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Pradip Debnath
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Fixed Point Theory and Fractional Calculus

Recent Advances and Applications

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Editors

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Preface

Fixed point theory and fractional calculus emerged as two indispensable and interrelated tools in the mathematical modelling of various experiments in nonlinear sciences and engineering over the last few decades. This book will present the most recent developments in these two fields through contributions from eminent scientists and mathematicians worldwide.

Readers will find several useful tools and techniques to develop their skills and expertise in these areas. New research directions are also indicated in chapters. This book is meant for graduate students, faculty and researchers willing to expand their knowledge in fixed point theory and fractional calculus. The readers of this book will require minimum prerequisites of analysis, topology and functional analysis.

This book is an effort towards presenting the two very important topics in modern mathematics and their applications in science and engineering. It consists of 18 chapters. The first half of the book will deal with applications of fixed point theory, whereas the second half will discuss fractional calculus. "Best Proximity Points for Some Multivalued Contractive Mappings" and "Best Proximity Point Theorems via Some Generalized Notions" investigate best proximity theorems under some new contractive conditions. "New Fixed-Figure Results on Metric Spaces" deals with fixed-figure results and discontinuity at fixed points, and "Some Fixed Point Results for Suzuki W-Contractions Involving Quadratic Terms in Modular b-Metric Spaces" is about some fixed point results for the Suzuki–Wardowski contractions. Common fixed point results for JS-contraction-type mappings are discussed in "Some Common Fixed Point Results via α-Series for a Family of JS-Contraction-type Mappings". "Solution of Nonlinear First-Order Hybrid Integro-Differential Equations via Fixed Point Theorem" presents an algorithm for the solution of nonlinear first-order hybrid integro-differential equations by using the fixed point theorem. In "Application of Darbo's Fixed Point Theorem for Existence Result of Generalized 2D Functional Integral Equations", an application of Darbo's fixed point theorem for the solution of two-dimensional functional integral equations has been explored. "Results on Generalized Tripled Fuzzy b-Metric Spaces" and "A Novel Controlled Picture Fuzzy Metric Space and Some Related Fixed Point Results" elucidate some new fixed point results in fuzzy metric spaces. From "Theoretical Analysis for a Generalized Fractional-Order Boundary Value Problem" onwards, the studies in fractional calculus have been listed. "Theoretical Analysis for a Generalized Fractional-Order Boundary Value Problem" presents a theoretical analysis for a generalized fractional order boundary value problem, while "On Well-posed Variational Problems Involving Multidimensional Integral Functionals " describes well-posed variational problems involving multidimensional integral functionals. The coupled system of tempered fractional differential equations with anti-periodic boundary conditions is the subject of study in "On the Coupled System of Tempered Fractional Differential Equations with Anti-periodic Boundary Conditions". "Application of Measure of Noncompactness on the Infinite System of Hadamard Fractional Integral Equations" and "Observability, Reachability, Trajectory Reachability and Optimal Reachability of Fractional Dynamical Systems using Riemann–Liouville Fractional Derivative" deal with the measure of noncompactness on the infinite system of Hadamard fractional integral equations and optimal reachability of fractional dynamical systems using the Reimann–Liouville fractional derivative, respectively. "Fractional Calculus Approach to Logistic Equation and its Applications" is devoted to the study of the fractional calculus approach to the logistic equation, and "Hermite–Hadamard Type Inequalities for Coordinated Quasi-Convex Functions via Generalized Fractional Integrals" studies the Hermite–Hadamard type inequalities via fractional integrals. The Leray–Schauder theorem for the implicit fractional differential equation is investigated in "Leray–Schauder Theorem for Implicit Fractional Differential Equation and Nonlocal Multi-Point Conditions". Finally, "The q-Deformed Hamiltonian, Lagrangian, Entropy and Fisher Information" explores the q-deformed Hamiltonian, Lagrangian, entropy and Fisher information.

Fixed point theory and fractional calculus consist of a diverse collection of topics. The main sources of information on these topics are scattered among a variety of journals and proceedings. As such, consulting all the information by amateur learners is seldom possible. Our book is an attempt in this direction of providing a simple interface for learners and researchers in the theory and applications of fixed points and fractional calculus.

This book may be used as a reference book for a broad range of readers interested in studying fixed point theory and fractional calculus. In each chapter, the preliminaries have been listed first and then the advanced discussion takes place.

Silchar, India — Pradip Debnath
Victoria, Canada — H. M. Srivastava
Bangkok, Thailand — Poom Kumam
Guwahati, India — Bipan Hazarika

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About the Editors

Dr. Pradip Debnath is Assistant Professor of Mathematics at the Department of Applied Science and Humanities, Assam University, Silchar, India. He received his Ph.D. in Mathematics from the National Institute of Technology Silchar, India. His research interests include fixed-point theory, nonlinear analysis, soft computing, and mathematical statistics. He has published over 60 papers in various journals of international repute and is Reviewer for more than 30 renowned international journals including "Mathematical Reviews" (published by the American Mathematical Society). He is the lead Editor of the books "Metric Fixed Point Theory - Applications in Science, Engineering and Behavioural Sciences" (2021, Springer Nature) and "Soft Computing Techniques in Engineering, Health, Mathematical and Social Sciences" (2021, CRC Press). He has successfully guided several Ph.D. students in the areas of fixed-point theory and soft computing. At present, he is working on a major basic science research project on fixed-point theory funded by the UGC, the Government of India. Having been an academic gold medalist during his post-graduation studies, Dr. Debnath has qualified several national-level examinations in mathematics in India.

Prof. H. M. Srivastava is Professor Emeritus at the Department of Mathematics and Statistics, University of Victoria, Canada. He joined the University of Victoria in 1969 as Associate Professor (1969–1974) and then became Full Professor (1974–2006). Professor Srivastava has held numerous visiting positions including (for example) those at West Virginia University in the U.S.A (1967–1969), Universite Laval in Canada (1975), and the University of Glasgow in the U.K. (1975–1976) and indeed also at many other universities and research institutes in different parts of the world. Professor Srivastava has published 21 books (monographs and edited volumes), 30 book chapters, 45 papers in international conference proceedings, and over 1,000 scientific research journal articles on various topics of mathematical analysis and applicable mathematics. He also has edited (and contributed to) many volumes, which are dedicated to the memories of famous mathematical scientists. Citations of his research contributions can be found in many books and monographs, Ph.D. and D.Sc. theses, and scientific journal articles, much too numerous to be recorded here.

Currently, he is actively associated (as editor, honorary editor, senior editor, associate editor, or editorial board member) with over 200 international scientific research journals. Professor Srivastava is a Clarivate Analytics (Web of Science) Highly-Cited Researcher. His biographical sketches (many of which are illustrated with his photograph) have appeared in various issues of more than 50 international biographies, directories, and Who's Who's. Professor Srivastava is a Clarivate Analytics (Web of Science) Highly-Cited Researcher

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He is awarded the TRF-CHE-Scopus Young Researcher Award in 2010, which is awarded by the corporation of three organizations—Thailand Research Fund (TRF), the Commission of Higher Education (CHE), and Elsevier Publisher (Scopus). He received the TWAS Prize for Young Scientist in Thailand, in 2012, which is given by the Academy of Sciences for the Developing World TWAS (UNESCO) together with the National Research Council of Thailand. In 2014, he received the Fellowship Award for Outstanding Contribution to Mathematics from the International Academy of Physical Science, Allahabad, India. In 2015, He has been awarded the Thailand Frontier Author Award for his outstanding research. In 2016, he has been awarded Thailand Frontier Researcher Award on Innovation Forum: Discovery, Protection, Commercialization by Intellectual Property and Science and Thomson Reuters. He also has been Highly Cited Researcher in 2015, 2016, and 2017. He received KMUTT-Hall of Fame 2017, in honour of the recipients of Academic Awards, KMUTT Young Researcher Awards, Excellence in Teaching Awards for 2016. In 2019, he received CMMSE Prize for his contributions to the developments of numerical methods for physics, chemistry, engineering, and economics from the CMMSE Conference. He has also been listed and ranked in the 197th place in general mathematics among the top 2% scientists in the world 2021 (published by Stanford University, U.S.A.).

Prof. Bipan Hazarika is Professor at the Department of Mathematics, Gauhati University, Guwahati, Assam, India. Earlier, he was Professor at Rajiv Gandhi University, Rono Hills, Doimukh, Arunachal Pradesh, India, from 2005–2018. He received his Ph.D. degree from Gauhati University, Guwahati, India. His main research interests are in sequences spaces, summability theory, applications fixed-point theory, fuzzy analysis, and function spaces of non-absolute integrable functions. He has published more than 150 research papers in several international journals. He is Regular Reviewer of more than 50 different journals published by Springer Nature, Elsevier, Taylor and Fancis, Wiley, IOS Press, World Scientific, American Mathematical Society, and De Gruyter. He has published books on differential equations, differential calculus, and integral calculus. He is Editorial Board Member for more than five international journals and Guest Editor of special issue "Sequence Spaces, Function Spaces and Approximation Theory", of *Journal of Function Spaces.*

Best Proximity Points for Some Multivalued Contractive Mappings

Pradip Debnath and Boško Damjanović

Abstract Best proximity point theorems provide assurance to the existence of approximate solutions to some particular equations of the type $f(x) = x$, when exact solution of such equations do not exist. As such, these theorems are of paramount importance in the theory of optimization and approximation. In this chapter, we present some common best proximity point (CBPP) results for multivalued mappings using $\mathfrak{F}$-contractions.

1 Introduction

The prime objective of fixed point theory is to establish methodologies those lead to solutions of nonlinear equations of the type $f(x) = x$ such that f is a self-mapping defined on a subset of a topological space or metric space or normed linear space. However, this equation not necessarily admits a solution when it fails to be a self-map. Thus, a method of approximation turns out to be more feasible and appropriate which attempts to find an element x which is in close vicinity of $f(x)$ instead of being exactly equal to $f(x)$. Such problems motivate the study of best proximity theorems and best approximation theorems. Even though best approximation theorems can provide approximate solution to the equations of the type $f(x) = x$, such a solution need not be optimal. Alternately, an approximate solution produced by the best proximity theorem happens to be optimal. The aim of this chapter is to address a more general problem of similar context on the existence of CBPP in the framework of metric spaces.

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Nadler [13] commenced the investigation of fixed points for multivalued mappings with the help of Pompeiu-Hausdorff metric $\mathfrak{H}$ as follows.

Let $(\mathfrak{X}, \varpi)$ be a complete metric space (MS) and let $CB(\mathfrak{X})$ denote the collection of all non-empty bounded and closed subsets of the non-empty set $\mathfrak{X}$. Then for $\mathcal{P}, \mathcal{Q} \in CB(\mathfrak{X})$, define the map $\mathfrak{H} : CB(\mathfrak{X}) \times CB(\mathfrak{X}) \to [0, \infty)$ by

$$\mathfrak{H}(\mathcal{P}, \mathcal{Q}) = \max\{\sup_{\xi \in \mathcal{Q}} \Delta(\xi, \mathcal{P}), \sup_{\delta \in \mathcal{P}} \Delta(\delta, \mathcal{Q})\},$$

where $\Delta(\delta, \mathcal{Q}) = \inf_{\xi \in \mathcal{Q}} \varpi(\delta, \xi)$. Then $(CB(\mathfrak{X}), \mathfrak{H})$ is a complete MS.

For any two non-empty subsets $\mathcal{P}, \mathcal{Q}$ of the MS $(\mathfrak{X}, \varpi)$, we shall use the following notations:

$$\mathcal{P}_0 = \{x \in \mathcal{P} : \varpi(x, y) = \varpi(\mathcal{P}, \mathcal{Q}) \text{ for some } y \in \mathcal{Q}\},$$

$$\mathcal{Q}_0 = \{y \in \mathcal{Q} : \varpi(x, y) = \varpi(\mathcal{P}, \mathcal{Q}) \text{ for some } x \in \mathcal{P}\},$$

where $\varpi(\mathcal{P}, \mathcal{Q}) = \inf\{\varpi(x, y) : x \in \mathcal{P}, y \in \mathcal{Q}\}$.

For $\mathcal{P}, \mathcal{Q} \in CB(\mathfrak{X})$, we have

$$\varpi(\mathcal{P}, \mathcal{Q}) \leq \mathfrak{H}(\mathcal{P}, \mathcal{Q}).$$

$\mu \in \mathfrak{X}$ is said to be a BPP of the multivalued map $S : \mathfrak{X} \to CB(\mathfrak{X})$ if $\Delta(x, Sx) = \varpi(\mathcal{P}, \mathcal{Q})$. $u \in \mathfrak{X}$ is a fixed point of the multivalued map $S : \mathfrak{X} \to CB(\mathfrak{X})$ if $u \in Su$.

Remark 1 1. In the MS $(CB(\mathfrak{X}), \mathfrak{H})$, $\upsilon \in \mathfrak{X}$ is a fixed point of S if and only if $\Delta(\upsilon, S\upsilon) = 0$.
2. If $\varpi(\mathcal{P}, \mathcal{Q}) = 0$, then a fixed point and a BPP coincide.
3. The metric function $\varpi : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ is continuous in the sense that if $\{u_n\}, \{v_n\}$ are two sequences in $\mathfrak{X}$ with $(u_n, v_n) \to (u, v)$ for some $u, v \in \mathfrak{X}$, as $n \to \infty$, then $\varpi(u_n, v_n) \to \varpi(u, v)$ as $n \to \infty$. Thus, we obtain that the function Δ is continuous considering the fact that if $u_n \to u$ as $n \to \infty$, then $\Delta(u_n, \mathcal{P}) \to \Delta(u, \mathcal{P})$ as $n \to \infty$ for any $\mathcal{P} \subseteq \mathfrak{X}$.

The following results are useful in the present context.

Lemma 1 ([3, 5]) *Let $(\mathfrak{X}, \varpi)$ be a MS and $\mathcal{P}, \mathcal{Q}, C \in CB(\mathfrak{X})$. Then*

1. $\Delta(x, \mathcal{Q}) \leq \varpi(x, y)$ *for any* $y \in \mathcal{Q}$ *and* $x \in \mathfrak{X}$*;*
2. $\Delta(x, \mathcal{Q}) \leq \mathfrak{H}(\mathcal{P}, \mathcal{Q})$ *for any* $x \in \mathcal{P}$*.*

Lemma 2 ([13]) *Let $\mathcal{P}, \mathcal{Q} \in CB(\mathfrak{X})$ and let $x \in \mathcal{P}$, then for any $r > 0$, there exists $y \in \mathcal{Q}$ such that*

$$\varpi(x, y) \leq \mathfrak{H}(\mathcal{P}, \mathcal{Q}) + r.$$

However, a point $y \in \mathcal{Q}$ such that

$$\varpi(x, y) \leq \mathfrak{H}(\mathcal{P}, \mathcal{Q}),$$

may not exist.

If $\mathcal{Q}$ is compact, then such a point y exists, i.e., $\varpi(x, y) \leq \mathfrak{H}(\mathcal{P}, \mathcal{Q})$.

The concept of $\mathfrak{H}$-continuity for multivalued mapping is defined below.

Definition 1 ([9]) Let $(\mathfrak{X}, \varpi)$ be a MS. A multivalued map $S : \mathfrak{X} \to CB(\mathfrak{X})$ is said to be $\mathfrak{H}$-continuous at a point x_0, if for each sequence $\{x_n\} \subset \mathfrak{X}$, such that $\lim_{n\to\infty} \varpi(x_n, x_0) = 0$, we have $\lim_{n\to\infty} \mathfrak{H}(Sx_n, Sx_0) = 0$ (i.e., if $x_n \to x_0$, then $Sx_n \to Sx_0$ as $n \to \infty$).

Definition 2 ([13]) Let $S : \mathfrak{X} \to CB(\mathfrak{X})$ be a multivalued map. S is said to be a multivalued contraction if $\mathfrak{H}(Sx, Sy) \leq \lambda\varpi(x, y)$ for all $x, y \in \mathfrak{X}$, where $\lambda \in [0, 1)$.

Remark 2 1. S is $\mathfrak{H}$-continuous on a subset $\mathcal{P}$ of $\mathfrak{X}$ if it is continuous on every point of $\mathcal{P}$.
2. If S is a multivalued contraction, then it is $\mathfrak{H}$-continuous.

In 2012, Wardowski [17] defined the concept of $\mathfrak{F}$-contraction as follows.

Definition 3 Let $\mathfrak{F} : (0, +\infty) \to (-\infty, +\infty)$ be a mapping satisfying:
(F1) $\mathfrak{F}$ is strictly increasing;
(F2) For each sequence $\{u_n\}_{n\in\mathbb{N}} \subset (0, +\infty)$, $\lim_{n\to+\infty} u_n = 0$ if and only if $\lim_{n\to+\infty} \mathfrak{F}(u_n) = -\infty$;
(F3) There exists $t \in (0, 1)$ such that $\lim_{u\to 0^+} u^t \mathfrak{F}(u) = 0$.
Let $\mathcal{F}$ denote the class of all such functions $\mathfrak{F}$. If $(\mathfrak{X}, \varpi)$ is a metric space, then a self-map $S : \mathfrak{X} \to \mathfrak{X}$ is said to be an $\mathfrak{F}-$contraction if there exist $\lambda > 0, \mathfrak{F} \in \mathfrak{F}$, such that for all $x, y \in \mathfrak{X}$,

$$\varpi(Sx, Sy) > 0 \Rightarrow \lambda + \mathfrak{F}(\varpi(Sx, Sy)) \leq \mathfrak{F}(\varpi(x, y)).$$

Multivalued $\mathfrak{F}$-contractions were defined by Altun et al. [2] as follows.

Definition 4 ([2]) Let $(\mathfrak{X}, \varpi)$ be a MS. A multivalued map $S : \mathfrak{I} \to CB(\mathfrak{X})$ is said to be a multivalued $\mathfrak{F}$-contraction (MVFC, in short) if there exist $\lambda > 0$ and $\mathfrak{F} \in \mathcal{F}$ such that

$$\lambda + \mathfrak{F}(\mathfrak{H}(Sx, Sy)) \leq \mathfrak{F}(\varpi(x, y)) \tag{1}$$

for all $x, y \in \mathfrak{I}$ with $Sx \neq Sy$.

Remark 3 An MVFC is $\mathfrak{H}$-continuous.

The concept of P-property was introduced by Sankar Raj [16]. The notion weak P property was put forward by Zhang et al. [18] to improve the results of Caballero et al. [4] on Geraghty-contractions.

Definition 5 ([16]) Let $(\mathfrak{X}, \varpi)$ be a MS and $\mathcal{P}, \mathcal{Q}$ be two non-empty subsets of $\mathfrak{X}$ such that $\mathcal{P}_0 \neq \phi$. The pair $(\mathcal{P}, \mathcal{Q})$ satisfies the P-property if and only if $\varpi(x_1, y_1) = \varpi(\mathcal{P}, \mathcal{Q}) = \varpi(x_2, y_2)$ implies $\varpi(x_1, x_2) = \varpi(y_1, y_2)$, where $x_1, x_2 \in \mathcal{P}_0$ and $y_1, y_2 \in \mathcal{Q}_0$.

Definition 6 ([18]) Let $(\mathfrak{X}, \varpi)$ be a MS and $\mathcal{P}, \mathcal{Q}$ be two non-empty subsets of $\mathfrak{X}$ such that $\mathcal{P}_0 \neq \phi$. The pair $(\mathcal{P}, \mathcal{Q})$ satisfies the weak P-property if and only if $\varpi(x_1, y_1) = \varpi(\mathcal{P}, \mathcal{Q}) = \varpi(x_2, y_2)$ implies $\varpi(x_1, y_2) \leq \varpi(y_1, y_2)$, where $x_1, x_2 \in \mathcal{P}_0$ and $y_1, y_2 \in \mathcal{Q}_0$.

For some significant research related to P-property, we refer to the works of Omidvari et al. [15] and Nazari [14]. Some interesting results in the present context were recently established by Debnath [7, 8] and Debnath and Srivastava [10, 11]. For more relevant literature, we refer to [1, 6].

Using the concepts of the weak P property and $\mathfrak{F}$-contraction, in the current chapter, we establish a CBPP result for multivalued mappings. An example has also been provided in which the P property is not satisfied although the weak P property holds true.

2 Best Proximity Point for MVFC

In this section, we establish our main results.

First we define a multivalued $\mathfrak{F}$-contractive pair of mappings.

Definition 7 Let $(\mathfrak{X}, \varpi)$ be a MS and $\mathcal{P}, \mathcal{Q}$ be two non-empty subsets of $\mathfrak{X}$. The pair of mappings $S, T : \mathcal{P} \to CB(\mathcal{Q})$ is said to be a multivalued $\mathfrak{F}$-contraction pair (MVFCP) if there exist $\tau > 0$ and $\mathfrak{F} \in \mathcal{F}$ such that

$$\tau + \mathfrak{F}(\mathfrak{H}(Sx, Ty)) \leq \mathfrak{F}(\varpi(x, y)) \tag{2}$$

for all $x, y \in \Im$ with $Sx \neq Ty$.

Theorem 1 *Let $(\mathfrak{X}, \varpi)$ be a complete MS and $\mathcal{P}, \mathcal{Q}$ be two non-empty closed subsets of $\mathfrak{X}$ such that $\mathcal{P}_0 \neq \phi$ and that the pair $(\mathcal{P}, \mathcal{Q})$ satisfies the weak P-property. Suppose $S, T : \mathcal{P} \to CB(\mathcal{Q})$ be a MVFCP such that Sx and Tx are compact for each $x \in \mathcal{P}$ and $Sx, Tx \subseteq \mathcal{Q}_0$ for all $x \in \mathcal{P}_0$. Further, assume that S, T are $\mathfrak{H}$-continuous. Then S, T have a CBPP.*

Proof Fix $x_0 \in \mathcal{P}_0$ and choose $y_0 \in Tx_0 \subseteq \mathcal{Q}_0$. By the definition of $\mathcal{Q}_0$, we choose $x_1 \in \mathcal{P}_0$ such that

$$\varpi(x_1, y_0) = \varpi(\mathcal{P}, \mathcal{Q}). \tag{3}$$

If $y_0 \in Tx_1 \cap Sx_1$, then

$$\varpi(\mathcal{P}, \mathcal{Q}) \leq \Delta(x_1, Sx_1) \leq \varpi(x_1, y_0) = \varpi(\mathcal{P}, \mathcal{Q}) \text{ (since } y_0 \in Sx_1),$$

and

$$\varpi(\mathcal{P}, \mathcal{Q}) \leq \Delta(x_1, Tx_1) \leq \varpi(x_1, y_0) = \varpi(\mathcal{P}, \mathcal{Q}) \text{ (since } y_0 \in Tx_1).$$

Thus $\varpi(\mathcal{P}, \mathcal{Q}) = \Delta(x_1, Sx_1) = \Delta(x_1, Tx_1)$, i.e., x_1 is a CBPP of S and T. So, assume that $y_0 \notin Tx_1 \cap Sx_1$.

First we consider the case $y_0 \notin Sx_1$.

Since Sx_1 is compact, by Lemma 2 and Definition 7 there exists $y_1 \in Sx_1 \subseteq \mathcal{Q}_0$ and $\lambda \in [0, 1)$ such that

$$0 < \Delta(y_0, Sx_1) < \varpi(y_0, y_1) \leq \mathfrak{H}(Tx_0, Sx_1). \tag{4}$$

Since $\mathfrak{F}$ is strictly increasing, from (4), we have

$$\begin{aligned} \mathfrak{F}(\varpi(y_0, y_1)) &\leq \mathfrak{F}(\mathfrak{H}(Tx_0, Sx_1)) \\ &\leq \mathfrak{F}(\varpi(x_0, x_1)) - \lambda. \end{aligned} \tag{5}$$

Since $y_1 \in \mathcal{Q}_0$, there exists $x_2 \in \mathcal{P}_0$ such that

$$\varpi(x_2, y_1) = \varpi(\mathcal{P}, \mathcal{Q}). \tag{6}$$

From (3) and (6) and using weak $P-$property, we have that

$$\varpi(x_1, x_2) \leq \varpi(y_0, y_1). \tag{7}$$

From (5) and (7), we have

$$\mathfrak{F}(\varpi(x_1, x_2)) \leq \mathfrak{F}(\varpi(y_0, y_1)) \leq \mathfrak{F}(\varpi(x_0, x_1)) - \lambda. \tag{8}$$

If $y_1 \in Tx_2 \cap Sx_2$, like earlier we can show that x_2 is a CBPP of T and S. So, assume that $y_1 \notin Tx_2 \cap Sx_2$.

Consider the case $y_1 \notin Tx_2$.

Since Tx_2 is compact, by Lemma 2, there exists $y_2 \in Tx_2$ such that

$$0 < \Delta(y_1, Tx_2) < \varpi(y_1, y_2) \leq \mathfrak{H}(Tx_2, Sx_1).$$

Using the fact that $\mathfrak{F}$ is strictly increasing, we have that

$$\begin{aligned}\mathfrak{F}(\varpi(y_1, y_2)) &\leq \mathfrak{F}(\mathfrak{H}(Tx_2, Sx_1)) \\ &\leq \mathfrak{F}(\varpi(x_2, x_1)) - \lambda \\ &\leq \mathfrak{F}(\varpi(x_0, x_1)) - 2\lambda \text{ (using 8)}.\end{aligned} \tag{9}$$

Since $y_2 \in Tx_2 \subseteq \mathcal{Q}_0$, there exists $x_3 \in \mathcal{P}_0$ such that

$$\varpi(x_3, y_2) = \varpi(\mathcal{P}, \mathcal{Q}). \tag{10}$$

From (8) and (10) and using weak property P, we have that

$$\varpi(x_2, x_3) \leq \varpi(y_1, y_2). \tag{11}$$

From (10) and (11), we have

$$\mathfrak{F}(\varpi(x_2, x_3)) \leq \mathfrak{F}(\varpi(y_1, y_2)) \leq \mathfrak{F}(\varpi(x_0, x_1)) - 2\lambda. \tag{12}$$

Continuing in this way, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{P}_0$ and $\mathcal{Q}_0$ respectively, satisfying

(A) $y_{2n} \in Tx_{2n} \subseteq \mathcal{Q}_0$ and $y_{2n+1} \in Sx_{2n+1} \subseteq \mathcal{Q}_0$

(B) $\varpi(x_{n+1}, y_n) = \varpi(\mathcal{P}, \mathcal{Q})$,

(C) $\mathfrak{F}(\varpi(x_n, x_{n+1})) \leq \mathfrak{F}(\varpi(y_{n-1}, y_n)) \leq \mathfrak{F}(\varpi(x_0, x_1)) - n\lambda$,

for each $n = 0, 1, 2, \ldots$.

Put $\alpha_n = \varpi(x_n, x_{n+1})$ for each $n = 0, 1, 2, \ldots$. Taking limit on both sides of **(C)** as $n \to \infty$, we have

$$\lim_{n\to\infty} \mathfrak{F}(\alpha_n) = -\infty.$$

Using **(F2)**, we obtain

$$\lim_{n\to\infty} \alpha_n = 0. \tag{13}$$

Using **(F3)**, there exists $k \in (0, 1)$ such that

$$\alpha_n^k \mathfrak{F}(\alpha_n) \to 0 \text{ as } n \to \infty. \tag{14}$$

From **(C)**, for each $n \in \mathbb{N}$, we have that

$$\mathfrak{F}(\alpha_n) - \mathfrak{F}(\alpha_0) \leq -n\lambda.$$

This implies

$$\alpha_n^k \mathfrak{F}(\alpha_n) - \alpha_n^k \mathfrak{F}(\alpha_0) \leq -n\alpha_n^k \lambda \leq 0. \tag{15}$$

Letting $n \to \infty$ in (15) and using (13), (14), we obtain

$$\lim_{n\to\infty} n\alpha_n^k = 0.$$

Thus there exists $n_0 \in \mathbb{N}$ such that $n\alpha_n^k \leq 1$ for all $n \geq n_0$, i.e., $\alpha_n \leq \frac{1}{n^{\frac{1}{k}}}$ for all $n \geq n_0$.

Let $m, n \in \mathbb{N}$ with $m > n \geq n_0$. Then

$$\begin{aligned} \varpi(x_m, x_n) &\leq \sum_{i=n}^{m-1} \varpi(x_i, x_{i+1}) = \sum_{i=n}^{m-1} \alpha_i \\ &\leq \sum_{i=n}^{\infty} \alpha_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ is convergent for $k \in (0, 1)$, we have $\varpi(x_m, x_n) \to 0$ as $m, n \to \infty$. Hence, $\{x_n\}$ is Cauchy in $\mathcal{P}_0 \subseteq \mathcal{P}$. Since $(\mathfrak{X}, \varpi)$ is complete and $\mathcal{P}$ is closed, we have $\lim_{n\to\infty} x_n = u$ for some $u \in \mathcal{P}$.

Since T is $\mathfrak{H}$-continuous (for it is an MVFCP), we have

$$\lim_{n\to\infty} \mathfrak{H}(Tx_n, Tu) = 0. \tag{16}$$

Exactly in the similar manner as above, using **(C)**, we can prove that $\{y_n\}$ is Cauchy in $\mathcal{Q}$ and since $\mathcal{Q}$ is closed, there exists $v \in \mathcal{Q}$ such that $\lim_{n\to\infty} y_n = v$.

From (B), $\varpi(x_{n+1}, y_n) = \varpi(\mathcal{P}, \mathcal{Q})$ for all $n \in \mathbb{N}$ and thus we have

$$\lim_{n\to\infty} \varpi(x_{n+1}, y_n) = \varpi(u, v) = \varpi(\mathcal{P}, \mathcal{Q}). \tag{17}$$

We claim that $v \in Tu \cap Su$. Indeed, since $y_{2n} \in Tx_{2n}$ for all $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \Delta(y_{2n}, Tu) \leq \lim_{n\to\infty} \mathfrak{H}(Tx_{2n}, Tu) = 0.$$

Therefore, $\Delta(v, Tu) = 0$. Since Tu is closed, we have $v \in Tu$.

Also, since $y_{2n+1} \in Sx_{2n+1}$, we have

$$\lim_{n\to\infty} \Delta(y_{2n+1}, Su) \leq \lim_{n\to\infty} \mathfrak{H}(Sx_{2n+1}, Su) = 0.$$

Thus, we have $\Delta(v, Su) = 0$ and so $v \in Su$. Therefore, we have that

$$v \in Tu \cap Su. \tag{18}$$

Using (17) and (18), we have

$$\varpi(\mathcal{P}, \mathcal{Q}) \leq \Delta(u, Su) \leq \varpi(u, v) = \varpi(\mathcal{P}, \mathcal{Q}).$$

Thus

$$\Delta(u, Su) = \varpi(\mathcal{P}, \mathcal{Q}), \tag{19}$$

and

$$\varpi(\mathcal{P}, \mathcal{Q}) \leq \Delta(u, Tu) \leq \varpi(u, v) = \varpi(\mathcal{P}, \mathcal{Q}).$$

Thus

$$\Delta(u, Tu) = \varpi(\mathcal{P}, \mathcal{Q}). \tag{20}$$

From (19) and (20), we conclude that u is a CBPP of S and T.

Our previous theorem produces a Geraghty-type result as a consequence. Let $\mathcal{G}$ be the collection of functions $g : [0, \infty) \to [0, 1)$ those fulfill the condition: $g(t_n) \to 1$ implies $t_n \to 0$. A well-known example of this type of function is $g(t) = (1 + t)^{-1}$ for all $t > 0$ and $g(0) \in [0, 1)$. Geraghty [12] generalized Banach's contraction principle using the class $\mathcal{G}$.

Definition 8 Let $\mathcal{P}, \mathcal{Q}$ be two non-empty subsets of the MS $(\mathfrak{X}, \varpi)$. The pair of multivalued mappings $S, T : \mathcal{P} \to CB(\mathcal{Q})$ is said to be a multivalued Geraghty-type $\mathfrak{F}$-contraction pair (MVGFCP) if there exist $\lambda > 0$, $\mathfrak{F} \in \mathcal{F}$ and $g \in \mathcal{G}$ such that

$$\lambda + \mathfrak{F}(\mathfrak{H}(Sx, Ty)) \leq g(\varpi(x, y)) \cdot \mathfrak{F}(\varpi(x, y)) \tag{21}$$

for all $x, y \in \mathfrak{X}$ with $Sx \neq Ty$.

Corollary 1 *Let $(\mathfrak{X}, \varpi)$ be a complete MS and $\mathcal{P}, \mathcal{Q}$ be two non-empty closed subsets of $\mathfrak{X}$ such that $\mathcal{P}_0 \neq \phi$ and that the pair $(\mathcal{P}, \mathcal{Q})$ satisfies the weak P-property. Suppose $S, T : \mathcal{P} \to CB(\mathcal{Q})$ be a MVGFCP such that Sx, Ty are compact for each $x, y \in \mathcal{P}$ and $Sx, Ty \subseteq \mathcal{Q}_0$ for all $x, y \in \mathcal{P}_0$. Also, S, T are $\mathfrak{H}$-continuous. Then S, T have a CBPP.*

Proof For $g(t) \in [0, 1)$ for all $t \in [0, \infty)$, from (21), we obtain

$$\lambda + \mathfrak{F}(\mathfrak{H}(Sx, Ty)) \leq \mathfrak{F}(\varpi(x, y)) \tag{22}$$

for all $x, y \in \mathcal{P}$ with $Sx \neq Ty$. Thus, S, T is an MVFCP, and therefore, from Theorem 1 it follows that S, T have a CBPP.

Next, we present an example in which the pair $(\mathcal{P}, \mathcal{Q})$ satisfies only the weak P-property but not the P-property.

Example 1 Consider $\mathfrak{X} = \mathbb{R}^2$ with usual metric $\varpi(x, y) = |x - y|$ for all $x, y \in \mathfrak{X}$. Let $\mathcal{P} = \{(-7, 0), (0, 2), (7, 0)\}$ and $\mathcal{Q} = \{(x, y) : y = 3 + \sqrt{3 - x^2}, x \in [-\sqrt{3}, \sqrt{3}]\}$. Then $\varpi(\mathcal{P}, \mathcal{Q}) = 2$ and $\mathcal{P}_0 = \{(0, 2)\}$, $\mathcal{Q}_0 = \{(\sqrt{3}, 3), (-\sqrt{3}, 3)\}$.

Figure 1 illustrates the graph of the set $\mathcal{Q}$ (i.e., the function $y = 3 + \sqrt{3 - x^2}, x \in [-\sqrt{3}, \sqrt{3}]$).

Define two multivalued mappings $S, T : \mathcal{P} \to CB(\mathcal{Q})$ by

$$S(-7, 0) = \{(-\sqrt{3}, 3), (-\sqrt{2}, 4)\}, S(0, 2) = \{(\sqrt{3}, 3)\}, S(7, 0) = \{(\sqrt{3}, 3), (\sqrt{2}, 4)\}$$

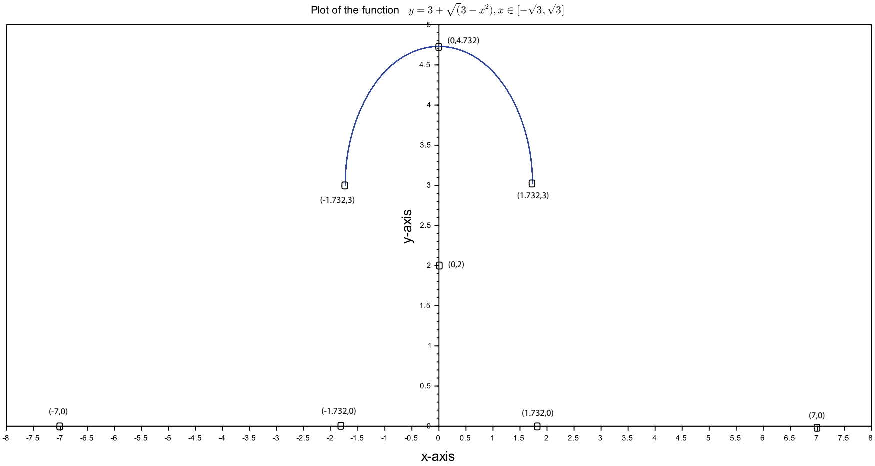

Fig. 1 Plot of the function $y = 3 + \sqrt{3 - x^2}$, $x \in [-\sqrt{3}, \sqrt{3}]$

and

$$T(-7, 0) = \{(0, 3 + \sqrt{3})\}, T(0, 2) = \{(\sqrt{3}, 3), (-\sqrt{3}, 3)\}, S(7, 0) = \{(\sqrt{2}, 4), (\sqrt{2}, 4)\}.$$

It can be verified that S, T is a MVFCP with $\mathfrak{F}(t) = \ln t$, $t > 0$ and $\lambda = \ln 2$ and also S, T are $\mathfrak{H}$-continuous.

Finally, we have that $\varpi((0, 2), (\sqrt{3}, 3)) = \varpi((0, 2), (-\sqrt{3}, 3)) = 2 = \varpi(\mathcal{P}, \mathcal{Q})$ but $\varpi((0, 2), (0, 2)) = 0 < \varpi((\sqrt{3}, 3), (-\sqrt{3}, 3)) = 2\sqrt{3}$.

Thus, $(\mathcal{P}, \mathcal{Q})$ has weak P-property but the P-property and all conditions of Theorem 1 are satisfied. Also, we have

$$\Delta((0, 2), T(0, 2)) = \Delta((0, 2), S(0, 2)) = \varpi(\mathcal{P}, \mathcal{Q}) = 2.$$

Hence $(0, 2)$ is a CBPP of S, T.

3 Conclusion

We have established some new CBPP results for multivalued mappings using $\mathfrak{F}$-contraction. The main result has been illustrated graphically with an example. Our results provide extensions of some renowned theorems in literature such as [4, 8, 18]. The results of these articles may be considered as particular cases of our work when both the mappings in the pair are identical or when single-valued mappings are considered.

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Best Proximity Point Theorems via Some Generalized Notions

Somayya Komal and Poom Kumam

Abstract In this chapter, we have obtained interesting results regarding best proximity points with the help of different routes in analysis. By using $\mathcal{Z}$-contraction, we defined a useful approach for finding best proximity points. Also, described some useful notions related to the existence of optimal approximate solutions with uniqueness. Furthermore, these results are extended to some existing results in the literature.

1 Introduction

During the last thirty years, the research in fixed points theory has attained a lot of importance in the system of nonlinear functional analysis. Especially, the ways, as well as the notions in functional analysis, have been applied in other areas of applied mathematics as well as the rest of the categories in science and engineering. The very fundamental theorem Banach Contraction Mapping Principle (BCMP) is introduced by Banach [1] in 1922.

However, in case of non-self mapping, the theorems in the field of fixed points are not specified to assure valid solutions for the equation $Fg = g$, where F is not a self mapping. Research in the era of fixed points for non-self mappings on various abstract spaces got a lot of attention of many mathematicians.

Accurately, for a provided closed subsets $G \neq \phi$ and $H \neq \phi$ in complete metric space (Y, d), it is not necessary that a contraction mapping $F : G \to H$ has a fixed point, that is, $d(Fg, g) \neq 0$. In this situation, it is simple to find a point $g \in Y$ such that $d(g, Fg)$ is least. Let G and H be closed subsets of a metric space (Y, d) and a

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mapping $F : G \to H$. An element $g \in G$ for which $d(g, Fg) = d(G, H)$ is called best proximity point of F. If $G \cap H \neq \phi$ then a best proximity point becomes a fixed point of F. One can say that the best proximity point is used to find a fixed point if the mapping under discussion is supposed to be self mapping. Best proximity point theorems are simple and easy generalizations of the BCMP. In corresponding approach, the first result was given by Fan [3] in 1969.

Now, by studying some historical literature review deeply, we came to know the following facts:
In 1968, Kannan [2] mentioned some results on fixed points.
In 1972, Chatterjea [4] also worked out for fixed point theorems.
In 2003, [5] Kirk, Srinivasan, and Veeramani discussed fixed points for mappings satisfying cyclical contractive conditions.
In 2008, Suzuki [6] introduced generalized Banach contraction principle which characteristics in metric spaces.
In 2011, Sadiq Basha [7, 8] described best proximity point theorems.
In 2012, Raj [9] and Wardowski [10] demonstrated nice best proximity point theorems in metric spaces.
In 2013, Basha, Shahzad and Jeyaraj [11] discussed about best proximity points briefly.
In 2015, Roldn-Lpez-de-Hierro et al. [12] constructed Coincidence point theorems on metric spaces by using simulation functions.
In 2015, Nastasi and Vetro [14] investigated Fixed point results.
In 2015, Argoubi, Samet and Vetro in [15] discussed nonlinear contractions involving simulation functions.
In 2016, Olgun [16] described nice expression regarding to Picard operators via simulation functions.
In 2016, Kasamsuk Ungchittrakool [17] introduced new contractions. In the same year, we developed some useful best proximity point theorems, see [18].
In 2019, Neog et al. [19] did wonderful work on common fixed point theorems.
In 2020, Debnath and Sirivastava [20, 21] proved fixed point results with different approach.
Debnath [22–24] has a number of quality workpieces in the area of fixed points and best proximity points. They proposed their results with the help of examples as well. Giving reference of researches above, it seems useful to study [17, 18].

2 Preliminaries

Now, let's discuss a few common, as well as useful, notations which are useful tools to study about optimal approximate solutions in this chapter. Let $G \neq \phi$ and $H \neq \phi$ are subsets of a metric space (Y, d).

$$G_0 = \{g \in G : d(g, h) = d(G, H) \text{ for some } h \in H\},$$

$$H_0 = \{h \in H : d(g,h) = d(G,H) \text{ for some } g \in G\},$$

$$d(G,H) = \inf\{d(g,h) : g \in G, h \in H\}.$$

Definition 1 ([18]) Let (Y, d) be a metric space and $G \neq \phi$, $H \neq \phi$ be subsets of Y. An element $g \in Y$ is said to be *best proximity point* for $F : G \to H$ if

$$d(g, Fg) = d(G, H).$$

Obviously, if $d(G, H) = 0$, then a best proximity point coincides a fixed point. Moreover, for all $g \in G$ then $d(g, Fg) \geq d(G, H)$, the function $g \mapsto d(g, Fg)$ attains its global minimum at a best proximity point.

Definition 2 ([17]) Given $G \neq \phi$ and $H \neq \phi$ be subsets of a metric space (Y, d). Let mappings $C : G \to H$ and $S : G \to H$, the pair (S, C) is called $K-$*cyclic contraction* if there exists a constant $0 \leq k < \frac{1}{2}$ such that

$$d(Cg, Sh) \leq k(d(g, Cg) + d(h, Sh)) + (1 - 2k)d(G, H) \tag{1}$$

for all $g \in G$ and $h \in H$.

Definition 3 ([13]) Let a mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$, then ζ is said to be simulation function if:

1. $\zeta(0, 0) = 0$;
2. $\zeta(q, p) < p - q$ for $q, p > 0$;
3. if $\{q_n\}, \{p_n\}$ are sequences in $(0, \infty)$ such that

$$\lim_{n\to\infty} q_n = \lim_{n\to\infty} p_n > 0,$$

then

$$\limsup_{n\to\infty} \zeta(q_n, p_n) < 0.$$

We denote the set of all simulation functions by $\mathcal{Z}$.

Definition 4 ([9]) Let (G, H) be a pair of non-empty subsets of a metric space (Y, d) with $G_0 \neq \phi$. Then the pair (G, H) is said to have the P-property if and only if

$$d(g_1, h_1) = d(G, H) \text{ and } d(g_2, h_2) = d(G, H) \Rightarrow d(g_1, g_2) = d(h_1, h_2),$$

where $g_1, g_2 \in G_0$ and $h_1, h_2 \in H_0$.

Theorem 1 ([18]) *Let (G, H) be the pair of closed subsets of a complete metric space $(Y, d) \neq \phi$ such that $G_0 \neq \phi$. Define a mapping $R : G \to H$ satisfying:*

1. *R is $\mathcal{Z}$-contraction with $R(G_0) \subseteq H_0$;*
2. *the pair (G, H) has weak P-property.*

Then there exists unique best proximity point in G and the iteration sequence $\{g_{2n}\}$ defined by

$$d(g_{2n+2}, g_{2n+1}) = d(G, H), \quad n = 0, 1, 2, \ldots$$

converges, to g^*, for every $g_0 \in G_0$.

Definition 5 ([17]) Let $G \neq \phi$ and $H \neq \phi$ be subsets of a metric space (Y, d) and let $S : H \to G$ be a mapping. A mapping $C : G \to H$ is called generalized non-self Kannan and Chatterjea mapping w.r.t the mapping S if

$$d(Cg, Ch) \leq k_1 d(g, h) + k_2(d(g, SCg) + d(h, SCh)) + k_3(d(g, SCh) + d(h, SCg))$$

for all $g, h \in G$ where k_1, k_2, k_3 non-negative constants such that $k_1 + 2k_2 + 2k_3 < 1$.

Theorem 2 ([17]) *Let $G \neq \phi$ and $H \neq \phi$ be closed subset of a complete metric space (Y, d) and assume $S : H \to G$ and $C : G \to H$ be two mappings such that S is Lipschitzian mapping with Lipschitz constant $L \geq 1$. The pair (S, C) forms a weak K-cyclic contraction. Also, C is generalized non-self Kannan and Chatterjea mapping w.r.t the mapping S. Then there exist elements $g \in G$ and $h \in H$ such that*

$$d(g, Cg) = d(G, H)$$

$$d(h, Sh) = d(G, H)$$

$$d(g, h) = d(G, H)$$

If g_0 is any fixed element in G with $g_{2n+1} = Cg_{2n}$ and $g_{2n} = Sg_{2n-1}$, then the sequences $\{g_{2n}\}$ and $\{g_{2n+1}\}$ converge to some best proximity points of C and S. Furthermore, if g^* is another best proximity point of C, then

$$d(g, g^*) \leq \frac{2(1 + 2(k_2 + k_3))}{1 - (k_1 + 2k_3)} d(G, H).$$

3 Best Proximity Results for $\mathcal{Z}$-Contraction and Its Generalizations

Now, here we try to attempt the best proximity theorems with beautiful conditions and nice proof patterns. Also, the idea of non-self $\mathcal{Z}$-contraction defined in [18] can be used here to extend the results of [17]. After recalling some notions of [18], we express our results in this section.

Definition 6 ([18]) Let (Y, d) be a metric space, $R : G \to H$ is a mapping and $\zeta \in \mathcal{Z}$. Then R is called a $\mathcal{Z}$-contraction with respect to ζ if

$$\zeta(d(Rg, Rh), d(g, h)) \geq 0 \tag{2}$$

where $G, H \subseteq X$ and $g, h \in G$, with $g \neq h$.

Definition 7 ([18]) Let (Y, d) be a metric space, $R : G \to H$ is a mapping and $\zeta \in \mathcal{Z}$. Then R is called a Suzuki type $\mathcal{Z}$-contraction with respect to ζ if the following condition holds:

$$\frac{1}{2}d(g, Rg) < d(g, h) \Rightarrow \zeta(d(Rg, Rh), d(g, h)) \geq 0 \tag{3}$$

where $G, H \subseteq Y$ and $g, h \in G$, with $g \neq h$.

Remark 1 ([18]) Since

$$\frac{1}{2}d(g, Rg) < d(g, h) \Rightarrow d(Rg, Rh) < d(g, h)$$

for some different $g, h \in G$.

Remark 2 ([18]) Each Suzuki type $\mathcal{Z}$-contraction is also a $\mathcal{Z}$-contraction.

With the interpretation of some above mentioned known notions, we are able to define our results

Theorem 3 *Let $G \neq \phi$ and $H \neq \phi$ be closed subsets of a complete metric space (Y, d). Define $R : G \to H$ and $Q : H \to G$ with $Q(H_0) \subseteq G_0$, where R is $\mathcal{Z}$-contraction with $R(G_0) \subseteq H_0$ and Q is contraction mapping with Lipschitz constant L that is, $0 \leq L < 1$ and the pair (G, H) has P-property such that G_0 is non-empty. Then there exists unique best proximity point g of R in G and h of Q in H, respectively, with $d(g, h) = d(G, H)$.*

Proof Since G_0 is non-empty, so pick up an element g_0 in G_0 be fixed. Then there exists h_0 in H such that $d(g_0, h_0) = d(G, H)$. Then for g_1 in G_0, we get h_1 in H_0 such that $d(g_1, h_1) = d(G, H)$. In a similar fashion, we able to construct a sequence $\{g_n\}$ in G_0 and $\{h_n\}$ in H_0. Since H is closed. For this, let us take $\{h_n\} \subseteq H_0$ a sequence such that $h_n \to h \in H$. Since the pair (G, H) has P-property, it clears from the P-property that

$$d(h_n, h_m) \to 0 \Rightarrow d(g_n, g_m) \to 0,$$

as $m, n \to \infty$, and $g_n, g_m \in G_0 \quad$ and $d(g_n, h_n) = d(g_m, h_m) = d(G, H)$.

Thus, $\{g_n\}$ is a Cauchy sequence in G_0 and converges strongly to a point $g \in G_0$ and $\{h_n\}$ is a Cauchy sequence in H_0 and converges strongly to a point $h \in H_0$. By the continuity of the metric d, we have $d(g, h) = d(G, H)$, that is $h \in H_0$ and $g \in G_0$. Hence, G_0 and H_0 are closed.

Since R is $\mathcal{Z}$-contraction, it shows that

$$\begin{aligned} 0 &\leq \zeta(d(Rg_1, Rg_2), d(g_1, g_2)) \\ &< d(g_1, g_2) - d(Rg_1, Rg_2), \end{aligned}$$

implies that

$$d(Rg_1, Rg_2) < d(g_1, g_2). \tag{4}$$

Furthermore, it can be seen that

$$d(QRg_{2n}, QRg) \leq Ld(Rg_{2n}, Rg) < d(g_{2n}, g)$$

By taking limit $n \to \infty$, with the fact that $0 \leq L < 1$, we can calculate as

$$\lim_{n\to\infty} d(Rg_{2n}, Rg) \to 0,$$

which means that $d(h, Rg) = 0$, where $R(G_0) \subseteq H_0$, then $Rg_{2n} = h \in H_0$ that is, $Cg = h$.

Furthermore, we also observed that

$$d(RQh_{2n}, RQh) < d(Qh_{2n}, Qh) \leq Ld(h_{2n}, h).$$

By taking limit $n \to \infty$, with the fact that $0 \leq L < 1$, we can calculate as

$$\lim_{n\to\infty} d(Qh_{2n}, Qh) \to 0,$$

which means that $d(g, Qh) = 0$, where $Q(H_0) \subseteq G_0$, implies that $Qh_{2n} = g \in G_0$ that is, $Qh = g$. Since the pair (G, H) has the P-property, then from $d(g_n, h_n) = d(g_m, h_m) = d(G, H)$ and after taking limit $n \to \infty$, we have $d(g, h) = d(G, H)$. Since $Rg = h$ and $Qh = g$, thus, from $d(g, h) = d(G, H)$, we got unique best proximity points of R and Q, respectively, as $d(g, Rg) = d(G, H)$ and $d(h, Rh) = d(G, H)$. This completes the proof.

Theorem 4 *Let $G \neq \phi$ and $H \neq \phi$ be closed subsets of a complete metric space (Y, d). Define $R : G \to H$ and $Q : H \to G$ with $Q(H_0) \subseteq H_0$, where R is Suzuki Type $\mathcal{Z}$-contraction with $R(G_0) \subseteq H_0$ and Q is a non-self contraction mapping with Lipschitz constant L that is, $0 \leq L < 1$ and the pair (G, H) has P-property such that G_0 is non-empty. Then there exists unique best proximity points xg of R in G and h of Q in H, respectively, with $d(g, h) = d(G, H)$.*

Proof Let g_0 be a fixed element G_0. Then there exists h_0 in H_0 such that $d(g_0, h_0) = d(G, H)$. Since G_0 is non-empty, so for g_1 in G_0, we get h_1 in H_0 such that $d(g_1, h_1) = d(G, H)$. In a similar fashion, we are able to construct a sequence $\{g_n\}$ in G and $\{h_n\}$ in H. Since H is closed. For this, let us take $\{h_n\} \subseteq H_0$ a sequence such that $h_n \to h \in H$. Since the pair (G, H) has P-property, it is clear from the P-property that

$$d(h_n, h_m) \to 0 \Rightarrow d(g_n, g_m) \to 0,$$

as $m, n \to \infty$, and $g_n, g_m \in G_0$ and $d(g_n, h_n) = d(g_m, h_m) = d(G, H)$.

Thus, $\{g_n\}$ is a Cauchy sequence in G_0 and converges strongly to a point $g \in G_0$ and $\{h_n\}$ is a Cauchy sequence in H_0 and converges strongly to a point $h \in H_0$. By the continuity of the metric d, we have $d(g, h) = d(G, H)$, that is $h \in H_0$ and $g \in G_0$. Hence, G_0 and H_0 are closed.

Since R is Suzuki Type $\mathcal{Z}$-contraction, such that for $\frac{1}{2}d(g_1, Rg_1) < d(g_1, h_1)$, we have

$$\begin{aligned} 0 &\leq \zeta(d(Rg_1, Rg_2), d(g_1, g_2)) \\ &< d(g_1, g_2) - d(Rg_1, Rg_2), \end{aligned}$$

implies that

$$d(Rg_1, Rg_2) < d(g_1, g_2). \tag{5}$$

Since every Suzuki type $\mathcal{Z}$-contraction is a $\mathcal{Z}$-contraction, following this note, we proceed with the proof in a similar way as for $\mathcal{Z}$-contraction. Furthermore, it can be seen that

$$d(QRg_{2n}, QRg) \leq Ld(Rg_{2n}, Rg) < d(g_{2n}, g)$$

By taking limit $n \to \infty$, with the fact that $0 \leq L < 1$, we can calculate

$$\lim_{n\to\infty} d(Rg_{2n}, Rg) \to 0,$$

which means that $d(h, Rg) = 0$, where $R(G_0) \subseteq H_0$, so $Rg_{2n} = h \in H_0$, that is, $Tg = h$.

On the other hand, we also observed that

$$d(RQh_{2n}, RQh) < d(Qh_{2n}, Qh) \leq Ld(h_{2n}, h).$$

By taking limit $n \to \infty$, with the fact that $0 \leq L < 1$, we can calculate

$$\lim_{n\to\infty} d(Qh_{2n}, Qh) \to 0,$$

which means that $d(g, Qh) = 0$, where $Q(H_0) \subseteq G_0$, then $Qh_{2n} = g \in G_0$, that is, $Qh = g$. Since the pair (G, H) has the P-property, then from $d(g_n, h_n) = d(g_m, h_m) = d(G, H)$ after taking limit $n \to \infty$, we have $d(g, h) = d(G, H)$. Since

$Rg = h$ and $Qh = g$. Thus, from $d(g, h) = d(G, H)$, we got unique best proximity point of R and Q, respectively, as $d(g, Rg) = d(G, H)$ and $d(h, Rh) = d(G, H)$. This completes the proof.

4 Best Proximity Results in Some Other Generalized Contractions

The extension of the results of [18] is not finished yet by the only way of $\mathcal{Z}$-contraction and Suzuki Type $\mathcal{Z}$-contraction. Next, in this chapter, with the help of [18], we introduce some new and generalized contractions and notions which will further mention the proof lines for best proximity points.

Definition 8 Let G and H be non-empty subsets of a metric space (Y, d) and let $S : H \to G$ be a mapping. A mapping $C : G \to H$ is said to be K-generalized non-self Kannan and Chatterjea mapping w.r.t the mapping S if

$$\begin{aligned} d(Cg, Ch) \leq k_1 d(g, h) + k_2(d(g, SCg) + d(h, SCh)) + k_3(d(g, SCh) \\ + d(h, SCg)) + k_4 d(SCg, SCh) \end{aligned}$$

for every $g, h \in G$, where k_1, k_2, k_3, k_4 are non-negative constants such that $k_1 + 2k_2 + 2k_3 + k_4 < 1$ and $d(Sg, Sh) \leq \lambda d(g, h)$, where $0 \leq \lambda = \frac{k}{1-k} < 1$ and $k = k_1 + k_2 + k_3 + k_4$.

Definition 9 Let $G \neq \phi$ and $H \neq \phi$ be subsets of a metric space (Y, d) and let $C : G \to H$ and $S : H \to G$ be two mappings. Then the pair (S, C) is said to form a generalized K-cyclic contraction if there exists a non-negative real number $k < \frac{1}{3}$ such that

$$d(Cg, Sh) \leq k[d(g, h) + d(g, Cg) + d(h, Sh)] + (1 - 3k)d(G, H)$$

for all $g \in G$ and $h \in H$.

Theorem 5 *Let G and H be non void closed subset of a complete metric space (Y, d) and assume $S : H \to G$ and $C : G \to H$ be two mappings such that C is K-generalized non-self Kannan and Chatterjea mapping w.r.t mapping S and the pair (S, C) constructs a generalized K-cyclic contraction. Then there exist elements $g \in G$ and $h \in H$ such that*

$$d(g, Cg) = d(G, H)$$

$$d(h, Sh) = d(G, H)$$

$$d(g, h) = d(G, H).$$

If g_0 is any fixed element in G with $g_{2n+1} = Cg_{2n}$ and $g_{2n} = Sg_{2n-1}$, then the sequences $\{g_{2n}\}$ and $\{g_{2n+1}\}$ converge to some best proximity points of C and S. Furthermore, if g^* is another best proximity point of T, then

$$d(g, g^*) \leq \frac{2(1 + 2(k_2 + k_3 + k_4))}{1 - (k_1 + 2k_3 + k_4)} d(G, H).$$

Proof Let us choose an element $g_0 \in G$ and then generate a sequence $\{g_{2n}\}$ in G and $\{g_{2n+1}\}$ in H by $g_{2n} = Sg_{2n-1}$ and $g_{2n+1} = Tg_{2n}$ for all $n \geq 1$ and $n \geq 0$, respectively. It is seen that

$$\begin{aligned}
d(g_{2n}, g_{2n+2}) &= d(Sg_{2n-1}, Sg_{2n+1}) \\
&\leq \lambda d(g_{2n-1}, g_{2n+1}) \\
&= \lambda d(Cg_{2n-2}, Cg_{2n}) \\
&\leq \lambda(k_1 d(g_{2n-2}, g_{2n}) \\
&+ k_2(d(g_{2n-2}, SCg_{2n-2}) + d(g_{2n}, SCg_{2n})) \\
&+ k_3(d(g_{2n-2}, SCg_{2n}) + d(g_{2n}, SCg_{2n-2})) \\
&+ k_4(d(SCg_{2n-2}, SCg_{2n}))) \\
&= \lambda(k_1 d(g_{2n-2}, g_{2n}) \\
&+ k_2(d(g_{2n-2}, g_{2n})) + d(g_{2n}, g_{2n+2})) \\
&+ k_3(d(g_{2n-2}, g_{2n+2}) + d(g_{2n}, g_{2n})) \\
&+ k_4(d(g_{2n+2}, g_{2n}))) \\
&\leq \lambda(k_1 d(g_{2n-2}, g_{2n}) \\
&+ k_2(d(g_{2n-2}, g_{2n}) + d(g_{2n}, g_{2n+2})) \\
&+ k_3(d(g_{2n-2}, g_{2n}) + d(g_{2n}, g_{2n+2})) \\
&+ k_4(d(g_{2n+2}, g_{2n}))) \\
&= \lambda(k_1 + k_2 + k_3) d(g_{2n-2}, g_{2n}) \\
&+ \lambda(k_2 + k_3 + k_4) d(g_{2n}, g_{2n+2})
\end{aligned}$$

From the above calculations we obtain

$$d(g_{2n}, g_{2n+2}) \leq \frac{\lambda(k_1 + k_2 + k_3)}{1 - \lambda(k_2 + k_3 + k_4)} d(g_{2n-2}, g_{2n}) \tag{6}$$

By using mathematical induction, we get

$$d(g_{2n}, g_{2n+2}) \leq \frac{\lambda(k_1 + k_2 + k_3)}{1 - \lambda(k_2 + k_3 + k_4)}^{n} d(g_0, g_2) \tag{7}$$

Since $0 \leq \lambda < 1$, it is clear that $\{g_{2n}\}$ is a Cauchy sequence in G and so converges to some element g in G. In the same fashion, it is seen that

$$
\begin{aligned}
d(g_{2n+1}, g_{2n+3}) &= d(Cg_{2n}, Cg_{2n+2}) \\
&\leq k_1 d(g_{2n}, g_{2n+2}) \\
&+ k_2(d(g_{2n}, SCg_{2n}) + d(g_{2n+2}, SCg_{2n+2})) \\
&+ k_3(d(g_{2n}, SCg_{2n+2}) + d(g_{2n+2}, SCg_{2n})) \\
&+ k_4 d(SCg_{2n}, SCg_{2n+2}) \\
&= k_1 d(Sg_{2n-1}, Sg_{2n+1}) \\
&+ k_2(d(Sg_{2n-1}, Sg_{2n+1}) + d(Sg_{2n+1}, Sg_{2n+3})) \\
&+ k_3(d(Sg_{2n-1}, Sg_{2n+3}) + d(Sg_{2n+1}, Sg_{2n+1})) \\
&+ k_4 d(Sg_{2n+1}, Sg_{2n+3}) \\
&\leq \lambda k_1 d(g_{2n-1}, g_{2n+1}) \\
&+ k_2\lambda(d(g_{2n-1}, g_{2n+1}) + d(g_{2n+1}, g_{2n+3})) \\
&+ k_3\lambda(d(g_{2n-1}, g_{2n+3}) \\
&+ k_4\lambda d(g_{2n+1}, g_{2n+3}) \\
&\leq \lambda k_1 d(g_{2n-1}, g_{2n+1}) \\
&+ k_2\lambda(d(g_{2n-1}, g_{2n+1})) + d(g_{2n+1}, g_{2n+3})) \\
&+ k_3\lambda(d(g_{2n-1}, g_{2n+1}) + d(g_{2n+1}, g_{2n+3})) \\
&+ k_4\lambda d(g_{2n+1}, g_{2n+3}) \\
&= \lambda(k_1 + k_2 + k_3) d(g_{2n-1}, g_{2n+1}) \\
&+ \lambda(k_2 + k_3 + k_4) d(g_{2n+1}, g_{2n+3})
\end{aligned}
$$

It follow that

$$d(g_{2n+1}, g_{2n+3}) \leq \frac{\lambda(k_1 + k_2 + k_3)}{1 - \lambda(k_2 + k_3 + k_4)} d(g_{2n-1}, g_{2n+1}) \tag{8}$$

Thus, from the mathematical induction, we can write

$$d(g_{2n+1}, g_{2n+3}) \leq \frac{\lambda(k_1 + k_2 + k_3)}{1 - \lambda(k_2 + k_3 + k_4)}^{n} d(g_1, g_3) \tag{9}$$

Thus, we obtain $\{g_{2n+1}\}$, which is a Cauchy sequence in H and hence converges to some element $h \in H$. Also, we can see that

$$\begin{aligned}
d(g_{2n+2}, SCg) &= d(SCg_{2n}, SCg) \leq \lambda d(Cg_{2n}, Cg) \\
&= \lambda d(g_{2n+1}, Cg) \leq \lambda(k_1 d(g_{2n}, g) \\
&+ k_2(d(g_{2n}, SCg_{2n}) + d(g, SCg)) \\
&+ k_3(d(g_{2n}, SCg) + d(g, SCg_{2n})) \\
&+ k_4 d(SCg, SCg_{2n})) \\
&= \lambda k_1 d(g_{2n}, g) \\
&+ k_2(d(g_{2n}, g_{2n+2}) + d(g, SCg)) \\
&+ k_3(d(g_{2n}, SCg) + d(g, g_{2n+2})) \\
&+ k_4 d(SCg, g_{2n+2})
\end{aligned}$$

Let us take limit $n \to \infty$, it calculates as follows:

$$d(g, SCg) \leq \lambda d(h, Cg) \leq \lambda(k_2 + k_3 + k_4) d(g, SCg). \tag{10}$$

Since $0 \leq \lambda(k_2 + k_3 + k_4) < 1$, which shows that $d(h, Tg) = 0$ and hence $Tg = h$. Similarly, we also investigate that

$$\begin{aligned}
d(g_{2n+3}, CSh) &= d(CSg_{2n+1}, CSh) \\
&\leq k_1 d(Sg_{2n+1}, Sh) \\
&+ k_2(d(Sg_{2n+1}, SCSg_{2n+1}) + d(Sh, SCSh)) \\
&+ k_3(d(Sg_{2n+1}, SCSh) + d(Sh, SCSg_{2n+1})) \\
&+ k_4 d(SCSg_{2n+1}, SCSh)) \\
&\leq k_1 \lambda d(g_{2n+1}, h) \\
&+ k_2 d(Sg_{2n+1}, g_{2n+4}) + k_2 \lambda d(h, CSh)) \\
&+ k_3 \lambda(d(g_{2n+1}, CSh) + d(h, g_{2n+3})) \\
&+ k_4 d(g_{2n+3}, CSh) \\
&= k_1 \lambda d(g_{2n+1}, h) \\
&+ k_2 d(Sg_{2n+1}, Sg_{2n+3}) + k_2 \lambda d(h, CSh)) \\
&+ k_3 \lambda(d(g_{2n+1}, CSh) + d(h, g_{2n+3})) \\
&+ k_4 d(g_{2n+3}, CSh) \\
&\leq k_1 \lambda d(g_{2n+1}, h) \\
&+ k_2 \lambda(d(g_{2n+1}, g_{2n+3}) + d(h, CSh)) \\
&+ k_3 \lambda(d(g_{2n+1}, CSh) + d(h, g_{2n+3})) \\
&+ k_4 d(g_{2n+3}, CSh)
\end{aligned}$$

Letting $n \to \infty$, it gives us as

$$\begin{aligned} d(h, CSh) &\leq k_2 d(Sh, g) + \lambda(k_2 + k_3 + k_4) d(h, CSh) \\ &\leq \lambda(k_2 + k_3 + k_4) d(h, CSh). \end{aligned}$$

This concludes that $d(Sh, g) = 0$ and so $Sh = g$. Since the pair (S, C) forms a generalized K-cyclic contraction, it follows that there exists $k \in [0, \frac{1}{3})$ such that

$$\begin{aligned} d(g, h) &= d(Cg, Sh) \\ &\leq k[d(g, Cg) + d(h, Sh) + d(g, h)] + (1 - 3k) d(G, H) \\ &= 3k d(g, h) + (1 - 3k) d(G, H). \end{aligned}$$

It guarantees that

$$(1 - 3k) d(g, h) \leq (1 - 3k) d(G, H)$$

hence we obtain

$$d(g, h) = d(G, H),$$

$$d(g, Cg) = d(G, H),$$

and

$$d(h, Sh) = d(G, H).$$

We can conclude that g and h are best proximity points of C and S, respectively. If we consider that g^* is one more best proximity point of C, then

$$d(G, H) = d(g^*, Cg^*) = d(Cg^*, SCg^*)$$

In this manner, observations are as follows:

$$\begin{aligned} d(g, g^*) &\leq d(g, Cg) + d(Cg, Cg^*) + d(g^*, Cg^*) \\ &\leq 2d(G, H) + k_1 d(g, g^*) + k_2 d(g, SCg) \\ &+ d(g^*, SCg^*) + k_3(d(g, SCg^*) + d(g^*, SCg)) \\ &+ k_4 d(SCg, SCg^*)) \\ &\leq 2d(G, H) + k_1 d(g, g^*) + k_2(d(g, Cg) \\ &+ d(Cg, SCg) + d(g^*, Cg^*) + d(Cg^*, SCg^*)) \\ &+ k_3(d(g, g^*) + d(g^*, Cg^*) + d(Cg^*, SCg^*) \\ &+ d(g^*, g) + d(g, Cg) + d(Cg, SCg)) \\ &+ k_4 d(SCg, Cg) + k_4 d(SCg^*, Cg^*) + d(Cg, Cg^*) \\ &\leq 2d(G, H) + k_1 d(g, g^*) + k_2(d(g, Cg) \\ &+ d(Cg, SCg) + d(g^*, Cg^*) + d(Cg^*, SCg^*)) \\ &+ k_3(d(g, g^*) + d(g^*, Cg^*) + d(Cg^*, SCg^*) \\ &+ d(g^*, g) + d(g, Cg) + d(Cg, SCg)) \\ &+ k_4 d(SCg, Cg) + k_4 d(SCg^*, Cg^*) + k_4 d(g, Cg) + k_4 d(g^*, Cg^*) + k_4 d(g, g^*) \\ &\leq 2d(G, H) + (k_1 + 2k_3 + k_4) d(g, g^*) \end{aligned}$$

$$
\begin{aligned}
&+ (k_2 + k_3 + k_4)[d(g, Tg) + d(Cg, SCg) + d(g^*, Cg^*) + d(Cg^*, SCg^*)] \\
&= 2d(G, H) + (k_1 + 2k_3 + k_4)(d(g, g^*) \\
&+ 4(k_2 + k_3 + k_4)d(G, H) \\
&= 2(1 + 2(k_2 + k_3 + k_4))d(G, H) + (k_1 + 2k_3 + k_4)d(g, g^*)
\end{aligned}
$$

Thus, $d(g, g^*) \leq \frac{2(1+2(k_2+k_3+k_4))}{1-(k_1+2k_3+k_4)} d(G, H)$. Hence, proved the theorem.

5 Example

Next, an example is proposed in the favor of our proposed work.

Example 1 Let $C[0, \pi] = \{[0, \pi] \to \mathbb{R} \mid f$ is continuous$\}$ with supermum norm. Let us construct

$$G = \{f_\alpha := 1 - \alpha \cos(\cdot) \| \alpha \in [0, 1]\}, H = \{g_\beta := \beta \cos(\cdot) - 1 \mid \beta \in [0, 1]\}. \tag{11}$$

It seems not difficult to prove that $G, H \subseteq C[0, \pi]$ and $G \cap H = \emptyset$ then $d(G, H) = 2$.

Define the mappings $C : G \to H$ and $S : H \to G$ by

$$
\begin{aligned}
C(f_\alpha) &= -f_{0.1\sqrt{\alpha}} = g_{0.1\sqrt{\alpha}} = 0.1\sqrt{\alpha}\cos(\cdot) - 1, \\
S(g_\beta) &= -g_{0.25+0.55\beta^2} = f_{0.25+0.55\beta^2} = 1 - (0.25 + 0.55\beta^2)\cos(\cdot)
\end{aligned} \tag{12}
$$

for all $\alpha, \beta \in [0, 1]$ as appeared in Fig. 1.

All conditions of Theorem (5) are satisfied here.

Solution: First, we consider

$$
\begin{aligned}
d(C(f_{\alpha_1}), C(f_{\alpha_2})) &= \|C(f_{\alpha_1}) - C(f_{\alpha_2})\| \\
&= \|0.1\sqrt{\alpha_1}\cos(\cdot) - 1 - (0.1\sqrt{\alpha_2}\cos(\cdot) - 1)\| \\
&= 0.1|\sqrt{\alpha_1} - \sqrt{\alpha_2}|\|\cos(\cdot)\| \\
&= 0.1|\sqrt{\alpha_1} - \sqrt{\alpha_2}|.
\end{aligned}
$$

So

$$d(C(f_{\alpha_1}), C(f_{\alpha_2})) = 0.1|\sqrt{\alpha_1} - \sqrt{\alpha_2}|. \tag{13}$$

On the other hand, let us assume

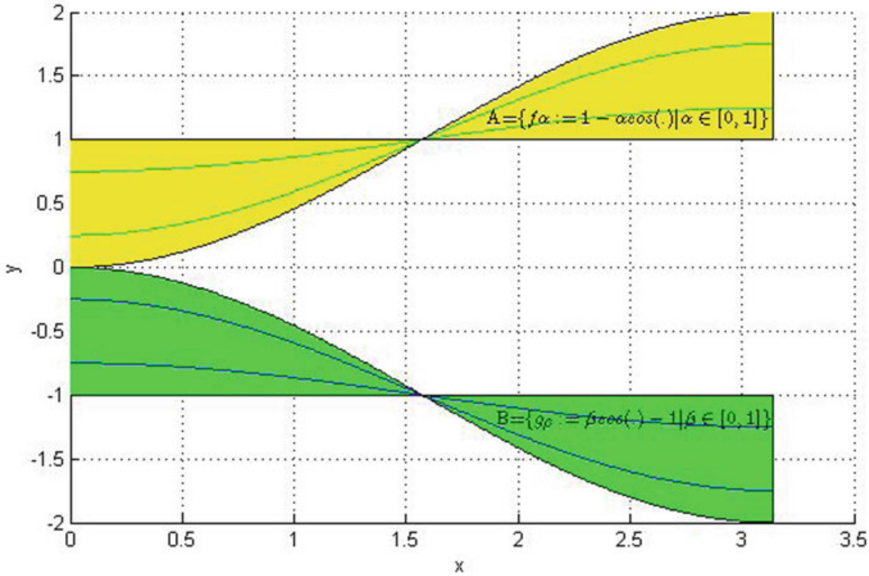


Fig. 1 G and H in the function space $(C[0,\pi], \|\cdot\|)$

$$
\begin{aligned}
d(f_{\alpha_1}, SC(f_{\alpha_1})) &= \|f_{\alpha_1} - SC(f_{\alpha_1})\| \\
&= \|1 - \alpha_1 \cos(\cdot) - SC(1 - \alpha_1 \cos(\cdot))\| \\
&= \|1 - \alpha_1 \cos(\cdot) - S(0.1\sqrt{\alpha_1}\cos(\cdot) - 1)\| \\
&= \|1 - \alpha_1 \cos(\cdot) - (1 - (0.25 + 0.55(0.01\alpha_1))\cos(\cdot))\| \\
&= \|1 - \alpha_1 \cos(\cdot) - (1 - (0.25 + 0.0055\alpha_1)\cos(\cdot))\| \\
&= |0.25 + 0.0055\alpha_1 - \alpha_1|\|\cos(\cdot)\| \\
&= |0.25 - 0.9945\alpha_1|\|\cos(\cdot)\| \\
&= |0.25 - 0.9945\alpha_1|.
\end{aligned}
$$

Hence

$$
d(f_{\alpha_1}, SC(f_{\alpha_1})) = |0.25 - 0.9945\alpha_1|. \tag{14}
$$

$$
\begin{aligned}
d(f_{\alpha_2}, SC(f_{\alpha_2})) &= \|f_{\alpha_2} - SC(f_{\alpha_2})\| \\
&= \|1 - \alpha_2 \cos(\cdot) - SC(1 - \alpha_2 \cos(\cdot))\| \\
&= \|1 - \alpha_2 \cos(\cdot) - S(0.1\sqrt{\alpha_2}\cos(\cdot) - 1)\| \\
&= \|1 - \alpha_2 \cos(\cdot) - (1 - (0.25 + 0.55(0.01\alpha_2))\cos(\cdot))\| \\
&= \|1 - \alpha_2 \cos(\cdot) - (1 - (0.25 + 0.0055\alpha_2)\cos(\cdot))\|
\end{aligned}
$$

$$
\begin{aligned}
&= |0.25 + 0.0055\alpha_2 - \alpha_2| \| \cos(\cdot) \| \\
&= |0.25 - 0.9945\alpha_2| \| \cos(\cdot) \| \\
&= |0.25 - 0.9945\alpha_2|.
\end{aligned}
$$

So

$$d(f_{\alpha_2}, SC(f_{\alpha_2})) = |0.25 - 0.9945\alpha_2|. \tag{15}$$

$$
\begin{aligned}
d(f_{\alpha_1}, SC(f_{\alpha_2})) &= \| f_{\alpha_1} - SC(f_{\alpha_2}) \| \\
&= \| 1 - \alpha_1 \cos(\cdot) - SC(1 - \alpha_2 \cos(\cdot)) \| \\
&= \| 1 - \alpha_1 \cos(\cdot) - S(0.1\sqrt{\alpha_2} \cos(\cdot) - 1) \| \\
&= \| 1 - \alpha_1 \cos(\cdot) - (1 - (0.25 + 0.55(0.01\alpha_2)) \cos(\cdot)) \| \\
&= \| 1 - \alpha_1 \cos(\cdot) - (1 - (0.25 + 0.0055\alpha_2) \cos(\cdot)) \| \\
&= |0.25 + 0.0055\alpha_2 - \alpha_1| \| \cos(\cdot) \| \\
&= |0.25 + 0.0055\alpha_2 - \alpha_1|.
\end{aligned}
$$

Thus

$$d(f_{\alpha_1}, SC(f_{\alpha_2})) = |0.25 + 0.0055\alpha_2 - \alpha_1|. \tag{16}$$

$$
\begin{aligned}
d(f_{\alpha_2}, SC(f_{\alpha_1})) &= \| f_{\alpha_1} - SC(f_{\alpha_1}) \| \\
&= \| 1 - \alpha_2 \cos(\cdot) - SC(1 - \alpha_1 \cos(\cdot)) \| \\
&= \| 1 - \alpha_2 \cos(\cdot) - S(0.1\sqrt{\alpha_1} \cos(\cdot) - 1) \| \\
&= \| 1 - \alpha_2 \cos(\cdot) - (1 - (0.25 + 0.55(0.01\alpha_1)) \cos(\cdot)) \| \\
&= \| 1 - \alpha_2 \cos(\cdot) - (1 - (0.25 + 0.0055\alpha_1) \cos(\cdot)) \| \\
&= |0.25 + 0.0055\alpha_1 - \alpha_2| \| \cos(\cdot) \| \\
&= |0.25 + 0.0055\alpha_1 - \alpha_2|,
\end{aligned}
$$

Thus

$$d(f_{\alpha_2}, SC(f_{\alpha_1})) = |0.25 + 0.0055\alpha_1 - \alpha_2|, \tag{17}$$

$$
\begin{aligned}
d(SC(f_{\alpha_1}), SC(f_{\alpha_2})) &= \| SC(f_{\alpha_1}) - SC(f_{\alpha_2}) \| \\
&= \| SC(1 - \alpha_1 \cos(\cdot)) \\
&\quad - SC(1 - \alpha_2 \cos(\cdot)) \|
\end{aligned}
$$

$$
\begin{aligned}
&= \|S(0.1\sqrt{\alpha_1}\cos(\cdot) - 1) \\
&- S(0.1\sqrt{\alpha_2}\cos(\cdot) - 1)\| \\
&= \|(1 - (0.25 + 0.55(0.01\alpha_1))\cos(\cdot)) \\
&- (1 - (0.25 + 0.55(0.01\alpha_2))\cos(\cdot))\| \\
&= 0.0055|\alpha_1 - \alpha_2|\|\cos(\cdot)\| \\
&= 0.0055|\alpha_1 - \alpha_2|.
\end{aligned}
$$

$$
d(SC(f_{\alpha_1}), SC(f_{\alpha_2})) = 0.0055|\alpha_1 - \alpha_2|. \tag{18}
$$

For convenience of writing, let $s = \alpha_1$ and $t = \alpha_2$. Now, it is sufficient to show that from (14), (15), (16), (17), and (18)

$$
\begin{aligned}
0.1|\sqrt{s} - \sqrt{t}| \leq 0.3445|s - t| &+ 0.15|0.25 - 0.9945s| + 0.15|0.25 - 0.9945t| \\
&+0.15|0.25 + 0.0055y - s| + 0.15|0.25 + 0.0055s - t| \\
&+0.0055|s - t|.
\end{aligned}
$$

or equivalently,

$$
\begin{aligned}
0.1|\sqrt{s} - \sqrt{t}| \leq 0.3|s - t| &+ 0.15|0.25 - 0.9945s| + 0.15|0.25 - 0.9945t| \\
&+0.15|0.25 + 0.0055t - s| + 0.15|0.25 + 0.0055s - t| \\
&+0.15|s - t|.
\end{aligned}
$$

We can write it as follows:

$$
\begin{aligned}
0.1|\sqrt{s} - \sqrt{t}| \leq 0.45|s - t| &+ 0.15|0.25 - 0.9945s| + 0.15|0.25 - 0.9945t| \\
&+0.15|0.25 + 0.0055t - s| + 0.15|0.25 + 0.0055s - t|.
\end{aligned}
$$

for all $s, t \in [0, 1]$. Next, we have

$$
\begin{aligned}
U(s, t) &= 0.1|\sqrt{s} - \sqrt{t}| \\
V(s, t) &= 0.45|s - t| + 0.15|0.25 - 0.9945s| + 0.15|0.25 - 0.9945t| \\
&\quad +0.15|0.25 + 0.0055t - s| + 0.15|0.25 + 0.0055s - t|
\end{aligned}
$$

for all $s, t \in [0, 1]$.

The two surfaces $U(s, t)$ and $V(s, t)$ can be illustrated as two-dimensional and three-dimensional in Fig. 2 and B, respectively.

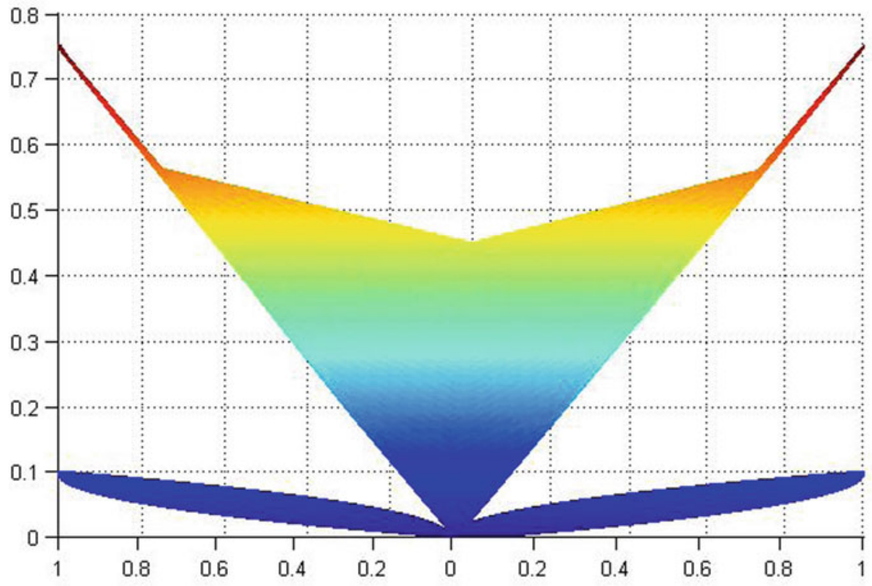
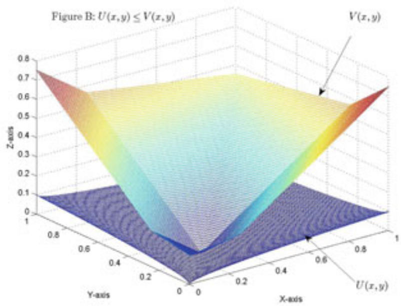


Fig. 2 Two surfaces $U(s,t)) \leq V(s,t)$ in 2-D, Two surfaces $U(s,t)) \leq V(s,t)$



Two surfaces $U(s,t)) \leq V(s,t)$.

To show the validity of (19), it is required to divide the unit square $[0,1] \times [0,1]$ of $st-$plane into six parts, for reference see Fig. 3.

Next, we will show that (19) is true on area G_1, G_2, G_3, G_4, G_5, and G_6.

(1) The area $G_1 = \{(s,t) \in [0,1]^2 \mid t \leq s \leq 0.25 + 0.0055t\}$.

Then we get $|\sqrt{s} - \sqrt{t}| = (\sqrt{s} - \sqrt{t})$, $|s - t| = (s - t)$ and $|0.25 + 0.0055t - s| = (0.25 + 0.0055t - s)$ for all $(s,t) \in G_1$.

Since $(s,t) \in G_1$ then $t \leq s \leq 0.25 + 0.0055t \leq 0.25 + 0.0055s$ which implies that $0.9945s \leq 0.25$, $0.9945t \leq 0.25$ and $t \leq 0.25 + 0.0055s$.

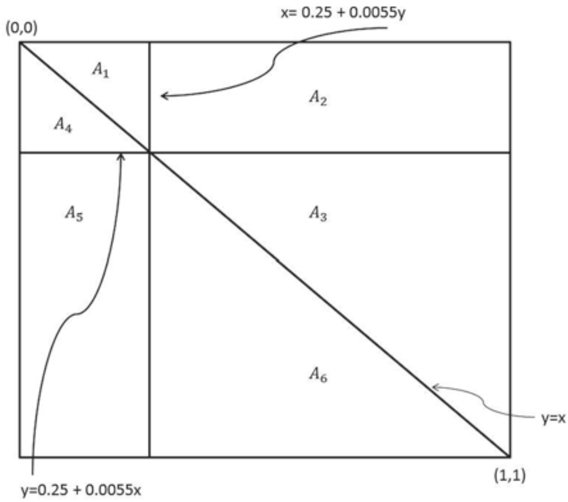


Fig. 3 $G_1 = A_1, G_2 = A_2, G_3 = A_3, G_4 = A_4, G_5 = A_5$ and $G_6 = A_7$ on the unit square

We obtain $|0.25 - 0.9945s| = (0.25 - 0.9945s)$, $|0.25 - 0.9945t| = (0.25 - 0.9945t)$, and $|0.25 + 0.0055s - t| = (0.25 + 0.0055s - t)$.

Thus, we consider

$$\begin{aligned}
0.45|s-t| + 0.15|0.25 - 0.9945s| & \\
+0.15|0.25 - 0.9945t| & \\
+0.15|0.25 + 0.0055t - s| & \\
+0.15|0.25 + 0.0055s - t| &= 0.45(s-t) + 0.15(0.25 - 0.9945s) \\
&\quad +0.15(0.25 - 0.9945t) + 0.15(0.25 + 0.0055t - s) \\
&\quad +0.15(0.25 + 0.0055s - t) \\
&= 0.45s - 0.45t + 0.0375 - 0.149175s \\
&\quad +0.0375 - 0.149175t + 0.0375 + 0.000825t - 0.15s \\
&\quad +0.0375 + 0.000825s - 0.15t \\
&= s(0.45 - 0.149175 - 0.15 + 0.000825) \\
&\quad +t(-0.45 - 0.149175 + 0.000825 - 0.15) \\
&\quad +(0.0375 + 0.0375 + 0.0375 + 0.375) \\
&= 0.15165s - 0.74835t + 0.15.
\end{aligned}$$

Thus, it is sufficient to show that

$$0.1(\sqrt{s} - \sqrt{t}) \leq 0.15165s - 0.74835t + 0.15$$

or equivalently to prove

$$0.1\sqrt{s} - 0.15165s \leq 0.1\sqrt{t} - 0.74835t + 0.15 \tag{19}$$

Let us define

$$\varphi(s) = 0.1\sqrt{s} - 0.15165s \tag{20}$$
$$\psi(t) = 0.1\sqrt{t} - 0.74835t + 0.15 \tag{21}$$

for all $s, t \in \left[0, \frac{500}{1989}\right]$. It is easy to prove that

$$\varphi'(s) = \frac{0.05}{\sqrt{s}} - 0.15165 > 0 \tag{22}$$

for all $s \in (0, \frac{500}{1989})$. Hence, φ is an increasing function on $\left[0, \frac{500}{1989}\right]$ which is implies that

$$\begin{aligned} 0.1\sqrt{s} - 0.15165s &\leq \varphi\left(\frac{500}{1989}\right) \\ &= 0.1\sqrt{\frac{500}{1989}} - 0.15165\left(\frac{500}{1989}\right) \\ &\approx 0.012016 \end{aligned}$$

for all $s \in \left[0, \frac{500}{1989}\right]$. On the other hand

$$\psi'(t) = \frac{0.05}{\sqrt{t}} - 0.74835 \tag{23}$$

for all $t \in (0, \frac{500}{1989})$. It is found that $t = (\frac{0.05}{0.74835})^2 \approx 0.004464 \in (0, \frac{500}{1989})$ is the critical point of ψ providing the absolute maximum value

$$\begin{aligned} \psi\left(\left(\frac{0.05}{0.74835}\right)^2\right) &= 0.1\left(\frac{0.05}{0.74835}\right) - 0.74835\left(\frac{0.05}{0.74835}\right)^2 + 0.15 \\ &\approx 0.153340 \end{aligned}$$

on $\left[0, \frac{500}{1989}\right]$. We can see that ψ is an increasing function on $\left[0, (\frac{0.05}{0.74835})^2\right]$ and ψ is decreasing function on $\left[(\frac{0.05}{0.74835})^2, \frac{500}{1989}\right]$. Notice that the function starting point $t = 0$ provides the value $\psi(0) = 0.15\big(> \varphi\big(\frac{500}{1989}\big) \approx 0.049728\big)$ and so $t = \frac{500}{1989}$ provides the absolute minimum value

$$\begin{aligned} \psi\left(\frac{500}{1989}\right) &= 0.1\left(\sqrt{\frac{500}{1989}}\right) - 0.74835\left(\frac{500}{1989}\right) + 0.15 \\ &\approx 0.012016 \\ &= \varphi\left(\frac{500}{1989}\right) \end{aligned}$$

on $\left[0, \frac{500}{1989}\right]$. Therefore $\psi\big(\frac{500}{1989}\big) \leq 0.1\sqrt{t} - 0.74835t + 0.15$ for all $t \in \left[0, \frac{500}{1989}\right]$.

Thus, we have

$$\begin{aligned}0.1\sqrt{s} - 0.15165s &\leq \varphi\left(\frac{500}{1989}\right)\\ &= \psi\left(\frac{500}{1989}\right)\\ &\leq 0.1\sqrt{t} - 0.74835t + 0.15\end{aligned}$$

for all $(s, t) \in G_1$.

(2) For the area $G_2 = \{(s, t) \in [0, 1]^2 \mid x \geq 0.25 + 0.0055t$ and $t \leq 0.25 + 0.0055s\}$, we can examine from G_1 to show that (19) satisfies for all $(s, t) \in G_2$.

(3) For the area $G_3 = \{(s, t) \in [0, 1]^2 \mid 0.25 + 0.0055s \leq t \leq s\}$, from G_1 we see that (19) holds for every $(s, t) \in G_3$.

It is being observed that the graph of $U(s, t)$ and $V(s, t)$ has symmetry. Therefore, (19) is also true for the areas

$$G_4 = \{(s, t) \in [0, 1]^2 \mid s \leq t \leq 0.25 + 0.0055s\}, \tag{24}$$

$$G_5 = \{(s, t) \in [0, 1]^2 \mid s \leq 0.25 + 0.0055t, t \geq 0.25 + 0.0055s\}, \tag{25}$$

$$G_6 = \{(s, t) \in [0, 1]^2 \mid 0.25 + 0.0055t \leq s \leq t\}. \tag{26}$$

So, we can say that (19) is true for every $s, t \in [0, 1]$. It proves that non-self K-generalized Kannan and Chatterjea mappings w.r.t the mapping S and constant $k_1 = 0.3, k_2 = k_3 = k_4 = 0.15$. As well as, we proved that

$$\lambda = \frac{k_1 + k_2 + k_3 + k_4}{1 - (k_1 + k_2 + k_3 + k_4)} = \frac{1 - (0.3 + 0.15 + 0.15 + 0.15)}{0.3 + 0.15 + 0.15 + 0.15} \approx 0.333 < 1. \tag{27}$$

Now, it is being proved that the pair (S, C) is a generalized $K - cyclic$ contraction. Thus, we assume

$$\begin{aligned}d(Cf_\alpha, Sg_\beta) &= \|Cf_\alpha - Sg_\beta\|\\ &= \|C(1 - \alpha\cos(\cdot)) - S(\beta\cos(\cdot)) - 1\|\\ &= \|(0.1\sqrt{\alpha}\cos(\cdot) - 1) - (1 - (0.25 + 0.55(\beta^2)\cos(\cdot))\|\\ &= \|(0.1\sqrt{\alpha} + 0.25 + 0.55\beta^2)\cos(\cdot) - 2\|\\ &= 2\end{aligned}$$

and then

$$\begin{aligned}d(f_\alpha, Cf_\alpha) + d(g_\beta, Sg_\beta) + d(f_\alpha, g_\beta) &= \|f_\alpha - Cf_\alpha\| + \|g_\beta - Sg_\beta\| + \|f_\alpha - g_\beta\|\\ &= \|1 - \alpha\cos(\cdot) - C(1 - \alpha\cos(\cdot))\|\\ &+ \|\beta\cos(\cdot) - 1 - S(\beta\cos(\cdot) - 1)\|\end{aligned}$$

$$
\begin{aligned}
&+ \|1 - \alpha\cos(\cdot) - \beta\cos(\cdot) + 1\| \\
&= \|1 - \alpha\cos(\cdot) - (0.1\sqrt{\alpha}\cos(\cdot) - 1)\| \\
&+ \|\beta\cos(\cdot) - 2 + (0.25 + 0.55\beta^2)\cos(\cdot)\| \\
&+ \|2 - (\alpha + \beta)\cos(\cdot)\| \\
&= \|2 - (\alpha + 0.1\sqrt{0.1})\cos(\cdot)\| + 2 + 2 \\
&= 6.
\end{aligned}
$$

Thus, for some $k \in [0, \frac{1}{3})$, we obtain

$$
\begin{aligned}
d(Cf_\alpha, Sg_\beta) &= 2 \\
&\leq 6k + 2(1 - 3k) \\
&= k(d(f_\alpha, Cf_\alpha) + d(g_\beta, Sg_\beta) + d(f_\alpha, g_\beta)) + (1 - 3k)d(A, B).
\end{aligned}
$$

Therefore, the pair (S, C) is a generalized $K - cyclic$ contraction. Hence proved.

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New Fixed-Figure Results on Metric Spaces

Nihal Taş and Nihal Özgür

Abstract Geometric properties of the non-unique fixed points of a self-mapping $\mathfrak{F}$ are investigated on a metric space in the framework of the fixed-figure problem. Mainly, we introduce new types of self-mappings of which fixed point set contains a certain geometric figure (e.g. an Apollonius circle, a Cassini curve, a circle, an ellipse or a hyperbola). This geometric figure is called a fixed figure (a fixed Apollonius circle, a fixed Cassini curve and so on) of the corresponding self-mapping. Then, using the classical techniques of fixed point theory and appropriate auxiliary numbers, we give new fixed-figure results. These kind geometric results are important in terms of applications in the cases of non-unique fixed points.

1 Introduction

Non-unique fixed points have been appeared in both theoretical and applied studies (see, for instance, [1, 3, 7–9, 14, 20, 31, 34, 38] and the references therein). Then, the determination of the geometric properties of non-unique fixed points appears as a natural problem in the non-unique fixed point theory and its applications. To this direction, a new approach called the *fixed-figure problem* has been studied via the known fixed point techniques in recent times. Briefly, this problem can be defined as the determination of new contractive conditions to assure a geometric figure is included in the fixed point set of a given self-mapping. Let $(\mathfrak{A}, \mathfrak{m})$ be a metric space, $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ a self-mapping on $\mathfrak{A}$. Let us consider the fixed point set $Fix(\mathfrak{F}) = \{\mathfrak{a} \in \mathfrak{A} : \mathfrak{F}\mathfrak{a} = \mathfrak{a}\}$ of $\mathfrak{F}$. First, we recall that the circle $C_{\mathfrak{a}_0,r} = \{\mathfrak{a} \in \mathfrak{A} : \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_0) = r\}$ (resp. the disc $D_{\mathfrak{a}_0,r} = \{\mathfrak{a} \in \mathfrak{A} : \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_0) \leq r\}$) is a fixed circle (resp. a fixed disc) of $\mathfrak{F}$ if $\mathfrak{F}\mathfrak{a} = \mathfrak{a}$ for all $\mathfrak{a} \in C_{\mathfrak{a}_0,r}$ (resp. for all $\mathfrak{a} \in D_{\mathfrak{a}_0,r}$) (see [19, 24, 25]). More

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generally, for a geometric figure $\mathcal{F}$, if we have $\mathcal{F} \subset Fix(\mathfrak{F})$ then $\mathcal{F}$ is called a *fixed figure* of $\mathfrak{F}$ [27]. Using the Caristi's inequality, the first solution of the fixed-figure problem was given for the fixed circle case on metric spaces in [24]. Then, several results have been investigated using appropriate auxiliary functions and approaches such as Wardowski type, Khan type, Meir-Keeler type contractions on metric and some generalized metric spaces (for instance, see [2, 5, 12, 17–19, 22, 23, 25, 28–30, 32, 33] and the references therein).

Theoretic results of the fixed point theory and geometric interpretations have found important applications in the study of neural networks. For example, in [11], by using fixed point theory, the stability of a stochastic neutral cellular neural network was considered and a new criteria for exponential stability in mean square of the considered stochastic neutral cellular neural network was obtained. Especially, Krasnoselskii fixed point theorem was used for the stability analysis. In [26], by using of a special activation function whose fixed point set is an ellipse, an application to a complex-valued Hopfield neural network (CVHNN) was given. For further studies see [1, 6, 16, 21, 31, 36, 38] and the references therein.

In some recent studies, the fixed-figure problem was considered for various special cases. The interested reader can refer to [10, 12, 13, 25, 27, 37]. In [25], some fixed-disc results were presented providing a new technique using the theory of simulation functions defined in [15] (see [4] and [15] for more details on the theory of simulation functions). In [37], new solutions were given to the fixed-circle problem (resp. fixed-disc problem) using the bilateral type contractions and the numbers

$$R_{\mathfrak{F}}(\mathfrak{a},\mathfrak{b}) = \max\left\{\mathfrak{m}(\mathfrak{a},\mathfrak{b}), \frac{\mathfrak{m}(\mathfrak{a},\mathfrak{F}\mathfrak{a})\mathfrak{m}(\mathfrak{b},\mathfrak{F}\mathfrak{b})}{\mathfrak{m}(\mathfrak{a},\mathfrak{b})}\right\} \tag{1}$$

and

$$Q_{\mathfrak{F}}(\mathfrak{a},\mathfrak{b}) = \max\left\{\mathfrak{m}(\mathfrak{a},\mathfrak{b}), \frac{(1+\mathfrak{m}(\mathfrak{a},\mathfrak{F}\mathfrak{a}))\,\mathfrak{m}(\mathfrak{b},\mathfrak{F}\mathfrak{b})}{1+\mathfrak{m}(\mathfrak{a},\mathfrak{b})}\right\}. \tag{2}$$

In [27], some fixed-ellipse theorems have been studied via the help of the numbers $M(\mathfrak{a},\mathfrak{b})$ and ρ defined by

$$M(\mathfrak{a},\mathfrak{b}) = \max\left\{\begin{array}{c}\alpha\mathfrak{m}(\mathfrak{a},\mathfrak{F}\mathfrak{a}) + (1-\alpha)\mathfrak{m}(\mathfrak{b},\mathfrak{F}\mathfrak{b}),\\ (1-\alpha)\mathfrak{m}(\mathfrak{a},\mathfrak{F}\mathfrak{a}) + \alpha\mathfrak{m}(\mathfrak{b},\mathfrak{F}\mathfrak{b}), \frac{\mathfrak{m}(\mathfrak{a},\mathfrak{F}\mathfrak{b})+\mathfrak{m}(\mathfrak{b},\mathfrak{F}\mathfrak{a})}{2}\end{array}\right\},\ 0 \le \alpha < 1, \tag{3}$$

and

$$\rho = \inf\left\{\mathfrak{m}(\mathfrak{a},\mathfrak{F}\mathfrak{a}) : \mathfrak{a} \in \mathfrak{A} \text{ and } \mathfrak{a} \notin Fix(\mathfrak{F})\right\}. \tag{4}$$

Motivated by the above studies on the set $Fix(\mathfrak{F})$ obtained by a geometric perspective, in this chapter, we define new types of contractions to obtain some fixed-figure results on metric spaces. For this, mainly, we take inspiration from the bilateral type contractions and use appropriate auxiliary numbers. Under the five subsections, we investigate new fixed-circle, fixed-ellipse, fixed-hyperbola, fixed-Cassini curve and fixed-Apollonius circle results using different techniques. Also, we mention some related consequences of the obtained geometric theorems. Theoretical results obtained in the paper are verified by illustrative examples.

2 Main Results

In this section, our main idea is to use some combined conditions and an auxiliary number. Let $\phi : \mathfrak{A} \to [0, \infty)$ and $\psi : \mathfrak{A} \times \mathfrak{A} \to [0, \infty)$ be two functions. We define new types of contractive conditions of the form

$$\mathfrak{m}\left(\mathfrak{F}\mathfrak{a}, \mathfrak{a}\right) \leq \left[\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})\right] \psi\left(\mathfrak{a}, \mathfrak{b}\right)$$

or

$$\mathfrak{m}\left(\mathfrak{F}\mathfrak{a}, \mathfrak{a}\right) \mathfrak{m}\left(\mathfrak{F}\mathfrak{a}, \mathfrak{b}\right) \leq \left[\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})\right] \psi\left(\mathfrak{a}, \mathfrak{b}\right)$$

and we use the number $r \in [0, \infty)$ defined by

$$r := \inf\left\{\frac{\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a})}{\phi(\mathfrak{a})} : \mathfrak{a} \in \mathfrak{A} \text{ and } \mathfrak{a} \notin Fix(\mathfrak{F})\right\}, \tag{5}$$

with the assumption $\phi\left(\mathfrak{a}\right) > 0$ for all $\mathfrak{a} \in \mathfrak{A}$ such that $\mathfrak{a} \notin Fix(\mathfrak{F})$.

2.1 Fixed-Circle Theorems and Related Consequences

In this subsection, we present some fixed-circle results.

Theorem 1 *Let* $(\mathfrak{A}, \mathfrak{m})$ *be a metric space,* $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ *and* r *as defined in (5) such that there exist a function* $\phi : \mathfrak{A} \to [0, \infty)$ *and a point* $\mathfrak{a}_0 \in \mathfrak{A}$ *satisfying conditions*

$$\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a})\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_0) \leq \left[\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})\right]\left(\mathfrak{m}\left(\mathfrak{a}, \mathfrak{a}_0\right)\right)^2 \tag{6}$$

and

$$\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_0) \geq r \tag{7}$$

for all $\mathfrak{a} \in \mathfrak{A}$ *with* $\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}) > 0$*. Then, we have* $\mathfrak{F}\mathfrak{a}_0 = \mathfrak{a}_0$ *and* $C_{\mathfrak{a}_0,r} \subset Fix(\mathfrak{F})$*.*

Proof If $\mathfrak{m}(\mathfrak{F}\mathfrak{a}_0, \mathfrak{a}_0) > 0$, then by (6), we get

$$\mathfrak{m}(\mathfrak{F}\mathfrak{a}_0, \mathfrak{a}_0)\mathfrak{m}(\mathfrak{F}\mathfrak{a}_0, \mathfrak{a}_0) \leq \left[\phi(\mathfrak{a}_0) - \phi(\mathfrak{F}\mathfrak{a}_0)\right]\left(\mathfrak{m}\left(\mathfrak{a}_0, \mathfrak{a}_0\right)\right)^2$$

and so

$$(\mathfrak{m}(\mathfrak{F}\mathfrak{a}_0, \mathfrak{a}_0))^2 \leq 0.$$

This implies $\mathfrak{F}\mathfrak{a}_0 = \mathfrak{a}_0$.

If $r = 0$, then we have $C_{\mathfrak{a}_0,r} = \{\mathfrak{a}_0\}$ and clearly, $C_{\mathfrak{a}_0,r} \subset Fix(\mathfrak{F})$.

Let $r > 0$ and $\mathfrak{a} \in C_{\mathfrak{a}_0,r}$ be any point such that $\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}) > 0$. By condition (6) and the definition of the number r, we obtain

$$\begin{aligned}
\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a})\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_0) &\leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\,(\mathfrak{m}\,(\mathfrak{a}, \mathfrak{a}_0))^2 \\
&= [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\,r^2 \\
&\leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\,\frac{\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a})}{\phi(\mathfrak{a})} r \\
&< \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) r
\end{aligned}$$

and hence $\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_0) < r$, a contradiction by the hypothesis (7). This contradiction leads us $\mathfrak{F}\mathfrak{a} = \mathfrak{a}$ and so, $C_{\mathfrak{a}_0,r}$ is a fixed circle of $\mathfrak{F}$.

It is clear that the disc $D_{u_0,r}$ is also fixed by $\mathfrak{F}$ in Theorem 1.

Throughout the paper, unless otherwise stated, we present our illustrative examples for self-mappings of the usual metric spaces $(\mathbb{R}, \mathfrak{m})$ or $(\mathbb{C}, \mathfrak{m})$. The following example illustrates Theorem 1.

Example 1 For any number $\alpha \in (0, \infty)$, define a self-mapping $\mathfrak{F}_a$ on $\mathbb{R}$ by

$$\mathfrak{F}_\alpha \mathfrak{a} = \begin{cases} \alpha \ ; \ \mathfrak{a} > \alpha \\ \mathfrak{a} \ ; \ \mathfrak{a} \leq \alpha \end{cases}.$$

Considering the function $\phi : \mathbb{R} \longrightarrow [0, \infty)$ defined by

$$\phi(\mathfrak{a}) = \frac{1}{\alpha} |\mathfrak{a} - \alpha|,$$

we see that $\mathfrak{F}_\alpha$ satisfies all conditions of Theorem 1 with $\mathfrak{a}_0 = 0$. Indeed, for all $\mathfrak{a} \in (\alpha, \infty)$, we obtain $\mathfrak{a} \neq \mathfrak{F}_\alpha \mathfrak{a}$ and

$$\mathfrak{m}\,(\mathfrak{F}_\alpha \mathfrak{a}, 0) = |\alpha - 0| = \alpha,$$

$$\begin{aligned}
\mathfrak{m}(\mathfrak{F}_\alpha \mathfrak{a}, \mathfrak{a})\mathfrak{m}(\mathfrak{F}_\alpha \mathfrak{a}, \mathfrak{a}_0) &= |\alpha - \mathfrak{a}|\,|\alpha - 0| = \alpha\,|\mathfrak{a} - \alpha| \\
&\leq \left[\frac{1}{\alpha} |\mathfrak{a} - \alpha| - 0\right] |\mathfrak{a} - 0|^2 \\
&= [\phi(\mathfrak{a}) - \phi(\mathfrak{F}_\alpha \mathfrak{a})]\,(\mathfrak{m}\,(\mathfrak{a}, \mathfrak{a}_0))^2\,.
\end{aligned}$$

Also we find

$$r = \inf\left\{\frac{\mathfrak{m}(\mathfrak{a}, \mathfrak{F}_\alpha\mathfrak{a})}{\phi(\mathfrak{a})} : \mathfrak{a} \neq \mathfrak{F}_\alpha\mathfrak{a},\, \mathfrak{a} \in \mathfrak{A}\right\} = \inf\left\{\frac{|\mathfrak{a}-\alpha|}{\frac{|\mathfrak{a}-\alpha|}{\alpha}} = \alpha : \mathfrak{a} > \alpha\right\} = \mathfrak{a}.$$

Clearly, we have $D_{0,\alpha} = [-\alpha, \alpha] \subset Fix\,(\mathfrak{F}_\alpha) = (-\infty, \alpha]$.

Example 2 Let $\mathbb{C}$ be the set of complex numbers and $(\mathbb{C}, \mathfrak{m})$ the usual metric space with the usual metric $\mathfrak{m}$ defined by $\mathfrak{m}(\mathfrak{z}, w) = |\mathfrak{z} - w|$ for all $\mathfrak{z}, w \in \mathbb{C}$. Define a self-mapping $\mathfrak{F} : \mathbb{C} \to \mathbb{C}$ by

$$\mathfrak{F}\mathfrak{z} = \begin{cases} \frac{9}{\overline{\mathfrak{z}}} \;;\; |\mathfrak{z}| \geq 3 \\ \mathfrak{z} \;;\; |\mathfrak{z}| < 3 \end{cases}$$

and a function $\phi : \mathbb{C} \longrightarrow [0, \infty)$ by

$$\phi(\mathfrak{z}) = \begin{cases} \frac{|\mathfrak{z}|^2-9}{3|\mathfrak{z}|} \;;\; |\mathfrak{z}| > 3 \\ 0 \quad\;;\; |\mathfrak{z}| \leq 3 \end{cases},$$

where $\overline{\mathfrak{z}} = \mathfrak{a} - i\mathfrak{b}$ is the complex conjugate of the complex number $\mathfrak{z} = \mathfrak{a} + i\mathfrak{b}$. For the complex numbers $\mathfrak{z}$ with $|\mathfrak{z}| > 3$, we have $\mathfrak{F}\mathfrak{z} \neq \mathfrak{z}$ and hence we get

$$r = \inf\left\{\frac{\mathfrak{m}(\mathfrak{z}, \mathfrak{F}\mathfrak{z})}{\phi(\mathfrak{z})} : |\mathfrak{z}| > 3\right\} = \inf\left\{\frac{\left|\frac{9}{\overline{\mathfrak{z}}} - \mathfrak{z}\right|}{\frac{|\mathfrak{z}|^2-9}{3|\mathfrak{z}|}} : |\mathfrak{z}| > 3\right\} = 3$$

and

$$\begin{aligned} \left|\frac{9}{\overline{\mathfrak{z}}} - \mathfrak{z}\right| \left|\frac{9}{\overline{\mathfrak{z}}} - 0\right| &= 9\frac{|\mathfrak{z}|^2 - 9}{|\mathfrak{z}|^2} \\ &\leq \left[\frac{|\mathfrak{z}|^2 - 9}{3\,|\mathfrak{z}|} - 0\right] |\mathfrak{z} - 0|^2 . \end{aligned}$$

This last inequality shows that $\mathfrak{F}$ satisfies condition (6) for the point $\mathfrak{z}_0 = 0$ (notice that $\left|\frac{9}{\overline{\mathfrak{z}}}\right| < 3$ when $|\mathfrak{z}| > 3$). Clearly, we have

$$Fix\,(\mathfrak{F}) = D_{0,3} = \{\mathfrak{z} \in \mathbb{C} : |\mathfrak{z}| \leq 3\},$$

that is, $\mathfrak{F}$ fixes the disc $D_{0,3}$. But, $\mathfrak{F}$ does not satisfy condition (7) of Theorem 1. Indeed, we have

$$|\mathfrak{F}\mathfrak{z} - 0| = \left|\frac{9}{\overline{\mathfrak{z}}} - 0\right| = \frac{9}{|\overline{\mathfrak{z}}|} = \frac{9}{|\mathfrak{z}|} < 3$$

for all $|\mathfrak{z}| > 3$.

This example shows that the converse statement of Theorem 1 does not hold in general.

Theorem 2 *Let* $(\mathfrak{A}, \mathfrak{m})$ *be a metric space,* $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ *and* r *as defined in (5) such that there exist a function* $\phi : \mathfrak{A} \to [0, \infty)$ *and a point* $\mathfrak{a}_0 \in \mathfrak{A}$ *satisfying*

$$\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}) \leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\, \mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_0) \tag{8}$$

and

$$\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_0) \leq \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_0) \tag{9}$$

for all $\mathfrak{a} \in \mathfrak{A}$ *with* $\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}) > 0$*. Then, all of the circles* $C_{\mathfrak{a}_0,\mu}$ *with* $\mu \leq r$ *are fixed circles of* $\mathfrak{F}$*, that is, we have* $D_{\mathfrak{a}_0,r} \subset Fix(\mathfrak{F})$*.*

Proof Let $\mathfrak{a} \in D_{\mathfrak{a}_0,r}$ be any point such that $\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}) > 0$. By conditions (8) and (9), we get

$$\begin{aligned}
\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}) &\leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\, \mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_0) \\
&\leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\, \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_0) \\
&= [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\, r \\
&\leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\, \frac{\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a})}{\phi(\mathfrak{a})} \\
&< \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}),
\end{aligned}$$

which is a contradiction. This contradiction leads us $\mathfrak{F}\mathfrak{a} = \mathfrak{a}$ and so, the disc $D_{\mathfrak{a}_0,r}$ is contained in the set $Fix(\mathfrak{F})$.

Example 3 Define a self-mapping $\mathfrak{F} : \mathbb{R} \to \mathbb{R}$ by

$$\mathfrak{F}\mathfrak{a} = \begin{cases} k\mathfrak{a} \ ; \ \mathfrak{a} \leq 0 \\ \ \mathfrak{a} \ \ ; \ \mathfrak{a} > 0 \end{cases}$$

and a function $\phi : \mathbb{R} \longrightarrow [0, \infty)$ by

$$\phi(\mathfrak{a}) = |\mathfrak{a}|\,,$$

where $k \in [0, 1]$. Then, $\mathfrak{F}$ satisfies all hypotheses of Theorem 2 with $\mathfrak{a}_0 = 1$. Indeed, we have

$$\begin{aligned}
r &= \inf \left\{ \frac{|k\mathfrak{a} - \mathfrak{a}|}{|\mathfrak{a}|} : \mathfrak{a} < 0 \right\} \\
&= 1 - k.
\end{aligned}$$

Since we have $\mathfrak{F}\mathfrak{a} \neq \mathfrak{a}$ for all $\mathfrak{a} < 0$, we get

$$\begin{aligned} |k\mathfrak{a} - \mathfrak{a}| &= (1-k)|\mathfrak{a}| \leq (1-k)|\mathfrak{a}||k\mathfrak{a} - 1| = [|\mathfrak{a}| - |k\mathfrak{a}|]|k\mathfrak{a} - 1| \\ &= [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]|k\mathfrak{a} - 1| \end{aligned}$$

and

$$|k\mathfrak{a} - 1| \leq |\mathfrak{a} - 1|.$$

Thus, both of conditions (8) and (9) are satisfied by $\mathfrak{F}$. Clearly, the set $Fix(\mathfrak{F})$ contains the disc $D_{1,1-k} = [k, 2-k]$.

We note that if a self-mapping f satisfies conditions (8) and (9) then it also satisfies condition (6). However, the converse is not true in general. We present an example of this situation.

Example 4 Define a self-mapping $\mathfrak{F} : \mathbb{R} \to \mathbb{R}$ by

$$\mathfrak{F}\mathfrak{a} = \begin{cases} \frac{1}{2}\mathfrak{a} \ ; \ \mathfrak{a} > 1 \\ \mathfrak{a} \ ; \ \mathfrak{a} \leq 1 \end{cases}$$

and a function $\phi : \mathbb{R} \longrightarrow [0, \infty)$ by

$$\phi(\mathfrak{a}) = |\mathfrak{a}|.$$

Then we have $r = \frac{1}{2}$ and $\mathfrak{F}$ satisfies all hypothesis of Theorem 1 with $\mathfrak{a}_0 = 0$. Indeed, we have

$$\left|\frac{1}{2}\mathfrak{a} - \mathfrak{a}\right|\left|\frac{1}{2}\mathfrak{a} - 0\right| \leq \left[|\mathfrak{a}| - \left|\frac{1}{2}\mathfrak{a}\right|\right]|\mathfrak{a} - 0|^2$$

and

$$\left|\frac{1}{2}\mathfrak{a} - 0\right| \geq \frac{1}{2}$$

for all $\mathfrak{a} \in (1, \infty)$. However, $\mathfrak{F}$ does not satisfy condition (8) of Theorem 2.

In the following, we define a self-mapping that satisfies all hypotheses of Theorem 2 but does not satisfy all conditions of Theorem 1.

Example 5 Define a self-mapping $\mathfrak{F} : \mathbb{R} \to \mathbb{R}$ by

$$\mathfrak{F}\mathfrak{a} = \begin{cases} -\frac{1}{2}\mathfrak{a} \ ; & \mathfrak{a} < -3 \\ \mathfrak{a} \ ; & -3 \leq \mathfrak{a} \leq 3 \\ 0 \ ; & \mathfrak{a} > 3 \end{cases}$$

and a function $\phi : \mathbb{R} \longrightarrow [0, \infty)$ by

$$\phi(\mathfrak{a}) = 2|\mathfrak{a}|.$$

Then, we have $r = \min\left\{\frac{3}{4}, \frac{1}{2}\right\} = \frac{1}{2}$ and $\mathfrak{F}$ satisfies all hypothesis of Theorem 2 for the point $\mathfrak{a}_0 = 0$. Therefore, $\mathfrak{F}$ satisfies condition (6). But, $\mathfrak{F}$ does not satisfy condition (7) of Theorem 1.

Remark 1 (1) We note that the self-mapping defined in Example 1 satisfies conditions of Theorem 2 and the self-mapping defined in Example 3 satisfies all hypothesis of Theorem 1.

(2) The self-mapping defined in Example 2 satisfies condition (9) but does not satisfy condition (8) of Theorem 2. Hence, we conclude that the converse statement of Theorem 2 does not hold in general.

2.2 Fixed-Ellipse Theorems and Related Consequences

In this subsection, we consider the ellipse $E_r(\mathfrak{a}_1, \mathfrak{a}_2)$ defined by

$$E_r(\mathfrak{a}_1, \mathfrak{a}_2) = \{\mathfrak{a} \in \mathfrak{A} : \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1) + \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2) = r\}$$

and the set

$$\overline{E}_r(\mathfrak{a}_1, \mathfrak{a}_2) = \{\mathfrak{a} \in \mathfrak{A} : \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1) + \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2) \leq r\}.$$

Clearly, we have

$$r = 0 \Rightarrow \mathfrak{a}_1 = \mathfrak{a}_2 \text{ and } E_r(\mathfrak{a}_1, \mathfrak{a}_2) = C_{\mathfrak{a}_1, r} = \{\mathfrak{a}_1\}.$$

Theorem 3 *Let* $(\mathfrak{A}, \mathfrak{m})$ *be a metric space,* $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ *and* r *as defined in (5) such that there exist a function* $\phi : \mathfrak{A} \to [0, \infty)$ *and the points* $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathfrak{A}$ *satisfying the conditions*

$$\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a})\left(\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_1) + \mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_2)\right) \leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\left(\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1) + \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)\right)^2 \tag{10}$$

and

$$\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_1) + \mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_2) \geq r \tag{11}$$

for all $\mathfrak{a} \in \mathfrak{A}$ *with* $\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}) > 0$. *Then the ellipse* $E_r(\mathfrak{a}_1, \mathfrak{a}_2)$ *is a fixed ellipse of* $\mathfrak{F}$. *Furthermore, we have* $\overline{E}_r(\mathfrak{a}_1, \mathfrak{a}_2) \subset Fix(\mathfrak{F})$.

Proof Let $\mathfrak{a} \in E_r(\mathfrak{a}_1, \mathfrak{a}_2)$ be any point such that $\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}) > 0$. Considering condition (10) and the definition of the number r, we get

$$\begin{aligned}\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a})\left(\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a}_1)+\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a}_2)\right) &\le [\phi(\mathfrak{a})-\phi(\mathfrak{F}\mathfrak{a})]\left(\mathfrak{m}(\mathfrak{a},\mathfrak{a}_1)+\mathfrak{m}(\mathfrak{a},\mathfrak{a}_2)\right)^2\\ &= [\phi(\mathfrak{a})-\phi(\mathfrak{F}\mathfrak{a})]\,r^2\\ &\le [\phi(\mathfrak{a})-\phi(\mathfrak{F}\mathfrak{a})]\,\frac{\mathfrak{m}(\mathfrak{a},\mathfrak{F}\mathfrak{a})}{\phi(\mathfrak{a})}r\\ &< \mathfrak{m}(\mathfrak{a},\mathfrak{F}\mathfrak{a})r\end{aligned}$$

and hence $\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a}_1)+\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a}_2)<r$, a contradiction by the hypothesis (11). Then, we have $\mathfrak{a}\in Fix(\mathfrak{F})$. Since $\mathfrak{a}\in E_r(\mathfrak{a}_1,\mathfrak{a}_2)$ is arbitrary, the ellipse $E_r(\mathfrak{a}_1,\mathfrak{a}_2)$ is a fixed ellipse of $\mathfrak{F}$.

Furthermore, for any $\mathfrak{a}\in\overline{E}_r(\mathfrak{a}_1,\mathfrak{a}_2)$, the above arguments are also true. Consequently, we have $\overline{E}_r(\mathfrak{a}_1,\mathfrak{a}_2)\subset Fix(\mathfrak{F})$.

Theorem 4 *Let* $(\mathfrak{A},\mathfrak{m})$ *be a metric space,* $\mathfrak{F}:\mathfrak{A}\to\mathfrak{A}$ *and* r *as defined in (5) such that there exist a function* $\phi:\mathfrak{A}\to[0,\infty)$ *and the points* $\mathfrak{a}_1,\mathfrak{a}_2\in\mathfrak{A}$ *satisfying the conditions*

$$\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a})\le[\phi(\mathfrak{a})-\phi(\mathfrak{F}\mathfrak{a})]\left(\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a}_1)+\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a}_2)\right) \tag{12}$$

and

$$\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a}_1)+\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a}_2)\le\mathfrak{m}(\mathfrak{a},\mathfrak{a}_1)+\mathfrak{m}(\mathfrak{a},\mathfrak{a}_2) \tag{13}$$

for all $\mathfrak{a}\in\mathfrak{A}$ *with* $\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a})>0$. *Then the ellipse* $E_r(\mathfrak{a}_1,\mathfrak{a}_2)$ *is a fixed ellipse of* $\mathfrak{F}$. *Furthermore, we have* $\overline{E}_r(\mathfrak{a}_1,\mathfrak{a}_2)\subset Fix(\mathfrak{F})$.

Proof For a point $\mathfrak{a}\in E_r(\mathfrak{a}_1,\mathfrak{a}_2)$ such that $\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a})>0$, using conditions (12), (13) and the definition of the number r, we get

$$\begin{aligned}\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a}) &\le [\phi(\mathfrak{a})-\phi(\mathfrak{F}\mathfrak{a})]\left(\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a}_1)+\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a}_2)\right)\\ &\le [\phi(\mathfrak{a})-\phi(\mathfrak{F}\mathfrak{a})]\left(\mathfrak{m}(\mathfrak{a},\mathfrak{a}_1)+\mathfrak{m}(\mathfrak{a},\mathfrak{a}_2)\right)\\ &= [\phi(\mathfrak{a})-\phi(\mathfrak{F}\mathfrak{a})]\,r\\ &\le [\phi(\mathfrak{a})-\phi(\mathfrak{F}\mathfrak{a})]\,\frac{\mathfrak{m}(\mathfrak{a},\mathfrak{F}\mathfrak{a})}{\varphi(\mathfrak{a})}\\ &< \mathfrak{m}(\mathfrak{a},\mathfrak{F}\mathfrak{a})\end{aligned}$$

this is a contradiction. Then, we have $\mathfrak{a}\in Fix(\mathfrak{F})$ and so, the ellipse $E_r(\mathfrak{a}_1,\mathfrak{a}_2)$ is a fixed ellipse of $\mathfrak{F}$.

Furthermore, for any $\mathfrak{a}\in\overline{E}_r(\mathfrak{a}_1,\mathfrak{a}_2)$, the above arguments are also true. Consequently, we have $\overline{E}_r(\mathfrak{a}_1,\mathfrak{a}_2)\subset Fix(\mathfrak{F})$.

Now, we obtain new fixed-ellipse results using different bilateral type contractive conditions via the numbers $R_{\mathfrak{F}}(\mathfrak{a},\mathfrak{b})$ and $Q_{\mathfrak{F}}(\mathfrak{a},\mathfrak{b})$ defined in (1) and (2), respectively.

Theorem 5 *Let* $(\mathfrak{A}, \mathfrak{m})$ *be a metric space,* $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ *a self-mapping and r defined as in (5). If there exist* $\mathfrak{a}_1, \mathfrak{a}_2 \in Fix(\mathfrak{F})$ *and a function* $\phi : \mathfrak{A} \to [0, \infty)$ *such that*

$$\mathfrak{F}\mathfrak{a} \neq \mathfrak{a} \Longrightarrow \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) \leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \left(R_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_1\right) + R_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_2)),$$

for all $\mathfrak{a} \in \mathfrak{A} - \{\mathfrak{a}_1, \mathfrak{a}_2\}$*, then the ellipse* $E_r(\mathfrak{a}_1, \mathfrak{a}_2)$ *is a fixed ellipse of* $\mathfrak{F}$*. Additionally, we have* $\overline{E}_r(\mathfrak{a}_1, \mathfrak{a}_2) \subset Fix(\mathfrak{F})$.

Proof Let $r = 0$. Then we have $\mathfrak{a}_1 = \mathfrak{a}_2$ and $E_r(\mathfrak{a}_1, \mathfrak{a}_2) = \{\mathfrak{a}_1\}$. From the hypothesis, we have $\mathfrak{F}\mathfrak{a}_1 = \mathfrak{a}_1$.

Now, assume that $r > 0$ and $\mathfrak{a} \in E_r(\mathfrak{a}_1, \mathfrak{a}_2)$ is an arbitrary point with $\mathfrak{a} \neq \mathfrak{F}\mathfrak{a}$, that is, $\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) > 0$. Using the hypothesis and the definition of the number r, we obtain

$$\begin{aligned}
\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) &\leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \left(R_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_1\right) + R_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_2)) \\
&= [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] (\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1) + \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)) \\
&= [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\, r \leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \frac{\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a})}{\phi(\mathfrak{a})} < \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}),
\end{aligned}$$

a contradiction. Thereby, we have $\mathfrak{a} \in Fix(\mathfrak{F})$ and hence, $E_r(\mathfrak{a}_1, \mathfrak{a}_2)$ is a fixed ellipse of $\mathfrak{F}$ since $\mathfrak{a} \in E_r(\mathfrak{a}_1, \mathfrak{a}_2)$ is an arbitrary point.

Similar arguments are valid for any $\mathfrak{a} \in \overline{E}_r(\mathfrak{a}_1, \mathfrak{a}_2)$ and we deduce that $\overline{E}_r(\mathfrak{a}_1, \mathfrak{a}_2) \subset Fix(\mathfrak{F})$.

Theorem 6 *Let* $(\mathfrak{A}, \mathfrak{m})$ *be a metric space,* $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ *a self-mapping and r defined as in (5). If there exist* $\mathfrak{a}_1, \mathfrak{a}_2 \in Fix(\mathfrak{F})$ *and a function* $\phi : \mathfrak{A} \to [0, \infty)$ *such that*

$$\mathfrak{F}\mathfrak{a} \neq \mathfrak{a} \Longrightarrow \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) \leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \left(Q_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_1\right) + Q_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_2)),$$

for all $\mathfrak{a} \in \mathfrak{A} - \{\mathfrak{a}_1, \mathfrak{a}_2\}$*, then* $E_r(\mathfrak{a}_1, \mathfrak{a}_2)$ *is a fixed ellipse of* $\mathfrak{F}$*. Additionally, we have* $\overline{E}_r(\mathfrak{a}_1, \mathfrak{a}_2) \subset Fix(\mathfrak{F})$.

Proof The proof is similar to that of Theorem 5 and is omitted.

As the consequences of Theorems 5 and 6, we give the following corollaries.

