilon_4 = \sigma_4 = 0, \text{ in region 4.} \quad (10.53)$$

Using the previous velocity, strain and stress fields, the three distinct regions of the rod for the time intervals $0 \leq t \leq l/c_e$ and $t > l/c_e$ are shown in Figure 10.3. The solution is valid until the plastic characteristic OB reaches the elastic unloading characteristic AB at point B. This happens at the time

$$t_B = \frac{x_B}{c_p} = \frac{2l - x_B}{c_e}. \quad (10.54)$$

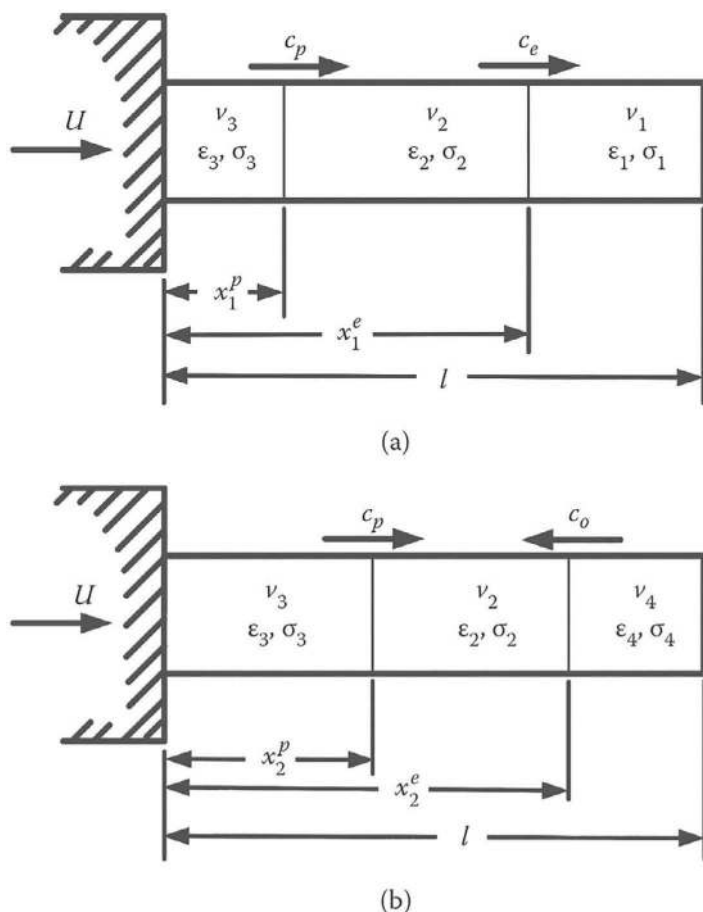


FIGURE 10.3 Three distinct regions in the rod at two different times: (a) $t < l/c_e$; (b) $t > l/c_e$.

The solution beyond this time can be continued in a similar manner. This solution is given in Chakrabarty (2010a).

Even if the material is not linearly hardening, in the present problem, it is possible to assume that the solution does not depend separately on x and t but depends on the combination x/t . Then, all the characteristics of the positive slope become straight lines having the equation

$$\frac{x}{t} = c. \quad (10.55)$$

The characteristic field corresponding to this case is shown in Figure 10.4. Like in Figure 10.2, the characteristics OA and AB are, respectively, the loading and unloading elastic characteristics with wave speed c_e . The velocity along the characteristic OB is U . The region OBC is a plastic region with a constant stress field. The value of stress in this region is the same as that along the characteristic OB. The region OAB represents the centered fan field of plastic waves since $d\sigma/d\epsilon$ is not a constant. The velocity and stress

change continuously in this region along the characteristics of the negative slope. In order to determine the velocity and stress in this region, the following procedure is used.

The velocity and stress along the elastic characteristic OA have already been determined. Let us denote them as v_e and σ_e . Then

$$v_e = \sigma_Y / \rho c_e, \quad \sigma_e = -\sigma_Y. \quad (10.56)$$

Now, the relation between the velocity v and stress σ in the region OAB can be obtained by integrating the differential relation between dv and $d\sigma$ (Equation 10.17) along the characteristics of the negative slope. Along the typical characteristics of the negative slope, this relation becomes

$$(v - v_e) = -\frac{1}{\rho} \int_{\sigma_e}^{\sigma} \frac{d\sigma}{c}. \quad (10.57)$$

Note that, for the intermediate characteristics of the positive slope lying between OA and OB, the velocity changes continuously from v_e to U . Now, choose a characteristic, say, OQ (see Figure 10.4) with a known velocity v between the range (v_e, U) .

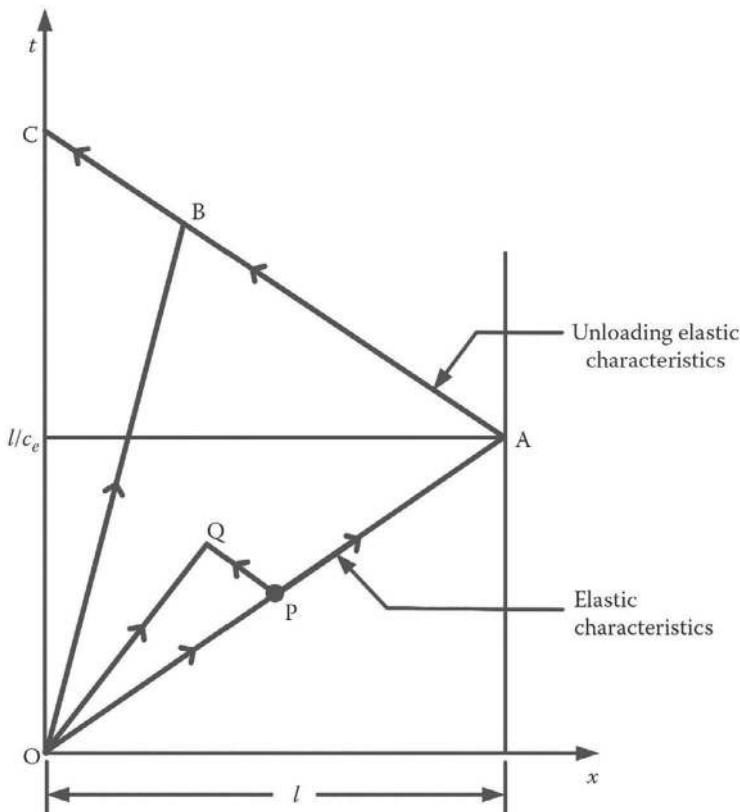


FIGURE 10.4 Partial characteristic field when the material is not linearly hardening.

Then, the stress along the characteristic OQ can be obtained using Equation 10.57. The complete characteristic field for this case is given in Lee (1953).

10.3 TAYLOR ROD PROBLEM (IMPACT OF CYLINDRICAL ROD AGAINST FLAT RIGID SURFACE, 1-D PROBLEM)

The problem of impacting a rod (i.e., a flat-ended cylindrical projectile) against a flat rigid surface or target was solved by Taylor (1948). It is called the Taylor rod problem. The Taylor rod test is used to study the effect of a high strain rate on the yield stress of a material.

Consider a rod of initial length l_0 and initial area of cross section A_0 impacting against a rigid surface with normal velocity U in the downward direction (Figure 10.5a). Usually, the impact velocity U is large enough that both the elastic and plastic waves get generated at the bottom impacted end. Since the elastic wave velocity is larger than the plastic wave velocity, the elastic wave front travels ahead of the plastic wave front. After some time, the elastic wave front gets reflected from the top free end and meets the advancing plastic wave front. Due to this, the top part of the rod decelerates and comes to rest. In the final configuration, the length of the rod shortens (i.e., the length l becomes smaller than l_0), the bottom impacted end bulges due to the plastic deformation (i.e., the area of cross section A becomes bigger than A_0), but the top end does not undergo any permanent plastic deformation.

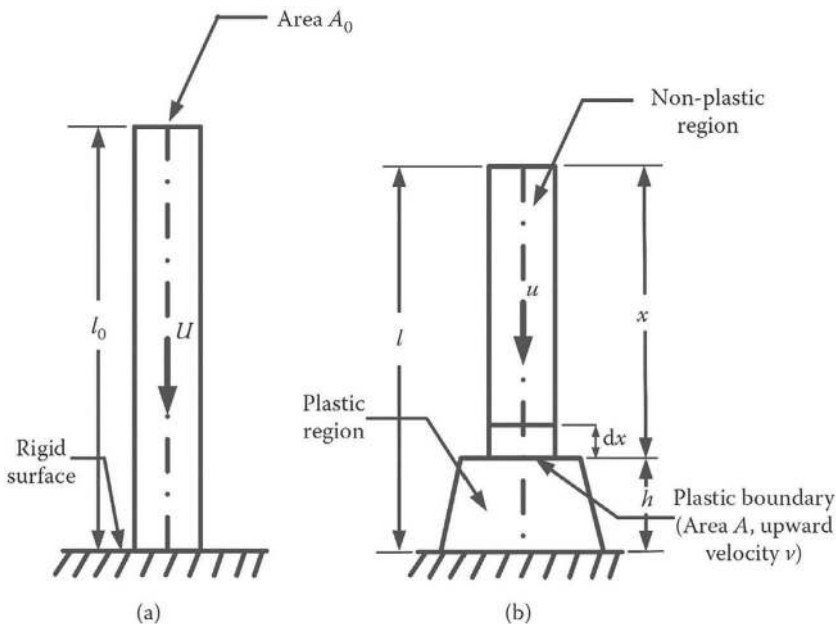


FIGURE 10.5 Impact of cylindrical rod against a flat rigid surface: (a) initial configuration; (b) intermediate configuration.

In order to find the final configuration, the following assumptions are made:

- The elastic deformation is negligible compared to the plastic deformation
- Axial stress is uniformly distributed over the cross section
- The material is perfectly plastic with σ_y as the yield stress

The following four governing equations normally need to be solved: (1) strain-displacement relation, (2) stress-strain relation, (3) equation of motion and (4) volume constancy equation since the elastic deformation is negligible. In this problem, instead of using the strain-displacement relation, the two kinematic relations between the rates of change of the lengths of the non-plastic and plastic regions and the velocities are used. Since the material is perfectly plastic, the stress-strain relation is not needed. Along with the equation of motion for the non-plastic region, the rate of change of momentum of a thin slice of the rod at the plastic boundary (an algebraic equation) is also used. The volume constancy relation is expressed as an algebraic equation between the areas of the cross section and the velocities. Finally, these equations are combined to find the lengths of the non-plastic and plastic regions as well as the shape of the deformed plastic region. For convenience, an implicit time parameter e is introduced as

$$e = 1 - \frac{A_0}{A}, \quad (10.58)$$

where A is the area of the cross section at the plastic boundary.

10.3.1 GOVERNING EQUATIONS

10.3.1.1 Kinematic Relations

Figure 10.5b shows an intermediate configuration in which x is the length of the non-plastic region and h is that of the plastic region (such that $x + h = l < l_0$). Further, u is the downward velocity of the non-plastic region, whereas v is the upward velocity of the plastic boundary. Thus, the velocity of the non-plastic region relative to the plastic boundary is $u + v$. Then, u and v are related to the rates of change of x and h in the following manner:

$$v = \frac{dh}{dt}, \quad (10.59)$$

$$u + v = -\frac{dx}{dt}. \quad (10.60)$$

Here, the negative sign in Equation 10.60 means x decreases with time t . The quantity $u + v$ is considered as the magnitude of the relative velocity.

10.3.1.2 Equation of Motion

Consider an element of length $l dx$ at the plastic boundary (Figure 10.5b). Before coming to rest, its area of the cross section is A_0 , and its downward velocity is u and

is acted on by the upward force $A_0\sigma_Y$ on the bottom surface. During the time dt , it comes to rest, and its bottom area becomes A and is acted on by an additional upward force of $(A - A_0)\sigma_Y$. Therefore, the rate of change of the momentum during dt is the additional force acting on it during that time:

$$\frac{0 - (-\rho A_0 |dx| u)}{dt} = (A - A_0) \sigma_Y. \quad (10.61)$$

Using the relation for $|dx|/dt$ from Equation 10.60, this equation becomes

$$\rho A_0 (u + v) u = (A - A_0) \sigma_Y. \quad (10.62)$$

An additional equation is needed for the whole of the non-plastic region of length x . This region has the area of cross section A_0 , downward velocity u and downward deceleration du/dt and is acted on by the upward force $A_0\sigma_Y$ on the bottom surface. Therefore, its equation of motion becomes

$$\rho x \left(-\frac{du}{dt} \right) = \sigma_Y. \quad (10.63)$$

10.3.1.3 Volume Constancy Condition

The length of an element at the plastic boundary (Figure 10.5b), after passing through the plastic boundary, changes from $|dx|$ to dh . Further, its area of the cross section changes from A_0 to A . Then, the volume constancy condition becomes

$$A_0 |dx| = A dh. \quad (10.64)$$

After substituting the expressions for $|dx|$ and dh from Equation 10.59 and 10.60, this relation becomes

$$A_0(u+v) = Av. \quad (10.65)$$

Equations 10.59, 10.60, 10.62, 10.63 and 10.65 are five equations for five unknown functions of time: x , h , u , v and A . The unknown A is to be replaced by e using Equation 10.58. Here, e is treated as an independent variable, and x and h are determined as functions of e (i.e., implicit functions of time t).

10.3.2 DETERMINATION OF x AS A FUNCTION OF e

In this section, the differential equation is developed for x with respect to e so that its integration would provide information about the change in the length of the non-plastic region from the beginning of the impact until the rod comes to rest. The derivation of the differential equation begins with expressing A in terms of e using Equation 10.58:

$$A = \frac{A_0}{1-e}. \quad (10.66)$$

Using this relation, Equations 10.62 and 10.65 can be expressed in terms of e :

$$\frac{\rho(u+v)u}{\sigma_Y} = \frac{e}{1-e}, \quad v = \frac{1-e}{e}u. \quad (10.67)$$

Elimination of v from the previous two relations leads to the following expression for u in terms of e :

$$\frac{\rho u^2}{\sigma_Y} = \frac{e^2}{1-e}. \quad (10.68)$$

Taking the differential of both sides, one gets

$$2 \frac{\rho u}{\sigma_Y} du = \frac{e(2-e)}{(1-e)^2} de. \quad (10.69)$$

Elimination of dt from Equations 10.60 and 10.63 gives the following expression for dx :

$$dx = \frac{x\rho(u+v)du}{\sigma_Y}. \quad (10.70)$$

Substitution of the expression for v from the second part of Equation 10.67 gives the following form of the previous equation:

$$dx = \frac{x\rho u du}{e\sigma_Y}. \quad (10.71)$$

Finally, the elimination of $(\rho u du/\sigma_Y)$ from Equations 10.69 and 10.71 gives the desired differential equation for x in terms of e :

$$DE : \frac{dx}{de} = \frac{x(2-e)}{2(1-e)^2}. \quad (10.72)$$

The initial condition for this differential equation can be obtained as follows. At the initial instant, that is, when $e = e_0$, the value of x is l_0 . Thus, the initial condition is

$$IC : x = l_0, \quad \text{at } e = e_0. \quad (10.73)$$

Before this differential equation is solved, it is necessary to express the initial value e_0 in terms of the known parameters. At the initial instant (i.e., at $e = e_0$), the value of

u is the impact velocity U . Then, Equation 10.68 gives the following expression for e_0 in terms of ρ , U and σ_Y :

$$\frac{e_0^2}{1-e_0} = \frac{\rho U^2}{\sigma_Y}. \quad (10.74)$$

In order to obtain the solution of the differential Equation 10.72, it is arranged as

$$2 \frac{dx}{x} = \frac{de}{(1-e)} + \frac{de}{(1-e)^2}. \quad (10.75)$$

Integration of each term leads to

$$2 \ln x = -\ln(1-e) + \frac{1}{(1-e)} + \ln C, \quad (10.76)$$

where C is a constant. The previous expression can be rearranged as

$$\ln x^2 = \ln \left[\frac{1}{(1-e)} \right] + \ln \left[\exp \left(\frac{1}{(1-e)} \right) \right] + \ln C. \quad (10.77)$$

Use of the initial condition 10.73 leads to the following expression for the constant C :

$$\ln C = \ln l_0^2 - \ln \left[\frac{1}{(1-e_0)} \right] - \ln \left[\exp \left(\frac{1}{(1-e_0)} \right) \right]. \quad (10.78)$$

Elimination of the constant $\ln C$ from Equations 10.77 and 10.78 and a certain simplification leads to

$$\ln \left(\frac{x}{l_0} \right)^2 = \ln \left[\frac{(1-e_0)}{(1-e)} \right] + \ln \left[\exp \left(\frac{1}{(1-e)} - \frac{1}{(1-e_0)} \right) \right]. \quad (10.79)$$

Further simplification leads to

$$\left(\frac{x}{l_0} \right)^2 = \frac{(1-e_0)}{(1-e)} \exp \left(\frac{e-e_0}{(1-e)(1-e_0)} \right) \quad (10.80)$$

At the final instant, when the rod comes to rest, the area of cross section A at the plastic boundary becomes exactly equal to A_0 . As a result, e becomes 0. Let x_f be the final length of the non-plastic region. Then, it can be obtained by substituting $e = 0$ in Equation 10.80:

$$\left(\frac{x_f}{l_0} \right)^2 = (1-e_0) \exp \left(\frac{-e_0}{(1-e_0)} \right). \quad (10.81)$$

10.3.3 DETERMINATION OF h AS A FUNCTION OF e

To determine the shape of the deformed plastic region of the rod, a differential equation for h with respect to e is developed. The derivation of the differential equation starts with the elimination of dt from Equations 10.59 and 10.60:

$$\frac{dh}{dx} = -\frac{v}{u+v}. \quad (10.82)$$

Then, the expression for v in terms of e (Equation 10.67) is substituted to get

$$\frac{dh}{dx} = -(1-e). \quad (10.83)$$

Thus, the differential equation for h with respect to e is

$$DE : dh = -(1-e)dx. \quad (10.84)$$

Note that, at the initial instant (i.e., when $e = e_0$), the value of h is zero. Thus, the initial condition is

$$IC : h = 0, \text{ at } e = e_0. \quad (10.85)$$

By integrating Equation 10.84 from the initial instant ($e = e_0$, $h = 0$, $x = l_0$) to the current instant, one gets

$$h = -\int_{l_0}^x (1-e)(1)dx. \quad (10.86)$$

In order to evaluate the previous integral, the integrand of the previous equation is treated as the product of $1-e$ and 1, and the integration is carried out with respect to x by parts. Then, one gets

$$h = -\left[(1-e)x\right]_{e=e_0, x=l_0}^{e,x} + \int_{l_0}^x \left(-\frac{de}{dx}\right)x dx. \quad (10.87)$$

Next, the limits in the first term are evaluated, and the second term in the previous equation is simplified. This gives

$$h = -\left[(1-e)x\right] + \left[(1-e_0)l_0\right] - \int_{e_0}^e x de. \quad (10.88)$$

Finally, the previous equation is modified by dividing each term by l_0 :

$$\frac{h}{l_0} = -\left[(1-e)\frac{x}{l_0}\right] + (1-e_0) - \int_{e_0}^e \frac{x}{l_0} de. \quad (10.89)$$

The integral in the previous equation needs to be evaluated numerically after substituting the expression for x/l_0 in terms of e (Equation 10.80). This would give us an expression for h as a function of e from the initial instant ($e = e_0$) until the final instant ($e = 0$).

As stated earlier, the value of e is zero at the final instant (i.e., when the rod comes to rest). Further, the final length of the non-plastic region (i.e., the value of x at the final instant) is x_f . Substituting $e = 0$ and $x = x_f$ (outside the integral) in the previous equation gives the final length (h_f) of the deformed region:

$$\frac{h_f}{l_0} = -\left(\frac{x_f}{l_0}\right) + (1 - e_0) - \int_{e_0}^0 \frac{x}{l_0} de. \quad (10.90)$$

10.3.4 DETERMINATION OF t AS A FUNCTION OF e

In this section, the differential equation is developed for t with respect to e . For this purpose, we start with the expression for du in terms of de (Equation 10.69). u is eliminated from this equation by using the expression 10.68 for u :

$$2 \frac{\rho}{\sigma_Y} \left(\sqrt{\frac{\sigma_Y}{i}} \frac{e}{\sqrt{1-e}} \right) du = \frac{e(2-e)}{(1-e)^2} de. \quad (10.91)$$

Then, this equation is simplified as follows:

$$\frac{de}{dt} = \left(2 \sqrt{\frac{\rho}{\sigma_Y}} \frac{(1-e)^{3/2}}{(2-e)} \right) \frac{du}{dt}. \quad (10.92)$$

Next, du/dt is eliminated from the previous equation by using Equation 10.63:

$$\frac{de}{dt} = \left(2 \sqrt{\frac{\rho}{\sigma_Y}} \frac{(1-e)^{3/2}}{(2-e)} \right) \left(-\frac{\sigma_Y}{\rho} \frac{1}{x} \right) = - \left(2 \sqrt{\frac{\sigma_Y}{\rho}} \frac{1}{x} \frac{(1-e)^{3/2}}{(2-e)} \right). \quad (10.93)$$

Finally, using Equation 10.74, we substitute for $\sqrt{\sigma_Y / \rho}$ in terms of e_0 .

$$\frac{de}{dt} = - \left(\frac{2U}{x} \frac{(1-e_0)^{1/2}}{e_0} \frac{(1-e)^{3/2}}{(2-e)} \right). \quad (10.94)$$

Thus, the differential equation for t with respect to e is

$$DE : \frac{dt}{de} = - \left(\frac{x}{2U} \frac{e_0}{(1-e_0)^{1/2}} \frac{(2-e)}{(1-e)^{3/2}} \right). \quad (10.95)$$

Note that, at the initial instant ($t = 0$), the value of e is e_0 . Thus, the initial condition is

$$IC : t = 0, \text{ at } e = e_0. \quad (10.96)$$

Integration of Equation 10.95 from the initial instant ($t = 0, e = e_0$) to the current instant gives

$$t = -\frac{1}{2U} \frac{e_0}{(1-e_0)^{1/2}} \int_{e_0}^e \left(x \frac{(2-e)}{(1-e)^{3/2}} \right) de. \quad (10.97)$$

The non-dimensional time can be defined as $t(U/l_0)$. Then, Equation 10.97 can be used to express the non-dimensional time as a function of e :

$$t \left(\frac{U}{l_0} \right) = -\frac{1}{2} \frac{e_0}{(1-e_0)^{1/2}} \int_{e_0}^e \left(\frac{x}{l_0} \frac{(2-e)}{(1-e)^{3/2}} \right) de. \quad (10.98)$$

The integral in the previous equation needs to be evaluated numerically after substituting the expression for x/l_0 in terms of e (Equation 10.80). This would give us an expression for t as a function of e from the initial instant ($e = e_0$) until the final instant ($e = 0$).

Using Equation 10.98, e can be eliminated from expression 10.80 for x and expression 10.89 for h to obtain them as functions of time t .

10.3.5 ENERGY METHOD

A typical deformed shape of the impacted rod obtained using the Taylor's (1948) solution, as described in Sections 10.3.1–10.3.4, is shown in Figure 10.6. Predictions

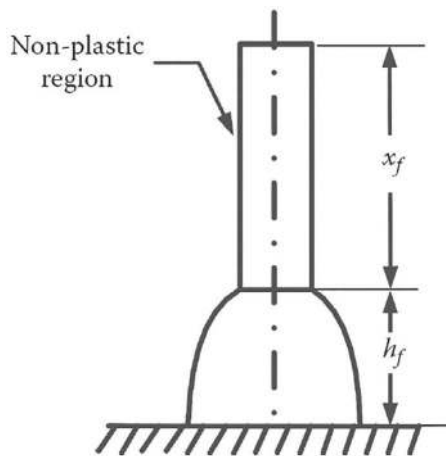


FIGURE 10.6 Typical deformed shape of the impacted rod obtained using the Taylor's (1948) solution.

of the variations of the lengths of the non-plastic (x) and deformed plastic regions (h) with time, as well as the final lengths (x_f, h_f) have been found to be in agreement with experimental values for relatively low-impact velocities ($\rho U^2 \sigma_Y \leq 0.5$). However, the experimentally observed shapes of the deformed plastic region are concave and not convex, as shown in Figure 10.6. Hawkyard (1969) used the energy method to obtain the solution of the Taylor rod. His predictions of the shape of the deformed plastic region are in better agreement with experimental results.

Exercises

1. For the problem of the longitudinal wave propagation in a rod (Section 10.2), choose the following non-linear hardening relation for the rod material:

$$\sigma = \sigma_Y + K \epsilon^n.$$

For this material, find the stress σ along the characteristics OB (Figure 10.4) in terms of v_e , U , ρ , σ_e , K and n using relations 10.57 and 10.6. Note that the velocity v along the characteristics OB is U .

2. For the Taylor rod problem (Section 10.3), choose the following geometric and material parameters and impact velocity:

$$l_0 = 300 \text{ mm}, \sigma_Y = 250 \text{ MPa}, \rho = 8000 \text{ kg/m}^3, U = 100 \text{ m/s}.$$

- a. Find the value of e_0 using Equation 10.74.
- b. Find the final length of the non-plastic part (x_f) using Equation 10.81.
- c. Find the variation of the height of the deformed plastic region (h) with e by integrating Equation 10.89 numerically using expression 10.80 for x/l_0 . Using this variation, plot the shape of the deformed plastic region. (You need to consider the values of e from the initial instant $e = e_0$ to the final instant $e = 0$.)

11 Continuum Damage Mechanics and Ductile Fracture*

11.1 INTRODUCTION

Fracture of materials occurs almost every day. Fracture can be either *inconsequential* or *consequential* depending upon the effect it has on our lives. Breaking a china dish, pencil lead, glass window, tearing off a notched lid of a cola, etc., can be cast in the first category. However, fracture of a cup during the drawing operation, armor penetration of tanks, crash of an automobile, grounding of ships, tearing of pipelines and aircraft fuselages, etc., are some examples where the fracture is often accompanied by huge economic loss and leads to loss of human life. This type of fracture can be classified into the *consequential* type. It is this type of fracture that requires considerable investigation. What causes the fracture of engineering structures through formation and propagation of cracks should be known. Also, with the development of new and advanced materials, more investigation is needed.

Fracture was initially analyzed using a linear elastic fracture mechanics approach and later by an elastic-plastic fracture mechanics approach. However, in 1958, Kachnov proposed a simple model of material damage, which has since been extended by many researchers (Lemaitre, 1996; Lemaitre and Desmorat, 2005; Murakami, 2012) to brittle, plastic and viscous materials. The model has now evolved into a full field, which is called ‘continuum damage mechanics’ (CDM). The models based on CDM can not only analyze but also predict the failure of materials through the evolution of internal damage before external macrocracks are visible.

The purpose of the present chapter is to present a concise introduction to CDM theory as applied to ductile fracture of plastic materials. For a detailed discussion and advanced topics on the subject, the reader is referred to excellent monographs by Lemaitre (1996), Lemaitre and Desmorat (2005) and Murakami (2012).

11.2 MOTIVATION

As stated earlier, failure of materials, especially of engineering structures, is a serious problem. In the following, some well-known examples of failure of real-life structures are briefly presented. The purpose is to bring forth the dangers of lapse of engineering judgments while designing structures. In each case, as will be presented next, the engineers failed to investigate the performance of the material under

* This chapter has been contributed by Dr. S.S. Gautam, Associate Professor, IIT Guwahati.

different loading conditions. This led to catastrophic failure of material leading to losses—human and economic.

11.2.1 FAILURE OF THE TITANIC

The sinking of the Titanic is probably one of the most famous disasters in recent times. At the time of its construction, the ship was thought as unsinkable. On its maiden voyage in 1912, three days after setting sail from Southampton, UK, the Titanic sank in the North Atlantic Ocean on April 15 after colliding with an iceberg. Out of 2600 people on board, 1500 people perished. It is now well established that the cold-water temperatures caused the steel, used in the hull, to crack in a brittle manner when it hit the iceberg. The steel, instead of behaving in a ductile manner, behaved like glass. It was only after 50 years in the year 1963, that the researchers at the US Naval Research Lab came up with a comprehensive analysis describing the phenomenon of ductile to brittle crack propagation in steel. Thus, it was the lack of detailed study of damage to steel at low temperatures that was directly the cause. The designer of the ship, in the absence of detailed knowledge, chose the wrong steel.

11.2.2 FAILURE OF LIBERTY SHIPS

During World War II, the United States built all-welded cargo vessels called the Liberty Ships. However, nearly 40% of the 2700 liberty ships constructed were damaged due to brittle fracture. The failure was caused by the development of brittle cracks due to a lack of fracture toughness of the welded joints. In fact, a number of ships suddenly broke into two pieces. This happened because of the ductile-to-brittle transition behavior of the steel at welded joints. The brittle fracture started at the weld cracks and other stress concentration points when the ships were operated in cold waters over a long period of time, leading to the failure of the ship.

11.2.3 FAILURE OF COMET PASSENGER AIRCRAFT

In the early 1950s, two de Havilland Comet aircraft disintegrated mid-flight. A number of lives were lost. The failure has been studied extensively in the engineering community. During the design, the actual size of the cracks was large enough to be detected in the regular inspection. However, the designers had failed to consider the effect of constant pressurization and depressurization of the fuselage on the small cracks. In these two particular cases, the cracks originated from the rivet hole near the square cabin windows. When the crack reached the window, the size of the window added to the crack length and effectively led to catastrophic failure. It is widely considered that the long dominance of American aircraft firms in civil aviation is in part due to the Comet failure.

11.2.4 FAILURE OF THE SPACE SHUTTLE CHALLENGER

The Challenger space shuttle disaster was the first in the history of NASA. It occurred after 55 successful shuttle launches. A total of seven human lives were lost.

The main cause of the failure was the failure of the O-ring rubber seals installed in the Challenger's booster rocket. The low temperature of the launch site on the day of the launch led the rubber to lose its elasticity and rendered it brittle. This caused the seal to shrink, developing cracks, which enabled the fuel burning inside to leak. This burning fuel came in contact with the tanks containing hydrogen and oxygen causing an explosion.

The preceding examples show that failure has both economic and human costs associated with it. With the advent of advanced analysis and simulation tools and a deeper understanding of material behavior, it is now possible to avoid such catastrophic failures. However, there is a constant effort toward the development of new advanced materials, for example, for automotive and aerospace applications. Hence, it becomes necessary for designers and engineers to become familiar with the background necessary to develop the tools to analyze and predict the behavior of material under various loading conditions. In the present chapter, only one form of failure is dealt with, that is, ductile failure.

11.3 OBJECTIVE AND PLAN OF THE CHAPTER

Based on the examples presented in Section 11.2, it is clear that the study of fracture is essential to prevent human and economic losses. The objective of the present chapter is to first discuss one of the most important fracture processes, ductile fracture, followed by a brief overview and formulation of CDM theory for its prediction.

The chapter is organized as follows. First, the classification of the different types of fracture processes is presented. Then, the differences between global and local approaches to studying fracture are outlined with the respective advantages and disadvantages of each approach. The phenomenon of ductile fracture is discussed next. The various phases of the ductile fracture process are discussed along with various mathematical models used to study each phase. A number of models for fracture initiation, including the Gurson porous plasticity model and the CDM model of Lemaitre, are presented. Finally, the detailed theory of CDM is presented. The effect of crack closure for the case of compressive loading is also discussed. Some experimental techniques to measure damage are presented, followed by the application of the CDM model to the simulation of ductile fracture in tensile test specimens.

11.4 CLASSIFICATION OF FRACTURE

The material of an engineering structure may suffer deterioration or damage in its mechanical properties owing to various reasons. The damaged material may subsequently suffer fracture. The damage and subsequent fracture can broadly be classified into the following subcategories.

When the nucleation of microvoids and microcracks and subsequent coalescence occurs after appreciable plastic deformation, the damage (fracture) is called ductile fracture. The plastic strain is above a certain threshold value. This type of fracture usually occurs in materials like mild steel, aluminum, copper, etc. Figure 11.1a shows a typical stress-strain curve for a ductile material.

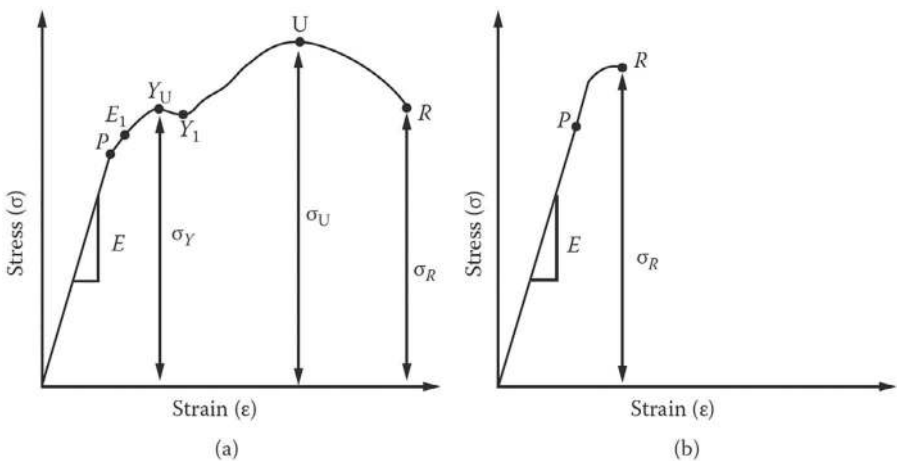


FIGURE 11.1 Schematic representation of tensile stress-strain curve up to rupture for (a) ductile material (for example, mild steel) and (b) brittle material (for example, cast iron or ceramic). E is the Young’s modulus, σ_Y is the yield stress, σ_U is the ultimate stress and σ_R is the rupture stress. Various points on the stress-strain curves are as follows: P is the proportional limit, E_1 is the elastic limit, Y_U is the upper yield limit, Y_L is the lower yield limit, U is the point of ultimate stress and R is the point of rupture.

TABLE 11.1
Classification of Types of Fatigue Fracture

S. no.	Type	Number of cycles to fracture
1	Very low cycle	$N_R < 10^2$
2	Low cycle	$10^2 < N_R < 10^4$
3	High cycle	$10^4 < N_R < 10^5$
4	Very high cycle	$N_R > 10^5$

When the nucleation of microvoids and microcracks and subsequent coalescence occurs without an appreciable amount of plastic deformation, the damage is categorized as brittle fracture. This type of fracture is usually observed in cases of concrete, glass, ceramics, composites, cast iron, etc. No appreciable plastic deformation occurs and the failure is mostly by debonding. Figure 11.1b shows a typical stress-strain curve for a brittle material.

Sometimes, the plastic deformation is accompanied by viscous effects. The nucleation of microvoids and microcracks and subsequent coalescence occurring under these conditions is called creep fracture. The viscous effects are appreciable when the material is subjected to temperatures above one-third of the absolute melting temperature T_m .

When the material is subjected to cyclic loading, it is called fatigue loading. Microvoids and microcracks nucleate when the material is subjected to a high level

of fatigue load. The coalescence of microvoids and microcracks leads to a fracture called the fatigue fracture. Fatigue is usually classified into four types depending on the number of cycles, N_R , to fracture (Murakami, 2012). This is shown in Table 11.1.

When the material is subjected to impact load, microvoids and microcracks may be nucleated by the interaction of the stress waves. The initial impact creates stress waves, which travel through the material. These stress waves get reflected from the boundaries and then interfere with each other. This causes zones of tensile and compressive loading. Afterward, reflection and interaction of these stress waves from various boundaries lead to microvoid and microcrack nucleation, which may later coalesce to form macrocracks. This type of fracture is called spallation.

11.5 GLOBAL AND LOCAL APPROACHES TO FRACTURE

Over the past 100 years, a great deal of effort has been devoted to modeling and analysis of the degradation of the strength-carrying capacity of structures under various types of loads. The science of fracture was started by Griffith (1921) who found that a crack in equilibrium must balance the driving force causing the crack to open with the surface tension forces tending to close the crack. A number of strategies have been proposed and are followed. However, all the strategies can broadly be classified into two categories. These are the *global approach* and *local approach* to fracture, respectively.

The global approach, that is, the fracture mechanics approach, has been one of the early tools to model pre-existing cracks (Knott, 1973; Kannien and Popelar, 1985; Anderson, 2005). The J-integral proposed by Rice (1968) has been used for industrial problems. Alternative approaches such as the crack tip opening displacement (CTOD) and the crack tip opening angle (CTOA) (Anderson, 2005) were proposed. However, CTOD is more difficult to compute than the J-integral in finite-element computations. Recently, to model an unknown crack path, advanced numerical techniques such as XFEM (Moes et al., 1999) and XFEM coupled with a cohesive zone model (Moes and Belytschko, 2002) have been proposed.

The limitations of the global or fracture mechanics approach have led to the development of a local approach or the continuum mechanics approach. This approach is based more on physical mechanisms that take place during the damage process. In this approach, the damage can be represented in the bulk using the *CDM models* or on the surface using *cohesive zone models*. The effect of displacement discontinuities (microvoids and microcracks) is modeled within the framework of continuum mechanics. A good review of continuum mechanics-based models to describe ductile fracture can be found in Besson (2010). The book by Berdin et al. (2004) provides detailed methodologies on the local approach to fracture.

11.5.1 LIMITATIONS OF GLOBAL AND LOCAL APPROACHES TO FRACTURE

It is essential to clearly state the difference between the two approaches (i.e., global and local). In the fracture mechanics approach, only a small number of cracks are considered. Hence, it is useful for modeling only after the macrocracks have already formed. Some of the specific limitations are as follows:

- It can be applied to bodies with pre-existing cracks
- It cannot be used for crack initiation and propagation
- Parameters like J-integral are not an intrinsic material property. This means that the value of J-integral depends strongly on the geometry of the specimen used
- It can be applied only to simple geometries
- Alternative approaches like CTOD and CTOA also suffer from similar limitations. In numerical settings, that is, in finite-element analysis, the results using J-integral, CTOD or CTOA are highly mesh-dependent and overestimate the crack areas
- Advanced techniques like XFEM are still mostly applied for two-dimensional (2-D) cases involving elastic solids or small-scale plastic deformation

In the local approach or the CDM approach, the material degradation of the body is modeled and analyzed. The body may or may not have initial cracks (i.e., damaged). The cracks are modeled by smearing them out continuously at various locations and length scales. Since it is based on the framework of continuum mechanics, it is easier to implement in a numerical setting, usually finite-element methods. However, issues like mesh-dependent results remain; see Murakami (2012) for a detailed discussion on this and other issues. Also, depending on the CDM model used, the number of material parameters required might be too large, and it might be difficult to obtain all of them from the experiments.

11.6 DUCTILE FRACTURE

Ductile fracture is one of the most important types of fracture as most engineering structures are made of ductile materials such as iron, aluminum, etc. Rather than cracks appearing suddenly, the material keeps ‘pulling apart’ for a long time, leaving a rough or ‘dimpled’ surface at fracture. Hence, a large amount of energy is absorbed before fracture. Some of the energy from stress concentration at the crack tips is dissipated by plastic deformation before the crack actually propagates. Metals, especially materials with high purity, can sustain very large deformation (around 50%–100% or more strain) before fracture. For example, pure iron can deform up to 80% strain before fracture at room temperature (Bonora, 1997), whereas cast iron or other high-carbon steels can fracture at as low as 3% strain.

Ductile fracture has also been defined as a mode of fracture in which voids—pre-existing or nucleated during deformation—grow under favorable conditions and coalesce to form a continuous fracture path (Garrison and Moody, 1987). Garrison and Moody (1987) have mentioned that ductile fracture is also called *fibrous fracture* or *dimpled fracture*. Figure 11.2 shows the ductile fracture process. A typical material consists of pre-existing microvoids, microcracks and inclusions of various sizes and shapes. When the material is subjected to loads—either static or dynamic—the microvoids enlarge. The inclusions also experience stress both inside and at the interface of the inclusion and matrix material. When the loading is significant, new voids nucleate either by the decohesion between the inclusion and

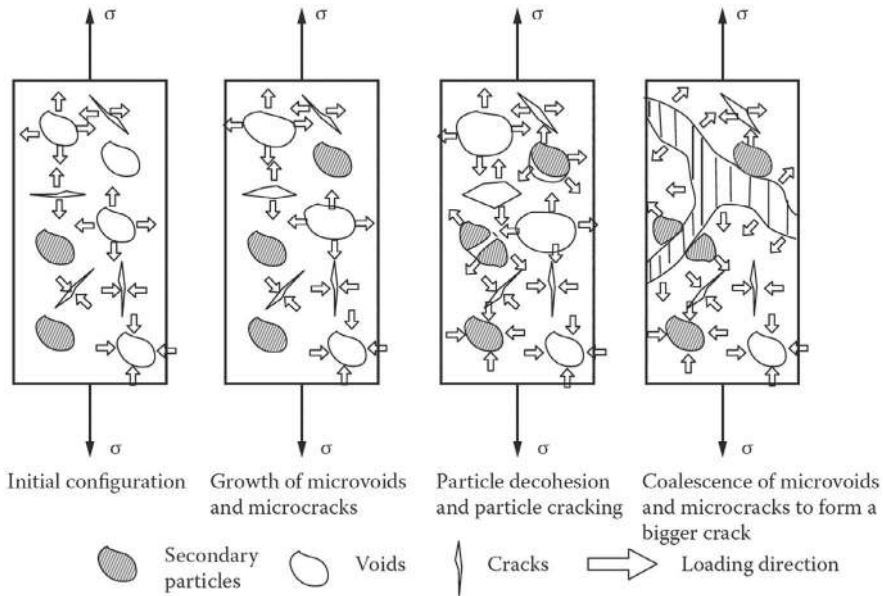


FIGURE 11.2 Schematic description of the ductile fracture process. The material has pre-existing cracks, voids and inclusion/secondary particles, which grow under applied loading or leading to eventual formation of a larger crack. Note that not all voids expand as some may experience compressive loading, as shown in the figure.

the matrix material or by splitting of the inclusion. The pre-existing microvoids also enlarge in size. Finally, the microvoids coalesce, leading to ductile fracture of the matrix. A much larger crack is now formed. A similar process repeats at various locations inside the material where the conditions are favorable. These large cracks then coalesce (even with smaller microvoids) together to form macrocracks that lead to eventual fracture.

Following the previous discussion, it is clear that the ductile fracture process involves three basic steps: void nucleation or initiation, void growth and void coalescence (also known as crack formation) leading to crack propagation and final fracture. However, in real situations, it is very difficult to define when one step ends and the other starts. Garrison and Moody (1987) have mentioned that there are two approaches to studying ductile fracture. One is to use a *continuum mechanics* approach and the other is the *classical metallurgical* approach of studying the microstructural changes and correlating these changes with the macroscopic properties. However, a complete solution to the problem of ductile fracture should involve both the continuum mechanics and microstructural approaches (Garrison and Moody, 1987).

Next, each of the three steps is described in detail along with modeling methods that have been proposed by various authors. For a detailed discussion on ductile fracture, the reader is referred to the excellent review paper on ductile fracture by Garrison and Moody (1987) and a monograph by Thomason (1990).

11.6.1 VOID NUCLEATION OR INITIATION

For materials with inclusions/secondary particles, void nucleation can occur by decohesion of the inclusion from the matrix material and/or by fracture of the inclusion. However, even if it is assumed that the material is composed of only one type of inclusion, void nucleation will not occur simultaneously at all the inclusions even if they are spherical. It has been observed that typically the void nucleation starts at the larger inclusion.

Void nucleation or initiation by decohesion. The inclusion-matrix decohesion does not occur simultaneously at all points along the inclusion-matrix interface even for spherical inclusions. Factors that influence the particle-matrix decohesion include particle shape and size, their orientation, nucleation strain, stress state, strength of interfacial cohesion, strength of the matrix material and particle volume fraction. The void initiation usually starts at larger particles within a given particle distribution (Garrison and Moody, 1987). This is because the nucleation strain is lower for larger particles. The void nucleation strain itself depends on the stress state. Two different approaches to modeling decohesion have been proposed (Thomason, 1990) for particles below and above 1 μm in size. For the particles below this size, it has been shown by Goods and Brown (1977) that the nucleation strain is directly proportional to the particle radius. Experimental support for this correlation is obtained from LeRoy et al. (1981). In a continuum context, a nucleation law based on attainment of a critical strain may be used to model the nucleation of voids from these particles. For particles with a size in excess of 1 μm , continuum plasticity models can be employed to describe the microvoid nucleation process, which can be phrased in terms of a critical stress criterion.

Void nucleation or initiation by inclusion fracture. The second mechanism of void nucleation is by fracture of the inclusion itself. This type of nucleation has been observed, for example, in aluminum alloys, cementite in steel as well as sulfides in steel. Gurland (1972) has shown that void nucleation occurred by cracking of spheroidized cementite in a 1.05% carbon steel under tensile, compressive and torsional loading (Garrison and Moody, 1987). Inclusion cracking occurred in a direction perpendicular to the maximum principal strain under all three loading conditions, which coincided with the direction of the highest tensile stress. Further, the largest inclusions/secondary particles were observed to fracture at the lowest strains. Factors that influence the tendency of void formation by particle cracking include particle size (Gurland, 1972), strength of the matrix-particle shape (Lindley et al., 1970) and the strength of the matrix-particle bond, which can be altered by segregation of impurity elements (Garrison and Moody, 1987).

Garrison and Moody (1987) have mentioned that void nucleation can start at sites other than the inclusion also, for example, in metastable β titanium alloy. Various attempts have been made to develop a criterion for void nucleation by inclusion

decohesion or fracture. All the models work on the premise that the stress at the interface or applied to the inclusion must exceed some critical value. A detailed discussion on various models for void nucleation can be found in the work of Garrison and Moody (1987). In summary, void nucleation at inclusions, by decohesion or cracking, is influenced by the shape of the inclusion, inclusion size, its orientation, interfacial stresses, strength of the inclusion-matrix interface and the strength of the matrix itself.

11.6.2 VOID GROWTH

The second stage of the ductile fracture process is the void growth, which occurs due to plastic deformation of the matrix material and is strongly influenced by the high triaxial tension. The resulting stress-free surface of the void causes a localized stress and strain concentration in the adjacent plastic field. With the continuing plastic flow of the matrix, the microvoid will therefore undergo a volumetric growth and shape change, which amplifies the distortion imposed by the remote uniform strain rate field (Thomason, 1990). If it is assumed that the microvoids are nucleated at positions sufficiently far apart so that there is virtually no initial interaction between their local stress and strain fields, it is possible to develop an adequate model for the early stages of microvoid growth in terms of a single void in an infinite plastic solid.

11.6.2.1 Analytical Models for Void Growth

Ashby (1966) proposed a dislocation-based void growth model. In his model, once the void is nucleated, the growth is given by

$$\frac{\Delta V_{\text{void}}}{V_{\text{void}}} \approx \frac{\epsilon_{\text{shear}}}{2} \quad (11.1)$$

where ΔV_{void} is the increase in volume, V_{void} is the volume of nucleating void and ϵ_{shear} is the shear strain on the primary slip band. However, this model predicts growth rates that are too small. Also, being based on dislocation mechanics, it causes high localized stresses around the void without suggesting the mechanism for void growth (Garrison and Moody, 1987).

The earliest continuum mechanics-based analytical study on the growth of voids was undertaken by McClintock (1968). He considered a circular void in a 2-D infinite medium subjected to remote uniform strain rate field $\dot{\epsilon}$ and stress field σ . He developed equations describing the growth of voids of different geometries (circular and elliptical) under different conditions. He then obtained solutions to these equations for the cases of non-hardening and linear hardening plastic materials. Finally, he postulated that the behavior of moderately strain-hardening materials could be obtained by interpolating the solutions for linear hardening and non-hardening plastic materials. McClintock's void growth equations have been summarized by Garrison and Moody (1987). The equation for intermediately hardening material for the cylindrical void of an elliptical cross section is given by

$$\ln \frac{R}{R_0} = \frac{\varepsilon_{eq}^p \sqrt{3}}{2(1-n)} \sinh \left[\frac{\sqrt{3}(1-n)(\sigma_a + \sigma_b)}{2\sigma_{eq}} + \left(\frac{\varepsilon_a + \varepsilon_b}{2} \right) \right], \quad (11.2)$$

where $R = (a + b)/2$ is the mean radius (a and b are the semimajor and semi-minor axes of the elliptical void, respectively), R_0 is the initial mean void radius, σ_{eq} and ε_{eq}^p are the equivalent stress and plastic strain, respectively, σ_a and σ_b are the constant applied stresses in the principal directions, ε_a and ε_b are the strains in the principal directions, and n is the hardening coefficient. However, it was found that McClintock's equations greatly underestimate the extent of void growth though confirming the importance of high-stress triaxiality on void growth.

Rice and Tracey (1969) have proposed another continuum-based model for void growth. They considered a spherical void in a rigid perfectly plastic material. The rate of change of the void radius expressed for high-stress triaxiality under the action of the tensile extension rate of $\dot{\varepsilon}_{eq}^p$ is given by

$$\frac{\dot{R}}{R} = \alpha \exp \left(\frac{\sqrt{3}\sigma_m}{2\sigma_y^m} \right) \dot{\varepsilon}_{eq}^p, \quad (11.3)$$

where R is the average void radius, σ_m is the mean stress, σ_y^m is the matrix yield strength in shear, and α is a numerical factor. Rice and Tracey (1969) initially proposed a value of 0.283 for α , which was later modified to achieve greater accuracy (Besson, 2010). Based on the model of Rice and Tracey, a simple criterion for fracture has been used, which states that fracture occurs when the normalized void radius reaches a critical value,

$$\left(\frac{R}{R_0} \right) = \left(\frac{R}{R_0} \right)_c, \quad (11.4)$$

where R_0 is the initial void radius, and $(R/R_0)_c$ is a material-dependent parameter, which characterizes the critical value of void growth.

The previous continuum-based models are based on the analysis of growth of an isolated void in an infinite medium. The drawback of the previous models is that they do not account for void interaction and effect of void growth on the matrix material behavior, that is, softening. Hence, they underpredict the extent of void growth. Any good model of void growth should account for the change in shape of the void and void interaction.

Gurson (1977) was the first to address this drawback. He used an upper bound-based approach to analyze the growth of a spherical void in a sphere made of rigid perfectly plastic material. Damage was represented using porosity f , that is, void volume fraction, which corresponds to the ratio of the volume of the void to the volume of the sphere. The effect of porosity was taken into account in the definition of plastic yield surface as

$$\Phi = \left(\frac{\sigma_{eq}}{\sigma_Y} \right)^2 + 2f \cosh \left(\frac{\text{tr}(\sigma)}{2\sigma_Y} \right) - 1 - f^2. \quad (11.5)$$

The use of the normality rule together with the mass conservation leads to the following porosity evolution equation due to void growth $\dot{f}_g$:

$$\dot{f}_g = (1 - f) \text{tr}(\dot{\epsilon}^P), \quad (11.6)$$

where $\dot{\epsilon}^P$ is the plastic part of the strain rate tensor.

11.6.3 VOID COALESCENCE

This is the final stage in the ductile fracture process. In this stage, the growing voids link together, and the actual fracture starts, either stably or unstably. There seem to be two different mechanisms by which the growing voids coalesce. The *first* occurs in a material containing only one population of void nucleating particles activated during the fracture process. It may also be possible that there exist other particles in the matrix, but they are bonded to the matrix very strongly. Hence, no void is able to nucleate from these particles. Coalescence occurs when the voids nucleated at these particles impinge. This is called *void impingement* as schematically represented in Figure 11.3. In the *second* case, the voids are nucleated first at the larger particles, which are weakly bonded with the matrix. These voids continue to grow as the matrix material deforms plastically. Eventually, voids are also nucleated at the second particle population. These particles are very strongly bonded to the matrix material so that the strain required for decohesion is much larger. Also, these voids are smaller in size and finely distributed. Finally, the voids growing around the larger particles coalesce by the sudden bridging of the ligament between them by the growth of the voids nucleated at the second particle population. This process is often referred to as *void sheet coalescence* (see Figure 11.4). Experiments suggest that the coalescence event is, in general, rapid, occurring over a small interval of macroscopic strain (Garrison and Moody, 1987).

In the process of coalescence by void impingement, two mechanisms have been proposed. In the first mechanism, the ligaments between the growing voids would simply neck down to a point. Thus, the fracture surface would exhibit voids of a rather uniform size and spacing with the ridges around the dimples representing material drawn to a knife-edge fracture of zero cross section. This mode of void coalescence is observed when the length of the elongated cavities is equal to their spacing (Garrison and Moody, 1987). The second mechanism, which results in direct impingement of the growing voids, is the slipping-off mechanism observed in the tensile fracture of single crystals (Garrison and Moody, 1987).

In the process of void sheet coalescence, strong strain localization occurs between the larger voids, which promotes the rapid nucleation of voids from the smaller particles. In addition, void sheet coalescence can connect larger voids by nucleating microvoids, which do not require a second particle population. Puttick

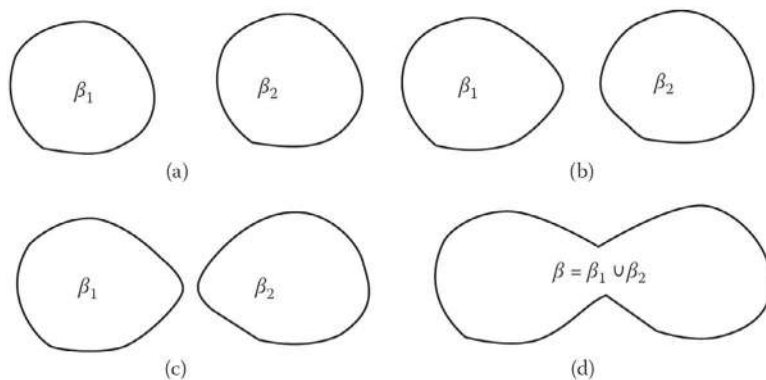


FIGURE 11.3 Schematic representation of the void coalescence through the process of void impingement. (a) Initial configuration, (b) necking starts, (c) necking progresses further and (d) final void impingement.

(1959) observed that, in smooth axisymmetric tensile specimens of oxygen-free high thermal conductivity copper, the voids form first at the center of the tensile specimen and coalesce to form an internal notch. The notch tends to concentrate or localize the deformation at its tip in narrow bands of high shear strain at an angle of 50° – 60° to the transverse plane. Under the combined action of the applied tensile stress and the resulting shear strain, ‘sheets’ of voids are nucleated in these bands, growing until coalescence (by impingement) occurs, producing local fracture of the ‘void’ sheet. Thus, the requirement for void sheet coalescence is that voids nucleate and coalesce on a nominally 2-D surface defined by a region of strain localization. In general, void sheet coalescence does not require a second population of void-nucleating particles.

Gologanu et al. (2001a, 2001b) studied the problem of theoretically predicting the coalescence of cavities in periodically voided ductile material. Two sets of loadings were considered: one in the axial direction and the other in the lateral direction on a cylindrical representative volume element (RVE). Recently, Zhang and Chen (2007) used a three-dimensional unit cell model to investigate the effect of stress triaxiality on the coalescence of voids. They showed, by considering five different stress states, that stress triaxiality greatly affects the onset of void coalescence. They suggested that a triaxiality-dependent critical void volume fraction should be used in the damage-based finite-element simulations.

