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# Ontology and the Foundations of Mathematics

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# ONTOLOGY AND THE FOUNDATIONS OF MATHEMATICS

*Talking Past Each Other*

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# Ontology and the Foundations of Mathematics

## Talking Past Each Other

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**Abstract:** This Element looks at the problem of inter-translation between mathematical realism and anti-realism and argues that so far as realism is inter-translatable with anti-realism, there is a burden on realists to show how their posited reality differs from that of the anti-realists. It also argues that an effective defence of just such a difference needs a commitment to the independence of mathematical reality, which in turn involves a commitment to the ontological access problem – the problem of how knowable mathematical truths are identifiable with a reality independent of us as knowers. Specifically, if the only access problem acknowledged is the epistemological problem – that is, the problem of how we come to know mathematical truths – then nothing is gained by the realist notion of an independent reality and in effect, nothing distinguishes realism from anti-realism in mathematics.

**Keywords:** realism, anti-realism, independence, inter-translatability, truth



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# Contents

# 1 What Are We Talking About?

There is an age-old question running through the philosophy of mathematics: are numbers (or any of the structures that mathematics studies) invented or discovered?[1] The primary focus of this Element will not be to cover or to contribute to this debate (at least, not directly) but rather to focus on the *question itself*, which, as we will see, is by no means as clear as it may at first seem.

We will look at a number of core concepts and issues, all of which need to be addressed in some way in order to even begin to understand what exactly we are talking about – what exactly we are asking – when we posit the question in the first place.

This discussion could range over a great deal – far more than can be addressed in an introductory Element such as this. Similarly, it could take many possible assumptions or positions as its starting point. For example, it could begin with the assumption that the question is fundamentally philosophical and so has not much to do with the actual practice of mathematics. Or, it could look at the question, in the first place, as primarily a question for linguistics. It could take the question to be ethical – about what we value and why; or it could situate the question as properly formal – perhaps one that should initially be asked within the realm of, say, logical and mathematical symbols. The list goes on. What we take as the initial range of the question, or where we begin when we begin to try to understand it, will affect what we take as the main concepts and issues with which it interacts.

I do not claim to have identified the only proper way to begin our investigation, but hope it will highlight its complexity and depth, and so the importance of examining it in the first place.

The initial focus here – what we might initially take the question to be about – will be on the status of that which provides the foundations of mathematics as it is practised, in particular as it is practised *as true.* An immediate point to note is that we cannot survey all practising mathematicians to determine just how many do indeed practise mathematics as though it is true; but it is safe to say that there is enough interest in the question among practitioners of both mathematics and the philosophy of mathematics, who accept that our usual inclination is to believe in the truth of such claims as $2 + 2 = 4$, to justify the focus on the truth of mathematics taken here.

To push the point, the sort of truth generally ascribed to mathematical statements is an especially trustworthy truth – scientific, logical, and

[1] For an introduction to this introductory question, see www.youtube.com/watch?v=ujvS2K06dg4&list=TLPQMTcwNjIwMjHV9PBJnA4ltA&index=4

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mathematical truths are generally held in higher regard than, say, social or sociological truths.

This focus will further narrow to the historically important and continuing debate between two main positions on just what might constitute this truth, namely ‘constructivism’ and ‘Platonism’. These positions within the philosophy of mathematics roughly line up with what has commonly been called, in other philosophical fields, ‘anti-realism’ and ‘realism’ respectively. In broad strokes, the former takes the truth of mathematics to be constituted by human activity, language, minds, and so on – that is to say, it takes the position that the truth of mathematics largely depends on us. The latter takes the truth of mathematics to be independent of humans – such that it is, as Michael Resnik puts it (swapping out the word ‘logical’ with the word ‘mathematical’),

> [t]he doctrine that statements attributing *mathematical* [author’s addition] . . . properties or relationships . . . are true or false independently of our holding them to be true, our psychology, our linguistic and inferential conventions, or other facts about human beings. (Boghossian and Peacocke, p. 334)

Both these characterisations have their problems, and both have been contested as to whether they are the best frameworks within which we can or should characterise the differences between the two positions. So the central focus here, within the focus on these two positions, will be on just how best to understand and articulate their differences.

An immediate impact of the broad-brush characterisation we’ve just offered is that each position so characterised comes with its own strengths and weaknesses regarding the core concerns of the philosophy of mathematics. For anti-realism, thought of as encompassing the idea that mathematical truth is constituted (or constructed) by humans somehow, there is an immediate problem regarding the strength of this truth – how, for instance, if we adopt this position, can we account for the notion that the truth of mathematical statements is especially trustworthy and somehow different to other apparently human-constructed or human-dependent truths?

On the other hand, mathematical realism, thought of as encompassing the idea that mathematical truth is broadly independent of humans, encounters an immediate problem of access – for instance, how, if we adopt this position, can we know that we’ve managed to discover the ‘real’ mathematical truths, as opposed to, say, those we find most useful to us or perhaps to a small set determined by our own intellectual, biological functioning? Worse, how can we know if we’re even close to discovering independent truth in an abstract field such as mathematics? If we’ve no way of accessing truth via empirical data, tests or our basic senses, how can that truth be confirmed or trusted?

But each position thus characterised also has its own particular advantages. Mathematical realism has the advantage of explaining, or at least articulating, the sort of strong, trustworthy truth generally attributed to mathematical claims. And, mathematical anti-realism has the advantage of explaining, or at least articulating the way we access mathematical truths – after all, if that truth is in essence our creation, all we need do is study relevant features of our own human biology, mind, and so on to see how we arrive at what we've come to believe as true in mathematics.

To better see why the relationship between these two positions is an important question within the philosophy of mathematics, note that while much has been written on the impacts of taking one or the other of realism or anti-realism regarding the truth of mathematics, there is a clear way in which the two positions themselves are, on closer inspection, quite difficult to distinguish from one another.

And so, our initial way into understanding the question itself runs quite quickly up against a thorny problem: what do you, the constructivist (or, broadly put, anti-realist), really mean when you ask whether mathematics is invented or discovered, and what do you, the realist, really mean when you ask the same thing? And how can we tell when we've understood each position as it would understand itself?

Of course, it is important to note that I intend 'constructivism' (which I take as broadly as possible here: i.e. as an instance of 'anti-realism' at large) and 'realism' to be cover terms for an immense array of particular positions within each camp, many of which are, again, quite different from each other. And some philosophers of mathematics would take this broad approach to be a mistake at the outset.[2] But the focus of this Element is not on separating the myriad specific positions in the literature, but on what happens at the very beginning – on how any one enquirer can (or cannot) understand the question of inquiry itself, and so whether or not 'the' question can make any sense in the first place.

Section 2 will focus on this conundrum – highlighting the extent to which the two (broad-brushed) positions may in fact be unable to be distinguished, or to be distinguished along the lines to which proponents on either side would agree. This is a significant problem: without agreed-on terms for the debate, the debate is stopped before it begins.

Stewart Shapiro notes this (1997), arguing that non-revisionist, anti-realist programs (i.e. programs that seek to retain and explain most of mathematics as it stands) are inter-translatable with classical logic, or traditional mathematics complete with its apparent realist commitments to numbers, sets and other

[2] Thanks to an anonymous referee for highlighting this point.

mathematical things. That's actually the way that anti-revisionist, anti-realist programs work in the first place – by translating apparently problematic number (etc.) talk into talk of other, apparently less problematic stuff. But, as we'll shortly explore in more depth, this translation works both ways: so anytime I like, I can equate your talk of mental constructions with my talk of numbers and vice versa.

Shapiro then argues that the problems associated with each program are inter-translatable also. So constructivism, for instance, ends up saddled with the same problems regarding epistemological access as traditional realism. This is all due to the fact that each non-revisionist, anti-realist program has to posit abstracts or use the notion of 'possibility' in a semi-concrete sense. But, the reference to semi-concrete abstract notions is as vexing a problem for the anti-realist as it is for the realist. That is, there is as much an access problem for anti-realism as for realism, just insofar as anti-realism inter-translates with traditional mathematics.

On the basis (mentioned earlier) that inter-translation works both ways, the problems each position encounters run in the opposite direction also: that is, so far as realism is inter-translatable with anti-realism, there is a burden on the mathematical realist to show how their posited reality differs from that of the anti-realist.

Ultimately, this Element argues that an effective defence of just such a difference needs a commitment to the notion of 'independence' regarding mathematical reality (or, at least a commitment to agree on what this might mean), which in itself involves a commitment to what we'll call the 'ontological' access problem *as a genuine problem* – essentially, this is the problem of how knowable mathematical truths are identifiable with a reality independent of us as knowers.

Section 3 outlines two ways in which we might understand what such 'access' might mean and explores some of the ways in which 'ontological' access has been conflated with what should, I suggest, be identified as something else entirely, namely 'epistemological' access. Ontology and Epistemology are two well established fields of study within the philosophical tradition. Broadly, the former deals with what is, and the latter with knowledge. The hope is that mapping the boundaries and potentially shared terms of Ontology and Epistemology along the potential division of constructivism and realism helps tease out what we may really be debating when we ask the question under investigation.

Section 3 specifically argues that if the ontological problem (recall, this is identified here as the problem of how knowable mathematical truths are identifiable with a reality independent of us as knowers) is defused, side-stepped or conflated with the 'epistemological' problem – that is, the problem of how *we come to know* mathematical truths – then it appears that nothing is gained by the realist notion of an independent reality and, in effect, nothing then distinguishes realism from anti-realism (or constructivism) in mathematics. That is, if the epistemological problem is the only playing ground for skirmishes between realism and anti-realism in mathematics, then the problem of effectively distinguishing the two positions remains.

That problem is not small. It boils down to this: that the 'what-is-perceived' and 'what-existed-prior-to-that-perception' must (initially at least) be distinguishable from each other, then shown to be (at least reliably to be believed as) one and the same when we 'get it right'. This is the ontological access problem for traditional (physical) realism and is precisely why such realism is the sceptic's favourite target. In the realist story, it is in the separability of independent reality (or, in Fregean terms, 'the referent') from our constructions or objects of belief, or our experiences of the 'mode of presentation', that the strong sense of objectivity and justification reside. The corresponding access problem is about overcoming this separateness while nonetheless retaining what is gained by that separateness itself.

Section 4 then examines how this all might help us agree on a notion of 'independence' able to effectively distinguish between the two sides of the debate and so overcome the twin problems of their talking past one another and their inter-translatability.

Section 5 argues that the notion of an independent mathematical reality will meet similar inter-translatability problems as covered in earlier sections, unless it does some sort of extra (or differentiable) work. This section argues that this differentiable work needs an appeal to a certain notion of what it is to be justified in mathematics.

The terrain here is very broad indeed, and is intended to be, as the Element is a discussion of the very broad strokes in which we might situate various positions regarding the foundations of mathematics, or better still with which each such position may be distinguished along our very general understanding of the initial question.

## 2 Inter-translatability

Looking at the 'inter-translatability' phenomenon in more detail, we return to what Shapiro argued are the best of the anti-realist programs – that is, those that seek to explain most of mathematics as it stands. To flesh out the argument a little more, note again that Shapiro argues that such programs involve at least as many of the epistemic and semantic problems as are involved in realism:

> In a sense the problems are equivalent – for example, a common maneuver today is to introduce a 'primitive' such as a modal operator, in order to reduce ontology. The proposal is to trade ontology for ideology. However in the context at hand – mathematics – the ideology introduces epistemic problems quite in line with the problems with realism. The epistemic difficulties with realism are generated by the richness of mathematics itself. (p. 5)

Recall that the specific sort of realism that Shapiro (1997) defends is holistic, which is why he does not address the ontological access problem from first principles, or 'from without' as it were.