Corollary 1 *Let* $(\mathfrak{A}, \mathfrak{m})$ *be a metric space,* $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ *a self-mapping and r defined as in (5). If there exist* $\mathfrak{a}_1, \mathfrak{a}_2 \in Fix(\mathfrak{F})$ *and a function* $\phi : \mathfrak{A} \to [0, \infty)$ *such that*

$$\mathfrak{F}\mathfrak{a} \neq \mathfrak{a} \Longrightarrow \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) \leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \begin{pmatrix} \alpha_1 \left(\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1) + \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)\right) \\ +\alpha_2 \left(\frac{\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a})\mathfrak{m}(\mathfrak{a}_1, \mathfrak{F}\mathfrak{a}_1)}{\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1)} + \frac{\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a})\mathfrak{m}(\mathfrak{a}_2, \mathfrak{F}\mathfrak{a}_2)}{\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)} \right) \end{pmatrix},$$

for all $\mathfrak{a} \in \mathfrak{A} - \{\mathfrak{a}_1, \mathfrak{a}_2\}$*, where* α_1, α_2 *are two nonnegative real numbers with a sum* 1*, then* $E_r(\mathfrak{a}_1, \mathfrak{a}_2)$ *is a fixed ellipse of* $\mathfrak{F}$ *and* $\overline{E}_r(\mathfrak{a}_1, \mathfrak{a}_2) \subset Fix(\mathfrak{F})$.

Proof Considering the inequality

$$\alpha_1 \left(\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1) + \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)\right) + \alpha_2 \left(\frac{\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a})\mathfrak{m}(\mathfrak{a}_1, \mathfrak{F}\mathfrak{a}_1)}{\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1)} + \frac{\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a})\mathfrak{m}(\mathfrak{a}_2, \mathfrak{F}\mathfrak{a}_2)}{\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)} \right)$$
$$\leq R_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_1) + R_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_2),$$

the proof follows easily.

Corollary 2 *Let* $(\mathfrak{A}, \mathfrak{m})$ *be a metric space,* $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ *a self-mapping and* r *defined as in (5). If there exist* $\mathfrak{a}_1, \mathfrak{a}_2 \in Fix(\mathfrak{F})$ *and a function* $\phi : \mathfrak{A} \to [0, \infty)$ *such that*

$$\mathfrak{F}\mathfrak{a} \neq \mathfrak{a} \Longrightarrow \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) \leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \left(\begin{array}{c} \alpha_1 \left(\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1) + \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)\right) \\ +\alpha_2 \left(\frac{(1+\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}))\mathfrak{m}(\mathfrak{a}_1, \mathfrak{F}\mathfrak{a}_1)}{1+\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1)} + \frac{(1+\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}))\mathfrak{m}(\mathfrak{a}_2, \mathfrak{F}\mathfrak{a}_2)}{1+\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)} \right) \end{array} \right),$$

for all $\mathfrak{a} \in \mathfrak{A} - \{\mathfrak{a}_1, \mathfrak{a}_2\}$ *, where* α_1, α_2 *are two nonnegative real numbers with a sum* 1*, then* $E_r(\mathfrak{a}_1, \mathfrak{a}_2)$ *is a fixed ellipse of* $\mathfrak{F}$ *and* $\overline{E}_r(\mathfrak{a}_1, \mathfrak{a}_2) \subset Fix(\mathfrak{F})$*.*

Proof Using the inequality

$$\alpha_1 \left(\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1) + \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)\right) + \alpha_2 \left(\frac{(1 + \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}))\mathfrak{m}(\mathfrak{a}_1, \mathfrak{F}\mathfrak{a}_1)}{1 + \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1)} + \frac{(1 + \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}))\mathfrak{m}(\mathfrak{a}_2, \mathfrak{F}\mathfrak{a}_2)}{1 + \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)} \right)$$
$$\leq Q_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_1) + Q_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_2),$$

the proof follows easily.

Corollary 3 *Let* $(\mathfrak{A}, \mathfrak{m})$ *be a metric space,* $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ *a self-mapping and* r *defined as in (5). If there exist* $\mathfrak{a}_1, \mathfrak{a}_2 \in Fix(\mathfrak{F})$ *and a function* $\phi : \mathfrak{A} \to [0, \infty)$ *such that*

$$\mathfrak{F}\mathfrak{a} \neq \mathfrak{a} \Longrightarrow \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) \leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \left(\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1) + \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)\right),$$

for all $\mathfrak{a} \in \mathfrak{A} - \{\mathfrak{a}_1, \mathfrak{a}_2\}$ *, then* $E_r(\mathfrak{a}_1, \mathfrak{a}_2)$ *is a fixed ellipse of* $\mathfrak{F}$ *and* $\overline{E}_r(\mathfrak{a}_1, \mathfrak{a}_2) \subset Fix(\mathfrak{F})$*.*

Proof Using the inequalities,

$$\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1) + \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2) \leq R_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_1) + R_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_2)$$

and

$$\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1) + \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2) \leq Q_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_1) + Q_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_2),$$

we can easily derive this corollary.

2.3 Fixed-Hyperbola Theorems and Related Consequences

In this subsection, we study the fixed hyperbola case of the fixed-figure problem with the assumption $r > 0$. We consider the hyperbola $H_r(\mathfrak{a}_1, \mathfrak{a}_2)$ defined by

$$H_r(\mathfrak{a}_1, \mathfrak{a}_2) = \{\mathfrak{a} \in \mathfrak{A} : |\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1) - \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)| = r\}$$

and the set

$$\overline{H}_r(\mathfrak{a}_1, \mathfrak{a}_2) = \{\mathfrak{a} \in \mathfrak{A} : |\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1) - \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)| \leq r\}.$$

Theorem 7 *Let* $(\mathfrak{A}, \mathfrak{m})$ *be a metric space,* $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ *and* r *as defined in (5) such that there exist a function* $\phi : \mathfrak{A} \to [0, \infty)$ *and the points* $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathfrak{A}$ *satisfying the conditions*

$$\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a})\,|\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_1) - \mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_2)| \leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\,|\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1) - \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)|^2 \tag{14}$$

and

$$|\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_1) - \mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_2)| \geq r \tag{15}$$

for all $\mathfrak{a} \in \mathfrak{A}$ *with* $\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}) > 0$*. Then the hyperbola* $H_r(\mathfrak{a}_1, \mathfrak{a}_2)$ *is a fixed hyperbola of* $\mathfrak{F}$*. Furthermore, we have* $\overline{H}_r(\mathfrak{a}_1, \mathfrak{a}_2) \subset Fix(\mathfrak{F})$*.*

Proof Let $r > 0$ and $\mathfrak{a} \in H_r(\mathfrak{a}_1, \mathfrak{a}_2)$ be any point such that $\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}) > 0$. By condition (14) and the definition of the number r, we obtain

$$\begin{aligned}
\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a})\,|\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_1) - \mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_2)| &\leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\,|\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1) - \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)|^2 \\
&= [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\,r^2 \\
&\leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\,\frac{\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a})}{\phi(\mathfrak{a})} r \\
&< \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) r
\end{aligned}$$

and hence $|\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_1) - \mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_2)| < r$. This is a contradiction by the hypothesis (15). Then, we have $\mathfrak{F}\mathfrak{a} = \mathfrak{a}$ and hence, the hyperbola $H_r(\mathfrak{a}_1, \mathfrak{a}_2)$ is a fixed hyperbola of $\mathfrak{F}$.

Clearly, the above arguments are also true for any $\mathfrak{a} \in \overline{H}_r(\mathfrak{a}_1, \mathfrak{a}_2)$. Consequently, we have $\overline{H}_r(\mathfrak{a}_1, \mathfrak{a}_2) \subset Fix(\mathfrak{F})$.

Theorem 8 *Let* $(\mathfrak{A}, \mathfrak{m})$ *be a metric space,* $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ *and* r *as defined in (5) such that there exist a function* $\phi : \mathfrak{A} \to [0, \infty)$ *and the points* $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathfrak{A}$ *satisfying the conditions*

$$\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}) \leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\,|\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_1) - d(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_2)| \tag{16}$$

and

$$|\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_1) - \mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_2)| \leq |\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1) - \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)| \tag{17}$$

for all $\mathfrak{a} \in \mathfrak{A}$ *with* $\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}) > 0$. *Then the hyperbola* $H_r(\mathfrak{a}_1, \mathfrak{a}_2)$ *is a fixed hyperbola of* $\mathfrak{F}$. *Furthermore, we have* $\overline{H}_r(\mathfrak{a}_1, \mathfrak{a}_2) \subset Fix(\mathfrak{F})$.

Proof Let $r > 0$ and $\mathfrak{a} \in H_r(\mathfrak{a}_1, \mathfrak{a}_2)$ be any point such that $\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}) > 0$. Then by the definition of the number r and using conditions (16), (17), we get

$$\begin{aligned}\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}) &\leq [\phi(\mathfrak{a}) - \varphi(\mathfrak{F}\mathfrak{a})] \left|\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_1) - \mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_2)\right| \\ &\leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \left|\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1) - \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)\right| \\ &= [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\, r \\ &\leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \frac{\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a})}{\phi(\mathfrak{a})} \\ &< \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}),\end{aligned}$$

which is a contradiction. Then, we have $\mathfrak{a} \in Fix(\mathfrak{F})$ and so, the hyperbola $H_r(\mathfrak{a}_1, \mathfrak{a}_2)$ is a fixed hyperbola of $\mathfrak{F}$.

Furthermore, for any $\mathfrak{a} \in \overline{H}_r(\mathfrak{a}_1, \mathfrak{a}_2)$, the above arguments are also true. Consequently, we have $\overline{H}_r(\mathfrak{a}_1, \mathfrak{a}_2) \subset Fix(\mathfrak{F})$.

Now, we obtain fixed-hyperbola results by means of new bilateral type contractive conditions.

Theorem 9 *Let* $(\mathfrak{A}, \mathfrak{m})$ *be a metric space,* $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ *a self-mapping and the number r defined as in (5). If there exist* $\mathfrak{a}_1, \mathfrak{a}_2 \in Fix(\mathfrak{F})$ *and a function* $\phi : \mathfrak{A} \to [0, \infty)$ *such that*

$$\mathfrak{F}\mathfrak{a} \neq \mathfrak{a} \Longrightarrow \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) \leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \left|R_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_1) - R_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_2)\right|,$$

for all $\mathfrak{a} \in \mathfrak{A} - \{\mathfrak{a}_1, \mathfrak{a}_2\}$, *then* $H_r(\mathfrak{a}_1, \mathfrak{a}_2)$ *is a fixed hyperbola of* $\mathfrak{F}$. *Furthermore, we have* $\overline{H}_r(\mathfrak{a}_1, \mathfrak{a}_2) \subset Fix(\mathfrak{F})$.

Proof Let $r > 0$ and $\mathfrak{a} \in H_r(\mathfrak{a}_1, \mathfrak{a}_2)$ be any point such that $\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) > 0$. Using the hypothesis and the definition of the number r, we obtain

$$\begin{aligned}\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) &\leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \left|R_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_1) - R_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_2)\right| \\ &= [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \left|\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1) - \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)\right| \\ &= [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\, r \leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \frac{\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a})}{\phi(\mathfrak{a})} < \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}),\end{aligned}$$

a contradiction. This contradiction implies $\mathfrak{F}\mathfrak{a} = \mathfrak{a}$ and so, $H_r(\mathfrak{a}_1, \mathfrak{a}_2)$ is a fixed hyperbola of $\mathfrak{F}$. Clearly, similar arguments are also valid for any $\mathfrak{a} \in \overline{H}_r(\mathfrak{a}_1, \mathfrak{a}_2)$ and hence we have $\overline{H}_r(\mathfrak{a}_1, \mathfrak{a}_2) \subset Fix(\mathfrak{F})$.

Theorem 10 *Let* $(\mathfrak{A}, \mathfrak{m})$ *be a metric space,* $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ *a self-mapping and r defined as in (5). If there exist* $\mathfrak{a}_1, \mathfrak{a}_2 \in Fix(f)$ *and a function* $\phi : \mathfrak{A} \to [0, \infty)$ *such that*

$$\mathfrak{F}\mathfrak{a} \neq \mathfrak{a} \Longrightarrow \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) \leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \left|Q_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_1) - Q_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_2)\right|,$$

for all $\mathfrak{a} \in \mathfrak{A} - \{\mathfrak{a}_1, \mathfrak{a}_2\}$*, then* $H_r(\mathfrak{a}_1, \mathfrak{a}_2)$ *is a fixed hyperbola of* $\mathfrak{F}$*. Furthermore, we have* $\overline{H}_r(\mathfrak{a}_1, \mathfrak{a}_2) \subset Fix(\mathfrak{F})$.

Proof The proof is omitted.

Corollary 4 *Let* $(\mathfrak{A}, \mathfrak{m})$ *be a metric space,* $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ *a self-mapping and r defined as in (5). If there exist* $\mathfrak{a}_1, \mathfrak{a}_2 \in Fix(\mathfrak{F})$ *and a function* $\phi : \mathfrak{A} \to [0, \infty)$ *such that* (i)

$$\mathfrak{F}\mathfrak{a} \neq \mathfrak{a} \Longrightarrow \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) \leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \begin{pmatrix} \alpha_1 \left|\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1) - \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)\right| \\ +\alpha_2 \left| \frac{\mathfrak{m}(\mathfrak{a},\mathfrak{F}\mathfrak{a})\mathfrak{m}(\mathfrak{a}_1,\mathfrak{F}\mathfrak{a}_1)}{\mathfrak{m}(\mathfrak{a},\mathfrak{a}_1)} - \frac{\mathfrak{m}(\mathfrak{a},\mathfrak{F}\mathfrak{a})\mathfrak{m}(\mathfrak{a}_2,\mathfrak{F}\mathfrak{a}_2)}{\mathfrak{m}(\mathfrak{a},\mathfrak{a}_2)} \right| \end{pmatrix},$$

for all $\mathfrak{a} \in \mathfrak{A} - \{\mathfrak{a}_1, \mathfrak{a}_2\}$*, where* α_1, α_2 *are two nonnegative real numbers with a sum* 1*, or* (ii)

$$\mathfrak{F}\mathfrak{a} \neq \mathfrak{a} \Longrightarrow \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) \leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \begin{pmatrix} \alpha_1 \left|\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1) - \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)\right| \\ +\alpha_2 \left| \frac{(1+\mathfrak{m}(\mathfrak{a},\mathfrak{F}\mathfrak{a}))\mathfrak{m}(\mathfrak{a}_1,\mathfrak{F}\mathfrak{a}_1)}{1+\mathfrak{m}(\mathfrak{a},\mathfrak{a}_1)} - \frac{(1+\mathfrak{m}(\mathfrak{a},\mathfrak{F}\mathfrak{a}))\mathfrak{m}(\mathfrak{a}_2,\mathfrak{F}\mathfrak{a}_2)}{1+\mathfrak{m}(\mathfrak{a},\mathfrak{a}_2)} \right| \end{pmatrix},$$

for all $\mathfrak{a} \in \mathfrak{A} - \{\mathfrak{a}_1, \mathfrak{a}_2\}$*, where* α_1, α_2 *are two nonnegative real numbers with a sum* 1*, or* (iii)

$$\mathfrak{F}\mathfrak{a} \neq \mathfrak{a} \Longrightarrow \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) \leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \left|\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1) - \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)\right|,$$

for all $\mathfrak{a} \in \mathfrak{A} - \{\mathfrak{a}_1, \mathfrak{a}_2\}$,
then $H_r(\mathfrak{a}_1, \mathfrak{a}_2)$ *is a fixed hyperbola of* $\mathfrak{F}$ *and* $\overline{H}_r(\mathfrak{a}_1, \mathfrak{a}_2) \subset Fix(\mathfrak{F})$.

Proof Using the definitions of the numbers $R_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{b})$ and $Q_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{b})$, the proof follows easily.

Remark 2 Let us examine the two cases below:

1. First, we consider the case $r = 0$ in detail. We have

$$H_r(\mathfrak{a}_1, \mathfrak{a}_2) = \begin{cases} \{\mathfrak{a} \in \mathfrak{A} : \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1) = \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)\} & \text{if } \mathfrak{a}_1 \neq \mathfrak{a}_2 \\ \mathfrak{A} & \text{if } \mathfrak{a}_1 = \mathfrak{a}_2 \end{cases}.$$

The hyperbola $H_r(\mathfrak{a}_1, \mathfrak{a}_2)$ does not have to consist of a single element. For example, let us consider the discrete metric on $\mathbb{R}$. Hence we get the followings:
• If $\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1) = \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2) = 0$ implies $\mathfrak{a}_1 = \mathfrak{a}_2$.
• If $\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1) = \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2) = 1$ implies $\mathfrak{a} \neq \mathfrak{a}_1$ and $\mathfrak{a} \neq \mathfrak{a}_2$.
Hence, we find

$$H_r(\mathfrak{a}_1, \mathfrak{a}_2) = \begin{cases} \mathbb{R} - \{\mathfrak{a}_1, \mathfrak{a}_2\} & \text{if } \mathfrak{a}_1 \neq \mathfrak{a}_2 \\ \mathbb{R} & \text{if } \mathfrak{a}_1 = \mathfrak{a}_2 \end{cases}.$$

2. If the conditions of Theorems 7–10 are satisfied in the case $r = 0$, then we deduce that the hyperbola $H_r(u_1, u_2)$ is fixed by $\mathfrak{F}$.

Example 6 Define the self-mapping $\mathfrak{F}$ by

$$\mathfrak{F}\mathfrak{a} = \begin{cases} 1 \; ; \; \mathfrak{a} > 1 \\ \mathfrak{a} \; ; \; \mathfrak{a} \leq 1 \end{cases} .$$

Considering the function ϕ defined by

$$\phi(\mathfrak{a}) = |\mathfrak{a}|,$$

we have

$$\begin{aligned} r &= \inf \left\{ \frac{\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a})}{\phi(\mathfrak{a})} : \mathfrak{a} \in \mathfrak{A} \text{ and } \mathfrak{a} \notin Fix(\mathfrak{F}) \right\} \\ &= \inf \left\{ \frac{|\mathfrak{a} - 1|}{|\mathfrak{a}|} : \mathfrak{a} > 1 \right\} = 0. \end{aligned}$$

Now we consider the hyperbola

$$H_r(-1, 1) = \{\mathfrak{a} \in \mathbb{R} : |u + 1| = |u - 1|\} = \{0\}.$$

Clearly, we have $H_r(-1, 1) \subset Fix(f) = (-\infty, 1]$. It is easy to check that conditions of Theorems 7–10 are satisfied by $\mathfrak{F}$.

2.4 Fixed-Cassini Curve Theorems and Related Consequences

In this subsection, we consider the Cassini curve $C_r(\mathfrak{a}_1, \mathfrak{a}_2)$ defined by

$$C_r(\mathfrak{a}_1, \mathfrak{a}_2) = \{\mathfrak{a} \in \mathfrak{A} : \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1)\, \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2) = r\}$$

and the set

$$\overline{C}_r(\mathfrak{a}_1, \mathfrak{a}_2) = \{\mathfrak{a} \in \mathfrak{A} : \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1)\, \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2) \leq r\}.$$

We present new fixed-figure results for the cases in which the set $Fix(\mathfrak{F})$ contains a Cassini curve.

Theorem 11 *Let* $(\mathfrak{A}, \mathfrak{m})$ *be a metric space,* $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ *and r as defined in (5) such that there exist a function* $\phi : \mathfrak{A} \to [0, \infty)$ *and the points* $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathfrak{A}$ *satisfying the conditions*

$$\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a})\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_1)\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_2) \leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\, (\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1)\, \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2))^2 \quad (18)$$

and

$$\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_1)\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_2) \geq r \tag{19}$$

for all $\mathfrak{a} \in \mathfrak{A}$ *with* $\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}) > 0$. *Then we have* $\overline{C}_r(\mathfrak{a}_1, \mathfrak{a}_2) \subset Fix(\mathfrak{F})$.

Proof Let $\mathfrak{a} \in \overline{C}_r(\mathfrak{a}_1, \mathfrak{a}_2)$ be any point such that $\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}) > 0$. By condition (18) and the definition of the number r, we obtain

$$\begin{aligned}
\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a})\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_1)\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_2) &\leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \left(\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1)\, \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)\right)^2 \\
&\leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\, r^2 \\
&\leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \frac{\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a})}{\phi(\mathfrak{a})} r \\
&< d(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) r
\end{aligned}$$

and hence $\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_1)\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_2) < r$, a contradiction by the hypothesis (19). Then, we have $\mathfrak{a} \in Fix(\mathfrak{F})$ and this implies $\overline{C}_r(\mathfrak{a}_1, \mathfrak{a}_2) \subset Fix(\mathfrak{F})$ since $\mathfrak{a}$ is an arbitrary point of $\overline{C}_r(\mathfrak{a}_1, \mathfrak{a}_2)$.

Theorem 12 *Let* $(\mathfrak{A}, \mathfrak{m})$ *be a metric space,* $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ *and r as defined in (5) such that there exist a function* $\phi : \mathfrak{A} \to [0, \infty)$ *and the points* $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathfrak{A}$ *satisfying the conditions*

$$\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}) \leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\, \mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_1)\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_2) \tag{20}$$

and

$$\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_1)\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_2) \leq \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1)\, \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2) \tag{21}$$

for all $\mathfrak{a} \in \mathfrak{A}$ *with* $\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}) > 0$. *Then we have* $\overline{C}_r(\mathfrak{a}_1, \mathfrak{a}_2) \subset Fix(\mathfrak{F})$.

Proof For a point $\mathfrak{a} \in \overline{C}_r(\mathfrak{a}_1, \mathfrak{a}_2)$ such that $\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}) > 0$, using conditions (20), (21) and the definition of the number r, we get

$$\begin{aligned}
\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}) &\leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\, \mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_1)\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_2) \\
&\leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\, \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1)\, \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2) \\
&= [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\, r \\
&\leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \frac{\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a})}{\phi(\mathfrak{a})} \\
&< \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}),
\end{aligned}$$

a contradiction. This contradiction implies $\mathfrak{F}\mathfrak{a} = \mathfrak{a}$ and so, the set $\overline{C}_r(\mathfrak{a}_1, \mathfrak{a}_2)$ contained in the set $Fix\,(\mathfrak{F})$.

Now, using new bilateral type contractive conditions, we obtain more fixed Cassini curve results.

Theorem 13 *Let* $(\mathfrak{A}, \mathfrak{m})$ *be a metric space,* $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ *a self-mapping and* r *defined as in (5). If there exist* $\mathfrak{a}_1, \mathfrak{a}_2 \in Fix(\mathfrak{F})$ *and a function* $\phi : \mathfrak{A} \to [0, \infty)$ *such that*

$$\mathfrak{F}\mathfrak{a} \neq \mathfrak{a} \Longrightarrow \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) \leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \left(R_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_1)\right) R_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_2)),$$

for all $\mathfrak{a} \in \mathfrak{A} - \{\mathfrak{a}_1, \mathfrak{a}_2\}$*, then* $C_r(\mathfrak{a}_1, \mathfrak{a}_2)$ *is a fixed Cassini curve of* $\mathfrak{F}$ *and* $\overline{C}_r(\mathfrak{a}_1, \mathfrak{a}_2) \subset Fix(\mathfrak{F})$.

Proof Let $r = 0$. Then we have

$$C_r(\mathfrak{a}_1, \mathfrak{a}_2) = \begin{cases} \{\mathfrak{a}_1, \mathfrak{a}_2\} \text{ if } \mathfrak{a}_1 \neq \mathfrak{a}_2 \\ \{\mathfrak{a}_1\} \quad \text{ if } \mathfrak{a}_1 = \mathfrak{a}_2 \end{cases},$$

that is, $C_r(\mathfrak{a}_1, \mathfrak{a}_2) \subseteq \{\mathfrak{a}_1, \mathfrak{a}_2\}$. From the hypothesis, we know $\mathfrak{F}\mathfrak{a}_1 = \mathfrak{a}_1$ and $\mathfrak{F}\mathfrak{a}_2 = \mathfrak{a}_2$.

Let $r > 0$ and $\mathfrak{a} \in C_r(\mathfrak{a}_1, \mathfrak{a}_2)$ be any point with $\mathfrak{a} \neq \mathfrak{F}\mathfrak{a}$. Using the hypothesis and the definition of the number r, we get

$$\begin{aligned} \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) &\leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \left(R_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_1)\right) R_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_2)) \\ &= [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] (\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1)\, \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)) \\ &= [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\, r \leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \frac{\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a})}{\phi(\mathfrak{a})} < \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}), \end{aligned}$$

a contradiction. So, we have $\mathfrak{a} \in Fix(\mathfrak{F})$ and consequently, $C_r(\mathfrak{a}_1, \mathfrak{a}_2)$ is a fixed Cassini curve of $\mathfrak{F}$. Clearly, by similar arguments, we get $\overline{C}_r(\mathfrak{a}_1, \mathfrak{a}_2) \subset Fix(\mathfrak{F})$.

Theorem 14 *Let* $(\mathfrak{A}, \mathfrak{m})$ *be a metric space,* $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ *a self-mapping and* r *defined as in (5). If there exist* $\mathfrak{a}_1, \mathfrak{a}_2 \in Fix(\mathfrak{F})$ *and a function* $\phi : \mathfrak{A} \to [0, \infty)$ *such that*

$$\mathfrak{F}\mathfrak{a} \neq \mathfrak{a} \Longrightarrow \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) \leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \left(Q_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_1)\right) Q_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_2)),$$

for all $\mathfrak{a} \in \mathfrak{A} - \{\mathfrak{a}_1, \mathfrak{a}_2\}$*, then* $C_r(\mathfrak{a}_1, \mathfrak{a}_2)$ *is a fixed Cassini curve of* $\mathfrak{F}$ *and* $\overline{C}_r(\mathfrak{a}_1, \mathfrak{a}_2) \subset Fix(\mathfrak{F})$.

Proof The proof is omitted.

Corollary 5 *Let* $(\mathfrak{A}, \mathfrak{m})$ *be a metric space,* $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ *a self-mapping and* r *defined as in (5). If there exist* $\mathfrak{a}_1, \mathfrak{a}_2 \in Fix(\mathfrak{F})$ *and a function* $\phi : \mathfrak{A} \to [0, \infty)$ *such that* (i)

$$\mathfrak{F}\mathfrak{a} \neq \mathfrak{a} \Longrightarrow \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) \leq [\varphi(\mathfrak{a}) - \varphi(\mathfrak{F}\mathfrak{a})] \begin{pmatrix} \alpha_1 \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1)\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2) \\ +\alpha_2 \frac{\mathfrak{m}(\mathfrak{a},\mathfrak{F}\mathfrak{a})\mathfrak{m}(\mathfrak{a}_1,\mathfrak{F}\mathfrak{a}_1)}{\mathfrak{m}(\mathfrak{a},\mathfrak{a}_1)} \frac{\mathfrak{m}(\mathfrak{a},\mathfrak{F}\mathfrak{a})\mathfrak{m}(\mathfrak{a}_2\mathfrak{F}\mathfrak{a}_2)}{\mathfrak{m}(\mathfrak{a},\mathfrak{a}_2)} \end{pmatrix},$$

for all $\mathfrak{a} \in \mathfrak{A} - \{\mathfrak{a}_1, \mathfrak{a}_2\}$ *, where* α_1, α_2 *are two nonnegative real numbers with a sum* 1*, or* (ii)

$$\mathfrak{F}\mathfrak{a} \neq \mathfrak{a} \Longrightarrow \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) \leq [\varphi(\mathfrak{a}) - \varphi(\mathfrak{F}\mathfrak{a})] \begin{pmatrix} \alpha_1 \mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1)\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2) \\ +\alpha_2 \frac{(1+\mathfrak{m}(\mathfrak{a},\mathfrak{F}\mathfrak{a}))\mathfrak{m}(\mathfrak{a}_1,\mathfrak{F}\mathfrak{a}_1)}{1+\mathfrak{m}(\mathfrak{a},\mathfrak{a}_1)} \frac{(1+\mathfrak{m}(\mathfrak{a},\mathfrak{F}\mathfrak{a}))\mathfrak{m}(\mathfrak{a}_2,\mathfrak{F}\mathfrak{a}_2)}{1+\mathfrak{m}(\mathfrak{a},\mathfrak{a}_2)} \end{pmatrix},$$

for all $\mathfrak{a} \in \mathfrak{A} - \{\mathfrak{a}_1, \mathfrak{a}_2\}$*, where* α_1, α_2 *are two nonnegative real numbers with a sum* 1*, or* (iii)

$$\mathfrak{F}\mathfrak{a} \neq \mathfrak{a} \Longrightarrow \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) \leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] (\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1)\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)),$$

for all $\mathfrak{a} \in \mathfrak{A} - \{\mathfrak{a}_1, \mathfrak{a}_2\}$ *with* $\mathfrak{F}\mathfrak{a}_1 = \mathfrak{a}_1$*,* $\mathfrak{F}\mathfrak{a}_2 = \mathfrak{a}_2$*,*
then $C_r(\mathfrak{a}_1, \mathfrak{a}_2)$ *is a fixed Cassini curve of* $\mathfrak{F}$ *and* $\overline{C}_r(\mathfrak{a}_1, \mathfrak{a}_2) \subset Fix(\mathfrak{F})$*.*

Proof We can easily prove this corollary using the definitions of the numbers $R_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{b})$ and $Q_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{b})$.

2.5 Fixed-Apollonius Circle Theorems and Related Consequences

In this subsection, we consider the Apollonius circle $A_r(\mathfrak{a}_1, \mathfrak{a}_2)$ defined by

$$A_r(\mathfrak{a}_1, \mathfrak{a}_2) = \left\{\mathfrak{a} \in \mathfrak{A} - \{\mathfrak{a}_2\} : \frac{\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1)}{\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)} = r\right\}$$

and the set

$$\overline{A}_r(\mathfrak{a}_1, \mathfrak{a}_2) = \left\{\mathfrak{a} \in \mathfrak{A} - \{\mathfrak{a}_2\} : \frac{\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1)}{\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)} \leq r\right\}.$$

Theorem 15 *Let* $(\mathfrak{A}, \mathfrak{m})$ *be a metric space,* $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ *and r as defined in (5) such that there exist a function* $\phi : \mathfrak{A} \to [0, \infty)$ *and the points* $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathfrak{A}$ *satisfying the conditions*

$$\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a})\frac{\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_1)}{\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_2)} \leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \left(\frac{\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1)}{\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)}\right)^2 \tag{22}$$

and

$$\frac{\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_1)}{\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}_2)} \geq r \tag{23}$$

for all $\mathfrak{a} \in \mathfrak{A} - \{\mathfrak{a}_2\}$ *with* $\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}) > 0$*. If* $\mathfrak{F}\mathfrak{a} \neq \mathfrak{a}_2$ *for all* $\mathfrak{a} \in A_\mu(\mathfrak{a}_1, \mathfrak{a}_2)$*, then,* $\mathfrak{F}$ *fixes the Apollonius circle* $A_\mu(\mathfrak{a}_1, \mathfrak{a}_2)$ *with* $\mu \leq r$*.*

Proof Let $\mathfrak{a} \in A_\mu(\mathfrak{a}_1, \mathfrak{a}_2)$ be any point such that $\mathfrak{m}(\mathfrak{F}\mathfrak{a}, \mathfrak{a}) > 0$. By condition (22) and the definition of the number r, we obtain

$$\begin{aligned}\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a})\frac{\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a}_1)}{\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a}_2)} &\leq [\phi(\mathfrak{a})-\phi(\mathfrak{F}\mathfrak{a})]\left(\frac{\mathfrak{m}(\mathfrak{a},\mathfrak{a}_1)}{\mathfrak{m}(\mathfrak{a},\mathfrak{a}_2)}\right)^2\\ &= [\phi(\mathfrak{a})-\phi(\mathfrak{F}\mathfrak{a})]\,r^2\\ &\leq [\phi(\mathfrak{a})-\phi(\mathfrak{F}\mathfrak{a})]\,\frac{\mathfrak{m}(\mathfrak{a},\mathfrak{F}\mathfrak{a})}{\phi(\mathfrak{a})}r\\ &< d(\mathfrak{a},\mathfrak{F}\mathfrak{a})r\end{aligned}$$

and hence $\frac{\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a}_1)}{\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a}_2)} < r$, a contradiction by the hypothesis (23). This shows that $\mathfrak{F}\mathfrak{a}=\mathfrak{a}$ and hence, the Apollonius circle $A_\mu(\mathfrak{a}_1,\mathfrak{a}_2)$ is fixed by $\mathfrak{F}$.

We note that the set $\overline{A}_r(\mathfrak{a}_1,\mathfrak{a}_2)$ is also fixed by $\mathfrak{F}$ in Theorem 15.

Theorem 16 *Let* $(\mathfrak{A},\mathfrak{m})$ *be a metric space,* $\mathfrak{F}:\mathfrak{A}\to\mathfrak{A}$ *and* r *as defined in (5) such that there exist a function* $\phi:\mathfrak{A}\to[0,\infty)$ *and the points* $\mathfrak{a}_1,\mathfrak{a}_2\in\mathfrak{A}$ *satisfying the conditions*

$$\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a})\leq[\phi(\mathfrak{a})-\phi(\mathfrak{F}\mathfrak{a})]\,\frac{\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a}_1)}{\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a}_2)} \tag{24}$$

and

$$\frac{\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a}_1)}{\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a}_2)}\leq\frac{\mathfrak{m}(\mathfrak{a},\mathfrak{a}_1)}{\mathfrak{m}(\mathfrak{a},\mathfrak{a}_2)} \tag{25}$$

for all $\mathfrak{a}\in\mathfrak{A}-\{\mathfrak{a}_2\}$ *with* $\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a})>0$. *If* $\mathfrak{F}\mathfrak{a}\neq\mathfrak{a}_2$ *for all* $\mathfrak{a}\in A_\mu(\mathfrak{a}_1,\mathfrak{a}_2)$*, then,* $\mathfrak{F}$ *fixes the Apollonius circle* $A_\mu(\mathfrak{a}_1,\mathfrak{a}_2)$ *with* $\mu\leq r$.

Proof Let $\mathfrak{a}\in A_\mu(\mathfrak{a}_1,\mathfrak{a}_2)$ be any point such that $\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a})>0$. By conditions (24) and (25), we get

$$\begin{aligned}\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a}) &\leq [\phi(\mathfrak{a})-\phi(\mathfrak{F}\mathfrak{a})]\,\frac{\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a}_1)}{\mathfrak{m}(\mathfrak{F}\mathfrak{a},\mathfrak{a}_2)}\\ &\leq [\phi(\mathfrak{a})-\phi(\mathfrak{F}\mathfrak{a})]\,\frac{\mathfrak{m}(\mathfrak{a},\mathfrak{a}_1)}{\mathfrak{m}(\mathfrak{a},\mathfrak{a}_2)}\\ &= [\phi(\mathfrak{a})-\phi(\mathfrak{F}\mathfrak{a})]\,r\\ &\leq [\phi(\mathfrak{a})-\phi(\mathfrak{F}\mathfrak{a})]\,\frac{\mathfrak{m}(\mathfrak{a},\mathfrak{F}\mathfrak{a})}{\phi(\mathfrak{a})}\\ &< \mathfrak{m}(\mathfrak{a},\mathfrak{F}\mathfrak{a}),\end{aligned}$$

which is a contradiction. This contradiction implies $\mathfrak{a}\in Fix(\mathfrak{F})$ and therefore, $\mathfrak{F}$ fixes the Apollonius circle $A_\mu(\mathfrak{a}_1,\mathfrak{a}_2)$.

It is clear from the proof of Theorem 16 that the set $\overline{A}_r(\mathfrak{a}_1,\mathfrak{a}_2)$ is also fixed by $\mathfrak{F}$ in Theorem 16.

Now, we investigate some fixed-Apollonius circle results by means of new bilateral type contractive conditions.

Theorem 17 *Let* $(\mathfrak{A}, \mathfrak{m})$ *be a metric space,* $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ *self-mapping and* r *defined as in (5). If there exist* $\mathfrak{a}_1, \mathfrak{a}_2 \in Fix(\mathfrak{F})$ *and a function* $\phi : \mathfrak{A} \to [0, \infty)$ *such that*

$$\mathfrak{F}\mathfrak{a} \neq \mathfrak{a} \Longrightarrow \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) \leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \frac{R_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_1)}{R_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_2)},$$

for all $\mathfrak{a} \in \mathfrak{A} - \{\mathfrak{a}_1, \mathfrak{a}_2\}$*, then* $A_r(\mathfrak{a}_1, \mathfrak{a}_2)$ *is a fixed Apollonius circle of* $\mathfrak{F}$ *and* $\overline{A}_r(\mathfrak{a}_1, \mathfrak{a}_2) \subset Fix(\mathfrak{F})$.

Proof Let $r = 0$. Then we have $A_r(\mathfrak{a}_1, \mathfrak{a}_2) = \{\mathfrak{a}_1\}$ and so using the hypothesis, we have $\mathfrak{F}\mathfrak{a}_1 = \mathfrak{a}_1$.

Let $r > 0$ and $\mathfrak{a} \in A_r(\mathfrak{a}_1, \mathfrak{a}_2)$ be an arbitrary point with $\mathfrak{a} \neq \mathfrak{F}\mathfrak{a}$. Using the hypothesis, we get

$$\begin{aligned}\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) &\leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \frac{R_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_1)}{R_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_2)} = [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \frac{\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1)}{\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)} \\ &= [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\, r \leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \frac{\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a})}{\phi(\mathfrak{a})} < \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}),\end{aligned}$$

a contradiction. This contradiction implies that $\mathfrak{a} = \mathfrak{F}\mathfrak{a}$ and consequently, $A_r(\mathfrak{a}_1, \mathfrak{a}_2)$ is a fixed Apollonius circle of $\mathfrak{F}$. By similar arguments, we deduce that $\overline{A}_r(\mathfrak{a}_1, \mathfrak{a}_2) \subset Fix(\mathfrak{F})$.

Theorem 18 *Let* $(\mathfrak{A}, \mathfrak{m})$ *be a metric space,* $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ *a self-mapping and* r *defined as in (5). If there exist* $\mathfrak{a}_1, \mathfrak{a}_2 \in Fix(\mathfrak{F})$ *and a function* $\phi : \mathfrak{A} \to [0, \infty)$ *such that*

$$\mathfrak{F}\mathfrak{a} \neq \mathfrak{a} \Longrightarrow \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) \leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \frac{Q_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_1)}{Q_{\mathfrak{F}}(\mathfrak{a}, \mathfrak{a}_2)},$$

for all $\mathfrak{a} \in \mathfrak{A} - \{\mathfrak{a}_1, \mathfrak{a}_2\}$*, then* $A_r(\mathfrak{a}_1, \mathfrak{a}_2)$ *is a fixed Apollonius circle of* $\mathfrak{F}$ *and* $\overline{A}_r(\mathfrak{a}_1, \mathfrak{a}_2) \subset Fix(\mathfrak{F})$.

Proof The proof is omitted.

As a consequence of Theorems 17 and 18, we give the following corollary.

Corollary 6 *Let* $(\mathfrak{A}, \mathfrak{m})$ *be a metric space,* $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ *a self-mapping and* r *defined as in (5). If there exist* $\mathfrak{a}_1, \mathfrak{a}_2 \in Fix(\mathfrak{F})$ *and a function* $\phi : \mathfrak{A} \to [0, \infty)$ *such that*

$$\mathfrak{F}\mathfrak{a} \neq \mathfrak{a} \Longrightarrow \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) \leq [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \frac{\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_1)}{\mathfrak{m}(\mathfrak{a}, \mathfrak{a}_2)},$$

for all $\mathfrak{a} \in \mathfrak{A} - \{\mathfrak{a}_1, \mathfrak{a}_2\}$*, then* $A_r(\mathfrak{a}_1, \mathfrak{a}_2)$ *is a fixed Apollonius circle of* $\mathfrak{F}$ *and* $\overline{A}_r(\mathfrak{a}_1, \mathfrak{a}_2) \subset Fix(\mathfrak{F})$.

Proof Using the following inequalities

$$\frac{\mathfrak{m}\,(\mathfrak{a},\mathfrak{a}_1)}{\mathfrak{m}\,(\mathfrak{a},\mathfrak{a}_2)} \leq \frac{R_{\mathfrak{F}}(\mathfrak{a},\mathfrak{a}_1)}{R_{\mathfrak{F}}(\mathfrak{a},\mathfrak{a}_2)}$$

and

$$\frac{\mathfrak{m}\,(\mathfrak{a},\mathfrak{a}_1)}{\mathfrak{m}\,(\mathfrak{a},\mathfrak{a}_2)} \leq \frac{Q_{\mathfrak{F}}(\mathfrak{a},\mathfrak{a}_1)}{Q_{\mathfrak{F}}(\mathfrak{a},\mathfrak{a}_2)},$$

we can easily prove this result.

2.6 Some Illustrative Examples

We emphasize the validity and importance of the main results obtained in the paper by presenting some illustrative examples. First, in the following example, we define a self-mapping that satisfies all conditions of Theorems 3, 4, 11 and 12.

Example 7 Define a self-mapping $\mathfrak{F} : \mathbb{C} \longrightarrow \mathbb{C}$ by

$$\mathfrak{F}\mathfrak{z} = \begin{cases} \overline{\mathfrak{z}} \ ; \ y > 2 \\ \mathfrak{z} \ ; \ y \leq 2 \end{cases}, \tag{26}$$

for all $\mathfrak{z} = x + iy \in \mathbb{C}$. Considering the function $\phi : \mathbb{C} \longrightarrow [0, \infty)$ defined by

$$\phi(\mathfrak{z}) = \begin{cases} \frac{2}{5}\,|y| \ ; \ y > 2 \\ \ \ 0 \ \ \ ; \ y \leq 2 \end{cases},$$

for all $\mathfrak{z} = x + iy$, we see that $\mathfrak{F}$ satisfies conditions (12) and (13) for the points $\mathfrak{z}_1 = -2$ and $\mathfrak{z}_2 = 2$. Indeed, for all $\mathfrak{z} = x + iy \in \mathbb{C}$ with $y > 2$, we have $\mathfrak{z} \neq \mathfrak{F}\mathfrak{z}$ and

$$|\overline{\mathfrak{z}} + 2| + |\overline{\mathfrak{z}} - 2| = |\mathfrak{z} + 2| + |\mathfrak{z} - 2| > 5,$$

$$\begin{aligned} |\mathfrak{z} - \overline{\mathfrak{z}}| = 2\,|y| &\leq \frac{2}{5}\,|y|\,(|\overline{\mathfrak{z}} + 2| + |\overline{\mathfrak{z}} - 2|) \\ &= \left[\frac{2}{5}\,|y| - 0\right](|\overline{\mathfrak{z}} + 2| + |\mathfrak{z} - 2|) \\ &= [\phi(\mathfrak{z}) - \phi(f\mathfrak{z})]\,(|\overline{\mathfrak{z}} + 2| + |\overline{\mathfrak{z}} - 2|)\,. \end{aligned}$$

Hence, $\mathfrak{F}$ also satisfies conditions (10) and (11). Also, we find

$$r = \inf\left\{\frac{|\mathfrak{z}-\bar{\mathfrak{z}}|}{\frac{2}{5}|y|} : y > 2\right\}$$
$$= \inf\left\{\frac{2|y|}{\frac{2}{5}|y|} : y > 2\right\} = 5.$$

Obviously, the set $Fix(\mathfrak{F})$ contains the ellipse $E = E_5(-2, 2)$ with the equation

$$E : |\mathfrak{z}+2| + |\mathfrak{z}-2| = 5.$$

Clearly, $\mathfrak{F}$ satisfies all conditions of Theorems 11 and 12 since we have

$$|\bar{\mathfrak{z}}+2| . |\bar{\mathfrak{z}}-2| = |\mathfrak{z}+2| . |\mathfrak{z}-2| > 5,$$

for all $\mathfrak{z} = x + iy \in \mathbb{C}$ with $y > 2$. The set $Fix(\mathfrak{F})$ contains also the Cassini curve $C = C_5(-2, 2)$ with the equation

$$C : |\mathfrak{z}+2| \, |\mathfrak{z}-2| = 5.$$

In Fig. 1, which is drawn using Mathematica [39], the sets $\{\mathfrak{z} \in \mathbb{C} : |\mathfrak{z}+2| + |\mathfrak{z}-2| \leq 5\}$ and $\{\mathfrak{z} \in \mathbb{C} : |\mathfrak{z}+2| \, |\mathfrak{z}-2| \leq 5\}$ contained in $Fix(\mathfrak{F})$ can be seen.

In the following example, we define a self-mapping that satisfying all conditions of Theorems 7 and 8.

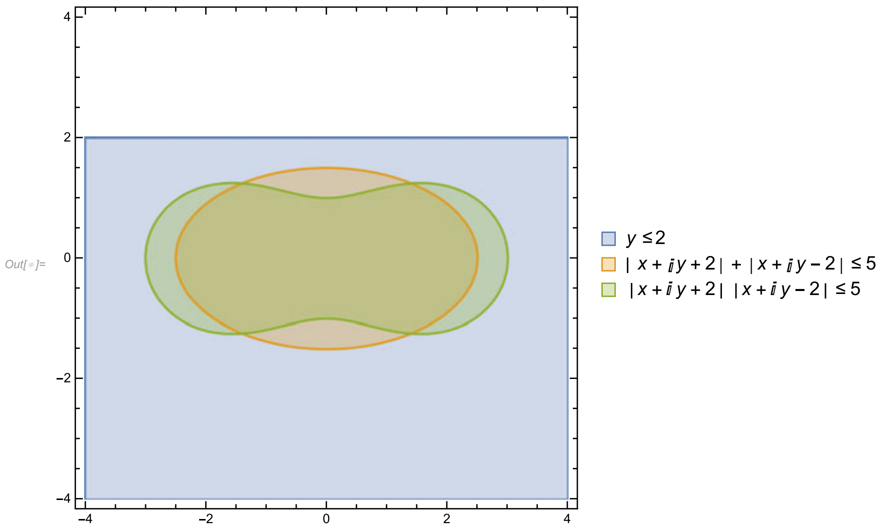

Fig. 1 The set $Fix(\mathfrak{F})$ of the self-mapping $\mathfrak{F}$ defined in (26)

Example 8 Define the self-mapping $\mathfrak{F} : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\mathfrak{F}\mathfrak{a} = \begin{cases} \mathfrak{a} & ; \mathfrak{a} \geq -1 \\ -\mathfrak{a} & ; \mathfrak{a} < -1 \end{cases} \tag{27}$$

and a function $\phi : \mathbb{R} \longrightarrow [0, \infty)$ by

$$\phi(\mathfrak{a}) = \begin{cases} 2|\mathfrak{a}| & ; \mathfrak{a} < -1 \\ 0 & ; \mathfrak{a} \geq -1 \end{cases} .$$

We find

$$\begin{aligned} r &= \inf\left\{\frac{|-\mathfrak{a} - \mathfrak{a}|}{2|\mathfrak{a}|} : \mathfrak{a} < -1\right\} \\ &= \inf\{1 : \mathfrak{a} < -1\} = 1. \end{aligned}$$

For all $\mathfrak{a} \in \mathbb{R}$ with $\mathfrak{a} < -1$, we have $\mathfrak{a} \neq \mathfrak{F}\mathfrak{a}$ and

$$||-\mathfrak{a} + 1| + |-\mathfrak{a} - 1|| = ||\mathfrak{a} + 1| + |\mathfrak{a} - 1|| = 2 > 1,$$

$$\begin{aligned} |-\mathfrak{a} - \mathfrak{a}| &= 2|\mathfrak{a}| \leq 4|\mathfrak{a}| \\ &= [2|\mathfrak{a}| - 0]\,2 \\ &= [\phi(\mathfrak{a}) - \phi(f\mathfrak{a})]\,||-\mathfrak{a} + 1| + |-\mathfrak{a} - 1||\,. \end{aligned}$$

Then, $\mathfrak{F}$ satisfies the conditions (16) and (17) for the points $\mathfrak{a}_1 = -1$ and $\mathfrak{a}_2 = 1$. Hence, $\mathfrak{F}$ also satisfies the conditions (14) and (15). Clearly, the set $Fix(\mathfrak{F}) = [-1, \infty)$ contains the hyperbola H with the equation

$$H_1(-1, 1) = \{\mathfrak{a} \in \mathbb{R} : ||\mathfrak{a} + 1| - |\mathfrak{a} - 1|| = 1\} = \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$

and the set

$$\overline{H}_1(-1, 1) = \{\mathfrak{a} \in \mathbb{R} : ||\mathfrak{a} + 1| - |\mathfrak{a} - 1|| \leq 1\} = \left\{\mathfrak{a} \in \mathbb{R} : |\mathfrak{a}| \leq \frac{1}{2}\right\}.$$

Example 9 Define a self-mapping $\mathfrak{F} : \mathbb{C} \longrightarrow \mathbb{C}$ by

$$\mathfrak{F}\mathfrak{z} = \begin{cases} \bar{\mathfrak{z}} & ; \left|\mathfrak{z} - \frac{2+9i}{3}\right| < \frac{4}{3} \\ \mathfrak{z} & ; \left|\mathfrak{z} - \frac{2+9i}{3}\right| \geq \frac{4}{3} \end{cases} , \tag{28}$$

and a function $\phi : \mathbb{C} \longrightarrow [0, \infty)$ by

$$\phi(\mathfrak{z}) = \begin{cases} |y| \; ; & \left|\mathfrak{z} - \frac{2+9i}{3}\right| < \frac{4}{3} \\ 0 \; ; & \left|\mathfrak{z} - \frac{2+9i}{3}\right| \geq \frac{4}{3} \end{cases},$$

for all $\mathfrak{z} = x + iy \in \mathbb{C}$. We find

$$\begin{aligned} r &= \inf \left\{ \frac{|\mathfrak{z} - \bar{\mathfrak{z}}|}{|y|} : \left|\mathfrak{z} - \frac{2+9i}{3}\right| < \frac{4}{3} \right\} \\ &= \inf \left\{ \frac{2\,|y|}{|y|} : \left|\mathfrak{z} - \frac{2+9i}{3}\right| < \frac{4}{3} \right\} = 2. \end{aligned}$$

Observe that the conditions of Theorems 15 and 16 are not satisfied by $\mathfrak{F}$ for the points $\mathfrak{z}_1 = -2 + 3i$ and $\mathfrak{z}_2 = 3i$. However, the set $Fix(\mathfrak{F})$ contains the Apollonius circle A with the equation

$$A : \frac{|\mathfrak{z} + 2 - 3i|}{|\mathfrak{z} - 3i|} = 2. \tag{29}$$

We note that the Apollonius circle A with the Eq. (29) is the Euclidean circle with equation $\left|\mathfrak{z} - \frac{2+9i}{3}\right| = \frac{4}{3}$ and we have

$$Fix(\mathfrak{F}) = \left\{ \mathfrak{z} \in \mathbb{C} : \frac{|\mathfrak{z} + 2 - 3i|}{|\mathfrak{z} - 3i|} \leq 2 \right\} = \left\{ \mathfrak{z} \in \mathbb{C} : \left|\mathfrak{z} - \frac{2+9i}{3}\right| \geq \frac{4}{3} \right\}.$$

The set $Fix(\mathfrak{F})$ can be seen in Fig. 2.

The above example indicates that the converses of Theorems 15 and 16 do not hold in general. Now, we present an example of a self-mapping satisfying all hypotheses of Theorems 15 and 16.

Example 10 Let $\mathfrak{a}_1 = -1$, $\mathfrak{a}_2 = 1$ and define a self-mapping $\mathfrak{F} : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\mathfrak{F}\mathfrak{a} = \begin{cases} 3 \; ; & \mathfrak{a} \in \left(\frac{1}{3}, 3\right) \\ \mathfrak{a} \; ; & \mathfrak{a} \in \left(-\infty, \frac{1}{3}\right] \cup [3, \infty) \end{cases} \tag{30}$$

and a function $\phi : \mathbb{R} \longrightarrow [0, \infty)$ by

$$\phi(\mathfrak{a}) = \begin{cases} \frac{|\mathfrak{a}-3|}{2} \; ; & \mathfrak{a} \in \left(\frac{1}{3}, 3\right) \\ 0 \; ; & \mathfrak{a} \in \left(-\infty, \frac{1}{3}\right] \cup [3, \infty) \end{cases}.$$

We find

$$\begin{aligned} r &= \inf \left\{ \frac{|\mathfrak{a} - 3|}{\frac{|\mathfrak{a}-3|}{2}} : \mathfrak{a} \in \left(\frac{1}{3}, 3\right) \right\} \\ &= \inf \left\{ 2 : \mathfrak{a} \in \left(\frac{1}{3}, 3\right) \right\} = 2. \end{aligned}$$

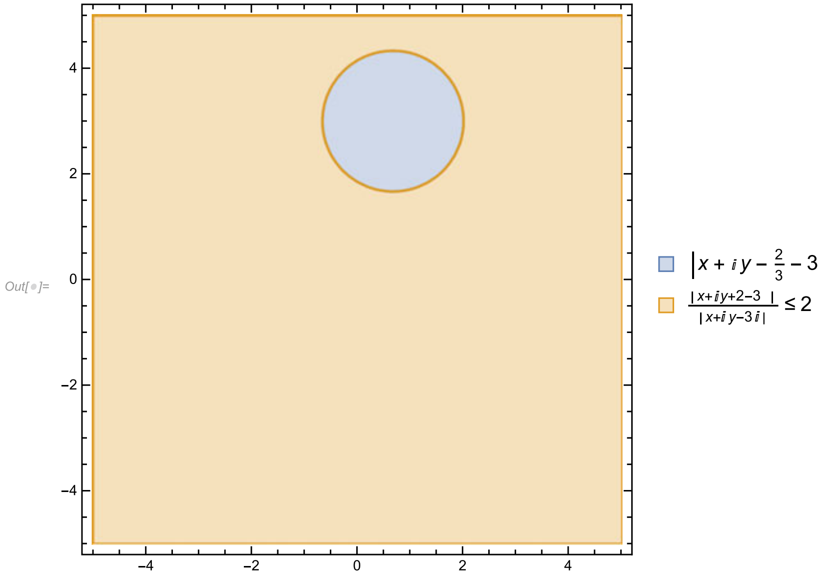

Fig. 2 The set $Fix(\mathfrak{F})$ of the self-mapping $\mathfrak{F}$ defined in (28)

For all $\mathfrak{a} \in \left(\frac{1}{3}, 3\right)$, we have $\mathfrak{a} \neq \mathfrak{F}\mathfrak{a}$ and

$$|\mathfrak{a} - 3| = \left[\frac{|\mathfrak{a} - 3|}{2} - 0\right].2$$
$$= [\phi(\mathfrak{a}) - \phi(3)]\frac{|3+2|}{|3-1|},$$

$$2 = \frac{|3+2|}{|3-1|} < \frac{|\mathfrak{a}+1|}{|\mathfrak{a}-1|}.$$

Then the conditions (24) and (25) are satisfied by $\mathfrak{F}$ for the points $\mathfrak{a}_1 = -1$ and $\mathfrak{a}_2 = 1$. Hence, $\mathfrak{F}$ also satisfies conditions (22) and (23).

Clearly, the set $Fix(\mathfrak{F}) = \left(-\infty, \frac{1}{3}\right] \cup [3, \infty)$ contains the Apollonius circle $A_2(-1, 1)$ with the equation

$$A_2(-1,1) = \left\{\mathfrak{a} \in \mathbb{R} : \frac{|\mathfrak{a}+1|}{|\mathfrak{a}-1|} = 2\right\} = \left\{\frac{1}{3}, 3\right\}$$

and the set

$$\overline{A}_2(-1,1) = \left\{\mathfrak{a} \in \mathbb{R} : \frac{|\mathfrak{a}+1|}{|\mathfrak{a}-1|} \leq 2\right\} = \left(-\infty, \frac{1}{3}\right] \cup [3, \infty).$$

We present a self-mapping satisfying conditions of Theorem 5 (resp. Theorems 6, 9, 10, 13, 14, 17 and 18) as follows:

Example 11 Define a self-mapping $\mathfrak{F} : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\mathfrak{F}\mathfrak{a} = \begin{cases} \mathfrak{a}\ ; & \mathfrak{a} \in [-2, 2] \\ 0\ ; & \mathfrak{a} \in (-\infty, 2) \cup (2, \infty) \end{cases},$$

for all $\mathfrak{a} \in \mathbb{R}$, and a function $\phi : \mathfrak{A} \longrightarrow [0, \infty)$ defined by

$$\phi(\mathfrak{a}) = \begin{cases} 1 \ ; & \mathfrak{a} \in [-2, 2] \\ 4\,|\mathfrak{a}|\ ; & \mathfrak{a} \in (-\infty, 2) \cup (2, \infty) \end{cases},$$

for all $\mathfrak{a} \in \mathbb{R}$. Then,

▹ $\mathfrak{F}$ satisfies all hypotheses of Theorem 5 with the points $\mathfrak{a}_1 = -\frac{1}{10}$ and $\mathfrak{a}_2 = \frac{1}{10}$. Indeed, we obtain

$$\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) = |\mathfrak{a}| > 0$$

and

$$\begin{aligned} \mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) = |\mathfrak{a}| &\leq [4\,|\mathfrak{a}| - 1] \left(\left| \mathfrak{a} + \frac{1}{10} \right| + \left| \mathfrak{a} - \frac{1}{10} \right| \right) \\ &= [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})] \left(R_{\mathfrak{F}} \left(\mathfrak{a}, -\frac{1}{10} \right) + R_{\mathfrak{F}} \left(\mathfrak{a}, \frac{1}{10} \right) \right), \end{aligned}$$

for all $\mathfrak{a} \in (-\infty, 2) \cup (2, \infty)$. Also,we have

$$\mathfrak{F}\left(-\frac{1}{10}\right) = -\frac{1}{10}, \mathfrak{F}\left(\frac{1}{10}\right) = \frac{1}{10}$$

and we get

$$\begin{aligned} r &= \inf \left\{ \frac{\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a})}{\phi(\mathfrak{a})} : \mathfrak{a} \neq \mathfrak{F}\mathfrak{a}, \mathfrak{a} \in \mathbb{R} \right\} \\ &= \inf \left\{ \frac{|\mathfrak{a}|}{4\,|\mathfrak{a}|} : \mathfrak{a} \in (-\infty, 2) \cup (2, \infty) \right\} = \frac{1}{4}. \end{aligned}$$

Similarly, $\mathfrak{F}$ satisfies all conditions of Theorem 6 with the points $\mathfrak{a}_1 = -\frac{1}{10}$, $\mathfrak{a}_2 = \frac{1}{10}$. Consequently, $E_{\frac{1}{4}}\left(-\frac{1}{10}, \frac{1}{10}\right) = \left\{-\frac{1}{8}, \frac{1}{8}\right\}$ is a fixed ellipse of $\mathfrak{F}$.

▹ $\mathfrak{F}$ satisfies all conditions of Theorem 9 with the points $\mathfrak{a}_1 = -1$ and $\mathfrak{a}_2 = 1$. Indeed, we obtain

$$\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) = |\mathfrak{a}| > 0$$

and

$$\begin{aligned}\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) &= |\mathfrak{a}| \leq [4|\mathfrak{a}| - 1]\,||\mathfrak{a}+1| - |\mathfrak{a}-1|| \\ &= [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\left|R_{\mathfrak{F}}(\mathfrak{a}, -1) - R_{\mathfrak{F}}(\mathfrak{a}, 1)\right|,\end{aligned}$$

for all $\mathfrak{a} \in (-\infty, 2) \cup (2, \infty)$. Also,we have

$$\mathfrak{F}(-1) = -1, \mathfrak{F}(1) = 1$$

and

$$r = \frac{1}{4}.$$

Similarly, $\mathfrak{F}$ satisfies conditions of Theorem 10 with the points $\mathfrak{a}_1 = -1$, $\mathfrak{a}_2 = 1$. Consequently, $H_{\frac{1}{4}}(-1, 1) = \left\{-\frac{1}{8}, \frac{1}{8}\right\}$ is a fixed hyperbola of $\mathfrak{F}$.

▹ $\mathfrak{F}$ satisfies all conditions of Theorem 13 with the points $\mathfrak{a}_1 = -1$ and $\mathfrak{a}_2 = 1$. Indeed, we have

$$\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) = |\mathfrak{a}| > 0$$

and

$$\begin{aligned}\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) &= |\mathfrak{a}| \leq [4|\mathfrak{a}| - 1]\,(|\mathfrak{a}+1|\,|\mathfrak{a}-1|) \\ &= [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\left(R_{\mathfrak{F}}(\mathfrak{a}, -1)\, R_{\mathfrak{F}}(\mathfrak{a}, 1)\right),\end{aligned}$$

for all $\mathfrak{a} \in (-\infty, 2) \cup (2, \infty)$. Also,we have

$$\mathfrak{F}(-1) = -1, \mathfrak{F}(1) = 1$$

and

$$r = \frac{1}{4}.$$

Similarly, $\mathfrak{F}$ satisfies all conditions of Theorem 14 with the points $\mathfrak{a}_1 = -1$, $\mathfrak{a}_2 = 1$. Consequently, $C_{\frac{1}{4}}(-1, 1) = \left\{-\frac{\sqrt{5}}{2}, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{5}}{2}\right\}$ is a fixed Cassini curve of $\mathfrak{F}$.

▹ $\mathfrak{F}$ satisfies all conditions of Theorem 17 with the points $\mathfrak{a}_1 = -1$ and $\mathfrak{a}_2 = 1$. Indeed, we have

$$\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) = |\mathfrak{a}| > 0$$

and

$$\begin{aligned}\mathfrak{m}(\mathfrak{a}, \mathfrak{F}\mathfrak{a}) &= |\mathfrak{a}| \leq [4|\mathfrak{a}| - 1]\,\frac{|\mathfrak{a}+1|}{|\mathfrak{a}-1|} \\ &= [\phi(\mathfrak{a}) - \phi(\mathfrak{F}\mathfrak{a})]\,\frac{R_{\mathfrak{F}}(\mathfrak{a}, -1)}{R_{\mathfrak{F}}(\mathfrak{a}, 1)},\end{aligned}$$

for all $\mathfrak{a} \in (-\infty, 2) \cup (2, \infty)$. Also, we obtain

$$\mathfrak{F}(-1) = -1, \mathfrak{F}(1) = 1$$

and

$$r = \frac{1}{4}.$$

Similarly, $\mathfrak{F}$ satisfies all conditions of Theorem 18 with the points $\mathfrak{a}_1 = -1, \mathfrak{a}_2 = 1$. Consequently, $A_{\frac{1}{4}}(-1, 1) = \left\{-\frac{5}{3}, -\frac{3}{5}\right\}$ is a fixed Apollonius circle of $\mathfrak{F}$.

Remark 3 The self-mapping $\mathfrak{F}$ defined in Example 11 also fixes the circle $C_{0,2} = \{-2, 2\}$ and the disc $D_{0,2} = [-2, 2]$. Thereby, we say that $\mathfrak{F}$ fixes at least six figures such as an Apollonius circle, a circle, a disc, an ellipse, a hyperbola and a Cassini curve.

Example 12 Let $\mathfrak{A} = \left\{-3, -\sqrt{5}, -2, -1, 0, \frac{3}{5}, 1, \frac{5}{3}, 2, \sqrt{5}, 3, 4\right\}$ and $(\mathfrak{A}, \mathfrak{m})$ be a metric space with the usual metric $\mathfrak{m}$. If we consider the self-mapping $\mathfrak{F}$ defined as

$$\mathfrak{F}\mathfrak{a} = \begin{cases} \mathfrak{a} \ ; \ \mathfrak{a} \in \left\{-\sqrt{5}, -2, \frac{3}{5}, \frac{5}{3}, 2, \sqrt{5}\right\} \\ 0 \ ; \quad \ \mathfrak{a} \in \{-3, -1, 0, 1, 3, 4\} \end{cases},$$

for all $\mathfrak{a} \in \mathfrak{A}$. Then $\mathfrak{F}$ fixes the ellipse $E_4(-1, 1) = \{-2, 2\}$, the hyperbola $H_4(-3, 3) = \{-2, 2\}$, the Cassini curve $C_4(-1, 1) = \{-\sqrt{5}, \sqrt{5}\}$ and the Apollonius circle $A_4(-1, 1) = \left\{\frac{3}{5}, \frac{5}{3}\right\}$, but f does not satisfies the conditions of Theorem 5 (resp. Theorems 6, 9, 10, 13, 14, 17 and 18).

Remark 4 If we consider Example 12, we see that the converse statement of Theorem 5 (resp. Theorems 6, 9, 10, 13, 14, 17 and 18) is not always true in general.

3 Conclusion

We have investigated the fixed-figure problem for some special cases. Non-unique fixed point results and geometric methods are important for both theoretical and applied studies. For example, in [1], it was stated that the number of fixed points of a Boolean network is a key feature of its dynamical behaviour. In [31], the notion of an Apollonius circle was used to present a geometric method in data point analysis. It was noted that the proposed method using Apollonius geometric and subtended arc methods have higher efficiency than the new algorithms on the majority of real data sets (see [31] for more details). Such studies show the efficiency of the non-unique fixed point results and geometric methods.

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Some Fixed Point Results for Suzuki $W-$Contractions Involving Quadratic Terms in Modular $b-$Metric Spaces

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Abstract In this study, we established some common fixed point theorems for two self-mappings and obtained some results for a self-mapping in the frame of modular $b-$metric space, which includes the $W-$contractive type condition introduced by Wardowski [15]. Also, some examples are given to illustrate the usability of the acquired consequences.

1 Introduction and Preliminaries

Throughout the study, the following illustrations are used:

- $\mathbf{N}$: the set of all positive natural numbers,
- $\mathbf{R}_+$: the set of all non-negative real numbers.

Let $\Im$ be a non-void set and $Q, \Lambda : \Im \to \Im$ be self-mappings. For an element ξ of $\Im$, it is defined as a fixed point of the mapping Q and the common fixed point of the mappings Q and Λ, if the following expressions are provided, respectively.

- $Q\xi = \xi$,
- $Q\xi = \Lambda\xi = \xi$.

Banach contraction principle (BCP) or Banach fixed point theorem provided a basis for metric fixed point theory and using this theorem to prove the existence and the uniqueness of a fixed point of operators or mappings has been a useful and valuable way. Up to now, several generalizations of the theorem have been made in various ways.

Banach in 1922 [1] proved that "When $(\Im, m)$ is a complete metric space and $Q : (\Im, m) \to (\Im, m)$ is a mapping which obeys the following inequality for all

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$\xi, \iota \in \Im$

$$m(Q\xi, Q\iota) \leq cm(\xi, \iota), \quad where \quad c \in (0, 1), \tag{1}$$

the mapping Q possesses a unique fixed point in $\Im$. Moreover, the iterative sequence $\{Q^n \xi_0\}$ is convergent to this fixed point for each element $\xi_0 \in \Im$."

In 2010, Chistyakov [2] introduced a new generalized metric space called a modular metric space.

Let $\Im$ be a non-empty set and $\sigma : (0, \infty) \times \Im \times \Im \to [0, \infty]$ be a function. For simplicity, we will write:

$$\sigma_\ell(\xi, \iota) = \sigma(\ell, \xi, \iota)$$

for all $\ell > 0$ and $\xi, \iota \in \Im$.

Definition 1 ([2]) Let $\Im$ be a non-empty set. A function $\sigma : (0, \infty) \times \Im \times \Im \to [0, \infty]$ is said to be a metric modular on $\Im$ if the following condition holds: for all $\xi, \iota, \nu \in \Im$

(σ_1) $\sigma_\ell(\xi, \iota) = 0$ for all $\ell > 0$ if and only if $\xi = \iota$,
(σ_2) $\sigma_\ell(\xi, \iota) = \sigma_\ell(\iota, \xi)$ for all $\ell > 0$,
(σ_3) $\sigma_{\ell+\mu}(\xi, \iota) \leq \sigma_\ell(\xi, \nu) + \sigma_\mu(\nu, \iota)$ for all $\ell, \mu > 0$.

If (σ_1) is replaced by the following condition

$\left(\sigma_1{'}\right)$ $\sigma_\ell(\xi, \xi) = 0$ for all $\ell > 0$, then σ is labelled as a (metric) pseudomodular on $\Im$.

Also, to understand more detail, see [2–8].

Firstly, in 1989, Bakhtin [9] reintroduced the concept of b-metric (also known as quasi-metric) and, Czerwik [10, 11] used this in metric fixed point theory. Likewise, several researchers have achieved numerous fixed point results, including some contractive mappings using this concept.

Definition 2 ([10]) Let $\Im$ be a non-void set and $\rho \geq 1$ ($\rho \in \mathbf{R}$). A function $\eta : \Im \times \Im \to \mathbf{R}^+$ is named as $b-$metric on $\Im$ provided to the following circumstances hold:

(η_1) $\eta(\xi, \iota) = 0 \Leftrightarrow \xi = \iota$,
(η_2) $\eta(\xi, \iota) = \eta(\iota, \xi)$,
(η_3) $\eta(\xi, \iota) \leq \rho[\eta(\xi, \nu) + \eta(\nu, \iota)]$,

for all $\xi, \iota, \nu \in \Im$. Also, the pair $(\Im, \eta)$ is a $b-$metric space.

Even if a standard metric is a continuous map, a $b-$metric is not all the time. Besides, apparently, for $\rho = 1$, it is clear that $b-$metric degrade to ordinary metric.

In general, the following lemma is required when the $b-$metric function is not continuous.

Lemma 1 *([12]) Let $(\Im, \eta)$ be a $b-$metric space with $\rho \geq 1$ and $\{\xi_n\}$ and $\{\iota_n\}$ be convergent to ξ and ι, respectively. Then,*

$$\frac{1}{\rho^2}\eta(\xi,\iota) \leq \liminf_{n\to\infty} \eta(\xi_n,\iota_n) \leq \limsup_{n\to\infty} \eta(\xi_n,\iota_n) \leq \rho^2\eta(\xi,\iota).$$

Especially, if $\xi = \iota$, *then* $\lim_{n\to\infty} \eta(\xi_n,\iota_n) = 0$. *Further, for* $z \in \Im$, *the following inequality is provided,*

$$\frac{1}{\rho}\eta(\xi,z) \leq \liminf_{n\to\infty} \eta(\xi_n,z) \leq \limsup_{n\to\infty} \eta(\xi_n,z) \leq \rho\eta(\xi,z).$$

In 2018 Ege and Alaca [13] introduced a new concept by inspiring the above descriptions and named as the modular $b-$metric space. Moreover, some fixed point theorems, including new notions, are proved in the new space that expressed as follows.

Definition 3 ([13]) Let $\Im$ be a non-empty set and let $\rho \geq 1$ ($\rho \in \mathbf{R}$). Then, a map $\varpi : (0,\infty) \times \Im \times \Im \to [0,\infty]$ is entitled as modular $b-$metric, provided that the following circumstances satisfies,

(ϖ_1) $\varpi_\ell(\xi,\iota) = 0$ for all $\ell > 0$ if and only if $\xi = \iota$,
(ϖ_2) $\varpi_\ell(\xi,\iota) = \varpi_\ell(\iota,\xi)$ for all $\ell > 0$,
(ϖ_3) $\varpi_{\ell+\mu}(\xi,\iota) \leq \rho\left[\varpi_\ell(\xi,\nu) + \varpi_\mu(\nu,\iota)\right]$ for all $\ell,\mu > 0$,

for all $\xi,\iota,\nu \in \Im$. So, $(\Im,\varpi)$ is a modular $b-$metric space, which denotes as MbMS.

In the definition of modular $b-$metric, if we choose $\rho = 1$, then it is abbreviated as a natural extension of modular metric.

Example 1 ([13]) Consider the space

$$l_p = \left\{(\xi_n) \subset \mathbf{R} : \sum_{n=1}^{\infty} |\xi_n|^p < \infty\right\} \quad 0 < p < 1,$$

$\ell \in (0,\infty)$ and $\varpi_\ell(\xi,\iota) = \frac{m(\xi,\iota)}{\ell}$ such that

$$m(\xi,\iota) = \left(\sum_{n=1}^{\infty} |\xi_n - \iota_n|^p\right)^{\frac{1}{p}}, \quad \xi = \xi_n, \iota = \iota_n \in l_p.$$

It could be easily seen that $(\Im,\varpi)$ is an MbMS.

Example 2 ([14]) Let $(\Im,\sigma)$ be a modular metric space and let $p \geq 1$ be a real number. Take $\upsilon_\ell(\xi,\iota) = (\sigma_\ell(\xi,\iota))^p$. Using the convexity of the function $Q(t) = t^p$ for $t \geq 0$ and also using Jensen inequality, we obtain

$$(\alpha+\beta)^p \leq 2^{p-1}\left(\alpha^p + \beta^p\right)$$

for $\alpha,\beta \geq 0$. Thus, $(\Im,\upsilon)$ is a MbMS with $\rho = 2^{p-1}$.

Definition 4 Let ϖ be a modular $b-$metric on a set $\Im$. For $\xi, \iota \in \Im$, the binary relation $\overset{\varpi}{\sim}$ on $\Im$ defined by

$$\xi \sim \iota \Leftrightarrow \lim_{\ell \to \infty} \varpi_\ell (\xi, \iota) = 0$$

is an equivalence relation. A modular set is defined by

$$\Im_\varpi = \left\{ \iota \in \Im : \iota \overset{\varpi}{\sim} \xi \right\}.$$

Note that the set

$$\Im_\varpi^* = \{\xi \in \Im : \exists \ell = \ell (\xi) > 0 \text{ such that } \varpi_\ell (\xi, \xi_0) < \infty\} \ \ (\xi_0 \in \Im)$$

are said to be modular metric spaces (around ξ_0).

Now, in the following, we express some essential topological properties of MbMS.

Definition 5 Let $(\Im, \varpi)$ be an MbMS.

(i) The sequence $(\xi_n)_{n \in \mathbf{N}}$ in $\Im_\varpi^*$ is said to be $\varpi-$convergent to $\xi \in \Im_\varpi^*$, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbf{N}$ such that $\varpi_\ell (\xi_n, \xi) < \varepsilon$ for all $n \geq n_0$, that is, $\varpi_\ell (\xi_n, \xi) \to 0$, as $n \to \infty$ for all $\xi > 0$.
(ii) The sequence $(\xi_n)_{n \in \mathbf{N}}$ in $\Im_\varpi^*$ is said to be $\varpi-$Cauchy if for every $\varepsilon > 0$ there exists $n_0 \in \mathbf{N}$ such that $\varpi_\ell (\xi_n, \xi_m) < \varepsilon$ for all $n, m \geq n_0$, namely if $\lim_{n \to \infty} \varpi_\ell (\xi_n, \xi_m) = 0$ for all $\xi > 0$.
(iii) A modular $b-$metric space $\Im_\varpi^*$ is $\varpi-$complete if each $\varpi-$Cauchy sequence in $\Im_\varpi^*$ is $\varpi-$convergent and the limit of the sequence belongs to $\Im_\varpi^*$.

On the other hand, in 2012, Wardowski [15] put forth a new notion called $W-$contraction. So, this notion has been a useful tool to get new type contractive mappings and fixed point results.

Definition 6 ([15]) Let $(\Im, m)$ be a metric space. The mapping $Q : \Im \to \Im$ is entitled as $W-$contraction on $(\Im, m)$ provided that there exists $W \in \mathscr{W}$ and $\kappa > 0$ such that for all $\xi, \iota \in \Im$,

$$m (Q\xi, Q\iota) > 0 \Rightarrow \kappa + W (m (Q\xi, Q\iota)) \leq W (m (\xi, \iota)),$$

where $\mathscr{W}$ is the set of functions $W : (0, \infty) \to \mathbf{R}$ satisfying the following ones:

(W_1) W is strictly increasing, that is, for all $\varsigma, \omega \in (0, \infty)$ such that $W (\varsigma) < W (\omega)$ whenever $\varsigma < \omega$,
(W_2)] For each sequence $\{a_n\}_{n \in \mathbf{N}}$ of positive numbers $\lim_{n \to \infty} a_n = 0 \Leftrightarrow \lim_{n \to \infty} W (a_n) = -\infty$,
(W_3) There exists $c \in (0, 1)$ such that $\lim_{a \to 0^+} a^c W (a) = 0$.

Next, Wardowski proved in [15] that any $W-$contraction mapping on a complete metric space $(\Im, m)$ possesses a unique fixed point.

Example 3 ([15]) The following functions $W : (0, \infty) \to \mathbf{R}$ belong to $\mathscr{W}$.

(i) $W_1(\varsigma) = \ln \varsigma$,
(ii) $W_2(\varsigma) = \ln \varsigma + \varsigma$,
(iii) $W_3(\varsigma) = -\frac{1}{\sqrt{\varsigma}}$,
(iv) $W_4(\varsigma) = \ln\left(\varsigma^2 + \varsigma\right)$.

Hussain and Salimi [16] presented a new family of functions as indicated below. Let Δ_Σ denotes the set of all functions $\Sigma : \mathbf{R}_+^4 \to \mathbf{R}$ satisfying:

(Σ) If $\zeta_1.\zeta_2.\zeta_3.\zeta_4 = 0$ for all $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \mathbf{R}_+$, then, there exists $\kappa > 0$ such that $\Sigma(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = \kappa$.

Next, considering the function Σ, they identified the $\alpha - \eta - \Sigma W-$contraction and also achieved new fixed point results in the sense of complete metric space, as declared follows.

Definition 7 ([16]) Let $(\Im, m)$ be a metric space and $\alpha, \eta : \Im \times \Im \to [0, \infty)$ be two functions. A self-mapping Q on $\Im$ is called an $\alpha - \eta - \Sigma W-$contraction if

$$\Sigma\left(m(\xi, Q\xi), m(\iota, Q\iota), m(\xi, Q\iota), m(\iota, Q\xi)\right) + W(m(Q\xi, Q\iota)) \leq W(m(\xi, \iota))$$

is provided for $\xi, \iota \in \Im$ with $\eta(\xi, Q\xi) \leq \alpha(\xi, \iota)$ and $m(Q\xi, Q\iota) > 0$, where $\Sigma \in \Delta_\Sigma$ and $W \in \mathscr{W}$.

Theorem 1 ([16]) *Let $(\Im, m)$ be a complete metric space. Let $Q : \Im \to \Im$ be a self-mapping satisfying the following assertions:*

(i) Q is an $\alpha-$admissible mapping with respect to η;
(ii) Q is an $\alpha - \eta - \Sigma W-$contraction;
(iii) there exists $\xi_0 \in \Im$ such that $\alpha(\xi_0, Q\xi_0) \geq \eta(\xi_0, Q\xi_0)$;
(iv) Q is an $\alpha - \eta-$continuous mapping.

Then Q admits a fixed point in $\Im$. Moreover, Q has a unique fixed point when $\alpha(\xi, \iota) \geq \eta(\xi, \xi)$ for all $\xi, \iota \in Fix(Q)$.

For more detail about the $W-$contraction mappings, refer to [17–26].

Lastly, the following fixed point results, mentioned as Suzuki contraction in literature, is one of the most interesting and efficient generalizations of the Banach fixed point theorem, which is given by Suzuki [27].

Theorem 2 ([27]) *Let $(\Im, m)$ be a compact metric space and $Q : \Im \to \Im$ be a self-mapping. Then, Q admits a unique fixed point in $\Im$ provided that the following inequality is satisfied.*

$$\frac{1}{2}m(\xi, Q\xi) < m(\xi, \iota) \Rightarrow m(Q\xi, Q\iota) < m(\xi, \iota)$$

for all $\xi, \iota \in \Im$ *with* $\xi \neq \iota$.

2 Main Results

Primarily, owing to the fact that the concept of metric modular does not have to be finite, the following circumstances are essential to guarantee the existence and uniqueness of fixed points of contractive mappings in the setting of a modular metric space and a modular $b-$metric space. In the sequel, the following two conditions will be needed to prove the theorems and corollaries:

(S_1) $\varpi_\ell(\xi, Q\xi) < \infty$ for all $\xi > 0$ where $\xi \in \Im^*_\varpi$.
(S_2) $\varpi_\ell(\xi, \iota) < \infty$ for all $\xi > 0$ where $\xi, \iota \in \Im^*_\varpi$.

In the continuation of this section, we demonstrate some common fixed point theorems using Suzuki contraction and $W-$contraction in modular $b-$metric spaces.

Theorem 3 *Let* $\Im^*_\varpi$ *be a* $\varpi-$*complete MbMS with* $\rho \geq 1$ *and* $Q, \Lambda : \Im^*_\varpi \to \Im^*_\varpi$ *be two self-mappings. Presume that the following conditions are satisfied:*

i. there exist $W \in \mathscr{W}$ *and* $\Sigma \in \Delta_\Sigma$ *such that*

$$\frac{1}{2\rho}\min\{\varpi_\ell(\xi, Q\xi), \varpi_\ell(Q\iota, \Lambda Q\iota)\} \leq \varpi_\ell(\xi, Q\iota)$$

implies

$$\Sigma(\varpi_\ell(\xi, Q\xi), \varpi_\ell(Q\iota, \Lambda Q\iota), \varpi_\ell(\xi, \Lambda Q\iota), \varpi_\ell(Q\iota, Q\xi)) + W\left(\rho^2\varpi_\ell(Q\xi, \Lambda Q\iota)\right) \leq$$
$$W\left(\gamma(\varpi_\ell(\xi, Q\iota))\max\left\{\varpi_\ell(\xi, Q\iota), \varpi_\ell(\xi, Q\xi), \varpi_\ell(Q\iota, \Lambda Q\iota), \frac{\varpi_{2\ell}(\xi, \Lambda Q\iota)+\varpi_{2\ell}(Q\iota, Q\xi)}{2\rho}\right\}\right) \tag{2}$$

for all $\xi, \iota \in \Im^*_\varpi$ *and all* $\ell > 0$ *with* $\varpi_\ell(Q\xi, \Lambda Q\iota) > 0$, *where* $\gamma : \bar{P} \to [0, 1)$ *is an upper semi-continuous function on* $\bar{P} := \left\{\varpi_\ell(\xi, \iota) : \quad \xi, \iota \in \Im^*_\varpi\right\}$,

ii. one of the mappings Q *or* W *is continuous.*

If the condition (S_1) *is satisfied, then* Q *and* Λ *hold a common fixed point. Moreover, if* (S_2) *is provided, the common fixed point of the mappings* Q *and* Λ *is unique.*

Proof Let $\xi_0 \in \Im^*_\varpi$. Then, there exists $\xi_1 \in \Im^*_\varpi$ such that $\xi_1 = Q\xi_0$. Likewise, there exists $\xi_2 \in \Im^*_\varpi$ such that $\xi_2 = \Lambda\xi_1$. By proceeding in this line, we constitute a sequence $\left\{\xi_j\right\}_{j\in\mathbf{N}}$ in $\Im^*_\varpi$ featured

$$\xi_{2j+1} = Q\xi_{2j} \quad \text{and} \quad \xi_{2j+2} = \Lambda\xi_{2j+1}.$$

Assume that $\varpi_\ell\left(\xi_j, \xi_{j+1}\right) = 0$ for some $j \in \mathbf{N}$ and for all $\ell > 0$. Without loss of generality if we select $j = 2k$ for some $k \in \mathbf{N}$, then we attain $\varpi_\ell\left(\xi_{2k}, \xi_{2k+1}\right) = 0$ for all $\ell > 0$. Now, we suppose $\varpi_\ell\left(\xi_{2k+1}, \xi_{2k+2}\right) > 0$. Since

$$\frac{1}{2\rho}\min\left\{\varpi_\ell\left(\xi_{2k}, Q\xi_{2k}\right), \varpi_\ell\left(Q\xi_{2k}, \Lambda Q\xi_{2k}\right)\right\} \leq \varpi_\ell\left(\xi_{2k}, Q\xi_{2k}\right),$$

by (2), we achieve

$$\begin{aligned}
&\Sigma\left(\varpi_\ell\left(\xi_{2k}, Q\xi_{2k}\right), \varpi_\ell\left(Q\xi_{2k}, \Lambda Q\xi_{2k}\right), \varpi_\ell\left(\xi_{2k}, \Lambda Q\xi_{2k}\right), \varpi_\ell\left(Q\xi_{2k}, Q\xi_{2k}\right)\right) + \\
&W\left(\rho^2\varpi_\ell\left(Q\xi_{2k}, \Lambda Q\xi_{2k}\right)\right) \leq W\left(\gamma\left(\varpi_\ell\left(\xi_{2k}, Q\xi_{2k}\right)\right)\max\left\{\varpi_\ell\left(\xi_{2k}, Q\xi_{2k}\right),\right.\right. \\
&\quad \left.\left.\varpi_\ell\left(\xi_{2k}, Q\xi_{2k}\right), \varpi_\ell\left(Q\xi_{2k}, \Lambda Q\xi_{2k}\right), \tfrac{\varpi_{2\ell}\left(\xi_{2k}, \Lambda Q\xi_{2k}\right)+\varpi_{2\ell}\left(Q\xi_{2k}, Q\xi_{2k}\right)}{2\rho}\right\}\right).
\end{aligned}$$

Because

$$\varpi_\ell\left(\xi_{2k}, \xi_{2k+1}\right).\varpi_\ell\left(\xi_{2k+1}, \xi_{2k+2}\right).\varpi_\ell\left(\xi_{2k}, \xi_{2k+2}\right).\varpi_\ell\left(\xi_{2k+1}, \xi_{2k+1}\right) = 0,$$

from the definition of the function (Σ), there exists $\kappa > 0$ such that

$$\Sigma\left(\varpi_\ell\left(\xi_{2k}, \xi_{2k+1}\right), \varpi_\ell\left(\xi_{2k+1}, \xi_{2k+2}\right), \varpi_\ell\left(\xi_{2k}, \xi_{2k+2}\right), 0\right) = \kappa.$$

Besides, by denoting $\eta_k = \varpi_\ell\left(\xi_{2k}, \xi_{2k+1}\right)$, the following precise expression can be written;

$$\kappa + W\left(\rho^2\eta_{k+1}\right) \leq W\left(\gamma\left(\eta_k\right)\max\left\{\eta_k, \eta_{k+1}, \frac{\varpi_{2\ell}\left(\xi_{2k}, \xi_{2k+2}\right) + \varpi_{2\ell}\left(\xi_{2k+1}, \xi_{2k+1}\right)}{2\rho}\right\}\right).$$

Note that $\varpi_{2\ell}\left(\xi_{2k}, \xi_{2k+2}\right) \leq \rho\left(\eta_k + \eta_{k+1}\right)$ and since $\eta_k = \varpi_\ell\left(\xi_{2k}, \xi_{2k+1}\right) = 0$, we get $\max\left\{0, \eta_{k+1}, \frac{0+\eta_{k+1}}{2}\right\} = \eta_{k+1}$. Hence we conclude that

$$\kappa + W\left(\rho^2\eta_{k+1}\right) \leq W\left(\gamma\left(0\right)\eta_{k+1}\right).$$

Using the strictly increasing property of the W function, the following result is obtained from the above expression, but a contradictory situation arises due to the fact that $\gamma\left(0\right) < 1$

$$\eta_{k+1} \leq \rho^2\eta_{k+1} < \gamma\left(0\right)\eta_{k+1}.$$

Then, we achieve that $\eta_{k+1} = 0$, i.e., $\xi_{2k+1} = \xi_{2k+2}$. Therefore, the equalities $\xi_{2k} = \xi_{2k+1} = \xi_{2k+2}$, and $\xi_{2k} = Q\xi_{2k} = \Lambda\xi_{2k}$ hold, which means that ξ_{2k} is a common fixed point of Q and Λ. For this reason, in the rest of the proof, we also assume that $\xi_j \neq \xi_{j+1}$.

Applying the considered condition (2) and keeping in mind what we obtain in the above, we get

$$\begin{aligned} \kappa + W\left(\rho^2\eta_{j+1}\right) &\le W\left(\gamma\left(\eta_j\right)\max\left\{\eta_j, \eta_j, \eta_{j+1}, \tfrac{\eta_j+\eta_{j+1}}{2}\right\}\right) \\ &= W\left(\gamma\left(\eta_j\right)\max\left\{\eta_j, \eta_{j+1}\right\}\right). \end{aligned} \tag{3}$$

Presume that $\max\left\{\eta_j, \eta_{j+1}\right\} = \eta_{j+1}$. Thereby, (3) reduced to the expression given below;

$$\kappa + W\left(\rho^2\eta_{j+1}\right) \le W\left(\gamma\left(\eta_j\right)\eta_{j+1}\right).$$

Once more, if we use the strictly increasing property of W, we come to the conclusion that

$$\eta_{j+1} \le \rho^2\eta_{j+1} < \gamma\left(\eta_j\right)\eta_{j+1} < \eta_{j+1},$$

and this causes a contradiction. $\max\left\{\eta_j, \eta_{j+1}\right\}$ must be equal to η_j. Hence

$$\kappa + W\left(\rho^2\eta_{j+1}\right) \le W\left(\gamma\left(\eta_j\right)\eta_j\right) < W\left(\eta_j\right), \tag{4}$$

for all $j \in \mathbf{N}$. Similarly, we attest that

$$\kappa + W\left(\rho^2\eta_j\right) < W\left(\eta_{j-1}\right), \tag{5}$$

for all $j \in \mathbf{N}$. By the expressions (4) and (5), we have

$$\begin{aligned} W\left(\rho^2\varpi_\ell\left(\xi_j, \xi_{j+1}\right)\right) &< W\left(\varpi_\ell\left(\xi_{j-1}, \xi_j\right)\right) - \kappa \\ &< W\left(\varpi_\ell\left(\xi_{j-2}, \xi_{j-1}\right)\right) - 2\kappa \\ &< \quad \cdots \\ &< W\left(\varpi_\ell\left(\xi_0, \xi_1\right)\right) - j\kappa \end{aligned} \tag{6}$$

for all $j \in \mathbf{N}$. It follows that $\lim\limits_{j\to+\infty} W\left(\rho^2\varpi_\ell\left(\xi_j, \xi_{j+1}\right)\right) = -\infty$. By the property (W_2), we get $\lim\limits_{j\to+\infty} \rho^2\varpi_\ell\left(\xi_j, \xi_{j+1}\right) = 0$. Since $\rho \ge 1$,

$$\lim_{j\to+\infty} \varpi_\ell\left(\xi_j, \xi_{j+1}\right) = 0, \tag{7}$$

is procured. Now, from (W_3), there exists $\tau \in (0, 1)$ such that

$$\lim_{j\to+\infty} \varpi_\ell\left(\xi_j, \xi_{j+1}\right)^\tau W\left(\varpi_\ell\left(\xi_j, \xi_{j+1}\right)\right) = 0. \tag{8}$$

By (6), the following holds, for all $j \in \mathbf{N}$ and for all $\ell > 0$:

$$\left(\varpi_{\ell}\left(\xi_{j},\xi_{j+1}\right)\right)^{\tau}\left(W\left(\varpi_{\ell}\left(\xi_{j},\xi_{j+1}\right)\right)-W\left(\varpi_{\ell}\left(\xi_{0},\xi_{01}\right)\right)\right)\leq-\left(\varpi_{\ell}\left(\xi_{j},\xi_{j+1}\right)\right)^{\tau}j\kappa\leq 0.$$

If we take the limit in the above, we have

$$\lim_{j\to+\infty} j\left(\varpi_{\ell}\left(\xi_{j},\xi_{j+1}\right)\right)^{\tau}=0.$$

So, there exists $j_1 \in \mathbf{N}$ such that $j\left(\varpi_{\ell}\left(\xi_{j},\xi_{j+1}\right)\right)^{\tau} \leq 1$ for all $j \geq j_1$. Thus, we have for all $j \geq j_1$

$$\varpi_{\ell}\left(\xi_{j},\xi_{j+1}\right)\leq\frac{1}{j^{\frac{1}{\tau}}}. \tag{9}$$

Subsequently, in order to show that $\left\{\xi_j\right\}_{j\in\mathbf{N}}$ is a $\varpi-$Cauchy sequence, consider $g, j \in \mathbf{N}$ such that $g > j \geq j_1$. By using (ϖ_3) and from (9), we deduce that

$$\varpi_{\ell}\left(\xi_{j},\xi_{g}\right)\leq\rho\varpi_{\frac{\ell}{2}}\left(\xi_{j},\xi_{j+1}\right)+\rho^{2}\varpi_{\frac{\ell}{4}}\left(\xi_{j+1},\xi_{j+2}\right)+\cdots+\rho^{g-j}\varpi_{\frac{\ell}{2^{g-j}}}\left(\xi_{g-1},\xi_{g}\right).$$

Without loss of generality, we have

$$\begin{aligned}\varpi_{\ell}\left(\xi_{j},\xi_{g}\right)&\leq\sum_{r=j}^{g-1}\rho^{r-j+1}\varpi_{\ell}\left(\xi_{r},\xi_{r+1}\right)\\&\leq\sum_{r=j}^{g-1}\rho^{r-j+1}\left(\frac{1}{r^{\frac{1}{\tau}}}\right)\leq\sum_{r=j}^{\infty}\rho^{r-j+1}\left(\frac{1}{r^{\frac{1}{\tau}}}\right).\end{aligned}$$

By the convergence of the series $\sum_{r=j}^{\infty}\frac{1}{r^{\frac{1}{\tau}}}$ as $r \to \infty$ and since multiplying a scalar number in a convergent series gives a convergent series, passing to limit $j \to \infty$, we have $\varpi_{\ell}\left(\xi_j, \xi_g\right) \to 0$ for all $\ell > 0$, which yields that $\left\{\xi_j\right\}$ is a $\varpi-$Cauchy sequence in $\Im^*_{\varpi}$. By the completeness of the space, we obtain that $\xi^* \in \Im^*_{\varpi}$ such that

$$\varpi_{\ell}\left(\xi_{j},\xi^{*}\right)\to 0, \tag{10}$$

as $j \to \infty$. Now, if Q is a continuous mapping, then we have

$$\varpi_{\ell}\left(\xi^{*},Q\xi^{*}\right)=\lim_{j\to\infty}\varpi_{\ell}\left(\xi_{2j},Q\xi_{2j}\right)=\lim_{j\to\infty}\varpi_{\ell}\left(\xi_{2j},\xi_{2j+1}\right)=0.$$

This last equation displays that ξ^* is the fixed point of Q. Assume that $\xi^* \neq \Lambda\xi^*$, i.e., $\varpi_{\ell}\left(\xi^*, \Lambda\xi^*\right) > 0$. Then, since

$$\frac{1}{2\rho}\min\left\{\varpi_\ell\left(\xi^*, Q\xi^*\right), \varpi_\ell\left(Q\xi^*, \Lambda Q\xi^*\right)\right\} \leq \varpi_\ell\left(\xi^*, Q\xi^*\right),$$

from (2), we get

$$\begin{aligned}&\Sigma\left(\varpi_\ell\left(\xi^*, Q\xi^*\right), \varpi_\ell\left(Q\xi^*, \Lambda Q\xi^*\right), \varpi_\ell\left(\xi^*, \Lambda Q\xi^*\right), \varpi_\ell\left(Q\xi^*, Q\xi^*\right)\right) + \\ &W\left(\rho^2\varpi_\ell\left(Q\xi^*, \Lambda Q\xi^*\right)\right) \leq W\left(\gamma\left(\varpi_\ell\left(\xi^*, Q\xi^*\right)\right)\max\left\{\varpi_\ell\left(\xi^*, Q\xi^*\right),\right.\right. \\ &\quad\left.\left.\varpi_\ell\left(\xi^*, Q\xi^*\right), \varpi_\ell\left(Q\xi^*, \Lambda Q\xi^*\right), \tfrac{\varpi_{2\ell}\left(\xi^*, \Lambda Q\xi^*\right)+\varpi_{2\ell}\left(Q\xi^*, Q\xi^*\right)}{2\rho}\right\}\right).\end{aligned}$$

Because of the definition of (Σ), we conclude that

$$\Sigma\left(0, \varpi_\ell\left(\xi^*, \Lambda\xi^*\right), \varpi_\ell\left(\xi^*, \Lambda\xi^*\right), 0\right) = \kappa.$$

Therefore, the following expression is obtained, but it signifies a contradiction;

$$\begin{aligned}\kappa + W\left(\rho^2\varpi_\ell\left(\xi^*, \Lambda\xi^*\right)\right) &\leq W\left(\gamma(0)\max\left\{0, 0, \varpi_\ell\left(\xi^*, \Lambda\xi^*\right), \tfrac{\varpi_{2\ell}\left(\xi^*, \Lambda\xi^*\right)}{2\rho}\right\}\right) \\ &\leq W\left(\gamma(0)\,\varpi_\ell\left(\xi^*, \Lambda\xi^*\right)\right) \\ &< W\left(\varpi_\ell\left(\xi^*, \Lambda\xi^*\right)\right),\end{aligned}$$

that is, $\xi^* = \Lambda\xi^*$. Hence, ξ^* is a common fixed point of the mappings Q and Λ in case of Q is a continuous mapping.

Now, let's assume that the function W be continuous. In this case, if $Q\xi_{2j} = Q\xi^*$ for infinite values of $j \in \mathbf{N}$, then we have

$$\xi^* = \lim_{j\to\infty}\xi_{2j+1} = \lim_{j\to\infty}Q\xi_{2j} = Q\xi^*.$$

This proves that ξ^* is a fixed point of Q.

Since $Q\xi_{2j} = Q\xi^* = \xi^*$, we conclude that $\Lambda Q\xi_{2j} = \Lambda\xi_{2j+1} = \Lambda\xi^*$ and also get

$$\xi^* = \lim_{j\to\infty}\xi_{2j+2} = \lim_{j\to\infty}\Lambda\xi_{2j+1} = \Lambda\xi^*,$$

which means that ξ^* is a fixed point of the mapping Λ.

Then suppose that $\xi_{2j+2} \neq Q\xi^*$ for all $j \in \mathbf{N}$. First of all, we shall prove that $\xi^* = Q\xi^*$.

To show this, we put forth for all $j \geq 0$, at least one of the following inequalities is true:

$$\frac{1}{2\rho}\varpi_\ell\left(\xi_{2j+1}, \xi_{2j+2}\right) \leq \varpi_\ell\left(\xi^*, \xi_{2j+1}\right), \tag{11}$$

or

$$\frac{1}{2\rho}\varpi_\ell\left(\xi_{2j+2},\xi_{2j+3}\right)\leq\varpi_\ell\left(\xi^*,\xi_{2j+1}\right). \tag{12}$$

Unlike, if for some $j_0 \geq 0$, both of them are not provided. Hence, using (11) and (12), we say

$$\begin{aligned}\varpi_\ell\left(\xi_{2j_0+1},\xi_{2j_0+2}\right) &\leq \rho\varpi_\ell\left(\xi_{2j_0+1},\xi^*\right)+\rho\varpi_\ell\left(\xi^*,\xi_{2j_0+2}\right)\\ &< \tfrac{1}{2}\varpi_\ell\left(\xi_{2j_0+1},\xi_{2j_0+2}\right)+\tfrac{1}{2}\varpi_\ell\left(\xi_{2j_0+2},\xi_{2j_0+3}\right)\\ &< \tfrac{1}{2}\varpi_\ell\left(\xi_{2j_0+1},\xi_{2j_0+2}\right)+\tfrac{1}{2}\varpi_\ell\left(\xi_{2j_0+1},\xi_{2j_0+2}\right)=\varpi_\ell\left(\xi_{2j_0+1},\xi_{2j_0+2}\right),\end{aligned}$$

which causes a contradiction. Therefore, our assertion is true. From this point, one can discuss the following two subcases.

Subcase (i): The inequality (11) holds for infinitely many $j \geq 0$. In this case, for infinitely many $j \geq 0$, we have

$$\begin{aligned}\tfrac{1}{2\rho}\min\left\{\varpi_\ell\left(\xi^*,Q\xi^*\right),\varpi_\ell\left(Q\xi_{2j},\Lambda Q\xi_{2j}\right)\right\} &= \tfrac{1}{2\rho}\min\left\{\varpi_\ell\left(\xi^*,Q\xi^*\right),\varpi_\ell\left(\xi_{2j+1},\xi_{2j+2}\right)\right\}\\ &\leq \varpi_\ell\left(\xi^*,\xi_{2j+1}\right).\end{aligned}$$

Then, by (2), we get

$$\begin{aligned}&\Sigma\left(\varpi_\ell\left(\xi^*,Q\xi^*\right),\varpi_\ell\left(Q\xi_{2j},\Lambda Q\xi_{2j}\right),\varpi_\ell\left(\xi^*,\Lambda Q\xi_{2j}\right),\varpi_\ell\left(Q\xi_{2j},Q\xi^*\right)\right)+\\ &W\left(\rho^2\varpi_\ell\left(Q\xi^*,\Lambda Q\xi_{2j}\right)\right)\leq W\left(\gamma\left(\varpi_\ell\left(\xi^*,Q\xi_{2j}\right)\right)\max\left\{\varpi_\ell\left(\xi^*,Q\xi_{2j}\right),\right.\right.\\ &\quad\left.\left.\varpi_\ell\left(\xi^*,Q\xi^*\right),\varpi_\ell\left(Q\xi_{2j},\Lambda Q\xi_{2j}\right)\ ,\frac{\varpi_{2\ell}\left(\xi^*,\Lambda Q\xi_{2j}\right)+\varpi_{2\ell}\left(Q\xi_{2j},Q\xi^*\right)}{2\rho}\right\}\right)\end{aligned}$$

and so, it implies that

$$\begin{aligned}&W\left(\rho^2\varpi_\ell\left(Q\xi^*,\xi_{2j+2}\right)\right)\leq W\left(\gamma\left(\varpi_\ell\left(\xi^*,\xi_{2j+1}\right)\right)\max\left\{\varpi_\ell\left(\xi^*,\xi_{2j+1}\right),\right.\right.\\ &\quad\left.\left.\varpi_\ell\left(\xi^*,Q\xi^*\right),\varpi_\ell\left(\xi_{2j+1},\xi_{2j+2}\right)\ ,\frac{\varpi_{2\ell}\left(\xi^*,\xi_{2j+2}\right)+\varpi_{2\ell}\left(\xi_{2j+1},Q\xi^*\right)}{2\rho}\right\}\right).\end{aligned} \tag{13}$$

Then, by the upper semi-continuity of γ, we have

$$\lim_{j\to\infty}\sup\gamma\left(\varpi_\ell\left(\xi^*,\xi_{2j+1}\right)\right)\leq\gamma\left(0\right).$$

Hence, taking the upper limit as $j \to \infty$ in (13),

$$W\left(\rho^2\varpi_\ell\left(Q\xi^*,\xi^*\right)\right)\le W\left(\lim_{j\to\infty}\sup\left[\gamma\left(\varpi_\ell\left(\xi^*,\xi_{2j+1}\right)\right)\max\left\{\varpi_\ell\left(\xi^*,\xi_{2j+1}\right),\right.\right.\right.$$

$$\left.\left.\left.\varpi_\ell\left(\xi^*,Q\xi^*\right),\varpi_\ell\left(\xi_{2j+1},\xi_{2j+2}\right),\frac{\varpi_{2\ell}\left(\xi^*,\xi_{2j+2}\right)+\rho\left[\varpi_\ell\left(\xi_{2j+1},\xi_{2j+2}\right)+\varpi_\ell\left(\xi_{2j+1},Q\xi^*\right)\right]}{2\rho}\right\}\right]\right)$$

$$\le W\left(\gamma\left(0\right)\varpi_\ell\left(\xi^*,Q\xi^*\right)\right),$$

is obtained. Since W is strictly increasing, we get

$$\varpi_\ell\left(Q\xi^*,\xi^*\right)\le\rho^2\varpi_\ell\left(Q\xi^*,\xi^*\right)\le\gamma\left(0\right)\varpi_\ell\left(\xi^*,Q\xi^*\right),$$

which yields $\xi^*=Q\xi^*$.

Similarly, taking $\xi_{2j+1}\neq\Lambda\xi^*$ for all $j\in\mathbf{N}$, we also procure $\Lambda\xi^*=\xi^*$.

Subcase (ii): The inequality (11) merely satisfies for finitely many $j\ge 0$. In this case, there exists $j_0\ge 0$ such that (12) holds for any $j\ge j_0$. Like in Subcase (i), one can prove that (12) also leads to a contradiction unless ξ^* is a common fixed point of the mappings Q or Λ.

Consequently, in both subcases, ξ^* is considered as the common fixed point of Q and Λ.

Finally, for uniqueness, let ξ^* and ξ_1^* be two distinct common fixed points of Q and Λ. Hence, $\varpi_\ell\left(Q\xi^*,\Lambda Q\xi_1^*\right)=\varpi_\ell\left(\xi^*,\xi_1^*\right)>0$ and also,

$$0=\frac{1}{2\rho}\min\{\varpi_\ell\left(\xi^*,Q\xi^*\right),\varpi_\ell\left(Q\xi_1^*,\Lambda Q\xi_1^*\right)\}\le\varpi_\ell\left(\xi^*,Q\xi_1^*\right)=\varpi_\ell\left(\xi^*,\xi_1^*\right),$$

which implies by (2) that

$$\Sigma\left(\varpi_\ell\left(\xi^*,Q\xi_1^*\right),\varpi_\ell\left(Q\xi_1^*,\Lambda Q\xi_1^*\right),\varpi_\ell\left(\xi^*,\Lambda Q\xi_1^*\right),\varpi_\ell\left(Q\xi_1^*,Q\xi^*\right)\right)+$$

$$W\left(\rho^2\varpi_\ell\left(Q\xi^*,\Lambda Q\xi_1^*\right)\right)\le W\left(\gamma\left(\varpi_\ell\left(\xi^*,Q\xi_1^*\right)\right)\max\left\{\varpi_\ell\left(\xi^*,Q\xi_1^*\right),\right.\right.$$

$$\left.\left.\varpi_\ell\left(\xi^*,Q\xi^*\right),\varpi_\ell\left(Q\xi_1^*,\Lambda Q\xi_1^*\right),\frac{\varpi_{2\ell}\left(\xi^*,\Lambda Q\xi_1^*\right)+\varpi_{2\ell}\left(Q\xi_1^*,Q\xi^*\right)}{2\rho}\right\}\right).$$

From (Σ), we obtain

$$\Sigma\left(0,0,\varpi_\ell\left(\xi^*,\xi_1^*\right),\varpi_\ell\left(\xi_1^*,\xi^*\right)\right)=\kappa,$$

and so, we conclude that

$$\kappa+W\left(\rho^2\varpi_\ell\left(\xi^*,\xi_1^*\right)\right)\le W\left(\gamma\left(\varpi_\ell\left(\xi^*,\xi_1^*\right)\right)\max\left\{\varpi_\ell\left(\xi^*,\xi_1^*\right),0,0,\frac{\varpi_{2\ell}\left(\xi^*,\xi_1^*\right)}{\rho}\right\}\right)$$

$$\le W\left(\gamma\left(\varpi_\ell\left(\xi^*,\xi_1^*\right)\right)\varpi_\ell\left(\xi^*,\xi_1^*\right)\right)$$

$$< W\left(\varpi_\ell\left(\xi^*,\xi_1^*\right)\right),$$

which is a contradiction, that is, $\xi^* = \xi_1^*$. This shows that the common fixed point of Q and Λ is unique. So this finishes the proof. □

Taking $Q = \Lambda$ in Theorem 3, then we achieve the following consequence.

Corollary 1 *Let $\Im_{\varpi}^*$ be a $\varpi-$complete MbMS with $\rho \geq 1$ and $Q : \Im_{\varpi}^* \to \Im_{\varpi}^*$ be a self-mapping. Presume that there exist $W \in \mathscr{W}$ and $\Sigma \in \Delta_{\Sigma}$ such that*

$$\frac{1}{2\rho}\varpi_{\ell}(\xi, Q\xi) \leq \varpi_{\ell}(\xi, Q\iota)$$

implies

$$\Sigma\left(\varpi_{\ell}(\xi, Q\xi), \varpi_{\ell}\left(Q\iota, Q^2\iota\right), \varpi_{\ell}\left(\xi, Q^2\iota\right), \varpi_{\ell}(Q\iota, Q\xi)\right) + W\left(\rho^2\varpi_{\ell}\left(Q\xi, Q^2\iota\right)\right) \leq$$
$$W\left(\gamma\left(\varpi_{\ell}(\xi, Q\iota)\right)\max\left\{\varpi_{\ell}(\xi, Q\iota), \varpi_{\ell}(\xi, Q\xi), \varpi_{\ell}\left(Q\iota, Q^2\iota\right), \frac{\varpi_{2\ell}\left(\xi, Q^2\iota\right)+\varpi_{2\ell}(Q\iota, Q\xi)}{2\rho}\right\}\right), \tag{14}$$

for all $\xi, \iota \in \Im_{\varpi}^*$ and all $\ell > 0$ with $\varpi_{\ell}\left(Q\xi, Q^2\iota\right) > 0$, where $\gamma : \bar{P} \to [0, 1)$ is an upper semi-continuous function on $\bar{P} := \left\{\varpi_{\ell}(\xi, \iota) : \quad \xi, \iota \in \Im_{\varpi}^*\right\}$. If Q or W is continuous, then under the conditions (S_1) and (S_2), Q holds a unique fixed point in $\Im_{\varpi}^*$.

The following theorem is proved with the same lines applied in the proof of Theorem 3.

Theorem 4 *Let $\Im_{\varpi}^*$ be a $\varpi-$complete MbMS with $\rho \geq 1$ and $Q, \Lambda : \Im_{\varpi}^* \to \Im_{\varpi}^*$ be two self-mappings. Assume that the following statements are satisfied:*

i. *there exist $W \in \mathscr{W}$ and $\Sigma \in \Delta_{\Sigma}$ such that*

$$\frac{1}{2\rho}\min\left\{\varpi_{\ell}(\xi, Q\xi), \varpi_{\ell}(Q\iota, \Lambda Q\iota)\right\} \leq \varpi_{\ell}(\xi, Q\iota)$$

implies

$$\Sigma\left(\varpi_{\ell}(\xi, Q\xi), \varpi_{\ell}(Q\iota, \Lambda Q\iota), \varpi_{\ell}(\xi, \Lambda Q\iota), \varpi_{\ell}(Q\iota, Q\xi)\right) + W\left(\rho^2\varpi_{\ell}(Q\xi, \Lambda Q\iota)\right) \leq$$
$$W\left(\alpha\max\left\{\varpi_{\ell}(\xi, Q\iota), \frac{\varpi_{\ell}(\xi, Q\xi)+\varpi_{\ell}(Q\iota, \Lambda Q\iota)}{2}, \frac{\varpi_{2\ell}(\xi, \Lambda Q\iota)+\varpi_{2\ell}(Q\iota, Q\xi)}{2\rho}\right\}\right) \tag{15}$$

for all $\xi, \iota \in \Im_{\varpi}^*$ and all $\ell > 0$ with $\varpi_{\ell}(Q\xi, \Lambda Q\iota) > 0$, where $\alpha \in [0, 1)$,

ii. *one of the mappings Q or W is continuous,*

iii. *(S_1) and (S_2) are provided.*

Then the mappings Q and Λ hold a unique common fixed point in $\Im_{\varpi}^*$.

Similarly, by taking $Q = \Lambda$ in Theorem 4, we acquire the below consequence.

Corollary 2 *Let $\Im_{\varpi}^*$ be a $\varpi-$complete MbMS with $\rho \geq 1$ and $Q : \Im_{\varpi}^* \to \Im_{\varpi}^*$ be a self-mapping. Suppose that there exist $W \in \mathscr{W}$ and $\Sigma \in \Delta_{\Sigma}$ such that*

$$\frac{1}{2\rho}\varpi_\ell(\xi, Q\xi) \leq \varpi_\ell(\xi, Q\iota)$$

implies

$$\begin{aligned}&\Sigma\left(\varpi_\ell(\xi, Q\xi), \varpi_\ell\left(Q\iota, Q^2\iota\right), \varpi_\ell\left(\xi, Q^2\iota\right), \varpi_\ell(Q\iota, Q\xi)\right) + W\left(\rho^2\varpi_\ell\left(Q\xi, Q^2\iota\right)\right) \leq \\ &W\left(\alpha\max\left\{\varpi_\ell(\xi, Q\iota), \frac{\varpi_\ell(\xi,Q\xi)+\varpi_\ell\left(Q\iota,Q^2\iota\right)}{2}, \frac{\varpi_{2\ell}\left(\xi,Q^2\iota\right)+\varpi_{2\ell}(Q\iota,Q\xi)}{2\rho}\right\}\right)\end{aligned} \tag{16}$$

for all $\xi, \iota \in \Im_\varpi^*$ and all $\ell > 0$ with $\varpi_\ell\left(Q\xi, Q^2\iota\right) > 0$, where $\alpha \in [0, 1)$. If Q or W is a continuous mapping, as well as the conditions (S_1) and (S_2) are provided, then Q holds a unique fixed point in $\Im_\varpi^*$.

In the rest of this article, it will be considered that $\tau \in \left(0, \frac{1}{2}\right)$ and we establish some fixed point theorems and their results for $W-$contractions involving product expressions.

Theorem 5 *Let $\Im_\varpi^*$ be a $\varpi-$complete MbMS with $\rho \geq 1$ and $Q, \Lambda : \Im_\varpi^* \to \Im_\varpi^*$ be two self-mappings. Suppose that the following statements hold:*

i. there exist $W \in \mathscr{W}$ and $\Sigma \in \Delta_\Sigma$ such that

$$\frac{1}{2\rho}\min\{\varpi_\ell(\xi, Q\xi), \varpi_\ell(\iota, \Lambda\iota)\} \leq \varpi_\ell(\xi, \iota)$$

implies

$$\begin{aligned}&\Sigma(\varpi_\ell(\xi, Q\xi), \varpi_\ell(\iota, \Lambda\iota), \varpi_\ell(\xi, \Lambda\iota), \varpi_\ell(\iota, Q\xi)) + \\ &W\left(\rho^2\varpi_\ell^2(Q\xi, \Lambda\iota)\right) \leq W\begin{pmatrix}\alpha\left[\varpi_\ell(\xi, Q\xi)\,\varpi_\ell(\iota, \Lambda\iota) + \frac{1}{\rho}\varpi_{2\ell}(\xi, \Lambda\iota)\,\varpi_{2\ell}(\iota, Q\xi)\right] + \\ \beta\left[\varpi_\ell(\xi, Q\xi)\,\varpi_{2\ell}(\iota, Q\xi) + \frac{1}{\rho}\varpi_{2\ell}(\xi, \Lambda\iota)\,\varpi_\ell(\iota, \Lambda\iota)\right]\end{pmatrix}\end{aligned} \tag{17}$$

for all $\xi, \iota \in \Im_\varpi^*$ and all $\ell > 0$ with $\varpi_\ell(Q\xi, \Lambda\iota) > 0$, where $\alpha, \beta \geq 0$, $\alpha + \beta < \frac{1}{\rho}$.

ii. Q or W is a continuous mapping.

iii. the conditions (S_1) and (S_2) are satisfied.

Then Q and Λ possess a unique common fixed point in $\Im_\varpi^*$.

Proof Specify $\xi_0 \in \Im_\varpi^*$ as an arbitrary element. Then there exists $\xi_1 \in \Im_\varpi^*$ such that $\xi_1 = Q\xi_0$. Likewise, there exists $\xi_2 \in \Im_\varpi^*$ such that $\xi_2 = \Lambda\xi_1$. If we carry over this way, we compose a sequence $\left\{\xi_j\right\}_{j\in\mathbf{N}}$ in $\Im_\varpi^*$ featured

$$\xi_{2j+1} = Q\xi_{2j} \quad \text{and} \quad \xi_{2j+2} = \Lambda\xi_{2j+1}.$$

Presume that $\varpi_\ell\left(\xi_j, \xi_{j+1}\right) = 0$ for all $\ell > 0$. Next, if we take $j = 2i$ for some $i \in \mathbf{N}$, then this yields that $\varpi_\ell(\xi_{2i}, \xi_{2i+1}) = 0$ for all $\ell > 0$. So, we suppose

$\varpi_\ell(\xi_{2i+1}, \xi_{2i+2}) > 0$. Due to the fact that

$$\frac{1}{2\rho}\min\{\varpi_\ell(\xi_{2i}, Q\xi_{2i}), \varpi_\ell(\xi_{2i+1}, \Lambda\xi_{2i+1})\} \le \varpi_\ell(\xi_{2i}, \xi_{2i+1}),$$

from (17), it implies

$$\begin{aligned}
&\Sigma\left(\varpi_\ell(\xi_{2i}, Q\xi_{2i}), \varpi_\ell\left(\xi_{2i+1}, \Lambda\xi_{2i+1}\right), \varpi_\ell\left(\xi_{2i}, \Lambda\xi_{2i+1}\right), \varpi_\ell\left(\xi_{2i+1}, Q\xi_{2i}\right)\right) + \\
&W\left(\rho^2\varpi_\ell^2\left(Q\xi_{2i}, \Lambda\xi_{2i+1}\right)\right) \le \\
&W\left(\alpha\left[\varpi_\ell(\xi_{2i}, Q\xi_{2i})\varpi_\ell\left(\xi_{2i+1}, \Lambda\xi_{2i+1}\right) + \tfrac{1}{\rho}\varpi_{2\ell}\left(\xi_{2i}, \Lambda\xi_{2i+1}\right)\varpi_{2\ell}\left(\xi_{2i+1}, Q\xi_{2i}\right)\right] + \right. \\
&\left.\beta\left[\varpi_\ell(\xi_{2i}, Q\xi_{2i})\varpi_{2\ell}\left(\xi_{2i+1}, Q\xi_{2i}\right) + \tfrac{1}{\rho}\varpi_{2\ell}\left(\xi_{2i}, \Lambda\xi_{2i+1}\right)\varpi_\ell\left(\xi_{2i+1}, \Lambda\xi_{2i+1}\right)\right]\right).
\end{aligned}$$

Now, as

$$\varpi_\ell(\xi_{2i}, \xi_{2i+1}).\varpi_\ell(\xi_{2i+1}, \xi_{2i+2}).\varpi_\ell(\xi_{2i}, \xi_{2i+2}).\varpi_\ell(\xi_{2i+1}, \xi_{2i+1}) = 0,$$

so, from (Σ), there exists $\kappa > 0$ such that

$$\Sigma(\varpi_\ell(\xi_{2i}, \xi_{2i+1}), \varpi_\ell(\xi_{2i+1}, \xi_{2i+2}), \varpi_\ell(\xi_{2i}, \xi_{2i+2}), 0) = \kappa.$$

Also, let $\eta_i = \varpi_\ell(\xi_{2i}, \xi_{2i+1})$. Then, we conclude that

$$\kappa + W\left(\rho^2{\eta_{i+1}}^2\right) \le W\left(\alpha\left[\eta_i\eta_{i+1}\right] + \beta\left[\frac{1}{\rho}\varpi_{2\ell}(\xi_{2i}, \xi_{2i+2})\,\eta_{i+1}\right]\right).$$

Note that $\varpi_{2\ell}(\xi_{2i}, \xi_{2i+2}) \le \rho(\eta_i + \eta_{i+1})$ and because $\eta_i = \varpi_\ell(\xi_{2i}, \xi_{2i+1}) = 0$, we obtain

$$\kappa + W\left(\rho^2{\eta_{i+1}}^2\right) \le W\left(\beta{\eta_{i+1}}^2\right).$$

Because of strictly increasing property of W, we determine $\rho^2{\eta_{i+1}}^2 < \beta{\eta_{i+1}}^2$, which is a contradiction. So, $\eta_{i+1} = 0$, that is, $\xi_{2i+1} = \xi_{2i+2}$. Consequently, we achieve $\xi_{2i} = Q\xi_{2i} = \Lambda\xi_{2i}$; namely, ξ_{2i} is a common fixed point of Q and Λ. In the rest of the proof, we also assume that $\xi_j \neq \xi_{j+1}$.

Applying the considered condition (17) and by keeping in mind what we obtain in the above, we get

$$\begin{aligned}
\kappa + W\left(\rho^2{\eta_{j+1}}^2\right) &\le W\left(\alpha\left[\eta_j\eta_{j+1}\right] + \beta\left[\left(\eta_j + \eta_{j+1}\right)\eta_{j+1}\right]\right) \\
&= W\left((\alpha+\beta)\,\eta_j\eta_{j+1} + \beta{\eta_{j+1}}^2\right).
\end{aligned}$$

By using the feature of W, we deduce that

$$\rho^2 {\eta_{j+1}}^2 < (\alpha + \beta)\, \eta_j \eta_{j+1} + \beta {\eta_{j+1}}^2$$

and, hence

$$\left(\rho^2 - \beta\right) \eta_{j+1} < (\alpha + \beta)\, \eta_j,$$

for all $j \in \mathbf{N}$. From $\alpha + \beta < \frac{1}{\rho}$, where $\rho \geq 1$, we obtain $\rho^2 - \beta > 0$ and so,

$$\varpi_\ell \left(\xi_{2j+1}, \xi_{2j+2}\right) < \left(\frac{\alpha + \beta}{\rho^2 - \beta}\right) \varpi_\ell \left(\xi_{2j}, \xi_{2j+1}\right) < \varpi_\ell \left(\xi_{2j}, \xi_{2j+1}\right).$$

Therefore, we have

$$\kappa + W\left(\rho^2 \varpi_\ell^2 \left(\xi_{2j+1}, \xi_{2j+2}\right)\right) \leq W\left(\varpi_\ell^2 \left(\xi_{2j}, \xi_{2j+1}\right)\right)$$

for all $j \in \mathbf{N}$ and, similarly, we can show

$$\kappa + W\left(\rho^2 \varpi_\ell^2 \left(\xi_{2j}, \xi_{2j+1}\right)\right) \leq W\left(\varpi_\ell^2 \left(\xi_{2j-1}, \xi_{2j}\right)\right)$$

for all $j \in \mathbf{N}$. From the above inequalities, we gain

$$W\left(\rho^2 \varpi_\ell^2 \left(\xi_j, \xi_{j+1}\right)\right) < W\left(\varpi_\ell^2 \left(\xi_{j-1}, \xi_j\right)\right) - \kappa < \ldots < W\left(\varpi_\ell^2 \left(\xi_0, \xi_1\right)\right) - j\kappa$$

for all $j \in \mathbf{N}$ and for all $\ell > 0$.

Now, step by step, if we continue as in the proof of Theorem 3, then it is easy to show that $\left\{\xi_j\right\}_{j \in \mathbf{N}}$ is a $\varpi-$Cauchy sequence in $\Im_\varpi^*$. Owing to the completeness of the space, we obtain that $\xi^* \in \Im_\varpi^*$ such that

$$\varpi_\ell \left(\xi_j, \xi^*\right) \to 0, \tag{18}$$

as $j \to \infty$. Now, if Q is continuous, then we have

$$\varpi_\ell \left(\xi^*, Q\xi^*\right) = \lim_{j \to \infty} \varpi_\ell \left(\xi_{2j}, Q\xi_{2j}\right) = \lim_{j \to \infty} \varpi_\ell \left(\xi_{2j}, \xi_{2j+1}\right) = 0.$$

It implies that ξ^* is a fixed point of Q. Assume that $\xi^* \neq \Lambda\xi^*$, i.e., $\varpi_\ell\left(\xi^*, \Lambda\xi^*\right) > 0$. Then, since

$$\frac{1}{2\rho} \min\left\{\varpi_\ell \left(\xi^*, Q\xi^*\right), \varpi_\ell \left(\xi^*, \Lambda\xi^*\right)\right\} \leq \varpi_\ell \left(\xi^*, \xi^*\right),$$

from (17), we get

$$\Sigma\left(\varpi_{\ell}\left(\xi^{*}, Q\xi^{*}\right), \varpi_{\ell}\left(\xi^{*}, \Lambda\xi^{*}\right), \varpi_{\ell}\left(\xi^{*}, \Lambda\xi^{*}\right), \varpi_{\ell}\left(\xi^{*}, Q\xi^{*}\right)\right)$$

$$+W\left(\rho^{2}\varpi_{\ell}^{2}\left(Q\xi^{*}, \Lambda\xi^{*}\right)\right) \leq$$

$$W\left(\begin{array}{l}\alpha\left[\varpi_{\ell}\left(\xi^{*}, Q\xi^{*}\right)\varpi_{\ell}\left(\xi^{*}, \Lambda\xi^{*}\right)+\frac{1}{\rho}\varpi_{2\ell}\left(\xi^{*}, \Lambda\xi^{*}\right)\varpi_{2\ell}\left(\xi^{*}, Q\xi^{*}\right)\right]+\\ \beta\left[\varpi_{\ell}\left(\xi^{*}, Q\xi^{*}\right)\varpi_{2\ell}\left(\xi^{*}, Q\xi^{*}\right)+\frac{1}{\rho}\varpi_{2\ell}\left(\xi^{*}, \Lambda\xi^{*}\right)\varpi_{\ell}\left(\xi^{*}, \Lambda\xi^{*}\right)\right]\end{array}\right).$$

Due to the definition of (Σ), we decide on

$$\Sigma\left(0, \varpi_{\ell}\left(\xi^{*}, \Lambda\xi^{*}\right), \varpi_{\ell}\left(\xi^{*}, \Lambda\xi^{*}\right), 0\right)=\kappa,$$

thus, we come by

$$\begin{aligned}\kappa+W\left(\rho^{2}\varpi_{\ell}^{2}\left(\xi^{*}, \Lambda\xi^{*}\right)\right) &\leq W\left(\beta\left[\tfrac{1}{\rho}\varpi_{2\ell}\left(\xi^{*}, \Lambda\xi^{*}\right)\varpi_{\ell}\left(\xi^{*}, \Lambda\xi^{*}\right)\right]\right)\\ &\leq W\left(\beta\varpi_{\ell}^{2}\left(\xi^{*}, \Lambda\xi^{*}\right)\right)\\ &< W\left(\varpi_{\ell}\left(\xi^{*}, \Lambda\xi^{*}\right)\right),\end{aligned}$$

so this conclusion causes a contradiction, i.e., $\xi^{*}=\Lambda\xi^{*}$. On the other hand, assume that W is continuous. In this case, if $Q\xi_{2j}=Q\xi^{*}$ for infinite values of $j\in\mathbf{N}$, then we have

$$\xi^{*}=\lim_{j\to\infty}\xi_{2j+1}=\lim_{j\to\infty}Q\xi_{2j}=Q\xi^{*}.$$

This proves that ξ^{*} is a fixed point of Q.

Since $Q\xi_{2j}=Q\xi^{*}=\xi^{*}$, we conclude that $\Lambda Q\xi_{2j}=\Lambda\xi_{2j+1}=\Lambda\xi^{*}$. Then, we get

$$\xi^{*}=\lim_{j\to\infty}\xi_{2j+2}=\lim_{j\to\infty}\Lambda\xi_{2j+1}=\Lambda\xi^{*}.$$

This shows that ξ^{*} is a fixed point of Λ.