11.7 MODELS OF FRACTURE INITIATION

As stated earlier, ductile fracture involves three stages, namely, void nucleation, void growth and void coalescence to form a microcrack. It is found from experiments that macroscopic material properties change due to void growth. As a result, the constitutive relation needs to be changed when the extent of void growth is large (more than 10%) (Brown et al., 1980). Thus, any realistic model for the prediction of

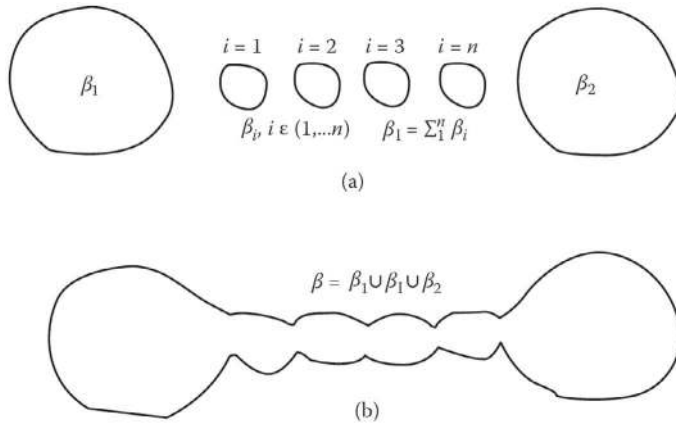


FIGURE 11.4 Schematic representation of the void coalescence through the process of void sheet formation. (a) Initial configuration and (b) after coalescence.

ductile fracture should include the void growth-dependent constitutive relation and a condition for void coalescence. The two most commonly used macroscopic models employed to predict the ductile fracture initiation (i.e., microcrack initiation) on the basis of void nucleation, growth and coalescence are:

1. Porous plasticity model of Berg and Gurson (Berg, 1970; Gurson, 1977), Gurson-Tvergaard-Needleman (GTN) model (Tvergaard and Needleman, 1984).
2. CDM model (Lemaitre, 1984, 1985a, 1985b; Lemaitre and Chaboche, 1990; Benallal et al., 1991; Lemaitre and Desmorat, 2005; Murakami, 2012).

In the first approach, based on Berg's (1970) theory of dilatational plasticity, Gurson (1977) proposed a plastic potential in terms of the porosity or void volume fraction. The void growth-dependent elastoplastic constitutive relation is obtained from this plastic potential. An evolution law for the porosity, which incorporates both the void nucleation as well as void growth, is needed while using the Berg-Gurson model. A critical value of the porosity or the void volume fraction is normally used as an indicator of fracture initiation in the Berg-Gurson model.

In the CDM model, the change in material behavior due to void growth is incorporated by introducing a damage variable as an internal variable in the plastic potential. The damage variable quantifies the intensity of microvoids. As a result, it is identified as the void volume fraction. For the case of isotropic damage, it can be related to the area void fraction. The theory of continuum thermodynamics is used to derive the void growth-dependent constitutive relation and the damage growth law (Lemaitre, 1984, 1985a, 1985b; Lemaitre and Chaboche, 1990; Benallal et al., 1991; Lemaitre and Desmorat, 2005; Murakami, 2012). A critical value of the damage variable, either based on an appropriate void coalescence model or determined experimentally, is used for predicting the fracture initiation.

11.7.1 POROUS PLASTICITY MODEL (GURSON AND GTN MODEL)

In the Gurson model (Gurson, 1977), the yield surface of a porous material, a material containing void, is obtained by the upper-bound model. It is given by Equation 11.5. It was later realized by Tvergaard and Needleman (1984) that the yield surface proposed by Gurson (1977) was unable to represent fracture and coalescence. Further, it does not include the plastic limit load failure of the intervoid matrix (Thomason, 1990). Also, numerical simulations using the unit cell showed that the void growth was not accurately predicted. Another major limitation of the Gurson model (Gurson, 1977) was that it could only handle the growth of spherical voids remaining spherical. Tvergaard and Needleman (1984) then proposed a modification to the original Gurson's model (Gurson, 1977) where the original yield function proposed by Gurson was rewritten as

$$\Phi = \left(\frac{\sigma_{eq}}{\sigma_Y} \right)^2 + 2q_1 f_* \cosh \left(\frac{\sqrt{3} q_2 \text{tr}(\sigma)}{2\sigma_Y} \right) - 1 - q_3 f_*^2, \quad (11.7)$$

where q_1 , q_2 , q_3 and f_* are the new material parameters. This model is often referred to as the GTN model. The yield function reduces to the von Mises yield function when $f_* = 0$. The parameters q_1 , q_2 , q_3 and f_* are chosen to describe the void growth computations more accurately and match the experimental results. The values $q_1 = 1.5$, $q_2 = 1.0$ and $q_3 = q_1$ are often used. Tvergaard found that $q_1 = 1.5$, $q_2 = 1.0$ and $q_3 = q_1^2 = 2.25$ represented the plain strain condition better. However, it has been found that these parameters depend on the hardening exponent, and the ratio of yield stress and Young's modulus (Besson, 2010). The effective porosity f_* was introduced to account effectively for the void coalescence. It was defined as a piecewise function given by

$$f_* = \begin{cases} f & \text{for } f \leq f_c \\ f_c + \left(\frac{1}{q_1} - f_c \right) \frac{f - f_c}{f_R - f_c} & \text{otherwise,} \end{cases}$$

where f_R is the fracture porosity. Fracture occurs when $f_* = 1/q_1$. It can be seen from the previous definition of f_* that once the porosity reaches the critical value f_c , an extra term is added that accounts for the increase in damage due to void coalescence. The evolution of total effective porosity is given by

$$\dot{f}_* = \dot{f}_g + \dot{f}_n, \quad (11.8)$$

which is the sum of the rates of change of porosity due to void growth $\dot{f}_g$ and void nucleation $\dot{f}_n$, respectively. The expression for change in porosity due to void growth (based on the conservation of mass) is given by Equation 11.6:

$$\dot{f}_g = (1 - f) \text{tr}(\dot{\epsilon}^p), \quad (11.9)$$

The rate of change of porosity due to void nucleation is introduced in a purely phenomenological manner and is given by

$$\dot{f}_n = A_n \dot{\epsilon}_{eq}^p, \quad (11.10)$$

where the nucleation strain A_n is expressed as a Gaussian function as (Besson, 2010)

$$A_n = \frac{f_N}{S_N \sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left(\frac{\epsilon_{eq}^p - \epsilon_N}{S_N} \right)^2 \right\}. \quad (11.11)$$

Here, ϵ_N is the nucleation strain at which 50% of the inclusions are broken, S_N is the standard deviation of the nucleation strain, and f_N is the void volume fraction of the nucleating particles over the entire volume of the material. Thus, f_N is much less than the total void volume fraction of the material. The nucleation strain A_n depends on the type of material and its chemical composition, etc. It should be noted that many forms of A_n can be chosen, in particular, those based on experimental measurements (Besson, 2010). Recently, Xue (2007a) has modified the GTN model to take into account the effect of hydrostatic pressure and Lode angle.

11.7.2 CDM-BASED MODEL: REVIEW OF LITERATURE

CDM studies the ductile plastic damage within the general framework of continuum thermodynamics of irreversible processes (Chaboche, 1981, 1984; Lemaitre, 1984, 1985a, 1985b; Rousselier, 1987; Lemaitre and Chaboche, 1990; Benallal et al., 1991; Lemaitre and Desmorat, 2005). Based on the idea by Kachnov of effective stress (see Section 11.9.5 for more discussion) and the principle of strain equivalence (see Section 11.9.7 for more discussion), Lemaitre (1984) proposed a damage mechanics model for elastic-plastic materials. Most of the current CDM-based approaches now follow Lemaitre's (1984) initial proposal. In Lemaitre's (Lemaitre, 1984, 1985a, 1985b; Lemaitre and Chaboche, 1990; Benallal et al., 1991; Lemaitre and Desmorat, 2005) CDM model, the plastic potential of a damaged material is assumed to consist of two parts—(1) the plastic potential associated with yielding and hardening and (2) the plastic potential associated with damage known as damage potential. The void growth-dependent elastic-plastic constitutive equation is obtained as the stress derivative of the former, whereas the damage growth law is obtained as the damage derivative of the latter. Lemaitre and coworkers (Lemaitre, 1984, 1985a, 1985b; Lemaitre and Chaboche, 1990; Benallal et al., 1991; Lemaitre and Desmorat, 2005) also proposed a form for the damage potential, which leads to a certain damage growth law. For the isothermal case, it contains only one material parameter, which can be expressed in terms of the ratio of the critical value of damage to the difference between the fracture strain and the threshold strain (i.e., equivalent plastic strain at which damage starts). This leads to a simple damage growth law in which the damage rate depends linearly on the plastic strain and the thermodynamic force corresponding to the damage. The thermodynamic force depends quadratically on the triaxiality.

Later, Tai and Yang (1986) modified the Lemaitre's damage growth law to make it dependent on the current value of damage. They obtained the material constants appearing in the damage growth law from the experimental results of LeRoy et al. (1981) on area void fraction measurements at different strain levels. Tie-Jun (1992) introduced non-linearity with respect to plastic strain in the damage growth law. Dhar (1995) and Dhar et al. (1996) proposed a damage growth law based on the experimental results of LeRoy et al. (1981), which depends non-linearly on plastic strain. Bonora (1997) proposed a damage potential that leads to a damage growth law depending non-linearly on the damage. This form is based on experimental observations in various metals and incorporates the void nucleation, growth and coalescence (through the critical value of damage). Bonora et al. (2005) used this non-linear damage growth law in a finite-element formulation and validated it against experimental results in A533 B low alloy steel. Thakkar and Pandey (2007) proposed a non-linear damage evolution law, which represents the experimental damage evolution pattern in aluminum, copper, spheroidized steel and steel-X C38. The damage growth law, however, requires six model parameters, which are difficult to obtain for many materials.

Celentano and Chaboche (2007) presented a combined experimental and numerical characterization scheme in steels for the damage growth law proposed by Lemaitre (Lemaitre, 1984, 1985a, 1985b; Lemaitre and Chaboche, 1990; Benallal et al., 1991; Lemaitre and Desmorat, 2005). The experimental scheme involves a set of loading-unloading cycles and a new damage identification scheme based on the evolution of Young's modulus. The numerical simulation on SAE1020 and SAE1045 steel shows a good agreement with experimental results. However, it is suggested that toward the final rupture, anisotropic damage should be considered since the assumption of spherical voids no longer holds. Lemaitre and Dufailly (1987) proposed eight different methods to measure damage. The methods include both direct and indirect measurement techniques. They suggested the applicability of different techniques for different material behaviors like brittle, ductile, creep and fatigue. Alves et al. (2001) studied numerically the effect of specimen geometry on the accuracy of Young's modulus measurements. They found that this error in the evaluation has an important effect on the value of the associated damage. They concluded that the actual damage values are larger than the values published in literature.

Rousselier (1987) proposed a damage potential that depends on the current density of the material and the mean stress. Xue (2007b) proposed a damage evolution law in the rate form through a 'cylindrical decomposition' of the damage by incorporating the effects of the pressure (i.e., the mean stress) and the Lode angle. He used this damage evolution law for the numerical simulation of compact tension specimen. Chaboche et al. (2006) introduced an additional damage variable, D_v , associated with the volume change. Advantage of such a formulation over Gurson's model (Gurson, 1977) and the Lemaitre's damage growth law has been outlined. Brünig (2003) proposed an anisotropic damage model for ductile metals where the strain rate tensor is decomposed into three parts: (1) elastic, (2) plastic and (3) damage. The damage growth law is expressed in terms of the damage part of the strain rate tensor. A damage potential is proposed whose stress derivative gives the damage growth law. Brünig (2006) employed this anisotropic damage model to study the

dynamic behavior of AL-6XN stainless steel by taking into account the temperature and strain rate-dependent material behavior.

11.7.3 OTHER MODELS OF FRACTURE INITIATION

As stated earlier, in Gurson's porous plasticity model, the critical void volume fraction is used as the fracture initiation criterion, and in Lemaitre's CDM model, the critical damage is used as the fracture initiation criterion. However, various other fracture initiation criteria have been proposed. In some of these, the damage is either defined differently or some quantity related to damage is considered critical at the fracture initiation. In other cases, the fracture criteria are expressed as a relation between some continuum parameters like the equivalent plastic strain, mean stress, stress triaxiality, etc. This relation is called the fracture locus or fracture envelope. In some of these criteria, the effect of void growth (and nucleation) on the constitutive relation is not considered.

Thomason (1990) combined the results of Goods and Brown (1977) on the void nucleation and those of Rice and Tracey (1969) on the void growth and his own on the void coalescence to arrive at a fracture initiation criterion in the form of a graph of fracture strain versus the mean stress. For numerical predictions, the graph can be used along with conventional elastic-plastic constitutive equations.

Zheng et al. (1992) proposed a new damage variable, called the macrodamage variable, whose growth depends on the plastic strain and mean stress. The fracture initiation is assumed to take place when this variable reaches the value unity. He applied this criterion to an upsetting process. Chaouadi et al. (1994) proposed a parameter called the damage work to predict fracture initiation in 18MND5 and 22NiMoCr37 steels. The fracture is assumed to initiate when the damage work reaches a critical value. The Rice and Tracey (1969) model is used to estimate the damage (i.e., the void volume fraction) growth. They used this criterion in the finite-element simulations of tension tests on notched specimens of 18MND5 and 22NiMoCr37 steels. Schiffmann et al. (1998, 2003) applied the critical damage work criterion in tension tests on (1) plane strain, notched and unnotched specimens of FeE690 steel; and (2) axisymmetric specimens of 9SMn28 and 9SMn28Te steels.

Komori (1999) proposed a fracture initiation criterion in the form of a graph of the plastic strain versus the void volume fraction. This graph is obtained from a modified version of the necking (or void coalescence) criterion proposed by Thomason (1968). Gurson's porous plasticity model is used to incorporate the effects of void nucleation and growth on the constitutive relation of the material. He applied this criterion to the multi-pass drawing process. Based on finite-element studies on representative material volume (i.e., RMV) containing a single spherical void at its center, Gao and Kim (2006) proposed a fracture initiation criterion as a relation between the equivalent plastic strain, triaxiality and Lode angle. The fracture initiation is said to occur when the stress in the simulated stress-strain curve suddenly drops.

Xue and Wierzbicki (2008) proposed a fracture initiation criterion in which the fracture strain depends on the pressure (i.e., the mean stress) and the Lode angle. They called this relation the fracture strain envelope. The fracture initiation is said to take place when the damage variable reaches unity. The damage variable is determined

from the damage plasticity theory of Bao and Wierzbicki (2008). In this theory, the damage growth law, besides depending on the plastic strain, also depends on the pressure and the Lode angle through the ‘cylindrical decomposition’ of damage. The criterion was applied to study the crack initiation and propagation in compact tension (CT) specimens and three-point bending tests of 2024-T351 aluminum alloy. Xue (2009) also proposed a fracture initiation criterion in the form of a fracture strain envelope. However, he obtained the fracture strain envelope from the fracture stress envelope using the stress-strain relation. The damage-coupled Tresca yield criterion is used to obtain the fracture stress envelope in which the damage is evaluated from the damage growth law of the damage plasticity theory. Numerical simulations of fracture initiation and propagation were carried out for tension test, plane strain test and upsetting process of 2024-T351 aluminum alloy using this criterion.

Based on experimental (i.e., tensile tests, shear tests and upsetting on 2024-T351 aluminum alloy) and numerical studies, Bao and Wierzbicki (2004) proposed a fracture initiation criterion in the form of a graph of the fracture strain versus the tri-axiality. This graph is called the fracture locus. They showed that there are three distinct branches of the fracture locus. For negative stress triaxialities, the fracture is shown to be governed by the shear mode. For larger triaxialities, the void growth is shown to be the dominant failure mode, while at low-stress triaxialities, the fracture may develop as a combination of the shear and void growth modes. Bao and Wierzbicki (2005) derived analytically, based on experimental results of upsetting tests, a cutoff value of the stress triaxiality equal to $-1/3$ below which the fracture never occurs. Numerical simulations performed with the cutoff value in fracture loci successfully captured the main features observed in tensile tests under hydrostatic pressure by Bridgman (1952). Experimental results have shown that fracture initiation in uncracked ductile solids is sensitive to the hydrostatic pressure and dependent on the Lode angle. Recently, Xue (2007a) has proposed a damage plasticity model for ductile fracture in uncracked solids taking into account the effect of hydrostatic pressure and Lode angle. The combined effects of pressure and Lode angle were used to define a fracture envelope in principal stress space. Damage is calculated by an integral of the damage rate measured at current loading and deformation state with respect to the fracture envelope. He proposed a power law damage rule to characterize the non-linearity in damage accumulation. The material parameters were calibrated from standard laboratory tests. The proposed model was used to numerically study fracture and crack path prediction in unnotched axisymmetric round bar, doubly grooved flat plate, tensile flat specimen and compact tension specimen. Erice and Gálvez (2014) have proposed a coupled elastoplastic-damage constitutive model with Lode angle-dependent failure criterion for high-strain rate situations.

11.8 THERMODYNAMICS OF CONTINUUM

A thermodynamic process is normally described by a set of kinematic variables $a_k (k = 1, 2, \dots, n)$ and the absolute temperature T (non-negative). These quantities are known as independent state variables. Any function of the state variables is referred to as the state function. Examples of state functions are internal energy $U(a_k, T)$, entropy $S(a_k, T)$, Helmholtz free energy $\Psi(a_k, T)$, etc. For a reversible process, the

free energy (called the thermodynamic potential) completely specifies the thermomechanical behavior of a continuum. By means of the first law of thermodynamics, one gets the following expression for the (conservative) thermodynamic force A_k^c corresponding to a_k :

$$A_k^c = \frac{\partial \Psi}{\partial a_k}. \quad (11.12)$$

The quantity A_k^c is also called the conjugate variable corresponding to a_k . As an example, consider the following expression for the specific Helmholtz free energy of a thermo-elastic process:

$$\Psi = \frac{1}{2\rho} \boldsymbol{\varepsilon}^e : \mathbf{C}^E : \boldsymbol{\varepsilon}^e - Ts. \quad (11.13)$$

Here, the components of the elastic part of the strain tensor $\boldsymbol{\varepsilon}^e$ are the kinematic variables. The quantity $\mathbf{C}^E$ is the fourth-order elasticity tensor, ρ is the density, and s is the specific entropy. The symbol ‘:’ denotes two indices contracted product of tensors. For example, $\mathbf{C}:\boldsymbol{\varepsilon}$ is $C_{ijkl} \varepsilon_{kl}$ in index notation. The Cauchy stress tensor is given by

$$\boldsymbol{\sigma} = \rho \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^e} = \mathbf{C}^E : \boldsymbol{\varepsilon}^e. \quad (11.14)$$

For an irreversible process, since the entropy production is non-zero, it is necessary to consider the dissipative thermodynamic forces to completely describe the thermomechanical behavior of a continuum. For such a process, the rate of entropy production $\dot{S}^i$ is not a state variable but, in general, a function of a_k , T , $\dot{a}_k$ and $\dot{T}$. The superscript ‘i’ stands for the irreversible process, whereas the superscript ‘diacritical mark’ denotes the time rate of the quantity. The dissipative power Φ is defined as the product of absolute temperature T and the rate of entropy production $\dot{S}^i$. Thus, for an irreversible process, the dissipative power Φ is expressed as

$$\Phi \equiv T\dot{S}^i = \Phi(\dot{a}_k, \dot{T}, a_k, T). \quad (11.15)$$

If it is assumed that the irreversible system is purely dissipative, then the dissipative thermodynamic forces A_k^d can be completely derived from Φ . Thus,

$$A_k^d = \frac{\partial \Phi}{\partial \dot{a}_k}. \quad (11.16)$$

The second law of thermodynamics states that

$$\Phi \geq 0. \quad (11.17)$$

11.8.1 THERMODYNAMIC PROCESS WITH INTERNAL VARIABLES

In the field of thermomechanics, all variables can be classified into two categories, namely, the variables that are measurable or controllable (like the kinematic variables, temperature, etc.) and the variables that cannot be measured or controlled directly. The first kind is termed the *external variable*, whereas the second one is called the *internal variable*. When they are included in the thermodynamic potential, they are called the external state variable and internal state variable, respectively.

For the dissipative process, the current state also depends on the past history. This history can only be represented by means of internal variables. For example, in a damaged elastic-plastic material, the history of deformation can be represented by accumulated plastic strain, accumulated damage, etc.

The thermodynamic potential or free energy as a function of internal state variables α_k is expressed as

$$\Psi = \Psi(a_k, \alpha_k, T), \quad (11.18)$$

and the corresponding thermodynamic forces are given by

$$A_k^c = \frac{\partial \Psi}{\partial a_k}, \quad (11.19)$$

$$\beta_k^c = \frac{\partial \Psi}{\partial \alpha_k}, \quad (11.20)$$

$$S = -\frac{\partial \Psi}{\partial T}. \quad (11.21)$$

Here, β_k^c is the conservative part of the thermodynamic force β_k corresponding to the internal state variable α_k .

When the dissipative potential $\dot{\Phi}(a_k, \dot{\alpha}_k, \dot{T}, a_k, \alpha_k, T)$ is also a function of internal state variables α_k and $\dot{\alpha}_k$, the dissipative forces are given by

$$A_k^d = \frac{\partial \Phi}{\partial \dot{\alpha}_k}, \quad (11.22)$$

and

$$\beta_k^d = \frac{\partial \Phi}{\partial \dot{\alpha}_k}. \quad (11.23)$$

Here, β_k^d is the dissipative part of the thermodynamic force β_k corresponding to the internal variable α_k . Since β_k is an internal force,

$$\beta_k^c + \beta_k^d = 0, \quad (11.24)$$

or

$$\beta_k^c = -\beta_k^d. \quad (11.25)$$

11.8.2 THERMO-ELASTIC-PLASTIC PROCESS

The dissipative potential for a thermo-elasto-plastic process is expressed as

$$\Phi = \Phi(\dot{\varepsilon}^p, \dot{p}, \dot{D}, \dot{T}, \varepsilon^p, p, D, T). \quad (11.26)$$

Here, ε^p , the plastic part of the strain tensor, is the external (or kinematic) variable, and p and D are the internal variables corresponding to isotropic hardening and damage, respectively. The physical identification of the damage variable with the void density is discussed in detail in Section 11.9. For the case of strain hardening, p is nothing but the equivalent plastic strain $\dot{\varepsilon}_{eq}^p$. Thus,

$$p \equiv \varepsilon_{eq}^p = \int_0^t \dot{\varepsilon}_{eq}^p dt, \quad (11.27)$$

where

$$\dot{\varepsilon}_{eq}^p = \sqrt{\frac{2}{3} \dot{\varepsilon}^p : \dot{\varepsilon}^p}, \quad (11.28)$$

and $\dot{\varepsilon}^p$ is the plastic part of the strain rate tensor $\dot{\varepsilon}$:

$$\dot{\varepsilon} = \frac{1}{2} \left\{ \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right\}. \quad (11.29)$$

Here, $\mathbf{v}$ is the velocity vector and ∇ denotes the differentiation with respect to the position vector $\mathbf{x}$. The integral in Equation 11.28 is to be carried along the particle path.

The thermodynamic forces corresponding to ε^p , p and D are obtained from the following relations:

$$\sigma = \frac{\partial \Phi}{\partial \varepsilon^p}, \quad (11.30)$$

$$-R = \frac{\partial \Phi}{\partial p}, \quad (11.31)$$

$$-Y = \frac{\partial \Phi}{\partial D}. \quad (11.32)$$

Here, σ is the Cauchy stress and $-R$ and $-Y$ are the dissipative parts of the thermodynamic forces corresponding to the internal variables p and D , respectively. These laws are called complementary laws, whereas Equations 11.19–11.21 are called state laws. Using the Legendre-Fenchel transformation (Haupt, 1952), one can transform

$\Phi(\dot{\varepsilon}^p, \dot{p}, \dot{D}, \dot{T}, \varepsilon^p, p, D, T)$ to its dual $\Phi^*(\sigma, -R, -Y, T)$ and write the complementary laws as evolution laws of the flux variables (i.e., the rates of external and internal variables). Thus,

$$\dot{\varepsilon}^p = \frac{\partial \Phi^*}{\partial \sigma}, \quad (11.33)$$

$$\dot{p} = \frac{\partial \Phi^*}{\partial (-R)}, \quad (11.34)$$

$$\dot{D} = \frac{\partial \Phi^*}{\partial (-Y)}. \quad (11.35)$$

Since Φ^* and Φ are governed by an inequality (see Equation 11.17), its actual value is usually difficult to determine. As a result, these laws are not useful from the computational point of view. To derive the useful relations, the dual dissipative potential Φ^* is maximized using the constraint that the yield function or the plastic potential $F(\sigma, -R, -Y, T)$ should be zero on the yield surface. Thus,

$$\delta(\Phi^* - \dot{\lambda} F) = 0, \quad (11.36)$$

where δ denotes the variation, and $\dot{\lambda}$ acts as a Lagrangian multiplier. The function F is also called an indicator function. Combining Equations 11.33–11.35 with Equation 11.36, one gets

$$\dot{\varepsilon}^p = \dot{\lambda} \frac{\partial F}{\partial \sigma}, \quad (11.37)$$

$$\dot{p} = \dot{\lambda} \frac{\partial F}{\partial (-R)}, \quad (11.38)$$

$$\dot{D} = \dot{\lambda} \frac{\partial F}{\partial (-Y)}. \quad (11.39)$$

The laws given by Equations 11.37–11.39 are called plastic flow rules, and they describe the mechanical behavior of an elastic-plastic material. For details on the theory of continuum thermodynamics, it is suggested to refer to the textbook by Tadmor et al. (2012).

11.9 CONTINUUM DAMAGE MECHANICS

Damage is defined as the deterioration of material properties caused by the development of microvoids and microcracks at various length scales under the action of external loads. CDM provides the framework for the analysis of material damage by coupling the plasticity and damage in the constitutive equation using the continuum thermodynamics. It deals with the initiation of microvoids and microcracks, their growth and their coalescence to form macrocracks and eventual fracture.

11.9.1 LENGTH SCALES OF DAMAGE

The final fracture of a material is preceded by material damage at various length scales. The damage is induced by the breakage of atomic bonds by either tensile or shear decohesion. The phenomena of damage occur at the following five length scales (Murakami, 2012), see Figure 11.5:

1. *Atomic scale.* At this scale, the material is seen to be composed of atoms. The mechanical properties of the material are determined by individual atoms and their interaction with other atoms through interatomic or intermolecular forces such as the van der Waals forces. Failure at this scale occurs by the breakage of atomic bonds.
2. *Microscopic scale.* When a material is observed at a scale such that it appears to be composed of a number of continuous regions separated by lines or planes, it is called the microscopic scale. The failure at this scale happens by the formation of displacement discontinuities in the form of microvoids and microcracks.
3. *Mesosopic scale.* This is defined as the scale at which the average mechanical property over a small volume, also called the representative volume element (RVE) (explained in Section 11.9.2), around a material point can be expressed as a function of the position $\mathbf{x}$ of the material point. This means that the material can be idealized as a continuum, and the mechanical state of the material at the point is the statistical average of the states over the small volume.
4. *Macroscopic scale.* This scale, on the other hand, is a scale at which every material point can be considered as a material point of a continuum.
5. *Structural scale.* It is the scale considered for engineering structures and mechanical components. At this scale, the crack is of the order of a few millimeters to a few centimeters.

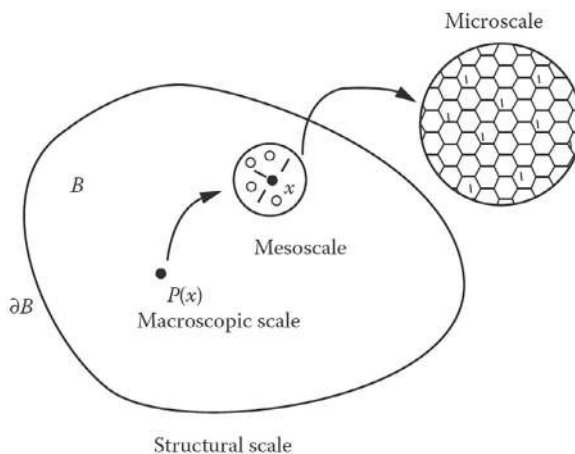


FIGURE 11.5 Different length scales in CDM.

It should be noted that CDM is mainly concerned with analysis of damage at the mesoscopic and macroscopic length scales. The microscopic mechanism of damage consists of the following four mechanisms: (a) cleavage, (b) glide plane decohesion, (c) growth and coalescence of microvoids and (d) growth of voids by grain boundary diffusion (refer to Murakami [2012] for a detailed discussion on these four mechanisms).

11.9.2 REPRESENTATIVE VOLUME ELEMENT

In real applications, hundreds to thousands of microvoids and microcracks are formed inside the material not necessarily simultaneously. It is impossible to describe the nucleation, growth and coalescence, that is, the evolution of each individual microvoid or microcrack using the models described in Section 11.6. In order to treat the effects of microvoids and microcracks within the framework of continuum mechanics, the micro effects need to be homogenized and presented as a macroscopic continuous field in the material.

Hence, it is essential to define the notion of an RVE at a material point $\mathbf{x}$. The mechanical effects of microscopic discontinuities like voids, cracks, inclusions, etc., are homogenized over the RVE and represented as a macroscopic continuous field in the material to be used within the continuum mechanics framework. For this, a small region γ of the material at the mesoscale is taken, as shown in Figure 11.5. It is assumed that the material with discontinuities like voids, cracks, inclusions, etc., in the region γ can be homogenized, and the mechanical state can be represented by the statistical average of the mechanical properties in the region γ . The volume of the region V for which this condition is satisfied is called the RVE.

If an RVE of appropriate size is selected, then the material with discontinuities such as voids, cracks, inclusions, etc., can be treated as a continuum. The mechanical state of the continuum is unique if the RVE represents the statistical average of the actual material. For this, the RVE should satisfy the following two conditions:

1. The size of the RVE should be small enough so that the variation of the macroscopic variable in the RVE is small.
2. The size of the RVE should be large enough to contain a sufficient number of discontinuities so that the material is statistically homogeneous.

The size of an RVE for some typical materials (Lemaitre, 1996; Murakami, 2012) is given in Table 11.2.

The size of an RVE depends on the microstructure of the material and the mechanical phenomena like ductile, brittle, creep or fatigue. Due to the highly localized nature of discontinuities in the case of brittle and fatigue damage, the size of an RVE is much larger when compared to ductile and creep damage.

11.9.3 REQUIREMENTS OF DAMAGE MODELING

To analyze a material undergoing damage and fracture when subjected to various loading conditions, within the framework of continuum mechanics, requires the following procedures (Murakami, 2012):

TABLE 11.2
Typical Size of RVE for Various Materials

S. No.	Material	RVE
1	Metal/ceramics	(0.1 mm) ³
2	Polymer/composites	(1 mm) ³
3	Wood	(10 mm) ³
4	Concrete	(100 mm) ³

- Representation of the damage state of the material by means of a damage variable $D(\mathbf{x})$ —can be scalar or a higher-order tensor such as a second- or fourth-order tensor. In the present chapter, the damage is assumed to be isotropic. Isotropic damage at a point $\mathbf{x}$ consists of defects that are oriented uniformly in all directions. Then, the damage variable can be treated as a scalar quantity, which greatly simplifies the modeling and analysis of damaged material. However, the concept of isotropic damage cannot be applied to a situation where considerable anisotropy is present, for example, in the case of brittle damage
- Formulation of the evolution equation for damage, that is, how the damage develops over time with applied loads
- Formulation of the constitutive equation describing the mechanical behavior of material with damage
- The resulting initial and boundary value problem needs to be solved—mostly through numerical techniques—using the equations developed in the previous steps

The solution of the initial and boundary value problem is out of the scope of this chapter. An excellent description on numerical techniques can be found in the monograph by Zhang and Cai (2010).

11.9.4 DEFINITION OF A SCALAR DAMAGE VARIABLE

Let ΔA be the area of a section of an RVE having a unit normal $\hat{\mathbf{n}}$ (Figure 11.6). The damage at this section may be interpreted as composed of the traces of the microvoids and microcracks formed by various void nucleation, growth and coalescence mechanisms discussed in Section 11.6. Also, let the total area of the traces of the microvoids and microcracks be given by ΔA_D . Then the value of damage $D(\mathbf{x}, \hat{\mathbf{n}})$ at point $\mathbf{x}$ in the direction $\hat{\mathbf{n}}$ is given by

$$D(\mathbf{x}, \hat{\mathbf{n}}) = \lim_{\Delta A \rightarrow 0} \frac{\Delta A_D}{\Delta A}. \quad (11.40)$$

As stated earlier, it is assumed that the voids are scattered in an isotropic way, that is, the microvoids and microcracks at a point are uniformly distributed in all directions. Then, D can be assumed as a scalar quantity. Thus, the damage variable is identified as an area void fraction at point $\mathbf{x}$ in any plane given by

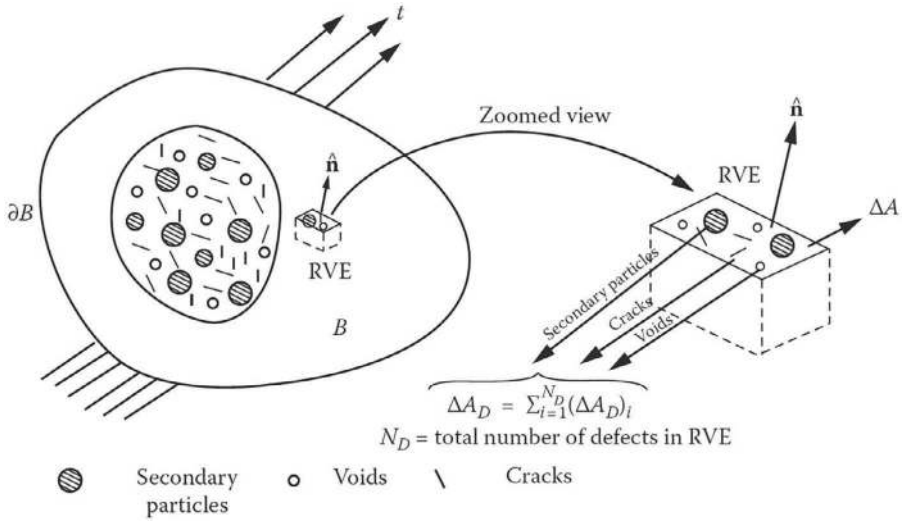


FIGURE 11.6 Damaged material and an RVE with microvoids and microcracks.

$$D = \lim_{\Delta A \rightarrow 0} \frac{\Delta A_D}{\Delta A}. \quad (11.41)$$

In other words, the damage variable D denotes the surface density of the microvoids and microcracks of the material in any plane at point $\mathbf{x}$. Based on the definition of the damage variable D defined in 11.41, it is clear that the maximum value of the total area of microvoids and microcracks ΔA_D can be ΔA . In an undamaged material, a material having absolutely no microvoids and microcracks, the value of ΔA_D will be 0. Thus, D is bounded between 0 and 1. D is equal to 0 for undamaged material and 1 for fully damaged material. In real cases, it has been found that failure occurs much before D can reach unity, that is, at failure, $D < 1$. This is because the material suffers from instability at high values of damage caused by the sudden separation of the remaining resisting area. The value of damage D at which this happens is called the critical value of damage D_c . This depends on the material and type of loading.

11.9.5 EFFECTIVE STRESS CONCEPT

Now, consider the situation when the RVE in Figure 11.6 is loaded uni-axially by a force

$$\mathbf{F} = F\hat{\mathbf{n}}. \quad (11.42)$$

The usual uni-axial stress is given by

$$\sigma = \frac{F}{\Delta A}. \quad (11.43)$$

The effective area of resistance ΔA^* to the applied load is given by

$$\Delta A^* = \Delta A - \Delta A_D. \quad (11.44)$$

For the damaged RVE loaded with uni-axial force F , the effective uni-axial stress σ^* is given by

$$\sigma^* = \frac{F}{\Delta A - \Delta A_D}. \quad (11.45)$$

Using

$$D = \frac{\Delta A_D}{\Delta A}, \quad (11.46)$$

and Equation 11.43 in Equation 11.45, the effective stress is given by

$$\sigma^* = \frac{\sigma}{1-D}. \quad (11.47)$$

This is the expression for Cauchy stress in a one-dimensional case. For a general three-dimensional case, the effective Cauchy stress tensor σ^* is defined as (Lemaitre, 1996)

$$\sigma^* = \frac{\sigma}{(1-D)}. \quad (11.48)$$

This definition of effective stress is valid for tensile loading. This means that the microvoids and microcracks that have formed do not close under compression. If some of the microvoids or microcracks close, then the effective area of resistance increases, $\Delta A^* > (\Delta A - \Delta A_D)$. Then, the *crack closure effects* need to be considered. This is explained in Section 11.9.12.

11.9.6 CRACK INITIATION CRITERION

Again, considering the uni-axial tension case, the maximum stress that can be applied is the ultimate fracture stress σ_u . Substituting σ_u for σ^* in Equation 11.47 and rearranging gives

$$D_c \approx 1 - \frac{\sigma}{\sigma_u}, \quad (11.49)$$

where D_c is the critical value of the damage for the mesocrack initiation for unidirectional stress σ . The critical value of damage D_c varies from $D_c \approx 0$ for brittle fracture to $D_c \approx 1$ for pure ductile fracture. For most practical cases of ductile fracture, D_c is of the order of 0.2 to 0.6. The condition for microcrack initiation is given by

$$D = D_c. \quad (11.50)$$

Application of Equation 11.49 to uni-axial *monotonic* tension test yields the following expression for the critical damage $(D_c)_{ID}$ (Lemaitre, 1996):

$$(D_c)_{ID} \approx 1 - \frac{\sigma_R}{\sigma_u}, \quad (11.51)$$

where σ_R is the rupture stress.

It has been observed that there is a threshold plastic strain ε_{pD} below which no damage occurs. This happens because the microvoids and microcracks have to nucleate by an accumulation of microstresses or dislocations. Hence, the following condition holds:

$$D = 0 \text{ if } \varepsilon_p < \varepsilon_{pD}.$$

11.9.7 STRAIN EQUIVALENCE PRINCIPLE

Another important concept is the principle of strain equivalence. It states that the deformation behavior of a damaged material can be represented by the constitutive laws of the undamaged material if the usual stress is replaced by the effective stress. This concept is represented in Figure 11.7.

To explain the concept, an RVE loaded uni-axially is considered. Then, the uni-axial elastic strain of the RVE is written as

$$\varepsilon_e = \frac{\sigma^*}{E} = \frac{\sigma}{E(1-D)}. \quad (11.52)$$

However, it assumes that different behaviors of the material such as elasticity, plasticity and viscoplasticity are affected in the same manner by damage. This is, in fact, not entirely correct in practical cases.

11.9.8 ELASTIC STRAIN ENERGY EQUIVALENCE PRINCIPLE (VOYIADJIS AND KATTAN, 2005; MURAKAMI, 2012)

Another important concept, that can be used instead of the principle of strain equivalence, is the concept of elastic strain energy equivalence. This principle assumes that the elastic energy of a damaged material is equivalent to that of the undamaged material except that the stress is replaced by the effective stress in the energy formulation. Considering again the uni-axial tension case, the elastic strain energy for the damaged material can be written as

$$W_e = \frac{1}{2} \sigma \varepsilon_e. \quad (11.53)$$

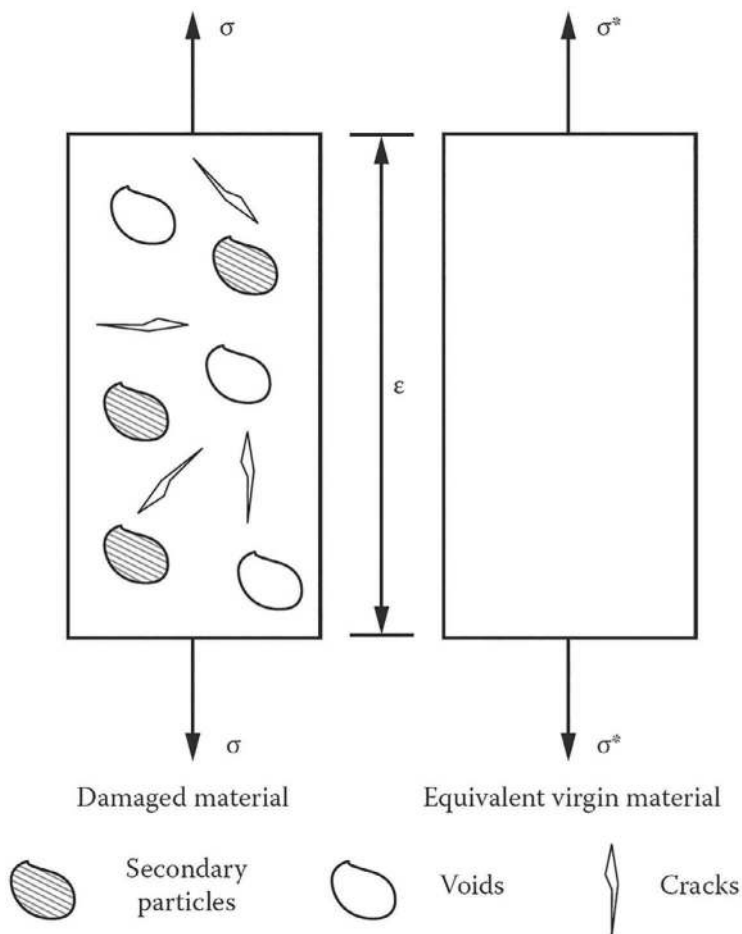


FIGURE 11.7 Strain equivalence principle.

The elastic strain energy for the effective undamaged material is given by

$$W_e^* = \frac{1}{2} \sigma^* \varepsilon_e^*, \quad (11.54)$$

where ε_e^* is the effective strain corresponding to ε_e . According to the elastic strain energy equivalence principle,

$$W_e = W_e^*. \quad (11.55)$$

Apart from the previous two principles, there also exist principles of the equivalence of complementary strain energy and total energy. For details, the reader should refer to the monograph by Murakami (2012).

11.9.9 THERMODYNAMIC FORCE CORRESPONDING TO DAMAGE

The thermodynamic force corresponding to damage is obtained from the thermodynamic potential or free energy. The thermodynamic potential is a continuous scalar function of state variables (see Section 11.8), which is concave with temperature and convex with other state variables (Lemaitre, 1996). It thus defines the energy involved in each physical process. The dissipative potential (see Section 11.8.2), expressed in terms of associated variables, gives the kinetic laws of the evolution of dissipative variables. For example, the constitutive equation for the damage gives the rate of change of the damage variable as a function of its associated variable (see Equation 11.39).

The Helmholtz free energy for a damaged material is written as

$$\Psi = \Psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^e, \boldsymbol{\varepsilon}^p, D, T), \quad (11.56)$$

where $\boldsymbol{\varepsilon}$ is the total strain tensor, $\boldsymbol{\varepsilon}^e$ is its elastic part, and $\boldsymbol{\varepsilon}^p$ is its plastic part given by the additive decomposition:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p. \quad (11.57)$$

Further, T is the temperature, and D is the damage variable. As damage is dimensionless and the product $Y\dot{D}$ is the power involved in the damage process, the conservative thermodynamic force Y corresponding to damage becomes the volume energy density (Lemaitre, 1996). Here, the kinematic hardening is not considered. The formulation using kinematic hardening can be found in the work of Lemaitre (1996) and Lemaitre and Desmorat (2005). Using Equation 11.57, the elastic strain can be written as

$$\boldsymbol{\varepsilon}^e = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p \quad (11.58)$$

and then the Helmholtz free energy for a damaged material becomes

$$\Psi = \Psi(\boldsymbol{\varepsilon}^e, T, D). \quad (11.59)$$

Assuming constant density ρ , which is approximately true for ductile damage, and using Equations 11.19–11.21, the state law is obtained as

$$\boldsymbol{\sigma} = \rho \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^e}, \quad (11.60)$$

$$\mathbf{S} = -\frac{\partial \Psi}{\partial T}, \quad (11.61)$$

$$Y = \rho \frac{\partial \Psi}{\partial D}. \quad (11.62)$$

The analytical expression for Ψ has to be obtained using the experimental observations and the micromechanical results. However, it can also be obtained from Equation 11.13 using the principle of strain equivalence. Thus, for a linearly elastic material, the Helmholtz (specific) free energy of a thermo-elastic process is given by

$$\Psi = \frac{1}{2\rho} \boldsymbol{\varepsilon}^e : \mathbf{C}^E : \boldsymbol{\varepsilon}^e (1-D) - TS. \quad (11.63)$$

Note that, in a thermo-elastic process, the elastic strain $\boldsymbol{\varepsilon}^e$ is obtained from the total strain tensor $\boldsymbol{\varepsilon}$ by subtracting the thermal strain:

$$\boldsymbol{\varepsilon}^e = \boldsymbol{\varepsilon} - \alpha (T - T_o) \mathbf{I}, \quad (T_o = \text{initial temperature}), \quad (11.64)$$

where α is the coefficient of thermal expansion, $T - T_o$ is the temperature rise, and the symbol $\mathbf{I}$ denotes the second-order identity tensor. From Equation 11.63, the Cauchy stress components are obtained using Equation 11.60 as

$$\boldsymbol{\sigma} = \rho \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^e} = (1-D) \mathbf{C}^E : \boldsymbol{\varepsilon}^e. \quad (11.65)$$

Further, the conservative part of the thermodynamic force Y corresponding to D is obtained from Equation 11.63 using Equation 11.62 as

$$Y = \rho \frac{\partial \Psi}{\partial D} = -\frac{1}{2} \boldsymbol{\varepsilon}^e : \mathbf{C}^E : \boldsymbol{\varepsilon}^e \quad (11.66)$$

Since it is an internal force, the dissipative counterpart of Y is obtained as (see Equation 11.25)

$$-Y = \frac{1}{2} \boldsymbol{\varepsilon}^e : \mathbf{C}^E : \boldsymbol{\varepsilon}^e, \quad (11.67)$$

For a better physical interpretation of Y , Equation 11.67 can be modified as follows. From Equation 11.65, one gets

$$\left. \frac{d\boldsymbol{\sigma}}{dD} \right|_{T=\text{constant}} = (1-D) \mathbf{C}^E : \left. \frac{d\boldsymbol{\varepsilon}^e}{dD} \right|_{T=\text{constant}} - \mathbf{C}^E : \boldsymbol{\varepsilon}^e. \quad (11.68)$$

For constant stress,

$$\left. \frac{d\boldsymbol{\sigma}}{dD} \right|_{T=\text{constant}} = 0, \quad (11.69)$$

and hence

$$\mathbf{C}^E : \boldsymbol{\varepsilon}^e = (1-D) \mathbf{C}^E : \frac{d\boldsymbol{\varepsilon}^e}{dD} \bigg|_{(\boldsymbol{\sigma}, T) = \text{constant}}. \quad (11.70)$$

Using Equation 11.70 in Equation 11.67, one gets

$$-Y = \frac{1}{2} (1-D) \boldsymbol{\varepsilon}^e : \mathbf{C}^E : \frac{d\boldsymbol{\varepsilon}^e}{dD} \bigg|_{(\boldsymbol{\sigma}, T) = \text{constant}}. \quad (11.71)$$

The use of the symmetry of $\mathbf{C}^E = (C_{ijkl}^E = C_{klij}^E)$ and substitution of Equation 11.65 leads to

$$-Y = \frac{1}{2} \boldsymbol{\sigma} : \frac{d\boldsymbol{\varepsilon}^e}{dD} = \frac{1}{2} \frac{dW_e}{dD} \bigg|_{(\boldsymbol{\sigma}, T) = \text{constant}}. \quad (11.72)$$

Here,

$$dW_e = \boldsymbol{\sigma} : d\boldsymbol{\varepsilon}^e, \quad (11.73)$$

is the elemental strain energy per unit volume. Therefore, $-Y$ is nothing but the change in strain energy due to damage when the stress and temperature remain constant.

An explicit expression for $-Y$, for an isotropic material, can be found in Equations 11.65 and 11.67, which will be useful in the analysis of problems involving isotropic damage. First, inverting Equation 11.65, one gets

$$\boldsymbol{\varepsilon}^e = \frac{1}{(1-D)} \mathbf{S}^E : \boldsymbol{\sigma}, \quad (11.74)$$

where $\mathbf{S}^E$ is the inverse of $\mathbf{C}^E$,

$$\mathbf{S}^E = (\mathbf{C}^E)^{-1}. \quad (11.75)$$

The previous expression, when substituted in Equation 11.67, gives Y in terms of the stress tensor:

$$-Y = \frac{1}{2(1-D)^2} \boldsymbol{\sigma} : \mathbf{S}^E : \boldsymbol{\sigma}. \quad (11.76)$$

Using $\mathbf{S}^E$ for an isotropic elastic material and decomposing the stress tensor in its deviatoric and mean components, the following can be written:

$$-Y = \frac{1}{2} \left[\frac{(1+\nu)}{E} \frac{\boldsymbol{\sigma}' : \boldsymbol{\sigma}'}{(1-D)^2} + 3 \frac{(1-2\nu)}{E} \frac{\sigma_m^2}{(1-D)^2} \right]. \quad (11.77)$$

Here, E is the Young's modulus, and ν is the Poisson's ratio of the material,

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \sigma_m \mathbf{I}, \quad (11.78)$$

is the deviatoric part of $\boldsymbol{\sigma}$ and

$$\sigma_m = \frac{1}{3} \text{tr}(\boldsymbol{\sigma}) \quad (11.79)$$

is the mean or hydrostatic part. Using the definition of equivalent stress,

$$\sigma_{eq} = \sqrt{\frac{3}{2} \boldsymbol{\sigma}' : \boldsymbol{\sigma}'}, \quad (11.80)$$

expression 11.77 can be written as

$$-Y = \frac{\sigma_{eq}^2}{2E(1-D)^2} \left[\frac{2}{3}(1+\nu) + 3(1-2\nu) \left(\frac{\sigma_m}{\sigma_{eq}} \right)^2 \right]. \quad (11.81)$$

The quantity $\left(\frac{\sigma_m}{\sigma_{eq}} \right)$ is called triaxiality, which plays a very important role in the fracture of materials. It is known that at a high value of triaxiality, materials behave in a brittle manner (Lemaitre, 1996). The term within the bracket in Equation 11.81 is termed a triaxiality function $f\left(\frac{\sigma_m}{\sigma_{eq}}\right)$. Hence, we obtain

$$-Y = \frac{\sigma_{eq}^2}{2E(1-D)^2} f\left(\frac{\sigma_m}{\sigma_{eq}}\right). \quad (11.82)$$

Using the definition of effective stress (Equation 11.48), we get

$$-Y = \frac{\sigma_{eq}^{*2}}{2E} f\left(\frac{\sigma_m}{\sigma_{eq}}\right). \quad (11.83)$$

Here, similar to Equation 11.80, the expression for damage equivalent stress σ_{eq}^* is given by

$$\sigma_{eq}^* = \sqrt{\frac{3}{2} \boldsymbol{\sigma}'^* : \boldsymbol{\sigma}'^*}, \quad (11.84)$$

where $\boldsymbol{\sigma}'^*$ is the deviatoric part of $\boldsymbol{\sigma}^*$. The damage equivalent stress is defined as the one-dimensional stress σ^* , which for the same value of damage yields the

same value of the strain energy density as for a three-dimensional case (Lemaitre, 1996).

Expression 11.83 for $-Y$ is derived for a thermo-elastic process. However, it remains valid even for a thermo-elasto-plastic process. In the latter case, the elastic strain appearing in Equation 11.63 is obtained by removing not just the thermal part (Equation 11.64) but also both the plastic and thermal parts.

11.9.10 CONSTITUTIVE EQUATIONS FOR THERMO-ELASTO-PLASTIC PROCESS IN A DAMAGED MATERIAL

The constitutive equations for the thermo-elasto-plastic behavior of a damaged material are derived from an appropriate plastic potential F . It is assumed that (Lemaitre, 1984, 1985a, 1985b), in metal plasticity where ductile damage is prominent, it is possible to decompose F as

$$F = F_1(\sigma, -R, D, T) + F_D(-Y, D, T), \quad (11.85)$$

where F_D is the plastic potential associated with damage such that it reduces to zero whenever $D = 0$. For a material yielding according to the von Mises criterion, the form of F_1 is

$$F_1 = \frac{\sigma_{eq} - R}{1 - D} - {}^0\sigma_Y = \sigma_{eq}^* - \sigma_Y, \sigma_Y = \left(\frac{R}{1 - D} + {}^0\sigma_Y \right), \quad (11.86)$$

where ${}^0\sigma_Y$ is the initial yield stress in tension. When $D = 0$, F_1 reduces to the original form due to the von Mises criterion. Now the plastic flow rules (Equations 11.37–11.39) become

$$\dot{\epsilon}^p = \dot{\lambda} \frac{\partial F_1}{\partial \sigma} = \frac{\dot{\lambda}}{1 - D} \frac{3}{2} \frac{\sigma'}{\sigma_{eq}}, \quad (11.87)$$

$$\dot{p} = \dot{\lambda} \frac{\partial F_1}{\partial (-R)} = \frac{\dot{\lambda}}{1 - D}, \quad (11.88)$$

$$\dot{D} = \dot{\lambda} \frac{\partial F_D}{\partial (-Y)}. \quad (11.89)$$

Combining Equations 11.27 and 11.88, we get the following expression for $\dot{\lambda}$:

$$\dot{\lambda} = (1 - D) \dot{\epsilon}_{eq}^p. \quad (11.90)$$

Substitution of Equation 11.90 into Equations 11.87 and 11.89 leads to the following constitutive equations (stress-strain rate relation and damage growth law):

$$\dot{\varepsilon}^p = \frac{3}{2} \frac{\dot{\varepsilon}_{eq}^p}{\sigma_{eq}} \sigma', \quad (11.91)$$

$$\dot{D} = (1 - D) \dot{\varepsilon}_{eq}^p \frac{\partial F_D}{\partial (-Y)}. \quad (11.92)$$

In Section 11.9.11, some expressions for F_D are discussed.

11.9.11 DAMAGE GROWTH LAWS

The expressions for F_D and $\dot{D}$ for some of the CDM models are presented next. For detailed discussions and derivations, the reader is referred to the cited references.

- *Lemaitre's damage model I* (Lemaitre, 1984). The expression for F_D is given by

$$F_D = \frac{S_0}{2} \left(-\frac{Y}{S_0} \right)^2 \dot{\varepsilon}_{eq}^p, \quad (11.93)$$

where S_0 is a temperature-dependent material coefficient. The damage evolution is given by

$$\dot{D} = -\frac{Y}{S_0} \dot{\varepsilon}_{eq}^p. \quad (11.94)$$

It can be observed that the damage evolution is linear in $\dot{\varepsilon}_{eq}^p$

- *Lemaitre's damage model II* (Lemaitre, 1985a). The expression for F_D is given by

$$F_D = \left(\frac{S_0}{S_0 + 1} \right) \left(-\frac{Y}{S_0} \right)^{s_0+1} \dot{\varepsilon}_{eq}^p, \quad (11.95)$$

where S_0 and s_0 are the temperature-dependent material coefficients. The damage evolution is given by

$$\dot{D} = \left(-\frac{Y}{S_0} \right)^{s_0} \dot{\varepsilon}_{eq}^p. \quad (11.96)$$

Again, it can be observed that the damage evolution is linear

- *Tai and Yang model* (Tai and Yang, 1986). The expression for F_D is given by

$$F_D = \frac{1}{2} S_0 \left(-\frac{Y}{S_0} \right)^2 D \dot{\varepsilon}_{eq}^p, \quad (11.97)$$

where S_0 is the temperature-dependent material coefficient. The damage growth rate is given by

$$\dot{D} = m \frac{1}{\varepsilon_R^m - \varepsilon_0^m} \ln \left(\frac{D_c}{D_0} \right) f \left(\frac{\sigma_m}{\sigma_{eq}} \right) \left(\varepsilon_{eq}^p \right)^{2/n} D \dot{\varepsilon}_{eq}^p, \quad (11.98)$$

where $f \left(\frac{\sigma_m}{\sigma_{eq}} \right)$ is given by Equation 11.82,

$$m = \frac{2+n}{n}, \quad (11.99)$$

and n is the material-hardening exponent. The damage evolution is exponential. It is suitable for low-carbon steel and small triaxiality ranges

- *Tie-Jun model* (Tie-Jun and Zhiwen, 1990; Tie-Jun, 1991, 1992). The expression for F_D is given by

$$F_D = \frac{S_0}{2} \left(-\frac{Y}{S_0} \right)^2 \frac{\dot{\varepsilon}_{eq}^p}{\left(\varepsilon_{eq}^p \right)^{2/n} \left(1 - \frac{\varepsilon_{eq}^p}{\left(\varepsilon_{eq}^p \right)_c} \right)^{1-\alpha}}, \quad (11.100)$$

where $\left(\varepsilon_{eq}^p \right)_c$ is the critical value of equivalent plastic strain, S_0 is the temperature-dependent material coefficient, n is the hardening exponent, and α is the damage coefficient that helps in understanding the accumulation of damage.