Rather, his defence of his ontology of structures is via an argument extending Wright's notion of entitlement to the successor principle and thence to finite cardinality structure and the natural number structure:

> Wright argues that 'Entitlement of cognitive project does not . . . extend to matters of ontology'. At least in the case at hand, basic arithmetic, I would say that it does. *S* [a generalized person interested in the nature of mathematics] has reason to believe in the existence of the (*ante rem*) finite cardinality structures, and also in the existence of the natural number structure. The entitlement to the successor principle is a key element in the justification for the existence claim. (Shapiro, 2011, 146)

Wright's notion of entitlement is a general one – very loosely put, it grants justification to a belief just so long as denying that belief would be irrational or harmful to our overall life and our attempt to understand the world we encounter.[3] Shapiro (2011) gives a similarly holistic argument in which it is argued that our ability to coherently discuss a mathematical structure is evidence that it exists (p. 147).

In line with the equivalence argument presented at the beginning of this section, this Element suggests that the difficulties generated by equivalence itself and in turn by the sort of holistic realism that the equivalence argument supports, are arguably greater than the difficulties generated by a more traditional Platonism. Granted – the burden on the latter is both ontological and epistemological. The burden on the former, however, is to show how the ontology/ideology trade-off works in the realist's favour, specifically how it differentiates realism from anti-realism, or minimally, if perhaps an independent reality is *stipulated*, how that stipulation (given equivalence) avoids the ontological access problem. On the other hand, insofar as the ontological problem can be set aside, the burden on the holistic realist is then to show how (given equivalence) the mathematical reality posited is any different from the ideological universe of mathematical objects in anti-realism.

[3] For more on this, see Wright (2004).

That is, Shapiro has shown that denying the existence of mathematical objects requires at least as much philosophical justification as asserting it. But to this we can add that asserting the holistic ante rem existence of mathematical structures similarly requires just as much philosophical justification as asserting a more traditional Platonist position (i.e. both do, just insofar as they stipulate the independence of those structures, encounter the ontological access problem), but that the latter has the added advantage of being more than an ideology/ontology swap. That is, the latter retains an emphasis on the principal means by which realism may be differentiated from anti-realism – the concept of independence as a live, continuing issue, one that is unresolvable via inter-translatability arguments. So long as the cat on the mat is taken to independently exist, the best explanation of our corresponding belief that the cat is on the mat is provided by an account in which the ontological access problem remains acute: that is, one in which there is an inherent gap between what we know, how we know, and what, independently, is so.

The argument here, then, in some more detail, begins where Shapiro seeks to establish that an anti- realist who proposes to avoid a commitment to the existence of abstract objects such as numbers, sets or structures, is ultimately no better off – particularly on the epistemic front – than a realist who embraces the notion that such objects actually exist.

Insofar as it challenges the anti-realist notion that mathematics is about no more than linguistic or logical notions we are already familiar with, this argument is a powerful tool for the realist. But just so far as it challenges these anti-realist notions it jeopardises the realist's own proposed notion that mathematics is about an actual rather than a possible reality, especially any kind of non-derivative, independent realm.

The argument that Shapiro puts forward begins by showing that the anti-realist programs he examines and the realist's commitment to real abstract objects are, in fact, equivalent. That is, he shows that 'any insight that modalists [the particular group of anti-realists to whom Shapiro directs his argument] claim for their system can be immediately appropriated by realists and vice versa.

Moreover, the epistemological problems get "translated" as well' (p. 219).1

Very basically rendered, the argument compares a possible mathematical realm (one built from a plural quantifier, a contractibility quantifier, or an operator for logical possibility) with a real mathematical realm (of structures, individual objects, or simply an independent reality). This comparison shows that each is as problematic as the other.

For example, in fictionalism, abstract objects – like sets and numbers – are exchanged for a primitive notion of logical possibility (and uncountably many points and regions; Shapiro, 1997, 223), but Shapiro notes that there are direct, trivial translations from the fictionalists' language into the realists' and vice versa. The translation from realism to fictionalism is simply a matter of inserting modal operators in appropriate places, and perhaps conjoining axioms. For example, the translation of a sentence (x) would be of the form ◊(alpha & (x)*) where alpha is a conjunction of axioms from the background mathematical theory, and (x)* is a variant of x. For the converse, simply replace possibility with satisfiability. A subformula of the form 'phi is possible' becomes 'phi is satisfiable'. Modal structuralism is similarly tackled on page 229, Charles Chihara's 'constructibility' quantifier is tackled on page 230, and George Boolos' plural quantification is tackled on page 233. Boolos proposes plural quantifiers as a way of interpreting second-order quantifiers – that is, instead of reading these as saying 'there is a class' or 'there is a property', to read them as saying something like 'there are objects' or 'there are people'. But 'here too there are straightforward translations between the standard metalanguage, with classes and no plurals, and the classless language with plural quantifiers' (p. 234).

In short, as mentioned earlier, typically an anti-realist tries to replace abstract entities (numbers, functions, set, etc.) in mathematics with talk of something apparently more acceptable. But so long as such attempts also try to account for all or most of mathematics as it is practised, they wind up having to quantify over all of whatever they talk about (or quantify over); for example, they may use a plural quantifier, a constructibility quantifier, or an operator for logical possibility, or they may quantify over proofs. However, all of these are also abstract objects, so we can ask all the same questions of these replacements as we did of the realist's numbers, structures, and so on. And this is just exactly why we wind up with inter-translatability.

Given this situation, Shapiro concludes that we might as well opt for the real realm. This is primarily because this option minimises the 'ideology' needed to explain the realm in question (given that a real realm takes mathematics at face value), is the 'most perspicuous and natural' interpretation and, not least, explains why mathematics is done as though it is about independently real objects.

This is an issue for these sorts of anti-revisionist programs, of course, because whatever type of 'mental constructions' you might want to propose *instead* of mathematical-type objects, you still wind up needing to talk of *these* directly and, in comparably problematic ways, to account for mathematics as we *know it*. Haim Gaifman (2012) makes the same point when he says that (potential anti-realist) mathematicians ask questions about 'the provability of a sentence' or 'the consistency of a theory' as though such questions have definite answers.

But then they're talking about proof in just the same way that mathematical realists talk about other abstract mathematical stuff like numbers or sets.

So if we try to reduce all number/set talk to talk of constructible or apparently more accessible things like proofs, we don't avoid talk of mathematical things; in a sense, we just rename them.

I accept the arguments here and take it as true that modal anti-realism (where 'modal anti-realism' encompasses all the programs Shapiro reviews) is, in an important sense, equivalent to ante rem structuralism.

But the arguments show more than this. Again, equivalence goes both ways. Unless something more than a reduced ideology sets it apart from its anti-realist counterparts, ante rem structuralism inherits the features of modal anti-realism, including, for example, the particular burden of explaining why mathematics can legitimately be taken at face value at all – how, for example, its objects are still as bona fide as any objects in the natural or physical world. And these sorts of questions mean that the realist camp now also has to take on board precisely what Shapiro hoped to avoid – an increased ideology.

So the proposed 'trade off . . . between a vast ontology and an increased ideology' (Shapiro 1997, 218) has more ramifications than are at first apparent. The trade-off does, as Shapiro claims, mean that any possible reality inherits all the problems of an actual reality, especially since a possible ontology is, in the end, no smaller or less problematic than its real counterpart.

But the trade-off also means that a real or actual ontology is, for all intents and purposes, equivalent to a possible ontology – unless something extra sets the former apart from the latter – and so is not (without further argument) real in quite the same way it intended, crucially including not as independent physical reality. That is, it is not easy to see how such an actual reality is in any way independent – either of the constructed or possible reality it is equivalent to or, by extension, of the quantifiers and concepts from which a possible ontology is derived.

## 3 Two Access Problems

In the introduction to his 1997 book, Shapiro outlines a number of desiderata for a philosophy of mathematics. These begin with the assertion that mathematical assertions ought to be taken literally, that is, 'at face value' (p. 3). This desideratum, for Shapiro, suggests realism in mathematics for two reasons. First, mathematical assertions tend to talk about mathematical objects as though they exist. The second is that scientific assertions tend to talk about scientific objects as though they exist, and scientific language is effectively inseparable from mathematical language.

Specifically, since model theoretic semantics applies to ordinary and scientific language, the argument is that this same model theoretic semantics ought to apply to mathematical language.

This means that the initial desideratum boils down to two separate desiderata. The first – realism in ontology – arises from the fact that model theoretic semantics has the singular terms of its language denoting objects and the variables ranging over a domain of discourse. Taking mathematics at face value, then, means taking mathematical objects to exist. The second – realism in truth-value – arises from the fact that the model theoretic framework attributes to each well-formed, meaningful sentence a determinate and non-vacuous truth-value, either true or false. Thus, the primary requirement that Shapiro places on his own account is 'to develop an epistemology for mathematics while maintaining the ontological and semantic commitments [above]' (p. 4).

Since then, Shapiro has reaffirmed this realism (2011, 130):

> According to my *ante rem* structuralism, the subject-matter of a branch of mathematics is a structure, or class of structures, that exist objectively, independently of the community of mathematicians and scientists, their minds, languages, forms of life, etc … So *ante rem* metaphysics of mathematics.

Frege gave similar arguments for the existence of mathematical objects and for the truth of mathematical theorems (see Linnebo (2018), section 2.1). Frege's argument points out that mathematical language seems to refer to objects in much the same way that ordinary language does. This argument rests on the notion that certain subject-predicate phrases in ordinary, or *natural* language can only apparently be tested for truth or falsity – and perhaps can only be understood at all if any given hearer or translator supposes that there is such a subject to be found in the 'real' world, and that the properties of that subject can be discovered through scientific or empirical investigation. For example, the truth of a claim that 'Dogs are four-legged' can be investigated by referring to actual dogs and their supposed properties. Frege points out that given a great many mathematical claims have the same semantic structure (e.g. 'Natural numbers are divisible'), the burden should be on anyone insisting these two types of claims are to be treated differently, rather than on anyone who treats them similarly – that is, rather than on anyone who treats both as referring to objects in the world to be investigated as to their supposed properties.

Of course, the hitch here is that the idea that we can 'discover' mathematical truths in the same way that we can discover empirical or physical truths begs the question – or the problem – of access.

We've still no real answer to the question, 'But *how* do we discover mathematical truths?'.

Note, though, that Frege (e.g. in his 1970 essay) separates the 'mode of presentation' (aka the 'sense') whereby an individual or a community comes to 'see' an object from the object itself (aka the 'referent'). His famous example notes that without such a distinction, identity claims such as 'Hesperus is Phosphorous' would be rendered inexplicable. It seems fair to expect that, given Frege's argument has weight, a similar arrangement should hold for the 'referring' to mathematical objects.

Frege gives a nice example (also in his 1970 essay) explaining the roles of the referent, the mode of presentation, and the 'knowing' human. His example is a human looking at the moon through a telescope. Frege asks us to understand the 'mode of presentation' here as that which lies between, or mediates the subjective experience of that human and the moon itself. So the mode of presentation, in this example, is what is provided by the telescope – linking a subjective human experience to an objective, real moon.

This is important because Shapiro's first desideratum incorporates a robust type of existence for mathematical reality – that is, the notion that the way in which mathematicians take mathematical objects to exist can be understood as the way in which we humans take objective reality to exist generally: that is, (at least) as independent of 'the community . . . their minds, languages, forms of life, etc.'.

## 3.1 The Problem of the 'Mode of Presentation'

Again, traditionally, the primary philosophical appeal of this sort of realism corresponds directly to its primary problem, that is, the access problem. According to MacBride (2008, 156, quoted in Shapiro (2011, 131)), the problem runs something as follows:

> a face-value or realist interpretation of mathematics appears to make a mystery of how mathematicians access truths about the mathematical domain. To access them it appears that mathematicians must do the impossible: they must transcend their own concrete natures to pass over to the abstract domain.