Then, we suppose that $\xi_{2j+2}\neq Q\xi^{*}$ for all $n\in\mathbf{N}$. Again, as in Theorem 3, we have

$$\frac{1}{2\rho}\min\left\{\varpi_{\ell}\left(\xi^{*}, Q\xi^{*}\right), \varpi_{\ell}\left(Q\xi_{2j}, \Lambda Q\xi_{2j}\right)\right\}\leq\varpi_{\ell}\left(\xi^{*}, Q\xi_{2j}\right).$$

Hence, by (17), we get

$$\Sigma\left(\varpi_\ell\left(\xi^*, Q\xi^*\right), \varpi_\ell\left(\xi_{2j+1}, \Lambda\xi_{2j+1}\right), \varpi_\ell\left(\xi^*, \Lambda\xi_{2j+1}\right), \varpi_\ell\left(\xi_{2j+1}, Q\xi^*\right)\right)$$
$$+W\left(\rho^2\varpi_\ell^2\left(Q\xi^*, \Lambda\xi_{2j+1}\right)\right) \leq$$
$$W\begin{pmatrix} \alpha\left[\varpi_\ell\left(\xi^*, Q\xi^*\right)\varpi_\ell\left(\xi_{2j+1}, \Lambda\xi_{2j+1}\right) + \frac{1}{\rho}\varpi_{2\ell}\left(\xi^*, \Lambda\xi_{2j+1}\right)\varpi_{2\ell}\left(\xi_{2j+1}, Q\xi^*\right)\right] + \\ \beta\left[\varpi_\ell\left(\xi^*, Q\xi^*\right)\varpi_{2\ell}\left(\xi_{2j+1}, Q\xi^*\right) + \frac{1}{\rho}\varpi_{2\ell}\left(\xi^*, \Lambda\xi_{2j+1}\right)\varpi_\ell\left(\xi_{2j+1}, \Lambda\xi_{2j+1}\right)\right] \end{pmatrix}$$

and so, it implies that

$$W\left(\rho^2\varpi_\ell^2\left(Q\xi^*, \xi_{2j+2}\right)\right) \leq$$
$$W\begin{pmatrix} \alpha\left[\varpi_\ell\left(\xi^*, Q\xi^*\right)\varpi_\ell\left(\xi_{2j+1}, \xi_{2j+2}\right) + \frac{1}{\rho}\varpi_{2\ell}\left(\xi^*, \xi_{2j+2}\right)\varpi_{2\ell}\left(\xi_{2j+1}, Q\xi^*\right)\right] + \\ \beta\left[\varpi_\ell\left(\xi^*, Q\xi^*\right)\varpi_{2\ell}\left(\xi_{2j+1}, Q\xi^*\right) + \frac{1}{\rho}\varpi_{2\ell}\left(\xi^*, \xi_{2j+2}\right)\varpi_\ell\left(\xi_{2j+1}, \xi_{2j+2}\right)\right] \end{pmatrix}. \tag{19}$$

Then, taking the limit as $j \to \infty$ in (19) and using the continuity of W, the following expression is acquired;

$$W\left(\rho^2\varpi_\ell^2\left(Q\xi^*, \xi^*\right)\right) \leq W\left(\lim_{j\to\infty}\beta\left[\varpi_\ell^2\left(\xi^*, Q\xi^*\right)\left(\rho\varpi_\ell\left(\xi_{2j+1}, \xi_{2j+2}\right) + \rho\varpi_\ell\left(\xi_{2j+2}, Q\xi^*\right)\right)\right]\right)$$
$$\leq W\left(\beta\rho\varpi_\ell^2\left(Q\xi^*, \xi^*\right)\right).$$

As W is a strictly increasing mapping, we derive

$$\rho\varpi_\ell^2\left(Q\xi^*, \xi^*\right) < \beta\varpi_\ell^2\left(Q\xi^*, \xi^*\right).$$

This means that $Q\xi^* = \xi^*$. Similarly, taking $\xi_{2j+1} \neq \Lambda\xi^*$ for all $j \in \mathbf{N}$, we also attain $\Lambda\xi^* = \xi^*$.

Consequently, ξ^* is a common fixed point of Q and Λ.

Finally, for uniqueness, let ξ^* and ξ_1^* be two distinct common fixed points of Q and Λ. Hence $\varpi_\ell\left(Q\xi^*, \Lambda\xi_1^*\right) = \varpi_\ell\left(\xi^*, \xi_1^*\right) > 0$ and the expression

$$0 = \frac{1}{2\rho}\min\left\{\varpi_\ell\left(\xi^*, Q\xi^*\right), \varpi_\ell\left(\xi_1^*, \Lambda\xi_1^*\right)\right\} \leq \varpi_\ell\left(\xi^*, \xi_1^*\right)$$

implies from the inequality (17)

$$\Sigma\left(\varpi_\ell\left(\xi^*, Q\xi^*\right), \varpi_\ell\left(\xi_1^*, \Lambda\xi_1^*\right), \varpi_\ell\left(\xi^*, \Lambda\xi_1^*\right), \varpi_\ell\left(\xi_1^*, Q\xi^*\right)\right) + W\left(\rho^2\varpi_\ell^2\left(Q\xi^*, \Lambda\xi_1^*\right)\right)$$
$$\leq W\begin{pmatrix} \alpha\left[\varpi_\ell\left(\xi^*, Q\xi^*\right)\varpi_\ell\left(\xi_1^*, \Lambda\xi_1^*\right) + \frac{1}{\rho}\varpi_{2\ell}\left(\xi^*, \Lambda\xi_1^*\right)\varpi_{2\ell}\left(\xi_1^*, Q\xi^*\right)\right] + \\ \beta\left[\varpi_\ell\left(\xi^*, Q\xi^*\right)\varpi_{2\ell}\left(\xi_1^*, Q\xi^*\right) + \frac{1}{\rho}\varpi_{2\ell}\left(\xi^*, \Lambda\xi_1^*\right)\varpi_\ell\left(\xi_1^*, \Lambda\xi_1^*\right)\right] \end{pmatrix}.$$

From the definition of (Σ),
$\Sigma\left(0, 0, \varpi_\ell\left(\xi^*, \xi_1^*\right), \varpi_\ell\left(\xi_1^*, \xi^*\right)\right) = \kappa$ holds and, we obtain

$$\kappa + W\left(\rho^2 \varpi_\ell^2\left(\xi^*, \xi_1^*\right)\right) \leq W\left(\alpha\left[\frac{1}{\rho}\varpi_{2\ell}^2\left(\xi^*, \xi_1^*\right)\right]\right)$$
$$\leq W\left(\varpi_\ell^2\left(\xi^*, \xi_1^*\right)\right).$$

This is a contradiction, i.e., $\xi^* = \xi_1^*$. Therefore, we say that the common fixed point of Q and Λ is unique and the proof ends. □

In the case of $Q = \Lambda$ in Theorem 5, the following result is procured.

Corollary 3 *Let $\Im_\varpi^*$ be a $\varpi-$complete MbMS with $\rho \geq 1$ and $Q : \Im_\varpi^* \to \Im_\varpi^*$ be a self-mapping. Assume that there exist $W \in \mathscr{W}$ and $\Sigma \in \Delta_\Sigma$ such that*

$$\frac{1}{2\rho}\varpi_\ell(\xi, Q\xi) \leq \varpi_\ell(\xi, \iota)$$

implies

$$\Sigma\left(\varpi_\ell(\xi, Q\xi), \varpi_\ell(\iota, Q\iota), \varpi_\ell(\xi, Q\iota), \varpi_\ell(\iota, Q\xi)\right) +$$
$$W\left(\rho^2\varpi_\ell^2(Q\xi, Q\iota)\right) \leq W\left(\begin{array}{c}\alpha\left[\varpi_\ell(\xi, Q\xi)\varpi_\ell(\iota, Q\iota) + \frac{1}{\rho}\varpi_{2\ell}(\xi, Q\iota)\varpi_{2\ell}(\iota, Q\xi)\right] + \\ \beta\left[\varpi_\ell(\xi, Q\xi)\varpi_{2\ell}(\iota, Q\xi) + \frac{1}{\rho}\varpi_{2\ell}(\xi, Q\iota)\varpi_\ell(\iota, Q\iota)\right]\end{array}\right) \tag{20}$$

for all $\xi, \iota \in \Im_\varpi^*$ and all $\ell > 0$ with $\varpi_\ell(Q\xi, Q\iota) > 0$, where $\alpha, \beta \geq 0$, $\alpha + \beta < \frac{1}{\rho}$. If Q or W is continuous and by adding the conditions (S_1) and (S_2), then Q holds a unique fixed point in $\Im_\varpi^*$.

Also, we establish the following common fixed point theorem.

Theorem 6 *Let $\Im_\varpi^*$ be a $\varpi-$complete MbMS with $\rho \geq 1$ and $Q, \Lambda : \Im_\varpi^* \to \Im_\varpi^*$ be two self-mappings. Suppose that the following circumstances hold:*

i. *there exist $W \in \mathscr{W}$ and $\Sigma \in \Delta_\Sigma$ such that*

$$\frac{1}{2\rho}\min\left\{\varpi_\ell(\xi, Q\xi), \varpi_\ell(Q\iota, \Lambda Q\iota)\right\} \leq \varpi_\ell(\xi, Q\iota)$$

implies

$$\Sigma\left(\varpi_\ell(\xi, Q\xi), \varpi_\ell(Q\iota, \Lambda Q\iota), \varpi_\ell(\xi, \Lambda Q\iota), \varpi_\ell(Q\iota, Q\xi)\right) + W\left(\rho^2 \varpi_\ell^2(Q\xi, \Lambda Q\iota)\right) \leq$$

$$W\left(\gamma\left(\varpi_\ell(\xi, Q\iota)\right) \max\left\{\begin{array}{l} \varpi_\ell(\xi, Q\xi)\varpi_\ell(Q\iota, \Lambda Q\iota), \frac{1}{2\rho^2}\varpi_{2\ell}(\xi, \Lambda Q\iota)\varpi_{2\ell}(Q\iota, Q\xi), \\ \frac{1}{2\rho}\varpi_\ell(\xi, Q\xi)\varpi_{2\ell}(Q\iota, Q\xi), \frac{1}{2\rho}\varpi_\ell(Q\iota, \Lambda Q\iota)\varpi_{2\ell}(\xi, \Lambda Q\iota) \end{array}\right\}\right) \tag{21}$$

for all $\xi, \iota \in \Im_\varpi^*$ *and for all* $\ell > 0$ *with* $\varpi_\ell(Q\xi, \Lambda Q\iota) > 0$, *where*

$$\gamma : \bar{P} \to \left[0, \frac{1}{2\rho}\right)$$

is an upper semi-continuous function on $\bar{P} := \left\{\varpi_\ell(\xi, \iota) : \quad \xi, \iota \in \Im_\varpi^*\right\}$,

ii. *if* Q *or* W *is a continuous mapping,*

iii. *the statements* (S_1) *and* (S_2) *are satisfied.*

Then the common fixed point of Q *and* Λ *is unique in* $\Im_\varpi^*$.

Proof Choose $\xi_0 \in \Im_\varpi^*$ as an arbitrary point. Then, one can find $\xi_1 \in \Im_\varpi^*$ such that $\xi_1 = Q\xi_0$. Similarly, there exists $\xi_2 \in \Im_\varpi^*$ such that $\xi_2 = \Lambda\xi_1$. If we carry on this way, we obtain a sequence $\left\{\xi_j\right\}_{j\in\mathbf{N}}$ in $\Im_\varpi^*$ such that

$$\xi_{2j+1} = Q\xi_{2j} \quad \text{and} \quad \xi_{2j+2} = \Lambda\xi_{2j+1}.$$

Assume that $\varpi_\ell\left(\xi_j, \xi_{j+1}\right) = 0$ for all $\ell > 0$. Next, if we select $j = 2r$ for some $r \in \mathbf{N}$, then we acquire that $\varpi_\ell(\xi_{2r}, \xi_{2r+1}) = 0$ for all $\ell > 0$. Now, we suppose $\varpi_\ell(\xi_{2r+1}, \xi_{2r+2}) > 0$. Since

$$\frac{1}{2\rho}\min\left\{\varpi_\ell(\xi_{2r}, Q\xi_{2r}), \varpi_\ell(Q\xi_{2r}, \Lambda Q\xi_{2r})\right\} \leq \varpi_\ell(\xi_{2r}, Q\xi_{2r}),$$

by (21), we obtain

$$\Sigma\left(\varpi_\ell(\xi_{2r}, Q\xi_{2r}), \varpi_\ell(Q\xi_{2r}, \Lambda Q\xi_{2r}), \varpi_\ell(\xi_{2r}, \Lambda Q\xi_{2r}), \varpi_\ell(Q\xi_{2r}, Q\xi_{2r})\right) +$$

$$W\left(\rho^2\varpi_\ell(Q\xi_{2r}, \Lambda Q\xi_{2r})\right) \leq W\left(\gamma\left(\varpi_\ell(\xi_{2r}, Q\xi_{2r})\right)\right.$$

$$\left.\max\left\{\begin{array}{l} \varpi_\ell(\xi_{2r}, Q\xi_{2r})\varpi_\ell(Q\xi_{2r}, \Lambda Q\xi_{2r}), \frac{1}{2\rho^2}\varpi_{2\ell}(\xi_{2r}, \Lambda Q\xi_{2r})\varpi_{2\ell}(Q\xi_{2r}, Q\xi_{2r}), \\ \frac{1}{2\rho}\varpi_\ell(\xi_{2r}, Q\xi_{2r})\varpi_{2\ell}(Q\xi_{2r}, Q\xi_{2r}), \frac{1}{2\rho}\varpi_\ell(Q\xi_{2r}, \Lambda Q\xi_{2r})\varpi_{2\ell}(\xi_{2r}, \Lambda Q\xi_{2r}) \end{array}\right\}\right).$$

As

$$\varpi_\ell(\xi_{2r}, \xi_{2r+1}).\varpi_\ell(\xi_{2r+1}, \xi_{2r+2}).\varpi_\ell(\xi_{2r}, \xi_{2r+2}).\varpi_\ell(\xi_{2r+1}, \xi_{2r+1}) = 0,$$

by the property of (Σ), there exists $\kappa > 0$ such that

$$\Sigma\left(\varpi_\ell(\xi_{2r}, \xi_{2r+1}), \varpi_\ell(\xi_{2r+1}, \xi_{2r+2}), \varpi_\ell(\xi_{2r}, \xi_{2r+2}), 0\right) = \kappa.$$

On the other hand, to simplify, let $\eta_r = \varpi_\ell(\xi_{2r}, \xi_{2r+1})$. Thus, we obtain

$$\kappa + W\left(\rho^2 {\eta_{r+1}}^2\right) \leq W\left(\gamma(\eta_r) \max\left\{\eta_r \eta_{r+1}, 0, 0, \frac{1}{2\rho}\eta_{r+1}\varpi_{2\ell}(\xi_{2r}, \xi_{2r+2})\right\}\right).$$

Note that $\varpi_{2\ell}(\xi_{2r}, \xi_{2r+2}) \leq \rho(\eta_r + \eta_{r+1})$ and in view of $\eta_r = \varpi_\ell(\xi_{2r}, \xi_{2r+1}) = 0$, we get

$$\kappa + W\left(\rho^2 {\eta_{r+1}}^2\right) \leq W\left(\gamma(0)\, {\eta_{r+1}}^2\right).$$

As W is strictly increasing, we deduce that

$$\rho^2 {\eta_{r+1}}^2 < \gamma(0)\, {\eta_{r+1}}^2$$

is contradiction inasmuch as $\gamma(0) < 1$. Then η_{r+1} is equal to 0, i.e., $\xi_{2r+1} = \xi_{2r+2}$. Thus we have $\xi_{2r} = \xi_{2r+1} = \xi_{2r+2}$. Therefore, $\xi_{2r} = Q\xi_{2r} = \Lambda\xi_{2r}$, which yields ξ_{2r} is a common fixed point of Q and Λ. So, in rest of the proof, we also assume that $\xi_j \neq \xi_{j+1}$.

Applying the considered condition (21) and keeping in mind what we obtain in the above, we get

$$\begin{aligned} \kappa + W\left(\rho^2 {\eta_{j+1}}^2\right) &\leq W\left(\gamma(\eta_j) \max\left\{\eta_j \eta_{j+1}, 0, 0, \tfrac{1}{2\rho}\eta_{j+1}\varpi_{2\ell}\left(\xi_{2j}, \Lambda Q\xi_{2j}\right)\right\}\right) \\ &\leq W\left(\gamma(\eta_j) \max\left\{\eta_j\eta_{j+1}, 0, 0, \tfrac{1}{2}\eta_{j+1}\left(\eta_j + \eta_{j+1}\right)\right\}\right). \end{aligned} \tag{22}$$

By simple calculations, it is clear that

$$\max\left\{\eta_j\eta_{j+1}, 0, 0, \frac{1}{2}\eta_{j+1}\left(\eta_j + \eta_{j+1}\right)\right\} = \frac{\eta_j\eta_{j+1} + {\eta_{j+1}}^2}{2}.$$

So, by (22),

$$\kappa + W\left(\rho^2 {\eta_{j+1}}^2\right) \leq W\left(\gamma(\eta_j)\frac{\eta_j\eta_{j+1} + {\eta_{j+1}}^2}{2}\right) < W\left(\frac{\eta_j\eta_{j+1} + {\eta_{j+1}}^2}{2}\right).$$

By using the properties of W, we deduce that

$${\eta_{j+1}}^2 \leq \rho^2 {\eta_{j+1}}^2 < \frac{\eta_j\eta_{j+1} + {\eta_{j+1}}^2}{2},$$

and the inequality $\eta_{j+1} < \eta_j$ is obtained, that is,

$$\kappa + W\left(\rho^2 \varpi_\ell^2\left(\xi_{2j+1}, \xi_{2j+2}\right)\right) \leq W\left(\varpi_\ell^2\left(\xi_{2j}, \xi_{2j+1}\right)\right),$$

for all $j \in \mathbf{N}$ and, similarly, we can show that

$$\kappa + W\left(\rho^2 \varpi_\ell^2\left(\xi_{2j}, \xi_{2j+1}\right)\right) \leq W\left(\varpi_\ell^2\left(\xi_{2j-1}, \xi_{2j}\right)\right),$$

for all $j \in \mathbf{N}$. From the above inequalities, we procure that

$$W\left(\rho^2 \varpi_\ell^2\left(\xi_j, \xi_{j+1}\right)\right) < W\left(\varpi_\ell^2\left(\xi_{j-1}, \xi_j\right)\right) - \kappa < \ldots < W\left(\varpi_\ell^2\left(\xi_0, \xi_1\right)\right) - j\kappa,$$

for all $j \in \mathbf{N}$ and for all $\ell > 0$.

If we continue as in the proof of Theorem 3 (also, similar expression as in Theorem 5), then it is easy to show that $\left\{\xi_j\right\}_{j\in\mathbf{N}}$ is a $\varpi-$Cauchy sequence in $\Im_\varpi^*$. Because the space is $\varpi-$complete, we obtain $\xi^* \in \Im_\varpi^*$ such that

$$\varpi_\ell\left(\xi_j, \xi^*\right) \to 0. \tag{23}$$

Now, if Q is continuous, then ξ^* is a fixed point of Q. Assume that $\xi^* \neq \Lambda\xi^*$, i.e., $\varpi_\ell\left(\xi^*, \Lambda\xi^*\right) > 0$. Since

$$\frac{1}{2\rho} \min\left\{\varpi_\ell\left(\xi^*, Q\xi^*\right), \varpi_\ell\left(Q\xi^*, \Lambda Q\xi^*\right)\right\} \leq \varpi_\ell\left(\xi^*, Q\xi^*\right),$$

from (21), we derive

$$\begin{aligned}
&\Sigma\left(\varpi_\ell\left(\xi^*, Q\xi^*\right), \varpi_\ell\left(Q\xi^*, \Lambda Q\xi^*\right), \varpi_\ell\left(\xi^*, \Lambda Q\xi^*\right), \varpi_\ell\left(Q\xi^*, Q\xi^*\right)\right) + \\
&W\left(\rho^2 \varpi_\ell^2\left(Q\xi^*, \Lambda Q\xi^*\right)\right) \leq W\left(\gamma\left(\varpi_\ell\left(\xi^*, Q\xi^*\right)\right)\right. \\
&\left.\max\left\{\begin{array}{l}\varpi_\ell\left(\xi^*, Q\xi^*\right)\varpi_\ell\left(Q\xi^*, \Lambda Q\xi^*\right), \frac{1}{2\rho^2}\varpi_{2\ell}\left(\xi^*, \Lambda Q\xi^*\right)\varpi_{2\ell}\left(Q\xi^*, Q\xi^*\right), \\ \frac{1}{2\rho}\varpi_\ell\left(\xi^*, Q\xi^*\right)\varpi_{2\ell}\left(Q\xi^*, Q\xi^*\right), \frac{1}{2\rho}\varpi_\ell\left(Q\xi^*, \Lambda Q\xi^*\right)\varpi_{2\ell}\left(\xi^*, \Lambda Q\xi^*\right)\end{array}\right\}\right).
\end{aligned}$$

Because of the definition of (Σ), we decide on

$$\Sigma\left(0, \varpi_\ell\left(\xi^*, \Lambda\xi^*\right), \varpi_\ell\left(\xi^*, \Lambda\xi^*\right), 0\right) = \kappa.$$

Hence

$$\begin{aligned}
\kappa + W\left(\rho^2 \varpi_\ell^2\left(\xi^*, \Lambda\xi^*\right)\right) &\leq W\left(\gamma(0)\max\left\{0, 0, 0, \tfrac{1}{2\rho}\varpi_\ell\left(\xi^*, \Lambda\xi^*\right)\varpi_{2\ell}\left(\xi^*, \Lambda\xi^*\right)\right\}\right) \\
&\leq W\left(\tfrac{1}{2}\gamma(0)\varpi_\ell^2\left(\xi^*, \Lambda\xi^*\right)\right) \\
&< W\left(\varpi_\ell^2\left(\xi^*, \Lambda\xi^*\right)\right),
\end{aligned}$$

is a contradiction, that is, $\xi^* = \Lambda\xi^*$. Therefore, ξ^* is a common fixed point of Q and Λ in case of the continuity of Q.

On the other hand, assume that W is continuous. In this case, if $Q\xi_{2j} = Q\xi^*$ for infinite values of $j \in \mathbf{N}$, then we have

$$\xi^* = \lim_{j\to\infty} \xi_{2j+1} = \lim_{j\to\infty} Q\xi_{2j} = Q\xi^*.$$

This proves that ξ^* is a fixed point of Q.

Since $Q\xi_{2j} = Q\xi^* = \xi^*$, we conclude that $\Lambda Q\xi_{2j} = \Lambda\xi_{2j+1} = \Lambda\xi^*$. Then, the following equality is true;

$$\xi^* = \lim_{j\to\infty} \xi_{2j+2} = \lim_{j\to\infty} \Lambda\xi_{2j+1} = \Lambda\xi^*.$$

This shows that ξ^* is a fixed point of Λ.

Then, we assume that $\xi_{2j+2} \neq Q\xi^*$ for all $n \in \mathbf{N}$. Again, as in Theorem 3, since

$$\frac{1}{2\rho}\min\left\{\varpi_\ell\left(\xi^*, Q\xi^*\right), \varpi_\ell\left(Q\xi_{2j}, \Lambda Q\xi_{2j}\right)\right\} \leq \varpi_\ell\left(\xi^*, Q\xi_{2j}\right),$$

by (21), we get

$$\begin{aligned}
&\Sigma\left(\varpi_\ell(\xi^*, Q\xi^*), \varpi_\ell\left(Q\xi_{2j}, \Lambda Q\xi_{2j}\right), \varpi_\ell\left(\xi^*, \Lambda Q\xi_{2j}\right), \varpi_\ell\left(Q\xi_{2j}, Q\xi^*\right)\right) + \\
&W\left(\rho^2\varpi_\ell^2\left(Q\xi^*, \Lambda Q\xi_{2j}\right)\right) \leq W\left(\gamma\left(\varpi_\ell\left(\xi^*, Q\xi_{2j}\right)\right)\right. \\
&\qquad \max\left\{\begin{array}{l}\varpi_\ell(\xi^*, Q\xi^*)\,\varpi_\ell\left(Q\xi_{2j}, \Lambda Q\xi_{2j}\right), \frac{1}{2\rho^2}\varpi_{2\ell}\left(\xi^*, \Lambda Q\xi_{2j}\right)\varpi_{2\ell}\left(Q\xi_{2j}, Q\xi^*\right), \\ \frac{1}{2\rho}\varpi_\ell(\xi^*, Q\xi^*)\,\varpi_{2\ell}\left(Q\xi_{2j}, Q\xi^*\right), \frac{1}{2\rho}\varpi_\ell\left(Q\xi_{2j}, \Lambda Q\xi_{2j}\right)\varpi_{2\ell}\left(\xi^*, \Lambda Q\xi_{2j}\right)\end{array}\right\}\Bigg)
\end{aligned}$$

and so, it implies that

$$\begin{aligned}
&W\left(\rho^2\varpi_\ell^2\left(Q\xi^*, \xi_{2j+2}\right)\right) \leq W\left(\gamma\left(\varpi_\ell\left(\xi^*, \xi_{2j+1}\right)\right)\right. \\
&\qquad \max\left\{\begin{array}{l}\varpi_\ell(\xi^*, Q\xi^*)\,\varpi_\ell\left(\xi_{2j+1}, \xi_{2j+2}\right), \frac{1}{2\rho^2}\varpi_{2\ell}\left(\xi^*, \xi_{2j+2}\right)\varpi_{2\ell}\left(\xi_{2j+1}, Q\xi^*\right), \\ \frac{1}{2\rho}\varpi_\ell(\xi^*, Q\xi^*)\,\varpi_{2\ell}\left(\xi_{2j+1}, Q\xi^*\right), \frac{1}{2\rho}\varpi_\ell\left(\xi_{2j+1}, \xi_{2j+2}\right)\varpi_{2\ell}\left(\xi^*, \xi_{2j+2}\right)\end{array}\right\}\Bigg).
\end{aligned} \tag{24}$$

Then, by the upper semi-continuity of γ, we have

$$\lim_{j\to\infty}\sup\gamma\left(\varpi_\ell\left(\xi^*, \xi_{2j+1}\right)\right) \leq \gamma(0).$$

Hence, taking upper limit as $j \to \infty$ in (24), we obtain

$$W\left(\rho^2\varpi_\ell^2\left(Q\xi^*, \xi^*\right)\right) \leq W\left(\frac{1}{2}\gamma(0)\,\varpi_\ell^2\left(\xi^*, Q\xi^*\right)\right).$$

Then, as in the proof of Theorem 3, we have a contradiction. This means that $\xi^* = Q\xi^*$.

Similarly, taking $\xi_{2j+1} \neq \Lambda\xi^*$ for all $j \in \mathbf{N}$, we also obtain $\Lambda\xi^* = \xi^*$.

Consequently, ξ^* is a common fixed point of Q and Λ.

Finally, for the uniqueness, let ξ^* and ξ_1^* be two distinct common fixed points of Q and Λ. Hence $\varpi_\ell\left(Q\xi^*, \Lambda Q\xi_1^*\right) = \varpi_\ell\left(\xi^*, \xi_1^*\right) > 0$ and also,

$$0 = \frac{1}{2\rho}\min\left\{\varpi_\ell\left(\xi^*, Q\xi^*\right), \varpi_\ell\left(Q\xi_1^*, \Lambda Q\xi_1^*\right)\right\} \leq \varpi_\ell\left(\xi^*, Q\xi_1^*\right) = \varpi_\ell\left(\xi^*, \xi_1^*\right)$$

which implies by (21)

$$\begin{aligned}
&\Sigma\left(\varpi_\ell\left(\xi^*, Q\xi^*\right), \varpi_\ell\left(Q\xi_1^*, \Lambda Q\xi_1^*\right), \varpi_\ell\left(\xi^*, \Lambda Q\xi_1^*\right), \varpi_\ell\left(Q\xi_1^*, Q\xi^*\right)\right) + \\
&W\left(\rho^2\varpi_\ell^2\left(Q\xi^*, \Lambda Q\xi_1^*\right)\right) \leq W\left(\gamma\left(\varpi_\ell\left(\xi^*, Q\xi_1^*\right)\right)\right. \\
&\qquad \left.\max\left\{\begin{array}{l} \varpi_\ell\left(\xi^*, Q\xi^*\right)\varpi_\ell\left(Q\xi_1^*, \Lambda Q\xi_1^*\right), \frac{1}{2\rho^2}\varpi_{2\ell}\left(\xi^*, \Lambda Q\xi_1^*\right)\varpi_{2\ell}\left(Q\xi_1^*, Q\xi^*\right), \\ \frac{1}{2\rho}\varpi_\ell\left(\xi^*, Q\xi^*\right)\varpi_{2\ell}\left(Q\xi_1^*, Q\xi^*\right), \frac{1}{2\rho}\varpi_\ell\left(Q\xi_1^*, \Lambda Q\xi_1^*\right)\varpi_{2\ell}\left(\xi^*, \Lambda Q\xi_1^*\right) \end{array}\right\}\right).
\end{aligned}$$

From (Σ), we obtain

$$\Sigma\left(0, 0, \varpi_\ell\left(\xi^*, \xi_1^*\right), \varpi_\ell\left(\xi_1^*, \xi^*\right)\right) = \kappa.$$

So, we conclude that

$$\begin{aligned}
\kappa + W\left(\rho^2\varpi_\ell^2\left(\xi^*, \xi_1^*\right)\right) &\leq W\left(\gamma\left(\varpi_\ell\left(\xi^*, \xi_1^*\right)\right)\varpi_\ell^2\left(\xi^*, \xi_1^*\right)\right) \\
&< W\left(\varpi_\ell^2\left(\xi^*, \xi_1^*\right)\right)
\end{aligned}$$

which is a contradiction, that is, $\xi^* = \xi_1^*$. This shows that the common fixed point of Q and Λ is unique. □

We see that if we take Q is equal to Λ in Theorem 6, the following corollary becomes a direct result.

Corollary 4 *Let $\Im_\varpi^*$ be a $\varpi-$complete MbMS with $\rho \geq 1$ and $Q : \Im_\varpi^* \to \Im_\varpi^*$ be a self-mapping. Suppose that there exist $W \in \mathscr{W}$ and $\Sigma \in \Delta_\Sigma$ such that*

$$\frac{1}{2\rho}\varpi_\ell\left(\xi, Q\xi\right) \leq \varpi_\ell\left(\xi, Q\iota\right)$$

implies

$$\begin{aligned}
&\Sigma\left(\varpi_\ell\left(\xi, Q\xi\right), \varpi_\ell\left(Q\iota, Q^2\iota\right), \varpi_\ell\left(\xi, Q^2\iota\right), \varpi_\ell\left(Q\iota, Q\xi\right)\right) + W\left(\rho^2\varpi_\ell^2\left(Q\xi, Q^2\iota\right)\right) \leq \\
&W\left(\gamma\left(\varpi_\ell\left(\xi, Q\iota\right)\right)\max\left\{\begin{array}{l} \varpi_\ell\left(\xi, Q\xi\right)\varpi_\ell\left(Q\iota, Q^2\iota\right), \frac{1}{2\rho^2}\varpi_{2\ell}\left(\xi, Q^2\iota\right)\varpi_{2\ell}\left(Q\iota, Q\xi\right), \\ \frac{1}{2\rho}\varpi_\ell\left(\xi, Q\xi\right)\varpi_{2\ell}\left(Q\iota, Q\xi\right), \frac{1}{2\rho}\varpi_\ell\left(Q\iota, Q^2\iota\right)\varpi_{2\ell}\left(\xi, Q^2\iota\right) \end{array}\right\}\right)
\end{aligned} \tag{25}$$

for all $\xi, \iota \in \Im_{\varpi}^{*}$ *and all* $\ell > 0$ *with* $\varpi_{\ell}\left(Q\xi, Q^{2}\iota\right) > 0$, *where* $\gamma : \bar{P} \to \left[0, \frac{1}{2\rho}\right)$ *is an upper semi-continuous function on* $\bar{P} := \left\{\varpi_{\ell}(\xi, \iota) : \quad \xi, \iota \in \Im_{\varpi}^{*}\right\}$. *If* Q *or* W *is continuous and the statements* (S_1) *and* (S_2) *are provided, then* Q *holds a unique fixed point in* $\Im_{\varpi}^{*}$.

The following theorem can be proved by using the same lines as in the proof of Theorem 5.

Theorem 7 *Let* $\Im_{\varpi}^{*}$ *be a* $\varpi-$*complete MbMS with* $\rho \geq 1$ *and* $Q, \Lambda : \Im_{\varpi}^{*} \to \Im_{\varpi}^{*}$ *be two self-mappings. Presume that there exist* $W \in \mathscr{W}$ *and* $\Sigma \in \Delta_{\Sigma}$ *such that*

$$\frac{1}{2\rho}\min\left\{\varpi_{\ell}(\xi, Q\xi), \varpi_{\ell}(\iota, \Lambda\iota)\right\} \leq \varpi_{\ell}(\xi, \iota)$$

implies

$$\begin{aligned} &\Sigma\left(\varpi_{\ell}(\xi, Q\xi), \varpi_{\ell}(\iota, \Lambda\iota), \varpi_{\ell}(\xi, \Lambda\iota), \varpi_{\ell}(\iota, Q\xi)\right) + W\left(\rho^{2}\varpi_{\ell}^{2}(Q\xi, \Lambda\iota)\right) \\ &\leq W\left(\alpha\max\left\{\begin{array}{l} \varpi_{\ell}(\xi, Q\xi)\,\varpi_{\ell}(\iota, \Lambda\iota), \varpi_{\ell}(\xi, \iota)\,\varpi_{\ell}(\xi, Q\xi), \varpi_{\ell}(\xi, \iota)\,\varpi_{\ell}(\iota, \Lambda\iota) \\ c\left(\frac{1}{\rho^{2}}\varpi_{2\ell}(\xi, \Lambda\iota)\,\varpi_{2\ell}(\iota, Q\xi)\right) \end{array}\right\}\right) \end{aligned} \tag{26}$$

for all $\xi, \iota \in \Im_{\varpi}^{*}$ *and all* $\ell > 0$ *with* $\varpi_{\ell}(Q\xi, \Lambda\iota) > 0$, *where* $0 < \alpha < 1$, $0 < c < 1$ *ve* $\alpha c < \frac{1}{2\rho}$, *and besides one of the mappings* Q *or* W *is continuous. Then, together with the conditions* (S_1) *and* (S_2), Q *and* Λ *possess a unique common fixed point in* $\Im_{\ell}^{*}$.

Putting $Q = \Lambda$ in Theorem 7, we obtain the following result.

Corollary 5 *Let* $\Im_{\varpi}^{*}$ *be a* $\varpi-$*complete MbMS with* $\rho \geq 1$ *and* $Q : \Im_{\varpi}^{*} \to \Im_{\varpi}^{*}$ *be a self-mapping. Presume that the following statements hold:*

i. *there exists* $W \in \mathscr{W}$ *and* $\Sigma \in \Delta_{\Sigma}$ *such that*

$$\frac{1}{2\rho}\varpi_{\ell}(\xi, Q\xi) \leq \varpi_{\ell}(\xi, \iota)$$

implies

$$\begin{aligned} &\Sigma\left(\varpi_{\ell}(\xi, Q\xi), \varpi_{\ell}(\iota, Q\iota), \varpi_{\ell}(\xi, Q\iota), \varpi_{\ell}(\iota, Q\xi)\right) + W\left(\rho^{2}\varpi_{\ell}^{2}(Q\xi, Q\iota)\right) \\ &\leq W\left(\alpha\max\left\{\begin{array}{l} \varpi_{\ell}(\xi, Q\xi)\,\varpi_{\ell}(\iota, Q\iota), \varpi_{\ell}(\xi, \iota)\,\varpi_{\ell}(\xi, Q\xi), \varpi_{\ell}(\xi, \iota)\,\varpi_{\ell}(\iota, Q\iota) \\ c\left(\frac{1}{\rho^{2}}\varpi_{2\ell}(\xi, Q\iota)\,\varpi_{2\ell}(\iota, Q\xi)\right) \end{array}\right\}\right) \end{aligned} \tag{27}$$

for all $\xi, \iota \in \Im_{\varpi}^{*}$ *and all* $\ell > 0$ *with* $\varpi_{\ell}(Q\xi, Q\iota) > 0$, *where* $0 < \alpha < 1$, $0 < c < 1$ *and* $\alpha c < \frac{1}{2\rho}$,

ii. Q or W is a continuous mapping,
iii. the statements (S_1) *and* (S_2) *are satisfied.*

Then Q holds a unique fixed point in $\Im_\ell^*$.

3 Examples

In this section, we furnish some examples illustrating the usability of the obtained results.

The following example demonstrates the validity of Theorem 3.

Example 4 Let $\Im_\varpi^* = [0, 1]$. Adopt the modular $b-$metric

$$\varpi_\ell(\xi, \iota) = \frac{|\xi - \iota|^k}{\ell}, \quad k \geq 1$$

for all $\xi, \iota \in \Im_\varpi^*$ and for all $\ell > 0$. Note that $\Im_\varpi^*$ is a $\varpi-$complete modular b-metric space with $\rho = 2^{k-1}$. Identify $Q, \Lambda : \Im \to \Im$, $\Sigma : R_+^4 \to R$, $W : (0, \infty) \to \mathbf{R}$ and $\gamma : \bar{P} \to [0, 1)$ where $\bar{P}$ as defined in Theorem 3, by

$$Q\xi = \frac{\xi}{16}, \text{ for all } \xi \in [0, 1] \quad \text{and} \quad \Lambda\xi = \begin{cases} 0 & , \text{ if } \xi \in [0, 1] - \{1/2\} \\ 2 & , \text{ if } \xi = 1/2 \end{cases}$$

$\Sigma(a_1, a_2, a_3, a_4) = \kappa$, where $\kappa > 0$, $W(\alpha) = \ln \alpha$ and $\gamma(\xi) = \left(\frac{4}{15}\right)^k$, $k \geq 1$, respectively. Here, without loss of generality, we may assume that $\xi \geq \iota \geq 0$.

We have the following possible cases.

Case (i): $\xi = \iota = \frac{1}{2}$.

In this case, $Q\xi = Q\iota = \frac{1}{32}$ and $\Lambda Q\iota = 0$. Next, we write the Suzuki condition given in Theorem 3.

$$\frac{1}{2.2^{k-1}} \min\left\{\varpi_\ell\left(\frac{1}{2}, \frac{1}{32}\right), \varpi_\ell\left(\frac{1}{32}, 0\right)\right\} \leq \varpi_\ell\left(\frac{1}{2}, \frac{1}{32}\right).$$

By simple calculations, we get

$$\frac{1}{2^k} \min\left\{\frac{15^k}{2^{5k}\ell}, \frac{1}{2^{5k}\ell}\right\} = \frac{1}{2^{6k}\ell} \leq \frac{15^k}{2^{5k}\ell}.$$

Thus, this implies that

$$\kappa + W\left(2^{2k-2}\varpi_\ell\left(\frac{1}{32},0\right)\right) \le W\left(\begin{array}{l}\gamma\left(\varpi_\ell\left(\frac{1}{2},\frac{1}{32}\right)\right)\max\left\{\varpi_\ell\left(\frac{1}{2},\frac{1}{32}\right),\varpi_\ell\left(\frac{1}{2},\frac{1}{32}\right),\right.\\ \left.\varpi_\ell\left(\frac{1}{32},0\right),\frac{\varpi_{2\ell}\left(\frac{1}{2},0\right)+\varpi_{2\ell}\left(\frac{1}{32},\frac{1}{32}\right)}{2.2^{k-1}}\right\}\end{array}\right).$$

Again, with similar calculations, we have

$$\kappa + W\left(\frac{1}{2^{3k+2}\ell}\right) \le W\left(\frac{4^k}{15^k}.\frac{15^k}{2^{5k}\ell}\right),$$

which yields that $\kappa - \ln 2^{3k+2}\ell \le \ln 4^k - \ln 2^{5k}\ell$. So, for $\kappa = \ln 4$, all conditions of Theorem 3 are provided.

Case (ii): $\xi \neq \frac{1}{2},\ \iota = \frac{1}{2}$.

In this case, $Q\xi = \frac{\xi}{16},\ Q\iota = \frac{1}{32}$ and $\Lambda Q\iota = 0$ such that

$$\frac{1}{2.2^{k-1}}\min\left\{\varpi_\ell\left(\xi,\frac{\xi}{16}\right),\varpi_\ell\left(\frac{1}{32},0\right)\right\} \le \varpi_\ell\left(\xi,\frac{1}{32}\right)$$

that is,

$$\frac{1}{2^k}\min\left\{\frac{15^k}{2^{4k}\ell},\frac{1}{2^{5k}\ell}\right\} = \frac{1}{2^{6k}\ell} \le \frac{(32\xi-1)^k}{2^{5k}\ell}$$

is satisfied. So, this implies that

$$\kappa + W\left(2^{2k-2}\varpi_\ell\left(\frac{\xi}{16},0\right)\right) \le W\left(\begin{array}{l}\gamma\left(\varpi_\ell\left(\xi,\frac{1}{32}\right)\right)\max\left\{\varpi_\ell\left(\xi,\frac{1}{32}\right),\varpi_\ell\left(\xi,\frac{\xi}{16}\right),\right.\\ \left.\varpi_\ell\left(\frac{1}{32},0\right),\frac{\varpi_{2\ell}(\xi,0)+\varpi_{2\ell}\left(\frac{1}{32},\frac{\xi}{16}\right)}{2.2^{k-1}}\right\}\end{array}\right).$$

Then we decide on

$$\kappa + W\left(\frac{\xi}{2^{2k+2}\ell}\right) \le W\left(\frac{4^k}{15^k}.\frac{(32\xi-1)^p}{2^{5k}\ell}\right)$$

and by using $W(\alpha) = \ln\alpha$, we get

$$\kappa \le \ln\frac{4^k}{15^k} + \ln\frac{(32\xi-1)^k}{\xi^k} + \ln\frac{2^{2k+2}\ell}{2^{5k}\ell} \le \ln\left(\frac{16}{15}\right)^k 4.$$

For $\kappa = \ln 4$, this case holds, too.

Case (iii): $\xi \neq \frac{1}{2},\ \iota \neq \frac{1}{2}$.

We have $Q\xi = \frac{\xi}{16}$, $Q\iota = \frac{\iota}{16}$ and $\Lambda Q\iota = 0$ such that

$$\frac{1}{2.2^{k-1}} \min\left\{\varpi_\ell\left(\xi, \frac{\xi}{16}\right), \varpi_\ell\left(\frac{\iota}{16}, 0\right)\right\} \le \varpi_\ell\left(\xi, \frac{\iota}{16}\right).$$

Thus, the following expression holds true,

$$\frac{1}{2^k} \min\left\{\frac{15^k}{2^{4k}\ell}, \frac{\iota^k}{2^{4k}\ell}\right\} = \frac{\iota^k}{2^{5k}\ell} \le \frac{(16\xi - \iota)^k}{2^{4k}\ell},$$

since $\xi > \frac{1}{2}$ and $\iota < \frac{1}{2}$. Therefore, we gain that

$$\kappa + W\left(2^{2k-2}\varpi_\ell\left(\frac{\xi}{16}, 0\right)\right) \le W\left(\begin{array}{l}\gamma\left(\varpi_\ell\left(\xi, \frac{\iota}{16}\right)\right) \max\left\{\varpi_\ell\left(\xi, \frac{\iota}{16}\right), \varpi_\ell\left(\xi, \frac{\xi}{16}\right),\right. \\ \left.\varpi_\ell\left(\frac{\iota}{16}, 0\right), \frac{\varpi_{2\ell}(\xi,0)+\varpi_{2\ell}\left(\frac{\iota}{16}, \frac{\xi}{16}\right)}{2.2^{k-1}}\right\}\end{array}\right)$$

and, with some calculations, the following inequality emerged

$$\kappa + W\left(\frac{\xi}{2^{2k+2}\ell}\right) \le W\left(\frac{4^k}{15^k}.\frac{(16\xi - 1)^k}{2^{4k}\ell}\right).$$

This signifies that

$$\kappa \le \ln\frac{4^k}{15^k} + \ln\frac{(16\xi - \iota)^k}{\xi^k} + \ln\frac{2^{2k+2}\ell}{2^{4k}\ell} \le \ln\left(\frac{16}{15}\right)^k 4,$$

and for $\kappa = \ln 4$, this case holds, too.

Case (iv): $\xi = \frac{1}{2}$, $\iota \ne \frac{1}{2}$.

We write $Q\xi = \frac{1}{32}$, $Q\iota = \frac{\iota}{16}$ and $\Lambda Q\iota = 0$, hence

$$\frac{1}{2.2^{k-1}} \min\left\{\varpi_\ell\left(\frac{1}{2}, \frac{1}{32}\right), \varpi_\ell\left(\frac{\iota}{16}, 0\right)\right\} \le \varpi_\ell\left(\frac{1}{2}, \frac{\iota}{16}\right)$$

that is,

$$\frac{1}{2^k} \min\left\{\frac{15^k}{2^{5k}\ell}, \frac{\iota^k}{2^{4k}\ell}\right\} = \frac{\iota^k}{2^{5k}\ell} \le \frac{(16 - 2\iota)^k}{2^{5k}\ell}.$$

This connotes that

$$\kappa + W\left(2^{2k-2}\varpi_\ell\left(\frac{1}{32},0\right)\right) \le W\left(\begin{array}{l}\gamma\left(\varpi_\ell\left(\frac{1}{2},\frac{\iota}{16}\right)\right)\max\left\{\varpi_\ell\left(\frac{1}{2},\frac{\iota}{16}\right),\varpi_\ell\left(\frac{1}{2},\frac{1}{32}\right),\right.\\ \left.\varpi_\ell\left(\frac{\iota}{16},0\right),\frac{\varpi_{2\ell}\left(\frac{1}{2},0\right)+\varpi_{2\ell}\left(\frac{\iota}{16},\frac{1}{32}\right)}{2.2^{k-1}}\right\}\end{array}\right).$$

Next, we reason out

$$\kappa + W\left(\frac{1}{2^{3k+2}\ell}\right) \le W\left(\frac{4^k}{15^k}.\frac{(8-\iota)^k}{2^{4k}\ell}\right),$$

and

$$\kappa \le \ln\frac{4^k}{15^k} + \ln(8-\iota)^k + \ln\frac{2^{3k+2}\ell}{2^{4k}\ell} \le \ln\left(\frac{16}{15}\right)^k.4$$

with $\kappa = \ln 4$.

Accordingly, in all cases, all the conditions of Theorem 3 are fulfilled and hence Q and Λ have a unique common fixed point. $\xi = 0$ is a unique common fixed point of the mappings Q and Λ.

Example 5 In the previous example, if we prefer $k = 1$, then we get

$$\varpi_\ell(\xi,\iota) = \frac{|\xi-\iota|}{\ell},$$

obviously $(\Im, \varpi)$ is a complete modular metric space. As well as all of the conditions of Theorem 3 hold. So, our example is valid in the setting of modular metric space provided to consider $W(\alpha) = \ln\alpha$, $\gamma(\xi) = \frac{4}{5}$ and $\kappa = \ln 2$.

Remark 1 Under the same conditions, Example 4 is still valid for Theorem 4, too. Moreover, this example remains valid within the results obtained from Theorem 4.

4 Conclusion

Consequently, we extend the result of Wardowski [15] by using some auxiliary functions and involving quadratic terms under the Suzuki type contractive condition in the sense of modular $b-$metric spaces. Finally, we also provide some examples to illustrate the usability of the acquired consequences.

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Some Common Fixed Point Results via α-Series for a Family of JS-Contraction Type Mappings

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Abstract In this chapter, we tend to prove some common fixed point theorems for families of JS-contractive type mappings using an α-series in complete metric spaces. In addition, we provide an example and an application to confirm the relevant results.

1 Introduction

The metric spaces (MS) and the principle of Banach contraction are two of the basic concepts of mathematical analysis. Most studies have been done on contraction mappings. For example, the contractions of Wardowski [20], Ćirić [5], Chatterjea [4], Kannan [13], Reich [18], Meir-Keeler [15], Jleli-Samet [11], Hardy-Rogers [9], etc., are of this type. In general, different contractive conditions have been considered for single and multi-valued maps (see [1, 3, 6, 12, 14]).

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Fixed point theory has attracted many researchers since 1922 when Banach's famous fixed point theorem has been introduced. There is a lot of materials in this field and it is currently a very active research field in mathematical analysis. A fixed point theorem is a theorem which states that if some conditions are met, it can be said that the function f has a fixed point such as x, that is, $f(x) = x$.

Also, the Banach fixed point theorem has many applications in many disciplines and branches of mathematics. Many authors generalized this classic result in non-linear analysis (see [7, 8, 16, 19] for more details).

In [10], sufficient conditions to the existence of a fixed point for a generalized contractive mapping were presented through a function $\theta \in \Theta$, as a control function, in the field of complete metric spaces and b-metric spaces.

In this chapter, we present the concept of JS-contractive type mappings [11], using a sequence of self-mappings via an α-series, which has been introduced in [19]. First, we provide the following basic definitions.

In 2014, the notion of an α-series was introduced by Sihag et al., which is given as follows:

Definition 1 ([19]) Let $\{\sigma_\nu\}$ be a sequence of non-negative real numbers. We say that a series $\sum_{\nu=1}^{+\infty} \sigma_\nu$ is an α-series, if there exist $0 < \alpha < 1$ and $\nu_\alpha \in \mathbb{N}$ such that $\sum_{\jmath=1}^{\kappa} \sigma_\jmath \leq \alpha\kappa$ for each $\kappa \geq \nu_\alpha$.

Example 1 Series $\sum_{\nu=1}^{+\infty} \frac{1}{2^{\nu-1}}$ and $\sum_{\nu=1}^{+\infty} \frac{1}{2^\nu}$ are α-series. Note that all convergent series with non-negative real sentences are α-series. Moreover, there exists a series like $\sum_{\nu=1}^{+\infty} \frac{1}{\nu}$ which is an α-series.

Definition 2 ([17]) The set of functions $\theta : (0, \infty) \to (1, \infty)$ represented by Θ, if θ satisfies the following assertions:

(θ1) θ is increasing;

(θ2) for any sequence $\{\sigma_\nu\} \subseteq (0, \infty)$

$$\lim_{\nu\to\infty} \theta(\sigma_\nu) = 1 \Leftrightarrow \lim_{\nu\to\infty} \sigma_\nu = 0;$$

(θ3) there exists $\alpha \in (0, 1)$ and $\iota \in (0, \infty]$ such that

$$\lim_{\upsilon\to 0^+} \frac{\theta(\upsilon) - 1}{\upsilon^\alpha} = \iota;$$

(θ4) $\theta(\jmath + j) \leq \theta(\jmath)\theta(j)$ for all $\jmath, j > 0$.

We denote the class of functions $\theta \in \Theta$ with $\Re$, without the condition (θ4).

Definition 3 Let (X, ρ) be a MS and mapping $\Upsilon : \mathsf{X} \to \mathsf{X}$ be given. Then

(i) Υ is a Banach contraction (see [2]) if there is $a \in [0, 1)$ such that for each $\iota, \varsigma \in X$

$$\rho(\Upsilon\iota, \Upsilon\varsigma) \leq a\rho(\iota, \varsigma).$$

(ii) Υ is a P-contraction (see [17]) If there exists $\theta \in \Re$ and $\iota_1, \iota_2, \iota_3, \iota_4 \geq 0$ with $\iota_1 + \iota_2 + \iota_3 + \iota_4 < 1$ such that the following holds:

$$\theta(\rho(\Upsilon\iota, \Upsilon\varsigma) \leq (\theta(\rho(\iota, \varsigma)))^{\iota_1}(\theta(\rho(\iota, \Upsilon\iota)))^{\iota_2}(\theta(\rho(\varsigma, \Upsilon\varsigma)))^{\iota_3} \left(\theta(\frac{\rho(\iota, \Upsilon_j\varsigma) + \rho(\varsigma, \Upsilon_\jmath\iota)}{2})\right)\Bigg]^{\iota_4},$$

for all $\iota, \varsigma \in X$.

In this chapter, we limit the conditions on θ as a control function. For this, the set of functions $\theta : (0, \infty) \to (1, \infty)$ is represented by $\Re$, so that

(θ1) θ is strictly increasing and continuous;
(θ2) for every sequence $\{\sigma_\nu\} \subseteq (0, \infty)$

$$\lim_{\nu\to\infty} \theta(\sigma_\nu) = 1 \Leftrightarrow \lim_{\nu\to\infty} \sigma_\nu = 0.$$

The following theorem states the result of Jleli and Samet:

Theorem 1 ([11]) *Let* (X, ρ) *be a MS. Let a mapping* $\Upsilon : \mathsf{X} \to \mathsf{X}$ *be given. Suppose that there exists* $\theta \in \Re$ *and* $\mu \in (0, 1)$ *such that for all* $\iota, \varsigma \in \mathsf{X}$,

$$\rho(\Upsilon\iota, \Upsilon\varsigma) \neq 0 \Rightarrow \theta(\rho(\Upsilon\iota, \Upsilon\varsigma)) \leq (\theta(\rho(\iota, \varsigma)))^\mu.$$

Then Υ *has a unique fixed point.*

The purpose of this chapter is to investigate the existence and uniqueness of common fixed point via an α-series for JS-contraction type mappings using a sequence of self-mappings $\Upsilon_\jmath$ on X, where (X, ρ) is a MS.

2 Main Results

In this section, we first provide the following definition.

Definition 4 Let $\{\Upsilon_\jmath\}_{\jmath\in\mathbb{N}}$ be a sequence of self-mappings on CMS (X, ρ). We call $\{\Upsilon_\jmath\}_{\jmath\in\mathbb{N}}$ a family of multiplicative H-contractions, if there is $\theta \in \Re$ such that for all $\iota, \varsigma \in \mathsf{X}$,

$$\theta(\rho(\Upsilon_\jmath\iota, \Upsilon_j\varsigma) \leq \left[\theta(\rho(\iota, \Upsilon_\jmath\iota))\theta(\rho(\varsigma, \Upsilon_j\varsigma)) \left(\theta(\frac{\rho(\iota, \Upsilon_j\varsigma) + \rho(\varsigma, \Upsilon_\jmath\iota)}{2})\right)\right]^{\pi_{\jmath,j}} (\theta(\rho(\iota, \varsigma)))^{\varpi_{\jmath,j}}, \quad (1)$$

where $0 \leq \pi_{\jmath,j}, \varpi_{\jmath,j} < 1$ for all $\jmath, j \in \mathbb{N}$; $\pi_{\jmath,\jmath+1} + \varpi_{\jmath,\jmath+1} < 1$, $2\pi_{n-1,\jmath} < 1$ and $\sum_{\jmath=1}^{+\infty} \left(\frac{2\pi_{\jmath,\jmath+1}+\varpi_{\jmath,\jmath+1}}{1-\pi_{\jmath,\jmath+1}}\right)$ is an α-series.

Now, we state the main result.

Theorem 2 *Each H-contractive mapping on a CMS has a unique common fixed point $\iota^* \in \mathsf{X}$.*

Proof Let $\iota_0 \in \mathsf{X}$ be an arbitrary point. We make sequence $\{\iota_n\}$ as follows:

$$\iota_n = \Upsilon_{n-1}\iota_{n-1}, \ \forall n \geq 1.$$

We assume that $\iota_n \neq \iota_{n+1}$ for each $n \geq 0$. We assert that

$$\lim_{n\to\infty} \rho(\iota_n, \iota_{n+1}) = 0.$$

Let $h_n := \rho(\iota_n, \iota_{n+1})$. By the hypothesis, we have

$$\begin{aligned}
\theta(h_n) &= \theta(\rho(\iota_n, \iota_{n+1})) \\
&= \theta(\rho(\Upsilon_{n-1}\iota_{n-1}, \Upsilon_n\iota_n)) \\
&\leq [\theta(\rho(\iota_{n-1}, \Upsilon_{n-1}\iota_{n-1}))\theta(\rho(\iota_n, \Upsilon_n\iota_n)) \\
&\quad \theta(\frac{\rho(\iota_{n-1}, \Upsilon_n\iota_n) + \rho(\iota_n, \Upsilon_{n-1}\iota_{n-1})}{2})]^{\pi_{n-1,n}} \\
&\quad (\theta(\rho(\iota_{n-1}, \iota_n)))^{\varpi_{n-1,n}} \\
&= [\theta(\rho(\iota_{n-1}, \iota_n))\theta(\rho(\iota_n, \iota_{n+1}))\theta(\frac{\rho(\iota_{n-1}, \iota_{n+1}) + \rho(\iota_n, \iota_n)}{2})]^{\pi_{n-1,n}} \\
&\quad (\theta(\rho(\iota_{n-1}, \iota_n)))^{\varpi_{n-1,n}} \\
&= [\theta(\rho(\iota_{n-1}, \iota_n))\theta(\rho(\iota_n, \iota_{n+1})) \\
&\quad (\theta(\max\{\rho(\iota_{n-1}, \iota_n), \rho(\iota_n, \iota_{n+1})\}]^{\pi_{n-1,n}}(\theta(\rho(\iota_{n-1}, \iota_n)))^{\varpi_{n-1,n}}. \qquad (2)
\end{aligned}$$

If for some L, one has

$$\rho(\iota_{L-1}, \iota_L) < \rho(\iota_L, \iota_{L+1}),$$

then according to (θ1), we obtain

$$\theta(\rho(\iota_{L-1}, \iota_L)) < \theta(\rho(\iota_L, \iota_{L+1})). \qquad (3)$$

Using (2), we have

$$\begin{aligned}
\theta(h_n) &\leq [\theta(h_{n-1})\theta(h_n)^2]^{\pi_{n-1,n}}(\theta(h_{n-1}))^{\varpi_{n-1,n}} \\
&= (\theta(h_{n-1}))^{(\pi_{n-1,n}+\varpi_{n-1,n})}(\theta(h_n))^{2\pi_{n-1,n}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\theta(h_n) &\leq (\theta(h_{n-1}))^{\left(\frac{\pi_{n-1,n}+\varpi_{n-1,n}}{1-2\pi_{n-1,n}}\right)} \\
&\leq (\theta(h_{n-1})),
\end{aligned}$$

which contradicts (3). As a result, for all $n \geq 1$,

$$\max\{\rho(\iota_{n-1}, \iota_n), \rho(\iota_n, \iota_{n+1})\} = \rho(\iota_{n-1}, \iota_n).$$

Hence, if α and ν_α be as in Definition 1, then, for all $n \geq \nu_\alpha$, according to the fact that the non-negative numbers geometric mean is less than or equal to the arithmetic mean, it follows that

$$\begin{aligned}\theta(h_n) &\leq (\theta(h_{n-1}))^{\left(\frac{2\pi_{n-1,n}+\varpi_{n-1,n}}{1-\pi_{n-1,n}}\right)} \\ &\leq (\theta(h_{n-2}))^{\left(\frac{2\pi_{n-1,n}+\varpi_{n-1,n}}{1-\pi_{n-1,n}}\right)\left(\frac{2\pi_{n-2,n-1}+\varpi_{n-2,n-1}}{1-\pi_{n-2,n-1}}\right)} \\ &\vdots \\ &\leq (\theta(h_0))^{\prod_{\jmath=1}^{n}\left(\frac{2\pi_{\jmath,\jmath+1}+\varpi_{\jmath,\jmath+1}}{1-\pi_{\jmath,\jmath+1}}\right)} \\ &\leq (\theta(h_0))^{\left(\frac{1}{n}\sum_{\jmath=1}^{n}\left(\frac{2\pi_{\jmath,\jmath+1}+\varpi_{\jmath,\jmath+1}}{1-\pi_{\jmath,\jmath+1}}\right)\right)^n} \\ &\leq (\theta(h_0))^{\alpha^n}. \end{aligned} \tag{4}$$

By ($\theta 2$), we have

$$\lim_{n\to\infty} h_n = 0. \tag{5}$$

Now, to show that $\{\iota_n\}$ is a Cauchy sequence, suppose that there exist $\varepsilon > 0$ so that for all $m_\jmath$ and $n_\jmath$ with $\jmath < m_\jmath < n_\jmath$, one has

$$\rho(\iota_{m_\jmath}, \iota_{n_\jmath}) \geq \varepsilon, \tag{6}$$

and

$$\rho(\iota_{m_\jmath}, \iota_{n_\jmath-1}) < \varepsilon. \tag{7}$$

By (6), we have

$$\rho(\iota_{m_\jmath-1}, \iota_{n_\jmath-1}) \leq \rho(\iota_{m_\jmath-1}, \iota_{m_\jmath}) + \rho(\iota_{m_\jmath}, \iota_{n_\jmath-1}).$$

From (5) and (7), we have

$$\limsup_{\jmath\to\infty} \rho(\iota_{m_\jmath-1}, \iota_{n_\jmath-1}) \leq \varepsilon. \tag{8}$$

As a result

$$\limsup_{\jmath\to\infty} \rho(\iota_{m_\jmath-1}, \iota_{n_\jmath}) \leq \varepsilon.$$

On the other hand, one has

$$\begin{aligned}\theta(\rho(\iota_{m_J},\iota_{n_J})) &= \theta(\rho(\Upsilon_{m_J-1}\iota_{m_J-1},\Upsilon_{n_J-1}\iota_{n_J-1}))\\ &\le [\theta(\rho(\iota_{m_J-1},\Upsilon_{m_J-1}\iota_{m_J-1}))\theta(\rho(\iota_{n_J-1},\Upsilon_{n_J-1}\iota_{n_J-1}))\\ &\quad \theta(\frac{\rho(\iota_{m_J-1},\Upsilon_{n_J-1}\iota_{n_J-1})+\rho(\iota_{n_J-1},\Upsilon_{m_J-1}\iota_{m_J-1})}{2})]^{\pi_{m_J-1,n_J-1}}\\ &\quad (\theta(\rho(\iota_{m_J-1},\iota_{n_J-1})))^{\varpi_{m_J-1,n_J-1}}\\ &= [\theta(\rho(\iota_{m_J-1},\iota_{m_J}))\theta(\rho(\iota_{n_J-1},\iota_{n_J}))\\ &\quad \theta(\frac{\rho(\iota_{m_J-1},\iota_{n_J})+\rho(\iota_{n_J-1},\iota_{m_J})}{2})]^{\pi_{m_J-1,n_J-1}}\\ &\quad (\theta(\rho(\iota_{m_J-1},\iota_{n_J-1})))^{\varpi_{m_J-1,n_J-1}}. \end{aligned} \tag{9}$$

Now using (θ1) and (5)–(8), one has

$$\begin{aligned}\theta(\varepsilon) &\le \theta(\limsup_{J\to\infty}\rho(\iota_{m_J},\iota_{n_J}))\\ &\le [\theta(\limsup_{J\to\infty}\rho(\iota_{m_J-1},\iota_{m_J}))\theta(\limsup_{J\to\infty}\rho(\iota_{n_J-1},\iota_{n_J}))\\ &\quad \theta(\limsup_{J\to\infty}(\frac{\rho(\iota_{m_J-1},\iota_{n_J})+\rho(\iota_{n_J-1},\iota_{m_J})}{2})]^{\pi_{m_J-1,n_J-1}}\\ &\quad (\theta(\limsup_{J\to\infty}\rho(\iota_{m_J-1},\iota_{n_J-1})))^{\varpi_{m_J-1,n_J-1}}\\ &\le (\theta(\varepsilon))^{\pi_{m_J-1,n_J-1}}(\theta(\varepsilon))^{\varpi_{m_J-1,n_J-1}}. \end{aligned} \tag{10}$$

This gives the result that

$$1 < (\theta(\varepsilon)) \le (\theta(\varepsilon))^{\pi_{m_J-1,n_J-1}+\varpi_{m_J-1,n_J-1}},$$

which is a contradiction. So, we have shown that $\{\iota_n\}$ is a Cauchy sequence in X.

Since (X,ρ) is compelet, $\iota_n \to \iota$ as $n\to\infty$, that is, $\lim_{n\to\infty}\iota_n=\iota$. On the other hand

$$\begin{aligned}\theta(\rho(\iota_n,\Upsilon_J\iota)) &= \theta(\rho(\Upsilon_{n-1}\iota_{n-1},\Upsilon_J\iota))\\ &\le [\theta(\rho(\iota_{n-1},\Upsilon_{n-1}\iota_{n-1}))\theta(\rho(\iota,\Upsilon_J\iota))\\ &\quad \theta(\frac{\rho(\iota_{n-1},\Upsilon_J\iota)+\rho(\iota,\Upsilon_{n-1}\iota_{n-1})}{2})]^{\pi_{n-1,J}}\\ &\quad (\theta(\rho(\iota_{n-1},\iota)))^{\varpi_{n-1,J}}. \end{aligned} \tag{11}$$

Using (θ1) and condition (5) as $n\to\infty$, one has

$$\theta(\rho(\iota,\Upsilon_\jmath\iota)) \le [\theta(\rho(\iota,\iota))\theta(\rho(\iota,\Upsilon_\jmath\iota))\theta(\frac{\rho(\iota,\Upsilon_\jmath\iota)+\rho(\iota,\iota)}{2})]^{\pi_{n-1,\jmath}} (\theta(\rho(\iota,\iota)))^{\varpi_{n-1,\jmath}} \le (\theta(\rho(\iota,\Upsilon_\jmath\iota)))^{2\pi_{n-1,\jmath}}. \tag{12}$$

We deduce that $\iota = \Upsilon_\jmath\iota$, as $2\pi_{n-1,\jmath} < 1$. So, ι is a common fixed point.

Now, to prove that the common fixed point is unique, suppose that $\Upsilon_m\varrho = \varrho \neq \varrho' = \Upsilon_{m'}\varrho'$. Hence

$$\begin{aligned}\theta(\rho(\Upsilon_m\varrho,\Upsilon_{m'}\varrho')) &\le [\theta(\rho(\varrho,\Upsilon_m\varrho))\theta(\rho(\varrho',\Upsilon_{m'}\varrho'))\\ &\quad \theta(\frac{\rho(\varrho,\Upsilon_{m'}\varrho')+\rho(\varrho',\Upsilon_m\varrho)}{2})]^{\pi_{m,m'}}\\ &\quad (\theta(\rho(\varrho,\varrho')))^{\varpi_{m,m'}}\\ &= (\theta(\rho(\varrho,\varrho')))^{\pi_{m,m'}}(\theta(\rho(\varrho,\varrho')))^{\varpi_{m,m'}}\\ &= (\theta(\rho(\varrho,\varrho')))^{\pi_{m,m'}+\varpi_{m,m'}},\end{aligned} \tag{13}$$

since $\pi_{m,m'} + \varpi_{m,m'} < 1$ then $\varrho = \varrho'$, so ϱ is the unique common fixed point of $\Upsilon_\jmath$.

We have the following corollary as a special case of Theorem 2.

Corollary 1 *Let $\{\Upsilon_\jmath\}_{\jmath\in\mathbb{N}}$ be a sequence of self-mappings on a CMS (X,ρ). Let there exists $\theta \in \Re$ so that for all $\iota,\varsigma \in \mathsf{X}$,*

$$\theta(\rho(\Upsilon_\jmath\iota,\Upsilon_j\varsigma)) \le [\theta(\rho(\iota,\Upsilon_\jmath\iota))\theta(\rho(\varsigma,\Upsilon_j\varsigma))\theta(\frac{\rho(\iota,\Upsilon_j\varsigma)+\rho(\varsigma,\Upsilon_\jmath\iota)}{2})]^{\pi_{\jmath,j}},$$

where $0 \le \pi_{\jmath,j} < 1$ for all $\jmath, j \in \mathbb{N}$ and $2\pi_{n-1,\jmath} < 1$. If $\sum_{\jmath=1}^{+\infty}\left(\frac{2\pi_{\jmath,\jmath+1}}{1-\pi_{\jmath,\jmath+1}}\right)$ is an α-series, then $\{\Upsilon_\jmath\}_{\jmath\in\mathbb{N}}$ has a unique common fixed point in X.

Now, by taking $\pi_{\jmath,\jmath+1} = 0$ in Theorem 2 and inspired by the Jelly-Samet result, we get the following result.

Corollary 2 *Let $\{\Upsilon_\jmath\}_{\jmath\in\mathbb{N}}$ be a sequence of self-mappings on a CMS (X,ρ). Let there exists $\theta \in \Re$ so that for all $\iota,\varsigma \in \mathsf{X}$,*

$$\theta(\rho(\Upsilon_\jmath\iota,\Upsilon_j\varsigma)) \le (\theta(\rho(\iota,\varsigma)))^{\varpi_{\jmath,j}},$$

where $0 \le \varpi_{\jmath,j} < 1$ for all $\jmath, j \in \mathbb{N}$. If $\sum_{\jmath=1}^{+\infty}\varpi_{\jmath,\jmath+1}$ is an α-series, then $\{\Upsilon_\jmath\}_{\jmath\in\mathbb{N}}$ has a unique common fixed point $\iota^* \in \mathsf{X}$.

Following similar concepts to prove the Theorem 2, we have the following theorem.

Theorem 3 *Let $\{\Upsilon_\jmath\}_{\jmath\in\mathbb{N}}$ be a sequence of self-mappings on a CMS (X,ρ). Let there exists $\theta \in \Re$ so that for all $\iota,\varsigma \in \mathsf{X}$,*

$$\theta(\rho(\Upsilon_J^p\iota, \Upsilon_j^p\varsigma)) \leq [\theta(\rho(\iota, \Upsilon_J^p\iota))\theta(\rho(\varsigma, \Upsilon_j^p\varsigma))\theta(\frac{\rho(\iota, \Upsilon_j^p\varsigma) + \rho(\varsigma, \Upsilon_J^p\iota)}{2})]^{\pi_{J,j}}$$
$$(\theta(\rho(\iota, \varsigma)))^{\varpi_{J,j}}, \tag{14}$$

where p is a positive integer, $0 \leq \pi_{J,j}, \varpi_{J,j} < 1$ for all $J, j \in \mathbb{N}$ and $\pi_{J,J+1} + \varpi_{J,J+1} < 1$ and $2\pi_{n-1,J} < 1$. If $\sum_{J=1}^{+\infty}\left(\frac{2\pi_{J,J+1}+\varpi_{J,J+1}}{1-\pi_{J,J+1}}\right)$ is an α-series, then $\{\Upsilon_J\}_{J\in\mathbb{N}}$ has a unique common fixed point $\iota^* \in \mathsf{X}$.

Proof Using Theorem 2 on the self-mapping $U := \Upsilon_J^p$, we deduce that U has a unique fixed point, such as ι^*, such that $\Upsilon_J^p(\iota^*) = U(\iota^*) = \iota^*$. Since $\Upsilon_J^{p+1}(\iota^*) = \Upsilon_J(\iota^*)$,

$$U\mathcal{P}_J(\iota^*) = \Upsilon_J^p(\Upsilon_J(\iota^*)) = \Upsilon_J^{p+1}(\iota^*) = \Upsilon_J(\iota^*).$$

So, $\Upsilon_J(\iota^*)$ is a fixed point of U. Due to the uniqueness of the fixed point of U, we obtain that $\Upsilon_J(\iota^*) = \iota^*$.

In Theorem 2, the continuity of θ can be replaced by the continuity of $\{\Upsilon_J\}$. Putting $\theta(t) = e^{\sqrt{t}}$, one has

Corollary 3 *Let $\{\Upsilon_J\}_{J\in\mathbb{N}}$ be a sequence of self-mappings on a CMS (X, ρ). Assume that for all $\iota, \varsigma \in \mathsf{X}$,*

$$\sqrt{\rho(\Upsilon_J\iota, \Upsilon_j\varsigma)} \leq \pi_{J,j}\sqrt{\rho(\iota, \Upsilon_J\iota) + \rho(\varsigma, \Upsilon_j\varsigma) + \frac{\rho(\iota, \Upsilon_j\varsigma) + \rho(\varsigma, \Upsilon_J\iota)}{2}}$$
$$+ \varpi_{J,j}\sqrt{\rho(\iota, \varsigma)}, \tag{15}$$

where $0 \leq \pi_{J,j}, \varpi_{J,j} < 1$ for all $J, j \in \mathbb{N}$, $\pi_{J,J+1} + \varpi_{J,J+1} < 1$ and $2\pi_{n-1,J} < 1$. If $\sum_{J=1}^{+\infty}\left(\frac{2\pi_{J,J+1}+\varpi_{J,J+1}}{1-\pi_{J,J+1}}\right)$ is an α-series, then $\{\Upsilon_J\}_{J\in\mathbb{N}}$ has a unique common fixed point in X.

Remark 1 Notice that condition (15) is equivalent to

$$\begin{aligned}&\rho(\Upsilon_J\iota, \Upsilon_j\varsigma)\\ &\leq {\pi_{J,j}}^2\left[\rho(\iota, \Upsilon_J\iota) + \rho(\varsigma, \Upsilon_j\varsigma) + \frac{\rho(\iota, \Upsilon_j\varsigma) + \rho(\varsigma, \Upsilon_J\iota)}{2}\right] + \varpi_{J,j}^2\rho(\iota, \varsigma)\\ &\quad + 2\pi_{J,j}\varpi_{J,j}\sqrt{\rho(\iota, \varsigma)\left[\rho(\iota, \Upsilon_J\iota) + \rho(\varsigma, \Upsilon_j\varsigma) + \frac{\rho(\iota, \Upsilon_j\varsigma) + \rho(\varsigma, \Upsilon_J\iota)}{2}\right]}.\end{aligned} \tag{16}$$

Next, according to Remark 1, by taking $\pi_{J,j} = 0$ in Corollary 3, we get the following extension of Banach's result.

Corollary 4 *Let $\{\Upsilon_J\}_{J\in\mathbb{N}}$ be a sequence of self-mappings on a CMS (X, ρ). Assume that for all $\iota, \varsigma \in \mathsf{X}$,*

$$\rho(\Upsilon_\jmath \iota, \Upsilon_j \varsigma) \leq \varpi_{\jmath,j}^2 \rho(\iota, \varsigma),$$

where $0 \leq \varpi_{\jmath,j} < 1$ *for all* $\jmath, j \in \mathbb{N}$. *If* $\sum_{\jmath=1}^{+\infty} \varpi_{\jmath,\jmath+1}$ *is an* α*-series, then* $\{\Upsilon_\jmath\}_{\jmath\in\mathbb{N}}$ *has a unique common fixed point in* X.

On the other hand, by taking $\varpi_{\jmath,\jmath+1} = 0$ in Corollary 3, we get the following result.

Corollary 5 *Let* $\{\Upsilon_\jmath\}_{\jmath\in\mathbb{N}}$ *be a sequence of self-mappings on a CMS* (X, ρ). *Assume that for all* $\iota, \varsigma \in \mathsf{X}$,

$$\rho(\Upsilon_\jmath \iota, \Upsilon_j \varsigma) \leq {\pi_{\jmath,j}}^2 \left[\rho(\iota, \Upsilon_\jmath \iota) + \rho(\varsigma, \Upsilon_j \varsigma) + \frac{\rho(\iota, \Upsilon_j \varsigma) + \rho(\varsigma, \Upsilon_\jmath \iota)}{2} \right],$$

where $0 \leq \pi_{\jmath,j} < 1$ *for all* $\jmath, j \in \mathbb{N}$ *and* $2\pi_{n-1,\jmath} < 1$. *If* $\sum_{\jmath=1}^{+\infty} \left(\frac{2\pi_{\jmath,\jmath+1}}{1-\pi_{\jmath,\jmath+1}} \right)$ *is an* α*-series, then* $\{\Upsilon_\jmath\}_{\jmath\in\mathbb{N}}$ *has a unique common fixed point in* X.

lastly, taking $\theta(t) = e^{\sqrt[n]{t}}$ in (1), we have the following corollary.

Corollary 6 *Let* $\{\Upsilon_\jmath\}_{\jmath\in\mathbb{N}}$ *be a sequence of self-mappings on a CMS* (X, ρ). *Assume that for all* $\iota, \varsigma \in \mathsf{X}$,

$$\sqrt[n]{\rho(\Upsilon_\jmath \iota, \Upsilon_j \varsigma)} \leq \pi_{\jmath,j} \sqrt[n]{\rho(\iota, \Upsilon_\jmath \iota) + \rho(\varsigma, \Upsilon_j \varsigma) + \frac{\rho(\iota, \Upsilon_j \varsigma) + \rho(\varsigma, \Upsilon_\jmath \iota)}{2}} + \varpi_{\jmath,j} \sqrt[n]{\rho(\iota, \varsigma)}, \tag{17}$$

where $0 \leq \pi_{\jmath,j}, \varpi_{\jmath,j} < 1$ *for all* $\jmath, j \in \mathbb{N}$, $\pi_{\jmath,\jmath+1} + \varpi_{\jmath,\jmath+1} < 1$ *and* $2\pi_{n-1,\jmath} < 1$. *If* $\sum_{\jmath=1}^{+\infty} \left(\frac{2\pi_{\jmath,\jmath+1} + \varpi_{\jmath,\jmath+1}}{1-\pi_{\jmath,\jmath+1}} \right)$ *is an* α*-series, then* $\{\Upsilon_\jmath\}_{\jmath\in\mathbb{N}}$ *has a unique common fixed point in* X.

Example 2 Let $\mathsf{X} = [0, 1]$, and let $\rho : \mathsf{X}^2 \to R_+$ with $\rho(\iota, \varsigma) = |\iota - \varsigma|$ for all $\iota, \varsigma \in \mathsf{X}$. Then (X, ρ) is a CMS. Let $\Upsilon_\jmath : \mathsf{X} \to \mathsf{X}$ be defined by

$$\Upsilon_\jmath(\iota) = \frac{\sqrt{\iota}}{2^\jmath},$$

for all $\iota \in \mathsf{X}$ and for all $\jmath = 1, 2, \ldots$. Define $\pi_{\jmath,j} = \frac{1}{2^{\jmath+2}}$ and $\varpi_{\jmath,j} = \frac{1}{2^{2i+1}}$ for all $\jmath, j = 1, 2, \ldots$. Consider $\theta : [0, \infty) \to [1, \infty)$ be such that $\theta(\rho) = e^\rho$. Then all conditions of Theorem 2 are satisfied.

Using mathematical induction, we show that $\Upsilon_\jmath$ satisfies the condition (1). If $\iota > \varsigma$ and $\jmath < j$, then we have

$$\theta(\rho(\frac{\sqrt{\iota}}{2^\jmath}, \frac{\sqrt{\varsigma}}{2^j}) \leq [\theta(\rho(\iota, \frac{\sqrt{\iota}}{2^\jmath}))\theta(\rho(\varsigma, \frac{\sqrt{\varsigma}}{2^j}))\theta(\frac{\rho(\iota, \frac{\sqrt{\varsigma}}{2^j}) + \rho(\varsigma, \frac{\sqrt{\iota}}{2^\jmath})}{2})]^{\frac{1}{2^{\jmath+2}}} (\theta(\rho(\iota, \varsigma)))^{\frac{1}{2^{2i+1}}}.$$

We know that the greatest value of the first side in (1) is when $\jmath = 1, j \to \infty$. Suppose that for $\jmath = 1$ and $j = k$. So, we have

$$e^{(\frac{\sqrt{\iota}}{2}-\frac{\sqrt{\varsigma}}{2^k})} \leq [e^{(\iota-\frac{\sqrt{\iota}}{2})}e^{(\varsigma-\frac{\sqrt{\varsigma}}{2^k})}e^{\frac{(\iota-\frac{\sqrt{\varsigma}}{2^k})+(\varsigma-\frac{\sqrt{\iota}}{2})}{2}}]^{\frac{1}{2^3}}(e^{(\iota-\varsigma)})^{\frac{1}{2^3}}$$
$$= e^{\frac{1}{8}[(\iota-\frac{\sqrt{\iota}}{2})+(\varsigma-\frac{\sqrt{\varsigma}}{2^k})+\frac{(\iota-\frac{\sqrt{\varsigma}}{2^k})+(\varsigma-\frac{\sqrt{\iota}}{2})}{2}]+\frac{1}{8}(\iota-\varsigma)}.$$

For $j = k + 1$, we obtain

$$\mathcal{A} := e^{(\frac{\sqrt{\iota}}{2}-\frac{1}{2}\frac{\sqrt{\varsigma}}{2^k})}$$
$$\leq [e^{(\iota-\frac{\sqrt{\iota}}{2})}e^{(\frac{\varsigma}{2}-\frac{1}{2}\frac{\sqrt{\varsigma}}{2^k})}e^{\frac{(\iota-\frac{1}{2}\frac{\sqrt{\varsigma}}{2^k})+(\frac{\varsigma}{2}-\frac{\sqrt{\iota}}{2})}{2}}]^{\frac{1}{2^3}}(e^{(\iota-\frac{\varsigma}{2})})^{\frac{1}{2^3}}$$
$$= e^{\frac{1}{8}[(\iota-\frac{\sqrt{\iota}}{2})+(\frac{\varsigma}{2}-\frac{1}{2}\frac{\sqrt{\varsigma}}{2^k})+\frac{(\iota-\frac{1}{2}\frac{\sqrt{\varsigma}}{2^k})+(\frac{\varsigma}{2}-\frac{\sqrt{\iota}}{2})}{2}]+\frac{1}{8}(\iota-\frac{\varsigma}{2}))} := \mathcal{B},$$

so

$$\mathcal{A} \leq e^{\frac{1}{2}(\frac{\sqrt{\iota}}{2}-\frac{\sqrt{\varsigma}}{2^k})+\frac{1}{2}(\frac{\sqrt{\iota}}{2})}$$
$$\leq e^{\frac{1}{2}[\frac{1}{8}[(\iota-\frac{\sqrt{\iota}}{2})+(\varsigma-\frac{\sqrt{\varsigma}}{2^k})+\frac{(\iota-\frac{\sqrt{\varsigma}}{2^k})+(\varsigma-\frac{\sqrt{\iota}}{2})}{2}]+\frac{1}{8}(\iota-\varsigma)]+\frac{\sqrt{\iota}}{4}}$$
$$\leq \mathcal{B}.$$

Since $\rho(\iota, \varsigma)$ is symmetric, the role of $\jmath, j$ can be changed together and a similar result can be reached. Then, the condition (1) is satisfied for all $\jmath, j$. Moreover, the series $\sum_{\jmath=1}^{+\infty} \frac{2^{\jmath+1}+2}{2^{2i+2}-2^{\jmath}}$ is an α-series with $\alpha = \frac{1}{2}$. Therefore, all conditions of Theorem 2 are hold and $\Upsilon_\jmath$ has unique common fixed point 0 in X.

3 Application

Integral equations are equations that involve a function $f(x)$ and the integral of that function is solved for $f(x)$. When the integral's bounds are constant, the integral equation is known as the Fredholm integral equation. A Volterra integral equation is one in which one of the limits is variable. If the unknown function is solely beneath the integral sign, the equation is said to be of the first kind. If the function is both inside and outside, the equation is called of the second type. When the outside unknown function is equal to 0, it is said to be homogenous.

We consider a system of Fredholm integral equations of the second type as follows:

$$\iota(t) = \int_0^\ell (f_\jmath(t, \iota(v))dv, \quad t \in [0, \ell], \quad \ell > 0, \tag{18}$$

where $f_J : [0, \ell] \times R \to R$ is integrable. Let $\mathsf{X} = C([0, \ell], R)$ be the set of real continuous functions defined on $[0, \ell]$, and the sequence of mappings $\Upsilon_J : \mathsf{X} \to \mathsf{X}$ be defined by

$$\Upsilon_J(\iota)(t) = \int_0^\ell (f_J(t, \iota(v))dv, \quad t \in [0, \ell].$$

Theorem 4 *Let* $\mathsf{X} = C([0, \ell], R)$ *and* $\rho : \mathsf{X}^2 \to [0, \infty)$ *be defined by* $\rho(\iota, \varsigma) = \sup_{t\in[0,\ell]} |\iota(t) - \varsigma(t)|$ *for every* $\iota, \varsigma \in \mathsf{X}$. *Suppose that there exsits a function* $\theta \in \Re$ *so that* $\theta(\int_0^\ell |f_J(t, \iota(v)) - f_j(t, \varsigma(v))|dv) \leq \int_0^\ell \theta(|f_J(t, \iota(v)) - f_j(t, \varsigma(v))|)dv$ *for arbitrary functions* f_J *and suppose that*

$$\begin{aligned}\theta|f_J(t, \iota(v)) - f_j(t, \varsigma(v))| \leq \frac{1}{\ell}[(\theta(|\iota(v) - \Upsilon_J\iota(v)|)\theta(|\varsigma(v) - \Upsilon_j\varsigma(v)|)\\ \theta(\frac{|\iota(v) - \Upsilon_j\varsigma(v)| + |\varsigma(v) - \Upsilon_J\iota(v)|}{2})^{\pi_{J,j}}\\ (\theta(|\iota(v) - \varsigma(v)|))^{\varpi_{J,j}}],\end{aligned}$$

for all $\iota, \varsigma \in \mathbb{R}$, *where* $\pi_{J,j} = \frac{1}{2^{J+2}}$, *and* $\varpi_{J,j} = \frac{1}{2^{2i+1}}$. *Then, the integral equation (18) has a unique solution in* X.

Proof Clearly, (X, ρ) is a CMS. We have

$$\begin{aligned}&\theta(|\Upsilon_J(\iota)(t) - \Upsilon_j(\varsigma)(t)|) = \theta(|\int_0^\ell f_J(t, \iota(v))dv - \int_0^\ell f_j(t, \varsigma(v))dv|)\\ &\leq \int_0^\ell \theta(|f_J(t, \iota(v)) - f_j(t, \varsigma(v))|)dv\\ &\leq \frac{1}{\ell}\int_0^\ell \Big(\big[\theta(|\iota(t) - \int_0^\ell f_J(t, \iota(v))dv|)\theta(|\varsigma(t) - \int_0^\ell f_j(t, \varsigma(v))dv|)\\ &\theta(\frac{|\iota(t) - \int_0^\ell f_j(t, \varsigma(v)dv| + |\varsigma(t) - \int_0^\ell f_J(t, \iota(v))dv|}{2}]^{\pi_{J,j}}\\ &(\theta(|\iota(t) - \varsigma(t)|))^{\varpi_{J,j}}dv\Big),\\ &\leq \frac{1}{\ell}\int_0^\ell [\theta(\rho(\iota, \Upsilon_J\iota)\theta(\rho(\varsigma, \Upsilon_j\varsigma))\theta(\frac{\rho(\iota, \Upsilon_j\varsigma) + \rho(\varsigma, \Upsilon_J\iota)}{2})]^{\pi_{J,j}}(\theta(\rho(\iota, \varsigma)))^{\varpi_{J,j}}dv\\ &\leq \frac{1}{\ell}[\theta(\rho(\iota, \Upsilon_J\iota))\theta(\rho(\varsigma, \Upsilon_j\varsigma))\theta(\frac{\rho(\iota, \Upsilon_j\varsigma) + \rho(\varsigma, \Upsilon_J\iota)}{2})]^{\pi_{J,j}}(\theta(\rho(\iota, \varsigma)))^{\varpi_{J,j}}\int_0^\ell dv\\ &= [\theta(\rho(\iota, \Upsilon_J\iota)\theta(\rho(\varsigma, \Upsilon_j\varsigma))\theta(\frac{\rho(\iota, \Upsilon_j\varsigma) + \rho(\varsigma, \Upsilon_J\iota)}{2})]^{\pi_{J,j}}(\theta(\rho(\iota, \varsigma)))^{\varpi_{J,j}}.\end{aligned}$$

So, the system of integral equations (18) has a unique solution in X. In other words, Υ_J has a unique common fixed point.

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Solution of Nonlinear First-Order Hybrid Integro-Differential Equations via Fixed Point Theorem

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Abstract The idea of a $(\mathcal{F}, \beta, \psi)$-contractive condition for a pair of maps is introduced in this chapter, and several common fixed-point findings for $\alpha - \eta-$admissible mappings in a Banach space are based on it. We give two good examples to back up the fixed-point result. We use this proven fixed-point finding to a pair of first-order ordinary nonlinear hybrid integro-differential equations of the type to get a shared solution of the form

$$\begin{cases} u'(t) + \lambda u(t) = f_j\left(t, u(t), \int_{\theta_0}^{t} g_j(s, u(s))ds\right), \; t \in I = [\theta_0, \theta_0 + \hbar] \subset \mathbb{R}, \theta_0 \geq 0, \hbar > 0, \text{ j=1,2} \\ u(\theta_0) = \mu_0 \in \mathbb{R}, \end{cases}$$

for some $\lambda \in \mathbb{R}$, $\lambda > 0$, where $g_j : I \times \mathbb{R} \to \mathbb{R}$ and $f_j : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

1 Introduction

A natural extension of fixed-point findings for various forms of contractions in metric spaces may be obtained by adding a (partial) ordering structure to the metric space (X, d). This is accomplished by adding an ordering structure to the metric space (X, d). Turinici [12, 13] published articles in 1986 that established some of the first results in this approach. It should be noted that their beginning points were the

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"amorphous" contributions to the field made by Matkowski [3, 4] and other authors. These findings have been rediscovered by Ran and Reurings [8]; also see Nieto and Rodriguez-Lopez [6] and Nieto and R'odriguez-Lopez [6] for further information. The findings of Turinici were further developed and enhanced in articles [6, 7]. Nashine and Samet [5], as well as many others, went on to expand the results of [8] and offer applications to ordinary differential equations and integral equations, respectively.

Similar to this, Samet et al. [10] developed the notion of α-admissible mappings and established a number of fixed-point theorems for (α, ψ)-contractive mappings fulfilling the α-admissibility condition in full metric spaces, among other things. In the context of fixed-point findings, this idea (or other versions of it, such as [1, 9, 11]) has been used by several writers to prove various variants.

In this work, we derive some common fixed-point results under $(\mathcal{F}, \beta, \psi)$-contractive conditions with an $\alpha - \eta-$admissible mapping for a pair of mappings in the setting of normed spaces. We extend and generalize the results of Mohammadi et al. [2] for a pair of maps and generalize admissible mappings. We provide two justify examples to validate the main fixed-point result. Further, we drive some consequences of the main results and apply this established fixed-point result to get a common solution for a pair of first-order ordinary nonlinear hybrid integro-differential equations of the form

$$\begin{cases} u'(t) + \lambda u(t) = f_j\left(t, u(t), \int_{\theta_0}^t g_j(s, u(s))ds\right), \; t \in I = [\theta_0, \theta_0 + \hbar] \subset \mathbb{R}, \theta_0 \geq 0, \hbar > 0, \text{j} = 1{,}2 \\ u(\theta_0) = \mu_0 \in \mathbb{R}, \end{cases}$$

for some $\lambda \in \mathbb{R}$, $\lambda > 0$, where $g_j : I \times \mathbb{R} \to \mathbb{R}$ and $f_j : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

2 Preliminaries

Denote $\mathbb{R}$:= the set of real numbers, $\mathbb{R}_+ := [0, +\infty)$, $\mathbb{N}$:= the set of natural numbers, and $\mathbb{N}^* := \mathbb{N} \cup \{0\}$.

Definition 1 Let $\mathcal{Z}$ be a nonempty set. Let $\alpha, \eta : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}_+$ and $\mathcal{P}, \mathcal{Q} : \mathcal{Z} \to \mathcal{Z}$ be given mappings. Then

- Reference [10] $\mathcal{P}$ is said to be α-admissible if for all $p, q \in \mathcal{Z}$ with $\alpha(p, q) \geq 1$ implies $\alpha(\mathcal{P}p, \mathcal{P}q) \geq 1$.
- Reference [9] $\mathcal{P}$ is said to be an α-admissible mapping with respect to η if for all $p, q \in \mathcal{Z}$ with $\alpha(p, q) \geq \eta(p, q)$ implies $\alpha(\mathcal{P}p, \mathcal{P}q) \geq \eta(\mathcal{P}p, \mathcal{P}q)$.
- Reference [1] The pair $(\mathcal{P}, \mathcal{Q})$ is said to be a generalized α-admissible pair if for all $p, q \in \mathcal{Z}$, $\alpha(p, q) \geq 1$ implies $\alpha(\mathcal{P}p, \mathcal{Q}q) \geq 1$ and $\alpha(\mathcal{Q}\mathcal{P}p, \mathcal{P}\mathcal{Q}q) \geq 1$.

Next, we define an $\alpha - \eta-$admissible mapping for a pair of maps as a generalization of all above given concepts.

Definition 2 ([11]) Let $\mathcal{Z}$ be a nonempty set. Let $\alpha, \eta : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}_+$ be given mappings. We say that the pair $\mathcal{P}, \mathcal{Q}$: $\mathcal{Z} \to \mathcal{Z}$ is a generalized $\alpha - \eta$-admissible pair if for all $p, q \in \mathcal{Z}$, $\alpha(p, q) \geq \eta(p, q)$ implies $\alpha(\mathcal{P}p, \mathcal{Q}q) \geq \eta(\mathcal{P}p, \mathcal{Q}q)$ and $\alpha(\mathcal{Q}\mathcal{P}p, \mathcal{P}\mathcal{Q}q) \geq \eta(\mathcal{Q}\mathcal{P}p, \mathcal{P}\mathcal{Q}q)$.

Remark 1
- If we take $\eta(p, q) = 1$, then we say that the pair $(\mathcal{P}, \mathcal{Q})$ is a generalized α-admissible map. Also, if we take $\alpha(p, q) = 1$, then we say that the pair $(\mathcal{P}, \mathcal{Q})$ is an η-subadmissible mapping.
- If $\mathcal{P} = \mathcal{P}^{-1}$ with $\mathcal{Q} = \mathcal{P}$, then we get the α-admissible with respect to η notion. Thus, the class of mappings $(\mathcal{P}, \mathcal{Q}) \neq \emptyset$.
- If $\mathcal{P}$ is α-admissible with respect to η, it is obvious that $(\mathcal{P}, \mathcal{P})$ is a generalized $\alpha - \eta$-admissible pair.

Next, we define triangular $\alpha - \eta$-admissible property for a pair $(\mathcal{P}, \mathcal{Q})$.

Definition 3 ([11]) Let $\mathcal{P}, \mathcal{Q}$ be two self-mappings on $\mathcal{Z}$. Let $\alpha, \eta : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}_+$ be mappings. A pair $(\mathcal{P}, \mathcal{Q})$ is called triangular $\alpha - \eta$-admissible if

(i) the pair $(\mathcal{P}, \mathcal{Q})$ be a generalized $\alpha - \eta$-admissible pair; and
(ii) for any $p, q, r \in \mathcal{Z}, \alpha(p, q) \geq \eta(p, q), \alpha(q, r) \geq \eta(q, r) \Rightarrow \alpha(p, r) \geq \eta(p, r)$.

We denote by $\Xi(\mathcal{P}, \mathcal{Q}, \alpha, \eta)$, the collection of all triangular $\alpha - \eta$-admissible pairs $(\mathcal{P}, \mathcal{Q})$.

For sake of completion, we proof the following lemma.

Lemma 1 ([11]) *Let $(\mathcal{P}, \mathcal{Q}) \in \Xi(\mathcal{P}, \mathcal{Q}, \alpha, \eta)$. Assume that there exists $\theta_0 \in \mathcal{Z}$ such that $\alpha(\theta_0, \mathcal{P}\theta_0) \geq \eta(\theta_0, \mathcal{P}\theta_0)$. Define a sequence $\{\theta_n\}$ by $\theta_{2n+1} = \mathcal{P}\theta_{2n}$ and $\theta_{2n+2} = \mathcal{Q}\theta_{2n+1}$ where $n \in \mathbb{N}^*$. Then $\alpha(\theta_n, \theta_m) \geq \eta(\theta_n, \theta_m)$ for all $m, n \in \mathbb{N}$ with $m > n$.*

Proof Given $\theta_0 \in \mathcal{Z}$ satisfying $\alpha(\theta_0, \theta_1) = \alpha(\theta_0, \mathcal{P}\theta_0) \geq \eta(\theta_0, \mathcal{P}\theta_0) = \eta(\theta_0, \theta_1)$. Since the pair $(\mathcal{P}, \mathcal{Q})$ is a generalized $\alpha - \eta$-admissible pair, we choose $\theta_2, \theta_3 \in \mathcal{Z}$ such that

$$\alpha(\mathcal{P}\theta_0, \mathcal{Q}\theta_1) \geq \eta(\mathcal{P}\theta_0, \mathcal{Q}\theta_1) \text{ and } \alpha(\mathcal{Q}\mathcal{P}\theta_0, \mathcal{P}\mathcal{Q}\theta_1) \geq \eta(\mathcal{Q}\mathcal{P}\theta_0, \mathcal{P}\mathcal{Q}\theta_1),$$

that is,

$$\alpha(\theta_1, \theta_2) \geq \eta(\theta_1, \theta_2) \text{ and } \alpha(\theta_2, \theta_3) \geq \eta(\theta_2, \theta_3).$$

Repeating this process, we have $\alpha(\theta_n, \theta_{n+1}) \geq \eta(\theta_n, \theta_{n+1})$ for all $n \in \mathbb{N}^*$. Suppose that $\alpha(\theta_n, \theta_m) \geq \eta(\theta_n, \theta_m)$. We will prove that $\alpha(\theta_n, \theta_{m+1}) \geq \eta(\theta_n, \theta_{m+1})$, where $m > n$. Since $\alpha(\theta_m, \theta_{m+1}) \geq \eta(\theta_m, \theta_{m+1})$, $(\mathcal{P}, \mathcal{Q}) \in \Xi(\mathcal{P}, \mathcal{Q}, \alpha, \eta)$ implies that $\alpha(\theta_n, \theta_{m+1}) \geq \eta(\theta_n, \theta_{m+1})$ for all $m, n \in \mathbb{N}^*$ with $m > n$.

3 Main Results

We start with defining some control functions discussed in [2].

Denote $\mathbb{F} := \{\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}\}$ satisfying $(\mathcal{F}_1)$ $\mathcal{F}$ is strictly increasing and continuous; $(\mathcal{F}_2)$ $\mathcal{F}(r) = 0 \Leftrightarrow r = 1$.

Examples of $\mathbb{F}$ for $r \in \mathbb{R}^+$ are (i) $\mathcal{F}_1(r) = \ln(r)$, (ii) $\mathcal{F}_3(r) = -\frac{1}{\sqrt{r}} + 1$.

Denote $\Lambda := \{\beta : (0, \infty) \longrightarrow [0, 1) \ \ | \ \ \limsup_{\omega \longrightarrow t^+} \beta(\omega) < 1, \text{ for any } t \geq 0\}$.

Denote $\Psi := \{\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+\}$ satisfying (ψ_1) $\psi(r) = 0 \Leftrightarrow r = 0$; (ψ_2) ψ is nondecreasing and continuous.

We now introduce the notion of an $(\mathcal{F}, \beta, \psi)$-contractive mapping in a normed space.

Definition 4 Let $(\mathcal{Z}, \|.\|)$ be a normed space and $\mathcal{P}, \mathcal{Q} \colon \mathcal{Z} \to \mathcal{Z}$ be given mappings. We say that $(\mathcal{P}, \mathcal{Q})$ is a $(\mathcal{F}, \beta, \psi)$-contractive pair if there exist $\mathcal{F} \in \mathbb{F}, \beta \in \Lambda, \psi \in \Psi$ such that for all $p, q \in \mathcal{Z}$,

$$\alpha(p,q) \geq \eta(p,q) \Rightarrow \mathcal{F}(\psi(\|\mathcal{P}p - \mathcal{Q}q\|)) \leq \mathcal{F}(\beta(\psi(\Delta(p,q)))) + \mathcal{F}(\psi(\Delta(p,q))), \quad (1)$$

where

$$\Delta(p,q) = \max\{\|p - q\|, \|p - \mathcal{P}p\|, \|q - \mathcal{Q}q\|\}.$$

The set of all fixed (common) points of a self-mapping $\mathcal{P}$ (and self-mapping $\mathcal{Q}$) on a set $\mathcal{Z} \neq \emptyset$ is denoted by $Fix(\mathcal{P})$ $(CFP(\mathcal{P}, \mathcal{Q}))$.

Theorem 1 *Let $\mathcal{Z}$ be a closed subset of Banach space $\mathfrak{X}$, and let $\mathcal{P}, \mathcal{Q} : \mathcal{Z} \to \mathcal{Z}$ be $(\mathcal{F}, \beta, \psi)$-contractive mappings. The following hypotheses are assumed:*

$(\mathcal{H}_1)$ there exists $\upsilon_0 \in \mathcal{Z}$ such that $\alpha(\upsilon_0, \mathcal{P}\upsilon_0) \geq \eta(\upsilon_0, \mathcal{P}\upsilon_0)$;
$(\mathcal{H}_2)$ $(\mathcal{P}, \mathcal{Q}) \in \Xi(\mathcal{P}, \mathcal{Q}, \alpha, \eta)$;
$(\mathcal{H}_3)$ $\alpha(\mathcal{Q}\mathcal{P}p, \mathcal{P}p) \geq \eta(\mathcal{Q}\mathcal{P}p, \mathcal{P}p)$ for all $p \in \mathcal{Z}$;
$(\mathcal{H}_4)$ $\mathcal{P}$ and $\mathcal{Q}$ are continuous, or
$(\mathcal{H}_4')$ if $\{\upsilon_n\}$ be a sequence in $\mathcal{Z}$ such that $\alpha(\upsilon_n, \upsilon_{n+1}) \geq \eta(\upsilon_n, \upsilon_{n+1})$ and $\alpha(\upsilon_{n+1}, \upsilon_n) \geq \eta(\upsilon_{n+1}, \upsilon_n)$ for all n and $\upsilon_n \to \omega \in \mathcal{Z}$ as $n \to \infty$, then there exists a subsequence $\{\upsilon_{n_k}\}$ of $\{\upsilon_n\}$ such that $\alpha(\upsilon_{n_k}, \omega) \geq \eta(\upsilon_{n_k}, \omega)$ and $\alpha(\omega, \upsilon_{n_k}) \geq \eta(\omega, \upsilon_{n_k})$ for all k.

Then $\mathcal{P}$ and $\mathcal{Q}$ have a common fixed point. If

$(\mathcal{H}_5)$ for all $\zeta, \xi \in CFP(\mathcal{Q}, \mathcal{P})$, we have $\alpha(\zeta, \xi) \geq \eta(\zeta, \xi)$

hold, then $\mathcal{P}$ and $\mathcal{Q}$ have a unique common fixed point.

Proof Start with $\upsilon_0 \in \mathcal{Z}$, an arbitrarily element, we construct a sequence $\{\upsilon_m\}$ in $\mathcal{Z}$ as:

$$\upsilon_{2m+1} = \mathcal{P}\upsilon_{2m} \text{ and } \upsilon_{2m+2} = \mathcal{Q}\upsilon_{2m+1} \text{ where } m \in \mathbb{N}^*.$$

For $\upsilon_0 \in \mathcal{Z}$, using $(\mathcal{H}_1)$, we have $\alpha(\upsilon_0, \mathcal{P}\nu_0) \geq \eta(\upsilon_0, \mathcal{P}\upsilon_0)$. Also from $(\mathcal{H}_2)$, we have $(\mathcal{P}, \mathcal{Q}) \in \Xi(\mathcal{P}, \mathcal{Q}, \alpha, \eta)$. Therefore, for $\upsilon_{2n} \neq \upsilon_{2n+1}$ for all $n \in \mathbb{N}^*$, by Lemma 1,

$$\alpha(\upsilon_{2n}, \upsilon_{2n+1}) \geq \eta(\upsilon_{2n}, \upsilon_{2n+1}) \text{ for all } n \in \mathbb{N}^*. \tag{2}$$

Also, $(\mathcal{H}_3)$ implies that

$$\begin{aligned} &\alpha(\upsilon_2, \upsilon_1) = \alpha(\mathcal{QP}\upsilon_0, \mathcal{P}\upsilon_0) \geq \eta(\mathcal{QP}\upsilon_0, \mathcal{P}\upsilon_0) = \eta(\upsilon_2, \upsilon_1) \\ &\text{and} \\ &\alpha(\upsilon_4, \upsilon_3) = \alpha(\mathcal{QP}\upsilon_2, \mathcal{P}\upsilon_2) \geq \eta(\mathcal{QP}\upsilon_2, \mathcal{P}\upsilon_2) = \eta(\upsilon_4, \upsilon_3). \end{aligned}$$

Continuing this process, we get

$$\alpha(\upsilon_{2n}, \upsilon_{2n-1}) \geq \eta(\upsilon_{2n}, \upsilon_{2n-1}), \ \ for\ all\ n \in \mathbb{N}. \tag{3}$$

Keeping generality in mind, we consider $\upsilon_m \neq \upsilon_{m+1}$ for each $m \in \mathbb{N}^*$. Indeed, if $\upsilon_{m_0} = \upsilon_{m_0+1}$ for some $m_0 \in \mathbb{N}^*$, then $\vartheta^* = \upsilon_{m_0}$ will be in $CFP(\mathcal{P}, \mathcal{Q})$ which is complete the proof. More precisely, to see that $\vartheta^* \in CFP(\mathcal{P}, \mathcal{Q})$, we can discuss two cases. First, we assume that m_0 is even, that is, $m_0 = 2r$. In this case, we have $\upsilon_{2r} = \upsilon_{2r+1} = \mathcal{P}\upsilon_{2r}$, that is, υ_{2r} is a fixed point of $\mathcal{P}$. Now, we shall prove that $\upsilon_{2r} = \upsilon_{2r+1} = \mathcal{P}\upsilon_{2r} = \mathcal{Q}\upsilon_{2r+1}$. Suppose on the contrary that $\|\mathcal{P}\upsilon_{2r} - \mathcal{Q}\upsilon_{2r+1}\| > 0$. Using (2), we can apply (1) for $p = \upsilon_{2r}$ and $q = \upsilon_{2r+1}$ which imples

$$\begin{aligned} \mathcal{F}(\psi(\|\upsilon_{2r+1} - \upsilon_{2r+2}\|)) &= \mathcal{F}(\psi(\|\mathcal{P}\upsilon_{2r} - \mathcal{Q}\upsilon_{2r+1}\|)) \\ &\leq \mathcal{F}(\beta(\psi(\Delta(\upsilon_{2r}, \upsilon_{2r+1})))) + \mathcal{F}(\psi(\Delta(\upsilon_{2r}, \upsilon_{2r+1}))), \end{aligned}$$

where

$$\begin{aligned} \Delta(\upsilon_{2r}, \upsilon_{2r+1}) &= \max\{\|\upsilon_{2r} - \upsilon_{2r+1}\|, \|\upsilon_{2r} - \mathcal{P}\upsilon_{2r}\|, \|\upsilon_{2r+1} - \mathcal{Q}\upsilon_{2r+1}\|\} \\ &= \max\{\|\upsilon_{2r} - \upsilon_{2r+1}\|, \|\upsilon_{2r} - \upsilon_{2r+1}\|, \|\upsilon_{2r+1} - \upsilon_{2r+2}\|\} \\ &= \|\upsilon_{2r+1} - \upsilon_{2r+2}\|. \end{aligned}$$

Therefore,

$$\mathcal{F}(\psi(\|\upsilon_{2r+1} - \upsilon_{2r+2}\|)) \leq \mathcal{F}(\beta(\psi(\|\upsilon_{2r+1} - \upsilon_{2r+2}\|))) + \mathcal{F}(\psi(\|\upsilon_{2r+1} - \upsilon_{2r+2}\|)).$$

Since $\beta(\psi(\|\upsilon_{2r+1} - \upsilon_{2r+2}\|)) < 1$ and $\mathcal{F}$ is strictly increasing, we get $\mathcal{F}(\beta(\psi(\|\upsilon_{2r+1} - \upsilon_{2r+2}\|)) < \mathcal{F}(1) = 0$. Therefore,

$$\begin{aligned} \mathcal{F}(\psi(\|\upsilon_{2r+1} - \upsilon_{2r+2}\|)) &\leq \mathcal{F}(\beta(\psi(\|\upsilon_{2r+1} - \upsilon_{2r+2}\|))) + \mathcal{F}(\psi(\|\upsilon_{2r+1} - \upsilon_{2r+2}\|)) \\ &< \mathcal{F}(\psi(\|\upsilon_{2r+1} - \upsilon_{2r+2}\|)), \end{aligned}$$

a contradiction, and hence $\|\mathcal{P}\upsilon_{2r} - \mathcal{Q}\upsilon_{2r+1}\| = 0$, $\upsilon_{2r} = \upsilon_{2r+1} = \mathcal{P}\upsilon_{2r} = \mathcal{Q}\upsilon_{2r+1}$, that is, $\upsilon_{2r} = \upsilon_{2r+1} = \vartheta^* \in CFP(\mathcal{P}, \mathcal{Q})$. Similarly, it can be shown when m_0 is odd, say, $m_0 = 2r - 1$.

Thus, we assume $\upsilon_m \neq \upsilon_{m+1}$ for each $m \in \mathbb{N}^*$ and let $\varrho_m = \psi(\|\upsilon_m - \upsilon_{m+1}\|) = \psi(\sigma_m)$ for $m \in \mathbb{N}^*$. Then $\varrho_m > 0$ for all $m \in \mathbb{N}^*$. We will prove that $\lim_{n\to\infty} \sigma_m = 0$.

Following (2), we can apply (1) for $p = \upsilon_{2n-1}$ and $q = \upsilon_{2n}$

$$\mathcal{F}(\varrho_{2n}) = \mathcal{F}(\psi(\|\mathcal{P}\upsilon_{2n-1} - \mathcal{Q}\upsilon_{2n}\|)) \leq \mathcal{F}(\beta(\psi(\Delta(\upsilon_{2n-1}, \upsilon_{2n})) + \mathcal{F}(\psi(\Delta(\upsilon_{2n-1}, \upsilon_{2n})))$$

where

$$\begin{aligned}\Delta(\upsilon_{2n-1}, \upsilon_{2n}) &= \max\{\|\upsilon_{2n-1} - \upsilon_{2n}\|, \|\upsilon_{2n-1} - \mathcal{P}\upsilon_{2n-1}\|, \|\upsilon_{2n} - \mathcal{Q}\upsilon_{2n}\|\} \\ &= \max\{\|\upsilon_{2n-1} - \upsilon_{2n}\|, \|\upsilon_{2n-1} - \upsilon_{2n}\|, \|\upsilon_{2n} - \upsilon_{2n+1}\|\} \\ &= \max\{\|\upsilon_{2n-1} - \upsilon_{2n}\|, \|\upsilon_{2n} - \upsilon_{2n+1}\|\} \\ &= \max\{\sigma_{2n-1}, \sigma_{2n}\}.\end{aligned}$$

Therefore,

$$\mathcal{F}(\varrho_{2n}) \leq \mathcal{F}(\beta(\psi(\max\{\sigma_{2n-1}, \sigma_{2n}\}))) + \mathcal{F}(\psi(\max\{\sigma_{2n-1}, \sigma_{2n}\})). \tag{4}$$

We shall show that $\{\varrho_n\}$ is a nonincreasing sequence. Indeed, if $\varrho_{2n-1} < \varrho_{2n}$ for some $n \in \mathbb{N}$, then (4) implies that

$$\mathcal{F}(\varrho_{2n}) \leq \mathcal{F}(\beta(\varrho_{2n})) + \mathcal{F}(\varrho_{2n}).$$

Since $\beta(\varrho_{2n}) < 1$ and $\mathcal{F}$ is strictly increasing, we get $\mathcal{F}(\beta(\varrho_{2n})) < \mathcal{F}(1) = 0$. Therefore, we have

$$\mathcal{F}(\varrho_{2n}) \leq \mathcal{F}(\beta(\varrho_{2n})) + \mathcal{F}(\varrho_{2n}) < \mathcal{F}(\varrho_{2n}),$$

a contradiction. Hence, $\varrho_{2n} \leq \varrho_{2n-1}$ for all $n \in \mathbb{N}$. Similarly, using (1) and (3), we have $\varrho_{2n+1} \leq \varrho_{2n}$ for all $n \in \mathbb{N}_0$. Therefore, for all $n \in \mathbb{N}$, $\varrho_n \leq \varrho_{n-1}$ and thus the sequence $\{\varrho_n\} = \psi(\sigma_n)$ is a nonincreasing sequence of positive real numbers, and as ψ is increasing, so $\{\sigma_n\}$ is decreasing. Then there is $\omega \geq 0$ so that $\{\sigma_n\}$ converges to ω. Since ψ is continuous,

$$\psi(\omega) = \lim_{n\longrightarrow\infty} \psi(\sigma_n) = 0. \tag{5}$$

Therefore, $\omega = 0$. Hence, $\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \|\upsilon_n - \upsilon_{n+1}\| = 0$. Similarly, it can be shown that

$$\lim_{n\to\infty} \|\upsilon_{n+1} - \upsilon_n\| = 0. \tag{6}$$

Following that, we shall demonstrate that $\{\upsilon_n\}$ is a Cauchy sequence in the closed subset $\mathcal{Z}$ of the Banach space $\mathfrak{X}$. It is sufficient to demonstrate that $\{\upsilon_{2n}\}$ is a Cauchy sequence in the closed subset $\mathcal{Z}$ of Banach space $\mathfrak{X}$ using (6). We begin with negation and assume that $\{\upsilon_{2n}\}$ is not Cauchy. Then, for any even integer $2k$, we can discover a $\eta > 0$ such that there are even integers $2m_k > 2n_k > 2k$ such that

$$\|\upsilon_{2n_k} - \upsilon_{2m_k}\| \geq \eta \text{ for } k \in \{1, 2, \dots\}. \quad (7)$$

We may also assume

$$\|\upsilon_{2m_k-2} - \upsilon_{2n_k}\| < \eta \quad (8)$$

by choosing $2m_k$ to be the smallest number exceeding $2n_k$ for which (7) holds. Now (6)–(8) imply

$$\begin{aligned}0 < \eta &\leq \|\upsilon_{2n_k} - \upsilon_{2m_k}\| \\ &\leq \|\upsilon_{2n_k} - \upsilon_{2m_k-2}\| + \|\upsilon_{2m_k-2} - \upsilon_{2m_k-1}\| + \|\upsilon_{2m_k-1} - \upsilon_{2m_k}\| \\ &\leq \eta + \|\upsilon_{2m_k-2} - \upsilon_{2m_k-1}\| + \|\upsilon_{2m_k-1} - \upsilon_{2m_k}\|\end{aligned}$$

and so

$$\lim_{k\to\infty} \|\upsilon_{2n_k} - \upsilon_{2m_k}\| = \eta. \quad (9)$$

Also, by the triangular inequality,

$$\left|\|\upsilon_{2n_k} - \upsilon_{2m_k-1}\| - \|\upsilon_{2n_k} - \upsilon_{2m_k}\|\right| \leq \|\upsilon_{2m_k-1} - \upsilon_{2m_k}\|$$

and

$$\left|\|\upsilon_{2n_k+1} - \upsilon_{2m_k-1}\| - \|\upsilon_{2n_k} - \upsilon_{2m_k}\|\right| \leq \|\upsilon_{2m_k-1} - \upsilon_{2m_k}\| + \|\upsilon_{2n_k} - \upsilon_{2n_k+1}\|.$$

Therefore, we get

$$\lim_{k\to\infty} \|\upsilon_{2n_k} - \upsilon_{2m_k-1}\| = \eta \quad (10)$$

and

$$\lim_{k\to\infty} \|\upsilon_{2n_k+1} - \upsilon_{2m_k-1}\| = \eta. \quad (11)$$

Applying condition (1) to the pair $v = 2m_k + 1$ and $u = 2n_k + 1$

$$\begin{aligned}\mathcal{F}(\psi(\|\upsilon_{2n_k+1} - \upsilon_{2m_k+1}\|) &\leq \mathcal{F}(\psi(\|\upsilon_{2n_k+1} - \upsilon_{2m_k+1}\|)) \\ &\leq \mathcal{F}(\beta(\psi(\Delta(\upsilon_{2n_k}, \upsilon_{2m_k})) + \mathcal{F}(\psi(\Delta(\upsilon_{2n_k}, \upsilon_{2m_k})),\end{aligned} \quad (12)$$

where

$$\Delta(\upsilon_{2n_k}, \upsilon_{2m_k}) = \max\left\{\|\upsilon_{2n_k} - \upsilon_{2m_k}\|, \|\upsilon_{2n_k} - \upsilon_{2n_k+1}\|, \|\upsilon_{2m_k} - \upsilon_{2m_k+1}\|\right\},$$

implying that

$$\lim_{k\to\infty} \Delta(\upsilon_{2n_k}, \upsilon_{2m_k}) = \max\{0, \eta\} = \eta.$$

Taking the limit in (12), and using properties of the control functions,

$$\begin{aligned}\mathcal{F}(\psi(\eta)) &\le \liminf_{k\to\infty} \mathcal{F}(\psi(\|\upsilon_{2n_k+1}-\upsilon_{2m_k+1}\|)) \le \limsup_{k\to\infty} \mathcal{F}(\psi(\|\upsilon_{2n_k+1}-\upsilon_{2m_k+1}\|))\\ &\le \limsup_{k\to\infty}[\mathcal{F}(\beta(\psi(\Delta_1(\upsilon_{2n_k},\upsilon_{2m_k})))) + \mathcal{F}(\psi(\Delta_1(\upsilon_{2n_k},\upsilon_{2m_k})))]\\ &\le \mathcal{F}(\limsup_{k\to\infty} \beta(\psi(\|\upsilon_{2n_k}-\upsilon_{2m_k}\|))) + \mathcal{F}(\psi(\eta)). \end{aligned} \tag{13}$$

Since $\|\upsilon_{2n_k}-\upsilon_{2m_k}\| \to \eta^+$ and ψ is increasing, thus $\psi(\|\upsilon_{2n_k}-\upsilon_{2m_k}\|) \to \psi(\eta)^+$. So, $\limsup_{k\to\infty} \beta(\psi(\|\upsilon_{2n_k}-\upsilon_{2m_k}\|)) < 1$. Therefore, $\mathcal{F}(\limsup_{k\to\infty} \beta(\psi(\|\upsilon_{2n_k}-\upsilon_{2m_k}\|))) < 0$. Thus, inequality (13) implies that $\mathcal{F}(\psi(\eta)) < \mathcal{F}(\psi(\eta))$, a contradiction. Hence, $\eta = 0$. Thus, $\{\upsilon_{2n}\}$ is a Cauchy sequence in the closed subset $\mathcal{Z}$ of the Banach space $\mathfrak{X}$ and so convergent to some $\vartheta^* \in \mathcal{Z}$ with $\lim_{n\to\infty}\upsilon_n = \vartheta^*$. Therefore, the two subsequences $\{\upsilon_{2n}\}$ and $\{\upsilon_{2n+1}\}$ converge to ϑ^*.

Assume $(\mathcal{H}_4)$ holds. The continuity of $\mathcal{P}$ and $\mathcal{Q}$ imply that

$$\upsilon_{2n+1} = \mathcal{P}(\upsilon_{2n}) \to_{n\to\infty} \mathcal{P}(\vartheta^*) \text{ and } \upsilon_{2n+2} = \mathcal{Q}(\upsilon_{2n+1}) \to_{n\to\infty} \mathcal{Q}(\vartheta^*).$$

Uniqueness of limit of the sequence $\{\upsilon_n\}$ implies the result.

Assume $(\mathcal{H}_4)$' hold. As $\lim_{n\to\infty}\upsilon_n = \vartheta^*$, $(\mathcal{H}_4)$' implies that there exists a subsequence $\{\upsilon_{n_k}\}$ of $\{\upsilon_n\}$ such that $\alpha(\upsilon_{2n_k},\vartheta^*) \ge (\upsilon_{2n_k},\vartheta^*)$ and $\alpha(\vartheta^*,\upsilon_{2n_k-1}) \ge (\vartheta^*,\upsilon_{2n_k-1})$ for all k. Making use of (1) for $p = \upsilon_{2n_k}$ and $q = \vartheta^*$ with $\alpha(\upsilon_{2n_k},\vartheta^*) \ge (\upsilon_{2n_k},\vartheta^*)$,

$$\begin{aligned}\mathcal{F}(\psi(\|\upsilon_{2n_k+1}-\mathcal{Q}\vartheta^*\|)) &= \mathcal{F}(\psi(\|\mathcal{P}\upsilon_{2n_k}-\mathcal{Q}\vartheta^*\|))\\ &\le \mathcal{F}(\beta(\psi(\max\{\|\upsilon_{2n_k}-\vartheta^*\|, \|\upsilon_{2n_k}-\mathcal{P}\upsilon_{2n_k}\|, \|\vartheta^*-\mathcal{Q}\vartheta^*\|\})))\\ &\quad + \mathcal{F}(\psi(\max\{\|\upsilon_{2n_k}-\vartheta^*\|, \|\upsilon_{2n_k}-\mathcal{P}\upsilon_{2n_k}\|, \|\vartheta^*-\mathcal{Q}\vartheta^*\|\}))\\ &= \mathcal{F}(\beta(\psi(\max\{\|\upsilon_{2n_k}-\vartheta^*\|, \|\upsilon_{2n_k}-\upsilon_{2n_k+1}\|, \|\vartheta^*-\mathcal{Q}\vartheta^*\|\})))\\ &\quad + \mathcal{F}(\psi(\max\{\|\upsilon_{2n_k}-\vartheta^*\|, \|\upsilon_{2n_k}-\upsilon_{2n_k+1}\|, \|\vartheta^*-\mathcal{Q}\vartheta^*\|\})).\end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality, we infer

$$\begin{aligned}\mathcal{F}(\psi(\vartheta^*-\mathcal{Q}\vartheta^*\|)) &\le \mathcal{F}(\beta(\psi(\|\vartheta^*-\mathcal{Q}\vartheta^*\|)))\\ &\quad + \mathcal{F}(\psi(\|\vartheta^*-\mathcal{Q}\vartheta^*\|)).\end{aligned} \tag{14}$$

Since $\psi(\|\vartheta^*-\mathcal{Q}\vartheta^*\|) > 0$, we have $\beta(\psi(\|\vartheta^*-\mathcal{Q}\vartheta^*\|)) < 1 \Rightarrow \mathcal{F}(\beta(\psi(\|\vartheta^*-\mathcal{Q}\vartheta^*\|))) < 0$. Thus, inequality (14) implies that $\mathcal{F}(\psi(\|\vartheta^*-\mathcal{Q}\vartheta^*\|)) < \mathcal{F}(\psi(\|\vartheta^*-\mathcal{Q}\vartheta^*\|))$, a contradiction. Hence, the $Fix(\mathcal{Q}) = \{\vartheta^*\}$.

On the similar arguments, using (1) for $q = \upsilon_{2n_k-1}$ and $p = \vartheta^*$ with $\alpha(\vartheta^*, \upsilon_{2n_k-1}) \ge (\vartheta^*,\upsilon_{2n_k-1})$, we get $Fix(\mathcal{P}) = \{\vartheta^*\}$.

Assume $(\mathcal{H}_5)$ hold. To check singleness of the $CFP(\mathcal{P},\mathcal{Q}) = \{\vartheta^*\}$, let $\mu \ne \vartheta^*$ be such that $\mu = \mathcal{P}\mu = \mathcal{Q}\mu$. Using $\alpha(\vartheta^*,\mu) \ge \eta(\vartheta^*,\mu)$ with (1) for $p = \vartheta^*$, $q = \mu$, we have

$$\begin{aligned}\mathcal{F}(\psi(\|\vartheta^* - \mu\|)) &= \mathcal{F}(\psi(\|\mathcal{P}\vartheta^* - \mathcal{Q}\mu\|))\\ &\leq \mathcal{F}(\beta(\psi(\max\{\|\vartheta^* - \mu\|, \|\vartheta^* - \mathcal{P}\vartheta^*\|, \|\mu - \mathcal{Q}\mu\|\})))\\ &\quad + \mathcal{F}(\psi(\max\{\|\vartheta^* - \mu\|, \|\vartheta^* - \mathcal{P}\vartheta^*\|, \|\mu - \mathcal{Q}\mu\|\})) \qquad (15)\\ &= \mathcal{F}(\beta(\psi(\|\vartheta^* - \mu\|))) + \mathcal{F}(\psi(\|\vartheta^* - \mu\|)). \qquad (16)\end{aligned}$$

Since $\psi(\|\vartheta^* - \mu\|) > 0$, we have $\beta(\psi(\|\vartheta^* - \mu\|)) < 1 \Rightarrow \mathcal{F}(\beta(\psi(\|\vartheta^* - \mu\|))) < 0$. Thus, inequality (15) implies that $\mathcal{F}(\psi(\|\vartheta^* - \mu\|)) < \mathcal{F}(\psi(\|\vartheta^* - \mu\|))$, a contradiction. Hence, the $CFP(\mathcal{P}, \mathcal{Q}) = \{\vartheta^*\}$ is unique.

Taking $\mathcal{Q} = \mathcal{P}$ in Theorem 1, we can state the following result.

Corollary 1 *Let $\mathcal{Z}$ be a closed subset of Banach space $\mathfrak{X}$, and let $\mathcal{P} : \mathcal{Z} \to \mathcal{Z}$ be a mapping. The following hypotheses are assumed:*

($\mathcal{H}_1$) there exists $\upsilon_0 \in \mathcal{Z}$ such that $\alpha(\upsilon_0, \mathcal{P}\upsilon_0) \geq \eta(\upsilon_0, \mathcal{P}\upsilon_0)$;
($\mathcal{H}_2'$) $\mathcal{P}$ is an α-admissible mapping and $\mathcal{F} \in \mathbb{F}$, $\beta \in \Lambda$, $\psi \in \Psi$ such that for all $p, q \in \mathcal{Z}$,

$$\alpha(p, q) \geq \eta(p, q) \Rightarrow \mathcal{F}(\psi(\|\mathcal{P}p - \mathcal{P}q\|)) \leq \mathcal{F}(\beta(\psi(\Delta'(p, q)))) + \mathcal{F}(\psi(\Delta'(p, q))),$$

where

$$\Delta'(p, q) = \max\{\|p - q\|, \|p - \mathcal{P}p\|, \|q - \mathcal{P}q\|\}$$

($\mathcal{H}_3'$) $\mathcal{P}$ is continuous, or
($\mathcal{H}_4'$) if $\{\upsilon_n\}$ is a sequence in $\mathcal{Z}$ such that if $\alpha(\upsilon_n, \upsilon_{n+1}) \geq \eta(\upsilon_n, \upsilon_{n+1})$ for all n and $\upsilon_n \to \omega \in \mathcal{Z}$ as $n \to \infty$, then there exists a subsequence $\{\upsilon_{n_k}\}$ of $\{\upsilon_n\}$ such that $\alpha(\upsilon_{n_k}, \omega) \geq \eta(\upsilon_{n_k}, \omega)$ for all k.