The damage evolution is given by

$$\dot{D} = \frac{(D_c - D_0)}{\varepsilon_R} \left(1 - \frac{\varepsilon_0}{\varepsilon_R} \right)^{-\alpha} f \left(\frac{\sigma_m}{\sigma_{eq}} \right) \frac{\dot{\varepsilon}_{eq}^p}{\left(1 - \frac{\varepsilon_{eq}^p}{\left(\varepsilon_{eq}^p \right)_c} \right)^\alpha}, \quad (11.101)$$

where D_0 is the initial damage at strain ε_0 , and D_c is the critical damage at rupture strain ε_R . The coefficient α is introduced to account for non-linearity. It can be seen that $\alpha = 1$ gives a linear model

- *Chandranth and Pandey exponential model* (Chandranth and Pandey, 1993). The expression for F_D is given by

$$F_D = \frac{S_0}{2} \left(\frac{-Y}{S_0} \right)^2 \left[\frac{1}{(1-D)^n} - (1-D) \right] \dot{\epsilon}_{eq}^p. \quad (11.102)$$

The damage evolution is given by

$$\dot{D} = A \left(\frac{2+n}{n(n+1)} \right) f \left(\frac{\sigma_m}{\sigma_{eq}} \right) \left[\frac{1}{(1-D)^n} - (1-D) \right] \left(\epsilon_{eq}^p \right)^{2/n} \dot{\epsilon}_{eq}^p, \quad (11.103)$$

where

$$A = \frac{\ln \left\{ \left(1 - (1 - D_c)^{n+1} \right) / \left(1 - (1 - D_0)^{n+1} \right) \right\}}{\epsilon_R^m \epsilon_0^m} \quad (11.104)$$

and m is given by Equation 11.99

- *Chandrankanth and Pandey parabolic model* (Chandrankanth and Pandey, 1995). The expression for F_D is given by

$$F_D = \frac{S_0}{2} \left(\frac{-Y}{S_0} \right)^2 \frac{1}{D^{\alpha/n} \left(\epsilon_{eq}^p \right)^{2/n}} \dot{\epsilon}_{eq}^p. \quad (11.105)$$

The damage evolution is given by

$$\dot{D} = \frac{D_c^M - D_0^M}{\epsilon_R - \epsilon_0} \left(\frac{n}{\alpha + n} \right) f \left(\frac{\sigma_m}{\sigma_{eq}} \right) D^{-\alpha/n} \dot{\epsilon}_{eq}^p, \quad (11.106)$$

where $M = \alpha/n + 1$

- *Bonora's non-linear model* (Bonora, 1997). The expression for F_D is given by

$$F_D = \left[\frac{1}{2} \left(\frac{-Y}{S_0} \right)^2 \frac{S_0}{1-D} \right] \frac{(D_c - D)^{(\alpha-1)/\alpha}}{\left(\epsilon_{eq}^p \right)^{(2+n)/n}}, \quad (11.107)$$

where S_0 is a material constant, α is the damage exponent characteristic to the material, and n is the material-hardening exponent. The damage evolution is given by

$$\dot{D} = \left(\frac{\sigma_{eq}^2}{(1-D)^2} \right) f \left(\frac{\sigma_m}{\sigma_{eq}} \right) \frac{1}{2ES_0} \frac{(D_c - D)^{(\alpha-1)/\alpha}}{(\varepsilon_{eq}^p)^{(2+n)/n}} \frac{1}{1-D} \dot{\varepsilon}_{eq}^p. \quad (11.108)$$

- *Thakkar and Pandey model* (Thakkar and Pandey, 2007). The expression for F_D is given by

$$F_D = -\frac{1}{2} S_0 \left(\frac{-Y}{S_0} \right)^2 \frac{k_1 (\varepsilon_{eq}^p)^{(\alpha/n)-1} + k_2 (\varepsilon_{eq}^p)^{(\beta/n)-1} + k_3}{D^{(\alpha/n)-1} (\varepsilon_{eq}^p)^{2/n}} \dot{\varepsilon}_{eq}^p, \quad (11.109)$$

where k_1, k_2, k_3, α and β are the constants used to fit various damage growth models.

The damage evolution is given by

$$\dot{D} = \frac{-Y}{S_0} \frac{k_1 (\varepsilon_{eq}^p)^{(\alpha/n)-1} + k_2 (\varepsilon_{eq}^p)^{(\beta/n)-1} + k_3}{D^{(\alpha/n)-1} (\varepsilon_{eq}^p)^{2/n}} \dot{\varepsilon}_{eq}^p. \quad (11.110)$$

- Selecting suitable values for the constants k_1, k_2 and k_3 , the damage models of Lemaitre (1984), Tai and Yang (1986) and Chandrakanth and Pandey (1993, 1995) can be obtained. For details, see Thakkar and Pandey (2007).

11.9.12 MICROCRACK CLOSURE EFFECT

The model described so far is suitable to predict damage growth in materials subjected to simple tensile loads or strain paths. However, in more realistic situations, a material may be subjected to complex loads or strain paths leading to both tensile and compressive loading. The effective stress, given by Equation 11.47, has been derived considering tensile loading. However, it has often been observed that in the case of compression, the microvoids or microcracks may partially close leading to an increase in the load-bearing area and the stiffness of the material. Hence, Equation 11.47 is only valid for the case of tensile loading (e.g., in the one-dimensional case, the stress is positive, i.e., $\sigma \geq 0$). Lemaitre (1996) has derived the following modification to take into account the partial crack-closing effect:

$$\sigma^* = \frac{\sigma}{1-hD}, \quad (11.111)$$

where h is an experimentally determined scalar constant. This constant takes into account the partial crack closure effect of microvoids or microcracks. The value of h lies in the range

$$0 \leq h \leq 1. \quad (11.112)$$

It has been shown by Lemaitre (1996) that the value of h is of the same order as the critical value of the damage variable, that is, D_c . However, in practice, h is taken to be constant. A value of $h \approx 0.2$ has been suggested, which usually gives a good correlation with experimental results. However, the value of h can also be identified by the measurement of Young's modulus E of a damaged material subjected to tensile and compressive loading.

Splitting of stress tensor into tensile and compressive components: It is easy to define unidirectional stress as tension or compression by the sign of its magnitude. However, an extension of such a simple procedure to the general three-dimensional state of stress at a point is not trivial. The problem is how to identify the tensile and compressive components of the stress tensor. If the stress tensor can be expressed in terms of some scalars, independent of each other, then the stress can be decomposed into positive and negative components. One such convenient form has been suggested by Lemaitre (1996) where the stress tensor is written in terms of principal values σ_1 , σ_2 and σ_3 . The procedure has been applied by Pires et al. (2003) to study the effect of crack closure effect in bulk metal forming. They demonstrated that the crack closure effect has a strong influence on damage evolution. This can be crucial, especially in the case of multiaxial loading.

The procedure described by Lemaitre (1996) is as follows. The stress tensor is first written as

$$[\sigma_{\text{principal}}] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}, \quad (11.113)$$

where σ_1 , σ_2 and σ_3 are the principal stress components. Then, the stress tensor is decomposed into positive and negative parts using the Macaulay brackets:

$$\begin{aligned} \langle x \rangle &= x & \text{if } x \geq 0, \\ \langle x \rangle &= 0 & \text{if } x < 0. \end{aligned} \quad (11.114)$$

Thus,

$$[\sigma_{\text{principal}}] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} - \begin{bmatrix} -\sigma_1 & 0 & 0 \\ 0 & -\sigma_2 & 0 \\ 0 & 0 & -\sigma_3 \end{bmatrix}, \quad (11.115)$$

or

$$\sigma_{\text{principal}} = \langle \sigma \rangle - \langle -\sigma \rangle. \quad (11.116)$$

The definition of the multi-dimensional damaged model with crack closure effect is obtained by modifying the standard three-dimensional stress-strain relation by including the tensile/compressive split described previously.

The expression for the damage energy release rate $-Y$ (see Equation 11.77) is then replaced with

$$-Y = \frac{1+\varpi}{2E} \left[\frac{\langle \sigma \rangle : \langle \sigma \rangle}{(1-D)^2} + \frac{h \langle -\sigma \rangle : \langle -\sigma \rangle}{(1-hD)^2} \right] - \frac{\nu}{2E} \left[\left(\frac{\text{trace}(\langle \sigma \rangle)}{(1-D)} \right)^2 + h \left(\frac{\text{trace}(\langle \sigma \rangle)}{(1-hD)} \right)^2 \right]. \quad (11.117)$$

11.10 TECHNIQUES FOR DAMAGE MEASUREMENT

Damage, microvoids and microcracks, create new surfaces in the material. These new surfaces act as points of discontinuity and reduce the values of the following material properties (Lemaitre and Desmorat, 2005):

1. Elastic modulus (E) of the material.
2. Density (ρ) of the material.
3. Yield stress (σ_y) of the material.
4. Hardness of the material.
5. Ultrasonic wave velocity in the material.

However, damage increases the electrical resistance. Lemaitre (1996) has discussed a number of techniques for the measurement of damage. Some of them are mentioned next along with their suitability for various loading conditions.

1. *Direct measurements.* Damage is defined by Equation 11.40. Based on this definition, damage can be calculated directly by measuring the total crack area ΔA_D contained in the surface area ΔA at the mesoscale. LeRoy et al. (1981) measured the damage directly by studying the fractured surfaces of the polished cross sections of the broken damaged specimens. However, the recent development of X-ray tomography (Morgeneyer et al., 2008) allows for direct observation of bulk damage. Using these techniques, the error in measurement of damage by surface preparation can be avoided, and direct three-dimensional information of the damage can be obtained (Besson, 2010). This direct measurement method gives fairly good results for ductile and creep applications but is not suitable for brittle and fatigue applications.
2. *Variation of elastic modulus.* This is an indirect method of measuring the damage. From Equation 11.52 the effective Young's modulus E can be written as

$$\tilde{E} = E(1-D). \quad (11.118)$$

Rearranging the previous equation yields

$$D = 1 - \frac{\tilde{E}}{E}. \quad (11.119)$$

This is a destructive method and requires very accurate measurement of strains, which can be tedious. The specimens are mechanically loaded and unloaded to generate stress-strain data from which the effective Young's modulus $\tilde{E}$ is measured. It is suitable for the measurement of damage in ductile, creep and low-cycle fatigue applications. However, this method is not recommended for brittle and high-cycle fatigue damage measurements as the damage can be highly localized.

3. *Variation of microhardness.* This is also an indirect method. In this method, a diamond indenter is inserted in the material. The load of the indenter is chosen to achieve a specific indented area of the same order of magnitude as the RVE. The hardness H of the material is defined as the mean stress over the indented area. The damage is then defined in terms of the effective hardness $\tilde{H}$ and the hardness of the undamaged material H as

$$D = 1 - \frac{\tilde{H}}{H}. \quad (11.120)$$

This method is a quasi-destructive method as indentation is very small. It is suitable for the measurement of damage in ductile and low-cycle fatigue applications and partially for brittle and creep applications. However, this method is not recommended for the measurement of damage in high-cycle fatigue applications.

4. *Other methods.* Lemaitre (1996) has outlined some other methods for the measurement of damage. They are (1) measuring ultrasonic wave velocity (suitable for the measurement of damage in brittle applications), (2) change in density of the damaged material and (3) change in electrical resistance. However, most of these methods have been found to be not as good as the first three methods and hence are not recommended.

For further details, derivations and practical aspects of damage measurements, we refer to the detailed discussion given in the monograph by Lemaitre (1996).

11.11 APPLICATION OF A CDM MODEL^{2*}

In the following, we present an example of ductile fracture simulation using a damage growth law proposed by Dhar (1995) and further analyzed in the work of Gautam (2010) and Gautam and Dixit (2010). The objective of this section is to show the effect of damage on the response of the material when it is considered in the formulation. The finite-element simulations are performed using the formulation developed in the work of Gautam (2010).

* This section has been adapted from Gautam and Dixit (2010), with permission from the International Journal of Computational Methods.

We recall from Equation 11.92 that the damage evolution equation for the case of isotropic damage growth is given by^{3†*}

$$\dot{D} = (1 - D) \dot{\epsilon}_{eq}^{pL} \frac{\partial F_D}{\partial (-Y)}. \quad (11.121)$$

Unlike the plastic potential F_1 , the plastic potential F_D associated with damage is not well established in the literature. As a result, instead of using Equation 11.121 for the damage growth law, experimental results on void measurement at different deformation levels are used to propose a damage growth law. Based on the experimental results of LeRoy et al. (1981) for spheroidized steel, the following growth law proposed by Dhar (1995) is used:

$$\dot{D} = c_d \dot{\epsilon}_{eq}^{pL} + (a_1 + a_2 D)(-Y) \dot{\epsilon}_{eq}^{pL}. \quad (11.122)$$

Here, the coefficients c_d , a_1 and a_2 are material constants, which are evaluated from the experimental results of LeRoy et al. (1981). It is observed that the graph of area void fraction versus strain is almost linear at a low strain level but nonlinear at higher values of strain. This non-linearity is taken care of by the third term involving the product of D and $\dot{\epsilon}_{eq}^{pL}$ in Equation 11.122. One can add more non-linear terms like $a_3 D^2$, $a_4 D^3$, etc. However, it is observed that, for the material under study, such terms have insignificant contributions. The first term in Equation 11.122 represents the void nucleation, whereas the other two terms represent the void growth. In the damage growth law of Lemaitre (1984, 1985a, 1985b) (Equation 11.94), the terms corresponding to c_d and a_2 are not considered.

The damage growth law given in Equation 11.122 is used to study the static damage growth and the subsequent ductile fracture in cylindrical specimens. Two spheroidized steels, namely, AISI1090 and AISI1045 (LeRoy et al., 1981), are considered for the numerical simulation. The chemical compositions of the steels as mentioned in the reference (LeRoy et al., 1981) are given in Table 11.3. The material properties are given in Table 11.4.

11.11.1 PROCEDURE FOR DETERMINING DAMAGE LAW COEFFICIENTS IN EQUATION 11.122

LeRoy et al. (1981) conducted tensile tests on a set of cylindrical steel specimens. Different specimens were stretched to different strain levels. Each stretched specimen was cut at a particular cross section, and the number of voids was counted and measured by scanning electron microscopy. The mean void diameter was determined by the intercept method. The void volume fraction corresponding to the strain level of the specimen was evaluated from the mean void diameter and the number of voids. From these data, the graph of the area void fraction versus the strain was generated.

† In references Gautam (2010) and Gautam and Dixit (2010) logarithmic strain is used. Hence, in the present section, a superscript 'L' is added to the equivalent plastic strain rate ...

TABLE 11.3
Chemical Composition of Steels Used for Tensile Testing

Material	C (wt%)	Mn (wt%)	P (wt%)	S (wt%)	Si (wt%)	Fe (wt%)
AISI1090	0.92	0.72	0.009	0.022	0.20	Rest
AISI1045	0.46	0.72	0.011	0.018	0.23	Rest

Source: Adapted from Gautam, S. S. and Dixit, P. M., *International Journal of Computational Methods*, 7(2), 319–348, 2010. With permission.

TABLE 11.4
Material Properties of Steels Used for Simulation of Tensile Testing

Material	E (GPa)	ν	σ_Y (MPa)	K (MPa)	n
AISI1090	210	0.30	464	816	0.73
AISI1045	210	0.30	302	796	0.59

Source: Adapted from Gautam, S. S. and Dixit, P. M., *International Journal of Computational Methods*, 7(2), 319–348, 2010. With permission.

TABLE 11.5
Coefficient Values in Damage Growth Law and Critical Damage Values of Steels Used for Tensile Testing

Material	c_d	a_1 (MPa ⁻¹)	a_2 (MPa ⁻¹)	D_c
AISI1090	3.80×10^{-03}	9.80×10^{-04}	3.70	0.60
AISI1045	5.83×10^{-03}	4.04×10^{-04}	2.60	0.50

Source: Adapted from Gautam, S. S. and Dixit, P. M., *International Journal of Computational Methods*, 7(2), 319–348, 2010. With permission.

The values of the coefficients c_d , a_1 and a_2 in the damage growth law (Equation 11.122) are determined using the following procedure. First, Equation 11.122 is rearranged as a relation between ${}^t(\partial D / \partial \epsilon_{eq}^{pL})$ and ${}^t\epsilon_{eq}^{pL}$ using the Bridgman’s (1952) relation between the triaxiality $({}^t\sigma_m / {}^t\sigma_{eq})$ and equivalent plastic strain. Then, from the experimental results of LeRoy et al. (1981), the slopes ${}^t(\partial D / \partial \epsilon_{eq}^{pL})$ are calculated at different levels of ${}^t\epsilon_{eq}^{pL}$. Finally, the coefficients c_d , a_1 and a_2 are obtained using the method of least squares curve fitting by minimizing the error with respect to each coefficient. The values of the coefficients in the damage growth law (Equation 11.122) are given in Table 11.5. The effect of various terms in the damage growth law has been discussed in detail in the work of Gautam (2010) and Gautam and Dixit (2010). The critical values of the damage variable (D_c) are estimated from the experimental results of LeRoy et al. (1981). The value of the damage at which the graph of area void fraction versus strain becomes almost vertical is taken as the critical value. The critical values are also shown in Table 11.5.

11.11.2 TENSILE TESTING AND DUCTILE FRACTURE OF CYLINDRICAL SPECIMEN

Figure 11.8 shows the specimen with the geometric data. Only one-eighth of the specimen is considered for the analysis because of symmetry. The finite-element mesh consists of 1890 eight-node brick elements and 2627 nodes. A small imperfection of 0.1% is introduced in the diameter of the mid-section to simulate necking. The damage increment given by Equation 11.122 is computed (at a Gauss point) only when the triaxiality is greater than $-1/3$, as suggested by Bao and Wierzbicki (2005). Damage at a Gauss point is updated until it reaches the critical value (D_c). When the damage at a Gauss point exceeds the critical value, the damage increment is set to zero. An element is deemed to have failed when the average damage inside the element reaches the critical value (D_c). The spread of fracture is simulated by tracing the path of the failed elements. The failed elements are assumed to have zero stiffness.

To check the accuracy of the results obtained from the computer simulation, the results for the cylindrical specimen are compared with the experimental results of LeRoy et al. (1981). Figure 11.9a and b show the simulated as well as the experimental damage growth curves for the two steel specimens. The simulated curves are obtained by integrating the damage growth law (Equation 11.122) up to the point of failure at the center and at the outer surface on the midsection of the cylindrical specimen. The experimental damage values are the average values over the cross section of the specimen. It is observed that many experimental damage points lie close to the band formed by the two simulated damage curves, one at the center and the other at the outer surface. Thus, there is a reasonable agreement between the experimental and simulated damage curves.

It is seen that, initially, the damage growth is slow when the deformation is primarily stretching along the axis of the rod. Further, the damage rises nearly at the same rate at the center and at the outer surface for the initial stage of the deformation. But, once the necking starts, the rate of increase in the damage at the center is much faster than at the outer surface. This is due to a significant rise in the triaxiality at the center (Figure 11.10a and b). The damage thus reaches the critical value (D_c) at the center much before it reaches the critical value at the outer surface of the specimen. This is consistent with the results of Tvergaard and Needleman (1984) and Pires et al. (2003).

Figure 11.11a and b shows the comparison between the simulated true stress-true strain curves with the experimental results of LeRoy et al. (1981) for both steel specimens. The true stress has been computed by dividing the total load by the current area of the necked region. The true strain is calculated from the following equation:

$$\epsilon = 2 \ln \left(\frac{d_0}{d} \right), \quad (11.123)$$

where d_0 and d are the initial and the current diameters of the test specimen in the necked portion. The true stress-true strain curves for the case with no damage are also shown in Figure 11.11a and b. It is observed that the material softening due to

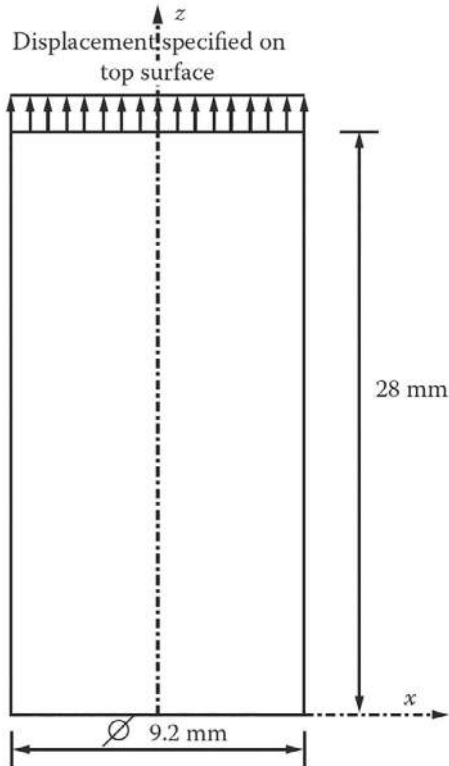


FIGURE 11.8 Domain of the problem for cylindrical specimen. (Adapted from Gautam, S. S. and Dixit, P. M., International Journal of Computational Methods, 7(2), 319–348, 2010. With permission.)

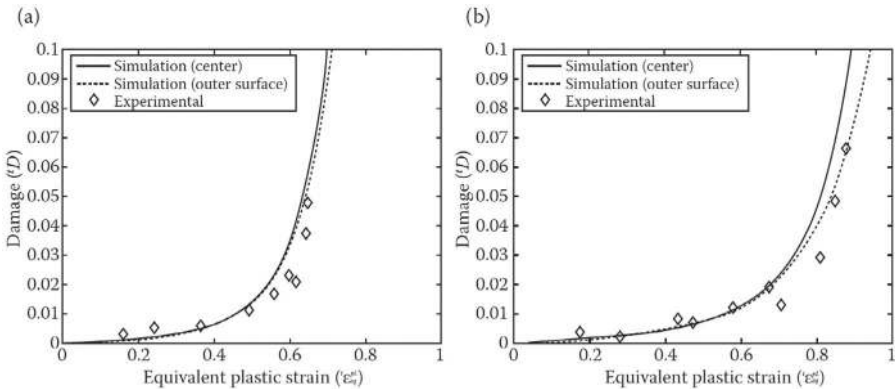


FIGURE 11.9 Damage growth curves for (a) AISI1090 and (b) AISI1045 steels. (Adapted from Gautam, S. S. and Dixit, P. M., International Journal of Computational Methods, 7(2), 319–348, 2010. With permission.)

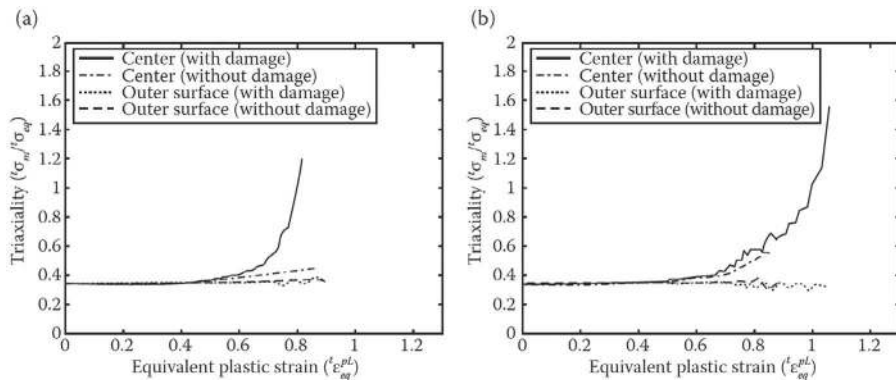


FIGURE 11.10 Growth of triaxiality with equivalent plastic strain for both the steels. (a) AISI1090. (b) AISI1045. (Adapted from Gautam, S. S. and Dixit, P. M., International Journal of Computational Methods, 7(2), 319–348, 2010. With permission.)

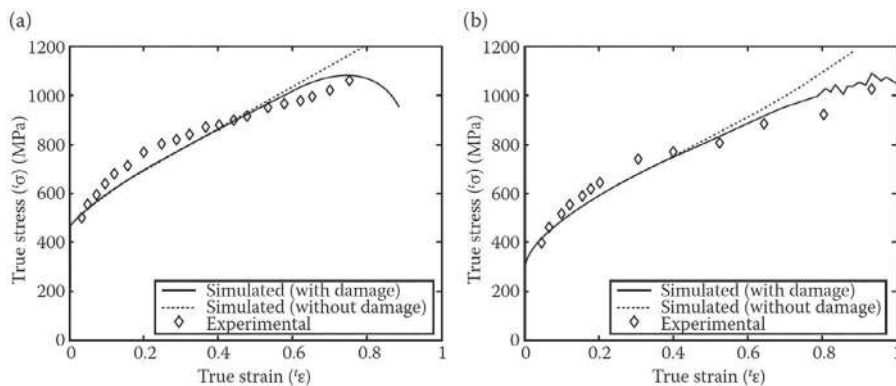


FIGURE 11.11 True stress-true strain curves for (a) AISI1090 and (b) AISI1045 steels. (Adapted from Gautam, S. S. and Dixit, P. M., International Journal of Computational Methods, 7(2), 319–348, 2010. With permission.)

damage occurs at a high strain (>0.5). Below this strain, the behavior is hardly distinguishable. It is observed that the simulated true stress-true strain curves (with damage) follow the experimental trends quite well. This shows that the damage growth law (Equation 11.122) is consistent with the experimental results.

Figure 11.10a and b show the variation of triaxiality with equivalent plastic strain for the two steels at the center and at the outer surface. The results without damage are also presented. It is seen that, in the initial stage, the triaxiality at the center of the specimen is one-third, manifesting a uni-axial state of stress. After the necking, the state of stress becomes triaxial, and the triaxiality rises very quickly at the center for both the steels. For the case without damage, the triaxiality rises much slower, than the case with damage, at the center for both the steels. At the outer surface, the triaxiality remains more or less close to one-third for both of the cases, that is, with and without damage.

Figure 11.12a and b show the damage versus triaxiality curves for the two steels at the center and at the outer surface. It is seen that, at the center, the rise in the damage is accompanied by a rise in the triaxiality. However, a different picture emerges at the outer surface. The damage rises with almost no (for AISI1090 steel) or very small (for AISI1045 steel) change in the triaxiality. Hence, while the failure at the center is dominated by the growth in the triaxiality, it is only the increase in the equivalent plastic strain that leads to the failure at the outer surface.

Figure 11.13a and b show the load-displacement curves for the two steels until failure. The load-displacement curves for the case without damage are also shown. It is seen that before the necking occurs in each steel, the difference between the values with and without damage is not distinguishable. Once the necking initiates, the load-displacement curves with damage fall almost vertically. This manifests the decrease in the strength of the steel due to the damage growth.

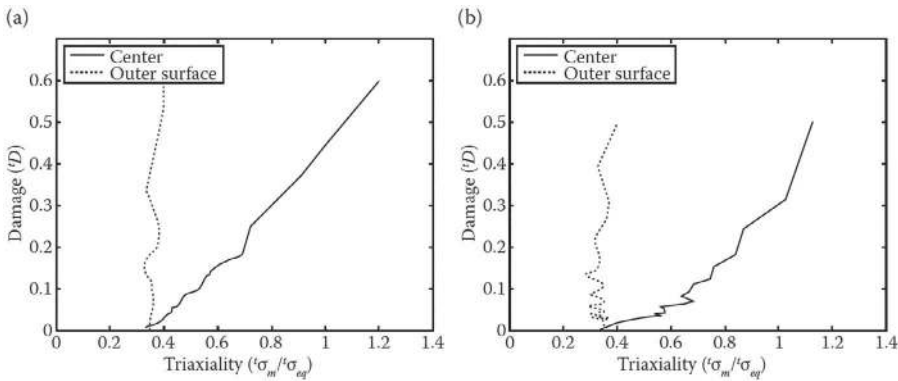


FIGURE 11.12 Growth of damage with triaxiality for both the steels. (a) AISI1090. (b) AISI1045. (Adapted from Gautam, S. S. and Dixit, P. M., International Journal of Computational Methods, 7(2), 319–348, 2010. With permission.)

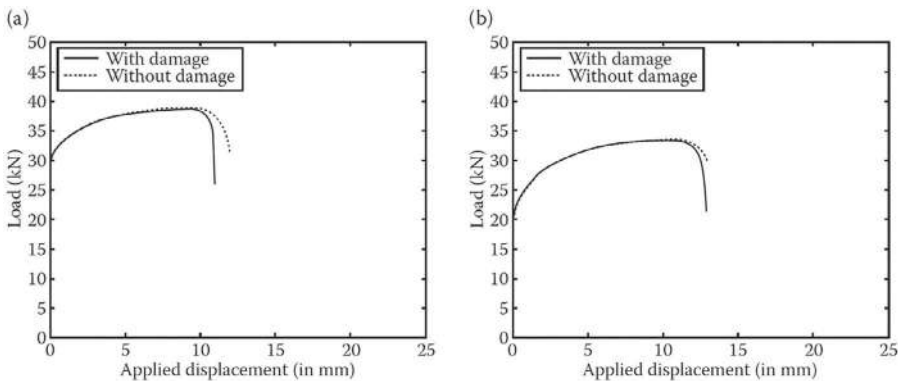


FIGURE 11.13 Load displacement curves for both the steels. (a) AISI1090. (b) AISI1045. (Adapted from Gautam, S. S. and Dixit, P. M., International Journal of Computational Methods, 7(2), 319–348, 2010. With permission.)

The damage growth law (Equation 11.122) has been further used to study the damage growth and ductile fracture in pre-notched specimens (Gautam and Dixit, 2010). It has been further employed to simulate damage growth and subsequent ductile fracture of Taylor rod impact specimens (Gautam et al., 2011) and thin-walled cylindrical tubes (Gautam and Dixit, 2012).

11.12 EFFECT OF LODE ANGLE ON DUCTILE FRACTURE

Xue (2008) and Xue and Wierzbicki (2008) reported that for some class of ductile materials, it has been experimentally observed that the fracture strain for simple shear is less than the fracture strain for simple tension. Hence, it was stated that for such materials the ductile fracture depends on the third stress invariant apart from the first and second stress invariants. However, this dependency is often neglected in the ductile fracture modeling. It can be incorporated by considering the effect of the Lode angle in the yield criterion expressed in principal stress space. The Lode angle, as will be shown in the following text, characterizes the deviatoric state of stress on the octahedral plane (see Section 3.11) in the principal stress space. The Lode angle effect on ductile fracture is phenomenologically taken care by introducing a Lode angle-dependent function in the yield criterion.

In what follows next, first a brief description of the principal stress space and its invariants are discussed followed by yield function representation in terms of stress invariants. This is followed by a detailed discussion on the concept of the Lode angle and its characteristics. Finally, some studies on ductile fracture considering the Lode angle effect are discussed. For a detailed discussion on the effect of the Lode angle on ductile fracture process, the reader is referred to the excellent review article on the subject by Algarni et al. (2021) and the references cited therein.

11.12.1 PRINCIPAL STRESS SPACE

Let us consider a coordinate system with the principal stresses σ_1 , σ_2 , and σ_3 (see Section 3.9) as its three mutually perpendicular axis. This is called the principal stress space. The plane of equation

$$f(\sigma_1, \sigma_2, \sigma_3) = \sigma_1 + \sigma_2 + \sigma_3, \quad (11.124)$$

is known as the octahedral plane. As defined in Section 3.11 the octahedral plane is equally inclined to three principal planes. When the octahedral plane contains the origin O it is called the π -plane. As stated in Section 5.4.3, the state of stress at point A can be geometrically represented by a vector **OA** in the three-dimensional principal stress space as shown in Figure 11.14. Then, the vector **OA** can be considered as the sum of two vectors—one which is perpendicular to the octahedral plane, that is, **OB** and the other which is in the octahedral plane, that is, **BA**. The vector **OB** represents the hydrostatic part of the stress ($\sigma_1 = \sigma_2 = \sigma_3$) and **BA** represents the deviatoric part of the stress (see Section 3.12). Therefore, the octahedral plane is also called the deviatoric plane. It can be easily shown that the magnitude of **OB** is

$$OB = -\sqrt{3}p, \quad (11.125)$$

where p is the hydrostatic part defined by Equation 3.109 and that of $\mathbf{BA}$ is

$$BA = \sqrt{\frac{2}{3}}\sigma_{eq}, \quad (11.126)$$

where the equivalent stress σ_{eq} is given by Equation 5.43. The reader is referred to Xue (2007a) for detailed derivations.

The yield criterion can be expressed in terms of the principal stresses as

$$f(\sigma_1, \sigma_2, \sigma_3) = 0, \quad (11.127)$$

where the functional form of f is given by Equation 5.48. The cylindrical surface of Figure 11.14 is the geometrical representation of Equation 5.48 in the principal stress space. This figure shows that starting from the origin O , each point $A(\sigma_1, \sigma_2, \sigma_3)$ on the surface can be located using two linear distances and one angular distance. These distances are as follows. The first linear distance is the distance of the deviatoric plane from the origin O (denoted by OB in Figure 11.14). The second linear distance (denoted by BA in Figure 11.15) is the radius of the circular cross-section of the cylinder in the deviatoric plane on which point A lies. The angular distance (denoted by θ in Figure 11.15) is the angle made by BA with the unit vector $\mathbf{m}$ along the projection of σ_1 -axis on the deviatoric plane. This angle is called the *Lode angle*. In the next subsection, it is shown that the distance OB depends on I_1 (i.e., the first principal invariant of the stress tensor), the distance BA depends on J_2 (i.e.,

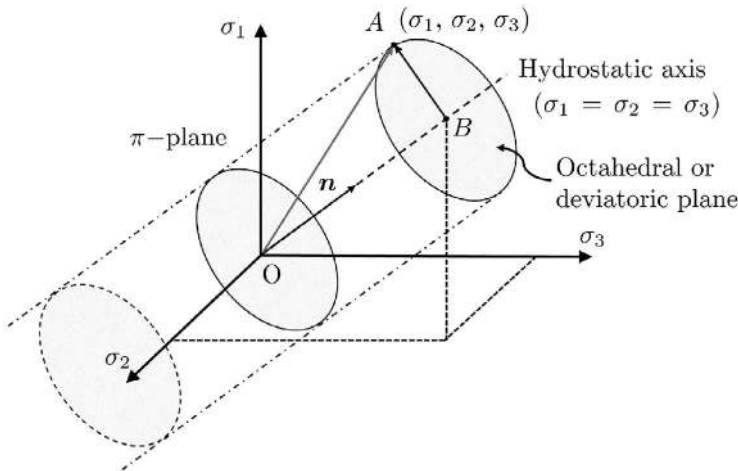


FIGURE 11.14 Stress vector in three-dimensional space of principal stresses.

the second principal invariant of the deviatoric part of the stress tensor) and $\cos\theta$ depends on J_2 and J_3 (i.e., the third principal invariant of the deviatoric part of the stress tensor). Thus, the yield criterion of Equation (11.127) can also be expressed in terms of the invariants I_1 , J_2 and $\cos\theta$:

$$f(I_1, J_2, \cos\theta) = 0. \quad (11.128)$$

For more details of these derivations, the reader is referred to Ottosen and Ristinmaa (2005).

11.12.2 DERIVATION OF OB, BA AND LOBE ANGLE IN TERMS OF PRINCIPAL STRESS INVARIANTS

In Figure 11.14 the length OB can be obtained by taking the projection of vector **OA** as

$$OB = \mathbf{n} \cdot \mathbf{OA}. \quad (11.129)$$

Here,

$$\mathbf{n} = \frac{1}{\sqrt{3}}(1, 1, 1), \quad (11.130)$$

is the unit normal vector directed along the hydrostatic axis and

$$\mathbf{OA} = (\sigma_1, \sigma_2, \sigma_3). \quad (11.131)$$

Using Equation 11.130 and Equation 11.131 in Equation 11.129 gives

$$OB = \frac{I_1}{\sqrt{3}}, \quad (11.132)$$

where

$$I_1 = (\sigma_1 + \sigma_2 + \sigma_3). \quad (11.133)$$

Then, the vector

$$\mathbf{OB} = OB \mathbf{n}. \quad (11.134)$$

Also, it can be shown that the vector

$$\mathbf{BA} = \mathbf{OA} - \mathbf{OB}, \quad (11.135)$$

can be expressed as

$$\mathbf{BA} = (s_1, s_2, s_3) \quad (11.136)$$

where s_i ($i = 1, 2, 3$) are the principal deviatoric stresses given by

$$s_i = \sigma_i - I_1/3, \quad (11.137)$$

see Ottosen and Ristinmaa (2005). The magnitude of $\mathbf{BA}$ is given by

$$BA^2 = s_1^2 + s_2^2 + s_3^2. \quad (11.138)$$

It is well known that

$$J_2 = \frac{1}{2}(s_1^2 + s_2^2 + s_3^2). \quad (11.139)$$

Then, using Equation 11.138 for BA^2 and Equation 11.139 for J_2 , it can be shown that

$$BA = \sqrt{2J_2}. \quad (11.140)$$

Finally, the expression for Lode angle θ in terms of J_2 and J_3 is derived. For this, first refer to Figure 11.15. In the figure unit vector $\mathbf{m}$, which lies on the deviatoric plane, is directed along the principal stress axis σ_1 . Using the fact that vector $\mathbf{m}$ is perpendicular to $\mathbf{n}$ it can be shown that

$$\mathbf{m} = \frac{1}{\sqrt{6}}(2, -1, -1), \quad (11.141)$$

see Ottosen and Ristinmaa (2005) for detailed derivations. The angle θ , called the *Lode angle*, is then obtained using the fact that the projection of vector $\mathbf{BA}$ along σ_1 axis is equal to $BA \cos\theta$. This gives

$$\mathbf{BA} \cdot \mathbf{m} = BA \cos\theta. \quad (11.142)$$

It is known that

$$s_1 + s_2 + s_3 = 0. \quad (11.143)$$

Then, using Equations 11.136, 11.140, 11.141, and 11.143 in Equation 11.142 it can be shown that

$$\cos\theta = \frac{\sqrt{3}}{2} \frac{s_1}{\sqrt{J_2}}. \quad (11.144)$$

Also,

$$J_3 = s_1 s_2 s_3, \quad (11.145)$$

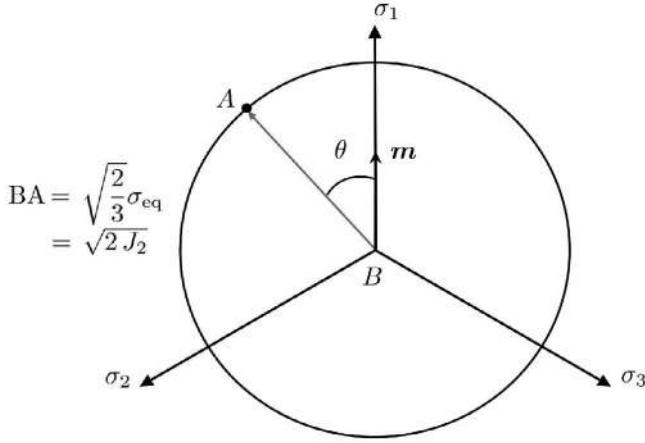


FIGURE 11.15 Projection of the state A on the deviatoric or octahedral plane when viewed along the hydrostatic axis BO.

and

$$\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta \quad (11.146)$$

Substitution of Equations 11.139 and 11.143–11.145 in Equation 11.146 provides the expression for Lode angle θ in terms of J_2 and J_3 as

$$\cos 3\theta = \frac{3\sqrt{3}}{2} \frac{J_3}{\sqrt{J_2^3}}. \quad (11.147)$$

11.12.3 EFFECT OF LODE ANGLE

The Lode angle effect on ductile fracture has been often ignored in a number of widely used fracture models like Johnson and Cook (1985) model and Gurson (1977) model. Experimental results show that material is easier to break under plane strain conditions than in other stress conditions on an octahedral plane. Since there is very little work done on exploring the Lode angle dependence of ductile fracture, the exact shape of the Lode angle dependence function is difficult to calibrate. However, with the appropriate hypothesis, a reasonably good quality yield criterion can be obtained from the analysis of experimental results, see Xue (2007a, 2008).

Wilkins et al. (1980) proposed a linear damage model where the damage is obtained by integral along the plastic strain path of the product of two functions $w_1(p)$ and $w_2(A)$. The function $w_1(p)$ incorporated the effect of hydrostatic pressure p and $w_2(A)$, called the asymmetric-strain weighting function, incorporated the effect of Lode angle through parameter A which depends on the stress state. Bai and Wierzbicki (2008) proposed a nonlinear damage/fracture model that is similar to the damage model of Johnson and Cook (1985) but also incorporated the effect of

stress triaxiality and the normalized Lode angle $\bar{\theta}$. The fracture model has six material constants that must be calibrated from the experiments. These two models are examples of uncoupled damage models, the effect of damage accumulation is not considered on the elasto-plastic constitutive relations. Next, some recently proposed coupled damage models are discussed where the effect of damage accumulation and Lode angle is considered on the elasto-plastic constitutive relations through a parameter.

Xue (2007a) and Xue and Wierzbicki (2008) proposed a simple and easy-to-use coupled nonlinear damage model which considered the effect of hydrostatic pressure p and Lode angle θ through equivalent plastic strain. Echávarri (2012), Erice and Gálvez (2014) and Erice et al. (2014) proposed a new nonlinear coupled damage elasto-plastic constitutive model called the Johnson-Cook, Xue, and damage model (JCXd model) which collated two models—J-C model (1985) and Xue and Wierzbicki (2008) model. The effect of the Lode angle was incorporated in the failure strain through a Lode angle-dependent function $\mu_\theta(\theta)$. The JCXd model is the most comprehensive damage model which incorporates the effect of stress triaxiality, Lode angle, strain rate and temperature effects on the ductile fracture process. Recently Bora et al. (2019) employed the JCXd model to study the effect of including the Lode angle on the ductile fracture of thin-walled cylindrical tubes impacted against a rigid surface using Abaqus. It was shown that there is a significant difference in the fracture pattern when the Lode angle is incorporated in the fracture model. For more discussion, the reader is referred to Algarni et al. (2021) and cited references therein for a detailed description of the Lode angle effect on ductile fracture.

11.13 CLOSURE AND FURTHER READING

In the present chapter, first, the phenomenon of ductile fracture is discussed as a three-stage process, namely, void nucleation, growth and coalescence. Some analytical models for void growth are presented. The factors affecting each of the three stages are also discussed. Two different processes of void coalescence—void impingement and void sheet formation—are described. Then, a brief review of different models of fracture initiation is presented. The porous plasticity model of Gurson is outlined together with its modification by Tvergaard and Needleman (1984). Various CDM models from the literature are presented followed by the thermodynamical theory of CDM. In this chapter, we have restricted coverage to isotropic damage. However, for advanced topics such as the damage evolution equations for anisotropic damage, viscoelastoplastic damage mechanics, damage mechanics of composite materials, dynamic damage mechanics and numerical applications to fatigue, creep, combined creep-fatigue, brittle and ductile-brittle damage, the interested reader is referred to excellent monographs on the subject like Murakami (2012), Kattan and Voyiadjis (2002), Allix and Hild (2002) and Zhang and Cai (2010). As a closing remark, it is essential to mention that the study of damage and especially CDM remains an active area of research. Current research topics include the simulation of slant fractures,

the development of damage growth laws with temperature and strain rate effects, coupling of x-FEM (Moes et al., 1999) and CDM.

Exercises

1. Show that

$$D = 1 - \frac{\tilde{E}}{E},$$

using the principle of strain equivalence.

2. Show the following using the elastic strain energy equivalence principle (see Section 11.9.8):

$$\varepsilon_e^* = (1 - D)\varepsilon_e, D = \left(1 - \frac{\sqrt{E^*}}{E}\right),$$

where E^* is the effective Young's modulus.

3. Some techniques for damage measurement were discussed in Section 11.10. Show that
 - a. The expression for damage in terms of longitudinal wave speed for damaged material $\tilde{v}_L$ and undamaged material v_L is given by $D \approx 1 - \left(\frac{\tilde{v}_L}{v_L}\right)^2$. Assume that the damage consists mainly of small microvoids and microcracks.
 - b. The expression for pure ductile damage in terms of the density of damaged material $\tilde{\rho}$ and undamaged material ρ is given by $D = \left(1 - \frac{\tilde{\rho}}{\rho}\right)^{2/3}$. Assume spherical cavities.
4. Show that the integration of Equation 11.73 using the laws of elasticity and assuming no variation of damage, that is, $D = \text{constant}$, yields

$$Y = \frac{W_e}{(1 - D)}.$$

5. Show that for isotropic case, Equation 11.74 has the following form:

$$\varepsilon^e = \frac{1 + \nu}{E} \frac{\sigma}{1 - D} - \frac{\nu}{E} \frac{\text{tr}(\sigma)}{1 - D} \mathbf{I}.$$

6. Derive Equation 11.77 from Equation 11.76.

7.

- a. Show that for a one-dimensional case, the damage equivalent stress σ^* is related to the equivalent stress σ_{eq} by

$$\sigma_{eq}^* = \sigma_{eq} f\left(\frac{\sigma_m}{\sigma_{eq}}\right)^{1/2},$$

where $f\left(\frac{\sigma_m}{\sigma_{eq}}\right)$ is the triaxiality function defined in Equation 11.82.

- b. Using the previous expression, establish the relation between σ^* and the in-plane stresses σ_1 and σ_2 for plane stress problem. Plot the graph of $\frac{\sigma_1}{\sigma^*}$ versus $\frac{\sigma_2}{\sigma^*}$.

12 Plastic Anisotropy

12.1 INTRODUCTION

An isotropic material has the same properties in all directions. An anisotropic material may have different properties in different directions. There are many materials that have anisotropic material behavior. These materials may follow linear elasticity. If an infinitesimal strain measure is assumed, there are six independent components of strain. In the absence of body moment, there are six stress components. If stress and strain components are linearly related, 36 coefficients are needed to express stress components in terms of the strain components. However, using the first law of thermodynamics and assuming the existence of a strain energy density function, it can be shown that there are only 21 distinct coefficients (Boresi et al., 1993). The elastic anisotropy is well described in many elasticity books such as Boresi et al. (1993). It will not be described in this chapter. The scope of this chapter is plastic anisotropy.

Plastic anisotropy means that plastic properties are direction-dependent. This means the flow stress and hardening behavior will vary with directions. A metal is generally macroscopically homogeneous and isotropic when the small crystal grains forming the aggregate are distributed with random orientations. Due to plastic deformation, the crystallographic directions rotate toward a common axis and a preferred orientation. Thus, initially isotropic metal may become anisotropic. Sometimes anisotropic metal is prepared to meet special design and manufacturing requirements.

12.2 NORMAL AND PLANAR ANISOTROPY

In a sheet, plastic properties may differ along a thickness direction in comparison to properties in the plane of the sheet. This is called normal anisotropy. Normal anisotropy is good for deep drawability of the sheet. If the sheet has less in-plane flow stress and high flow stress along the thickness, it has a lesser tendency for tearing during deep drawing. In-plane properties may also be different along different directions. This type of anisotropy is called planar anisotropy. Planar anisotropy leads to the formation of ears in deep drawing. The earing defect causes a wavy edge on a drawn cup. A sketch of the earing defect is shown in Figure 12.1.

A measure of normal anisotropy is the strain ratio, which can be defined as follows. In a steel sheet, let x be the rolling direction, y be the transverse direction and z be the normal or thickness direction. Now, suppose a tensile test sample is cut from the sheet such that the longitudinal axis x' makes an angle θ with the rolling direction, as shown in Figure 12.2. If the transverse axis of the sample is denoted as y'

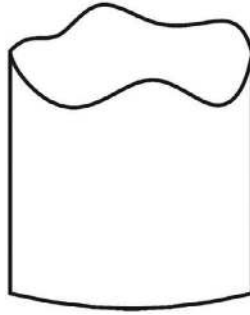


FIGURE 12.1 Earing defect.

and the normal axis as z' (which of course coincides with z), then the strain ratio is defined as

$$r_{\theta} = \frac{\varepsilon_{y'y'}^p}{\varepsilon_{z'z'}^p}, \quad (12.1)$$

where $\varepsilon_{y'y'}^p$ and $\varepsilon_{z'z'}^p$ are the plastic strains along y' and z' . The strain ratio is also called the r -value and the Lankford coefficient. It generally changes with the amount of strain, and it is common to measure the strain ratio at 20% elongation of the sheet. For the case of proportional loading in flow formulation, the strain rate ratio can be similarly defined.

Most materials show planar anisotropy, that is r_{θ} varies with the θ -direction in the plane of the sheet. In many metals, the variation is such that r_{θ} decreases from 0° to 45° and then increases from 45° to 90° . An average measure of the r -ratio is defined as

$$\bar{r} = \frac{r_0 + 2r_{45} + r_{90}}{4}, \quad (12.2)$$

where r_0 and r_{90} are the r -values along rolling and transverse directions, respectively, and r_{45} is taken at 45° to these axes. This relation can be derived based on the trapezoidal rule of numerical integration. Considering the variation of r from 0° to 90° and taking two divisions of this range,

$$\int_0^{\pi/2} r d\theta \simeq \frac{\pi}{8}(r_0 + r_{45}) + \frac{\pi}{8}(r_{45} + r_{90}) = \frac{\pi}{8}(r_0 + 2r_{45} + r_{90}). \quad (12.3)$$

Hence, the average value of the strain ratio is obtained as

$$\bar{r} = \frac{\int_0^{\pi/2} r d\theta}{\int_0^{\pi/2} d\theta} = \frac{\frac{\pi}{8}(r_0 + 2r_{45} + r_{90})}{\frac{\pi}{2}} = \frac{(r_0 + 2r_{45} + r_{90})}{4}. \quad (12.4)$$

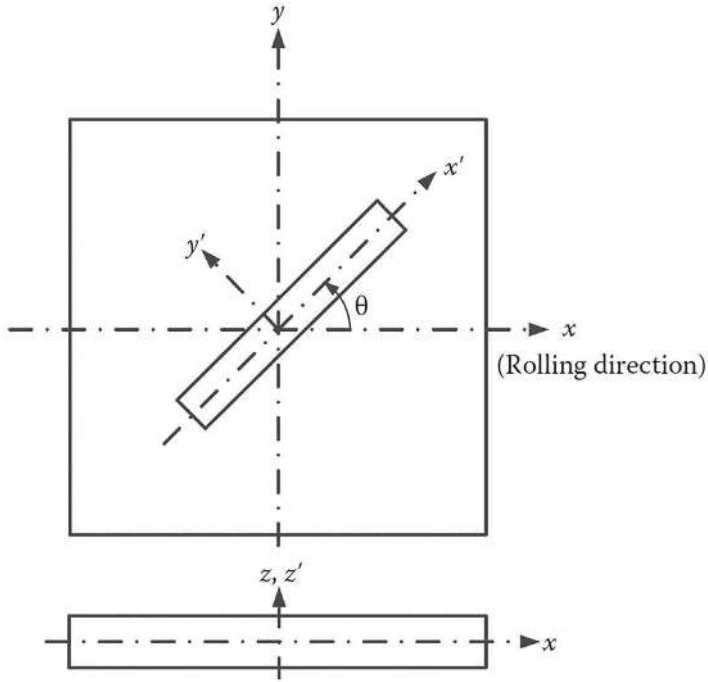


FIGURE 12.2 Tensile specimen for finding strain ratio.

The planar anisotropy coefficient is defined as

$$r_p = \frac{1}{2}(r_0 - 2r_{45} + r_{90}). \quad (12.5)$$

This is basically the average of the variation of r in two segments (0° to 45° and 45° to 90°). Thus,

$$r_p = \frac{(r_0 - r_{45}) + (r_{90} - r_{45})}{2}. \quad (12.6)$$

When the planar anisotropy is negligible, $\bar{r}$ is considered as a measure of the normal anisotropy. The automobile manufacturers need a high value of $\bar{r}$ in sheets for better formability. This means they want that the sheet should deform more in the plane and lesser along the thickness direction.

Example 12.1

Given $r_0 = 1.4$, $r_{30} = 1.3$ and $r_{90} = 1$, find the average strain ratio and the planar anisotropy coefficient.

SOLUTION

Here,

$$\begin{aligned}\int_0^{\pi/2} r d\theta &\approx \frac{\pi}{12}(r_0 + r_{30}) + \frac{\pi}{6}(r_{30} + r_{90}) \\ &= \frac{\pi}{12}(r_0 + 3r_{30} + 2r_{90}) = \frac{\pi}{12}(1.4 + 3.9 + 2).\end{aligned}$$

Hence,

$$\bar{r} = \frac{\int_0^{\pi/2} r d\theta}{\int_0^{\pi/2} d\theta} \approx \frac{\frac{\pi}{12}(1.4 + 3.9 + 2)}{\frac{\pi}{2}} = 1.22.$$

The variation of r from 0° to 30° is 0.1 and that from 30° to 90° is -0.3 . Considering that the planar anisotropy coefficient is the average variation in a 45° segment, the planar anisotropy coefficient is calculated as

$$r_p = \frac{1}{2} \left(\frac{0.1}{30} + \frac{-0.3}{60} \right) \times 45 = -0.0375.$$

Note that this is not the standard way of calculating the average strain ratio and the planar anisotropy coefficient. As far as possible, measurements should be carried out at 0° , 45° and 90° .

12.3 HILL'S ANISOTROPIC YIELD CRITERIA

Hill (1948) provided the following yield criterion for the anisotropic materials:

$$\begin{aligned}F(\sigma_{ij}) &= f(\sigma_{yy} - \sigma_{zz})^2 + g(\sigma_{zz} - \sigma_{xx})^2 + h(\sigma_{xx} - \sigma_{yy})^2 \\ &\quad + 2l\sigma_{yz}^2 + 2m\sigma_{zx}^2 + 2n\sigma_{xy}^2 - \sigma_0^2 = 0,\end{aligned}\tag{12.7}$$

where σ_{ij} are stress components with respect to the orthogonal coordinate system, and σ_0 is a scaling factor, which can be taken as equal to uni-axial yield stress in any direction. Six constants f, g, h, l, m and n can be determined by three uni-axial tests and three shear tests along the three orthogonal principal anisotropy axes x, y and z . A principal axis of anisotropy is the intersection of two planes of symmetry. In a rolled sheet, the rolling direction is taken as one principal axis. The other directions are the (in-plane) transverse direction and the normal (to the plane of the sheet) direction.

Six constants f, g, h, l, m and n can be determined by conducting three uni-axial tests along the principal axes and three shear tests. Suppose a uni-axial tensile test is

carried out along the x -axis. Then the material will start yielding at a yield stress of σ_{Yx} . Hence, Equation 12.7 provides

$$g\sigma_{Yx}^2 + h\sigma_{Yx}^2 = \sigma_0^2. \quad (12.8)$$

Similarly,

$$f\sigma_{Yy}^2 + h\sigma_{Yy}^2 = \sigma_0^2 \quad (12.9)$$

and

$$f\sigma_{Yz}^2 + g\sigma_{Yz}^2 = \sigma_0^2. \quad (12.10)$$

Solving Equations 12.8–12.10,

$$f = \frac{\sigma_0^2}{2} \left(\frac{1}{\sigma_{Yy}^2} + \frac{1}{\sigma_{Yz}^2} - \frac{1}{\sigma_{Yx}^2} \right), \quad (12.11a)$$

$$g = \frac{\sigma_0^2}{2} \left(\frac{1}{\sigma_{Yx}^2} + \frac{1}{\sigma_{Yz}^2} - \frac{1}{\sigma_{Yy}^2} \right), \quad (12.11b)$$

$$h = \frac{\sigma_0^2}{2} \left(\frac{1}{\sigma_{Yx}^2} + \frac{1}{\sigma_{Yy}^2} - \frac{1}{\sigma_{Yz}^2} \right). \quad (12.11c)$$

If τ_{Yxy} , τ_{Yyz} and τ_{Yzx} are the yield shear stresses with respect to anisotropy axes, then

$$l = \frac{\sigma_0^2}{2\tau_{Yyz}^2}, m = \frac{\sigma_0^2}{2\tau_{Yzx}^2}, n = \frac{\sigma_0^2}{2\tau_{Yxy}^2}. \quad (12.12)$$

It is easily seen that in the case of an isotropic material with yield stress as σ_y and shear yield stress as $\sigma_y/\sqrt{3}$,

$$f = g = h = \frac{l}{3} = \frac{m}{3} = \frac{n}{3} = \frac{\sigma_0^2}{2\sigma_y^2} \quad (12.13)$$

and the criterion reduces to the von Mises yield criterion.