But there is more than one way to interpret the access problem for realism. Shapiro (2011, 132) interprets it thus:

> When proposing a philosophy of mathematics, one must surely say something about epistemology. That is, one must indicate how it is that mathematics is known. This epistemology should dovetail with the metaphysics and ontology of the presented view. If you say that mathematics is about *p*s, then

> your epistemology should show how it is that mathematicians manage to know things *about* these *p*s. An ontological realist has it that mathematics is about a realm of abstract objects, places in *ante rem* structures . . . So an advocate of such a view . . . should say something about how the mathematician comes to know things about places in *ante rem* structures. Hartry Field correctly says that the realist must explain how it is that mathematicians manage to have reliable beliefs about mathematical objects. Hence the access problem.

But this interpretation, as we can see from noting Frege's three-way distinction, actually contains more than one problem. There is, on the one hand, the problem of accessing a reality the nature of which is as posited: real, objective, abstract and independent. Call this the ontological problem. On the other hand, there is the nature of our access itself: of our beliefs and knowledge of such objects. Call this the epistemological problem. To return to the moon for a moment: our telescope-viewing human could ask herself two questions. One: is the moon itself an independently existing object with properties all of its own, regardless of how it happens to appear to me? And two: can I rely on the telescope and my own capabilities to provide me with sensible, coherent beliefs about what is true of the moon itself?

The ontological problem takes seriously the notion that the status of the posited reality as accessed and as simply *there* – that is, as part of our ontology – is an important part of the realist's position and so needs direct defence. Here, this problem is generated as much (if not more) by the status of the posited reality as independent as it is by its status as *mathematical* (abstract, seemingly not apt for empirical investigation, etc.).

Note that this problem is still an access problem, but it focuses on the type of thing we are supposed to access rather than on the mode of access itself: that is, if the type of thing we are supposed to access is *mathematical* – independent of ourselves, abstract, and so on – then knowing when we have actually accessed *it* (as opposed, say, to some construction or projection of our own making or to a moonish smear on a telescope's lens) is the problem.

One typical objection or formulation of the ontological access problem runs along these lines: if the posited realm is indeed real, we should be able to know about it; but how could we possibly know about such a reality, one which is independent, abstract, apparently acausal and objective? If no satisfactory answer is found, so this argument goes, then we must conclude that that realm is not real, or at least not in the specific way proposed (or, alternatively, that the type of reality or non- reality it has is moot).

Of course, if the realist position on mathematical reality is already that it does not actually exist as independent, abstract, objective and so on, or only possibly so exists (or that its actual reality or non-reality is moot), then this formulation of the ontological access problem will not arise. But if a realist position does incorporate a claim that mathematical reality is real in the specific way most often associated with Platonism, that is, as independent, objective and abstract (and so on), then that realist position is vulnerable to a version of the ontological problem – just so long as its proponent wishes to retain or defend that claim.

On the other hand, there is the epistemological problem. This is what Shapiro focuses on most in his cashing out of the access problem. The focus of the epistemological problem is on how we can defend our belief that we know truths or indeed anything about the mathematical realm posited.

This problem is addressed by showing that we can reasonably believe or ought to believe in the existence of the type of reality posited. Shapiro (1997, 2011) sets out to answer this particular access problem. Regarding the ontological access problem, he says (2011, 135):

> The goal of my book was not to provide a deductive argument, from commonly accepted premises, whether mathematical or non-mathematical, to the conclusion that *ante rem* structures exist and that we have the appropriate 'access' to them.

But arguably this approach emphasises the epistemological problem and sets aside the component of the ontological problem which addresses whether or not we know that what we experience (or intuit, or believe in as real, independent and objective) is indeed what we may reasonably deduce it is. Without directly addressing this problem, realist accounts run up against the problem of intertranslatability and so ultimately carry with them a burden to defend the posited reality *as* real, independent, objective and so on: that is, as precisely the sort of thing Shapiro posits and the positing of which most characterises the realist position in philosophy. But so long as such a reality is defended, the further ontological problem – whether or not what we 'see' or access is what *is* (aside from the end result of any of our means and methods of such 'seeing') – cannot be set aside. This is why Frege's distinction is important to a realist: the referent itself is what is, and is apart from its mode of presentation – aka what we reasonably believe is true.

The telescope-viewing human takes herself to reasonably believe that she sees *the* moon: a real, independent subject that she can investigate. But she has at her disposal only the apparatus of the telescope. She might ask whether her faith in that apparatus and in what it tells her of what she believes is the moon itself is reliable or trustworthy. This is the epistemological access problem.

Were she then to ask whether what she believes she is seeing – an independently, objectively real moon – is in fact what she is seeing, something that in fact does exist apart from any and all of her endeavours to understand it, then she'd be engaging with the ontological problem.

Hartry Field, in a discussion of Hilary Putnam's view on the ontological problem for realists regarding the mathematical realm, spells this out clearly:

In basic outline [for mathematics], the argument is this:

> [ia] There is nothing in our inferential practice that could determine that our term 'set' singles out the entire set theoretic universe V rather than a suitably closed subset of V.
> [ib] Even on the assumption that it singles out the whole set theoretic universe V, there is nothing in our inferential practice that could determine that our term [ε] singles out the membership relation E on V as opposed to some other relation on V that obeys the axioms we have laid down.
> [ii] The indeterminacy in [ia] and [ib] is sufficient to leave indeterminate the truth value of typical undecidable sentences of set theory (Field 2001, 317).

This argument suggests that there is no way, akin to the empirical process by which the morning star and the evening star (Hesperus and Phosphorous) were identified as one and the same, that might decide which realm of mathematical referents we should or do refer to, because there is no way of settling on which 'modes of presentation' that afford us 'access' to that referent.

As Field (2001, 334) puts it:

> [T]here are lots of properties and relations that the mathematical objects in this universe can stand in; and there isn't a whole lot [at least not once we get beyond a certain finite conception of natural numbers] to determine what we should take our mathematical predicates to stand for, beyond that they make the mathematical sentences we accept true . . . for example, whatever choice we make for the size of the continuum (as long as it's consistent with the rest of our set theory), we can find properties and relations for our set- theoretic vocabulary to stand for that will make that choice true and the others false . . . so the continuum could be aleph 23 or aleph 817.

Field (2001, 335) adds:

> the 'full blooded platonism' of Balaguer . . . [says that there] isn't just a single universe of sets, but many different ones existing side-by-side: some in which the continuum has size 23, some in which it has size 817, and so on [for more on this, see Balaguer 1998]. Also, as we've seen, Putnam's argument has it that even a view on which there is a single universe of sets, independent of the mind and existing prior to our probing, ultimately must yield the same anti-objectivist consequence as do these other views: for even if there is a single universe of pre-existing sets, there are multiple relations on it that are candidates for what we mean by membership, so that the effect of many universes is achievable in a single universe. 'Multiple

universe' views and 'mind-dependent objects' views merely have the virtue of making the anti-objectivist consequence manifest.

One more view with the same anti-objectivist consequence is the fictionalist view, mentioned in Section 2, in which there are no mathematical objects at all. Again, this gives rise to the same limitation on objectivity: without a clear defence of mathematical objects, separable from their 'mode of presentation', the realist philosopher of mathematics cannot adequately address the ontological access problem – that is, the problem of how we might see through the mode of presentation to the thing in itself, as itself. But, as Field argues, without a clear defence of mathematical objects as independently and objectively existing, then anything goes, so long as that 'anything' meets the requirements of consistency (in a broad sense), though, of course, some fictions are more useful or beautiful than others.

Tangentially, Mount Everest has (at least) two names, 'Chomolungma' and 'Sagarmatha' (MEGS, 2020), reflecting two distinct epistemological experiences of one and the same reality. And so our being able to understand that the two names refer to the one mountain presents just the sort of identity claim Frege was concerned to explain via his 'mode of presentation'. The Putnam/Field argument just presented suggests that we have no analogous 'mode of presentation' (separable from the real referent itself) to explore when it comes to mathematical referents.[4]

## 3.2 Why Realism?

One way to defend realism is to appeal to its capacity to minimise ideology and to explain the practice of mathematicians and their beliefs. But, as is implicit in Shapiro's inter-translatability arguments discussed earlier, these aims can be served as well by a belief in an independent reality as by a direct defence of the existence of that reality itself. Given that the ontological problem does indeed arise whenever an independently existing reality is posited, why posit such a realm in the first place?

Traditionally, the appeal of realism lies in the explanatory and justificatory power of its proposed connection between what we know and what exists independently of us. In essence, such a reality justifies our beliefs about it (presuming access is achieved) simply because, since it is not us, we know we did not make it up, and we can hope we did not get our beliefs about it wrong. The justification provided by the traditional realist account of reality is not just

[4] Interestingly, Mt Everest has recently been declared two feet taller than previously believed (smithsonianmab.com, 10 December 2020), a result of China and Nepal's 'joint measuring' of the mountain.

of our practices and beliefs but of these practices and beliefs as independently correct (as opposed to workable or justified on other, perhaps pragmatic, perhaps deductive grounds.)

To anticipate: due to the inter-translatability of our belief in an independent realm with its anti- realist counterparts, perhaps it is not in our belief or even in our knowledge of such a realm, but in the particular sort of justificatory work such an independent reality offers, that the principal gain of realism over other accounts resides. There may be other gains, but these need to be defended against inter-translatability – that is, to be shown to be distinctly realist advantages, not also able to be claimed by anti-realist accounts.

It seems reasonable to propose that the work done by the positing of this type of reality at least balances the problems it causes. That is, it is hard to see how a position positing a Platonist mathematical reality is, in the end, any better off than one which does not, without some direct utilisation and defence of the reality in question. In fact, once we take into account the ontological problem and the phenomenon of inter-translatability, then we can note that a realist's account of mathematical reality, if left as a largely undefended stipulation, not only carries with it all of the problems of both realism and anti-realism, but it does so without the principal gain traditionally associated with the Platonist status of mathematical reality as genuinely independently existent.

For the sake of providing a simple shorthand concept to refer to the particular status that mathematical realism might confer on mathematical reality,[5] we can look at a footnote in Benacerraf's (1973) article on mathematical truth about his *causal theory* for everyday objects (in this case, Hermione seeing a cat on a mat):

> I would like to avoid taking any stand on the cluster of issues in the philosophy of mind or psychology concerning the nature of psychological states. Any view in which Hermione can learn that the cat is on the mat by looking at a real cat on a real mat will do for my purposes. If looking at a cat on a mat puts Hermione into a state and you wish to call that state a physical, or psychological, or even physiological state, I will not object so long as it is understood that such a state, if it is in her state of knowledge, is causally related in an appropriate way to the cat's having been on the mat when she looked. If there is no such state, then so much the worse for my view. (p. 412)

The direction of causality is important here. Benacerraf is at pains to note that causality, while it may run from us to reality, also crucially runs from reality to us. The existence of the cat causes our relevant knowledge or (to circumvent the

[5] As well as some evidence that Benacerraf was concerned with both versions (ontological and epistemological) of the access problem.

issues surrounding the notion of causality) our relevant knowledge depends on the existence of the cat. Whatever it is that our knowledge might cause (or that depends on our knowledge) – say the physical or physiological state Hermione is in (incorporating the perceptual image of the cat, the belief that the cat is truly, independently there, etc.) – this has to be 'related in an appropriate way' to the actual cat. If the cat independently, objectively exists, then our beliefs about it are justified in a way in which they are not if it does not.