Then $\mathcal{P}$ possesses a fixed point. If

($\mathcal{H}_5'$) for all $\zeta, \xi \in Fix(\mathcal{P})$, we have $\alpha(\zeta, \xi) \geq \eta(\zeta, \xi)$

hold, then $\mathcal{P}$ possesses a unique fixed point.

Let $\mathcal{Z}$ be a nonempty set. As has became standard, $(\mathcal{Z}, d, \preceq)$ will be called an ordered normed space if

(i) $(\mathcal{X}, d)$ is a normed space, and,
(ii) $(\mathcal{X}, \preceq)$ is a partially ordered set.

Theorem 2 *Let $\mathcal{Z}$ be a closed subset of ordered Banach space $\mathfrak{X}$, and let $\mathcal{P}, \mathcal{Q} : \mathcal{Z} \to \mathcal{Z}$ be mappings. The following hypotheses are assumed:*

($\mathcal{B}_1$) there exists $\upsilon_0 \in \mathcal{Z}$ such that $\upsilon_0 \preceq \mathcal{P}\upsilon_0$;
($\mathcal{B}_2$) $p \preceq q$ implies $\mathcal{P}p \preceq \mathcal{Q}q$ and $\mathcal{Q}\mathcal{P}p \preceq \mathcal{P}\mathcal{Q}q$ for all $p, q \in \mathcal{Z}$,
($\mathcal{B}_3$) there exist $\mathcal{F} \in \mathbb{F}$, $\beta \in \Lambda$, $\psi \in \Psi$ such that for all $p, q \in \mathcal{Z}$ with $p \preceq q$

$$\mathcal{F}(\psi(\|\mathcal{P}p - \mathcal{Q}q\|)) \leq \mathcal{F}(\beta(\psi(\Delta(p, q)))) + \mathcal{F}(\psi(\Delta(p, q))), \qquad (17)$$

($\mathcal{B}_4$) $\mathcal{QP}p \preceq \mathcal{P}p$ *for all* $p \in \mathcal{Z}$;
($\mathcal{B}_5$) $\mathcal{P}$ *and* $\mathcal{Q}$ *are continuous, or*
($\mathcal{B}_5'$) *if* $\{\upsilon_n\}$ *is a sequence in* $\mathcal{Z}$ *such that if* $\upsilon_n \preceq \upsilon_{n+1}$ *and* $\upsilon_{n+1} \preceq \upsilon_n$ *for all* n *and* $\upsilon_n \to \omega \in \mathcal{Z}$ *as* $n \to \infty$, *then there exists a subsequence* $\{\upsilon_{n_k}\}$ *of* $\{\upsilon_n\}$ *such that* $\upsilon_{n_k} \preceq \omega$ *and* $\omega \preceq \upsilon_{n_k}$ *for all* k.

Then $\mathcal{P}$ *and* $\mathcal{Q}$ *have a common fixed point. If*

($\mathcal{B}_6$) *for all* $\zeta, \xi \in CFP(\mathcal{Q}, \mathcal{P})$, *we have* $\zeta \preceq \xi$

hold, then $\mathcal{P}$ *and* $\mathcal{Q}$ *have a unique common fixed point.*

Proof Define functions $\alpha, \eta \colon \mathcal{Z} \times \mathcal{Z} \to [0, +\infty)$ by

$$\alpha(u, v) \geq \begin{cases} \eta(u, v), & \text{if } u \preceq v, \\ -\eta(u, v), & \text{otherwise}, \end{cases}$$

for all $u, v \in \mathcal{Z}$.

Then, all hypotheses of Theorem 1 are satisfied and hence the pair $(\mathcal{P}, \mathcal{Q})$ has a common fixed point in $\mathcal{Z}$.

We demonstrate the Theorem 1 by following modified example [11].

Example 1 Let $\mathcal{Z} = [-2, 2]$ with usual distance $d(p, q) = \|p - q\| = |p - q|$ for all $p, q \in \mathcal{Z}$. Define the mappings $\mathcal{P}, \mathcal{Q} : \mathcal{Z} \to \mathcal{Z}$ by

$$\mathcal{P}p = \begin{cases} \frac{p}{9}, & p \in [0, 1] \\ \frac{4p}{3}, & \text{otherwise} \end{cases}, \quad \mathcal{Q}p = \begin{cases} \frac{p}{3}, & p \in [0, 1] \\ \frac{3p}{2}, & \text{otherwise}. \end{cases}$$

Also, we define the functions $\alpha, \eta : \mathcal{Z}^2 \to [0, +\infty)$ by

$$\alpha(p, q) = \begin{cases} 3, & p, y \in [0, 1] \\ \frac{1}{4}, & \text{otherwise} \end{cases}, \quad \eta(p, q) = \begin{cases} 1, & p \in [0, 1] \\ 4, & \text{otherwise}. \end{cases}$$

Obviously, $\mathcal{Z}$ is a Banach metric space, and $\mathcal{P}$ and $\mathcal{Q}$ are continuous mappings on $\mathcal{Z}$.

Consider $F(t) = log(t), t > 0$, $\beta(t) = k \in (0, 1)$ and $\psi(t) = t \geq 0$. To check $(\mathcal{H}_2)$, we consider two cases:

- When $p, q \in [0, 1]$, condition $(\mathcal{H}_2)$ would be

$$\left|\frac{p}{9} - \frac{q}{3}\right| \leq k \max\left\{|p - q|, \frac{8p}{9}, \frac{2q}{3}\right\}$$

  which is true for $k = 9/10$ for all $p, q \in [0, 1]$. Also, in this case $\alpha(p, q) = 3 \geq 1 = \eta(p, q)$.

- When $p, q \in D := [-2, 0) \cup (1, 2]$, condition $(\mathcal{H}_2)$ would be

$$\left|\frac{4p}{3} - \frac{3q}{2}\right| \leq k \;\max\left\{|p-q|, \left|\frac{p}{3}\right|, \left|\frac{q}{2}\right|\right\}$$

  which is not true for any $k \in (0, 1)$ in general in D; in particular at $p = 2, q = -2$. Also, in this case $\alpha(p, q) = 1/4 \ngeq 4 = \eta(p, q)$.

From the above discussion, it is clear that $(\mathcal{P}, \mathcal{Q})$ is a $(\mathcal{F}, \beta, \psi)$-contractive mapping.

Here, $\upsilon_0 = 0$ so that $\alpha(\upsilon_0, \mathcal{P}\upsilon_0) = \alpha(0, 0) = 3 \geq 1 = \eta(0, 0) = \eta(\upsilon_0, \mathcal{P}\upsilon_0)$. Also, it is easy to check that $(\mathcal{P}, \mathcal{Q})$ is a generalized $\alpha - \eta$-admissible pair and satisfy condition $(\mathcal{H}_3)$ in $[0, 1]$. Thus, all assumptions of Theorem 1 are satisfied. Hence, the pair $(\mathcal{P}, \mathcal{Q})$ has a unique common fixed point $\vartheta^* = 0$.

Example 2 Let $\mathcal{Z} = [0, \infty)$ with usual distance $d(p, q) = \|p - q\| = |p - q|$ for all $p, q \in \mathcal{Z}$. Define the mappings $\mathcal{P}, \mathcal{Q} : \mathcal{Z} \to \mathcal{Z}$ by

$$\mathcal{P}p = \begin{cases} \frac{p^2}{2}, & p \in [0,1] \\ 2p, & p \in (1, \infty) \end{cases}, \quad \mathcal{Q}p = \begin{cases} \frac{p^3}{3}, & p \in [0,1] \\ 3p, & p \in (1, \infty). \end{cases}$$

Also, we define the functions $\alpha, \eta : \mathcal{Z} \times \mathcal{Z} \to [0, +\infty)$ by

$$\alpha(p, q) = \begin{cases} 2, & p, q \in [0,1] \\ \frac{1}{2}, & \text{otherwise} \end{cases}, \quad \eta(p, q) = \begin{cases} \frac{5}{4}, & p \in [0,1] \\ 3, & \text{otherwise.} \end{cases}$$

Obviously, $\mathcal{Z}$ is a Banach metric space.

Consider $F(t) = log(t), t > 0$, $\beta(t) = k \in (0, 1)$ and $\psi(t) = t \geq 0$. To check $(\mathcal{H}_2)$, we consider two cases:

- When $p, q \in [0, 1]$, condition $(\mathcal{H}_2)$ would be

$$\left|\frac{p^2}{2} - \frac{q^3}{3}\right| \leq k \;\max\left\{|p-q|, \left|p - \frac{p^2}{2}\right|, \left|q - \frac{q^3}{3}\right|\right\}$$

  which is true for $k = 9/10$ for all $p, q \in [0, 1]$. Also, in this case $\alpha(p, q) = 2 \geq 5/4 = \eta(p, q)$.
- When $p \in [0, 1], q \in (1, \infty)$, condition $(\mathcal{H}_2)$ would be

$$\left|\frac{p^2}{2} - 3q\right| \leq k \;\max\left\{|p-q|, \left|p - \frac{p^2}{2}\right|, 2q\right\}$$

  which is not true for any $k \in (0, 1)$. For instance at $p = 0, q = 2$,

$$|\mathcal{P}p - \mathcal{Q}q| = 6 \nleq 4k = k \max\{|p - q|, |p - \mathcal{P}p|, |q - \mathcal{Q}q|\}.$$

  Also, in this case $\alpha(p, q) = 2 \ngeq 3 = \eta(p, q)$.

- When $q \in [0, 1]$, $p \in (1, \infty)$, condition $(\mathcal{H}_2)$ would be

$$\left|2p - \frac{q^3}{3}\right| \leq k \ \max\left\{|p-q|, p, \left|q - \frac{q^3}{3}\right|\right\}$$

which is not true for any $k \in (0, 1)$. For instance at $p = 3, q = 1$,

$$|\mathcal{P}p - \mathcal{Q}q| = \frac{17}{3} \nleq 3k = k \max\{|p-q|, |p - \mathcal{P}p|, |q - \mathcal{Q}q|\}.$$

Also, in this case $\alpha(p, q) = 1/2 \ngeq 5/4 = \eta(p, q)$.

From the above discussion, it is clear that $(\mathcal{P}, \mathcal{Q})$ is a $(\mathcal{F}, \beta, \psi)$-contractive mapping.

Here, $\upsilon_0 = 0$ so that $\alpha(\upsilon_0, \mathcal{P}\upsilon_0) = \alpha(0, 0) = 2 \geq 5/4 = \eta(0, 0) = \eta(\upsilon_0, \mathcal{P}\upsilon_0)$. To check that $(\mathcal{P}, \mathcal{Q})$ is a generalized $\alpha - \eta$-admissible pair, we take $\alpha(p, q) = 2 \geq 5/4 = \eta(p, q)$ in $p, q \in [0, 1]$, then

$$\alpha(\mathcal{P}p, \mathcal{Q}q) = \alpha\left(\frac{p^2}{2}, \frac{q^3}{3}\right) = 2 \geq \frac{5}{4} = \eta\left(\frac{p^2}{2}, \frac{q^3}{3}\right) = \eta(\mathcal{P}p, \mathcal{Q}q)$$

and

$$\alpha(\mathcal{Q}\mathcal{P}p, \mathcal{P}\mathcal{Q}q) = \alpha\left(\frac{p^6}{24}, \frac{q^6}{16}\right) = 2 \geq \frac{5}{4} = \eta\left(\frac{p^6}{24}, \frac{q^6}{16}\right) = \eta(\mathcal{Q}\mathcal{P}p, \mathcal{P}\mathcal{Q}q).$$

Thus, $(\mathcal{P}, \mathcal{Q})$ is a generalized $\alpha - \eta$-admissible pair in $[0, 1]$. But if take $p, q \in (1, \infty)$, then it will not satisfy in general; for instance, $p = 2$ and $q = 3$. Similarly, it is easy to check condition $(\mathcal{H}_3)$ and $(\mathcal{H}_4')$ in $[0, 1]$. Thus, all assumptions of Theorem 1 are satisfied. Hence, the pair $(\mathcal{P}, \mathcal{Q})$ has a unique common fixed point $\vartheta^* = 0$.

4 Application to a Nonlinear First-Order Hybrid Integro-Differential Equations

Consider a pair of first-order ordinary nonlinear hybrid integro-differential equations

$$\begin{cases} u'(t) + \lambda u(t) = f_j\left(t, u(t), \int_{\theta_0}^t g_j(s, u(s))ds\right), \ t \in I = [\theta_0, \theta_0 + \hbar] \subset \mathbb{R}, \theta_0 \geq 0, \hbar > 0, \mathrm{j} = 1, 2 \\ u(\theta_0) = \mu_0 \in \mathbb{R}, \end{cases} \tag{18}$$

for some $\lambda \in \mathbb{R}$, $\lambda > 0$, where $g_j : I \times \mathbb{R} \to \mathbb{R}$ and $f_j : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. This challenge is the same as the integral equation

$$u(t) = ce^{-\lambda t} + e^{-\lambda t} \int_{\theta_0}^t e^{\lambda s} f_j\left(s, u(s), \int_{\theta_0}^s g_j(\tau, u(\tau))d\tau\right) ds, \ \ t \in I, j = 1, 2 \tag{19}$$

where $c = \mu_0 e^{\lambda\theta_0}$ is a constant.

Let $\mathcal{Z} = C(I, \mathbb{R})$ be the usual Banach space (with the supremum norm). Define a distance on $\mathcal{Z}$ by $d(x, y) = \|x - y\|_\infty$ for all $x, y \in \mathcal{Z}$.

Define the mappings $\mathcal{S}, \mathcal{T} : \mathcal{Z} \to \mathcal{Z}$ by

$$\mathcal{S}u(t) = ce^{-\lambda t} + e^{-\lambda t}\int_{\theta_0}^{t} e^{\lambda s} f_1\left(s, u(s), \int_{\theta_0}^{s} g_1(\tau, u(\tau))d\tau\right)ds, \text{ for all } u \in \mathcal{Z},\ t \in I. \tag{20}$$

and

$$\mathcal{T}u(t) = ce^{-\lambda t} + e^{-\lambda t}\int_{\theta_0}^{t} e^{\lambda s} f_2\left(s, u(s), \int_{\theta_0}^{s} g_2(\tau, u(\tau))d\tau\right)ds, \text{ for all } u \in \mathcal{Z},\ t \in I. \tag{21}$$

Note that by a solution of the problem (18), we mean a function $u \in C^1(I, \mathbb{R})$ that satisfies the conditions (18), where $C^1(I, \mathbb{R})$ is the space of continuously differentiable real-valued functions defined on I. Also, $u \in C^1(I, \mathbb{R})$ is a common fixed point of the pair $(\mathcal{S}, \mathcal{T})$ iff $u \in C^1(I, \mathbb{R})$ is a solution of the problem (18).

Theorem 3 *Consider the problem* (18) *and let* $\xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be a given function. Assume the following statements are hold:*

(A1) *For all* $t \in I$, *the function* $f(t, \cdot, \cdot)$ *is nondecreasing in second and third variables;*
(A2) *For all* $t \in I$, *the function* $g(t, \cdot)$ *is nondecreasing in second variable;*
(A3) *there exists an* $x_0 \in \mathcal{Z}^2$ *such that* $(x_0, \mathcal{S}x_0) \in \mathcal{Z}^2$ *and* $\xi(x_0(t), \mathcal{S}x_0(t)) \geq 0$ *for all* $t \in I$;
(A4) *for each* $t \in I$ *and* $(x, y) \in \mathcal{Z}^2$, $\xi(x(t), y(t)) \geq 0$ *implies that* $\xi(\mathcal{S}x(t), \mathcal{T}y(t)) \geq 0$ *and* $\xi(\mathcal{T}\mathcal{S}x(t), \mathcal{S}\mathcal{T}y(t)) \geq 0$;
(A5) *for each* $t \in I$ *and* $x \in \mathcal{Z}$, $\xi(\mathcal{T}\mathcal{S}x(t), \mathcal{S}x(t)) \geq 0$;
(A6) *for each* $t \in I$, *if* $\{x_n\}$ *is a sequence in* $\mathcal{Z}$ *such that* $x_n \to x$ *in* $\mathcal{Z}$ *and* $\xi(x_n(t), x_{n+1}(t)) \geq 0$ *and* $\xi(x_{n+1}(t), x_n(t)) \geq 0$ *for all* $n \in \mathbb{N}$, *then* $\xi(x_n(t), x(t)) \geq 0$ *and* $\xi(x(t), x_n(t)) \geq 0$ *for all* $n \in \mathbb{N}$;
(A7) *for all* $u_1, u_2, v_1, v_2 \in \mathbb{R}$ *and* $t \in I$ *with* $(u_1, v_1) \in \mathcal{Z}^2$ *and* $(u_2, v_2) \in \mathcal{Z}^2$ *with* $\xi(u_1, v_1) \geq 0$ *and* $\xi(u_2, v_2) \geq 0$, *the functions* f *and* g *satisfy*

$$0 \leq |f_1(t, u_1, u_2) - f_2(t, v_1, v_2)| \leq |u_1 - v_1| + |u_2 - v_2|$$

and

$$0 \leq |g_1(t, u_1) - g_2(t, v_1)| \leq |u_1 - v_1|;$$

(A8) *there exists* $k \in (0, 1)$ *such that* $\sup_{t\in I}\int_{\theta_0}^{t} e^{\lambda s}ds \leq \frac{k}{1+\hbar}$.

Then the problem (18) *has a unique solution.*

Proof Define functions $\alpha, \eta \colon \mathcal{Z} \times \mathcal{Z} \to [0, +\infty)$ by

$$\alpha(u, v) \geq \begin{cases} \eta(u, v), & \text{if } \xi(u(t), v(t)) \geq 0, \text{ for all } u, v \in \mathcal{Z}, \text{ and for all } t \in I \\ -\eta(u, v), & \text{otherwise.} \end{cases} \tag{22}$$

Let $u, v \in \mathcal{Z}$ such that $\xi(u(t), v(t)) \geq 0$ for all $t \in [0, 1]$. Then by (22), we have $\alpha(u, v) \geq \eta(u, v)$ which implies from $(A7)$ that

$$\begin{aligned}
&\|\mathcal{S}u - \mathcal{T}v\|_\infty \\
&= \sup_{t \in I} |\mathcal{S}u(t) - \mathcal{T}v(t)| \\
&\leq \sup_{t \in I} \left| \begin{array}{l} e^{-\lambda t} \int_{\theta_0}^t e^{\lambda s} f_1 \left(s, u(s), \int_{\theta_0}^s g_1(\tau, u(\tau))\, d\tau\right) ds \\ -e^{-\lambda t} \int_{\theta_0}^t e^{\lambda s} f_2 \left(s, v(s), \int_{\theta_0}^s g_2(\tau, v(\tau))\, d\tau\right) ds \end{array} \right| \\
&\leq \sup_{t \in I} |e^{-\lambda t}| \left[\int_{\theta_0}^t e^{\lambda s} \left| f_1 \left(s, u(s), \int_{\theta_0}^s g_1(\tau, u(\tau))\, d\tau\right) - f_2 \left(s, v(s), \int_{\theta_0}^s g_2(\tau, v(\tau))\, d\tau\right) \right| ds \right] \\
&\leq \sup_{t \in I} \int_{\theta_0}^t e^{\lambda s} \left[(|u(s) - v(s)|) + \left| \int_{\theta_0}^s g_1(\tau, u(\tau))\, d\tau - \int_{\theta_0}^s g_2(\tau, v(\tau))\, d\tau \right| \right] ds \\
&\leq \sup_{t \in I} \int_{\theta_0}^t e^{\lambda s} \left[(|u(s) - v(s)|) + \int_{\theta_0}^s |g_1(\tau, u(\tau)) - g_2(\tau, v(\tau))|\, d\tau \right] ds \\
&\leq \sup_{t \in I} \int_{\theta_0}^t e^{\lambda s} \left[(|u(s) - v(s)|) + \int_{\theta_0}^s |u(\tau) - v(\tau)|\, d\tau \right] ds \\
&\leq \sup_{t \in I} \int_{\theta_0}^t e^{\lambda s} \left[(1 + \hbar)(\|u - v\|_\infty) \right] ds \\
&\leq k(\|u - v\|_\infty) \\
&\leq k \max \left\{ \|u - v\|_\infty, \|u - \mathcal{S}u\|_\infty, \|v - \mathcal{T}v\|_\infty \right\}.
\end{aligned}$$

This implies that

$$\|\mathcal{S}u - \mathcal{T}v\|_\infty \leq k \max \left\{ \|u - v\|_\infty, \|u - \mathcal{S}u\|_\infty, \|v - \mathcal{T}v\|_\infty \right\}.$$

Now, by considering the control functions $\mathcal{F} \in \mathbb{F}$, $\beta \in \Lambda$, $\psi \in \Psi$ described by

$$\mathcal{F}(t) = log(t), \ t > 0, \ \beta(t) = k \in (0, 1), \ \psi(t) = t \geq 0,$$

we get

$$\begin{aligned} \mathcal{F}(\psi(\|\mathcal{S}u - \mathcal{T}v\|)) &\leq \mathcal{F}(\beta(\psi\,(\max \{\|u - v\|_\infty, \|u - \mathcal{S}u\|_\infty, \|v - \mathcal{T}v\|_\infty\}))) \\ &\quad + \mathcal{F}(\psi\,(\max \{\|u - v\|_\infty, \|u - \mathcal{S}u\|_\infty, \|v - \mathcal{T}v\|_\infty\})). \end{aligned}$$

This implies that for each $(u, v) \in \mathcal{Z}^2$ with $\alpha(u, v) \geq \eta(u, v)$, we have

$$\mathcal{F}(\psi(\|\mathcal{S}u-\mathcal{T}v\|)) \leq \mathcal{F}(\beta(\psi(\max\{\|u-v\|_\infty, \|u-\mathcal{S}u\|_\infty, \|v-\mathcal{T}v\|_\infty\}))) + \mathcal{F}(\psi(\max\{\|u-v\|_\infty, \|u-\mathcal{S}u\|_\infty, \|v-\mathcal{T}v\|_\infty\})).$$

Using (22) and conditions (A3)-(A5), the following assertions hold for all $x, y \in \mathcal{Z}$:

$$\xi(x(t), \mathcal{S}x(t)) \geq 0 \text{ for all } t \in I \Longrightarrow \alpha(x_0, \mathcal{S}x_0) \geq \eta(x_0, \mathcal{S}x_0) \text{ for } x_0 \in \mathcal{Z},$$

$$\begin{aligned}\alpha(x, y) \geq \eta(x, y) &\Longrightarrow \xi(x(t), y(t)) \geq 0 \text{ for all } t \in I\\ &\Longrightarrow \xi(\mathcal{S}x(t), \mathcal{T}y(t)) \geq 0 \text{ and } \xi(\mathcal{T}\mathcal{S}x(t), \mathcal{S}\mathcal{T}y(t)) \geq 0;\ \text{ for all } t \in I\\ &\Longrightarrow \alpha(\mathcal{S}x, \mathcal{T}y) \geq \eta(\mathcal{S}x, \mathcal{T}y) \text{ and } \alpha(\mathcal{T}\mathcal{S}x, \mathcal{S}\mathcal{T}y) \geq \eta(\mathcal{T}\mathcal{S}x, \mathcal{S}\mathcal{T}y)\\ &\quad \text{for all } x, y \in \mathcal{Z}\end{aligned}$$

and

$$\xi(\mathcal{T}\mathcal{S}x(t), \mathcal{S}x(t)) \geq 0;\ \text{ for all } t \in I \Rightarrow \alpha(\mathcal{T}\mathcal{S}x, \mathcal{S}x) \geq \eta(\mathcal{T}\mathcal{S}x, \mathcal{S}x) \text{ for } x \in \mathcal{Z}.$$

Next, from (22) and condition (A6), it easily follows that

$$\begin{cases} \text{for any sequence } \{x_n\} \text{ in } \mathcal{Z} \text{ if } \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})\\ \text{and } \alpha(x_{n+1}, x_n) \geq \eta(x_{n+1}, x_n) \text{ for all } n \in \mathbb{N} \text{ and } x_n \to x \in \mathcal{Z}\\ \text{as } n \to \infty,\ \text{ then } \alpha(x_n, x) \geq \eta(x_n, x) \text{ and } \alpha(x_{n+1}, x) \geq \eta(x_{n+1}, x) \text{ for all } n \in \mathbb{N}. \end{cases}$$

Therefore, from Theorem 1, the pair $(\mathcal{S}, \mathcal{T})$ has a unique common fixed point, that is, the problem (18) has a unique solution.

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Application of Darbo's Fixed Point Theorem for Existence Result of Generalized 2D Functional Integral Equations

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Abstract In this chapter, we establish some existence of solutions for $2D$ functional integral equations concerning Darbo's fixed point theorem in Banach algebra. This existence of solutions involves various obtained from earlier studies. Some examples are introduced to confirm the applicability of our results.

1 Introduction

Many non-linear problems that rising from the fields of the real world, such as basic sciences, can be described with operator equations. Mainly, FIEs perform a very powerful and important part of the non-linear analysis and have several applications in real-world problems. For illustration, some problems in biology, economics, physics and different fields can be specified with the help of integral and integro-differential equations (see [4, 14, 17, 20]). Recently, there have been many successful efforts to use the theory of MNC and different fixed point theorems in the study of solvability of FIEs (see [6–13, 15, 16, 18, 19, 24–27]). Here, we study the solvability of $2D$ FIE of the form:

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$$z(s,\varphi) = P\left(s,\varphi,\int_0^s\int_0^\varphi u(s,\varphi,g,\zeta,z(g,\zeta))d\zeta dg,\int_0^c\int_0^d f(s,\varphi,g,\zeta,z(g,\zeta))d\zeta dg, z(s,\varphi)\right)$$
$$\times F\left(s,\varphi,\int_0^s\int_0^\varphi p(s,\varphi,g,\zeta,z(g,\zeta))d\zeta dg,\int_0^c\int_0^d q(s,\varphi,g,\zeta,z(g,\zeta))d\zeta dg, z(s,\varphi)\right), \quad (1)$$

for $(s,\varphi) \in I = [0,c] \times [0,d]$.

Das et al. [5] studied the existence result for 2D FIE

$$z(s,\varphi) = h(s,\varphi) + P\left(s,\varphi,z(s,\varphi),\int_0^s\int_0^\varphi u(s,\varphi,g,\zeta,z(g,\zeta))d\zeta dg\right) \quad (2)$$

for $(s,\varphi) \in [0,1] \times [0,1]$.

Mishra et al. [22] studied the existence of solutions for 2D FIE

$$z(\varphi,t) = P\left(s,\varphi,\int_0^s\int_0^\varphi u(s,\varphi,g,\zeta,z(g,\zeta))d\zeta dg\right) \times F\left(s,\varphi,\int_0^c\int_0^d q(s,\varphi,g,\zeta,z(g,\zeta))d\zeta dg\right).$$

for $(s,\varphi) \in I$.

Further, a famous $2D$ FIE of Hammerstein type [23] has the form

$$z(s,\varphi) = h(s,\varphi) + \int_0^s\int_0^\varphi u_1(s,\varphi,g,\zeta)u_2(s,\varphi,z(g,\zeta))d\zeta dg.$$

The $2D$ FIE (1) cover various special type of FIEs. The aim of this work is to investigate the method to prove the solvability of equation (1) with the help of MNC in $[0,c] \times [0,d]$. The main advantage of Darbo's fixed point theorem is that the compactness of the domain of operator which is required in Schauder's fixed point theorem has been rested.

2 Preliminaries

In this study, the following symbols are used:

- S : Banach space;
- $B(z,r)$: Closed ball at centre z with radius r;
- $co\bar{E}$: Closed convex hull of a set S;
- coE : Convex hull of a set S;
- N_S: Set of all relatively compact subsets of S
- M_S: Set of all bounded subsets of S.

Definition 1 ([21]) Let $E \in M_S$ and

$$\psi(E) = \inf\left\{\sigma > 0 : E = \bigcup_{i=1}^n E_i \text{ with } \operatorname{diam} E_i \le \sigma,\ i = 1,2,...,n\right\}.$$

where,

$$\text{diam } E = \sup\{\|z - \hat{z}\| : z, \hat{z} \in E\}.$$

Hence, $0 \leq \psi(E) < \infty$. $\psi(E)$ is called the Kuratowski MNC.

Theorem 1 ([2]) *Suppose that $E, \hat{E} \in M_S$ and $\lambda \in \mathbb{R}$. Then*

(i) $\psi(E) = 0$ *if and only if* $E \in N_S$;
(ii) $E \subseteq \hat{E} \implies \psi(E) \leq \psi(\hat{E})$;
(iii) $\psi(Conv E) = \psi(E)$;
(iv) $\psi(E \cup \hat{E}) = \max\{\psi(E), \psi(\hat{E})\}$;
(v) $\psi(\lambda E) = |\lambda|\psi(E)$;
(vi) $\psi(E + \hat{E}) \leq \psi(E) + \psi(\hat{E})$;

Definition 2 ([2]) Let D be a non-empty, closed, convex and bounded subset of S and let $T : D \to D$ be continuous mapping such that there exists a constant $k \in [0, 1)$, with

$$\psi(TE) \leq k\psi(E)$$

for any subset of E of D. Then T has a fixed point in D.

Theorem 2 ([3]) *Assume that Ω is a non-empty, convex, bounded and close subset of S and the operators U and V, which transform continuously the set Ω into S such that $U(\Omega)$ and $V(\Omega)$ are bounded. Again, let operator $T = U.V$ transform Ω into itself. If U and V fulfils Darbo's condition on Ω with the constant k_1 and k_2, respectively, then T fulfils Darbo's condition on Ω with the constant*

$$||U(\Omega)||k_2 + ||V\Omega)||k_1 < 1.$$

If T is a contraction with respect to ψ and has at least one fixed point in Ω.

Let $C(I, \mathbb{R})$ be Banach algebra consisting of real valued continuous functions on the set I with usual norm

$$||z|| = sup\{|z(s, \varphi)| : (s, \varphi) \in I\}.$$

Let $E \subset S = C(I, \mathbb{R})$ and for $z \in E$, $\sigma > 0$, denoted by $\omega(z, \sigma)$ modulus of continuity of z

$$\omega(z, \sigma) = sup\{|z(s, \varphi) - z(w, \hat{w})| : s, w \in [0, c]; \varphi, \hat{w} \in [0, d]; |s - w| \leq \sigma, |\varphi - \hat{w}| \leq \sigma\},$$

$$\omega(E, \sigma) = sup\{\omega(z, \sigma) : z \in E\},\ \omega_0(E) = \lim_{\sigma \to 0} \omega(E, \sigma).$$

In [2], $\omega_0(E)$ is a regular MNC in $C(I, \mathbb{R})$.

3 Main Results

Now, we study the solvability of the FIE (1) for $z \in C[0, c] \times [0, d]$ under the following assumptions.

(1) $P, F : I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions and there exists $N \geq 0$, such that

$$|P(s, \varphi, 0, 0, 0)| \leq N \text{ and } |F(s, \varphi, 0, 0, 0)| \leq N.$$

(2) Let $K_i : I \to \mathbb{R}_+$ $(i = 1, ..., 6)$ be continuous functions such that

$$\begin{aligned} |P(s, \varphi, z_1, x_1, y_1) - P(s, \varphi, z_2, x_2, y_2)| &\leq K_1(s, \varphi)|z_1 - z_2| + K_2(s, \varphi)|x_1 - x_2| \\ &\quad + K_3(s, \varphi)|y_1 - y_2|, \\ |F(s, \varphi, z_1, x_1, y_1) - F(s, \varphi, z_2, x_2, y_2)| &\leq K_4(s, \varphi)|z_1 - z_2| + K_5(s, \varphi)|x_1 - x_2| \\ &\quad + K_6(s, \varphi)|y_1 - y_2|, \end{aligned}$$

for all $(s, \varphi) \in I$ and $z_1, z_2, x_1, x_2, y_1, y_2 \in \mathbb{R}$ and,

$$K = max\Big\{K_i(s, \varphi) : i = 1, ..., 6;\ (s, \varphi) \in I\Big\}.$$

(3) u, f, p, and $q : I \times I \times \mathbb{R} \to \mathbb{R}$. are continuous functions and there exist non-negative constants β_1 and β_2 such that

$$\begin{aligned} |u(s, \varphi, g, \zeta, z(g, \zeta))| &\leq \beta_1 + \beta_2|z|, \\ |f(u(s, \varphi, g, \zeta, z(g, \zeta))| &\leq \beta_1 + \beta_2|z|, \\ |p(s, \varphi, g, \zeta, z(g, \zeta))| &\leq \beta_1 + \beta_2|z|, \\ |q(u(s, \varphi, g, \zeta, z(g, \zeta))| &\leq \beta_1 + \beta_2|z|, \end{aligned}$$

for all $(s, \varphi) \in I,\ z \in \mathbb{R}$.

Furthermore,
$4h_1h_2 < 1$ for $h_1 = 2Kcd\beta_2, h_2 = 2Kcd\beta_1 + N$.

Theorem 3 *From Assumptions (1)–(3), Eq. (1) has at least one solution in I.*

Proof Putting operators U and V defined on I such that

$$(Uz)(s, \varphi) = P\Big(s, \varphi, \int_0^s \int_0^\varphi u(s, \varphi, g, \zeta, z(g, \zeta))d\zeta dg, \int_0^c \int_0^d f(s, \varphi, g, \zeta, z(g, \zeta))d\zeta dg, z(s, \varphi)\Big),$$
$$(Vz)(s, \varphi) = F\Big(s, \varphi, \int_0^s \int_0^\varphi p(s, \varphi, g, \zeta, z(g, \zeta))d\zeta dg, \int_0^c \int_0^d q(s, \varphi, g, \zeta, z(g, \zeta))d\zeta dg, z(s, \varphi)\Big),$$

for $(s, \varphi) \in I$.

By (1) and (3), we get U and V transform I into itself.

Step-I Now, we put

$$Tz = (Uz)(Vz).$$

Clearly, T transform I into itself. Now, fix $z \in I$. Then,

$$
\begin{aligned}
&|(Tz)(s,\varphi)| \\
&= |(Uz)(s,\varphi)|.|(Vz)(s,\varphi)| \\
&= \Bigg(\Big|P\Big(s,\varphi,\int_0^s\int_0^\varphi u(s,\varphi,g,\zeta,z(g,\zeta))d\zeta dg,\int_0^c\int_0^d f(s,\varphi,g,\zeta,z(g,\zeta))d\zeta dg,z(s,\varphi)\Big)\Big| \\
&\quad\times\Big|F\Big(s,\varphi,\int_0^s\int_0^\varphi p(s,\varphi,g,\zeta,z(g,\zeta))d\zeta dg,\int_0^c\int_0^d q(s,\varphi,g,\zeta,z(g,\zeta))d\zeta dg,z(s,\varphi)\Big)\Big|\Bigg), \\
&\le \Bigg(\Big|P\Big(s,\varphi,\int_0^s\int_0^\varphi u(s,\varphi,g,\zeta,z(g,\zeta))d\zeta dg,\int_0^c\int_0^d f(s,\varphi,g,\zeta,z(g,\zeta))d\zeta dg,z(s,\varphi)\Big) \\
&\quad -P(s,\varphi,0,0,0)\Big| + |P(s,\varphi,0,0,0)|\Bigg) \\
&\quad\times\Bigg(\Big|F\Big(s,\varphi,\int_0^s\int_0^\varphi p(s,\varphi,g,\zeta,z(g,\zeta))d\zeta dg,\int_0^c\int_0^d q(s,\varphi,g,\zeta,z(g,\zeta))d\zeta dg,z(s,\varphi)\Big) \\
&\quad -F(s,\varphi,0,0,0)\Big| + |F(s,\varphi,0,0,0)|\Bigg), \\
&\le \Bigg(K_1(s,\varphi)\int_0^s\int_0^\varphi |u(s,\varphi,g,\zeta,z(g,\zeta))|d\zeta dg + K_2(s,\varphi)\int_0^c\int_0^d |f(s,\varphi,g,\zeta,z(g,\zeta))|d\zeta dg \\
&\quad +K_3|(z(s,\varphi))| + N\Bigg) \\
&\quad\times\Bigg(K_4(s,\varphi)\int_0^s\int_0^\varphi |p(s,\varphi,g,\zeta,z(g,\zeta))|d\zeta dg + K_5(s,\varphi)\int_0^c\int_0^d |q(s,\varphi,g,\zeta,z(g,\zeta))|d\zeta dg \\
&\quad +K_6|(z(s,\varphi))| + N\Bigg) \\
&\le \Big(2Kcd(\beta_1+\beta_2||z||)+N\Big)\times\Big(2Kcd(\beta_1+\beta_2||z||)+N\Big) \\
&\le \Big(2Kcd\beta_2||z|| + 2Kcd\beta_1 + N\Big)^2
\end{aligned}
$$

Taking $h_1 = 2Kcd\beta_2$ and $h_2 = 2Kcd\beta_1 + N$ then,

$$||Uz|| \le h_1||z|| + h_2, \tag{3}$$

$$||Vz|| \le h_2||z|| + h_2, \tag{4}$$

$$||Tz|| \le (h_1||z|| + h_2)^2, \tag{5}$$

for $z \in I$. From Eq. (5), the operator T maps the ball $B_r \subset I$ into itself for$r_1 \le r \le r_2$, where

$$r_1 = \frac{(1-2h_1h_2)-\sqrt{1-4h_1h_2}}{2h_1^2}.$$

$$r_2 = \frac{(1-2h_1h_2)+\sqrt{1-4h_1h_2}}{2h_1^2}.$$

Using Eqs. (3) and (4),

$$||UB_r|| \le h_1\, r + h_2, \tag{6}$$
$$||VB_r|| \le h_1\, r + h_2. \tag{7}$$

Step-2 We prove that T is continuous on B_r. For this, fixed $\sigma > 0$ and arbitrary $z, x \in B_r$ such that $||z - x|| \le \sigma$. Then for $(s, \varphi) \in I$,

$$\begin{aligned}
&|(Uz)(s,\varphi) - (Ux)(s,\varphi)| \\
&= \Bigg| P\Big(s, \varphi, \int_0^s \int_0^\varphi u(s,\varphi,g,\zeta,z(g,\zeta))d\zeta dg, \int_0^c \int_0^d f(s,\varphi,g,\zeta,z(g,\zeta))d\zeta dg, z(s,\varphi)\Big) \\
&\quad -P\Big(s, \varphi, \int_0^s \int_0^\varphi u(s,\varphi,g,\zeta,x(g,\zeta))d\zeta dg, \int_0^c \int_0^d f(s,\varphi,g,\zeta,x(g,\zeta))d\zeta dg, x(s,\varphi)\Big) \Bigg| \\
&\le K_1(s,\varphi) \int_0^s \int_0^\varphi |u(s,\varphi,g,\zeta,z(g,\zeta)) - u(s,\varphi,g,\zeta,x(g,\zeta))| d\zeta dg \\
&\quad + K_2(s,\varphi) \int_0^c \int_0^d |f(s,\varphi,g,\zeta,z(g,\zeta)) - f(s,\varphi,g,\zeta,x(g,\zeta))| d\zeta dg \\
&\quad + K_3(s,\varphi)|z(s,\varphi) - x(s,\varphi)| \\
&\le Kcd\omega(u,\sigma) + Kcd\omega(f,\sigma) + K||z - x||, \\
&\le Kcd\omega(u,\sigma) + Kcd\omega(f,\sigma) + K\sigma,
\end{aligned}$$

□

where

$$\begin{aligned}
\omega(u,\sigma) = sup\Big\{ &|u(s,\varphi,g,\zeta,z) - u(s,\varphi,g,\zeta,x)| : (s,\varphi) \in I, \\
&z, x \in [-r, r], ||z - x|| \le \sigma \Big\}, \\
\omega(f,\sigma) = sup\Big\{ &|f(s,\varphi,g,\zeta,z) - f(s,\varphi,g,\zeta,x)| : (s,\varphi) \in I, \\
&z, x \in [-r, r], ||z - x|| \le \sigma \Big\},
\end{aligned}$$

The functions $u = u(s,\varphi,g,\zeta,z)$ and $f = f(s,\varphi,g,\zeta,z)$ are uniform continuous on the bounded subset $I \times I \times [-r, r]$, then $\omega(u,\sigma)$ and $\omega(f,\sigma) \to 0$ as $\sigma \to 0$. Thus, U is continuous operator on B_r. Similarly,

$$\begin{aligned}
&|(Vz)(s,\varphi)-(Vx)(s,\varphi)| \\
&= \Big|F\Big(s,\varphi,\int_0^s\int_0^\varphi p(s,\varphi,g,\zeta,z(g,\zeta))d\zeta dg,\int_0^c\int_0^d q(s,\varphi,g,\zeta,z(g,\zeta))d\zeta dg, z(s,\varphi)\Big) \\
&\quad -F\Big(s,\varphi,\int_0^s\int_0^\varphi p(s,\varphi,g,\zeta,x(g,\zeta))d\zeta dg,\int_0^c\int_0^d q(s,\varphi,g,\zeta,x(g,\zeta))d\zeta dg, x(s,\varphi)\Big)\Big| \\
&\le K_4(s,\varphi)\int_0^s\int_0^\varphi |p(s,\varphi,g,\zeta,z(g,\zeta))-p(s,\varphi,g,\zeta,x(g,\zeta))|d\zeta dg \\
&\quad +K_5(s,\varphi)\int_0^c\int_0^d |q(s,\varphi,g,\zeta,z(g,\zeta))-q(s,\varphi,g,\zeta,x(g,\zeta))|d\zeta dg \\
&\quad +K_6(s,\varphi)|z(s,\varphi)-x(s,\varphi)| \\
&\le Kcd\omega(p,\sigma)+Kcd\omega(q,\sigma)+K||z-x||, \\
&\le Kcd\omega(p,\sigma)+Kcd\omega(q,\sigma)+K\sigma,
\end{aligned}$$

where

$$\omega(p,\sigma)=sup\Big\{|p(s,\varphi,g,\zeta,z)-p(s,\varphi,g,\zeta,x)|:(s,\varphi)\in I,$$
$$z,x\in[-r,r],||z-x||\le\sigma\Big\},$$
$$\omega(q,\sigma)=sup\Big\{|q(s,\varphi,g,\zeta,z)-q(s,\varphi,g,\zeta,x)|:(s,\varphi)\in I,$$
$$z,x\in[-r,r],||z-x||\le\sigma\Big\},$$

The function $p=p(s,\varphi,g,\zeta,z)$ and $q=q(s,\varphi,g,\zeta,z)$ are uniform continuous on the bounded subset $I\times I\times[-r,r]$, then $\omega(p,\sigma)$ and $\omega(q,\sigma)\to 0$ as $\sigma\to 0$. Thus, V is continuous operator on B_r. Hence, T is a continuous operator on B_r.

Step-3 We prove that the U and V fulfil Darbo's condition with respect ω_0, in B_r. Assume that a subset E of B_r and $z\in E$, let $\sigma>0$ be fixed and $s_1,\varphi_1,s_2,\varphi_2\in I$ such that $|s_1-s_2|\le\sigma$, and $|\varphi_1-\varphi_2|\le\sigma$. We have

$$
\begin{aligned}
&|(Uz)(s_2,\varphi_2)-(Uz)(s_1,\varphi_1)|\\
=&\Big|P\Big(s_2,\varphi_2,\int_0^{s_2}\int_0^{\varphi_2}u(s_2,\varphi_2,g,\zeta,z(g,\zeta))d\zeta dg,\int_0^{c}\int_0^{d}f(s_2,\varphi_2,g,\zeta,z(g,\zeta))d\zeta dg,z(s_2,\varphi_2)\Big)\\
&-P\Big(s_1,\varphi_1,\int_0^{s_1}\int_0^{\varphi_1}u(s_1,\varphi_1,g,\zeta,z(g,\zeta))d\zeta dg,\int_0^{c}\int_0^{d}f(s_1,\varphi_1,g,\zeta,z(g,\zeta))d\zeta dg,z(s_1,\varphi_1)\Big)\Big|\\
\le&\Big|P\Big(s_2,\varphi_2,\int_0^{s_2}\int_0^{\varphi_2}u(s_2,\varphi_2,g,\zeta,z(g,\zeta))d\zeta dg,\int_0^{c}\int_0^{d}f(s_2,\varphi_2,g,\zeta,z(g,\zeta))d\zeta dg,z(s_2,\varphi_2)\Big)\\
&-P\Big(s_2,\varphi_2,\int_0^{s_2}\int_0^{\varphi_2}u(s_2,\varphi_2,g,\zeta,z(g,\zeta))d\zeta dg,\int_0^{c}\int_0^{d}f(s_2,\varphi_2,g,\zeta,z(g,\zeta))d\zeta dg,z(s_1,\varphi_1)\Big)\Big|\\
&+\Big|P\Big(s_2,\varphi_2,\int_0^{s_2}\int_0^{\varphi_2}u(s_2,\varphi_2,g,\zeta,z(g,\zeta))d\zeta dg,\int_0^{c}\int_0^{d}f(s_2,\varphi_2,g,\zeta,z(g,\zeta))d\zeta dg,z(s_1,\varphi_1)\Big)\\
&-P\Big(s_2,\varphi_2,\int_0^{s_2}\int_0^{\varphi_2}u(s_2,\varphi_2,g,\zeta,z(g,\zeta))d\zeta dg,\int_0^{c}\int_0^{d}f(s_1,\varphi_1,g,\zeta,z(g,\zeta))d\zeta dg,z(s_1,\varphi_1)\Big)\Big)\Big|\\
&+\Big|P\Big(s_2,\varphi_2,\int_0^{s_2}\int_0^{\varphi_2}u(s_2,\varphi_2,g,\zeta,z(g,\zeta))d\zeta dg,\int_0^{c}\int_0^{d}f(s_1,\varphi_1,g,\zeta,z(g,\zeta))d\zeta dg,z(s_1,\varphi_1)\Big)\\
&-P\Big(s_2,\varphi_2,\int_0^{s_1}\int_0^{\varphi_1}u(s_1,\varphi_1,g,\zeta,z(g,\zeta))d\zeta dg,\int_0^{c}\int_0^{d}f(s_1,\varphi_1,g,\zeta,z(g,\zeta))d\zeta dg,z(s_1,\varphi_1)\Big)\Big)\Big|\\
&+\Big|P\Big(s_2,\varphi_2,\int_0^{s_1}\int_0^{\varphi_1}u(s_1,\varphi_1,g,\zeta,z(g,\zeta))d\zeta dg,\int_0^{c}\int_0^{d}f(s_1,\varphi_1,g,\zeta,z(g,\zeta))d\zeta dg,z(s_1,\varphi_1)\Big)\\
&-P\Big(s_1,\varphi_1,\int_0^{s_1}\int_0^{\varphi_1}u(s_1,\varphi_1,g,\zeta,z(g,\zeta))d\zeta dg,\int_0^{c}\int_0^{d}f(s_1,\varphi_1,g,\zeta,z(g,\zeta))d\zeta dg,z(s_1,\varphi_1)\Big)\Big)\Big|\\
\le&\ K_3(s,\varphi)|z(s_2,\varphi_2)-z(s_1,\varphi_1)|\\
&+K_2(s,\varphi)\Big|\int_0^{c}\int_0^{d}f(s_2,\varphi_2,g,\zeta,z(g,\zeta))d\zeta dg-\int_0^{c}\int_0^{d}f(s_1,\varphi_1,g,\zeta,z(g,\zeta))d\zeta dg\Big|\\
&+K_1(s,\varphi)\Big|\int_0^{s_2}\int_0^{\varphi_2}u(s_2,\varphi_2,g,\zeta,z(g,\zeta))d\zeta dg-\int_0^{s_1}\int_0^{\varphi_1}u(s_1,\varphi_1,g,\zeta,z(g,\zeta))d\zeta dg\Big|+\omega_P(I,\sigma)\\
\le&\ K|z(s_2,\varphi_2)-z(s_1,\varphi_1)|+K\int_0^{c}\int_0^{d}|f(s_2,\varphi_2,g,\zeta,z(g,\zeta))-f(s_1,\varphi_1,g,\zeta,z(g,\zeta))|d\zeta dg\\
&+K\int_0^{s_1}\int_0^{\varphi_1}|u(s_2,\varphi_2,g,\zeta,z(g,\zeta))-u(s_1,\varphi_1,g,\zeta,z(g,\zeta))|d\zeta dg\\
&+K\int_{s_1}^{s_2}\int_{\varphi_1}^{\varphi_2}|u(s_2,\varphi_2,g,\zeta,z(g,\zeta))|d\zeta dg+K\int_{s_1}^{s_2}\int_0^{\varphi_1}|u(s_2,\varphi_2,g,\zeta,z(g,\zeta))|d\zeta dg\\
&+K\int_0^{s_1}\int_{\varphi_1}^{\varphi_2}|u(s_2,\varphi_2,g,\zeta,z(g,\zeta))|d\zeta dg,
\end{aligned}
$$

where

$$\omega_u(I,\sigma) = sup\Big\{|u(s_2,\varphi_2,g,\zeta,z) - u(s_1,\varphi_1,g,\zeta,z)| : s_1,s_2 \in [0,c], \varphi_1,\varphi_2 \in [0,d],$$
$$z \in [-r,r], |s_2 - s_1| \le \sigma, |\varphi_2 - \varphi_1| \le \sigma\Big\},$$
$$\omega_f(I,\sigma) = sup\Big\{|f(s_2,\varphi_2,g,\zeta,z) - f(s_1,\varphi_1,g,\zeta,z)| : s_1,s_2 \in [0,c], \varphi_1,\varphi_2 \in [0,d],$$
$$z \in [-r,r], |s_2 - s_1| \le \sigma, |\varphi_2 - \varphi_1| \le \sigma\Big\},$$
$$L = sup\Big\{|u(s,\varphi,g,\zeta,z)| : (s,\varphi),(g,\zeta) \in I, z \in [-r,r]\Big\},$$

and

$$\omega_P(I,\sigma) = sup\Big\{|P(s_2,\varphi_2,z,x,y) - P(s_1,\varphi_1,z,x,y)| : s_1,s_2 \in [0,c],\ \varphi_1,\varphi_2 \in [0,d],$$
$$y \in [-r,r], |s_2 - s_1| \le \sigma, |\varphi_2 - \varphi_1| \le \sigma, z,x \in [-Lcd, Lcd]\Big\}.$$

Then using above relation, we obtain

$$|(Uz)(s_2,\varphi_2) - (Uz)(s_1,\varphi_1)| \le K|z(s_2,\varphi_2) - z(s_1,\varphi_1)| + K(cd\omega_u(I,\sigma) + L\sigma^2 + L\sigma d + L\sigma c)$$
$$+\omega_P(I,\sigma) + Kcd\omega_f(I,\sigma).$$

From the assumptions, P, u and f are uniformly continuous on $I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, $I \times I \times \mathbb{R}$ and $I \times I \times \mathbb{R}$. Hence, deduce that

$$\omega_P(I,\sigma) \to 0\ \omega_u(I,\sigma) \to 0\ and\ \omega_f(I,\sigma) \to 0\ as\ \sigma \to 0.$$

Thus

$$\omega_0(UH) \le K\omega_0(H). \tag{8}$$

Similarly, we write

$$
\begin{aligned}
&|(Vz)(s_2,\varphi_2)-(Vz)(s_1,\varphi_1)|\\
=&\left|F\Big(s_2,\varphi_2,\int_0^{s_2}\int_0^{\varphi_2}p(s_2,\varphi_2,g,\zeta,z(g,\zeta))d\zeta dg,\int_0^{c}\int_0^{d}q(s_2,\varphi_2,g,\zeta,z(g,\zeta))d\zeta dg,z(s_2,\varphi_2)\Big)\right.\\
&\left.-F\Big(s_1,\varphi_1,\int_0^{s_1}\int_0^{\varphi_1}p(s_1,\varphi_1,g,\zeta,z(g,\zeta))d\zeta dg,\int_0^{c}\int_0^{d}q(s_1,\varphi_1,g,\zeta,z(g,\zeta))d\zeta dg,z(s_1,\varphi_1)\Big)\right|\\
\le&\left|F\Big(s_2,\varphi_2,\int_0^{s_2}\int_0^{\varphi_2}p(s_2,\varphi_2,g,\zeta,z(g,\zeta))d\zeta dg,\int_0^{c}\int_0^{d}q(s_2,\varphi_2,g,\zeta,z(g,\zeta))d\zeta dg,z(s_2,\varphi_2)\Big)\right.\\
&\left.-F\Big(s_2,\varphi_2,\int_0^{s_2}\int_0^{\varphi_2}p(s_2,\varphi_2,g,\zeta,z(g,\zeta))d\zeta dg,\int_0^{c}\int_0^{d}q(s_2,\varphi_2,g,\zeta,z(g,\zeta))d\zeta dg,z(s_1,\varphi_1)\Big)\right|\\
&+\left|F\Big(s_2,\varphi_2,\int_0^{s_2}\int_0^{\varphi_2}p(s_2,\varphi_2,g,\zeta,z(g,\zeta))d\zeta dg,\int_0^{c}\int_0^{d}q(s_2,\varphi_2,g,\zeta,z(g,\zeta))d\zeta dg,z(s_1,\varphi_1)\Big)\right.\\
&\left.-F\Big(s_2,\varphi_2,\int_0^{s_2}\int_0^{\varphi_2}p(s_2,\varphi_2,g,\zeta,z(g,\zeta))d\zeta dg,\int_0^{c}\int_0^{d}q(s_1,\varphi_1,g,\zeta,z(g,\zeta))d\zeta dg,z(s_1,\varphi_1)\Big)\Big)\right|\\
&+\left|F\Big(s_2,\varphi_2,\int_0^{s_2}\int_0^{\varphi_2}p(s_2,\varphi_2,g,\zeta,z(g,\zeta))d\zeta dg,\int_0^{c}\int_0^{d}q(s_1,\varphi_1,g,\zeta,z(g,\zeta))d\zeta dg,z(s_1,\varphi_1)\Big)\right.\\
&\left.-F\Big(s_2,\varphi_2,\int_0^{s_1}\int_0^{\varphi_1}p(s_1,\varphi_1,g,\zeta,z(g,\zeta))d\zeta dg,\int_0^{c}\int_0^{d}q(s_1,\varphi_1,g,\zeta,z(g,\zeta))d\zeta dg,z(s_1,\varphi_1)\Big)\Big)\right|\\
&+\left|F\Big(s_2,\varphi_2,\int_0^{s_1}\int_0^{\varphi_1}p(s_1,\varphi_1,g,\zeta,z(g,\zeta))d\zeta dg,\int_0^{c}\int_0^{d}q(s_1,\varphi_1,g,\zeta,z(g,\zeta))d\zeta dg,z(s_1,\varphi_1)\Big)\right.\\
&\left.-F\Big(s_1,\varphi_1,\int_0^{s_1}\int_0^{\varphi_1}p(s_1,\varphi_1,g,\zeta,z(g,\zeta))d\zeta dg,\int_0^{c}\int_0^{d}q(s_1,\varphi_1,g,\zeta,z(g,\zeta))d\zeta dg,z(s_1,\varphi_1)\Big)\Big)\right|\\
\le&\ K_6(s,\varphi)|z(s_2,\varphi_2)-z(s_1,\varphi_1)|\\
&+K_5(s,\varphi)\left|\int_0^{c}\int_0^{d}q(s_2,\varphi_2,g,\zeta,z(g,\zeta))d\zeta dg-\int_0^{c}\int_0^{d}q(s_1,\varphi_1,g,\zeta,z(g,\zeta))d\zeta dg\right|\\
&+K_4(s,\varphi)\left|\int_0^{s_2}\int_0^{\varphi_2}p(s_2,\varphi_2,g,\zeta,z(g,\zeta))d\zeta dg-\int_0^{s_1}\int_0^{\varphi_1}p(s_1,\varphi_1,g,\zeta,z(g,\zeta))d\zeta dg\right|+\omega_F(I,\sigma)\\
\le&\ K|z(s_2,\varphi_2)-z(s_1,\varphi_1)|+K\int_0^{c}\int_0^{d}|q(s_2,\varphi_2,g,\zeta,z(g,\zeta))-q(s_1,\varphi_1,g,\zeta,z(g,\zeta))|d\zeta dg\\
&+K\int_0^{s_1}\int_0^{\varphi_1}|p(s_2,\varphi_2,g,\zeta,z(g,\zeta))-p(s_1,\varphi_1,g,\zeta,z(g,\zeta))|d\zeta dg\\
&+K\int_{s_1}^{s_2}\int_{\varphi_1}^{\varphi_2}|p(s_2,\varphi_2,g,\zeta,z(g,\zeta))|d\zeta dg+K\int_{s_1}^{s_2}\int_0^{\varphi_1}|p(s_2,\varphi_2,g,\zeta,z(g,\zeta))|d\zeta dg\\
&+K\int_0^{s_1}\int_{\varphi_1}^{\varphi_2}|p(s_2,\varphi_2,g,\zeta,z(g,\zeta))|d\zeta dg,
\end{aligned}
$$

where

$$\omega_p(I,\sigma) = sup\Big\{|p(s_2,\varphi_2,g,\zeta,z) - p(s_1,\varphi_1,g,\zeta,z)| : s_1,s_2 \in [0,c], \varphi_1,\varphi_2 \in [0,d], z \in [-r,r], |s_2 - s_1| \le \sigma, |\varphi_2 - \varphi_1| \le \sigma\Big\},$$

$$\omega_q(I,\sigma) = sup\Big\{|q(s_2,\varphi_2,g,\zeta,z) - q(s_1,\varphi_1,g,\zeta,z)| : s_1,s_2 \in [0,c], \varphi_1,\varphi_2 \in [0,d], z \in [-r,r], |s_2 - s_1| \le \sigma, |\varphi_2 - \varphi_1| \le \sigma\Big\},$$

$$L = sup\Big\{|p(s,\varphi,g,\zeta,z)| : (s,\varphi),(g,\zeta) \in I, z \in [-r,r]\Big\},$$

and

$$\omega_F(I,\sigma) = sup\Big\{|F(s_2,\varphi_2,z,x,y) - F(s_1,\varphi_1,z,x,y)| : s_1,s_2 \in [0,c],\ \varphi_1,\varphi_2 \in [0,d], y \in [-r,r], |s_2 - s_1| \le \sigma, |\varphi_2 - \varphi_1| \le \sigma, z,x \in [-Lcd, Lcd]\Big\}.$$

Then using above relation, we obtain

$$|(Vz)(s_2,\varphi_2) - (Vz)(s_1,\varphi_1)| \le K|z(s_2,\varphi_2) - z(s_1,\varphi_1)| + K(cd\omega_p(I,\sigma) + L\sigma^2 + L\sigma d + L\sigma c) + \omega_F(I,\sigma) + Kcd\omega_q(I,\sigma).$$

From the assumptions, functions F, p and q are uniformly continuous on $I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, $I \times I \times \mathbb{R}$ and $I \times I \times \mathbb{R}$. Hence, deduce that

$$\omega_F(I,\sigma) \to 0\ \omega_p(I,\sigma) \to 0\ and\ \omega_q(I,\sigma) \to 0\ as\ \sigma \to 0.$$

Thus

$$\omega_0(VH) \le K\omega_0(H). \tag{9}$$

From Eqs. (8)–(9) and Theorem 3, T satisfies Darbo's condition on B_r with respect to ω_0 with the following constant:

$$\begin{aligned}(h_1 r + h_2)\ K + (h_1 r + h_2)\ K &= 2K(h_1 r + h_2)\\ &= 2K(h_1 r_1 + h_2)\\ &= 2K\left(h_1\left(\frac{(1 - 2h_1h_2) - \sqrt{1 - 4h_1h_2}}{2h_1^2}\right) + h_2\right)\\ &= 2K\left(\frac{1 - \sqrt{1 - 4h_1h_2}}{h_1}\right)\\ &< 1.\end{aligned}$$

Hence, T is a contraction on B_r with respect to ω_0. So, by Theorem 3, Eq. (1) has at least one solution in B_r.

Corollary 1 *Putting* $P(s,\varphi,z,x,y)=P(s,\varphi,z,y)$, *and* $F(s,\varphi,z,x,y)=F(s,\varphi,z,y)$.

Then Eq. (1) can be reduced into the following FIE

$$z(s,\varphi) = P\Big(s,\varphi,\int_0^s\int_0^\varphi u(s,\varphi,g,\zeta,z(g,\zeta))d\zeta dg, z(s,\varphi)\Big) \\ \times F\Big(s,\varphi,\int_0^s\int_0^\varphi p(s,\varphi,g,\zeta,z(g,\zeta))d\zeta dg, z(s,\varphi)\Big), \tag{10}$$

has at least one solution in $I=[0,c]\times[0,d]$.

Proof The proof is linked to Theorem 3 and leaves these details. □

Corollary 2 *Putting* $P(s,\varphi,z,x,y)=P(s,\varphi,x,y)$, *and* $F(s,\varphi,z,x,y)=F(s,\varphi,x,y)$.

Then Eq. (1) can be reduced into the following FIE

$$z(s,\varphi) = P\Big(s,\varphi,\int_0^c\int_0^d u(s,\varphi,g,\zeta,z(g,\zeta))d\zeta dg, z(s,\varphi)\Big) \\ \times F\Big(s,\varphi,\int_0^c\int_0^d p(s,\varphi,g,\zeta,z(g,\zeta))d\zeta dvg, z(s,\varphi)\Big), \tag{11}$$

has at least one solution in $I=[0,c]\times[0,d]$.

Proof The proof is linked to Theorem 3 and leaves these details. □

4 Applications

In this part, we give some examples of FIEs to explain the applications of our results.

Example 1

$$z(s,\varphi)=g(s,\varphi)+\int_0^s\int_0^\varphi u_1(s,\varphi,g,\zeta)u_2(g,\zeta,u(g,\zeta))d\zeta dg,$$

for $u(s,\varphi,g,\zeta,z)=u_1(s,\varphi,g,\zeta)u_2(g,\zeta,u(g,v))$, $F(s,\varphi,z,x,y)=1$, and $P(s,\varphi,z,x,y)=g(s,\varphi)+z$, which may be viewed like a two independent variables generalization of the famous Hammerstein type integral equation [23]

$$z(s,\varphi)=g(s,\varphi)+\int_0^1\int_0^1 q(s,\varphi,g,\zeta,z(g,\zeta))d\zeta dg,$$

which is the famous two-dimensional Fredholm integral equation analysed by various authors in history [1].

Example 2 Putting $P(s,\varphi,z,x,y)=g(s,\varphi)+z+x$, and $F(s,\varphi,z,x,y)=1$ and then Eq. (1) converts to the following equation

$$z(s,\varphi)=g(s,\varphi)+\int_0^s\int_0^\varphi p(s,\varphi,g,\zeta,z(g,\zeta))d\zeta dg+\int_0^1\int_0^1 q(s,\varphi,g,\zeta,z(g,\zeta))d\zeta dg. \tag{12}$$

Eq. (12) is studied by many authors, one can see [4, 23].

Example 3 Let the following non-linear FIE:

$$z(s,\varphi)=\left[\frac{1}{13}\int_0^s\int_0^\varphi\left(\frac{s\varphi}{3+s^2\varphi}sin(1+z(g,\zeta))+(1+s\varphi)arctan\frac{|z(g,\zeta)|}{2+|z(g,\zeta)|}\right)dtdv\right.$$
$$\left.+\frac{1}{15}\int_0^1\int_0^1\left(\frac{s\varphi}{2(1+s\varphi^2)}sin\frac{z(g,v)}{4+z(g,\zeta)}+3\ln(1+|z(g,\zeta)|)\right)d\zeta dg\right]$$
$$\times\left[\frac{1}{18}\int_0^s\int_0^\varphi\left(\frac{1}{2(1+s^2\varphi^3)}cos\frac{z^2(g,\zeta)}{1+z^2(g,\zeta)}+3s^3\varphi\ arctan|z^2(g,\zeta)|\right)d\zeta dg\right.$$
$$\left.+\frac{1}{17}\int_0^1\int_0^1\left(\frac{s^2\varphi}{3+s\varphi^2}sin(1+z(g^2,\zeta^2))+(2+s\varphi)arctan\frac{g^2\zeta|z(g^2,\zeta^2|}{1+|z(g^2,\zeta^2|}\right)d\zeta dg\right],\quad (13)$$

where $(s,\varphi)\in[0,1]\times[0,1]$. Eq. (13) is a particular form of Eq. (1).

$$P(s,\varphi,z_1,x_1,y_1)=\frac{1}{13}z_1+\frac{1}{15}x_1,\ F(s,\varphi,z_1,x_1,y_1)=\frac{1}{18}z_1+\frac{1}{17}x_1,$$
$$u(s,\varphi,g,\zeta,z)=\frac{s\varphi}{3+s^2\varphi}sin(1+z(g,\zeta))+(1+s\varphi)arctan\frac{|z(g,\zeta)|}{2+|z(g,\zeta)|},$$
$$f(s,\varphi,g,\zeta,z)=\frac{s\varphi}{2(1+s\varphi^2)}sin\frac{z(g,\zeta)}{4+z(g,\zeta)}+3\ln(1+|z(g,\zeta)|),$$
$$p(s,\varphi,g,\zeta,z)=\frac{1}{2(1+s^2\varphi^3)}cos\frac{z^2(g,\zeta)}{1+z^2(g,\zeta)}+3s^3\varphi\ arctan|z^2(g,\zeta)|,$$
$$q(s,\varphi,g,\zeta,z)=\frac{s^2\varphi}{3+s\varphi^2}sin(1+z(g^2,\zeta^2))+(2+s\varphi)arctan\frac{g^2\zeta|z(g^2,\zeta^2|}{1+|z(g^2,\zeta^2|}.$$

Now, we prove that all assumptions (1)–(3) for Theorem 3 are satisfied. Here $K_1=\frac{1}{13}$, $K_2=\frac{1}{15}$, $K_3=0$, $K_4=\frac{1}{18}$, $K_5=\frac{1}{17}$, and $K_6=0$. Then $K=max\{\frac{1}{13},\frac{1}{15},\frac{1}{18},\frac{1}{17}\}=\frac{1}{13}$. Moreover,

$$|P(s,\varphi,0,0,0)|=0,\ |F(s,\varphi,0,0,0)|=0,\ |u(s,\varphi,g,\zeta,z)|=\frac{1}{4}+3|z|,\ |f(s,\varphi,g,\zeta,z)|=\frac{1}{4}+3|z|,$$

$$|p(s,\varphi,g,\zeta,z)|=\frac{1}{4}+3|z|,\ |q(s,\varphi,g,\zeta,z)|=\frac{1}{4}+3|z|.$$

We also have $N=0$, $\beta_1=\frac{1}{4}$, $\beta_2=3$ and c, d = 1.

Finally, we have

$$4h_1h_2<1.$$

Hence, all assumptions (1) to (3) are satisfied. Eq. (13) has at least one solution in $[0,1]\times[0,1]$.

5 Conclusion

We explained the existence result of Eq. (1) in the composition form of the FIE in the Banach algebra using Darbo's fixed point theorem associated with MNC. The interested researchers can obtain the existence of solution of Eq. (1) for different methods as well as different spaces, e.g. Sobolev space, Hölder space, Orlicz space, etc.

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Results on Generalized Tripled Fuzzy b-Metric Spaces

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Abstract The aim of this chapter is to introduce the concept of tripled fuzzy b-metric space (TFbMS) and establish some new results on TFbMS. We put forward the concept of tripled fuzzy $\psi - b$-contraction mapping and tripled fuzzy $\psi - b$-contraction sequence. We also investigate the uniqueness of fixed point on TFbMS. In order to validate the non-triviality of results, some examples are provided.

1 Introduction

In analytical mathematics, the notion of metric space performs an important role to explain the existence of fixed point and common fixed point of mappings. In analysis, the existence of the fixed point of a mapping often implies the existence of a solution of that mapping within that metric space. Hence, the study of fixed point theory in several generalized metric spaces is of paramount importance for the development and modeling of different areas of science and engineering. In certain situations, the construction of a modeling system becomes difficult due to the inadequate measure of distance between two elements or points. For such situations, fuzzy sets or fuzzy logic contribute a consequential platform to construct and improve the modeling and designing systems.

The notion of generalized b-metric space also plays an important role to the investigation of fixed point theory. Such types of generalization can control more complex situations efficiently and also reduce the complexity of modeling systems for higher order sets. They also extend a feasible platform for scientific modeling and designing. The flexible nature of such type of models helps to improve the applications of fixed point theory in the areas of science, engineering, and mathematics and also motives several future work.

In order to elaborate the situations where data or elements are imprecise or vague and to represent a mathematical structure for such types of situations, a new con-

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cept of set theory called fuzzy set theory was established by Zadeh [24] in 1965. Simultaneously, Kaleva and Seikkala [16] initiated the notion of fuzzy metric space. Consequently, the notion of fuzzy metric space was modified by several mathematicians like-Kramosil and Michalek [17], George and Veeramani [9], etc. Jungck and Rhoads [15] established the concept of weakly compatible maps of metric spaces. Development of metric space in various ways is the overriding concern of the mathematician. Due to this concern Huang and Zhang [13] established the idea of cone metric space. After this Abbas and Jungck [1] introduced some results on non-commuting mapping in cone metric spaces. By considering a weaker condition, in place of triangular inequality, Bakhtin [5] and Czerwike [8] introduced the notion of b metric space.

Initially, Heilpern [12] introduced the notion of fixed point theory and established an extended version of Banach's contraction principle in the setting of fuzzy metric space. The concept of fuzzy cone metric space was initiated by Oner et al. [19]. They also established Banach contraction principle and some basic properties for fixed point. This famous work has been further generalized and extended by many mathematicians in the settings of fuzzy set [2, 4, 6, 7, 11, 18, 20–23].

The primary aim of this current chapter is to explain and extend the notion of tripled fuzzy b-metric space (TFbMS) and to introduce some new results which are useful for further generalization of this work. We also establish the concept of tripled fuzzy $\psi - b$-contraction mapping and tripled fuzzy $\psi - b$-contraction sequence. Main motive of this chapter is to develop the generalized metric space in order to establish some analytic results and explain the properties with some examples.

2 Some Definition

Below we discuss a few preliminary definitions which are essential for our main result.

Definition 1 Consider a binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$. Then $*$ is known as a continuous *t-norm* if it satisfies the condition:

(i) $*$ is associative and commutative,
(ii) $*$ is continuous,
(iii) $\alpha * 1 = \alpha$ for all $\alpha \in [0, 1]$,
(iv) $\alpha * b \leq \beta * d$ whenever $\alpha \leq \beta$ and $b \leq d$ and $\alpha, \beta, c, d \in [0, 1]$.

Definition 2 Consider a function $d : S \times S \longrightarrow \mathbb{R}$, where $S \neq \phi$. Then $\forall s_1, s_2, s_3 \in S$, (S, d) is called a metric space if it satisfies the following condition:

(i) $d(s_1, s_2) \geq 0$ and $d(s_1, s_2) = 0$ iff $s_1 = s_2$.
(ii) $d(s_1, s_2) = d(s_2, s_1)$.
(iii) $d(s_1, s_3) \leq d(s_1, s_2) + d(s_2, s_3)$.

Definition 3 Suppose X is a classical set of data, called the universe and $A \in X$. Then membership of A is considered as a characteristic function μ_A from X to $\{0, 1\}$ such that

$$\mu_A(x) = \begin{cases} 1 \text{ iff } x \in A \\ 0 \text{ iff } x \notin A. \end{cases}$$

$\{0, 1\}$ is called a valuation set. If $\{0, 1\}$ is allowed to be $[0, 1]$, A is said to be a fuzzy set.

Kramosil and Michalek [17] defined fuzzy metric space as follows:

Definition 4 ([17]) Consider a set $X \neq \phi$ and a continuous t-norm $*$. Suppose M is a fuzzy set on $X^2 \times \mathbb{R}$. Then $\forall\, a_1, a_2, a_3 \in X$ and $t, s \in \mathbb{R}$, $(X, M, *)$ is called fuzzy metric space if it satisfies the following axioms:

(i) $M(a_1, a_2, t) = 0 \;\forall\, t \leq 0$.
(ii) $M(a_1, a_2, t) = 1 \;\forall\, t > 0$ iff $a_1 = a_2$.
(iii) $M(a_1, a_2, t) = M(a_2, a_1, t)$.
(iv) $M(a_1, a_2, t) * M(a_2, a_3, s) \leq M(a_1, a_3, t + s)$.
(v) $M(a_1, a_2, t) : (0, \infty) \to [0, 1]$ is left continuous.
(vi) $\lim_{t\to\infty} M(a_1, a_2, t) = 1$.

George and Veeramani [9, 10] made an appealing modification of fuzzy metric spaces in the following way:

Definition 5 ([9]) Consider a set $X \neq \phi$ and a continuous t-norm $*$. Suppose M is a fuzzy set on $X^2 \times (0, \infty)$. Then $\forall\, a_1, a_2, a_3 \in X$ and $t, s \in \mathbb{R}$, $(X, M, *)$ is called fuzzy metric space if it satisfies the following axioms:

(i) $M(a_1, a_2, t) > 0$.
(ii) $M(a_1, a_2, t) = 1 \;\forall\, t > 0$ iff $a_1 = a_2$.
(iii) $M(a_1, a_2, t) = M(a_2, a_1, t)$.
(iv) $M(a_1, a_2, t) * M(a_2, a_3, s) \leq M(a_1, a_3, t + s)$.
(v) $M(a_1, a_2, t) : (0, \infty) \to [0, 1]$ is continuous.

Definition 6 ([14]) Consider a non-empty set S and a continuous t-norm $*$. Suppose P is a fuzzy set on $S \times S \times \times(0, \infty)$ such that $\forall\, u, v, w \in S$ and $\alpha, \beta > 0$ following conditions hold:

(i) $P(\hbar_1, \hbar_2, \alpha) > 0$,
(ii) $P(\hbar_1, \hbar_2, \alpha) = 1$ iff $\hbar_1 = \hbar_2$,
(iii) $P(\hbar_1, \hbar_2, \alpha) = P(\hbar_2, \hbar_1, \alpha)$,
(iv) $P(\hbar_1, \hbar_2, \cdot) : (0, \infty) \to (0, 1]$ is continuous,
(v) $P(\hbar_1, \hbar_3, \alpha + \beta) \geq *(P(\hbar_1, \hbar_2, \frac{\alpha}{b}), P(\hbar_2, \hbar_3, \frac{\beta}{b}))$

Then $(S, P, *)$ is called a FbMS.

Definition 7 ([3]) Consider a metric space (Y, d) and a function $T : Y \to Y$. Then T is called a contraction mapping or contraction if $\exists$ a constant α (called constant of contraction), with $0 \leq \alpha < 1$, such that

$$d(T(y_1), T(y_2)) \leq \alpha d(y_1, y_2), \forall y_1, y_2 \in Y.$$

Definition 8 Suppose S and f are two self-maps of a set X. Then, a point $v \in X$ is said to be a coincidence point of S and f if we have $v = S(v) = f(v)$. The self-mappings S and f are said to be weakly compatible if they commute at their coincidence point, i.e., for some $v \in X$ $S(v) = f(v)$ we have $Sf(v) = fS(v)$.

Proposition 1 *Suppose that S and f are two weakly compatible self-maps of a set X. If S and f have a unique point of coincidence $v = S(v) = f(v)$, then v is the unique common fixed point of S and f.*

Next we elaborate on the results of the chapter.

3 Major Work of the Chapter

In this section, we put forward the definition of Tripled fuzzy b-metric space(shortly, TFbMS). After defining the main concept, we provide some propositions and examples. We also establish the fixed point theorem in TFbMS.

3.1 Definition and Example of TFbMS

Definition 9 Consider an arbitrary set S, a fuzzy set T on $S \times S \times S \times (0, \infty)$ and a continuous t-norm $*$. Then $\forall$ $\hbar_1, \hbar_2, \hbar_3, \hbar_4 \in S$, $(S, T, *)$ is said to be TFbMS if the following properties holds:

(i) $T_{\hbar_1,\hbar_2,\hbar_3}(\alpha) > 0$;
(ii) $T_{\hbar_1,\hbar_2,\hbar_3}(\alpha) = 1$ if and only if $\hbar_1 = \hbar_2 = \hbar_3$;
(iii) $T_{\hbar_1,\hbar_1,\hbar_2}(\alpha) \geq T_{\hbar_1,\hbar_2,\hbar_3}(\alpha)$ for $\hbar_3 \neq \hbar_2$;
(iv) $T_{\hbar_1,\hbar_2,\hbar_3}(\alpha) = T_{\hbar_1,\hbar_3,\hbar_2}(\alpha) = T_{\hbar_2,\hbar_1,\hbar_3}(\alpha) = \cdots$;
(v) $T_{\hbar_1,\hbar_2,\hbar_3}(\cdot) : (0, \infty) \to (0, 1]$ is continuous;
(vi) $T_{\hbar_1,\hbar_2,\hbar_3}(\alpha + \beta) \geq *(T_{\hbar_1,\hbar_4,\hbar_4}(\frac{\alpha}{b}), T_{\hbar_4,\hbar_2,\hbar_3}(\frac{\beta}{b}))$

Example 1 Consider the set of real number $\mathbb{R}$. For all $\hbar_1, \hbar_2, \hbar_3 \in \mathbb{R}$ and $\alpha > 0$ construct the function $T : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times (0, 1]$ such that

$$T_{\hbar_1,\hbar_2,\hbar_3}(\alpha) = [e^{(\frac{|\hbar_1-\hbar_2|+|\hbar_2-\hbar_3|+|\hbar_3-\hbar_1|}{\alpha})}]^{-1}$$

then $(\mathbb{R}, T, *)$ is a TFbMS.

Proof Since $e^{(\frac{|\hbar_1-\hbar_2|+|\hbar_2-\hbar_3|+|\hbar_3-\hbar_1|}{\alpha})} > 0$ therefore $T_{\hbar_1,\hbar_2,\hbar_3}(\alpha) > 0$.
Condition (i) satisfied.

If we consider $\hbar_1 = \hbar_2 = \hbar_3$ then $e^{(\frac{|\hbar_1-\hbar_2|+|\hbar_2-\hbar_3|+|\hbar_3-\hbar_1|}{\alpha})} = 1$ and hence $T_{\hbar_1,\hbar_2,\hbar_3}(\alpha) = 1$.
Condition (ii) is satisfied.

In similar manner we can see that conditions (iii), (iv), and (v) are also satisfied.

Finally, we need to verify the condition:

$$T_{\hbar_1,\hbar_2,\hbar_3}(\beta+\alpha) \geq *\left(T_{\hbar_1,\hbar_4,\hbar_4}\left(\frac{\beta}{b}\right), T_{\hbar_4,\hbar_2,\hbar_3}\left(\frac{\alpha}{b}\right)\right)$$

for all $\hbar_1, \hbar_2, \hbar_3, \hbar_4 \in \mathbb{R}$ and $\beta, \alpha > 0$.

Now, we have

$$\begin{aligned}
&\frac{|\hbar_1-\hbar_2|+|\hbar_2-\hbar_3|+|\hbar_3-\hbar_1|}{\frac{\beta+\alpha}{b}} \\
&\leq \frac{|\hbar_1-\hbar_4|+|\hbar_4-\hbar_2|+|\hbar_2-\hbar_3|+|\hbar_3-\hbar_4|+|\hbar_4-\hbar_1|}{\frac{\beta+\alpha}{b}} \\
&= \frac{2|\hbar_1-\hbar_4|}{\frac{\beta+\alpha}{b}} + \frac{|\hbar_4-\hbar_2|+|\hbar_2-\hbar_3|+|\hbar_3-\hbar_4|}{\frac{\beta+\alpha}{b}} \\
&< \frac{2|\hbar_1-\hbar_4|}{\frac{\beta}{b}} + \frac{|\hbar_4-\hbar_2|+|\hbar_2-\hbar_3|+|\hbar_3-\hbar_4|}{\frac{\alpha}{b}}
\end{aligned}$$

Hence, for some $b > 0 \in \mathbb{R}$, we have

$$\begin{aligned}
T_{\hbar_1,\hbar_2,\hbar_3}(\beta+\alpha) &= e^{-(\frac{|\hbar_1-\hbar_2|+|\hbar_2-\hbar_3|+|\hbar_3-\hbar_1|}{\beta+\alpha})} \\
&\geq e^{-(\frac{|\hbar_1-\hbar_2|+|\hbar_2-\hbar_3|+|\hbar_3-\hbar_1|}{\frac{\beta+\alpha}{b}})} \\
&\geq e^{\frac{2|\hbar_1-\hbar_4|}{\frac{\beta}{b}} + \frac{|\hbar_4-\hbar_2|+|\hbar_2-\hbar_3|+|\hbar_3-\hbar_4|}{\frac{\alpha}{b}}} \\
&= T_{\hbar_1,\hbar_4,\hbar_4}\left(\frac{\beta}{b}\right) T_{\hbar_4,\hbar_2,\hbar_3}\left(\frac{\alpha}{b}\right) \\
&= *\left(T_{\hbar_1,\hbar_4,\hbar_4}\left(\frac{\beta}{b}\right), T_{\hbar_4,\hbar_2,\hbar_3}\left(\frac{\alpha}{b}\right)\right)
\end{aligned}$$

Therefore, $(\mathbb{R}, T, *)$ is a TF*b*MS.

3.2 Some Results in TFbMS

Proposition 2 *Consider a FbMS* $(S, P, *)$. *For some* $\hbar_1, \hbar_2, \hbar_3 \in S$ *and* $\alpha > 0$ *a function is constructed as* $T : S \times S \times S \times (0, \infty) \to (0, 1]$ *and defined by*

$$T_{\hbar_1,\hbar_2,\hbar_3}(\alpha) = *\left(*\left(P\left(\hbar_1, \hbar_2, \frac{\alpha}{b}\right), P\left(\hbar_2, \hbar_3, \frac{\alpha}{b}\right)\right), P\left(\hbar_1, \hbar_3, \frac{\alpha}{b}\right)\right)$$

Then $(S, T, *)$ *is a TFbMS.*

Proof Since the triple $(S, P, *)$ is a FbMS therefore it satisfies the properties of Definition 9 from (i) to (v). Hence, we have to verify the property (vi) of Definition 9 for the proof, i.e., for all $\hbar_1, \hbar_2, \hbar_3 \in S$ and $\alpha, \beta > 0$ we have to verify:

$$T_{\hbar_1,\hbar_2,\hbar_3}(\beta + \alpha) \geq *\left(T_{\hbar_1,\hbar_4,\hbar_4}\left(\frac{\beta}{b}\right), T_{\hbar_4,\hbar_2,\hbar_3}\left(\frac{\alpha}{b}\right)\right)$$

This implies

$$\begin{aligned}
&*\left(*\left(P\left(\hbar_1, \hbar_2, \frac{\beta+\alpha}{b}\right), P\left(\hbar_2, \hbar_3, \frac{\beta+\alpha}{b}\right)\right), P\left(\hbar_1, \hbar_3, \frac{\beta+\alpha}{b}\right)\right) \\
&\quad \geq *\left(*\left(*\left(P\left(\hbar_1, \hbar_4, \frac{\beta}{b^2}\right), P\left(\hbar_4, \hbar_4, \frac{\beta}{b^2}\right)\right), P\left(\hbar_4, \hbar_1 \frac{\beta}{b^2}\right)\right),\right. \\
&\quad \left. *\left(*\left(P\left(\hbar_4, \hbar_2, \frac{\alpha}{b^2}\right), P\left(\hbar_2, \hbar_3, \frac{\alpha}{b^2}\right)\right), P\left(\hbar_3, \hbar_4, \frac{\alpha}{b^2}\right)\right)\right) \\
&\quad = *\left(*\left(P\left(\hbar_1, \hbar_4, \frac{\beta}{b^2}\right), P\left(\hbar_4, \hbar_1, \frac{\beta}{b^2}\right)\right), *\left(*\left(P\left(\hbar_4, \hbar_2, \frac{\alpha}{b^2}\right),\right.\right.\right. \\
&\quad \left.\left.\left. P\left(\hbar_2, \hbar_3, \frac{\alpha}{b^2}\right)\right), P\left(\hbar_3, \hbar_4, \frac{\alpha}{b^2}\right)\right)\right)
\end{aligned}$$

Next applying the property of FbMS, we have

$$\begin{aligned}
P\left(\hbar_1, \hbar_2, \frac{\beta+\alpha}{b}\right) &\geq *\left(P\left(\hbar_1, \hbar_4, \frac{\beta}{b^2}\right), P\left(\hbar_4, \hbar_2, \frac{\alpha}{b^2}\right)\right) \\
&\geq *\left(\min\left\{P\left(\hbar_1, \hbar_4, \frac{\beta}{b^2}\right), P\left(\hbar_4, \hbar_1, \frac{\beta}{b^2}\right)\right\},\right. \\
&\quad \left.\min\left\{\min\left\{P\left(\hbar_4, \hbar_2, \frac{\alpha}{b^2}\right), P\left(\hbar_2, \hbar_3, \frac{\alpha}{b^2}\right)\right\}, P\left(\hbar_4, \hbar_3, \frac{\alpha}{b^2}\right)\right\}\right) \\
&\geq *\left(*\left(P\left(\hbar_1, \hbar_4, \frac{\beta}{b^2}\right), P\left(\hbar_4, \hbar_1, \frac{\beta}{b^2}\right)\right),\right. \\
&\quad \left. *\left(*\left(P\left(\hbar_4, \hbar_2, \frac{\alpha}{b^2}\right), P\left(\hbar_2, \hbar_3, \frac{\alpha}{b^2}\right)\right), P\left(\hbar_4, \hbar_3, \frac{\alpha}{b^2}\right)\right)\right) \qquad (1)
\end{aligned}$$

$$\begin{aligned}
P\left(\hbar_2, \hbar_3, \frac{\beta+\alpha}{b}\right) &\geq P\left(\hbar_2, \hbar_3, \frac{\alpha}{b^2}\right) \\
&\geq \min\left\{P\left(\hbar_4, \hbar_2, \frac{\alpha}{b^2}\right), P\left(\hbar_2, \hbar_3, \frac{\alpha}{b^2}\right), P\left(\hbar_4, \hbar_3, \frac{\beta}{b^2}\right)\right\} \\
&\geq *\left(P\left(\hbar_1, \hbar_1, \frac{\beta}{b^2}\right), \min\left\{P\left(\hbar_4, \hbar_2, \frac{\alpha}{b^2}\right), P\left(\hbar_2, \hbar_3, \frac{\alpha}{b^2}\right), P\left(\hbar_4, \hbar_3, \frac{\beta}{b^2}\right)\right\}\right) \\
&\geq *\left(*\left(P\left(\hbar_1, \hbar_4, \frac{\beta}{b^2}\right), P\left(\hbar_4, \hbar_1, \frac{\beta}{b^2}\right)\right),\right. \\
&\left.\min\left\{\min\left\{P\left(\hbar_4, \hbar_2, \frac{\alpha}{b^2}\right), P\left(\hbar_4, \hbar_3, \frac{\beta}{b^2}\right)\right\}, P\left(\hbar_2, \hbar_3, \frac{\alpha}{b^2}\right)\right\}\right) \\
&\geq *\left(*\left(P\left(\hbar_1, \hbar_4, \frac{\beta}{b^2}\right), P\left(\hbar_4, \hbar_1, \frac{\beta}{b^2}\right)\right),\right. \\
&\left. *\left(*\left(P\left(\hbar_4, \hbar_2, \frac{\alpha}{b^2}\right), P\left(\hbar_4, \hbar_3, \frac{\beta}{b^2}\right)\right), P\left(\hbar_2, \hbar_3, \frac{\alpha}{b^2}\right)\right)\right) \qquad (2)
\end{aligned}$$

and

$$\begin{aligned}
P\left(\hbar_1, \hbar_3, \frac{\beta+\alpha}{b}\right) &\geq *\left(P\left(\hbar_1, \hbar_4, \frac{\beta}{b^2}\right), P\left(\hbar_4, \hbar_3, \frac{\alpha}{b^2}\right)\right) \\
&\geq *\left(\min\left\{P\left(\hbar_1, \hbar_4, \frac{\beta}{b^2}\right), P\left(\hbar_4, \hbar_1, \frac{\beta}{b^2}\right)\right\},\right. \\
&\left.\min\left\{\min\left\{P\left(\hbar_4, \hbar_2, \frac{\alpha}{b^2}\right), P\left(\hbar_2, \hbar_3, \frac{\alpha}{b^2}\right)\right\}, P\left(\hbar_4, \hbar_3, \frac{\alpha}{b^2}\right)\right\}\right) \\
&\geq *\left(*\left(P\left(\hbar_1, \hbar_4, \frac{\beta}{b^2}\right), P\left(\hbar_4, \hbar_1, \frac{\beta}{b^2}\right)\right),\right. \\
&\left. *\left(*\left(P\left(\hbar_4, \hbar_2, \frac{\alpha}{b^2}\right), P\left(\hbar_2, \hbar_3, \frac{\alpha}{b^2}\right)\right), P\left(\hbar_4, \hbar_3, \frac{\alpha}{b^2}\right)\right)\right) \qquad (3)
\end{aligned}$$

Therefore, we have from (1) to (3)

$$\begin{aligned}
*\left(*\left(P\left(\hbar_1, \hbar_2, \frac{\beta+\alpha}{b}\right), P\left(\hbar_2, \hbar_3, \frac{\beta+\alpha}{b}\right)\right), P\left(\hbar_1, \hbar_3, \frac{\beta+\alpha}{b}\right)\right)& \\
\geq *\left(*\left(P\left(\hbar_1, \hbar_4, \frac{\beta}{b^2}\right), P\left(\hbar_4, \hbar_1, \frac{\beta}{b^2}\right)\right),\right.& \\
\left. *\left(*\left(P\left(\hbar_4, \hbar_2, \frac{\alpha}{b^2}\right), P\left(\hbar_2, \hbar_3, \frac{\alpha}{b^2}\right)\right), P\left(\hbar_3, \hbar_4, \frac{\alpha}{b^2}\right)\right)\right)&
\end{aligned}$$

Hence

$$T_{\hbar_1,\hbar_2,\hbar_3}(\beta+\alpha) \geq *\left(T_{\hbar_1,\hbar_4,\hbar_4}\left(\frac{\beta}{b}\right), T_{\hbar_4,\hbar_2,\hbar_3}\left(\frac{\alpha}{b}\right)\right)$$

Thus $(S, T, *)$ is a TFbMS.

Proposition 3 *Consider a TFbMS $(S, T, *)$. For some $\hbar_1, \hbar_2, \hbar_3 \in S$ and $\alpha > 0$, a function P is constructed as $P : S \times S \times (0, \infty) \to (0, 1]$ such that*

$$P(\hbar_1, \hbar_2, \alpha) = *\left(T_{\hbar_1,\hbar_2,\hbar_2}\left(\frac{\alpha}{b}\right), T_{\hbar_2,\hbar_1,\hbar_1}\left(\frac{\alpha}{b}\right)\right)$$

Then P is a FbM.

Proof From the Definition 6 it is seen that P satisfies the properties of Definition 6 from (i) to (iv) very easily. Hence, we have to verify the property (v) for the proof, i.e, for some $\hbar_1, \hbar_2, \hbar_4 \in S$ and $\beta, \alpha > 0$ we have to show that

$$P(\hbar_1, \hbar_2, \beta + \alpha) \geq \left(P\left(\hbar_1, \hbar_3, \frac{\beta}{b}\right), P\left(\hbar_3, \hbar_2, \frac{\alpha}{b}\right)\right)$$

Now from the given condition we have for some $\hbar_1, \hbar_2, \hbar_4 \in S$ and $\beta, \alpha > 0$,

$$P(\hbar_1, \hbar_2, \beta + \alpha) = *\left(T_{\hbar_1,\hbar_2,\hbar_2}\left(\frac{\beta + \alpha}{b}\right), T_{\hbar_2,\hbar_1,\hbar_1}\left(\frac{\beta + \alpha}{b}\right)\right)$$

From the definition of TFbMS, we have

$$*\left(T_{\hbar_1,\hbar_2,\hbar_2}\left(\frac{\beta + \alpha}{b}\right), T_{\hbar_2,\hbar_1,\hbar_1}\left(\frac{\beta + \alpha}{b}\right)\right) \geq *\left(* \left(T_{\hbar_1,\hbar_3,\hbar_3}\left(\frac{\beta}{b^2}\right), T_{\hbar_3,\hbar_2,\hbar_2}\left(\frac{\alpha}{b^2}\right)\right), * \left(T_{\hbar_2,\hbar_3,\hbar_3}\left(\frac{\alpha}{b^2}\right), T_{\hbar_3,\hbar_1,\hbar_1}\left(\frac{\beta}{b^2}\right)\right)\right)$$

Next we have

$$\begin{aligned} T_{\hbar_1,\hbar_3,\hbar_3}\left(\frac{\beta}{b^2}\right) &\geq \min\left\{T_{\hbar_1,\hbar_3,\hbar_3}\left(\frac{\beta}{b^2}\right), T_{\hbar_3,\hbar_1,\hbar_1}\left(\frac{\beta}{b^2}\right)\right\} \\ &\geq *\left(T_{\hbar_1,\hbar_3,\hbar_3}\left(\frac{\beta}{b^2}\right), T_{\hbar_3,\hbar_1,\hbar_1}\left(\frac{\beta}{b^2}\right)\right) \end{aligned}$$

$$\begin{aligned} T_{\hbar_3,\hbar_2,\hbar_2}\left(\frac{\alpha}{b^2}\right) &\geq \min\left\{T_{\hbar_3,\hbar_2,\hbar_2}\left(\frac{\alpha}{b^2}\right), T_{\hbar_2,\hbar_3,\hbar_3}\left(\frac{\alpha}{b^2}\right)\right\} \\ &\geq *\left(T_{\hbar_3,\hbar_2,\hbar_2}\left(\frac{\alpha}{b^2}\right), T_{\hbar_2,\hbar_3,\hbar_3}\left(\frac{\alpha}{b^2}\right)\right) \end{aligned}$$

$$T_{\hbar_2,\hbar_3,\hbar_3}\left(\frac{\beta}{b^2}\right) \geq \min\left\{T_{\hbar_3,\hbar_2,\hbar_2}\left(\frac{\alpha}{b^2}\right), T_{\hbar_2,\hbar_3,\hbar_3}\left(\frac{\alpha}{b^2}\right)\right\}$$
$$\geq *\left(T_{\hbar_3,\hbar_2,\hbar_2}\left(\frac{\alpha}{b^2}\right), T_{\hbar_2,\hbar_3,\hbar_3}\left(\frac{\alpha}{b^2}\right)\right)$$

and

$$T_{\hbar_3,\hbar_1,\hbar_1}\left(\frac{\beta}{b^2}\right) \geq \min\left\{T_{\hbar_1,\hbar_3,\hbar_3}\left(\frac{\beta}{b^2}\right), T_{\hbar_3,\hbar_1,\hbar_1}\left(\frac{\beta}{b^2}\right)\right\}$$
$$\geq *\left(T_{\hbar_1,\hbar_3,\hbar_3}\left(\frac{\beta}{b^2}\right), T_{\hbar_3,\hbar_1,\hbar_1}\left(\frac{\beta}{b^2}\right)\right)$$

Again from the property of Definition 9, we have

$$T_{\hbar_1,\hbar_2,\hbar_2}\left(\frac{\beta+\alpha}{b}\right) \geq *\left(T_{\hbar_1,\hbar_3,\hbar_3}\left(\frac{\beta}{b^2}\right), T_{\hbar_3,\hbar_2,\hbar_2}\left(\frac{\alpha}{b^2}\right)\right)$$
$$\geq *\left(* \left(T_{\hbar_1,\hbar_3,\hbar_3}\left(\frac{\beta}{b^2}\right), T_{\hbar_3,\hbar_1,\hbar_1}\left(\frac{\beta}{b^2}\right)\right),\right.$$
$$\left. * \left(T_{\hbar_3,\hbar_2,\hbar_2}\left(\frac{\alpha}{b^2}\right), T_{\hbar_2,\hbar_3,\hbar_3}\left(\frac{\alpha}{b^2}\right)\right)\right)$$
$$= *\left(P\left(\hbar_1, \hbar_3, \frac{\beta}{b}\right), P\left(\hbar_3, \hbar_2, \frac{\alpha}{b}\right)\right)$$

and

$$T_{\hbar_2,\hbar_1,\hbar_1}\left(\frac{\beta+\alpha}{b}\right) \geq *\left(T_{\hbar_2,\hbar_3,\hbar_3}\left(\frac{\alpha}{b^2}\right), T_{\hbar_3,\hbar_1,\hbar_1}\left(\frac{\beta}{b^2}\right)\right)$$
$$\geq *\left(* (T_{\hbar_3,\hbar_2,\hbar_2}\left(\frac{\alpha}{b^2}\right), T_{\hbar_2,\hbar_3,\hbar_3}\left(\frac{\alpha}{b^2}\right)\right),$$
$$\left. * \left(T_{\hbar_1,\hbar_3,\hbar_3}\left(\frac{\beta}{b^2}\right), T_{\hbar_3,\hbar_1,\hbar_1}\left(\frac{\beta}{b^2}\right)\right)\right)$$
$$= *\left(P\left(\hbar_3, \hbar_2, \frac{\alpha}{b}\right), P\left(\hbar_1, \hbar_3, \frac{\beta}{b}\right)\right)$$

Therefore, we have

$$P(\hbar_1, \hbar_2, \beta + \alpha) = * \left(T_{\hbar_1,\hbar_2,\hbar_2}\left(\frac{\beta + \alpha}{b}\right), T_{\hbar_2,\hbar_1,\hbar_1}\left(\frac{\beta + \alpha}{b}\right)\right)$$
$$\geq \left(P\left(\hbar_1, \hbar_3, \frac{\beta}{b}\right), P\left(\hbar_3, \hbar_2, \frac{\alpha}{b}\right)\right)$$

This implies that P satisfies the (v) property of Definition 6.
Thus P is a FbM.

Remark 1 Suppose $(S, T, *)$ is a TFbMS. Then $T_{\hbar_1,\hbar_2,\hbar_2}(\cdot)$ is nondecreasing for all $\hbar_1, \hbar_2 \in S$.

3.3 Fixed Point and Related Proposition in TFbMS

Next we are going to define the concept of generalized tripled fuzzy $\psi - b$-contraction mapping (shortly, $TF\psi - b$-C mapping) and generalized tripled fuzzy ψ-contraction sequence(shortly, $F\psi$-C sequence) in TFbMS. With the help of this newly defined contraction mapping, we are going to establish unique fixed point in TFbMS.

Throughout the section, we consider that Ψ is the set of all mappings ψ such that $\psi : (0, 1] \to (0, 1]$ is continuous, nondecreasing and $\psi(\alpha) > \alpha$ for any $\alpha \in (0, 1)$.

Definition 10 Consider a TFbMS $(S, T, *)$ and suppose $\psi \in \Psi$. For some $\alpha > 0$ and $\forall \hbar_1, \hbar_2, \hbar_3 \in S$ the mapping $f : S \to S$ is defined by $T_{f(\hbar_1),f(\hbar_2),f(\hbar_3)}(\alpha) \geq \psi(T_{\hbar_1,\hbar_2,\hbar_3}(\frac{\alpha}{b}))$. Then f is called a generalized $TF\psi - b - C$ mapping.

Definition 11 Consider a TFbMS $(S, T, *)$ and suppose $\psi \in \Psi$. For some $\alpha > 0$ and $\forall n \in \mathbb{N}$ the sequence $\{\hbar_n\}$ in S defined as

$$T_{\hbar_{n+1},\hbar_{n+2},\hbar_{n+3}}(\alpha) \geq \psi\left(T_{\hbar_n,\hbar_{n+1},\hbar_{n+2}}\left(\frac{\alpha}{b}\right)\right).$$

Then $\{\hbar_n\}$ is called a generalized $TF\psi - b - C$ sequence.

Definition 12 A sequence $\{\hbar_n\}$ in a TFbMS $(S, T, *)$ is called a generalized $TF\psi - b - C$ sequence if $\forall\, n \in \mathbb{N}$ and $\alpha > 0$ it satisfies

$$T_{\hbar_{n+1},\hbar_{n+2},\hbar_{n+2}}(\alpha) \geq \psi\left(T_{\hbar_n,\hbar_{n+1},\hbar_{n+1}}\left(\frac{\alpha}{b}\right)\right).$$

With the help of the above definition, we are going to establish some proposition and lemma on $TFbMS$. Proof of the below proposition and theorem can be easily obtained using the above results.

Lemma 1 *If $\psi \in \Psi$, then for each $\alpha \in (0, 1]$, $\lim_n \psi^n(\alpha) = 1$.*

Proposition 4 *Consider a TFbMS $(S, T, *)$ and a generalized F$\psi - b$-C sequence $\{\hbar_n\}$ in S. Then $\{\hbar_n\}$ is a convergent sequence if $\wedge_{\alpha>0} T_{\hbar_0,\hbar_1,\hbar_1}(\alpha) > 0$ and $\wedge_{\alpha>0} T_{\hbar_1,\hbar_0,\hbar_0}(\alpha) > 0$.*

Theorem 1 *Consider a TFbMS $(S, T, *)$ and a generalized F$\psi - b$-C mapping $f : S \to S$. Then the mapping f has a unique fixed point if and only if there is a $\hbar \in S$ satisfying $\wedge_{\alpha>0} T_{\hbar,f(\hbar),f(\hbar)}(\alpha) > 0$ and $\wedge_{\alpha>0} T_{f(\hbar),\hbar,\hbar}(\alpha) > 0$.*

Proof Suppose f has a unique fixed point, then $\exists$ a $\hbar \in S$ such that $f(\hbar) = \hbar$.

For each $\alpha > 0$,

$$T_{\hbar,f(\hbar),f(\hbar)}(\alpha) = T_{\hbar,\hbar,\hbar}(\alpha) = 1$$

and

$$T_{f(\hbar),\hbar,\hbar}(\alpha) = T_{\hbar,\hbar,\hbar}(\alpha) = 1$$

Therefore

$$\wedge_{\alpha>0} T_{\hbar,f(\hbar),f(\hbar)}(\alpha) = 1 > 0 \text{ and } \wedge_{\alpha>0} T_{f(\hbar),\hbar,\hbar}(\alpha) = 1 > 0$$

Conversely, suppose that $\exists$ a $\hbar \in S$ satisfying the condition

$$\wedge_{\alpha>0} T_{\hbar,f(\hbar),f(\hbar)}(\alpha) > 0 \text{ and } \wedge_{\alpha>0} T_{f(\hbar),\hbar,\hbar}(\alpha) > 0$$

Consider $\hbar_0 = \hbar$ and for each $n \geq 1$, $\hbar_n = f^n(\hbar)$. Then, we have

$$\begin{aligned} T_{\hbar_{n+1},\hbar_{n+2},\hbar_{n+3}}(\alpha) &= T_{f^{n+1}(\hbar),f^{n+2}(\hbar),f^{n+3}(\hbar)}(\alpha) \\ &= T_{f.f^n(\hbar),f.f^{n+1}(\hbar),f.f^{n+2}(\hbar)}(\alpha) \\ &= T_{f(\hbar_n),f(\hbar_{n+1}),f(\hbar_{n+2})}(\alpha) \\ &\geq \psi\left(T_{\hbar_n,\hbar_{n+1},\hbar_{n+2}}\left(\frac{\alpha}{b}\right)\right) \end{aligned}$$

Therefore, $\{\hbar_n\}$ is a generalized F$\psi - b$-C sequence.

Next

$$\wedge_{\alpha>0} T_{\hbar_0,\hbar_1,\hbar_1}(\alpha) = \wedge_{\alpha>0} T_{\hbar,f(\hbar),f(\hbar)}(\alpha) > 0$$

and

$$\wedge_{\alpha>0} T_{\hbar_1,\hbar_0,\hbar_0}(\alpha) = \wedge_{\alpha>0} T_{f(\hbar),\hbar,\hbar}(\alpha) > 0$$

Then from the Proposition 4 $\hbar_n$ is a convergence sequence.
Since $(S, T, *)$ is complete, therefore $\exists\ \overline{\hbar} \in S$ satisfying

$$\lim_n T_{\hbar_n,\overline{\hbar},\overline{\hbar}}(\alpha) = 1,$$

for each $\alpha > 0$.
Again for each $n \in \mathbb{N}$ and each $\alpha > 0$ we have

$$T_{f(\overline{\hbar}),\hbar_{n+1},\hbar_{n+1}}(\alpha) \geq \psi\left(T_{\overline{\hbar},\hbar_n,\hbar_n}\left(\frac{\alpha}{b}\right)\right)$$

Therefore, for each $\alpha > 0$

$$\begin{aligned} T_{f(\overline{\hbar}),\overline{\hbar},\overline{\hbar}}(\alpha) &= \lim_n T_{f(\overline{\hbar}),\hbar_{n+1},\hbar_{n+1}}(\alpha) \\ &\geq \lim_n \psi(T_{\overline{\hbar},\hbar_n,\hbar_n}\left(\frac{\alpha}{b}\right) \\ &= 1 \end{aligned}$$

Hence $\overline{\hbar}$ is a fixed point of f.
Next we have to verify that $\overline{\hbar}$ is unique.
If possible suppose $\overline{\hbar_1}$ is another fixed point of f, then for any $\alpha > 0$.

$$\begin{aligned} T_{\overline{\hbar},\overline{\hbar},\overline{\hbar_1}}(\alpha) &= T_{f(\overline{\hbar}),f(\overline{\hbar}),f(\overline{\hbar_1})}(\alpha) \\ &\geq \psi(T_{\overline{\hbar},\overline{\hbar},\overline{\hbar_1}}\left(\frac{\alpha}{b}\right) \end{aligned}$$

If $\overline{\hbar} \neq \overline{\hbar_1}$, then for some $\beta > 0$

$$T_{\overline{\hbar},\overline{\hbar},\overline{\hbar_1}}(\beta) < 1 \Rightarrow 0 < T_{\overline{\hbar},\overline{\hbar},\overline{\hbar_1}}(\beta) < 1$$

This implies

$$\begin{aligned} T_{\overline{\hbar},\overline{\hbar},\overline{\hbar_1}}(\beta) &= T_{f(\overline{\hbar}),f(\overline{\hbar}),f(\overline{\hbar_1})}(\beta) \\ &\geq \psi(T_{\overline{\hbar},\overline{\hbar},\overline{\hbar_1}}\left(\frac{\beta}{b}\right) \\ &> T_{\overline{\hbar},\overline{\hbar},\overline{\hbar_1}}(\beta), \end{aligned}$$

which is a contradiction.

Therefore, $\overline{\hbar} = \overline{\hbar_1}$.

Hence, f has a unique fixed point in $(S, T, *)$.

4 Conclusion

Throughout the chapter, we have generalized and introduced the notion of TFbMS. Some related properties have been studied in this area. A new generalized form of contraction mapping and sequence have also been developed in the settings of TFbMS. With the help of this new generalized form of contraction mapping, we have also developed the unique fixed point in TFbMS. The work of this chapter gives researchers an incentive to improve and develop the research area of b-metric space in a new manner. Eventual outcome of this work will help the researcher to develop a new way to establish the fixed point theory and analytical properties of fuzzy b-metric space. One can also improve the fuzzy iteration schemes with the help of fuzzy integral and differential equations. Since TFbMS is a generalized form of fuzzy b-metric space, one can introduce and establish the extension version of Pythagorean fuzzy sets in a new manner. In the area of decision-making and bio-informatics, a huge application of this work will be established in future.

Appendix

ϕ	The empty set
$\mathbb{N}$	The set of natural numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^+$	The set of positive real numbers
i.e.	That is
$\Rightarrow$	Implies
$\Leftrightarrow$	Implies and implied by
$\forall$	For all

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A Novel Controlled Picture Fuzzy Metric Space and Some Related Fixed Point Results

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Abstract In this manuscript, we establish the novel notion of controlled picture fuzzy metric spaces as a generalization of picture fuzzy metric spaces by using a control function in triangle inequalities and establish fixed point theorems related to them in the setting of controlled picture fuzzy metric space. Our results are improvement of many well-known results existing in the literature. An extensive set of non-trivial examples are imparted to validate the feasibility of our results.

Keywords Fuzzy sets · Picture fuzzy sets · Controlled picture fuzzy metric space

1 Introduction

Since the introduction of fuzzy sets (FSs) by Zadeh [1], several scientists have looked into this concept more closely. Kramosil and Michalek [2] initiated the concept of fuzzy metric spaces by generalizing the notion of probabilistic metric spaces to fuzzy metric spaces. Cuong [3] introduced the concept of picture fuzzy sets (PFS), which are direct extensions of the fuzzy sets and the intuitionistic fuzzy sets. In addition, Mlaiki [4] introduced the controlled metric type spaces, by employing a control function $\alpha(x, y)$ of the right-hand side of the b-triangle inequality. Recently, Sezen [5] introduced a new extension in the subject of fuzzy metric, called controlled fuzzy

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metric space (CFMS). This notion is a generalization of fuzzy b-metric spaces. For details, please check the most interesting generalizations can be found in [6–11, 14, 15].

Integrating the notions of PFMS and controlled function, in this paper, we investigate the discussion of a new generalization, the CPFMS. The topological properties, i.e., Hausdorff picture fuzzy boundedness, open sets, completeness, compactness, and nowhere dense sets are defined accordingly. Most importantly, the Uniform Convergence Theorem and Baire's Category Theorem are established in CPFMSs. We provide some non-trivial examples to validate the superiority of our results to those in the existing literature. We also give some fixed point (FP) results and provide an application.

2 Preliminaries

First, we provide some basic definitions and related concepts, which would be essential for our discussion in this article.

Definition 1 ([12]) A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous triangle norm (briefly CTN) if:

(1) $\pi * \mu = \mu * \pi, \forall \pi, \mu \in [0, 1]$;
(2) $*$ is continuous;
(3) $\pi * 1 = \pi, \forall \pi \in [0, 1]$;
(4) $(\pi * \mu) * \varkappa = \pi * (\mu * \varkappa), \forall \pi, \mu, \varkappa \in [0, 1]$;
(5) If $\pi \leq \varkappa$ and $\mu \leq d$, with $\pi, \mu, \varkappa, d \in [0, 1]$, then $\pi * \mu \leq \varkappa * d$.

Example 1 ([12]) Some fundamental examples of t-norms are: $\pi * \mu = \pi \cdot \mu$, $\pi * \mu = \min\{\pi, \mu\}$ and $\pi * \mu = \max\{\pi + \mu - 1, 0\}$.

Definition 2 ([12]) A binary operation $\circ : [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous triangle conorm (briefly CTCN) if it meets the below assertions:

(1) $\pi \circ \mu = \mu \circ \pi$, for all $\pi, \mu \in [0, 1]$;
(2) $\circ$ is continuous;
(3) $\pi \circ 0 = 0$, for all $\pi \in [0, 1]$;
(4) $(\pi \circ \mu) \circ \varkappa = \pi \circ (\mu \circ \varkappa)$, for all $\pi, \mu, \varkappa \in [0, 1]$;
(5) If $\pi \leq \varkappa$ and $\mu \leq d$, with $\pi, \mu, \varkappa, d \in [0, 1]$, then $\pi \circ \mu \leq \varkappa \circ d$.

Example 2 ([12]) $\pi \circ \mu = \max\{\pi, \mu\}$ and $\pi \circ \mu = \min\{\pi + \mu, 1\}$ are examples of CTCMs.

Definition 3 ([13]) Take $\Xi \neq \emptyset$. Let $*$ be a CTN, $\circ$ be a CTCN, $b \geq 1$ and $\aleph_b$, $\Im_b$ be FSs on $\Xi \times \Xi \times (0, \infty)$. If $(\Xi, \aleph_b, \Im_b, *, \circ)$ verifies the following for all $\alpha, \beta \in \Xi$ and $\mathcal{S}, \mathcal{T} > 0$:

(1) $\aleph_b(\alpha, \beta, \mathcal{T}) + \Im_b(\alpha, \beta, \mathcal{T}) \leq 1$;
(2) $\aleph_b(\alpha, \beta, \mathcal{T}) > 0$;
(3) $\aleph_b(\alpha, \beta, \mathcal{T}) = 1 \Longleftrightarrow \alpha = \beta$;

(4) $\aleph_b(\alpha, \beta, \mathcal{T}) = \aleph_b(\beta, \alpha, \mathcal{T})$;
(5) $\aleph_b(\alpha, \lambda, b(\mathcal{T} + \mathcal{S})) \geq \aleph_b(\alpha, \beta, \mathcal{T}) * \aleph_b(\beta, \lambda, \mathcal{S})$;
(6) $\aleph_b(\alpha, \beta, \cdot)$ is a non decreasing (ND) function of $\mathbb{R}^+$ and $\lim_{\mathcal{T}\to\infty} \aleph_b(\alpha, \beta, \mathcal{T}) = 1$;
(7) $\Im_b(\alpha, \beta, \mathcal{T}) > 0$;
(8) $\Im_b(\alpha, \beta, \mathcal{T}) = 0 \Longleftrightarrow \alpha = \beta$;
(9) $\Im_b(\alpha, \beta, \mathcal{T}) = \Im_b(\beta, \alpha, \mathcal{T})$;
(10) $\Im_b(\alpha, \lambda, b(\mathcal{T} + \mathcal{S})) \leq \Im_b(\alpha, \beta, \mathcal{T}) \circ \Im_b(\beta, \lambda, \mathcal{S})$;
(11) $\Im_b(\alpha, \beta, \cdot)$ is a non increasing (NI) function of $\mathbb{R}^+$ and $\lim_{\mathcal{T}\to\infty} \Im_b(\alpha, \beta, \mathcal{T}) = 0$, then $(\Xi, \aleph_b, \Im_b, *, \circ)$ is an intuitionistic fuzzy b-metric space (IFBMS).

We generalize the intuitionistic fuzzy metric space with the help of picture fuzzy theory.

Definition 4 Let $\Xi \neq \emptyset$ and $*$ is a CTN, $\circ$ be a CTCN and $\mathcal{L}, \Im, S$ are picture fuzzy sets (PFSs) on $\Xi \times \Xi \times (0, \infty)$. If for all $\alpha, \beta, \lambda \in \Xi$, the below circumstances fulfil:
(1) $\mathcal{L}(\alpha, \beta, \mathcal{T}) + \Im(\alpha, \beta, \mathcal{T}) + S(\alpha, \beta, \mathcal{T}) \leq 1$;
(2) $\mathcal{L}(\alpha, \beta, \mathcal{T}) > 0$;
(3) $\mathcal{L}(\alpha, \beta, \mathcal{T}) = 1$ for all $\mathcal{T} > 0$, iff $\alpha = \beta$;
(4) $\mathcal{L}(\alpha, \beta, \mathcal{T}) = \mathcal{L}(\beta, \alpha, \mathcal{T})$;
(5) $\mathcal{L}(\alpha, \lambda, \mathcal{T} + \mathcal{S}) \geq \mathcal{L}(\alpha, \beta, \mathcal{T}) * \mathcal{L}(\beta, \lambda, \mathcal{S})$;
(6) $\mathcal{L}(\alpha, \beta, \cdot) : (0, \infty) \to [0, 1]$ is continuous and $\lim_{\mathcal{T}\to\infty} \mathcal{L}(\alpha, \beta, \mathcal{T}) = 1$;
(7) $\Im(\alpha, \beta, \mathcal{T}) < 1$;
(8) $\Im(\alpha, \beta, \mathcal{T}) = 0$ for all $\mathcal{T} > 0$, iff $\alpha = \beta$;
(9) $\Im(\alpha, \beta, \mathcal{T}) = \Im(\beta, \alpha, \mathcal{T})$;
(10) $\Im(\alpha, \lambda, \mathcal{T} + \mathcal{S}) \leq \Im(\alpha, \beta, \mathcal{T}) \circ \Im(\beta, \lambda, \mathcal{S})$;
(11) $\Im(\alpha, \beta, \cdot) : (0, \infty) \to [0, 1]$ is continuous and $\lim_{\mathcal{T}\to\infty} \Im(\alpha, \beta, \mathcal{T}) = 0$;
(12) $S(\alpha, \beta, \mathcal{T}) < 1$;
(13) $S(\alpha, \beta, \mathcal{T}) = 0$ for all $\mathcal{T} > 0$ iff $\alpha = \beta$;
(14) $S(\alpha, \beta, \mathcal{T}) = S(\beta, \alpha, \mathcal{T})$;
(15) $S(\alpha, \lambda, \mathcal{T} + \mathcal{S}) \leq S(\alpha, \beta, \mathcal{T}) \circ S(\beta, \lambda, \mathcal{S})$;
(16) $S(\alpha, \beta, \cdot) : (0, \infty) \to [0, 1]$ is continuous and $\lim_{\mathcal{T}\to\infty} S(\alpha, \beta, \mathcal{T}) = 0$;
(17) If $\mathcal{T} \leq 0$, then $\mathcal{L}(\alpha, \beta, \mathcal{T}) = 0$, $\Im(\alpha, \beta, \mathcal{T}) = 1$ and $S(\alpha, \beta, \mathcal{T}) = 1$,
then $(\Xi, \mathcal{L}, \Im, S, *, \circ)$ is called picture fuzzy metric space (PFMS).

3 Main Results

In this section, we introduce the concept of Controlled Picture fuzzy metric spaces and prove the existence and uniqueness of fixed point results in this framework.

Definition 5 Suppose that $\Xi \neq \emptyset$, and we have a six tuple $(\Xi, \aleph_\phi, \Im_\phi, \Re_\phi, *, \circ)$ where $*$ is a CTN, $\circ$ is a CTCN, $\phi : \Xi \times \Xi \to [1, \infty)$ and $\aleph_\phi, \Im_\phi, \Re_\phi$ are picture fuzzy sets (PFSs) on $\Xi \times \Xi \times (0, \infty)$. If $(\Xi, \aleph_\phi, \Im_\phi, \Re_\phi, *, \circ)$ meet the below circumstances for all $\alpha, \beta, \lambda \in \Xi$ and $\mathcal{S}, \mathcal{T} > 0$:

(1) $\aleph_\phi(\alpha, \beta, \mathcal{T}) + \Im_\phi(\alpha, \beta, \mathcal{T}) + \Re_\phi(\alpha, \beta, \mathcal{T}) \leq 1$,
(2) $\aleph_\phi(\alpha, \beta, \mathcal{T}) > 0$,
(3) $\aleph_\phi(\alpha, \beta, \mathcal{T}) = 1 \Longleftrightarrow \alpha = \beta$,
(4) $\aleph_\phi(\alpha, \beta, \mathcal{T}) = \aleph_\phi(\beta, \alpha, \mathcal{T})$,
(5) $\aleph_\phi(\alpha, \lambda, (\mathcal{T} + \mathcal{S})) \geq \aleph_\phi\left(\alpha, \beta, \frac{\mathcal{T}}{\phi(\alpha,\beta)}\right) * \aleph_\phi\left(\beta, \lambda, \frac{\mathcal{S}}{\phi(\beta,\lambda)}\right)$,
(6) $\aleph_\phi(\alpha, \beta, \cdot)$ is a function of $\mathbb{R}^+$ and $\lim_{\mathcal{T}\to\infty} \aleph_\phi(\alpha, \beta, \mathcal{T}) = 1$,
(7) $\Im_\phi(\alpha, \beta, \mathcal{T}) > 0$,
(8) $\Im_\phi(\alpha, \beta, \mathcal{T}) = 0 \Longleftrightarrow \alpha = \beta$,
(9) $\Im_\phi(\alpha, \beta, \mathcal{T}) = \Im_\phi(\beta, \alpha, \mathcal{T})$,
(10) $\Im_\phi(\alpha, \lambda, (\mathcal{T} + \mathcal{S})) \leq \Im_\phi\left(\alpha, \beta, \frac{\mathcal{T}}{\phi(\alpha,\beta)}\right) \circ \Im_\phi\left(\beta, \lambda, \frac{\mathcal{S}}{\phi(\beta,\lambda)}\right)$,
(11) $\Im_\phi(\alpha, \beta, \circ)$ is a function of $\mathbb{R}^+$ and $\lim_{\mathcal{T}\to\infty} \Im_\phi(\alpha, \beta, \mathcal{T}) = 0$,
(12) $\Re_\phi(\alpha, \beta, \mathcal{T}) > 0$,
(13) $\Re_\phi(\alpha, \beta, \mathcal{T}) = 0 \Longleftrightarrow \alpha = \beta$,
(14) $\Re_\phi(\alpha, \beta, \mathcal{T}) = \Re_\phi(\beta, \alpha, \mathcal{T})$,
(15) $\Re_\phi(\alpha, \lambda, (\mathcal{T} + \mathcal{S})) \leq \Re_\phi\left(\alpha, \beta, \frac{\mathcal{T}}{\phi(\alpha,\beta)}\right) \circ \Re_\phi\left(\beta, \lambda, \frac{\mathcal{S}}{\phi(\beta,\lambda)}\right)$,
(16) $\Re_\phi(\alpha, \beta, \cdot)$ is a function of $\mathbb{R}^+$ and $\lim_{\mathcal{T}\to\infty} \Re_\phi(\alpha, \beta, \mathcal{T}) = 0$,
(17) If $\mathcal{T} \leq 0$, then $\aleph_\phi(\alpha, \beta, \mathcal{T}) = 0$, $\Im_\phi(\alpha, \beta, \mathcal{T}) = 1$ and $\Re_\phi(\alpha, \beta, \mathcal{T}) = 1$,

then $(\Xi, \aleph_\phi, \Im_\phi, \Re_\phi, *, \circ)$ is a CPFMS.

Remark 1 If we take $\phi(\alpha, \beta) = \phi(\beta, \lambda) = 1$, for all $\alpha, \beta, \lambda \in \Xi$, then $(\Xi, \aleph_\phi, \Im_\phi, \Re_\phi, *, \circ)$ is an PFMS.

Example 3 Let $\Xi = (0, \infty)$. Define $\aleph_\phi, \Im_\phi, \Re_\phi : \Xi \times \Xi \times (0, \infty) \to [0, 1]$ by

$$\aleph_\phi(\alpha, \beta, \mathcal{T}) = \frac{\mathcal{T}}{\mathcal{T} + |\alpha - \beta|^2}, \Im_\phi(\alpha, \beta, \mathcal{T}) = \frac{|\alpha - \beta|^2}{\mathcal{T} + |\alpha - \beta|^2}, \Re_\phi(\alpha, \beta, \mathcal{T}) = \frac{|\alpha - \beta|^2}{\mathcal{T}}$$

for all $\alpha, \beta \in \Xi$ and $\mathcal{T} > 0$. Define CTN "$*$" by $\pi * \mu = \pi \cdot \mu$ and CTCN $\circ$ by $\pi \circ \mu = \max\{\pi, \mu\}$ and define ϕ by

$$\phi(\alpha, \beta) = \begin{cases} 1, & if \quad \alpha = \beta, \\ \frac{1+\max\{\alpha,\beta\}}{\min\{\alpha,\beta\}}, & if \quad \alpha \neq \beta. \end{cases}$$

Then $\left(\Xi, \aleph_\phi, \Im_\phi, \Re_\phi, *, \circ\right)$ is a CPFMS.