Using the associated flow rule,

$$\begin{aligned} d\varepsilon_{yy}^p &= d\lambda \frac{\partial F}{\partial \sigma_{yy}} = 2d\lambda \left[f(\sigma_{yy} - \sigma_{zz}) - h(\sigma_{xx} - \sigma_{yy}) \right], \\ d\varepsilon_{zz}^p &= d\lambda \frac{\partial F}{\partial \sigma_{zz}} = 2d\lambda \left[-f(\sigma_{yy} - \sigma_{zz}) + g(\sigma_{zz} - \sigma_{xx}) \right]. \end{aligned} \quad (12.14)$$

The strain ratio is defined as the ratio of width plastic strain to thickness plastic strain in the case of a uni-axial tensile test. In the uni-axial tensile test, only non-zero stress is $\sigma_{xx} = \sigma_{yy}$. Thus, the average value of the strain ratio is given by (considering the planar anisotropy to be negligible)

$$\bar{r} = \frac{d\epsilon_{yy}^p}{d\epsilon_{zz}^p} = \frac{h}{g}. \quad (12.15)$$

From Equation 12.8,

$$g + h = \sigma_0^2 / \sigma_{Yx}^2 \quad (12.16)$$

and from Equation 12.10

$$f + g = \sigma_0^2 / \sigma_{Yz}^2. \quad (12.17)$$

If the planar anisotropy is negligible, f will be approximately equal to g . Hence Equation 12.17 can be written as

$$2g = \sigma_0^2 / \sigma_{Yz}^2. \quad (12.18)$$

From Equations 12.18 and 12.16

$$\frac{g+h}{2g} = \sigma_{Yz}^2 / \sigma_{Yx}^2. \quad (12.19)$$

Using Equation 12.15 in Equation 12.19 and rearranging

$$\frac{\sigma_{Yz}}{\sigma_{Yx}} = \sqrt{\frac{1+\bar{r}}{2}}. \quad (12.20)$$

It is difficult to conduct tensile tests along the thickness direction. For most metals, compressive and tensile yield strengths are equal. Therefore, σ_{Yz} can be considered equal to σ_c , the compressive yield strength along the thickness direction, and we have

$$\frac{\sigma_c}{\sigma_{Yx}} = \sqrt{\frac{1+\bar{r}}{2}}. \quad (12.21)$$

The hydrostatic state of stress does not have any influence on the plastic deformation. Assume that a sheet is subjected to compressive stress σ_c in the z -direction. If a hydrostatic tensile stress of magnitude σ_c is imposed on it, the net result is a state of stress in which $\sigma_x = \sigma_y = \sigma_c$ and all other stress components are zero. This state of stress is called the balanced bi-axial state of stress. There are machines that can apply the tensile loads in two directions simultaneously, and a balanced bi-axial tensile test can be conducted on these machines. The balanced bi-axial tension σ_b at which the

yielding starts is called yield stress in the balanced bi-axial test and is equal to σ_c , the compressive yield strength along the thickness direction. Thus, Equation 12.21 can be written as

$$\frac{\sigma_b}{\sigma_{Yx}} = \sqrt{\frac{1+\bar{r}}{2}}. \quad (12.22)$$

Equation 12.21 shows that if the average strain ratio is less than 1, σ_b will be less than σ_{Yx} . However, Woodthorpe and Pearce (1970) obtained anomalous behavior in commercial-purity aluminum sheets. There, σ_b was greater than σ_{Yx} even when the average strain ratio was less than 1. This is the anomaly exhibited by Hill's 1948 criterion.

Hill (1979) proposed another anisotropic criterion for initial yielding, which does not exhibit the abovementioned anomaly. This criterion can be stated as

$$\begin{aligned} F(\sigma_{ij}) \equiv & f|\sigma_2 - \sigma_3|^m + g|\sigma_3 - \sigma_1|^m + h|\sigma_1 - \sigma_2|^m + a|2\sigma_1 - \sigma_2 - \sigma_3|^m \\ & + b|2\sigma_2 - \sigma_3 - \sigma_1|^m + c|2\sigma_3 - \sigma_1 - \sigma_2|^m - \sigma_0^m = 0, \end{aligned} \quad (12.23)$$

where σ_i are the principal stresses, σ_0 is a scaling factor, and the material constants f , g , h , a , b and c are determined through experiments. The parameter m is assumed to be known depending on the crystal structure and is generally more than one. Here, it is assumed that the principal directions of the stress tensor coincide with the principal axes of anisotropy. Thus, this criterion does not contain the shear stress terms. As a result, it is restricted to loading along the principal axes of orthotropy only. Further, this criterion does not always satisfy the convexity condition, which is a requirement for every yield function. In Equation 12.23, σ_0 can be taken as equal to σ_b , the yield stress in balanced bi-axial tension. The yield criterion is satisfied for $\sigma_1 = \sigma_2 = \sigma_b$ and all the other stress components as zero. Hence,

$$f + g + a + b + 2^m c = 1. \quad (12.24)$$

Five special cases for plane stress deducible from Equation 12.23 are as follows:

Case 1: $a = b = h = 0$ and $f = g$

$$f|\sigma_2|^m + f|\sigma_1|^m + c|\sigma_1 + \sigma_2|^m = \sigma_0^m, \quad (12.25)$$

Case 2: $c = f = g = 0$ and $a = b$

$$h|\sigma_1 - \sigma_2|^m + a|2\sigma_1 - \sigma_2|^m + a|2\sigma_2 - \sigma_1|^m = \sigma_0^m, \quad (12.26)$$

Case 3: $c = h = 0$, $f = g$ and $a = b$

$$f|\sigma_2|^m + f|\sigma_1|^m + a|2\sigma_1 - \sigma_2|^m + a|2\sigma_2 - \sigma_1|^m = \sigma_0^m, \quad (12.27)$$

Case 4: $a = b = f = g = 0$

$$h|\sigma_1 - \sigma_2|^m + c|\sigma_1 + \sigma_2|^m = \sigma_0^m, \quad (12.28)$$

Case 5: $a = b = f = g = 0$ and $f = g$

$$f|\sigma_2|^m + f|\sigma_1|^m + h|\sigma_1 - \sigma_2|^m = \sigma_0^m \quad (12.29)$$

Case 5 is called Hosford's criterion. Most of the applications of Hill's 1979 criterion pertain to cases 4 and 5. Hill showed that in the case of planar anisotropy

$$r = \frac{(2^{m-1} + 2)a - c + h}{(2^{m-1} - 1)a + 2c + f}. \quad (12.30)$$

Hill (1990) suggested a function of the type

$$\begin{aligned} & \left| \frac{1}{2}(\sigma_1 + \sigma_2) \right|^m + h|\sigma_1 - \sigma_2|^m \\ & + (\sigma_1^2 + \sigma_2^2)^{(m-2)/2} (\sigma_1 - \sigma_2)(k\sigma_1 - l\sigma_2) = \sigma_b^m, \end{aligned} \quad (12.31)$$

where $m > 1$. When there is in-plane isotropy, Equation 12.31 simplifies to

$$\left| \frac{1}{2}(\sigma_1 + \sigma_2) \right|^m + h|\sigma_1 - \sigma_2|^m = \sigma_b^m. \quad (12.32)$$

In a general coordinate system, the criterion is written as

$$\begin{aligned} & \left| \sigma_{xx} + \sigma_{yy} \right|^m + \frac{\sigma_b^m}{\tau_y^m} \left| (\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2 \right|^{m/2} \\ & + \left| \sigma_{xx}^2 + \sigma_{yy}^2 + 2\sigma_{xy}^2 \right|^{(m/2-1)} \left\{ -2a(\sigma_{xx}^2 - \sigma_{yy}^2) + b(\sigma_{xx} - \sigma_{yy})^2 \right\} = (2\sigma_b)^m, \end{aligned} \quad (12.33)$$

where τ_y is the yield stress in pure shear deformation ($\sigma_1 = -\sigma_2$), and a and b are material constants.

Some aluminum and brass materials have almost equal yield stresses but different strain ratios in rolling and transverse direction. This type of behavior is called 'anomalous behavior of second order.' Hill (1993) proposed a yield criterion that takes into account anomalous behavior of $r < 1$ though $\sigma_b/\sigma_{Yx} > 1$. In the first quadrant (bi-axial tension), the criterion is given by

$$\frac{\sigma_1^2}{\sigma_0^2} - \frac{c\sigma_1\sigma_2}{\sigma_0\sigma_{90}} + \frac{\sigma_2^2}{\sigma_{90}^2} + \left\{ (p+q) - \frac{(p\sigma_1 + q\sigma_2)}{\sigma_b} \right\} \frac{\sigma_1\sigma_2}{\sigma_0\sigma_{90}} = 1, \quad (12.34)$$

where

$$\frac{c}{\sigma_0 \sigma_{90}} = \frac{1}{\sigma_0^2} + \frac{1}{\sigma_{90}^2} - \frac{1}{\sigma_b^2}. \quad (12.35)$$

Consider the case when stress is uni-axial ($\sigma_1 = \sigma_0$, $\sigma_2 = 0$). Note that here, σ_0 is the yield stress along the 0° direction (rolling direction). In this case, the right-hand side and left-hand side of Equation 12.34 are equal. Similarly, when $\sigma_1 = 0$ and $\sigma_2 = \sigma_{90}$, then also the right-hand side and left-hand side of Equation 12.34 are equal. Thus, the criterion is satisfied identically when the stress is uni-axial. When stress is equibiaxial ($\sigma_1 = \sigma_2 = \sigma_b$), Equation 12.34 provides

$$\frac{\sigma_b^2}{\sigma_0^2} - \frac{c\sigma_b^2}{\sigma_0 \sigma_{90}} + \frac{\sigma_b^2}{\sigma_{90}^2} + \left\{ (p+q) - \frac{(p\sigma_b + q\sigma_b)}{\sigma_b} \right\} \frac{\sigma_b^2}{\sigma_0 \sigma_{90}} = 1. \quad (12.36)$$

Rearranging Equation 12.36,

$$\frac{1}{\sigma_0^2} - \frac{c}{\sigma_0 \sigma_{90}} + \frac{1}{\sigma_{90}^2} = \frac{1}{\sigma_b^2}, \quad (12.37)$$

which is true in view of Equation 12.35. Thus, the criterion is identically satisfied for the equibiaxial stress case.

The yield criterion given by Equations 12.34 and 12.35 requires five material constants. These may be σ_0 , σ_{90} , σ_b , p and q . There can be various ways to determine constants p and q . One way is to measure strain ratios r_0 and r_{90} . The derivation of the expressions relating strain ratios and material constants is described in the following paragraphs.

At the yield surface $F(\sigma_1, \sigma_2) = C$, $dF = 0$. Hence,

$$dF = \frac{\partial F}{\partial \sigma_1} d\sigma_1 + \frac{\partial F}{\partial \sigma_2} d\sigma_2 = 0. \quad (12.38)$$

Hence,

$$\frac{d\sigma_2}{d\sigma_1} = - \frac{\partial F / \partial \sigma_1}{\partial F / \partial \sigma_2} = - \frac{d\epsilon_1^p}{d\epsilon_2^p}. \quad (12.39)$$

We also have

$$d\epsilon_1^p + d\epsilon_2^p + d\epsilon_3^p = 0. \quad (12.40)$$

Now, consider the case when yielding takes place at the uni-axial stress situation with $\sigma_1 = \sigma_0$ and $\sigma_2 = 0$. Using Equations 12.39 and 12.40,

$$\frac{d\sigma_1}{d\sigma_2} = - \frac{d\epsilon_2^p}{d\epsilon_1^p} = \frac{d\epsilon_2^p}{d\epsilon_2^p + d\epsilon_3^p} = \frac{d\epsilon_2^p / d\epsilon_3^p}{1 + d\epsilon_2^p / d\epsilon_3^p} = \frac{r_0}{1 + r_0}. \quad (12.41)$$

Consider another case when yielding takes place with $\sigma_1 = 0$ and $\sigma_2 = \sigma_{90}$. In this case,

$$\frac{d\sigma_2}{d\sigma_1} = -\frac{d\varepsilon_1^p}{d\varepsilon_2^p} = \frac{d\varepsilon_1^p}{d\varepsilon_1^p + d\varepsilon_3^p} = \frac{d\varepsilon_1^p / d\varepsilon_3^p}{1 + d\varepsilon_1^p / d\varepsilon_3^p} = \frac{r_{90}}{1 + r_{90}}. \quad (12.42)$$

Now, differentiating the expression in Equation 12.34 with respect to σ_2 ,

$$\begin{aligned} & \frac{2\sigma_1}{\sigma_0^2} \frac{d\sigma_1}{d\sigma_2} - \frac{c\sigma_2}{\sigma_0\sigma_{90}} \frac{d\sigma_1}{d\sigma_2} - \frac{c\sigma_1}{\sigma_0\sigma_{90}} + \frac{2\sigma_2}{\sigma_{90}^2} \\ & + \left\{ -\frac{p}{\sigma_b} \frac{d\sigma_1}{d\sigma_2} - \frac{q}{\sigma_b} \right\} \frac{\sigma_1\sigma_2}{\sigma_0\sigma_{90}} + \left\{ (p+q) - \frac{(p\sigma_1 + q\sigma_2)}{\sigma_b} \right\} \frac{1}{\sigma_0\sigma_{90}} \left(\sigma_2 \frac{d\sigma_1}{d\sigma_2} + \sigma_1 \right) = 0. \end{aligned} \quad (12.43)$$

Putting $\sigma_1 = \sigma_0$, $\sigma_2 = 0$ and the expression for $d\sigma_1/d\sigma_2$ from Equation 12.41,

$$\frac{2}{\sigma_0} \frac{r_0}{(1+r_0)} - \frac{c}{\sigma_{90}} + \left\{ (p+q) - \frac{p\sigma_0}{\sigma_b} \right\} \frac{1}{\sigma_{90}} = 0. \quad (12.44)$$

Similarly, putting $\sigma_1 = 0$, $\sigma_2 = \sigma_{90}$ and the expression for $d\sigma_1/d\sigma_2$ from Equation 12.42,

$$-\frac{c(1+r_{90})}{\sigma_0 r_{90}} + \frac{2}{\sigma_{90}} + \left\{ (p+q) - \frac{q\sigma_{90}}{\sigma_b} \right\} \frac{1}{\sigma_0\sigma_{90}} \left(\sigma_{90} \frac{(1+r_{90})}{r_{90}} \right) = 0. \quad (12.45)$$

Equations 12.44 and 12.45 are two simultaneous equations in p and q , which can be easily solved. Thus, parameters σ_0 , σ_{90} , σ_b , r_0 and r_{90} are sufficient for Hill's 1993 yield criterion.

When (σ_1, σ_2) are not in the first quadrant, Equation 12.34 needs modification to maintain the continuity of the first partial derivative. A general expression is given by

$$\frac{\sigma_1^2}{\sigma_0^2} - \frac{c\sigma_1\sigma_2}{\sigma_0\sigma_{90}} + \frac{\sigma_2^2}{\sigma_{90}^2} + \left\{ (p+q) - \frac{(p|\sigma_1| + q|\sigma_2|)}{\sigma_b} \right\} \frac{\sigma_1\sigma_2}{\sigma_0\sigma_{90}} = 1. \quad (12.46)$$

This is used in conjunction with Equation 12.35. Hill's 1993 criterion can be used only if the directions of principal stresses are coincidental with the orthotropic axes. It does not allow describing the variation of the anisotropy coefficient and of the uniaxial yield stress in the plane of the sheet. Moreover, the yield surface predicted by this function is far from that obtained from polycrystal theories.

Example 12.2

Consider the plane stress yield criterion given by

$$c_1 |\sigma_1 + \sigma_2|^m + c_2 |\sigma_1 - \sigma_2|^m = \sigma_0^m,$$

where σ_0 is the yield stress along any direction in the plane of the sheet assuming planar isotropy and c_1 , c_2 and m are the material parameters. Show that

$$c_1 = \frac{1}{2(1+r)}, \quad c_2 = \frac{1+2r}{2(1+r)},$$

where r is the Lankford coefficient.

SOLUTION

The criterion should be identically satisfied when $\sigma_1 = \sigma_0$, $\sigma_2 = 0$. Hence,

$$c_1 |\sigma_0|^m + c_2 |\sigma_0|^m = \sigma_0^m,$$

or

$$c_1 + c_2 = 1.$$

By definition,

$$r = \frac{d\varepsilon_2^p}{d\varepsilon_3^p} = -\frac{d\varepsilon_2^p}{d\varepsilon_1^p + d\varepsilon_2^p}, \text{ for } \sigma_1 = \sigma_0, \sigma_2 = 0.$$

Let

$$F = c_1 |\sigma_1 + \sigma_2|^m + c_2 |\sigma_1 - \sigma_2|^m - \sigma_0^m = 0.$$

Using flow rule (assuming $\sigma_1 > \sigma_2$),

$$\begin{aligned} d\varepsilon_1^p &= d\lambda \frac{\partial F}{\partial \sigma_2} = d\lambda \left\{ c_1 m (\sigma_1 + \sigma_2)^{m-1} + c_2 m (\sigma_1 - \sigma_2)^{m-1} \right\} \\ d\varepsilon_2^p &= d\lambda \frac{\partial F}{\partial \sigma_2} = d\lambda \left\{ c_1 m (\sigma_1 + \sigma_2)^{m-1} - c_2 m (\sigma_1 - \sigma_2)^{m-1} \right\}. \end{aligned}$$

For $\sigma_1 = \sigma_0$, $\sigma_2 = 0$,

$$\begin{aligned} d\varepsilon_1^p &= d\lambda \left\{ c_1 m (\sigma_0)^{m-1} + c_2 m (\sigma_0)^{m-1} \right\} \\ d\varepsilon_2^p &= d\lambda \left\{ c_1 m (\sigma_0)^{m-1} - c_2 m (\sigma_0)^{m-1} \right\}. \end{aligned}$$

Hence,

$$r = -\frac{d\varepsilon_2^p}{d\varepsilon_1^p + d\varepsilon_2^p} = -\frac{c_1 - c_2}{c_1 + c_2 + c_1 - c_2} = \frac{c_2 - c_1}{2c_1} = \frac{1 - c_1 - c_1}{2c_1},$$

or

$$2c_1r = 1 - 2c_1 \Rightarrow c_1 = \frac{1}{2(1+r)} \quad \text{and} \quad c_2 = 1 - c_1 = \frac{1+2r}{2(1+r)}.$$

Example 12.3

Hill's 1948 yield criterion is given by

$$\begin{aligned} \phi = & F(\sigma_{yy} - \sigma_{zz})^2 + G(\sigma_{zz} - \sigma_{xx})^2 + H(\sigma_{xx} - \sigma_{yy})^2 \\ & + 2L\sigma_{yz}^2 + 2M\sigma_{zx}^2 + 2N\sigma_{xy}^2 = 1. \end{aligned}$$

Show that for the plane stress case (σ_{xx} , σ_{yy} and σ_{xy} are non-zero),

$$r_{45} = \frac{N}{F+G} - \frac{1}{2},$$

where r_{45} is the strain ratio along 45° from the x -axis.

SOLUTION

By definition,

$$r_{45} = \frac{d\varepsilon_{135}^p}{d\varepsilon_{zz}^p} = -\frac{d\varepsilon_{135}^p}{d\varepsilon_{xx}^p + d\varepsilon_{yy}^p}, \quad (\text{i})$$

where ε_{135}^p is the normal strain along a direction that makes an angle of 135° from the x -axis, when uni-axial loading is along a direction making an angle of 45° from the x -axis. It is seen that by the flow rule, $d\varepsilon_{yz}^p = d\varepsilon_{zx}^p = 0$. The expression of normal strain along any direction making an angle θ from the x -axis in the x - y plane is

$$\varepsilon_\theta = \varepsilon_{xx} \cos^2 \theta + \varepsilon_{yy} \sin^2 \theta + 2\varepsilon_{xy} \cos \theta \sin \theta.$$

Hence,

$$\begin{aligned} \varepsilon_{135}^p &= \varepsilon_{xx}^p \cos^2 135^\circ + \varepsilon_{yy}^p \sin^2 135^\circ + 2\varepsilon_{xy}^p \cos 135^\circ \sin 135^\circ \\ &= \varepsilon_{xx}^p / 2 + \varepsilon_{yy}^p / 2 - \varepsilon_{xy}^p. \end{aligned}$$

Thus, expression (i) becomes

$$r_{45} = -\frac{d\varepsilon_{135}^p}{d\varepsilon_{xx}^p + d\varepsilon_{yy}^p} = \frac{d\varepsilon_{xy}^p - (d\varepsilon_{xx}^p + d\varepsilon_{yy}^p)/2}{d\varepsilon_{xx}^p + d\varepsilon_{yy}^p} = \frac{d\varepsilon_{xy}^p}{d\varepsilon_{xx}^p + d\varepsilon_{yy}^p} - \frac{1}{2}. \quad (\text{ii})$$

Now, the associated flow rule is applied. For this purpose, it is essential to express the yield criterion in the following way:

$$= F(\sigma_{yy} - \sigma_{zz})^2 + G(\sigma_{zz} - \sigma_{xx})^2 + H(\sigma_{xx} - \sigma_{yy})^2 \\ + L\sigma_{yz}^2 + L\sigma_{zy}^2 + M\sigma_{zx}^2 + M\sigma_{xz}^2 + N\sigma_{xy}^2 + N\sigma_{yx}^2 = 1.$$

In this form, ϕ is a function of nine stress components, and nine incremental strain components can be obtained. Later on, the symmetry condition can be employed. Now,

$$d\varepsilon_{xx}^p = d\lambda \frac{\partial \phi}{\partial \sigma_{xx}} = 2 d\lambda G(\sigma_{xx} - \sigma_{zz}) + 2 d\lambda H(\sigma_{xx} - \sigma_{yy}), \quad (\text{iii})$$

$$d\varepsilon_{yy}^p = d\lambda \frac{\partial \phi}{\partial \sigma_{yy}} = 2 d\lambda F(\sigma_{yy} - \sigma_{zz}) - 2 d\lambda H(\sigma_{xx} - \sigma_{yy}), \quad (\text{iv})$$

$$d\varepsilon_{xy}^p = d\lambda \frac{\partial \phi}{\partial \sigma_{xy}} = 2 d\lambda N \sigma_{xy}. \quad (\text{v})$$

Substituting Equations (iii), (iv) and (v) in Equation (ii) and noting that $\sigma_{zz} = 0$,

$$r_{45} = \frac{N\sigma_{xy}}{F\sigma_{yy} + G\sigma_{xx}} - \frac{1}{2}. \quad (\text{vi})$$

The values of σ_{xx} , σ_{yy} and σ_{xy} need to be substituted in Equation (vi). At this stage, it is known that a uni-axial stress σ_{45} sufficient to cause yielding has been applied along a direction making an angle 45° from the x -axis in the plane of the sheet, that is the x - y plane. The transformation rule for stress tensor provides

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \cos 45^\circ & \cos 135^\circ & 0 \\ \cos 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{45} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \cos 45^\circ & \cos 45^\circ & 0 \\ \cos 135^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \sigma_{45} & \sigma_{45} & 0 \\ \sigma_{45} & \sigma_{45} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, $\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = \sigma_{45}/2$. Substituting the values of stress components in Equation (vi),

$$r_{45} = \frac{N}{F + G} - \frac{1}{2}.$$

Q.E.D.

12.4 PLANE STRESS ANISOTROPIC YIELD CRITERION OF BARLAT AND LIAN

The basis of Barlat and Lian's (1989) yield criterion is the isotropic yield criterion due to Hosford. Hosford's (1972) isotropic yield criterion is given by

$$|\sigma_2 - \sigma_3|^m + |\sigma_3 - \sigma_1|^m + |\sigma_1 - \sigma_2|^m = 2\sigma_Y^m. \quad (12.47)$$

The value of m is taken as six for BCC metals and eight for FCC metals. For $m = 2$, the criterion converts to the von Mises criterion. For the plane stress case, Equation 12.47 can be written as

$$|\sigma_2|^m + |\sigma_3|^m + |\sigma_1 - \sigma_2|^m = 2\sigma_Y^m. \quad (12.48)$$

Consider any general state of plane stress with components σ_{xx} , σ_{yy} and σ_{xy} . The principal stress can be written as

$$\sigma_1 = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \sqrt{\frac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2}, \quad (12.49a)$$

$$\sigma_2 = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) - \sqrt{\frac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2}. \quad (12.49b)$$

It is noted that the following are the invariants of the plane stress tensor:

$$K_1 = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}), \quad (12.50)$$

$$K_2 = \sqrt{\frac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2}. \quad (12.51)$$

In terms of invariants,

$$\sigma_1 = K_1 + K_2, \sigma_2 = K_1 - K_2. \quad (12.52)$$

Therefore, Equation 12.48 may be written as

$$|K_1 + K_2|^m + |K_1 - K_2|^m + |2K_2|^m = 2\sigma_Y^m. \quad (12.53)$$

For an anisotropic material having planar isotropy, Barlat and Lian provided the following yield criterion:

$$a|K_1 + K_2|^m + b|K_1 - K_2|^m + c|2K_2|^m = 2\sigma_Y^m, \quad (12.54)$$

where a , b and c are material-dependent parameters and σ_Y is the in-plane flow stress. The continuity of the derivative of the yield function for equal bi-axial stresses is obtained when $a = b$. Consider the uni-axial yield case in which $\sigma_{xx} = \sigma_Y$ and other components of the stress tensor are zero. In that case, Equation 12.55 provides

$$a|\sigma_Y|^m + c|2\sigma_Y|^m = 2\sigma_Y^m, \quad (12.55)$$

which gives $a = 2 - c$. Thus, Barlat and Lian's yield criterion for planar isotropy needs only parameters a , m and σ_Y . Both a and σ_Y can be found from a uni-axial yield test. The parameter a can be found by measuring the strain ratio in the uni-axial yield test, as explained in the following.

Let us express the yield function as

$$f = a|\sigma_1|^m + a|\sigma_2|^m + (2-a)|\sigma_1 - \sigma_2|^m = 2\sigma_Y^m. \quad (12.56)$$

If x - and y -directions are the principal stress directions, then

$$f = a|\sigma_x|^m + a|\sigma_y|^m + (2-a)|\sigma_x - \sigma_y|^m = 2\sigma_Y^m. \quad (12.57)$$

Using flow rate,

$$\begin{aligned} d\varepsilon_{xx}^p &= d\lambda \frac{\partial f}{\partial \sigma_{xx}} = d\lambda \left(am\sigma_{xx}^{m-1} + (2-a)m|\sigma_{xx} - \sigma_{yy}|^{m-1} \right), \\ d\varepsilon_{yy}^p &= d\lambda \frac{\partial f}{\partial \sigma_{yy}} = d\lambda \left(am\sigma_{yy}^{m-1} - (2-a)m|\sigma_{xx} - \sigma_{yy}|^{m-1} \right). \end{aligned} \quad (12.58)$$

For the uni-axial case, once the plastic deformation starts ($\sigma_x = \sigma_Y$)

$$\begin{aligned} d\varepsilon_{xx}^p &= 2d\lambda m\sigma_Y^{m-1}, \\ d\varepsilon_{yy}^p &= -d\lambda (2-a)m\sigma_Y^{m-1}. \end{aligned} \quad (12.59)$$

From the volume incompressibility condition, one gets

$$d\varepsilon_{zz}^p = -d\lambda am\sigma_Y^{m-1}. \quad (12.60)$$

By the definition of the Lankford ratio,

$$r = \frac{d\varepsilon_{yy}}{d\varepsilon_{zz}} = \frac{(2-a)}{a}. \quad (12.61)$$

Hence,

$$a = \frac{2}{r+1}. \quad (12.62)$$

The parameter m may be chosen to match the prediction of this criterion with experimental results.

Barlat and Lian (1989) extended the criterion to the case of planar anisotropy by modifying the expressions of K_1 and K_2 in the following manner:

$$K_1 = \frac{1}{2}(\sigma_{xx} + h\sigma_{yy}), \quad (12.63)$$

$$K_2 = \sqrt{\frac{1}{4}(\sigma_{xx} - h\sigma_{yy})^2 + p^2\sigma_{xy}^2}. \quad (12.64)$$

Essentially, these expressions are obtained after a linear transformation of stress components and the convexity of the yield function is preserved. Now, the yield criterion can be written as

$$a|K_1 + K_2|^m + a|K_1 - K_2|^m + (2-a)|2K_2|^m = 2\sigma_0^m, \quad (12.65)$$

where σ_0 is the flow stress in the x -direction. Let σ_{90} be the uni-axial yield stress in the transverse direction, τ_{s2} be the shear yield stress such that $\sigma_{yy} = -\sigma_{xx} = \tau_{s2}$, $\sigma_{xy} = 0$ and τ_{s1} be the shear yield stress such that $\sigma_{xx} = \sigma_{yy} = 0$, $\sigma_{xy} = \tau_{s1}$. It can be shown that

$$a = \frac{2\left(\frac{\sigma_0}{\tau_{s2}}\right)^m - 2\left(1 + \frac{\sigma_0}{\sigma_{90}}\right)^m}{1 + \left(\frac{\sigma_0}{\sigma_{90}}\right)^m - \left(1 + \frac{\sigma_0}{\sigma_{90}}\right)^m}, \quad (12.66)$$

$$h = \frac{\sigma_0}{\sigma_{90}},$$

$$p = \frac{\sigma_0}{\tau_{s1}} \left(\frac{2}{2a + 2^m(2-a)} \right)^{1/m}.$$

The material parameters a , h and p can also be determined from the measurements of the strain rate ratio r . The parameters a and h are given by

$$a = 2 - 2\left(\frac{r_0}{1+r_0} \frac{r_{90}}{1+r_{90}}\right)^{1/2}, \quad h = \left(\frac{r_0}{1+r_0} \frac{1+r_{90}}{r_{90}}\right)^{1/2}. \quad (12.67)$$

However, there is no analytical expression for p in terms of r . Therefore, the variation of p with r needs to be determined either numerically or graphically. A relationship between the Lankford coefficient for a direction making an angle with rolling direction and p exists when other parameters are known, which can be determined graphically and numerically. Since the graphical (or numerical) method gives only an approximate variation of p with r , the two methods of calculating the

parameters a , h and p do not give an identical set of values. The parameter m is chosen to match the predictions of this criterion with experimental results.

12.5 THREE-DIMENSIONAL ANISOTROPIC YIELD CRITERIA OF BARLAT AND COWORKERS

Barlat and his group have provided a series of yield criterion to represent anisotropic material behavior (Barlat et al., 1991, 2005). Here, only two criteria are described. The starting point is Hosford's (1972) *isotropic* yield criterion expressed in terms of the principal values σ'_i of the *stress deviator* σ' (i.e. the deviatoric part of the Cauchy stress tensor):

$$f(\sigma_{ij}) \equiv |\sigma'_2 - \sigma'_3|^m + |\sigma'_3 - \sigma'_1|^m + |\sigma'_1 - \sigma'_2|^m - 2\sigma_Y^m = 0. \quad (12.68)$$

Barlat et al. (1991) introduced anisotropy in the above criterion by expressing it in terms of the principal values S'_i of the *modified stress deviator* S' :

$$f(\sigma_{ij}) \equiv |S'_2 - S'_3|^m + |S'_3 - S'_1|^m + |S'_1 - S'_2|^m - 2\sigma_0^m = 0, \quad (12.69)$$

where S' is obtained from the *stress deviator* σ' by the following linear transformation:

$$S' = C\sigma', \quad (12.70)$$

where

$$\{S'\} = \begin{Bmatrix} S'_{xx} \\ S'_{yy} \\ S'_{zz} \\ S'_{xy} \\ S'_{yz} \\ S'_{zx} \end{Bmatrix}, \quad \{\sigma'\} = \begin{Bmatrix} \sigma'_{xx} \\ \sigma'_{yy} \\ \sigma'_{zz} \\ \sigma'_{xy} \\ \sigma'_{yz} \\ \sigma'_{zx} \end{Bmatrix}. \quad (12.71)$$

In Equation 12.69, σ_0 is a scaling factor, which may be taken equal to yield stress in the uni-axial tensile test. The parameter m may be chosen to match the predictions of this criterion with experimental results.

It can be easily shown that deviatoric stress components are related to Cauchy stress components in the following way:

$$\begin{Bmatrix} \sigma'_{xx} \\ \sigma'_{yy} \\ \sigma'_{zz} \\ \sigma'_{xy} \\ \sigma'_{yz} \\ \sigma'_{zx} \end{Bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}. \quad (12.72)$$

In compact notation, it can be written as

$$\sigma' = T\sigma. \quad (12.73)$$

The tensor S' can be expressed in terms of the Cauchy stress tensor σ by the following relation:

$$S' = L\sigma, \quad L = CT. \quad (12.74)$$

The tensors C , T and L are all fourth-order tensors. Since the tensors S' , σ' and σ are all symmetric tensors, the tensors C , T and L can have at the most 36 independent components.

Barlat et al. (1991) assumed that the matrix $[C]$ contains nine entries as non-zero and is represented as

$$[C] = \begin{bmatrix} 0 & -c_{12} & -c_{13} & 0 & 0 & 0 \\ -c_{21} & 0 & -c_{23} & 0 & 0 & 0 \\ -c_{31} & -c_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix}, \quad (12.75)$$

Further, they assumed that the matrix $[C]$ is symmetric:

$$c_{12} = c_{21}, \quad c_{23} = c_{32}, \quad c_{31} = c_{13}. \quad (12.76)$$

Thus, in this anisotropic yield criterion, besides the parameter m , there are six more parameters (c_{12} , c_{23} , c_{31} , c_{44} , c_{55} , c_{66}) that characterize the material anisotropy. This anisotropic yield criterion has been labeled as Yld91 by Barlat et al. (1991).

When all the anisotropic parameters are equal to one, and m is equal to two, Yld91 reduces to the Mises yield criterion. Six anisotropic parameters may be determined using three uni-axial yield stresses and three shear yield stresses associated with three axes of anisotropy. The parameter m is determined to match the predictions of this criterion with experimental results.

To represent the anisotropy of aluminum sheets to a better degree of accuracy, Barlat et al. (2005) used *two* linear transformations by introducing *two modified stress deviators*:

$$S' = C'\sigma', \quad S'' = C''\sigma'. \quad (12.77)$$

The yield criterion is expressed as

$$\begin{aligned} f(\sigma_{ij}) = & |S'_1 - S''_1|^m + |S'_1 - S''_2|^m + |S'_1 - S''_3|^m + |S'_1 - S''_1|^m + |S'_1 - S''_2|^m + |S'_2 - S''_3|^m \\ & + |S'_3 - S''_1|^m + |S'_3 - S''_2|^m + |S'_3 - S''_3|^m - 4\sigma_0^m = 0, \end{aligned} \quad (12.78)$$

where S'_i and S''_i are the principal values of the *modified stress deviators* S' and S'' . The associated linear transformation matrices are

$$[C'] = \begin{bmatrix} 0 & -c'_{12} & -c'_{13} & 0 & 0 & 0 \\ -c'_{21} & 0 & -c'_{23} & 0 & 0 & 0 \\ -c'_{31} & -c'_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c'_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c'_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c'_{66} \end{bmatrix}, \quad (12.79)$$

$$[C''] = \begin{bmatrix} 0 & -c''_{12} & -c''_{13} & 0 & 0 & 0 \\ -c''_{21} & 0 & -c''_{23} & 0 & 0 & 0 \\ -c''_{31} & -c''_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c''_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c''_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c''_{66} \end{bmatrix}. \quad (12.80)$$

This anisotropic yield criterion has been labeled as Yld2004-18p by Barlat et al. (2005). When the matrices $[C']$ and $[C'']$ are identical and symmetric, this criterion reduces to Yld91. Further, when all the anisotropic parameters are equal to one, and m is equal to two, Yld2004-18p reduces to the Mises yield criterion. It can be shown that the yield function given by Equation 12.78 is convex.

Barlat et al. (2005) determined the 18 anisotropic parameters of Yld2004-18p for the aluminum alloys 2090-T3 and 6111-T4 (mildly anisotropic) by minimizing the following weighted error function:

$$E(c'_{ij}, c''_{ij}) = \sum_p w_p \left(\frac{\sigma_p^{pr}}{\sigma_p^{ex}} - 1 \right)^2 + \sum_q w_q \left(\frac{r_q^{pr}}{r_q^{ex}} - 1 \right)^2, \quad (12.81)$$

where w_p and w_q are the weight functions. The indices p and q denote, respectively, the index number of experimental stresses (uni-axial, bi-axial or shear yield) and experimental strain rate ratios (r). Further, the superscripts pr and ex indicate whether it is a predicted value or an experimental value.

12.6 PLANE STRAIN ANISOTROPIC YIELD CRITERION

An anisotropic yield criterion for the plane strain rolling problem was proposed by Dixit and Dixit (1997), as also described in Dixit and Dixit (2008). It is assumed that z is the thickness direction. For plane strain problems in the x - z plane, the anisotropy is restricted to the x - z plane only, and therefore, there is planar *isotropy* in the plane of the sheet, that is in the x - y plane. For this case, the coefficients f , g , a and b in Hill's 1979 anisotropic criterion (Equation 12.23) satisfy

the relation $f = g$ and $a = b$. It is further assumed that $f = 0$ and $a = 0$. Then, the criterion reduces to

$$f(\sigma_{ij}) \equiv c|\sigma_1 + \sigma_2 - 2\sigma_3|^m + h|\sigma_1 - \sigma_2|^m - \sigma_0^m = 0. \quad (12.82)$$

Rewriting Equation 12.82 in terms of σ_{xx} , σ_{yy} and σ_{zz} ,

$$f(\sigma_{ij}) \equiv c|\sigma_{xx} + \sigma_{yy} - 2\sigma_{zz}|^m + h|\sigma_{xx} - \sigma_{yy}|^m - \sigma_0^m = 0. \quad (12.83)$$

When the loading is not along the principal axes of anisotropy, one needs to include the shear stress terms in the above criterion. For the plane strain problems in the x - z plane, the only non-zero shear stress components are σ_{xz} and σ_{zx} . Of course, due to symmetry, $\sigma_{zx} = \sigma_{xz}$. Thus, to get an anisotropic yield criterion for plane strain problems in the x - z plane, an extra term containing σ_{xz} is added to Equation 12.83. Thus, one gets

$$f(\sigma_{ij}) \equiv c|\sigma_{xx} + \sigma_{yy} - 2\sigma_{zz}|^m + h|\sigma_{xx} - \sigma_{yy}|^m + N(|\sigma_{xz}|^m + |\sigma_{zx}|^m) - \sigma_0^m = 0. \quad (12.84)$$

Here, the scaling factor σ_0 may be taken equal to the uni-axial yield stress in the rolling direction, and the coefficient N accounts for the presence of in-plane shear stress component σ_{xz} . Thus, there are four parameters that characterize the material anisotropy: c , h , N and m .

The material parameters c , h , N and m need to be determined in terms of measurable quantities like the uni-axial yield stresses and the strain rate ratios. If there is only one non-zero component of stress ($\sigma_{xx} = \sigma_0$), then Equation 12.84 provides

$$c + h = 1. \quad (12.85)$$

Using the associated flow rule for $\dot{\epsilon}_{yy}^p$ and $\dot{\epsilon}_{zz}^p$ in the tension test along the x -axis, the strain rate ratio r_0 can be obtained in terms of c and h . The associated flow rule provides

$$\begin{aligned} \dot{\epsilon}_{yy}^p &= \dot{\lambda} \left\{ cm(\sigma_{xx} + \sigma_{yy} - 2\sigma_{zz})^{m-1} - hm(\sigma_{xx} - \sigma_{yy})^{m-1} \right\}, \\ \dot{\epsilon}_{zz}^p &= \dot{\lambda} \left\{ -2cm(\sigma_{xx} + \sigma_{yy} - 2\sigma_{zz})^{m-1} \right\}. \end{aligned} \quad (12.86)$$

During the uni-axial tensile test, $\sigma_{xx} = \sigma_0$ and other components are zero. Hence,

$$r_0 = \frac{\dot{\epsilon}_{yy}^p}{\dot{\epsilon}_{zz}^p} \bigg|_{\text{uni-axial test}} = \frac{c-h}{-2c} = \frac{1}{2} \left(\frac{h}{c} - 1 \right). \quad (12.87)$$

As there is *planar isotropy*, the ratios r_0 , r_{45} and r_{90} may all be considered equal to the average strain rate ratio $\bar{r}$. Thus, Equation 12.87 can be written as

$$\bar{r} = \frac{1}{2} \left(\frac{h}{c} - 1 \right). \quad (12.88)$$

From Equations 12.85 and 12.88, the following expressions for the material parameters c and h in terms of $\bar{r}$ are obtained:

$$c = \frac{1}{2(\bar{r} + 1)}; \quad h = \frac{2\bar{r} + 1}{2(\bar{r} + 1)}. \quad (12.89)$$

As the average strain rate ratio is always positive, the material parameters c and h are also positive. Now, consider a coordinate system (x', y', z') such that $y' = y$ and the axes (x', z') are in the x - z plane. Further, the x' -axis makes a counterclockwise angle of 45° with the rolling direction x . In the uni-axial state of stress along the x' direction, the stress components with respect to the (x', y', z') system at yielding are given by

$$\sigma_{x'x'} = \sigma_{45}, \quad \sigma_{z'z'} = 0, \quad \sigma_{x'z'} = 0, \quad \sigma_{y'y'} = 0, \quad (12.90)$$

where σ_{45} is the uni-axial yield stress along the 45° direction to the rolling direction. Transforming these components to the (x, y, z) system, one gets

$$\begin{aligned} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} &= \begin{bmatrix} \cos 45^\circ & 0 & \cos 135^\circ \\ 0 & 1 & 0 \\ \cos 45^\circ & 0 & \cos 45^\circ \end{bmatrix} \begin{bmatrix} \sigma_{45} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} \cos 45^\circ & 0 & \cos 45^\circ \\ 0 & 1 & 0 \\ \cos 135^\circ & 0 & \cos 45^\circ \end{bmatrix} \begin{bmatrix} \sigma_{45} & 0 & \sigma_{45} \\ 0 & 0 & 0 \\ \sigma_{45} & 0 & \sigma_{45} \end{bmatrix} \end{aligned} \quad (12.91)$$

Substituting these values in the anisotropic yield criterion of Equation 12.84,

$$c |\sigma_{45} / 2 - \sigma_{45}|^m + h |\sigma_{45} / 2|^m + 2N \left(|\sigma_{45}|^m / 2 \right) - \sigma_0^m = 0, \quad (12.92)$$

or

$$c (\sigma_{45} / 2)^m + (1 - c) (\sigma_{45} / 2)^m + 2N \left(|\sigma_{45}|^m / 2 \right) - \sigma_0^m = 0.$$

This provides the following expression for the material parameter N :

$$N = \frac{\sigma_0^m - (\sigma_{45}/2)^m}{2(\sigma_{45}/2)^m}. \quad (12.93)$$

Finally, the parameter m is evaluated from the bi-axial test in the x - y plane. Equation 12.84 provides

$$c|\sigma_b + \sigma_b|^m - \sigma_0^m = 0. \quad (12.94)$$

Substituting the value of c from Equation 12.89,

$$\frac{1}{2(\bar{r} + 1)} 2^m \sigma_b^m = \sigma_0^m. \quad (12.95)$$

The above expression can be used for finding the value of m .

Dixit and Dixit (1997) have employed this criterion in the analysis of anisotropic plane strain rolling. However, the criterion could not be popular as most of the sheet metal forming problems are solved by plane stress assumption.

12.7 CONSTITUTIVE RELATIONS FOR ANISOTROPIC MATERIALS

Constitutive relation for anisotropic material is also derived from the associated flow rule. If the yield function is expressed by $f = 0$, then

$$d\varepsilon_{ij}^p = d\lambda \frac{\partial f}{\partial \sigma_{ij}}; \quad \dot{\varepsilon}_{ij}^p = \dot{\lambda} \frac{\partial f}{\partial \sigma_{ij}}. \quad (12.96)$$

The question arises of how $d\lambda$ or $d\lambda/dt$ should be determined. One way is to express the stress in terms of incremental strain or strain rate in the yield criterion. From that, $d\lambda$ or $d\lambda/dt$ can be expressed in terms of incremental strain or strain rate. For example, consider the plane strain yield function

$$f \equiv c|\sigma_{xx} + \sigma_{yy} - 2\sigma_{zz}|^m + h|\sigma_{xx} - \sigma_{yy}|^m + N\left(|\sigma_{xz}|^m + |\sigma_{zx}|^m\right) - \sigma_0^m = 0. \quad (12.97)$$

Assuming $\sigma_{xx} > \sigma_{yy}$ and $\sigma_{xx} + \sigma_{yy} > 2\sigma_{zz}$ and applying the associated flow rule,

$$\dot{\varepsilon}_{xx}^p = \dot{\lambda} \left[cm(\sigma_{xx} + \sigma_{yy} - 2\sigma_{zz})^{m-1} + hm(\sigma_{xx} - \sigma_{yy})^{m-1} \right], \quad (12.98a)$$

$$\dot{\varepsilon}_{yy}^p = \dot{\lambda} \left[cm(\sigma_{xx} + \sigma_{yy} - 2\sigma_{zz})^{m-1} - hm(\sigma_{xx} - \sigma_{yy})^{m-1} \right], \quad (12.98b)$$

$$\dot{\varepsilon}_{zz}^p = \dot{\lambda} \left[-2cm(\sigma_{xx} + \sigma_{yy} - 2\sigma_{zz})^{m-1} \right], \quad (12.98c)$$

$$\dot{\epsilon}_{xz}^p = \dot{\lambda} \left[(sign \sigma_{xz}) Nm |\sigma_{xz}|^{m-1} \right]. \quad (12.98d)$$

These relations need to be expressed only in terms of σ_{xx} , σ_{zz} and σ_{xz} . To eliminate σ_{yy} from these relations, the plane strain condition $\dot{\epsilon}_{yy}^p = 0$ is used. Substitution of this condition in Equation 12.98b leads to

$$\sigma_{yy} = \frac{1-d}{1+d} \sigma_{xx} + \frac{2d}{1+d} \sigma_{zz}, \quad (12.99)$$

where

$$d = \left(\frac{c}{h} \right)^{1/(m-1)}. \quad (12.100)$$

Now, Equation 12.98 may be written as

$$\dot{\epsilon}_{xx}^p = 2\dot{\lambda} cm \left(\frac{2}{1+d} \right)^{m-1} (\sigma_{xx} - \sigma_{zz})^{m-1}, \quad (12.101a)$$

$$\dot{\epsilon}_{zz}^p = -2\dot{\lambda}_{cm} \left(\frac{2}{1+d} \right)^{m-1} (\sigma_{xx} - \sigma_{zz})^{m-1}, \quad (12.101b)$$

$$\dot{\epsilon}_{xz}^p = \dot{\lambda} \left[(sign \sigma_{xz}) Nm |\sigma_{xz}|^{m-1} \right]. \quad (12.101c)$$

Because of volume constancy, the x and z components of strain rate are equal. Using this fact and Equation 12.101, the yield function can be expressed as

$$c \left(\frac{|\dot{\epsilon}_{xx}^p|}{2cm\dot{\lambda}} \right)^{m/(m-1)} + h \left(\frac{|\dot{\epsilon}_{xx}^p|}{2hm\dot{\lambda}} \right)^{m/(m-1)} + 2N \left(\frac{|\dot{\epsilon}_{xz}^p|}{Nm\dot{\lambda}} \right)^{m/(m-1)} - \sigma_0^m = 0 \quad (12.102)$$

This provides

$$\dot{\lambda} = \frac{\left[\left(\frac{1}{c^{1/(m-1)}} + \frac{1}{h^{1/(m-1)}} \right) \left(\frac{|\dot{\epsilon}_{xx}^p|}{2m} \right)^{m/(m-1)} + \frac{2}{N^{1/(m-1)}} \left(\frac{|\dot{\epsilon}_{xz}^p|}{m} \right)^{m/(m-1)} \right]^{(m-1)/m}}{\sigma_0^{m-1}}. \quad (12.103)$$

This provides $d\lambda/dt$ in terms of plastic strain-rate components. If strain hardening is to be considered, then σ_0 can be expressed using some hardening law, which

expresses it in terms of some equivalent plastic strain. It is clear that constitutive relation is non-linear and the problem needs to be solved iteratively. Initially, some value of $d\lambda/dt$ can be assumed, and the problem can be solved to get the plastic strain component. The value of $d\lambda/dt$ can then be updated, and the problem can be repeated.

For obtaining the complete elastic-plastic strain rate relation, it can be assumed that plastic and elastic strain rates are additive. Therefore, elastic and plastic constitutive relations can be combined in an additive manner.

Example 12.4

Consider the plane stress yield criterion given by

$$c_1 (\sigma_{xx} + \sigma_{yy})^2 + (1 - c_1) (\sigma_{xx} - \sigma_{yy})^2 = \sigma_0^2,$$

where σ_0 is the yield stress along any direction in the plane of the sheet assuming planar isotropy, and c_1 is the material parameter. Obtain the incremental plastic strain and stress relation.

SOLUTION

Using the associated flow rule,

$$\begin{aligned} d\varepsilon_{xx}^p &= d\lambda \left\{ 2c_1 (\sigma_{xx} + \sigma_{yy}) + (2 - 2c_2) (\sigma_{xx} - \sigma_{yy}) \right\} \\ d\varepsilon_{yy}^p &= d\lambda \left\{ 2c_1 (\sigma_{xx} + \sigma_{yy}) - (2 - 2c_2) (\sigma_{xx} - \sigma_{yy}) \right\}. \end{aligned} \quad (i)$$

Solving these simultaneous equations,

$$\sigma_{xx} + \sigma_{yy} = \frac{(d\varepsilon_{xx}^p + d\varepsilon_{yy}^p)}{4c_1 d\lambda}; \quad \sigma_{xx} - \sigma_{yy} = \frac{(d\varepsilon_{xx}^p - d\varepsilon_{yy}^p)}{4(1 - c_1) d\lambda}. \quad (ii)$$

Substituting these relations in the yield criterion,

$$\frac{(d\varepsilon_{xx}^p + d\varepsilon_{yy}^p)^2}{16c_1 (d\lambda)^2} + \frac{(d\varepsilon_{xx}^p - d\varepsilon_{yy}^p)^2}{16(1 - c_1) (d\lambda)^2} = \sigma_0^2. \quad (iii)$$

Hence,

$$d\lambda = \left\{ \frac{(d\varepsilon_{xx}^p + d\varepsilon_{yy}^p)^2}{16c_1 \sigma_0^2} + \frac{(d\varepsilon_{xx}^p - d\varepsilon_{yy}^p)^2}{16(1 - c_1) \sigma_0^2} \right\}^{1/2}. \quad (iv)$$

Equation (iv) can be substituted in Equation (i) to obtain the relation between the incremental plastic strain and stress relation.

12.8 KINEMATIC HARDENING

Isotropic hardening models do not take into account the Bauschinger effect. Prager (1955) was the first to model the Bauschinger effect by employing a rigid translation of the initial yield surface without incorporating either the change in size or shape of the yield locus. He used the phrase ‘kinematic hardening’ for this type of hardening behavior.

Assume that the initial yield criterion is given by

$$f(\sigma_{ij}) = 0. \quad (12.104)$$

After a certain amount of yielding, the yield criterion becomes

$$f(\sigma_{ij} - d\alpha_{ij}) = 0, \quad (12.105)$$

where $d\alpha_{ij}$ is called the incremental back stress and represents the incremental translation of the yield surface. For materials following the von Mises criterion, the kinematic hardening rule can be written as

$$\frac{3}{2}(\sigma'_{ij} - d\alpha_{ij})(\sigma'_{ij} - d\alpha_{ij}) = \sigma_Y^2, \quad (12.106)$$

where σ_Y is the yield stress in uni-axial tension.

Prager assumed that incremental translation of the yield surface takes place along the direction of the plastic part of the incremental linear strain tensor. Thus,

$$d\alpha_{ij} = cd\varepsilon_{ij}^p. \quad (12.107)$$

In the case of linear kinematic hardening, c is constant. If c is a function of deformation history, the kinematic hardening becomes non-linear. For materials initially obeying the von Mises criterion,

$$\frac{3}{2}(\sigma'_{ij} - cd\varepsilon_{ij}^p)(\sigma'_{ij} - cd\varepsilon_{ij}^p) = \sigma_Y^2. \quad (12.108)$$

For a uni-axial state of stress,

$$\sigma'_{xx} = \frac{2}{3}\sigma_{xx}, \sigma'_{yy} = -\frac{1}{3}\sigma_{xx}, \sigma'_{zz} = -\frac{1}{3}\sigma_{xx}, d\varepsilon_{yy}^p = d\varepsilon_{zz}^p = -\frac{1}{2}d\varepsilon_{xx}^p \quad (12.109)$$

with the other stress and incremental strain components being zero. Hence, for the uni-axial state of stress,

$$\begin{aligned} & \frac{3}{2} \left\{ \left(\frac{2}{3}\sigma_{xx} - cd\varepsilon_{xx}^p \right) \left(\frac{2}{3}\sigma_{xx} - cd\varepsilon_{xx}^p \right) \right. \\ & \left. + 2 \left(-\frac{1}{3}\sigma_{xx} + \frac{c}{2}d\varepsilon_{xx}^p \right) \left(-\frac{1}{3}\sigma_{xx} + \frac{c}{2}d\varepsilon_{xx}^p \right) \right\} = \sigma_Y^2, \end{aligned} \quad (12.110)$$

$$9 \left(\frac{1}{3} \sigma_{xx} - \frac{c}{2} d\varepsilon_{xx}^p \right)^2 = \sigma_{y'}^2, \quad (12.111)$$

or

$$\frac{1}{3} \sigma_{xx} - \frac{c}{2} d\varepsilon_{xx}^p = \frac{\sigma_Y}{3}. \quad (12.112)$$

This provides

$$d\sigma_{xx} = \sigma_{xx} - \sigma_Y = \frac{3c}{2} d\varepsilon_{xx}^p. \quad (12.113)$$

Hence,

$$c = \frac{2}{3} \frac{d\sigma_{xx}}{d\varepsilon_{xx}^p}. \quad (12.114)$$

Thus, as per Prager's model, the material parameter c is equal to $2/3$ times the slope of the uni-axial stress-plastic strain curve. However, Prager's model shows that in the x -direction, the yield locus moves in the positive direction of the incremental longitudinal plastic strain, but in the transverse direction, it moves opposite to the incremental longitudinal plastic strain. This is not supported by the experiments.

Ziegler (1959) proposed a certain modification in Prager's model. The modification can be represented by the following mathematical expression:

$$d\alpha_{ij} = (\sigma_{ij} - \alpha_{ij}) d\mu, \quad (12.115)$$

where $d\mu$ is a material parameter representing the kinematic hardening. Employing the condition that stress point remains on the yield surface,

$$(d\sigma_{ij} - d\alpha_{ij}) d\varepsilon_{ij}^p = 0. \quad (12.116)$$

This provides

$$d\mu = \frac{d\sigma_{ij} d\varepsilon_{ij}^p}{(\sigma_{kl} - \alpha_{kl}) d\varepsilon_{kl}^p}. \quad (12.117)$$

The following non-linear modifications have been proposed to Prager's and Ziegler's models (Rees, 2006):

$$d\alpha_{ij} = c_1 d\varepsilon_{ij}^p - \frac{c_2}{\sigma_Y} d\varepsilon_{eq}^p \alpha_{ij}, \quad (12.118)$$

and

$$d\alpha_{ij} = \frac{c_1}{\sigma_Y} d\varepsilon_{eq}^p (\sigma_{ij} - \alpha_{ij}) - \frac{c_2}{\sigma_Y} d\varepsilon_{eq}^p \alpha_{ij} \quad (12.119)$$

Here, σ_Y is the yield stress, $d\varepsilon_{eq}^p$ is the equivalent plastic strain increment and c_1 and c_2 are functions of the integrals of the second and third invariants of $d\varepsilon_{ij}^p$.