The analogous problem for Platonism is not in showing how we can reasonably believe that mathematical reality exists but in showing or arguing that if it does exist, then our beliefs about it are justified in a particular way and that this particular type of justification is worth retaining and defending in our account.

Note that a traditional realist about the cat on the mat would hold that ordinary physical objects have an existence independent of our own – that they exist prior to and separate from us, and so are not dependent on us. But, in order to defend the traditional realist justification of our beliefs about these sorts of objects, such a realist needs to show (at least) that what we 'perceive' as physical reality is in fact, or in some sense, the same reality that existed prior to, or independent of our perception of it, or again, that such a notion of justification is worth retaining and defending – that is, balances or outweighs the problem it brings along with it.

As mentioned earlier, this problem is not small.

If the independent reality posited by the realist is ultimately inseparable or indistinguishable from our perception (or knowledge or belief) of it, then in turn, the realist account has little to distinguish it from an anti-realist one. This is why the ontological problem of access remains and should remain a real one for the realist. In short, mathematical realists need to underline the role of the actual, independent cat on the mat: despite its significant attendant problem, perhaps the sort of realism in which it is the cat itself that justifies (perhaps among other things) is the sort – likely the only sort – that can overcome the inter-translatability problem.

## 3.3 The Caesar Problem

Another part of Shapiro's main argument from his 1997 book, and indeed the main sort of argument generally given for the prioritising of some version of the epistemological access problem over the ontological access problem, is that the so-called 'Caesar problem' is not a real problem for mathematical realism. In short, the Caesar problem is a more targeted version of Frege's identity problem. It asks precisely what the real or original objects of mathematics are and how can these be identified, specifically, how numbers can be distinguished from any other kinds of object, such as the object that is Julius Caesar. Since there is no 'in principle' way of deciding between the various candidates, it is tempting to conclude with Benacerraf (see Benacerraf (1965)) that the singular terms of mathematical language do not denote objects at all, at least nothing akin to ordinary or scientific objects. Shapiro (1997) argues that the acceptability of this conclusion 'depends on what it is to *be an object* . . . on what sort of questions can be legitimately asked about objects and what sort of questions have determinate answers waiting to be discovered' (p. 5).

Shapiro's argument says that there is no real fact determining the matter of whether the places of a structure and other objects are identifiable, or whether the places of two different structures are identifiable; so the only questions of identity with a determinate answer are those asked within the context of the single structure. That is, there is no determinate answer to the question, 'Is the 2 of the natural numbers identifiable with the 2 of the real numbers?', but there is a determinate answer to the question, 'Is the 2 of the natural numbers identifiable with the 4 of the natural numbers?'. So whether or not the places of the natural number structure exist as objects in the universe, or existed as objects before we identified them as such, is a non-question.

This has the consequence that our identification or 'discovery' of mathematical structures may or may not be a discovery or identification of objects that existed previous to whatever procedure enabled it. But this means that the structures of Shapiro's structuralism are either entirely new objects whose existence depends on the various epistemic routes by which they are encountered, in which case their inter-translatability with anti-realist objects undermines their status as independent; or are some other sort of reality which is unable to be identified with any unique realm of real, pre-existing mathematical objects. In this case further argument is needed to defend the stipulation of such a mathematical realm and its relevance to the epistemologically accessible structures.

To recap: It seems that one of the best accounts we have of the independent reality of an independent mathematical realm – Shapiro's structuralism – can address the epistemological access problem only, so perhaps we have to set the ontological problem to one side.[6] It remains unclear just what structures are and how such things can be said to occupy a place in the realm of independent, objective reality.

[6] At least insofar as it seeks to go beyond criterion 4, in Hellman and Shapiro (2019, 54). But see their p. 54 for an argument that pushes the ontological problem up a level – that is, from the ontological problem of the existence of set-theoretic structures to ante rem structures.

# 4 Independence

It seems that the strongest sort of independence that we can attribute to mathematical structures – the ‘ante rem’ existence of structures – may not, in the end, do enough explanatory or justificatory work to balance the first principles, asked from the ‘outside-in’ ontological problem it encounters. That is, ‘ante rem’ structures cannot serve the realist desideratum outlined in Section 3 without further defence – that is, without further defence for the relevance and role of the independence claim itself and without acknowledging the ontological access problem it encounters.

In particular, the notion of ‘ante rem’ existence does not settle traditional realist problems, such as the various offshoots of Benacerraf’s problem. As we saw earlier, these most notably include problems of reference and correctness: for instance, the problem of showing that mathematical names and formalisms correctly correspond to or ‘pick out’ their genuine counterparts in the independently existing mathematical realm, if these counterparts are, as stipulated, indeed independent of their instances.

## 4.1 Traditional Ante Rem Independence

In his reworking of the structuralist slogan ‘that mathematical objects are places in structures’, Shapiro points out that there is an intuitive difference between an object and the place or space it occupies. An object is an ‘office-holder’, and the place in a structure that object might occupy is an office. The intuition that these are indeed two separate items boils down, for Shapiro, to a distinction in linguistic practice, depending on where (or on what) your focus lies.

For Shapiro, the slogan ‘mathematical objects are places in structures’ is to be interpreted from the ‘places-are-objects’ perspective. So when the structuralist says that sets are objects, he means that each place in the set theoretic structure is an object – at least grammatically and at most literally. Arithmetic, then, is about the natural number structure, and its domain of discourse consists of the places in this structure. The same goes for the other non-algebraic fields, such as real and complex analysis, Euclidean geometry, and perhaps set theory (p. 89).

This distinction is important to Shapiro’s account for a number of reasons, one of which is the articulation of his particular realism, particularly to his claim that structures are indeed ‘ante rem’. That is, Shapiro’s realism takes the ‘places-are-offices’ perspective literally. For Shapiro, ‘bona fide singular terms, like “Vice-President,” “short-stop,” and “2,” denote bona fide objects’ (p. 83). Likewise: ‘structures exist whether or not they are exemplified in a nonstructural realm’ (p. 89).

Another reason that this particular distinction is important to Shapiro's account is that he sees it as enabling his program both to commit to numbers as objects and to resolve (or, more accurately, to resign) Frege's 'Caesar problem' via ontological, specifically interstructural relativity.[7]

This second reason will be dealt with in some depth later, but is worth touching upon now. Recall that, in a nutshell, the Caesar problem asks how numbers can be objects (as Frege argued they were) when, to repeat an earlier point, it seems that there is no way in principle of either distinguishing between or identifying numbers and any other of their modes of presentation, nor indeed of any other kind of object (recall the moon and telescope example presented in Section 3).

Shapiro's solution uses the distinction between the 'places-are-offices' and the 'places-are-objects' perspectives by arguing that each perspective creates its own context for these kinds of identity questions, and it is only within the confines of one of these contexts at any one time that such questions can even make sense.

For instance, from the 'places-are-objects' perspective, we may ask whether or not the object '2' is the same or different from the object '4'. And because this question is asked within the confines of only one of the two possible perspectives, we may expect to discover a definite answer. It is determinate, Shapiro argues, that '2' is identical to '2' and not to '4'. But he adds:

> [I]t makes no sense to pursue the identity between a place in the natural number structure and some other object, expecting there to be a fact of the matter. Identity between natural numbers is determinate; identity between numbers and other sorts of objects is not, and neither is identity between numbers and the positions of other structures (p. 79).

That is, these latter questions, rather than staying within the confines of only one or the other of the two perspectives, commit the error of taking both perspectives at once – of trying to compare their different foci within one question. Asking whether 'the shortstop is Ozzie Smith' (p. 79) commits a similar mixing of perspectives, or crossing of contexts. Shapiro argues that in this case the person and the position share no criteria by which to establish or reject their identity. Of course, Ozzie Smith can take the position of shortstop and so, in this sense, 'be' that position; but according to Shapiro, this is not a question of identity. And it is in just this same way – that Ozzie Smith can be the shortstop, but can neither be nor not be identical to the shortstop – that Julius

[7] Again, though, see Hellman and Shapiro (2019, 54) for a summary of a higher-level distinction between set-theoretic structures themselves, based on existence criteria for the ante rem structures they may instantiate.

Caesar can be '2', but is neither identical nor non-identical to '2'. Thus, Shapiro's solution to the Caesar problem is effectively to ban the question it asks – at the level of offices and set-theoretic structures.

Arguably, though, the same problem arises at the higher level wherein we may wish to ask which set-theoretic structures exemplify which ante rem structures and how to identify or distinguish between the various cases of the latter themselves.[8] For example, questions, such as Cantor's,[9] about the 'true' or 'real' nature of infinity arguably rely on a separation between perspectives at the most fundamental level of mathematics: we have (at least) two notions of infinity – that which we understand as 'counting' (i.e. the natural numbers – I, 2, 3, . . . etc.), and that which we understand as somehow in-between each of our numbers (the real numbers – 1.00001, 1.00002, etc.). How are we to decide whether questions about the relationship between these (e.g. Cantor's theorem) mix perspectives or whether they do not?

Or, perhaps a better example, should we understand the notion of 'set' as a simple fundamental akin to 'any bunch of stuff' (as does naïve set theory), or should we constrain the notion of 'set' to that which we can somehow characterise (as Fefermen's predicative set theory,[10] or Russell's type theory[11]).

Perhaps questions of identity like these that (even potentially) mix perspectives 'do not have definitive answers, and they do not need them' (p. 80).

This solution though, in order to be consistent, has to apply to questions of identity between different structures as well. This is because the places of one structure can be the placeholders of another, in the same way that Ozzie Smith can be the shortstop.

The argument here (presented more fully below) is that this particular application of a general ban on mixing perspectives threatens a realist solution to the Caesar problem. To anticipate, there is something intuitively quite different between asking whether Julius Caesar is identical to the real number 2 and asking whether the natural number 2 is identical to the real number 2 (also again, at the higher level offered in Hellman and Shapiro (2019), it equally seems that there is something intuitively different about identity questions between ante rem structures themselves, and those within particular structures exemplifying them – see p. 49). The latter sort of question is not so easily dismissed as nonsensical or wholly indeterminate.

[8] For more on this, see Andrew David Irvine and Harry Deutsch, 'Russell's Paradox', *The Stanford Encyclopedia of Philosophy* (Spring 2021 Edition), Edward N. Zalta (ed.), https://plato.stanford.edu/archives/spr2021/entries/russell-paradox.

[9] See https://plato.stanford.edu/entries/settheory-alternative.

[10] See https://plato.stanford.edu/entries/set-theory-constructive/#PreConSetThe.

[11] See https://plato.stanford.edu/entries/type-theory.

The important point for now though is that, from what we have looked at so far, it can be argued that the nature or degree of independence that Shapiro wants for structures is a significant one. For Shapiro, taking the 'places-are-offices' perspective literally means that structures exist before and apart from any non-structural exemplification in any realm whatsoever (p. 75).

But this could mean a couple of different things. It might mean that the existence of structures is an existence that is independent of the theories – mathematical or otherwise – that we have about them. It might also mean that their existence is independent of both the act (since they exist before exemplification) and objects (since they exist apart from any exemplification) of our comprehension or understanding.

The benefits of an ante rem structuralism, then, include '[the fact that this kind of structuralism is] the most perspicuous and least artificial . . . It comes closest to capturing how mathematical theories are conceived' (p. 90). But we could argue that, given both the literal interpretation of the 'places- are-offices' perspective, and the intuitive difference between the Caesar problem and that of differentiating pre-existing mathematical objects (or, indeed, ante rem structures) from one another, this claim only holds provided that by 'ante rem structuralism' Shapiro means a structuralism which includes both types of independence outlined earlier in this section.

Even if this proviso is granted, though, and we conclude that the most perspicuous structuralism posits structures with this degree of independence, there still remains the question of how to accommodate it in a structuralist or realist account.