Proof $(1)-(4)$, $(6)-(9)$, $(9)-(14)$, (16) and (17) are obvious. Here, we prove (5), (10) and (15). We have

$$|\alpha - \lambda|^2 \leq \phi(\alpha, \beta)\left[|\alpha - \beta|^2 + \phi(\beta, \lambda)|\beta - \lambda|^2\right].$$

Therefore,

$$\begin{aligned}
&\mathcal{T}\mathcal{S}|\alpha-\lambda|^2 \le \phi(\alpha,\beta)\left(\mathcal{T}\mathcal{S}+\mathcal{S}^2\right)|\alpha-\beta|^2+\phi(\beta,\lambda)\left(\mathcal{T}\mathcal{S}+\mathcal{T}^2\right)|\beta-\lambda|^2\\
&\Rightarrow \mathcal{T}\mathcal{S}|\alpha-\lambda|^2 \le \phi(\alpha,\beta)(\mathcal{T}+\mathcal{S})\mathcal{S}|\alpha-\beta|^2+\phi(\beta,\lambda)(\mathcal{T}+\mathcal{S})\mathcal{T}|\beta-\lambda|^2\\
&\Rightarrow \mathcal{T}\mathcal{S}(\mathcal{T}+\mathcal{S})+\mathcal{T}\mathcal{S}|\alpha-\lambda|^2 \le \mathcal{T}\mathcal{S}(\mathcal{T}+\mathcal{S})+\phi(\alpha,\beta)(\mathcal{T}+\mathcal{S})\mathcal{S}|\alpha-\beta|^2+\phi(\beta,\lambda)(\mathcal{T}+\mathcal{S})\mathcal{T}|\beta-\lambda|^2.
\end{aligned}$$

That is,

$$\mathcal{T}\mathcal{S}\left[(\mathcal{T}+\mathcal{S})+|\alpha-\lambda|^2\right] \le (\mathcal{T}+\mathcal{S})\left[\mathcal{T}\mathcal{S}+\phi(\alpha,\beta)\mathcal{S}|\alpha-\beta|^2+\phi(\beta,\lambda)\mathcal{T}|\beta-\lambda|^2\right]$$

$$\Rightarrow \mathcal{T}\mathcal{S}\left[(\mathcal{T}+\mathcal{S})+|\alpha-\lambda|^2\right] \le (\mathcal{T}+\mathcal{S})\left[\begin{array}{c}\mathcal{T}\mathcal{S}+\phi(\alpha,\beta)\mathcal{S}|\alpha-\beta|^2\\ +\phi(\beta,\lambda)\mathcal{T}|\beta-\lambda|^2+\phi(\alpha,\beta)\phi(\beta,\lambda)|\alpha-\beta|^2|\beta-\lambda|^2\end{array}\right]$$

$$\Rightarrow \mathcal{T}\mathcal{S}\left[(\mathcal{T}+\mathcal{S})+|\alpha-\lambda|^2\right] \le (\mathcal{T}+\mathcal{S})\left[\mathcal{T}+\phi(\alpha,\beta)|\alpha-\beta|^2][\mathcal{S}+\phi(\beta,\lambda)|\beta-\lambda|^2\right].$$

Then,

$$\begin{aligned}
\frac{(\mathcal{T}+\mathcal{S})}{(\mathcal{T}+\mathcal{S})+|\alpha-\lambda|^2} &\ge \frac{\mathcal{T}\mathcal{S}}{\left[\mathcal{T}+\phi(\alpha,\beta)|\alpha-\beta|^2][\mathcal{S}+\phi(\beta,\lambda)|\beta-\lambda|^2\right]}\\
&\Rightarrow \frac{(\mathcal{T}+\mathcal{S})}{(\mathcal{T}+\mathcal{S})+|\alpha-\lambda|^2} \ge \frac{\mathcal{T}}{\mathcal{T}+\phi(\alpha,\beta)|\alpha-\beta|^2}.\frac{\mathcal{S}}{\mathcal{S}+\phi(\beta,\lambda)|\beta-\lambda|^2}\\
&\Rightarrow \frac{(\mathcal{T}+\mathcal{S})}{(\mathcal{T}+\mathcal{S})+|\alpha-\lambda|^2} \ge \frac{\frac{\mathcal{T}}{\phi(\alpha,\beta)}}{\frac{\mathcal{T}}{\phi(\alpha,\beta)}+|\alpha-\beta|^2}.\frac{\frac{\mathcal{S}}{\phi(\beta,\lambda)}}{\frac{\mathcal{S}}{\phi(\beta,\lambda)}+|\beta-\lambda|^2}.
\end{aligned}$$

Hence,

$$\aleph_\phi(\alpha,\lambda,(\mathcal{T}+\mathcal{S})) \ge \aleph_\phi\left(\alpha,\beta,\frac{\mathcal{T}}{\phi(\alpha,\beta)}\right) * \aleph_\phi\left(\beta,\lambda,\frac{\mathcal{S}}{\phi(\beta,\lambda)}\right).$$

So, (v) is satisfied. Also,

$$|\alpha-\lambda|^2 = |\alpha-\lambda|^2\max\{1,1\}.$$

Therefore,

$$\begin{aligned}
|\alpha-\lambda|^2 &= |\alpha-\lambda|^2\max\left\{\frac{|\alpha-\beta|^2}{|\alpha-\beta|^2},\frac{|\beta-\lambda|^2}{|\beta-\lambda|^2}\right\}\\
&\Rightarrow |\alpha-\lambda|^2 \le \left[(\mathcal{T}+\mathcal{S})+|\alpha-\lambda|^2\right]\max\left\{\frac{|\alpha-\beta|^2}{|\alpha-\beta|^2},\frac{|\beta-\lambda|^2}{|\beta-\lambda|^2}\right\}\\
&\Rightarrow |\alpha-\lambda|^2 \le \left[(\mathcal{T}+\mathcal{S})+|\alpha-\lambda|^2\right]\max\left\{\frac{\phi(\alpha,\beta)|\alpha-\beta|^2}{\phi(\alpha,\beta)|\alpha-\beta|^2},\frac{\phi(\beta,\lambda)|\beta-\lambda|^2}{\phi(\beta,\lambda)|\beta-\lambda|^2}\right\}.
\end{aligned}$$

Then,

$$\frac{|\alpha-\lambda|^2}{(\mathcal{T}+\mathcal{S})+|\alpha-\lambda|^2} \leq \max\left\{\frac{\phi(\alpha,\beta)|\alpha-\beta|^2}{\mathcal{T}+\phi(\alpha,\beta)|\alpha-\beta|^2}, \frac{\phi(\beta,\lambda)|\beta-\lambda|^2}{\mathcal{S}+\phi(\beta,\lambda)|\beta-\lambda|^2}\right\}.$$

That is,

$$\frac{|\alpha-\lambda|^2}{(\mathcal{T}+\mathcal{S})+|\alpha-\lambda|^2} \leq \max\left\{\frac{|\alpha-\beta|^2}{\frac{\mathcal{T}}{\phi(\alpha,\beta)}+|\alpha-\beta|^2}, \frac{|\beta-\lambda|^2}{\frac{\mathcal{S}}{\phi(\beta,\lambda)}+|\beta-\lambda|^2}\right\}.$$

Hence,

$$\Im_\phi(\alpha,\lambda,(\mathcal{T}+\mathcal{S})) \leq \Im_\phi\left(\alpha,\beta,\frac{\mathcal{T}}{\phi(\alpha,\beta)}\right) * \Im_\phi\left(\beta,\lambda,\frac{\mathcal{S}}{\phi(\beta,\lambda)}\right).$$

That is, (10) is satisfied.

It is easy to see that

$$\frac{|\alpha-\lambda|^2}{\mathcal{T}+\mathcal{S}} \leq \max\left\{\frac{\phi(\alpha,\beta)|\alpha-\beta|^2}{\mathcal{T}}, \frac{\phi(\beta,\lambda)|\beta-\lambda|^2}{\mathcal{S}}\right\}.$$

That is,

$$\frac{|\alpha-\lambda|^2}{(\mathcal{T}+\mathcal{S})} \leq \max\left\{\frac{|\alpha-\beta|^2}{\frac{\mathcal{T}}{\phi(\alpha,\beta)}}, \frac{|\beta-\lambda|^2}{\frac{\mathcal{S}}{\phi(\beta,\lambda)}}\right\}.$$

Hence,

$$\Re_\phi(\alpha,\lambda,(\mathcal{T}+\mathcal{S})) \leq \Re_\phi\left(\alpha,\beta,\frac{\mathcal{T}}{\phi(\alpha,\beta)}\right) * \Re_\phi\left(\beta,\lambda,\frac{\mathcal{S}}{\phi(\beta,\lambda)}\right).$$

So, (15) is satisfied.

Remark 2 The above example also satisfied for CTN $\pi * \mu = \min\{\pi, \mu\}$ and CTCN $\pi \circ \mu = \max\{\pi, \mu\}$.

Example 4 Let $\Xi = (0, \infty)$. Define $\aleph_\phi, \Im_\phi, \Re_\phi : \Xi \times \Xi \times (0, \infty) \to [0, 1]$ by

$$\aleph_\phi(\alpha,\beta,\mathcal{T}) = \begin{cases} 1, & if \quad \alpha=\beta \\ \frac{\mathcal{T}}{\mathcal{T}+\max\{\alpha,\beta\}}, & \text{if otherwise} \end{cases},$$

$$\Im_\phi(\alpha,\beta,\mathcal{T}) = \begin{cases} 0, & if \quad \alpha=\beta \\ \frac{\max\{\alpha,\beta\}}{\mathcal{T}+\max\{\alpha,\beta\}}, & \text{if otherwise} \end{cases},$$

and

$$\Re_{\phi}(\alpha, \beta, \mathcal{T}) = \begin{cases} 0, & if \quad \alpha = \beta \\ \frac{\max\{\alpha,\beta\}}{\mathcal{T}}, & \text{if otherwise} \end{cases},$$

for all $\alpha, \beta \in \Xi$ and $\mathcal{T} > 0$. Define CTN $*$ by $\pi * \mu = \pi \cdot \mu$ and CTCN $\circ$ by $\pi \circ \mu = \max\{\pi, \mu\}$ and define ϕ by

$$\phi(\alpha, \beta) = 1 + \alpha + \beta.$$

Then $(\Xi, \aleph_{\phi}, \Im_{\phi}, \Re_{\phi}, *, \circ)$ is a CPFMS.

Definition 6 Let $(\Xi, \aleph_{\phi}, \Im_{\phi}, \Re_{\phi}, *, \circ)$ be a CPFMS. We define an open ball $B(\alpha, r, \mathcal{T})$ with centre α, radius $0 < r < 1$ as follows:

$$B(\alpha, r, \mathcal{T}) = \{\beta \in \Xi : \mathcal{L}(\alpha, \beta, \mathcal{T}) > 1 - r, \Im(\alpha, \beta, \mathcal{T}) < r, \Re(\alpha, \beta, \mathcal{T}) < r\}.$$

Definition 7 Let $(\Xi, \aleph_{\phi}, \Im_{\phi}, \Re_{\phi}, *, \circ)$ be a CPFMS.

(1) A sequence $\{\alpha_n\}$ in Ξ is named to be CPF-Cauchy sequence (CPFCS) if and only if for all $q > 0$ and $\mathcal{T} > 0$,

$$\lim_{n\to\infty} \aleph_{\phi}(\alpha_n, \alpha_{n+q}, \mathcal{T}) = 1, \lim_{n\to\infty} \Im_{\phi}(\alpha_n, \alpha_{n+q}, \mathcal{T}) = 0 \text{ and } \lim_{n\to\infty} \Re_{\phi}(\alpha_n, \alpha_{n+q}, \mathcal{T}) = 0.$$

(2) A sequence $\{\alpha_n\}$ in Ξ is named to be CPF-convergent (CPFC) to α in Ξ, if and only if for all $\mathcal{T} > 0$,

$$\lim_{n\to\infty} \aleph_{\phi}(\alpha_n, \alpha, \mathcal{T}) = 1, \lim_{n\to\infty} \Im_{\phi}(\alpha_n, \alpha, \mathcal{T}) = 0 \text{ and } \lim_{n\to\infty} \Re_{\phi}(\alpha_n, \alpha, \mathcal{T}) = 0.$$

(3) A CPFMS is named to be complete iff each CPFCS is CPF-convergent.

At this time we prove the CPF-Banach contraction result.

Theorem 1 *Suppose that* $(\Xi, \aleph_{\phi}, \Im_{\phi}, \Re_{\phi}, *, \circ)$ *is a Complete CPFMS and* $\phi : \Xi \times \Xi \to [1, \infty)$. *Suppose that*

$$\lim_{\mathcal{T}\to\infty} \aleph_{\phi}(\alpha, \beta, \mathcal{T}) = 1, \lim_{\mathcal{T}\to\infty} \Im_{\phi}(\alpha, \beta, \mathcal{T}) = 0 \textit{ and } \lim_{\mathcal{T}\to\infty} \Re_{\phi}(\alpha, \beta, \mathcal{T}) = 0 \quad (1)$$

for all $\alpha, \beta \in \Xi$ *and* $\mathcal{T} > 0$. *Let* $\xi : \Xi \to \Xi$ *be a mapping satisfying*

$$\aleph_{\phi}(\xi\alpha, \xi\beta, \aleph\mathcal{T}) \geq \aleph_{\phi}(\alpha, \beta, \mathcal{T}), \quad (2)$$

$$\Im_{\phi}(\xi\alpha, \xi\beta, \aleph\mathcal{T}) \leq \Im_{\phi}(\alpha, \beta, \mathcal{T}), \quad (3)$$

$$\textit{and } \Re_{\phi}(\xi\alpha, \xi\beta, \aleph\mathcal{T}) \leq \Re_{\phi}(\alpha, \beta, \mathcal{T}), \quad (4)$$

for all $\alpha, \beta \in \Xi$, $\mathcal{T} > 0$ *and for some* $0 < \aleph < 1$. *Also, assume that for every* $\alpha \in Z$,

$$\lim_{n\to\infty} \phi(\alpha_n, \beta) = \lim_{n\to\infty} \phi(\beta, \alpha_n) \quad (5)$$

exist and finite. Then ζ has a unique fixed point in Z.

Proof Let α_0 be a random element of Ξ and describe a sequence α_n by $\alpha_n = \xi^n \alpha_0 = \xi \alpha_{n-1}$, $n \in \mathbb{N}$. By using (2) for all $\mathcal{T} > 0$, we have

$$\begin{aligned}\aleph_\phi\left(\alpha_n, \alpha_{n+1}, \aleph\mathcal{T}\right) &= \aleph_\phi\left(\xi\alpha_{n-1}, \xi\alpha_n, \aleph\mathcal{T}\right) \geq \aleph_\phi\left(\alpha_{n-1}, \alpha_n, \mathcal{T}\right) \geq \aleph_\phi\left(\alpha_{n-2}, \alpha_{n-1}, \frac{\mathcal{T}}{\aleph}\right)\\ &\geq \aleph_\phi\left(\alpha_{n-3}, \alpha_{n-2}, \frac{\mathcal{T}}{\aleph^2}\right) \geq \cdots \geq \aleph_\phi\left(\alpha_0, \alpha_1, \frac{\mathcal{T}}{\aleph^{n-1}}\right),\\ \Im_\phi\left(\alpha_n, \alpha_{n+1}, \aleph\mathcal{T}\right) &= \Im_\phi\left(\xi\alpha_{n-1}, \xi\alpha_n, \aleph\mathcal{T}\right) \leq \Im_\phi\left(\alpha_{n-1}, \alpha_n, \mathcal{T}\right) \leq \Im_\phi\left(\alpha_{n-2}, \alpha_{n-1}, \frac{\mathcal{T}}{\aleph}\right)\\ &\leq \Im_\phi\left(\alpha_{n-3}, \alpha_{n-2}, \frac{\mathcal{T}}{\aleph^2}\right) \leq \cdots \leq \Im_\phi\left(\alpha_0, \alpha_1, \frac{\mathcal{T}}{\aleph^{n-1}}\right)\end{aligned}$$

and

$$\begin{aligned}\Re_\phi\left(\alpha_n, \alpha_{n+1}, \aleph\mathcal{T}\right) &= \Re_\phi\left(\xi\alpha_{n-1}, \xi\alpha_n, \aleph\mathcal{T}\right) \leq \Re_\phi\left(\alpha_{n-1}, \alpha_n, \mathcal{T}\right) \leq \Re_\phi\left(\alpha_{n-2}, \alpha_{n-1}, \frac{\mathcal{T}}{\aleph}\right)\\ &\leq \Re_\phi\left(\alpha_{n-3}, \alpha_{n-2}, \frac{\mathcal{T}}{\aleph^2}\right) \leq \cdots \leq \Re_\phi\left(\alpha_0, \alpha_1, \frac{\mathcal{T}}{\aleph^{n-1}}\right).\end{aligned}$$

Hence, we obtain that

$$\begin{aligned}\aleph_\phi\left(\alpha_n, \alpha_{n+1}, \aleph\mathcal{T}\right) &\geq \aleph_\phi\left(\alpha_0, \alpha_1, \frac{\mathcal{T}}{\aleph^{n-1}}\right),\\ \Im_\phi\left(\alpha_n, \alpha_{n+1}, \aleph\mathcal{T}\right) &\leq \Im_\phi\left(\alpha_0, \alpha_1, \frac{\mathcal{T}}{\aleph^{n-1}}\right),\\ \Re_\phi\left(\alpha_n, \alpha_{n+1}, \aleph\mathcal{T}\right) &\leq \Re_\phi\left(\alpha_0, \alpha_1, \frac{\mathcal{T}}{\aleph^{n-1}}\right).\end{aligned} \tag{6}$$

For any $q \in \mathbb{N}$, using (v) , (x) and (xv), we deduce that

$$\begin{aligned}&\aleph_\phi\left(\alpha_n, \alpha_{n+q}, \mathcal{T}\right)\\ &\geq \aleph_\phi\left(\alpha_n, \alpha_{n+1}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_n, \alpha_{n+1}\right)\right)}\right) * \aleph_\phi\left(\alpha_{n+1}, \alpha_{n+q}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\right)}\right)\\ &\geq \aleph_\phi\left(\alpha_n, \alpha_{n+1}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_n, \alpha_{n+1}\right)\right)}\right) * \aleph_\phi\left(\alpha_{n+1}, \alpha_{n+2}, \frac{\mathcal{T}}{(2)^2\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+1}, \alpha_{n+2}\right)\right)}\right)\\ &* \aleph_\phi\left(\alpha_{n+2}, \alpha_{n+q}, \frac{\mathcal{T}}{(2)^2\left(\phi\left(\alpha_{n+1}, \alpha_{n+\Im}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\right)}\right)\\ &\geq \aleph_\phi\left(\alpha_n, \alpha_{n+1}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_n, \alpha_{n+1}\right)\right)}\right) * \aleph_\phi\left(\alpha_{n+1}, \alpha_{n+2}, \frac{\mathcal{T}}{(2)^2\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+1}, \alpha_{n+2}\right)\right)}\right)\\ &* \aleph_\phi\left(\alpha_{n+2}, \alpha_{n+3}, \frac{\mathcal{T}}{(2)^3\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+3}\right)\right)}\right)\end{aligned}$$

$$* \aleph_\phi\left(\alpha_{n+3}, \alpha_{n+\Im}, \frac{\mathcal{T}}{(2)^3\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+q}\right)\right)}\right)$$

$$\geq \aleph_\phi\left(\alpha_n, \alpha_{n+1}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_n, \alpha_{n+1}\right)\right)}\right) * \aleph_\phi\left(\alpha_{n+1}, \alpha_{n+2}, \frac{\mathcal{T}}{(2)^2\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+1}, \alpha_{n+2}\right)\right)}\right)$$

$$* \aleph_\phi\left(\alpha_{n+2}, \alpha_{n+3}, \frac{\mathcal{T}}{(2)^3\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+\Im}\right)\phi\left(\alpha_{n+2}, \alpha_{n+3}\right)\right)}\right)$$

$$* \aleph_\phi\left(\alpha_{n+3}, \alpha_{n+4}, \frac{\mathcal{T}}{(2)^4\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+4}\right)\right)}\right) * \cdots *$$

$$\aleph_\phi\left(\alpha_{n+q-2}, \alpha_{n+q-1}, \frac{\mathcal{T}}{(2)^{q-1}\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+q}\right)\cdots\phi\left(\alpha_{n+q-2}, \alpha_{n+q-1}\right)\right)}\right)$$

$$* \aleph_\phi\left(\alpha_{n+q-1}, \alpha_{n+q}, \frac{\mathcal{T}}{(2)^{q-1}\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+q}\right)\cdots\phi\left(\alpha_{n+q-1}, \alpha_{n+q}\right)\right)}\right),$$

$$\Im_\phi\left(\alpha_n, \alpha_{n+q}, \mathcal{T}\right)$$

$$\leq \Im_\phi\left(\alpha_n, \alpha_{n+1}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_n, \alpha_{n+1}\right)\right)}\right) \circ \Im_\phi\left(\alpha_{n+1}, \alpha_{n+q}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\right)}\right)$$

$$\leq \Im_\phi\left(\alpha_n, \alpha_{n+1}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_n, \alpha_{n+1}\right)\right)}\right) \circ \Im_\phi\left(\alpha_{n+1}, \alpha_{n+2}, \frac{\mathcal{T}}{(2)^2\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+1}, \alpha_{n+2}\right)\right)}\right)$$

$$\circ \Im_\phi\left(\alpha_{n+2}, \alpha_{n+q}, \frac{\mathcal{T}}{(2)^2\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\right)}\right)$$

$$\leq \Im_\phi\left(\alpha_n, \alpha_{n+1}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_n, \alpha_{n+1}\right)\right)}\right) \circ \Im_\phi\left(\alpha_{n+1}, \alpha_{n+2}, \frac{\mathcal{T}}{(2)^2\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+1}, \alpha_{n+2}\right)\right)}\right)$$

$$\circ \Im_\phi\left(\alpha_{n+2}, \alpha_{n+3}, \frac{\mathcal{T}}{(2)^3\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+3}\right)\right)}\right)$$

$$\circ \Im_\phi\left(\alpha_{n+3}, \alpha_{n+q}, \frac{\mathcal{T}}{(2)^3\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+q}\right)\right)}\right)$$

$$\leq \Im_\phi\left(\alpha_n, \alpha_{n+1}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_n, \alpha_{n+1}\right)\right)}\right) \circ \Im_\phi\left(\alpha_{n+1}, \alpha_{n+2}, \frac{\mathcal{T}}{(2)^2\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+1}, \alpha_{n+2}\right)\right)}\right)$$

$$\circ \Im_\phi\left(\alpha_{n+2}, \alpha_{n+3}, \frac{\mathcal{T}}{(2)^3\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+3}\right)\right)}\right)$$

$$\circ \Im_\phi\left(\alpha_{n+3}, \alpha_{n+4}, \frac{\mathcal{T}}{(2)^4\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+4}\right)\right)}\right) \circ \cdots \circ$$

$$\Im_\phi\left(\alpha_{n+q-2}, \alpha_{n+q-1}, \frac{\mathcal{T}}{(2)^{q-1}\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+q}\right)\cdots\phi\left(\alpha_{n+q-2}, \alpha_{n+q-1}\right)\right)}\right)$$

$$\circ \Im_\phi\left(\alpha_{n+q-1}, \alpha_{n+q}, \frac{\mathcal{T}}{(2)^{q-1}\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+q}\right)\cdots\phi\left(\alpha_{n+q-1}, \alpha_{n+q}\right)\right)}\right)$$

and

$$
\begin{aligned}
&\Re_\phi\left(\alpha_n,\alpha_{n+q},\mathcal{T}\right)\\
&\leq \Re_\phi\left(\alpha_n,\alpha_{n+1},\frac{\mathcal{T}}{2\left(\phi\left(\alpha_n,\alpha_{n+1}\right)\right)}\right)\circ\Re_\phi\left(\alpha_{n+1},\alpha_{n+q},\frac{\mathcal{T}}{2\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\right)}\right)\\
&\leq \Re_\phi\left(\alpha_n,\alpha_{n+1},\frac{\mathcal{T}}{2\left(\phi\left(\alpha_n,\alpha_{n+1}\right)\right)}\right)\circ\Re_\phi\left(\alpha_{n+1},\alpha_{n+2},\frac{\mathcal{T}}{(2)^2\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\phi\left(\alpha_{n+1},\alpha_{n+2}\right)\right)}\right)\\
&\circ\Re_\phi\left(\alpha_{n+2},\alpha_{n+q},\frac{\mathcal{T}}{(2)^2\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\phi\left(\alpha_{n+2},\alpha_{n+q}\right)\right)}\right)\\
&\leq \Re_\phi\left(\alpha_n,\alpha_{n+1},\frac{\mathcal{T}}{2\left(\phi\left(\alpha_n,\alpha_{n+1}\right)\right)}\right)\circ\Re_\phi\left(\alpha_{n+1},\alpha_{n+2},\frac{\mathcal{T}}{(2)^2\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\phi\left(\alpha_{n+1},\alpha_{n+2}\right)\right)}\right)\\
&\circ\Re_\phi\left(\alpha_{n+2},\alpha_{n+3},\frac{\mathcal{T}}{(2)^3\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\phi\left(\alpha_{n+2},\alpha_{n+q}\right)\phi\left(\alpha_{n+2},\alpha_{n+3}\right)\right)}\right)\\
&\circ\Re_\phi\left(\alpha_{n+3},\alpha_{n+q},\frac{\mathcal{T}}{(2)^3\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\phi\left(\alpha_{n+2},\alpha_{n+q}\right)\phi\left(\alpha_{n+3},\alpha_{n+q}\right)\right)}\right)\\
&\leq \Re_\phi\left(\alpha_n,\alpha_{n+1},\frac{\mathcal{T}}{2\left(\phi\left(\alpha_n,\alpha_{n+1}\right)\right)}\right)\circ\Re_\phi\left(\alpha_{n+1},\alpha_{n+2},\frac{\mathcal{T}}{(2)^2\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\phi\left(\alpha_{n+1},\alpha_{n+2}\right)\right)}\right)\\
&\circ\Re_\phi\left(\alpha_{n+2},\alpha_{n+3},\frac{\mathcal{T}}{(2)^3\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\phi\left(\alpha_{n+2},\alpha_{n+q}\right)\phi\left(\alpha_{n+2},\alpha_{n+3}\right)\right)}\right)\\
&\circ\Re_\phi\left(\alpha_{n+3},\alpha_{n+4},\frac{\mathcal{T}}{(2)^4\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\phi\left(\alpha_{n+2},\alpha_{n+q}\right)\phi\left(\alpha_{n+3},\alpha_{n+q}\right)\phi\left(\alpha_{n+3},\alpha_{n+4}\right)\right)}\right)\circ\cdots\circ\\
&\Re_\phi\left(\alpha_{n+q-2},\alpha_{n+q-1},\frac{\mathcal{T}}{(2)^{q-1}\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\phi\left(\alpha_{n+2},\alpha_{n+q}\right)\phi\left(\alpha_{n+3},\alpha_{n+q}\right)\cdots\phi\left(\alpha_{n+q-2},\alpha_{n+q-1}\right)\right)}\right)\\
&\circ\Re_\phi\left(\alpha_{n+q-1},\alpha_{n+q},\frac{\mathcal{T}}{(2)^{q-1}\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\phi\left(\alpha_{n+2},\alpha_{n+q}\right)\phi\left(\alpha_{n+3},\alpha_{n+q}\right)\cdots\phi\left(\alpha_{n+q-1},\alpha_{n+q}\right)\right)}\right).
\end{aligned}
$$

Using (4) in the above inequalities, we deduce

$$
\begin{aligned}
&\aleph_\phi\left(\alpha_n,\alpha_{n+q},\mathcal{T}\right)\\
&\geq \aleph_\phi\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{2(\pounds)^{n-1}\left(\phi\left(\alpha_n,\alpha_{n+1}\right)\right)}\right)*\aleph_\phi\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^2(\pounds)^{n}\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\phi\left(\alpha_{n+1},\alpha_{n+2}\right)\right)}\right)\\
&*\aleph_\phi\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^3(\pounds)^{n+1}\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\phi\left(\alpha_{n+2},\alpha_{n+q}\right)\phi\left(\alpha_{n+2},\alpha_{n+3}\right)\right)}\right)\\
&*\aleph_\phi\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^4(\pounds)^{n+2}\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\phi\left(\alpha_{n+2},\alpha_{n+q}\right)\phi\left(\alpha_{n+3},\alpha_{n+q}\right)\phi\left(\alpha_{n+3},\alpha_{n+4}\right)\right)}\right)\\
&*\cdots*\\
&\aleph_\phi\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^{q-1}(\pounds)^{n+q-2}\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\phi\left(\alpha_{n+2},\alpha_{n+q}\right)\phi\left(\alpha_{n+3},\alpha q\right)\cdots\phi\left(\alpha_{n+q-2},\alpha_{n+q-1}\right)\right)}\right)\\
&*\aleph_\phi\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^{q-1}(\pounds)^{n+q-1}\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\phi\left(\alpha_{n+2},\alpha_{n+q}\right)\phi\left(\alpha_{n+3},\alpha_{n+q}\right)\cdots\phi\left(\alpha_{n+q-1},\alpha_{n+q}\right)\right)}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\Im_\phi\left(\alpha_n, \alpha_{n+q}, \mathcal{T}\right)\\
&\leq \Im_\phi\left(\alpha_0, \alpha_1, \frac{\mathcal{T}}{2(£)^{n-1}\left(\phi\left(\alpha_n, \alpha_{n+1}\right)\right)}\right) \circ \Im_\phi\left(\alpha_0, \alpha_1, \frac{\mathcal{T}}{(2)^2(£)^n\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+1}, \alpha_{n+2}\right)\right)}\right)\\
&\circ \Im_\phi\left(\alpha_0, \alpha_1, \frac{\mathcal{T}}{(2)^3(£)^{n+1}\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+3}\right)\right)}\right)\\
&\circ \Im_\phi\left(\alpha_0, \alpha_1, \frac{\mathcal{T}}{(2)^4(£)^{n+2}\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+4}\right)\right)}\right)\\
&\circ \cdots \circ\\
&\Im_\phi\left(\alpha_0, \alpha_1, \frac{\mathcal{T}}{(2)^{q-1}(£)^{n+q-2}\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+q}\right)\cdots\phi\left(\alpha_{n+q-2}, \alpha_{n+q-1}\right)\right)}\right)\\
&\circ \Im_\phi\left(\alpha_0, \alpha_1, \frac{\mathcal{T}}{(2)^{q-1}(£)^{n+q-1}\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+q}\right)\cdots\phi\left(\alpha_{n+q-1}, \alpha_{n+q}\right)\right)}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Re_\phi\left(\alpha_n, \alpha_{n+q}, \mathcal{T}\right)\\
&\leq \Re_\phi\left(\alpha_0, \alpha_1, \frac{\mathcal{T}}{2(£)^{n-1}\left(\phi\left(\alpha_n, \alpha_{n+1}\right)\right)}\right) \circ \Re_\phi\left(\alpha_0, \alpha_1, \frac{\mathcal{T}}{(2)^2(£)^n\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+1}, \alpha_{n+2}\right)\right)}\right)\\
&\circ \Re_\phi\left(\alpha_0, \alpha_1, \frac{\mathcal{T}}{(2)^3(£)^{n+1}\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+3}\right)\right)}\right)\\
&\circ \Re_\phi\left(\alpha_0, \alpha_1, \frac{\mathcal{T}}{(2)^4(£)^{n+2}\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+4}\right)\right)}\right)\\
&\circ \cdots \circ\\
&\Re_\phi\left(\alpha_0, \alpha_1, \frac{\mathcal{T}}{(2)^{q-1}(£)^{n+q-2}\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+q}\right)\cdots\phi\left(\alpha_{n+q-2}, \alpha_{n+q-1}\right)\right)}\right)
\end{aligned}
$$

$$
\circ \Re_\phi\left(\alpha_0, \alpha_1, \frac{\mathcal{T}}{(2)^{q-1}(£)^{n+q-1}\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+q}\right)\cdots\phi\left(\alpha_{n+q-1}, \alpha_{n+q}\right)\right)}\right).
$$

Using (1), for $n \to \infty$, we deduce

$$
\lim_{n\to\infty} \aleph_\phi\left(\alpha_n,\ \alpha_{n+q}, \mathcal{T}\right) = 1 * 1 * \cdots * 1 = 1,
$$

$$
\lim_{n\to\infty} \left(\Im_\phi \alpha_n, \alpha_{n+q}, \mathcal{T}\right) = 0 \circ 0 \circ \cdots \circ 0 = 0,
$$

and

$$
\lim_{n\to\infty} \Re_\phi\left(\alpha_n, \alpha_{n+q}, \mathcal{T}\right) = 0 \circ 0 \circ \cdots \circ 0 = 0.
$$

That is, $\{\alpha_n\}$ is a CPFCS. Since $\left(\Xi, \aleph_\phi, \Im_\phi, \Re_\phi, *, \circ\right)$ is a CPF-complete CPFMS, there exists

$$\lim_{n\to\infty} \alpha_n = \alpha.$$

Now investigate that α is an FP of ξ. Using (5) , (10) , (15) and (1) we obtain

$$\aleph_\phi\left(\alpha, \xi\alpha, \mathcal{T}\right) \geq \aleph_\phi\left(\alpha, \alpha_{n+1}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha, \alpha_{n+1}\right)\right)}\right) * \aleph_\phi\left(\alpha_{n+1}, \xi\alpha, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_{n+1}, \xi\alpha\right)\right)}\right)$$
$$\aleph_\phi\left(\alpha, \xi\alpha, \mathcal{T}\right) \geq \aleph_\phi\left(\alpha, \alpha_{n+1}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha, \alpha_{n+1}\right)\right)}\right) * \aleph_\phi\left(\xi\alpha_n, \xi\alpha, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_{n+1}, \xi\alpha\right)\right)}\right)$$
$$\aleph_\phi\left(\alpha, \xi\alpha, \mathcal{T}\right) \geq \aleph_\phi\left(\alpha, \alpha_{n+1}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha, \alpha_{n+1}\right)\right)}\right) * \aleph_\phi\left(\alpha_n, \alpha, \frac{\mathcal{T}}{2\pounds\left(\phi\left(\alpha_{n+1}, \xi\alpha\right)\right)}\right) \to 1 * 1 = 1$$

as $n \to \infty$,

$$\Im_\phi\left(\alpha, \xi\alpha, \mathcal{T}\right) \leq \Im_\phi\left(\alpha, \alpha_{n+1}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha, \alpha_{n+1}\right)\right)}\right) \circ \Im_\phi\left(\alpha_{n+1}, \xi\alpha, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_{n+1}, \xi\alpha\right)\right)}\right)$$
$$\Im_\phi\left(\alpha, \xi\alpha, \mathcal{T}\right) \leq \Im_\phi\left(\alpha, \alpha_{n+1}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha, \alpha_{n+1}\right)\right)}\right) \circ \Im_\phi\left(\xi\alpha_n, \xi\alpha, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_{n+1}, \xi\alpha\right)\right)}\right)$$
$$\Im_\phi\left(\alpha, \xi\alpha, \mathcal{T}\right) \leq \Im_\phi\left(\alpha, \alpha_{n+1}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha, \alpha_{n+1}\right)\right)}\right) \circ \Im_\phi\left(\alpha_n, \alpha, \frac{\mathcal{T}}{2\pounds\left(\phi\left(\alpha_{n+1}, \xi\alpha\right)\right)}\right) \to 0 \circ 0 = 0$$

as $n \to \infty$, and

$$\Re_\phi\left(\alpha, \xi\alpha, \mathcal{T}\right) \leq \Re_\phi\left(\alpha, \alpha_{n+1}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha, \alpha_{n+1}\right)\right)}\right) \circ \Re_\phi\left(\alpha_{n+1}, \xi\alpha, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_{n+1}, \xi\alpha\right)\right)}\right)$$
$$\Re_\phi\left(\alpha, \xi\alpha, \mathcal{T}\right) \leq \Re_\phi\left(\alpha, \alpha_{n+1}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha, \alpha_{n+1}\right)\right)}\right) \circ \Re_\phi\left(\xi\alpha_n, \xi\alpha, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_{n+1}, \xi\alpha\right)\right)}\right)$$
$$\Re_\phi\left(\alpha, \xi\alpha, \mathcal{T}\right) \leq \Re_\phi\left(\alpha, \alpha_{n+1}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha, \alpha_{n+1}\right)\right)}\right) \circ \Re_\phi\left(\alpha_n, \alpha, \frac{\mathcal{T}}{2\pounds\left(\phi\left(\alpha_{n+1}, \xi\alpha\right)\right)}\right) \to 0 \circ 0 = 0$$

as $n \to \infty$. This implies that $\xi\alpha = \alpha$ is an FP. Now, we show the uniqueness. Suppose that $\xi c = c$ for some $c \in \Xi$. Then

$$1 \geq \aleph_\phi\left(c, \alpha, \mathcal{T}\right) = \aleph_\phi\left(\xi c, \xi\alpha, \mathcal{T}\right) \geq \aleph_\phi\left(c, \alpha, \frac{\mathcal{T}}{\pounds}\right) = \aleph_\phi\left(\xi c, \xi\alpha, \frac{\mathcal{T}}{\pounds}\right)$$
$$\geq \aleph_\phi\left(c, \alpha, \frac{\mathcal{T}}{\pounds^2}\right) \geq \cdots \geq \aleph_\phi\left(c, \alpha, \frac{\mathcal{T}}{\pounds^n}\right) \to 1 \text{ as } n \to \infty,$$
$$0 \leq \Im_\phi\left(c, \alpha, \mathcal{T}\right) = \Im_\phi\left(\xi c, \xi\alpha, \mathcal{T}\right) \leq \Im_\phi\left(c, \alpha, \frac{\mathcal{T}}{\pounds}\right) = \Im_\phi\left(\xi c, \xi\alpha, \frac{\mathcal{T}}{\pounds}\right)$$
$$\leq \Im_\phi\left(c, \alpha, \frac{\mathcal{T}}{\pounds^2}\right) \leq \cdots \leq \Im_\phi\left(c, \alpha, \frac{\mathcal{T}}{\pounds^n}\right) \to 0 \text{ as } n \to \infty$$

and

$$0 \leq \Re_\phi (c, \alpha, \mathcal{T}) = \Re_\phi (\xi c, \xi \alpha, \mathcal{T}) \leq \Re_\phi \left(c, \alpha, \frac{\mathcal{T}}{£} \right) = \Re_\phi \left(\xi c, \xi \alpha, \frac{\mathcal{T}}{£} \right)$$
$$\leq \Re_\phi \left(c, \alpha, \frac{\mathcal{T}}{£^2} \right) \leq \cdots \leq \Re_\phi \left(c, \alpha, \frac{\mathcal{T}}{£^n} \right) \to 0 \text{ as } n \to \infty.$$

Using (3) , (8) and (12) , we conclude that $\alpha = c$.

Definition 8 Let $\left(\Xi, \aleph_\phi, \Im_\phi, \Re_\phi, *, \circ\right)$ be a CPFMS. A map $\xi : \Xi \to \Xi$ is a CPF-contraction if there exists $0 < £ < 1$, such that

$$\frac{1}{\aleph_\phi (\xi\alpha, \xi\beta, \mathcal{T})} - 1 \leq £ \left[\frac{1}{\aleph_\phi (\alpha, \beta, \mathcal{T})} - 1 \right], \tag{7}$$

$$\Im_\phi (\xi\alpha, \xi\beta, \mathcal{T}) \leq £ \Im_\phi (\alpha, \beta, \mathcal{T}) \tag{8}$$

and

$$\Re_\phi (\xi\alpha, \xi\beta, \mathcal{T}) \leq £ \Re_\phi (\alpha, \beta, \mathcal{T}), \tag{9}$$

for all $\alpha, \beta \in \Xi$ and $\mathcal{T} > 0$.

Now we prove the following theorem for CPF contractions.

Theorem 2 *Let* $\left(\Xi, \aleph_\phi, \Im_\phi, \Re_\phi, *, \circ\right)$ *be a CPF-complete CPFMS with* $\phi : \Xi \times \Xi \to [1, \infty)$ *and suppose that*

$$\lim_{\mathcal{T}\to\infty} \aleph_\phi(\alpha, \beta, \mathcal{T}) = 1, \lim_{\mathcal{T}\to\infty} \Im_\phi (\alpha, \beta, \mathcal{T}) = 0 \textit{ and } \lim_{\mathcal{T}\to\infty} \Re_\phi (\alpha, \beta, \mathcal{T}) = 0 \tag{10}$$

for all $\alpha, \beta \in \Xi$ *and* $\mathcal{T} > 0$. *Let* $\xi : \Xi \to \Xi$ *be a CPF contraction. Further, suppose that for an arbitrary* $\alpha_0 \in \Xi$, *and* $n \in \mathbb{N}$, *where* $\alpha_n = \xi^n \alpha_0 = \xi\alpha_{n-1}$ $\lim_{n\to\infty} \phi(\alpha_n, \beta)$ *and* $\lim_{n\to\infty} \phi(\beta, \alpha_n)$ *exists and finite. Then* ξ *has a unique FP.*

Proof Let α_0 be a random element of Ξ and describe a sequence α_n by $\alpha_n = \xi^n \alpha_0 = \xi\alpha_{n-1}$, $n \in \mathbb{N}$. Using (5) and (6) for all $\mathcal{T} > 0$, and for all n, we have

$$\begin{aligned}
\frac{1}{\aleph_\phi (\alpha_n, \alpha_{n+1}, \mathcal{T})} - 1 &= \frac{1}{\aleph_\phi \left(\xi\alpha_{n-1}, \alpha_n, \mathcal{T}\right)} - 1 \\
&\leq £ \left[\frac{1}{\aleph_\phi (\alpha_{n-1}, \alpha_n, \mathcal{T})} - 1 \right] = \frac{£}{\aleph_\phi (\alpha_{n-1}, \alpha_n, \mathcal{T})} - £ \\
&\Rightarrow \frac{1}{\aleph_\phi (\alpha_n, \alpha_{n+1}, \mathcal{T})} \leq \frac{£}{\aleph_\phi (\alpha_{n-1}, \alpha_n, \mathcal{T})} + (1 - £) \\
&\leq \frac{£^2}{\aleph_\phi (\alpha_{n-2}, \alpha_{n-1}, \mathcal{T})} + £ (1 - £) + (1 - £).
\end{aligned}$$

Continuing in this way, we get

$$\begin{aligned}\frac{1}{\aleph_\phi(\alpha_n,\alpha_{n+1},\mathcal{T})} &\le \frac{£^n}{\aleph_\phi(\alpha_0,\alpha_1,\mathcal{T})}+£^{n-1}(1-£)\\ &\quad+£^{n-2}(1-£)+\cdots+£(1-£)+(1-£)\\ &\le \frac{£^n}{\aleph_\phi(\alpha_0,\alpha_1,\mathcal{T})}+\left(£^{n-1}+£^{n-2}+\cdots+1\right)(1-£)\\ &\le \frac{£^n}{\aleph_\phi(\alpha_0,\alpha_1,\mathcal{T})}+\left(1-£^n\right).\end{aligned}$$

We obtain that

$$\frac{1}{\frac{£^n}{\aleph_\phi(\alpha_0,\alpha_1,\mathcal{T})}+(1-£^n)} \le \aleph_\phi(\alpha_n,\alpha_{n+1},\mathcal{T}) \tag{11}$$

and

$$\begin{aligned}\Im_\phi(\alpha_n,\alpha_{n+1},\mathcal{T}) &= \Im_\phi(\xi\alpha_{n-1},\alpha_n,\mathcal{T})\\ &\le £\Im_\phi(\alpha_{n-1},\alpha_n,\mathcal{T})\\ &= \Im_\phi(\xi\alpha_{n-2},\alpha_{n-1},\mathcal{T})\\ &\le £^2\Im_\phi(\alpha_{n-2},\alpha_{n-1},\mathcal{T})\\ &\le \cdots\\ &\le £^n\Im_\phi(\alpha_0,\alpha_1,\mathcal{T})\end{aligned}$$

and

$$\begin{aligned}\Re_\phi(\alpha_n,\alpha_{n+1},\mathcal{T}) &= \Re_\phi(\xi\alpha_{n-1},\alpha_n,\mathcal{T})\\ &\le £\Re_\phi(\alpha_{n-1},\alpha_n,\mathcal{T})\\ &= \Re_\phi(\xi\alpha_{n-2},\alpha_{n-1},\mathcal{T})\\ &\le £^2\Re_\phi(\alpha_{n-2},\alpha_{n-1},\mathcal{T})\\ &\le \cdots\le £^n\Re_\phi(\alpha_0,\alpha_1,\mathcal{T}).\end{aligned} \tag{12}$$

So, for any $q\in\mathbb{N}$, using (5) , (10) and (15), we deduce that

$$\begin{aligned}&\aleph_\phi\left(\alpha_n,\alpha_{n+q},\mathcal{T}\right)\\ &\ge \aleph_\phi\left(\alpha_n,\alpha_{n+1},\frac{\mathcal{T}}{2(\phi(\alpha_n,\alpha_{n+1}))}\right)*\aleph_\phi\left(\alpha_{n+1},\alpha_{n+q},\frac{\mathcal{T}}{2\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\right)}\right)\\ &\ge \aleph_\phi\left(\alpha_n,\alpha_{n+1},\frac{\mathcal{T}}{2(\phi(\alpha_n,\alpha_{n+1}))}\right)*\aleph_\phi\left(\alpha_{n+1},\alpha_{n+2},\frac{\mathcal{T}}{(2)^2\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\phi(\alpha_{n+1},\alpha_{n+2})\right)}\right)\\ &\quad*\aleph_\phi\left(\alpha_{n+2},\alpha_{n+q},\frac{\mathcal{T}}{(2)^2\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\phi\left(\alpha_{n+2},\alpha_{n+q}\right)\right)}\right)\\ &\ge \aleph_\phi\left(\alpha_n,\alpha_{n+1},\frac{\mathcal{T}}{2(\phi(\alpha_n,\alpha_{n+1}))}\right)*\aleph_\phi\left(\alpha_{n+1},\alpha_{n+2},\frac{\mathcal{T}}{(2)^2\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\phi(\alpha_{n+1},\alpha_{n+2})\right)}\right)\end{aligned}$$

$$*\aleph_\phi\left(\alpha_{n+2}, \alpha_{n+3}, \frac{\mathcal{T}}{(2)^3\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+3}\right)\right)}\right)$$
$$*\aleph_\phi\left(\alpha_{n+3}, \alpha_{n+q}, \frac{\mathcal{T}}{(2)^3\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+q}\right)\right)}\right)$$

$$\geq \aleph_\phi\left(\alpha_n, \alpha_{n+1}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_n, \alpha_{n+1}\right)\right)}\right) * \aleph_\phi\left(\alpha_{n+1}, \alpha_{n+2}, \frac{\mathcal{T}}{(2)^2\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+1}, \alpha_{n+2}\right)\right)}\right)$$
$$* \aleph_\phi\left(\alpha_{n+2}, \alpha_{n+3}, \frac{\mathcal{T}}{(2)^3\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+3}\right)\right)}\right)$$
$$* \aleph_\phi\left(\alpha_{n+3}, \alpha_{n+4}, \frac{\mathcal{T}}{(2)^4\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+4}\right)\right)}\right) * \cdots *$$
$$\aleph_\phi\left(\alpha_{n+q-2}, \alpha_{n+q-1}, \frac{\mathcal{T}}{(2)^{q-1}\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+q}\right)\cdots\phi\left(\alpha_{n+q-2}, \alpha_{n+q-1}\right)\right)}\right)$$
$$* \aleph_\phi\left(\alpha_{n+q-1}, \alpha_{n+q}, \frac{\mathcal{T}}{(2)^{q-1}\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+q}\right)\cdots\phi\left(\alpha_{n+q-1}, \alpha_{n+q}\right)\right)}\right),$$

$$\Im_\phi\left(\alpha_n, \alpha_{n+q}, \mathcal{T}\right)$$
$$\leq \Im_\phi\left(\alpha_n, \alpha_{n+1}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_n, \alpha_{n+1}\right)\right)}\right) \circ \Im_\phi\left(\alpha_{n+1}, \alpha_{n+q}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\right)}\right)$$
$$\leq \Im_\phi\left(\alpha_n, \alpha_{n+1}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_n, \alpha_{n+1}\right)\right)}\right) \circ \Im_\phi\left(\alpha_{n+1}, \alpha_{n+2}, \frac{\mathcal{T}}{(2)^2\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+1}, \alpha_{n+2}\right)\right)}\right)$$
$$\circ \Im_\phi\left(\alpha_{n+2}, \alpha_{n+q}, \frac{\mathcal{T}}{(2)^2\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\right)}\right)$$
$$\leq \Im_\phi\left(\alpha_n, \alpha_{n+1}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_n, \alpha_{n+1}\right)\right)}\right) \circ \Im_\phi\left(\alpha_{n+1}, \alpha_{n+2}, \frac{\mathcal{T}}{(2)^2\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+1}, \alpha_{n+2}\right)\right)}\right)$$
$$\circ \Im_\phi\left(\alpha_{n+2}, \alpha_{n+3}, \frac{\mathcal{T}}{(2)^3\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+3}\right)\right)}\right)$$
$$\circ \Im_\phi\left(\alpha_{n+3}, \alpha_{n+q}, \frac{\mathcal{T}}{(2)^3\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+q}\right)\right)}\right)$$

$$\leq \Im_\phi\left(\alpha_n, \alpha_{n+1}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_n, \alpha_{n+1}\right)\right)}\right) \circ \Im_\phi\left(\alpha_{n+1}, \alpha_{n+2}, \frac{\mathcal{T}}{(2)^2\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+1}, \alpha_{n+2}\right)\right)}\right)$$
$$\circ \Im_\phi\left(\alpha_{n+2}, \alpha_{n+3}, \frac{\mathcal{T}}{(2)^3\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+3}\right)\right)}\right)$$
$$\circ \Im_\phi\left(\alpha_{n+3}, \alpha_{n+4}, \frac{\mathcal{T}}{(2)^4\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+4}\right)\right)}\right) \circ \cdots \circ$$
$$\Im_\phi\left(\alpha_{n+q-2}, \alpha_{n+q-1}, \frac{\mathcal{T}}{(2)^{q-1}\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+q}\right)\cdots\phi\left(\alpha_{n+q-2}, \alpha_{n+q-1}\right)\right)}\right)$$

$$\circ \Im_\phi\left(\alpha_{n+q-1}, \alpha_{n+q}, \frac{\mathcal{T}}{(2)^{q-1}\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+q}\right)\cdots\phi\left(\alpha_{n+q-1}, \alpha_{n+q}\right)\right)}\right)$$

and

$$\begin{aligned}
&\Re_\phi\left(\alpha_n, \alpha_{n+q}, \mathcal{T}\right)\\
&\leq \Re_\phi\left(\alpha_n, \alpha_{n+1}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_n, \alpha_{n+1}\right)\right)}\right) \circ \Re_\phi\left(\alpha_{n+1}, \alpha_{n+q}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\right)}\right)\\
&\leq \Re_\phi\left(\alpha_n, \alpha_{n+1}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_n, \alpha_{n+1}\right)\right)}\right) \circ \Re_\phi\left(\alpha_{n+1}, \alpha_{n+2}, \frac{\mathcal{T}}{(2)^2\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+1}, \alpha_{n+2}\right)\right)}\right)\\
&\circ \Re_\phi\left(\alpha_{n+2}, \alpha_{n+q}, \frac{\mathcal{T}}{(2)^2\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\right)}\right)\\
&\leq \Re_\phi\left(\alpha_n, \alpha_{n+1}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_n, \alpha_{n+1}\right)\right)}\right) \circ \Re_\phi\left(\alpha_{n+1}, \alpha_{n+2}, \frac{\mathcal{T}}{(2)^2\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+1}, \alpha_{n+2}\right)\right)}\right)\\
&\circ \Re_\phi\left(\alpha_{n+2}, \alpha_{n+3}, \frac{\mathcal{T}}{(2)^3\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+3}\right)\right)}\right)\\
&\circ \Re_\phi\left(\alpha_{n+3}, \alpha_{n+q}, \frac{\mathcal{T}}{(2)^3\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+q}\right)\right)}\right)\\
&\leq \Re_\phi\left(\alpha_n, \alpha_{n+1}, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_n, \alpha_{n+1}\right)\right)}\right) \circ \Re_\phi\left(\alpha_{n+1}, \alpha_{n+2}, \frac{\mathcal{T}}{(2)^2\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+1}, \alpha_{n+2}\right)\right)}\right)\\
&\circ \Re_\phi\left(\alpha_{n+2}, \alpha_{n+3}, \frac{\mathcal{T}}{(2)^3\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+3}\right)\right)}\right)\\
&\circ \Re_\phi\left(\alpha_{n+3}, \alpha_{n+4}, \frac{\mathcal{T}}{(2)^4\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+4}\right)\right)}\right) \circ \cdots \circ\\
&\Re_\phi\left(\alpha_{n+q-2}, \alpha_{n+q-1}, \frac{\mathcal{T}}{(2)^{q-1}\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+q}\right)\cdots\phi\left(\alpha_{n+q-2}, \alpha_{n+q-1}\right)\right)}\right)\\
&\circ \Re_\phi\left(\alpha_{n+q-1}, \alpha_{n+q}, \frac{\mathcal{T}}{(2)^{q-1}\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+q}\right)\cdots\phi\left(\alpha_{n+q-1}, \alpha_{n+q}\right)\right)}\right).
\end{aligned}$$

$$\begin{aligned}
&So, \aleph_\phi\left(\alpha_n, \alpha_{n+q}, \mathcal{T}\right)\\
&\quad\geq \frac{1}{\frac{\pounds^n}{\aleph_\phi\left(\alpha_0, \alpha_1, \frac{\mathcal{T}}{2\left(\phi\left(\alpha_n, \alpha_{n+1}\right)\right)}\right)} + (1-\pounds^n)} * \frac{1}{\frac{\pounds^{n+1}}{\aleph_\phi\left(\alpha_0, \alpha_1, \frac{\mathcal{T}}{(2)^2\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+1}, \alpha_{n+2}\right)\right)}\right)} + \left(1-\pounds^{n+1}\right)}\\
&\quad * \frac{1}{\frac{\pounds^{n+2}}{\aleph_\phi\left(\alpha_0, \alpha_1, \frac{\mathcal{T}}{(2)^3\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+3}\right)\right)}\right)} + \left(1-\pounds^{n+2}\right)} * \cdots *\\
&\quad \frac{1}{\frac{\pounds^{n+q-2}}{\aleph_\phi\left(\alpha_0, \alpha_1, \frac{\mathcal{T}}{(2)^{q-1}\left(\phi\left(\alpha_{n+1}, \alpha_{n+q}\right)\phi\left(\alpha_{n+2}, \alpha_{n+q}\right)\phi\left(\alpha_{n+3}, \alpha_{n+q}\right)\cdots\phi\left(\alpha_{n+q-2}, \alpha_{n+q-1}\right)\right)}\right)} + \left(1-\pounds^{n+q-2}\right)}
\end{aligned}$$

$$* \frac{1}{\frac{\pounds^{n+q-1}}{\aleph_{\phi}\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^{q-1}\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\phi\left(\alpha_{n+2},\alpha_{n+q}\right)\phi\left(\alpha_{n+3},\alpha_{n+q}\right)\cdots\phi\left(\alpha_{n+q-1},\alpha_{n+q}\right)\right)}\right)}+\left(1-\pounds^{n+q-1}\right)},$$

$$\begin{aligned}
&\Im_{\phi}\left(\alpha_n,\alpha_{n+q},\mathcal{T}\right)\\
&\leq \pounds^{n}\Im_{\phi}\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{2\left(\phi\left(\alpha_n,\alpha_{n+1}\right)\right)}\right)\circ \pounds^{n+1}\Im_{\phi}\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^2\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\phi\left(\alpha_{n+1},\alpha_{n+2}\right)\right)}\right)\\
&\circ \pounds^{n+2}\Im_{\phi}\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^3\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\phi\left(\alpha_{n+2},\alpha_{n+q}\right)\phi\left(\alpha_{n+2},\alpha_{n+3}\right)\right)}\right)\circ\cdots\circ\\
&\pounds^{n+q-2}\Im_{\phi}\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^{q-1}\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\phi\left(\alpha_{n+2},\alpha_{n+q}\right)\phi\left(\alpha_{n+3},\alpha_{n+q}\right)\cdots\phi\left(\alpha_{n+q-2},\alpha_{n+q-1}\right)\right)}\right)\\
&\circ \pounds^{n+q-1}\Im_{\phi}\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^{q-1}\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\phi\left(\alpha_{n+2},\alpha_{n+q}\right)\phi\left(\alpha_{n+3},\alpha_{n+q}\right)\cdots\phi\left(\alpha_{n+q-1},\alpha_{n+q}\right)\right)}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\Re_{\phi}\left(\alpha_n,\alpha_{n+q},\mathcal{T}\right)\\
&\leq \pounds^{n}\Re_{\phi}\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{2\left(\phi\left(\alpha_n,\alpha_{n+1}\right)\right)}\right)\circ \pounds^{n+1}\Re_{\phi}\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^2\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\phi\left(\alpha_{n+1},\alpha_{n+2}\right)\right)}\right)\\
&\circ \pounds^{n+2}\Re_{\phi}\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^3\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\phi\left(\alpha_{n+2},\alpha_{n+q}\right)\phi\left(\alpha_{n+2},\alpha_{n+3}\right)\right)}\right)\circ\cdots\circ\\
&\pounds^{n+q-2}\Re_{\phi}\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^{q-1}\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\phi\left(\alpha_{n+2},\alpha_{n+q}\right)\phi\left(\alpha_{n+3},\alpha_{n+q}\right)\cdots\phi\left(\alpha_{n+q-2},\alpha_{n+q-1}\right)\right)}\right)\\
&\circ \pounds^{n+q-1}\Re_{\phi}\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^{q-1}\left(\phi\left(\alpha_{n+1},\alpha_{n+q}\right)\phi\left(\alpha_{n+2},\alpha_{n+q}\right)\phi\left(\alpha_{n+3},\alpha_{n+q}\right)\cdots\phi\left(\alpha_{n+q-1},\alpha_{n+q}\right)\right)}\right).
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\aleph_{\phi}\left(\alpha_n,\alpha_{n+q},\mathcal{T}\right)=1*1*\cdots*1=1,$$

and

$$\lim_{n\to\infty}\Im_{\phi}\left(\alpha_n,\alpha_{n+q},\mathcal{T}\right)=0\circ 0\circ\cdots\circ 0=0,$$

$$\lim_{n\to\infty}\Re_{\phi}\left(\alpha_n,\alpha_{n+q},\mathcal{T}\right)=0\circ 0\circ\cdots\circ 0=0.$$

That is, $\{\alpha_n\}$ is a CPFCS. Since $\left(\Xi,\aleph_{\phi},\Im_{\phi},\Re_{\phi},*,\circ\right)$ is a CPF-complete CPFMS, there exists

$$\lim_{n\to\infty}\alpha_n=\alpha.$$

Now, investigate that α is a FP of ξ. Using (v) , (x) and (xv) , we obtain

$$\frac{1}{\aleph_{\phi}\left(\xi\alpha_{n},\xi\alpha,\mathcal{T}\right)}-1\leq £\left[\frac{1}{\aleph_{\phi}\left(\alpha_{n},\alpha,\mathcal{T}\right)}-1\right]=\frac{£}{\aleph_{\phi}\left(\alpha_{n},\alpha,\mathcal{T}\right)}-£.$$

So,

$$\frac{1}{\frac{£}{\aleph_{\phi}(\alpha_{n},\alpha,\mathcal{T})}+(1-£)}\leq\aleph_{\phi}\left(\xi\alpha_{n},\xi\alpha,\mathcal{T}\right).$$

Using the above inequality, we obtain

$$\begin{aligned}\aleph_{\phi}\left(\alpha,\xi\alpha,\mathcal{T}\right) &\geq \aleph_{\phi}\left(\alpha,\alpha_{n+1},\frac{\mathcal{T}}{2\phi\left(\alpha,\alpha_{n+1}\right)}\right)*\aleph_{\phi}\left(\alpha_{n+1},\xi\alpha,\frac{\mathcal{T}}{2\phi\left(\alpha_{n+1},\xi\alpha\right)}\right)\\ &\geq \aleph_{\phi}\left(\alpha,\alpha_{n+1},\frac{\mathcal{T}}{2\phi\left(\alpha,\alpha_{n+1}\right)}\right)*\aleph_{\phi}\left(\xi\alpha_{n},\xi\alpha,\frac{\mathcal{T}}{2\phi\left(\alpha_{n+1},\xi\alpha\right)}\right)\\ &\geq \aleph_{\phi}\left(\alpha_{n},\alpha_{n+1},\frac{\mathcal{T}}{2\phi\left(2\phi\left(\alpha,\alpha_{n+1}\right)\right)}\right)*\frac{1}{\frac{£}{\aleph_{\phi}\left(\alpha_{n},\alpha,\frac{\mathcal{T}}{2\phi\left(\alpha_{n+1},\xi\alpha\right)}\right)}+(1-£)}\rightarrow 1*1=1\end{aligned}$$

as $n\rightarrow\infty$,

$$\begin{aligned}\Im_{\phi}\left(\alpha,\xi\alpha,\mathcal{T}\right) &\leq \aleph_{\phi}\left(\alpha,\alpha_{n+1},\frac{\mathcal{T}}{2\phi\left(\alpha,\alpha_{n+1}\right)}\right)\circ\Im_{\phi}\left(\alpha_{n+1},\xi\alpha,\frac{\mathcal{T}}{2\phi\left(\alpha_{n+1},\xi\alpha\right)}\right)\\ &\leq \Im_{\phi}\left(\alpha,\alpha_{n+1},\frac{\mathcal{T}}{2\phi\left(\alpha,\alpha_{n+1}\right)}\right)\circ\Im_{\phi}\left(\xi\alpha_{n},\xi\alpha,\frac{\mathcal{T}}{2\phi\left(\alpha_{n+1},\xi\alpha\right)}\right)\\ &\leq \Im_{\phi}\left(\alpha_{n},\alpha_{n+1},\frac{\mathcal{T}}{2\phi\left(\alpha,\alpha_{n+1}\right)}\right)\circ£\Im_{\phi}\left(\alpha_{n},\alpha,\frac{\mathcal{T}}{2\phi\left(\alpha_{n+1},\xi\alpha\right)}\right)\rightarrow 0\circ 0=0\end{aligned}$$

as $n\rightarrow\infty$,

$$\begin{aligned}\Re_{\phi}\left(\alpha,\xi\alpha,\mathcal{T}\right) &\leq \Re_{\phi}\left(\alpha,\alpha_{n+1},\frac{\mathcal{T}}{2\phi\left(\alpha,\alpha_{n+1}\right)}\right)\circ\Re_{\phi}\left(\alpha_{n+1},\xi\alpha,\frac{\mathcal{T}}{2\phi\left(\alpha_{n+1},\xi\alpha\right)}\right)\\ &\leq \Re_{\phi}\left(\alpha,\alpha_{n+1},\frac{\mathcal{T}}{2\phi\left(\alpha,\alpha_{n+1}\right)}\right)\circ\Re_{\phi}\left(\xi\alpha_{n},\xi\alpha,\frac{\mathcal{T}}{2\phi\left(\alpha_{n+1},\xi\alpha\right)}\right)\\ &\leq \Re_{\phi}\left(\alpha_{n},\alpha_{n+1},\frac{\mathcal{T}}{2\phi\left(\alpha,\alpha_{n+1}\right)}\right)\circ£\Re_{\phi}\left(\alpha_{n},\alpha,\frac{\mathcal{T}}{2\phi\left(\alpha_{n+1},\xi\alpha\right)}\right)\rightarrow 0\circ 0=0 \text{ as } n\rightarrow\infty.\end{aligned}$$

This implies that $\xi\alpha=\alpha$. Now, we show the uniqueness. Suppose that $\xi c=c$ for some $c\in\Xi$. Then

$$\begin{aligned}\frac{1}{\aleph_{\phi}\left(\alpha,c,\mathcal{T}\right)}-1 &= \frac{1}{\aleph_{\phi}\left(\xi\alpha,\xi c,\mathcal{T}\right)}-1\\ &\leq £\left[\frac{1}{\aleph_{\phi}\left(\alpha,c,\mathcal{T}\right)}-1\right]<\frac{1}{\aleph_{\phi}\left(\alpha,c,\mathcal{T}\right)}-1\end{aligned}$$

a contradiction. Also,

$$\Im_{\phi}\left(\alpha,c,\mathcal{T}\right)=\Im_{\phi}\left(\xi\alpha,\xi c,\mathcal{T}\right)\leq£\Im_{\phi}\left(\alpha,c,\mathcal{T}\right)<\Im_{\phi}\left(\alpha,c,\mathcal{T}\right),$$

and

$$\Re_\phi(\alpha, c, \mathcal{T}) = \Re_\phi(\xi\alpha, \xi c, \mathcal{T}) \leq £\Re_\phi(\alpha, c, \mathcal{T}) < \Re_\phi(\alpha, c, \mathcal{T})$$

which are contradictions.

Therefore, we must have $\aleph_\phi(\alpha, c, \mathcal{T}) = 1$, $\Im_\phi(\alpha, c, \mathcal{T}) = 0$ and $\Re_\phi(\alpha, c, \mathcal{T}) = 0$, hence $\alpha = c$.

Example 5 Let $\Xi = [0, 1]$. Define ϕ by

$$\phi(\alpha, \beta) = \begin{cases} 1, & \text{if } \alpha = \beta, \\ \frac{1+\max\{\alpha,\beta\}}{\min\{\alpha,\beta\}}, & \text{if } \alpha \neq \beta \neq 0. \end{cases}$$

Also, take

$$\aleph_\phi(\alpha, \beta, \mathcal{T}) = \begin{cases} 1, & \text{if } \alpha = \beta \\ \frac{\mathcal{T}}{\mathcal{T}+\max\{\alpha,\beta\}}, & \text{if otherwise} \end{cases},$$

$$\Im_\phi(\alpha, \beta, \mathcal{T}) = \begin{cases} 0, & \text{if } \alpha = \beta \\ \frac{\max\{\alpha,\beta\}}{\mathcal{T}+\max\{\alpha,\beta\}}, & \text{if otherwise} \end{cases},$$

and

$$\Re_\phi(\alpha, \beta, \mathcal{T}) = \begin{cases} 0, & \text{if } \alpha = \beta \\ \frac{\max\{\alpha,\beta\}}{\mathcal{T}}, & \text{if otherwise} \end{cases}$$

with $\pi * \mu = \pi \cdot \mu$ and $\pi \circ \mu = \max\{\pi, \mu\}$. Then $(\Xi, \aleph_\phi, \Im_\phi, \Re_\phi, *, \circ)$ is a CPF-complete CPFMS. Observe that $\lim_{\mathcal{T}\to\infty} \aleph_\phi(\alpha, \beta, \mathcal{T}) = 1$, $\lim_{\mathcal{T}\to\infty} \Im_\phi(\alpha, \beta, \mathcal{T}) = 0$ and $\lim_{\mathcal{T}\to\infty} \Re_\phi(\alpha, \beta, \mathcal{T}) = 0$, satisfied. Define $\xi : \Xi \to \Xi$ by

$$\xi(\alpha) = \begin{cases} 0, & \text{if } \alpha \in \left[0, \frac{1}{2}\right], \\ \frac{\alpha}{4}, & \text{if } \alpha \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Then we have four cases:

(1) If $\alpha, \beta \in \left[0, \frac{1}{2}\right]$, then $\xi\alpha = \xi\beta = 0$;
(2) If $\alpha \in \left[0, \frac{1}{2}\right]$ and $\beta \in \left(\frac{1}{2}, 1\right]$, then $\xi\alpha = 0$ and $\xi\beta = \frac{\beta}{4}$;
(3) If $\beta \in \left[0, \frac{1}{2}\right]$ and $\alpha \in \left(\frac{1}{2}, 1\right]$, then $\xi\beta = 0$ and $\xi\alpha = \frac{\beta}{4}$;
(4) If $\alpha, \beta \in \left(\frac{1}{2}, 1\right]$, then $\xi\alpha = \frac{\beta}{4}$ and $\xi\beta = \frac{\beta}{4}$;

In all above $(1 - 4)$ cases,

$$\aleph_\phi(\xi\alpha, \xi\beta, £\mathcal{T}) \geq \aleph_\phi(\alpha, \beta, \mathcal{T}),$$

$$\Im_\phi(\xi\alpha, \xi\beta, £\mathcal{T}) \leq \Im_\phi(\alpha, \beta, \mathcal{T}) \text{ and } \Re_\phi(\xi\alpha, \xi\beta, £\mathcal{T}) \leq \Re_\phi(\alpha, \beta, \mathcal{T})$$

are satisfied for $£ \in \left[\frac{1}{2}, 1\right)$, and also

$$\frac{1}{\aleph_{\phi}\left(\xi\alpha,\xi\beta,\mathcal{T}\right)}-1\leq \pounds\left[\frac{1}{\aleph_{\phi}\left(\alpha,\beta,\mathcal{T}\right)}-1\right],$$

$$\Im_{\phi}\left(\xi\alpha,\xi\beta,\mathcal{T}\right)\leq \pounds\Im_{\phi}\left(\alpha,\beta,\mathcal{T}\right),\Re_{\phi}\left(\xi\alpha,\xi\beta,\mathcal{T}\right)\leq \pounds\Re_{\phi}\left(\alpha,\beta,\mathcal{T}\right)$$

satisfied for $\pounds\in\left[\frac{1}{2},1\right)$.

We can easily see that $\lim_{n\to\infty}\phi\left(\alpha_n,\beta\right)$ and $\lim_{n\to\infty}\phi\left(\beta,\alpha_n\right)$ exist and finite. Observe that all circumstances of Theorems 1 and 2 are fulfilled, and 0 is a unique FP of ξ.

4 Application to Fuzzy Fredholm Integral Equation

Let $\Xi=C([e,g],\mathbb{R})$ be the set of the entire continuous functions with domain of real values and defined on $[e,g]$.

Now, we let the fuzzy integral equation:

$$\alpha\left(l\right)=f\left(j\right)+\beta\int_{e}^{g}F\left(l,\ j\right)\alpha\left(l\right)dj \text{ for all } l,j\in[e,g] \tag{1}$$

Note that $\beta>0$ and $f(j)$ is a fuzzy function of j where $j\in[e,g]$ and $F\in\Xi$. Define $\aleph_{\phi}$ and $\Im_{\phi}$ by

$$\aleph_{\phi}\left(\alpha\left(l\right),\beta\left(l\right),\mathcal{T}\right)=\sup_{l\in[e,g]}\frac{\mathcal{T}}{\mathcal{T}+\left|\acute{a}\left(l\right)-\hat{a}\left(l\right)\right|^{2}} \text{ for all } \alpha,\beta\in\Xi \text{ and } \mathcal{T}>0,$$

$$\Im_{\phi}\left(\alpha\left(l\right),\beta\left(l\right),\ \mathcal{T}\right)=1-\sup_{l\in[e,g]}\frac{\mathcal{T}}{\mathcal{T}+\left|\acute{a}\left(l\right)-\hat{a}\left(l\right)\right|^{2}} \text{ for all } \alpha,\beta\in\Xi \text{ and } \mathcal{T}>0,$$

and

$$\Re_{\phi}\left(\alpha\left(l\right),\beta\left(l\right),\ \mathcal{T}\right)=\sup_{l\in[e,g]}\frac{\left|\acute{a}\left(l\right)-\hat{a}\left(l\right)\right|^{2}}{\mathcal{T}} \text{for all } \alpha,\beta\in\Xi \text{ and } \mathcal{T}>0$$

with CTN and CTCN define by $\pi*\mu=\pi\cdot\mu$ and $\pi\circ\mu=\max\{\pi,\mu\}$. Define $\phi:\Xi\times\Xi\to[1,\infty)$ as

$$\phi\left(\alpha,\beta\right)=\begin{cases}1, & \text{if } \alpha=\beta;\\ \frac{1+\max\{\alpha,\beta\}}{\min\{\alpha,\beta\}} & \text{if } \alpha\neq\beta\neq 0.\end{cases}$$

Then $(\Xi,\aleph_{\phi},\Im_{\phi},\Re_{\phi},*,\circ)$ is a complete CPFMS. Assume that $|F\left(l,j\right)\alpha\left(l\right)-F\left(l,j\right)\beta\left(l\right)|=|\alpha\left(l\right)-\beta\left(l\right)|$ for all $\alpha,\beta\in\Xi$, $\pounds\in(0,1)$ and $\forall l,j\in[e,g]$. Consider $\left(\beta\int_{e}^{g}dj\right)^{2}\leq\pounds<1$.

Theorem 3 *The fuzzy integral equation in* (1) *has a unique solution.*

Proof Define $\xi : \Xi \to \Xi$ by
$\xi\alpha(l) = f(j) + \beta \int_e^g F(l, j)\, e(l)\, dj$ for all $l, j \in [e, g]$.

Note that the survival of an FP of the operator ξ is equivalent to that of a solution of the fuzzy integral equation.

Now, for all $\alpha, \beta \in \Xi$, we obtain

$$\begin{aligned}
&\aleph_\phi\left(\xi\alpha(l), \hat{\imath}\beta(l), £\mathcal{T}\right)\\
&= \sup_{l\in[e,g]} \frac{£\mathcal{T}}{£\mathcal{T} + \left|\hat{\imath}\acute{a}(l) - \hat{\imath}\hat{a}(l)\right|^2}\\
&= \sup_{l\in[e,g]} \frac{£\mathcal{T}}{£\mathcal{T} + \left|f(j) + \beta\int_e^g F(l, j)\, e(l)\, dj - f(j) - \beta\int_e^g F(l, j)\, e(l)\, dj\right|^2}\\
&= \sup_{l\in[e,g]} \frac{£\mathcal{T}}{£\mathcal{T} + \left|\beta\int_e^g F(l, j)\, e(l)\, dj - \beta\int_e^g F(l, j)\, e(l)\, dj\right|^2}\\
&= \sup_{l\in[e,g]} \frac{£\mathcal{T}}{£\mathcal{T} + |F(l, j)\,\alpha(l) - F(l, j)\,\beta(l)|^2\left(\beta\int_e^g dj\right)^2}\\
&\geq \sup_{l\in[e,g]} \frac{\mathcal{T}}{\mathcal{T} + |\alpha(l) - \beta(l)|^2}\\
&\geq \aleph_\phi(\alpha(l), \beta(l), \mathcal{T}),
\end{aligned}$$

$$\begin{aligned}
&\Im_\phi\left(\xi\alpha(l), \hat{\imath}\beta(l), £\,\mathcal{T}\right)\\
&= 1 - \sup_{l\in[e,g]} \frac{£\mathcal{T}}{£\mathcal{T} + \left|\hat{\imath}\acute{a}(l) - \hat{\imath}\hat{a}(l)\right|^2}\\
&= 1 - \sup_{l\in[e,g]} \frac{£\mathcal{T}}{£\mathcal{T} + \left|f(j) + \beta\int_e^g F(l, j)\, e(l)\, dj - f(j) - \beta\int_e^g F(l, j)\, e(l)\, dj\right|^2}\\
&= 1 - \sup_{l\in[e,g]} \frac{£\mathcal{T}}{£\mathcal{T} + \left|\beta\int_e^g F(l, j)\, e(l)\, dj - \beta\int_e^g F(l, j)\, e(l)\, dj\right|^2}\\
&= 1 - \sup_{l\in[e,g]} \frac{£\mathcal{T}}{£\mathcal{T} + |F(l, j)\,\alpha(l) - F(l, j)\,\beta(l)|^2\left(\beta\int_e^g dj\right)^2}\\
&\leq 1 - \sup_{l\in[e,g]} \frac{\mathcal{T}}{\mathcal{T} + |\alpha(l) - \beta(l)|^2}\\
&\leq \Im_\phi(\alpha(l), \beta(l), \mathcal{T})
\end{aligned}$$

and

$$\begin{aligned}
&\Re_{\phi}\left(\xi\alpha\left(l\right),\hat{\imath}\beta\left(l\right),£\mathcal{T}\right)\\
&=\sup_{l\in[e,g]}\frac{\left|\hat{\imath}\acute{a}\left(l\right)-\hat{\imath}\hat{a}\left(l\right)\right|^{2}}{£\mathcal{T}}\\
&=\sup_{l\in[e,g]}\frac{\left|f\left(j\right)+\beta\int_{e}^{g}F\left(l,j\right)e\left(l\right)dj-f\left(j\right)-\beta\int_{e}^{g}F\left(l,j\right)e\left(l\right)dj\right|^{2}}{£\mathcal{T}}\\
&=\sup_{l\in[e,g]}\frac{\left|\beta\int_{e}^{g}F\left(l,j\right)e\left(l\right)dj-\beta\int_{e}^{g}F\left(l,\ j\right)e\left(l\right)dj\right|^{2}}{£\mathcal{T}}\\
&=\sup_{l\in[e,g]}\frac{\left|F\left(l,\ j\right)\alpha\left(l\right)-F\left(l,j\right)\beta\left(l\right)\right|^{2}\left(\beta\int_{e}^{g}dj\right)^{2}}{£\mathcal{T}}\\
&\leq\sup_{l\in[e,g]}\frac{\left|\alpha\left(l\right)-\beta\left(l\right)\right|^{2}}{\mathcal{T}}\\
&\leq\Re_{\phi}\left(\alpha\left(l\right),\beta\left(l\right),\ \mathcal{T}\right).
\end{aligned}$$

Therefore, all circumstances of Theorem 1 are fulfilled. Hence, operator ξ has a single FP. This implies that the fuzzy integral equation (1) has a single solution.

Conclusion

Herein, we introduced the notion of controlled picture fuzzy metric space and some new types of fixed point theorems in this new setting. Moreover, we provided a non-trivial example to demonstrate the viability of the proposed methods. We have supplemented this work with an application that demonstrates how the built method outperforms those found in the literature. Since our structure is more general than the class of fuzzy and controlled fuzzy spaces, our results and notions expand and generalize a number of previously published results.

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Theoretical Analysis for a Generalized Fractional-Order Boundary Value Problem

Idris Ahmed, Poom Kumam, Jessada Tariboon, and Abdullahi Yusuf

Abstract Fixed point theorems are effective and reliable tools for investigating and analyzing several nonlinear problems. In this chapter, we formulate a fractional differential equation with a more general integral boundary condition in the setting of ψ-Caputo fractional derivatives. Based on the techniques of Green's function, an equivalent integral equation was established. Moreover, the existence and uniqueness of solutions of generalized boundary value problems have been investigated using Schaefer's and Banach's fixed point theorems.

1 Introduction

The main advantage of fractional-order models over the classical integer-order models is the description of memory and hereditary properties. Fractional differential equations emerge in many engineering and scientific disciplines because the mathematical modeling of systems and processes in physics, chemistry, and complex media electrodynamics required fractional-order derivatives. As a result, the subject of fractional differential equations is receiving a lot of attention; for further details, see [2, 6, 14, 16, 18]. However, the theory of boundary value problems for nonlinear fractional differential equations is still in its early stages, with many topics needing to be addressed.

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Researchers have become interested in boundary value problems because they exist in a variety of phenomena. In the previous two decades, similar problems have been applied to a wide range of scientific and engineering applications, including underground water flow, population dynamics, and blood flow, and many more, we refer readers to see [1, 17, 19–21]. Fixed point theorems are effective and reliable tools for investigating and analyzing several nonlinear problems; see [7–10, 15].

Ahmad and Nieto, in [13], studied a nonlinear fractional integro-differential equations of the form:

$$\begin{cases} {}^C D_{0^+}^{r} z(t) = f(t, z(t), (\chi z)(t)), \quad t \in (0,1),\ 1 < r \le 2, \\ az(0) + \beta z'(0) = \int_0^1 q_1(z(s))ds,\ az(1) + \beta z'(1) = \int_0^1 q_2(z(s))ds, \end{cases} \tag{1}$$

where ${}^C D_{0^+}^{\rho}(\cdot)$ is the Caputo fractional derivative, $f : [0,1] \times E \times E \to E$, for $\gamma : [0,1] \times [0,1] \to [0,\infty)$,

$$(\chi z(t)) = \int_0^1 \gamma(t,s) z(s) ds, \quad q_1, q_2 : E \to E,$$

and $\alpha > 0$, $\beta \ge 0$ are positive real numbers. The existence and uniqueness of solution were investigated via Schaefer's and Banach's fixed point theorems. Moreover, Idris et al. [3] examined a ψ-Caputo fractional derivative given by

$$\begin{cases} {}^C D_{0^+}^{p;\psi} z(t) = f(t, z(t), z(\chi t), {}^C D_{0^+}^{p;\psi} z(t)),\ t \in [0,b],\ 0 < \chi < 1,\ b > 0, \\ a_1 z(0) = -a_2 z(b),\ a_1 {}^C D_{0^+}^{q;\psi} z(0) = -a_2 {}^C D_{0^+}^{q;\psi} z(b), \\ a_1 {}^C D_{0^+}^{r;\psi} z(0) = -a_2 {}^C D_{0^+}^{r;\psi} z(b), \end{cases} \tag{2}$$

where ${}^C D_{0^+}^{p;\psi}(\cdot)$, ${}^C D_{0^+}^{q;\psi}(\cdot)$, and ${}^C D_{0^+}^{r;\psi}(\cdot)$ are ψ-Caputo fractional derivative of order $(2 < p \le 3)$, $(0 < q \le 1)$, and $(1 < r \le 2)$, respectively, with respect to $\psi \in C[0,b]$ such that $\psi'(t) > 0$, for all $t \in [0,b]$. Besides, the authors studied the nonlinear implicit fractional pantograph boundary value

$$\begin{cases} {}^C D_{0^+}^{p;\psi} z(t) = f(t, z(t), z(\chi t), {}^C D_{0^+}^{p;\psi} z(t)),\ t \in [0,b],\ 0 < \chi < 1,\ b > 0, \\ a_1 z(0) + a_2 z(b) = 0,\ a_1 \delta_\psi z(0) + a_2 \delta_\psi z(b) = 0, \end{cases} \tag{3}$$

where ${}^C D_{0^+}^{p;\psi}(\cdot)$ is a ψ-Caputo fractional derivative of order $(1 < p \le 2)$ with respect to another function $\psi \in C[0,b]$ such that $\psi'(t) > 0$, for all $t \in [0,b]$, $a_1 \ge a_2 > 0$, $\delta_\psi = \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right) z(t)$, and $f : [0,b] \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function. Existence and uniqueness of solutions using Schaefer's and Banach's fixed point theorem were proved.

Ali et al. [4] investigated the existence, uniqueness, and stability of the following fractional-order boundary value problem:

$$\begin{cases} {}^C D^{\alpha}_{0^+} z(t) = f(t, z(t), z(\chi t), {}^C D^{\alpha}_{0^+} z(t)), \ t \in [0, T], \ 2 < \alpha \le 3, \ 0 < \zeta < 1 \\ z(0) = -z(T), \ {}^C D^{p}_{0^+} z(0) = -{}^C D^{p}_{0^+} z(T), \ {}^C D^{q}_{0^+} z(0) = -{}^C D^{q}_{0^+} z(T), \end{cases} \tag{4}$$

where ${}^C D^{\alpha}_{0^+}(\cdot), \ {}^C D^{p}_{0^+}(\cdot), \ {}^C D^{q}_{0^+}(\cdot)$ are, respectively, Caputo fractional derivatives of order $(2 < \alpha \le 3), \ (0 < p \le 1)$ and $(1 < q \le 2)$, and $f \in C([0, T], \mathbb{R}^3, \mathbb{R})$.

Motivated by the above recent development, we aim to investigate the existence and uniqueness of solutions of the following nonlinear fractional-order differential equation of the form:

$$\begin{cases} {}^C\mathbb{D}^{\gamma;\psi}_{a^+} u(t) = f(t, u(t)), \qquad t \in J = [a, b], \ 1 < \gamma \le 2, \ 0 \le a < b, \\ \alpha u(a) + \beta \delta_{\psi} u(a) = \mathbb{I}^{\sigma;\psi}_{a^+} z_1(\theta_1, u(\theta_1)), \ \ \alpha u(b) + \beta \delta_{\psi} u(b) = \mathbb{I}^{\rho;\psi}_{a^+} z_2(\theta_2, u(\theta_2)), \end{cases} \tag{5}$$

where ${}^C\mathbb{D}^{\gamma;\psi}_{a+}$ is the generalized Caputo fractional derivative of order γ, $\mathbb{I}^{\sigma;\psi}_{a^+}, \ \mathbb{I}^{\rho;\psi}_{a^+}$ are generalized fractional integral of order σ and ρ, respectively, with respect to another function ψ such that $\psi(t) \ge 0$ and $\psi'(t) > 0$ for all $t \in [a, b], \ f, z_1, z_2 : J \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, $\theta_1, \theta_2 \in \mathbb{R}$ satisfying $\theta_1, \theta_2 \in (a, b)$ and $\delta_{\psi} u(t) = \frac{1}{\psi'(t)} \frac{d}{dt} u(t)$.

2 Preliminaries and Theoretical Results

This part reviewed various fundamental definitions and lemmas related to fractional operators.

Let $X = C([J, \mathbb{R}])$ be a Banach space equipped with the norm defined by

$$\|u\|_{\mathbb{X}} = \sup\{|u(t)| : t \in J\}.$$

The space of all absolutely continuous real valued function on J is denoted by $AC([J, \mathbb{R}])$. So, we define the space $AC^n_{\psi}([\mathcal{J}, \mathbb{R}])$ by

$$AC^n_{\Phi}([J, \mathbb{R}]) = \left\{ z : J \to \mathbb{R}; \ (\delta^{n-1}_{\psi} z)(t) \in AC([J, \mathbb{R}]), \ \delta_{\psi} = \frac{1}{\psi'(t)} \frac{d}{dt} \right\},$$

with the norm defined by

$$\|z\|_{AC^n_{\Phi}} = \sum_{k=0}^{n-1} \|\delta^k_{\psi} z\|_X,$$

where $\psi \in C^n([J, \mathbb{R}]), \ \psi'(t) > 0$ on J and $\delta^k_{\psi} = \underbrace{\delta_{\psi} \delta_{\psi} \cdots \delta_{\psi}}_{\text{k-times}}$.

Definition 1 ([12]) Let $z \in L^1[a,b]$ be a function. The Riemann–Liouville fractional integral of order r is given by

$$\mathbb{I}_a^{\gamma} z(t) = \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} z(s)ds, \quad \gamma > 0,\ t > a \geq 0, \tag{6}$$

provided the right-hand side of (6) exists.

Definition 2 ([12]) Let $n \in \mathbb{N}, r, t, a \in \mathbb{R}_+$ and the function $z \in \mathcal{C}^n[a,b]$. The Caputo's fractional derivative of order r is defined by

$$^{C}\mathbb{D}_a^{\gamma} z(t) = \begin{cases} \frac{1}{\Gamma(n-\gamma)} \int_a^t (t-s)^{n-\gamma-1} z^n(s)ds, & \gamma > 0,\ t > a \geq 0, \\ \frac{d^n}{dt^n} z(t), & \gamma = n, \end{cases} \tag{7}$$

where the right-hand side of (2) is point-wise defined on (a,∞).

Definition 3 ([12]) Suppose $(a,b] \subset \mathbb{R}_+$ be a finite or infinite interval. Let $f \in L^1[a,b]$ and $\psi(t) > 0$ be monotone function on $(a,b]$ such that $\psi'(t) \in C([(a,b), \mathbb{R}])$. The ψ-Riemann–Liouville fractional integral is defined by

$$(\mathbb{I}_{0^+}^{\gamma;\psi} z)(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\gamma-1} z(s)ds, \quad \gamma > 0,\ t > 0. \tag{8}$$

Definition 4 ([5, 11]) Let z, ψ be two functions such that $z \in AC_{\psi}^n([J,\mathbb{R}]),\ \psi \in C^n([J,\mathbb{R}]),\ \psi(t) > 0$, and $\psi'(t) \neq 0$ for all $t \in J$. The fractional operator

$$^{C}\mathbb{D}_{a^+}^{\gamma;\psi} z(t) = \begin{cases} \mathbb{I}_{a^+}^{(n-\gamma);\psi}(\delta_{\psi}^n z)(t), & \gamma > 0,\ n = [\gamma]+1,\ n \in \mathbb{N}, \\ (\delta_{\psi}^n z)(t), & \gamma = n \in \mathbb{N}, \end{cases} \tag{9}$$

is referred to left-sided ψ-Caputo fractional derivative of a function z of order γ with respect to another function ψ.

Lemma 1 ([5]) *Given* $z(t) = (\psi(t)-\psi(a))^k$ *and* $\gamma > 0$. *Thus,*

$$^{C}\mathbb{D}_{a^+}^{\gamma;\psi} z(t) = \begin{cases} \frac{\Gamma(k)}{\Gamma(k-\gamma)}(\psi(t)-\psi(a))^{k-\gamma-1}, & k \in \mathbb{R}, k > n, \\ \frac{k!}{\Gamma(k+1-\gamma)}(\psi(t)-\psi(a))^{k-\gamma}, & n \leq k \in \mathbb{N}, \\ 0, & n > k \in \mathbb{N} \cup \{0\}. \end{cases} \tag{10}$$

Lemma 2 ([11]) *Suppose* $z \in AC_{\psi}^n([J,\mathbb{R}])$ *and* $\gamma > 0$. *Then*

$$\mathbb{I}_{a^+}^{\gamma;\psi}\, {}^{C}\mathbb{D}_{a^+}^{\gamma;\psi} z(t) = z(t) - \sum_{k=0}^{n-1} \frac{(\psi(t)-\psi(a))^k}{k!}(\delta_{\psi}^n z)(0),$$

for all $t \in \mathcal{J}$. *Also, for* $2 < \gamma \leq 3$ *gives*

$$\mathbb{I}_{a^+}^{\gamma;\psi}\,{}^C\mathbb{D}_{a^+}^{\gamma;\psi} z(t) = z(t) - c_0 - c_1(\psi(t) - \psi(a)) - c_2(\psi(t) - \psi(a))^2,$$

where c_0, c_1, c_2 *are constants on* $\mathbb{R}$.

3 Main Results

In this part, we use the Green function to convert the proposed problem (5) into an analogous integral equation. Moreover, utilizing the techniques of Schaefer's and Banach's fixed point theorems, the existence and uniqueness of solutions to problem (5) were established.

The following lemma establishes the relationship between problem (5) and the mixed-type Volterra integral equation.

Lemma 3 *Let* $1 < \gamma \leq 2$ *and* $\sigma, \rho \geq 0$. *Suppose* $g, h_1, h_2 : J \to \mathbb{R}$ *then the equivalent integral equation of the following problem:*

$$\begin{cases} {}^C\mathbb{D}_{a^+}^{\gamma;\psi} u(t) = g(t), \qquad t \in \mathcal{I} = [a, b],\ b > a \geq 0, \\ \alpha u(a) + \beta \delta_\psi u(a) = \mathbb{I}_{a^+}^{\sigma;\psi} h_1(\theta_1), \ \ \alpha u(b) + \beta \delta_\psi u(b) = \mathbb{I}_{a^+}^{\rho;\psi} h_2(\theta_2), \end{cases} \tag{11}$$

is given by

$$\begin{aligned} u(t) = &\int_a^b \mathbb{G}(t, s)\psi'(s) g(s) ds \\ &+ \frac{1}{\alpha^2(\psi(b) - \psi(a))}\Big[\big(\beta + \alpha((\psi(b) - \psi(a)) - (\psi(t) - \psi(a)))\big)\mathbb{I}_{a^+}^{\sigma;\psi} h_1(\theta_1) \\ &+ (\alpha(\psi(t) - \psi(a)) + \beta)\mathbb{I}_{a^+}^{\rho;\psi} h_2(\theta_2)\Big], \end{aligned} \tag{12}$$

where

$$\mathbb{G}(t, s) = \begin{cases} \frac{\alpha(\psi(t)-\psi(s))^{\gamma-1} + (\beta - \alpha(\psi(b)-\psi(a)))(\psi(b)-\psi(s))^{\gamma-1}}{\alpha(\psi(b)-\psi(a))\Gamma(\gamma)} \\ \quad + \frac{\beta(\beta-\alpha(\psi(t)-\psi(a)))(\psi(b)-\psi(s))^{\gamma-2}}{\alpha^2(\psi(b)-\psi(a))\Gamma(\gamma-1)}, \ a \leq s \leq t \leq b; \\ \frac{(\beta-\alpha(\psi(b)-\psi(a)))(\psi(b)-\psi(s))^{\gamma-1}}{\alpha(\psi(b)-\psi(a))\Gamma(\gamma)} + \frac{\beta(\beta-\alpha(\psi(t)-\psi(a)))(\psi(b)-\psi(s))^{\gamma-2}}{\alpha^2(\psi(b)-\psi(a))\Gamma(\gamma-1)}, \\ a \leq t \leq s \leq b. \end{cases} \tag{13}$$

Proof Applying the operator $\mathbb{I}_{a^+}^{\gamma;\psi}$ to both sides of Eq. (11) and making use of Lemma 2 yield

$$u(t) = \mathbb{I}_{a^+}^{\gamma;\psi} g(t) - c_0 - c_1(\psi(t) - \psi(a)), \tag{14}$$

such that $c_1, c_1 \in \mathbb{R}$ are constants. Taking the δ_ψ-derivative of (14) gives

$$\delta_\psi u(t) = \mathbb{I}_{a^+}^{\gamma-1;\psi} g(t) - c_1. \tag{15}$$

Substituting the boundary conditions $\alpha u(a) + \beta\delta_\psi u(a) = \mathbb{I}_{a^+}^{\sigma;\psi} h_1(\theta_1)$ and $\alpha u(b) + \beta\delta_\psi u(b) = \mathbb{I}_{a^+}^{\rho;\psi} h_2(\theta_2)$, in Eqs. (14) and (15), yields

$$\alpha c_0 + \beta c_1 = -\mathbb{I}_{a^+}^{\sigma;\psi} h_1(\theta_1) \tag{16}$$

and

$$\alpha c_0 + c_1(\alpha(\psi(b) - \psi(a)) + \beta) = \alpha\mathbb{I}_{a^+}^{\gamma;\psi} g(b) + \mathbb{I}_{a^+}^{\gamma-1;\psi} g(b) - \mathbb{I}_{a^+}^{\rho;\psi} h_2(\theta_2). \tag{17}$$

Upon simplification of Eqs. (16) and (17), we get

$$\begin{aligned} c_0 =& \frac{1}{\alpha^2(\psi(b) - \psi(a))}\left[\beta\mathbb{I}_{a^+}^{\rho;\psi} h_2(\theta_2) - (\beta + \alpha(\psi(b) - \psi(a)))\mathbb{I}_{a^+}^{\sigma;\psi} h_1(\theta_1)\right] \\ & - \frac{\beta}{\alpha(\psi(b) - \psi(a))}\mathbb{I}_{a^+}^{\gamma;\psi} g(b) - \frac{\beta^2}{\alpha^2(\psi(b) - \psi(a))}\mathbb{I}_{a^+}^{\gamma-1;\psi} g(b), \end{aligned} \tag{18}$$

and

$$\begin{aligned} c_1 =& \frac{1}{\alpha(\psi(b) - \psi(a))}\left[\mathbb{I}_{a^+}^{\sigma;\psi} h_1(\theta_1) - \mathbb{I}_{a^+}^{\rho;\psi} h_2(\theta_2)\right] + \frac{1}{(\psi(b) - \psi(a))}\mathbb{I}_{a^+}^{\gamma;\psi} g(b) \\ & + \frac{\beta}{\alpha(\psi(b) - \psi(a))}\mathbb{I}_{a^+}^{\gamma-1;\psi} g(b). \end{aligned} \tag{19}$$

Inserting Eqs. (18) and (19), in Eq. (14), we obtain

$$\begin{aligned} u(t) =& \int_a^t \left[\frac{\alpha(\psi(t) - \psi(s))^{\gamma-1} + (\beta - \alpha(\psi(t) - \psi(a)))(\psi(b) - \psi(s))^{\gamma-1}}{\alpha(\psi(b) - \psi(a))\Gamma(\gamma)}\right. \\ & + \left.\frac{(\beta - \alpha(\psi(t) - \psi(a)))(\psi(b) - \psi(s))^{\gamma-2}}{\alpha^2(\psi(b) - \psi(a))\Gamma(\gamma - 1)}\right]\psi'(s)g(s)ds \\ & + \int_t^b \left[\frac{(\beta - \alpha(\psi(t) - \psi(a)))(\psi(b) - \psi(s))^{\gamma-1}}{\alpha(\psi(b) - \psi(a))\Gamma(\gamma)}\right. \\ & + \left.\frac{(\beta - \alpha(\psi(t) - \psi(a)))(\psi(b) - \psi(s))^{\gamma-2}}{\alpha^2(\psi(b) - \psi(a))\Gamma(\gamma - 1)}\right]\psi'(s)g(s)ds \\ & + \frac{1}{\alpha^2(\psi(b) - \psi(a))}\Big[\big(\beta + \alpha((\psi(b) - \psi(a)) - (\psi(t) - \psi(a)))\big)\mathbb{I}_{a^+}^{\sigma;\psi} h_1(\theta_1) \\ & + (\alpha(\psi(t) - \psi(a)) - \beta)\mathbb{I}_{a^+}^{\rho;\psi} h_2(\theta_2)\Big] \\ =& \int_a^b \mathbb{G}(t, s)\psi'(s)g(s)ds \\ & + \frac{1}{\alpha^2(\psi(b) - \psi(a))}\Big[\big(\beta + \alpha((\psi(b) - \psi(a)) - (\psi(t) - \psi(a)))\big)\mathbb{I}_{a^+}^{\sigma;\psi} h_1(\theta_1) \\ & + (\alpha(\psi(t) - \psi(a)) - \beta)\mathbb{I}_{a^+}^{\rho;\psi} h_2(\theta_2)\Big]. \end{aligned} \tag{20}$$

Hence, the proof is completed.

Lemma 4 *The function $\mathbb{G}(t, s)$ in (13) satisfies the following relations:*
$(\mathcal{C}_1)$ $\mathbb{G}(t, s)$ is continuous over J;

$(\mathcal{C}_2)$ $\sup_{t\in J}\int_a^b |\mathbb{G}(t,s)|\psi'(s)ds \leq \Delta,$
where

$$\Delta = \frac{[(\alpha+\beta)+\alpha(\psi(b)-\psi(a))]}{\alpha(\psi(b)-\psi(a))\Gamma(\gamma+1)}(\psi(b)-\psi(a))^{\gamma} + \frac{[\beta^2+\alpha\beta(\psi(b)-\psi(a))]}{\alpha^2(\psi(b)-\psi(a))^2\Gamma(\gamma)} \times (\psi(b)-\psi(a))^{\gamma}. \tag{21}$$

Proof $(\mathcal{C}_1)$ follows trivially. Thus, to prove $(\mathcal{C}_2)$, we have

$$\begin{aligned}
&\sup_{t\in J}\int_a^b |\mathbb{G}(t,s)|\psi'(s)ds = \\
&\sup_{t\in J}\Bigg(\frac{1}{(\psi(b)-\psi(a))\Gamma(\gamma)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\gamma-1}ds \\
&\quad + \frac{|\beta-\alpha(\psi(b)-\psi(a))|}{\alpha(\psi(b)-\psi(a))\Gamma(\gamma)}\int_a^b \psi'(s)(\psi(b)-\psi(s))^{\gamma-1}ds \\
&\quad + \frac{|(\beta-(\psi(t)-\psi(a)))|}{\alpha^2(\psi(T)-\psi(0))\Gamma(\gamma-1)}\int_a^b \psi'(s)(\psi(b)-\psi(s))^{\gamma-2}ds \\
&\leq \max_{t\in J}\Bigg(\frac{(\psi(t)-\psi(a))^{\gamma}}{(\psi(b)-\psi(a))\Gamma(\gamma+1)} + \frac{|\beta-\alpha(\psi(b)-\psi(a))|}{\alpha(\psi(b)-\psi(a))\Gamma(\gamma+1)}(\psi(b)-\psi(a))^{\gamma} \\
&\quad + \frac{|(\beta-(\psi(t)-\psi(a)))|}{\alpha^2(\psi(b)-\psi(a))\Gamma(\gamma)}(\psi(b)-\psi(a))^{\gamma-1}\Bigg) \\
&\leq \frac{[(\alpha+\beta)+\alpha(\psi(b)-\psi(a))]}{\alpha(\psi(b)-\psi(a))\Gamma(\gamma+1)}(\psi(b)-\psi(a))^{\gamma} + \frac{[\beta^2+\alpha\beta(\psi(b)-\psi(a))]}{\alpha^2(\psi(b)-\psi(a))^2\Gamma(\gamma)} \\
&\quad \times (\psi(b)-\psi(a))^{\gamma}.
\end{aligned}$$

Hence,

$$\sup_{t\in[a,b]}\int_a^b \mathbb{G}(t,s)\psi'(s)ds \leq \Delta.$$

We denote the following notations:

$$\Delta_1 = \frac{\beta}{\alpha^2\Gamma(\sigma+1)(\psi(b)-\psi(a))}(\psi(\theta_1)-\psi(a))^{\sigma}, \tag{22}$$

and

$$\Delta_2 = \frac{(\alpha(\psi(b)-\psi(a))-\beta)}{\alpha^2\Gamma(\rho+1)(\psi(b)-\psi(a))}(\psi(\theta_2)-\psi(a))^{\rho}, \tag{23}$$

for the sake of simplicity.

3.1 Existence Result

We state and prove the existence of at least one solution of problem (5) via the concepts of Schaefer's fixed point theorem. Before we proceed, the following hypotheses are needed.
$(\mathcal{C}_3)$ Suppose there exist constants $K_1, K_2, K_3, M_1, M_2 > 0$ such that $\|f(t,w) - f(t,\bar{w})\| \leq K_1\|w-\bar{w}\|$ for all $t \in J,\ w, \bar{w} \in X$.
$(\mathcal{C}_4)$ $\|z_1(t,w) - z_1(t,\bar{w})\| \leq K_2\|w-\bar{w}\|,\quad \|z_2(t,w) - z_2(t,\bar{w})\| \leq K_3\|w-\bar{w}\|$ with $\|z_1(t,w)\| \leq M_1,\ \|z_2(t,w)\| \leq M_2$ for all $t \in J,\ w, \bar{w} \in X$.

Theorem 1 *Suppose that* $(\mathcal{C}_3) - (\mathcal{C}_4)$ *holds and there exists* $m \in L^{\infty}([a,b], \mathbb{R}^+)$ *such that*

$$\|f(t,u(t))\| \leq m(t), \quad \textit{for all } (t,u) \in [a,b] \times X.$$

Then, problem (5) has at least one solution on $[a,b]$ *provided that*

$$\begin{aligned}
&\frac{\beta + \alpha\left(\psi(b) - \psi(a)\right)}{\alpha^2\left(\psi(b) - \psi(a)\right)}\Big(K_2\frac{\left(\psi(b) - \psi(a)\right)^{\sigma}}{\Gamma(\sigma+1)} + K_3\frac{\left(\psi(b) - \psi(a)\right)^{\rho}}{\Gamma(\rho+1)}\Big)\\
&+ K_1\Big(\frac{\beta + \alpha\left(\psi(b) - \psi(a)\right)}{\alpha\left(\psi(b) - \psi(a)\right)\Gamma(\gamma+1)}\left(\psi(b) - \psi(a)\right)^{\gamma}\\
&+ \frac{\beta^2 + \alpha\beta\left(\psi(b) - \psi(a)\right)}{\alpha^2\left(\psi(b) - \psi(a)\right)\Gamma(\gamma)}\left(\psi(b) - \psi(a)\right)^{\gamma-1}\Big) < 1.
\end{aligned}$$

Proof For any $R > 0$, we let

$$B_R = \{u \in C([a,b], X) : \|u\| \leq R\}$$

with $R \geq \dfrac{\left(\beta + \alpha\left(\psi(b) - \psi(a)\right)\right)}{\alpha^2\left(\psi(b) - \psi(a)\right)}\left(M_1 + M_2\right) + M\Big(\dfrac{\beta + 2\alpha\left(\psi(b) - \psi(a)\right)}{\alpha\left(\psi(b) - \psi(a)\right)\Gamma(\gamma+1)}$ $\left(\psi(b) - \psi(a)\right)^{\gamma} + \dfrac{\beta^2 + \alpha\beta\left(\psi(b) - \psi(a)\right)}{\alpha^2\left(\psi(b) - \psi(a)\right)\Gamma(\gamma)}\left(\psi(b) - \psi(a)\right)^{\gamma-1}\Big)$.

We define the operators Φ and Ψ on B_R as

$$\begin{aligned}
\left(\Phi u\right)(t) &= \frac{1}{\Gamma(\gamma)}\int_a^t \left(\psi(t) - \psi(s)\right)^{\gamma-1}\psi'(s) f(s,u(s))ds\\
\left(\Psi u\right)(t) &= \frac{1}{\alpha^2\left(\psi(b) - \psi(a)\right)}\Big[\left(\beta + \alpha\left(\psi(b) - \psi(t)\right)\right)\mathbb{I}_{a^+}^{\sigma;\psi} z_1(\theta_1, u(\theta_1))\\
&\quad + \left(\alpha\left(\psi(t) - \psi(a)\right) + \beta\right)\mathbb{I}_{a^+}^{\rho;\psi} z_2(\theta_2, u(\theta_2))\Big]\\
&\quad + \int_a^b\Big[\frac{\left(\beta - \alpha\left(\psi(t) - \psi(a)\right)\right)\left(\psi(b) - \psi(s)\right)^{\gamma-1}}{\alpha\left(\psi(b) - \psi(a)\right)\Gamma(\gamma)}\\
&\quad + \frac{\beta\left(\beta - \alpha\left(\psi(t) - \psi(a)\right)\right)\left(\psi(b) - \psi(s)\right)^{\gamma-2}}{\alpha^2\left(\psi(b) - \psi(a)\right)\Gamma(\gamma-1)}\Big]\psi'(s) f(s,u(s))ds.
\end{aligned}$$

For $u, v \in B_R$, we can find that

$$\begin{aligned}\|\Phi u + \Psi v\| \leq & \frac{(\beta + \alpha(\psi(b) - \psi(a)))}{\alpha^2(\psi(b) - \psi(a))}(M_1 + M_2) \\ & + M\Big(\frac{\beta + 2\alpha(\psi(b) - \psi(a))}{\alpha(\psi(b) - \psi(a))\,\Gamma(\gamma+1)}(\psi(b) - \psi(a))^{\gamma} \\ & + \frac{\beta^2 + \alpha\beta(\psi(b) - \psi(a))}{\alpha^2(\psi(b) - \psi(a))\,\Gamma(\gamma)}(\psi(b) - \psi(a))^{\gamma-1}\Big) \leq R\end{aligned}$$

where $M = \|m\|_{\infty}$. Then, we obtain $\Phi u + \Psi v \in B_R$.

Next, we will show that Ψ is a contraction mapping. For any $u, v \in B_R$ and for each $t \in [a, b]$, we have

$$\begin{aligned}& \|(\Psi u)(t) - (\Psi v)(t)\| \\ & \leq \frac{1}{\alpha^2(\psi(b) - \psi(a))}\Big[(\beta + \alpha(\psi(b) - \psi(t)))\,\|\mathbb{I}_{a^+}^{\sigma;\psi} z_1(\theta_1, u(\theta_1)) - \mathbb{I}_{a^+}^{\sigma;\psi} z_1(\theta_1, v(\theta_1))\| \\ & \quad + (\alpha(\psi(t) - \psi(a)) + \beta)\,\|\mathbb{I}_{a^+}^{\rho;\psi} z_2(\theta_2, u(\theta_2)) - \mathbb{I}_{a^+}^{\rho;\psi} z_2(\theta_2, v(\theta_2))\|\Big] \\ & \quad + \int_a^b \Big[\frac{(\beta + \alpha(\psi(t) - \psi(a)))(\psi(b) - \psi(s))^{\gamma-1}}{\alpha(\psi(b) - \psi(a))\,\Gamma(\gamma)} \\ & \quad + \frac{\beta(\beta + \alpha(\psi(t) - \psi(a)))(\psi(b) - \psi(s))^{\gamma-2}}{\alpha^2(\psi(b) - \psi(a))\,\Gamma(\gamma-1)}\Big]\psi'(s)\|f(s, u(s)) - f(s, v(s))\|ds \\ & \leq \frac{\beta + \alpha(\psi(b) - \psi(a))}{\alpha^2(\psi(b) - \psi(a))}\Big(K_2\left(\mathbb{I}_{a^+}^{\sigma;\psi}\|u - v\|\right) + K_3\left(\mathbb{I}_{a^+}^{\rho;\psi}\|u - v\|\right)\Big) \\ & \quad + K_1\Big(\frac{\beta + \alpha(\psi(b) - \psi(a))}{\alpha(\psi(b) - \psi(a))\,\Gamma(\gamma+1)}(\psi(b) - \psi(a))^{\gamma} \\ & \quad + \frac{\beta^2 + \alpha\beta(\psi(b) - \psi(a))}{\alpha^2(\psi(b) - \psi(a))\,\Gamma(\gamma)}(\psi(b) - \psi(a))^{\gamma-1}\Big)\|u - v\| \\ & \leq \frac{\beta + \alpha(\psi(b) - \psi(a))}{\alpha^2(\psi(b) - \psi(a))}\Big(K_2\frac{(\psi(b) - \psi(a))^{\sigma}}{\Gamma(\sigma+1)} + K_3\frac{(\psi(b) - \psi(a))^{\rho}}{\Gamma(\rho+1)}\Big)\|u - v\| \\ & \quad + K_1\Big(\frac{\beta + \alpha(\psi(b) - \psi(a))}{\alpha(\psi(b) - \psi(a))\,\Gamma(\gamma+1)}(\psi(b) - \psi(a))^{\gamma} \\ & \quad + \frac{\beta^2 + \alpha\beta(\psi(b) - \psi(a))}{\alpha^2(\psi(b) - \psi(a))\,\Gamma(\gamma)}(\psi(b) - \psi(a))^{\gamma-1}\Big)\|u - v\| \\ & = \Lambda_{\alpha,\beta,\gamma,\rho,K_1,K_2,K_3}\|u - v\|,\end{aligned}$$

where

$$\begin{aligned}& \Lambda_{\alpha,\beta,\gamma,\rho,K_1,K_2,K_3} \\ & = \frac{\beta + \alpha(\psi(b) - \psi(a))}{\alpha^2(\psi(b) - \psi(a))}\Big(K_2\frac{(\psi(b) - \psi(a))^{\sigma}}{\Gamma(\sigma+1)} + K_3\frac{(\psi(b) - \psi(a))^{\rho}}{\Gamma(\rho+1)}\Big) \\ & \quad + K_1\Big(\frac{\beta + \alpha(\psi(b) - \psi(a))}{\alpha(\psi(b) - \psi(a))\,\Gamma(\gamma+1)}(\psi(b) - \psi(a))^{\gamma} \\ & \quad + \frac{\beta^2 + \alpha\beta(\psi(b) - \psi(a))}{\alpha^2(\psi(b) - \psi(a))\,\Gamma(\gamma)}(\psi(b) - \psi(a))^{\gamma-1}\Big).\end{aligned}$$

By the continuity of f, we obtain that the operator Φ is also continuous. Moreover, the operator Φ is uniformly bounded on B_r with

$$\|\Phi u\| \leq \frac{M\,(\psi(b)-\psi(a))^{\gamma}}{\Gamma(\gamma+1)}.$$

Now, we prove the compactness of the operator Φ. For each $a \leq t_1 \leq t_2 \leq b$, we have

$$\begin{aligned}
\|(\Phi u)(t_2)-(\Phi u)(t_1)\| &\leq \frac{1}{\Gamma(\gamma)}\bigg\| \int_a^{t_2} (\psi(t_2)-\psi(s))^{\gamma-1}\,\psi'(s) f(s,u(s))ds \\
&\quad - \int_a^{t_1} (\psi(t_1)-\psi(s))^{\gamma-1}\,\psi'(s) f(s,u(s))ds \bigg\| \\
&\leq \frac{1}{\Gamma(\gamma)}\bigg\| \int_a^{t_1} \left[(\psi(t_2)-\psi(s))^{\gamma-1}-(\psi(t_1)-\psi(s))^{\gamma-1}\right]\psi'(s) f(s,u(s))ds \\
&\quad + \int_{t_1}^{t_2} (\psi(t_2)-\psi(s))^{\gamma-1}\,\psi'(s) f(s,u(s))ds \bigg\| \\
&\leq \frac{M}{\Gamma(\gamma)}\bigg| \int_a^{t_1} \left[(\psi(t_2)-\psi(s))^{\gamma-1}-(\psi(t_1)-\psi(s))^{\gamma-1}\right]\psi'(s) ds \\
&\quad + \int_{t_1}^{t_2} (\psi(t_2)-\psi(s))^{\gamma-1}\,\psi'(s) ds \bigg| \\
&\leq \frac{M}{\Gamma(\gamma+1)}\left(|(\psi(t_2)-\psi(a))^{\gamma}-(\psi(t_2)-\psi(t_1))^{\gamma}-(\psi(t_1)-\psi(a))^{\gamma}|\right. \\
&\quad \left. + |(\psi(t_2)-\psi(t_1))^{\gamma}|\right) \\
&\leq \frac{M}{\Gamma(\gamma+1)}\left((\psi(t_2)-\psi(a))^{\gamma}-(\psi(t_1)-\psi(a))^{\gamma}+2\,(\psi(t_2)-\psi(t_1))^{\gamma}\right)
\end{aligned}$$

which is independent of u. Thus, Φ is relatively compact on B_R. Hence, by the Arzela–Ascoli Theorem, Φ is compact on B_R. Thus, as a consequences of Schaefer's fixed point theorem, there exists at least one solution of problem (5) on $[a, b]$

3.2 *Uniqueness Result*

In this subsection, we state and prove the uniqueness results of the problem (5) via the concepts of the Banach contraction principle.

Theorem 2 *Suppose that $f : J \times X \to X$ is jointly continuous and maps bounded subsets of $J \times X$ into relatively compact subsets of X and $z_1, z_1 : X \to X$ are continuous functions. Then, problem (5) has a unique solution on J provided that*

$$K_1\Delta + K_2\Delta_1 + K_3\Delta_3 < 1,$$

where $\Delta,\ \Delta_1$, *and* Δ_2 *are defined in (22), (23), and (21), respectively.*

Proof Reformulate problem (5) into a fixed point problem defined by

$$\begin{aligned}(\mathbb{P}u)(t) = &\int_a^b \mathbb{G}(t,s)\psi'(s)f(s,u(s))ds \\ &+ \frac{1}{\alpha^2(\psi(b)-\psi(a))}\Big[\big(\beta + \alpha((\psi(b)-\psi(a)) - (\psi(t)-\psi(a)))\big) \\ &\times \mathbb{I}_{a^+}^{\sigma;\psi} z_1(\theta_1, u(\theta_1)) + (\alpha(\psi(t)-\psi(a)) - \beta)\mathbb{I}_{a^+}^{\rho;\psi} z_2(\theta_2, u(\theta_2))\Big],\end{aligned} \tag{24}$$

where the operator $\mathbb{P} : X \longrightarrow X$. It is well-known that the fixed points of the operator $\mathbb{P}$ are just the solution of the proposed problem (5).

Now, let $\sup\limits_{t\in[a,b]} \|f(t,0)\| = M_3$ and choosing $r \geq M_3\Delta + M_1\Delta_1 + M_2\Delta_2$. It is enough to show that $\mathbb{P}B_k \subset B_k$, where $B_k = \{u \in C : \|u\| \leq k\}$. For $u \in B_k$ gives

$$\begin{aligned}\|(\mathbb{P}u)(t)\| \leq &\int_a^b \mathbb{G}(t,s)\psi'(s)\|f(s,u(s))\|ds \\ &+ \frac{1}{\alpha^2(\psi(b)-\psi(a))}\Big[\big(\beta + \alpha((\psi(b)-\psi(a)) - (\psi(t)-\psi(a)))\big) \\ &+ \times\mathbb{I}_{a^+}^{\sigma;\psi}\|z_1(\theta_1, u(\theta_1))\|(\alpha(\psi(t)-\psi(a)) - \beta)\mathbb{I}_{a^+}^{\rho;\psi}\|z_2(\theta_2, u(\theta_2))\|\Big] \\ \leq &\int_a^b \mathbb{G}(t,s)\psi'(s)[\|f(s,u(s)) - f(s,0)\| + \|f(s,0)\|]ds \\ &+ \frac{1}{\alpha^2(\psi(b)-\psi(a))}\left[\frac{\beta M_1}{\Gamma(\sigma)}\int_a^{\theta_1}(\psi(\theta_1)-\psi(s))^{\sigma-1}\psi'(s)ds\right. \\ &\left.+ \frac{(\alpha(\psi(b)-\psi(a)) - \beta)M_2}{\Gamma(\rho)}\int_a^{\theta_2}(\psi(\theta_2)-\psi(s))^{\rho-1}\psi'(s)ds\right] \\ \leq &(K_1k + M_3)\Delta + M_1\Delta_1 + M_2\Delta_2 \\ \leq &k.\end{aligned} \tag{25}$$

Moreover, for each $t \in [a,b]$ and any $u_1, u_2 \in B_k$ yields

$$
\begin{aligned}
\|(\mathbb{P}u_1)(t)-(\mathbb{P}u_2)(t)\| &\le \int_a^b \mathbb{G}(t,s)\psi'(s)\|f(s,u_1(s))-f(s,u_2(s))\|ds \\
&\quad + \frac{1}{\alpha^2(\psi(b)-\psi(a))}\Big[\big(\beta+\alpha((\psi(b)-\psi(a))-(\psi(t)-\psi(a)))\big) \\
&\quad \times \mathbb{I}_{a^+}^{\sigma;\psi}\|z_1(\theta_1,u_1(\theta_1))-z_1(\theta_1,u_2(\theta_1))\| \\
&\quad + (\alpha(\psi(t)-\psi(a))-\beta)\mathbb{I}_{a^+}^{\rho;\psi}\|z_2(\theta_2,u_1(\theta_2))-z_2(\theta_2,u_2(\theta_2))\|\Big] \\
&\le K_1\int_a^b \mathbb{G}(t,s)\psi'(s)ds\|u_1-u_2\| \\
&\quad + \frac{\beta K_2}{\alpha^2(\psi(b)-\psi(a))}\frac{1}{\Gamma(\sigma)}\int_a^{\theta_1}(\psi(\theta_1)-\psi(s))^{\sigma-1}\psi'(s)ds\|u_1-u_2\| \\
&\quad + \frac{(\alpha(\psi(b)-\psi(a))-\beta)K_3}{\alpha^2(\psi(b)-\psi(a))\Gamma(\rho)}\int_a^{\theta_2}(\psi(\theta_2)-\psi(s))^{\rho-1}\psi'(s)ds\|u_1-u_2\| \\
&\le (K_1\Delta+K_2\Delta_1+K_3\Delta_2)\|u_1-u_2\|.
\end{aligned}
\tag{26}
$$

Thus, it follows that the operator $\mathbb{P}$ is a contraction, and hence, there exists a unique solution of the proposed problem (5).

4 Concluding Remarks

Nonlinear analysis is one of the most effective methodologies for analyzing applied problems. In this chapter, we develop nonlinear fractional differential equations with more general integral boundary conditions in the context of ψ-Caputo fractional derivatives. By constructing Green's functions, we derive an equivalent mixed-type Volterra integral equation for the proposed problem. Using Schaefer's and Banach's fixed point theorems, the existence and uniqueness of solutions to the proposed problems were examined. Moreover,

- If $\Phi(t)=t,\ \psi(t)=\ln t$, and $\psi(t)=\frac{t^\sigma}{\sigma}$, the proposed problem (5) reduces to Riemann–Liouville, Caputo, Caputo–Hadamard, and Caputo–Erdélyi–Kober fractional differential equations, respectively.

Therefore, the results obtained are new and generalized some of the existing results in the literature.

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On Well-posed Variational Problems Involving Multidimensional Integral Functionals

Savin Treanţă

Abstract In this chapter, based on the notions of monotonicity, pseudomonotonicity, and hemicontinuity associated with the considered path-independent curvilinear integral functional, we study the well-posedness and well-posedness in generalized sense for a new class of controlled variational inequality problems governed by second-order partial derivatives. To this aim, we introduce the approximating solution set and the concept of approximating sequence for the considered controlled variational inequality problem. Further, by using the aforementioned new mathematical tools, we formulate and prove some characterization results on well-posedness and well-posedness in generalized sense. Also, the theoretical elements included in the chapter are accompanied by some illustrative examples.

1 Introduction

As it is well-known, sometimes, it is very difficult to find out the solution associated with some optimization problems by using certain methods which may or may not ensure the exactness of the solutions. In this regard, the well-posedness of an optimization problem becomes important because this condition ensures the convergence of the sequence of approximate solutions obtained through iterative techniques. Over time, many researchers have studied this concept for unconstrained optimization problems (see Tykhonov [23]) and, moreover, they introduced various kinds of well-posedness, such as Levitin-Polyak well-posedness [10], and extended well-posedness (see Chen et al. [3]). The concept of Tykhonov well-posedness has also been extended to variational inequalities (see Huang et al. [6], Fang et al. [4], Virmani and Srivastava [25], Hu et al. [5]), and thereafter, to some other problems like fixed point problems, hemivariational inequality problems (see Ceng et al. [2], Wang et al. [26], Shu et al. [15]), equilibrium problems, complementary problems,

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and Nash equilibrium problems (see Lignola and Morgan [11]). Lin and Chuang [12] proposed the generalized well-posedness for variational disclusion problems, inclusion problems and the minimization problems involving variational disclusion problems, inclusion problems as constraints. Moreover, Lalita and Bhatia [9] studied the well-posedness and generalized well-posedness for parametric type quasi-variational inequality problems and minimization problems. Recently, Jayswal and Shalini [7] introduced the well-posedness for generalized mixed vector variational-like inequality and optimization problems including this inequality as a constraint. In the last few years, the multi-time variational inequality problem appeared as an interesting generalization of variational inequality (see Treanţă [19, 21, 22]). Also, the multi-dimensional optimization problems have been investigated, with remarkable results, by Treanţă [16–18, 20]. For other different but connected ideas on well-posedness, the reader is directed to Antonelli et al. [1], Sawano [14], and Lakzian and Munn [8].

In this chapter, motivated by the aforementioned research works, we study the well-posedness associated with a class of controlled variational inequality problems. More specifically, by considering the concepts of monotonicity, pseudomonotonicity and hemicontinuity for path-independent curvilinear integral functionals, and by defining the approximating solution set of the considered controlled variational inequality problem, we establish some characterization results on well-posedness. The main novelty elements of this chapter are given by the presence of the curvilinear integral functionals governed by second-order partial derivatives, and of the mathematical framework determined by infinite-dimensional function spaces (the former works are studied in the classical finite-dimensional spaces). Besides totally new elements mentioned above and thanks to the physical significance of the integral functionals (the path-independent curvilinear integrals represent the mechanical work performed by a variable force to move its point of application along a given piecewise smooth curve), this chapter represents a reference work for researchers in the field of abstract and applied mathematics.

The present chapter is structured as follows. In Sect. 2, we introduce the preliminary mathematical tools, namely, the notions of monotonicity, pseudomonotonicity, and hemicontinuity associated with a curvilinear integral functional, and an auxiliary lemma. The well-posedness of the problem under study is investigated in Sect. 3 by considering the approximating solution set of the considered class of controlled variational inequality problems. Concretely, we establish that well-posedness is characterized in the terms of existence and uniqueness of the solution. Also, in order to validate the theoretical developments included in this chapter, we also provide some illustrative examples. In Sect. 4, we conclude the chapter.

2 Preliminaries

In this chapter, we consider the following notations and mathematical tools: Ω is a compact domain in $\mathbb{R}^m$, $\Omega \ni s = (s^\mu)$, $\mu = \overline{1, m}$, is a multi-parameter of evolution, and $\Omega \supset \Xi$ is a piecewise smooth curve joining two different points

$s_1 = (s_1^1, \dots, s_1^m)$, $s_2 = (s_2^1, \dots, s_2^m)$ in Ω. Consider $\mathscr{A}$ is the space of all C^4-class *state* functions $a : \Omega \to \mathbb{R}^n$ and $a_\kappa := \dfrac{\partial a}{\partial t^\kappa}$, $a_{\alpha\beta} := \dfrac{\partial^2 a}{\partial t^\alpha \partial t^\beta}$ denote the *partial speed* and *partial acceleration*, respectively. Also, let $\mathscr{U}$ be the space of C^1-class *control* functions $u : \Omega \to \mathbb{R}^k$. Assume that $A \times U$ is a nonempty, closed and convex subset of $\mathscr{A} \times \mathscr{U}$, equipped with the inner product

$$\langle (a, u), (b, w) \rangle = \int_\Xi [a(s) \cdot b(s) + u(s) \cdot w(s)] ds^\mu$$

$$= \int_\Xi [\sum_{i=1}^n a^i(s) b^i(s) + \sum_{j=1}^k u^j(s) w^j(s)] ds^\mu$$

$$= \int_\Xi [\sum_{i=1}^n a^i(s) b^i(s) + \sum_{j=1}^k u^j(s) w^j(s)] ds^1 + \dots + [\sum_{i=1}^n a^i(s) b^i(s) + \sum_{j=1}^k u^j(s) w^j(s)] ds^m,$$

$$\forall (a, u), (b, w) \in \mathscr{A} \times \mathscr{U}$$

and the induced norm.

Let $J^2(\mathbb{R}^m, \mathbb{R}^n)$ be the jet bundle of second-order associated with $\mathbb{R}^m$ and $\mathbb{R}^n$. Assume that the following multi-time controlled second-order Lagrangians $f_\mu : J^2(\mathbb{R}^m, \mathbb{R}^n) \times \mathbb{R}^k \to \mathbb{R}$, $\mu = \overline{1, m}$, determine a controlled closed (complete integrable) Lagrange 1-form (see summation over the repeated indices, Einstein summation)

$$f_\mu(s, a(s), a_\kappa(s), a_{\alpha\beta}(s), u(s)) ds^\mu,$$

which generates the following controlled path-independent curvilinear integral functional

$$\mathbb{F} : \mathscr{A} \times \mathscr{U} \to \mathbb{R}, \quad \mathbb{F}(a, u) = \int_\Xi f_\mu \left(s, a(s), a_\kappa(s), a_{\alpha\beta}(s), u(s)\right) ds^\mu$$

$$= \int_\Xi f_1 \left(s, a(s), a_\kappa(s), a_{\alpha\beta}(s), u(s)\right) ds^1 + \dots + f_m \left(s, a(s), a_\kappa(s), a_{\alpha\beta}(s), u(s)\right) ds^m.$$

To formulate the problem under study, we shall introduce Saunders's multi-index notation (see Saunders [13]). A *multi-index* is an m-tuple I of natural numbers. The components of I are denoted $I(\alpha)$, where α is an ordinary index, $1 \le \alpha \le m$. The multi-index 1_α is defined by $1_\alpha(\alpha) = 1$, $1_\alpha(\beta) = 0$ for $\alpha \ne \beta$. The addition and the substraction of the multi-indexes are defined componentwise (although the result of a substraction might not be a multi-index): $(I \pm V)(\alpha) = I(\alpha) \pm V(\alpha)$. The length of a multi-index is $| I |= \sum_{\alpha=1}^{m} I(\alpha)$, and its factorial is $I! = \prod_{\alpha=1}^{m}(I(\alpha))!$. The number of distinct indices represented by $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, $\alpha_j \in \{1, 2, \ldots, m\}$, $j = \overline{1, k}$, is

$$n(\alpha_1, \alpha_2, \ldots, \alpha_k) = \frac{| 1_{\alpha_1} + 1_{\alpha_2} + \cdots + 1_{\alpha_k} |!}{(1_{\alpha_1} + 1_{\alpha_2} + \cdots + 1_{\alpha_k})!}.$$

By using the above mathematical tools, we introduce the following controlled variational inequality problem (for short, CVIP), formulated as find $(a, u) \in A \times U$ such that

$$\int_{\Xi} \left[\frac{\partial f_\mu}{\partial a}(\chi_{a,u}(s))(b(s) - a(s)) + \frac{\partial f_\mu}{\partial a_\kappa}(\chi_{a,u}(s))D_\kappa(b(s) - a(s))\right] ds^\mu \quad (CVIP)$$

$$+ \int_{\Xi} \left[\frac{1}{n(\alpha, \beta)} \frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{a,u}(s))D^2_{\alpha\beta}(b(s) - a(s))\right] ds^\mu$$

$$+ \int_{\Xi} \left[\frac{\partial f_\mu}{\partial u}(\chi_{a,u}(s))(w(s) - u(s))\right] ds^\mu \geq 0, \quad \forall (b, w) \in A \times U,$$

where $D_\kappa := \frac{\partial}{\partial s^\kappa}$ is the total derivative operator, $D^2_{\alpha\beta} := D_\alpha(D_\beta)$, and $(\chi_{a,u}(s)) := (s, a(s), a_\kappa(s), a_{\alpha\beta}(s), u(s))$.

Let Π be the set of all feasible solutions of (CVIP), that is,

$$\begin{aligned}\Pi = \Big\{(a, u) \in A \times U : \int_{\Xi} [&(b(s) - a(s))\frac{\partial f_\mu}{\partial a}(\chi_{a,u}(s)) \\ &+ D_\kappa(b(s) - a(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{a,u}(s)) \\ &+ \frac{1}{n(\alpha, \beta)} D^2_{\alpha\beta}(b(s) - a(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{a,u}(s)) \\ &+ (w(s) - u(s))\frac{\partial f_\mu}{\partial u}(\chi_{a,u}(s))] ds^\mu \geq 0, \\ &\forall (b, w) \in A \times U\Big\}.\end{aligned}$$

Assumption 1 In this chapter, we assume the following working hypothesis:

$$dG := D_\kappa \left[\frac{\partial f_\mu}{\partial a_\kappa}(a - b)\right] ds^\mu \qquad (H)$$

is an exact total differential satisfying $G(s_1) = G(s_2)$.