Exercises

1. A sheet metal specimen of gauge length l_0 is subjected to longitudinal tensile load to deform it plastically. After deformation, the gauge length becomes l . The initial width and thickness of the specimen are w_0 and t_0 , respectively, whereas the final width and thickness are l and t , respectively. Prove that the strain ratio (also called the anisotropy coefficient) is given by

$$r = \frac{\ln(w / w_0)}{\ln(l_0 w_0 / lw)}.$$

2. Consider Hill's 1948 yield criterion given by

$$F(\sigma_{yy} - \sigma_{zz})^2 + G(\sigma_{zz} - \sigma_{xx})^2 + H(\sigma_{xx} - \sigma_{yy})^2 + 2L\sigma_{yz}^2 + 2M\sigma_{zx}^2 + 2N\sigma_{xy}^2 = 1.$$

Show that for any material, only one of the parameters out of F , G and H can be negative.

3. Consider Hill's 1948 yield criterion given by

$$F(\sigma_{yy} - \sigma_{zz})^2 + G(\sigma_{zz} - \sigma_{xx})^2 + H(\sigma_{xx} - \sigma_{yy})^2 + 2L\sigma_{yz}^2 + 2M\sigma_{zx}^2 + 2N\sigma_{xy}^2 = 1.$$

Show that strain ratios with uni-axial stresses applied along the x -axis and y -axis, respectively, are given by

$$r_0 = \frac{H}{G}, \quad r_{90} = \frac{H}{F}.$$

4. Show that Hill's 1948 yield criterion provides

$$\frac{\sigma_0}{\sigma_{90}} = \sqrt{\frac{r_0(1+r_{90})}{r_{90}(1+r_0)}}.$$

5. Prove that triaxial loading $(\sigma_1, \sigma_2, \sigma_3)$ is plastically equivalent to plane stress loading $(\sigma_1 - \sigma_3, \sigma_2 - \sigma_3)$.
6. Consider the yield criterion given by

$$\sigma_1^2 - \left(2 - \frac{\sigma_u^2}{\sigma_b^2}\right) \sigma_1 \sigma_2 + \sigma_2^2 + \left\{p + q - \frac{(p\sigma_1 + q\sigma_2)}{\sigma_b}\right\} \sigma_1 \sigma_2 = \sigma_u^2,$$

where σ_u is the uni-axial yield stress, and σ_b is the balanced bi-axial yield stress with the following condition:

$$\frac{c}{\sigma_0 \sigma_{90}} = \frac{1}{\sigma_0^2} + \frac{1}{\sigma_{90}^2} - \frac{1}{\sigma_b^2}.$$

Show that the criterion is satisfied by the uni-axial and equibiaxial load.

7. Show that for the criterion given in Equation 12.34, that is for Hill's 1993 criterion,

$$\begin{aligned} \frac{p\sigma_u}{\sigma_b} - (p + q) &= \frac{2r_0}{1 + r_0} - \left(2 - \frac{\sigma_u^2}{\sigma_b^2}\right), \\ \frac{q\sigma_u}{\sigma_b} - (p + q) &= \frac{2r_{90}}{1 + r_{90}} - \left(2 - \frac{\sigma_u^2}{\sigma_b^2}\right), \end{aligned}$$

where $\sigma_0 = \sigma_{90} = \sigma_u$

8. Consider Hill's 1979 criterion given by

$$\begin{aligned} F(\sigma_{ij}) &\equiv f|\sigma_2 - \sigma_3|^m + g|\sigma_3 - \sigma_1|^m + h|\sigma_1 - \sigma_2|^m + a|2\sigma_1 - \sigma_2 - \sigma_3|^m \\ &\quad + b|2\sigma_2 - \sigma_3 - \sigma_1|^m + c|2\sigma_3 - \sigma_1 - \sigma_2|^m - \sigma_0^m = 0. \end{aligned}$$

For this criterion, show that

$$r = \frac{(2^{m-1} + 2)a - c + h}{(2^{m-1} - 1)a + 2c + f}.$$

9. Hosford's isotropic yield criterion is given by

$$|\sigma_2 - \sigma_3|^m + |\sigma_3 - \sigma_1|^m + |\sigma_1 - \sigma_2|^m = 2\sigma_Y^m.$$

Show that it reduces to the Tresca criterion for $m = 1$.

13 Thermal Considerations in Plasticity*

13.1 INTRODUCTION

The essence of carrying out an elastic-plastic analysis is to study the cause-and-effect phenomenon between strain and stress. In the previous chapters, governing equations of elasticity and plasticity have been confined to the key assumptions that the strain is caused by external forces and the process is isothermal, that is the material properties are temperature-independent. In many applications and processes, the deformation domain may be subjected to extreme temperatures and fluctuations with or without external forces. Take the example of a structural engineer who is interested in designing a steel structure that can withstand sufficiently high temperatures for a prolonged duration without failure in the event of a fire. Here the core part of the design is the elastic-plastic analysis of the structure with a special focus on the thermal effects through which a factor of safety can be confidently assigned. Another example is the hot rolling that is carried out above the recrystallization temperature. Here, the prediction of the roll forces and the stresses generated in the material depends on a sound thermomechanical analysis. Steels, subjected to thermal processing such as heat treatment undergo a continuous cycle of heating and cooling during which microstructural, thermal and mechanical transformations take place at different length scales. Thus, depending on the range of the temperature, thermal effects on material behavior comprise a complex interaction of microstructural, mechanical and thermal processes as depicted in Figure 13.1 for a typical steel. Some of the interactions are as follows:

- *Thermo-mechanical.* When steel is heated, thermal strains are generated. Conversely, when steel is subjected to plastic deformation, heat is generated. In addition, an increase in temperature alters the mechanical properties
- *Thermo-microstructural.* When steel is heated above a certain threshold temperature, generally the lower or upper critical temperatures, phase transformation takes place. Conversely, phase changes release latent heat that increases the temperature. Further, the thermal properties vary according to the phase fractions
- *Microstructural-mechanical.* When steel is subjected to plastic deformation, microstructural changes take place in the form of grain refinement. The plastic deformation, which also drives the movement and interaction

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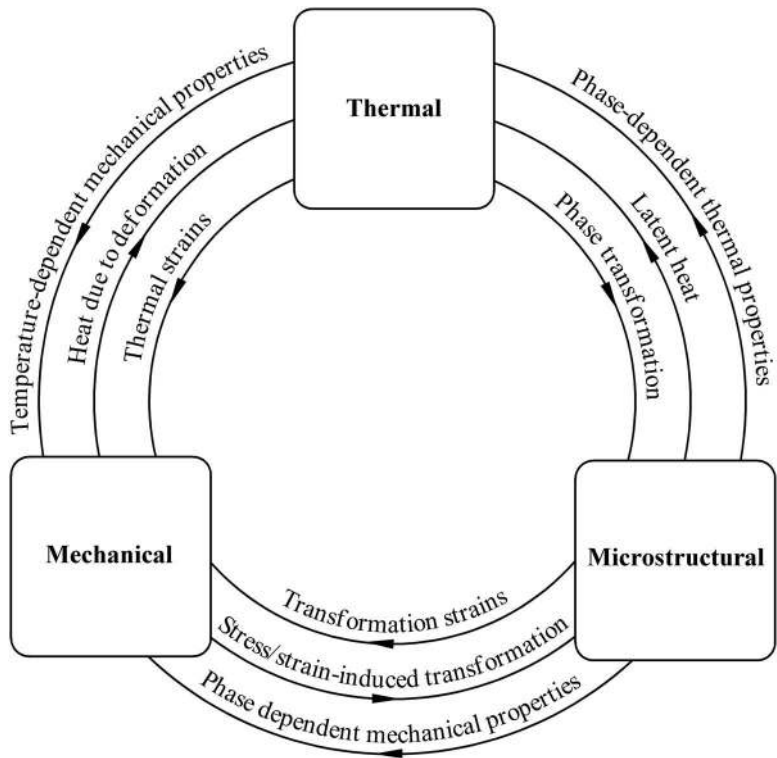


FIGURE 13.1 Interaction of various physical mechanisms during the thermal processing of steel.

of dislocations in the steel, causes strain hardening of the steel. Both these phenomena are often utilized as a strengthening mechanism. On the other hand, phase transformation causes different phases to contract and expand at different rates. This causes dilatational strain due to changes in volume and shape along with transformation-induced plasticity (TRIP). Thus, the mechanical strains induce the phase transformation and vice versa. Additionally, the phase transformation alters the mechanical properties

For temperatures well below the recrystallization temperature, only the thermo-mechanical interaction is significant. In general, the plastic behavior of steel involving microstructural transformations can be classified into two types—(1) classical plasticity, in which the plastic flow arises due to applied external forces or the temperature, and (2) transformation-induced plasticity (TRIP), where the plastic flow occurs due to phase transformation.

Formulations of elastic-plastic, thermal and microstructural analysis of steel subjected to thermal processing are very difficult to solve by using a conventional approach. Often, engineers and physicists take recourse to numerical techniques like

finite difference method (FDM) and finite element method (FEM). This chapter discusses the governing equations for thermo-mechanical-microstructural analysis of steel subjected to thermal processing. The organization of the chapter is as follows. Section 13.2 presents a brief introduction to FEM. Section 13.3 presents the governing equations for the multi-physics of thermal, mechanical and microstructural interactions. Section 13.4 presents an example of thermal analysis of hot rolling. Section 13.5 presents an example of thermal analysis of laser forming. Section 13.6 presents an example of inverse analysis of laser forming. Section 13.7 presents an analysis of heat treatment of steel coupled with thermal autofrettage. Section 13.8 concludes the chapter.

13.2 A BRIEF INTRODUCTION OF FINITE ELEMENT METHOD (FEM)

In general, a deformation domain can be considered as either continuous or discrete. A continuous system can be discretized into a number of small elements. The ideal spring-mass system offers a classic example of a discrete system, in which each mass is considered a particle (or node); hypothetically, spring is considered massless and represents force only as a function of displacement. A system may have several springs and several masses (nodes). The displacements of the nodes are functions of stiffness values and forces. If the springs are linear, the formulation results in a system of linear equations that can be solved algebraically. On the other hand, a continuous system is one having an infinite number of degrees of freedom. A continuous system is modeled by linear or non-linear differential equations, which are solved by applying certain boundary conditions. In comparison, a discrete system can be modeled by algebraic equations, which are easier to solve. However, a discrete system with a limited number of nodes is only an approximation of a continuous system. It is expected that as the number of nodes in the approximation increase, the results from the discrete system will be closer to those from the continuous system. However, in certain cases, even lesser number of nodes can provide same results as obtained by solving differential equations of the continuous system.

The finite element method (FEM) is a technique where a continuous system is converted to a discrete system by discretizing the continuous domain into many parts called elements. For each element, algebraic equations are developed in terms of the unknown values of the variables at specific points of the elements called nodes. In essence, the differential equations are converted into a system of algebraic equations. A global system of algebraic equations is constructed by assembling algebraic elemental equations and then solved after applying the boundary conditions. The solution is obtained in the form of nodal values, but post-processing can convert it into a continuous solution (using interpolation or extrapolation).

FEM is applied in many fields of engineering, science and mathematics. It is essentially a general numerical technique of solving differential equations. It comprises the following steps—preprocessing, developing elemental equations, assembling the elemental equations, applying the boundary conditions, solving the equations and post-processing. The steps of FEM are explained in the following subsections.

13.2.1 PREPROCESSING

In this step, the continuous domain is discretized into finite elements. Elements vary in form and size. The accuracy of the results increases with the number of elements but at the cost of computational time and resources. Thus, the size of each element or the number of elements is usually decided by a mesh sensitivity analysis. Some commonly used elements are shown in Figures 13.2–13.4. In each element, the variable is expressed as a known function (generally polynomial) of the unknown values of the variables at the nodes. A condition of continuity is invoked at the junction nodes of adjacent elements. The known values at some specific nodes are called essential boundary conditions.

Consider a 2-node line element shown in Figure 13.5. To predict the displacement field in the element, let the approximating polynomial for displacement be

$$u = ax + b, \tag{13.1}$$



FIGURE 13.2 One-dimensional (a) 2-node and (b) 3-node line element.

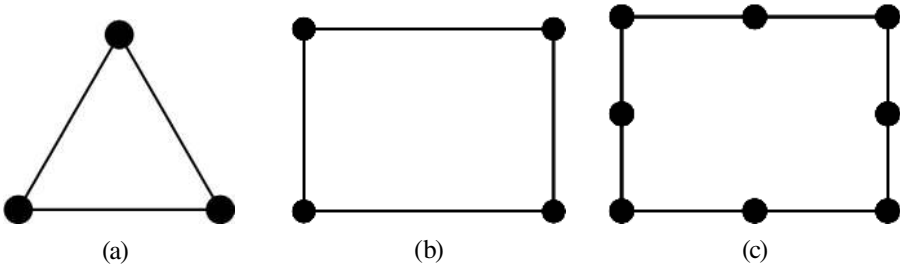


FIGURE 13.3 Two-dimensional (a) 3-node triangular, (b) 4-node rectangular and (c) 8-node rectangular element.

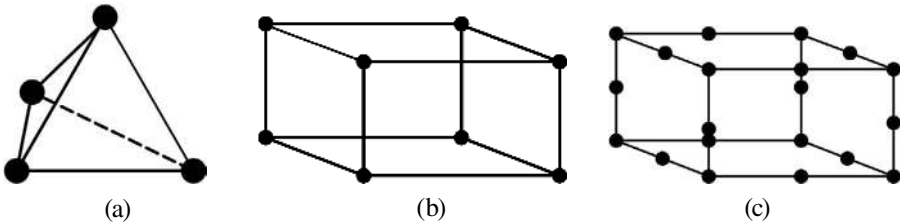


FIGURE 13.4 Three-dimensional (a) 4-node tetrahedral (b) 8-node brick and (c) 20-node brick element.

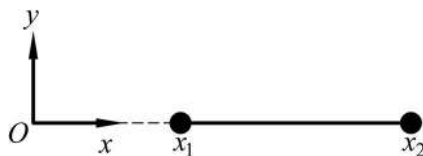


FIGURE 13.5 A 2-node line element with spatial nodal coordinates.

where x is the spatial coordinate and a and b are the element-specific constants. If the displacement at x_1 and x_2 are known, then substituting the nodal values, $u(x_1) = u_1$ and $u(x_2) = u_2$ in Equation 13.1 provides the following system of equations:

$$\begin{aligned} u_1 &= ax_1 + b, \\ u_2 &= ax_2 + b. \end{aligned} \quad (13.2)$$

The solution provides

$$a = \frac{u_1 - u_2}{x_1 - x_2} \quad \text{and} \quad b = \frac{u_2 x_1 - u_1 x_2}{x_1 - x_2}.$$

Substituting the expressions of a and b in Equation 13.1 and rearranging provides

$$u = N_1 u_1 + N_2 u_2, \quad (13.3)$$

where

$$N_1 = \left(\frac{x - x_2}{x_1 - x_2} \right) \quad \text{and} \quad N_2 = \left(\frac{x - x_1}{x_2 - x_1} \right) \quad (13.4)$$

are called shape functions because they convey an idea of the shape of the approximating function. The expressions of the shape functions are called Lagrangian polynomial interpolation functions. These polynomial expressions ensure the automatic continuity of the function at each junction node of adjacent elements. Similarly, for a 3-node line element, the approximating function for u can be derived and expressed as follows:

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3, \quad (13.5)$$

where

$$N_1 = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}, \quad N_2 = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} \quad \text{and} \quad N_3 = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}. \quad (13.6)$$

The Lagrangian shape functions possess the following characteristics:

- The degree of the polynomial representing the shape function is one less than the number of nodes in the element
- The sum of all the shape functions is always one
- The value of a shape function at the assigned node is one, while other shape functions are zero at that node

13.2.2 DEVELOPING ELEMENTAL EQUATIONS

There are various ways of obtaining the elemental equations. Some common approaches are the direct stiffness method, method of virtual work, principle of minimum energy and Galerkin's method. For the sake of brevity, only the direct stiffness method and the Galerkin's method are explained in this chapter.

13.2.2.1 Direct Stiffness Method

The direct stiffness method is suitable for deriving the elemental equations for one-dimensional elements like bars and beams. Figure 13.6 shows a bar subjected to uni-axial force F . The bar is discretized into three equal elements as shown in Figure 13.7(a). The free body diagram for an element with internal forces F_i and F_j acting at the end nodes i and j , respectively, is shown in Figure 13.7(b).

The compressive strain ε_i at the i^{th} node is

$$\varepsilon_i = \frac{du}{dx} = \frac{u_i - u_j}{l}, \quad (13.7)$$

where l is the length of the element. The corresponding compressive stress σ_i is

$$\sigma_i = E\varepsilon_i, \quad (13.8)$$

Hence, the compressive force F_i is

$$F_i = \sigma_i A, \quad (13.9)$$

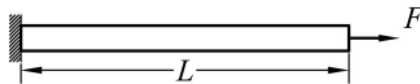


FIGURE 13.6 Axially loaded bar.

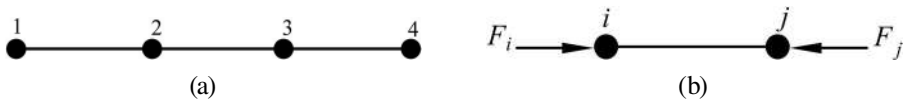


FIGURE 13.7 Pre-processing step: (a) discretization of the bar into three elements and (b) free body diagram of a typical 2-node element.

where A is the cross-sectional area of the element. Substituting Equations 13.7 and 13.8 in Equation 13.9 provides

$$F_i = \frac{EA}{l}(u_i - u_j). \quad (13.10)$$

Similarly, the nodal tensile force F_j can be expressed as

$$F_j = \frac{EA}{l}(u_j - u_i). \quad (13.11)$$

Equations (13.10) and (13.11) can be arranged in the following form:

$$\frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} F_i \\ F_j \end{Bmatrix}. \quad (13.12)$$

The coefficient matrix of Equation 13.12 is called the elemental coefficient matrix and the right-hand side vector is called the elemental right-hand side vector. In solid mechanics, the elemental coefficient matrix represents the stiffness of the system and therefore it is called the elemental stiffness matrix. Further, the elemental right-hand side vector represents the nodal internal loads in the system and therefore it is called the elemental internal load vector.

13.2.2.2 Galerkin Method

The Galerkin method belongs to a general class of method for developing elemental equation called the weighted residual method. First, the weighted residual method is explained followed by the special case of Galerkin. In the weighted residual method, differential equations of a problem are solved approximately using an assumed solution called the trial solution. Consider a differential equation

$$D\phi_f + q_f = 0, \quad (13.13)$$

where D is a differential operator, ϕ_f is the primary variable function to be determined and q_f is a known function. If a trial function $\bar{\phi}_f$ is chosen as the approximate solution, then Equation 13.13 becomes

$$D\bar{\phi}_f + q_f = R, \quad (13.14)$$

where R is called residual. It is zero if the trial function is the same as the exact solution.

In addition to satisfying the boundary conditions, the trial function $\bar{\phi}_f$ must also be differentiable up to the order of the differential equation. Thus, its requirement for differentiability is strong and Equation 13.14 is called strong form. The skill is to choose a trial function such that it minimizes the weighted residual. Depending on the chosen weight function, different types of weighted residual methods are obtained. In collocation method, the residual is set to zero at certain specific points.

In sub-domain method, the integral of residual is made zero at selected intervals. In the least square method, the integral of R^2 is minimized over the domain. The Galerkin method, which is discussed in details, involves setting of a weighted integral of R over the domain to zero. The method is explained in the following paragraph.

The residual is first multiplied by a weight function and the integral is equated to zero. Thus,

$$\int_V wR \, dV = \int_V w(D\bar{\phi}_f + q_f) \, dV = 0, \quad (13.15)$$

where V is the volume of the domain and w is the weight function. Integration by parts reduces the order of derivatives and the trial function with a lesser requirement for differentiability can be selected. This form of the equation with a weaker requirement for differentiability is called weak form. In Galerkin method, the weight function is the linear combination of the shape functions of the trial function. The coefficients of the shape function in the weight function are often called nodal weights and are akin to nodal variables. This procedure yields a system of algebraic equations.

The Galerkin method is illustrated here with a simple example. Consider a uni-axial loaded bar with a load intensity q having a length L and a cross-sectional area A as shown in Figure 13.8. For the sake of the generality, it is assumed that the cross-sectional area A as well as Young's modulus E may vary with coordinate x . The problem is governed by the following differential equation:

$$\frac{d}{dx} \left(EA \frac{du}{dx} \right) + q = 0, \quad (13.16)$$

with boundary conditions

$$u(0) = 0, \quad \left. \frac{du}{dx} \right|_{x=L} = 0. \quad (13.17)$$

The bar is discretized into three 2-node elements each of length $l = L/3$ as shown in Figure 13.9. Here each element has two nodes called Node 1 and Node 2. However, a global numbering scheme is adopted in Figure 13.9, in which each node is associated with a distinct node number. Global node number 3 corresponds to Node 2 of Element 2 as well as Node 1 of Element 3. A similar correspondence exists between other global nodes and local elemental nodes.

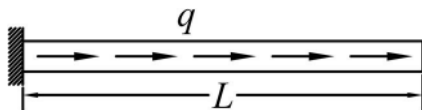


FIGURE 13.8 Uni-axial loaded bar with load intensity q . Cross-sectional area and Young's modulus may be non-uniform.

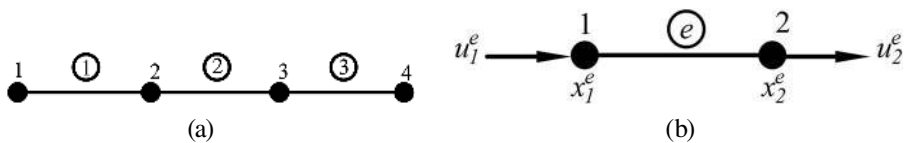


FIGURE 13.9 Discretization: (a) finite element mesh and (b) a typical element e showing local nodes 1 and 2 with their coordinates and nodal displacements.

To derive the elemental equation with local notations for the nodal coordinates as x_1^e and x_2^e , the following procedure is followed. Multiplying the differential equation (Eq. 13.16) by a weight function w and integrating by parts yields

$$\left. \begin{aligned} \int_{x_1^e}^{x_2^e} w \left\{ \frac{d}{dx} \left(EA \frac{du^e}{dx} \right) + q \right\} dx &= 0 \\ \Rightarrow \int_{x_1^e}^{x_2^e} \left(-EA \frac{dw}{dx} \frac{du^e}{dx} + qw \right) dx + wEA \frac{du^e}{dx} \Big|_{x_1^e}^{x_2^e} &= 0 \end{aligned} \right\}, \quad (13.18)$$

where u^e is the displacement function for the element. The linear approximation of the displacement is

$$u_e = \begin{bmatrix} N_1^e & N_2^e \end{bmatrix} \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix}, \quad (13.19)$$

where u_1^e and u_2^e denote the local notations for the nodal displacements (degrees of freedom) and N_1^e and N_2^e are the shape functions defined in terms of the nodal coordinates as follows:

$$\begin{aligned} N_1^e &= \frac{(x - x_2^e)}{(-x_2^e + x_1^e)} = -\frac{(x - x_2^e)}{l}, \\ N_2^e &= \frac{(x - x_1^e)}{(x_2^e - x_1^e)} = \frac{(x - x_1^e)}{l}. \end{aligned} \quad (13.20)$$

From Equation 13.19,

$$\frac{du^e}{dx} = \begin{bmatrix} \frac{dN_1^e}{dx} & \frac{dN_2^e}{dx} \end{bmatrix} \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \begin{bmatrix} N_{1,x}^e & N_{2,x}^e \end{bmatrix} \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix}. \quad (13.21)$$

In the Galerkin method, the weight function is of the same form as the approximating function and is taken as a linear combination of the shape functions. It is expressed as

$$w = \begin{bmatrix} w_1^e & w_2^e \end{bmatrix} \begin{Bmatrix} N_1^e \\ N_2^e \end{Bmatrix}, \quad (13.22)$$

where w_1^e and w_2^e denote the local notation for the arbitrary nodal weights. Differentiation of Equation 13.20 gives

$$\begin{aligned} \frac{dN_1^e}{dx} &= \frac{-1}{l}, \\ \frac{dN_2^e}{dx} &= \frac{1}{l}. \end{aligned} \quad (13.23)$$

Substitution of Equations 13.21–13.22 in Equation 13.18 provides

$$\begin{aligned} - \int_{x_1^e}^{x_2^e} \left(\begin{bmatrix} w_1^e & w_2^e \end{bmatrix} EA \begin{Bmatrix} N_{1,x}^e \\ N_{2,x}^e \end{Bmatrix} \begin{bmatrix} N_{1,x}^e & N_{2,x}^e \end{bmatrix} \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} \right) dx + \int_{x_1^e}^{x_2^e} \left(q \begin{bmatrix} w_1^e & w_2^e \end{bmatrix} \begin{Bmatrix} N_1^e \\ N_2^e \end{Bmatrix} \right) dx \\ + \begin{bmatrix} w_1^e & w_2^e \end{bmatrix} \begin{Bmatrix} -E_1 A_1 \left(\frac{du^e}{dx} \right)_1 \\ E_2 A_2 \left(\frac{du^e}{dx} \right)_2 \end{Bmatrix} = 0, \end{aligned} \quad (13.24)$$

where the values of the first derivative of the displacement functions are $(du^e/dx)_1$ at the local Node 1 and $(du^e/dx)_2$ at the local Node 2. The values of EA at Node 1 and Node 2 are $E_1 A_1$ and $E_2 A_2$, respectively. Since the weights are arbitrary, Equation 13.24 reduces to

$$\int_{x_1^e}^{x_2^e} \left(EA \begin{Bmatrix} N_{1,x}^e \\ N_{2,x}^e \end{Bmatrix} \begin{bmatrix} N_{1,x}^e & N_{2,x}^e \end{bmatrix} \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} \right) dx = \int_{x_1^e}^{x_2^e} \left(q \begin{Bmatrix} N_1^e \\ N_2^e \end{Bmatrix} \right) dx + \begin{Bmatrix} -E_1 A_1 \left(\frac{du^e}{dx} \right)_1 \\ E_2 A_2 \left(\frac{du^e}{dx} \right)_2 \end{Bmatrix}. \quad (13.25)$$

Substitution of Equation 13.20 and 13.23 in Equation 13.25 provides

$$\int_{x_1^e}^{x_2^e} EA \begin{Bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{Bmatrix} \begin{bmatrix} -1 & 1 \\ l & l \end{bmatrix} \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} dx = \int_{x_1^e}^{x_2^e} q \begin{Bmatrix} -\frac{(x-x_2^e)}{l} \\ \frac{(x-x_1^e)}{l} \end{Bmatrix} dx + \begin{Bmatrix} -E_1 A_1 \left(\frac{du^e}{dx} \right)_1 \\ E_2 A_2 \left(\frac{du^e}{dx} \right)_2 \end{Bmatrix}. \quad (13.26)$$

Equation (13.26) gets simplified to

$$\int_{x_1^e}^{x_2^e} \left(\frac{EA}{l^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} \right) dx = \frac{q}{l} \begin{Bmatrix} -\frac{(x-x_2^e)^2}{2} \bigg|_{x_1^e}^{x_2^e} \\ \frac{(x-x_1^e)^2}{2} \bigg|_{x_1^e}^{x_2^e} \end{Bmatrix} + \begin{Bmatrix} -E_1 A_1 \left(\frac{du^e}{dx} \right)_1 \\ E_2 A_2 \left(\frac{du^e}{dx} \right)_2 \end{Bmatrix}. \quad (13.27)$$

For a constant EA , Equation 13.27 reduces to

$$\frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \begin{Bmatrix} \frac{ql}{2} \\ \frac{ql}{2} \end{Bmatrix} + \begin{Bmatrix} -EA \left(\frac{du^e}{dx} \right)_1 \\ EA \left(\frac{du^e}{dx} \right)_2 \end{Bmatrix}. \quad (13.28)$$

Equation 13.28 is the final elemental equation. The next step is to assemble the elemental equations of all the elements.

13.2.3 ASSEMBLING THE ELEMENTAL EQUATIONS

The global equation is the equation of the entire domain. The size of the global stiffness matrix is equal to the total number of degrees of freedom at all the nodes. For example, if a bar comprises three 2-node elements with one degree of freedom per node, the order of the global stiffness matrix is 4×4 . The relationship between the global node number and local node number of an element is determined using the connectivity matrix where the columns represent the element number and the rows represent the global node numbers. The connectivity matrix of the example from Section 13.2.2 with the mesh shown in Figure 13.9 is given in Figure 13.10.

	Local node 1	Local node 2
Element 1	1	2
Element 2	2	3
Element 3	3	4

FIGURE 13.10 A typical connectivity matrix.

For four nodes having one degree of freedom per node, the order of the global stiffness matrix is 4×4 . With the help of the connective matrix, the elemental equations with the global node locations for each element can be expressed as follows (assuming constant EA for ease of illustration):

$$\text{Element } e=1: u_1^e = u_1, u_2^e = u_2, \left(\frac{du^e}{dx} \right)_1 = \left(\frac{du}{dx} \right)_1, \left(\frac{du^e}{dx} \right)_2 = \left(\frac{du}{dx} \right)_2,$$

$$\frac{EA}{l} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} \frac{ql}{2} \\ \frac{ql}{2} \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} -EA \left(\frac{du}{dx} \right)_1 \\ EA \left(\frac{du}{dx} \right)_2 \\ 0 \\ 0 \end{Bmatrix}, \quad (13.29)$$

$$\text{Element } e=2: u_1^e = u_2, u_2^e = u_3, \left(\frac{du^e}{dx} \right)_1 = \left(\frac{du}{dx} \right)_2, \left(\frac{du^e}{dx} \right)_2 = \left(\frac{du}{dx} \right)_3,$$

$$\frac{EA}{l} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{ql}{2} \\ \frac{ql}{2} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -EA \left(\frac{du}{dx} \right)_2 \\ EA \left(\frac{du}{dx} \right)_3 \\ 0 \end{Bmatrix}, \quad (13.30)$$

$$\text{Element } e=3: u_1^e = u_3, u_2^e = u_4, \left(\frac{du^e}{dx} \right)_1 = \left(\frac{du}{dx} \right)_3, \left(\frac{du^e}{dx} \right)_2 = \left(\frac{du}{dx} \right)_4,$$

$$\frac{EA}{l} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \frac{ql}{2} \\ \frac{ql}{2} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -EA \left(\frac{du}{dx} \right)_3 \\ EA \left(\frac{du}{dx} \right)_4 \end{Bmatrix}. \quad (13.31)$$

It is to be mentioned that while mapping of elemental nodal displacement to global nodal displacement is exact, the same cannot be said about derivatives because of its discontinuous nature. Thus, for example, for Element 1, $u_1^e = u_1$, but $\left(\frac{du^e}{dx} \right)_1 \approx \left(\frac{du}{dx} \right)_1$ and similarly for other nodes. Adding Equations 13.29–13.31, after accepting this approximation, provides the following system of global equations:

$$\frac{EA}{l} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} \frac{ql}{2} \\ ql \\ ql \\ \frac{ql}{2} \end{Bmatrix} + \begin{Bmatrix} -EA \left(\frac{du}{dx} \right)_1 \\ 0 \\ 0 \\ EA \left(\frac{du}{dx} \right)_4 \end{Bmatrix}. \quad (13.32)$$

13.2.4 APPLYING BOUNDARY CONDITIONS AND SOLVING THE SYSTEM OF EQUATIONS

The boundary conditions for the example of Section 13.2.2.1 with the mesh shown in Figure 13.7 are given in Equation 13.17. In the nodal form, the boundary conditions are written as

$$u_1 = 0, \quad \left(\frac{du}{dx} \right)_4 = 0. \quad (13.33)$$

The first boundary condition with the known value of the primary variable is called essential boundary condition. The second boundary condition providing traction-free condition is called natural or force boundary condition. The boundary conditions can be inserted in Equation 13.32 as follows:

$$\frac{EA}{l} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ ql \\ ql \\ \frac{ql}{2} \end{Bmatrix}. \quad (13.34)$$

It is to be noted that the first equation in Equation 13.34 is now replaced by the essential boundary condition equation $u_1=0$. The natural boundary condition is applied by substituting $(du/dx)_4 = 0$ in the fourth equation. Solving provides

$$u_1 = 0, \quad u_2 = \frac{5ql^2}{2EA}, \quad u_3 = \frac{4ql^2}{EA} \quad \text{and} \quad u_4 = \frac{9ql^2}{2EA}. \quad (13.35)$$

Note that as the bar of length L has been discretised into three elements, $l = L/3$.

13.2.5 POST-PROCESSING

In this step, the nodal displacements in Equation 13.35 are used to determine strain and stress in the bar. Strain in element 1 shown in Figure 13.8 is calculated as follows:

$$\varepsilon^{(1)} = \frac{u_2 - u_1}{l} = \frac{1}{l} \left(\frac{5ql^2}{2EA} - 0 \right) = \frac{5ql}{2EA}. \quad (13.36)$$

Using the strain, stresses can also be determined. Post-processing also includes the representation of the results in the form of tables, graphs and contours, etc.

The finite element method as a numerical technique can be used to solve the differential equations pertaining to problems of areas other than solid mechanics. Terminology like the stiffness matrix used in Equation 13.12 sounds biased as if the method is confined to solid mechanics but it can be used to solve thermal and electrical equations as well. All types of FEM procedures essentially follow the previously explained five steps. Nowadays, many commercial as well as open-source packages are available that can be used for solving various physical problems.

13.3 GOVERNING EQUATIONS FOR THERMO-MECHANICAL PROBLEMS

This section presents the governing equations for the multi-physics of thermal, mechanical and microstructural interactions. First, the microstructural equations are presented followed by the thermal and mechanical equations.

13.3.1 MICROSTRUCTURAL EQUATIONS

This section presents the equations for the transformation of steel into various phases at different temperatures. For a better understanding, the elementary concept of the phase diagram of steel is first explained followed by the mathematical equations. Chemically, steel is an alloy of pure iron with a maximum carbon content of 2.1% by weight. Steel at a given temperature is composed of different phases. A phase, in brief, is a part of the alloy that has similar physical and chemical properties. Under conditions of constant temperature, pressure and composition, all the phases remain in equilibrium. The effect of pressure in general is neglected and only that of temperature and composition are taken into account. The pictorial graph that is

obtained by mapping the evolution of the phases at different temperatures for all possible compositions of pure iron and carbon is popularly known as the iron-carbon phase diagram. Since the discussions of the iron-carbon alloy system for the present chapter is solely based on steel; the portion of the phase diagram pertaining only to steel is shown in Figure 13.11.

At room temperature, steel exists as an alloy of ferrite and cementite. Ferrite, also referred to as the α -iron, is a phase of iron in its crystal structure with a body-centered cubic (BCC) structure, while cementite, with the chemical formula Fe_3C , is a metastable compound of iron and carbon. In some discussions, ferrite is also referred to as pure iron; however, it is a phase of iron with a very low carbon content due to the low solubility in BCC structure. When steel is heated, it goes through various phase transformations. After a certain temperature, the steel structure transforms to a single solid phase called austenite or γ -iron, which is a phase with a face-centered cubic structure (FCC). Because of the FCC structure, carbon has better solubility in austenite, hence, it can have a higher content of carbon. In heat treatment of steel, steel is generally heated until full austenization and then cooled down to obtain different types of phases. Referring to Figure 13.11, the temperatures represented by points on the lines A_3 and A_{cm} are called upper critical temperatures (A_T). Upper critical temperature is the temperature above which the full austenization occurs during heating. The lines A_3 and A_{cm} meet at the point where the carbon concentration is 0.76% at a temperature of 727 °C. This point is called the eutectoid point and the type of steel with this particular concentration of carbon (0.76%) is called eutectoid steel. The horizontal line A_1 passing through the eutectoid point is called the eutectoid isotherm. The temperature of 727 °C is also called the lower critical temperature or eutectoid temperature. A eutectoid steel on reaching the eutectoid temperature undergoes a reversible phase reaction called the eutectoid reaction, $\gamma(0.76\% \text{C}) \xrightleftharpoons[\text{cooling}]{\text{heating}} \alpha(0.022\% \text{C}) + \text{Fe}_3\text{C}(6.7\% \text{C})$, where austenite decomposes into two phases—ferrite and cementite.

The ferrite obtained from the eutectoid reaction is called eutectoid ferrite. If the eutectoid reaction takes place during cooling, the microstructure consists of alternating layers of ferrite and cementite called pearlite. Pure pearlite is obtained in eutectoid steel at a very slow rate of cooling considering isothermal conditions. The iron-carbon diagram shows the evolution of the phases during heating and cooling of iron-carbon alloys under the idealized condition when each phase remains in equilibrium. With this understanding, some depictions of the iron-carbon diagram show pearlite. However, it could lead to the impression that pearlite is formed whenever eutectoid steel is cooled down at any rate, which is not true. Thus, it is omitted from Figure 13.11 and instead shown in Figure 13.11, which depicts the evolution of phases with time and temperature.

The eutectoid point vertically demarcates the phase diagram into two regions based on the content of carbon, that is to say, one where it is less than 0.76% and one where it is more than 0.76%. Thus, steel is classified as eutectoid (carbon content equal to 0.76%), hypoeutectoid (carbon content less than 0.76%) and hypereutectoid (carbon content more than 0.76%) steel. When hypoeutectoid or hypereutectoid steel is cooled, phase transformation takes place through the A_3 and A_{cm} lines,

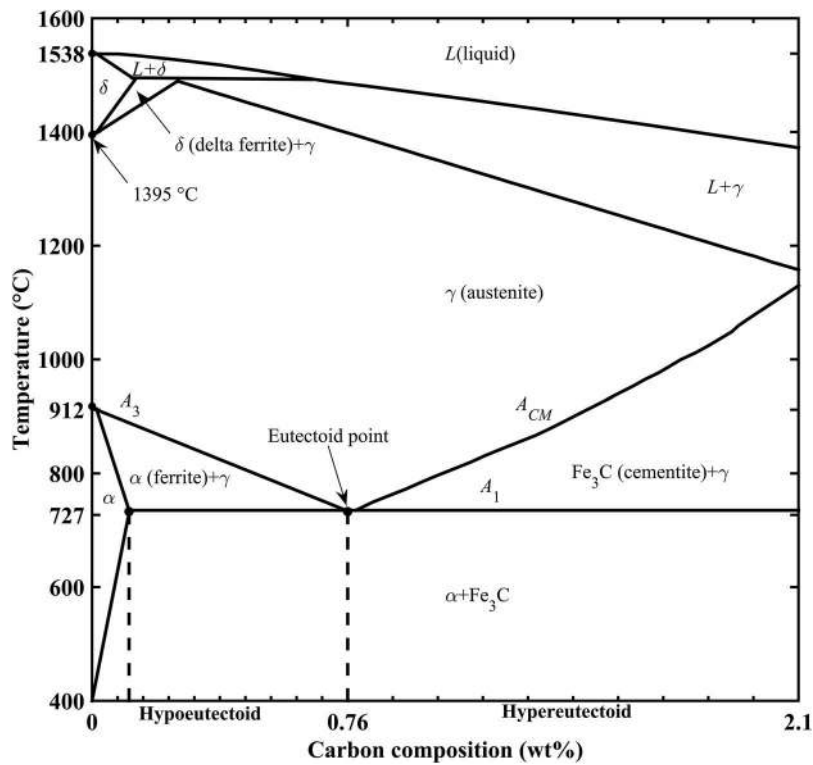


FIGURE 13.11 Iron-carbon phase diagram for steel (not to scale).

respectively before reaching the eutectoid temperature A_1 . This leads to the early formation of ferrite in hypoeutectoid steel and cementite in hypereutectoid steel prior to commencing the eutectoid reaction. These pre-eutectoid inclusions are thus termed as proeutectoid ferrite and proeutectoid cementite phases.

The physics that describes the rate at which a parent phase transforms into its microstructural constituents at different temperatures is called phase transformation kinetics. Based on how the atoms move in the alloy system, phase transformation can be of two types—diffusional and displacive. In brief, a diffusional transformation is where each atom moves and rearranges to form a new microstructural lattice; here the atomic bonds break due to heating. A displacive transformation is where well-ordered groups of atoms move without rupture of the atomic bonds. The governing equations for the two types of transformation are explained as follows.

13.3.1.1 Diffusional Transformation

Heat treatment of steel is carried out by heating above the austenization temperature and cooling at a suitable rate. If the transformation is diffusional, a mixture of the following phase constituents is formed, that is to say, proeutectoid ferrite (in hypoeutectoid steel), proeutectoid cementite (in hypereutectoid steel), pearlite and

bainite. As a first mention, bainite is a microstructural state of steel that is obtained at cooling rates faster than those required for pearlite. The transformation from austenite to bainite involves diffusion-controlled processes at temperatures below the pearlite transformation temperature. The microstructure of bainite features acicular or lath-like structures of ferrite and cementite. Diffusional transformation may either be isothermal or anisothermal depending on the variation of temperature with time. First, the kinetics of the isothermal case is explained followed by its extension to the non-isothermal one.

The governing equation for the kinetics of isothermal diffusional transformation is the Johnson-Mehl-Avrami-Kolmogorov (JMAK) equation expressed as (Callister and Rethwisch, 2009):

$$X_k(t) = X_k^{\max} \left\{ 1 - \exp(-b_k t^{n_k}) \right\}, \quad (13.37)$$

where t denotes the time, $X_k(t)$ denotes the volume fraction of the k^{th} phase constituent that transforms at the rate defined by the temperature-dependent rate constants n_k and b_k . The quantity $X_k^{\max}$ is the maximum transformable volume fraction of the k^{th} phase whose value is always equal to one for pearlite and bainite. For hypoeutectoid or hypereutectoid phases, $X_k^{\max}$ is computed as (Phadke et al., 2004):

$$X_k^{\max} = \begin{cases} 0, & T > A_1 \\ X_k^{eq} \left(\frac{A_T - T}{A_T - A_1} \right), & A_1 \geq T \geq A_T \\ X_k^{eq}, & T < A_T \end{cases} \quad (13.38)$$

where A_T is the upper critical temperature whose value is A_3 for hypoeutectoid steel and A_{cm} for hypereutectoid steel and X_k^{eq} is the equilibrium volume fraction of the proeutectoid phase. The equilibrium volume fraction X_k^{eq} is computed from the iron-carbon diagram using the lever rule, which is stated as follows (Callister and Rethwisch, 2009):

$$X_k^{eq} = \begin{cases} \left(\frac{0.76 - C}{0.76 - 0.022} \right), & \text{for proeutectoid ferrite} \\ \left(\frac{2.1 - C}{2.1 - 0.76} \right), & \text{for proeutectoid cementite} \end{cases} \quad (13.39)$$

where C is the percentage of carbon content. By convention, the rate of a transformation of any phase is defined as the reciprocal of time required to attain 50% completion, $t_{0.5}$. Thus,

$$\text{rate} = \frac{1}{t_{0.5}}. \quad (13.40)$$

At the start and finish of a transformation of the k^{th} phase $X_k = 0$ and $X_k = X_k^{\text{max}}$, respectively. From Equation 13.37, at $t = 0$, $X_k = 0$ and as t tends to infinity, $X_k = X_k^{\text{max}}$. It is very difficult to record the exact time when the volume fraction of the k^{th} phase constituent starts growing from zero. Hence, the beginning of the transformation is taken from the time when the volume fraction has become 1% of X_k^{max} . The ending of the transformation is taken as the time when the volume fraction becomes 99% of X_k^{max} . The graph of X_k versus t is a characteristic S-shaped curve shown in Figure 13.12 for

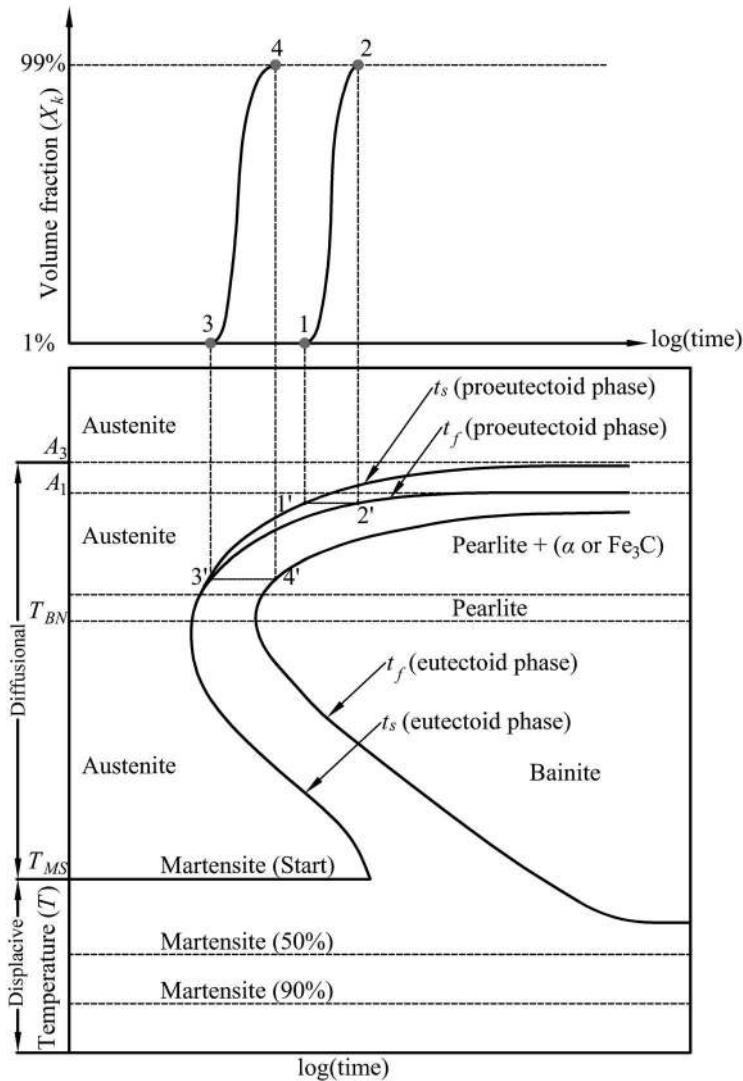


FIGURE 13.12 Schematic of a typical JMAK equation and corresponding temperature time transformation (TTT) diagram obtained by projection of start and finish times of several phase transformation curves under isothermal conditions for a typical steel alloy.

two sets of rate constants n_k and b_k . When the start and finish times of several JMAK equations with different sets of rate constants n_k and b_k are projected on a time-temperature plane, a pair of C-shaped curves are obtained as shown in Figure 13.12, which is termed as the time-temperature-transformation (TTT) diagram.

The TTT diagram for a typical steel with a proeutectoid phase is depicted in Figure 13.12, where t_s and t_f , respectively, denote the start and end time of the transformation of any phase. Two JMAK curves represented by Equation 13.37 with typical values of X_k^{max} (Equation 13.38) and X_k^{eq} (Equation 13.39), are plotted on the volume fraction versus $\log(\text{time})$ plot. Referring to Figure 13.12, the JMAK curve 1–2 corresponds to the transformation of a proeutectoid phase whereas 3–4 corresponds to that of a eutectoid phase. When the start and end points (like 1 and 2, respectively in Figure 13.12) of all the possible JMAK curves of the proeutectoid phase for the particular steel are plotted on the temperature versus $\log(\text{time})$ plot, two continuous curves containing the start and end times of the transformation of the proeutectoid phase are obtained. Likewise, the two continuous curves containing the start and end times of the eutectoid phases are obtained when the start and end points (like 3 and 4, respectively in Figure 13.12) of all the possible JMAK curves of the eutectoid phase are plotted on the temperature versus $\log(\text{time})$ plot. There are three characteristic curves—(1) a start time curve for the proeutectoid phase, (2) a finish time curve for the proeutectoid phase that coincides with the start time curve for the eutectoid phase and (3) a finish time curve for the eutectoid phase. For eutectoid steel, there would be no proeutectoid phase and the TTT would comprise only two curves for the start and finish times of the eutectoid phase. Some characteristic transformations during cooling are as follows.

In the temperature range ($A_1 < T \leq A_3$), austenite transforms into the proeutectoid phase. Hypothetically, the maximum proeutectoid phase (Equation 13.38) is obtained when the temperature is held within the range for an infinite duration. At a temperature equal to or below A_1 but above the temperature for bainite transformation (T_{BN}), that is ($T_{BN} < T \leq A_1$), austenite transforms into pearlite with the possible inclusion of a proeutectoid phase like ferrite (α) or cementite (Fe_3C) depending on the temperature. A 100% pearlite without any proeutectoid phase is also possible at some isotherms as indicated in Figure 13.12. At a temperature equal to or below T_{BN} but above the martensitic temperature (T_{MS}), that is ($T_{MS} < T \leq T_{BN}$), austenite transforms into bainite without any inclusion of a proeutectoid phase. At temperatures at or below T_{MS} , the transformation is displacive and austenite transforms into martensite. It is worth emphasizing to the reader that, even at temperatures below A_3 , it is assumed that the alloy remains entirely in an austenitic state as indicated on the left side TTT curve. This is because the start of the transformation is assumed not to have commenced until the transformation has reached 1%.

The volume fraction of the phase constituents under the anisothermal condition is computed by assuming that the transformation takes place in increments of subsequent isothermal transformations. The volume fraction of the k^{th} constituent $X_k(t + \Delta t)$ as a function of time at the end of each time increment Δt is calculated by (De Oliveira et al., 2010)

$$X_k(t + \Delta t) = X_k^{max} X_\gamma \left[1 - \exp \left\{ -b_k (\tau + \Delta t)^{n_k} \right\} \right]. \quad (13.41)$$

The remaining volume fraction of austenite available for subsequent transformation represented by X_γ is updated at the end of each time increment using

$$X_\gamma = 1 - \sum_{i \neq k} X_i. \quad (13.42)$$

where X_i is the volume fraction of the already transformed i^{th} phase. The parameter τ is a fictitious time derived from the JMAK equation for isothermal diffusional transformation and is expressed as

$$\tau = \left\{ \frac{-1}{b_k} \ln \left(1 - \frac{X_k}{X_k^{\max}} \right) \right\}^{\frac{1}{n_k}}. \quad (13.43)$$

The fictitious time parameter τ provides the time required to achieve a known volume fraction X_k . The temperature-dependent constants b_k and n_k are determined from Equation 13.37 by substituting the $X_k = 0.01$ and $X_k = 0.99$ corresponding to the start and finish of the transformation. The value of $X_k^{\max}$ is one for pearlite and bainite whereas it is calculated using Equation 13.38 for any proeutectoid phase. For pearlite and bainite, the expressions for the rate constants reduce to

$$n_k = \frac{6.127}{\ln \left(\frac{t_f}{t_s} \right)}, \quad (13.44)$$

$$b_k = \frac{0.01}{t_s^{n_k}} \quad \text{or} \quad b_k = \frac{4.60}{t_f^{n_k}}. \quad (13.45)$$

The start and finish times denoted by t_s and t_f , respectively are taken from the TTT diagram of the material.

The schematic of an anisothermal transformation considering a k^{th} constituent is depicted in Figure 13.13. The continuous decrease of temperature is approximated as a series of small isotherms each of duration Δt . Each temperature point is obtained on the TTT diagram and the projection of the endpoints of the TTT diagram on the time-volume fraction Cartesian plane provides the JMAK curve governing the isothermal transformation at the temperature. The time t_o is the datum time at the temperature T_o at which the volume fraction of the k^{th} constituent has reached 1% of the maximum obtainable volume fraction. For temperature T_o , the isothermal transformation in the increment Δt occurs along the JMAK curve traced by ab resulting in a volume fraction X_k^0 . When the temperature changes to T_1 , the rate constants n_k and b_k are calculated using the t_s and t_f for T_1 in Equation 13.44–13.45, respectively. Then Equation 13.43 is applied to calculate the fictitious time τ_1 required to attain the volume fraction X_k^0 . Next, the increment Δt is added to τ_1 . The times τ_1 and $\tau_1 + \Delta t$ on the log(time) axis of Figure 13.13 correspond to points b' and c on the JMAK curve for T_1 . Finally, the point c is used to find the corresponding volume fraction X_k^1 on the volume fraction axis of Figure 13.13. Likewise, in the next increment of time, the temperature decreases to T_2 . The fictitious time τ_2 is first calculated and the volume

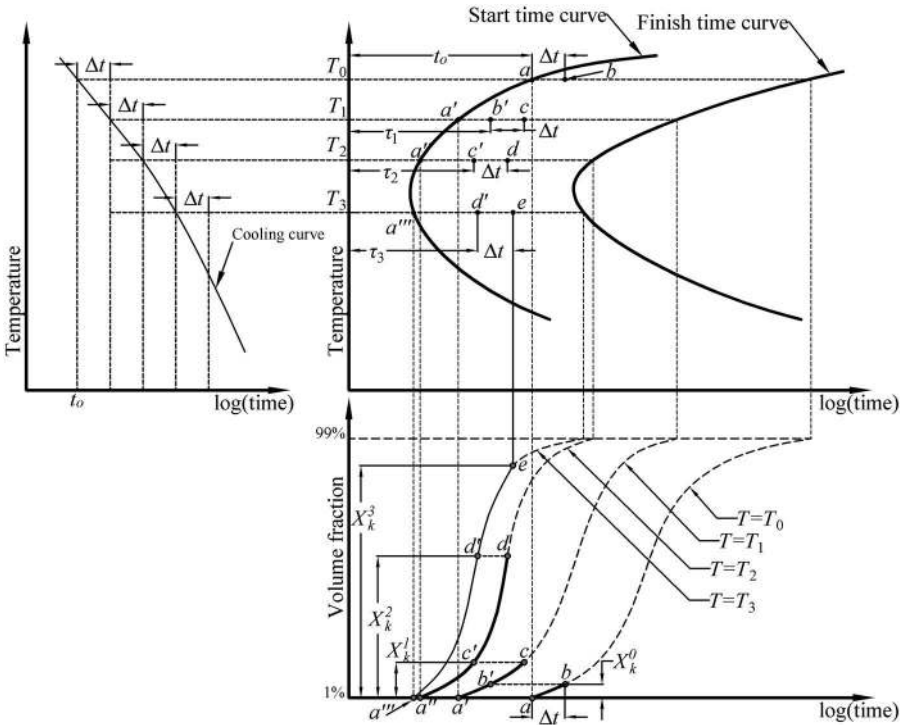


FIGURE 13.13 Schematic of anisothermal cooling curve and the projected temperature points on the TTT diagram with the incremental transformation along different JMAK curves.

fraction is computed defined by the path $a''c'd$ resulting in the volume fraction X_k^2 . The computation of the volume fraction in the next increment at the temperature of T_3 along the path $a''d'e$ is also computer using a similar approach.

13.3.1.2 Displacive Transformation

If the transformation is displacive, only martensite is formed. Martensite is a crystal-line structure that is formed when the steel from an austenitic state is rapidly cooled from a high temperature in water, oil or air, a process generally called quenching. The kinetics of displacive transformation are also indicated in the TTT diagram in Figure 13.12. The start of the displacive transformation is indicated in the diagram by Martensite (start). Lines indicating the completion of 50% and 90% transformation are also indicated. The volume fraction of martensite represented by X_{MS} is only a function of temperature and is expressed as follows

$$X_{MS} = X_{\gamma} \left[1 - \exp \left\{ -0.011 \left(T_{MS} - T_j \right) \right\} \right], \tag{13.46}$$

where T_{MS} is the temperature below which the transformation of austenite to mar-tensite commences and T_j is the current temperature. The temperature T_{MS} is 225 °C in typical steels.

Example 13.1

The kinetics of transformation of an alloy is governed by the JMAK equation with rate constant $n_k = 3.1$. Out of a maximum transformable fraction of 100%, if 30% is transformed after 15 minutes, determine the rate of transformation.