## 4.2 Independence in Mathematics

The notion of 'independence', much like that of 'objectivity', is often presumed understood for a given realm of discourse, but rarely explicated beyond a few well-known slogans, such as that appealed to in Resnik's formulation, and (Øystein, 2018, section 3.2):

> Independence says that mathematical objects, if there are any, are independent of intelligent agents and their language, thought and practices.

Taken as it stands, though, this characterisation invites an inter-translatability problem all of its own – just one that occurs yet another level up from those we've been exploring. Such slogans around the notion of independence do little to distinguish a realist from an anti-realist account of mathematics. To see this, we can refer to Wright's characterisation of objectivity (in his (1992)), which he explicitly seeks to render an acceptable, metaphysically neutral account of objectivity for any given realm – that is, Wright proposes a sound, thoughtful, and insightful account of objectivity that both the realist and the anti-realist can, he hopes, agree on. But I would argue that this only pushes the problem of 'inter-translatability' another level above that we've already encountered. Similarly, a notion of independence, such as that appealed to earlier, insofar as it can be encompassed by both an anti-realist and a realist account of mathematical reality, faces the same problem. If we are to have a thorough, in-depth, and instructive conversation about the status of mathematical reality, we need some way to distinguish the two primary positions one can take here, such that each does not simply and easily translate to the other. (Of course, the flip side of this desideratum is that such a conversation cannot be had *unless* there is a clear, shared language in which it can be conducted – this is the tightrope on which we balance.)

An example here is Wright's first criterion for objectivity, 'epistemic constraint' (Shapiro, 2000a, 357). A realist position embracing epistemic constraint effectively lines up with the realist position which adopts 'real' to avoid ideology, yet whose account is constrained by what is possible given by classical logic and widely accepted mathematics. Epistemic constraint for a discourse is defined thus: 'A discourse is epistemically constrained if it is not possible for there to be unknowable truths in the discourse' (Shapiro, 2000a, 357). Were a position to grant the existence of unknowable truths, this would prima facie distinguish it from any anti-realist counterparts, in the same way that a position stipulating that even known 'truth' goes beyond whatever it is that is constrained by any particular method, means, ability, and so on by which we come to know it is prima facie distinct from one simply allowing the proper translation from truth to 'possibly knowable truth'. But often, the stipulation of such 'transcendent' truth itself leaves the realm of any 'shared language' in which the existence of such a thing can be debated.

A key point of this Element is that nowhere is this conundrum more acute than in the realm of discourse regarding mathematical truth, given the Field/Putnam underdetermination touched on in Section 3. In the mathematical realm, it seems we feel we have fewer 'hooks' tying our methods and means of knowing than we do in the physical, or at least the middle-sized portion of the physical in which we can collect empirical data to attest to the reasonableness of such claims as that there *really is* a cat on the mat. We can, for instance, attest to object permanence regarding the cat (provided he stays on the mat when our backs are turned) and to shareable, empirically testable data establishing his real, objective existence. But mathematics apparently lacks such hooks, and without such hooks, the distinction between the object of knowledge and the object itself seems more problematic.

## 4.3 Who Cares?

One response to the inter-translatability problem is, essentially: who cares? We can be either realist or not without much changing at all (is the thought). Interestingly, Shapiro and Gaifman take inter- translatability to be a reason to accept realism (because at least then you preserve something of the sense that mathematics is objective, and you seem to be capturing the way mathematicians actually do maths). But Field (and others) have taken the same sort of argument to be a reason to accept anti-realism/constructivism (because, if anything goes, nothing is actually objective here, and pretending so just creates a burden we don't need).

So this is interesting – we have a situation with mathematics that we don't have (at least not obviously) elsewhere: mathematical definitions, relations, and properties seem to so wholly circumscribe, or even in some sense *be* what they are about, that going on to offer some further description (say, these are actually proofs, or these are actually structures, objects, possibilities, or whatever) can just seem a little beside the point.

On the other hand, there's another response, essentially: we should care, there is much at stake. Often this feeling is put down to the sense that there is a determinate answer to our mathematical questions: if anything is objective, mathematics is. We must have a good account of what that objectivity actually amounts to, because mathematics is just (objectively) *true.*

A mathematical realist, remember, is someone who thinks mathematical structures, numbers, sets, or whatever your poison, exists. So, again, perhaps what is at stake for a mathematical realist is *existence*, and not just any existence, but independent existence; that is, the sort of reality touted is the sort that ***is*** (at least as much as anything is) regardless of whether we – that is any human – is, was, or would be around to discover it. But this means that what's at stake for a realist then (well, ok, one conception which is interesting (to me) to imagine) is not objectivity *as such*, but objectivity as *a result of*, or perhaps in some way *because of* independent existence.

The trouble with, and the interest in this particular thought is that – many philosophers throw up their hands at this point – this is what Hume disparagingly called metaphysics. It's beyond Kant's transcendental categories: it's being theological and mystical and just somehow plain silly. Existence *that* independent is (it seems) beyond any effective control – past any means we might have of either confirming or denying knowledge of it or any interaction with it at all – so the anti- metaphysics claims tend to go.

And, yes, metaphysics is weird and, as touched on here, it's hard enough to sensibly talk about the idea that ordinary physical stuff might have this sort of independent existence (let's focus on middle-sized stuff such as the cat on the mat, to dodge quantum questions for the minute). But it gets extra interesting when you try it out with mathematical stuff. Mathematical reality, remember, is (in some sense) 'wholly determined' by whatever makes the axioms true – that's it. There is, so far as we know, no 'more' to this stuff than that. There are ways we can talk about *physical stuff* to try to make sense of the idea of independent existence *there*; appeals, say, to our perspective or our acquaintance with whatever makes physical stuff whatever it is – the idea that somehow there's maybe more to the story than we know or can know, because, after all, we're not **it** and we see things a certain way, constrained by our own physical limitations, and so on.

But, when it comes to mathematical stuff – so far as we know, there's nowhere (or barely anywhere) to go. We can't say there's somehow more beyond our perspective (whatever that might be) of mathematical objects – or our proofs and our axioms; in a very real sense there **is** no more to the number 4 than the sum of 1 and 3 (and the product of 2 and 2, etc.).

I think this sort of weirdness – the strange character of mathematical abstract stuff as compared with other abstract stuff – is why a lot of modern-day philosophers of mathematics turn the original thought about independent existence being what's at stake upside down. Rather than having what's at stake being the way in which objectivity *comes from* independent reality, they retreat back to objectivity itself and then bring independent existence back into the story (if they do at all) by *incorporating* these weird metaphysical ideas into the notion and criteria of 'objectivity' or 'real' and so on, and so they wind up with something that effectively says 'independent reality *comes from* objectivity – or at least from an entitlement to speak, believe, and act as if the mathematical realm were objective, real', and so on.

So a good majority agree with Gaifman when he says: if we think questions about mathematics have a 'factually meaningful' answer (independently of whether or not we can find it out), then we're 'realists' about mathematics (Gaifman, 2012). Since most mathematicians do work (in something like) this way, they are (or are as good as) 'realists' in this sense. And as we've seen, a lot of what we've called anti-realist positions in mathematics want to retain mathematics as it stands, or to account for it as it is practised (rather than to revise it), and, in this sense, they can be aligned with realism too.

So Gaifman (and a lot of modern philosophers with him) have the idea that if we have to talk of existence at all, it is best characterised as something like this: if you think there's a yes-or-no answer to a question about something (independently of whether we can find it out or not), you think that thing independently exists.

The thing that bothers/interests me here is: is this sort of idea *all* there really is to thinking something exists, or (if we dodge the notion of existence for a minute) is this all there is to the realist's idea of an 'independent' way things are or 'what it is for something to be (independently) the case'? Surely this is not what a realist intends (or what all versions of realism boil down to). What happened to metaphysics? It seems it's been swallowed up by a shareable or neutral notion of objectivity.

But what if we *still* feel there's more to what's at stake in the original debate than whether we believe there's a yes-or-no answer to mathematical questions independently of our ability to find them? Well then, again, this is the tightrope we walk – metaphysics sneaks back in.

Why might we want to invite metaphysics back to the table? Well, let's look again at what might be going on here. Gaifman says *meaning* for the realist has to do with the notion of a determinate answer: he starts with that notion just because he takes it to be more primitive and easily understood than weird things like existence or objects – remember, for Gaifman 'a *factually meaningful question* is a question that has a determinate answer, which is independent of our knowing it or our abilities to find it' (p. 481)

But, as we've seen, there's nothing stopping a fictionalist or constructivist thinking there may be mathematical truths independent of our knowing them or being able to find them (perhaps we could *in principle,* or, if we were infinite, but their point can still stand in this case – maths can still be *about* non-existent, non-independent things)

So there's a collection of questions for which anti-realists and realists agree there *is* an answer independent of our ability to find it (supposing for a minute that these are just questions asked within the main body of mathematics as practised); but, if this is all there is to thinking something independently exists, then all realists can be read as anti-realists or all anti-realists can be read as realists. Again, there is nothing to distinguish a realist stance about these questions from one where the answer is understood as or argued to be *dependent* on our (possibly) knowing it and our abilities to find it. I say (possible) constructible proof, you say independent realm – it doesn't really matter at all, and methodologically it need make no difference. We are back to inter-translatability again.

And, this may mean that 'independence', if it's taken as something more than this – that is, more than as what can be translated as 'real', 'objective', and so on, according to a set of criteria, acceptable to realism and anti-realism alike, there is a risk that this 'extra' or 'over and above' notion of independence might be seen as a sort of 'by the way' add-on; a bit ad hoc. Indeed, for Gaifman, if something like Hilbert's programme (whose goal, very briefly stated, was to prove that the system in which we ask mathematical questions is consistent and that every such question will have a determinate (yes-or-no) answer) was actually accomplished – that is, if 'the mathematician could . . . apply "true" and "false" across the board, [then] conjectures arising in mathematical practice are true if they can be proved, false if they can be refuted, and it is either one or the other. The appeal to finitist reasoning is needed in order to guarantee consistency and base mathematics on firm foundations. *But once this is done, the problem of mathematical realism becomes moot*' (p. 497).

But if you do want to resist this situation, or want to bring metaphysics back in somehow, or want to restipulate a separate or prior role for independence and existence – if you think inter-translatable objectivity is *not* all there is to thinking (mathematical) somethings exist, you're going to find it hard to express this in any clear way that's effectively *different* from what Gaifman says. This is because, in a way, an appeal to objectivity as characterised by bivalence does no more, and a lot less, than does an appeal to the Wrightian objectivity mentioned in Section 4.2.

Appealing to bivalence shifts the emphasis. Gaifman can still talk of independent existence, that is, just by adding that what it strictly meant (or what it can be reduced to) is this notion of an objective answer to a question. And, yes, on the whole, realists *do* agree that there being an independent yes–or-no answer is what they meant, or it's part of it somehow; but some may mean more, and it seems, in this case, all they can do is lamely add 'but I actually meant something more than that too'. it seems that the only way you can say so is to jump up and down a lot. This is because we've now got a weird kind of *deeper* inter-translation going on – your 'independent existence' is translatable (or reducible) to my 'objective answer', and vice versa. Saying 'well, we wanted the objectivity to depend on the independent existence, not the other way around' achieves very little, because so long as these concepts are tied to objectivity at all, it doesn't really matter which depends on what – we can all be construed as saying the same thing, just with different emphasis.

## 4.4 What Is the Case?

There is another option, though, and this is for mathematical realists to try to explicitly 'write in' to their account what exactly they meant by there being 'something else' involved.

An initially appealing option here might be to somehow figure out a way of spelling out the way mathematical reality is more, perhaps by drawing analogies with physical stuff, or perhaps in some other way.