Further, in accordance to assumption (H) and by considering the notion of monotonicity associated with variational inequality problems, we introduce the concepts of monotonicity and pseudomonotonicity of the aforementioned curvilinear integral functional $\mathbb{F}$.

Definition 1 The curvilinear integral functional $\mathbb{F}$ is called *monotone* on $A \times U$ if the following inequality holds:

$$\int_{\Xi} \Big[(a(s) - b(s)) \left(\frac{\partial f_\mu}{\partial a}(\chi_{a,u}(s)) - \frac{\partial f_\mu}{\partial a}(\chi_{b,w}(s)) \right) + (u(s) - w(s)) \left(\frac{\partial f_\mu}{\partial u}(\chi_{a,u}(s)) - \frac{\partial f_\mu}{\partial u}(\chi_{b,w}(s)) \right) + D_\kappa (a(s) - b(s)) \left(\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{a,u}(s)) - \frac{\partial f_\mu}{\partial a_\kappa}(\chi_{b,w}(s)) \right) + \frac{1}{n(\alpha, \beta)} D^2_{\alpha\beta}(a(s) - b(s)) \left(\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{a,u}(s)) - \frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{b,w}(s)) \right) \Big] ds^\mu \geq 0,$$

$$\forall (a, u), (b, w) \in A \times U.$$

Example 1 Let $\mu \in \{1, 2\}$, $\Omega = [0, 1]^2$ and $\Omega \supset \Xi$ be a piecewise smooth curve joining the points $(0, 0)$, $(1, 1)$ in Ω. Consider

$$f_\mu(\chi_{a,u}(s)) ds^\mu = f_1(\chi_{a,u}(s)) ds^1 + f_2(\chi_{a,u}(s)) ds^2$$

$$= \left[\frac{\partial a}{\partial s^1} + u(s) \right] ds^1 + (e^{a(s)} - 1) ds^2.$$

Now, we show that the curvilinear integral functional $\int_{\Xi} f_\mu(\chi_{a,u}(s)) ds^\mu$ is monotone on $A \times U = C^4(\Omega, \mathbb{R}) \times C^1(\Omega, \mathbb{R})$. Indeed, we have

$$\int_{\Xi} \Big[(a(s) - b(s)) \left(\frac{\partial f_\mu}{\partial a}(\chi_{a,u}(s)) - \frac{\partial f_\mu}{\partial a}(\chi_{b,w}(s)) \right) + (u(s) - w(s)) \left(\frac{\partial f_\mu}{\partial u}(\chi_{a,u}(s)) - \frac{\partial f_\mu}{\partial u}(\chi_{b,w}(s)) \right) + D_\kappa (a(s) - b(s)) \left(\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{a,u}(s)) - \frac{\partial f_\mu}{\partial a_\kappa}(\chi_{b,w}(s)) \right) + \frac{1}{n(\alpha, \beta)} D^2_{\alpha\beta}(a(s) - b(s)) \left(\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{a,u}(s)) - \frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{b,w}(s)) \right) \Big] ds^\mu$$

$$= \int_{\Xi} (a(s) - b(s))(e^{a(s)} - e^{b(s)}) ds^2 \geq 0, \ \forall (a, u), (b, w) \in A \times U.$$

Definition 2 The curvilinear integral functional $\mathbb{F}$ is called *pseudomonotone* on $A \times U$ if the following implication holds:

$$\int_{\Xi} [(a(s) - b(s))\frac{\partial f_\mu}{\partial a}(\chi_{b,w}(s)) + (u(s) - w(s))\frac{\partial f_\mu}{\partial u}(\chi_{b,w}(s))$$

$$+D_\kappa(a(s) - b(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{b,w}(s))]ds^\mu$$

$$+\frac{1}{n(\alpha, \beta)}D^2_{\alpha\beta}(a(s) - b(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{b,w}(s))]ds^\mu \geq 0$$

$$\Rightarrow \int_{\Xi} [(a(s) - b(s))\frac{\partial f_\mu}{\partial a}(\chi_{a,u}(s)) + (u(s) - w(s))\frac{\partial f_\mu}{\partial u}(\chi_{a,u}(s))$$

$$+D_\kappa(a(s) - b(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{a,u}(s))]ds^\mu$$

$$+\frac{1}{n(\alpha, \beta)}D^2_{\alpha\beta}(a(s) - b(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{a,u}(s))]ds^\mu \geq 0,$$

$$\forall (a, u), (b, w) \in A \times U.$$

In the following, we present an example of curvilinear integral functional which is pseudomonotone but not monotone.

Example 2 Let $\mu \in \{1, 2\}$, $\Omega = [0, 1]^2$ and $\Omega \supset \Xi$ be a piecewise smooth curve joining the points $(0, 0)$, $(1, 1)$ in Ω. Consider

$$f_\mu(\chi_{a,u}(s))ds^\mu = f_1(\chi_{a,u}(s))ds^1 + f_2(\chi_{a,u}(s))ds^2$$

$$= \left[\frac{\partial a}{\partial s^1} + \sin u(s)\right]ds^1 + a(s)e^{a(s)}ds^2.$$

Now, we show that the curvilinear integral functional $\int_{\Xi} f_\mu(\chi_{a,u}(s))ds^\mu$ is pseudomonotone on $A \times U = C^4(\Omega, [-1, 1]) \times C^1(\Omega, [-1, 1])$. Indeed, we have

$$\int_{\Xi} [(a(s) - b(s))\frac{\partial f_\mu}{\partial a}(\chi_{b,w}(s)) + (u(s) - w(s))\frac{\partial f_\mu}{\partial u}(\chi_{b,w}(s))$$

$$+D_\kappa(a(s) - b(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{b,w}(s))$$

$$+\frac{1}{n(\alpha, \beta)}D^2_{\alpha\beta}(a(s) - b(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{b,w}(s))]ds^\mu$$

$$= \int_{\Xi} [(u(s) - w(s)) \cos w(s) + D_1(a(s) - b(s))]\, ds^1$$
$$+ (a(s) - b(s))(e^{b(s)} + b(s)e^{b(s)})ds^2 \geq 0, \quad \forall (a, u), (b, w) \in A \times U$$

$$\Rightarrow \int_{\Xi} [(a(s) - b(s)) \frac{\partial f_\mu}{\partial a}(\chi_{a,u}(s)) + (u(s) - w(s)) \frac{\partial f_\mu}{\partial u}(\chi_{a,u}(s))$$
$$+ D_\kappa(a(s) - b(s)) \frac{\partial f_\mu}{\partial a_\kappa}(\chi_{a,u}(s))$$
$$+ \frac{1}{n(\alpha, \beta)} D^2_{\alpha\beta}(a(s) - b(s)) \frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{a,u}(s))] ds^\mu$$

$$= \int_{\Xi} [(u(s) - w(s)) \cos u(s) + D_1(a(s) - b(s))]\, ds^1$$
$$+ (a(s) - b(s))(e^{a(s)} + a(s)e^{a(s)})ds^2 \geq 0, \quad \forall (a, u), (b, w) \in A \times U.$$

But it is not monotone on $A \times U$, because

$$\int_{\Xi} \Big[(a(s) - b(s)) \left(\frac{\partial f_\mu}{\partial a}(\chi_{a,u}(s)) - \frac{\partial f_\mu}{\partial a}(\chi_{b,w}(s)) \right)$$
$$+ (u(s) - w(s)) \left(\frac{\partial f_\mu}{\partial u}(\chi_{a,u}(s)) - \frac{\partial f_\mu}{\partial u}(\chi_{b,w}(s)) \right)$$
$$+ D_\kappa(a(s) - b(s)) \left(\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{a,u}(s)) - \frac{\partial f_\mu}{\partial a_\kappa}(\chi_{b,w}(s)) \right)$$
$$+ \frac{1}{n(\alpha, \beta)} D^2_{\alpha\beta}(a(s) - b(s)) \left(\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{a,u}(s)) - \frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{b,w}(s)) \right) \Big] ds^\mu$$
$$= \int_{\Xi} (u(s) - w(s))(\cos u(s) - \cos w(s)) ds^1$$
$$+ (a(s) - b(s))(a(s)e^{a(s)} + e^{a(s)} - b(s)e^{b(s)} - e^{b(s)})ds^2 \ngeq 0,$$
$$\forall (a, u), (b, w) \in A \times U.$$

Inspired by Usman and Khan [24], we introduce the following definition of hemicontinuity for the aforementioned curvilinear integral functional $\mathbb{F}$.

Definition 3 The curvilinear integral functional $\mathbb{F}$ is said to be *hemicontinuous* on $A \times U$ if, for $\forall (a, u), (b, w) \in A \times U$, the application

$$\lambda \to \left\langle ((a(s), u(s)) - (b(s), w(s)) \,, \left(\frac{\delta_\mu \mathbb{F}}{\delta a_\lambda}, \frac{\delta_\mu \mathbb{F}}{\delta u_\lambda} \right) \right\rangle, \quad 0 \leq \lambda \leq 1$$

is continuous at 0^+, where

$$\frac{\delta_\mu \mathbb{F}}{\delta a_\lambda} := \frac{\partial f_\mu}{\partial a}(\chi_{a_\lambda,u_\lambda}(s)) - D_\kappa \frac{\partial f_\mu}{\partial a_\kappa}(\chi_{a_\lambda,u_\lambda}(s)) + \frac{1}{n(\alpha,\beta)} D^2_{\alpha\beta} \frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{a_\lambda,u_\lambda}(s)) \in A,$$

$$\frac{\delta_\mu \mathbb{F}}{\delta u_\lambda} := \frac{\partial f_\mu}{\partial u}(\chi_{a_\lambda,u_\lambda}(s)) \in U,$$

$$a_\lambda := \lambda a + (1-\lambda) b, \quad u_\lambda := \lambda u + (1-\lambda) w.$$

The following lemma is an auxiliary result for proving the main results derived in the present chapter.

Lemma 1 *Consider the curvilinear integral functional* $\mathbb{F}$ *is pseudomonotone and hemicontinuous on* $A \times U$. *A pair* $(a, u) \in A \times U$ *is solution of (CVIP) if and only if it is solution for*

$$\int_{\Xi} [(b(s) - a(s)) \frac{\partial f_\mu}{\partial a}(\chi_{b,w}(s)) + (w(s) - u(s)) \frac{\partial f_\mu}{\partial u}(\chi_{b,w}(s))$$

$$+ D_\kappa (b(s) - a(s)) \frac{\partial f_\mu}{\partial a_\kappa}(\chi_{b,w}(s))$$

$$+ \frac{1}{n(\alpha,\beta)} D^2_{\alpha\beta}(b(s) - a(s)) \frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{b,w}(s))] ds^\mu \geq 0, \quad \forall (b, w) \in A \times U.$$

Proof Firstly, we consider the pair $(a, u) \in A \times U$ solves (CVIP), that is,

$$\int_{\Xi} [(b(s) - a(s)) \frac{\partial f_\mu}{\partial a}(\chi_{a,u}(s)) + (w(s) - u(s)) \frac{\partial f_\mu}{\partial u}(\chi_{a,u}(s))$$

$$+ D_\kappa (b(s) - a(s)) \frac{\partial f_\mu}{\partial a_\kappa}(\chi_{a,u}(s))$$

$$+ \frac{1}{n(\alpha,\beta)} D^2_{\alpha\beta}(b(s) - a(s)) \frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{a,u}(s))] ds^\mu \geq 0, \quad \forall (b, w) \in A \times U.$$

By using the definition of pseudomonotonicity, the above inequality implies

$$\int_{\Xi} [(b(s) - a(s)) \frac{\partial f_\mu}{\partial a}(\chi_{b,w}(s)) + (w(s) - u(s)) \frac{\partial f_\mu}{\partial u}(\chi_{b,w}(s))$$

$$+ D_\kappa (b(s) - a(s)) \frac{\partial f_\mu}{\partial a_\kappa}(\chi_{b,w}(s))$$

$$+ \frac{1}{n(\alpha,\beta)} D^2_{\alpha\beta}(b(s) - a(s)) \frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{b,w}(s))] ds^\mu \geq 0, \quad \forall (b, w) \in A \times U.$$

Conversely, assume that

$$\int_{\Xi}[(b(s)-a(s))\frac{\partial f_{\mu}}{\partial a}(\chi_{b,w}(s))+(w(s)-u(s))\frac{\partial f_{\mu}}{\partial u}(\chi_{b,w}(s))$$

$$+D_{\kappa}(b(s)-a(s))\frac{\partial f_{\mu}}{\partial a_{\kappa}}(\chi_{b,w}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}(b(s)-a(s))\frac{\partial f_{\mu}}{\partial a_{\alpha\beta}}(\chi_{b,w}(s))]ds^{\mu}\geq 0,\quad \forall(b,w)\in A\times U.$$

Now, for $(b, w) \in A \times U$ and $\lambda \in (0, 1]$, we define

$$(b_{\lambda}, w_{\lambda}) = ((1-\lambda)a+\lambda b, (1-\lambda)u+\lambda w)\in A\times U.$$

Thus, the above inequality can be rewritten as

$$\int_{\Xi}[(b_{\lambda}(s)-a(s))\frac{\partial f_{\mu}}{\partial a}(\chi_{b_{\lambda},w_{\lambda}}(s))+(w_{\lambda}(s)-u(s))\frac{\partial f_{\mu}}{\partial u}(\chi_{b_{\lambda},w_{\lambda}}(s))$$

$$+D_{\kappa}(b_{\lambda}(s)-a(s))\frac{\partial f_{\mu}}{\partial a_{\kappa}}(\chi_{b_{\lambda},w_{\lambda}}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}(b_{\lambda}(s)-a(s))\frac{\partial f_{\mu}}{\partial a_{\alpha\beta}}(\chi_{b_{\lambda},w_{\lambda}}(s))]ds^{\mu}\geq 0,\quad (b,w)\in A\times U.$$

Taking $\lambda \to 0$ and using the hemicontinuity property associated with the considered curvilinear integral functional, we have

$$\int_{\Xi}[(b(s)-a(s))\frac{\partial f_{\mu}}{\partial a}(\chi_{a,u}(s))+(w(s)-u(s))\frac{\partial f_{\mu}}{\partial u}(\chi_{a,u}(s))$$

$$+D_{\kappa}(b(s)-a(s))\frac{\partial f_{\mu}}{\partial a_{\kappa}}(\chi_{a,u}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}(b(s)-a(s))\frac{\partial f_{\mu}}{\partial a_{\alpha\beta}}(\chi_{a,u}(s))]ds^{\mu}\geq 0,\quad \forall(b,w)\in A\times U,$$

which shows that (a, u) solves (CVIP). □

3 Well-Posedness Associated with (CVIP)

In this section, by using the notions of monotonicity and hemicontinuity introduced in Sect. 2, we investigate the well-posedness of the considered class of controlled variational inequality problems involving second-order PDEs. In this regard, we introduce the following definitions.

Definition 4 The sequence $\{(a_n, u_n)\} \in A \times U$ is called an *approximating sequence* of (CVIP) if there exists a sequence of positive real numbers $\iota_n \to 0$ as $n \to \infty$, such that the following inequality holds:

$$\int_{\Xi} [(b(s) - a_n(s)) \frac{\partial f_\mu}{\partial a}(\chi_{a_n,u_n}(s)) + (w(s) - u_n(s)) \frac{\partial f_\mu}{\partial u}(\chi_{a_n,u_n}(s))$$

$$+ D_\kappa (b(s) - a_n(s)) \frac{\partial f_\mu}{\partial a_\kappa}(\chi_{a_n,u_n}(s))$$

$$+ \frac{1}{n(\alpha,\beta)} D^2_{\alpha\beta}(b(s) - a_n(s)) \frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{a_n,u_n}(s))] ds^\mu + \iota_n \geq 0, \quad \forall (b,w) \in A \times U.$$

Definition 5 The problem (CVIP) is called *well-posed* if the following two conditions hold:
(i) the problem (CVIP) possesses a single solution (a_0, u_0);
(ii) each approximating sequence of (CVIP) will converge to this single solution (a_0, u_0).

In order to investigate the well-posedness of (CVIP), we introduce the definition of *approximating solution set* of (CVIP) as follows:

$$\Pi_\iota = \Big\{ (a,u) \in A \times U : \int_{\Xi} [(b(s) - a(s)) \frac{\partial f_\mu}{\partial a}(\chi_{a,u}(s)) + (w(s) - u(s)) \frac{\partial f_\mu}{\partial u}(\chi_{a,u}(s))$$
$$+ D_\kappa (b(s) - a(s)) \frac{\partial f_\mu}{\partial a_\kappa}(\chi_{a,u}(s))$$
$$+ \frac{1}{n(\alpha,\beta)} D^2_{\alpha\beta}(b(s) - a(s)) \frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{a,u}(s))] ds^\mu + \iota \geq 0, \ \forall (b,w) \in A \times U \Big\}.$$

Remark 1 Clearly, $\Pi = \Pi_\iota$, when $\iota = 0$ and $\Pi \subseteq \Pi_\iota, \ \forall \iota > 0$.

Further, for a set B, the symbol "diam B" stands for the *diameter* of B, and it is defined as follows:

$$\text{diam } B = \sup_{\xi,\eta \in B} \|\xi - \eta\|.$$

Now, we are able to formulate and prove a first characterization result on the well-posedness of (CVIP).

Theorem 1 *Assume the curvilinear integral functional $\mathbb{F}$ is monotone and hemicontinuous on $A \times U$. Then, the problem (CVIP) is well-posed if and only if*

$$\Pi_\iota \neq \emptyset, \forall \iota > 0 \text{ and } \operatorname{diam} \Pi_\iota \to 0 \text{ as } \iota \to 0.$$

Proof Suppose the problem (CVIP) is well-posed. Then, by Definition 5, it possesses a single solution $(\bar{a}, \bar{u}) \in \Pi$. Since $\Pi \subseteq \Pi_\iota,\ \forall \iota > 0$, therefore, $\Pi_\iota \neq \emptyset,\ \forall \iota > 0$. Contrary to the result, we consider that diam $\Pi_\iota \nrightarrow 0$ as $\iota \to 0$. Then there exist $r > 0$, a positive integer m, a sequence of real numbers $\iota_n > 0$ with $\iota_n \to 0$, and two elements (a_n, u_n) and $(a'_n, u'_n) \in \Pi_{\iota_n}$ such that

$$\|(a_n(s), u_n(s)) - (a'_n(s), u'_n(s))\| > r, \quad \forall n \geq m. \tag{1}$$

Since $(a_n, u_n),\ (a'_n, u'_n) \in \Pi_{\iota_n}$, we get

$$\int_\Xi [(b(s) - a_n(s))\frac{\partial f_\mu}{\partial a}(\chi_{a_n,u_n}(s)) + (w(s) - u_n(s))\frac{\partial f_\mu}{\partial u}(\chi_{a_n,u_n}(s))$$

$$+D_\kappa(b(s) - a_n(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{a_n,u_n}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}(b(s) - a_n(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{a_n,u_n}(s))]ds^\mu + \iota_n \geq 0, \quad \forall (b, w) \in A \times U$$

and

$$\int_\Xi [(b(s) - a'_n(s))\frac{\partial f_\mu}{\partial a}(\chi_{a'_n,u'_n}(s)) + (w(s) - u'_n(s))\frac{\partial f_\mu}{\partial u}(\chi_{a'_n,u'_n}(s))$$

$$+D_\kappa(b(s) - a'_n(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{a'_n,u'_n}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}(b(s) - a'_n(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{a'_n,u'_n}(s))]ds^\mu + \iota_n \geq 0, \quad \forall (b, w) \in A \times U.$$

Clearly, it follows that $\{(a_n, u_n)\}$ and $\{(a'_n, u'_n)\}$ are approximating sequences of (CVIP) which converge to $(\bar{a}, \bar{u})$ since, by hypothesis, the problem (CVIP) is well-posed. By direct computation, we get

$$\|(a_n(s), u_n(s)) - (a'_n(s), u'_n(s))\|$$

$$= \|(a_n(s), u_n(s)) - (\bar{a}(s), \bar{u}(s)) + (\bar{a}(s), \bar{u}(s)) - (a'_n(s), u'_n(s))\|$$

$$\leq \|(a_n(s), u_n(s)) - (\bar{a}(s), \bar{u}(s))\| + \|(\bar{a}(s), \bar{u}(s)) - (a'_n(s), u'_n(s))\| \leq \iota,$$

which contradicts (1), for some $\iota = r$.

Conversely, consider $\{(a_n, u_n)\}$ is an approximating sequence of (CVIP). Then there exists a sequence of positive real numbers $\iota_n \to 0$ as $n \to \infty$ such that

$$\int_{\Xi} [(b(s) - a_n(s)) \frac{\partial f_\mu}{\partial a}(\chi_{a_n,u_n}(s)) + (w(s) - u_n(s)) \frac{\partial f_\mu}{\partial u}(\chi_{a_n,u_n}(s))$$

$$+ D_\kappa (b(s) - a_n(s)) \frac{\partial f_\mu}{\partial a_\kappa}(\chi_{a_n,u_n}(s))$$

$$+ \frac{1}{n(\alpha, \beta)} D^2_{\alpha\beta}((b(s) - a_n(s)) \frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{a_n,u_n}(s))] ds^\mu + \iota_n \geq 0, \quad \forall (b, w) \in A \times U \tag{2}$$

holds, involving that $(a_n, u_n) \in \Pi_{\iota_n}$. Since diam $\Pi_{\iota_n} \to 0$ as $\iota_n \to 0$, therefore $\{(a_n, u_n)\}$ is a Cauchy sequence which converges to some $(\bar{a}, \bar{u}) \in A \times U$ as $A \times U$ is a closed set.

By hypothesis, the curvilinear integral functional $\int_{\Xi} f_\mu(\chi_{a,u}(s)) ds^\mu$ is monotone on $A \times U$. Therefore, by Definition 1, for $(\bar{a}, \bar{u}), (b, w) \in A \times U$, we have

$$\int_{\Xi} \Big[(\bar{a}(s) - b(s)) \left(\frac{\partial f_\mu}{\partial a}(\chi_{\bar{a},\bar{u}}(s)) - \frac{\partial f_\mu}{\partial a}(\chi_{b,w}(s)) \right)$$

$$+ (\bar{u}(s) - w(s)) \left(\frac{\partial f_\mu}{\partial u}(\chi_{\bar{a},\bar{u}}(s)) - \frac{\partial f_\mu}{\partial a}(\chi_{b,w}(s)) \right)$$

$$+ D_\kappa (\bar{a}(s) - b(s)) \left(\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{\bar{a},\bar{u}}(s)) - \frac{\partial f_\mu}{\partial a_\kappa}(\chi_{b,w}(s)) \right)$$

$$+ \frac{1}{n(\alpha, \beta)} D^2_{\alpha\beta} (\bar{a}(s) - b(s)) \left(\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{\bar{a},\bar{u}}(s)) - \frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{b,w}(s)) \right) \Big] ds^\mu \geq 0$$

or, equivalently,

$$\int_{\Xi} \Big[(\bar{a}(s) - b(s)) \frac{\partial f_\mu}{\partial a}(\chi_{\bar{a},\bar{u}}(s)) + (\bar{u}(s) - w(s)) \frac{\partial f_\mu}{\partial u}(\chi_{\bar{a},\bar{u}}(s))$$

$$+ D_\kappa (\bar{a}(s) - b(s)) \frac{\partial f_\mu}{\partial a_\kappa}(\chi_{\bar{a},\bar{u}}(s))$$

$$+ \frac{1}{n(\alpha, \beta)} D^2_{\alpha\beta} (\bar{a}(s) - b(s)) \frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{\bar{a},\bar{u}}(s)) \Big] ds^\mu$$

$$\geq \int_{\Xi} \Big[(\bar{a}(s) - b(s)) \frac{\partial f_\mu}{\partial a}(\chi_{b,w}(s)) + (\bar{u}(s) - w(s)) \frac{\partial f_\mu}{\partial u}(\chi_{b,w}(s))$$

$$+D_{\kappa}(\bar{a}(s)-b(s))\frac{\partial f_{\mu}}{\partial a_{\kappa}}(\chi_{b,w}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^{2}_{\alpha\beta}(\bar{a}(s)-b(s))\frac{\partial f_{\mu}}{\partial a_{\alpha\beta}}(\chi_{b,w}(s))\Big]ds^{\mu}. \tag{3}$$

Taking limit in inequality (2), we have

$$\int_{\Xi}\Big[(\bar{a}(s)-b(s))\frac{\partial f_{\mu}}{\partial a}(\chi_{\bar{a},\bar{u}}(s))+(\bar{u}(s)-w(s))\frac{\partial f_{\mu}}{\partial u}(\chi_{\bar{a},\bar{u}}(s))$$

$$+D_{\kappa}(\bar{a}(s)-b(s))\frac{\partial f_{\mu}}{\partial a_{\kappa}}(\chi_{\bar{a},\bar{u}}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^{2}_{\alpha\beta}(\bar{a}(s)-b(s))\frac{\partial f_{\mu}}{\partial a_{\alpha\beta}}(\chi_{\bar{a},\bar{u}}(s))\Big]ds^{\mu}\leq 0,\quad \forall (b,w)\in A\times U. \tag{4}$$

On combining (3) and (4), we obtain

$$\int_{\Xi}\Big[(b(s)-\bar{a}(s))\frac{\partial f_{\mu}}{\partial a}(\chi_{b,w}(s))+(w(s)-\bar{u}(s))\frac{\partial f_{\mu}}{\partial u}(\chi_{b,w}(s))$$

$$+D_{\kappa}(b(s)-\bar{a}(s))\frac{\partial f_{\mu}}{\partial a_{\kappa}}(\chi_{b,w}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^{2}_{\alpha\beta}(b(s)-\bar{a}(s))\frac{\partial f_{\mu}}{\partial a_{\alpha\beta}}(\chi_{b,w}(s))\Big]ds^{\mu}\geq 0,\quad \forall (b,w)\in A\times U.$$

Further, taking into account Lemma 1, it follows

$$\int_{\Xi}\Big[(b(s)-\bar{a}(s))\frac{\partial f_{\mu}}{\partial a}(\chi_{\bar{a},\bar{u}}(s))+(w(s)-\bar{u}(s))\frac{\partial f_{\mu}}{\partial u}(\chi_{\bar{a},\bar{u}}(s))$$

$$+D_{\kappa}(b(s)-\bar{a}(s))\frac{\partial f_{\mu}}{\partial a_{\kappa}}(\chi_{\bar{a},\bar{u}}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^{2}_{\alpha\beta}(b(s)-\bar{a}(s))\frac{\partial f_{\mu}}{\partial a_{\alpha\beta}}(\chi_{\bar{a},\bar{u}}(s))\Big]ds^{\mu}\geq 0,\quad \forall (b,w)\in A\times U,$$

which implies that $(\bar{a},\bar{u})\in\Pi$. It remains to prove that $(\bar{a},\bar{u})$ is a single solution of (CVIP). Contrarily, we suppose $(a_1,u_1),(a_2,u_2)$ are two distinct solutions of (CVIP). Then

$$0<\|(a_1(s),u_1(s))-(a_2(s),u_2(s))\|\leq \operatorname{diam}\Pi_{\iota}\to 0 \text{ as } \iota\to 0,$$

which is not possible, and the proof is now complete. □

In the next theorem, we establish that the well-posedness of (CVIP) is equivalent to the existence and uniqueness of solution.

Theorem 2 *Assume the curvilinear integral functional* $\mathbb{F}$ *is monotone and hemicontinuous on* $A \times U$*. Then, (CVIP) is well-posed if and only if it possesses a single solution.*

Proof Assume that the problem (CVIP) is well-posed. In consequence, by Definition 5, it possesses a single solution (a_0, u_0). Conversely, suppose that (CVIP) has a single solution (a_0, u_0) but it is not well-posed. Then, there exists an approximating sequence $\{(a_n, u_n)\}$ of (CVIP) which does not converge to (a_0, u_0). Since $\{(a_n, u_n)\}$ is an approximating sequence of (CVIP), there must exist a sequence of positive real numbers $\iota_n \to 0$ as $n \to \infty$ such that

$$\int_{\Xi} [(b(s) - a_n(s))\frac{\partial f_\mu}{\partial a}(\chi_{a_n,u_n}(s)) + (w(s) - u_n(s))\frac{\partial f_\mu}{\partial u}(\chi_{a_n,u_n}(s))$$

$$+D_\kappa(b(s) - a_n(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{a_n,u_n}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}(b(s) - a_n(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{a_n,u_n}(s))]ds^\mu + \iota_n \geq 0, \quad \forall (b, w) \in A \times U. \tag{5}$$

Further, in order to prove the boundedness of $\{(a_n, u_n)\}$, we start by reductio ad absurdum. Suppose $\{(a_n, u_n)\}$ is not bounded, that is, $\|(a_n(s), u_n(s))\| \to +\infty$ as $n \to +\infty$. Consider $\delta_n(s) = \dfrac{1}{\|(a_n(s), u_n(s)) - (a_0(s), u_0(s))\|}$ and $(\mathsf{a}_n, \mathsf{u}_n) = (a_0, u_0) + \delta_n[(a_n, u_n) - (a_0, u_0)]$.

We can see that $\{(\mathsf{a}_n, \mathsf{u}_n)\}$ is bounded in $A \times U$. So, passing to a subsequence if necessary, we may assume that

$$(\mathsf{a}_n, \mathsf{u}_n) \to (\mathsf{a}, \mathsf{u}) \text{ weakly in } A \times U \neq (a_0, u_0).$$

It is not difficult to verify that $(\mathsf{a}, \mathsf{u}) \neq (a_0, u_0)$, thanks to

$$\|\delta_n(s)[(a_n(s), u_n(s)) - (a_0(s), u_0(s))]\| = 1,$$

for all $n \in \mathbb{N}$. Since (a_0, u_0) is solution of (CVIP), therefore

$$\int_{\Xi} \Big[(b(s) - a_0(s))\frac{\partial f_\mu}{\partial a}(\chi_{a_0,u_0}(s)) + (w(s) - u_0(s))\frac{\partial f_\mu}{\partial u}(\chi_{a_0,u_0}(s))$$

$$+D_\kappa(b(s) - a_0(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{a_0,u_0}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}(b(s) - a_0(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{a_0,u_0}(s))\Big]ds^\mu \geq 0, \quad \forall (b, w) \in A \times U.$$

Thus, by Lemma 1, the above inequality implies that

$$\int_{\Xi}\Big[(b(s)-a_0(s))\frac{\partial f_\mu}{\partial a}(\chi_{b,w}(s))+(w(s)-u_0(s))\frac{\partial f_\mu}{\partial u}(\chi_{b,w}(s))$$

$$+D_\kappa(b(s)-a_0(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{b,w}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}(b(s)-a_0(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{b,w}(s))\Big]ds^\mu \geq 0, \quad \forall (b,w)\in A\times U. \tag{6}$$

By hypothesis, the curvilinear integral functional $\mathbb{F}$ is monotone on $A\times U$, therefore, for $(a_n,u_n),(b,w)\in A\times U$, we have

$$\int_{\Xi}\Big[(a_n(s)-b(s))\left(\frac{\partial f_\mu}{\partial a}(\chi_{a_n,u_n}(s))-\frac{\partial f_\mu}{\partial a}(\chi_{b,w}(s))\right)$$

$$+(u_n(s)-w(s))\left(\frac{\partial f_\mu}{\partial u}(\chi_{a_n,u_n}(s))-\frac{\partial f_\mu}{\partial a}(\chi_{b,w}(s))\right)$$

$$+D_\kappa(a_n(s)-b(s))\left(\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{a_n,u_n}(s))-\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{b,w}(s))\right)$$

$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}(a_n(s)-b(s))\left(\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{a_n,u_n}(s))-\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{b,w}(s))\right)\Big]ds^\mu \geq 0,$$

or, equivalently,

$$\int_{\Xi}\Big[(b(s)-a_n(s))\frac{\partial f_\mu}{\partial a}(\chi_{a_n,u_n}(s))+(w(s)-u_n(s))\frac{\partial f_\mu}{\partial u}(\chi_{a_n,u_n}(s))$$

$$+D_\kappa(b(s)-a_n(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{a_n,u_n}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}(b(s)-a_n(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{a_n,u_n}(s))\Big]ds^\mu$$

$$\leq \int_{\Xi}\Big[(b(s)-a_n(s))\frac{\partial f_\mu}{\partial a}(\chi_{b,w}(s))+(w(s)-u_n(s))\frac{\partial f_\mu}{\partial u}(\chi_{b,w}(s))$$

$$+D_\kappa(b(s)-a_n(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{b,w}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}(b(s)-a_n(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{b,w}(s))\Big]ds^\mu. \tag{7}$$

Combining with (5) and (7), we have

$$\int_{\Xi}\Big[(b(s)-a_n(s))\frac{\partial f_\mu}{\partial a}(\chi_{b,w}(s))+(w(s)-u_n(s))\frac{\partial f_\mu}{\partial u}(\chi_{b,w}(s))$$

$$+D_\kappa(b(s)-a_n(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{b,w}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}(b(s)-a_n(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{b,w}(s))\Big]ds^\mu\geq-\iota_n,\quad\forall(b,w)\in A\times U.$$

Because of $\delta_n\to 0$ as $n\to\infty$ (by the assumption that $\{(a_n,u_n)\}$ is not bounded), so, we can take $n_0\in\mathbb{N}$ be large enough such that $\delta_n<1$ for all $n\geq n_0$. Multiplying the above inequality and (6) by $\delta_n>0$ and $1-\delta_n>0$, respectively, we sum the resulting inequalities to get

$$\int_{\Xi}\Big[(b(s)-\mathsf{a}_n(s))\frac{\partial f_\mu}{\partial a}(\chi_{b,w}(s))+(w(s)-\mathsf{u}_n(s))\frac{\partial f_\mu}{\partial u}(\chi_{b,w}(s))$$

$$+D_\kappa(b(s)-\mathsf{a}_n(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{b,w}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}(b(s)-\mathsf{a}_n(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{b,w}(s))\Big]ds^\mu\geq-\iota_n,\quad\forall(b,w)\in A\times U,\ \forall n\geq n_0.$$

Since $(\mathsf{a}_n,\mathsf{u}_n)\to(\mathsf{a},\mathsf{u})\neq(a_0,u_0)$ and $(\mathsf{a}_n,\mathsf{u}_n)=(a_0,u_0)+\delta_n[(a_n,u_n)-(a_0,u_0)]$, we have

$$\int_{\Xi}\Big[(b(s)-\mathsf{a}(s))\frac{\partial f_\mu}{\partial a}(\chi_{b,w}(s))+(w(s)-\mathsf{u}(s))\frac{\partial f_\mu}{\partial u}(\chi_{b,w}(s))$$

$$+D_\kappa(b(s)-\mathsf{a}(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{b,w}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}(b(s)-\mathsf{a}(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{b,w}(s))\Big]ds^\mu$$

$$=\lim_{n\to\infty}\int_{\Xi}\Big[(b(s)-\mathsf{a}_n(s))\frac{\partial f_\mu}{\partial a}(\chi_{b,w}(s))+(w(s)-\mathsf{u}_n(s))\frac{\partial f_\mu}{\partial u}(\chi_{b,w}(s))$$

$$+D_\kappa(b(s)-\mathsf{a}_n(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{b,w}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}(b(s)-\mathsf{a}_n(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{b,w}(s))\Big]ds^\mu$$

$$\geq -\lim_{n\to\infty} \iota_n = 0, \quad \forall (b, w) \in A \times U.$$

Thus, by Lemma 1, we have

$$\int_{\Xi} \Big[(b(s) - \mathsf{a}(s))\frac{\partial f_\mu}{\partial a}(\chi_{\mathsf{a},\mathsf{u}}(s)) + (w(s) - \mathsf{u}(s))\frac{\partial f_\mu}{\partial u}(\chi_{\mathsf{a},\mathsf{u}}(s))$$

$$+D_\kappa(b(s) - \mathsf{a}(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{\mathsf{a},\mathsf{u}}(s))$$

$$+ \frac{1}{n(\alpha, \beta)} D^2_{\alpha\beta}(b(s) - \mathsf{a}(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{\mathsf{a},\mathsf{u}}(s))\Big] ds^\mu \geq 0, \quad \forall (b, w) \in A \times U. \tag{8}$$

This implies that (a, u) is solution of (CVIP), which contradicts the uniqueness of (a_0, u_0). Therefore, $\{(a_n, u_n)\}$ is a bounded sequence having convergent subsequence $\{(a_{n_k}, u_{n_k})\}$, which converges to $(\bar{a}, \bar{u}) \in A \times U$ as $k \to \infty$. Again, from the definition of monotonicity, for $(a_{n_k}, u_{n_k}), (b, w) \in A \times U$, we have (see (7))

$$\int_{\Xi} \Big[(b(s) - a_{n_k}(s))\frac{\partial f_\mu}{\partial a}(\chi_{a_{n_k},u_{n_k}}(s)) + (w(s) - u_{n_k}(s))\frac{\partial f_\mu}{\partial u}(\chi_{a_{n_k},u_{n_k}}(s))$$

$$+D_\kappa(b(s) - a_{n_k}(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{a_{n_k},u_{n_k}}(s))$$

$$+\frac{1}{n(\alpha, \beta)} D^2_{\alpha\beta}(b(s) - a_{n_k}(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{a_{n_k},u_{n_k}}(s))\Big] ds^\mu$$

$$\leq \int_{\Xi} \Big[(b(s) - a_{n_k}(s))\frac{\partial f_\mu}{\partial a}(\chi_{b,w}(s)) + (w(s) - u_{n_k}(s))\frac{\partial f_\mu}{\partial u}(\chi_{b,w}(s))$$

$$+D_\kappa(b(s) - a_{n_k}(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{b,w}(s))$$

$$+ \frac{1}{n(\alpha, \beta)} D^2_{\alpha\beta}(b(s) - a_{n_k}(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{b,w}(s))\Big] ds^\mu. \tag{9}$$

Also, on behalf of (5), we can write

$$\lim_{k\to\infty} \int_{\Xi} \Big[(b(s) - a_{n_k}(s))\frac{\partial f_\mu}{\partial a}(\chi_{a_{n_k},u_{n_k}}(s)) + (w(s) - u_{n_k}(s))\frac{\partial f_\mu}{\partial u}(\chi_{a_{n_k},u_{n_k}}(s))$$

$$+D_\kappa(b(s) - a_{n_k}(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{a_{n_k},u_{n_k}}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}(b(s)-a_{n_k}(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{a_{n_k},u_{n_k}}(s))\Big]ds^\mu \geq 0. \tag{10}$$

Combining (9) and (10), we have

$$\lim_{k\to\infty}\int_\Xi \Big[(b(s)-a_{n_k}(s))\frac{\partial f_\mu}{\partial a}(\chi_{b,w}(s))+(w(s)-u_{n_k}(s))\frac{\partial f_\mu}{\partial u}(\chi_{b,w}(s))$$

$$+D_\kappa(b(s)-a_{n_k}(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{b,w}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}(b(s)-a_{n_k}(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{b,w}(s))\Big]ds^\mu \geq 0$$

$$\Rightarrow \int_\Xi \Big[(b(s)-\bar{a}(s))\frac{\partial f_\mu}{\partial a}(\chi_{b,w}(s))+(w(s)-\bar{u}(s))\frac{\partial f_\mu}{\partial u}(\chi_{b,w}(s))$$

$$+D_\kappa(b(s)-\bar{a}(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{b,w}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}(b(s)-\bar{a}(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{b,w}(s))\Big]ds^\mu \geq 0$$

Thus, by Lemma 1, the above inequality implies that

$$\int_\Xi \Big[(b(s)-\bar{a}(s))\frac{\partial f_\mu}{\partial a}(\chi_{\bar{a},\bar{u}}(s))+(w(s)-\bar{u}(s))\frac{\partial f_\mu}{\partial u}(\chi_{\bar{a},\bar{u}}(s))$$

$$+D_\kappa(b(s)-\bar{a}(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{\bar{a},\bar{u}}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}(b(s)-\bar{a}(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{\bar{a},\bar{u}}(s))\Big]ds^\mu \geq 0,$$

which shows that $(\bar{a},\bar{u})$ is solution of (CVIP). Hence, $(a_{n_k},u_{n_k})\to(\bar{a},\bar{u})$, that is, $(a_{n_k},u_{n_k})\to(a_0,u_0)$, involving $(a_n,u_n)\to(a_0,u_0)$ and the proof is complete. □

Further, we provide an illustrative application of the previous theoretical results.

Example 3 As in the previous section, let $\mu\in\{1,2\}$, $\Omega=[0,1]^2$ and $\Omega\supset\Xi$ be a piecewise smooth curve joining the points $(0,0)$, $(1,1)$ in Ω. Consider

$$f_\mu(\chi_{a,u}(s))ds^\mu = f_1(\chi_{a,u}(s))ds^1+f_2(\chi_{a,u}(s))ds^2 = u^2(s)ds^1+(e^{a(s)}-a(s))ds^2.$$

(CVIP-1): Find $(a,u)\in A\times U = C^4(\Omega,[-10,10])\times C^1(\Omega,[-10,10])$ so that

$$\int_{\Xi} 2(w(s) - u(s))u(s)ds^1 + (b(s) - a(s))(e^{a(s)} - 1)ds^2 \geq 0, \ \forall (b, w) \in A \times U.$$

Clearly, $\Pi = \{(0, 0)\}$. It can be easily verified that the functional $\int_{\Xi} f_\mu(\chi_{a,u}(s))ds^\mu$ is monotone and hemicontinuous on the nonempty, closed and convex set $A \times U = C^4(\Omega, [-10, 10]) \times C^1(\Omega, [-10, 10])$. Since all the hypotheses of Theorem 2 hold, the problem (CVIP-1) is well-posed. Further, $\Pi_\iota = \{(0, 0)\}$ and consequently, $\Pi_\iota \neq \emptyset$ and diam $\Pi_\iota \to 0$ as $\iota \to 0$. Thus, by Theorem 1, the problem (CVIP-1) is well-posed.

4 Well-Posedness in Generalized Sense Associated with (CVIP)

In this section, we extend the notion of well-posedness to well-posedness in generalized sense associated with (CVIP). To this aim, we introduce the following definition.

Definition 6 The problem (CVIP) is called *well-posed in generalized sense* if the following two conditions are satisfied:
(i) $\Pi \neq \emptyset$;
(ii) each approximating sequence of (CVIP) possesses a subsequence that will converge to some pair of Π.

The next result establishes that the well-posedness in generalized sense of (CVIP) is equivalent to the non-emptiness of its solution set.

Theorem 3 *Suppose the curvilinear integral functional* $\mathbb{F}$ *is monotone and hemicontinuous on the nonempty compact set* $A \times U$. *Then (CVIP) is well-posed in generalized sense if and only if the solution set* Π *is nonempty.*

Proof Consider the problem (CVIP) is well-posed in generalized sense. Consequently, its solution set Π is nonempty. Conversely, let $\{(a_n, u_n)\}$ be an approximating sequence of (CVIP). Then, there exists a sequence of positive real numbers $\iota_n \to 0$ such that

$$\int_{\Xi} [(b(s) - a_n(s))\frac{\partial f_\mu}{\partial a}(\chi_{a_n,u_n}(s)) + (w(s) - u_n(s))\frac{\partial f_\mu}{\partial u}(\chi_{a_n,u_n}(s))$$
$$+D_\kappa(b(s) - a_n(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{a_n,u_n}(s))$$
$$+ \frac{1}{n(\alpha, \beta)} D^2_{\alpha\beta}(b(s) - a_n(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{a_n,u_n}(s))]ds^\mu + \iota_n \geq 0, \quad \forall (b, w) \in A \times U. \tag{11}$$

By hypothesis, $A \times U$ is a compact set and, therefore, $\{(a_n, u_n)\}$ has a subsequence $\{(a_{n_k}, u_{n_k})\}$ which converges to some pair $(a_0, u_0) \in A \times U$. Since the integral functional $\mathbb{F}$ is monotone on $A \times U$, for $(a_{n_k}, u_{n_k}), (b, w) \in A \times U$, we have

$$\int_{\Xi} \Big[(b(s) - a_{n_k}(s))\frac{\partial f_\mu}{\partial a}(\chi_{a_{n_k},u_{n_k}}(s)) + (w(s) - u_{n_k}(s))\frac{\partial f_\mu}{\partial u}(\chi_{a_{n_k},u_{n_k}}(s))$$

$$+D_\kappa(b(s) - a_{n_k}(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{a_{n_k},u_{n_k}}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}(b(s) - a_{n_k}(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{a_{n_k},u_{n_k}}(s))\Big]ds^\mu$$

$$\leq \int_{\Xi} \Big[(b(s) - a_{n_k}(s))\frac{\partial f_\mu}{\partial a}(\chi_{b,w}(s)) + (w(s) - u_{n_k}(s))\frac{\partial f_\mu}{\partial u}(\chi_{b,w}(s))$$

$$+D_\kappa(b(s) - a_{n_k}(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{b,w}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}(b(s) - a_{n_k}(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{b,w}(s))\Big]ds^\mu.$$

Taking limit $k \to \infty$ in the above inequality, we have

$$\lim_{k\to\infty}\int_{\Xi} \Big[(b(s) - a_{n_k}(s))\frac{\partial f_\mu}{\partial a}(\chi_{a_{n_k},u_{n_k}}(s)) + (w(s) - u_{n_k}(s))\frac{\partial f_\mu}{\partial u}(\chi_{a_{n_k},u_{n_k}}(s))$$

$$+D_\kappa(b(s) - a_{n_k}(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{a_{n_k},u_{n_k}}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}(b(s) - a_{n_k}(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{a_{n_k},u_{n_k}}(s))\Big]ds^\mu$$

$$\leq \lim_{k\to\infty}\int_{\Xi} \Big[(b(s) - a_{n_k}(s))\frac{\partial f_\mu}{\partial a}(\chi_{b,w}(s)) + (w(s) - u_{n_k}(s))\frac{\partial f_\mu}{\partial u}(\chi_{b,w}(s))$$

$$+D_\kappa(b(s) - a_{n_k}(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{b,w}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}(b(s) - a_{n_k}(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{b,w}(s))\Big]ds^\mu. \tag{12}$$

Since $\{(a_{n_k}, u_{n_k})\}$ is an approximating subsequence in $A \times U$, therefore, on behalf of (11), we have

$$\lim_{k\to\infty}\int_{\Xi}\Big[(b(s)-a_{n_k}(s))\frac{\partial f_\mu}{\partial a}(\chi_{a_{n_k},u_{n_k}}(s))+(w(s)-u_{n_k}(s))\frac{\partial f_\mu}{\partial u}(\chi_{a_{n_k},u_{n_k}}(s))$$

$$+D_\kappa(b(s)-a_{n_k}(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{a_{n_k},u_{n_k}}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}(b(s)-a_{n_k}(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{a_{n_k},u_{n_k}}(s))\Big]ds^\mu \geq 0, \quad \forall (b,w)\in A\times U. \tag{13}$$

Combining (12) and (13), we get

$$\lim_{k\to\infty}\int_{\Xi}\Big[(b(s)-a_{n_k}(s))\frac{\partial f_\mu}{\partial a}(\chi_{b,w}(s))+(w(s)-u_{n_k}(s))\frac{\partial f_\mu}{\partial u}(\chi_{b,w}(s))$$

$$+D_\kappa(b(s)-a_{n_k}(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{b,w}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}(b(s)-a_{n_k}(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{b,w}(s))\Big]ds^\mu \geq 0, \quad \forall (b,w)\in A\times U$$

$$\Rightarrow \int_{\Xi}\Big[(b(s)-a_0(s))\frac{\partial f_\mu}{\partial a}(\chi_{b,w}(s))+(w(s)-u_0(s))\frac{\partial f_\mu}{\partial u}(\chi_{b,w}(s))$$

$$+D_\kappa(b(s)-a_0(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{b,w}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}(b(s)-a_0(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{b,w}(s))\Big]ds^\mu \geq 0, \quad \forall (b,w)\in A\times U.$$

Thus, by considering Lemma 1, we can also write

$$\int_{\Xi}\Big[(b(s)-a_0(s))\frac{\partial f_\mu}{\partial a}(\chi_{a_0,u_0}(s))+(w(s)-u_0(s))\frac{\partial f_\mu}{\partial u}(\chi_{a_0,u_0}(s))$$

$$+D_\kappa(b(s)-a_0(s))\frac{\partial f_\mu}{\partial a_\kappa}(\chi_{a_0,u_0}(s))$$

$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}(b(s)-a_0(s))\frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{a_0,u_0}(s))\Big]ds^\mu \geq 0, \quad \forall (b,w)\in A\times U,$$

which shows that $(a_0, u_0) \in \Pi$ and the proof is complete. □

In the following, we present the sufficiency of well-posedness in generalized sense of (CVIP).

Theorem 4 *Suppose the curvilinear integral functional* $\mathbb{F}$ *is monotone and hemicontinuous on the nonempty compact set* $A \times U$. *Then (CVIP) is well-posed in generalized sense if there exists* $\iota > 0$ *such that* Π_ι *is a nonempty bounded set.*

Proof Let $\iota > 0$ such that Π_ι is a nonempty and bounded set. Let $\{(a_n, u_n)\}$ be an approximating sequence of (CVIP). Then, there exists a sequence of positive real numbers $\iota_n \to 0$ such that the following inequality

$$\int_{\Xi} [(b(s) - a_n(s)) \frac{\partial f_\mu}{\partial a}(\chi_{a_n,u_n}(s)) + (w(s) - u_n(s)) \frac{\partial f_\mu}{\partial u}(\chi_{a_n,u_n}(s))$$

$$+ D_\kappa(b(s) - a_n(s)) \frac{\partial f_\mu}{\partial a_\kappa}(\chi_{a_n,u_n}(s))$$

$$+ \frac{1}{n(\alpha, \beta)} D^2_{\alpha\beta}(b(s) - a_n(s)) \frac{\partial f_\mu}{\partial a_{\alpha\beta}}(\chi_{a_n,u_n}(s))] ds^\mu + \iota_n \geq 0, \quad \forall (b, w) \in A \times U$$

holds, which implies that $(a_n, u_n) \in \Pi_\iota$, $\forall n > m$ (see m as a positive integer depending on ι). Therefore, $\{(a_n, u_n)\}$ is a bounded sequence having a convergent subsequence $\{(a_{n_k}, u_{n_k})\}$ which weakly converges to (a_0, u_0) as $k \to \infty$. Proceeding in the similar lines of the proof of Theorem 3, we get $(a_0, u_0) \in \Pi$ and the proof is complete. □

5 Conclusions

In this chapter, by considering the concepts of monotonicity, pseudomonotonicity, and hemicontinuity associated with path-independent curvilinear integral functionals governed by second-order partial derivatives, we have analyzed the well-posedness and well-posedness in generalized sense for a new class of controlled variational inequality problems, named (CVIP). We have proved that, under suitable conditions, the well-posedness of (CVIP) is characterized in terms of existence and uniqueness of solution, and the well-posedness in generalized sense is insured by assuming the boundedness of approximating solution set. Moreover, in order to support the mathematical development, some illustrative examples have been formulated throughout the chapter.

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On the Coupled System of Tempered Fractional Differential Equations with Anti-periodic Boundary Conditions

Mohamed I. Abbas

Abstract In this chapter, we are concerned with the existence and uniqueness of solutions for a coupled system of tempered fractional differential equations (*TFDEs*) with anti-periodic boundary conditions. The main tools in this investigation are the Leray-Schauder alternative and the Banach fixed point theorem. Two examples are given to illustrate the main results.

1 Introduction

Fractional calculus has been strongly employed to model a huge variety of problems in physics, biology, chemistry, finance, and other dynamical processes in complex systems, e.g., [18, 19] and the references therein.

Tempered fractional calculus (*TFC*) is a generalization of classical fractional calculus by multiplying the initial kernel by the term $\exp -\lambda t$ for a real number $\lambda \geq 0$. The associated model of the *TFC* has been qualified clearly in [17, 22]. At the same time, much of the survey has concentrated on numerical techniques to study tempered fractional models, e.g., [2, 10, 11, 14, 26] and several authors have been exposed to many different models of *TFC*, e.g., [1, 16, 24].

In [12], Fernandez et al. studied some useful analytic features of *TFC*. In [25], Zaky discussed the existence, uniqueness, and stability analysis of the solutions of nonlinear *TFDEs*, then the author improved and analyzed a singularity preserving spectral collocation technique for the numerical solutions of the proposed equations. On the other hand, by using fixed point theorems, the existence and uniqueness of solutions to differential/integral equations involving fractional operators were studied by a huge number of researchers. For instance, see [3–9, 20].

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So as to enrich the research of the *TFC* field, in this chapter, we consider the following coupled system of *TFDEs* accompanied by anti-periodic boundary conditions:

$$\begin{cases} {}_{0}^{C}D_{\varsigma}^{(\alpha,\lambda)}\zeta(\varsigma) = \Phi(\varsigma, \zeta(\varsigma), \xi(\varsigma)), & \varsigma \in [0, b], \\ {}_{0}^{C}D_{\varsigma}^{(\beta,\nu)}\xi(\varsigma) = \Psi(\varsigma, \zeta(\varsigma), \xi(\varsigma)), & \varsigma \in [0, b], \\ \zeta(0) + \zeta(b) = 0, \\ \xi(0) + \xi(b) = 0, \end{cases} \tag{1}$$

where $\lambda, \nu \geq 0$ are real numbers, $\Phi, \Psi : [0, b] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, and ${}_{0}^{C}D_{\varsigma}^{(\alpha,\lambda)}$, ${}_{0}^{C}D_{\varsigma}^{(\beta,\nu)}$ are the Caputo tempered fractional derivative of order $\alpha, \beta \in (0, 1)$ given in the following definition:

Definition 1 ([17, 23]) For $n - 1 < \alpha < 1, n \in \mathbb{N}^+, \lambda \geq 0$. The Caputo tempered fractional derivative is defined as

$$ {}_{0}^{C}D_{\varsigma}^{(\alpha,\lambda)}y(\varsigma) = e^{-\lambda\varsigma}\ {}_{0}^{C}D_{\varsigma}^{\alpha}(e^{\lambda\varsigma}y(\varsigma)) = \frac{e^{-\lambda\varsigma}}{\Gamma(n-\alpha)}\int_{0}^{\varsigma}\frac{1}{(\varsigma - s)^{\alpha-n+1}}\frac{d^n(e^{\lambda s}y(s))}{ds^n}ds, \tag{2}$$

where ${}_{0}^{C}D_{\varsigma}^{\alpha}(e^{\lambda\varsigma}y(\varsigma))$ is the Caputo fractional derivative [15, 21]

$$ {}_{0}^{C}D_{\varsigma}^{\alpha}(e^{\lambda\varsigma}y(\varsigma)) = \frac{1}{\Gamma(n-\alpha)}\int_{0}^{\varsigma}\frac{1}{(\varsigma - s)^{\alpha-n+1}}\frac{d^n(e^{\lambda s}y(s))}{ds^n}ds. \tag{3}$$

2 An Auxiliary Lemma

In this section, we shall provide a helpful lemma of the coupled system (1).

Lemma 1 (Lemma 3.2 [17]) *Let $h(\varsigma, \zeta)$ be a continuous function. The function $\zeta(\varsigma)$ is a solution of the Cauchy problem*

$$\begin{cases} {}_{0}^{C}D_{\varsigma}^{(\alpha,\lambda)}\zeta(\varsigma) = h(\varsigma, \zeta(\varsigma)), & \varsigma \in [0, b],\ \alpha \in (0, 1),\ \lambda \geq 0, \\ \zeta(0) = \zeta_0, \end{cases} \tag{4}$$

if and only if it satisfies the integral equation

$$\zeta(\varsigma) = \zeta_0\, e^{-\lambda\varsigma} + \frac{1}{\Gamma(\alpha)}\int_0^{\varsigma} e^{-\lambda(\varsigma - s)}(\varsigma - s)^{\alpha-1}h(s, \zeta(s))\, ds. \tag{5}$$

Lemma 2 *Let $\phi, \psi \in C([0, b], \mathbb{R})$. Then, the unique solution of the coupled system*

$$\begin{cases} {}_0^C D_\varsigma^{(\alpha,\lambda)} \zeta(\varsigma) = \phi(\varsigma), \quad \varsigma \in [0,b], \ \alpha \in (0,1), \ \lambda \geq 0, \\ {}_0^C D_\varsigma^{(\beta,\nu)} \xi(t) = \psi(\varsigma), \quad \varsigma \in [0,b], \ \beta \in (0,1), \ \nu \geq 0, \\ \zeta(0) + \zeta(b) = 0, \\ \xi(0) + \xi(b) = 0, \end{cases} \tag{6}$$

is equivalent to the integral equations

$$\zeta(\varsigma) = \frac{1}{\Gamma(\alpha)} \int_0^\varsigma e^{-\lambda(\varsigma-s)} (\varsigma - s)^{\alpha-1} \phi(s) ds - \frac{e^{-\lambda\varsigma}}{(1+e^{-\lambda b})\Gamma(\alpha)} \int_0^b e^{-\lambda(b-s)} (b-s)^{\alpha-1} \phi(s) ds, \tag{7}$$

and

$$\xi(\varsigma) = \frac{1}{\Gamma(\beta)} \int_0^\varsigma e^{-\nu(\varsigma-s)} (\varsigma - s)^{\beta-1} \psi(s) ds - \frac{e^{-\nu\varsigma}}{(1+e^{-\nu b})\Gamma(\beta)} \int_0^b e^{-\nu(b-s)} (b-s)^{\beta-1} \psi(s) ds. \tag{8}$$

Proof If $\zeta(\varsigma)$ is a solution of (7), then from (7), one has

$$\zeta(0) = \frac{-1}{(1+e^{-\lambda b})\Gamma(\alpha)} \int_0^b e^{-\lambda(b-s)} (b-s)^{\alpha-1} \phi(s) ds, \tag{9}$$

and

$$\zeta(b) = \frac{1}{(1+e^{-\lambda b})\Gamma(\alpha)} \int_0^b e^{-\lambda(b-s)} (b-s)^{\alpha-1} \phi(s) ds, \tag{10}$$

which prove that $\zeta(0) + \zeta(b) = 0$. Applying the Caputo tempered fractional derivative ${}_0^C D_\varsigma^{(\alpha,\lambda)}$ on both sides of Eq. (7) yields the first equation of (6). Therefore, ζ is a solution of the first equation of (6) with the boundary condition $\zeta(0) + \zeta(b) = 0$.

Conversely, if ζ is a solution of the first equation of (6), then by Lemma 1, we obtain

$$\zeta(\varsigma) = \zeta(0) e^{-\lambda\varsigma} + \frac{1}{\Gamma(\alpha)} \int_0^\varsigma e^{-\lambda(\varsigma-s)} (\varsigma - s)^{\alpha-1} \phi(s) ds. \tag{11}$$

Taking $\varsigma = b$ in the Eq. (11), we get

$$\zeta(b) = \zeta(0) e^{-\lambda b} + \frac{1}{\Gamma(\alpha)} \int_0^b e^{-\lambda(b-s)} (b-s)^{\alpha-1} \phi(s) ds. \tag{12}$$

Thus, the boundary condition $\zeta(0) + \zeta(b) = 0$ gives

$$\zeta(0) = e^{-\lambda b} + \frac{-1}{(1+e^{-\lambda b})\Gamma(\alpha)} \int_0^b e^{-\lambda(b-s)} (b-s)^{\alpha-1} \phi(s) ds. \tag{13}$$

Hence, by the substitution the value of $\zeta(0)$ in Eq. (11), we get the integral equation (7). The proof is finished.

3 Main Results

Consider the space $\mathbf{U} = \{\zeta(\varsigma) : \zeta(\varsigma) \in C([0, b], \mathbb{R})\}$ with the norm $\|\zeta\|_{\mathbf{U}} = \max_{\varsigma \in [0,b]} |u(\varsigma)|$. It is clear that $(\mathbf{U}, \|\cdot\|_{\mathbf{U}})$ is a Banach space. Also let $\mathbf{V} = \{\xi(\varsigma) : \xi(\varsigma) \in C([0, b], \mathbb{R})\}$ endowed with the norm $\|\xi\|_{\mathbf{V}} = \max_{t \in [0,b]} |\xi(t)|$. The product space $(\mathbf{U} \times \mathbf{V}, \|(\zeta, \xi)\|_{\mathbf{U}\times\mathbf{V}})$ is also a Banach space with the norm $\|(\zeta, \xi)\|_{\mathbf{U}\times\mathbf{V}} = \|\zeta\|_{\mathbf{U}} + \|\xi\|_{\mathbf{V}}$.

Define the operator $\mathscr{T} : \mathbf{U} \times \mathbf{V} \to \mathbf{U} \times \mathbf{V}$ by

$$\mathscr{T}(\zeta, \xi)(\varsigma) = \begin{pmatrix} \mathscr{T}_1(\zeta, \xi)(\varsigma) \\ \mathscr{T}_2(\zeta, \xi)(\varsigma), \end{pmatrix}$$

where

$$\begin{aligned} \mathscr{T}_1(\zeta, \xi)(\varsigma) = & \frac{1}{\Gamma(\alpha)} \int_0^{\varsigma} e^{-\lambda(\varsigma - s)} (\varsigma - s)^{\alpha-1} \Phi(s, \zeta(s), \xi(s)) ds \\ & - \frac{e^{-\lambda\varsigma}}{(1 + e^{-\lambda b})\Gamma(\alpha)} \int_0^b e^{-\lambda(b-s)} (b - s)^{\alpha-1} \Phi(s, \zeta(s), \xi(s)) ds, \end{aligned} \tag{14}$$

and

$$\begin{aligned} \mathscr{T}_2(\zeta, \xi)(\varsigma) = & \frac{1}{\Gamma(\beta)} \int_0^{\varsigma} e^{-\nu(\varsigma - s)} (\varsigma - s)^{\beta-1} \Psi(s, \zeta(s), \xi(s))\, ds \\ & - \frac{e^{-\nu t}}{(1 + e^{-\nu b})\Gamma(\beta)} \int_0^b e^{-\nu(b-s)} (b - s)^{\beta-1} \Psi(s, \zeta(s), \xi(s))\, ds. \end{aligned} \tag{15}$$

Lemma 3 (Leray-Schauder alternative [13]) *Let $\mathscr{T} : \mathbf{E} \to \mathbf{E}$ be a completely continuous operator, i.e., $\mathscr{T}$ is continuous and maps any bounded subset of $\mathbf{D} \subset \mathbf{E}$ into a relatively compact subset. Consider the set*

$$\mathscr{V}(\mathscr{T}) = \{x \in \mathbf{E} : x = \mu \mathscr{T}(x), \text{ for some } 0 < \mu < 1\}.$$

Then, either the set $\mathscr{V}(\mathscr{T})$ is unbounded or $\mathscr{T}$ has at least one fixed point.

Theorem 1 *Assume that*

(H1) *$\Phi, \Psi : [0, b] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions and there exist real constants $k_i, l_i \geq 0,\ i = 1, 2$ and $k_0 > 0, l_0 > 0$ such that $\forall \zeta_i, \xi_i \in \mathbb{R},\ i = 1, 2,$*

$$|\Phi(s, \zeta, \xi)| \leq k_0 + k_1|\zeta| + k_2|\xi|, \quad |\Psi(s, u, v)| \leq l_0 + l_1|\zeta| + l_2|\xi|.$$

Then, the coupled system (1) possesses at least one solution on $[0, b]$.

Proof First we show that the operator $\mathscr{T} : \mathbf{U} \times \mathbf{V} \to \mathbf{U} \times \mathbf{V}$ is completely continuous. By the continuity of functions Φ and Ψ, the operator $\mathscr{T}$ is continuous.

Let $\mathscr{K} \in \mathbf{U} \times \mathbf{V}$ be bounded. Then, there exist constants $L_1 > 0,\ L_2 > 0$ such that $|\Phi(t, \zeta(\varsigma), \xi(\varsigma))| \leq L_1$ and $|\Psi(t, \zeta(\varsigma), \xi(\varsigma))| \leq L_2$. Then, for any $(\zeta, \xi) \in \mathscr{K}$, one has

$$\begin{aligned}
|\mathscr{T}_1(\zeta, \xi)(\varsigma)| &\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^{\varsigma} e^{-\lambda(\varsigma - s)} (\varsigma - s)^{\alpha-1} \Phi(s, \zeta(s), \xi(s))\, ds \right| \\
&+ \left| \frac{e^{-\lambda\varsigma}}{(1 + e^{-\lambda b})\Gamma(\alpha)} \int_0^{b} e^{-\lambda(b - s)} (b - s)^{\alpha-1} \Phi(s, \zeta(s), \xi(s))\, ds \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{\varsigma} e^{\lambda s} (\varsigma - s)^{\alpha-1} |\Phi(s, \zeta(s), \xi(s))|\, ds \\
&+ \left| \frac{e^{-\lambda b}}{1 + e^{-\lambda b}} \right| \frac{1}{\Gamma(\alpha)} \int_0^{b} e^{\lambda s} (b - s)^{\alpha-1} |\Phi(s, \zeta(s), \xi(s))|\, ds \\
&\leq \frac{2e^{\lambda b} b^{\alpha} L_1}{\Gamma(\alpha + 1)}.
\end{aligned}$$

Thus,

$$\|\mathscr{T}_1(\zeta, \xi)(\varsigma)\|_{\mathbf{U}} \leq \frac{2e^{\lambda b} b^{\alpha} L_1}{\Gamma(\alpha + 1)}. \tag{16}$$

Similarly, we get

$$\|\mathscr{T}_2(\zeta, \xi)(\varsigma)\|_{\mathbf{V}} \leq \frac{2e^{\nu b} b^{\beta} L_2}{\Gamma(\beta + 1)}. \tag{17}$$

From inequalities (16), (17), we conclude that the operator $\mathscr{T}$ is uniformly bounded. Next, we show that the operator $\mathscr{T}$ is equicontinuous. For $\varsigma_1, \varsigma_2 \in [0, b]$ with $\varsigma_1 < \varsigma_2$, one has

$$\begin{aligned}
|\mathscr{T}_1(\zeta, \xi)(\varsigma_2) - \mathscr{T}_1(\zeta, \xi)(\varsigma_1)| &\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^{\varsigma_2} e^{-\lambda(\varsigma_2 - s)} (\varsigma_2 - s)^{\alpha-1} \Phi(s, \zeta(s), \xi(s))\, ds \right. \\
&- \frac{e^{-\lambda\varsigma_2}}{(1 + e^{-\lambda b})\Gamma(\alpha)} \int_0^{b} e^{-\lambda(b - s)} (b - s)^{\alpha-1} \Phi(s, \zeta(s), \xi(s))\, ds \\
&- \frac{1}{\Gamma(\alpha)} \int_0^{\varsigma_1} e^{-\lambda(\varsigma_1 - s)} (\varsigma_1 - s)^{\alpha-1} \Phi(s, \zeta(s), \xi(s))\, ds \\
&+ \left. \frac{e^{-\lambda\varsigma_1}}{(1 + e^{-\lambda b})\Gamma(\alpha)} \int_0^{b} e^{-\lambda(b - s)} (b - s)^{\alpha-1} \Phi(s, \zeta(s), \xi(s))\, ds \right|
\end{aligned}$$

$$\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^{\varsigma_1} \left(e^{-\lambda(\varsigma_2 - s)} (\varsigma_2 - s)^{\alpha-1} - e^{-\lambda(\varsigma_1 - s)} (\varsigma_1 - s)^{\alpha-1} \right) \Phi(s, \zeta(s), \xi(s))\, ds \right.$$
$$\left. + \frac{1}{\Gamma(\alpha)} \int_{\varsigma_1}^{\varsigma_2} e^{-\lambda(\varsigma_2 - s)} (\varsigma_2 - s)^{\alpha-1} \Phi(s, \zeta(s), \xi(s))\, ds \right|$$
$$+ \left| \frac{e^{-\lambda\varsigma_1} - e^{-\lambda\varsigma_2}}{(1 + e^{-\lambda b})\Gamma(\alpha)} \int_0^b e^{-\lambda(b-s)} (b - s)^{\alpha-1} \Phi(s, \zeta(s), \xi(s))\, ds \right|$$
$$\leq \frac{e^{\lambda b} L_1}{\Gamma(\alpha)} \int_0^{\varsigma_1} \left((\varsigma_2 - s)^{\alpha-1} - (\varsigma_1 - s)^{\alpha-1} \right) ds$$
$$+ \frac{e^{\lambda b} L_1}{\Gamma(\alpha)} \int_{\varsigma_1}^{\varsigma_2} (\varsigma_2 - s)^{\alpha-1} ds + \frac{e^{\lambda b} L_1}{\Gamma(\alpha)} (e^{-\lambda\varsigma_1} - e^{-\lambda\varsigma_2}) \int_0^b (b - s)^{\alpha-1} ds$$
$$= \frac{e^{\lambda b} L_1}{\Gamma(\alpha + 1)} \left((\varsigma_1^\alpha - \varsigma_2^\alpha) - (e^{-\lambda\varsigma_1} - e^{-\lambda\varsigma_2}) b^\alpha \right) \to 0,$$

independent of $(\zeta, \xi) \in \mathscr{K}$ as $\varsigma_2 - \varsigma_1 \to 0$. Also

$$|\mathscr{T}_2(\zeta, \xi)(\varsigma_2) - \mathscr{T}_2(\zeta, \xi)(\varsigma_1)| \leq \frac{e^{\nu b} L_2}{\Gamma(\beta + 1)} \left((\varsigma_1^\beta - \varsigma_2^\beta) - (e^{-\nu\varsigma_1} - e^{-\nu\varsigma_2}) b^\beta \right) \to 0,$$

independent of $(\zeta, \xi) \in \mathscr{K}$ as $\varsigma_2 - \varsigma_1 \to 0$. Analogously, we get

$$|\mathscr{T}_1(\zeta, \xi)(\varsigma_2) - \mathscr{T}_1(\zeta, \xi)(\varsigma_1)| \to 0, \ |\mathscr{T}_2(\zeta, \xi)(\varsigma_2) - \mathscr{T}_2(\zeta, \xi)(\varsigma_1)| \to 0,$$

independent of $(\zeta, \xi) \in \mathscr{K}$ as $\varsigma_2 - \varsigma_1 \to 0$. This shows that the operator $\mathscr{T}$ is equicontinuous. By Arzelà-Ascoli's theorem, we infer that $\mathscr{T}$ is completely continuous.

Finally, we show that the set

$$\mathscr{V} = \{(\zeta, \xi) \in \mathbf{U} \times \mathbf{V} : (\zeta, \xi) = \mu \mathscr{T}(\zeta, \xi), \ 0 < \mu < 1\}$$

is bounded.
Let $(\zeta, \xi) \in \mathscr{V}$, then $(\zeta, \xi) = \mu \mathscr{T}(\zeta, \xi)$. For any $\varsigma \in [0, b]$, one has

$$\zeta(\varsigma) = \mu \mathscr{T}_1(\zeta, \xi)(\varsigma), \ \ \xi(\varsigma) = \mu \mathscr{T}_2(\zeta, \xi)(\varsigma).$$

Then we have

$$|\zeta(\varsigma)| = |\mu \mathscr{T}_1(\zeta, \xi)(\varsigma)| \leq |\mathscr{T}_1(\zeta, \xi)(\varsigma)|$$
$$\leq \frac{2 e^{\lambda b} b^\alpha}{\Gamma(\alpha + 1)} (k_0 + k_1 |\zeta| + k_2 |\xi|),$$

and

$$|\xi(\varsigma)| = |\mu \mathscr{T}_2(\zeta,\xi)(\varsigma)| \leq |\mathscr{T}_2(\zeta,\xi)(\varsigma)| \leq \frac{2e^{\nu b}b^{\beta}}{\Gamma(\beta+1)}(l_0 + l_1|\zeta| + l_2|\xi|).$$

Hence, we get

$$\|\zeta\|_{\mathbf{U}} \leq \frac{2e^{\lambda b}b^{\alpha}}{\Gamma(\alpha+1)}(k_0 + k_1\|\zeta\|_{\mathbf{U}} + k_2\|\xi\|_{\mathbf{V}}),$$
$$\|\xi\|_{\mathbf{V}} \leq \frac{2e^{\nu b}b^{\beta}}{\Gamma(\beta+1)}(l_0 + l_1\|\zeta\|_{\mathbf{U}} + l_2\|\xi\|_{\mathbf{V}}),$$

which imply that

$$\|\zeta\|_{\mathbf{U}} + \|\xi\|_{\mathbf{V}} \leq (\Lambda_0 + \Lambda_1\|\zeta\|_{\mathbf{U}} + \Lambda_2\|\xi\|_{\mathbf{V}}),$$

where

$$\begin{aligned}
\Lambda_0 &= \frac{2e^{\lambda b}b^{\alpha}}{\Gamma(\alpha+1)}k_0 + \frac{2e^{\nu b}b^{\beta}}{\Gamma(\beta+1)}l_0,\\
\Lambda_1 &= \frac{2e^{\lambda b}b^{\alpha}}{\Gamma(\alpha+1)}k_1 + \frac{2e^{\nu b}b^{\beta}}{\Gamma(\beta+1)}l_1,\\
\Lambda_2 &= \frac{2e^{\lambda b}b^{\alpha}}{\Gamma(\alpha+1)}k_2 + \frac{2e^{\nu b}b^{\beta}}{\Gamma(\beta+1)}l_2.
\end{aligned} \tag{18}$$

Consequently, we get

$$\|(\zeta,\xi)\|_{\mathbf{U}\times\mathbf{V}} \leq \frac{\Lambda_0}{\Lambda}, \quad \Lambda = \min\{1-\Lambda_1, 1-\Lambda_2\}. \tag{19}$$

Therefore, the set $\mathscr{V}$ is bounded. Hence, by Lemma 3, the operator $\mathscr{T}$ possesses at least one fixed point which is a solution of the coupled system (1). The proof is finished.

Theorem 2 *Let $\Phi, \Psi : [0,b]\times\mathbb{R}^2 \to \mathbb{R}$ be continuous functions satisfying the following Lipschitz condition:*

(H2) $$|\Phi(\varsigma,\zeta_1,\xi_1) - \Phi(\varsigma,\zeta_2,\xi_2)| \leq L_{\Phi}(|\zeta_1-\zeta_2| + |\xi_1-\xi_2|),$$

$$|\Psi(\varsigma,\zeta_1,\xi_1) - \Psi(\varsigma,\zeta_2,\xi_2)| \leq L_{\Psi}(|\zeta_1-\zeta_2| + |\xi_1-\xi_2|),$$

$$L_{\Phi}, L_{\Psi} > 0,\ \forall\varsigma\in[0,b],\ \zeta_i,\xi_i\in\mathbb{R},\ i=1,2.$$

Then, the coupled system (1) has a unique solution on $[0,b]$ *provided that*

$$\left(\frac{2e^{\lambda b}b^{\alpha}L_{\Phi}}{\Gamma(\alpha+1)} + \frac{2e^{\nu b}b^{\beta}L_{\Psi}}{\Gamma(\beta+1)}\right) < 1. \tag{20}$$

Proof Let us set $M_{\Phi} = \max_{\varsigma\in[0,b]} |\Phi(\varsigma, 0, 0)| < \infty$ and $M_{\Psi} = \max_{\varsigma\in[0,b]} |\Psi(\varsigma, 0, 0)| < \infty$ and define

$$r \geq \max\left\{ \frac{\frac{2e^{\lambda b}b^{\alpha}M_{\Phi}}{\Gamma(\alpha+1)}}{\frac{1}{2} - \frac{2e^{\lambda b}b^{\alpha}L_{\Phi}}{\Gamma(\alpha+1)}}, \frac{\frac{2e^{\nu b}b^{\beta}M_{\Psi}}{\Gamma(\beta+1)}}{\frac{1}{2} - \frac{2e^{\nu b}b^{\beta}L_{\Psi}}{\Gamma(\beta+1)}} \right\},$$

with

$$\frac{2e^{\lambda b}b^{\alpha}L_{\Phi}}{\Gamma(\alpha+1)} < \frac{1}{2}, \quad \frac{2e^{\nu b}b^{\beta}L_{\Psi}}{\Gamma(\beta+1)} < \frac{1}{2}.$$

We first show that $\mathscr{T}\mathscr{B}_r \subset \mathscr{B}_r$, where

$$\mathscr{B}_r = \{(\zeta, \xi) \in \mathbf{U} \times \mathbf{V} : \|(\zeta, \xi)\|_{\mathbf{U}\times\mathbf{V}} \leq r\}.$$

For $(\zeta, \xi) \in \mathscr{B}_r,\ \varsigma \in [0, b]$, we get

$$\begin{aligned}
|\mathscr{T}_1(\zeta, \xi)(\varsigma)| &\leq \left|\frac{1}{\Gamma(\alpha)}\int_0^{\varsigma} e^{-\lambda(\varsigma-s)}(\varsigma - s)^{\alpha-1}\Phi(s, \zeta(s), \xi(s))\,ds\right| \\
&+ \left|\frac{e^{-\lambda\varsigma}}{(1+e^{-\lambda b})\Gamma(\alpha)}\int_0^{b} e^{-\lambda(b-s)}(b - s)^{\alpha-1}\Phi(s, \zeta(s), \xi(s))ds\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{\varsigma} e^{\lambda s}(\varsigma - s)^{\alpha-1}(|\Phi(s, \zeta(s), \xi(s)) - \Phi(s, 0, 0)| + |\Phi(s, 0, 0)|)ds \\
&+ \left|\frac{e^{-\lambda b}}{1+e^{-\lambda b}}\right|\frac{1}{\Gamma(\alpha)}\int_0^{b} e^{\lambda s}(b - s)^{\alpha-1}(|\Phi(s, \zeta(s), \xi(s)) - \Phi(s, 0, 0)| + |\Phi(s, 0, 0)|)ds \\
&\leq \frac{2e^{\lambda b}b^{\alpha}}{\Gamma(\alpha+1)}(L_{\Phi}(\|\zeta\|_{\mathbf{U}} + \|\xi\|_{\mathbf{V}}) + M_{\Phi}) \\
&\leq \frac{2e^{\lambda b}b^{\alpha}}{\Gamma(\alpha+1)}(L_{\Phi}\|(\zeta, \xi)\|_{\mathbf{U}\times\mathbf{V}} + M_{\Phi}) \\
&\leq \frac{2e^{\lambda b}b^{\alpha}}{\Gamma(\alpha+1)}(L_{\Phi}\, r + M_{\Phi}).
\end{aligned}$$

Thus,

$$\|\mathscr{T}_1(\zeta, \xi)\|_{\mathbf{U}} \leq \frac{2e^{\lambda b}b^{\alpha}}{\Gamma(\alpha+1)}(L_{\Phi}\, r + M_{\Phi}) \leq \frac{r}{2}. \tag{21}$$

Similarly, we get

$$\|\mathscr{T}_2(\zeta, \xi)\|_{\mathbf{V}} \leq \frac{2e^{\nu b}b^{\beta}}{\Gamma(\beta+1)}(L_{\Psi}\, r + M_{\Psi}) \leq \frac{r}{2}. \tag{22}$$

From (21) to (22), it follows that $\|\mathscr{T}(\zeta, \xi)\|_{\mathbf{U}\times\mathbf{V}} \leq r$, which proves that $\mathscr{T}\mathscr{B}_r \subset \mathscr{B}_r$.

For $\zeta_i, \xi_i \in \mathscr{B}_r,\ i = 1, 2$ and for each $\varsigma \in [0, b]$, one has

$$
\begin{aligned}
&|\mathscr{T}_1(\zeta_1, \xi_1)(\varsigma) - \mathscr{T}_1(\zeta_2, \xi_2)(\varsigma)| \\
&\quad \leq \left| \frac{1}{\Gamma(\alpha)} \int_0^{\varsigma} e^{-\lambda(\varsigma - s)} (\varsigma - s)^{\alpha - 1} (\Phi(s, \zeta_1(s), \xi_1(s) - \Phi(s, \zeta_2(s), \xi_2(s)) ds \right| \\
&\quad + \left| \frac{e^{-\lambda\varsigma}}{(1 + e^{-\lambda b}) \Gamma(\alpha)} \int_0^b e^{-\lambda(b - s)} (b - s)^{\alpha - 1} (\Phi(s, \zeta_1(s), \xi_1(s) - \Phi(s, \zeta_2(s), \xi_2(s)) ds \right| \\
&\quad \leq \frac{2 e^{\lambda b} b^{\alpha} L_{\Phi}}{\Gamma(\alpha + 1)} (\|\zeta_1 - \zeta_2\| + \|\xi_1 - \xi_2\|).
\end{aligned}
$$

This gives

$$
\|\mathscr{T}_1(\zeta_1, \xi_1) - \mathscr{T}_1(\zeta_2, \xi_2)\|_{\mathbf{U}} \leq \frac{2 e^{\lambda b} b^{\alpha} L_{\Phi}}{\Gamma(\alpha + 1)} (\|\zeta_1 - \eta_2\|_{\mathbf{U}} + \|\xi_1 - \xi_2\|_{\mathbf{V}}).
$$

Also, in a similar way, we get

$$
\|\mathscr{T}_2(\zeta_1, \xi_1) - \mathscr{T}_2(\zeta_2, \xi_2)\|_{\mathbf{V}} \leq \frac{2 e^{\nu b} b^{\beta} L_{\Psi}}{\Gamma(\beta + 1)} (\|\zeta_1 - \zeta_2\|_{\mathbf{U}} + \|\xi_1 - \xi_2\|_{\mathbf{V}}).
$$

Consequently, we get

$$
\|\mathscr{T}(\zeta_1, \xi_1) - \mathscr{T}(\zeta_2, \xi_2)\|_{\mathbf{U} \times \mathbf{V}} \leq \left(\frac{2 e^{\lambda b} b^{\alpha} L_{\Phi}}{\Gamma(\alpha + 1)} + \frac{2 e^{\nu b} b^{\beta} L_{\Psi}}{\Gamma(\beta + 1)} \right) (\|\zeta_1 - \zeta_2\|_{\mathbf{U}} + \|\xi_1 - \xi_2\|_{\mathbf{V}}).
$$

In view of the condition (20), we deduce that the operator $\mathscr{T}$ is a contraction. Hence, by virtue of Banach's fixed point theorem, the operator $\mathscr{T}$ has a unique fixed point which corresponds to the unique solution of the coupled system (1). This completes the proof.

4 Examples

Example 1 Consider the following coupled system of *TFDEs*

$$
\begin{cases}
{}^{C}D^{(\frac{1}{3},1)} \zeta(\varsigma) = \frac{e^{-3\varsigma}}{75+\varsigma} (\sin(\zeta(\varsigma)) + |\xi(\varsigma)|) + \frac{e^{-\varsigma}}{1+\varsigma^2}, \quad \varsigma \in [0, 1], \\
{}^{C}D^{(\frac{1}{4},2)} \xi(\varsigma) = \frac{1}{2\varsigma^2+100} \left(\frac{|\zeta(\varsigma)|}{1+|\zeta(\varsigma)|} + \sin(\xi(\varsigma)) \right) + \sin\varsigma + 1, \\
\zeta(0) + \zeta(1) = 0, \\
\xi(0) + \xi(1) = 0.
\end{cases}
\tag{23}
$$

Here, $\alpha = \frac{1}{3}$, $\beta = \frac{1}{4}$, $\lambda = 1$, $\nu = 2$, $b = 1$, $\Phi(\varsigma, \zeta, \xi) = \frac{e^{-3\varsigma}}{75+\varsigma} (\sin(\zeta(\varsigma)) + |\xi(\varsigma)|) + \frac{e^{-\varsigma}}{1+\varsigma^2}$ and $\Psi(\varsigma, \zeta, \xi) = \frac{1}{2\varsigma^2+100} \left(\frac{|\zeta(\varsigma)|}{1+|\zeta(\varsigma)|} + \sin(\xi(\varsigma)) \right) + \sin\varsigma + 1$.

We have

$$|\Phi(\varsigma,\zeta_1,\xi_1)-\Phi(\varsigma,\zeta_2,\xi_2)|\leq\frac{1}{75}(|\zeta_1(\varsigma)-\zeta_2(\varsigma)|+|\xi_1(\varsigma)-\xi_2(\varsigma)|),$$

$$|\Psi(\varsigma,\zeta_1,\xi_1)-\Psi(\varsigma,\zeta_2,\xi_2)|\leq\frac{1}{100}(|\zeta_1(\varsigma)-\zeta_2(\varsigma)|+|\xi_1(\varsigma)-\xi_2(\varsigma)|),$$

from which, we get $L_\Phi=\frac{1}{75}$ and $L_\Psi=\frac{1}{100}$.

Using the given data, the condition (20) becomes

$$\left(\frac{2e^{\lambda b}b^\alpha L_\Phi}{\Gamma(\alpha+1)}+\frac{2e^{\nu b}b^\beta L_\Psi}{\Gamma(\beta+1)}\right)=\frac{2e}{75\Gamma\left(\frac{4}{3}\right)}+\frac{2e^2}{100\Gamma\left(\frac{5}{4}\right)}\approx 0.244216<1.$$

Further,

$$\left(\frac{2e^{\lambda b}b^\alpha L_\Phi}{\Gamma(\alpha+1)}\right)\approx 0.0811748<\frac{1}{2},\quad\left(\frac{2e^{\nu b}b^\beta L_{|psi}}{\Gamma(\beta+1)}\right)\approx 0.163041<\frac{1}{2}.$$

Thus, all the conditions of Theorem 2 hold true. Hence, it follows by the conclusion of Theorem 2 that there exists a unique solution for the coupled system (23) on [0, 1].

Example 2 Consider the following coupled system

$$\begin{cases} {}^CD^{(\frac{1}{3},1)}\zeta(\varsigma)=\frac{1}{\sqrt{625+\varsigma}}\cos\varsigma+\frac{e^{-\varsigma}}{200}\sin(\zeta(\varsigma))+\frac{1}{300}\frac{\xi(\varsigma)|\zeta(\varsigma)|}{1+|\zeta(\varsigma)|},\quad \varsigma\in[0,1],\\ {}^CD^{(\frac{1}{4},2)}\xi(\varsigma)=\frac{e^{-2\varsigma}}{2\sqrt{1600+\varsigma}}+\frac{1}{270}\sin(\zeta(\varsigma))+\frac{1}{3(60+\varsigma)}\sin(\xi(\varsigma)),\\ \zeta(0)+\zeta(1)=0,\\ \xi(0)+\xi(1)=0. \end{cases}\tag{24}$$

Obviously,

$$|\Phi(\varsigma,\zeta(\varsigma),\xi(\varsigma))|\leq\frac{1}{25}+\frac{1}{200}\|\zeta\|+\frac{1}{300}\|\xi\|,$$

$$|\Psi(\varsigma,\zeta(\varsigma),\xi(\varsigma))|\leq\frac{1}{80}+\frac{1}{270}\|\zeta\|+\frac{1}{180}\|\xi\|.$$

Thus $k_0=\frac{1}{25},\ k_1=\frac{1}{200},\ k_2=\frac{1}{300},\ l_0=\frac{1}{80},\ l_1=\frac{1}{270},\ l_2=\frac{1}{180}$.
Using (18) and (19), we find that

$$\Lambda_0=\frac{2e}{25\Gamma(4/3)}+\frac{2e^2}{80\Gamma(5/4)}=0.4473264167,$$
$$\Lambda_1=\frac{2e}{200\Gamma(4/3)}+\frac{2e^2}{270\Gamma(5/4)}=0.09082628542,$$
$$\Lambda_2=\frac{2e}{300\Gamma(4/3)}+\frac{2e^2}{180\Gamma(5/4)}=0.2935157814,$$

and $\Lambda = \min\{0.9091737146, 0.7064842186\} = 0.7064842186$.

Therefore, $\|(u, v)\|_{\mathrm{U}\times V} \leq \frac{\Lambda_0}{\Lambda} = 0.6331725535$. Thus, the assumptions of Theorem 1 hold true. Hence, there exists a solution for the system (24) on [0, 1].

Conclusions

A coupled system of nonlinear *TFDEs* with anti-periodic boundary conditions is investigated. The existence and uniqueness theorems for the proposed system are established by using Leray-Schauder alternative and Banach's fixed point theorem. At the end of this work, we present two examples in order to verify the obtained theoretical results.

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Application of Measure of Noncompactness on the Infinite System of Hadamard Fractional Integral Equations

Anupam Das and Bipan Hazarika

Abstract In this chapter, Darbo's fixed point theorem (DFPT) is applied to check the existence of solution of infinite system of Hadamard fractional integral equations in $\mathscr{C}(Z, c_0)$, $\mathscr{C}(Z, \ell_1)$ and $\mathscr{C}(Z, \ell_p^\alpha)$, where $Z = [1, \tau]$. Also justify the results with the help of an example.

Keywords Fractional integral equation (FIE) · Measure of noncompactness (MNC) · Darbo's fixed point theorem

MSC subject classification: 45G05 · 26A33 · 74H20.

1 Introduction

Fractional calculus play a vital role to study the applications of integral and differential equations in the field of mathematical analysis. The different types of fractional integral equations were studied with the help of fixed point theorems. For solving different types of integral and differential equations for overcoming of different real life problems, numerous researchers have applied fixed point theorem connecting measure of noncompactness (see [2, 8, 9, 13–17, 19, 23]). Schauder and Darbo's fixed point theorems play a significant role for solving functional integral equations.

In this chapter, we use the concept of MNC to check the existence of solutions of infinite system Hadamard fractional integral equations. Also, verify our main results with the help of examples. Throughout the chapter, we consider

- $(\mathfrak{X}, \|\,.\,\|)$ is a real Banach space.
- $B(\gamma, \delta)$ is a closed ball in $\mathfrak{X}$ with radius δ and centered at γ.

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- $\bar{\mathscr{A}}$ and Conv$\mathscr{A}$ is the closure and convex closure of $\mathscr{A}$, respectively, where $\mathscr{A}(\neq \emptyset) \subset \mathfrak{T}$.
- $\mathfrak{K}_{\mathfrak{T}}$ is the family of all nonempty and bounded subsets of $\mathfrak{T}$.
- $\mathfrak{L}_{\mathfrak{T}}$ is the subfamily consisting of all relatively compact sets.

Definition 1 [3] A function $\varphi : \mathfrak{K}_{\mathfrak{T}} \to [0, \infty)$ is called a MNC in $\mathfrak{T}$ if the following conditions hold:

(i) $\varphi(\mathscr{H}) = 0 \implies \mathscr{H}$ is precompact, for all $\mathscr{H} \in \mathfrak{K}_{\mathfrak{T}}$.
(ii) the family $\ker \varphi = \{\mathscr{H} \in \mathfrak{K}_{\mathfrak{T}} : \varphi(\mathscr{H}) = 0\}$ is nonempty and $\ker \varphi \subset \mathfrak{L}_{\mathfrak{T}}$.
(iii) $\mathscr{H} \subseteq \mathscr{J} \implies \varphi(\mathscr{H}) \leq \varphi\left(\mathscr{J}\right)$.
(iv) $\varphi\left(\bar{\mathscr{H}}\right) = \varphi(\mathscr{H})$.
(v) $\varphi(Conv\mathscr{H}) = \varphi(\mathscr{H})$.
(vi) $\varphi\left(\beta\mathscr{H} + (1-\beta)\mathscr{J}\right) \leq \beta\varphi(\mathscr{H}) + (1-\beta)\varphi\left(\mathscr{J}\right)$ for $\beta \in [0, 1]$.
(vii) if $\mathscr{H}_n \in \mathfrak{K}_{\mathfrak{T}},\ \mathscr{H}_n = \bar{\mathscr{H}_n},\ \mathscr{H}_{n+1} \subset \mathscr{H}_n$ for $n = 1, 2, 3, \ldots$ and $\lim_{n\to\infty} \varphi(\mathscr{H}_n) = 0$ then $\bigcap_{n=1}^{\infty} \mathscr{H}_n \neq \phi$.

The family $ker\varphi$ is said to be the *kernel of measure* φ. It is clear that the intersection set $\mathscr{H}_\infty$ from (vii) is a member of the family $ker\varphi$. In fact, $\varphi(\mathscr{H}_\infty) \leq \varphi(\mathscr{H}_n)$ for any n, we infer that $\varphi(\mathscr{H}_\infty) = 0$. This gives $\mathscr{H}_\infty \in ker\varphi$.

For a bounded subset $\mathscr{S}$ of a metric space $\mathfrak{X}$, the Kuratowski measure of noncompactness is defined as [10]

$$\xi(\mathscr{S}) = \inf\left\{\delta > 0 : \mathscr{S} = \bigcup_{i=1}^{n} \mathscr{S}_i,\ \text{diam}(\mathscr{S}_i) \leq \delta\ for\ 1 \leq i \leq n \leq \infty\right\},$$

where diam $(\mathscr{S}_i)$ denotes the diameter of the set $\mathscr{S}_i$, that is,

$$\text{diam}(\mathscr{S}_i) = \sup\{d(x, y) : x, y \in \mathscr{S}_i\}.$$

The Hausdorff measure of noncompactness for a bounded set $\mathscr{S}$ is defined as

$$\zeta(\mathscr{S}) = \inf\{\epsilon > 0 : \mathscr{S} \text{ has finite } \epsilon - \text{net in } \mathfrak{X}\}.$$

The Hausdorff measure of noncompactness ζ for the Banach space $\left(c_0, \|\ .\ \|_{c_0}\right)$ is defined as (see [3]):

$$\zeta_{c_0}\left(\hat{D}\right) = \lim_{n\to\infty}\left[\sup_{u\in\hat{D}}\left(\max_{k\geq n} \mid u_k \mid\right)\right], \tag{1}$$

where $u = (u_i)_{i=1}^{\infty} \in c_0$ and $\hat{D} \in \mathscr{M}_{c_0}$.

The Hausdorff measure of noncompactness ζ, in the Banach space $\left(\ell_1, \|\ .\ \|_{\ell_1}\right)$ is defined as (see [3])

$$\zeta_{\ell_1}\left(\hat{D}\right) = \lim_{n\to\infty}\left[\sup_{u\in\hat{D}}\left(\sum_{k=n}^{\infty} \mid u_k \mid\right)\right], \tag{2}$$

where $u = (u_i)_{i=1}^{\infty} \in \ell_1$ and $\hat{D} \in \mathscr{M}_{\ell_1}$.

Denote $\mathscr{C}(Z, c_0)$, $Z = [a, \tau]$, $a \geq 0$, $\tau > 0$ is the space of all continuous functions on Z with values in c_0. Also, $\mathscr{C}(Z, c_0)$ becomes a Banach space with norm $\| x(t) \|_{\mathscr{C}(Z,c_0)} = \sup\left\{\| x(t) \|_{c_0} : t \in Z\right\}$, where $x(t) \in \mathscr{C}(Z, c_0)$.

If $\hat{\mathscr{F}}$ is any nonempty bounded subset of $\mathscr{C}(Z, c_0)$ and $t \in Z$, let $\hat{\mathscr{F}}(t) = \left\{x(t) : x \in \hat{\mathscr{F}}\right\}$. Now, using (1), we conclude that the Housdorff measure of noncompactness for $\hat{\mathscr{F}} \subset \mathscr{C}(Z, c_0)$ can be defined by

$$\zeta_{\mathscr{C}(Z,c_0)}(\hat{\mathscr{F}}) = \sup\left\{\zeta_{c_0}(\hat{\mathscr{F}}(t)) : t \in Z\right\}.$$

Similarly, $\mathscr{C}(Z, \ell_1)$ denotes the space of all continuous functions defined on Z with values in ℓ_1. Also, $\mathscr{C}(Z, \ell_1)$ is a Banach space with the norm

$$\| x(t) \|_{\mathscr{C}(Z,\ell_1)} = \sup\left\{\| x(t) \|_{\ell_1} : t \in Z\right\},$$

where $x(t) \in \mathscr{C}(Z, \ell_1)$.

Now, using (2), we conclude that the Housdorff measure of noncompactness for $\hat{\mathscr{F}} \subset \mathscr{C}(Z, \ell_1)$ can be defined by

$$\zeta_{\mathscr{C}(Z,\ell_1)}(\hat{\mathscr{F}}) = \sup\left\{\zeta_{\ell_1}(\hat{\mathscr{F}}(t)) : t \in Z\right\}.$$

Banaś and Krajewska [4] introduced tempered sequence spaces considering a fixed positive nonincreasing real sequence $\alpha = (\alpha_i)_{i=1}^{\infty}$ called the tempering sequence. Recently, Rebbani et al. [21] introduced the set $\mathscr{B}$ which consists of all real or complex sequences $y = (y_i)_{i=1}^{\infty}$ such that the sequence $\sum_{i=1}^{\infty} \alpha_i^p \, |y_i|^p < \infty$, $(1 \leq p < \infty)$. Clearly, $\mathscr{B}$ is a linear space over the field of real (or complex) numbers and to denote this space by $\mathscr{B} :\equiv \ell_p^\alpha$ for $1 \leq p < \infty$.

It is easy to observe that ℓ_p^α for $1 \leq p < \infty$ is a Banach space with the norm

$$\| y \|_{\ell_p^\alpha} = \left(\sum_{i=1}^{\infty} \alpha_i^p \, |y_i|^p\right)^{\frac{1}{p}}.$$

If we choose $\alpha_i = 1$ for all $i \in \mathbb{N}$ then $\ell_p^\alpha = \ell_p$ for $1 \leq p < \infty$.

The Hausdorff MNC $\zeta_{\ell_p^\alpha}$ for a nonempty bounded set B^α is given by (see [21])

$$\zeta_{\ell_p^\alpha}(B^\alpha) = \lim_{n\to\infty}\left[\sup_{y\in B^\alpha}\left(\sum_{k\geq n}\alpha_k^p \mid y_k \mid^p\right)^{\frac{1}{p}}\right]. \tag{3}$$

Let us denote $\mathscr{C}(Z, \ell_p^\alpha)$ be the space of all continuous functions on $Z = [a, \tau], a \geq 0, \tau > 0$ with the value on $\ell_p^\alpha(1 \leq p < \infty)$ and it is also a Banach space with the norm

$$\| y \|_{\mathscr{C}(Z,\ell_p^\alpha)} = \sup_{t\in I} \| y(t) \|_{\ell_p^\alpha},$$

where $y(t) = (y_i(t))_{i=1}^\infty \in \mathscr{C}(Z, \ell_p^\alpha)$.

Let E^α be any nonempty bounded subset of $\mathscr{C}(Z, \ell_p^\alpha)$ and for $t \in Z$, $E^\alpha(t) = \{y(t) : y(t) \in E^\alpha\}$. Thus, the measure of noncompactness for $E^\alpha \subset \mathscr{C}(Z, \ell_p^\alpha)$ can be defined by

$$\zeta_{\mathscr{C}(Z,\ell_p^\alpha)}(E^\alpha) = \sup_{t\in Z}\zeta_{\ell_p^\alpha}(E^\alpha(t)).$$

Definition 2 [3] Let $\mathfrak{X}$ be a nonempty subset of a Banach space $\mathfrak{T}$ and $\mathfrak{T} : \mathfrak{X} \to \mathfrak{T}$ is a continuous operator transforming bounded subset of $\mathfrak{X}$ to bounded ones. We say that $\mathfrak{T}$ satisfies the Darbo condition with a constant k with respect to measure φ provided $\varphi(\mathfrak{T}\mathfrak{Y}) \leq k\varphi(\mathscr{H})$ for each $\mathscr{H} \in \mathfrak{K}_{\mathfrak{T}}$ such that $\mathscr{H} \subset \mathfrak{X}$.

The following important theorems are used in our discussions:

Theorem 1 ([1, Schauder]) *Let $\mathfrak{D}$ be a nonempty, closed, and convex subset of a Banach space $\bar{\mathfrak{T}}$. Then every compact, continuous map $\mathfrak{T} : \mathfrak{D} \to \mathfrak{D}$ has at least one fixed point.*

Theorem 2 ([11, Darbo]) *Let $\mathscr{J}$ be a nonempty, bounded, closed and convex subset of a Banach space $\bar{\mathfrak{T}}$. Let $\mathfrak{T} : \mathscr{J} \to \mathscr{J}$ be a continuous mapping. Assume that there is a constant $k \in [0, 1)$ such that*

$$\varphi(\mathfrak{S}\mathfrak{M}) \leq k\varphi(\mathfrak{K}), \ \mathfrak{K} \subseteq \mathscr{J}.$$

Then, $\mathfrak{T}$ has a fixed point.

2 Application of MNC on Infinite System of Hadamard Fractional Integral Equations

Let $t \in [0, \infty)$ and $Re(\eta) > 0$. The Hadamard fractional integral of order η, applied to the function $f \in L^p[a, b],\ 1 \leq p < \infty,\ 0 < a < b < \infty$, for $t \in [a, b]$, is defined by [12],

$$J^\eta f(t) = \frac{1}{\Gamma(\eta)}\int_a^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} f(\nu)\frac{d\nu}{\nu}.$$

Therefore, we have

$$J^{\eta} f(t) = \frac{1}{\Gamma(\eta)} \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} f(\nu)\frac{d\nu}{\nu},\ \eta > 0,\ t > 1.$$

In this part, the existence of the solution of the following system of fractional integral equations are studied:

$$y_n(t) = \Theta_n\left(t, y(t), \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} \beta_n(t, \nu, y(\nu))\frac{d\nu}{\nu}\right),\ n \in \mathbb{N}, \tag{4}$$

where $0 < \eta < 1,\ t \in Z = [1, \tau],\ \tau > 1,\ y(t) = (y_n(t))_{n=1}^{\infty} \in E$ and E is a Banach sequence space.

3 Existence of Solution on $\mathscr{C}(Z, c_0)$

Consider the following assumptions

(1) The functions $\Theta_n : Z \times \mathscr{C}(Z, c_0) \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$|\Theta_n(t, y(t), l) - \Theta_n(t, \bar{y}(t), m)| \le A_n(t)\,|y_n(t) - \bar{y}_n(t)| + B_n(t)\,|l - m|$$

for $y(t) = (y_n(t))_{n=1}^{\infty}$, $\bar{y}(t) = (\bar{y}_n(t))_{n=1}^{\infty} \in \mathscr{C}(Z, c_0)$ and $A_n,\ B_n : Z \to [0, \infty)$ $(n \in \mathbb{N})$ are continuous functions. Also,

$$\bar{\Theta}_n = \sup\left\{\left|\Theta_n\left(t, y^0, 0\right)\right| : t \in Z\right\},$$

where $y^0 = \left(y_n^0(t)\right)_{n=1}^{\infty} \in \mathscr{C}(Z, c_0)$ such that $y_n^0(t) = 0$ for all $t \in Z,\ n \in \mathbb{N}$ and $\sup_{n\in\mathbb{N}} \bar{\Theta}_n = \bar{\Theta},\ \lim_{n\to\infty} \bar{\Theta}_n = 0$.

(2) The functions $\beta_n : Z \times Z \times \mathscr{C}(Z, c_0) \to \mathbb{R}(n \in \mathbb{N})$ are continuous and there exists

$$\hat{\beta} = \sup\left\{|\beta_n(t, \nu, y(\nu))| : t, \nu \in Z; n \in \mathbb{N}; y(\nu) \in \mathscr{C}(Z, c_0)\right\}.$$

(3) Define an operator T from $Z \times \mathscr{C}(Z, c_0) \times \mathbb{R}$ to $\mathscr{C}(Z, c_0)$ as follows:

$$(t, y(t)) \to (Ty)(t),$$

where

$$(Ty)(t) = \left(\Theta_n\left(t, y(t), \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} \beta_n(t, \nu, y(\nu))\frac{d\nu}{\nu}\right)\right)_{n=1}^{\infty}.$$

(4) Let

$$\sup_{t\in Z} A_n(t) = \hat{A}_n,$$

$$\sup_{t\in Z} B_n(t) = \hat{B}_n,$$

$$\sup_{n\in\mathbb{N}} \hat{A}_n = \hat{A},$$

$$\sup_{n\in\mathbb{N}} \hat{B}_n = \hat{B}.$$

Also,

$$\lim_{n\to\infty} \hat{B}_n = 0$$

and

$$0 < \hat{A} < 1.$$

Let $B = \left\{ y \in \mathscr{C}(Z, c_0) : \| y \|_{\mathscr{C}(Z,c_0)} \leq r \right\}$.

Theorem 3 *If the conditions (1)–(4) are satisfied, then Eq. (4) has at least one solution in* $\mathscr{C}(Z, c_0)$.

Proof For arbitrary fixed $t \in Z$,

$$\begin{aligned}
&\| y(t) \|_{c_0} \\
&= \sup_{n\geq 1} \left| \Theta_n \left(t, y(t), \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} \beta_n(t,\nu,y(\nu)) \frac{d\nu}{\nu} \right) \right| \\
&\leq \sup_{n\geq 1} \left| \Theta_n \left(t, y(t), \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} \beta_n(t,\nu,y(\nu)) \frac{d\nu}{\nu} \right) - \Theta_n(t, y^0(t), 0) \right| \\
&+ \sup_{n\geq 1} \left| \Theta_n(t, y^0(t), 0) \right| \\
&\leq \sup_{n\geq 1} \left[A_n(t) \left| y_n(t) \right| + B_n(t) \left| \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} \beta_n(t,\nu,y(\nu)) \frac{d\nu}{\nu} \right| \right] + + \bar{\Theta} \\
&\leq \hat{A} \| y(t) \|_{c_0} + \hat{B}\hat{\beta} \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} \frac{d\nu}{\nu} + \bar{\Theta} \\
&\leq \hat{A} \| y(t) \|_{c_0} + \frac{\hat{B}\hat{\beta}}{\eta} (\ln(T) - \ln(1))^{\eta} + \bar{\Theta}.
\end{aligned}$$

Therefore,

$$(1 - \hat{A}) \| y(t) \|_{c_0} \leq \frac{\hat{B}\hat{\beta}}{\alpha} (\ln(T))^{\eta} + \bar{\Theta}$$

implies

$$\| y(t) \|_{c_0} \leq \frac{\eta\Theta + \hat{B}\hat{\beta}\,(\ln(T))^{\eta}}{\eta(1-\hat{A})} = r(say).$$

Hence, $\| y \|_{\mathscr{C}(Z,c_0)} \leq r$.

Consider $T : Z \times B \to B$ be a operator given by

$$(Ty)(t) = \left(\Theta_n\left(t, y(t), \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} \beta_n(t,\nu,y(\nu))\frac{d\nu}{\nu}\right)\right)_{n=1}^{\infty} = ((T_n y)\,(t))_{n=1}^{\infty},$$

where $y(t) \in B,\ t \in Z$.

By assumption (3),

$$\lim_{n\to\infty} (T_n y)\,(t) = 0$$

hence $(Ty)\,(t) \in \mathscr{C}(Z, c_0)$.

Again,

$$\| Ty \|_{\mathscr{C}(Z,c_0)} \leq r$$

so T is self mapping on B.

Let $\bar{y}(t) = (\bar{y}_n(t))_{n=1}^{\infty} \in B$ and $\epsilon > 0$ such that $\| y - \bar{y} \|_{\mathscr{C}(Z,c_0)} < \frac{\epsilon}{2\hat{A}} = \delta$. Again for arbitrary fixed $t \in Z$,

$$\begin{aligned}
&|(T_n y)\,(t) - (T_n\bar{y})\,(t)| \\
&= \left|\Theta_n\left(t, y(t), \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} \beta_n(t,\nu,y(\nu))\frac{d\nu}{\nu}\right) - \Theta_n\left(t, \bar{y}(t), \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} \beta_n(t,\nu,\bar{y}(\nu))\frac{d\nu}{\nu}\right)\right| \\
&\leq A_n(x)\,|y_n(t) - \bar{y}_n(t)| + B_n(t)\int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} |\beta_n(t,\nu,y(\nu)) - \beta_n(t,\nu,\bar{y}(\nu))|\,\frac{d\nu}{\nu}.
\end{aligned}$$

As functions β_n are continuous for all $n \in \mathbb{N}$ so for $\| y - \bar{y} \|_{\mathscr{C}(Z,c_0)} < \frac{\epsilon}{2\hat{A}} = \delta$ we have for all $n \in \mathbb{N}$,

$$|\beta_n(x, w, z(w)) - \beta_n(x, w, \bar{z}(w))| < \frac{\eta\epsilon}{2\hat{B}(\ln(T))^{\eta}}.$$

Therefore,

$$\begin{aligned}
&|(T_n y)\,(t) - (T_n\bar{y})\,(t)| \\
&\leq \hat{A}\, \| y - \bar{y} \|_{\mathscr{C}(Z,c_0)} + \frac{\hat{B}\epsilon\eta}{2\hat{B}(\ln(T))^{\eta}}.\frac{(\ln(T))^{\eta}}{\eta} \\
&< \epsilon.
\end{aligned}$$

Therefore, $\| Ty - T\bar{y} \|_{\mathscr{C}(Z,c_0)} < \epsilon$ when $\| y - \bar{y} \|_{\mathscr{C}(Z,c_0)} < \delta$ hence T is continuous on B. Finally,

$$\begin{aligned}
&\zeta_{c_0}(TB)\\
&= \lim_{n\to\infty}\sup_{y\in B}\max_{k\geq n}\left|\Theta_n\left(t, y(t), \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1}\beta_n(t,\nu,y(\nu))\frac{d\nu}{\nu}\right)\right|\\
&\leq \lim_{n\to\infty}\sup_{y\in B}\max_{k\geq n}\left[\hat{A}\,|y_k(x)| + \frac{\hat{B}_k\hat{\beta}\,(\ln(T))^{\eta}}{\eta} + \bar{\Theta}_k\right].
\end{aligned}$$

i.e.,

$$\zeta_{c_0}(TB) \leq \hat{A}\zeta_{c_0}(B).$$

Therefore

$$\zeta_{\mathscr{C}(Z,c_0)}(TB) \leq \hat{A}\zeta_{\mathscr{C}(Z,c_0)}(B).$$

Thus, by assumption (4) and Theorem 2, T has at least one fixed point in $B \subseteq \mathscr{C}(Z, c_0)$. Hence, Eq. (4) has at least one solution in $\mathscr{C}(Z, c_0)$. This completes the proof.

Example 1

$$y_n(t) = \frac{y_n(t)}{2+n+t} + \frac{1}{n^2\left(1+n^2\right)}\int_1^t \left\{\ln\left(\frac{t}{\nu}\right)\right\}^{-\frac{1}{2}}(1+\sin(y_n(\nu)))\frac{d\nu}{\nu}, \tag{5}$$

where $t \in Z = [1, 2]$, $n \in \mathbb{N}$.

Here, $\Theta_n(t, y(t), l) = \frac{y_n(t)}{2+n+t} + \frac{l}{1+n^2}$, $\beta_n(t, \nu, y(\nu)) = \frac{1+\sin(y_n(\nu))}{n^2}$, $\eta = \frac{1}{2}$ and $\tau = 2$. It is obvious that Θ_n is continuous for all $n \in \mathbb{N}$ and

$$\begin{aligned}
&|\Theta_n(t, y(t), l) - \Theta_n(t, \bar{y}(t), m)|\\
&\leq \frac{1}{2+n+t}|y_n(t) - \bar{y}_n(t)| + \frac{1}{1+n^2}|l-m|.
\end{aligned}$$

Also,

$$A_n(t) = \frac{1}{2+n+t}, \hat{A}_n = \frac{1}{3+n}, \hat{A} = \frac{1}{4},$$

$$B_n(t) = \frac{1}{1+n^2}, \hat{B}_n = \frac{1}{1+n^2}, \lim_{n\to\infty}\hat{B}_n = 0, \hat{B} = \frac{1}{2},$$

$$\hat{\Theta}_n = 0, \hat{\Theta} = 0, \lim_{n\to\infty}\hat{\Theta}_n = 0.$$

Again, the functions β_n are continuous for all $n \in \mathbb{N}$.

If $y \in \mathscr{C}(Z, c_0)$ then as $n \to \infty$ and for all $t \in Z$, we get

$$y_n(t) \to 0, \frac{1}{n^2\left(1+n^2\right)}\int_1^t \left\{\ln\left(\frac{t}{\nu}\right)\right\}^{-\frac{1}{2}}(1+\sin(y_n(\nu)))\frac{d\nu}{\nu} \to 0.$$

Therefore, assumption (3) is satisfied.

Thus, all the assumptions of Theorem 3 are satisfied. Hence, Eq. (5) has a solution in $\mathscr{C}(Z, c_0)$.

4 Existence of Solution on $\mathscr{C}(Z, \ell_1)$

We consider the following assumptions

(1) The function $\Theta_n : Z \times \mathscr{C}(Z, \ell_1) \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$|\Theta_n(t, y(t), l) - \Theta_n(t, \bar{y}(t), m)| \le C_n(t)\,|y_n(t) - \bar{y}_n(t)| + D_n(t)\,|l - m|$$

for $y(t) = (y_n(t))_{n=1}^{\infty}$, $\bar{y}(t) = (\bar{y}_n(t))_{n=1}^{\infty} \in \mathscr{C}(Z, \ell_1)$ and $C_n, D_n : Z \to [0, \infty)$ $(n \in \mathbb{N})$ are continuous functions. Also,

$$\sum_{n=1}^{\infty} \left|\Theta_n\left(t, y^0, 0\right)\right|$$

converges to zero for all $t \in Z$, where $y^0 = \left(y_n^0(t)\right)_{n=1}^{\infty} \in \mathscr{C}(Z, \ell_1)$ such that $y_n^0(t) = 0$ for all $t \in Z$, $n \in \mathbb{N}$.

(2) The functions $\beta_n : Z \times Z \times \mathscr{C}(Z, \ell_1) \to \mathbb{R}(n \in \mathbb{N})$ are continuous and there exists

$$\bar{\beta}_k = \sup\left\{\sum_{n\ge k} |\beta_n(t, \nu, y(\nu))|\, t, \nu \in Z;\, y(\nu) \in \mathscr{C}(Z, \ell_1)\right\}$$

where $n, k \in \mathbb{N}$. Also $\sup_{k\in\mathbb{N}} \bar{\beta}_k = \bar{\beta}$ and $\lim_{k\to\infty} \bar{\beta}_k = 0$.

(3) Define an operator T from $Z \times \mathscr{C}(Z, \ell_1) \times \mathbb{R}$ to $\mathscr{C}(Z, \ell_1)$ as follows:

$$(t, y(t)) \to (Ty)(t),$$

where

$$(Ty)(t) = \left(\Theta_n\left(t, y(t), \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} \beta_n(t, \nu, y(\nu))\frac{d\nu}{\nu}\right)\right)_{n=1}^{\infty}.$$

(4) Let

$$\sup_{t\in Z} C_n(t) = \hat{C}_n,$$

$$\sup_{n\in\mathbb{N}} \hat{C}_n = \hat{C},\ 0 < \hat{C} < 1.$$

Also, for all $t \in Z$,

$$\sum_{n\geq 1} D_n(t) \leq \hat{D}.$$

Let $B_1 = \left\{ y \in \mathscr{C}(Z, \ell_1) : \parallel y \parallel_{\mathscr{C}(Z,\ell_1)} \leq \hat{r} \right\}.$

Theorem 4 *Under the hypothesis (1)–(4), Eq. (4) has at least one solution in* $\mathscr{C}(Z, \ell_1)$.

Proof For arbitrary $t \in Z$,

$$\begin{aligned}
&\parallel y(t) \parallel_{\ell_1} \\
&= \sum_{n\geq 1} \left| \Theta_n \left(t, y(t), \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} \beta_n(t,\nu,y(\nu)) \frac{d\nu}{\nu} \right) \right| \\
&\leq \sum_{n\geq 1} \left| \Theta_n \left(t, y(t), \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} \beta_n(t,\nu,y(\nu)) \frac{d\nu}{\nu} \right) - \Theta_n(t, y^0, 0) \right| \\
&+ \sum_{n\geq 1} \left| \Theta_n(t, y^0, 0) \right| \\
&\leq \sum_{n\geq 1} \left[C_n(t)\,|y_n(t)| + D_n(t) \left| \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} \beta_n(t,\nu,y(\nu)) \frac{d\nu}{\tau} \right| \right] \\
&\leq \hat{C} \parallel y(t) \parallel_{\ell_1} + \hat{D} \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} \sum_{n\geq 1} |\beta_n(t,\nu,y(\nu))| \frac{d\nu}{\nu} \\
&\leq \hat{C} \parallel y(t) \parallel_{\ell_1} + \hat{D}\bar{\beta} \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} \frac{d\tau}{\nu} \\
&\leq \hat{C} \parallel y(t) \parallel_{\ell_1} + \frac{\hat{D}\bar{\beta}}{\eta} (\ln(T))^{\eta}.
\end{aligned}$$

Therefore,

$$(1-\hat{C}) \parallel y(t) \parallel_{\ell_1} \leq \frac{\hat{D}\bar{\beta}}{\eta} (\ln(T))^{\eta}$$

implies

$$\parallel y(t) \parallel_{\ell_1} \leq \frac{\hat{D}\bar{\beta}\,(\ln(T))^{\eta}}{\eta(1-\hat{C})} = \hat{r}(say).$$

Hence, $\parallel y \parallel_{\mathscr{C}(Z,\ell_1)} \leq \hat{r}$.
Consider $T : Z \times B_1 \to B_1$ be a operator given by

$$(Ty)(t) = \left(\Theta_n \left(t, y(t), \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} \beta_n(t,\nu,y(\nu)) \frac{d\nu}{\nu} \right) \right)_{n=1}^{\infty} = ((T_n y)(t))_{n=1}^{\infty},$$

where $y(t) \in B_1,\ y \in I$.

By assumption (3),

$$\sum_{n\geq 1} |(T_n y)\,(t)|$$

is finite and unique hence $(Ty)\,(t) \in \mathscr{C}(Z, \ell_1)$. Again,

$$\| \, Ty \, \|_{\mathscr{C}(Z,\ell_1)} \leq \hat{r}$$

so T is self-mapping on B_1.

Let $\bar{y}(t) = (\bar{y}_n(t))_{n=1}^{\infty} \in B_1$ and $\epsilon > 0$ such that $\| \, y - \bar{y} \, \|_{\mathscr{C}(Z,\ell_1)} < \frac{\epsilon}{2\hat{C}} = \delta$. Again for arbitrary fixed $x \in I$,

$$\begin{aligned}
&|(T_n y)\,(t) - (T_n \bar{y})\,(t)| \\
&= \left| \Theta_n \left(t, y(t), \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} \beta_n(t,\nu,y(\nu)) \frac{d\nu}{\nu} \right) - \Theta_n \left(t, \bar{y}(t), \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} \beta_n(t,\nu,\bar{y}(\nu)) \frac{d\nu}{\nu} \right) \right| \\
&\leq C_n(t)\,|y_n(t) - \bar{y}_n(t)| + D_n(t) \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} |\beta_n(t,\nu,y(\nu)) - \beta_n(t,\nu,\bar{y}(\nu))| \, \frac{d\nu}{\nu}.
\end{aligned}$$

As functions β_n are continuous for all $n \in \mathbb{N}$ so for $\| \, y - \bar{y} \, \|_{\mathscr{C}(Z,\ell_1)} < \frac{\epsilon}{2\hat{C}}$, we have for all $n \in \mathbb{N}$,

$$|\beta_n(t,\nu,y(\nu)) - \beta_n(t,\nu,\bar{y}(\nu))| < \frac{\eta\epsilon}{2\hat{D}(\ln(T))^{\eta}}.$$

Therefore,

$$\begin{aligned}
&\sum_{n\geq 1} |(T_n y)\,(t) - (T_n \bar{y})\,(t)| \\
&\leq \hat{C} \sum_{n\geq 1} |y_n(t) - \bar{y}_n(t)| + \frac{\epsilon\eta}{2\hat{D}(\ln(T))^{\eta}} . \frac{(\ln(T))^{\eta}}{\eta} \sum_{n\geq 1} D_n(t) \\
&\leq \hat{C} \, \| \, y - \bar{y} \, \|_{\mathscr{C}(Z,\ell_1)} + \frac{\epsilon}{2\hat{D}} . \hat{D} \\
&< \epsilon.
\end{aligned}$$

Therefore, $\| \, Ty - T\bar{y} \, \|_{\mathscr{C}(Z,\ell_1)} < \epsilon$ when $\| \, y - \bar{y} \, \|_{\mathscr{C}(Z,\ell_1)} < \frac{\epsilon}{2\hat{C}}$ hence T is continuous on B_1. Finally,

$$\begin{aligned}
&\zeta_{\ell_1}\,(TB_1) \\
&= \lim_{n\to\infty} \sup_{y\in B_1} \sum_{k\geq n} \left| \Theta_n \left(t, y(t), \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} \beta_n(t,\nu,y(\tau)) \frac{d\nu}{\nu} \right) \right| \\
&\leq \lim_{n\to\infty} \sup_{y\in B_1} \sum_{k\geq n} \left[\hat{C} \sum_{k\geq n} |y_k(t)| + \frac{\bar{\beta}_k \hat{D}\,(\ln(T))^{\eta}}{\eta} \right].
\end{aligned}$$

i.e.,

$$\zeta_{\ell_1}(TB_1) \le \hat{C}\zeta_{\ell_1}(B_1).$$

Therefore,

$$\zeta_{\mathscr{C}(Z,\ell_1)}(TB_1) \le \hat{C}\zeta_{\mathscr{C}(Z,\ell_1)}(B_1).$$

By assumption (4) and Theorem 2, T has at least one fixed point in $B_1 \subseteq \mathscr{C}(Z, \ell_1)$. Hence, Eq. (4) has at least one solution in $\mathscr{C}(Z, \ell_1)$. This completes the proof.

Example 2

$$y_n(t) = \frac{y_n(t)}{n^3 + 3t} + \frac{1}{n^4} \int_1^t \frac{(\ln(\frac{t}{\nu}))^{-\frac{1}{2}} \cos(y_n(\nu))}{\nu n^4} \frac{d\nu}{\nu}, \tag{6}$$

where $t \in Z = [1, 2],\ n \in \mathbb{N}$.

Here, $\Theta_n(t, y(t), l) = \frac{y_n(t)}{n^3+3t} + \frac{l}{n^4},\ \beta_n(t, \nu, y(\nu)) = \frac{\cos(y_n(\nu))}{\nu n^4},\ \eta = \frac{1}{2}$ and $\tau = 2$. It is clear that Θ_n is continuous for all $n \in \mathbb{N}$ and

$$\begin{aligned} &|\Theta_n(t, y(t), l) - \Theta_n(t, \bar{y}(t), m)| \\ &\le \frac{1}{n^3 + 3t} |y_n(t) - \bar{y}_n(t)| + \frac{1}{n^4} |l - m| . \end{aligned}$$

Also,

$$C_n(t) = \frac{1}{n^3 + 3t}, \hat{C}_n = \frac{1}{n^3 + 3}, \hat{C} = \frac{1}{4} < 1,$$

$$D_n(t) = \frac{1}{n^4}, \sum_{n=1}^{\infty} D_n(t) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90},$$

and

$$\sum_{n=1}^{\infty} \left|\Theta_n(t, y^0, 0)\right| = 0.$$

Again, the functions β_n are continuous for all $n \in \mathbb{N}$ and

$$\sum_{n \ge k} |\beta_n(t, \tau, y(\nu))| \le \sum_{n \ge k} \frac{1}{n^4} \le \frac{\pi^4}{90}$$

which gives $\bar{\beta} = \frac{\pi^4}{90}$ and $\lim_{k\to\infty} \bar{\beta}_k = 0$.

If $z \in \mathscr{C}(Z, \ell_1)$ then

$$\sum_{n=1}^{\infty}\left|\frac{y_n(t)}{n^3+3t}+\frac{1}{n^4}\int_1^t \frac{(\ln(\frac{t}{\nu}))^{-\frac{1}{2}}\cos(y_n(\nu))}{\nu n^4}\frac{d\nu}{\nu}\right|$$
$$\leq \frac{1}{n^3+3t}\sum_{n=1}^{\infty}|y_n(t)|+\sum_{n=1}^{\infty}\frac{1}{n^8}\int_1^t (\ln(\frac{t}{\tau}))^{-\frac{1}{2}}\frac{d\tau}{\tau}$$
$$\leq \frac{1}{4}\|y\|_{\mathscr{C}(Z,\ell_1)}+2(\ln(T))^{\frac{1}{2}}\sum_{n=1}^{\infty}\frac{1}{n^8}.$$

Since $\sum_{n=1}^{\infty}\frac{1}{n^8}$ is convergent and $\| \ y \ \|_{\mathscr{C}(Z,\ell_1)}$ is finite and unique therefore

$$\sum_{n=1}^{\infty}\left|\frac{y_n(t)}{n^3+3t}+\frac{1}{n^4}\int_1^t \frac{(\ln(\frac{t}{\tau}))^{-\frac{1}{2}}\cos(y_n(\nu))}{\nu n^4}\frac{d\nu}{\nu}\right|$$

is convergent, hence assumption (3) is satisfied.

Thus, all the assumptions of Theorem 4 are satisfied. Hence, Eq. (6) has a solution in $\mathscr{C}(Z,\ell_1)$.

5 Existence of Solution in $\mathscr{C}(Z,\ell_p^{\alpha})$

We consider the following assumptions

(1) The function $\Theta_n : Z\times\mathscr{C}(Z,\ell_p^{\alpha}))\times\mathbb{R}\to\mathbb{R}$ is continuous and satisfies

$$|\Theta_n(t,y(t),l)-\Theta_n(t,\bar{y}(t),m)|^p\leq U_n(t)\,|y_n(t)-\bar{y}_n(t)|^p+V_n(t)\,|l-m|^p$$

for $y(t)=(y_n(t))_{n=1}^{\infty}$, $\bar{y}(t)=(\bar{y}_n(t))_{n=1}^{\infty}\in\mathscr{C}(Z,\ell_p^{\alpha})$ and $U_n,\ V_n : Z\to[0,\infty)$ $(n\in\mathbb{N})$ are continuous functions. Also,

$$\sum_{n=1}^{\infty}\alpha_n^p\left|\Theta_n\left(t,y^0,0\right)\right|^p$$

converges to zero for all $t\in Z$, where $y^0=\left(y_n^0(t)\right)_{n=1}^{\infty}\in\mathscr{C}(Z,\ell_p^{\alpha})$ such that $y_n^0(t)=0$ for all $t\in Z,\ n\in\mathbb{N}$.

(2) The functions $\beta_n : Z\times Z\times\mathscr{C}(Z,\ell_p^{\alpha})\to\mathbb{R}(n\in\mathbb{N})$ are continuous and there exists

$$P_n=\sup\left\{|\beta_n(t,\nu,y(\nu))|\,t,\nu\in Z;\,y(\nu)\in\mathscr{C}(Z,\ell_p^{\alpha})\right\},$$

for $n\in\mathbb{N}$. Also $\sup_{n\in\mathbb{N}}P_n=\hat{P}$ and $\lim_{n\to\infty}P_n=0$.