Solution

The JMAK Equation 13.37 for $X_k^{max} = 1$ is

$$X_k(t) = 1 - \exp(-b_k t^{n_k}).$$

Substituting $n_k = 3.1$ and $X_k(15) = 0.30$ provides the value of b_k as follows:

$$\begin{aligned} 0.3 &= 1 - \exp(-b_k t^{n_k}) \\ \Rightarrow \frac{\ln 0.7}{15^{3.1}} &= b_k \\ \Rightarrow b_k &= 8.06 \times 10^{-5} \end{aligned}$$

Using the rate constants, the time required for 50% completion is computed by substituting $X_k(t_{0.5}) = 0.5$ as follows,

$$\begin{aligned} 0.5 &= 1 - \exp(-8.06 \times 10^{-5} \times t_{0.5}^{3.1}) \\ \Rightarrow 0.5 &= \exp(-8.06 \times 10^{-5} \times t_{0.5}^{3.1}) \\ \Rightarrow \ln 0.5 &= -8.06 \times 10^{-5} \times t_{0.5}^{3.1} \\ \Rightarrow \frac{\ln 0.5}{-8.06 \times 10^{-5}} &= t_{0.5}^{3.1} \\ \Rightarrow 8599.84^{\frac{1}{3.1}} &= t_{0.5} \\ \Rightarrow t_{0.5} &= 18.59 \text{ minute} \end{aligned}$$

Thus the rate of transformation (Equation 13.40), is

$$\text{rate} = \frac{1}{t_{0.5}} = \frac{1}{18.59} = 0.054 (\text{minute})^{-1}.$$

Example 13.2

Consider an anisothermal transformation of a phase with $X_k^{max} = 1$, where 10% transformation has already occurred. Determine the fraction transformed in the next 1 minute along a JMAK curve whose start time is 2 minutes and finish time is 14.56 minutes.

Solution

First, the remaining fraction of austenite is updated from Equation 13.42
 $X_\gamma = 1 - \sum X_i = 1 - 0.1 = 0.9$.

The rate constants for the JMAK curve with $t_s = 2$ and $t_f = 14.56$ are obtained by substituting in Equations 13.44–13.45 as follows

$$n_k = \frac{6.127}{\ln\left(\frac{t_f}{t_s}\right)} = \frac{6.127}{\ln\left(\frac{14.56}{2}\right)} = 3.09$$

$$b_k = \frac{4.60}{t_f^{n_k}} = \frac{4.60}{14.56^{3.09}} = 1.17 \times 10^{-3}$$

Subsequently, the fictitious time corresponding to the 10% volume fraction on the given JMAK curve calculated using Equation 13.43 is

$$\begin{aligned} \tau &= \left\{ \frac{-1}{b_k} \ln(1 - X_k) \right\}^{\frac{1}{n_k}} \\ \Rightarrow \tau &= \left\{ \frac{-\ln(1 - 0.1)}{1.17 \times 10^{-3}} \right\}^{\frac{1}{3.09}} \\ \Rightarrow \tau &= \left\{ \frac{-\ln(0.9)}{1.17 \times 10^{-3}} \right\}^{\frac{1}{3.09}} \\ \Rightarrow \tau &= 4.29 \text{ min} \end{aligned}$$

Thus, for an increment of 1 minute, that is $\Delta t = 1$ minute, the acquired volume fraction is calculated using Equation 13.41 as follows

$$\begin{aligned} X_k &= X_\gamma \left[1 - \exp \left\{ -b_k (\tau + \Delta t)^{n_k} \right\} \right], \\ \Rightarrow X_k &= 0.9 \left[1 - \exp \left\{ -1.17 \times 10^{-3} \times (4.29 + 1)^{3.09} \right\} \right] \\ \Rightarrow X_k &= 0.9 \left[1 - \exp \left\{ -1.17 \times 10^{-3} \times (5.29)^{3.09} \right\} \right] \\ \Rightarrow X_k &= 0.9(0.18) = 0.1641 \\ \Rightarrow X_k &= 16.41\% \end{aligned}$$

13.3.2 THERMAL EQUATIONS

The thermal equation is governed by Fourier's law of heat conduction, which is

$$\rho c \frac{\partial T}{\partial t} = \nabla \bullet \{ \nabla (kT) \} + \dot{q}, \quad (13.47)$$

where ∇ is the gradient operator, ρ is the density, c is the specific heat, T is the temperature, k is the thermal conductivity and $\dot{q}$ is the rate of internal latent heat generated per unit volume due to phase transformation. The thermal properties of the material that is composed of different phases at a given temperature are calculated using the linear law of mixture expressed as:

$$P = \sum_{k=1}^N X_k P_k, \quad (13.48)$$

where P is a thermal property of mixture, P_k is the property, X_k is the volume fraction of the k^{th} constituent, respectively, and N is the number of phases in the material. The rate of internal latent heat is calculated as

$$\dot{q} = \sum_{k=1}^{N-1} \Delta H_k \frac{dX_k}{dt}, \quad (13.49)$$

where ΔH_k is the enthalpy of formation per unit volume when austenite transforms into a k^{th} constituent, and dX_k is the volume fraction of the phase transformed during time step dt . The effect of latent heat may also be incorporated by using a fictitious specific heat c_p^* , defined as

$$c_p^* = \sum_{k=1}^N c_p^k X_k + \sum_{k=1}^{N-1} \left(\frac{dX_k}{dT} \right) \Delta H_k = \sum_{k=1}^N c_p^k X_k + \sum_{k=1}^{N-1} \left(\frac{dX_k/dt}{dT/dt} \right) \Delta H_k, \quad (13.50)$$

where c_p^k is the specific heat capacity at constant pressure of a k^{th} constituent and dT/dt is the rate of temperature change.

13.3.3 MECHANICAL EQUATIONS

The constitutive equation is defined in the form of strain rates by adding the individual strain rates caused by different physics as

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^e + \dot{\varepsilon}_{ij}^p + \dot{\varepsilon}_{ij}^{th} + \dot{\varepsilon}_{ij}^{pt} + \dot{\varepsilon}_{ij}^{tp} \quad (13.51)$$

where $\dot{\varepsilon}_{ij}$, $\dot{\varepsilon}_{ij}^e$, $\dot{\varepsilon}_{ij}^p$, $\dot{\varepsilon}_{ij}^{th}$, $\dot{\varepsilon}_{ij}^{pt}$ and $\dot{\varepsilon}_{ij}^{tp}$ represent the total, elastic, plastic, thermal, phase transformation and transformation-induced plasticity (TRIP) strain rates, respectively. The stress rate is obtained by multiplying the elastic strain rate with the linear elastic tensor.

$$\dot{\sigma}_{ij} = C_{ijkl} \dot{\epsilon}_{ij}^e = \left(\dot{\epsilon}_{ij} - \dot{\epsilon}_{ij}^p - \dot{\epsilon}_{ij}^{th} - \dot{\epsilon}_{ij}^{pt} - \dot{\epsilon}_{ij}^{tp} \right) \quad (13.52)$$

Each term of the total strain rate (Equation 13.52) is explained as follows.

13.3.3.1 Elastic Strain

The elastic strain is already given in Equation 5.12 is repeated for convenience as

$$\epsilon_{ij}^e = \frac{1+\nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij}. \quad (13.53)$$

Differentiating with respect to t provides the rate form as follows:

$$\dot{\epsilon}_{ij}^e = \frac{1+\nu}{E} \dot{\sigma}_{ij} - \frac{\nu}{E} \dot{\sigma}_{kk} \delta_{ij}. \quad (13.54)$$

13.3.3.2 Plastic Strain

For heat processing such as quenching of steel, the initial yield stress is a function of volume fraction of the phases, which is given by the linear law of mixture similar to Equation 13.48 as follows:

$$\sigma_0 = \sum_{k=1}^N X_k \sigma_k^0, \quad (13.55)$$

where σ_k^0 is the yield strength of the k^{th} constituent phase. The flow stress for subsequent yielding is provided in Equation 5.74 is also modified by considering the effects of temperature, volume fraction of the phase constituents and plastic history of phases as follows:

$$\sigma_f = \sum_{k=1}^N X_k \sigma_k^0 + \sum_{k=1}^N X_k H_k \kappa_k = \sigma_0 + \sum_{k=1}^N X_k H_k \kappa_k \quad (13.56)$$

where H_k is the plastic hardening modulus of the k^{th} phase. The strain hardening parameter κ_k is called the transformation plasticity constant that accounts for the loss of plastic memory due to reconstruction of the lattice structure during transformation and is calculated as

$$\kappa_k(\tau + \Delta\tau) = \int_{\tau=0}^{\tau} \left\{ d\bar{\epsilon}^p - \frac{1}{X_k} \frac{dX_k}{dt} \kappa_k(\tau) \right\} d\tau. \quad (13.57)$$

Here, $d\bar{\epsilon}^p$ is the equivalent plastic strain rate defined by Equation 5.116. Hence, the yield function $f(\sigma_{ij})$ based on the Mises criterion for the subsequent yielding provided in Equation 5.74 is modified by defining a new yield function Φ by considering the effects of the Cauchy stress, temperature, volume fraction of the phase and plastic history of phases as follows:

$$\Phi = \Phi(T, \sigma_{ij}, X_k, \kappa_k) \quad (13.58)$$

The associated flow rule is modified from Equation 5.87 by considering the effect of phase transformations on flow stress is suggested as

$$d\varepsilon_{ij}^p = d\lambda \frac{\partial \Phi}{\partial \sigma_{ij}} \quad (13.59)$$

where $d\varepsilon_{ij}^p$ is the incremental plastic strain and $d\lambda$ is a positive scalar. By expressing the incremental plastic strain in terms of the plastic strain rate (Refer to Equation 5.115) and rearranging, the rate form of Equation 13.60 is expressed as follows

$$\dot{\varepsilon}_{ij}^p = \frac{d\lambda}{dt} \frac{\partial \Phi}{\partial \sigma_{ij}} \quad (13.60)$$

13.3.3.3 Thermal Strain

The thermal strain ε_T , generated due to a change of temperature from a reference temperature to a final temperature T is given by

$$\varepsilon_{ij}^{th} = \left(\int_0^T \alpha_T dT \right) \delta_{ij}. \quad (13.61)$$

where α_T is the coefficient of thermal expansion of the steel due to all the phases. The rate form is

$$\dot{\varepsilon}_{ij}^{th} = \alpha_T \dot{T} \delta_{ij}. \quad (13.62)$$

The presence of thermal strain does not imply thermal stress. There are two sources of thermal strain in a material subjected to fluctuations in temperature, namely, (1) when the expansion or contraction of the material is constrained and (2) when there is a significant temperature gradient in the material where the temperature of one part differs from that of the rest.

13.3.3.4 Volumetric Phase Transformation Strain

Volumetric strain due to phase transformation is

$$\varepsilon_{ij}^{pt} = \sum_{k=1}^{N-1} \frac{1}{3} \delta_{ij} \Delta_k X_k \quad (13.63)$$

where Δ_k is the structural dilatation due to transformation of the austenite to the k^{th} phase. Differentiating Equation 13.63 with respect to time provides the rate form as

$$\dot{\varepsilon}_{ij}^{pt} = \sum_{k=1}^{N-1} \frac{1}{3} \delta_{ij} \Delta_k \dot{X}_k \quad (13.64)$$

13.3.3.5 Transformation Induced Plasticity (TRIP) Strain

Transformation induced plasticity (TRIP) can be defined as the plasticity induced by phase change. Due to TRIP effect, plastic deformation may be observed even when the equivalent stress is less than the flow stress of the material. When phase transformation takes place, a certain part of a material for example a micro-region may alter its volume and shape. To achieve compatibility between the transforming micro-region and the neighboring material in the vicinity, the misfit must be compensated by an eigenstress state that may vary between grains or within a grain of a polycrystalline material. An eigenstress is a self-equilibrated internal stress state caused by eigenstrains in a material that is free from any other external force and surface constraint. Eigenstrain is any mechanical deformation in a material that is not caused by an external mechanical stress, for example, thermal strain and strain due to phase transformation. In many cases especially during quenching of steel, the misfit leads at least to a plastic deformation of the material in the vicinity of the micro-regions or that of the micro-regions themselves. The eigenstress state is also influenced by a stress state due to an externally applied force. The combination of these two stressing or straining mechanisms may also initiate plastic deformation. Hence, a macroscopic plastic deformation of the specimen can be observed.

Some advanced high-strength steels based on the TRIP phenomenon are called TRIP steels (Zhao and Jiang, 2017). Typically, they contain some amount of retained austenite (>5% by volume) embedded in the soft ferrite matrix (50–60% by volume) and some hard phases like bainite and martensite. As per the identification standard adopted by the Ultralight Steel Auto Body (ULSAB) Consortium, an advanced high-strength steel can be specified as

$$XX \text{ aaa/bbb}, \quad (13.65)$$

where XX is a set of alphabets denoting the type of steel, aaa is a three-digit integer denoting the minimum yield strength (YS) in MPa and bbb is a three-digit integer denoting the minimum ultimate tensile strength (UTS) in MPa (Keeler et al., 2017). For example, DP 210/440 is a dual-phase steel with YS of 201 MPa and UTS of 440 MPa; HSLA 350/450 is a high strength low alloy steel with YS of 350 MPa and UTS of 450 MPa and TRIP 400/700 is a transformation induced plasticity steel with YS of 400 MPa and UTS of 700 MPa.

To find the strain rate in TRIP, microstructural modeling is required. In the TRIP process, retained austenite is transformed into martensite. This depends on the stability of the austenite. The stability of the austenite can be quantified as

$$f_{\gamma} = f_{\gamma_0} \exp(-k\varepsilon), \quad (13.66)$$

where f_{γ_0} , f_{γ} and k are the initial austenite fraction, the austenite fraction at strain ε and k is the stability constant dependent on several factors.

13.4 EXAMPLE OF HEAT TRANSFER ANALYSIS OF ROLLING PROBLEM

The flat rolling process is a bulk metal forming process where the thickness of a strip or sheet is reduced by passing it between two counter-rotating rolls. Based on the working temperature, the process can be of three types—cold, warm and hot. Cold rolling is performed at a temperature less than $0.3T_m$, warm rolling is performed within a temperature range of 0.3 – $0.6 T_m$ and hot rolling is performed at a temperature greater than $0.6T_m$, where T_m is the melting point of the metal in Kelvin (Hawkins, 1985). The correct prediction of temperature is important in any type of rolling process as it significantly influences the lubricant, contact behavior, roll wear, microstructure and properties of the strip (Khan et al., 2004; Louaisil et al., 2009). In hot rolling, consideration of temperature is a must as the mechanical and microstructural properties are greatly affected by the temperature. The knowledge of the strip and roll temperature can provide insight into the microstructure of the strip and the lubricant behavior (Tseng, 1984). As a significant amount (around 90%) of the work is converted into heat due to friction and plastic deformation, it is necessary to carry out proper deformation modeling to predict the temperature distribution. This section presents the mathematical and numerical techniques for the estimation of the temperature distributions in the roll and sheet.

In this section, a two-dimensional thermo-mechanical analysis of the flat rolling process is described. The analysis comprises two modules—deformation module and thermal module. The distribution of temperature within the roll is obtained by solving the two-dimensional heat transfer equation considering the Eulerian reference frame. The following assumptions are considered:

1. The axial direction to the roll is insulated, hence, a two-dimensional problem.
2. Thermal properties of the material of the roll are independent of the temperature and the properties are considered at average temperature.
3. The input heat flux into the roll is uniform along the roll-strip interface.
4. The roll rotates at a uniform speed.

A rotating hollow roll with heat flux at a fixed location is shown in Figure 13.14. Heat conduction of a plane configuration is considered by representing the roll as

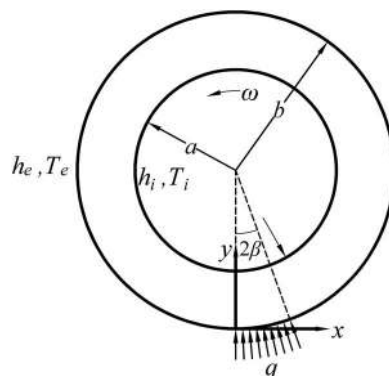


FIGURE 13.14 A stationary heat source with rotating hollow roll.

a circular disk satisfying a plain strain condition. Two methods that are commonly used to predict temperature in rolling are the finite difference method (FDM) and finite element method (FEM). Some analytical solutions formulated as a sum of an infinite series containing Bessel functions have also been proposed (Patula, 1981; Yiannopoulos et al., 1997). Although these methods are useful accurate analyses, they are computationally intensive and costly. In this chapter, an approximate method for the quick prediction of the temperature distribution proposed by Yadav et al. (2014) is presented.

Here, the distribution of temperature in the roll is computed based on the superposition of steady-state temperature from the work of Fischer et al. (2004) and the average temperature of the roll at a given radial location calculated from the heat balance equation. Superposition provides the steady-state distribution of temperature within the roll along the radial as well as circumferential directions. An Eulerian reference frame is applied with x - and y -axes shown in Figure 13.14. The roll is considered a half-plane, which is justifiable for a bigger-sized roll. The input heat source q acting along the surface in the interval $[0, 2\beta]$ is assumed to be spatially fixed. The governing equation for the steady-state heat conduction is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{\omega r}{\alpha_r} \frac{\partial T}{\partial x}, \quad (13.67)$$

where ω is the constant angular velocity of roll, r is the radial location of a point in the roll and α_r is the thermal diffusivity of the roll. The Peclet number (P_e), which is the ratio of advective transport rate to diffusive transport rate, is expressed as

$$P_e = \frac{\omega b^2 \beta}{2\alpha_r}. \quad (13.68)$$

For a large Peclet number, only the heat conduction along the y direction (or radial) is relevant (Fischer et al., 2004) and Equation 13.67 reduces to

$$\frac{\partial^2 T}{\partial y^2} = \frac{\omega r}{\alpha_r} \frac{\partial T}{\partial x}, \quad (13.69)$$

subject to the initial condition

$$T|_{t=0} = T_0 \quad (13.70)$$

and boundary condition

$$k \frac{\partial T}{\partial y} = \begin{cases} -q & \text{for } 0 \leq x \leq 2b\beta, y = 0, \\ 0 & \text{for } x > 2b\beta, y = 0. \end{cases} \quad (13.71)$$

Considering non-dimensional coordinates $x = 2\beta b\zeta$ and $y = \delta\zeta$ where δ is the depth of heat penetration given by (Fischer et al., 2004)

$$\delta = \sqrt{\frac{2b\beta\alpha_r}{\omega b}}. \quad (13.72)$$

The temperature distribution for a case with a zero initial temperature of the roll is obtained as (Fischer et al., 2004)

$$T(\xi, \zeta) = T_{\max} \left[\sqrt{\xi} \exp\left(-\frac{\zeta^2}{4\xi}\right) - \frac{\sqrt{\pi}}{2} \zeta \left\{ 1 - \operatorname{erf}\left(\frac{\zeta}{2\sqrt{\xi}}\right) \right\} \right] \quad \text{for } 0 \leq \xi \leq 1 \quad (13.73)$$

$$T(\xi, \zeta) = T_{\max} \left[\begin{aligned} &\sqrt{\xi} \exp\left(-\frac{\zeta^2}{4\xi}\right) - \frac{\sqrt{\pi}}{2} \zeta \left\{ 1 - \operatorname{erf}\left(\frac{\zeta}{2\sqrt{\xi}}\right) \right\} \\ &- \left\{ \sqrt{\xi-1} \exp\left(-\frac{\zeta^2}{4(\xi-1)}\right) \right\} - \frac{\sqrt{\pi}}{2} \zeta \left\{ 1 - \operatorname{erf}\left(\frac{\zeta}{2\sqrt{(\xi-1)}}\right) \right\} \end{aligned} \right] \quad \text{for } \xi \geq 1, \quad (13.74)$$

where $T_{\max}$ is given by

$$T_{\max} = \sqrt{\frac{8b\beta\alpha_r}{\pi\omega b}} \frac{q}{k} \quad (13.75)$$

and q is the constant heat flux on the surface defined in the interval $[0, 2\beta]$ and k is the thermal conductivity of the roll. The term erf represents the error function corresponding to the diffusion equation of heat transfer and is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-\eta^2) d\eta. \quad (13.76)$$

For a steady-state condition, a constant temperature $T_0(R)$ is imposed at the radial location corresponding to R such that

$$T_0(R) + \frac{\int_0^{\left(\frac{\pi R}{\beta b}\right)} T(\xi, (b-R)/\delta) d\xi}{\{\pi R / (\beta b)\}} = T_{\text{avg}}(R) \quad (13.77)$$

where $T_{\text{avg}}(R)$ is the average temperature of the surface at the radial location corresponding to R whose calculation is as follows. (The second term in Equation 13.77 is the average of the temperature distribution over the whole periphery corresponding

to radius R . As $x = 2\beta b\zeta$, the upper limit in the integral of the second term corresponds to $x = 2\beta b\zeta = 2\pi R$.)

Yadav et al. (2014) considered the following heat balance equation:

$$2\pi ah_i(T_a - T_i) + (2\pi - 2\beta)bh_e(T_b - T_e) = 2\beta b\dot{q} \quad (13.78)$$

where $\dot{q}$, a , b , T_a , T_b , T_i , T_e , h_i and h_e are the heat flux, inner radius of roll, outer radius of roll, average temperature of inner radius, average temperature of outer radius, ambient temperature at inner roll periphery, ambient temperature at outer roll periphery, convective heat transfer coefficient at the inner radius and convective heat transfer coefficient at the outer radius of the roll, respectively (Yadav et al., 2014). To solve for the two unknown values T_a and T_b , one additional equation is required that is provided by the boundary condition at the inner periphery of the roll as

$$-k\left(\frac{\partial T}{\partial r}\right)_{r=a} + h(T_a - T_i) = 0. \quad (13.79)$$

Equating the heat flows at the inner and outer peripheries gives the following expression for the temperature gradient at the outer periphery:

$$2\pi ak\left(\frac{\partial T}{\partial r}\right)_{r=a} = 2\pi bk\left(\frac{\partial T}{\partial r}\right)_{r=b} \Rightarrow \left(\frac{\partial T}{\partial r}\right)_{r=b} = \frac{a}{b}\left(\frac{\partial T}{\partial r}\right)_{r=a}. \quad (13.80)$$

Using the above expression, the average temperature gradient becomes

$$\left(\frac{\partial T}{\partial r}\right)_{avg} \equiv \frac{\left(\frac{\partial T}{\partial r}\right)_{r=a} + \left(\frac{\partial T}{\partial r}\right)_{r=b}}{2} = \frac{b+a}{2b}\left(\frac{\partial T}{\partial r}\right)_{r=a}. \quad (13.81)$$

Equating the above expression for the average temperature gradient with its finite difference expression

$$\left(\frac{\partial T}{\partial r}\right)_{avg} \equiv \frac{T_b - T_a}{b - a} = \frac{b+a}{2b}\left(\frac{\partial T}{\partial r}\right)_{r=a} \quad (13.82)$$

gives the following expression for the temperature gradient at the inner periphery:

$$\left(\frac{\partial T}{\partial r}\right)_{r=a} = 2b \frac{T_b - T_a}{b^2 - a^2}. \quad (13.83)$$

Substitution of Equation 13.83 in Equation 13.79 provides the additional equation between T_a and T_b :

$$h(T_a - T_i) = 2bk \frac{T_b - T_a}{b^2 - a^2}. \quad (13.84)$$

Solving Equations 13.78 and 13.84 provides the average temperatures T_a and T_b along the roll periphery. Assuming that T_a and T_b are linear functions of the radial distance, the average temperature $T_{avg}(R)$ at any radial distance can be calculated. Subsequently, Equation 13.77 can provide the temperature $T_0(R)$, which can be superimposed on the temperature distribution of Equation 13.73 and Equation 13.74 to provide the temperature distribution for different radial locations.

13.5 EXAMPLE OF LASER FORMING

Laser forming is a sheet metal forming process in which thermal stresses are induced using a controlled defocused high-power density laser beam. Here, a laser scans the sheet along a predefined path based on the desired shape. The process is similar to flame bending where an oxy-acetylene torch is used as a heat source (Hemmati and Shin, 2007). Although flame bending is suitable for bending large and thick sections, it is inconvenient for bending small and geometrically complex parts as it is difficult to focus and control the flame on a small area. For such complex geometries, laser bending is a better alternative due to its controllability.

The schematic of a laser bending process is depicted in Figure 13.15 (Kumar and Dixit, 2020). An edge along the width of the sheet is clamped on the laser machine bed using a fixture. The laser beam irradiates the workpiece along a path parallel to the width. The localized heating creates a temperature gradient across the thickness and induces thermal stresses. The increase in temperature results in the thermal softening and reduction of yield strength of the material. Plastic deformation commences when the thermal stress reaches the yield stress of the material. When the material is allowed to cool naturally or by forced cooling, the hotter upper layer of the material shrinks faster than the lower layer due to Newton's law of cooling which results in permanent sheet bending towards the laser beam.

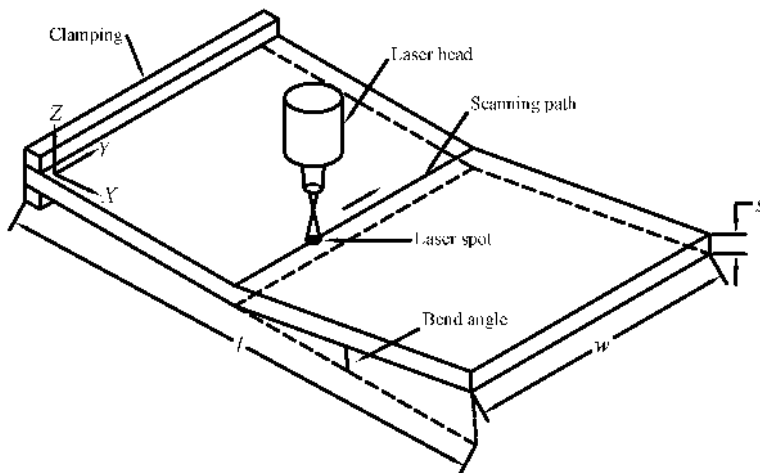


FIGURE 13.15 The schematic diagram for laser bending process. (With permission from (Kumar and Dixit, 2020). Copyright Elsevier.)

This section presents an analysis to predict the distribution of hardness in a laser-bent strip. The model incorporates the effects of phase transformation and mechanical strain hardening. Some key assumptions in the model are as follows:

1. The material of the sheet is homogeneous and isotropic.
2. The thermal heat flux follows a Gaussian distribution.
3. The heat generation due to inelastic deformation is neglected.
4. The strains due to TRIP are neglected.
5. Effect of phase dilation is neglected.

13.5.1 GOVERNING EQUATION

The microstructural equations are explained in detail in Section 13.3.1. The thermal equation is governed by Fourier's law of conduction already given in Equation 13.47. The constitutive relationship for the mechanical behavior is the same as that explained in Section 13.3.3. The total strain rate in this model is

$$\dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^e + \dot{\epsilon}_{ij}^p + \dot{\epsilon}_{ij}^{th}. \quad (13.85)$$

Subsequently,

$$\dot{\sigma}_{ij} = C_{ijkl} \dot{\epsilon}_{ij}^e = \left(\dot{\epsilon}_{ij} - \dot{\epsilon}_{ij}^p - \dot{\epsilon}_{ij}^{th} \right). \quad (13.86)$$

13.5.2 INITIAL AND BOUNDARY CONDITION

The material for the laser bending model is a hypoeutectoid steel having a carbon content of 0.02%. The equilibrium volume fraction of the proeutectoid ferrite X_{α}^{eq} calculated from Equation 13.39 for $C = 0.02$ is

$$X_{\alpha}^{\text{eq}} = \frac{0.76 - 0.02}{0.76 - 0.022} = 1.0027 \approx 1. \quad (13.87)$$

Hence, the maximum possible transformable fraction of the proeutectoid ferrite X_{α}^{max} is 100%. The initial phase fraction of the sheet is taken as 100% ferrite. Thus,

$$X_{\alpha} \Big|_{t=0} = 1. \quad (13.88)$$

The upper critical temperature from the A_3 line is 907 °C as per Figure 13.11, where a full austenization is assumed. Thus

$$X_{\gamma} \Big|_{T=925^{\circ}\text{C}} = 1. \quad (13.89)$$

The heat transfer boundary conditions are governed by the following convective boundary condition:

$$-k \frac{\partial T}{\partial n} = h(T_s - T_{amb}), \quad (13.90)$$

where n is the outward normal to the surface, h is the convective heat transfer coefficient, T_s is the surface temperature and T_{amb} is the ambient temperature. Here the effect of radiation is incorporated in that of convection and its small influence is included by adjusting the parameter h , which is determined inversely. It is assumed that surfaces without contact with the cooling medium are insulated, that is,

$$\left. \frac{\partial T}{\partial n} \right|_t = 0. \quad (13.91)$$

A mechanical boundary condition is also given to the strip where one end is fixed as shown in Figure 13.16.

13.5.3 LOADING CONDITION

The thermal load is given as a thermal heat flux (q) that follows a normal Gaussian distribution. The control of the laser beam radius (r) is defined as a function of the standoff distance between the tip of the nozzle and the top surface of the workpiece, which is expressed as (Sun, 1998)

$$r = w_0 \left\{ 1 + \left(\frac{M^2 \lambda_l H_s}{\pi w_0^2} \right) \right\}, \quad (13.92)$$

$$w_0 = \frac{2M^2 f \lambda_l}{\pi D_L}, \quad (13.93)$$

where w_0 called the laser beam waist radius is the minimum beam radius at the focal point of the lens, f is the focal length, λ_l is the wavelength of the laser beam, D_L is the laser beam diameter before the lens, H_s is the stand-off distance (distance from laser beam waist to irradiated spot), M^2 is the beam quality factor whose value is equal to one for a perfect Gaussian beam. In real practice, M^2 is more than one; here $M^2 = 1.4$. The Gaussian distribution of heat flux (q) is given by

$$q(x, y) = \frac{2\eta P}{\pi r^2} \exp \left\{ -2 \frac{(x^2 + y^2)}{r^2} \right\}, \quad (13.94)$$

where η and P are the absorptivity of the material and laser power, respectively. It is to be emphasized that x and y are the local coordinates different than those shown in Figure 13.15 and r is the beam radius given by Equation 13.94; in the local coordinate system, the origin is taken at the center of the laser spot; global Y coordinate is obtained by adding the product of scan speed and time to local y coordinate, but the local x and X coordinates are the same (see Figure 13.15).

13.5.4 ESTIMATION OF HARDNESS

During laser bending, the high temperature that leads to phase transformation and thermal strain may cause strain hardening of the material. This affects the hardness of the material. The prediction of Vickers hardness distribution due to phase transformation is based on the empirical model by Maynier et al. (1978), where the hardness values of proeutectoid ferrite, pearlite, bainite and martensite are calculated as a function of the chemical composition (in wt.%) of the workpiece material and cooling rate. The empirical expressions for the hardness value of each phase are as follows:

$$\begin{aligned} HV_{bain} = & -323 + 185(\%C) + 330(\%Si) + 153(\%Mn) \\ & + 65(\%Ni) + 144(\%Cr) + 191(\%Mo) \\ & + \{89 + 53(\%C) - 55(\%Si) - 22(\%Mn) \\ & + 10(\%Ni) - 20(\%Cr) - 33(\%Mo)\} \log V_R, \end{aligned} \quad (13.95)$$

$$\begin{aligned} HV_{fer/per} = & 42 + 223(\%C) + 53(\%Si) + 30(\%Mn) \\ & + 12.6(\%Ni) + 7(\%Cr) + 19(\%Mo) \\ & + \{10 - 1953(\%Si) + 4(\%Mn) + 8(\%Cr) \\ & + 130(\%V)\} \log V_R, \end{aligned} \quad (13.96)$$

$$\begin{aligned} HV_{mar} = & 127 + 949(\%C) + 53(\%Si) + 11(\%Mn) \\ & + 8(\%Ni) + 16(\%Cr) + 21 \log V_R, \end{aligned} \quad (13.97)$$

where $HV_{fer-per}$, HV_{mar} and HV_{bain} represent the Vickers hardness values of ferrite or pearlite, martensite and bainite, respectively; V_R is the cooling rate in $^{\circ}\text{C/h}$ at 700°C . The total Vickers hardness (HV) of the steel is calculated using the linear mixture law as follows

$$HV = \sum_{k=1}^{N-1} X_k HV_k, \quad (13.98)$$

where HV_k is the Vickers hardness of the k^{th} phase excluding austenite.

13.5.5 SOLUTION PROCEDURE AND TYPICAL RESULTS

A numerical procedure in the form of FEM is used to solve the problem. This involves a coupled thermo-microstructural-mechanical analysis to incorporate phase transformation effects. The governing thermal equation written in the finite element thermal form can be written as

$$[H]\{\dot{T}\} + [C_T]\{T\} = \{Q\}, \quad (13.99)$$

where $[H]$ and $[C_T]$ are specific heat and conductivity matrices, respectively. Further, $\{T\}$ and $\{\dot{T}\}$ are the nodal temperature and temperature rate vectors, respectively. For a typical strip having a length of 200 mm, width of 40 mm and thickness of 2 mm, the model discretized to finite elements is shown in Figure 13.16. Note that the mesh has been made finer in the vicinity where the laser scans the workpiece for better accuracy.

The distributions of the phase fractions measured along the line XX' (refer Figure 13.16) at a depth of 0.2 mm from the top surface are shown in Figure 13.17 for various strip thicknesses. On average, the surface comprises 92% bainite, 5% martensite and 3% retained austenite, at the point O. At the point farthest away from O, the microstructural state has a fully ferritic structure. Since bainite is harder than ferrite, the hardness is the maximum at O. The hardness gradually reduces by moving away from O.

The variation of temperature along the line OX' (refer to Figure 13.16) when the laser beam reaches point O in the fifth pass is shown in Figure 13.18. The temperature is the maximum at the laser-irradiated spot O and gradually decreases away from O. The horizontal line shows the lower critical temperature (725 °C) above which microstructure changes.

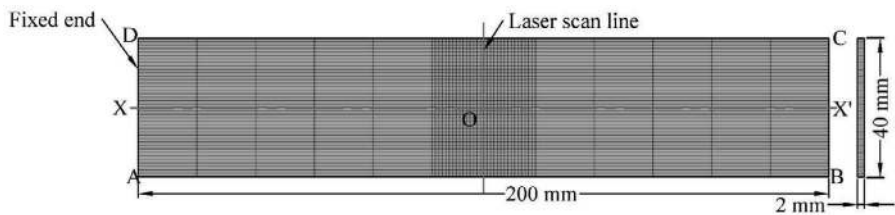


FIGURE 13.16 Geometry of the part model with a typical mesh. (With permission from Kumar and Dixit (2018a). Copyright Elsevier.)

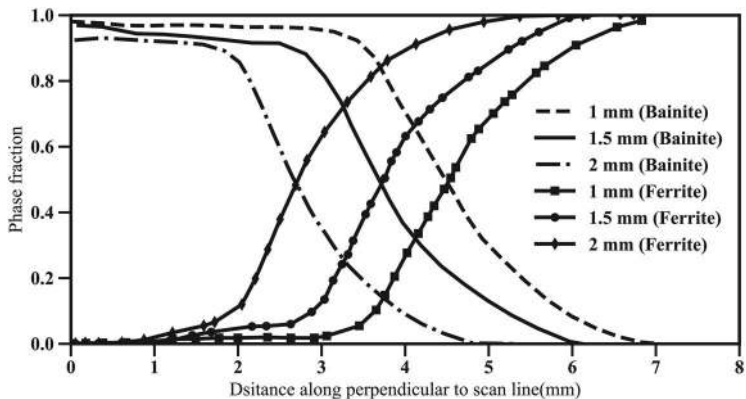


FIGURE 13.17 Variation of phase fractions of bainite and ferrite along perpendicular to laser scan line for workpiece of different thicknesses (1, 1.5 and 2 mm) (power: 700 W, scan speed: 900 mm/min, laser beam diameter: 9.6 mm, number of scans: 5). (With permission from Kumar and Dixit (2018a). Copyright Elsevier.)

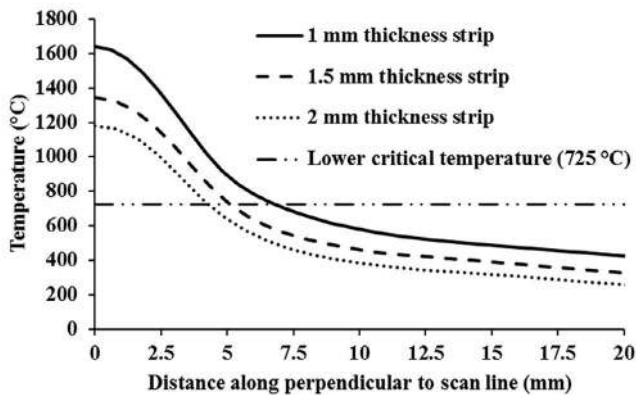


FIGURE 13.18 Temperature distribution on the top surface along perpendicular to scan line (power: 700 W, scan speed: 900 mm/min, laser beam diameter: 9.6 mm, 5th scan, position of laser: midpoint of the scan line). (With permission from Kumar and Dixit (2018a). Copyright Elsevier.)

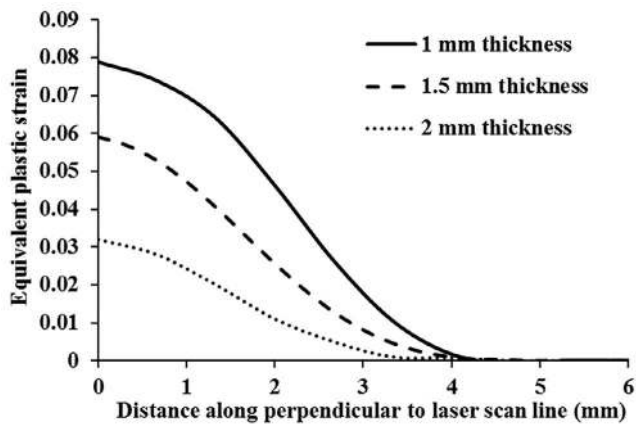


FIGURE 13.19 Equivalent plastic strain of the laser-scanned sheet along perpendicular to laser scan line (power: 700 W, scan speed: 900 mm/min, laser beam diameter: 9.6 mm, number of scans: 5). (With permission from Kumar and Dixit (2018a). Copyright Elsevier.)

The distribution of the equivalent plastic strain along OX' for various thicknesses of the laser-bent strip while keeping other dimensions and laser parameters constant is shown in Figure 13.19. The equivalent plastic strain is the maximum at the laser-irradiated spot O and decreases away along the line OX'. The variation of the equivalent plastic strain with temperature and time at point O from the start to finish of the laser bending process is shown in Figure 13.20. The temperature increases with each subsequent pass leading to a corresponding increase in the equivalent plastic strain. The final bend angle and maximum value of the hardness obtained for typical cases of geometrical and laser parameters are presented in Tables 13.1 and 13.2. These results have been obtained from FEM.

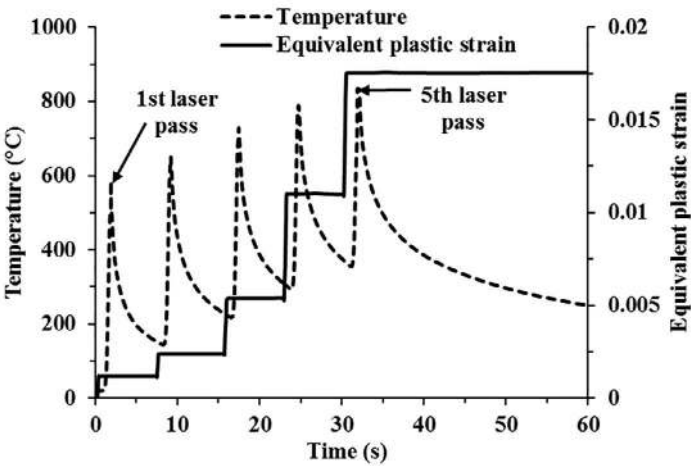


FIGURE 13.20 Variation of equivalent plastic strain and temperature with time at midpoint of the scan line (power: 500 W, scan speed:900 mm/min, beam diameter: 11.4 mm, number of scans: 5). (With permission from Kumar and Dixit (2018a). Copyright Elsevier.)

TABLE 13.1
Bend Angle and the Maximum Hardness of the Laser Bent Strips After Five Passes

Power (W)	Scan speed (mm/min)	Laser beam diameter (mm)	Width of strip (mm)	Thickness (mm)	Bend angle (°)	Hardness HV
700	900	9.6	30	1	4.05	243
				1.5	3.48	234
				2	3.14	220

Source: With permission from Kumar and Dixit (2018a). Copyright Elsevier.

TABLE 13.2
Evolution of Bend Angle and the Maximum Hardness with Laser Scans For a 2 mm Thick Strip

Power (W)	Scan speed (mm/min)	Laser beam diameter (mm)	Width of strip (mm)	Number of scans	Bend angle (°)	Hardness HV
500	900	11.4	40	1	0.13	141
				3	0.35	170
				5	1.06	202

Source: With permission from Kumar and Dixit (2018a). Copyright Elsevier.

The results obtained show that temperature and strain hardening significantly affect the hardness. A microstructure that comprises a higher fraction of hard phases like bainite (as in the present study) results in greater surface hardness. The resulting microstructure is a direct consequence of the temperature conditions during the laser bending process. The temperature is maximum at points along the laser scan line and gradually decreases while moving away from it in a perpendicular direction. The hardness of the workpiece is also affected by the strain hardening that occurs during cooling of the workpiece, which affects the equivalent plastic strain. The equivalent plastic strain is maximum at the laser scan line and reduces with the distance along the perpendicular direction.

13.6 INVERSE ANALYSIS

This section presents an inverse analysis to determine the material properties, that is, absorptivity, thermal conductivity and heat transfer coefficient of a laser irradiated sheet to estimate the transient temperature distribution in laser-based manufacturing processes. The inverse model uses the temperature history computed by a FEM-based direct model. The model estimates the properties by minimizing the deviation between the actual and model-predicted temperatures of the laser-irradiated sheet.

13.6.1 PROBLEM DEFINITION

Consider a rectangular workpiece having a particular set of length, width and thickness held at an initial ambient temperature T_{amb} . With no loss in generality, let the length be 100 mm, width 50 mm and thickness be s . The workpiece is irradiated at a point on the top surface with a prescribed laser heat flux q_1 for time t_1 , and then cooled down up to time t_2 . The front and bottom projections of the workpiece are shown in Figure 13.20. In the chosen coordinate system, the center of the laser spot diameter is at (45 mm, 25 mm, s). The y -axis is oriented along the width where one edge is clamped up to a length of 5 mm along the x -direction oriented along the length of the workpiece. The temperature variation with time is measured at two points B_1 (45 mm, 25 mm, 0) and B_2 (55 mm, 35 mm, 0). The convective heat transfer coefficient h is defined as the rate of heat transfer per unit difference in the temperatures between the surface of the workpiece and surroundings. This includes the effect of radiation. Although the radiation heat transfer coefficient is proportional to the fourth power of temperature difference, a linearized form is a good enough approximation for a small range of surface temperatures.

13.6.2 METHODOLOGY

The inverse estimation relies on a direct model. Here, a transient-heat transfer FEM is used as a direct model. The inverse procedure developed on the basis of the FEM model is as follows. The nodal temperature $T(x, y, z, t)$ as a function of position and time is recorded at two locations by considering the values of absorptivity η , thermal conductivity k and heat transfer coefficient as input. The inverse procedure uses the temperature measured at these two locations and averages the properties by using a heuristic algorithm (Kumar et al., 2019a).

13.6.3 THE INVERSE METHOD

The temperature measured at two locations can provide an approximate of thermal conductivity k as

$$(\theta_{B1} - \theta_{B2}) \propto \frac{1}{k}, \quad (13.100)$$

where θ_{B1} and θ_{B2} are the excess temperatures (rise in temperatures with respect to the ambient temperature) at the two locations of the workpiece (see Figure 13.21). For inverse estimation of k , θ_{B1} is taken as the peak temperature at B1 over the entire time duration and θ_{B2} is the temperature at B2 at the instant when the location B1 attains the peak temperature. Hence, the thermal conductivity is updated as

$$k_{new} = k_{old} \frac{(\theta_{B1} - \theta_{B2})_{pred}}{(\theta_{B1} - \theta_{B2})_{act}} = k_{old} \frac{(\theta_{B1B2})_{pred}}{(\theta_{B1B2})_{act}}, \quad (13.101)$$

where $(\theta_{B1B2})_{pred}$ is the temperature difference at the two locations predicted from FEM model using the old values of thermal conductivity k_{old} , absorptivity θ_{old} and

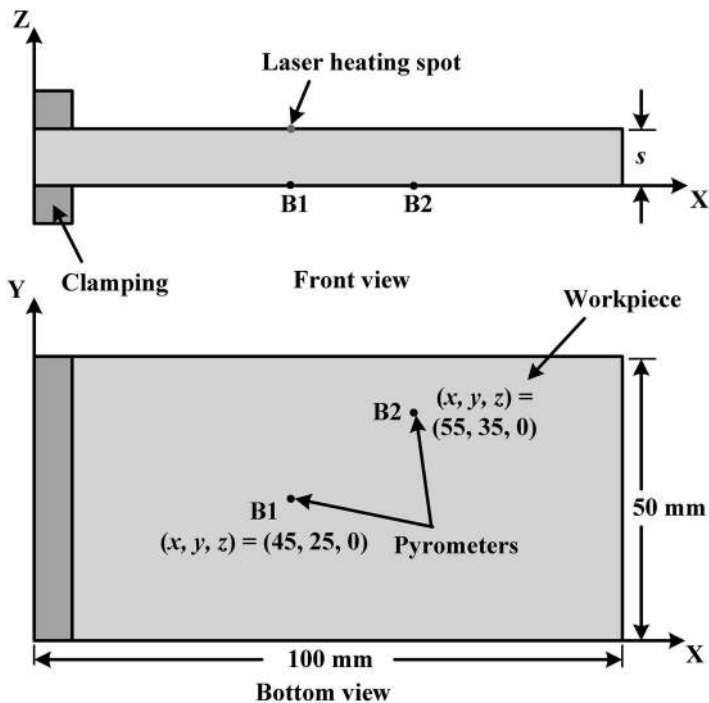


FIGURE 13.21 The front and bottom views of the workpiece. (With permission from (Kumar et al., 2019a). Copyright Elsevier.)

heat transfer coefficient h_{old} , and $(\theta_{B1B2})_{act}$ is the actual temperature difference. Equation 13.101 merely serves to provide a guess value of thermal conductivity, which is refined using classical Newton's method as described in sequel.

First, an empirical model is provided, which is expressed as follows (Kumar et al., 2019a).

$$\eta_{new} = \eta_{old} \frac{(\theta_{B1})_{act}}{(\theta_{B1})_{pred}} \left(\frac{k_{new}}{k_{old}} \right)^{0.55}. \quad (13.102)$$

Although Equation 13.102 is highly arbitrary for a general situation, it suffices for at least a guess value absorptivity. From Equations 13.101 and 13.102, the thermal conductivity and absorptivity values can be updated (following Newton's method) as

$$\begin{Bmatrix} k_{i+1} - k_i \\ \eta_{i+1} - \eta_i \end{Bmatrix} = \begin{bmatrix} \frac{\partial \theta_{B1}}{\partial k} & \frac{\partial \theta_{B1}}{\partial \eta} \\ \frac{\partial \theta_{B1B2}}{\partial k} & \frac{\partial \theta_{B1B2}}{\partial \eta} \end{bmatrix}^{-1} \begin{Bmatrix} (\theta_{B1})_{act} - (\theta_{B1})_{pred} \\ (\theta_{B1B2})_{act} - (\theta_{B1B2})_{pred} \end{Bmatrix}, \quad (13.103)$$

where $(\theta_{B1})_{pred}$ and $(\theta_{B1B2})_{pred}$ are the values of θ_{B1} and θ_{B1B2} predicted from the FEM model using the values k_i , η_i and h_{old} . This method is repeated until the convergence criterion is reached. The convergence criterion is

$$|E_{\theta, B1}| = \left| \frac{(\theta_{B1})_{pred} - (\theta_{B1})_{act}}{(\theta_{B1})_{act}} \right| \times 100 \leq \varepsilon_1, \quad (13.104)$$

$$|E_{\theta, B1B2}| = \left| \frac{(\theta_{B1B2})_{pred} - (\theta_{B1B2})_{act}}{(\theta_{B1B2})_{act}} \right| \times 100 \leq \varepsilon_1, \quad (13.105)$$

where ε_1 is the percentage error in temperature. The method will converge fast if the guess values of k and η are close to a solution. In Equation 13.103, the partial derivatives of θ_{B1} and θ_{B1B2} can be obtained using three available guess points in the following two equations (Kumar and Dixit, 2018b):

$$\Delta \theta_{B1} \cong \frac{\partial \theta_{B1}}{\partial k} \Delta k + \frac{\partial \theta_{B1}}{\partial \eta} \Delta \eta \quad (13.106)$$

$$\Delta \theta_{B1B2} \cong \frac{\partial \theta_{B1B2}}{\partial k} \Delta k + \frac{\partial \theta_{B1B2}}{\partial \eta} \Delta \eta \quad (13.107)$$

After the convergence, the converged values of k and η are used to carry out the inverse estimation to update the heat transfer coefficient. It is required to refer to the transient temperature data for which, a signed error E_{rr} is defined as

$$E_{rr} = \varepsilon_{\theta,B1} + \varepsilon_{\theta,B2}, \quad (13.108)$$

where

$$\varepsilon_{\theta,B1} = \sum_{n=1}^{N_s} \left| \frac{(\theta_{B1})_{act,n} - (\theta_{B1})_{pred,n}}{(\theta_{B1})_{act,n}} \right|, \quad (13.109)$$

$$\varepsilon_{\theta,B2} = \sum_{n=1}^{N_s} \left| \frac{(\theta_{B2})_{act,n} - (\theta_{B2})_{pred,n}}{(\theta_{B2})_{act,n}} \right|, \quad (13.110)$$

for N_s sampling times including heating and cooling. For the two values of heat transfer coefficients h_i and h_{i+1} (that are chosen such that the actual values of heat transfer coefficient are likely to lie within these two values) with corresponding signed errors as E_{rri} and E_{rri1} , respectively, the new value of heat transfer coefficient is calculated from secant method (Valencia and Quested, 2008) as

$$h_{new} = h_i - \left(\frac{h_{i+1} - h_i}{E_{rri1} - E_{rri}} \right) E_{rri}. \quad (13.111)$$

It is preferred that the chosen values of h_i and h_{i+1} result in oppositely signed E_{rri} and E_{rri1} . Based on this principle, h_i or h_{i+1} is updated with the new value of the heat transfer coefficient, and the method is repeated till the convergence is achieved. The convergence criterion is

$$|h_{i+1} - h_i| < \varepsilon_2, \quad (13.112)$$

where ε_2 is a convergence parameter, whose value is taken as 0.5 W/m² °C in this example. The converged value is denoted by h_{new} .

If the convergence criterion in Equation 13.112 is achieved, the value of $|h_{new} - h_{old}|$ is checked where h_{old} is the initially assumed value of h . If the deviation is greater than ε_2 , then the value of h_{old} is updated with h_{new} and the entire inverse procedure from Equations 13.101–13.112 is repeated; else, the inverse values of k , η and h are declared as the final predictions. The numerical procedure is summarized in the form of a flow chart in Figure 13.22.

13.6.4 APPLICATION EXAMPLE

The proposed method is demonstrated with an example. The selected material is aluminum alloy A319. The temperature-dependent material properties, namely, thermal conductivity, specific heat capacity and density are based on Valencia and Quested (2008). The average coefficient of thermal expansion is 2.2×10^{-5} (°C)⁻¹ (Hidnert and Krider, 1952). A plywood of size 50 mm × 5 mm × 5 mm is used to clamp the work-piece. The thermal conductivity, specific heat capacity, density and coefficient of thermal expansion of the plywood are 0.15 W/m °C, 1300 J/kg °C (TenWolde et al., 1988),

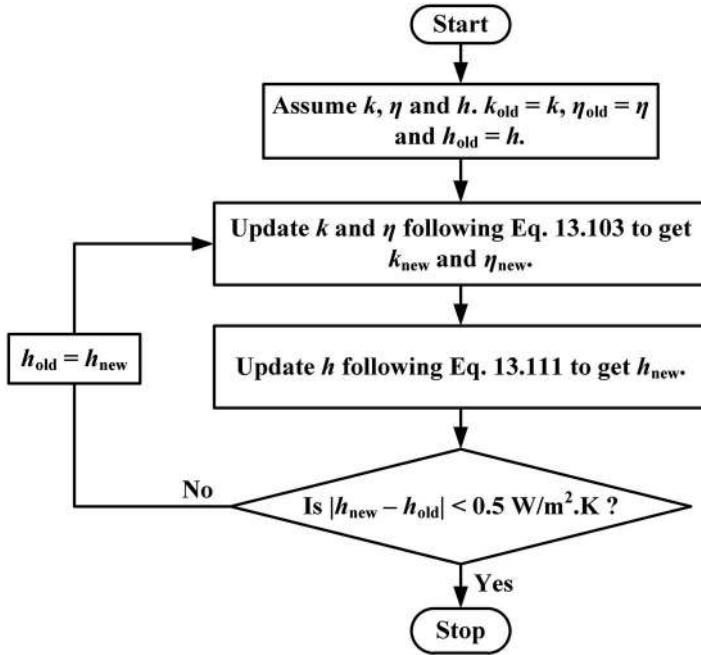


FIGURE 13.22 Flow chart of the inverse process. (With permission from (Kumar et al., 2019a). Copyright Elsevier.)

776 kg/m³ and $7.03 \times 10^{-6} \text{ (}^\circ\text{C)}^{-1}$ (Weatherwax and Stamm, 1947), respectively. The desired accuracy ε_i is set as 1%. The value of heat transfer coefficient is assumed between a range of 5 W/m² °C and 100 W/m² °C. Simulations are carried out until the irradiated surface of the workpiece reaches room temperature. The results are recorded at intervals of 120 equally spaced observations (NS) during the solution to compute the signed error E_{rr} defined by Equation 13.108. The inverse methodology is applied to predict the average value of thermal conductivity, absorptivity and heat transfer coefficient. Essentially, the inverse methodology comprises two steps. The first step is to obtain converged k and η for a constant value of h and the second step is to obtain converged h for constant values of k and η . The two steps are repeated until the convergence is achieved for all three parameters, which is k , η and h .

As stated in Section 13.6.1, the workpiece is 100 mm long and 50 mm wide. It has a thickness of 3 mm with η , h , laser power, laser beam (spot) diameter as 0.6, 20 W/m² °C, 400 W and 6 mm, respectively. The heating time is 3 s, total heating and cooling time for the simulation is 120 s. The actual temperature differences at two bottom surface points obtained from the FEM simulation are $(\theta_{B1})_{act} = 182.32 \text{ }^\circ\text{C}$ and $(\theta_{B2})_{act} = 43.77 \text{ }^\circ\text{C}$, respectively. These values are taken in lieu of experimental values. Thus, to demonstrate the method, recourse is taken to virtual experimentation.

Initial guesses of k , η and h are $k_{old} = 100 \text{ W/m }^\circ\text{C}$, $\eta_{old} = 0.3$ and $h_{old} = 100 \text{ W/m}^2 \text{ }^\circ\text{C}$, respectively. With the fixed value of the heat transfer coefficient, the values of k

and η are updated using Equations 13.101 and 13.103, respectively, for the first three iterations. Later on, for fast convergence, the values of k and η are updated using Newton’s method. The values of k and η converged after five iterations with both $|E_{\theta,B1}|$ and $|E_{\theta,B1B2}|$ less than 1%. The iterations are shown in Table 13.3.

In the next step, the value of the heat transfer coefficient is updated for constant values of k and η . Initially, the signed error E_{rr} is calculated for an initial value of $h = 100 \text{ W/m}^2 \text{ }^\circ\text{C}$ using Equation 13.108. Since the signed error E_{rr} is positive, the heat transfer coefficient is reduced to the lower limit of $5 \text{ W/m}^2 \text{ }^\circ\text{C}$ and FEM simulation is further carried out. Once the signed error E_{rr} becomes negative the next estimate of h is obtained using the secant method, which provides the convergence in five iterations as shown in Table 13.4.