Carrie Jenkins gives a thoughtful picture outlining one way in which whether or not you accept bivalence across a given domain can be influenced by whether or not you are realist about that realm – but acceptance or rejection of bivalence is not by itself at issue. For Jenkins, what's at issue is rather the idea that 'it is no part of *what it is* for *p* to be the case that our mental lives be a certain way' (Jenkins 2005, 200).

Jenkins has had a good go at spelling things out this way. First, she casts the whole debate so that it's not necessarily about existence (per se) or objects (per se) nor about any particular characterisation of an independent realm or reality or what have you, but just about the question of *what it is for (something) to be the case*.

So on Jenkin's account, we have anti-realism (for a given area) characterised as claiming that something's being the case (or not being the case) in this area depends in some way or other on humans (or ideal extensions of humans) – specifically on humans' mental states (according to Jenkins), but in principle, it could be on any sort of thing we tie to us – to our *own* 'being the case'.

So then, realism about an area (for Jenkins) is the idea that it is no part of humanness (specifically our mental lives or states being a certain way) that something is the case (or is not the case) in this area. You can still put this in more commonly used parlance; for example, 'what something essentially is or what something essentially is not'.

I quite like this characterisation (Jenkins calls it 'essential realism') – it shows how the difference between realism and anti-realism need not come down to whether or not we posit *existing* (in some standard sense of that word) *objects*, or an eternal, acausal, or abstract realm (which realists do often get landed with). This seems right, especially when we remember that the realist attitude is one of *potential discovery* of the way things are; so, in a way, deciding beforehand that the stuff we might discover is such and such a way (including eternal, abstract, meeting (our human) criteria for existence, being an object, or any such thing) seems a bit contrary to a central motivation of realism. Having just *independence* as what's at issue, with the difference between realism and anti- realism expressed in a Jenkinsy way, seems quite nice to me.

But again, if that's *all* we say, how will this make any practical difference between realism and anti-realism? How, by saying mathematical stuff is 'essentially independent', do we say anything non- translatable into the opposite position? I think, ultimately, Jenkins' characterisation won't work – at least for mathematics – unless it is supplemented somehow: that is, unless something else is said, her position too can be directly translated into a constructivist, or epistemically constrained-type account of objectivity/independence.

And remember, there is this curious thing about mathematical stuff – it's hard to see how it could be any more than what makes our axioms (and so on) true. Jenkins, and my realist, attempting to distinguish her account from anti-realism in mathematics, will have a very hard time spelling out what mathematical objects or structures *essentially are* without direct appeal to what we make of them – that is, without direct appeal to how we axiomatise them, what we believe is true of them, and so on. It is one thing to say that our mental state when we do mathematics is no part of what mathematical objects are, but it is quite another to then go on to say that independent mathematical reality is what's *left* if what we believe makes the axioms true is taken out of the equation.

## 4.5 Meaningfulness

Perhaps we can try to pin down the difference by looking at a bit more at what might make mathematics meaningful to an anti-realist and what makes mathematics meaningful to a realist. Let's check if there might be more we could say about Gaifman's 'factually meaningful answer'.

In general, by 'meaningful' anti-realists mean *constrained or trackable* (recall Wright's 'epistemic constraint' condition).

Gaifman, on the other hand (as we've seen), ties mathematical 'meaningfulness' to 'factually meaningful' – and cashes this notion out as being apt for yes-or-no answers. This characterisation arguably does capture the way mathematics is done.

The catch here is the subsequent fork in the road: if this way of describing mathematical meaningfulness gives no tie to construction, proof, methods, and means of knowing, then it inherits the realist problems without the realist advantage. If, on the other hand, appeal is made to our *believing* there's a factually meaningful yes-or-no answer, then this itself is a constraint that can be inter-translated with an anti-realist position – that is, we may then be taken as saying that the real or true is just that bit about which we think there is such an answer to be found.

We can attempt to contrast this with our 'differentiable' realist, for whom 'meaningful' may mean something like 'accessing or discovering or corresponding to independent reality'. As we've seen, in this formulation, objectivity is somehow external: tied to something beyond our constructions, that is, beyond our proof, justifications, and so on.

In a well-used example, Hilbert can represent the anti-realist here – he definitely had a conception of 'factually meaningful answer' in which meaning was tied to constructible proof. And, for Hilbert, something was 'meaningful' only in the sense it referred to a finite, and so in some clear way, *graspable* (and so accessible) totality. Hilbert wanted to show how all of mathematics is as irrefutable and clear and solvable as elementary number theory, so he came up with a program in which our natural conception of finiteness underpinned (or ultimately justified) the rest of mathematics.

Compare this to Gödel's conception of meaningful mathematics, taking Gödel here as representing realists, since he definitely had a conception of 'factually meaningful answer' in which meaning relied on an *external* justification for our proof. But now note what Gödel says about Hilbert's program:

> [I]t . . . acknowledge[s] that the truth of the axioms from which mathematics starts out cannot be justified or recognised in any way, and therefore the drawing of the consequences from them has meaning only in a hypothetical sense, whereby this drawing of consequences itself . . . is construed as a mere game with symbols according to certain rules, likewise not [supported by] insight. (Gödel 2001, 151)

The truth of the axioms, for Gödel, was attained by virtue of their referring to, or correctly capturing the ways things independently are. This comparison between the two underscores that, for the (Gödelian) realist, meaning *comes from* independence, or from something external to the axioms, proofs, and so on themselves.

Perhaps we could sum it up this way: anti-realist meaning comes from the idea of a coherent, effective system – that is, an effective constraint.

Whereas (real) meaning for realism comes from (as Folina puts it (1994)) a difference between what the facts are and what we take them to be, so, perhaps, the difference comes down to an in principle possible access to the truth/facts versus an in-principle falliblism.

So, if we're looking for a way to articulate realism such that it is non-translatable (i.e. effectively distinguishable from anti realism), perhaps we need to look seriously at the way in which, for realists, factually meaningful answers and objectivity come from the way in which we could have gotten it wrong – from the gap between what we construct and what independently *is*: again, between what the facts are and what we take them to be.

## 4.6 Wright, Continued

As we've seen, someone who has already gone to a lot of trouble to spell this all out in detail and to really pin down what anti-realism is saying about factually meaningful answers and objectivity versus what realism is saying, is Crispin Wright.

In his (2000), Wright proposes (Q) Q->(p is true <-> Z(p)) as a good formal encoding of (a good, or 'moderate') anti-realism, which he also calls 'direct realism'.

'Q' here is a general epistemic idealisation such as ideal conditions holding for maximally coherent set of beliefs, or ideal thinkers, and Z is a general alethic operator such as ' being judged to be true' (that is, Z could be read as 'known', 'accessible', etc.). So (according to Wright) what antirealists (in general) are trying to say is that ideal thinkers would believe (Z) a thing, if they happily inhabited the land of ideal conditions Q, if and only if it that thing was true.

Wright spells out Q this way:

> were p to be appraised under (topic-specified) sufficiently good epistemic conditions, p would be true if and only if p would be believed. (p. 347)

Note that this is what Gaifman is saying, really – for Gaifman, it's (possibly or allowably) just our actual epistemic capacities that prevent us from considering all mathematical questions in ever stronger systems. So Gaifman's view (which, remember, he calls 'realism') fits with Wright's encoding here (which, remember, Wright calls 'anti-realism'), which is exactly what this Element focuses on, is and why the terms 'realism' and 'anti-realism' have come to mean approximately nothing.

Wright says that so long as 'sufficiently good' is understood as conditions that incorporate whatever features it might take to enable a thinker to track the relevant facts, 'then no realism, however extreme, need hesitate to accept (Q's) provisional biconditionals everywhere' (p. 349. Also see footnote, p. 348). If there were no obligation for a constructive specification of what these facts might actually be, a mathematical realist might agree with Wright here, if what Wright means is that we could incorporate the whole of reality, not just us – that is, postulating what global ideal conditions might look like *if* they obtained. But then inter-translatability looms again. To differentiate realism from anti-realism here, the realist would perhaps have to postulate a *hypothetical* tracking of truth, just so long as it doesn't tie this tracking *in principle* to human mental states or humanness.

Wright isolates two scenarios in particular which are 'incompatible' with a constructible-type interpretation of Q: both are associated with what he calls 'metaphysical realism'.

The first scenario (incompatible with a constructive reading of Q) is the possibility that (in at least one case) there is *no* ideal or sufficiently good theory that would deliver knowledge; and the second is the possibility that, even if there were ideal or sufficiently good theories (in all cases), some of these (or, again, at least one) may actually have got it wrong, or (as Wright puts it) may be false.[12]

[12] This idea is contentious – 'false' and 'truth' both can be utilised *as if* by the realist. So it is better, perhaps, to say 'may not be (independently) the case'.

So, Wright says, it is not the notion of recognition-transcendent truth as such that anti-realists reject, but 'the notion that recognition-transcendent truth can arise for some reason other than the unavailability of sufficiently good conditions' (p. 355).

So then, maybe realism (or at least the kind that there's any point in an anti-realist taking the trouble to reject) says something like this: it's possible that there are sufficiently good conditions and that truth transcends the recognition or obtaining of these. But again, this *could* possibly make no practical difference – unless the realist could somehow point out (or ensure that our formal expressions brings it about) that the attaining of these sufficiently good conditions and the independent truth of something (or its 'being the case') are different things. And it seems to me that the only way to say that (without, again, making no difference) is to grasp the nettle and allow the full consequences of that scenario (as Wright has already outlined) – that, yes, on this account*, we could have it wrong*: even if we've produced a proof whose correctness is beyond reasonable doubt or that, say, in principle, it is possible that in some cases no theory, however ideal, *could* give us knowledge.

This is all to say that perhaps realism in mathematics must be a negative option; that perhaps the only means available to articulate the 'something else' is that which is not directly translatable into mathematical anti-realism.

So, perhaps the mathematical realist is stuck with saying 'it's *not* that' (i.e. with saying it's *not* simply what makes the axioms true, it's not whatever has an objective yes-or-no answer, and so on) because, again, there's that sense in which there's no *more*, no 'extra bit' to mathematical reality than what fits our mathematical theories. It seems that the mathematical realist is stuck with saying something like 'well yes, there's what fits our theories, and I can't point to anything richer or outline a sense in which our perspective on this stuff is only part of a wider story (the way I might, apparently, be able to with physical stuff). It seems that all I can say here is that independent mathematical reality is also *not* (or not identical) with that.'

Putting the 'something else' in this negative way and responding thus to Wright's scenario itself runs close to being paradoxical – just so long as the realist wants to say that we can *also* have (real) knowledge of the ways things independently are (and surely realists do want to retain this possibility).

The paradox, or contradiction, might be something like this:

> A realist, from the perspective of modern philosophy, is basically someone who claims to *think that which is where there is no thought*. That is to say, a realist is someone who keeps doing the opposite of what he says he's doing: he speaks of *thinking* a world in itself and independent of thought. But in saying this, does he not precisely speak of a world to which thought is given,

> and thus of a world dependent on our relation-to-the-world? (Meillassoux, 2008)

Or this:

> [T]he problem, for anyone inclined to suppose a 'transcendent' or 'independent' realm, is how cognition can reach that which is transcendent . . . [i.e.] the correlation between cognition as mental process, its referent and what objectively is . . . the source of the deepest and most difficult problems. Taken collectively, they are the problem of the possibility of cognition. (Husserl, 1964, 10–15)

that is, there's a sense in which we know and we don't know, we access and we don't access, all at the same time.