(3) Define an operator T from $Z\times\mathscr{C}(Z,\ell_p^{\alpha}))$ to $\mathscr{C}(Z,\ell_p^{\alpha}))$ as follows:

$$(t, y(t)) \to (Ty)(t),$$

where

$$(Ty)(t) = \left(\Theta_n\left(t, y(t), \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} \beta_n(t, \nu, y(\nu))\frac{d\nu}{\nu}\right)\right)_{n=1}^{\infty}.$$

(4) Let

$$\sup_{t\in I} U_n(t) = \hat{U}_n,$$

$$\sup_{n\in\mathbb{N}} \hat{U}_n = \hat{U},\ 0 < 2^{1-\frac{1}{p}}\hat{U}^{\frac{1}{p}} < 1.$$

Also, for all $t \in Z$,

$$\sum_{n\geq 1} \alpha_n^p V_n(t) \leq \hat{V}.$$

Let $B_{p,\alpha} = \left\{y \in \mathscr{C}(Z, \ell_p^\alpha) : \| y \|_{\mathscr{C}(Z,\ell_p^\alpha)} \leq \hat{r}\right\}$.

Theorem 5 *Under the hypothesis (1)–(4), Eq. (4) has at least one solution in* $\mathscr{C}(Z, \ell_p^\alpha)$.

Proof For arbitrary $t \in Z$,

$$\begin{aligned}
&\| y(t) \|_{\ell_p^\alpha}^p \\
&= \sum_{n\geq 1} \alpha_n^p \left|\Theta_n\left(t, y(t), \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} \beta_n(t, \nu, y(\nu))\frac{d\nu}{\nu}\right)\right|^p \\
&\leq 2^{p-1} \sum_{n\geq 1} \alpha_n^p \left|\Theta_n\left(t, y(t), \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} \beta_n(t, \nu, y(\nu))\frac{d\nu}{\nu}\right) - \Theta_n(t, y^0, 0)\right|^p \\
&+ 2^{p-1} \sum_{n\geq 1} \alpha_n^p \left|\Theta_n(t, y^0, 0)\right|^p \\
&\leq 2^{p-1} \sum_{n\geq 1} \alpha_n^p \left[U_n(t)\, |y_n(t)|^p + V_n(t) \left|\int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} \beta_n(t, \nu, y(\nu))\frac{d\nu}{\nu}\right|^p\right] \\
&\leq 2^{p-1}\hat{U} \| y(t) \|_{\ell_p^\alpha}^p + 2^{p-1} \sum_{n\geq 1} \alpha_n^p V_n(t) \left\{\int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} |\beta_n(t, \nu, y(\nu))| \frac{d\nu}{\nu}\right\}^p \\
&\leq 2^{p-1}\hat{U} \| y(t) \|_{\ell_p^\alpha}^p + 2^{p-1}\hat{P}^p \sum_{n\geq 1} \left[\alpha_n^p V_n(t) \left\{\frac{(\ln(T))^\eta}{\eta}\right\}^p\right] \\
&\leq 2^{p-1}\hat{U} \| y(t) \|_{\ell_p^\alpha}^p + 2^{p-1}\hat{P}^p \hat{V} \left\{\frac{(\ln(T))^\eta}{\eta}\right\}^p.
\end{aligned}$$

Therefore,

$$(1 - 2^{p-1}\hat{U}) \parallel y(t) \parallel_{\ell_p^\alpha}^p \leq 2^{p-1}\hat{P}^p\hat{V}\left\{\frac{(\ln(T))^\eta}{\eta}\right\}^p$$

implies

$$\parallel y(t) \parallel_{\ell_p^\alpha}^p \leq \frac{2^{p-1}\hat{P}^p\hat{V}\,(\ln(T))^{p\eta}}{\eta^p(1 - 2^{p-1}\hat{U})} = \hat{r}^p(say).$$

Hence, $\parallel y \parallel_{\mathscr{C}(Z,\ell_p^\alpha)} \leq \hat{r}$.

Consider $T : Z \times B_{p,\alpha} \to B_{p,\alpha}$ be a operator given by

$$(Ty)(t) = \left(\Theta_n\left(t, y(t), \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} \beta_n(t,\nu,y(\nu))\frac{d\nu}{\nu}\right)\right)_{n=1}^\infty = ((T_n y)\,(t))_{n=1}^\infty\,,$$

where $y(t) \in B_{p,\alpha},\ t \in Z$.

By assumption (3),

$$\sum_{n\geq 1} \alpha_n^p\,|(T_n y)\,(t)|^p$$

is finite and unique hence $(Ty)\,(t) \in \mathscr{C}(Z, \ell_p^\alpha)$.

Again,

$$\parallel Ty \parallel_{\mathscr{C}(Z,\ell_p^\alpha)} \leq \hat{r}$$

so T is self-mapping on $B_{p,\alpha}$.

Let $\bar{y}(t) = (\bar{y}_n(t))_{n=1}^\infty \in B_{p,\alpha}$ and $\epsilon > 0$ such that $\parallel y - \bar{y} \parallel_{\mathscr{C}(Z,\ell_p^\alpha)} < \frac{\epsilon}{2^{\frac{1}{p}}\hat{U}^{\frac{1}{p}}} = \delta$.

For arbitrary $t \in Z$,

$$\begin{aligned}
&|(T_n z)\,(t) - (T_n\bar{y})\,(t)|^p \\
&= \left|\Theta_n\left(t, y(t), \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} \beta_n(t,\nu,y(\nu))\frac{d\nu}{\nu}\right) - \Theta_n\left(t, \bar{y}(t), \int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\alpha-1} \beta_n(t,\nu,\bar{y}(\nu))\frac{d\nu}{\nu}\right)\right|^p \\
&\leq U_n(t)\,|y_n(t) - \bar{y}_n(t)|^p + V_n(t)\left\{\int_1^t \left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1} |\beta_n(t,\nu,y(\nu)) - \beta_n(t,\nu,\bar{y}(\nu))|\,\frac{d\nu}{\nu}\right\}^p.
\end{aligned}$$

As functions β_n are continuous for all $n \in \mathbb{N}$ so for $\parallel y - \bar{y} \parallel_{\mathscr{C}(Z,\ell_p^\alpha)}^p < \delta$ we have for all $n \in \mathbb{N}$,

$$|\beta_n(t,\nu,y(\nu)) - \beta_n(t,\nu,\bar{y}(\nu))| < \frac{\eta\epsilon}{2^{\frac{1}{p}}\hat{V}^{\frac{1}{p}}(\ln(T))^\eta}.$$

Therefore,

$$\begin{aligned}
&\sum_{n\geq 1}\alpha_n^p \left|(T_n y)(t)-(T_n\bar{y})(t)\right|^p\\
&\leq \hat{U}\sum_{n\geq 1}\alpha_n^p\left|y_n(t)-\bar{y}_n(t)\right|^p+\sum_{n\geq 1}\alpha_n^p V_n(t)\left\{\frac{(\ln(T))^\eta}{\eta}.\frac{\eta\epsilon}{2^{\frac{1}{p}}\hat{V}^{\frac{1}{p}}(\ln(T))^\eta}\right\}^p\\
&\leq \hat{U}\parallel y-\bar{y}\parallel^p_{\mathscr{C}(Z,\ell_p^\alpha)}+\frac{\epsilon^p}{2}\\
&\leq \epsilon^p.
\end{aligned}$$

Therefore, $\parallel Ty-T\bar{y}\parallel^p_{\mathscr{C}(Z,\ell_p^\alpha)}<\epsilon^p$ when $\parallel y-\bar{y}\parallel^p_{\mathscr{C}(Z,\ell_p^\alpha)}<\frac{\epsilon^p}{2\hat{U}}$ hence T is continuous on $B_{p,\alpha}$. Finally,

$$\begin{aligned}
&\zeta_{\ell_p^\alpha}\left(TB_{p,\alpha}\right)\\
&=\lim_{n\to\infty}\sup_{y\in B_{p,\alpha}}\left\{\sum_{k\geq n}\alpha_k^p\left|\Theta_k\left(t,y(t),\int_1^t\left(\ln\left(\frac{t}{\nu}\right)\right)^{\eta-1}\beta_k(t,\nu,y(\nu))\frac{d\nu}{\nu}\right)\right|^p\right\}^{\frac{1}{p}}\\
&\leq \lim_{n\to\infty}\sup_{y\in B_{p,\eta}}\left\{2^{p-1}\sum_{k\geq n}\alpha_k^p\left[\hat{U}\left|y_k(t)\right|^p+V_k(t)P_k^p.\frac{(\ln(T))^{p\alpha}}{\alpha^p}\right]\right\}^{\frac{1}{p}}.
\end{aligned}$$

i.e.,

$$\zeta_{\ell_p^\alpha}\left(TB_{p,\alpha}\right)\leq 2^{1-\frac{1}{p}}\hat{U}^{\frac{1}{p}}\zeta_{\ell_p^\alpha}\left(B_{p,\alpha}\right).$$

Therefore

$$\zeta_{\mathscr{C}(Z,\ell_p^\alpha)}\left(\mathfrak{T}B_{p,\alpha}\right)\leq 2^{1-\frac{1}{p}}\hat{U}^{\frac{1}{p}}\zeta_{\mathscr{C}(Z,\ell_p^\alpha)}\left(B_{p,\alpha}\right).$$

Thus, by assumption (4) and Theorem 2, $\mathfrak{T}$ has at least one fixed point in $B_{p,\alpha}\subseteq\mathscr{C}(Z,\ell_p^\alpha)$. Hence, Eq. (4) has at least one solution in $\mathscr{C}(Z,\ell_p^\alpha)$. This completes the proof.

Example 3

$$y_n(t)=\frac{y_n(t)}{6n^2t}+\frac{1}{n^2}\int_1^t\frac{\left(\ln(\frac{t}{\nu})\right)^{-\frac{1}{2}}\cos^2\left(y_n(\nu)\right)}{\tau+n^2}.\frac{d\nu}{\nu},\tag{7}$$

where $t\in Z=[1,2],\ n\in\mathbb{N}$. Here,

$$\Theta_n(t,y(t),l)=\frac{y_n(t)}{6n^2t}+\frac{l}{n^2},$$

$$\beta_n(t,\nu,y(\tau))=\frac{\cos^2\left(y_n(\nu)\right)}{\nu+n^2},$$

$$\eta = \frac{1}{2},$$

$$\alpha_n = \frac{1}{n}$$

and

$$\tau = 2.$$

Let $y(t) \in \mathscr{C}\left(Z, \ell_p^\alpha\right)$ for some fixed $t \in Z$, then

$$\begin{aligned}
&\sum_{n\geq 1} \alpha_n^p \left| \frac{y_n(t)}{6n^2 t} + \frac{1}{n^2}\int_1^t \frac{\left(\ln(\frac{t}{\nu})\right)^{-\frac{1}{2}} \cos^2\left(y_n(\nu)\right)}{\nu + n^2}.\frac{d\nu}{\nu} \right|^p \\
&= \sum_{n\geq 1} \frac{1}{n^p} \left| \frac{y_n(t)}{6n^2 t} + \frac{1}{n^2}\int_1^t \frac{\left(\ln(\frac{t}{\nu})\right)^{-\frac{1}{2}} \cos^2\left(y_n(\nu)\right)}{\nu + n^2}.\frac{d\nu}{\nu} \right|^p \\
&\leq 2^{p-1} \sum_{n\geq 1} \frac{1}{6n^p} |y_n(t)|^p + 2^{p-1} \sum_{n\geq 1} \frac{1}{n^{2p}} \left\{ \int_1^t \frac{\left(\ln\left(\frac{t}{\nu}\right)\right)^{-\frac{1}{2}}}{\nu + n^2}.\frac{d\nu}{\nu} \right\}^p \\
&\leq \frac{2^{p-1}}{6} \parallel y(t) \parallel_{\ell_p^\eta}^p + 2^{p-1} \sum_{n\geq 1} \frac{1}{n^{2p}}.\left\{ 2(\ln 2)^{\frac{1}{2}}.\frac{1}{2} \right\}^p \\
&= \frac{2^{p-1}}{6} \parallel y \parallel_{\mathscr{C}(Z,\ell_p^\alpha)}^p + 2^{p-1} (\ln 2)^{\frac{p}{2}} \sum_{n\geq 1} \frac{1}{n^{2p}}.
\end{aligned}$$

is finite and unique as both $\sum_{n\geq 1} \frac{1}{n^{2p}}$ are convergent for $p \geq 1$.

Therefore for $t \in Z$,

$$\left\{ \frac{y_n(t)}{6n^2 t} + \frac{1}{n^2}\int_1^t \frac{\left(\ln(\frac{t}{\nu})\right)^{-\frac{1}{2}} \cos^2\left(y_n(\nu)\right)}{\nu + n^2}.\frac{d\nu}{\nu} \right\}_{n=1}^\infty \in \ell_p^\alpha.$$

i.e.,

$$\left\{ \frac{y_n(t)}{6n^2 t} + \frac{1}{n^2}\int_1^t \frac{\left(\ln(\frac{t}{\nu})\right)^{-\frac{1}{2}} \cos^2\left(y_n(\nu)\right)}{\nu + n^2}.\frac{d\nu}{\nu} \right\}_{n=1}^\infty \in \mathscr{C}(Z, \ell_p^\alpha).$$

It is obvious that Θ_n is continuous for all $n \in \mathbb{N}$ and

$$\begin{aligned}&|\Theta_n(t, y(t), l)) - \Theta_n(t, \bar{y}(t), m)|^p \\ &= \left| \frac{1}{6n^2 t}(y_n(t) - \bar{y}_n(t)) + \frac{1}{n^2}(l - m) \right|^p \\ &\leq \frac{2^{p-1}}{6^p n^{2p}} |y_n(t) - \bar{y}_n(t)|^p + \frac{2^{p-1}}{n^{2p}} |l - m|^p .\end{aligned}$$

Here both U_n and V_n are continuous functions for all $n \in \mathbb{N}$ and

$$U_n(t) = \frac{2^{p-1}}{6^p n^{2p}}, \hat{U}_n = \frac{2^{p-1}}{6^p n^{2p}}, \hat{U} = \frac{2^{p-1}}{6^p},$$

$$V_n(t) = \frac{2^{p-1}}{n^{2p}}, \sum_{n=1}^{\infty} \alpha^n V_n(t) = \sum_{n=1}^{\infty} \frac{2^{p-1}}{n^{3p}} = 2^{p-1} \sum_{n=1}^{\infty} \frac{1}{n^{3p}}$$

is convergent for $p \geq 1$. Also,

$$\sum_{n=1}^{\infty} \alpha_n^p \left| \Theta_n(t, y^0, 0) \right|^p = 0$$

and $2^{1-\frac{1}{p}} \hat{U}^{\frac{1}{p}} = 2^{1-\frac{1}{p}}.2^{1-\frac{1}{p}}.6^{-1} < 1$, i.e., $0 < 2^{1-\frac{1}{p}} \hat{U}^{\frac{1}{p}} < 1$.

Again, each H_n is continuous for all $n \in \mathbb{N}$ and

$$|\beta_n(t, \nu, y(\nu))| = \frac{1}{\nu + n^2},$$

$$P_n = \frac{1}{1 + n^2}$$

which gives $\hat{P} = \frac{1}{2}$ and $\lim_{n \to \infty} P_n = 0$.

Thus, all the assumptions (1)–(4) of Theorem 5 are satisfied. Hence, Eq. (7) has a solution in $\mathscr{C}(Z, \ell_p^\alpha)$.

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Observability, Reachability, Trajectory Reachability and Optimal Reachability of Fractional Dynamical Systems using Riemann–Liouville Fractional Derivative

Madasamy Vellappandi and Venkatesan Govindaraj

Abstract The objective of this paper is to investigate the qualitative properties of the fractional dynamical system in terms of the Riemann–Liouville fractional derivative. The observability, reachability, trajectory—reachability and optimal reachability problems are discussed for the linear fractional-order dynamical systems characterised by the Riemann–Liouville fractional derivative using Grammian matrix technique, set-valued functions and the concepts of functional analysis.

1 Introduction

Fixed point theory is one of the most active areas of research in the last 50 years, with applications in nonlinear analysis, differential and integral equations, dynamic systems theory, fractal mathematics, game theory, optimisation problems and mathematical modelling. Also, it is a useful tool for investigating the qualitative properties of nonlinear systems. Fractional-order calculus (FOC) is a non-integer order development of classical calculus, and it is a powerful tool for many researchers working in various disciplines of engineering and science in recent decades. Various real-world systems are better characterised by FOC differential equations, which are an excellent tool for analysing problems of fractal dimension, long-term memory and chaotic behaviour. Fixed point theory and fractional calculus have recently received a lot of attention, attracting a lot of researchers to work in this field. Several noteworthy findings have recently been published in [1–8].

Observability is one of the specific concepts in control theory and it is based on the ability to infer the system's initial state from its input-output behaviour. This means that the behaviour of the entire system may be predicted based on the system's output. Bettayeb and Djennoune [9], Matignon and d'Andréa-Novel [10] and Shamardan and Moubarak [11] have analysed the results for observability of continuous-time linear non-integer order systems in a Caputo sense using observability Grammian

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matrix and rank condition. Above all, one can refer to the monograph [12] for the observability of both discrete- and continuous-time linear fractional-order systems via rank condition.

Controllability refers to the ability of a dynamical system to steer from any initial state to any random final state using the set of acceptable input functions. However, reachability means that the dynamical system is possible to lead from a zero initial state to any random final state by taking the set of favourable input functions. This implies reachability implies controllability. The necessary and sufficient conditions for the positivity, controllability and reachability to zero for fractional positive linear discrete-time system with single delay in state were studied by Trzasko in [13]. Recently, Kaczorek explored the connection between the reachability of positive standard system and positive continuous-time linear system in [14] and he analysed the reachability of positive fractional-order system in [15]. The necessary and sufficient conditions for the observability, reachability and minimum energy control problems for the positive linear fractional continuous-time systems with two various non-integer orders are investigated by Sajewski in [16]. He derived the solution to the minimum energy control problem and mentioned an electrical circuits' example. For more study, the interested learner can look into [17–23] and references there in.

In case of trajectory reachability, we aim for an input function that guides the system along with a specified trajectory rather than one that leads the system from a starting state to a demanded final state. Like, while launching a rocket into orbit, it may be advantageous to have an exact path that leads to the target location in order to save money and avoid collisions. Trajectory reachability is a stronger notion of reachability. Information on trajectory controllability can be searched in [24–26] and references there in.

Optimal reachability and its implications have applications in a variety of industries, including bioengineering, process control, finance, robotics, economics, aerospace and management science, and it is still hot topic in control theory. The optimal reachability problem is the cost function minimisation problem with an input constraint $r(t)$ given by dynamical system and a zero initial state. For more information on optimal reachability, the interested readers can refer to [27–31].

It is observed from the above theory, the qualitative properties of the fractional dynamical systems in the form of Riemann–Liouville derivative are still in the development process. Motivated by this fact, the observability, reachability, trajectory—reachability and optimal reachability problems are discussed for the linear fractional dynamical systems in the form of Riemann–Liouville derivative by applying Grammian matrix technique, set-valued functions and the concepts of functional analysis.

2 Preliminaries

In this portion, we provide some general definitions and properties which are needed to construct our results.

Definition 1 Let $[t_0, t_1]$ be a finite interval on the real line $\mathbb{R}$. The Riemann–Liouville (R-L)-type fractional-order integrals $I^{\beta}_{t_0+}g$ (left sided) and $I^{\beta}_{t_1-}g$ (right sided) of order $\beta > 0$, $n-1 < \beta \leq n$ and $n \in \mathbb{N}$ are given by

$$I^{\beta}_{t_0+}g(t) = \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-\zeta)^{\beta-1}g(\zeta)\mathrm{d}\zeta, \quad t > t_0$$
$$I^{\beta}_{t_1-}g(t) = \frac{1}{\Gamma(\beta)}\int_{t}^{t_1}(\zeta-t)^{\beta-1}g(\zeta)\mathrm{d}\zeta, \quad t < t_1,$$

where the function $g(t)$ is a suitable function.

Definition 2 The R-L fractional derivative operators ${}^{RL}D^{\beta}_{t_0+}g$ (left sided) and ${}^{RL}D^{\beta}_{t_1-}g$ (right sided) of order $\beta > 0$, $n-1 < \beta \leq n$ and $n \in \mathbb{N}$ are given by

$${}^{RL}D^{\beta}_{t_0+}g(t) = \frac{d^n}{dt^n}(I^{n-\beta}_{t_0+}g)(t), \quad t > t_0$$
$${}^{RL}D^{\beta}_{t_1-}g(t) = (-1)^n\frac{d^n}{dt^n}(I^{n-\beta}_{t_1-}g)(t), \quad t < t_1,$$

where the function $g(t)$ is a suitable function.

Definition 3 For any arbitrary square matrix A, the Mittag-Leffler matrix function is given by

$$E_{\beta,\gamma}(A) = \sum_{k=0}^{\infty}\frac{A^k}{\Gamma(\beta k+\gamma)},\ \beta,\gamma > 0,$$
$$E_{\beta,1}(A) = E_{\beta}(A) \text{ with } \gamma = 1.$$

3 Observability

Let us take the time-invariant fractional dynamical system

$${}^{RL}D^{\beta}_{t_0+}h(t) = Ah(t),\ 0 < \beta \leq 1,\ t \in [t_0, t_1]. \tag{1}$$

Here $h \in \mathbb{R}^n$ and A is a matrix of dimension $n \times n$. In addition to (1), we also have a linear observation

$$y(t) = Hh(t), \tag{2}$$

where $y \in \mathbb{R}^m$ and H is a matrix of order $m \times n$.

Definition 4 (*Observable*) The system (1) with a linear observation (2) is called to be observable on a time span $[t_0, t_1]$ if it is doable to find a unique initial state $I_{t_0+}^{1-\beta} h(t)\big|_{t=t_0} = h_0$ from the idea of the output $y(t)$ on $[t_0, t_1]$, i.e. the total state of the system is observable if initial state h_0 is observable.

Theorem 1 (Observablility Grammian) *The linear system* (1) *and* (2) *is observable on* $[t_0, t_1]$ *iff the observability Grammian matrix*

$$M[t_0, t_1] = \int_{t_0}^{t_1} (t - t_0)^{2\beta-2} E_{\beta,\beta}(A^*(t - t_0)^\beta) H^* H E_{\beta,\beta}(A(t - t_0)^\beta) \mathrm{d}t$$

is positive definite.

Proof The solution representation of (1) with the initial constraint $I_{t_0+}^{1-\beta} h(t)\big|_{t=t_0} = h_0$ is given by

$$h(t) = (t - t_0)^{\beta-1} E_{\beta,\beta}(A(t - t_0)^\beta) h_0.$$

and we have $y(t) = H h(t) = (t - t_0)^{\beta-1} H E_{\beta,\beta}(A(t - t_0)^\beta) h_0$.

$$\begin{aligned} \|y\|^2 &= \int_{t_0}^{t_1} y^*(t) y(t) \mathrm{d}t \\ &= h_0^* \int_{t_0}^{t_1} (t - t_0)^{2\beta-2} E_{\beta,\beta}(A^*(t - t_0)^\beta) H^* H E_{\beta,\beta}(A(t - t_0)^\beta) \mathrm{d}t \ h_0 \\ &= h_0^* M[t_0, t_1] h_0 \end{aligned}$$

a quadratic form in h_0. It is clear that matrix $M[t_0, t_1]$ is $n \times n$ symmetric. If $M[t_0, t_1]$ is positive definite, then $y = 0$ gives $h_0^* M[t_0, t_1] h_0 = 0$. Therefore, it means $h_0 = 0$. Hence, systems (1) and (2) are observable on $[t_0, t_1]$. If $M[t_0, t_1]$ is not positive definite, then there is some $h_0 \neq 0$ such that $h_0^* M[t_0, t_1] h_0 = 0$. Then $h(t) = (t - t_0)^{\beta-1} E_{\beta,\beta}(A(t - t_0)^\beta) h_0 \neq 0$, for $t \in [t_0, t_1]$, but $\|y\|^2 = 0$, so $y = 0$ and we get that systems (1) and (2) are not observable on $[t_0, t_1]$. So, we receive the necessary result.

If the linear systems (1) and (2) are observable on the range $[t_0, t_1]$, then h_0 is re-simulated straightly from the observation $y(t)$.

Definition 5 The $n \times n$ matrix function $R_k(t)$ defined on $[t_0, t_1]$ is a reconstruction kernel iff

$$\int_{t_0}^{t_1} (t - t_0)^{\beta-1} R_k(t) H E_{\beta,\beta}(A(t - t_0)^\beta) \mathrm{d}t = I.$$

Theorem 2 *There exists a reconstruction kernel* $R_k(t)$ *on* $[t_0, t_1]$ *iff systems* (1) *and* (2) *are observable on* $[t_0, t_1]$.

Proof If a reconstruction kernel exists and follows

$$\int_{t_0}^{t_1} R_k(t)y(t)\mathrm{d}t = \int_{t_0}^{t_1} (t-t_0)^{\beta-1} R_k(t) H E_{\beta,\beta}(A(t-t_0)^{\beta})\mathrm{d}t\, h_0 = h_0$$

and $y(t) = 0$, then $h_0 = 0$. So $h(t) = 0, \forall t \in [t_0, t_1]$ and we find that systems (1) and (2) are observable on $[t_0, t_1]$.

Another side, if systems (1) and (2) are observable on $[t_0, t_1]$, then from Theorem 1

$$M[t_0, t_1] = \int_{t_0}^{t_1} (t-t_0)^{2\beta-2} E_{\beta,\beta}(A^*(t-t_0)^{\beta}) H^* H E_{\beta,\beta}(A(t-t_0)^{\beta})\mathrm{d}t$$

is positive definite. Let

$$R_k^0(t) = (t-t_0)^{\beta-1} M^{-1}[t_0, t_1] E_{\beta,\beta}(A^*(t-t_0)^{\beta}) H^*, \quad \forall t \in [t_0, t_1]. \tag{3}$$

Then we have

$$\begin{aligned}
&\int_{t_0}^{t_1} R_k^0(t) H E_{\beta,\beta}(A(t-t_0)^{\beta})\mathrm{d}t \\
&= M^{-1}[t_0, t_1] \int_{t_0}^{t_1} (t-t_0)^{2\beta-2} E_{\beta,\beta}(A^*(t-t_0)^{\beta}) H^* H E_{\beta,\beta}(A(t-t_0)^{\beta})\mathrm{d}t = I,
\end{aligned}$$

so that (3) is a reconstruction kernel on $[t_0, t_1]$.

4 Reachability

Let us take the dynamical system

$$\begin{aligned}
{}^{RL}D_{t_0+}^{\beta} h(t) &= Ah(t) + Br(t), \quad 0 < \beta \le 1,\ t \in [t_0, t_1] \\
I_{t_0+}^{1-\beta} h(t)\big|_{t=t_0} &= 0,
\end{aligned} \tag{4}$$

where the state vector $h \in \mathbb{R}^n$, the input vector $r \in \mathbb{R}^m$ and A and B are the constant matrices having $n \times n$ and $n \times m$ dimensions, respectively. The solution representation of (4) is

$$h(t) = \int_{t_0}^{t} (t-\zeta)^{\beta-1} E_{\beta,\beta}(A(t-\zeta)^{\beta}) Br(\zeta)\mathrm{d}\zeta. \tag{5}$$

Clearly, $h(t_0) = 0$.

Definition 6 (*Reachable*) The linear system (4) is said to be reachable in time t_1 if for each vector $h_1 \in \mathbb{R}^n$ there exists an input function $r(t) \in L^2([t_0, t_1]; \mathbb{R}^n)$, which leads the system state (4) from the zero starting state $h(t_0) = 0$ to the final state $h(t_1) = h_1$.

Theorem 3 (Reachability Grammian) *System* (4) *is reachable in time* $t \in [t_0, t_1]$ *iff the Reachability Grammian*

$$R[t_0, t_1] = \int_{t_0}^{t_1} E_{\beta,\beta}(A(t_1 - \zeta)^{\beta}) B B^* E_{\beta,\beta}(A^*(t_1 - \zeta)^{\beta}) \mathrm{d}\zeta \tag{6}$$

is positive definite.

Proof Let us assume that the reachability Grammian $R[t_0, t_1]$ is invertible, then define the input function

$$r(t) = (t_1 - t)^{\beta-1} B^* E_{\beta,\beta}(A^*(t_1 - t)^{\beta}) R^{-1}[t_0, t_1] h_1. \tag{7}$$

Using (5), (6) and (7), we obtain

$$\begin{aligned} h(t_1) &= \int_{t_0}^{t_1} (t_1 - \zeta)^{\beta-1} E_{\beta,\beta}(A(t_1 - \zeta)^{\beta}) B r(\zeta) \mathrm{d}\zeta \\ &= \int_{t_0}^{t_1} E_{\beta,\beta}(A(t_1 - \zeta)^{\beta}) B B^* E_{\beta,\beta}(A^*(t_1 - t)^{\beta}) \mathrm{d}\zeta \times R^{-1}[t_0, t_1] h_1 \\ &= R[t_0, t_1] R^{-1}[t_0, t_1] h_1 \\ &= h_1. \end{aligned}$$

Therefore, the input function (7) leads the state of system (4) from 0 to h_1.

Let the reachability Grammian $R[t_0, t_1]$ is not positive definite, in that case, there exists a non-zero z such that $z^* R[t_0, t_1] z = 0$. It means that

$$z^* \int_{t_0}^{t_1} E_{\beta,\beta}(A(t_1 - \zeta)^{\beta}) B B^* E_{\beta,\beta}(A^*(t_1 - \zeta)^{\beta}) \mathrm{d}\zeta \, z = 0.$$

This means

$$z^* E_{\beta,\beta}(A(t_1 - t)^{\beta}) B = 0 \text{ on } [t_0, t_1].$$

From the assumptions, there exist an input function $r(t)$ such that the system state leads from origin to z in $[t_0, t_1]$. It follows that

$$h(t_1) = z = \int_{t_0}^{t_1} (t_1 - \zeta)^{\beta-1} E_{\beta,\beta}(A(t_1 - \zeta)^{\beta}) B r(\zeta) \mathrm{d}\zeta.$$

Then

$$z^*z = z^* \int_{t_0}^{t_1} (t_1 - \zeta)^{\beta-1} E_{\beta,\beta}(A(t_1 - \zeta)^{\beta}) Br(\zeta) \mathrm{d}\zeta.$$

We receive that $z^*z = 0$, since the term given in right side is zero. This gives a contradiction that $z \neq 0$. Thus $R[t_0, t_1]$ is positive definite.

5 Trajectory—Reachability

In this section, we study about the trajectory—reachability analysis for the given dynamical system

$$\begin{aligned} {}^{RL}D^{\beta}_{t_0+}h(t) &= ah(t) + b(t, r(t)), \quad 0 < \beta \leq 1,\ t \in [t_0, t_1] \\ I^{1-\beta}_{t_0+}h(t)\big|_{t=t_0} &= 0, \end{aligned} \tag{8}$$

where the state $h \in \mathbb{R}$, the input $r \in \mathbb{R}$, $a \in \mathbb{R}$ and $b : [t_0, t_1] \times \mathbb{R} \to \mathbb{R}$.

Remark 1 The term "reachability" refers to the ability to direct the system's state from zero to any specific final state h_1. However, it does not follow the path that we desire. In practise, it may be desired to direct the system along a predetermined path from its initial state of zero to its final state of h_1. Depending on the path chosen, it may reduce some of the costs associated with steering the system.

Let $\mathcal{T}_{\mathcal{R}}$ be the collection of all continuous functions $z(\cdot)$ given on $[t_0, t_1]$ such that $z(t_0) = 0$ and $z(t_1) = x_1$ and let the fractional operator ${}^{RL}D^{\beta}_{t_0+}z$ exist almost everywhere in $[t_0, t_1]$ and $I^{1-\beta}_{t_0+}z(t)\big|_{t=t_0} = 0$. We denote $\mathcal{T}_{\mathcal{R}}$, the set of all trajectories of system (8).

Definition 7 System (8) is called to be trajectory—reachability on $[t_0, t_1]$ if, for any $z \in \mathcal{T}_{\mathcal{R}}$, there exists an input $r \in L^2([t_0, t_1]; \mathbb{R})$ such that the respective solution $h(\cdot)$ of (8) agrees to $h(t) = z(t)$ on $[t_0, t_1]$.

Clearly trajectory—reachability implies reachability. First, let us explore trajectory—reachability of system (8), where the input $r(t)$ presents linearly:

$$\begin{aligned} {}^{RL}D^{\beta}_{t_0+}h(t) &= ah(t) + b(t)r(t), \quad 0 < \beta \leq 1,\ t \in [t_0, t_1] \\ I^{1-\beta}_{t_0+}h(t)\big|_{t=t_0} &= 0. \end{aligned} \tag{9}$$

Theorem 4 *Suppose that the function $b(t)$ is continuous and $b(t) \neq 0, \forall t \in [t_0, t_1]$, then system (9) is trajectory—reachability.*

Proof Assume that $z(t)$ be a proposed trajectory in $\mathcal{T}_{\mathcal{R}}$. We derive an input function $r(t)$ by

$$r(t) = \frac{{}^{RL}D^{\beta}_{t_0+}z(t) - az(t)}{b(t)} \tag{10}$$

with this input function, (9) becomes

$$\begin{aligned} {}^{RL}D^{\beta}_{t_0+}h(t) &= ah(t) + {}^{RL}D^{\beta}_{t_0+}z(t) - az(t) \\ I^{1-\beta}_{t_0+}h(t)\big|_{t=t_0} &= 0. \end{aligned}$$

Fixing $w(t) = h(t) - z(t)$, we have

$$\begin{aligned} {}^{RL}D^{\beta}_{t_0+}w(t) &= aw(t) \\ I^{1-\beta}_{t_0+}w(t)\big|_{t=t_0} &= 0. \end{aligned} \tag{11}$$

The solution of (11) is $w(t) = 0, \forall t \in [t_0, t_1]$. This proves the trajectory—reachability of system (9).

Let us now consider the case in which the input function $r(t)$ appears nonlinearly in (8). To prove the trajectory—reachability of (8), we will need to make the following assumptions:

[**A1**] $b(t, r(t))$ is continuous on $[t_0, t_1] \times \mathbb{R}$.
[**A2**] $b(t, r(t))$ is coercive in the second variable, that is,

$$b(t, r(t)) \to \pm\infty \text{ as } r(t) \to \pm\infty, t \in [t_0, t_1].$$

Theorem 5 *Let assumptions* **[A1]** *and* **[A2]** *are fulfilled, then the nonlinear system* (8) *is trajectory—reachability.*

Proof The solution of (8) is

$$h(t) = \int_{t_0}^{t} (t-\zeta)^{\beta-1} E_{\beta,\beta}(a(t-\zeta)^{\beta}) b(\zeta, r(\zeta)) \mathrm{d}\zeta. \tag{12}$$

Let $z \in \mathcal{T}_{\mathcal{R}}$ be the derived trajectory with $z(t_0) = 0$. We need to find an input function $r(t)$ satisfying

$$z(t) = \int_{t_0}^{t} (t-s)^{\beta-1} E_{\beta,\beta}(a(t-s)^{\beta}) b(s, r(s)) \mathrm{d}s.$$

Taking Riemann–Liouville derivative operator of order $\beta \in (0, 1]$ versus t on both sides, we have

$$\begin{aligned}
{}^{RL}D^{\beta}_{t_0+}z(t) &= {}^{RL}D^{\beta}_{t_0+}\left(\int_{t_0}^{t}(t-\zeta)^{\beta-1}E_{\beta,\beta}(a(t-\zeta)^{\beta})b(\zeta,r(\zeta))\mathrm{d}\zeta\right)(t)\\
&= \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\left[\int_{t_0}^{t}(t-\zeta)^{-\beta}\mathrm{d}s\int_{t_0}^{\zeta}(\zeta-\eta)^{\beta-1}E_{\beta,\beta}(a(\zeta-\eta)^{\beta})b(\eta,r(\eta))\mathrm{d}\eta\right]\\
&= \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\left[\int_{t_0}^{t}b(\eta,r(\eta))\mathrm{d}\eta\int_{\eta}^{\zeta}(t-\zeta)^{-\beta}(\zeta-\eta)^{\beta-1}E_{\beta,\beta}(a(\zeta-\eta)^{\beta})\mathrm{d}\zeta\right]\\
&= \frac{d}{dt}\int_{t_0}^{t}E_{\beta}(a(t-\eta)^{\beta})b(\eta,r(\eta))\mathrm{d}\eta\\
&= b(t,r(t))+a\int_{t_0}^{t}(t-\eta)^{\beta-1}E_{\beta,\beta}(a(t-\eta)^{\beta})b(\eta,r(\eta))\mathrm{d}\eta. \qquad (13)
\end{aligned}$$

Equation (13) becomes

$$w(t)=\int_{t_0}^{t}k(t,\eta)w(\eta)\mathrm{d}+\mathrm{w}_0(\mathrm{t}). \qquad (14)$$

Here $w(t)=b(t,r(t))$, $k(t,\zeta)=-a(t-\zeta)^{\beta-1}E_{\beta,\beta}(a(t-\zeta)^{\beta})$ and $w_0(t)={}^{RL}D^{\beta}z(t)$.

Equation (14) is a second kind of linear Volterra integral equation and it has a unique solution $w(t)$ for each given $w_0(t)$(refer [32]). Since from the solution of $w(t)$, it suffices to extract $r(t)$ by using the way of Deimling [33, 34].

Let $\mathcal{K}:[t_0,t_1]\to 2^{\mathbb{R}}$ be the multi-valued function given by $\mathcal{K}(t)=\{r\in\mathbb{R}: b(t,r)=w(t)\}$. Since $w(\cdot)$ and $b(\cdot,\cdot)$ are continuous, by assumption [**A2**], $\mathcal{K}(t)$ is non-empty and upper semicontinuous for all t. That is, $t_n\to 0$ gives $\mathcal{K}(t_n)\subset\mathcal{K}(0)+\bar{B}_{\epsilon}(0)$, for each $n\geq n(\epsilon,0)$. Further $\mathcal{K}$ has compact values. So, $\mathcal{K}$ is Lebesgue measurable and then has a measurable selection $r(\cdot)$. This function $r(t)$ is the desired input function which leads the nonlinear system along with the given trajectory $z(\cdot)$ and hence the system is trajectory—reachability.

We now extend the trajectory—reachability results of fractional dynamical systems in one dimension to n-dimensional non-integer order dynamical systems characterised by the differential equation of following texture:

$$\begin{aligned}
&{}^{RL}D^{\beta}_{t_0+}h(t)=Ah(t)+B(t,r(t)),\ 0<\beta\leq 1,\ t_0<t\leq t_1<\infty, \qquad (15)\\
&I^{1-\beta}_{t_0+}h(t)\big|_{t=t_0}=0,
\end{aligned}$$

where the state vector $h\in\mathbb{R}^n$, $n>1$, the input vector $r\in L^2([t_0,t_1]):=U$ and the matrix $A\in\mathbb{R}^{n\times n}$, $B:[0,T]\times U\to\mathbb{R}^n$.

[**C1**] Assume that $B(t,r(t))$ is continuous with respect to $r(t)$ for almost all $t\in[t_0,t_1]$ and it is measurable with respect to t for all $r\in U$ and it follows the growth condition

$$\|B(t,r)\|_{\mathbb{R}^n}\leq b_0(t)+b_1\|r\|_U,\forall r\in U,t\in[t_0,t_1].$$

The solution $h(t)$ of the system (15) is

$$h(t) = \int_{t_0}^{t} (t-\zeta)^{\beta-1} E_{\beta,\beta}(A(t-\zeta)^{\beta}) B(\zeta, r(\zeta)) \mathrm{d}\zeta. \tag{16}$$

Let $\mathcal{T}_{\mathcal{R}}$ be the set of all continuous functions $z(\cdot)$ given on $[t_0, t_1]$ such that $z(t_0) = 0$ and $z(t_1) = h_1$ and let ${}^{RL}D^{\beta}_{t_0+}z$ exist almost everywhere and $I^{1-\beta}_{t_0+}z(t)\big|_{t=t_0} = 0$. We say $\mathcal{T}_{\mathcal{R}}$ the set of all trajectories of system (15).

Definition 8 System (15) is called to be trajectory—reachability if, for any $z \in \mathcal{G}$, there exists an L^2-function $r : [t_0, t_1] \to \mathbb{R}^n$ such that the respective solution $h(\cdot)$ of (15) follows $h(t) = z(t)$ on $[t_0, t_1]$.

Now we are intended to show the trajectory—reachability results for system (15).

Theorem 6 *Let $B(t, u)$ satisfy coercivity and monotonicity conditions. That is,*

$$\langle B(t, r) - B(t, \zeta), r - \zeta \rangle \geq 0, \quad \textit{for all } r, \zeta \in U, t \in [t_0, t_1]$$

and

$$\lim_{\|r\| \to \infty} \frac{\langle B(t, r), r \rangle}{\|r\|} = \infty.$$

Then system (15) is trajectory—reachability by a measurable input function $r : [t_0, t_1] \to R^n$.

Proof Suppose z be any trajectory in $\mathcal{G}$. From Theorem 5, we look for an input function $r(t)$ satisfying

$$z(t) = \int_{t_0}^{t} (t-\zeta)^{\beta-1} E_{\beta,\beta}(A(t-\zeta)^{\beta}) B(\zeta, r(\zeta)) \mathrm{d}\zeta.$$

Taking R-L derivative of order $0 < \beta \leq 1$ versus t on both sides of the above equation, we receive

$${}^{RL}D^{\beta}_{t_0+}z(t) = B(t, r(t)) + A \int_{t_0}^{t} (t-\eta)^{\beta-1} E_{\beta,\beta}(A(t-\eta)^{\beta}) B(\eta, r(\eta)) \mathrm{d}\eta.$$

The above equation becomes

$$y(t) = \int_{t_0}^{t} k(t, \eta) y(\eta) \mathrm{d} + \mathrm{y_0(t)}, \tag{17}$$

where $y(t) = B(t, r(t)), k(t, \eta) = -A(t-\eta)^{\beta-1} E_{\beta,\beta}(A(t-\eta)^{\beta})$ and $y_0(t) = ^{RL} D^{\beta} z(t)$. Define an operator $K : L^2([t_0, t_1], \mathbb{R}^n) \to L^2([t_0, t_1], \mathbb{R}^n)$ by

$$(Ky)(t) = \int_{t_0}^{t} k(t, \zeta) y(\zeta) \mathrm{d}\zeta.$$

Definition 3 gives that K is a bounded linear operator. Also this can be shown that K^n is a contraction for sufficiently big n [33]. As a result of generalised Banach contraction principle, there exists a unique solution y for (17) for proposed $y_0 \in C([0, T], \mathbb{R}^n)$. Therefore trajectory—reachability follows if we can quote $r(t)$ from the equation

$$B(t, r(t)) = y(t), \ t \in [t_0, t_1]. \tag{18}$$

For checking it, derive an operator $N : L^2([t_0, t_1], \mathbb{R}^n) \to L^2([t_0, t_1], \mathbb{R}^n)$ by $(Nr)(t) = B(t, r(t))$. Assumption **[C1]** implies that N is well defined, bounded and continuous operator. Given hypothesis shows that N is coercive and monotone. A hemi-continuous monotone mapping is of type(M) [35]. Therefore, the nonlinear map N is onto by Theorem 3.6.9 of [35]. Hence there exists an input r satisfying (18). The measurability of $r(t)$ follows as r is in $L^2([t_0, t_1], \mathbb{R}^n)$. This proves the trajectory—reachability of system (15).

6 Optimal Reachability

In this part, we research some functional analysis concepts to investigate the optimal reachability of the fractional dynamical system (4). Throughout this section, we assume that X and Y are reflexive Banach spaces and U be a weakly compact subset of Y.

Consider the cost functional to be minimised over the class of inputs $r(t)$ which is given by

$$J(r) = \phi(r, h) = \int_{t_0}^{t_1} g(\zeta, h(\zeta), r(\zeta)) d\zeta.$$

For a given r, the solution representation of (4) can be expressed as

$$h(t) = \int_{t_0}^{t} (t-\zeta)^{\beta-1} E_{\beta,\beta}(A(t-\zeta)^{\beta}) Br(\zeta) d\zeta.$$

Let us define the space $X = L^2([t_0, t_1], R^n)$, $Y = AC([t_0, t_1], U)$ and the operators $L : X^* \to X$ and $T : Y \to X$ as follows:

$$(Lh)(t) = \int_{t_0}^{t} (t-\zeta)^{\beta-1} E_{\beta,\beta}(A(t-\zeta)^{\beta}) h(\zeta) d\zeta$$

$$(Tr)(t) = \int_{t_0}^{t} (t-\zeta)^{\beta-1} E_{\beta,\beta}(A(t-\zeta)^{\beta}) Br(\zeta) d\zeta.$$

By using the above definition, we can reduce equation (4) into an equivalent operator equation of the form

$$h = Tr, \tag{19}$$

where $h \in X$ for a fixed input r in Y.
Assumption [I]:

(i). The operator T is completely continuous.
(ii). The operator T is Gateaux differentiable with derivative $\delta T(r) = T$ for all $r \in Y$.
(iii). For a fixed h, $J(r) = \phi(r, h)$ is convex for all $r \in Y$.
(iv). The input set U is closed, convex and bounded.

Theorem 7 *Under Assumption [I], there exist an optimal pair* $(r^*, h^*) \in U \times X$ *for the abstract system* (19).

Proof Since the operator T is Gateaux differentiable, we can easily show that the cost functional J(r) is Gateaux differentiable on U.

Next, we have to prove that J(r) is weakly sequentially lower semicontinuous. Let $r, r^* \in U$. By the convexity of J, we can write

$$\beta J(r) + (1-\beta)J(r^*) \geq J(\beta r + (1-\beta r^*))$$
$$J(r) - J(r^*) \geq \frac{1}{\beta}[J(r^* + \beta(r - r^*)) - J(r^*)] \quad \forall r, r^* \in U.$$

This implies that

$$J(r) - J(r^*) \geq \left\langle \delta J(r^*), r - r^* \right\rangle_{X^* \times X} \quad \forall r, r^* \in U$$
$$J(r_n^*) - J(r^*) \geq \left\langle \delta J(r^*), r_n - r^* \right\rangle_{X^* \times X} \quad \forall r^* \in U, \quad \forall n \in N.$$

Then we can write

$$\lim_{n\to\infty} inf(J(r_n^*) - J(r^*)) \geq \lim_{n\to\infty} \left\langle \delta J(r^*), r_n^* - r^* \right\rangle = 0.$$

Since the operator T is completely continuous which implies that $Tr_n \to Tr^*$ strongly in X.

$$\lim_n inf(\phi(r_n^*, Tr_n^*) - \phi(r^*, Tr^*)) \geq 0$$

which implies

$$\phi(r^*, Tr^*) \leq \lim_n inf\, \phi(r_n^*, Tr_n^*).$$

As $\phi(r, h)$ is convex, weakly sequentially lower semicontinuous and U is closed, convex and bounded, then there exists an optimal pair $(r^*, h^*) \in U \times X$ such that

$$J(r^*) = \phi(r^*, h^*) \leq \inf_{r\in U} \phi(r, h) = \inf_{r\in U} J(r).$$

Hence, (r^*, h^*) is an optimal pair of the abstract system (19).

To derive an optimality system for the abstract system (19), we consider the quadratic cost functional of the form

$$J(r) = \langle r, Rr\rangle + \langle h, Qh\rangle \tag{20}$$

where R is symmetric bounded linear strictly monotone and coercive operator and Q is symmetric bounded linear monotone operator.

Lemma 1 *If T is Gateaux differentiable, then the cost functional* (20) *is Gateaux differentiable and* $\delta J(r)$ *is given by*

$$\delta J(r)x = 2\,\langle \delta T(r)x, QT(r)\rangle + 2\,\langle x, Rr\rangle \quad r \in Y.$$

Proof Let us take cost functional of the form

$$J(r) = \langle r, Rr\rangle + \langle h, Qh\rangle\,.$$

Since R and Q are symmetric bounded linear operators, we get

$$\begin{aligned}
J(r+\beta x) - J(r) &= \langle r+\beta x, R(r+\beta x)\rangle + \langle T(r+\beta x), QT(r+\beta x)\rangle - \langle r, Rr\rangle - \langle Tr, QTr\rangle \\
&= 2\beta\,\langle x, Rr\rangle + \beta^2\,\langle x, Rx\rangle + \langle T(r+\beta x) - T(r), QT(r)\rangle + \beta^2\,\langle T(x), QT(x)\rangle \\
&\quad +\beta\,\langle T(r), QT(r)\rangle \\
\frac{J(r+\beta x) - J(r)}{\beta} &= 2\,\langle x, Rr\rangle + \beta\,\langle x, Rx\rangle + 2\left\langle \frac{T(r+\beta x) - T(r)}{\beta}, QT(r)\right\rangle + \beta\,\langle T(x), QT(x)\rangle\,.
\end{aligned}$$

As $\beta \to 0$, the above equation becomes

$$\delta J(r)x = 2\,\langle x, Rr\rangle + 2\,\langle \delta T(r)x, QT(r)\rangle\,.$$

Theorem 8 *Under Assumption [I], the optimality system for the abstract system* (19) *is given by*

$$\begin{aligned}
h^* &= Tr^* \\
r^* &= -R^{-1}T^*Qh^*.
\end{aligned}$$

Proof The necessary condition for r^* is an optimal input which is $\delta J(r^*) = 0$. By using Lemma 1, we have

$$\delta J(r^*)x = 0 \implies 2\left\langle \delta T(r^*)x, QTr^*\right\rangle + 2\left\langle x, Rr^*\right\rangle = 0 \quad \forall h \in Y.$$

Taking adjoint of the derivative of the operator T, we get

$$\left\langle x, Rr^* \right\rangle + \left\langle x, [\delta T(r^*)]^* Qh^* \right\rangle = 0 \quad \forall x \in Y.$$

This implies that

$$[\delta T(r^*)]^* Qh^* = -Rr^*$$

which gives

$$r^* = -R^{-1}T^*Qh^*.$$

Hence, the optimal pair (r^*, h^*) satisfies the operator equations

$$\begin{aligned} h^* &= Tr^* \\ r^* &= -R^{-1}T^*Qh^*. \end{aligned}$$

This completes the proof.

Now we show that Hamiltonian system in Pontryagin's minimum principle satisfied by the optimal pair can be deduced from the optimality system derived in Theorem 8.

The Hamiltonian H(h, r, p) of the system is specified by

$$H(h, r, p) = \frac{1}{2}\langle h, Qh\rangle + \frac{1}{2}\langle r, Rr\rangle + \langle Ah, p\rangle + \langle Br, p\rangle$$

where p(t) denotes the costate of the system. From the definition of the operators, we have $T = LB$ and the optimality system in Theorem 8 can be written as

$$\begin{aligned} h^* &= Tr^* \\ r^* &= -R^{-1}B^*p^*, \end{aligned}$$

where $p^* = L^*Qh^*$.

Thus the optimal pair $(r^*(t), h^*(t))$ satisfies

$$\begin{aligned} h^*(t) &= \int_{t_0}^{t} (t-\zeta)^{\beta-1}E_{\beta,\beta}(A(t-\zeta)^{\beta})Br^*(\zeta)d\zeta \qquad (21) \\ r^*(t) &= -R^{-1}B^*p^* \end{aligned}$$

and costate

$$p^*(t) = \int_{t}^{t_1} (\zeta - t)^{\beta-1}E_{\beta,\beta}(A(\zeta - t)^{\beta})Qh^*(\zeta)d\zeta. \qquad (22)$$

Taking left- and right-sided R-L derivative operators of order β on both sides of (21) and (22), respectively, we get

$$
\begin{aligned}
{}^{RL}D^{\beta}_{t_0+}h^*(t) &= Ah^*(t) + Br^*(t) \equiv \frac{\partial}{\partial p} H(h^*(t), r^*(t), p^*(t)) \\
{}^{RL}D^{\beta}_{t_1-}p^*(t) &= -A^*p^*(t) - Qh^*(t) \equiv -\frac{\partial}{\partial h} H(h^*(t), r^*(t), p^*(t)) \\
I^{1-\beta}_{t_0+}h^*(t)|_{t=t_0} &= 0, \quad p^*(t_1) = 0.
\end{aligned}
\tag{23}
$$

Hence the pair of Eq. (23) is the Hamiltonian system satisfied by the optimal pair in Pontryagin's minimum principle. If the Hamiltonian system satisfied by the optimal pair is known, then it is also possible to derive the optimality system for the fractional dynamical systems.

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Fractional Calculus Approach to Logistic Equation and its Application

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Abstract In this study, we extend the properties of the Mittag-Leffler function that occurs as the solution of a fractional differential equation. Also, we use the properties to solve a fractional order mathematical model of epidemiology and offer a novel technique for obtaining an approximate solution to a fractional logistic equation.

1 Background of Study

The majority of real-world events may be mathematically represented using differential equations, which typically characterize the rate of change. Differential equations are created with the use of two fundamental tools, the derivative and the integral. When we formulate a problem, we almost always end up with a non-linear differential equation. Mathematical modeling using non-linear ordinary differentials has a wide variety of applications in fields such as fluid mechanics, biology, and engineering. As a result, the researchers are interested in solving non-linear ordinary differential equations and creating novel techniques for their practical applications. The majority of non-linear ordinary differential equations have a disadvantage in that they lack precise closed forms, necessitating the use of approximation techniques.

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However, a new type of derivative, known as a fractional derivative, was introduced three centuries ago. Earlier in ordinary differential equations, the order is given by whole numbers only whereas fractional differential equations attempt to promote the use of any real number or fraction as the order of the integral or derivative. Fractional calculus was studied mainly theoretically. After the theory of fractional differential equations has been established, many researchers currently concentrate on its application in solving practical problems. Since then, it has been investigated in various studies that fractional differential equations, unlike conventional differential equations, may successfully simulate various natural processes. In addition, in cases where both the classical and fractional methods are suitable for solving the problem, it is usually observed that the fractional method will minimize the error compared to the classical method. Fractional differential equations play an important role in analyzing mathematical models. Problems in epidemiology and finance can be solved using fractional differential equations. It has recently been shown that integral and fractional derivatives can be used to simulate numerous processes in viscoelasticity, fluid mechanics, physics, and other branches of science and technology. Mathai et al. [9] have reported various numerical and analytical techniques to obtain the solution of differential and integral equations with fractional order. The most used methods are the Adomain decomposition method (ADM), differential transformation method, and homotopy perturbation method (HPM). There are also some classic solutions, such as the Laplace transform method, fractional Green's function method, Mellin transform method, orthogonal polynomial method, and so on. Among these solving techniques, the VIM and the ADM are the most used techniques for solving fractional differential equations, as they provide the immediately visible symbolic form of the analytical solution, along with the linear and non-linear approximate numerical solutions.

The exponential function involved in the solution of differential equations plays an important role in describing the growth and decay of many real-world problems. In the fractional differential equation, the exponential function loses the property of describing the solution and is replaced by the Mittag-Leffler function. The Mittag-Leffler function appears as the solution of integral and differential equations with arbitrary order, especially in the fractional extension of dynamic equations, random walks, flight Levy, super diffusion transport research, and complex research system. The Mittag-Leffler functions interpolate between the purely exponential law and the power-law behavior, which is governed by ordinary dynamic equations and their fractional counterparts.

The present chapter is an extension of properties of the Mittag-Leffler function, and they are used to find the solution to non-linear logistic equations in arbitrary order. Pierre-Francois Verhulst's (1844–1845) pioneering research was the first to propose the logistic equation. To characterize the self-limiting expansion of biological populations, Verhulst [15] developed the logistic equation. Interestingly, Sweilam et al. [14] assert that the logistic equation is described by a first-order ordinary differential equation. Since then, the logistic equation is used extensively in many scientific research fields of chemistry, physics, ecology, population dynamism, political science, geoscience, economics, statistics, and sociology. In ecology, this equation is often used to describe population increase, where the reproduction rate is directly proportional to both the existing population and the number of available resources [15].

Verhulst [15] first published the population growth model in terms of non-linear differential equation subsequently termed as logistic equation,

$$\frac{d}{dt}u(t) = k\,u(t)(1 - u\,(t)), \quad t \geq 0, \tag{1}$$

whose precise solution is given by

$$u\,(t) = \frac{u_0}{u_0 + (1 - u_0)\exp(-kt)}, \tag{2}$$

where the value of u at initial stage given by u_0 is for time $t = 0$. Aforesaid equation occurs while formulating epidemics, neural networks, ecology, sociology, and Fermi distribution. In economics also, the equation plays an important role, which is a great motivation to study logistic equation by generalizing (1) it to its arbitrary order.

In 2015, West [16] investigated the fractional form of the non-linear logistic equation as follows:

$$D_t^{\alpha}\,[u\,(t)] = k^{\alpha}u\,(t)\,[1 - u\,(t)]\,. \tag{3}$$

He used Carleman embedding technique to get the solution of (3) as

$$u\,(t) = \sum_{n=0}^{\infty}\left(\frac{u_0 - 1}{u_0}\right)^n E_{\alpha}\,(-nk^{\alpha}t^{\alpha}), \quad t \geq 0. \tag{4}$$

In 2016, Area et al. [2] proved that the solution in (3) is not an accurate solution of the fractional logistic equation. Further, in 2017, Ortigueira et al. [3] also stated the exact solution of the fractional logistic equation and represented it in form of fractional Taylor series. In 2021, Area et al. [1] demonstrated the solution of fractional logistic equation in terms of power series and Nieto [11] has used non-singular kernel for fractional logistic equation. In this chapter, we propose a unique method for solving fractional logistic equations based on the Jumarie [6] idea.

2 Preliminaries

2.1 Mittag-Leffler Function

It is an generalized form of exponential function named after mathematician Gösta Mittag-Leffler [5] defined as follows:

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0. \tag{5}$$

2.2 Mittag-Leffler Function in Two Parameters

The two-parameter *Mittag-Leffler function* [5] is defined as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \quad , \tag{6}$$

where $\alpha, \beta \in C$; $Re(\alpha) > 0$, $Re(\beta) > 0$.

The Mittag-Leffler function is applicable in many areas of science and technology due to direct involvement in the solution of ordinary differential equation when extended to its non-integer order.

2.3 Riemann–Liouville Fractional Derivative

A fractional derivative of order α is given using the Riemann–Liouville definition [10] as

$$D^{\alpha}[x(t)] = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left[\int_0^t \frac{x(\tau)}{(t-\tau)^{\alpha}} d\tau \right], \tag{7}$$

where $\alpha \in R$ is order of fractional derivative, $n - 1 < \alpha \leq n$ and $n \in N = \{1, 2, 3, \ldots\}$, and $\Gamma(\cdot)$ is the Euler Gamma function.

2.4 Caputo's Fractional Derivative

Caputo's [13] definition of fractional derivative is given by

$${}_{0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{8}$$

where $\alpha \in R$ is order of fractional derivative, $n - 1 < \alpha \leq n$ and $n \in N = \{1, 2, 3, \ldots\}$, $f^n(\tau) = \frac{d^n}{dt^n} f(\tau)$, and $\Gamma(.)$ is the Euler Gamma function.

3 Main Results

3.1 *Mittag-Leffler Function (E_α) as the Base for Logarithm Function*

Consider a one-one and onto function $E_\alpha(x)$ (if exists) such that $E_\alpha : \mathscr{R} \to \mathscr{R}^+$ has its inverse $L_\alpha(\log_{E_\alpha})$, then for every L_α, (Re $\alpha > 0$), we have

$$\left.\begin{array}{lll} E_\alpha\left[L_\alpha(y)\right] = y, & \text{where} & y > 0 \\ L_\alpha\left[E_\alpha(x)\right] = x, & \text{where} & x \in \mathscr{R} \\ E_\alpha(x) = y & or & L_\alpha(y) = x \\ E_\alpha(x) \to +\infty & \text{when} & x \to +\infty \\ E_\alpha(x) \to 0 & \text{when} & x \to -\infty \\ L_\alpha(x) \to +\infty & \text{when} & x \to +\infty \\ L_\alpha(x) \to -\infty & \text{when} & x \to 0 \end{array}\right\}, \quad \text{where } \operatorname{Re}(\alpha) > 0. \tag{9}$$

Also, the Mittag-Leffler function $E_\alpha(x)$ is differentiable along with its inverse $L_\alpha(\log_{E_\alpha})$ is also differentiable. Further, it is important to note that $E'_\alpha(x) \neq E_\alpha(x)$. However, many authors have proved some properties of the Mittag-Leffler function by considering $E_\alpha\left(a(x_1+x_2)^a\right) = E_\alpha\left(ax_1^a\right) E_\alpha\left(ax_2^a\right)$, where $x_1,\ x_2 \geq 0$, a is real constant, and $\alpha > 0$. Various interesting results on the Mittag-Leffler function were given by Nieto [12]. Jumarie [6] also gave various definitions of fractional calculus. Gorenflo et al. [5] also reported many important results in their book. Chauhan et al. [4] have also given interesting results on the Mittag-Leffler function.

In this chapter, we have established some new results on the Mittag-Leffler function in the form of the following proposition.

Proposition: *Let* $\xi = E_\alpha(k_1)$ *and* $\eta = E_\alpha(k_2)$, *where* $\alpha > 0$ *with above conditions* (9). *Then*

(i) $E_\alpha(k_1 \oplus k_2) = E_\alpha(k_1) \odot E_\alpha(k_2)$.
(ii) $\log_{E_\alpha}(\xi \odot \eta) = k_1 \oplus k_2 = \log_{E_\alpha}(\xi) \oplus \log_{E_\alpha}(\eta)$.
(iii) $\log_{E_\alpha}(\xi \oplus \eta) = k_1 \ominus k_2 = \log_{E_\alpha}(\xi) \ominus \log_{E_\alpha}(\eta)$.

Proof (i) Considering

$$k_1 \oplus k_2 = L_\alpha(\xi \odot \eta), \tag{10}$$

with the conditions (9), then

$$E_\alpha\left(k_1 \oplus k_2\right) = \xi \odot \eta = E_\alpha\left(k_1\right) \odot E_\alpha\left(k_2\right). \tag{11}$$

Considering $\alpha \to 1$, then (11) reduces to

$$E_1\left(k_1 \oplus k_2\right) = E_1\left(k_1\right) \odot E_1\left(k_2\right), \tag{12}$$

which is equivalent to

$$\exp\left(k_1 + k_2\right) = \exp\left(k_1\right) \cdot \exp\left(k_2\right). \quad (13)$$

(ii) Now considering

$$\xi \odot \eta = E_\alpha\left(k_1 \oplus k_2\right), \quad (14)$$

taking logarithm with base E_α on both sides,

$$\log_{E_\alpha}\left(\xi \odot \eta\right) = k_1 \oplus k_2 = \log_{E_\alpha}(\xi) \oplus \log_{E_\alpha}(\eta).$$

On applying the limit $\alpha \to 1$, then (3.1) reduces to

$$\log_{E_1}\left(\xi \odot \eta\right) = \log_{E_1}(\xi) \oplus \log_{E_1}(\eta), \quad (15)$$

which is equivalent to

$$\log_e(\xi \cdot \eta) = \log_e(\xi) + \log_e(\eta). \quad (16)$$

(iii) Similarly, we can easily show

$$\log_{E_\alpha}\left(\xi \oslash \eta\right) = k_1 \ominus k_2 = \log_{E_\alpha}(\xi) \ominus \log_{E_\alpha}(\eta). \quad (17)$$

On applying the limit $\alpha \to 1$, then (17) reduces to

$$\log_{E_1}\left(\xi \oslash \eta\right) = k_1 \ominus k_2 = \log_{E_1}(\xi) \ominus \log_{E_1}(\eta), \quad (18)$$

or

$$\log_e\left(\frac{\xi}{\eta}\right) = \log_e(\xi) - \log_e(\eta). \quad (19)$$

Note: For distinct values of α, k_1 and k_2, the introduced operators $\oplus$, $\odot$, $\ominus$, and $\oslash$ operate very similarly as classical addition, multiplication, subtraction, and division operators. For distinct values of α, $0 < \alpha \leq 1$, the preceding argument can be verified numerically from Tables 1 and 2, where $\log_{E_\alpha}$ is logarithm function having base as the Mittag-Leffler function. Some properties of logarithmic function with the base Mittag-Leffler function $E_\alpha (= E_\alpha(1))$ are identical to logarithmic function having base as 'e'. For different choices of α, the graph of '$\log_{E_\alpha}$' is demonstrated in Fig. 1.

3.2 Fractional Logistic Equation and Its Solution

Extension of logistic equation (1) with arbitrary order α, $\alpha \in (0, 1]$ is given by

$$\frac{d^\alpha u(t)}{dt^\alpha} = k^\alpha u\left[1 - u(t)\right]. \quad (20)$$

Table 1 Values of $\log_{E_\alpha}(x)$ for different values of x and α

α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$E_\alpha(= E_\alpha(1))$	23.1605	11.823	8.0407	6.1471	5.009	4.2486	3.7041	3.2946	2.9749	2.7183
$\log_{E_\alpha}(0.1)$	−0.7327	−0.9322	−1.1046	−1.2680	−1.4291	−1.5917	−1.7584	−1.9312	−2.1120	−2.3026
$\log_{E_\alpha}(0.2)$	−0.5122	−0.6516	−0.7721	−0.8863	−0.9989	−1.1126	−1.2291	−1.3499	−1.4763	−1.6094
$\log_{E_\alpha}(0.3)$	−0.3831	−0.4874	−0.5776	−0.6630	−0.7472	−0.8323	−0.9194	−1.0098	−1.1043	−1.2040
$\log_{E_\alpha}(0.4)$	−0.2916	−0.3710	−0.4396	−0.5046	−0.5687	−0.6334	−0.6998	−0.7685	−0.8405	−0.9163
$\log_{E_\alpha}(0.5)$	−0.2206	−0.2806	−0.3325	−0.3817	−0.4302	−0.4792	−0.5293	−0.5814	−0.6358	−0.6931
$\log_{E_\alpha}(0.6)$	−0.1626	−0.2068	−0.2451	−0.2813	−0.3170	−0.3531	−0.3901	−0.4284	−0.4686	−0.5108
$\log_{E_\alpha}(0.7)$	−0.1135	−0.1444	−0.1711	−0.1964	−0.2214	−0.2466	−0.2724	−0.2992	−0.3272	−0.3567
$\log_{E_\alpha}(0.8)$	−0.0710	−0.0903	−0.1070	−0.1229	−0.1385	−0.1542	−0.1704	−0.1872	−0.2047	−0.2231
$\log_{E_\alpha}(0.9)$	−0.03353	−0.0426	−0.0505	−0.05802	−0.0654	−0.0728	−0.0805	−0.0884	−0.0966	−0.1054
$\log_{E_\alpha}(1)$	0	0	0	0	0	0	0	0	0	0
$\log_{E_\alpha}(2)$	0.2206	0.2806	0.3325	0.38169	0.4302	0.4792	0.5293	0.5814	0.6358	0.6931
$\log_{E_\alpha}(3)$	0.3496	0.4448	0.5270	0.6049	0.6818	0.7594	0.8390	0.9214	1.0077	1.0986
$\log_{E_\alpha}(4)$	0.4412	0.5612	0.6650	0.7634	0.8604	0.9583	1.0587	1.1627	1.2716	1.3863
$\log_{E_\alpha}(5)$	0.5122	0.6516	0.7721	0.8863	0.9989	1.1126	1.2291	1.3499	1.4763	1.6094
$\log_{E_\alpha}(6)$	0.5702	0.7254	0.8596	0.9867	1.1120	1.2386	1.3683	1.5028	1.6435	1.7917
$\log_{E_\alpha}(7)$	0.6192	0.7878	0.9335	1.0715	1.2077	1.3452	1.4861	1.6321	1.7849	1.9459
$\log_{E_\alpha}(8)$	0.6617	0.8419	0.9976	1.1451	1.2906	1.4375	1.5880	1.7441	1.9074	2.0794
$\log_{E_\alpha}(9)$	0.6992	0.8895	1.0541	1.2099	1.3637	1.5189	1.6780	1.8429	2.0154	2.1972
$\log_{E_\alpha}(10)$	0.7327	0.9322	1.1046	1.2680	1.4291	1.5917	1.7584	1.9312	2.1120	2.3026

Table 2 Examples of Propositions 1 and 2

x_1	x_2	α	$\log_{E_\alpha}(x_1 \cdot x_2)$	$\log_{E_\alpha}\left(\frac{x_1}{x_2}\right)$	$\log_{E_\alpha}(x_1)$	$\log_{E_\alpha}(x_2)$	$\log_{E_\alpha}(x_1) + \log_{E_\alpha}(x_2)$	$\log_{E_\alpha}(x_1) - \log_{E_\alpha}(x_2)$
0.2	1	0.1	−0.5122	−0.5122	−0.5122	0.00	−0.5122	−0.5122
0.2	1	0.2	−0.6516	−0.6516	−0.6516	0.00	−0.6516	−0.6516
0.2	1	0.3	−0.7721	−0.7721	−0.7721	0.00	−0.7721	−0.7721
0.75	0.35	0.1	−0.4256	0.2426	−0.0915	−0.3341	−0.4256	0.2426
0.75	0.35	0.5	−0.8301	0.4731	−0.1785	−0.6516	−0.8301	0.4731
0.75	0.35	0.9	−1.2268	0.6991	−0.2639	−0.9630	−1.2269	0.6991
0.81	0.4	0.2	−0.4563	0.2857	−0.0853	−0.3709	−0.4562	0.2856
0.81	0.4	0.7	−0.8607	0.5388	−0.1609	−0.6998	−0.8607	0.5389
0.81	0.4	0.8	−0.9453	0.5918	−0.1767	−0.7685	−0.9453	0.5918
0.93	0.5	0.5	−0.4752	0.3852	−0.0450	−0.4302	−0.4752	0.3852
2	3	0.6	1.2386	−0.2802	0.4792	0.7594	1.2386	−0.2802
3	5	0.7	2.0681	−0.3901	0.8390	1.2291	2.0681	−0.3901
6	7	0.8	3.1349	−0.1293	1.5028	1.6321	3.1349	−0.1293
10	2	0.9	2.7478	1.4763	2.1121	0.6358	2.7479	1.4763

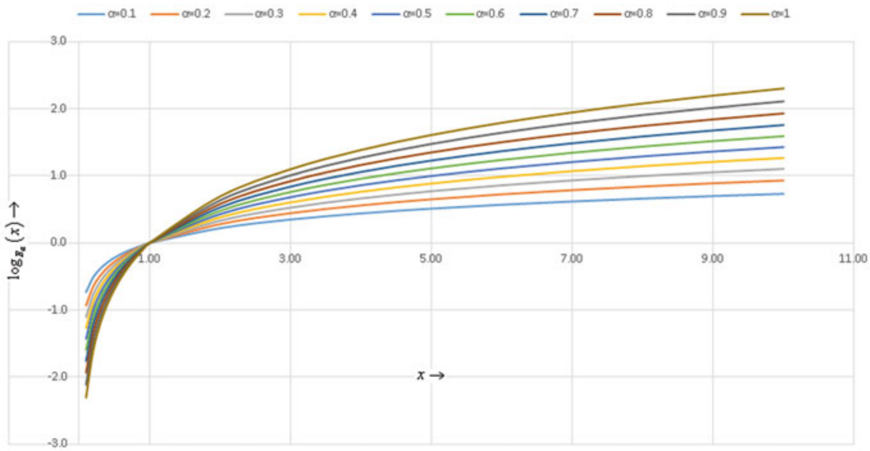

Fig. 1 Results of Log_{E_α} for distinct values of x and α

On separating the variables, (20) can be written as

$$\int \frac{d^\alpha u}{u(t)} \oplus \int \frac{d^\alpha u(t)}{1-u(t)} = k^\alpha \int dt^\alpha. \tag{21}$$

Solving above equation using the proposed proposition (21), we have

$$\log_{E_\alpha} u(t) \ominus \log_{E_\alpha} (1-u(t)) = \frac{k^\alpha}{\Gamma(2-\alpha)} \int t^{1-\alpha} dt^\alpha \oplus C, \tag{22}$$

with constant of integration as C. On using (9) and (18), we get

$$\frac{u(t)}{1-u(t)} = C E_\alpha \left[\frac{k^\alpha}{\Gamma(2-\alpha)} \int t^{1-\alpha} dt^\alpha \right]. \tag{23}$$

For $\alpha = 1$, we have $\log_{E_\alpha} x = \log_e x$. Here, we also show that the value of $u(t) \oslash (1 - u(t))$ is approximately equivalent to $\frac{u(t)}{1-u(t)}$ numerically.

Considering the initial state, i.e., when $t = 0$, we have $u(0) = u_0$, and hence, the value of $C = \frac{u_0}{1-u_0}$. Substituting C in (23), we have

$$\frac{u(t)}{1-u(t)} = \frac{u_0}{1-u_0} E_\alpha \left[\frac{k^\alpha}{\Gamma(2-\alpha)} \int t^{1-\alpha} dt^\alpha \right], \tag{24}$$

or

$$u(t) = \frac{1}{1 + \frac{1-u_0}{u_0} \left[E_\alpha \{ -\frac{k^\alpha}{\Gamma(2-\alpha)} \int t^{1-\alpha} dt^\alpha \} \right]}. \tag{25}$$

On setting $\alpha = 1$, Eq. (25) reduces to (2).

4 Mathematical Modeling

Mathematical modeling is an integrated process of translating real-world problem into the mathematical problem, which is based on mathematical concepts, i.e., functions, variables, constants, inequalities, etc. taken from algebra, geometry, calculus, and every other branch of mathematics (Fig. 2).

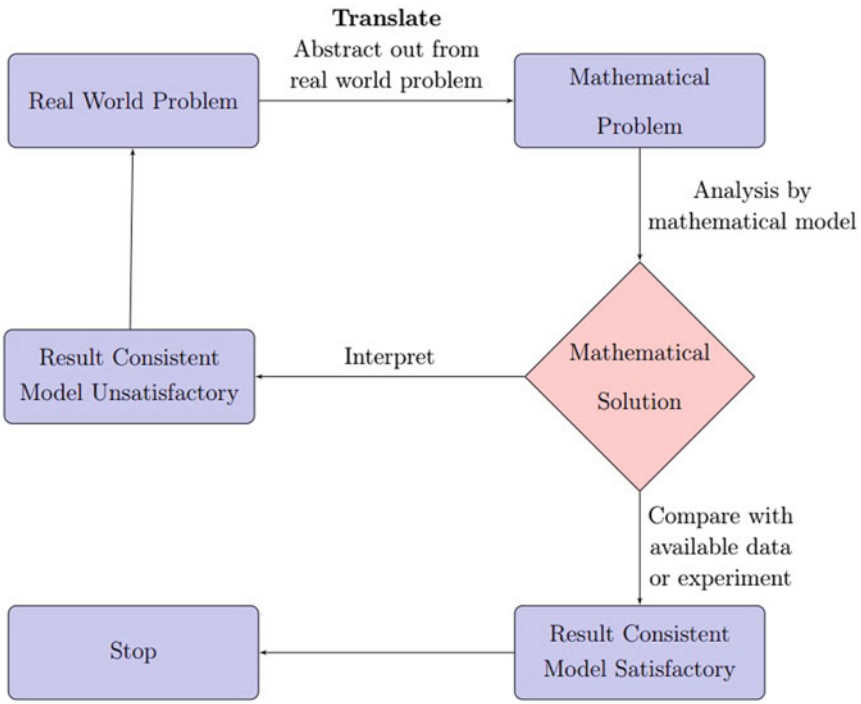

Fig. 2 Schematic diagram of mathematical modeling

With the help of mathematical modeling, one can describe the dynamical and stochastic processes. Usually, differential equations are used to describe the physical processes where the variables of the system change with time. The present section is devoted to the application of fractional differential equation to epidemiology which is described by generalizing logistic equation.

5 Application in Epidemiology

Epidemiology is the scientific study of disease transmission and its human effects. This encompasses a broad variety of disciplines, from biology to sociology and philosophy, all of which contribute to a thorough knowledge of epidemic transmission and control methods.

For gonorrhoea and encephalitis epidemics, standard epidemic transmission models were employed to analyze the epidemic breakout after an infectious stage. Daniel Bernoulli published the first report of mathematical modeling for epidemic spread in 1766 which was used to explain how part of the infected population requiring medical attention varies during an epidemic spread. Kermack and McKendrick [8] contributed to the mathematical theory of epidemics with their contribution. Epidemics can be analytically modeled with certain mathematical assumptions in order to explore the severity and preventative techniques by which diseases spread. This also aids in the prediction of an outbreak's future course and the evaluation of epidemic-control tactics. We're going to presume that the entire population given by N is partitioned into three main categories viz. S, the susceptible ones; I, the infected; and R, recovered patients during an epidemic. With closed demography, this model assumes that the whole population remains constant, i.e., there are no births or natural deaths. Any sickness that causes death, on the other hand, can be included in R. Many researchers have recently investigated and constructed epidemic models utilizing bifurcation approaches [8]. The ultimate goal is to simulate the problem of a saturated susceptible population, the time it takes for infected individuals to become infectious, and the stability of equilibrium solutions.

We identify the independent and dependent variables as the initial step in the modeling procedure. Time t, measured in days, is the independent variable and we focus on the two sets of dependent variables defined by S, I, and R as susceptible, infected, and recovered people, respectively.

Considering total population during epidemic N, we have

$$N(t) = S(t) + I(t) + R(t). \tag{26}$$

Here we analyze three simple epidemic models with some plausible assumptions, which are as follows.

5.1 SI Model

We assume that the entire population is made up of just susceptible and infected people in our model. As a result, the total population is calculated as follows [7]:

$$N(t) = S(t) + I(t).$$

Let the disease spread with I_0 infected persons during initial state, i.e., $t = 0$. Then the rate at which susceptible people gets infected with respect to time t is given by

$$\frac{dS}{dt} = -\beta IS, \tag{27}$$

where both S and I are functions of t, $t > 0$, and β is the positive constant and initially at time $t = 0$, $I(0) = I_0$. Also, the rate of change of infected is given by

$$\frac{dI}{dt} = \beta I(t)(N - I), \quad I(0) = I_0 \tag{28}$$

and

$$\frac{dS}{dt} = -\beta S(t)(N - S), \quad S(0) = N - I_0. \tag{29}$$

5.1.1 Fractional SI Model for Epidemic Spread

Here, we will analyze the fractional differential equation model for epidemics while considering only susceptible and infected persons and determine its solution.

On writing (28) with arbitrary order α, as

$$\frac{d^\alpha I}{dt^\alpha} = \beta I(N - I), \text{ where } 0 < \alpha \leq 1 \tag{30}$$

$$\Rightarrow \int \frac{d^\alpha I}{I} + \int \frac{d^\alpha I}{N - I} = N\beta \int dt^\alpha. \tag{31}$$

On using (18), the solution of (31) is given by

$$\log_{E_\alpha} I \ominus \log_{E_\alpha}(N - I) = \frac{N\beta}{\Gamma(2-\alpha)} \int t^{1-\alpha} dt^\alpha + c, \tag{32}$$

$$\Rightarrow \frac{I}{N - I} = CE_\alpha \left[\frac{N\beta}{\Gamma(2-\alpha)} \int t^{1-\alpha} dt^\alpha \right], \tag{33}$$

where $0 < \alpha < 1$ and $\lim_{\alpha \to 1} \log_{E_\alpha} x = \log_e x$.

Considering the initial condition, i.e., for time $t = 0$, we get $I(0) = I_0$, which gives $C = \frac{I_0}{N - I_0}$, or

$$\frac{I}{N - I} = \frac{I_0}{N - I_0} E_\alpha \left[\frac{N\beta}{\Gamma(2-\alpha)} \int t^{1-\alpha} dt^\alpha \right] \tag{34}$$

$$\Rightarrow I = \frac{N}{1 + \frac{N - I_0}{I_0}\left[E_\alpha\{-\frac{N\beta}{\Gamma(2-\alpha)}\int t^{1-\alpha}dt^\alpha\}\right]}, \tag{35}$$

when $\alpha \to 1$ and $t \to \infty$, it gives $I = N$.

5.2 SIS Model

The population in the SIS model is separated into two distinct classes: susceptible class to infected class and infected class to susceptible class. The disease's dynamics distribution is defined by two major functions viz. the contact rate and the infective period. People from the susceptible class enter to infective class after coming into contact with infectious people. After an infective phase, infective individuals revert to the susceptible class.

In this model, the infected person recovers and becomes vulnerable again determined by λI, where λ is a positive integer. As a result, we have differential equations as follows:

$$\frac{dI}{dt} = \beta I\,(N - I) - \lambda I \frac{dS}{dt} = -\beta S\,(N - S) + \lambda I. \tag{36}$$

5.2.1 Fractional SIS Model for Epidemic Spread

In this section, we propose the solution of the fractional ordered SIS model.

Rewriting Eq. (36),

$$\frac{dI}{dt} = \beta I\,[A - I]\,, \qquad I\,(0) = I_0,\ A = \left(N - \frac{\lambda}{\beta}\right). \tag{37}$$

Generalizing (36) to its arbitrary order with $\alpha \in (0, 1]$,

$$\frac{d^\alpha I}{dt^\alpha} = \beta I\,(A - I)\,, \qquad I\,(0) = I_0. \tag{38}$$

The solution of (38) can also be obtained using the similar way as demonstrated in the preceding section, i.e.,

$$\log_{E_\alpha} I \ominus \log_{E_\alpha}(A - I) = \frac{A\beta}{\Gamma(2-\alpha)}\int t^{1-\alpha}dt^\alpha + c \tag{39}$$

$$or \qquad \frac{I}{A - I} = C E_\alpha\left[\frac{A\beta}{\Gamma(2-\alpha)}\int t^{1-\alpha}dt^\alpha\right], \tag{40}$$

where $0 < \alpha < 1$ and $\lim_{\alpha \to 1} \log_{E_\alpha} x = \log_e x$. During initial stage, i.e., at time $t = 0$, we have $I(0) = I_0$, due to which $C = \frac{I_0}{A - I_0}$. Thus, (40) can be written as

$$\frac{I}{A - I} = \frac{I_0}{A - I_0} E_\alpha \left[\frac{A\beta}{\Gamma(2-\alpha)} \int t^{1-\alpha} dt^\alpha \right], \tag{41}$$

$$or \qquad I = \frac{A}{1 + \frac{A - I_0}{I_0} \left[E_\alpha \{ -\frac{A\beta}{\Gamma(2-\alpha)} \int t^{1-\alpha} dt^\alpha \} \right]}. \tag{42}$$

When $\alpha \to 1$ and $t \to \infty$, the above equation reduces $I = (N - \lambda/\beta)$, as reported by Kapur [7].

6 Conclusion

The novel method for solving logistic equations in an arbitrary order is simple and may be useful in the future study for addressing a variety of issues in physics and other branches of mathematics involving fractional differential equations. This article makes a significant addition to the theory of fractional calculus.

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Hermite–Hadamard Type Inequalities for Coordinated Quasi-Convex Functions via Generalized Fractional Integrals

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Abstract In this research, we used a generalized fractional integral to create a new Hermite–Hadamard-type integral inequality for functions of two independent variables that are quasi-convex on the coordinates. We also introduce additional inequalities of the Hermite–Hadamard type for functions of two variables that are twice partially differentiable and whose mixed-order partial derivatives in absolute value to specified powers are quasi-convex on the coordinates. Our findings are two-variable expansions of previous findings.

1 Introduction

One of the most fruitful concepts in Mathematics is the Convex Function, not only because of its theoretical impact in various areas but also because of the multiplicity of applications that have been developed in recent times.

A function $\psi : I \to \mathbb{R}$, $I := [a, b]$ is said to be convex if $\psi\big(\tau\xi + (1-\tau)\varsigma\big) \leq \tau\psi(\xi) + (1-\tau)\psi(\varsigma)$ holds $\forall\, \xi, \varsigma \in I, \tau \in [0, 1]$. And they say that the function ψ is concave on $[a, b]$ if the inequality is the opposite.

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Readers interested in the aforementioned development, can consult [43], where a panorama, practically complete, of these branches is presented.

The following double inequality is the well-known Hermite–Hadamard inequality:

Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then the following double inequalities hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Because of its many uses, the Hermite–Hadamard inequality has piqued the interest of many authors in recent decades. In the literature, there have been several generalizations, improvements and expansions of the Hermite–Hadamard inequality. We recommend the interested reader to the following papers [8, 9, 17, 18, 24, 27, 32–34, 39, 44, 45, 47, 48, 53–57].

Definition 1 (*See* [35]) The left and right Katugampola fractional integrals of f are

$${}^{\rho}I_{a+}^{\beta} f(x) = \frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_a^x (x^{\rho} - t^{\rho})^{\beta-1} t^{\rho-1} f(t)dt$$

and

$${}^{\rho}I_{b-}^{\beta} f(x) = \frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_x^b (t^{\rho} - x^{\rho})^{\beta-1} t^{\rho-1} f(t)dt,$$

where $\beta, \rho > 0$, the function f is real valued and defined on the interval $[a, b]$, with $a < b$, and the Gamma function, $\Gamma(\cdot)$, is

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt.$$

We recommend the interested reader to the next references for some recent results linked to the Katugampola fractional integral [18, 30–32, 34–36].

Recently, Chen and Katugampola [18] obtained the following generalizations of the Hermite–Hadamard inequality via the Katugampola fractional integrals.

Theorem 1 *Let $\beta, \rho > 0$ and $f : [a^{\rho}, b^{\rho}] \to \mathbb{R}$ be a positive function with $0 \leq a < b$. If f is a convex function on $[a^{\rho}, b^{\rho}]$, then the following inequalities hold:*

$$f\left(\frac{a^{\rho}+b^{\rho}}{2}\right) \leq \frac{\rho^{\beta}\Gamma(\beta+1)}{2(b^{\rho}-a^{\rho})^{\beta}}[{}^{\rho}I_{a+}^{\beta} f(b^{\rho}) + {}^{\rho}I_{b-}^{\beta} f(a^{\rho})]$$
$$\leq \frac{f(a^{\rho})+f(b^{\rho})}{2}.$$

Theorem 2 *Let $\beta, \rho > 0$ and $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable function on (a^ρ, b^ρ) with $0 \leq a < b$. If $|f'|$ is a convex function on $[a^\rho, b^\rho]$, then the following inequality holds:*

$$\left|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\rho^\beta \Gamma(\beta+1)}{2(b^\rho-a^\rho)^\beta}[{}^\rho I^\beta_{a+} f(b^\rho) + {}^\rho I^\beta_{b-} f(a^\rho)]\right| \leq \frac{b^\rho - a^\rho}{2(b^\rho-a^\rho)^\beta}[|f'(a^\rho)| + |f'(b^\rho)|].$$

The concept of quasi-convex functions defined below is a generalization of convex functions.

Definition 2 (*See* [28]) A function $f : [a, b] \to \mathbb{R}$ is said to be quasi-convex on $[a, b]$, if

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\},$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$.

In [51], the authors established the following Hermite–Hadamard-type inequalities for quasi-convex functions via the Katugampola fraction integrals.

Theorem 3 *Let $\beta, \rho > 0$ and $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a positive function with $0 \leq a < b$. If f is a quasi-convex function on $[a^\rho, b^\rho]$, then the following inequality holds:*

$$\frac{\rho^\beta \Gamma(\beta+1)}{2(b^\rho-a^\rho)^\beta}[{}^\rho I^\beta_{a+} f(b^\rho) + {}^\rho I^\beta_{b-} f(a^\rho)] \leq \max\{f(a^\rho), f(b^\rho)\}.$$

Theorem 4 *Let $\beta, \rho > 0$ and $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable function on (a^ρ, b^ρ) with $0 \leq a < b$. If $|f'|$ is a quasi-convex function on $[a^\rho, b^\rho]$, then the following inequality holds:*

$$\left|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\rho^\beta \Gamma(\beta+1)}{2(b^\rho-a^\rho)^\beta}[{}^\rho I^\beta_{a+} f(b^\rho) + {}^\rho I^\beta_{b-} f(a^\rho)]\right| \leq \frac{b^\rho - a^\rho}{\rho(\beta+1)}\left(1 - \frac{1}{2^{\rho(\beta+1)}}\right)\max\{|f'(a^\rho)|, |f'(b^\rho)|\}.$$

Theorem 5 *Let $\beta, \rho > 0$ and $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable function on (a^ρ, b^ρ) with $0 \leq a < b$. If $|f'|^q$ is a quasi-convex function on $[a^\rho, b^\rho]$ and $q \geq 1$, then the following inequality holds:*

$$\left|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\rho^\beta \Gamma(\beta+1)}{2(b^\rho-a^\rho)^\beta}[{}^\rho I^\beta_{a+} f(b^\rho) + {}^\rho I^\beta_{b-} f(a^\rho)]\right| \leq \frac{b^\rho - a^\rho}{\rho(\beta+1)}\left(1 - \frac{1}{2^\beta}\right)\left(\max\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\}\right)^{\frac{1}{q}}.$$

Our goal in this study is to extend Theorem 3 for functions of two variables that are quasi-convex on the coordinates, which is motivated by recent research on the Hermite–Hadamard inequality. We also show additional expansions of Theorems 4 and 5 for two-variable functions whose second-order mixed partial derivatives in absolute value at certain powers are quasi-convex on the coordinates. We will present some early definitions that will be relevant in our work in the next sections.

The Katugampola fractional integrals in Definition 1 are natural expansions of the following fractional integrals for functions of two independent variables (see [29]).

Definition 3 Let $\beta_1, \beta_2, \rho_1, \rho_2 > 0$ and f be a function of two independent variables. We define, the Katugampola fractional integrals of f on the coordinates as follows:

$$^{\rho_1}I_{a+}^{\beta_1} f(x,y) := \frac{\rho_1^{1-\beta_1}}{\Gamma(\beta_1)} \int_a^x (x^{\rho_1} - u^{\rho_1})^{\beta_1 - 1} u^{\rho_1 - 1} f(u,y)du$$

$$^{\rho_1}I_{b-}^{\beta_1} f(x,y) := \frac{\rho_1^{1-\beta_1}}{\Gamma(\beta_1)} \int_x^b (u^{\rho_1} - x^{\rho_1})^{\beta_1 - 1} u^{\rho_1 - 1} f(u,y)du$$

$$^{\rho_2}I_{c+}^{\beta_2} f(x,y) := \frac{\rho_2^{1-\beta_2}}{\Gamma(\beta_2)} \int_c^y (y^{\rho_2} - v^{\rho_2})^{\beta_2 - 1} v^{\rho_2 - 1} f(x,v)dv$$

$$^{\rho_2}I_{d-}^{\beta_2} f(x,y) := \frac{\rho_2^{1-\beta_2}}{\Gamma(\beta_2)} \int_y^d (v^{\rho_2} - y^{\rho_2})^{\beta_2 - 1} v^{\rho_2 - 1} f(x,v)dv$$

The Katugampola fractional integrals of f in the two variables are defined as follows:

$$^{\rho_1,\rho_2}I_{a+,c+}^{\beta_1,\beta_2} f(x,y) := \frac{\rho_1^{1-\beta_1}\rho_2^{1-\beta_2}}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_a^x \int_c^y (x^{\rho_1} - u^{\rho_1})^{\beta_1 - 1} (y^{\rho_2} - v^{\rho_2})^{\beta_2 - 1} \times u^{\rho_1 - 1} v^{\rho_2 - 1} f(u,v)dvdu$$

$$^{\rho_1,\rho_2}I_{a+,d-}^{\beta_1,\beta_2} f(x,y) := \frac{\rho_1^{1-\beta_1}\rho_2^{1-\beta_2}}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_a^x \int_y^d (x^{\rho_1} - u^{\rho_1})^{\beta_1 - 1} (v^{\rho_2} - y^{\rho_2})^{\beta_2 - 1} u^{\rho_1 - 1} v^{\rho_2 - 1} f(u,v)dvdu$$

$$^{\rho_1,\rho_2}I_{b-,c+}^{\beta_1,\beta_2} f(x,y) := \frac{\rho_1^{1-\beta_1}\rho_2^{1-\beta_2}}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_x^b \int_c^y (u^{\rho_1} - x^{\rho_1})^{\beta_1 - 1} (x^{\rho_2} - v^{\rho_2})^{\beta_2 - 1} u^{\rho_1 - 1} v^{\rho_2 - 1} f(u,v)dvdu$$

$$ {}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{b-,d-}f(x,y) := \frac{\rho_1^{1-\beta_1}\rho_2^{1-\beta_2}}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_x^b\int_y^d (u^{\rho_1}-x^{\rho_1})^{\beta_1-1}(v^{\rho_2}-y^{\rho_2})^{\beta_2-1} u^{\rho_1-1}v^{\rho_2-1}f(u,v)dvdu $$

Definition 4 (*See* [46]) A real-valued function f defined on the rectangle $\Lambda := [a,b]\times[c,d]$, with $a<b$ and $c<d$, is called quasi-convex on Λ if the following inequality:

$$ f(tx+(1-t)z, sy+(1-t)w) \le \max\{f(x,y), f(z,w)\} $$

holds for all $(x,y), (z,w) \in \Lambda$ and $(s,t)\in[0,1]\times[0,1]$.

A function $f : \Lambda := [a,b]\times[c,d] \to \mathbb{R}$ will be called quasi-convex on the coordinates on Λ if the partial mappings

$$ f_y : [a,b] \to \mathbb{R},\ f_y(u) := f(u,y) $$

and

$$ f_x : [c,d] \to \mathbb{R},\ f_x(v) := f(x,v) $$

defined for all $x\in[a,b]$ and $y\in[c,d]$ are quasi-convex. We have the following formal definition of quasi-convexity on the coordinates.

Definition 5 (*See also* [49]) A function $f : \Lambda := [a,b]\times[c,d] \to \mathbb{R}$ is said to be quasi-convex on the coordinates on Λ, if the following inequality:

$$ \begin{aligned} f(tx+(1-t)z,&sy+(1-t)w) \\ &\le \max\{f(x,y), f(x,w), f(z,y)f(z,w)\} \end{aligned} $$

holds for all $(x,y), (x,w), (z,y), (z,w) \in \Lambda$ and $(s,t)\in[0,1]\times[0,1]$.

We suggest interested readers to the following papers for more information on quasi-convex functions on coordinates [46, 49].

2 Main Results

The following identities will be very useful to our results.

Lemma 1 *Let $\beta_1, \beta_2, \rho_1, \rho_2 > 0$, and $f : [a^{\rho_1}, b^{\rho_1}]\times[c^{\rho_2}, d^{\rho_2}] \to \mathbb{R}$ be a real-valued function of two independent variables. The following identities hold:*

$$\int_0^1 s^{\beta_2\rho_2-1} f(x^{\rho_1}, s^{\rho_2}d^{\rho_2} + (1-s^{\rho_2})c^{\rho_2})ds = \frac{\rho_2^{\beta_2-1}\Gamma(\beta_2)}{(d^{\rho_2}-c^{\rho_2})^{\beta_2}} \, {}^{\rho_2}I_{d-}^{\beta_2} f(x^{\rho_1}, c^{\rho_2}), \tag{1}$$

$$\int_0^1 s^{\beta_2\rho_2-1} f(x^{\rho_1}, s^{\rho_2}c^{\rho_2} + (1-s^{\rho_2})d^{\rho_2})ds = \frac{\rho_2^{\beta_2-1}\Gamma(\beta_2)}{(d^{\rho_2}-c^{\rho_2})^{\beta_2}} \, {}^{\rho_2}I_{c+}^{\beta_2} f(x^{\rho_1}, d^{\rho_2}), \tag{2}$$

$$\int_0^1 t^{\beta_1\rho_1-1} f(t^{\rho_1}b^{\rho_1} + (1-t^{\rho_1})a^{\rho_1}, y^{\rho_2})dt = \frac{\rho_1^{\beta_1-1}\Gamma(\beta_1)}{(b^{\rho_1}-a^{\rho_1})^{\beta_1}} \, {}^{\rho_1}I_{b-}^{\beta_1} f(a^{\rho_1}, y^{\rho_2}), \tag{3}$$

$$\int_0^1 t^{\beta_1\rho_1-1} f(t^{\rho_1}a^{\rho_1} + (1-t^{\rho_1})b^{\rho_1}, y^{\rho_2})dt = \frac{\rho_1^{\beta_1-1}\Gamma(\beta_1)}{(b^{\rho_1}-a^{\rho_1})^{\beta_1}} \, {}^{\rho_1}I_{a+}^{\beta_1} f(b^{\rho_1}, y^{\rho_2}), \tag{4}$$

$$\begin{aligned}\int_0^1\int_0^1 & t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1} f(t^{\rho_1}x^{\rho_1} + (1-t^{\rho_1})b^{\rho_1}, s^{\rho_2}y^{\rho_2} + (1-s^{\rho_2})d^{\rho_2})dsdt \\ &= \frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{(b^{\rho_1}-x^{\rho_1})^{\beta_1}(d^{\rho_2}-y^{\rho_2})^{\beta_2}} \, {}^{\rho_1,\rho_2}I_{x+,y+}^{\beta_1,\beta_2} f(b^{\rho_1}, d^{\rho_2}),\end{aligned} \tag{5}$$

$$\begin{aligned}\int_0^1\int_0^1 & t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1} f(t^{\rho_1}x^{\rho_1} + (1-t^{\rho_1})b_1^{\rho}, s^{\rho_2}y^{\rho_2} + (1-s^{\rho_2})c^{\rho_2})dsdt \\ &= \frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{(b^{\rho_1}-x^{\rho_1})^{\beta_1}(y^{\rho_2}-c^{\rho_2})^{\beta_2}} \, {}^{\rho_1,\rho_2}I_{x+,y-}^{\beta_1,\beta_2} f(b^{\rho_1}, c^{\rho_2}),\end{aligned} \tag{6}$$

$$\begin{aligned}\int_0^1\int_0^1 & t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1} f(t^{\rho_1}x^{\rho_1} + (1-t^{\rho_1})a^{\rho_1}, s^{\rho_2}y^{\rho_2} + (1-s^{\rho_2})d^{\rho_2})dsdt \\ &= \frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{(x^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-y^{\rho_2})^{\beta_2}} \, {}^{\rho_1,\rho_2}I_{x-,y+}^{\beta_1,\beta_2} f(a^{\rho_1}, d^{\rho_2})\end{aligned} \tag{7}$$

and

$$\int_0^1\int_0^1 t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1}\times$$
$$f(t^{\rho_1}x^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}, s^{\rho_2}y^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})dsdt$$
$$=\frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{(x^{\rho_1}-a^{\rho_1})^{\beta_1}(y^{\rho_2}-c^{\rho_2})^{\beta_2}}\ {}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{x-,y-}f(a^{\rho_1},c^{\rho_2}). \tag{8}$$

Proof Using the change of variables and Definition 3, the results follow directly. □

Theorem 6 *Let $\beta_1, \beta_2, \rho_1, \rho_2 > 0$, and $f : [a^{\rho_1}, b^{\rho_1}] \times [c^{\rho_2}, d^{\rho_2}] \to \mathbb{R}$ be a quasi-convex function on the coordinates with $0 \le a < b$ and $0 \le c < d$. Then the following inequality holds:*

$$\frac{\rho_1^{\beta_1}\rho_2^{\beta_2}\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{4(b^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\Big[{}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{a+,c+}f(b^{\rho_1},d^{\rho_2})+{}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{a+,d-}f(b^{\rho_1},c^{\rho_2}) \tag{9}$$
$$+{}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{b-,c+}f(a^{\rho_1},d^{\rho_2})+{}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{b-,d-}f(a^{\rho_1},c^{\rho_2})\Big]$$
$$\le \max\Big\{f(a^{\rho_1},c^{\rho_2}), f(a^{\rho_1},d^{\rho_2}), f(b^{\rho_1},c^{\rho_2}), f(b^{\rho_1},d^{\rho_2})\Big\}.$$

Proof By using the quasi-convexity on the coordinates of f, we have

$$\int_0^1\int_0^1 t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1}f(t^{\rho_1}a^{\rho_1}+(1-t^{\rho_1})b^{\rho_1}, s^{\rho_2}c^{\rho_2}+(1-s^{\rho_2})d^{\rho_2})dsdt$$
$$\le \max\Big\{f(a^{\rho_1},c^{\rho_2}), f(a^{\rho_1},d^{\rho_2}), f(b^{\rho_1},c^{\rho_2}), f(b^{\rho_1},d^{\rho_2})\Big\}\times$$
$$\int_0^1\int_0^1 t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1}dsdt. \tag{10}$$

That is,

$$\int_0^1\int_0^1 t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1}f(t^{\rho_1}a^{\rho_1}+(1-t^{\rho_1})b_1^{\rho}, s^{\rho_2}c^{\rho_2}+(1-s^{\rho_2})d^{\rho_2})dsdt$$
$$\le \frac{1}{\beta_1\beta_2\rho_1\rho_2}\max\Big\{f(a^{\rho_1},c^{\rho_2}), f(a^{\rho_1},d^{\rho_2}), f(b^{\rho_1},c^{\rho_2}), f(b^{\rho_1},d^{\rho_2})\Big\}. \tag{11}$$

Similarly, we have

$$\int_0^1\int_0^1 t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1}f(t^{\rho_1}a^{\rho_1}+(1-t^{\rho_1})b^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})dsdt$$
$$\le \frac{1}{\beta_1\beta_2\rho_1\rho_2}\max\Big\{f(a^{\rho_1},c^{\rho_2}), f(a^{\rho_1},d^{\rho_2}), f(b^{\rho_1},c^{\rho_2}), f(b^{\rho_1},d^{\rho_2})\Big\}, \tag{12}$$

$$\int_0^1 \int_0^1 t^{\beta_1\rho_1-1} s^{\beta_2\rho_2-1} f\left(t^{\rho_1} b^{\rho_1} + (1-t^{\rho_1})a^{\rho_1}, s^{\rho_2} c^{\rho_2} + (1-s^{\rho_2})d^{\rho_2}\right) dsdt$$
$$\leq \frac{1}{\beta_1\beta_2\rho_1\rho_2} \max\left\{ f(a^{\rho_1}, c^{\rho_2}), f(a^{\rho_1}, d^{\rho_2}), f(b^{\rho_1}, c^{\rho_2}), f(b^{\rho_1}, d^{\rho_2}) \right\} \quad (13)$$

and

$$\int_0^1 \int_0^1 t^{\beta_1\rho_1-1} s^{\beta_2\rho_2-1} f\left(t^{\rho_1} b^{\rho_1} + (1-t^{\rho_1})a^{\rho_1}, s^{\rho_2} d^{\rho_2} + (1-s^{\rho_2})c^{\rho_2}\right) dsdt$$
$$\leq \frac{1}{\beta_1\beta_2\rho_1\rho_2} \max\left\{ f(a^{\rho_1}, c^{\rho_2}), f(a^{\rho_1}, d^{\rho_2}), f(b^{\rho_1}, c^{\rho_2}), f(b^{\rho_1}, d^{\rho_2}) \right\}. \quad (14)$$

By adding (11), (12), (13) and (14), and then applying Lemma 1, we get

$$\frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{(b^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}} \, {}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{a+,c+} f(b^{\rho_1}, d^{\rho_2})$$
$$+ \frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{(b^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}} \, {}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{a+,d-} f(b^{\rho_1}, c^{\rho_2})$$
$$+ \frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{(b^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}} \, {}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{b-,c+} f(a^{\rho_1}, d^{\rho_2})$$
$$+ \frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{(b^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}} \, {}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{b-,d-} f(a^{\rho_1}, c^{\rho_2})$$
$$\leq \frac{4}{\beta_1\beta_2\rho_1\rho_2} \max\left\{ f(a^{\rho_1}, c^{\rho_2}), f(a^{\rho_1}, d^{\rho_2}), f(b^{\rho_1}, c^{\rho_2}), f(b^{\rho_1}, d^{\rho_2}) \right\}. \quad (15)$$

The desired inequality follows from (15). □

Lemma 2 *Let* $\beta_1, \beta_2, \rho_1, \rho_2 > 0$ *and* $f : [a^{\rho_1}, b^{\rho_1}] \times [c^{\rho_2}, d^{\rho_2}] \to \mathbb{R}$ *be a twice partially differentiable mapping on* $(a^{\rho_1}, b^{\rho_1}) \times (c^{\rho_2}, d^{\rho_2})$ *with* $0 \leq a < b$, $0 \leq c < d$ *and* $\frac{\partial^2 f}{\partial t \partial s} \in L_1\left([a^{\rho_1}, b^{\rho_1}] \times [c^{\rho_2}, d^{\rho_2}]\right)$. *Then the following equality holds:*

$$\frac{f(a^{\rho_1}, c^{\rho_2}) + f(a^{\rho_1}, d^{\rho_2}) + f(b^{\rho_1}, c^{\rho_2}) + f(b^{\rho_1}, d^{\rho_2})}{4}$$
$$- \frac{\rho_2^{\beta_2}\Gamma(\beta_2+1)}{4(d^{\rho_2}-c^{\rho_2})^{\beta_2}} \Big[{}^{\rho_2}I^{\beta_2}_{c+} f(a^{\rho_1}, d^{\rho_2}) + {}^{\rho_2}I^{\beta_2}_{c+} f(b^{\rho_1}, d^{\rho_2})$$
$$+ {}^{\rho_2}I^{\beta_2}_{d-} f(a^{\rho_1}, c^{\rho_2}) + {}^{\rho_2}I^{\beta_2}_{d-} f(b^{\rho_1}, c^{\rho_2}) \Big]$$
$$- \frac{\rho_1^{\beta_1}\Gamma(\beta_1+1)}{4(b^{\rho_1}-a^{\rho_1})^{\beta_1}} \Big[{}^{\rho_1}I^{\beta_1}_{a+} f(b^{\rho_1}, c^{\rho_2}) + {}^{\rho_1}I^{\beta_1}_{a+} f(b^{\rho_1}, d^{\rho_2})$$
$$+ {}^{\rho_1}I^{\beta_1}_{b-} f(a^{\rho_1}, c^{\rho_2}) + {}^{\rho_1}I^{\beta_1}_{b-} f(a^{\rho_1}, d^{\rho_2}) \Big]$$

$$+\frac{\rho_1^{\beta_1}\rho_2^{\beta_2}\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{4(b^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\times$$
$$\Big[{}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{a+,c+}f(b^{\rho_1},d^{\rho_2})+{}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{a+,d-}f(b^{\rho_1},c^{\rho_2})$$
$$+{}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{b-,c+}f(c^{\rho_1},d^{\rho_2})+{}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{b-,d-}f(a^{\rho_1},c^{\rho_2})\Big]$$
$$=\frac{\rho_1\rho_2(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}{4}\Big(I_1-I_2-I_3+I_4\Big),$$

where

$$I_1=\int_0^1\int_0^1 s^{(\beta_2+1)\rho_2-1}t^{(\beta_1+1)\rho_1-1}\times$$
$$\frac{\partial^2}{\partial t\partial s}f(t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1},s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})dtds,$$

$$I_2=\int_0^1\int_0^1 s^{(\beta_2+1)\rho_2-1}t^{(\beta_1+1)\rho_1-1}\times$$
$$\frac{\partial^2}{\partial t\partial s}f(t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1},s^{\rho_2}c^{\rho_2}+(1-s^{\rho_2})d^{\rho_2})dtds,$$

$$I_3=\int_0^1\int_0^1 s^{(\beta_2+1)\rho_2-1}t^{(\beta_1+1)\rho_1-1}\times$$
$$\frac{\partial^2}{\partial t\partial s}f(t^{\rho_1}a^{\rho_1}+(1-t^{\rho_1})b^{\rho_1},s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})dtds$$

and

$$I_4=\int_0^1\int_0^1 s^{(\beta_2+1)\rho_2-1}t^{(\beta_1+1)\rho_1-1}\times$$
$$\frac{\partial^2}{\partial t\partial s}f(t^{\rho_1}a^{\rho_1}+(1-t^{\rho_1})b^{\rho_1},s^{\rho_2}c^{\rho_2}+(1-s^{\rho_2})d^{\rho_2})dtds.$$

Proof By using integration by parts, we have

$$I_1=\int_0^1 s^{(\beta_2+1)\rho_2-1}\left[\int_0^1 t^{\beta_1\rho_1}t^{\rho_1-1}\frac{\partial^2}{\partial t\partial s}f(t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1},s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})dt\right]ds$$

$$= \int_0^1 s^{(\beta_2+1)\rho_2-1}\Bigg[\frac{1}{(b^{\rho_1}-a^{\rho_1})\rho_1} t^{\beta_1\rho_1}\frac{\partial}{\partial s} f(t^{\rho}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})\Big|_0^1$$
$$- \frac{\beta_1}{(b^{\rho_1}-a^{\rho_1})}\int_0^1 t^{\beta_1\rho_1-1}\frac{\partial}{\partial s} f(t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})dt\Bigg]ds$$

$$= \int_0^1 s^{(\beta_2+1)\rho_2-1}\Bigg[\frac{1}{(b^{\rho_1}-a^{\rho_1})\rho_1}\frac{\partial}{\partial s} f(b^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})$$
$$- \frac{\beta_1}{(b^{\rho_1}-a^{\rho_1})}\int_0^1 t^{\beta_1\rho_1-1}\frac{\partial}{\partial s} f(t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})dt\Bigg]ds$$

$$= \frac{1}{(b^{\rho_1}-a^{\rho_1})\rho_1}\int_0^1 s^{\beta_2\rho_2}s^{\rho_2-1}\frac{\partial}{\partial s} f(b^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})ds$$
$$- \frac{\beta_1}{(b^{\rho_1}-a^{\rho_1})}\int_0^1 t^{\beta_1\rho_1-1}\Bigg[\int_0^1 s^{\beta_2\rho_2}s^{\rho_2-1}\frac{\partial}{\partial s} f(t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})ds\Bigg]dt$$

$$= \frac{1}{(b^{\rho_1}-a^{\rho_1})(y^{\rho_2}-c^{\rho_2})\rho_1\rho_2} s^{\beta_2\rho_2} f(b^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})\Big|_{s=0}^{s=1}$$
$$- \frac{\beta_2}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})\rho_1}\int_0^1 s^{\beta_2\rho_2-1} f(b^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})ds$$
$$- \frac{\beta_1}{(b^{\rho_1}-a^{\rho_1})}\int_0^1 t^{\beta_1\rho_1-1}\times$$
$$\Bigg[\frac{1}{(d^{\rho_2}-c^{\rho_2})\rho_2} s^{\beta_2\rho_2} f(t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})\Big|_{s=0}^{s=1}$$
$$- \frac{\beta_2}{(d^{\rho_2}-c^{\rho_2})}\times$$
$$\int_0^1 s^{\beta_2\rho_2-1} f(t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})ds\Bigg]dt$$
$$= \frac{1}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})\rho_1\rho_2} f(b^{\rho_1}, d^{\rho_2})$$
$$- \frac{\beta_2}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})\rho_1}\int_0^1 s^{\beta_2\rho_2-1} f(b^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})ds$$
$$- \frac{\beta_1}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})\rho_2}\int_0^1 t^{\beta_1\rho_1-1} f(t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}, d^{\rho_2})dt$$
$$+ \frac{\beta_1\beta_2}{(b^{\rho_1}-a^{\rho_1})(b^{\rho_2}-c^{\rho_2})}\times$$
$$\int_0^1\int_0^1 t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1} f(t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})dsdt.$$

That is,

$$I_1 = \frac{1}{(b^{\rho_1} - a^{\rho_1})(d^{\rho_2} - c^{\rho_2})\rho_1\rho_2} f(b^{\rho_1}, d^{\rho_2})$$
$$- \frac{\beta_2}{(b^{\rho_1} - a^{\rho_1})(d^{\rho_2} - c^{\rho_2})\rho_1} \int_0^1 s^{\beta_2\rho_2 - 1} f(b^{\rho_1}, s^{\rho_2} d^{\rho_2} + (1 - s^{\rho_2})c^{\rho_2}) ds$$
$$- \frac{\beta_1}{(b^{\rho_1} - a^{\rho_1})(d^{\rho_2} - c^{\rho_2})\rho_2} \int_0^1 t^{\beta_1\rho_1 - 1} f(t^{\rho_1} b^{\rho_1} + (1 - t^{\rho_1})a^{\rho_1}, d^{\rho_2}) dt$$
$$+ \frac{\beta_1\beta_2}{(b^{\rho_1} - a^{\rho_1})(b^{\rho_2} - c^{\rho_2})} \times$$
$$\int_0^1 \int_0^1 t^{\beta_1\rho_1 - 1} s^{\beta_2\rho_2 - 1} f(t^{\rho_1} b^{\rho_1} + (1 - t^{\rho_1})a^{\rho_1}, s^{\rho_2} d^{\rho_2} + (1 - s^{\rho_2})c^{\rho_2}) ds dt$$

By using Lemma 1, we have

$$I_1 = \frac{1}{(b^{\rho_1} - a^{\rho_1})(d^{\rho_2} - c^{\rho_2})\rho_1\rho_2} f(b^{\rho_1}, d^{\rho_2})$$
$$- \frac{\rho_2^{\beta_2 - 1}\Gamma(\beta_2 + 1)}{(b^{\rho_1} - a^{\rho_1})(d^{\rho_2} - c^{\rho_2})^{\beta_2+1}\rho_1} \, {}^{\rho_2}I_{d-}^{\beta_2} f(b^{\rho_1}, c^{\rho_2})$$
$$- \frac{\rho_1^{\beta_1 - 1}\Gamma(\beta_1 + 1)}{(b^{\rho_1} - a^{\rho_1})^{\beta_1+1}(d^{\rho_2} - c^{\rho_2})\rho_2} \, {}^{\rho_1}I_{b-}^{\beta_1} f(a^{\rho_1}, d^{\rho_2})$$
$$+ \frac{\rho_1^{\beta_1 - 1}\rho_2^{\beta_2 - 1}\Gamma(\beta_1 + 1)\Gamma(\beta_2 + 1)}{(b^{\rho_1} - a^{\rho_1})^{\beta_1+1}(d^{\rho_2} - c^{\rho_2})^{\beta_2+1}} \, {}^{\rho_1,\rho_2}I_{b-,d-}^{\beta_1,\beta_2} f(a^{\rho_1}, c^{\rho_2}).$$

So, it follows that

$$(b^{\rho_1} - a^{\rho_1})(d^{\rho_2} - c^{\rho_2})\rho_1\rho_2 I_1 = f(b^{\rho_1}, d^{\rho_2}) - \frac{\rho_2^{\beta_2}\Gamma(\beta_2 + 1)}{(d^{\rho_2} - c^{\rho_2})^{\beta_2}} \, {}^{\rho_2}I_{d-}^{\beta_2} f(b^{\rho_1}, c^{\rho_2})$$
$$- \frac{\rho_1^{\beta_1}\Gamma(\beta_1 + 1)}{(b^{\rho_1} - a^{\rho_1})^{\beta_1}} \, {}^{\rho_1}I_{b-}^{\beta_1} f(a^{\rho_1}, d^{\rho_2})$$
$$+ \frac{\rho_1^{\beta_1}\rho_2^{\beta_2}\Gamma(\beta_1 + 1)\Gamma(\beta_2 + 1)}{(b^{\rho_1} - a^{\rho_1})^{\beta_1}(d^{\rho_2} - c^{\rho_2})^{\beta_2}} \, {}^{\rho_1,\rho_2}I_{b-,d-}^{\beta_1,\beta_2} f(a^{\rho_1}, c^{\rho_2}). \tag{16}$$

By using similar arguments as in the above, we obtained the following:

$$(b^{\rho_1} - a^{\rho_1})(d^{\rho_2} - c^{\rho_2})\rho_1\rho_2 I_2 = -f(b^{\rho_1}, c^{\rho_2}) + \frac{\rho_2^{\beta_2}\Gamma(\beta_2 + 1)}{(d^{\rho_2} - c^{\rho_2})^{\beta_2}} \, {}^{\rho_2}I_{c+}^{\beta_2} f(b^{\rho_1}, d^{\rho_2})$$
$$+ \frac{\rho_1^{\beta_1}\Gamma(\beta_1 + 1)}{(b^{\rho_1} - a^{\rho_1})^{\beta_1}} \, {}^{\rho_1}I_{b-}^{\beta_1} f(a^{\rho_1}, c^{\rho_2}) - \frac{\rho_1^{\beta_1}\rho_2^{\beta_2}\Gamma(\beta_1 + 1)\Gamma(\beta_2 + 1)}{(b^{\rho_1} - a^{\rho_1})^{\beta_1}(d^{\rho_2} - c^{\rho_2})^{\beta_2}} \, {}^{\rho_1,\rho_2}I_{b-,c+}^{\beta_1,\beta_2} f(a^{\rho_1}, d^{\rho_2}), \tag{17}$$

$$(b^{\rho_1} - a^{\rho_1})(d^{\rho_2} - c^{\rho_2})\rho_1\rho_2 I_3 = -f(a^{\rho_1}, d^{\rho_2}) + \frac{\rho_2^{\beta_2}\Gamma(\beta_2 + 1)}{(d^{\rho_2} - c^{\rho_2})^{\beta_2}} \, {}^{\rho_2}I_{d-}^{\beta_2} f(a^{\rho_1}, c^{\rho_2})$$
$$+ \frac{\rho_1^{\beta_1}\Gamma(\beta_1 + 1)}{(b^{\rho_1} - a^{\rho_1})^{\beta_1}} \, {}^{\rho_1}I_{a+}^{\beta_1} f(b^{\rho_1}, d^{\rho_2}) - \frac{\rho_1^{\beta_1}\rho_2^{\beta_2}\Gamma(\beta_1 + 1)\Gamma(\beta_2 + 1)}{(b^{\rho_1} - a^{\rho_1})^{\beta_1}(d^{\rho_2} - c^{\rho_2})^{\beta_2}} \, {}^{\rho_1,\rho_2}I_{a+,d-}^{\beta_1,\beta_2} f(b^{\rho_1}, c^{\rho_2}), \tag{18}$$

and

$$(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})\rho_1\rho_2 I_4 = f(a^{\rho_1},c^{\rho_2}) - \frac{\rho_2^{\beta_2}\Gamma(\beta_2+1)}{(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\, {}^{\rho_2}I_{c+}^{\beta_2} f(a^{\rho_1},d^{\rho_2})$$
$$- \frac{\rho_1^{\beta_1}\Gamma(\beta_1+1)}{(b^{\rho_1}-a^{\rho_1})^{\beta_1}}\, {}^{\rho_1}I_{a+}^{\beta_1} f(b^{\rho_1},c^{\rho_2}) + \frac{\rho_1^{\beta_1}\rho_2^{\beta_2}\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{(b^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\, {}^{\rho_1,\rho_2}I_{a+,c+}^{\beta_1,\beta_2} f(b^{\rho_1},d^{\rho_2}). \tag{19}$$

The desired identity follows from adding (16), (17), (18) and (19).

Theorem 7 *Let* $\beta_1,\beta_2,\rho_1,\rho_2>0$, $0\le a<b, 0\le c<d$ *and* $f:[a^{\rho_1},b^{\rho_1}]\times[c^{\rho_2},d^{\rho_2}]\to\mathbb{R}$ *be a twice partially differentiable mapping on* $(a^{\rho_1},b^{\rho_1})\times(c^{\rho_2},d^{\rho_2})$ *and* $\frac{\partial^2 f}{\partial t\partial s}\in L_1([a^{\rho_1},b^{\rho_1}]\times[c^{\rho_2},d^{\rho_2}])$. *If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ *is quasi-convex on the coordinates on* $(a^{\rho_1},b^{\rho_1})\times(c^{\rho_2},d^{\rho_2})$, *then the inequality*

$$\Bigg|\frac{f(a^{\rho_1},c^{\rho_2})+f(a^{\rho_1},d^{\rho_2})+f(b^{\rho_1},c^{\rho_2})+f(b^{\rho_1},d^{\rho_2})}{4}$$
$$-\frac{\rho_2^{\beta_2}\Gamma(\beta_2+1)}{4(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\Big[{}^{\rho_2}I_{c+}^{\beta_2} f(a^{\rho_1},d^{\rho_2}) + {}^{\rho_2}I_{c+}^{\beta_2} f(b^{\rho_1},d^{\rho_2})$$
$$+ {}^{\rho_2}I_{d-}^{\beta_2} f(a^{\rho_1},c^{\rho_2}) + {}^{\rho_2}I_{d-}^{\beta_2} f(b^{\rho_1},c^{\rho_2})\Big]$$
$$-\frac{\rho_1^{\beta_1}\Gamma(\beta_1+1)}{4(b^{\rho_1}-a^{\rho_1})^{\beta_1}}\Big[{}^{\rho_1}I_{a+}^{\beta_1} f(b^{\rho_1},c^{\rho_2}) + {}^{\rho_1}I_{a+}^{\beta_1} f(b^{\rho_1},d^{\rho_2})$$
$$+ {}^{\rho_1}I_{b-}^{\beta_1} f(a^{\rho_1},c^{\rho_2}) + {}^{\rho_1}I_{b-}^{\beta_1} f(a^{\rho_1},d^{\rho_2})\Big]$$
$$+\frac{\rho_1^{\beta_1}\rho_2^{\beta_2}\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{4(b^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\times$$
$$\Big[{}^{\rho_1,\rho_2}I_{a+,c+}^{\beta_1,\beta_2} f(b^{\rho_1},d^{\rho_2}) + {}^{\rho_1,\rho_2}I_{a+,d-}^{\beta_1,\beta_2} f(b^{\rho_1},c^{\rho_2})$$
$$+ {}^{\rho_1,\rho_2}I_{b-,c+}^{\beta_1,\beta_2} f(c^{\rho_1},d^{\rho_2}) + {}^{\rho_1,\rho_2}I_{b-,d-}^{\beta_1,\beta_2} f(a^{\rho_1},c^{\rho_2})\Big]\Bigg|$$
$$\le \frac{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}{(\beta_1+1)(\beta_2+1)}\times$$
$$\max\left\{\left|\frac{\partial^2}{\partial t\partial s}f(b^{\rho_1},d^{\rho_2})\right|, \left|\frac{\partial^2}{\partial t\partial s}f\left(b^{\rho_1},c^{\rho_2}\right)\right|,\right.$$
$$\left.\left|\frac{\partial^2}{\partial t\partial s}f\left(a^{\rho_1},d^{\rho_2}\right)\right|, \left|\frac{\partial^2}{\partial t\partial s}f\left(a^{\rho_1},c^{\rho_2}\right)\right|\right\}.$$

holds.

Proof By using Lemma 2 and the properties of the absolute value, we obtain

$$\Bigg|\frac{f(a^{\rho_1},c^{\rho_2})+f(a^{\rho_1},d^{\rho_2})+f(b^{\rho_1},c^{\rho_2})+f(b^{\rho_1},d^{\rho_2})}{4}$$
$$-\frac{\rho_2^{\beta_2}\Gamma(\beta_2+1)}{4(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\Big[{}^{\rho_2}I_{c+}^{\beta_2}f(a^{\rho_1},d^{\rho_2})+{}^{\rho_2}I_{c+}^{\beta_2}f(b^{\rho_1},d^{\rho_2})$$
$$+{}^{\rho_2}I_{d-}^{\beta_2}f(a^{\rho_1},c^{\rho_2})+{}^{\rho_2}I_{d-}^{\beta_2}f(b^{\rho_1},c^{\rho_2})\Big]$$
$$-\frac{\rho_1^{\beta_1}\Gamma(\beta_1+1)}{4(b^{\rho_1}-a^{\rho_1})^{\beta_1}}\Big[{}^{\rho_1}I_{a+}^{\beta_1}f(b^{\rho_1},c^{\rho_2})+{}^{\rho_1}I_{a+}^{\beta_1}f(b^{\rho_1},d^{\rho_2})$$
$$+{}^{\rho_1}I_{b-}^{\beta_1}f(a^{\rho_1},c^{\rho_2})+{}^{\rho_1}I_{b-}^{\beta_1}f(a^{\rho_1},d^{\rho_2})\Big]$$
$$+\frac{\rho_1^{\beta_1}\rho_2^{\beta_2}\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{4(b^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\Big[{}^{\rho_1,\rho_2}I_{a+,c+}^{\beta_1,\beta_2}f(b^{\rho_1},d^{\rho_2})+{}^{\rho_1,\rho_2}I_{a+,d-}^{\beta_1,\beta_2}f(b^{\rho_1},c^{\rho_2})$$
$$+{}^{\rho_1,\rho_2}I_{b-,c+}^{\beta_1,\beta_2}f(c^{\rho_1},d^{\rho_2})+{}^{\rho_1,\rho_2}I_{b-,d-}^{\beta_1,\beta_2}f(a^{\rho_1},c^{\rho_2})\Big]\Bigg|$$
$$\leq\frac{\rho_1\rho_2(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}{4}\Big(|I_1|+|I_2|+|I_3|+I_4|\Big). \tag{20}$$

By using the quasi-convexity of $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ on the coordinates, we have

$$|I_1|\leq\int_0^1\int_0^1 s^{(\beta_2+1)\rho_2-1}t^{(\beta_1+1)\rho_1-1}\left|\frac{\partial^2}{\partial t\partial s}f(t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1},s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})\right|dtds$$
$$\leq\int_0^1\int_0^1 s^{(\beta_2+1)\rho_2-1}t^{(\beta_1+1)\rho_1-1}dtds\max\left\{\left|\frac{\partial^2}{\partial t\partial s}f(b^{\rho_1},d^{\rho_2})\right|,\left|\frac{\partial^2}{\partial t\partial s}f\left(b^{\rho_1},c^{\rho_2}\right)\right|,\right.$$
$$\left.\left|\frac{\partial^2}{\partial t\partial s}f\left(a^{\rho_1},d^{\rho_2}\right)\right|,\left|\frac{\partial^2}{\partial t\partial s}f\left(a^{\rho_1},c^{\rho_2}\right)\right|\right\}.$$

Thus,

$$|I_1|\leq\frac{1}{(\beta_1+1)(\beta_2+1)\rho_1\rho_2}\max\left\{\left|\frac{\partial^2}{\partial t\partial s}f(b^{\rho_1},d^{\rho_2})\right|,\left|\frac{\partial^2}{\partial t\partial s}f\left(b^{\rho_1},c^{\rho_2}\right)\right|,\right.$$
$$\left.\left|\frac{\partial^2}{\partial t\partial s}f\left(a^{\rho_1},d^{\rho_2}\right)\right|,\left|\frac{\partial^2}{\partial t\partial s}f\left(a^{\rho_1},c^{\rho_2}\right)\right|\right\}. \tag{21}$$

By using similarly arguments, we have

$$|I_2|\leq\frac{1}{(\beta_1+1)(\beta_2+1)\rho_1\rho_2}\max\left\{\left|\frac{\partial^2}{\partial t\partial s}f(b^{\rho_1},d^{\rho_2})\right|,\left|\frac{\partial^2}{\partial t\partial s}f\left(b^{\rho_1},c^{\rho_2}\right)\right|,\right.$$
$$\left.\left|\frac{\partial^2}{\partial t\partial s}f\left(a^{\rho_1},d^{\rho_2}\right)\right|,\left|\frac{\partial^2}{\partial t\partial s}f\left(a^{\rho_1},c^{\rho_2}\right)\right|\right\}, \tag{22}$$

$$|I_3| \leq \frac{1}{(\beta_1+1)(\beta_2+1)\rho_1\rho_2} \max\left\{\left|\frac{\partial^2}{\partial t \partial s} f(b^{\rho_1}, d^{\rho_2})\right|, \left|\frac{\partial^2}{\partial t \partial s} f(b^{\rho_1}, c^{\rho_2})\right|, \left|\frac{\partial^2}{\partial t \partial s} f(a^{\rho_1}, d^{\rho_2})\right|, \left|\frac{\partial^2}{\partial t \partial s} f(a^{\rho_1}, c^{\rho_2})\right|\right\} \tag{23}$$

and

$$|I_4| \leq \frac{1}{(\beta_1+1)(\beta_2+1)\rho_1\rho_2} \max\left\{\left|\frac{\partial^2}{\partial t \partial s} f(b^{\rho_1}, d^{\rho_2})\right|, \left|\frac{\partial^2}{\partial t \partial s} f(b^{\rho_1}, c^{\rho_2})\right|, \left|\frac{\partial^2}{\partial t \partial s} f(a^{\rho_1}, d^{\rho_2})\right|, \left|\frac{\partial^2}{\partial t \partial s} f(a^{\rho_1}, c^{\rho_2})\right|\right\}. \tag{24}$$

The desired inequality is deduced from (20), (21), (22), (23) and (24).