As per the initially assumed value of h , that is, $h_{old} = 100 \text{ W/m}^2 \text{ }^\circ\text{C}$, $|h_{new} - h_{old}|$ is $77 \text{ W/m}^2 \text{ }^\circ\text{C}$. As $|h_{new} - h_{old}|$ is greater than $0.5 \text{ W/m}^2 \text{ }^\circ\text{C}$, the values of k and η are updated with $h_{old} = 23 \text{ W/m}^2 \text{ }^\circ\text{C}$ and the inverse procedure is repeated. The following solution provides the values $k = 161 \text{ W/m}^2 \text{ }^\circ\text{C}$ and $\eta = 0.63$ for a fixed value of h after

TABLE 13.3
Iterations of FEM Simulation For Estimating k and η For $h = 100 \text{ W/m}^2 \text{ }^\circ\text{C}$

Iterations	k (W/m $^\circ\text{C}$)	η	$(\theta_{B1})_{pred}$ ($^\circ\text{C}$)	$(\theta_{B2})_{pred}$ ($^\circ\text{C}$)	$ E_{\theta,B1} $ (%)	$ E_{\theta,B1B2} $ (%)
1	71.7	0.377	193.3	166.87	6	20.4
2	86.3	0.394	176.82	148.62	3	7.3
3	92.6	0.422	179.56	149.43	1.5	7.85
4	144.9	0.615	187.98	145.37	3.1	4.9
5	157.7	0.639	183.3	139.64	0.5	0.8

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TABLE 13.4
Iterations of FEM Simulation For Estimating h For $\eta = 0.639$ and $k = 157.7 \text{ W/m}^2 \text{ }^\circ\text{C}$

h (W/m ² $^\circ\text{C}$)	Signed error (Err)	$ h_{i+1} - h_i $ (W/m ² $^\circ\text{C}$)
100	157.96	—
5	−101.3	95
42.12	67.5	37.12
27.27	18.18	14.85
23.88	4.19	3.39
23.13	0.8	0.75
23	−0.11	0.13

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two iterations of simulations. Further, the value of h is updated with fixed values of k and η . The convergence criterion is met after three iterations of simulations with the final results being $h = 22 \text{ W/m}^2 \text{ }^\circ\text{C}$. The data for these internal steps are not shown for brevity. The final results after convergence are $k = 161 \text{ W/m }^\circ\text{C}$, $\eta = 0.63$ and $h = 22 \text{ W/m}^2 \text{ }^\circ\text{C}$ for which $|E_{\theta,B1}| = 0.35\%$, $|E_{\theta,B1B2}| = 0.13\%$, $Err = 0.226$ and $|h_{new} - h_{old}| = 0.21 \text{ W/m}^2 \text{ }^\circ\text{C}$. The total number of simulations performed to obtain the final solution is 17. The total solution time is about 4 hours.

For validation of the inverse procedure, nine FEM simulations based on L_9 orthogonal array design (Logothetis and Wynn, 1995) are carried out. Processing parameters and conditions for the simulations are shown in Table 13.5. The actual values of absorptivity and heat transfer coefficient are taken as 0.6 and $20 \text{ W/m}^2 \text{ }^\circ\text{C}$, respectively; whereas the actual temperature-dependent thermal conductivity is based on Valencia and Quested (2008). The temperature resulting from the simulations based on actual values of absorptivity, thermal conductivity and heat transfer coefficient are taken as a benchmark. As stated in the last paragraph, the values of absorptivity, heat transfer coefficient and average thermal conductivity predicted by the inverse method are 0.63, $22 \text{ W/m}^2 \text{ }^\circ\text{C}$ and $161 \text{ W/m }^\circ\text{C}$, respectively. The comparison of the maximum temperature and temperature history based on the actual and inverse estimated parameters are described in the following four paragraphs.

The maximum temperature based on actual process parameters ($\eta = 0.6$, $h = 20 \text{ W/m}^2 \text{ }^\circ\text{C}$, k : temperature dependent) and inverse estimated process parameters ($\eta = 0.63$, $h = 22 \text{ W/m}^2 \text{ }^\circ\text{C}$, $k = 161 \text{ W/m }^\circ\text{C}$) at B1 and B2 (refer Figure 13.21), respectively are shown in Figs 13.23 (a) and (b). As observed, the actual and predicted values agree well. The maximum error in temperature prediction is 3.71% at B1 for Case 9 and 2.41% at B2 for Case 8. Figure 13.23 (c) shows the comparison of the maximum temperatures for actual and inversely estimated process parameters at the top surface laser heating point. The maximum error in temperature prediction is 4.21% for Case 9 (Figure 13.23).

TABLE 13.5
Orthogonal Array Design of the Three Levels of Material and Process Parameters

Case number	Power (W)	Laser beam diameter (mm)	Heating time (s)	Workpiece thickness (mm)
1	200	4	2	2
2	200	6	3	3
3	200	8	4	4
4	400	4	3	4
5	400	6	4	2
6	400	8	2	3
7	600	4	4	3
8	600	6	2	4
9	600	8	3	2

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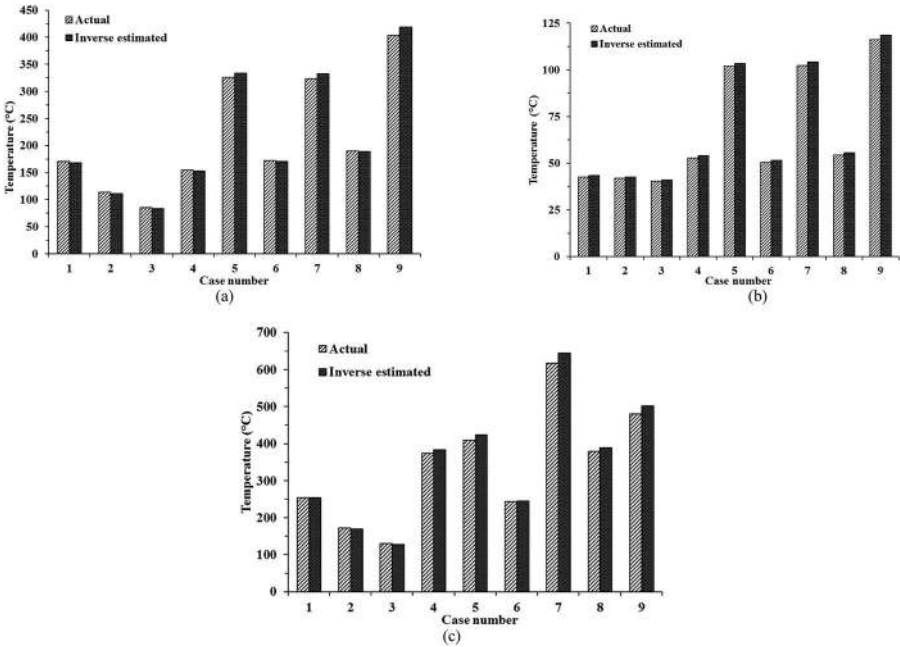


FIGURE 13.23 Results of the maximum temperature predicted by the actual and inverse estimated parameters for various cases (Table 13.5) of laser heating of a workpiece of size 100 mm × 50 mm at points (a) B1 of coordinate (x, y, z) = (45, 25, 0) (b) B2 of coordinate (x, y, z) = (55, 35, 0) and (c) laser heating point. (With permission from (Kumar et al., 2019a). Copyright Elsevier.)

The temperature history based on the actual parameters is compared with that based on the inverse estimated parameters. The percentage error of the maximum temperature being maximum at B1 and B2 for Case numbers 9 and 8, respectively, a comparison of temperature histories for these two cases are shown in Figures 13.24 and 13.25, respectively. It is found that the temperature history based on the actual parameters agrees well with that based on the inverse estimated parameters at B1, B2 and the heating point in both cases. In fact, all the data points of inverse estimation almost overlap with the actual temperature-time curve.

The sensitivity of the inverse method is further checked by measuring the error in the prediction of the transient temperature histories of two bottom points based on actual process parameters. The temperature after measurement error is expressed as

$$\theta_{meas} = \theta_{act} + \lambda, \tag{13.113}$$

where θ_{meas} and θ_{act} are the measured and actual temperatures, respectively and λ is the measurement error. The transient temperatures with errors $\pm 1\%$ and $\pm 5\%$ compared to the actual transient temperatures are obtained at B1 and B2, according to Equation 13.113. These errors are assumed based on $\pm 3\sigma_d$

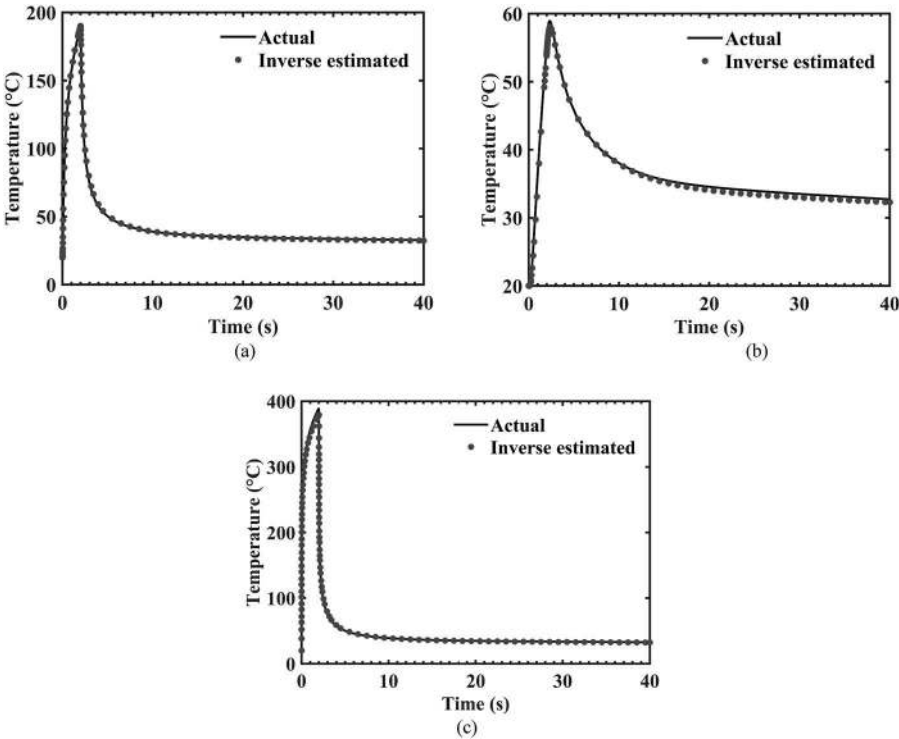


FIGURE 13.24 Comparison of temperature history of the laser-scanned sheet predicted with actual and inverse estimated parameters at points (a) $(x, y, z) = (45, 25, 0)$ (b) $(x, y, z) = (55, 35, 0)$ and (c) $(x, y, z) = (45, 25, 2)$ (Power: 600 W, laser beam diameter: 6 mm, heating time: 2 s, workpiece thickness: 4 mm). (With permission from (Kumar et al., 2019a). Copyright Elsevier.)

deviation, where σ_d is the standard deviation. Following the proposed procedure, the inverse solutions are obtained with various cases of errors. The uncertainties in results are shown in Table 13.4 considering extreme cases, that is positive error at one point and negative error at the other. As observed, $\pm 5\%$ measurement error leads to a lot of error in prediction. However, the errors are small with $\pm 1\%$ prediction. In fact, sometimes errors in η and h seem to improve with $\pm 1\%$ error. This may be because of an average value of k instead of temperature-dependent k , which may get adjusted in some cases to provide a lesser error in η and h .

Encouragingly, the temperature distribution predicted is not very sensitive to small errors in η , h and k . To demonstrate this, the values of maximum temperature at B1 are also shown in Table 13.6, which deviate by about $\pm 5\%$ for $\pm 5\%$ error in the measurement of temperatures. Thus, the inverse procedure is capable of predicting temperature distributions with the estimated parameters with an accuracy consistent enough with that based on temperature measuring instruments.

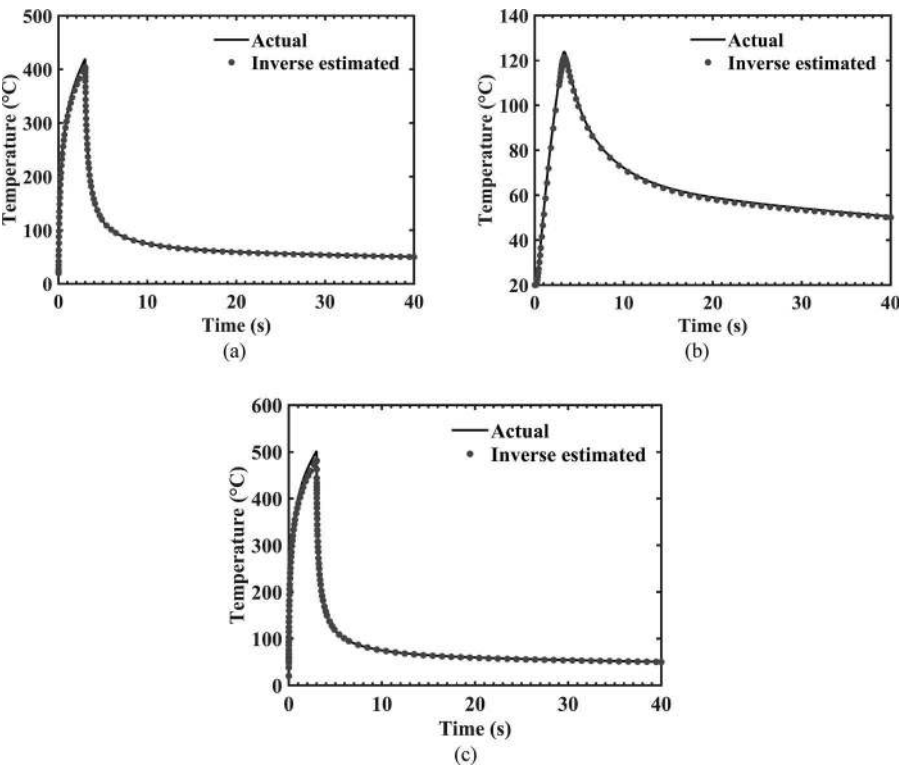


FIGURE 13.25 Comparison of temperature history of the laser-scanned sheet predicted with actual and inverse estimated parameters at points (a) $(x, y, z) = (45, 25, 0)$, (b) $(x, y, z) = (55, 35, 0)$ and (c) $(x, y, z) = (45, 25, 2)$ (Power: 600 W, laser beam diameter: 8 mm, heating time: 3 s, workpiece thickness: 2 mm). (With permission from (Kumar et al., 2019b). Copyright Elsevier.)

TABLE 13.6
Uncertainty Analysis For the Typical Case ($\eta_{act} = 0.6$ and $h_{act} = 20 \text{ W/m}^2 \text{ }^\circ\text{C}$)

Error at B1 (%)	Error at B2 (%)		Error in η (%)	h_{pred} (W/m $^\circ\text{C}$)	$ h_{pred} - h_{act} $ (W/m $^\circ\text{C}$)	k_{pred} (W/m $^\circ\text{C}$)	$(\theta_{B1})_{pred}$ $^\circ\text{C}$
		$(\eta)_{pred}$					
0	0	0.63	5	22.01	2.01	161.03	182.32
1	-1	0.624	4	21.34	1.34	154.84	184.14
-1	1	0.642	7	23.02	3.02	166.21	179.66
5	-5	0.559	-6.83	16	4.00	126.91	191.61
-5	5	0.711	18.5	28.11	8.11	201.4	172.67

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13.7 MODELING OF THERMAL AUTOFRETTAGE WITH HEAT TREATMENT

While the compressive residual stresses are beneficial, the tensile residual stresses reduce the strength of the autofrettaged vessel in the presence of external surface flaws like cracks (Parker, 1981). Tensile residual stresses are also detrimental for the fatigue strength and may promote stress corrosion cracking (Seifi and Babalhavaeji, 2012). Among a number of techniques to reduce the magnitude of the tensile residual stresses such as external wire winding (Sedighi and Jabbari, 2013), shot peening of the outer wall (Koh et al., 1997; Koh and Stephens, 1991) and post-heat treatment (Shufen and Dixit, 2018), the post heat treatment is the most recent. This section presents the FEM analysis of thermal autofrettage coupled with a heat treatment process. First, the concept of thermal autofrettage is briefly explained and then the FEM model of the coupled process is presented.

Thermal autofrettage is a process where a thermal load is applied to induce the residual stresses in the thick-walled cylinder or sphere. The process comprises two steps—a loading step and an unloading step. In the loading step, a temperature gradient is set up radially across the wall of the cylinder by keeping the inner wall at a temperature T_a while heating the outer wall to a higher temperature T_b as shown in Figure 13.26(a). The step creates an inner plastic zone and an outer elastic zone in the cylinder. In the unloading step, the temperature gradient is removed by cooling down the cylinder to ambient temperature, T_{amb} . The induced internal radial pressure after autofrettage is shown in Figure 13.26(b). This induces residual stresses, which are compressive at the vicinity of the inner wall and tensile at the vicinity of the outer wall. As a means to nullify the outer wall tensile stresses, Shufen and Dixit (2018)

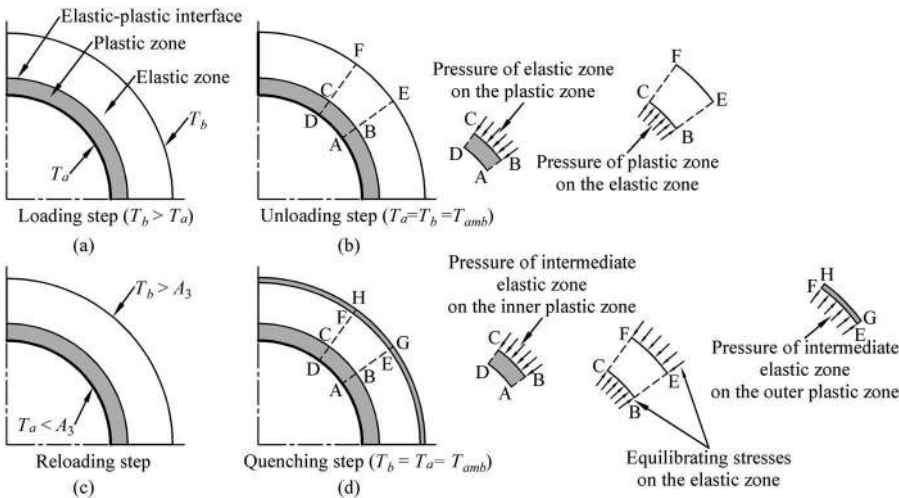


FIGURE 13.26 A typical heat treatment coupled thermal autofrettage process.

suggested a technique where the thermal autofrettage process is combined with a post-heat treatment. In the process, the cylinder is subjected to two additional steps after the thermal autofrettage, a reloading step and a quenching step.

In the reloading step, the thermally autofrettaged cylinder is again subjected to a temperature gradient by heating the outer wall to a temperature $T_b > A_3$ whereas the inner wall is kept at a temperature $T_a < A_3$ as shown in Figure 13.26(c). This step austenizes the vicinity of the outer wall. In the quenching step, the cylinder is cooled rapidly to ambient temperature T_{amb} as shown in Figure 13.26(d) causing a sudden contraction of the outer wall and a plastic deformation. This creates an outer plastic zone only in the vicinity of the outer wall. With further cooling, the continued contraction of the underlying material takes place. This underlying material pulls the plastically deformed outer wall and reduces the initial tensile residual stress at the vicinity of the outer wall as the temperature approaches room temperature. Eventually, the outer surface attains compressive residual stresses.

13.7.1 DETAILS OF FEM ANALYSIS FOR SIMULATING EXPERIMENTS

An axisymmetric model is used for the analysis as shown in Figure 13.27. The model simplifies the analysis from a three-dimensional cylinder to a two-dimensional rectangular region. This approach results in a significant reduction of the computational time and cost.

The microstructural equations are explained in detail in Section 13.3.1. The thermal equation is governed by Fourier's law of conduction already given in Equation 13.47. The constitutive relationship for the mechanical behavior is the same as that explained in Section 13.3.3. However, the strains due to TRIP and phase dilatation

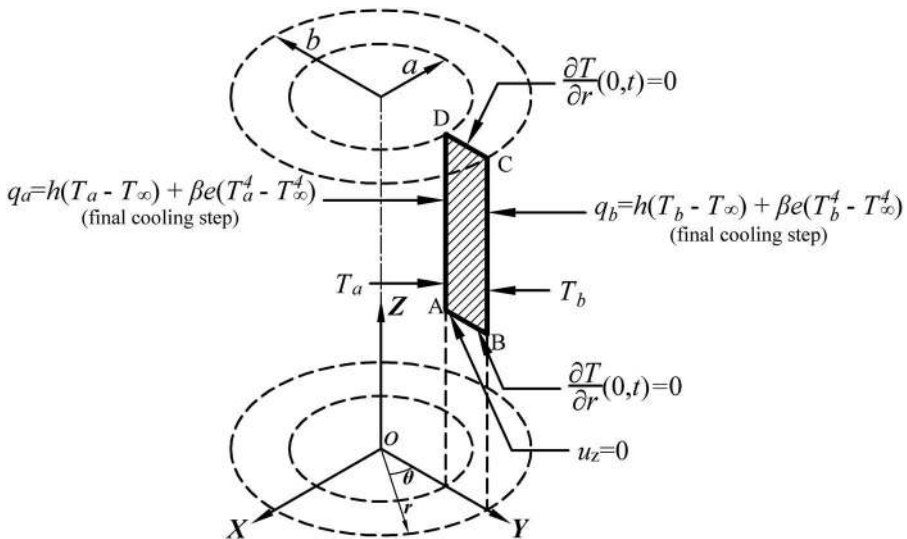


FIGURE 13.27 Axisymmetric model of the thick-walled cylinder. (With permission from Shufen and Dixit (2018). Copyright Elsevier.)

are neglected from the total strain of Equation 13.51. Thus, the total strain rate in this model is

$$\dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^e + \dot{\epsilon}_{ij}^p + \dot{\epsilon}_{ij}^{th}. \quad (13.114)$$

Subsequently,

$$\dot{\sigma}_{ij} = C_{ijkl} \dot{\epsilon}_{ij}^e = C_{ijkl} (\dot{\epsilon}_{ij} - \dot{\epsilon}_{ij}^p - \dot{\epsilon}_{ij}^{th}). \quad (13.115)$$

The overall FEM analysis can be divided into four successive analysis steps on the basis of the type of analysis and the thermal loading conditions subjected to the cylinder. The thermal and mechanical boundary conditions subjected to the part geometry of the FEM model are depicted in Figure 13.27. The inner wall (AD), outer wall (BC) and the free end (CD) all remain traction free. Due to the longitudinal symmetry of the cylinder specimen, the FEM part model has been reduced to half the length of the cylinder. This condition is applied using the mechanical boundary condition $u_z = 0$ at the mid-section (represented by AB in Figure 13.27) of the cylinder. In all the analysis steps, the ends of the cylinder are insulated. The insulation boundary condition is expressed as follows:

$$\frac{\partial T}{\partial r}(0, t) = 0, \quad (13.116)$$

which is assigned to the edges AB and CD as depicted in Figure 13.27. Other thermal boundary conditions exclusive to each analysis step are described as follows:

13.7.1.1 First and Second Step (Thermal Autofrettage)

In the first step, the inner wall (AD in Figure 6.8) is assigned the ambient temperature $T_a = T_{amb}$, while the outer wall (BC in Figure 13.27) is assigned a higher temperature $T_b = T_a + \Delta T$, where ΔT is the required temperature difference for the desired level of autofrettage. In the second step, the temperature of the outer wall BC is lowered to $T_b = 20^\circ\text{C}$, while retaining the temperature of the inner wall AD. Since the two steps are not dependent on time, a steady state analysis is sufficient for executing Steps 1 and 2.

13.7.1.2 Third Step (Austenization)

In this step, the temperature of the outer wall BC is assigned a temperature more than the upper critical temperature while the inner wall is kept at a lower temperature such that the difference does not result reyielding of the inner wall. Like in the case of the first two steps, a steady-state analysis is carried out.

13.7.1.3 Fourth Step (Quenching)

The phase transformation kinetics are incorporated in this step and hence it is required to carry out a transient analysis. Transient analysis takes the results at the end of Step 3 as the initial condition. The cylinder is cooled at a fast rate from

both the inner and outer walls simultaneously. A combined convection and radiation boundary condition is assigned to both the inner (AD) and outer (BC) walls, which is mathematically expressed as follows:

$$\left. \begin{aligned} q_a &= h(T_a - T_{amb}) + \beta e(T_a^4 - T_{amb}^4) \\ q_b &= h(T_b - T_{amb}) + \beta e(T_b^4 - T_{amb}^4) \end{aligned} \right\}, \quad (13.117)$$

where q_a , q_b are the heat fluxes at the inner and outer cylinder walls, respectively, e is the thermal emissivity and $\beta = 5.67 \times 10^{-8} \text{ W/m}^2$ is the Stefan-Boltzmann constant.

13.7.2 RESULTS AND DISCUSSION

The results of the FEM analysis are based on the work presented by Shufen and Dixit (2018). The autofrettage specimen was a 120 mm long cylinder with an inner radius of 10 mm and an outer radius of 30 mm made-up of 1080 steel, which is a eutectoid steel with a carbon content of 0.7%. Thus, the possible phases in the alloy were pearlite, bainite and martensite. It was assumed that the cylinder was initially composed of 55% pearlite and 45% bainite in its microstructure. The initial thermal autofrettage was carried out using a temperature difference of 350 °C, with an inner wall temperature of 20 °C and an outer wall temperature of 370 °C. The cylinder yielded at the inner wall when the temperature difference reached 229 °C. With a further increase in the temperature gradient, the compressive stresses increased. When the temperature difference reached 323 °C, the outer wall of the cylinder commenced yielding due to compressive stresses. The distribution of stresses in the cylinder after the thermal loading step and residual stresses after the unloading step are depicted in Figures 13.28(a) and 13.28(b), respectively. The cylinder comprises an inner plastic zone I up to a radius of 11.41 mm and an elastic zone II almost up to the outer wall. The maximum hoop and axial compressive residual stresses at the inner wall of the cylinder are 231.94 MPa and 260.17 MPa, respectively.

The thermally autofrettaged cylinder was subjected to a radial temperature gradient that austenitized the outer wall. The outer wall was heated to 725 °C and the temperature of the inner wall was based on a criterion that the resulting temperature gradient did not further yield the thermally autofrettaged cylinder. This minimum temperature difference was about 60 °C. Thus, the inner wall was heated to 700 °C. The stress distribution after the reheating step is depicted in Figure 13.29. As can be inferred, there is no further yielding in the cylinder and it remains in the same deformed state as obtained from the preceding thermal autofrettage process step.

The austenized cylinder was simultaneously cooled from both the inner and outer walls by subjecting it to the heat transfer boundary conditions given in Equation 13.117. A thermal emissivity e of 0.3 was used as the typical value. Two cases of cooling rates due to $h = 300 \text{ W/(m}^2\text{K)}$ and $h = 500 \text{ W/(m}^2\text{K)}$ were considered. The variation of the hoop and axial residual stresses at the outer surface with

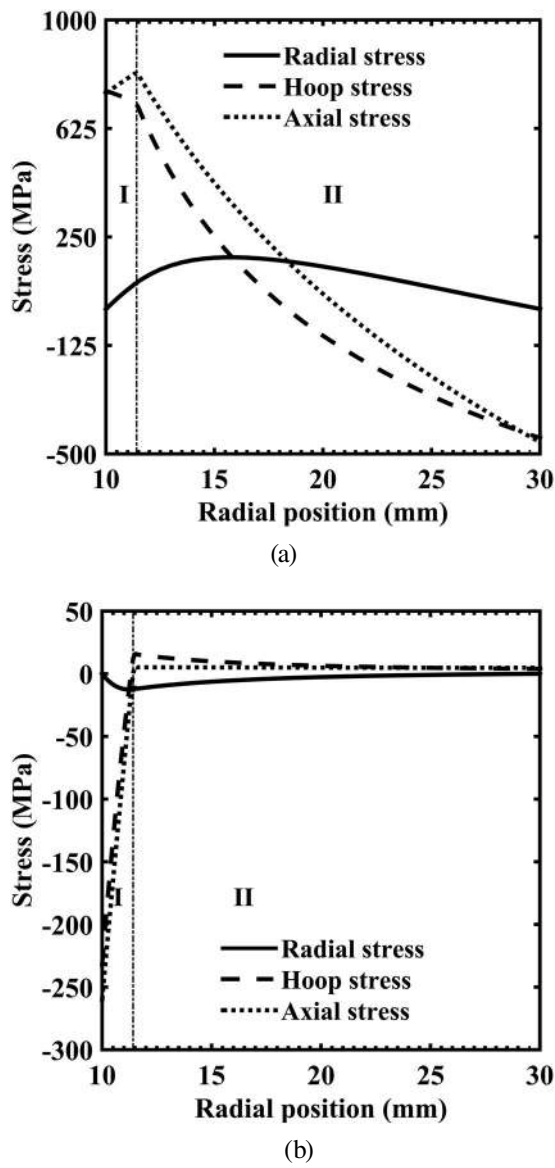


FIGURE 13.28 Radial distribution of (a) stresses after loading step and (b) residual stresses after unloading step in the cylinder thermally autofrettaged with wall temperatures $T_a = 20^\circ\text{C}$ and $T_b = 345^\circ\text{C}$.

time and temperature in each case of cooling the rate is depicted in Figure 13.30. The behavior of residual stresses with time and temperature may be divided into two stages—tensile and compressive. These two stages are explained in the following subsections.

13.7.2.1 Stage of Tensile Behavior

When the austenized outer wall is rapidly cooled, the surface temperature of the outer wall drops instantly. This leads to an immediate contraction of the outer wall surface. The underneath material resists this contraction due to which tensile stresses

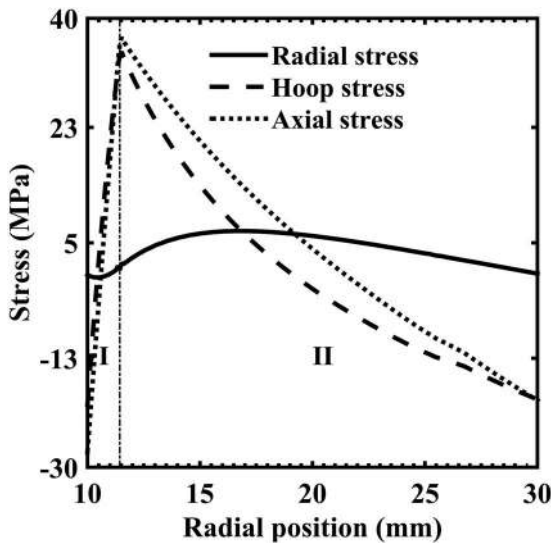


FIGURE 13.29 Distribution of stresses at the end of the reloading step with the temperature boundary conditions $T_a = 700\text{ }^{\circ}\text{C}$, $T_b = 725\text{ }^{\circ}\text{C}$.

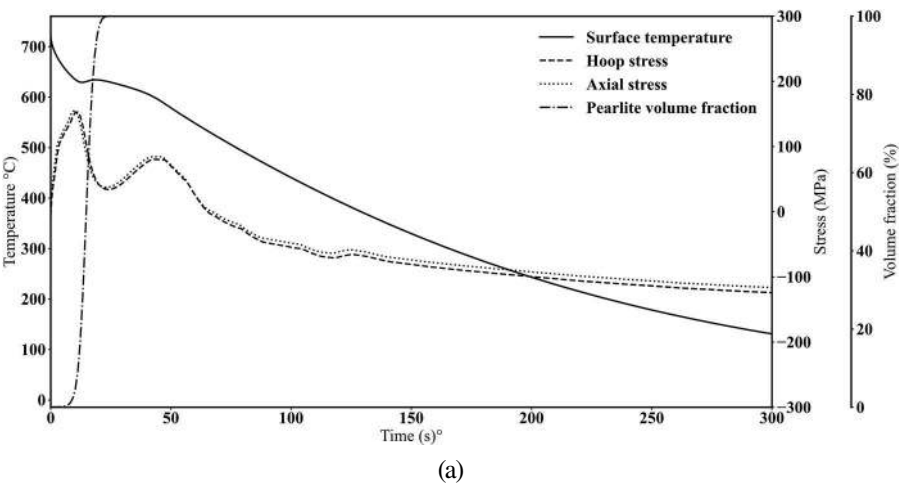


FIGURE 13.30 Variation of temperature, pearlite volume fraction, hoop and axial stresses with time at the outer wall of the cylinder for (a) $h = 300\text{ W/(m}^2\text{K)}$ and (b) $h = 500\text{ W/(m}^2\text{K)}$.

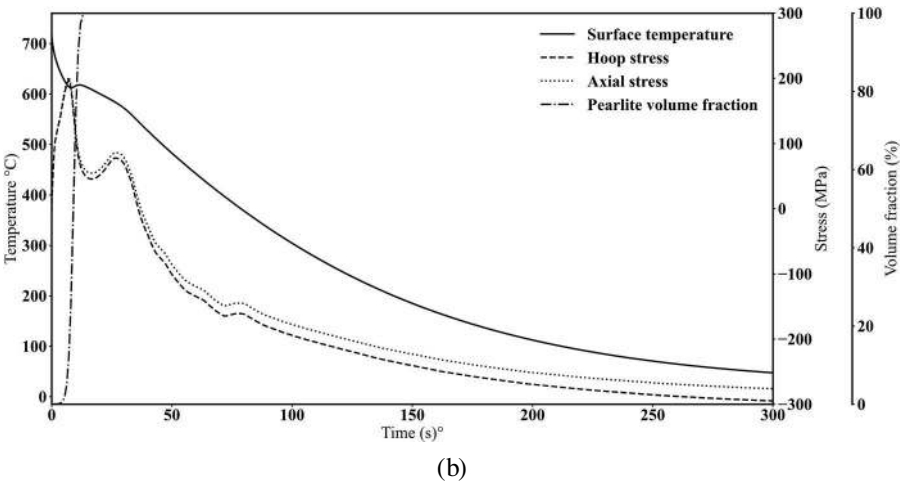


FIGURE 13.30 (Continued)

are generated at the outer wall surface. With further cooling, the temperature of the outer wall surface reaches the lower critical temperature 715 °C. At this temperature, pearlite starts to form as can be inferred from the transformation JMAK curves in Figures 13.30(a) and 13.30(b).

This causes a sudden increase in temperature due to the release of latent heat as shown by the temperature variation curve in Figures 13.30(a) and 13.30(b). The momentary increase in temperature causes expansion of the outer wall surface, which counteracts the contraction process and decreases the tensile hoop and axial stresses at the outer wall. This is also evident from the Figures 13.30(a) and 13.30(b). With time, the rate of pearlite transformation decreases leading to a decrease in the release of latent heat. Afterward, the contraction of the outer surface dominates and the tensile stresses again start to increase for some time.

13.7.2.2 Stage of Compressive Behavior

With further cooling, the rest of the cylinder material underneath the outer wall begins to contract gradually. Because of the larger volume, the contraction of the underlying material pulls the already cooled and hardened outer wall surface. This decreases the tensile residual stresses at the outer surface. As the temperature approaches room temperature, the outer initial tensile residual stresses become compressive.

The distribution of residual stresses at the end of the final cooling step with the cooling rates provided by $h = 300 \text{ W/(m}^2\text{K)}$ and $h = 500 \text{ W/(m}^2\text{K)}$ are depicted in Figures 13.31(a) and 13.31(b), respectively. The comparison of the maximum compressive residual stresses induced in the cylinder for these two cases is tabulated in Table 13.7. The corresponding decrease in the hoop and the axial stresses at the inner wall compared to the magnitudes, which is 231.94 MPa (for hoop stress) and 260.17 MPa (for axial stress) shown in Figure 3.28(b)

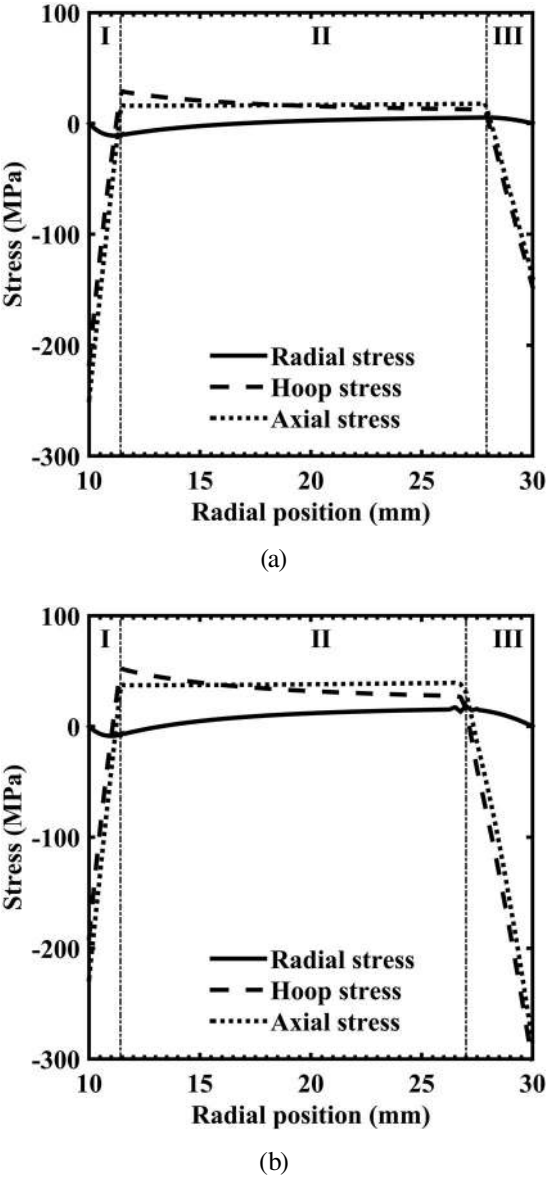


FIGURE 13.31 Distribution of residual stresses due to elastic recovery in the unloading step for various cases of cooling rates given by (a) $h = 300 \text{ W/(m}^2\text{K)}$ and (b) $h = 500 \text{ W/(m}^2\text{K)}$.

is also calculated. For the case of $h = 300 \text{ W/(m}^2\text{K)}$, the percentage decrease in the maximum hoop and axial residual stresses at the inner wall are 6.5% and 4.18%, respectively. For $h = 500 \text{ W/(m}^2\text{K)}$, the percentage decrease in the maximum hoop and axial residual stresses at the inner wall are 17.79% and 12.39%,

TABLE 13.7
Comparison of Residual Stresses and Increase in Pressure Bearing Capacity At Different Cooling Rates

Cylinder	h W/(m2.K)	Residual stress (MPa)				Yield pressure (MPa)	Increase in pressure-bearing capacity (%)
		Inner wall		Outer wall			
		Hoop	Axial	Hoop	Axial		
Original	–	0	0	0	0	372	–
Thermally autofrettagged	–	321.94	260.17	4.27	5.17	465	25%
Thermally autofrettagged	300	–216.86	–249.29	–147.59	–139.27	450	20.97%
and heat-treated	500	–190.67	–227.94	–304.77	–285.94	447	20.16%

respectively. This is because of plastic deformation of the outer wall that created the outer plastic zone III, which decreased the size of the elastic zone II, available at the end of the thermal autofrettage. As a result, the volume of material that suppressed the inner plastic zone decreased, which also reduced the inner wall compressive residual stresses.

To determine the changes in the pressure bearing capacity and the compressive residual stresses at the inner wall under different conditions of the cylinder, a pressure test was carried out in three types of cylinders—one without autofrettage, one after thermal autofrettage and one after thermal autofrettage and post heat treatment. The pressure test was carried out by pressurizing the cylinder with 600 MPa, which was applied gradually in increments of 1% of the load. The original cylinder without autofrettage started yielding at 62% of the load, that is at about 372 MPa. Thus, the pressure-bearing capacity of the cylinder without autofrettage was recorded as 372 MPa. Similarly, the thermally autofrettagged cylinder yielded at 465 MPa meaning a 25% increase in the pressure-bearing capacity compared to the 372 MPa of the original cylinder. For the thermally autofrettagged cylinder with the coupled heat treatment, the bearing capacity increased compared to the original cylinder but decreased compared to the cylinder with only thermal autofrettage. For the case of cooling with $h = 300 \text{ W/m}^2 \text{ K}$, the cylinder yielded at 450 MPa, which is an increase of 20.97% in the pressure bearing capacity whereas for that cooled with $h = 500 \text{ W/m}^2 \text{ K}$, the cylinder yielded at 447 MPa meaning an increase of 20.16% in the pressure bearing capacity. These along with the changes in the compressive stresses at the inner and outer wall are shown in Table 13.7. The distribution of microstructural phases in the cylinder after the process is depicted in Figure 13.32. The microstructure in the cylinder retains the initially assumed 55% pearlitic and 45% bainitic state up to a radius of about 20 mm while the remainder of the cylinder comprises a 100% pearlitic state.

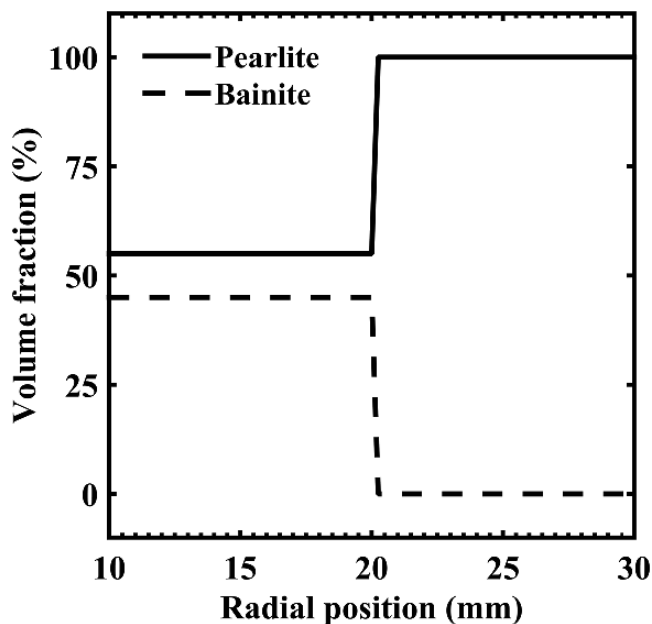


FIGURE 13.32 Distribution of microstructural constituents for the cooling rates corresponding to $h = 300 \text{ W/(m}^2\text{K)}$ and $h = 500 \text{ W/(m}^2\text{K)}$.

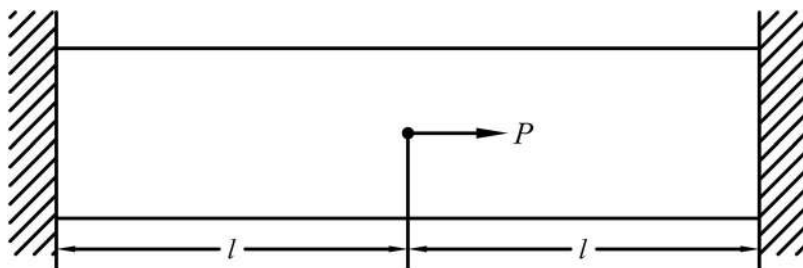


FIGURE 13.E1 Rod fixed at both ends with a point load at the mid section.

Exercises

1. A rod of length $2l$ rigidly constrained at both ends is subjected to an axial load P at the mid-length. Considering two equal finite elements, find the displacement at the point of application of load. Also, find the support reactions. Take the cross-sectional area of the rod as A and Young's modulus of elasticity as E .
2. It is given that the kinetics of transformation of an alloy is governed by the JMAK equation with rate constant $n_k = 3.1$. Out of a maximum transformable

fraction of 100%, if 30% is transformed after 10 minutes, determine the rate of transformation.

3. For a hypoeutectoid steel with a carbon content of 0.18%, calculate the maximum transformable fraction of the proeutectoid ferrite.
4. Consider an anisothermal transformation of a phase with $X_k^{max} = 1$, where 20% transformation has already occurred. Determine the fraction transformed in the next 1 minute along a JMAK curve whose start time is 1.01 minutes and finish time is 2.1 minutes.
5. Considering a diffusional transformation where 20% of the transformation has been completed, a displacive transformation commences at 225 °C. Calculate the fraction of the martensite when the temperature reaches 50 °C.

ANSWERS

1. Displacement is $\frac{Pl}{2AE}$. Both the support reactions are $-P/2$.
2. 0.08 min^{-1}
3. 50.11%
4. 0.8
5. 0.6832

14 Introduction to Crystal Plasticity*

14.1 INTRODUCTION

A crystalline material is characterized by a well-ordered repeating atomic structure. In a crystal, the constituent particles are arranged in a three-dimensional lattice pattern, which extends throughout the material volume. Interrogating the substructure of a typical engineering metallic component, we observe the existence of individual distinct grains of the material of varying shape, size and orientation. Due to the often random nature of the grain orientation, the macroscopic mechanical response of the material is nominally isotropic. The plasticity theory described in the earlier chapters is essentially the plasticity of ductile materials composed of a large ensemble of randomly oriented single-crystalline grains. In general, both instantaneous as well as time-dependent deformation of crystalline aggregates depends on the direction of the mechanical loads, its geometric constraints and the specific orientation of the crystal. As the size of a component reduces, deviation from isotropic behavior increases.

Crystal plasticity is a field of study that focuses on understanding the deformation mechanisms of a single or an ensemble of crystals under loads. The discipline is rooted in the fundamental principles of solid mechanics, crystallography and materials science. The goal of crystal plasticity is to accurately predict the mechanical behavior of materials at the relevant component level length-scale. It also helps to determine the behavior of the material under complex loading conditions and predict the potential location of component damage and failure. The mechanical properties of materials are strongly influenced by the crystal structure and orientation. In metals, plastic deformation occurs through the motion of dislocations. These are line defects in the crystal lattice. Dislocations move through the crystal lattice due to external stress, causing plastic deformation. The motion of dislocations is influenced by several factors, including the crystal structure, the type and density of defects, the existence of impurities, imposed deformation rates, process temperature and the crystallographic orientation with respect to the applied stress.

The plastic deformation of single crystals under homogeneous stress usually occurs by dislocation glide on specific crystallographic planes. These are known as primary glide systems or slip systems, which are inherent to the type of crystals being studied. Microstructures evolve based on thermodynamic potentials and their gradients. Thermodynamic potential indicates the state of a system. Four thermodynamic functions are internal energy, enthalpy, Gibbs free energy and Helmholtz free energy. The equilibrium phase has the lowest Gibbs free energy. However, the

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actual evolution path is strongly influenced not only by Gibbs' free energy but also by kinetics. This implies that microstructures form based on thermodynamic transients and as a rule not in full thermodynamic equilibrium. Thus, one needs to study non-equilibrium thermodynamics for understanding the evolution of microstructure. The significant impact of thermodynamic non-equilibrium mechanisms on the thermodynamic process results in a diverse and complex range of microstructures commonly found in engineering materials. Often, microstructures that correspond to a non-equilibrium state offer particularly advantageous material properties. These observations have contributed to and enhanced the field of product miniaturization where the nuances of the material microstructure in addition to the knowledge of the specifics of crystal plasticity are exploited to ensure the structural integrity of parts. Before delving into the details of constitutive laws appropriate for crystalline materials, some basics of crystallography and the conventions typically used to depict orientations and directions are reviewed.

14.2 COMMON CRYSTAL STRUCTURES

Most metal crystals of practical interest have face-centered cubic (FCC), body-centered cubic (BCC), or hexagonal crystal (HCP) structures (Callister and Rethwisch, 2009). A 'unit cell' of these crystal structures is shown in Figure 14.1. Aluminum, copper, nickel and gold are some typical FCC crystals; chromium, niobium, β -titanium, α -iron are BCC; whereas zinc, magnesium and cadmium are HCP. In this chapter only FCC crystal structures are discussed; models of BCC and HCP existing in the wider literature may be perused should the reader be interested.

For a cubic crystal, three basis vectors (directions) are $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, which are orthogonal to each other (Figure 14.1). Various planes and directions in a crystal are identified through Miller index notation. In this notation, coordinates of the unit cell are normalized such that each edge is of size one. Thus, if the edge along x -axis is of size a , the x -coordinates in physical space will be divided by a . Similarly, if the edge along y -axis is of size b , the y -coordinates in physical space will be divided by b . Finally, if the edge along z -axis is of size c , the z -coordinates in physical space will be divided by c . Thus, the normalized coordinates of a unit cell are bounded between -1 and 1 . Brackets $[]$, $()$, $\langle \rangle$ and $\{ \}$ are used to represent some critical aspect of the crystal orientation in Miller index notation. To indicate a direction, the notation $[lmn]$ is used. Here l, m, n are positive integers denoting a magnitude of l in the $\mathbf{e}_1$ direction, m in the $\mathbf{e}_2$ direction and n in the $\mathbf{e}_3$ direction. Thus, in the vector form direction is $(le_1 + me_2 + ne_3)$ that can be normalized by dividing it by $\sqrt{l^2 + m^2 + n^2}$. An index represented with a bar on top implies a negative direction. Thus $[\bar{l}mn]$ represents direction $(le_1 - me_2 + ne_3)$. The symbol $\langle lmn \rangle$ denotes the family of $[lmn]$ directions that are identical due to the symmetry of the crystal. For example, $\langle 111 \rangle$ in a cubic crystal includes all the following directions: $[111], [1\bar{1}\bar{1}], [\bar{1}1\bar{1}], [\bar{1}\bar{1}1], [\bar{1}11], [1\bar{1}1], [11\bar{1}], [11\bar{1}]$. As a convention, Miller indices l, m and n are three prime integers, namely, their common factor can be only 1 or -1 . Thus, the direction $(2e_1 + 2e_2 - 2e_3)$ will not have the Miller indices as $[222]$, but the Miller indices will be obtained by dividing with a common factor 2 to get $[\bar{1}1\bar{1}]$.

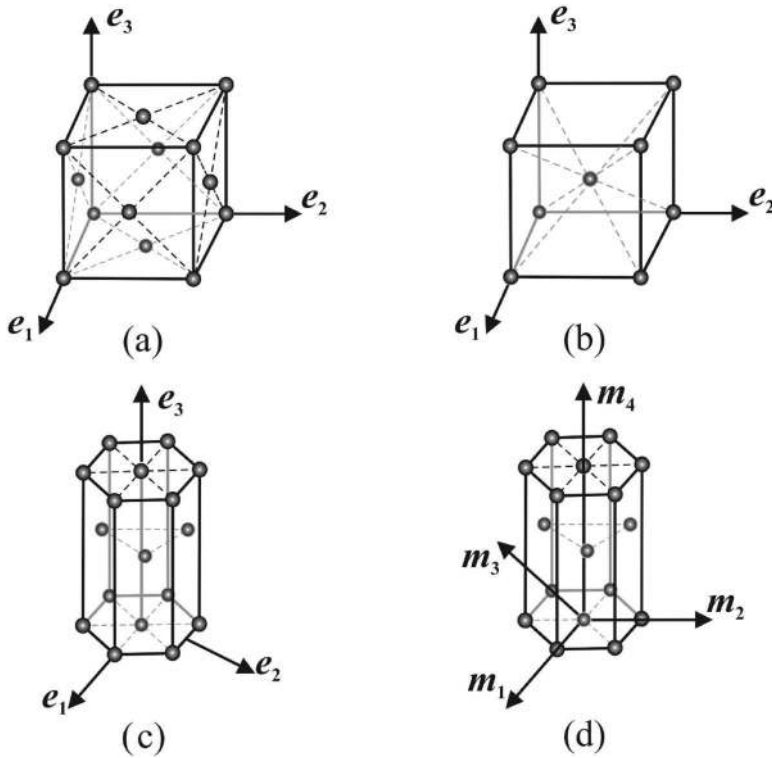


FIGURE 14.1 Common crystal structures: (a) FCC, (b) BCC, (c) HCP with orthonormal basis, (d) HCP with Miller-Bravais system.

The symbolic notation (lmn) denotes a plane that is perpendicular to a unit vector given by

$$\hat{n} = \frac{(le_1 + me_2 + ne_3)}{\sqrt{l^2 + m^2 + n^2}}. \quad (14.1)$$

As per three-dimensional coordinate geometry, the plane represented by Miller indices (lmn) is parallel to the plane

$$lx + my + nz = 1. \quad (14.2)$$

The coordinate system should be so chosen that it is possible to write the equation of the plane in the above form. The Miller indices should be integer only; hence, sometimes the right-hand term is a rational number other than 1 (Liu et al., 2020). It is emphasized that Miller indices of all parallel planes will be the same. In fact, Liu et al. (2020) have mentioned that in the literature Miller indices for a plane have been

misunderstood sometimes. As an example, in a paper by Sun et al. (2020) published in the same year, the procedure for identifying the Miller indices of a crystallographic plane is described based in the book by Askeland and Fulay (2009):

- (1) Identify the points at which the crystallographic plane intercepts the x , y and z coordinates in terms of the number of lattice parameters. Notably, the coordinate origin cannot be located on the above crystal facet. If the crystallographic plane passes through the coordinate origin, the origin of the coordinate system must be redefined.
- (2) Take reciprocals of the three intercepts, respectively.
- (3) Clear fractions but do not reduce to lowest integers, which are denoted as $h : k : l$.
- (4) Enclose the resulting number in parentheses (hkl).

Step 3 of the procedure contradicts the standard procedure, in which Miller indices are reduced to three prime integers. Thus, if a plane intercepts a cubic crystal of side a at $a/2$ at all axes, the Miller indices will be (222) by the procedure of Askeland and Fulay (2009). As per Liu et al. (2000) and a majority of researchers, Miller indices will be expressed as (111) only. Readers should carefully understand the definition of Miller indices while referring to a research article.

The notation $(\bar{l}\bar{m}n)$ denotes a plane that is perpendicular to a unit vector

$$\hat{n} = \frac{(le_1 - me_2 + ne_3)}{\sqrt{l^2 + m^2 + n^2}}. \quad (14.3)$$

Finally, the notation $\{lmn\}$ denotes the family of (lmn) planes that are crystallographically identical by symmetry. Notice that no comma is used between the indices unless an index is in double digits. However, the comma is used between the coordinates. For example, (1,2,3) denotes a point with coordinates as 1, 2 and 3, whereas (123) denotes a plane parallel to

$$x + 2y + 3z = 1. \quad (14.4)$$

The notation (10,23) denotes a plane parallel to

$$10x + 2y + 3z = 1. \quad (14.5)$$

In a hexagonal closed pack (HCP) crystal, the notations are different. Here, planes and directions are defined by introducing four auxiliary unit vectors m_1 , m_2 , m_3 and m_4 (Figure 14.1). This system is called the Miller-Bravais system; the main advantage of this system over the Miller system (three-index notation) for HCP is that the planes of the same family can be easily identified. The first three unit vectors lie in the basal plane and are oriented parallel to the three shortest distances between neighboring atoms. The angle between any two unit vectors in the basal

plane is 120° . The fourth vector is perpendicular to the basal plane. These vectors are related to an orthonormal basis (e_1, e_2, e_3) by the following relation:

$$m_1 = e_1, m_2 = \frac{-e_1 + \sqrt{3}e_2}{2}, m_3 = \frac{-e_1 - \sqrt{3}e_2}{2}, m_4 = e_3.$$

(14.6)

Here, the symbolic notation $[lmnp]$ denotes a vector direction $(lm_1 + mm_2 + nm_3 + pm_4)$. The symbol $\langle lmnp \rangle$ denotes the family of $[lmnp]$ directions that are crystallographically identical by symmetry. The notation $(lmnp)$ represents a plane that is perpendicular to the unit vector given by

$$\frac{lm_1 + mm_2 + nm_3 + pm_4}{\sqrt{l^2 + m^2 + n^2 + p^2 - lm - nl - mn}}.$$

(14.7)

The notation $\{lmnp\}$ denotes the family of $(lmnp)$ planes that are crystallographically identical. Notice that these indices are normalized with respect to edge length. Thus, $p = 1$ in $(lmnp)$ means that the reciprocal of the intercept of the plane is 1 on the edge parallel to vector m_4 . In the Miller-Bravais system, the sum of the first three indices is 0, which is $l + m + n = 0$.

In a crystal, a slip plane is the plane of greatest atomic density. The slip direction is the closest packed direction within the slip plane. The slip plane and slip direction together form a slip system. The movement of dislocation occurs on the slip plane along the slip direction. Slip planes and directions for common crystals are listed in Table 14.1.