## 5 Justification

After outlining his ante rem stance, Shapiro proposes a possible anti-realist response asking how we are justified in our beliefs about mathematical objects. In response to this question, Shapiro offers the theory that we can, through physical perception, 'recognise', 'apprehend', and 'attain knowledge of' abstract patterns – hence also of abstract structures. Those structures not directly attainable through physical perception, he argues, 'are in fact apprehended (but not perceived) by abstraction'. Specifically, this theory says, 'pattern recognition . . . is a faculty . . . leading to an apprehension of freestanding, ante rem structures' (all quotations on p. 113).[13]

But imagine, again, a traditional physical realism regarding Benacerraf's cat: suppose that this realism is simply the ordinary belief that the cat is real. Such a belief might easily incorporate the notion that ordinary physical objects have an existence independent of our own – that is, they exist prior to and separate from us and so are not dependent on us. But, in order to use that notion of independent reality to independently justify our beliefs about these sorts of objects, even such an ordinary realist needs to show (at least) that what we 'perceive' as physical reality is in fact, or in some sense, the same reality that existed prior to or independent of our perception of it. Note again that in order to show this, the 'what-is-perceived' and 'what-existed-prior-to-that-perception' must (initially at least) be distinguishable from each other. If they are not distinguishable, then 'the cat I see' cannot, at some later stage, be *shown* to be

[13] This theory gains traction given some recent research. See the following BBC articles: www.bbc.com/future/article/20200907-the-remarkable-ways-animals-understand-numbers www.bbc.com/future/article/20191121-why-you-might-be-counting-in-the-wrong-language Both of these seem to indicate ways in which a realist about the middle-sized physical world can have a good epistemology – biology adapts to its environment, it does not create it, nor does the environment depend on the biology of those adapting; or maybe the two are intertwined.

identical to, or in any way the same as 'the cat that exists prior to and independently of my seeing it' – this, again, is the thrust of Frege's (aka 'Hesperus is Phosphorous') identity problem.

This active, later identification is, in fact, a necessary part of the justification required if any independence claim is to be maintained or utilised. Simply assuming that the 'what-is-perceived' and 'what-existed-prior-to-that-perception' are the same thing from the beginning, with no initial distinction between them, begs the same question: how our beliefs about this previously existing independent reality are justified. But this reasoning can be applied to Shapiro's theory, wherein nothing distinguishes his 'apprehended structure' from his 'independently existing structures'.

Shapiro gives this problem some attention in his (1997), specifically by imagining and answering another possible objection to his theory – this time of pattern recognition:

> Somewhere along the line anti-realists might concede that pattern recognition and the other psycholinguistic mechanisms lead to a *belief* in (perhaps ante rem) structures, and they may concede that we have an ability to coherently discuss these structures. But anti-realists will maintain that these mechanisms do not yield *knowledge* unless the structures (or at least their places) exist. Have we established this last, ontological claim? Can this be done without begging the question?
>
> . . . I present an account of the existence of structures according to which an ability to coherently discuss a structure is evidence that it exists . . . This account is perspicuous and accounts for much of the 'data' – mathematical practice and common intuitions about mathematical and ordinary objects. The argument for realism is an inference to the best explanation. The nature of structures guarantees that certain experiences count as evidence for their existence. (p. 118)

As mentioned earlier, the same argument is given in (2011, 147). This argument is that certain mechanisms, most notably our ability to coherently discuss them, provide evidence for ante rem structures. The problem for our realist seeking to distinguish her position from any position inter- translatable or interchangeable with anti-realism is that, so long as these structures are supposed to be independent of such mechanisms, nothing about the mechanisms themselves can establish that what they provide evidence for are indeed what they appear to be: independent, previously existing structures. The mechanisms lead perhaps to a belief in Shapiro's ante rem structures but not necessarily to those structures themselves. Evidence that an abstracted or psycholinguistically apprehended structure exists is not evidence that the same structure existed prior to, or independently of the mechanism that led us to believe in it. The evidence, then, only supports the existence of the *apprehended* structure, and without any specific reason to believe that the structure is indeed ante rem, evidence for the existence of what we apprehend – even if this evidence is the ability to coherently discuss the object of our apprehension – is not evidence for its existence independent, or prior to, our apprehension of it.

Again, this is just because an independence claim (if it is to provide a distinguishing feature preventing proper inter-translation between realist and anti-realist positions) necessarily incorporates the ontological access problem: the realist conundrum that these two objects – the object of our apprehension and the independent object existing prior to our apprehension – need to be distinguished before they can be shown to be the same, similar, or strictly identical.

Suppose, though, that we give up trying to maintain any initial separation between the two, and we do simply assume or try directly to show that the apprehended structure is one and the same as the independent structure. A question remains: in what *sense* is the yielded structure's existence something more than – that is, above and beyond – the existence of the end result of the process of abstraction or psycholinguistic apprehension that yielded it, or those structures for which coherence is evidence? That is, does the idea of independent or prior existence attach naturally to the structures arrived at via abstraction? If so, how? What is it about the psycholinguistically yielded structures that makes it reasonable to ascribe independence or priority to their existence?

As we've seen, if this same question were applied to physical reality, one possible answer would be the apparent persistent existence of the objects when we close our eyes, walk away from them, and so on. What sort of things can we say about abstract mathematical structures such that our experience of them gives evidence for their independence? There certainly are compelling features of mathematical reality we can appeal to that support the idea that it is independent and existed prior to our perception of it. The undeniable force of mathematical truth suggests itself as one such feature. The point, though, is just that we can ask for this sort of evidence.

All this is to say that it is reasonable to ask what sort of existence the apprehended structures can naturally or reasonably be said to have – given only what we do know and experience of them. That is, we can still ask: does independent or prior existence sit naturally with the structures arrived at via abstraction? Or is such an ascription of independence, in this case, somehow ad hoc?

## 5.1 The Natural/Least Artificial Attitude

How, then, can we accommodate independent ante rem structures in a relatively natural – that is, in a 'non-ad hoc' – way? One way of judging whether the inclusion of this sort of independence is ad hoc or not would be to assess how well the proposed independence sits with the account overall. This could be done, first of all, by assessing whether or not the account has 'room' (or enough room) for this sort of independence and, if it does, by assessing whether this room provides comfortable or forced accommodation.

Before thus assessing Shapiro's account, note that another occasion upon which a proposed independence might appear ad hoc is when its inclusion in an account is simply stipulated. But perhaps we can grant that a significant independence, such as Shapiro's ante rem structuralism involves, probably does have to be simply stipulated at some stage or other, if it is to be included in an account at all. As Shapiro notes, independence cannot, after all, be positively 'shown' (see, again, Hellman and Shapiro (2019) for a response to this which shifts the question up to the existence of ante rem structures themselves). Generally, we establish that something is independent of something else by showing that it is not dependent on the something else, rather than by attempting to positively illustrate some inherent independent nature it might have. Independence is 'arrived at' (insofar as it is 'arrived at' at all) negatively, and so also by stipulation rather than by proof. So the fact that something is simply stipulated in an account does not automatically render its inclusion ad hoc. I take it that we can suspect the idea of independence to be ad hoc when its setting (i.e. the rest of the account) renders its inclusion somehow unnatural or, at worst, if there is nothing in an account able to interpret or make some sense of such a stipulation.

Now recall that, even if we agree with Shapiro's claim that the ability to discuss a structure coherently is itself evidence that that structure exists, the issue of the nature of that structure's existence still remains untouched. At best, all we have is evidence that an abstracted or psycholinguistically apprehended structure exists (independent or otherwise). But it remains unclear how this evidence supports the independent existence of that same abstracted structure.

We can get some idea of how Shapiro regards this problem from his brief discussion of the ambiguity, inherent in his account, in the reference of numerals (and across mathematics in general). Shapiro (1997) points out that his structuralism entails that:

> '4', for example, denotes a place in the natural number structure, a finite cardinal structure, a finite ordinal structure, a place in the real number structure, a place in the integer structure, a place in the complex number structure and a place in the set theoretic hierarchy. (p. 120)

Now recall that for Shapiro, structures exist before and apart from any of their exemplifications. But if we take this to mean that for Shapiro there is an independent structure that existed before our denoting term '4' did, then it seems that the ambiguity ought to be able to be resolved. That is, we ought to be able to say whether or not the denoting term '4' picks out that particular pre-existing structure, or a place in that structure (similarly, for Hellman and Shapiro (2019), that ante rem structures can be identified or distinguished from one another, as well as from their set-theoretic exemplifications).

## 5.2 The Caesar Problem Revisited

This same problem can be approached from a number of different angles, and the Caesar problem is just one of these. In chapter 4 of (1997), Shapiro utilises Robert Kraut's work, 'Indiscernibility and Ontology', to provide another 'epistemic route' to mathematical structures. Kraut considers an imaginary economist who speaks a version of impoverished English, a language that does not have the resources to distinguish between two people with the same income (p. 19). Where person P has the same income as person Q, anything that the speaker of this impoverished language can say about P applies equally to Q. That is, an interpreter of the economist's language could apply the Leibniz principle of the identity of indiscernibles that, for the economist, P = Q, since the two items cannot be distinguished. From the economist's point of view, they are treated as a single object.

Shapiro gives the following scenario to illustrate:

> if the interpreter sees a certain woman [who earns $35,000 per annum], he might say (on behalf of the economist), 'there is the $35,000'. If a man with the same income walks by, the interpreter might remark, 'There it is again.' The economist herself might make the identification if she knows that the two are indiscernible and does not envision a framework for distinguishing them. (p. 19)

Even at this stage, there is a natural ambiguity that threatens to undermine any potential realist stance regarding the objects proposed. So long as the economist's 'knowledge' that the two are indiscernible consists in her not being able to envision a framework in which they might be distinguished (in this case, the full background language, ordinary English), the object's existence could, in an important sense, be regarded either as given by the language itself or as prior to the existence of this particular language. This early ambiguity could be understood as highlighting the shortcomings of the Leibniz principle, specifically, the possibility that the principle by itself is not enough to establish true identity; that is, objects that are able to be 'taken as' one rather than two, are not necessarily one rather than two in point of fact. An alternative understanding of the ambiguity might simply be that an application of the principle that results in identity relations between things on this basis is, in fact, a *mis*application.

Shapiro makes two points, both of which can be used to support the idea that an object's existence is given by the (object) language itself. First, he notes that 'nothing is lost by interpreting [the economist's] language as about income levels and not people [such that] . . . A singular term, like "the Jones woman", denotes an income level' (p. 20). But if we were presented with this interpretation alone, we'd have a free-standing language, without the history from which it evolved, and in this case something more than history is lost.

Specifically, the missing history takes with it a potential item of evidence that the object's (or at least *an* object's) existence predated the language being used to refer to those objects. Indeed Shapiro agrees that the only way we can understand examples like Kraut's economist is by possessing the 'background language' ourselves. This same point applies to the mathematician and to the 'number- person' that Shapiro models on Kraut's example (the latter is someone who speaks an impoverished version of English in which equinumerous collections of objects are indiscernible.