Example 14.1

Determine the Miller indices of the shaded plane shown in Figure 14.2 in a cubic lattice.

SOLUTION

This plane is passing through the point (0, 0, 0) and hence cannot be written in the form given by Equation 14.2. Hence, the coordinate system is translated to another corner as shown in Figure 14.3. In this figure, coordinates are also normalized such that the unit cell becomes a cube of unit size. Notice that the

TABLE 14.1

Common Slip Plane and Slip Directions of Crystals

Crystal structure	Slip direction	Slip plane
FCC	$\langle 110 \rangle$	$\{111\}$
BCC	$\langle 111 \rangle$	$\{110\}, \{112\}$
HCP	$11\bar{2}0$	$(0001), \{11\bar{1}1\}$
	$11\bar{2}3$	$\{11\bar{1}0\}, \{11\bar{0}1\}.$

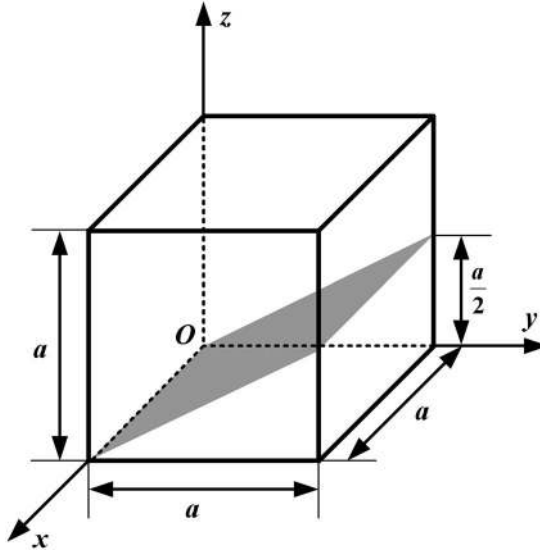


FIGURE 14.2 Shaded plane in a cubic crystal.

axes of the new coordinate system remain parallel to the original coordinate system. In physical space, three non-collinear points in the plane will be $(0, 0, a/2)$, $(a/2, 0, a/2)$ and $(0, -a, 0)$. Dividing the coordinates by a provides normalized coordinates, which are $(0, 0, 1/2)$, $(1/2, 0, 1/2)$ and $(0, -1, 0)$ as per the geometry of Figure 14.3. Hence, the equation of the plane in normalized coordinates is

$$\begin{vmatrix} (x-0) & (y-0) & \left(z-\frac{1}{2}\right) \\ \left(\frac{1}{2}-0\right) & (0-0) & \left(\frac{1}{2}-\frac{1}{2}\right) \\ \left(0-\frac{1}{2}\right) & (-1-0) & \left(0-\frac{1}{2}\right) \end{vmatrix} = 0. \quad (14.8)$$

Simplifying it,

$$\frac{y}{4} - \frac{1}{2}z + \frac{1}{4} = 0. \quad (14.9)$$

Arranging Equation 14.9 in the form of Equation 14.2, that is making the right-hand side equal to 1, the equation of the plane becomes

$$-y + 2z = 1. \quad (14.10)$$

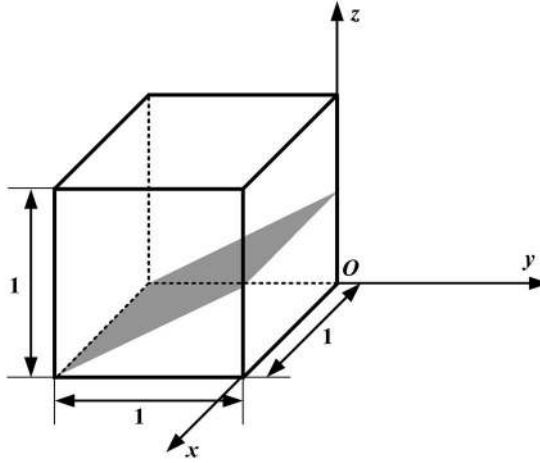


FIGURE 14.3 Coordinate system shifted to avoid the passing of the shaded plane from the origin.

The Miller indices are obtained as the coefficients of the coordinates. Hence, the Miller indices are $(0\bar{1}2)$.

Example 14.2

Find out the Miller indices of the directions of body diagonal OA and face diagonal OB in the unit cell shown in Figure 14.4. Also, indicate direction $[22\bar{1}]$ in the cell.

SOLUTION

The coordinates of O and A are $(0, 0, 0)$ and (a, a, a) , respectively. The corresponding normalized coordinates are $(0, 0, 0)$ and $(1, 1, 1)$. The vector $\overrightarrow{OA}$ can be written as

$$\overrightarrow{OA} = (1-0)\mathbf{e}_1 + (1-0)\mathbf{e}_2 + (1-0)\mathbf{e}_3 = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3. \quad (14.11)$$

Hence, the Miller indices of body diagonal OA are $[111]$.

The coordinates of B are $(a, a, 0)$. The normalized coordinates of B are $(1, 1, 0)$. The vector $\overrightarrow{OB}$ is given by

$$\overrightarrow{OB} = (1-0)\mathbf{e}_1 + (1-0)\mathbf{e}_2 + (0-0)\mathbf{e}_3 = \mathbf{e}_1 + \mathbf{e}_2. \quad (14.12)$$

Hence, the Miller indices of face diagonal OB are $[110]$.

The Miller indices $[22\bar{1}]$ indicate the direction $2\mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3$. Dividing by two, it is obtained as $\mathbf{e}_1 + \mathbf{e}_2 - 0.5\mathbf{e}_3$. In this form, coefficients associated with unit vectors do not exceed one. If vector $\overrightarrow{OC}$ represents this vector, the

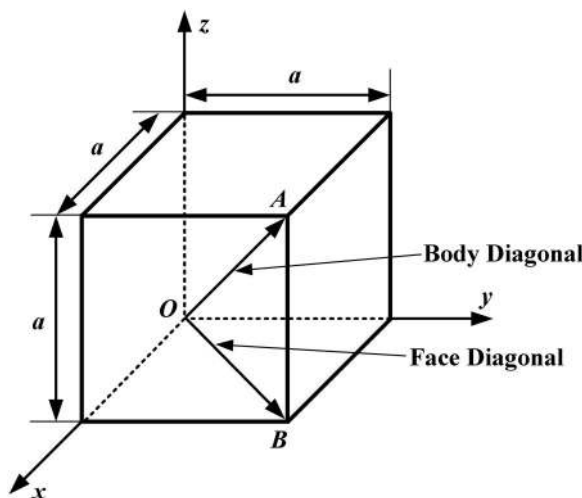


FIGURE 14.4 A cubic unit cell showing body and face diagonals.

coordinates of C will be given by $(1, 1, -0.5)$. This is in normalized form. In the physical space, the coordinates are $(a, a, -0.5a)$. Thus, the Miller indices $[22\bar{1}]$ will correspond to direction OC as shown in Figure 14.5. Notice that in this figure, the axis system shown in Figure 14.4 has been translated by distance a in the z -direction, such that origin O is now at the top-left corner. This is merely for the convenience of graphical representation so that point C can be shown in the unit cell. However, for all parallel directions, the Miller indices will be the same.

Example 14.3

The Miller-Bravais indices of a plane in HCP crystal are $(11\bar{2}0)$. Sketch the plane in a unit lattice.

SOLUTION

Intercepts of the plane on four axes can be written as the inverse of the Miller-Bravais indices. Thus, intercepts are $1, 1, -1/2$ and ∞ . It means that the plane cuts the m_4 axis at infinity. Hence, the plane is parallel to the axis normal to the basal plane. It cuts axes m_1 and m_2 at the corners because in the normalized coordinates, corners on axes m_1 and m_2 are given by $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$. This information is sufficient to sketch the plane. The plane is shown in Figure 14.6. However, let us find the intercept on the m_3 axis also.

From geometry, it is clear that m_3 axis bisects the line joining corners on m_1 and m_2 . A segment of the basal plane is shown in Figure 14.7. Plane $(11\bar{2}0)$ intersects the basal plane at line AB . Axes m_1, m_2 and m_3 make 120° with one another. From the symmetry, it is clear that triangles AOC and OCB

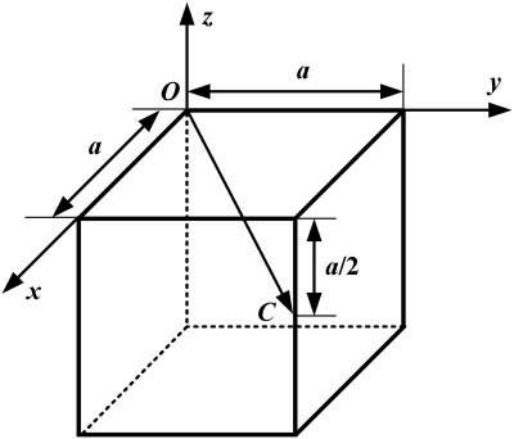


FIGURE 14.5 Direction OC represented by Miller indices $[22\bar{1}]$.

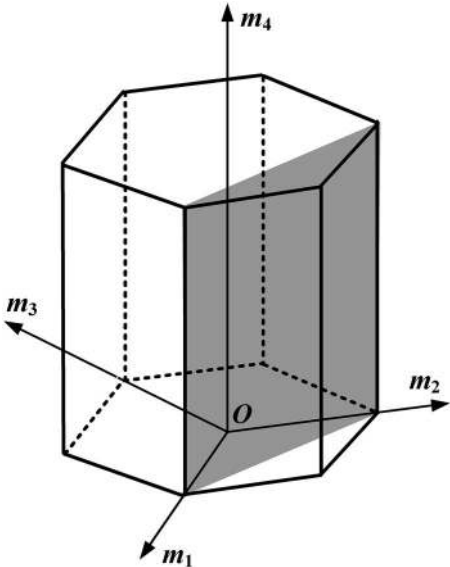


FIGURE 14.6 A shaded plane $(11\bar{2}0)$ in an HCP unit lattice.

are congruent triangles. Therefore, $\angle AOC = 60^\circ$ in the right-angle triangle AOC. Thus, $OC = OA/2 = 0.5$ in the normalized coordinate system. Hence, the intercept of line AB is $-1/2$ on m_3 axis and the third Miller-Bravais index will be $\bar{2}$.

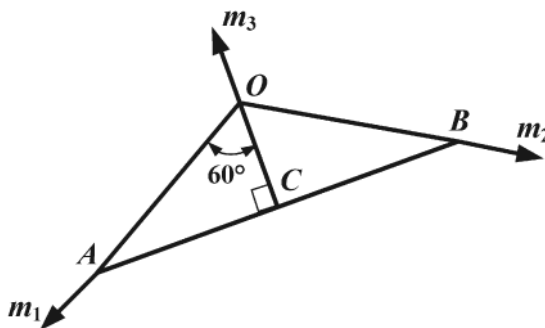


FIGURE 14.7 Intersection of the plane $(11\bar{2}0)$ with the basal plane shown by line AB.

Example 14.4

In the basal plane of a HCP crystal (Figure 14.7) direction OA is along the unit vector $\mathbf{m}_1$. Find the Miller-Bravais indices of this direction.

SOLUTION

Consider the normalized length of OA as one. The projection of OA on $\mathbf{m}_3$ is OC, whose length is 0.5; direction of OC is opposite to $-\mathbf{m}_3$. Similarly, the projection of OA on $\mathbf{m}_2$ is also $-\mathbf{m}_2$, which is evident from symmetry. Thus, projections of OA on unit vectors of the basal plane are 1, $-\mathbf{m}_2$ and $-\mathbf{m}_2$. Multiplying by two to make all of these integers, we get 2, $-\mathbf{m}_2$ and $-\mathbf{m}_2$. The projection of OA on $\mathbf{m}_4$ is 0. Hence, the Miller-Bravais indices of OA are $[2\bar{1}\bar{1}0]$.

14.3 PLASTIC FLOW IN SINGLE CRYSTALS

In the early 20th century, several scientists such as M. Polanyi (1891–1976), E. Schmid (1896–1983), E. Orowan (1902–1989) and G.I. Taylor (1886–1975) discussed the nature of plastic slip in crystalline materials. Their pioneering works formed the foundation for developing constitutive laws governing plastic flow in single and polycrystalline materials. A brief account of their works with references has been presented by Rodriguez (1996). For the last two decades, growing large sizes of single crystals have been pursued extensively in the aerospace industry. Understanding the plasticity in such single crystals becomes crucial for the design of components made of them. Single crystal components can span sizes in meters down to the microscale. The crystal structure affects the plastic behavior. It is not possible to explain experimentally observed phenomena like the Bauschinger effect and sample size effect without understanding the crystal structure.

Plastic flow in a crystal is most often measured by conducting a tensile test with the loading axis parallel to a chosen crystallographic direction. Experiments show that plastic deformation is the result of relative motion on slip planes, in response to shear stress along these planes. In particular, in a tensile specimen of a single crystal,

the tensile stress σ acts along an orientation vector at an angle of ϕ with the slip plane normal and at an angle of θ with the slip direction vector (Figure 14.8). Note that the sum of θ and ϕ need not be equal to 90° . Denoting τ as the resolved shear stress on the slip plane and in the slip direction,

$$\sigma = \tau (\cos \phi \cos \theta)^{-1}. \quad (14.13)$$

For three-dimensional loading, the resolved shear stress can be obtained as follows. Let σ be the stress tensor at a point. By Cauchy's theorem, the traction vector on a plane with normal $\mathbf{n}$ is given by

$$\mathbf{t}_n = \sigma^T \mathbf{n}. \quad (14.14)$$

If the slip direction on that plane is $\mathbf{s}$, then the magnitude of resolved shear stress will be given by

$$\tau = \mathbf{t}_n \cdot \mathbf{s} = (\sigma^T \mathbf{n}) \cdot \mathbf{s}. \quad (14.15)$$

For a symmetric stress tensor, Equation 14.15 can be written as

$$\tau = \mathbf{t}_n \cdot \mathbf{s} = (\sigma \mathbf{n}) \cdot \mathbf{s}. \quad (14.16)$$

In index notation, it is expressed as

$$\tau = \sigma_{ij} n_j s_i. \quad (14.17)$$

Equation 14.17 can also be written as

$$\tau = (s_i n_j) \sigma_{ij} = (\mathbf{s} \otimes \mathbf{n}) : \sigma. \quad (14.18)$$

Several experimental studies have confirmed that as the resolved shear stress exceeds a critical value, which is a constant for a given material at a given temperature, slip is initiated in the slip system. This observation is referred to as Schmid's law. The critical resolved shear stress (or the strength of the slip system) as a rule, is much higher for BCC metals than that for FCC or HCP metals.

HCP crystals have three primary slip systems, while FCC crystals have twelve primary slip systems. FCC has four slip planes corresponding to $\{111\}$ system. In fact, $\{111\}$ system represents eight planes, but only four of them are distinct. For example, $(\bar{1}\bar{1}\bar{1})$ the plane is the same as (111) plane. With reference to a coordinate system, if (111) represents one face of the plane, then $(\bar{1}\bar{1}\bar{1})$ can represent the other face of the plane. On a typical plane of $\{111\}$ family, there are three slip directions corresponding to three face diagonals. Note that by multiplying the Miller indices for direction by -1 , only the direction gets reversed; hence, no new slip direction is obtained. Thus, $[0\bar{1}1]$ and $[01\bar{1}]$ represent the same slip direction. As a result, in

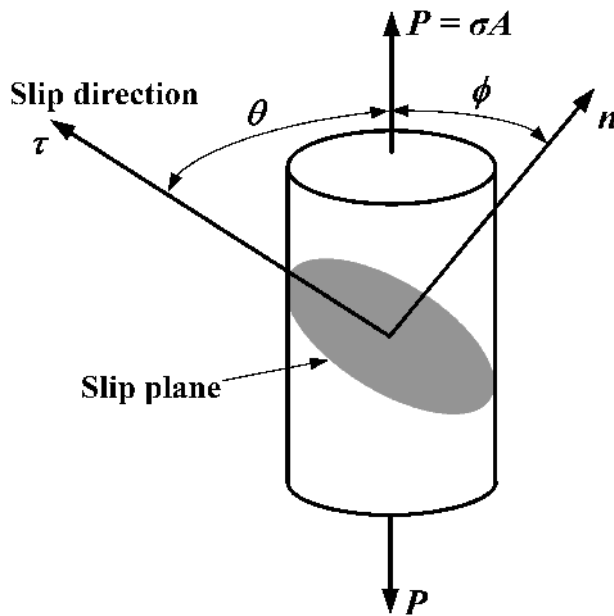


FIGURE 14.8 Resolved shear stress along slip direction.

12 independent slip systems, FCC metals exhibit considerably more ductility than HCP metals. In BCC crystals there are twelve slip systems. However, in some crystals, the difference in packing density between the closest-packed planes and certain other planes is not great, due to which additional slip systems become available even at ordinary temperatures. Metals having a BCC structure have a ductility similar to that of FCC metals.

An important aspect to keep in mind is that this perceived correlation between plastic deformation and crystal type is true only theoretically. In practice, almost every crystalline material contains impurities and inclusions that significantly influence the response of the crystal to external loads. It is widely accepted that all inelastic characteristics, including the hardening response, are influenced by this aspect. Elastic parameters depend only on the lattice structure and are assumed to be insensitive to impurities in the crystal lattice. Thus, for practical applications, it becomes imperative to assess all inelastic material parameters comprehensively via experiments to obtain numerically predictive computational models.

The subsequent discussion is based on several references. As additional readings, the book by Dixit and Dixit (2008) and a research paper by Liu et al. (2016) are recommended.

14.3.1 KINEMATICS: A STEP TOWARD LARGE DEFORMATION CONSTITUTIVE LAWS

In Section 5.5, the additive decomposition of the infinitesimal strain tensor into elastic and plastic parts was introduced. For strains that are too large to be treated as

TABLE 14.2
Slip System For FCC

Slip plane	Slip directions
(111)	$[0\bar{1}1], [10\bar{1}], [\bar{1}10]$
$(\bar{1}\bar{1}1)$	$[011], [\bar{1}0\bar{1}], [1\bar{1}0]$
$(\bar{1}11)$	$[0\bar{1}1], [\bar{1}0\bar{1}], [110]$
$(1\bar{1}1)$	$[011], [10\bar{1}], [\bar{1}\bar{1}0]$

infinitesimal, the additive decomposition is valid for logarithmic longitudinal strain only. To formulate a theory of plasticity for finite (large) deformation, the concept of deformation gradient introduced in Section 4.3 is used. The theory was developed notably by Lee (Lee, 1969) and Mandel (Mandel, 1973) based on considerations of the behavior of crystals. The theory first considers an infinitesimal neighborhood $(\mathbf{x}, \mathbf{x} + d\mathbf{x})$ in a plastically deformed crystal that could be cut out and instantaneously relieved of all stresses. This would be mapped into $(\hat{\mathbf{x}}, \hat{\mathbf{x}} + d\hat{\mathbf{x}})$, the transformation being composed of a rigid-body displacement and a purely elastic deformation (Figure 14.9). The position $\hat{\mathbf{x}}$ is arbitrary, but a linear relationship between $d\mathbf{x}$ and $d\hat{\mathbf{x}}$, may be assumed, that is,

$$d\hat{\mathbf{x}} = F_e^{-1} d\mathbf{x}. \quad (14.19)$$

Here F_e is not to be interpreted as a deformation gradient, since the transformation is not a displacement of the whole body. As F_e is non-singular, it may be subjected to polar decompositions

$$F_e = R_e U_e = V_e R_e. \quad (14.20)$$

The symmetric tensors U_e and V_e represent the elastic deformation that is removed along with the removal of the stresses. The tensor R_e is a rotation tensor. Now consider a stress-free reference configuration of the crystal, such that

$$d\mathbf{x} = F d\mathbf{x}_0, \quad (14.21)$$

then

$$d\hat{\mathbf{x}} = F_e^{-1} F d\mathbf{x}_0. \quad (14.22)$$

Defining

$$F_p = F_e^{-1} F, \quad (14.23)$$

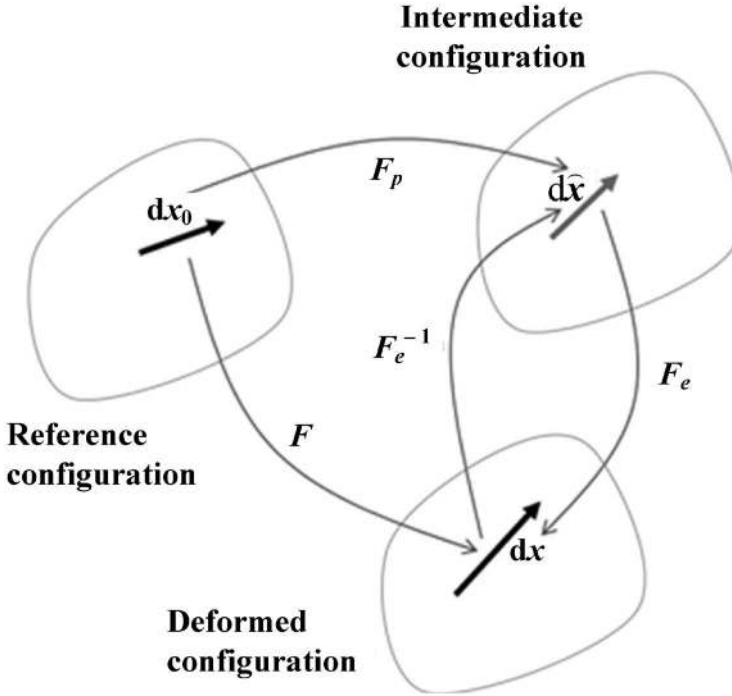


FIGURE 14.9 Deformation from a reference to a deformed configuration.

we can write

$$F = F_e F_p. \quad (14.24)$$

The process that moves the body from the reference configuration to the current one is responsible, in general, for both elastic and plastic deformations. By hypothesis, the elastic deformation is removed in the intermediate configuration. Consequently, the intermediate configuration may be said to be only plastically deformed, with the effects of the internal processes leading to plastic deformation embodied via the transformation F_p .

Example 14.5

A material undergoes deformation in the following manner:

$$x = 2x_0 + 0.070711y_0; y = 0.70711y_0 + 0.070711z_0; z = 0.2x_0 + 0.70711z_0,$$

where x , y and z are the current coordinates and x_0 , y_0 , z_0 are the initial (and also reference) coordinates.

1. Find the deformation gradient.

2. If the plastic part of the deformation gradient is given as

$$[F_p] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix},$$

find out the elastic part of the deformation gradient.

3. Show that the plastic part of deformation is isochoric, while the elastic part of deformation is non-isochoric.

SOLUTION

1. Using Equation 4.51, the deformation gradient is found as

$$[F] = \begin{bmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} & \frac{\partial x}{\partial z_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} & \frac{\partial y}{\partial z_0} \\ \frac{\partial z}{\partial x_0} & \frac{\partial z}{\partial y_0} & \frac{\partial z}{\partial z_0} \end{bmatrix} = \begin{bmatrix} 2 & 0.070711 & 0 \\ 0 & 0.70711 & 0.070711 \\ 0.2 & 0 & 0.70711 \end{bmatrix}.$$

2. As $F = F_e F_p$, elastic part of the deformation gradient is given as

$$\begin{aligned} [F_e] &= [F][F_p]^{-1} = \begin{bmatrix} 2 & 0.070711 & 0 \\ 0 & 0.70711 & 0.070711 \\ 0.2 & 0 & 0.70711 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 0.1 & 0 \\ 0 & 1 & 0.1 \\ 0.1 & 0 & 1 \end{bmatrix}. \end{aligned}$$

3. Here, the determinant of F_p is one. Hence, the plastic part is isochoric. The determinant of F_e is 1.001. Hence, it is not isochoric. In fact, the volume increases.

14.3.2 KINEMATICS AND CONSTITUTIVE LAW FOR SINGLE CRYSTAL PLASTICITY

Now let us consider a crystal lattice (FCC, BCC or HCP). The slip systems in the undeformed crystal are identified by unit vectors $\mathbf{M}^{(\alpha)}$ normal to the slip plane, together with unit vectors $\mathbf{S}^{(\alpha)}$ parallel to the slip direction (Figure 14.10). Here, $\alpha = 1, 2, \dots, N$ is the index used to denote a specific slip system with N the total number of slip systems (for FCC, $N = 12$). In the deformed solid, the slip plane normal and slip directions are denoted by $\mathbf{m}^{(\alpha)}$ and $\mathbf{s}^{(\alpha)}$, respectively. As discussed before, slip on the system initiates when the resolved shear stress exceeds a critical magnitude (the slip resistance), which is for a slip system α this occurs when $\tau^{(\alpha)} > g^{(\alpha)}$. The strength of the slip system ($g^{(\alpha)}$) increases with plastic straining, which is known as strain hardening. At the atomic scale, entanglement or pinning of dislocations causes strain hardening. Dislocations are caused by plastic deformation. An obvious outcome of such a behavior is that, depending on the direction of loads imposed and strain hardening, more than one slip system may be activated. If the material encounters excessive resistance in one direction, it tends to deform in another favorable direction. Sometimes several slip systems may be activated at the same time. As an example, if an FCC crystal is loaded parallel to a 100 direction, eight slip systems are subjected to the same resolved shear stress and so all of them are active at the same time. To design a comprehensive kinematical theory for the mechanics of elastic-plastic deformation

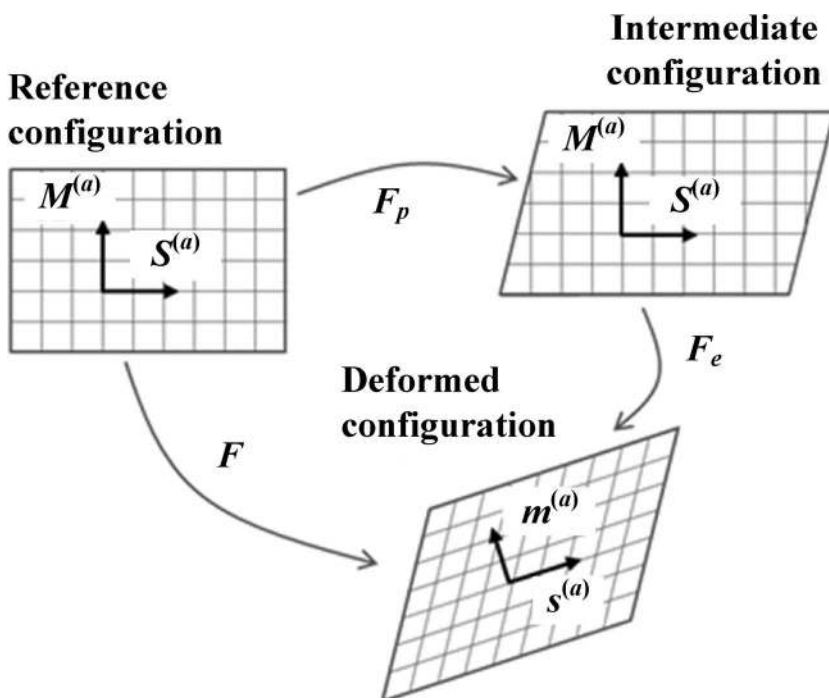


FIGURE 14.10 Deformation in a single crystal.

in crystals, it is accepted that the lattice itself undergoes elastic deformation and rotations (see Figure 14.10).

To determine the total rate of deformation gradient $\dot{\mathbf{F}}$, the product rule of differentiation is employed. First, the elastic part is differentiated with respect to time, keeping the plastic component constant. Next, the plastic part is differentiated with respect to time, with the elastic component unchanged. Thus, the rate of total deformation gradient is obtained by differentiating Equation 14.24 as

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}_e \mathbf{F}_p + \mathbf{F}_e \dot{\mathbf{F}}_p. \quad (14.25)$$

The velocity gradient $\mathbf{L}$ comprising the elastic and plastic part of the deformation gradient is given by

$$\begin{aligned} \mathbf{L} &= \dot{\mathbf{F}} \mathbf{F}^{-1} = (\dot{\mathbf{F}}_e \mathbf{F}_p + \mathbf{F}_e \dot{\mathbf{F}}_p) (\mathbf{F}_e \mathbf{F}_p)^{-1} = (\dot{\mathbf{F}}_e \mathbf{F}_p + \mathbf{F}_e \dot{\mathbf{F}}_p) \mathbf{F}_p^{-1} \mathbf{F}_e^{-1} \\ &= \dot{\mathbf{F}}_e \mathbf{F}_p \mathbf{F}_p^{-1} \mathbf{F}_e^{-1} + \mathbf{F}_e \dot{\mathbf{F}}_p \mathbf{F}_p^{-1} \mathbf{F}_e^{-1} = \dot{\mathbf{F}}_e \mathbf{F}_e^{-1} + \mathbf{F}_e (\dot{\mathbf{F}}_p \mathbf{F}_p^{-1}) \mathbf{F}_e^{-1} \\ &= \mathbf{L}_e + \mathbf{L}_p. \end{aligned} \quad (14.26)$$

Here, subscripts e and p indicate the elastic and plastic components, respectively. It is to be noted that the elastic component of the velocity gradient $\mathbf{L}_e$ comprises only the elastic part of the deformation gradient. However, the plastic component of the velocity gradient $\mathbf{L}_p$ comprises both the elastic as well as the plastic part of the deformation gradient. The plastic component of the velocity gradient, $\mathbf{L}_p$, is an effect induced by mechanical shearing on each available slip system. Thus, $\mathbf{L}_p$ is expressed as a cumulative summation of the shear rate on all slip systems:

$$\mathbf{L}_p = \sum_{\alpha=1}^N \dot{\gamma}^{(\alpha)} \mathbf{s}^{(\alpha)} \otimes \mathbf{m}^{(\alpha)}, \quad (14.27)$$

where α is the identifier of a slip system and $\dot{\gamma}^{(\alpha)}$ is the shear slip rate on the specific slip system. It is to be noted that any tensor can be decomposed into a symmetric and skew-symmetric part. The term ‘skew-symmetric’ is used interchangeably with the term ‘antisymmetric’. Here, a rate of deformation tensor $\mathbf{D}$ (also indicated by $\dot{\boldsymbol{\varepsilon}}$ as in Section 4.17) and a skew-symmetric spin $\mathbf{W}$ (also indicated by $\dot{\boldsymbol{\omega}}$ as in Section 4.18) are introduced to decompose velocity gradient $\mathbf{L}$ as

$$\mathbf{L} = \mathbf{D} + \mathbf{W} = (\mathbf{D}_e + \mathbf{W}_e) + (\mathbf{D}_p + \mathbf{W}_p). \quad (14.28)$$

Using Equations (14.26), (14.27) and (14.28), the following expressions are obtained:

$$\mathbf{D}_e + \mathbf{W}_e = \mathbf{L}_e = \dot{\mathbf{F}}_e \mathbf{F}_e^{-1}, \quad \mathbf{D}_p + \mathbf{W}_p = \mathbf{L}_p = \sum_{\alpha=1}^N \dot{\gamma}^{(\alpha)} \mathbf{s}^{(\alpha)} \otimes \mathbf{m}^{(\alpha)}. \quad (14.29)$$

Next, the Cauchy stress is obtained from (Huang, 1991),

$$\overset{o}{\sigma} + \sigma (\mathbf{I} : \mathbf{D}_e) = \mathbf{C}(\mathbf{D} - \mathbf{D}_p), \quad (14.30)$$

where $\mathbf{I}$ is the second-order unit tensor and $\mathbf{C}$ is the fourth-order, possibly anisotropic, elasticity tensor. The symbol ‘:’ indicates double contraction. Thus, $\mathbf{I}:\mathbf{D}_e$ is a scalar product of two tensors. The term $\mathbf{C}(\mathbf{D} - \mathbf{D}_p)$ is a second-order tensor (refer to Equation 5.4 for a similar term in index notation); some books write it as $\mathbf{C}:(\mathbf{D} - \mathbf{D}_p)$ to emphasize that this operation is a double contraction. The Jaumann stress rate was explained in Section 5.7.5. It is expressed as

$$\overset{o}{\sigma} = \dot{\sigma} - \mathbf{W}\sigma - \sigma\mathbf{W}^T. \quad (14.31)$$

The resolved shear stress, $\tau^{(\alpha)}$, on each slip system is obtained via the Schmid law,

$$\tau^{(\alpha)} = \left(\mathbf{s}^{(\alpha)} \otimes \mathbf{m}^{(\alpha)} \right) : \sigma \quad (14.32)$$

The shear rate, $\dot{\gamma}^{(\alpha)}$, is obtained by the widely used power law (Hutchinson, 1976):

$$\dot{\gamma}^{(\alpha)} = \dot{\gamma}_0 \left| \frac{\tau^{(\alpha)}}{g^{(\alpha)}} \right|^n \text{sgn}(\tau^{(\alpha)}), \quad (14.33)$$

where $\dot{\gamma}_0$ is the reference shear rate, $g^{(\alpha)}$ is the slip resistance on the specific slip system and n is the rate sensitivity parameter. A hardening law is needed which specifies the relationship between $g^{(\alpha)}$ and the plastic strain as a function of time. At the time $t = 0$, each slip system has the same initial strength g_0 . The slip does not take place simultaneously in all systems. The system at which slip takes place is called an active slip system and other slip systems are called inactive slip systems. Active slip systems have lower resistance to dislocation motion compared to other crystallographic planes and directions. These systems are the preferred paths for dislocation motion during plastic deformation. On the other hand, inactive slip systems have higher resistance to dislocation motion and are activated only when the active slip systems are blocked or saturated. Latent hardening is the hardening of inactive slip systems due to active slip systems. Due to plastic shearing, the slip resistance increases according to

$$\dot{g}^{(\alpha)} = \sum_{\beta=1}^N h_{\alpha\beta} \left| \dot{\gamma}^{(\beta)} \right|, \quad (14.34)$$

where $h_{\alpha\beta}$ is the hardening modulus that can be calculated by modifying the form proposed by Asaro (1983). The hardening modulus refers to a material parameter

that describes the rate at which a material strengthens or hardens under plastic deformation. The rate is calculated with respect to plastic strain. Thus, the hardening modulus has the same unit as that of stress. It quantifies the resistance of a crystal to plastic deformation as a function of the accumulated plastic strain. The notation $h_{\alpha\beta}$ indicates hardening modulus on α slip system due to strain at β slip system. If the two indices α and β are equal, then the hardening modulus corresponds to self-hardening, otherwise, it corresponds to the latent hardening. Self-hardening is the hardening of the active slip system due to the deformation of that system. The modified form for the hardening moduli is

$$h_{\alpha\alpha} = (h_0 - h_s) \operatorname{sech}^2 \left(\frac{h_0 \gamma}{\tau_s - \tau_0} \right) + h_s, \\ h_{\alpha\beta} = q h_{\alpha\alpha}, (\text{for } \alpha \neq \beta), \gamma = \sum_{\alpha} \int_0^t \left| \dot{\gamma}^{(\alpha)} \right| dt. \quad (14.35)$$

Here, h_0 and h_s represent the initial and saturated hardening moduli, respectively and γ is the cumulative shear strain over all the slip systems, τ_0 and τ_s represent the shear stresses at the onset of yield and the saturation of hardening, respectively, and q is the ratio of latent and self-hardening.

This formulation of crystalline plasticity is considered to be a classical theory of single-crystal metals. There have been several successful implementations of the model detailing the nuances of plastic deformation typically at fundamental length scales pertaining to the sample sizes used to assess the mechanical properties of the material in question. It is well established via experimental studies that single crystals at the micro and meso length scale show distinct mechanical behavior as compared to their bulk counterpart. The well-established phenomenon of ‘smaller is stronger’ was observed experimentally (Greer and De Hosson, 2011). In a phenomenological way, the sample size effect is often described by

$$\sigma_f = \sigma_0 + Kd^{-n}, \quad (14.36)$$

where σ_f is the measured flow stress of the material, d is the characteristic sample size and σ_0 , K and n are parameters. This power-law relationship is similar to the Hall-Petch effect. The Hall-Petch relation predicts that as the grain size reduces, the yield strength increases. The Hall-Petch relation is similar to Equation 14.36, but in that relation d is the grain size and n is 0.5. Equation 14.36 can be further simplified as

$$\sigma_f = Kd^{-n}. \quad (14.37)$$

Experimental results obtained via micro-pillar compression tests show that the value of n is 0.6–1.0 for FCC metals. For BCC metals, the value of n is less than or equal to 0.5. (Tarleton et al., 2015). HCP crystals show a variation in n for specific slip

systems. Experimental data show that for the prismatic slip in titanium, n is ~ 0.5 (Sun et al., 2011). For Mg, the value of n is 0.8 (Ye et al., 2011) or 0.4 (Byer and Ramesh, 2013) for basal slip and 0.2 for the pyramidal slip (Byer and Ramesh, 2013). Suppose d is reduced by half, then for $n = 0.8$, the flow stress increases by 1.7 times, while for $n = 0.2$, it increases by 1.1 times. Thus, it is important to determine the proper value of n , either by direct method or by inverse method.

The power-law relation of size effect observed in single crystals is observed to be different when considering different loading conditions, which implies that such an approach of expressing size effects using a power law is fundamentally inaccurate. For example, implementing the power-law relationship similar to Equation 14.36 or Equation 14.37 to characterize the sample size effect in a Cu single crystal, the measured value of n is around 0.8 for cantilever beam experiments (Motz et al., 2005). For micro-pillar compression tests, the value of n is 0.4 (Kiener et al., 2006). Thus, from Equation (14.37), reducing the size to half, increases the flow stress by 1.3 times, an increase of 30%. However, the increase is much more in the bending. In titanium single crystals, the value of K is about 354 Pa-m for prismatic slip as obtained from cantilever-beam experiments (Gong and Wilkinson, 2011). However, based on micro-pillar compression tests, the value of K is 131 Pa-m (Sun et al., 2011). Hence, Equations 14.36 and 14.37 are of limited utility and there is heavy dependence on the experimental values.

This shows that size effects emanate from two independent mechanisms. They are categorized into (1) intrinsic effects (first-order effects), which are due to the specifics of the microstructure and (2) extrinsic effects (second-order effect), which are due to the effect from gradients of deformation (often interpreted as strain gradient or gradient of slip). Experimental studies indicate that in bending, both first-order as well as second-order effects (due to bending-induced deformation gradient) are important in contrast to only first-order effects in uni-axial deformations. In bending strain gradient also becomes important. Conventional crystal plasticity-based formulations usually account for the first-order effects from a modeling perspective, while strain-gradient-based models may be used to describe the second-order effect. Non-uniform plastic deformation can occur due to the presence of geometrically necessary dislocations (GNDs). These are different from statistically stored dislocations (SSDs) and are independent of the plastic strain gradient. GNDs are formed during plastic deformation in regions where the deformation is highly localized, such as at a sharp corner or at the interface between materials with different crystallographic orientations. In these areas, the crystal lattice is significantly distorted, and the generated dislocations cannot move away easily due to the surrounding crystal structure. When the storage of GNDs is akin to SSDs, size-dependent work-hardening becomes significant. This results in the second-order effect. To introduce this effect of strain-gradient associated with GNDs into a crystal plasticity model, the theory considers the introduction of GND storage into the evolution of SSDs and the estimation of slip-system resistance (Ma et al., 2006).

Example 14.6

For the single crystal of a material, the following parameters are provided: initial shear stress τ_0 of 5 MPa, the saturated shear stress τ_s of 50 MPa, initial hardening modulus h_0 of 200 MPa, saturated hardening modulus h_s of 20 MPa and the ratio

of latent hardening to self-hardening q of 1.2. Find the self-hardening modulus and latent hardening modulus for cumulative shear strains of 0.1 and 1.

SOLUTION

Using Equation 14.35, the self-hardening modulus is given as

$$h_{\alpha\alpha} = (h_0 - h_s) \operatorname{sech}^2 \left(\frac{h_0 \gamma}{\tau_s - \tau_0} \right) + h_s.$$

Substituting the given data,

$$h_{\alpha\alpha} = (200 - 20) \operatorname{sech}^2 \left(\frac{200\gamma}{50 - 5} \right) + h_s = \{180 \operatorname{sech}^2 (4.4444\gamma) + 20\} \text{ MPa}.$$

For $\gamma = 0.1$, $h_{\alpha\alpha} = 168.65$ MPa and for $\gamma = 1$, $h_{\alpha\alpha} = 20.1$ MPa. We observe that the hardening modulus saturates at high cumulative shear strain. From Equation 14.22, the latent hardening modulus is q times the self-hardening modulus. Hence,

$$h_{\alpha\beta} = \begin{cases} 202.38 \text{ MPa for } \gamma = 0.1 \\ 24.12 \text{ MPa for } \gamma = 0.1 \end{cases}.$$

14.3.3 FIRST-ORDER EFFECT

When strain gradient is not present, it is commonly acknowledged that the slip resistance in a component is determined by the amount of statistically stored dislocations (SSDs). These are random dislocations that are generated during solidification and plastic deformation. It is to be noted that there is no fundamental difference between statistically stored dislocations and geometrically necessary dislocations. The difference is only in the way they are produced. SSDs trap each other randomly. In a single crystal at the macro-scale, dislocation nucleation is relatively simple (El-Awady, 2015) because there are numerous nucleation sites available. Therefore, the empirical Taylor model (El-Awady, 2015) can be used to describe slip resistance at the macro-scale, where slip resistance changes proportionally to the square root of dislocation density. Dislocation density is a measure of the number of dislocations in a unit volume of a crystalline material. It can be calculated by measuring the total length of the dislocation line in a unit volume. In the SI system, the unit of dislocation density is m^{-2} which translates to the dislocation line content per volume. Typically in metals, dislocation density varies from 10^9 m^{-2} to 10^{16} m^{-2} .

According to the source-limitation strengthening mechanism, higher nucleation stress is required at smaller length scales (Kiener et al., 2006; Tarleton et al., 2015). It is because as the size reduces, the number of dislocation sources decreases; thereby the flow stress of the material increases. Furthermore, the presence of

single-arm sources (which have one pinning point inside the crystal and a free end on the surface) makes it difficult to operate dislocation sources due to a dislocation-truncation mechanism (Parthasarathy et al., 2007). This mechanism assumes that smaller samples contain shorter sources which require higher flow stress to activate. Subsequently, such strengthening mechanisms lead to higher initial yield stress. The strengthening mechanisms have been confirmed by Laue diffraction analysis (Maass et al., 2009). Laue diffraction analysis is an X-ray diffraction technique that can find the crystal orientation and structure. It can also give an idea about local dislocation structure. To account for phenomena at a small length scale, Gong and Wilkinson (2011) incorporated a size-dependent term in the expression of slip resistance $g^{(\alpha)}$. Based on the literature, the slip resistance is given by

$$g^{(\alpha)} = \tau_D + \tau_S, \quad \tau_D = \chi \mu b \sqrt{\sum_{\beta} \rho_{SSD}^{(\beta)}}, \tau_S = \tau_0 \frac{D_1^0}{d} \quad (14.38)$$

The contribution of slip resistance from dislocation interactions is denoted by τ_D while τ_S takes into account the increase in nucleation stress resulting from source-limitation strengthening and dislocation-truncation mechanisms; χ denotes the hardening coefficient; μ and b represent the shear modulus and Burgers vector, respectively; $\rho_{SSD}^{(\beta)}$ is the density of statistically stored dislocations in the slip system β . The parameter d denotes the characteristic size of a single crystal sample and D_1^0 is the reference sample size; τ_0 represents the initial slip resistance which is independent of sample size. The nucleation stress increases considerably as the sample size decreases to the magnitude of D_1^0 . The initial slip resistance $g_0^{(\alpha)}$ can therefore be given by

$$g^{(\alpha)} = \tau_D + \tau_S = \tau_0 \left(1 + \frac{D_1^0}{d} \right), \quad \tau_D = \chi \mu b \sqrt{\sum_{\beta} \rho_{SSD,0}^{(\beta)}}, \tau_S = \chi \mu b \sqrt{\sum_{\beta} \rho_{SSD,0}^{(\beta)}} \times \left(\frac{D_1^0}{d} \right). \quad (14.39)$$

$\rho_{SSD,0}^{(\beta)}$ denotes the initial dislocation density. It is obtained by measuring τ_0 . This implies, that the size-dependent term, τ_S , only contributes to the magnitude of the initial yield stress.

The source-limitation, dislocation-starvation and source-truncation (dislocation-truncation) hardening mechanisms have been widely utilized to account for the intrinsically first-order effect (El-Awady, 2015; Kiener et al., 2006). The source-limitation and source-truncation hardening mechanisms result in an increase in dislocation nucleation stress (Kiener et al., 2006; Parthasarathy et al., 2007; Tarleton et al., 2015), meaning that greater stress is needed to generate dislocations. Meanwhile, the dislocation-starvation and source-truncation mechanisms impact dislocation

multiplication (Parthasarathy et al., 2007; Tang et al., 2007). As per the dislocation-starvation hardening mechanism, an increase in flow stress is expected due to the limited mobile dislocations in a small-scaled single-crystal component (Parthasarathy et al., 2007). In addition, discrete dynamics dislocation (DDD) simulations have shown that the existence of single-arm sources results in a smaller number of long dislocation segments and junctions at smaller scales (Tang et al., 2007). Micro-pillar compression studies in Cu have demonstrated that the hardening rate increases with smaller sample sizes (Kiener et al., 2011). Hence, it can be deduced that limited mobile dislocations and initial single-arm sources in the component can affect the hardening behavior. To represent the increase in hardening rate with a reduction in size, the hardening coefficient χ in Equation 14.38 and Equation 14.39 is assumed to be size-dependent as

$$\chi = \chi_0 \left(1 + \frac{D_1^0}{d} \right), \quad (14.40)$$

where χ_0 is the hardening coefficient at the macro-scale. Equation 14.40 suggests that as the size of a single crystal sample decreases, a higher stress level is required to generate and multiply dislocations for a fixed dislocation density magnitude. Notably, the value of τ_0 remains constant regardless of the size. Thus, as the size decreases, the initial density of SSDs reduces, consistent with the phenomenological model proposed by El-Awady (El-Awady, 2015).

Additionally, DDD simulations indicate that there is a maximum number of dislocations that can be sustained in small crystalline components (Deshpande et al., 2005; Kiener et al., 2011). Consequently, the dislocation content in components reaches a saturation point that is dependent on the size of the sample. The evolution of SSDs can be described as

$$\dot{\rho}_{SSD}^{(\alpha)} \left(k_1 \sqrt{\sum_{\beta} \rho_{SSD}^{(\beta)}} - k_2 \rho_{SSD}^{(\alpha)} \right) \left| \dot{\gamma}^{(\alpha)} \right|, \quad (14.41)$$

where k_1 and k_2 are the material parameters. It represents the generation and annihilation of dislocations, respectively. When $\rho_{SSD}^{(\alpha)} = 0$ is achieved, the balance between dislocation generation and annihilation is attained, resulting in a saturation of dislocation density ($\rho_{SSD,sat}^{(\alpha)}$). It can be inferred that this is proportional to $(k_1/k_2)^2$ when $\rho_{SSD}^{(\alpha)} = 0$. In the macro scale, it is expected that $\rho_{SSD,sat}^{(\alpha)}$ is constant. However, for sample sizes in the range of microns and sub microns, dislocations tend to escape from a free surface, which is expected to increase as the sample size decreases. This results in a decrease in $\rho_{SSD,sat}^{(\alpha)}$ as the sample size reduces, as confirmed by DDD simulations (Deshpande et al., 2005; Kiener et al., 2011). The annihilation of dislocations, represented by the parameter k_2 , should depend on the sample size to reflect the loss of dislocations from the free surface. This is described by

$$k_2 = k_2^0 \left[1 + \left(\frac{D_2^0}{d} \right)^\eta \right], \quad (14.42)$$

where k_2^0 is defined as the annihilation coefficient of dislocations at the macro-scale. A geometric parameter indicating the reference sample size is D_2^0 . It is assumed that dislocations can escape from the free surface (of this reference sample size). The parameter η is dependent on sample geometry. Equation 14.42 implies that, with sample sizes comparable to D_2^0 , the dislocation density at a saturated state becomes size-dependent. With the sample size becoming smaller the sample contains a relatively high number of dislocation sources producing a high mobile dislocation density at yield. Hence, D_2^0 should be larger than D_1^0 which defines the scale of the crystal with limited source or mobile dislocations. A similar conclusion was also found by El-Awady (2015). From Equation 14.41, it can be inferred that $\rho_{SSD,sat}^{(\alpha)}$ is proportional to $(k_1/k_2)^2$. It is assumed that $\rho_{SSD,sat}^{(\alpha)}$ decreases with the volume of a single-crystal sample, v , that is

$$\rho_{SSD,sat}^{(\alpha)} \propto \left(\frac{k_1}{k_2} \right)^2 \propto V. \quad (14.43)$$

In general, V is proportional to $\sim d^2$ for a sample with a large aspect ratio. However, for a sample with a small aspect ratio, V is proportional to $\sim d^3$. In Equation 14.42, the value η should lie in the range of 1.0~1.5 for a typical sample geometry.

Example 14.7

As per Taylor's hardening model, the shear flow strength depends on the dislocation density in the following manner:

$$\tau = \tau_{df} + \alpha \mu b \sqrt{\rho}, \quad (14.44)$$

where τ_{df} is the intrinsic flow strength for the dislocation-free material, α is a constant (0.2 for FCC and 0.4 for BCC structure), μ is the shear modulus and b is the length of Burgers vector. If after cold working, dislocation density in an aluminum sample increases from $0.5 \times 10^{13} \text{ m}^{-2}$ to $1.5 \times 10^{13} \text{ m}^{-2}$, find the increase in shear flow strength. Take $b = 0.286 \text{ nm}$ and $\mu = 26 \text{ GPa}$.

SOLUTION

Let the dislocation density change from ρ_1 to ρ_2 . Then the increase in shear flow strength is given by

$$\Delta \tau = \alpha \mu b \sqrt{\rho_2 - \rho_1}. \quad (14.45)$$

Substituting the values for aluminum:

$$\Delta\tau = 0.2 \times 26 \times 10^9 \times 0.286 \times 10^{-9} \times \sqrt{1 \times 10^{13}} = 4.7 \times 10^6 \text{ N/m}^2. \text{ Hence, shear flow strength will increase by 4.7 MPa.}$$

14.3.4 SECOND-ORDER EFFECT

During the deformation of a component, strain gradients arise within the material volume, resulting in geometrically necessary dislocations (GNDs). Therefore, the impact of GNDs on the evolution of SSDs and slip resistance must be taken into account (van Beers et al., 2015). As a result, the scalar density of GNDs, denoted as, $\rho_{GND}^{(\beta)}$, is incorporated into Equations 14.38 and 14.41 as

$$g^{(\alpha)} = \chi \mu b \sqrt{\sum_{\beta} \left(\rho_{SSD}^{(\beta)} + \rho_{GND}^{(\beta)} \right)} + \tau_0 \frac{D_1^0}{d} \quad (14.46)$$

and

$$\dot{\rho}_{SSD}^{(\alpha)} = \left(k_1 \sqrt{\sum_{\beta} \left(\rho_{SSD}^{(\beta)} + \rho_{GND}^{(\beta)} \right)} - k_2 \rho_{SSD}^{(\alpha)} \right) \left| \dot{\gamma}^{(\alpha)} \right| \quad (14.47)$$

Here, the GND density is defined as vector ‘density’ based on the work of Ma *et al.* (Ma et al., 2006). The rate of GND density evolution, $\dot{\rho}_{GND}^{(\alpha)}$, is represented as

$$\dot{\rho}_{GND}^{(\alpha)} = \frac{1}{b} \left[\nabla \times \left\{ \dot{\gamma}^{(\alpha)} \mathbf{F}_P^T \mathbf{m}^{(\alpha)} \right\} \right] \quad (14.48)$$

The gradient is taken with respect to reference coordinates. The scalar measure of GND density is $\rho_{GND}^{(\alpha)} = \left| \rho_{GND}^{(\alpha)} \right|$. The conventional understanding of GND density is that it increases with a rise in plastic deformation gradient. However, it is important to note that there exists a maximum limit to the dislocation density; dislocation density cannot keep increasing. This implies that there is a physical limit to the saturated GND density; saturated GND density cannot cross that limit. Huang *et al.* (Huang et al., 2006) carried out experiments that demonstrated a saturated GND density, $\rho_{GND}^{(\max)}$. In view of these experiments, Equation 14.48 can be written as

$$\dot{\rho}_{GND}^{(\alpha)} = \begin{cases} \dot{\rho}_{GND}^{(\alpha)} = \frac{1}{b} \left[\nabla \times \left\{ \dot{\gamma}^{(\alpha)} \mathbf{F}_P^T \mathbf{m}^{(\alpha)} \right\} \right], & \text{if } \rho_{GND}^{(\alpha)} \leq \rho_{GND}^{\max} \\ 0, & \text{otherwise} \end{cases} \quad (14.49)$$

The modeling approach proposed here is a practical extension of a strain-gradient crystal-plasticity formulation. Strain gradient plasticity theory takes into

account the spatial gradient of the strains. The modeling accuracy is dependent on the standard tests. It may be sufficient to get the uni-axial test data for incorporating the first-order effects. However, the second-order effects cannot be captured by uni-axial test data alone. For that purpose, a bending test may be needed because it induces inhomogeneous deformation and macroscopically quantifiable strain gradients. With this approach, it is possible to capture size effects in compression as well as bending deformation regimes. A lot of experimental studies are further needed to ascertain if uni-axial and bending tests can provide enough data to model any complex loading situation.

14.4 CONCLUSION

In this chapter, some of the basics of crystal plasticity theory are discussed. The constitutive models described are highly nonlinear (geometrically as well as constitutively). The review of first- and second-order effects is largely based on the article of Liu et al. (2016). Simple illustrative examples have been provided for a better appreciation of the concepts. Considering the complexity of the underlying mechanical interactions in a crystal responding to imposed loads and boundary conditions, which are external and in the case of poly-crystals also internal, a robust mathematical framework for computations is needed. It is difficult to find analytical solutions to problems involving crystal plasticity and recourse has to be taken in favor of numerical techniques.

Recently, the finite element modeling (FEM) approximation has been used to solve the nuances of crystalline plasticity across varying length scales. FEM solves the equilibrium equations by considering the constitutive behavior of the material and appropriate strain-displacement relations. The computational advantage of FEM allows for further research into crystalline physics as it seamlessly allows for tackling complex boundary value problems that are physically relevant based on the specifics of the material microstructure under study. There have been numerous applications of these techniques. Some areas are micro-machining, micro-forming and design of micro-electromechanical systems.

Exercises

1. In a cubic unit cell, depict the planes and directions having the following Miller indices:

$$(a)\left(1\bar{1}1\right)(b)(123)(c)\left[110\right](d)\left[1\bar{2}3\right]$$

2. For a hexagonal crystal system, show the following planes and directions:

$$(a)\left(1\bar{2}10\right)(b)(0001)(c)\left[\bar{1}100\right](d)\left[11\bar{2}3\right]$$

3. Considering $\mathbf{F} = \mathbf{F}_e \mathbf{F}_p$, the expression for velocity gradient is given by

$$\mathbf{L} = \dot{\mathbf{F}}_e \mathbf{F}_e^{-1} + \mathbf{F}_e \left(\dot{\mathbf{F}}_p \mathbf{F}_p^{-1} \right) \mathbf{F}_e^{-1},$$

where the dot indicates the derivative with respect to time. Show that this expression can also be written as

$$\mathbf{L} = \dot{\mathbf{F}}_e \mathbf{F}_e^{-1} + \mathbf{R}_e \left(\dot{\mathbf{F}}_p \mathbf{F}_p^{-1} \right) \mathbf{R}_e^{-1},$$

where $\mathbf{R}_e$ is the rotation tensor obtained after polar decomposition of $\mathbf{F}_e$. Further, show that for small rotations

$$\mathbf{L} = \dot{\mathbf{F}}_e \mathbf{F}_e^{-1} + \dot{\mathbf{F}}_p \mathbf{F}_p^{-1}.$$

4. Deformation gradients at time t and $t+\Delta t$ are given as $\mathbf{F}_t$ and $\mathbf{F}_{t+\Delta t}$, show that

$$\Delta \mathbf{F} = \mathbf{F}_{t+\Delta t} \mathbf{F}_t^{-1}.$$

5. Distinguish between statistically stored dislocations and geometrically necessary dislocations.

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