The second point Shapiro makes that appears to lend support to the idea that the object's existence arises from language, rather than the other way around, is the following: one way of understanding the thrust of the sublanguage examples is through Frege's *Grundlagen* result (taken here from Coffa (1991)) that 'a wide range of statements previously regarded as extralogical and involving an appeal to either empirical intuition (Mill) or to pure intuition (Kant) involved only reference to concepts' (p. 75). That is, 'number statements' make no sense so long as we try to understand them as being about objects, but they do make sense if we understand them as having a concept as their 'target'. Or, as Shapiro (1997) puts it:

> Suppose that our number-person looks at two decks of cards and we interpret him as saying 'There is two'. Then we assume that, at some level, the number-person knows that it is the decks and not the cards or the colours that are being counted. Nothing in the hunk of mass determines that it is 2, 104, or any other number for that matter. If the subject loses the use of sortals like 'deck', he will not see the stuff as 2. To accomplish this feat, the subject must be aware of the decks and must distinguish the two decks from each other. (pp. 20–1)

Frege's point (according to Coffa (1991)) is that since it is only when the concept is determined that the number attribute is fixed, regarding the concept itself as the topic of the number statement is natural. But Frege was sensitive to the possibility that this idea introduced an ambiguity between understanding and acquaintance. He not only addressed this possibility; he went to great pains to avoid it by drawing a sharp distinction between sense and the real world, pointing out that the things we understand, be they concepts, universals, properties, or essences, are something *altogether different* from the 'real world'. That is, 'He denied that understanding is a glorified form of "seeing" aimed at these entities . . . [His] point was that while understanding does involve "giving" the object in question, it need not necessarily be "given" in the mode of acquaintance' (p. 81).

Contra Frege, and in defence of structuralism, Shapiro offers that what he argues is a better way to understand the Kraut sublanguage examples; namely, that what they show us is that income levels, numbers, and so on are 'places in structures'. (Recall Shapiro's distinction, in linguistic practice, between an object, an office holder; and a place in a structure, an office.)

Although Shapiro acknowledges that the division between the places-are-offices perspective and the places-are-objects perspective is both relative and not sharp, its existence is nonetheless a crucial part of his structuralist interpretation of the sublanguage examples. This is because if the distinction is not drawn at some point, Shapiro's program becomes an eliminative program, whereby the places-are-objects perspective is reduced to the places-are-offices perspective – specifically by calling the former a generalisation over the latter. It is purposely to avoid this scenario that Shapiro gives a direct stipulation that at some point the places-are-objects perspective has to be taken literally. The problem then becomes whether, given the rest of his account, this perspective indeed can be taken literally and, if it can, whether the stipulation is somehow ad hoc.

Note that a Fregean-type response suggests not only that the places-are-objects perspective can indeed be taken literally but also that the stipulation that it *ought* to be is something that needs defence or at least is problematic, insofar as this type of answer represents a concerted effort to *retain* the realist idea that the truth or falsity of mathematical statements in some way depends on the presence or absence of the corresponding object in mathematical reality. It turns out, though, that the cost of this effort is the retention of 'the Caesar problem'.

A Shapiro-type answer also argues that the places-are-objects perspective can be taken literally, but makes a concerted effort to avoid the Caesar problem. But it turns out that the cost of avoiding the problem is the loss of the proposed realist's desiderata – a non–ad hoc notion of independence.

## 5.3 Frege, Shapiro, and Caesar

Frege's original Caesar problem came about via his belief that the real world comes to us *already* divided up into separate elements: namely those elements that constitute a given claim's topic and determine its truth value.

Coffa (1991) illustrates the Fregean approach with the following example. Whether the claim

*The author of 'Waverley' is tall*

is true or false depends entirely on whether a single object in the universe, namely Scott, has a single property, namely a certain height. The grammatical units

> 'the author of "Waverley"' and 'x is tall'

are associated with the truth-relevant parts of the world, that is, with those elements of the world that are the only ones relevant to the determination of whether what (A) says is true. That is, the grammatical units of a claim are associated with the elements that constitute its topic and determine its truth value.

The real world, then, includes all such elements: 'all that we talk about when we don't talk about talking' (p. 80). These elements, according to Frege, are themselves what is *meant* by a given claim. That is, they are a given claim's 'significance' or 'meaning' (p. 80). So, grammatical units, effectively or ineffectively, correctly or incorrectly, sort the world into those elements that matter and those that don't as far as the truth of a given claim is concerned.

The following argument, also Coffa's, clarifies Frege's belief in an original, previously partitioned ontology (the real world), going on to show how this belief leads directly to the Caesar problem. Coffa begins by noting that:

> All that is required for the purposes of communication or responsible discourse in general is that what we say be intelligible, and this has little to do with the possession of effective methods to identify either its referents or truth-values. (p. 81)

But, as Coffa observes, sense in Frege's account is nonetheless intimately related to the truth value–relevant elements in the world. This, Coffa notes, is strongly suggested by a seldom-noticed fact: Frege took it as self-evident that the grammatical analysis appropriate to the study of sense coincides with the grammatical analysis appropriate to the study of meaning. An understanding of the sense of a sentence is not, for Frege, a holistic phenomenon but comes about through an understanding of the sense of its parts:

> Notice that there is, in principle, no reason the grammatical units that provide the building blocks of propositional sense should be the same that provide the truth-relevant features of the world, *unless* sense (and therefore understanding) are essentially a matter of doing something with those worldly units. (p. 81)

For Frege, any language, including natural language, is capable of producing true claims only when it contains a device establishing a correlation, however ineffective, with objects of the original ontology – in this case, of the real world. That is, procedures such as Kraut's 'sublanguage procedure' from the Fregean viewpoint would be capable of true claims only to the extent that they are capable of establishing a link between the original elements of the real world and the elements 'produced' by that language. Indeed, for Frege (according to Coffa), it is only in this way that language can be seen as a means by which we understand claims concerning the objects language produces.

So for Frege, whether or not our understanding is effective – which, for Frege, amounts to whether what we understand is capable of being true or false – depends upon the absence or presence of objects in the original ontology, the real world. It seems reasonable to suppose that this dependence of effective understanding on the existence of the relevant objects in the real world comes about directly as a result of Frege's commitment to the realist notion of independence. After all, it is just when the idea of a single fixed universe is retained that the problem of identifying all the items to be found in the original ontology presents itself. And so we arrive back at the Caesar problem, so named because Frege's account seems simultaneously to demand, and to lack, a means by which to determine how and why each number is the same or different from any object whatsoever.

Shapiro agrees that we could not understand claims about objects which are themselves produced by a language without *an* original ontology. But, in contrast to the Fregean approach, he argues that there is not something that could be called *the* background ontology. For Shapiro, 'the idea of a single, fixed universe, divided into objects *a priori*, is rejected' (p. 28).

In mathematics, for Shapiro, there are no objects simpliciter (p. 15). That is, there is no structure- independent answer to such questions as whether Julius Caesar = 2' or '2 = $\{\{\varnothing\}\}$'. A mathematical object is a place in a fixed structure. So a number, for example, is a place in the natural number structure. Thus, as mentioned in Section 5.2, identity between objects within a given structure *is* determinate, but identity between numbers and other sorts of objects is not (p. 14).

Thus, for Shapiro, only statements internal to a single structure have a determinate truth value, so the truth value of this (internal) sort of statement is to be discovered. By contrast, the truth value of statements like the latter is a matter of invention or stipulation, which is why there is no answer waiting to be discovered to the Caesar problem.

To delve further into the Kraut sublanguage procedure, viewing the level-roles as played by groups of people or by individual people is taking the 'places are offices' perspective. But, 'when we focus on the impoverished sub-languages and interpret them with the Leibniz principle, we take the places of the structure – income levels and numbers – as objects in their own right. This is the "places are objects" perspective' (p. 21).

Now, the process by which we arrive at this 'impoverished sublanguage' involves, first of all, taking a framework with objects already within the range of its variables – that is, a framework that already has ontology. The next step is to

> focus on an equivalence relation over the ontology of the base language [which] . . . divides its domain into mutually exclusive collections – called 'equivalence classes'. . . . The idea here is to see the equivalence classes as exemplifying a structure and to treat the places of this structure as objects. (p. 21)

A sublanguage is then formulated for which the equivalence is a congruence such that where two items are equivalent they are indiscernible. Shapiro suggests that:

> in such cases . . . in the sub-language, the equivalence relation is the identity relation. The idea here is that the language and sub-language together characterise a structure, the structure exemplified by the equivalence classes and the relations between them formable in the sub-language. It is thus possible to invoke the places-are-objects orientation, in which the places in this structure are rightly taken to be its objects. (p. 22)

The final step in Shapiro's argument is to introduce the idea that the framework of pure mathematics, along with that of his own 'structure theory', might allow

> sub-languages in which either isomorphism or structure equivalence is a congruence. In this way, structures themselves are to be seen as objects in their own right and, according perfectly with the structuralist's credo, mathematics is the science of structure.

From the Fregean perspective introduced earlier, the problem with this is again that it boils down to the claim that it follows from the coherence of a structure characterisation that there are things which satisfy it. Indeed, on Shapiro's account, coherence just means satisfiability. And this means that coherence cannot be characterised in a non-circular way. Shapiro points out that mathematics itself involves an inherent circularity born of the common mathematical practice of settling questions of coherence just by modelling one structure in another – for example, complex numbers are coherent just because they can be modelled in the real number structure. The final port of call, then, would be set theory – in which all other mathematical structures can be modelled (p. 22).

But then, set theory itself is satisfied because it is coherent and is coherent because it is satisfied. We are now in a better position to examine how the notion of independence sits with the rest of Shapiro's account. Specifically, the circularity from satisfaction to coherence for set theory does not sit well with a claim that there is an independent realm of mathematical objects. That is to say, the notion that set theory is satisfied because it is coherent clashes with the notion of independence and, in particular, with the notion of the dependence of mathematical formalisms, proof, truth, and coherence on mathematical reality. On the other hand, the notion that coherence depends on satisfaction supports the independence claim. But, as is highlighted in Hellman and Shapiro, it is hard to see how anyone could argue that a structure is coherent because it is satisfied without a determinate background ontology of mathematical objects – or without the idea of a single, real mathematical universe that exists not because of, but before, our notion of mathematical coherence. But (as suggested earlier) simply shifting the problem up to another level, wherein the notion of a set-theoretic structure itself is taken as an instantiation of ante rem structures, seems to me to encounter the same problem, albeit in a more sophisticated, less direct way.

## 5.4 Structures, Realist and Otherwise

The idea that mathematical systems and concepts themselves are founded or depend on a pre- existing mathematical reality (and not the other way around) is one of the central realist notions underpinning the notion of justification discussed earlier. The appeal of such a notion gives a realist account some balance, insofar as it goes some way towards offsetting the cost of the problems entailed by the postulation of an independent reality. But it gets trickier. For the realist seeking to retain this particular sort of justification, there are other features that a structure characterisation would need to possess before it could serve as any sort of foundation or 'final court of appeal' for mathematical theories and questions of mathematical existence.

Specifically, it seems that if such a foundation were to serve to substantially differentiate realism from anti-realism, that foundation would need to be in some sense metaphysical, at least to the extent that existence questions are resolvable only by reference to a reality independent of the systems it grounds. That is, for the realist seeking to retain the traditional notion of justification, existence questions need to be resolved by reference to a supposed, or stipulated abstract reality, as opposed to the coherent characterisation of a structure or indeed to any sort of a reality whose existence can likewise be attributed to ourselves or to our ability to define or construct it. Indeed, without reference to

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a metaphysical independent reality, the line between construction and discovery – or in this case, between a coherent structure and a satisfied structure – will always be blurred.

The realist can grant that we can create coherent structures and say they are satisfied, or she can say that we can discover satisfied structures and show that they are coherent. But the term 'discover' and the idea that the structures are 'ante rem' both lose their specific realist force if, in the final analysis, each of these ideas is defined in terms of the other.

To conclude: there is, in the end, no reason to stipulate an independent reality unless it does some work. But the primary work such a notion can do is to provide a specific sort of justification for our knowledge of mathematical reality and truths. Without this, it is hard to see how the reality posited differs from the anti-realist realms to which it is equivalent, and the stipulation of an independent mathematical reality itself risks being somewhat ad hoc. With it, though, an account at least encounters the ontological access problem, and, at most – that is, just so long as it works to provide independent justification – it necessitates a commitment to that same problem's insolubility.

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