Theorem 8 *Let* $\beta_1, \beta_2, \rho_1, \rho_2 > 0$, $0 \leq a < b$, $0 \leq c < d$ *and* $f : [a^{\rho_1}, b^{\rho_1}] \times [c^{\rho_2}, d^{\rho_2}] \to \mathbb{R}$ *be a twice partially differentiable mapping on* $(a^{\rho_1}, b^{\rho_1}) \times (c^{\rho_2}, d^{\rho_2})$ *and* $\frac{\partial^2 f}{\partial t \partial s} \in L_1([a^{\rho_1}, b^{\rho_1}] \times [c^{\rho_2}, d^{\rho_2}])$. *If* $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$ *is quasi-convex on the coordinates on* $(a^{\rho_1}, b^{\rho_1}) \times (c^{\rho_2}, d^{\rho_2})$, *then the inequality*

$$\begin{aligned}
&\Bigg|\frac{f(a^{\rho_1}, c^{\rho_2}) + f(a^{\rho_1}, d^{\rho_2}) + f(b^{\rho_1}, c^{\rho_2}) + f(b^{\rho_1}, d^{\rho_2})}{4} \\
&- \frac{\rho_2^{\beta_2}\Gamma(\beta_2+1)}{4(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\Big[{}^{\rho_2}I_{c+}^{\beta_2} f(a^{\rho_1}, d^{\rho_2}) + {}^{\rho_2}I_{c+}^{\beta_2} f(b^{\rho_1}, d^{\rho_2}) \\
&\qquad + {}^{\rho_2}I_{d-}^{\beta_2} f(a^{\rho_1}, c^{\rho_2}) + {}^{\rho_2}I_{d-}^{\beta_2} f(b^{\rho_1}, c^{\rho_2})\Big] \\
&- \frac{\rho_1^{\beta_1}\Gamma(\beta_1+1)}{4(b^{\rho_1}-a^{\rho_1})^{\beta_1}}\Big[{}^{\rho_1}I_{a+}^{\beta_1} f(b^{\rho_1}, c^{\rho_2}) + {}^{\rho_1}I_{a+}^{\beta_1} f(b^{\rho_1}, d^{\rho_2}) \\
&\qquad + {}^{\rho_1}I_{b-}^{\beta_1} f(a^{\rho_1}, c^{\rho_2}) + {}^{\rho_1}I_{b-}^{\beta_1} f(a^{\rho_1}, d^{\rho_2})\Big] \\
&+ \frac{\rho_1^{\beta_1}\rho_2^{\beta_2}\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{4(b^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\Big[{}^{\rho_1,\rho_2}I_{a+,c+}^{\beta_1,\beta_2} f(b^{\rho_1}, d^{\rho_2}) + {}^{\rho_1,\rho_2}I_{a+,d-}^{\beta_1,\beta_2} f(b^{\rho_1}, c^{\rho_2}) \\
&\qquad + {}^{\rho_1,\rho_2}I_{b-,c+}^{\beta_1,\beta_2} f(c^{\rho_1}, d^{\rho_2}) + {}^{\rho_1,\rho_2}I_{b-,d-}^{\beta_1,\beta_2} f(a^{\rho_1}, c^{\rho_2})\Big]\Bigg| \\
&\leq \frac{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}{(\beta_1+1)(\beta_2+1)}\left(\max\left\{\left|\frac{\partial^2}{\partial t \partial s} f(b^{\rho_1}, d^{\rho_2})\right|^q, \left|\frac{\partial^2}{\partial t \partial s} f(b^{\rho_1}, c^{\rho_2})\right|^q, \right.\right. \\
&\qquad \left.\left. \left|\frac{\partial^2}{\partial t \partial s} f(a^{\rho_1}, d^{\rho_2})\right|^q, \left|\frac{\partial^2}{\partial t \partial s} f(a^{\rho_1}, c^{\rho_2})\right|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned} \tag{25}$$

holds for $q > 1$.

Proof By an application of Lemma 2 and the absolute value properties, we obtain

$$\Bigg| \frac{f(a^{\rho_1}, c^{\rho_2}) + f(a^{\rho_1}, d^{\rho_2}) + f(b^{\rho_1}, c^{\rho_2}) + f(b^{\rho_1}, d^{\rho_2})}{4}$$

$$- \frac{\rho_2^{\beta_2} \Gamma(\beta_2 + 1)}{4(d^{\rho_2} - c^{\rho_2})^{\beta_2}} \Big[{}^{\rho_2}I_{c+}^{\beta_2} f(a^{\rho_1}, d^{\rho_2}) + {}^{\rho_2}I_{c+}^{\beta_2} f(b^{\rho_1}, d^{\rho_2})$$

$$+ {}^{\rho_2}I_{d-}^{\beta_2} f(a^{\rho_1}, c^{\rho_2}) + {}^{\rho_2}I_{d-}^{\beta_2} f(b^{\rho_1}, c^{\rho_2}) \Big]$$

$$- \frac{\rho_1^{\beta_1} \Gamma(\beta_1 + 1)}{4(b^{\rho_1} - a^{\rho_1})^{\beta_1}} \Big[{}^{\rho_1}I_{a+}^{\beta_1} f(b^{\rho_1}, c^{\rho_2}) + {}^{\rho_1}I_{a+}^{\beta_1} f(b^{\rho_1}, d^{\rho_2})$$

$$+ {}^{\rho_1}I_{b-}^{\beta_1} f(a^{\rho_1}, c^{\rho_2}) + {}^{\rho_1}I_{b-}^{\beta_1} f(a^{\rho_1}, d^{\rho_2}) \Big]$$

$$+ \frac{\rho_1^{\beta_1} \rho_2^{\beta_2} \Gamma(\beta_1 + 1) \Gamma(\beta_2 + 1)}{4(b^{\rho_1} - a^{\rho_1})^{\beta_1} (d^{\rho_2} - c^{\rho_2})^{\beta_2}} \Big[{}^{\rho_1,\rho_2}I_{a+,c+}^{\beta_1,\beta_2} f(b^{\rho_1}, d^{\rho_2}) + {}^{\rho_1,\rho_2}I_{a+,d-}^{\beta_1,\beta_2} f(b^{\rho_1}, c^{\rho_2})$$

$$+ {}^{\rho_1,\rho_2}I_{b-,c+}^{\beta_1,\beta_2} f(c^{\rho_1}, d^{\rho_2}) + {}^{\rho_1,\rho_2}I_{b-,d-}^{\beta_1,\beta_2} f(a^{\rho_1}, c^{\rho_2}) \Big] \Bigg|$$

$$\leq \frac{\rho_1 \rho_2 (b^{\rho_1} - a^{\rho_1})(d^{\rho_2} - c^{\rho_2})}{4} \Big(|I_1| + |I_2| + |I_3| + I_4| \Big). \tag{26}$$

By using the Hölder's inequality with $\frac{1}{r} + \frac{1}{q} = 1$ and the quasi-convexity of $\left| \frac{\partial^2 f}{\partial t \partial s} \right|^q$ on the coordinates, we have

$$|I_1| \leq \left(\int_0^1 \int_0^1 s^{(\beta_2+1)\rho_2 - 1} t^{(\beta_1+1)\rho_1 - 1} ds dt \right)^{\frac{1}{r}} \times$$

$$\left(\int_0^1 \int_0^1 s^{(\beta_2+1)\rho_2 - 1} t^{(\beta_1+1)\rho_1 - 1} \left| \frac{\partial^2}{\partial t \partial s} f(t^{\rho_1} b^{\rho_1} + (1 - t^{\rho_1}) a^{\rho_1}, s^{\rho_2} d^{\rho_2} + (1 - s^{\rho_2}) c^{\rho_2}) \right|^q dt ds \right)^{\frac{1}{q}}$$

$$\leq \left(\frac{1}{(\beta_1 + 1)(\beta_2 + 1)\rho_1 \rho_2} \right)^{\frac{1}{r}} \times$$

$$\left(\int_0^1 \int_0^1 s^{(\beta_2+1)\rho_2 - 1} t^{(\beta_1+1)\rho_1 - 1} dt ds \max \left\{ \left| \frac{\partial^2}{\partial t \partial s} f(b^{\rho_1}, d^{\rho_2}) \right|^q, \left| \frac{\partial^2}{\partial t \partial s} f\left(b^{\rho_1}, c^{\rho_2}\right) \right|^q, \right.\right.$$

$$\left.\left. \left| \frac{\partial^2}{\partial t \partial s} f\left(a^{\rho_1}, d^{\rho_2}\right) \right|^q, \left| \frac{\partial^2}{\partial t \partial s} f\left(a^{\rho_1}, c^{\rho_2}\right) \right|^q \right\} \right)^{\frac{1}{q}}$$

$$= \left(\frac{1}{(\beta_1 + 1)(\beta_2 + 1)\rho_1 \rho_2} \right)^{\frac{1}{r}} \times$$

$$\left(\frac{1}{(\beta_1 + 1)(\beta_2 + 1)\rho_1 \rho_2} \max \left\{ \left| \frac{\partial^2}{\partial t \partial s} f(b^{\rho_1}, d^{\rho_2}) \right|^q, \left| \frac{\partial^2}{\partial t \partial s} f\left(b^{\rho_1}, c^{\rho_2}\right) \right|^q, \right.\right.$$

$$\left.\left. \left| \frac{\partial^2}{\partial t \partial s} f\left(a^{\rho_1}, d^{\rho_2}\right) \right|^q, \left| \frac{\partial^2}{\partial t \partial s} f\left(a^{\rho_1}, c^{\rho_2}\right) \right|^q \right\} \right)^{\frac{1}{q}}.$$

That is,

$$|I_1| \leq \frac{1}{(\beta_1+1)(\beta_2+1)\rho_1\rho_2}\left(\max\left\{\left|\frac{\partial^2}{\partial t\partial s}f(b^{\rho_1},d^{\rho_2})\right|^q,\left|\frac{\partial^2}{\partial t\partial s}f(b^{\rho_1},c^{\rho_2})\right|^q,\right.\right.$$
$$\left.\left.\left|\frac{\partial^2}{\partial t\partial s}f(a^{\rho_1},d^{\rho_2})\right|^q,\left|\frac{\partial^2}{\partial t\partial s}f(a^{\rho_1},c^{\rho_2})\right|^q\right\}\right)^{\frac{1}{q}}. \tag{27}$$

Using similar argument, we have

$$|I_2| \leq \frac{1}{(\beta_1+1)(\beta_2+1)\rho_1\rho_2}\left(\max\left\{\left|\frac{\partial^2}{\partial t\partial s}f(b^{\rho_1},d^{\rho_2})\right|^q,\left|\frac{\partial^2}{\partial t\partial s}f(b^{\rho_1},c^{\rho_2})\right|^q,\right.\right.$$
$$\left.\left.\left|\frac{\partial^2}{\partial t\partial s}f(a^{\rho_1},d^{\rho_2})\right|^q,\left|\frac{\partial^2}{\partial t\partial s}f(a^{\rho_1},c^{\rho_2})\right|^q\right\}\right)^{\frac{1}{q}}, \tag{28}$$

$$|I_3| \leq \frac{1}{(\beta_1+1)(\beta_2+1)\rho_1\rho_2}\left(\max\left\{\left|\frac{\partial^2}{\partial t\partial s}f(b^{\rho_1},d^{\rho_2})\right|^q,\left|\frac{\partial^2}{\partial t\partial s}f(b^{\rho_1},c^{\rho_2})\right|^q,\right.\right.$$
$$\left.\left.\left|\frac{\partial^2}{\partial t\partial s}f(a^{\rho_1},d^{\rho_2})\right|^q,\left|\frac{\partial^2}{\partial t\partial s}f(a^{\rho_1},c^{\rho_2})\right|^q\right\}\right)^{\frac{1}{q}} \tag{29}$$

and

$$|I_4| \leq \frac{1}{(\beta_1+1)(\beta_2+1)\rho_1\rho_2}\left(\max\left\{\left|\frac{\partial^2}{\partial t\partial s}f(b^{\rho_1},d^{\rho_2})\right|^q,\left|\frac{\partial^2}{\partial t\partial s}f(b^{\rho_1},c^{\rho_2})\right|^q,\right.\right.$$
$$\left.\left.\left|\frac{\partial^2}{\partial t\partial s}f(a^{\rho_1},d^{\rho_2})\right|^q,\left|\frac{\partial^2}{\partial t\partial s}f(a^{\rho_1},c^{\rho_2})\right|^q\right\}\right)^{\frac{1}{q}}. \tag{30}$$

The desired inequality follows from (31) and using (32)–(35). □

Theorem 9 *Let* $\beta_1, \beta_2, \rho_1, \rho_2 > 0,\ 0 \leq a < b,\ 0 \leq c < d$ *and* $f : [a^{\rho_1}, b^{\rho_1}] \times [c^{\rho_2}, d^{\rho_2}] \to \mathbb{R}$ *be a twice partially differentiable mapping on* $(a^{\rho_1}, b^{\rho_1}) \times (c^{\rho_2}, d^{\rho_2})$ *and* $\frac{\partial^2 f}{\partial t\partial s} \in L_1\left([a^{\rho_1}, b^{\rho_1}] \times [c^{\rho_2}, d^{\rho_2}]\right)$. *If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ *is quasi-convex on the coordinates on* $(a^{\rho_1}, b^{\rho_1}) \times (c^{\rho_2}, d^{\rho_2})$*, then the inequality*

$$\Bigg|\frac{f(a^{\rho_1},c^{\rho_2})+f(a^{\rho_1},d^{\rho_2})+f(b^{\rho_1},c^{\rho_2})+f(b^{\rho_1},d^{\rho_2})}{4}$$
$$-\frac{\rho_2^{\beta_2}\Gamma(\beta_2+1)}{4(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\Big[{}^{\rho_2}I_{c+}^{\beta_2}f(a^{\rho_1},d^{\rho_2})+{}^{\rho_2}I_{c+}^{\beta_2}f(b^{\rho_1},d^{\rho_2})$$
$$+{}^{\rho_2}I_{d-}^{\beta_2}f(a^{\rho_1},c^{\rho_2})+{}^{\rho_2}I_{d-}^{\beta_2}f(b^{\rho_1},c^{\rho_2})\Big]$$
$$-\frac{\rho_1^{\beta_1}\Gamma(\beta_1+1)}{4(b^{\rho_1}-a^{\rho_1})^{\beta_1}}\Big[{}^{\rho_1}I_{a+}^{\beta_1}f(b^{\rho_1},c^{\rho_2})+{}^{\rho_1}I_{a+}^{\beta_1}f(b^{\rho_1},d^{\rho_2})$$
$$+{}^{\rho_1}I_{b-}^{\beta_1}f(a^{\rho_1},c^{\rho_2})+{}^{\rho_1}I_{b-}^{\beta_1}f(a^{\rho_1},d^{\rho_2})\Big]$$
$$+\frac{\rho_1^{\beta_1}\rho_2^{\beta_2}\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{4(b^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\Big[{}^{\rho_1,\rho_2}I_{a+,c+}^{\beta_1,\beta_2}f(b^{\rho_1},d^{\rho_2})+{}^{\rho_1,\rho_2}I_{a+,d-}^{\beta_1,\beta_2}f(b^{\rho_1},c^{\rho_2})$$
$$+{}^{\rho_1,\rho_2}I_{b-,c+}^{\beta_1,\beta_2}f(c^{\rho_1},d^{\rho_2})+{}^{\rho_1,\rho_2}I_{b-,d-}^{\beta_1,\beta_2}f(a^{\rho_1},c^{\rho_2})\Big]\Bigg|$$
$$\le(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})\left(\frac{1}{(\beta_1r+1)(\beta_2r+1)}\right)^{\frac{1}{r}}$$
$$\times\Bigg(\max\Bigg\{\left|\frac{\partial^2}{\partial t\partial s}f(b^{\rho_1},d^{\rho_2})\right|^q,\left|\frac{\partial^2}{\partial t\partial s}f\,(b^{\rho_1},c^{\rho_2})\right|^q,$$
$$\left|\frac{\partial^2}{\partial t\partial s}f\,(a^{\rho_1},d^{\rho_2})\right|^q,\left|\frac{\partial^2}{\partial t\partial s}f\,(a^{\rho_1},c^{\rho_2})\right|^q\Bigg\}\Bigg)^{\frac{1}{q}},$$

holds for $q>1$, *where* $\frac{1}{r}+\frac{1}{q}=1$.

Proof By an application of Lemma 2 and the absolute value properties, we obtain

$$\Bigg|\frac{f(a^{\rho_1},c^{\rho_2})+f(a^{\rho_1},d^{\rho_2})+f(b^{\rho_1},c^{\rho_2})+f(b^{\rho_1},d^{\rho_2})}{4}$$
$$-\frac{\rho_2^{\beta_2}\Gamma(\beta_2+1)}{4(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\Big[{}^{\rho_2}I_{c+}^{\beta_2}f(a^{\rho_1},d^{\rho_2})+{}^{\rho_2}I_{c+}^{\beta_2}f(b^{\rho_1},d^{\rho_2})$$
$$+{}^{\rho_2}I_{d-}^{\beta_2}f(a^{\rho_1},c^{\rho_2})+{}^{\rho_2}I_{d-}^{\beta_2}f(b^{\rho_1},c^{\rho_2})\Big]$$
$$-\frac{\rho_1^{\beta_1}\Gamma(\beta_1+1)}{4(b^{\rho_1}-a^{\rho_1})^{\beta_1}}\Big[{}^{\rho_1}I_{a+}^{\beta_1}f(b^{\rho_1},c^{\rho_2})+{}^{\rho_1}I_{a+}^{\beta_1}f(b^{\rho_1},d^{\rho_2})$$
$$+{}^{\rho_1}I_{b-}^{\beta_1}f(a^{\rho_1},c^{\rho_2})+{}^{\rho_1}I_{b-}^{\beta_1}f(a^{\rho_1},d^{\rho_2})\Big]$$
$$+\frac{\rho_1^{\beta_1}\rho_2^{\beta_2}\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{4(b^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\Big[{}^{\rho_1,\rho_2}I_{a+,c+}^{\beta_1,\beta_2}f(b^{\rho_1},d^{\rho_2})+{}^{\rho_1,\rho_2}I_{a+,d-}^{\beta_1,\beta_2}f(b^{\rho_1},c^{\rho_2})$$

$$+ {}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{b-,c+} f(c^{\rho_1}, d^{\rho_2}) + {}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{b-,d-} f(a^{\rho_1}, c^{\rho_2})\Big]\Big|$$

$$\leq \frac{\rho_1\rho_2(b^{\rho_1} - a^{\rho_1})(d^{\rho_2} - c^{\rho_2})}{4}\Big(|I_1| + |I_2| + |I_3| + I_4|\Big). \tag{31}$$

By using the Hölder's inequality and the quasi-convexity of $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$ on the coordinates, we have

$$\begin{aligned}
|I_1| &\leq \left(\int_0^1\int_0^1 s^{\beta_2 r\rho_2} t^{\beta_1 r\rho_1} s^{\rho_2-1} t^{\rho_1-1} ds dt\right)^{\frac{1}{r}} \\
&\quad \times \left(\int_0^1\int_0^1 s^{\rho_2-1} t^{\rho_1-1} \left|\frac{\partial^2}{\partial t\partial s} f(t^{\rho_1} b^{\rho_1} + (1-t^{\rho_1})a^{\rho_1}, s^{\rho_2} d^{\rho_2} + (1-s^{\rho_2})c^{\rho_2})\right|^q dt ds\right)^{\frac{1}{q}} \\
&\leq \left(\frac{1}{(\beta_1 r+1)(\beta_2 r+1)\rho_1\rho_2}\right)^{\frac{1}{r}} \\
&\quad \times \left(\max\left\{\left|\frac{\partial^2}{\partial t\partial s} f(b^{\rho_1}, d^{\rho_2})\right|^q, \left|\frac{\partial^2}{\partial t\partial s} f\left(b^{\rho_1}, c^{\rho_2}\right)\right|^q,\right.\right. \\
&\qquad \left.\left.\left|\frac{\partial^2}{\partial t\partial s} f\left(a^{\rho_1}, d^{\rho_2}\right)\right|^q, \left|\frac{\partial^2}{\partial t\partial s} f\left(a^{\rho_1}, c^{\rho_2}\right)\right|^q\right\}\int_0^1\int_0^1 s^{\rho_2-1} t^{\rho_1-1} dt ds\right)^{\frac{1}{q}} \\
&= \left(\frac{1}{(\beta_1 r+1)(\beta_2 r+1)\rho_1\rho_2}\right)^{\frac{1}{r}} \\
&\quad \times \left(\frac{1}{\rho_1\rho_2}\max\left\{\left|\frac{\partial^2}{\partial t\partial s} f(b^{\rho_1}, d^{\rho_2})\right|^q, \left|\frac{\partial^2}{\partial t\partial s} f\left(b^{\rho_1}, c^{\rho_2}\right)\right|^q,\right.\right. \\
&\qquad \left.\left.\left|\frac{\partial^2}{\partial t\partial s} f\left(a^{\rho_1}, d^{\rho_2}\right)\right|^q, \left|\frac{\partial^2}{\partial t\partial s} f\left(a^{\rho_1}, c^{\rho_2}\right)\right|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned}$$

That is,

$$\begin{aligned}
|I_1| &\leq \frac{1}{\rho_1\rho_2}\left(\frac{1}{(\beta_1 r+1)(\beta_2 r+1)}\right)^{\frac{1}{r}} \\
&\quad \times \left(\max\left\{\left|\frac{\partial^2}{\partial t\partial s} f(b^{\rho_1}, d^{\rho_2})\right|^q, \left|\frac{\partial^2}{\partial t\partial s} f\left(b^{\rho_1}, c^{\rho_2}\right)\right|^q,\right.\right. \\
&\qquad \left.\left.\left|\frac{\partial^2}{\partial t\partial s} f\left(a^{\rho_1}, d^{\rho_2}\right)\right|^q, \left|\frac{\partial^2}{\partial t\partial s} f\left(a^{\rho_1}, c^{\rho_2}\right)\right|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned} \tag{32}$$

Using similar argument, we have

$$|I_2| \leq \frac{1}{\rho_1\rho_2}\left(\frac{1}{(\beta_1 r+1)(\beta_2 r+1)}\right)^{\frac{1}{r}} \times \left(\max\left\{\left|\frac{\partial^2}{\partial t\partial s}f(b^{\rho_1},d^{\rho_2})\right|^q, \left|\frac{\partial^2}{\partial t\partial s}f(b^{\rho_1},c^{\rho_2})\right|^q, \left|\frac{\partial^2}{\partial t\partial s}f(a^{\rho_1},d^{\rho_2})\right|^q, \left|\frac{\partial^2}{\partial t\partial s}f(a^{\rho_1},c^{\rho_2})\right|^q\right\}\right)^{\frac{1}{q}}, \tag{33}$$

$$|I_3| \leq \frac{1}{\rho_1\rho_2}\left(\frac{1}{(\beta_1 r+1)(\beta_2 r+1)}\right)^{\frac{1}{r}} \times \left(\max\left\{\left|\frac{\partial^2}{\partial t\partial s}f(b^{\rho_1},d^{\rho_2})\right|^q, \left|\frac{\partial^2}{\partial t\partial s}f(b^{\rho_1},c^{\rho_2})\right|^q, \left|\frac{\partial^2}{\partial t\partial s}f(a^{\rho_1},d^{\rho_2})\right|^q, \left|\frac{\partial^2}{\partial t\partial s}f(a^{\rho_1},c^{\rho_2})\right|^q\right\}\right)^{\frac{1}{q}} \tag{34}$$

and

$$|I_4| \leq \frac{1}{\rho_1\rho_2}\left(\frac{1}{(\beta_1 r+1)(\beta_2 r+1)}\right)^{\frac{1}{r}} \times \left(\max\left\{\left|\frac{\partial^2}{\partial t\partial s}f(b^{\rho_1},d^{\rho_2})\right|^q, \left|\frac{\partial^2}{\partial t\partial s}f(b^{\rho_1},c^{\rho_2})\right|^q, \left|\frac{\partial^2}{\partial t\partial s}f(a^{\rho_1},d^{\rho_2})\right|^q, \left|\frac{\partial^2}{\partial t\partial s}f(a^{\rho_1},c^{\rho_2})\right|^q\right\}\right)^{\frac{1}{q}}. \tag{35}$$

The desired inequality follows from (31) and using (32)–(35). □

3 Conclusion

We used an extended Katugampola-type fractional integral to develop a two-dimensional Hermite–Hadamard-type integral inequality for functions that are quasi-convex on the coordinates. We also found three Hermite–Hadamard-type integral inequalities for two-variable functions whose mixed-order partial derivatives in absolute value at particular powers are quasi-convex on the coordinates. Using the Riemann–Liouville fractional integrals, similar conclusions might be obtained by setting $\rho_1 = \rho_2 = 1$.

The findings of this work, we feel, will add to the literature on fractional integral inequalities using generalized concepts of convexity on the coordinates.

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Leray–Schauder Theorem for Implicit Fractional Differential Equation and Nonlocal Multi-Point Conditions

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Abstract The research work in this paper attempts to investigate and consider the existence and uniqueness of solution for a family of implicit fractional differential equations with nonlocal multi-point conditions of the form

$$
\begin{aligned}
&{}^C D^q_{0+}\mu(\tau) = F(\tau, \mu(\tau), {}^C D^q_{0+}\mu(\tau)),\ \tau \in [0, 1],\\
&\mu^{(k)}(0) = \xi_k,\ u(1) = \sum_{i=1}^{m} \alpha_i \mu(\eta_i),\ 0 < \eta_i < 1,
\end{aligned}
$$

where $q \in (n-1, n),\ n \geq 2,\ k = 0, 1, \ldots, n-2,\ m, n \in \mathbb{N}, \xi_k,\ \alpha_i \in \mathbb{R},\ {}^C D^q_{0+}$ is Caputo fractional derivative of order q, $F : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. The problem is solved using a specific type of generalized fractional derivative. The Leray–Schauder degree and the Boyd–Wong nonlinear contraction fixed point theorems are used to prove the existence and uniqueness of the problem. Finally, an illustration of the results is provided.

1 Introduction

Fractional differential equations (FDEs) demonstrate a variety of interesting and important results concerning existence and uniqueness of solutions. Implicit fractional differential equations (IFDEs) are a type of FDE that is particularly important. The significance of the implicit ordinary differential equation (IODE) of the form

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$$H\left(\tau, y(\tau), \frac{dy(\tau)}{d\tau} \dots, \frac{d^{n-1}y(\tau)}{d\tau^{n-1}}\right) = 0$$

under different initial and boundary conditions motivates this study. This type of equation is used in a variety of areas, including tools, engineering, chemistry, physics, networking, dynamics, fluid mechanics, electromagnetic theory, viscoelasticity, electrochemistry, control theory, movement through porous media etc. (see [1–7, 14] for more details).

In 2017, Sathiyanathan and Krishnaveni [8] investigate the presence of the implicit FDE and integral boundary condition

$$\begin{cases} {}^{C}D^{q}_{0^+}\mu(\tau) = -F(\tau, \mu(\tau), {}^{C}D^{q}_{0^+}\mu(\tau)),\ 0 < \tau < 1, \\ a\mu(0) - b\mu'(0) = 0,\ \mu(1) = \int_0^1 k(s)g(\tau, u(s))ds + u, \end{cases}$$

where $1 < q < 2$, ${}^{C}D^{q}_{0^+}$ is the Caputo fraction derivative order q, $(\mathcal{B}, \|\cdot\|)$ be Banach space, $F : [0, 1] \times \mathcal{C}([0, 1], \mathcal{B}) \times \mathcal{B} \to \mathcal{B}$, $k \in \mathcal{C}([0, 1], \mathcal{B})$, $k \neq 0$, $a, b \in \mathbb{R}^+$, $a + b > 0$ and $\frac{a}{a+b} < q - 1$.

In 2017, Tidke and Mahajan [9] studied the existence and uniqueness of solutions for the following implicit fractional differential equations with Riemann–Liouville derivative:

$$\begin{cases} {}_{RL}D^{q}_{0^+}\mu(\tau) = F(\tau, \mu(\tau), {}_{RL}D^{q}_{0^+}\mu(\tau)),\ 0 < \tau < b, \\ {}_{RL}D^{q-1}_{0^+}\mu(0) = \mu_0 \in \mathbb{R}, \end{cases}$$

where ${}_{RL}D^{q}_{0^+}$, $(0 < q < 1)$, $b > 0$ denotes Riemann–Liouville fractional derivative operator and F is a continuous function on $[0, b] \times \mathbb{R}^2$ into $\mathbb{R}$; $\mathbb{R}$ denotes the real space.

In 2019, Borisut et al. [10] analyzed and did research on the ψ-Hilfer fractional differential equation with nonlocal multi-point condition of the form

$$\begin{cases} D^{q,p;\psi}_{a^+}\mu(\tau) = F(\tau, \mu(\tau), D^{q,p;\psi}_{a^+}\mu(\tau)), \\ \mathcal{I}^{1-r;\psi}_{a^+}\mu(a) = \sum_{i=1}^{m} \beta_i \mu(\eta_i), \end{cases}$$

where $q \leq r = q + p - qp < 1$, $\tau \in [a, b]$, $\eta_i \in [a, b]$, $q \in (0, 1)$, $p \in [0, 1]$, $m \in \mathbb{N}$, $\beta_i \in \mathbb{R}$, $i = 1, 2, \dots, m$, $-\infty < a < b < \infty$, $D^{q,p;\psi}_{a^+}$ is the ψ-Hilfer fractional derivative, $F : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and $\mathcal{I}^{1-r;\psi}_{a^+}$ is the ψ-Riemann–Liouville fractional integral of order $1 - r$.

The purpose of this article is to deduce the existence and uniqueness solution of a nonlinear fractional differential equation and a nonlocal multi-point condition, as inspired by the paper in [8–11]

$$\begin{cases} {}^{C}D^{q}_{0^+}\mu(\tau) = F(\tau, \mu(\tau), {}^{C}D^{q}_{0^+}\mu(\tau)), \ \tau \in [0, 1] \\ \mu^{(k)}(0) = \xi_k, \ \mu(1) = \sum_{i=1}^{m} \alpha_i \mu(\eta_i), \ 0 < \eta_i < 1, \end{cases} \tag{1}$$

where $q \in (n-1, n), \ n \geq 2, \ k = 0, 1, \ldots, n-2, \ m, n \in \mathbb{N}, \xi_k, \ \alpha_i \in \mathbb{R}, {}^{C}D^{q}_{0^+}$ is Caputo fractional derivative of order q, $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$.

The following is a summary of the paper. Section 2 covers the fundamentals of fractional derivatives. In Sect. 3, we use the Leray–Schauder degree and the Boyd and Wong fixed point theorem to illustrate our major findings. We also offer an illustration of the major findings.

2 Background Materials

We will review some fundamental notations, definitions, lemmas and theorems that will be used to establish the major result in this section.

Definition 1 ([12]) Let $\Gamma(\cdot)$ be a gamma function which is given by

$$\Gamma(q) = \int_0^\infty e^{-s} s^{q-1} ds.$$

The Riemann–Liouville fractional integral of order $q > 0$ for a function $F : (0, \infty) \to \mathbb{R}$ is defined by

$${}_{RL}\mathcal{I}^{q}_{0^+} F(\tau) = \frac{1}{\Gamma(q)} \int_0^\tau (\tau - s)^{q-1} F(s) ds.$$

Definition 2 ([13]) The Caputo fractional derivative of order $q > 0$ of a function $F : (0, \infty) \to \mathbb{R}$ is defined by

$$\left({}^{C}D^{q}_{0^+} F\right)(\tau) = \frac{1}{\Gamma(n-q)} \int_0^\tau (\tau - s)^{n-q-1} F^{(n)}(s) ds,$$

where $n \in \mathbb{N}$ is the smallest number with $n \geq q$.

Lemma 1 ([12]) *Let $n - 1 < q < n$. If $F \in \mathcal{C}^n([a, b])$, then*

$${}_{RL}\mathcal{I}^{q}_{0^+}({}^{C}D^{q}_{0^+}\mu)(\tau) = \mu(\tau) + c_0 + c_1\tau + c_2\tau^2 + \cdots + c_{n-1}\tau^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, $n \in \mathbb{N}$ is the smallest number with $n \geq q$.

Lemma 2 ([13] Arzela–Ascoli theorem) *Let $\mathcal{M} \subseteq \mathcal{C}[a, b]$. $\mathcal{M}$ is relatively compact in $\mathcal{C}[a, b]$ if and only if $\mathcal{M}$ is*

1. *uniformly bounded (meaning that it is a bounded set in* $\mathcal{C}[a, b]$*),*
2. *equicontinuous on* $[a, b]$*, for any* $\epsilon > 0$ *there exists* $\delta > 0$ *such that* $|t_2 - t_1| < \delta$ *implies* $|F(t_2) - F(t_1)| < \epsilon$ *for any* $F \in \mathcal{M}$*.*

Definition 3 ([13]) Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces, $E \subset \mathcal{X}$ is bounded. We called the mapping $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ is completely continuous if $\mathcal{A}(E) \subset \mathcal{Y}$ is relatively compact.

Definition 4 ([13]) Let $\Delta \subset \mathcal{X}$. A continuous bounded mapping $\mathcal{F} : \Delta \to \mathcal{X}$ is called α-Lipschitz if there exists $k \geq 0$ such that $\alpha(\mathcal{F}(E)) \leq k\alpha(E), \forall E \subset \Delta$. If in addition, $k < 1$ then we say that $\mathcal{F}$ is a strict α-contraction. Let $\sigma = \{(I \setminus \mathcal{F}, \Delta, v) : \Delta \subset \mathcal{X}$ open and bounded. $\mathcal{F} \in C_\infty(\overline{\Delta}),\ v \in \mathcal{X} \setminus (I \setminus \mathcal{F})(\partial\Delta)\}$ be the family of admissible triplets.

Theorem 1 ([13]) *The properties of degree function* $\mathcal{D} : \sigma \to \mathbb{N}_0$ *are introduced as the following.*

1. *Normalization:* $\mathcal{D}(I, \Delta, v) = 1$ *for every* $v \in \Delta$*.*
2. *Additivity on domain: For every disjoint, open set* $\Delta_1, \Delta_2 \subset \Delta$ *and* v *does not belong to* $(I \setminus \mathcal{F})(\overline{\Delta} \setminus (\Delta_1 \cup \Delta_2))$ *we have* $\mathcal{D}(I \setminus \mathcal{F}, \Delta, v) = \mathcal{D}(I \setminus \mathcal{F}, \Delta_1, v) + \mathcal{D}(I \setminus \mathcal{F}, \Delta_2, v)$*.*
3. *Invariance under homotopy:* $\mathcal{D}(I \setminus \mathcal{G}(\tau, \cdot), \Delta, v(\tau))$ *is* $\mathcal{G} : [0, 1] \times \overline{\Delta} \to \mathcal{X}$ *which satisfies* $\alpha(\mathcal{G}([0, 1] \times E)) < \alpha(E),\ \forall E \subset \overline{\Delta}$ *with* $\alpha(E) > 0$ *and every continuous function* $v : [0, 1] \to \mathcal{X}$ *which satisfies* $v(\tau) \neq \mu - \mathcal{G}(\tau, \mu)\ \forall \tau \in [0, 1],\ \forall \mu \in \partial\Delta$*.*
4. *Existence:* $\mathcal{D}(I \setminus \mathcal{F}, \Delta, v) \neq 0$ *implies* $v \in (I \setminus \mathcal{F})(\Delta)$*.*
5. *Excision:* $\mathcal{D}(I \setminus \mathcal{F}, \Delta, v) = \mathcal{D}(I \setminus \mathcal{F}, \Delta_1, v)$ *for every open set* $\Delta_1 \subset \Delta$ *and every* v *does not belong to* $(I \setminus \mathcal{F})(\overline{\Delta} \setminus \Delta_1)$*.*

As a result of degree function defined on σ, we gather the usefulness of the previous estimate method by virtue of this degree.

Theorem 2 ([13]) *Suppose that* $\mathcal{A} : \overline{\Delta} \to \mathcal{B}$ *is a completely continuous operator and that* $\mathcal{A}u \neq \lambda u,\ \forall u \in \partial\Delta, \lambda \geq 1$*. Then* $\mathcal{D}(I \setminus \mathcal{A}), \Delta, \theta) = 1$ *and* $\mathcal{A}$ *has at least one fixed point in* $\overline{\Delta}$*.*

Definition 5 ([12]) Let $\mathcal{A} : \mathcal{B} \to \mathcal{B}$ be a mapping and $\mathcal{B}$ be a Banach space. $\mathcal{A}$ is said to be a nonlinear contraction if there exists a nondecreasing continuous function $\Pi : (0, +\infty) \to (0, +\infty)$ such that $\Pi(0) = 0$ and $\Pi(\epsilon) < \epsilon$ for all $\epsilon > 0$ with the following property:

$$\|\mathcal{A}\mu - \mathcal{A}v\| \leq \Pi(\|\mu - v\|),\ \forall \mu, v \in \mathcal{B}.$$

Theorem 3 ([12] Boyd and Wong fixed point theorem)
Let $\mathcal{B}$ *be a Banach space and let* $\mathcal{A} : \mathcal{B} \to \mathcal{B}$ *be a nonlinear contraction. Then* $\mathcal{A}$ *has a unique fixed point in* $\mathcal{B}$*.*

3 Main Results

In this section we consider the solutions of nonlinear fractional differential equation (1).

Lemma 3 *Let function $h^* \in \mathcal{C}([0,1],\mathcal{B})$, where $\mathcal{B}$ be a Banach space. Suppose that the function $\mu \in \mathcal{C}([0,1],\mathcal{B})$ is a solution of the following linear FDE and conditions*

$$\begin{cases} {}^C D^q_{0^+}\mu(\tau) = h^*(\tau),\ \tau \in [0,1], \\ \mu^{(k)}(0) = \xi_k,\ \mu(1) = \sum_{i=1}^{m} \alpha_i\ \mu(\eta_i), \end{cases} \tag{2}$$

where $q \in (n-1,n)$, $n \geq 2$, $k = 0,1,\dots,n-2$, $m,n \in \mathbb{N}$, ξ_k, $\alpha_i \in \mathbb{R}$. Here ${}^C D^q_{0^+}$ is the Caputo fractional derivative of order q and assume that

$$\Lambda := 1 - \sum_{i=1}^{m} \alpha_i \eta_i^{n-1} \neq 0.$$

Then, the solution of (2) is unique and given by

$$\mu(\tau) = {}_{RL}\mathcal{I}^q_{0^+}h^*(\tau) + \sum_{k=0}^{n-2}\frac{\xi_k\tau^k}{k!} + \frac{\tau^{n-1}}{\Lambda}\Big[\sum_{i=1}^{m}\alpha_i\,{}_{RL}\mathcal{I}^q_{0^+}h^*(\eta_i) + \sum_{i=1}^{m}\sum_{k=0}^{n-2}\frac{\alpha_i\xi_k\eta_i^k}{k!} - {}_{RL}\mathcal{I}^q_{0^+}h^*(1) - \sum_{k=0}^{n-2}\frac{\xi_k}{k!}\Big].$$

Proof From Lemma 1, we get constants $c_0, c_1, \dots, c_{n-1}$ belong to $\mathcal{B}$

$$\mu(\tau) = {}_{RL}\ \mathcal{I}^q_{0^+}h^*(\tau) + c_0 + c_1\tau + c_2\tau^2 + \cdots + c_{n-1}\tau^{n-1}.$$

We select from the first condition,

$$c_0 = \xi_0,\ c_1 = \xi_1,\ c_2 = \frac{\xi_2}{2!}\ \dots,\ c_{n-2} = \frac{\xi_{n-2}}{(n-2)!},$$

and so

$$\mu(\tau) = {}_{RL}\ \mathcal{I}^q_{0^+}h^*(\tau) + \sum_{k=0}^{n-2}\frac{\xi_k\tau^k}{k!} + c_{n-1}\tau^{n-1}. \tag{3}$$

The substitution $\tau = 1$ yield

$$\mu(1) =_{RL} \mathcal{I}^q_{0^+} h^*(1) + \sum_{k=0}^{n-2} \frac{\xi_k}{k!} + c_{n-1}.$$

Substitution of $\tau = \eta_i$ and applying $\sum_{i=1}^{m} \alpha_i \mu(\eta_i)$ in (3), we get

$$\sum_{i=1}^{m} \alpha_i \mu(\eta_i) = \sum_{i=1}^{m} \alpha_i \, {}_{RL}\mathcal{I}^q_{0^+} h^*(\eta_i) + \sum_{i=1}^{m} \sum_{k=0}^{n-2} \frac{\alpha_i \xi_k \eta_i^k}{k!} + c_{n-1} \sum_{i=1}^{m} \alpha_i \eta_i^{n-1}.$$

By the second condition, we have

$$c_{n-1} = \frac{1}{\Lambda}\Big[\sum_{i=1}^{m} \alpha_i \, {}_{RL}\mathcal{I}^q_{0^+} h^*(\eta_i) + \sum_{i=1}^{m} \sum_{k=0}^{n-2} \frac{\alpha_i \xi_k \eta_i^k}{k!} -_{RL} \mathcal{I}^q_{0^+} h^*(1) - \sum_{k=0}^{n-2} \frac{\xi_k}{k!} \Big],$$

$$\mu(\tau) = {}_{RL}\mathcal{I}^q_{0^+} h^*(\tau) + \sum_{k=0}^{n-2} \frac{\xi_k \tau^k}{k!} + \frac{\tau^{n-1}}{\Lambda}\Big[\sum_{i=1}^{m} \alpha_i \, {}_{RL}\mathcal{I}^q_{0^+} h^*(\eta_i) + \sum_{i=1}^{m} \sum_{k=0}^{n-2} \frac{\alpha_i \xi_k \eta_i^k}{k!} - {}_{RL}\mathcal{I}^q_{0^+} h^*(1) - \sum_{k=0}^{n-2} \frac{\xi_k}{k!} \Big].$$

Let $\mathcal{B}$ be the space of all continuous functions defined on $[0, 1]$, that is $\mathcal{B} = \mathcal{C}([0, 1], \mathbb{R})$ with the supremum norm $\|\mu\|_\infty := \sup_{\tau \in [0,1]} |\mu(\tau)|$, $\mu \in \mathcal{B}$ and the space $\mathcal{B}$ is a Banach space, let

$$\mathcal{K}\mu(\tau) =^C D^q_{0^+} \mu(\tau) = F(\tau, \mu(\tau), ^C D^q_{0^+} \mu(\tau)).$$

Define the nonlinear operator $\mathcal{A} : \mathcal{B} \to \mathcal{B}$ as follows:

$$\begin{cases} (\mathcal{A}\mu)(\tau) & =_{RL} \mathcal{I}^q_{0^+} F(s, \mu(s), \mathcal{K}\mu(s))(\tau) \\ & + \sum_{k=0}^{n-2} \frac{\xi_k \tau^k}{k!} + \frac{\tau^{n-1}}{\Lambda}\Big[\sum_{i=1}^{m} \alpha_i \, {}_{RL}\mathcal{I}^q_{0^+} F(s, \mu(s), \mathcal{K}\mu(s))(\eta_i) \\ & + \sum_{i=1}^{m} \sum_{k=0}^{n-2} \frac{\alpha_i \xi_k \eta_i^k}{k!} -_{RL} \mathcal{I}^q_{0^+} F(s, \mu(s), \mathcal{K}\mu(s))(1) - \sum_{k=0}^{n-2} \frac{\xi_k}{k!} \Big]. \end{cases} \tag{4}$$

Then the operator $\mathcal{A}$ has fixed point if and only if (1) has a solution. Now we prove and consider the existence and uniqueness of solution for problem (1) via Leray–Schauder degree, Boyd and Wong fixed point theorems.

3.1 Existence Result via Leray–Schauder Degree

Theorem 4 *Let* $F : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ *be a continuous function. Assume that there exist constants* $0 \le \kappa_1 \le \kappa_2 \le \omega^{-1}$ *and* $M_1 > 0$ *such that*

$$|F(\tau, \mu, v)| \le \kappa_1|\mu| + \kappa_2|v| + M_1,$$

for all $(\tau, \mu, v) \in [0, 1] \times \mathbb{R} \times \mathbb{R}$ *and*

$$\omega = \frac{1}{\Gamma(q+1)} + \frac{1}{|\Lambda|\Gamma(q+1)} + \frac{1}{|\Lambda|}\sum_{i=1}^{m}|\alpha_i|\frac{\eta_i^q}{\Gamma(q+1)}.$$

Then the functional boundary value problem (1) *has at least one solution on* [0, 1].

Proof As the operator $\mathcal{A} : \mathcal{B} \to \mathcal{B}$ is shown in (4), we next consider the fixed point problem

$$\mu = \mathcal{A}\mu. \tag{5}$$

We now show that there exists a fixed point $\mu \in \mathcal{B}$ satisfying (5). It is sufficient to show that $\mathcal{A} : \overline{B}_R \to \mathcal{B}$ satisfies

$$\mu \ne \lambda\mathcal{A}\mu, \ \forall \mu \in \partial B_R, \lambda > 1, \tag{6}$$

where $B_R = \{\mu \in \mathcal{B} : \max_{\tau\in[0,1]}|\mu(\tau)| < R, \ \ R > 0\}$, we define $\mathcal{H}(\lambda, \mu) = \lambda\mathcal{A}\mu$. We show the operator $\mathcal{A}$ is continuous, uniformly bounded and equicontinuous.

Step 1. The operator $\mathcal{A}$ is continuous. Let $\{\mu_n\}$ be a sequence such that $\lim_{n\to\infty} \mu_n = \mu$ in $\mathcal{B}$. If $\tau \in [0, 1]$, we have

$$\begin{aligned}
\|(\mathcal{A}\mu_n)(\tau) - (\mathcal{A}\mu)(\tau)\| &= {}_{RL}\mathcal{I}_{0^+}^q|F(s, \mu_n(s), \mathcal{K}\mu_n(s)) - F(s, \mu(s), \mathcal{K}\mu(s))|(\tau) \\
&\quad + \frac{1}{|\Lambda|}{}_{RL}\mathcal{I}_{0^+}^q|F(s, \mu_n(s), \mathcal{K}\mu_n(s)) - F(s, \mu(s), \mathcal{K}\mu(s))|(1) \\
&\quad + \frac{1}{|\Lambda|}\sum_{i=1}^{m}|\alpha_i|{}_{RL}\mathcal{I}_{0^+}^q|F(s, \mu_n(s), \mathcal{K}\mu_n(s)) \\
&\quad - F(s, \mu(s), \mathcal{K}\mu(s))|(\eta_i) \\
&\le \|F(s, \mu_n(s), \mathcal{K}\mu_n(s)) - F(s, \mu(s), \mathcal{K}\mu(s))\|\Big\{{}_{RL}\mathcal{I}_{0^+}^q(\tau) \\
&\quad + \frac{1}{|\Lambda|}\Big({}_{RL}\mathcal{I}_{0^+}^q(1) + \sum_{i=1}^{m}|\alpha_i|{}_{RL}\mathcal{I}_{0^+}^q(\eta_i)\Big)\Big\} \\
&\le \|F(s, \mu_n(s), \mathcal{K}\mu_n(s)) - F(s, \mu(s), \mathcal{K}\mu(s))\|\Big\{\frac{1}{\Gamma(q+1)} \\
&\quad + \frac{1}{|\Lambda|\Gamma(q+1)} + \frac{1}{|\Lambda|}\sum_{i=1}^{m}|\alpha_i|\frac{\eta_i^q}{\Gamma(q+1)}\Big\}
\end{aligned}$$

$$\leq \Big\{\frac{1}{\Gamma(q+1)} + \frac{1}{|\Lambda|\Gamma(q+1)} + \frac{1}{|\Lambda|}\sum_{i=1}^{m}|\alpha_i|\frac{\eta_i^q}{\Gamma(q+1)}\Big\}\|\mathcal{K}\mu_n(\cdot) - \mathcal{K}\mu(\cdot)\|.$$

Since F is continuous implies that $\mathcal{K}\mu$ is continuous. Thus $\|\mathcal{A}\mu_n - \mathcal{A}\mu\| \to 0$ as $n \to \infty$. Hence the operator $\mathcal{A}$ is continuous.

Step 2. Claim that $\mathcal{A}(B_R) \subset B_R$. Define $B_R = \Big\{\mu \in \mathcal{B} : \|\mu\| \leq R\Big\}$, where

$$R \geq \frac{\sum_{k=0}^{n-2}\frac{\xi_k}{k!}\left(1+\frac{1}{\Lambda}\right)(1-\kappa_1)\Gamma(q-1) + \left(1+\frac{1}{\Lambda}\left[\sum_{i=1}^{m}\alpha_i\eta_i^q + 1\right]\right)M}{(1-\kappa_2)\Gamma(q+1) - \kappa_1\left(1+\frac{1}{\Lambda}\left[\sum_{i=1}^{m}\alpha_i\eta_i^q + 1\right]\right)}.$$

Let μ belong to B_R. In order to prove that $\mathcal{A}\mu \in B_R$, it suffices to show that $|\mathcal{A}\mu(\tau)| \leq R$ for $\tau \in [0, 1]$, we get $|\mathcal{K}\mu(\tau)| \leq \dfrac{\kappa_1|\mu(\tau)| + M_1}{1-\kappa_2}$ and

$$\begin{aligned}
|\mathcal{A}\mu(\tau)| &\leq {}_{RL}\mathcal{I}_{0+}^q|\mathcal{K}\mu(\tau)| + \frac{\tau^{n+1}}{\Lambda}\Big[\sum_{i=1}^{m}\alpha_i\, {}_{RL}\mathcal{I}_{0+}^q|\mathcal{K}\mu(\eta_i)| + {}_{RL}\mathcal{I}_{0+}^q|\mathcal{K}\mu(1)|\Big] \\
&\quad + \sum_{k=0}^{n-2}\frac{\xi_k}{k!}|\tau^k - \frac{\tau^{n-1}}{\Lambda}| \\
&\leq {}_{RL}\mathcal{I}_{0+}^q\frac{\kappa_1|\mu(\tau)| + M_1}{1-\kappa_2} + \frac{\tau^{n+1}}{\Lambda}\Big[\sum_{i=1}^{m}\alpha_i\, {}_{RL}\mathcal{I}_{0+}^q\frac{\kappa_1|\mu(\eta_i)| + M_1}{1-\kappa_2} \\
&\quad + {}_{RL}\mathcal{I}_{0+}^q\frac{\kappa_1|\mu(1)| + M_1}{1-\kappa_2}|\Big] + \sum_{k=0}^{n-2}\frac{\xi_k}{k!}|\tau^k - \frac{\tau^{n-1}}{\Lambda}| \\
&\leq \frac{1}{(1-\kappa_2)\Gamma(q+1)}\Big\{\kappa_1\|\mu\| + M_1\Big\} \\
&\quad + \frac{\tau^{n-1}}{\Lambda}\Big[\sum_{k=1}^{m}\frac{\alpha_i\eta_i^q}{(1-\kappa_2)\Gamma(q+1)}\Big\{\kappa_1\|\mu\| + M_1\Big\} \\
&\quad + \frac{1}{\Gamma(q+1)}\Big\{\kappa_1\|\mu\| + M_1\Big\}\Big] + \sum_{k=0}^{n-2}\frac{\xi_k}{k!}|\tau^k + \frac{\tau^{n-1}}{\Lambda}| \\
&\leq \Big(1 + \frac{1}{\Lambda}\Big[\sum_{i=1}^{m}\alpha_i\eta_i^q + 1\Big]\Big)\Big\{\frac{\kappa_1 R + M_1}{(1-\kappa_2)\Gamma(q+1)}\Big\} \\
&\quad + \sum_{k=0}^{n-2}\frac{\xi_k}{k!}\left(1 + \frac{1}{\Lambda}\right) \\
&\leq R.
\end{aligned}$$

Hence $\mathcal{A}(B_R) \subset B_R$.

Step 3. Show that $\mathcal{A}(B_R)$ is uniformly bounded and equicontinuous.
By using step 2, we have $\mathcal{A}(B_R) = \{\mathcal{A}\mu : \mu \in B_R\}$. This gives us that $\|\mathcal{A}\mu\| \leq R$, for each $\mu \in B_R$. We can now conclude that $\mathcal{A}(B_R)$ is uniformly bounded. Let $\tau_1, \tau_2 \in [0, 1]$. Define

$$\sup_{(\tau,\mu,\mathcal{K}\mu)\in[0,1]\times B_R\times B_R} |F(\tau, \mu, \mathcal{K}\mu)| < C < \infty.$$

By choosing $\mu \in B_R$, we then obtain that

$$\begin{aligned}
|\mathcal{A}\mu(\tau_2) - \mathcal{A}\mu(\tau_1)| \leq & \Big|\frac{1}{\Gamma(q)}\int_0^{\tau_2}(\tau_2 - s)^{q-1}F(s, \mu(s), \mathcal{K}\mu(s))ds \\
& -\frac{1}{\Gamma(q)}\int_0^{\tau_1}(\tau_1 - s)^{q-1}F(s, \mu(s), \mathcal{K}\mu(s))ds\Big| \\
& +\frac{\tau_2^{n-1} - \tau_1^{n-1}}{\Lambda}\Big[\sum_{i=1}^{m}\alpha_i\, {}_{RL}\mathcal{I}_{0^+}^{q}F(s, \mu(s), \mathcal{K}\mu(s))(\eta_i) \\
& -{}_{RL}\mathcal{I}_{0^+}^{q}F(s, \mu(s), \mathcal{K}\mu(s))(1)\Big] \\
& +\sum_{i=1}^{m}\frac{\xi_k}{k!}\Big[\Big(\tau_2^k - \frac{\tau_2^{n-1}}{\Lambda}\Big) - \Big(\tau_1^k - \frac{\tau_1^{n-1}}{\Lambda}\Big)\Big]
\end{aligned}$$

$$\begin{aligned}
|\mathcal{A}\mu(\tau_2) - \mathcal{A}\mu(\tau_1)| \leq & \frac{C}{\Gamma(q+1)}\Big[(\tau_2^q - \tau_1^q) - (\tau_2 - \tau_1)^q\Big] + \frac{C}{\Gamma(q+1)}(\tau_2 - \tau_1)^q \\
& +\frac{\tau_2^{n-1} - \tau_1^{n-1}}{|\Lambda|}\Big[\sum_{i=1}^{m}\alpha_i\, {}_{RL}\mathcal{I}_{0^+}^{q}F(s, \mu(s), \mathcal{K}\mu(s))(\eta_i) \\
& -{}_{RL}\mathcal{I}_{0^+}^{q}F(s, \mu(s), \mathcal{K}\mu(s))(1)\Big] \\
& +\sum_{i=1}^{m}\frac{|\xi_k|}{k!}\Big[(\tau_2^k - \tau_1^k) - (\tau_2^{n-1} - \tau_1^{n-1})\frac{1}{|\Lambda|}\Big].
\end{aligned}$$

We see that the right-hand side converge to 0, as τ_2 converge to τ_1. Thus, $\mathcal{A}(B_R)$ is equicontinuous and uniformly bounded. Hence, from the Arzela–Ascoli theorem, this implies that the set $\mathcal{A}(B_R)$ is relatively compact in B_R.

Step 4. Define g_λ by $g_\lambda(\mu) = \mu - \mathcal{G}(\lambda, \mu) = \mu - \mathcal{A}\mu$ is completely continuous. In order to prove (6), we assume that $\mu = \lambda\mathcal{A}\mu$ for some $\lambda \in [0, 1]$ and for all $\tau \in [0, 1]$. Then with $\|\mu\| = \sup_{\tau\in[0,1]} |\mu(\tau)|$, we define an operator $\mathcal{A} : \mathcal{B} \to \mathcal{B}$ by (4). Consider

$$(\mathcal{A}\mu)(\tau) = {}_{RL}\mathcal{I}^{q}_{0^+} F(s, \mu(s), \mathcal{K}\mu(s))(\tau) + \sum_{k=0}^{n-2} \frac{\xi_k \tau^k}{k!}$$
$$+\frac{\tau^{n-1}}{\Lambda}\Big[\sum_{i=1}^{m} \alpha_i \, {}_{RL}\mathcal{I}^{q}_{0^+} F(s, \mu(s), \mathcal{K}\mu(s))(\eta_i)$$
$$+\sum_{i=1}^{m}\sum_{k=0}^{n-2} \frac{\alpha_i \xi_k \eta_i^k}{k!} - {}_{RL}\,\mathcal{I}^{q}_{0^+} F(s, \mu(s), K\mu(s))(1) - \sum_{k=0}^{n-2} \frac{\xi_k}{k!}\Big].$$

By applying step 2, we have $|\mu| = |\lambda\mathcal{A}\mu| \leq \rho$ which, on solving for $\|\mu\|$, yields $\|\mu\| \leq \rho$. If $\rho = R + 1$, inequality (6) holds. And we define $\mathcal{G}(\lambda, \mu) = \lambda\mathcal{A}\mu$. A continuous map g_λ defined by $g_\lambda(\mu) = \mu - \mathcal{G}(\lambda, \mu) = \mu - \lambda\mathcal{A}(\mu)$ is completely continuous. If (6) is true, then the following Laray–Schauder degree is well defined and by the monotony invariance of topological degree, we consider

$$\begin{aligned}\mathcal{D}(g_\lambda, B_R, 0) &= \mathcal{D}(I - \lambda\mathcal{A}, B_R, 0)\\ &= \mathcal{D}(g_1, B_R, 0) = \mathcal{D}(g_0, B_R, 0)\\ &= \mathcal{D}(I, B_R, 0) = 1 \neq 0, 0 \in B_R,\end{aligned}$$

where I stands for the identity operator. Because of the nonzero property of the Leray–Schauder degree $g_1(\mu) = \mu - \mu\mathcal{A} = 0$, for at least fixed point $\mu \in B_R$.

3.2 Existence and Uniqueness Result via Boyd and Wong Fixed Point Theorem

Theorem 5 *Let $F : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and $\|\mathcal{K}\mu - \mathcal{K}v\| \leq \frac{M}{1-N}\|\mu - v\|$ satisfying the assumption*

$$|F(\tau, \mu, \mathcal{K}\mu) - F(\tau, v, \mathcal{K}v)| \leq \frac{t(\tau)|\mu - v|}{T^* + |\mu - v|},$$

where $\tau \in [0, 1]$, $\mu, v \geq 0$, $M > 0, 0 < N < 1$ and $t(\tau) : [0, 1] \to \mathbb{R}^+$ is continuous. The constant T^* is defined by

$$T^* := {}_{RL}\,\mathcal{I}^{q}_{0^+} t(1) + \frac{1}{|\Lambda|}\Big[\sum_{i=1}^{m} |\alpha_i| \, {}_{RL}\mathcal{I}^{q}_{0^+} t(\eta_i) + {}_{RL}\,\mathcal{I}^{q}_{0^+} t(1)\Big] \neq 0.$$

Then the problem (1) *has a unique solution on* $[0, 1]$.

Proof Consider the nondecreasing continuous function $\Pi : \mathbb{R}^+ \to \mathbb{R}^+$ given by $\Pi(\epsilon) = \frac{T^*\epsilon}{T^*+\epsilon}$, $\forall \epsilon > 0$, such that $\Pi(0) = 0$ and $\Pi(\epsilon) > \epsilon$, $\forall \epsilon > 0$. For any $\mu, v \in \mathcal{B}$ and for each $\tau \in [0, 1]$, we have

$$
\begin{aligned}
|\mathcal{A}\mu(\tau)-\mathcal{A}v(\tau)| \le\ & {}_{RL}\mathcal{I}^q_{0+}\Big|F(s,\mu(s),\mathcal{K}\mu(s))-F(s,v(s),\mathcal{K}v(s))\Big|(\tau)\\
&+\frac{\tau^{n-1}}{|\Lambda|}\Big[\sum_{i=1}^{m}|\alpha_i|{}_{RL}\mathcal{I}^q_{0+}\Big|F(s,\mu(s),\mathcal{K}\mu(s))-F(s,v(s),\mathcal{K}v(s))\Big|(\eta_i)\\
&+{}_{RL}\mathcal{I}^q_{0+}\Big|F(s,\mu(s),\mathcal{K}\mu(s))-F(s,v(s),\mathcal{K}v(s))\Big|(1)\Big]\\
\le\ & \frac{\Pi(\|\mu-v\|)}{T^*}\Big\{{}_{RL}\mathcal{I}^q_{0+}t(\tau)+\frac{1}{|\Lambda|}\Big[\sum_{i=1}^{m}|\alpha_i|{}_{RL}\mathcal{I}^q_{0+}t(\eta_i)\\
&+{}_{RL}\mathcal{I}^q_{0+}t(1)\Big]\Big\}\\
\le\ & \Pi(\|\mu-v\|).
\end{aligned}
$$

This indicates that $\|\mathcal{A}\mu-\mathcal{A}v\|\le \Pi(\|\mu-v\|)$. By utilizing Boyd and Wong's fixed point theorem, we may conclude that the operator $\mathcal{A}$ is a nonlinear contraction and has a unique solution to the problem (1).

4 Application

Consider the following FDE and nonlocal multi-point conditions

$$
\begin{cases}
{}^C D^{\frac{9}{2}}_{0^+}\mu(\tau)=\frac{2\tau}{\pi(99+e^{\tau})}\left(\frac{|\mu(\tau)|}{|\mu(\tau)|+1}\right)+\frac{\cos^2\tau}{99+10^{\tau}}\left(\frac{|{}^C D^{\frac{9}{2}}_{0^+}\mu(\tau)|}{|{}^C D^{\frac{9}{2}}_{0^+}\mu(\tau)+1|}\right)+\sqrt{\pi},\\
\mu(0)=\pi,\ \mu'(0)=\frac{\pi}{2},\ \mu''(0)=\frac{\pi}{3},\ \mu'''(0)=\frac{\pi}{4},\\
\mu(1)=\frac{1}{3}\mu(\frac{1}{3})+\frac{1}{4}\mu(\frac{1}{4}),\ \tau\in[0,1].
\end{cases}
\tag{7}
$$

By comparing problem (1) and (7), we obtain the following parameters: $q=9/2,\ n=5,\ \xi_1=\pi,\ \xi_2=\pi/2,\ \xi_3=\pi/4,\ \alpha_1=1/3,\ \alpha_2=1/4,\ \eta_1=1/3,\ \eta_2=1/4,\quad F(\tau,\mu(\tau),{}^C D^{\frac{9}{2}}_{0^+}\mu(\tau))=\frac{2\tau}{\pi(99+e^{\tau})}\left(\frac{|\mu(\tau)|}{|\mu(\tau)|+1}\right)+\frac{\cos^2\tau}{99+10^{\tau}}\left(\frac{|{}^C D^{\frac{9}{2}}_{0^+}\mu(\tau)|}{|{}^C D^{\frac{9}{2}}_{0^+}\mu(\tau)+1|}\right)+\sqrt{\pi}$. From $\omega=\frac{1}{\Gamma(q+1)}+\frac{1}{|\Lambda|\Gamma(q+1)}+\frac{1}{|\Lambda|}(\alpha_1\eta_1^q+\alpha_2\eta_2^q)$, we now consider

$$
\begin{aligned}
\Big|F(\tau,\mu(\tau),{}^C D^{\frac{9}{2}}_{0^+}\mu(\tau))\Big| \le\ & \Big|\frac{2\tau}{\pi(99+e^{\tau})}\Big(\frac{|\mu(\tau)|}{|\mu(\tau)|+1}\Big)\\
&+\frac{\cos^2\tau}{99+10^{\tau}}\Big(\frac{|{}^C D^{\frac{9}{2}}_{0^+}\mu(\tau)|}{|{}^C D^{\frac{9}{2}}_{0^+}\mu(\tau)+1|}\Big)+\sqrt{\pi}\Big|\\
\le\ & \frac{1}{100}\Big|\mu(\tau)\Big|+\frac{1}{100}\Big|{}^C D^{\frac{9}{2}}_{0^+}\mu(\tau)\Big|+\sqrt{\pi}.
\end{aligned}
$$

Therefore the condition of Theorem 4 is satisfied with $\omega=0.0412$ and $M_1=\sqrt{\pi}$. Note that $\kappa_1=\kappa_2=1/100<1/\omega=24.27$. By choosing $t(\tau)=1/50=0.02$, we then obtain that $T^* = 0.0003851$. Next, we consider

$$\begin{aligned}
&\left|F(\tau, \mu(\tau), {}^C D_{0^+}^{\frac{9}{2}} \mu(\tau)) - F(\tau, v(\tau), {}^C D_{0^+}^{\frac{9}{2}} v(\tau))\right| \\
&\leq \frac{1}{100} \left(\frac{\left|\mu - v\right|}{1 + |\mu - v\left|\right.} + \frac{\left|{}^C D_{0^+}^{\frac{9}{2}} \mu(\tau) - {}^C D_{0^+}^{\frac{9}{2}} v(\tau)\right|}{1 + \left|{}^C D_{0^+}^{\frac{9}{2}} \mu(\tau) - {}^C D_{0^+}^{\frac{9}{2}} v(\tau)\right|} \right).
\end{aligned}$$

By setting $M, N = \frac{1}{100}$ and using $\|\mathcal{K}\mu - \mathcal{K}v\| \leq \frac{M}{1-N} \|\mu - v\|$. We now conclude that

$$\begin{aligned}
&\left|F(\tau, \mu(\tau), {}^C D_{0^+}^{\frac{9}{2}} \mu(\tau)) - F(\tau, v(\tau), {}^C D_{0^+}^{\frac{9}{2}} v(\tau))\right| \\
&\leq \frac{1}{100} \left(\frac{\left|\mu - v\right|}{1 + |\mu - v\left|\right.} + \frac{\frac{1}{99}\left|\mu - v\right|}{1 + \frac{1}{99}|\mu - v\left|\right.} \right) \\
&\leq \frac{1}{100} \left(\frac{\left|\mu - v\right|}{1 + |\mu - v\left|\right.} + \frac{\left|\mu - v\right|}{99 + |\mu - v\left|\right.} \right) \\
&\leq \frac{1}{50} \left(\frac{|\mu - v|}{0.0003851 + |\mu - v|} \right).
\end{aligned}$$

As a result, according to Theorem 5, the problem (7) has a unique solution on (0, 1).

5 Conclusions

In our study, first, we obtain the operator from Eq. (1). Secondly, Leray–Schauder degree and Boyd–Wong nonlinear contraction fixed point theorems were used to establish the existence and uniqueness solutions for implicit fractional differential equation which involves Caputo fractional derivatives order $q \in [n-1, n), n \in \mathbb{N}$ with nonlocal multi-point conditions. In addition, an example was given to illustrate our main results.

Competing Interests
The authors declare that they have no competing interests.

Authors' Contributions
All authors contributed equally and significantly in writing this article. All authors read and approved final manuscript.

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The q-Deformed Hamiltonian, Lagrangian, Entropy and Fisher Information

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Abstract The aim of this article is to give the first time how four separated subjects, namely, Hamiltonian, Lagrangian, entropy and Fisher information, are possibly connected through the method known as the q-deformation.

1 Introduction

In mathematical literature, the generalisation of the differentiation called the fractal calculus (a.k.a. q-analog or q-deformed) can be dated back to the time of Leibniz. Later, this subject has been seriously studied and has been expressed as a standard language by many scholars [1–9]. The interesting point is that the application of the fractional calculus has been recently discovered in many disciplines such as engineering [10, 14], physics [11, 12] and biology [13, 14], see also [15, 16]. The notion of entropy was first introduced by Clausius [17] in the thermodynamic context. Later, Boltzmann [18] proposed another entropy in the context of statistical mechanics. However, these two expressions of the entropy give the same description of the system. In different context, Shannon proposed a quantity known as the Shannon entropy to measure the information in the communication theory [19]. In recent years, the generalised concept of entropy, called the q-deformed entropy, has gained a mammoth attention [20–23] as well as a wide range of applications [24–33]. In this chapter, we would like to provide a big picture on how things, namely, Hamiltonian, Lagrangian, entropy and Fisher information, are possibly connected based on a collection of works [34–37]. This may sound questionable how these things would fit

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together. However, we hope every much that, after the reader completes the chapter, things would make sense. We also would like to note that the q-deformation in our context will come with only one parameter. However, things can be generalised more than one parameter. Then it would be reasonable to call the objects in the study as the n-parameter generalisation.

Let us here provide some basic mathematical concept of the q-analog and q-derivative which will be recalled later throughout the text. First we introduce the q-deformed logarithmic and exponential functions

$$\ln_q x = \frac{x^{1-q}-1}{1-q}\ ,\ (x>0),$$
$$e_q^x = [1+(1-q)x]^{\frac{1}{1-q}}\ ,\ (1+(1-q)x>0). \tag{1}$$

Here $\lim_{q\to 1}\ln_q x = \ln x$ and $\lim_{q\to 1} e_q^x = e^x$ are the standard functions which are recovered. Furthermore, we also give the definitions of the q-sum, q-difference, q-product and q-ratio [38, 39]

$$x \oplus_q y = x + y + (1-q)xy\ ,$$
$$x \ominus_q y = \frac{x-y}{1+(1-q)y}\ ,\quad 1+(1-q)y \neq 0\ ,$$
$$x \otimes_q y = \left[x^{1-q}+y^{1-q}-1\right]^{\frac{1}{1-q}}\ ,\ x>0,\ y>0 \text{ and } x^{1-q}+y^{1-q}-1>0\ ,$$
$$x \oslash_q y = \left[x^{1-q}-y^{1-q}-1\right]^{\frac{1}{1-q}}\ ,\ x>0,\ y>0 \text{ and } x^{1-q}-y^{1-q}-1>0\ .$$

With these relations, one can find that

$$\ln_q(xy) = \ln_q x \otimes_q \ln_q y\ ,\quad e_q(x)e_q(y) = e_q(x\otimes y)\ ,$$
$$\ln_q(x\otimes_q y) = \ln_q x + \ln_q y\ ,\quad e_q(x)\otimes_q e_q(y) = e_q(x+y)\ ,$$
$$\ln_q(x/y) = \ln_q x \oslash_q \ln_q y\ ,\quad e_q(x)/e_q(y) = e_q(x\oslash_q y)\ ,$$
$$\ln_q(x\oslash_q y) = \ln_q x - \ln_q y\ ,\quad e_q(x)\oslash_q e_q(y) = e_q(x-y)\ .$$

Moreover, the two possible q-derivatives can be introduced as follows:

$$D_x F(x) = \lim_{y\to x}\frac{F(x)-F(y)}{x\oslash_q y} = [1+(1-q)x]\frac{dF}{dx}\ ,$$
$$\tilde{D}_x F(x) = \lim_{y\to x}\frac{F(x)\oslash_q F(y)}{x-y} = \frac{1}{1+(1-q)F(x)}\frac{dF}{dx}\ .$$

It is not difficult to find that

$$\tilde{D}_x F(x) = \frac{1}{[1+(1-q)x][1+(1-q)F(x)]}D_x F(x)\ .$$

With these definitions of new derivatives, one can establish the multiplicative rule of the derivative as follows:

$$
\begin{aligned}
D_x[F(x)G(x)] &= D_x[F(x)]G(x) + F(x)D_x[G(x)] \,, \\
\tilde{D}_x[F(x)G(x)] &= \frac{1}{1+(1-q)F(x)G(x)}\Big([1+(1-q)F(x)]\tilde{D}_x[F(x)]G(x) \\
&\quad +[1+(1-q)G(x)]F(x)D_x[G(x)]\Big) \,.
\end{aligned}
$$

The organisation of the paper follows . In Sect. 2, the one-parameter generalisation of the Hamiltonian will be fully derived through the backward engineering to solving Hamilton's equation in the case of the one degree of freedom. One interesting point is that this one-parameter Hamiltonian is coincidentally in Tsallis's form. The one-parameter generalisation of the Lagrangian is immediately obtained by using the Legendre transformation. However, one-parameter generalised Lagrangian can also be directly obtained from the inverse problem of calculus of variation. The recipe for constructing the two and more parameters version of the Hamiltonian and Lagrangian will be provided. In Sect. 3, many types of entropies will be briefly reviewed, starting from the Boltzmann–Gibbs entropy which explained the uncertainty in the context of statistical mechanics to the Shannon entropy which indicates the uncertainty in the context of information theory. The Kullback–Leibler divergence or relative entropy will be also mentioned. After that, the one-parameter generalisation of the Shannon entropy called the Renyi entropy will be discussed. Also, the one-parameter generalisation of the Boltzmann–Gibbs entropy called the Tsallis entropy is discussed. The two-parameter generalisation of the Boltzmann–Gibbs entropy and Kullback–Leibler divergence will be immediately given right after. In Sect. 4, the basic Fisher information will firstly be introduced together with the Cramer–Rao inequality. The connection between the Fisher information and the action functional in the case of one degree of freedom will be discussed. Later, employing the result in Sect. 2, the one-parameter Fisher information will be derived as well as its properties. In Sect. 5, the conclusion and outlook will be provided.

2 The *n*-Parameter Generalised Hamiltonian and Lagrangian

In physics, Hamiltonian and Lagrangian are commonly used to study the dynamics of the system. On one hand, the Hamiltonian mechanics gives a picture of the trajectory on the cotangent bundle(phase space) constituted by a set of conjugate momentum and generalised coordinates $(\boldsymbol{p}, \boldsymbol{x})$. On the other hand, the Lagrangian mechanics provides the dynamics of the system on the configuration space subject to the least action principle.

With a set of generalised coordinates, one can construct the dual space of the cotangent bundle called the tangent bundle by replacing the conjugate momentum

$\boldsymbol{p}$ with the generalised velocities $\dot{\boldsymbol{x}}$. However, these two approaches provide the equivalent description to the system.

2.1 One-Parameter Hamiltonian

In this subsection, we will provide a construction to obtain the one-parameter generalisation of the Hamiltonian. Here, we are interested only in the case of a system of mass m with one degree of freedom. With the given Hamiltonian $H(p, x) = T(p) + V(x)$, where $T(p) = p^2/2m$ is the kinetic energy and $V(x)$ is the potential energy, we have a system of two coupled first-order equations

$$\dot{q} = \frac{\partial H}{\partial p}\ , \quad -\dot{p} = \frac{\partial H}{\partial x}\ . \tag{2}$$

These two equations would give the Newton equation

$$\ddot{x} = \frac{dV}{dx}\ . \tag{3}$$

However, (2) could be combined into a single equation

$$\frac{\partial^2 H}{\partial x \partial p}\frac{\partial H}{\partial p} - \frac{\partial^2 H}{\partial^2 p}\frac{\partial H}{\partial x} + \frac{1}{m}\frac{\partial H}{\partial x} = 0\ . \tag{4}$$

It is not difficult to check that the Hamiltonian $H(p, x) = T(p) + V(x)$ is a solution of (4). What we are going to look for is whether there exist other Hamiltonians satisfying (4) subject to the equation of the motion (3). Now we take the Hamiltonian in the form $\widetilde{H}(p, x) = K(p)W(x)$, where $K(p)$ and $W(x)$ are to be determined [40]. Inserting $\widetilde{H}(p, x)$ into (4), one obtains

$$\frac{d^2K}{dp^2} + \frac{1}{m\dot{p}W}\frac{dW}{dx}\left(p\frac{dK}{dp} + K\right) = 0\ . \tag{5}$$

To solve (5), we set

$$\frac{1}{m\dot{p}W}\frac{dW}{dx} = C\ ,$$

where C is a constant to be determined. Imposing the equation of motion (3), it is easy to obtain the W given by

$$W(x) = \beta_1 e^{-mCV(x)}\ ,$$

where β_1 is another constant to be determined. Now (5) becomes

$$\frac{d^2K}{dp^2} + C\left(p\frac{dK}{dp} + K\right) = 0$$
$$\frac{dK}{dp} + CpK = 0$$
$$\int \frac{dK}{K} = -W\int pdp$$
$$K(p) = \beta_2 e^{-\frac{Cp^2}{2}} \ ,$$

where β_2 is also a constant to be determined. What we have now for the Hamiltonian $\widetilde{H}(p,x)$

$$\widetilde{H}(p,x) = \kappa e^{-\frac{Cp^2}{2} - mCV(x)} \ , \tag{6}$$

where $\kappa = \beta_1\beta_2$ is a new constant. We find that with the energy unit constraint the suitable choice for constants are $\kappa = -m\lambda^2$ and $C = \frac{1}{m^2\lambda^2}$, where λ has the velocity unit. Then now the Hamiltonian (6) becomes

$$H_\lambda(p,x) \equiv \widetilde{H}(p,x) = -m\lambda^2 e^{-\frac{H(p,x)}{m\lambda^2}} \ . \tag{7}$$

To see the choice of these constants consistence, we consider the limit on the parameter λ approaching to infinity such that

$$\lim_{\lambda\to\infty} H_\lambda(p,x) = H(p,x) + m\lambda^2 \ .$$

We find that the standard Hamiltonian can be recovered. The extra constant $m\lambda^2$ does not contribute to the dynamics of the system. It can also be seen that the Hamiltonian (7) gives the equation of motion (3) by substituting H_λ into (4). Here we can treat the Hamiltonian $H_\lambda(p,x)$ as the one-parameter generalisation of the standard Hamiltonian $H(p,x)$.

2.2 One-Parameter Lagrangian

In this subsection, we continue to construct the one-parameter generalisation of the Lagrangian. We shall start by given the Euler–Lagrange equation

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \ , \tag{8}$$

where $L(\dot{x},x) = T(\dot{x}) - V(x)$ is the Lagrangian and $T(\dot{x}) = m\dot{x}^2/2$ is the kinetic energy. Equation (8) can be rewritten as

$$\frac{\partial L}{\partial x} - \frac{\partial^2 L}{\partial \dot{x}^2}\ddot{x} - \frac{\partial^2 L}{\partial \dot{x}\partial x}\dot{x} = 0\ . \tag{9}$$

Of course, the Lagrangian $L(\dot{x}, x)$ is a solution of (9). We again ask the question whether there exist other Lagrangians satisfying (9). We introduce the Lagrangian $\widetilde{L}(\dot{x}, x) = F(\dot{x})G(x)$, where $F(\dot{x})$ and $G(x)$ have be to be determined. Substituting $\widetilde{L}(\dot{x}, x)$ into (9), one obtains

$$\frac{1}{\ddot{x}G}\frac{dG}{dx}\left(F - \dot{x}\frac{dF}{d\dot{x}}\right) - \frac{d^2F}{d\dot{x}^2} = 0\ . \tag{10}$$

Next, we set

$$\frac{1}{\ddot{x}G}\frac{dG}{dx} = A\ , \tag{11}$$

where A is a constant to be determined. Equation (11) can be easily solved constrained with the equation of motion (3) resulting in

$$G(x) = \alpha_1 e^{-\frac{AV(x)}{m}}\ ,$$

where α_1 is a constant to be determined. Now Eq. (10) becomes

$$A\left(F - \dot{x}\frac{dF}{d\dot{x}}\right) - \frac{d^2F}{d\dot{x}^2} = 0\ . \tag{12}$$

To solve Eq. (12), one can try a generating function technique. We find that $F = \dot{x}$ is a particular solution of (12). Then, we propose another solution $\widetilde{F} = w(\dot{x})\dot{x}$, where $w(\dot{x})$ is to be determined. Inserting $\widetilde{F}$ into (12), we obtain

$$(2 + A\dot{x}^2)w' + \dot{x}\frac{dw'}{d\dot{x}} = 0, \quad \text{where} \quad w' = \frac{dw}{dx}\ .$$

One can solve for w' as

$$w' = \frac{\alpha_2 e^{-\frac{A\dot{x}^2}{2}}}{\dot{x}^2}\ ,$$

and of course

$$w = \alpha_2\left(\frac{e^{-\frac{A\dot{x}^2}{2}}}{\dot{x}} + A\int_0^{\dot{x}} e^{-\frac{A\dot{v}^2}{2}}dv\right) + \alpha_3\ ,$$

where α_2 and α_3 are constants to be determined. Then the function $F(\dot{x})$ is

$$F(\dot{x}) = \alpha_3\dot{x} + \alpha_2\left(e^{-\frac{A\dot{x}^2}{2}} + A\dot{x}\int_0^{\dot{x}} e^{-\frac{A\dot{v}^2}{2}}dv\right)\ ,$$

and the Lagrangian $\widetilde{L}(\dot{x}, x)$ is

$$\widetilde{L}(\dot{x}, x) = \left[k_1\dot{x} + k_2\left(e^{-\frac{A\dot{x}^2}{2}} + A\dot{x}\int_0^{\dot{x}} e^{-\frac{A\dot{v}^2}{2}}dv\right)\right]e^{\frac{AV(x)}{m}} ,$$

where $k_1 = \alpha_1\alpha_2$ and $k_2 = \alpha_1\alpha_3$ are constants to be determined. Again with the energy unit constraint, one may choose $A = \lambda^{-2}$, k_1 and $k_2 = -m\lambda^2$ resulting in

$$L_\lambda(\dot{x}, x) \equiv \widetilde{L}(\dot{x}, x) = m\lambda^2\left(e^{-\frac{\dot{x}^2}{2\lambda^2}} + \frac{\dot{x}}{\lambda^2}\int_0^{\dot{x}} e^{-\frac{\dot{v}^2}{2\lambda^2}}dv\right)e^{\frac{V(x)}{m\lambda^2}} .$$

If we consider the limit on the parameter λ approaching to infinity such that

$$\lim_{\lambda\to\infty} = L(\dot{x}, x) + m\lambda^2$$

the standard Lagrangian is recovered. The Lagrangian $L_\lambda(\dot{x}, x)$ can be treated as the one-parameter generalisation of the standard Lagrangian $L(\dot{x}, x)$ and it is also not difficult to see that this Lagrangian produces the equation of motion (3).

Now we introduce the momentum variable

$$p_\lambda = \frac{\partial L_\lambda}{\partial \dot{x}} = m\left(\int_0^{\dot{x}} e^{-\frac{\dot{v}^2}{2\lambda^2}}dv\right)e^{\frac{V(x)}{m\lambda^2}} ,$$

where $\lim_{\lambda\to\infty} p_\lambda = p = m\dot{x}$. With this new momentum variable, we introduce the Legendre transformation

$$H_\lambda(p, x) = p_\lambda\dot{x} - L_\lambda(\dot{x}, x) ,$$

where $H_\lambda(p, x)$ is the one-parameter Hamiltonian obtained in the previous section.

2.3 Lagrangian and Hamiltonian Hierarchies

In the previous subsections, we obtain the one-parameter generalisation of the Hamiltonian and Lagrangian. With the appropriate limit on the parameter, the standard Hamiltonian and Lagrangian can be recovered. Here in this section, we are going to consider the expansion the $H_\lambda(p, x)$ and $L_\lambda(\dot{x}, x)$ with respect to the parameter λ. In doing so, we first consider the expansion of the Lagrangian $L_\lambda(\dot{x}, x)$ and we obtain

$$L_\lambda(\dot{x}, x) = \sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{-1}{m\lambda^2}\right)^{j-1} L_j(\dot{x}, x) ,$$

where

$$L_j(\dot{x}, x) = \sum_{k=0}^{j} \left(\frac{j! T(\dot{x})^{j-k} V(x)^k}{(j-k)! k! (2j - (2k+1))} \right) \ , \quad \text{where} \quad j = 0, 1, 2, 3, \ldots \quad (13)$$

The interesting point is that these Lagrangians all produce the equation of motion (3). This can be seen by substituting (13) into (9). Then we might consider the Lagrangian $L_\lambda(\dot{x}, x)$ as the generating function for the Lagrangian hierarchy (13). Furthermore, we also find that

$$L_{j-1} = \frac{1}{j}\frac{\partial L_j}{\partial V} \ , \quad \text{and} \quad L_1 = \frac{1}{j!}\frac{\partial^{j-1} L_j}{\partial V^{j-1}}$$

which provide the connection between Lagrangian member in the hierarchy. Of course, in the standard language, the Lagrangian is not unique up to addition of the total derivative of the function $\mathcal{F}(x, t)$ with respect to time. Here we provide alternative and systematic way to produce infinite Lagrangian producing the same equation of motion.

Similarly, we can also generate the Hamiltonian hierarchy by employing the Legendre transformation

$$H_j(p, x) = \dot{x}\frac{\partial L_j}{\partial \dot{x}} - L_j(\dot{x}, x) = (T(p) + V(x))^j = H^j(p, x)$$

and the Hamiltonian $H_\lambda(p, x)$ can be expressed as

$$H_\lambda(p, x) = \sum_{j=1}^{\infty} \frac{1}{j!} \left(\frac{-1}{m\lambda^2} \right)^{j-1} H_j(p, x) \ .$$

In the same line of thought, one could treat the Hamiltonian $H_\lambda(p, x)$ as a generating function for the Hamiltonian hierarchy giving the equation of motion (3).

Remark 1

We see that when we consider the limit on the parameter approaching to infinity for both Hamiltonian $H_\lambda(p, x)$ and $L_\lambda(\dot{x}, x)$, there will be a remaining extra constant $m\lambda^2$. This extra constant can be eliminated if we modify the $H_\lambda(p, x)$ and $L_\lambda(\dot{x}, x)$ as follows:

$$H_\lambda(p, x) = -m\lambda^2 \left(e^{-\frac{H(p,x)}{m\lambda^2}} - 1 \right) \ ,$$

$$L_\lambda(\dot{x}, x) = m\lambda^2 \left(e^{-\frac{E(\dot{x},x)}{m\lambda^2}} + \frac{\dot{x}}{\lambda^2} \int_0^{\dot{x}} e^{-\frac{E(\dot{v},x)}{m\lambda^2}} dv - 1 \right) \ ,$$

where $E(\dot{x}, x) = m\dot{x}^2/2 - V(x)$.

2.4 Two-Parameter Hamiltonian

In previous subsection, we show how to construct the one-parameter generalisation of the Hamiltonian from solving Eq. (4) with the constraint (3). In this subsection, we will push forward the idea of generalisation of the Hamiltonian to the two-parameter case. We now introduce an ansatz Hamiltonian given by

$$H_\kappa(p,x) = \frac{1}{\kappa}\left(e^{\kappa F(p,x)} - 1\right) , \tag{14}$$

where κ is constant to be determined and $F(p,x)$ is a function defined on the phase space and has to be also determined. Substituting (14) into (4), we obtain

$$0 = \frac{1}{m}\frac{\partial F}{\partial x} + \dot{p}\frac{\partial^2 F}{\partial p^2} + \frac{p}{m}\frac{\partial^2 F}{\partial p \partial x} + \kappa_1\left[\dot{p}\left(\frac{\partial F}{\partial p}\right)^2 + \frac{p}{m}\frac{\partial F}{\partial p}\frac{\partial F}{\partial x}\right] . \tag{15}$$

We find that if one takes $F(p,x) = H(p,x)$ as the standard Hamiltonian, the first three terms in (15) are identical to (4). The last bracket must vanish and gives an extra relation

$$0 = \dot{p}\frac{\partial H}{\partial p} + \frac{p}{m}\frac{\partial H}{\partial x} \mapsto 0 = \dot{p}\frac{\partial H}{\partial x} + \dot{x}\frac{\partial H}{\partial x} = \frac{dH}{dt} , \tag{16}$$

which is nothing but the conservation of the energy. Furthermore, (4) can be obtained from (16) by considering the partial derivative with respect to the p-variable

$$\frac{\partial}{\partial p}\left(\dot{p}\frac{\partial H}{\partial p} + \frac{p}{m}\frac{\partial H}{\partial x}\right) = \frac{\partial^2 H}{\partial x \partial p}\frac{\partial H}{\partial p} - \frac{\partial^2 H}{\partial^2 p}\frac{\partial H}{\partial x} + \frac{1}{m}\frac{\partial H}{\partial x} = 0 .$$

This means that (4) is a consequence of (16) and, of course, (16) is simpler to solve the Hamiltonian.

2.4.1 Case 1. Additive Case

If we take the function $F(p,x) = T(p) + V(x)$ as the standard Hamiltonian and substitute into (16), we obtain

$$0 = \dot{p}\frac{dT}{dp} + \frac{p}{m}\frac{dV}{dx} .$$

Using (3), one can write

$$0 = \dot{p}\left(\frac{dT}{dp} - \frac{p}{m}\right) .$$

Since $\dot{p} \neq 0$, the bracket must vanish resulting in

$$\int dT = \int \frac{pdp}{m} \mapsto T(p) = \frac{p^2}{2m} + D \ ,$$

where D is a constant of integration which can be set to be zero for convenience. Then the additive function $F(p, x)$ is nothing but the standard Hamiltonian.

2.4.2 Case 2. Multiplicative Case

If we take the function $F(p, x) = K(p)W(x)$ (as we did previously), we obtain

$$0 = W\left[\dot{p}\left(W\frac{dK}{dp}\right) + \frac{p}{m}\left(K\frac{dB}{dx}\right)\right]$$

$$\frac{m}{Kp}\frac{dK}{dp} = \frac{1}{B}\frac{dB}{dV} \ . \tag{17}$$

We see that both sides of (17) are independent. Then equation holds if both sides equal to a constant δ. We first consider the left-hand side of (17)

$$\frac{m}{Kp}\frac{dK}{dp} = \delta \mapsto \int \frac{dK}{K} = \delta \int \frac{pdp}{m} \mapsto K(p) = ae^{\delta T(p)} \ ,$$

where a is a constant to be determined. Next, we consider the right-hand side of (17)

$$\frac{1}{B}\frac{dB}{dV} = \delta \mapsto \int \frac{dB}{B} = \delta \int dV \mapsto B(x) = be^{\delta V(x)} \ ,$$

where b is a constant to be determined. Then the function $F(p, x)$ is

$$F(p, x) = ce^{\delta H(p,x)} \ , \tag{18}$$

where $c = ab$ is a new constant. In fact, the function $F(p, x)$ is the one-parameter Hamiltonian in (6) with the choice $c = -m\lambda^2$ and $\delta = -1/m\lambda^2 = 1/c$.

Remark 2

Then we now could write

$$F(p, x) = H_\delta(p, x) = \frac{1}{\delta}\left(e^{\delta H(p,x)} - 1\right) \ . \tag{19}$$

The extra -1 comes from the fact that we would like to eliminate the extra constant when the parameter δ is approaching to zero as we mentioned in the Remark 1. The interesting point is that if we set $\delta = 1 - q$, where q is a new parameter. The

Hamiltonian (19) becomes

$$H_q(p,x) = \frac{1}{1-q}\left(e^{(1-q)H(p,x)} - 1\right) \ . \tag{20}$$

One might recognise that actually this Hamiltonian (20) is in the same form with the Tsallis entropy [52]. Hence the Hamiltonian (20) can be considered as the q-deformation of the standard Hamiltonian. One more interesting fact is that, for two separated systems, the Hamiltonian takes the additive form $H^{12}(p_1, p_2, x_1, x_2) = H^1(p_1, x_1) + H^2(p_2, x_2)$. However, with the Hamiltonian (20), one finds that

$$H_q^{12}(p_1, p_2, x_1, x_2) = H_q^1(p_1, x_1) + H_q^2(p_2, x_2) + (1-q)H_q^1(p_1, x_1)H_q^2(p_2, x_2) \ ,$$

where the relations $e^{(1-q)H^j(p_j,x_j)} = (1-q)H_q^j(p_j, x_j) + 1, \quad j = 1, 2$ are applied. Here is nothing but the non-additive analog of the Tsallis entropy. However, we are going to discuss about the entropy in great detail later section.

Now we insert (19) into (14) and we obtain

$$H_{\delta_1,\delta_2} = \frac{1}{\delta_1}\left(e^{\frac{\delta_1}{\delta_2}\left(e^{\delta_2 H(p,x)}-1\right)} - 1\right) = \frac{1}{\delta_1}\left(e^{\delta_1 H_{\delta_2}} - 1\right) \ , \tag{21}$$

where $\kappa = \delta_1$ and $\delta = \delta_2$. Furthermore, we find that

$$\lim_{\delta_2\to\infty}\lim_{\delta_1\to\infty} H_{\delta_1,\delta_2}(p,x) = H(p,x)$$

the standard Hamiltonian can be recovered. Then we succeed to construct the two-parameter generalisation of the Hamiltonian. Furthermore, one may replace the exponential with the q-exponential (1) in the Hamiltonian (19) resulting in

$$H_{q,r}(p,x) = \frac{1}{1-r}\left(e_q^{(1-r)H(p,x)} - 1\right) = \frac{1}{1-r}\left[(q+r-qr)H^{\frac{1}{1-q}}(p,x) - 1\right] \ .$$

Therefore, this Hamiltonian can also be treated as the two-parameter generalisation of the standard Hamiltonian. For $q \to 1$, the one-parameter Hamiltonian (19) is recovered.

2.5 Two-Parameter Lagrangian

To obtain the two-parameter generalisation of the Lagrangian, we will employ the Legendre transformation instead of solving it directly. In doing so, we introduce a new momentum variable p_{δ_1,δ_2} which can be obtained from Hamilton's equation

$$-\dot{p}_{\delta_1,\delta_2} = \frac{\partial H_{\delta_1,\delta_2}}{\partial x} = e^{\delta_1 H_{\delta_2}} e^{\delta_2 H} \frac{dV}{dx} \ .$$

Using the equation of motion (3), we obtain

$$\dot{p}_{\delta_1,\delta_2} = \dot{p} e^{\delta_1 H_{\delta_2}} e^{\delta_2 H} \mapsto p_{\delta_1,\delta_2} = m \int_0^{\dot{x}} e^{\delta_2 E(v,x)} e^{\frac{\delta_1}{\delta_2}\left(e^{\delta_2 E(v,x)}-1\right)} dv \ . \tag{22}$$

Next, we introduce the Legendre transformation

$$L_{\delta_1,\delta_2}(\dot{x}, x) = p_{\delta_1,\delta_2}\dot{x} - H_{\delta_1,\delta_2}(p, x) \ .$$

Using (21) and (22), one obtains

$$L_{\delta_1,\delta_2}(\dot{x}, x) = \frac{1}{\delta_1}\Bigg[e^{\frac{\delta_1}{\delta_2}\left(e^{\delta_2 E(\dot{x},x)}-1\right)} + m\dot{x}\delta_1 \int_0^{\dot{x}} e^{\delta_2 E(v,x)} e^{\frac{\delta_1}{\delta_2}\left(e^{\delta_2 E(v,x)}-1\right)} dv - 1 \Bigg] .$$

Again, we find that

$$\lim_{\delta_2 \to \infty} \lim_{\delta_1 \to \infty} L_{\delta_1,\delta_2}(\dot{x}, x) = L(\dot{x}, x) \ . \tag{23}$$

Then the Lagrangian (1) can be treated as the two-parameter generalisation of the standard Lagrangian $L(\dot{x}, x)$.

2.6 n-Parameter Hamiltonian and Lagrangian

Up to this point, it might not be difficult to see how we could construct the three-parameter generalisation of the Hamiltonian as well as the Lagrangian. Again we start with the ansatz form of the Hamiltonian such that

$$H_{\delta_1,\delta_2} = \frac{1}{\delta_1}\left(e^{\frac{\delta_1}{\delta_2}\left(e^{\delta_2 F(p,x)}-1\right)} - 1 \right) \ , \tag{24}$$

where $F(p, x)$ is again the function defined on the phase space. Of course, we could use (16) to solve for $F(p, x)$. However, we could skip all the steps and the function $F(p, x)$ could take the form of the standard Hamiltonian $F(p, x) = H(p, x)$ and (24) is nothing the two-parameter Hamiltonian. In the case that the function $F(p, x) = H_{\delta_3}$ is one-parameter Hamiltonian, (24) could give

$$H_{\delta_1,\delta_2,\delta_3} = \frac{1}{\delta_1}\left(e^{\frac{\delta_1}{\delta_2}\left(e^{\frac{\delta_2}{\delta_3}H_{\delta_3}(p,x)}-1\right)} - 1\right)$$
$$= \frac{1}{\delta_1}\left(e^{\frac{\delta_1}{\delta_2}\left(e^{\frac{\delta_2}{\delta_3}\left(e^{\delta_3 H(p,x)}-1\right)}-1\right)} - 1\right) , \qquad (25)$$

and with the suitable sequence of the limit

$$\lim_{\delta_3\to\infty}\lim_{\delta_2\to\infty}\lim_{\delta_1\to\infty} H_{\delta_1,\delta_2,\delta_3}(p,x) = H(p,x) \qquad (26)$$

the standard Hamiltonian is recovered.

Now we give the Legendre transformation

$$L_{\delta_1,\delta_2,\delta_3}(\dot{x},x) = p_{\delta_1,\delta_2,\delta_3}\dot{x} - H_{\delta_1,\delta_2,\delta_3}(p,x) .$$

The momentum variable can be solved with the same method provided in the previous subsection resulting in

$$p_{\delta_1,\delta_2,\delta_3} = m\int_0^{\dot{x}} e^{\delta_3 E(v,x)} e^{\frac{\delta_3}{\delta_2}e^{\delta_2 E(v,x)}} e^{\frac{\delta_1}{\delta_2}e^{\frac{\delta_2}{\delta_3}(\delta_3 E(v,x)-1)}} dv . \qquad (27)$$

Then the three-parameter Lagrangian is given by

$$L_{\delta_1,\delta_2,\delta_3}(\dot{x},x) = \frac{1}{\delta_1}\Bigg[e^{\frac{\delta_1}{\delta_2}e^{\frac{\delta_2}{\delta_3}(\delta_3 E(\dot{x},x)-1)}} + m\dot{x}\delta_1\int_0^{\dot{x}} e^{\delta_3 E(v,x)} e^{\frac{\delta_3}{\delta_2}e^{\delta_2 E(v,x)}}$$
$$\times\, e^{\frac{\delta_1}{\delta_2}e^{\frac{\delta_2}{\delta_3}(\delta_3 E(v,x)-1)}} dv - 1\Bigg]$$

and

$$\lim_{\delta_3\to\infty}\lim_{\delta_2\to\infty}\lim_{\delta_1\to\infty} L_{\delta_1,\delta_2,\delta_3}(\dot{x},x) = L(\dot{x},x) \qquad (28)$$

the standard Lagrangian is recovered.

In order to construct the four-parameter generalisation of the Hamiltonian, one could repeat the whole process again with the ansatz Hamiltonian

$$H_{\delta_1,\delta_2,\delta_3} = \frac{1}{\delta_1}\left(e^{\frac{\delta_1}{\delta_2}\left(e^{\frac{\delta_2}{\delta_3}e^{\delta_3 F(p,x)}}-1\right)} - 1\right) \qquad (29)$$

and insert the function $F(p,x) = \frac{1}{\delta_4}e^{\delta_4 H(p,x)}$. Of course the four-parameter Lagrangian can be obtained by using the Legendre transformation

$$L_{\delta_1,\delta_2,\delta_3,\delta_4}(\dot{x},x) = p_{\delta_1,\delta_2,\delta_3,,\delta_4}\dot{x} - H_{\delta_1,\delta_2,\delta_,,\delta_4}(p,x)\ .$$

Then if we keep continuing this recursive process, the n-parameter generalisation of the Hamiltonian and Lagrangian can be obtained.

Remark 3

We would like to point out that there are possible possibilities to construct the one-parameter generalisation of the Hamiltonian. This can be seen if one takes the ansatz Hamiltonian

$$H_\sigma(p,x) = F(p,x)\left(e^{\sigma F(p,x)} - 1\right)\ , \tag{30}$$

where σ is a constant to be determined. Substituting (30) into (4), we obtain

$$0 = \frac{1}{m}\frac{\partial F}{\partial x} + \dot{p}\frac{\partial^2 F}{\partial p^2} + \frac{p}{m}\frac{\partial^2 F}{\partial p \partial x} + \sigma\frac{2+\sigma F}{1+\sigma F}\left(\dot{p}\left(\frac{\partial F}{\partial p}\right)^2 + \frac{\partial F}{\partial p}\frac{\partial F}{\partial x}\right)\ . \tag{31}$$

Again, if we take $F(p,x) = H(p,x)$ as the standard Hamiltonian, the first three terms in (31) give (4) and consequently the bracket gives (16). Then (30) reads

$$H_\sigma(p,x) = H(p,x)\left(e^{\sigma H(p,x)} - 1\right)\ , \tag{32}$$

where σ has an inverse energy unit. Obviously, the limit that the parameter σ is approaching to zero gives the standard Hamiltonian.

3 Generalised Entropies

One of the most important quantities in physics known as the entropy had been around since the dawn of the thermodynamics. In more specific, this entropy first was introduced by Clausius [44] in order to capture the second law of thermodynamics, which is the statement of the impossible process, measuring the amount of energy in a system that cannot be used to deliver work. This notion of thermodynamic entropy is a bit abstract involving heat getting in and going out between system and environment with a given temperature. Later, Boltzmann [45, 46] and Gibbs [47] provided intriguing insight for the system in the level of microscopic realm, pioneered understanding the system emerging from the probabilistic feature known as the statistical mechanics. The statistical entropy measures the variety of the microscopic configuration of the system. Both thermodynamic entropy and statistical entropy provide an equivalent picture, but from different perspectives, how to describe the system according to the second law of thermodynamics. Coming from remote area, Shannon [19] accidentally found a quantity, which later is known as

Shannon entropy, as a measure of surprisal or average information gained from the system when he had built his theory of communication.

3.1 Boltzmann–Gibbs Entropy

Suppose a system is described by a large number of microstates called an ensemble Ω in the phase space and the associated probability of the system being in microstate k is p_k given by

$$p_k = \frac{e^{-\frac{E_k}{k_B T}}}{Z} \ , \quad \text{with} \quad \sum_k p_k = 1 \ ,$$

where E_k is the energy of the microstate, k_B is Boltzmann's constant and Z is the partition function. The Gibbs entropy is given by

$$S_{BG}(p) = -k_B \sum_i p_i \ln p_i \ . \tag{33}$$

Here the summation is over all possible microstates available in the ensemble. In the case that all microstates have the same probability distribution in the ensemble, $p_i = 1/W$, where W is the statistical weight or the number of all possible microstates, the entropy in (33) becomes

$$S_B(W) = k_B \ln W \ , \tag{34}$$

which is the Boltzmann entropy. One can say that the Boltzmann entropy is the upper bound of the Gibbs entropy $S_G \leq S_B$. One could think that with only given a number of microstates, no internal structure provided which in this sense is the probability distribution, the Boltzmann entropy is the only available measure. On the other hand, if the probability associated with each microstate is provided, the Gibbs entropy is a suitable measure.

To make it clear, let us provide a simple example as illustrated in Fig. 1. There is 2-d box that is divided into four small rooms containing a single atom of gas. Obviously, there are four possible configurations of the atom that is located in the box, i.e. {room1, room2, room3, room4}. Then Boltzmann's entropy reads $S_B = k_B \ln 4$. Here we assume the probability that the atom will be in each room is the same or uniform distribution. However, if we deal with the bias probability distribution, such that $\{p_1 = 1/2, p_2 = 0, p_3 = 1/2, p_4 = 0\}$, Gibbs' entropy is $S_G = k_B \ln 2$ and of course $S_G < S_B$.

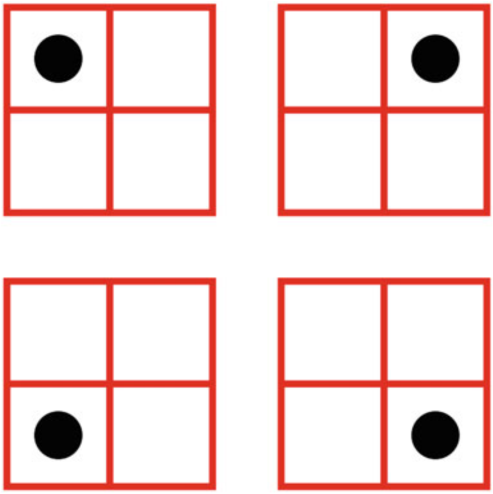

Fig. 1 Four possible configurations of a single atom

3.2 *Shannon Entropy*

Given a set of random variable $X = \{x_1, x_2, \ldots, x_W\}$ with associated probability distribution $P = \{p_1, p_2, \ldots, p_W\}$ and $\sum_k p_k = 1$, the Shannon entropy is given by

$$H(p) = -\sum_{i=1}^{W} p_i \log_2 p_i \ . \tag{35}$$

The unit of the Shannon entropy is "bit" as indicated by using the logarithm function base 2. Here if we define the surprisal or Shannon information for each of the ith outcome as $I_i = -\log_2 p_i$, what we have now is an ensemble of the surprisal $\{I_1, I_2, \ldots, I_W\}$. Then the Shannon entropy is a linear average of the surprisal

$$H(p) = \langle I_X \rangle = \sum_{i=1}^{W} p_i I_i \ . \tag{36}$$

Next, let us provide a nice example for understanding the role of the surprisal and the Shannon entropy, namely, tossing a coin game. There are two outcomes resulting in the set of random variable $X = \{\text{Head, Tail}\}$. If the coin is equally weighted both sides, 50% chance for getting head and 50% chance for getting tail, the surprisal for each outcomes is $I_i = -\log_2 2$, where $i = \text{Head, Tail}$. Then the average of the Shannon information or surprisal over all possible outcomes is $H = \log_2 2 = 1$ bit. This one bit of Shannon entropy is the maximum in this situation of the tossing coin game and it means that you are likely to be surprised because you hardly guess the outcome of the tossing coin. Next, we move to the situation with an unfair weighted coin saying 20% chance for getting head and 80% for getting tail. We find that $I_{\text{Head}} = -\log_2 5$ and $I_{\text{Tail}} = -(4/5)\log_2(5/4)$ and the Shannon entropy is $H \approx 0.721$ bit. With this value of the Shannon entropy, it is telling that you are likely to have a better chance to get a correct answer from guessing the outcome. In the extreme case biased coin, i.e. 100% chance for getting Head, the Shannon entropy is zero. This means that you learn noting from the outcome of tossing coin game since you perfectly guess for the outcome.

The last thing that we would like to mention is some properties of the Shannon entropy. It is not difficult to see that the Shannon entropy is a concave function of the probabilities p_i, where $i = 1, 2, \ldots, W$. To see this, we find

$$\frac{\partial H}{\partial p_k} = -\log_2 p_k - 1 \ , \quad \text{and} \quad \frac{\partial^2 H}{\partial p_l \partial p_k} = -\frac{1}{p_i}\delta_{ij} \ .$$

Since $p_i \geq 0, \forall i$, then the Shannon entropy is concave. One last interesting point is that the second partial derivative of the Shannon with respect to p gives the Fisher–Rao metric which will be greatly discussed later. Another interesting fact is that the Shannon entropy can be generated from a given function $F_\alpha(p) \equiv \sum_{i=1}^{W} p_i^\alpha$ as follows:

$$H(p) = -\frac{d}{dx}\sum_{i=1}^{W} p_i^x \Big|_{x=1} \ . \tag{37}$$

This kind of connection provides us a basic idea to extend the entropy later.

3.3 Kullback–Leibler Divergence

In this subsection, we will focus on how can we distinguish two probability distributions over the same set of random variable X? Given two set of probability distributions $P = (p_1, p_2, \ldots, p_W)$ and $Q = (q_1, q_2, \ldots, q_W)$, the measure, called the Kullback–Leibler divergence (KL), has been widely used in this context given by

$$D_{KL}(p(x)||q(x)) = \sum_{x \in X} p(x) \ln \frac{p(x)}{q(x)} \ . \tag{38}$$

We can say that the KL divergence (38) of $q(x)$ from $p(x)$ is a measure of the information lost when the probability distribution q is used to approximate the probability distribution $p(x)$. The important thing is that the KL divergence is not symmetric. In the case of the continuous variables, the KL divergence is expressed by

$$D_{KL}(p(x)||q(x)) = \int dx p(x) \ln \frac{p(x)}{q(x)} \ . \tag{39}$$

Intuitively, the KL divergence can be viewed as the "how does far away" between two distributions on the probability manifold. To see this, let's look at (39) and put $q(x) = p(x) + dp(x)$, where $dp(x)$ is small. Then KL divergence (39) reads

$$D_{KL}(p||p + dp) = \int dxp(x) \ln \frac{p(x)}{p(x) + dp(x)} \,. \tag{40}$$

Expanding (40) with respect to dp and keeping only the first order, we obtain

$$D_{KL}(p||p + dp) \approx I(p) = \int dxp(x) \left(\frac{dp(x)}{dx}\right)^2 \,. \tag{41}$$

Here $I(p)$ is known as the Fisher information which will be discussed in the next section. However, since KL divergence is not symmetric, the KL divergence is not a distance measure, even though the lowest order in (41) does not affect. Nevertheless, $D_{KL}(p(x)||q(x)) = 0$ if $P = Q$ and $D_{KL}(p(x)||q(x)) \geq 0$ for $P \neq Q$.

3.4 Rényi Entropy: One-Parameter Generalisation of the Shannon Entropy

In Sect. 3.2, one may see how we use the quantity called the Shannon entropy as an information measure. However, the question is that are there any other types of information measures? Before we go to the answer, it is better to introduce a set of axioms [48] as the criterion for information measures to satisfy.

Axiom 1: The information measure $E = E(p_1, p_2, \ldots, p_W)$ depends only on the probabilities p_i of the events and nothing else.

Axiom 2: The information measure E takes on an absolute minimum for the uniform distribution $(p_1 = 1/W, p_2 = 1/W, \ldots, p_W = 1/W)$, and other probability distribution has an information content that is larger or equal to that of the uniform distribution

$$E(p_1 = 1/W, p_2 = 1/W, \ldots, p_W = 1/W) \geq E(p_1, p_2, \ldots, p_W) \,.$$

Axiom 3: The information measure I should not change if the sample set of events is enlarged by another event that has probability zero

$$E(p_1, p_2, \ldots, p_W) \leq E(p_1, p_2, \ldots, p_W, 0) \,.$$

Axiom 4: Suppose the system is composed of two subsystems x and y, not necessarily independent. The probabilities of the system x are labelled by $p^x = \{p_1^x, p_2^x, \ldots, p_W^x\}$, those of the system y are $p^y = \{p_1^y, p_2^y, \ldots, p_W^y\}$. The joint probabilities $p_{ij}^{xy} = p_i^x p^y(j|i)$, where $p^y(j|i)$ is the conditional probability of event j

in system y under the condition that event i has already happened in system x. The quantity $E(\{p^y(j|i)\})$ is the conditional information of the system y formed with the conditional probabilities $p^y(j|i)$

$$E(\{p_{ij}^{xy}\}) = E(\{p_i^x\}) + \sum_i p_i^x E(\{p^y(j|i)\}) \ . \tag{42}$$

This axiom points out that the information measure should be independent of the way the information is collected. To make it a bit more clear, let's put it this way. We can fist collect the information in the subsystem y, assuming a given event i in subsystem x, and then sum the result over all possible events i in subsystem x weighting with the probabilities p_i^x. In the case that subsystems x and y are independent the joint probability can be factorised as $p_{ij}^{xy} = p_i^x p_j^y$ and (42) can be simply reduced to the addition of information for independent subsystems

$$E(\{p_{ij}^{xy}\}) = E(\{p_i^x\}) + E(\{p_j^y\}) \ . \tag{43}$$

Nonnegative functions I satisfying axioms 1–3 are called generalised entropies [49]. Now we look for the nonnegative function I follows the additive rule (this means that the axiom 4 has been replaced with (43))

$$E_g(p_1, p_2, \ldots, p_W) = \sum_{i=1}^{W} g(p_i) \ ,$$

where $g : [0, 1] \mapsto \mathscr{R}^+$ has to satisfy these conditions

- g is continuous,
- g is concave,
- $g(0) = 0$,

in order to be the generalised entropy. Obviously, the Shannon entropy $H(x) = -x \log_2 x$, where $0 \leq x \leq 1$ satisfies these conditions. Of course, the Shannon entropy is just the linear average of the surprisal. Rényi went one more step further providing another possible average [50]. According to the definition of the extended average, we now write

$$E(p_1, p_2, \ldots, p_W) = F^{-1}\left(\sum_{i=1}^{W} p_i F(I_i)\right) \ , \tag{44}$$

where F is a continuous monotonic function and invertible function known as the Kolmogorov–Nagumo (KN) function. Demanding on preserving the additive rule, there are only two possible solutions for the F-function. The first one is $F(x) = x$ which is just a common arithmetic mean. The second solution is

$$F(x) = c_1 b^{(1-q)x} + c_2 \ , \tag{45}$$

where q is a real parameter. The c_1 and c_2 are constants to be determined. Equation (45) is known as the exponential mean. Substituting (45) into (44), we obtain

$$R_q(p) = \frac{1}{1-q} \log_2 \sum_{i=1}^{W} p_i^q \ , \tag{46}$$

where a parameter b is set to be 2 and $c_1 = -c_2 = 1/(1-q)$. The summation is over all events i with $p_i \neq 0$. Then Rényi entropies follow the axioms 1–3 with additive rule (43). For the limit $\lim_{q\to 1} R_q(p) = H(p)$, the Shannon entropy is recovered. Therefore, the Rényi entropy can be considered as one-parameter generalisation of the Shannon entropy.

Now we will rewrite the Rényi entropy (46) as

$$\begin{aligned} R_q(p) &= \frac{1}{1-q} \ln \left(\sum_{i=1}^{W} p_i e^{-(1-q)\ln p_i} \right) \\ &= \frac{1}{1-q} \ln \left(\sum_{i=1}^{W} p_i e^{(1-q)I_i} \right) \\ &= \frac{1}{1-q} \ln \langle e^{(1-q)I} \rangle \ . \end{aligned} \tag{47}$$

Equation (47) is identical to

$$X_\alpha = \sum_{j=1}^{\infty} \frac{\kappa_j(X)}{n!} (\alpha-1)^{n-1} = \frac{1}{\alpha-1} \ln \langle e^{(\alpha-1)X} \rangle \ ,$$

where $\kappa_j(X)$ are called the cumulant. This implies that the negative Rényi entropies are the effective values of the negative surprisal $-I$. Effectively, (47) can be expressed as

$$-R_q(p) = \sum_{j=1}^{\infty} \frac{\kappa_j(-I)}{n!} (\alpha-1)^{n-1} = \frac{1}{\alpha-1} \ln \langle e^{(\alpha-1)(-I)} \rangle \ . \tag{48}$$

For $n = 1$, the cumulant $\kappa_1 = \langle -I \rangle = \langle \ln p_i \rangle = \sum_{i=1}^{W} p_i \ln p_i$ is indeed the Shannon entropy. Then (48) could be written as

$$-R_q(p) = \sum_{i=1}^{W} p_i \ln p_i + \sum_{j=2}^{\infty} \frac{\kappa_j(-I)}{n!} (\alpha-1)^{n-1} \ . \tag{49}$$

The second term in (49) represents the fluctuation in the uncertainty [51].

Alternatively, one can define $||p||_\alpha^\alpha = \sum_{j=1}^{W} p_j^\alpha$ as the α-norm. Then Rényi entropies become

$$R_q(p) = \frac{1}{1-q} \ln ||p||_\alpha^\alpha .$$

The α-norm is a measure of the distance between the origin and any particular point on the simplex space.

3.5 Tsallis Entropy: One-Parameter Generalisation of the Boltzmann–Gibbs Entropy

In the previous subsection, the one-parameter generalisation of the Shannon entropy was discussed. Here, in this subsection, we discuss the extension of the Boltzmann–Gibbs entropy. First, we shall look at an important property of the Boltzmann–Gibbs entropy, namely, the additivity. Suppose again there are two subsystems as we introduced in the previous subsections x and y and they are independent. The joint probability is given by $p_{ij}^{xy} = p_i^x p_j^y$. The Boltzmann–Gibbs entropy reads

$$\begin{aligned} S_{BG}(p^{xy}) &= -k_B \sum_{i,j=1}^{W} p_i^x p_j^y \ln p_i^x p_j^y \\ &= -k_B \left(\sum_i p_j^y \sum_i p_i^x \ln p_i^x + \sum_i p_i^x \sum_i p_j^y \ln p_j^y \right) \\ &= S_{BG}(p^x) + S_{BG}(p^y) . \end{aligned}$$

This is known as an extensive property of the Boltzmann–Gibbs entropy for the equilibrium system. However, if the system is out of equilibrium, the Boltzmann–Gibbs entropy is no longer applicable. Then one might need to look for a new type of entropy. Tsallis [52] set out for this task and proposed a new form of the entropy[1]

$$T_q(p) = \frac{1}{1-q} \left(\sum_{i=1}^{W} p_i^q - 1 \right) ,$$

where $q \in \mathscr{R}$ is called the Tsallis index. One thing that makes the Tsallis entropy different from the Rényi entropy is the logarithm function. A relation between Rényi entropy and Tsallis entropy can be easily seen by expressing[2]

$$\sum_{i=1}^{W} p_i^q = 1 - (q-1)T_q(p) = e^{(q-1)R_q(p)} ,$$

[1] We choose to ignore the Boltzmann constant k_B and we shall name the Boltzmann–Gibbs–Shannon entropy.

[2] Here at this point, we prefer to express the Rényi entropy in terms of the natural logarithm.

resulting in

$$T_q(p) = \frac{1}{q-1}\left(1 - e^{(q-1)R_q(p)}\right) .$$

What we can see is that Tsallis entropy is a monotonous function of the Rényi entropy. Then any maximum of the Tsallis entropy will be the same maximum of the Rényi entropy. However, the Tsallis entropy possesses many nice properties comparing with the Rényi entropy. One of them is the concavity. This can be illustrated by looking at the derivative the Tsallis entropy with respect to p

$$\frac{\partial T_q(p)}{\partial p_i} = -\frac{q}{q-1}p_i^{q-1} , \quad \frac{\partial^2 T_q(p)}{\partial p_i \partial p_j} = -q p_i^{q-1}\delta_{ij} .$$

This is obvious that the Tsallis entropy $T_q(p)$ is concave for all $q > 0$, while the Rényi entropy does not possess such property since the second derivative with respect to p can be positive or negative. Another thing is that the Boltzmann–Gibbs–Shannon entropy is a special case as $\lim_{q\to 1} T_q(p) = S_{BG}(p) = H(p)$. Moreover, this limit coincides with the Rényi entropy as $\lim_{q\to 1} T_q(p) = \lim_{q\to 1} R_q(p) = S_{BG}(p) = H(p)$. The last point is that, as we mentioned at the beginning of this subsection about the additive rule, the Tsallis entropy is not additive. To see this, let us recall again the joint probability $p_{ij}^{xy} = p_i^x p_j^y$ of the two independent subsystems x and y. What we have now is the Tsallis entropy of the joint system

$$T_q(\{p_{ij}^{xy}\}) = \frac{1}{1-q}\left(\sum_{i,j=1}^{W}(p_{ij}^{xy})^q - 1\right) = \frac{1}{1-q}\left(\sum_{i=1}^{W}(p_i^x)^q \sum_{j=1}^{W}(p_j^x)^q - 1\right) . \tag{50}$$

Using the fact that $\sum_{i=1}^{W}(p_i^x)^q = 1-(q-1)T_q(\{p_i^x\})$ and $\sum_{i=1}^{W}(p_i^y)^q = 1-(q-1)T_q(\{p_i^y\})$, then (50) becomes

$$T_q(p^{xy}) = T_q(p^x) + T_q(p^y) - (q-1)T_q(p^x)T_q(p^y) . \tag{51}$$

This non-additive relation (52) will give the additive relation (43) when $q = 1$ as the Tsallis entropy becomes the Boltzmann–Gibbs–Shannon entropy. The interesting point is that (52) gives rise to a research field named the non-extensive statistical mechanics [53].

Another point of view is that the Tsallis entropy can be considered as the q-deformation of the Boltzmann–Gibbs–Shannon entropy as follows. Then if we replace the logarithm with q-logarithm (1) in the Boltzmann–Gibbs–Shannon entropy, we obtain

$$T_q(p_1, p_2, \ldots, p_W) = -\sum_{i=1}^{W} p_i \ln_q p_i = \frac{1}{1-q}\left(\sum_{i=1}^{W} p_i^q - 1\right) , \tag{52}$$

which is nothing but the Tsallis entropy. One last thing is that the Tsallis entropy can be generated from the function $F_\alpha(p) = \sum_{i=1}^{W} p_i^\alpha$ as we do have in (37) with the replacement of the q-derivative

$$T_q(p_1, p_2, \ldots, p_W) = -D_q \sum_{i=1}^{W} p_i^\alpha \Big|_{\alpha=1} .$$

3.6 Sharma–Mittal Entropy: Two-Parameter Generalisation of the Boltzmann–Gibbs Entropy

In this subsection, we provide further generalisation of the Tsallis–Rényi entropy named two-parameter generalisation of the Boltzmann–Gibbs entropy known as the Sharma–Mittal entropy [61]

$$S_{qr}(p) = \frac{1}{1-r}\left[\left(\sum_i p_i^q\right)^{\frac{1-r}{1-q}} - 1\right] , \tag{53}$$

where $r \in \mathscr{R}$ and $q \in \mathscr{R}$. The derivation of the entropy (53) is the following. What we have seen is that, for the Tsallis entropy, the KN function takes the form $F(x) = x$ and averages over the elementary information $I_i = -\ln_q p_i$. Then the Tsallis entropy is a linear average $T_q(p) = \langle -\ln_q p_i \rangle$. While, for the Rényi entropy, the KN function takes the form $F(x) = \ln_q e^x$ and averages over the $I_i = -\ln p_i$. Then the Rényi entropy is an exponential average $R_q(p) = \langle -\ln p_i \rangle_{\mathrm{exp}}$. These two entropies share the same quantity $F(I_i) = -\ln_q p_i$. This suggests that a simplest further generalisation is to consider the KN function in the form

$$F(x) = \ln_q e_r^x ,$$

where e_q^x is the generalisation q-exponential function given by (1). We also choose to write $I_i = -\ln_r p_i$. With these two requirements, one obtains

$$S_{qr} = \ln_r e_q^{-\sum_i p_i \ln_q p_i} = \langle -\ln_r p_i \rangle_{\text{q-exp}} .$$

We find that for $r \to 1$, the Rényi entropy is recovered and for $r \to q$, the Tsallis entropy is obtained. Furthermore, we also find that, with joint probability $p_{ij}^{xy} = p_i^x p_j^y$, the Sharma–Mittal entropy reads

$$S_{qr}(p^{xy}) = S_{qr}(p^x) + S_{qr}(p^y) + (1-r) S_{qr}(p^x) S_{qr}(p^y) .$$

This non-additive property tells that the Sharma–Mittal entropy can be considered as the generalisation of the Rényi entropy to the non-extensive context. Here the parameter r plays a role of degree of non-extensivity, while the parameter q indicates the deformation of the probability distribution.

3.7 Generalisation of the Kullback–Leibler Divergence

In this section, we will provide one of many possible ways to generalise the Kullback–Leibler divergence. Before going for that we would like to introduce another type of the one-parameter generalisation of the Boltzmann–Gibbs–Shannon entropy by Wang [56] based on the effective probability $\sum_{i=1}^{W} p_i^q = 1$ to take into account incomplete information

$$S_{BG}(p^q) = \langle - \ln p_i \rangle_q = -\sum_{i=1}^{W} p_i^q \ln p_i \ , \tag{54}$$

where

$$\langle \mathcal{O} \rangle_q = \sum_{i=1}^{W} p_i^q \mathcal{O} \tag{55}$$

is the q-expectation value. Next, we introduce the fractional entropy [59]

$$S_q(p) = \sum_{i=1}^{W} p_i (- \ln p_i)^q \ .$$

With (54) and (55), the two-parameter generalisation of the Boltzmann–Gibbs–Shannon entropy is introduced [60]

$$S_{q,q'}(p) = \sum_{i=1}^{W} p_i^q (- \ln p_i)^{q'} \ , \tag{56}$$

where $q > 0$ and $q' > 0$. The standard Boltzmann–Gibbs–Shannon entropy can be recovered when both the parameters attain the value of unity $\lim_{q \to 1} \lim_{q' \to 1} S_{q,q'}$ $(p) = S_{BGS}(p)$. The two-parameter generalisation of the Boltzmann–Gibbs–Shannon entropy (56) satisfies all the Axioms 1–4. Furthermore, the concavity can be studied by extremising (56) with respect to $p_i = \exp(-q'/q)$ resulting in

$$\frac{\partial^2 S_{q,q'}}{\partial p_i^2} = p_i^{q-2} (- \ln p_i)^{q'-2} \Big[q(q-1)(- \ln p_i)^2 - q'(2q-1)(- \ln p_i) + q'(q'-1) \Big] \tag{57}$$

and

$$\frac{\partial^2 S_{q,q'}}{\partial p_i^2}\bigg|_{p_i=e^{-q'/q}} = -qe^{\frac{q'(2-q)}{q}}\left(\frac{q'}{q}\right)^{q'-1} . \tag{58}$$

We find that (58) is uniform throughout the range $q > 0$ and $q' > 0$ implying that the two-parameter entropy (56) is concave for $q', q \in \mathscr{R}^+$.

Next, with the definition of the two-parameter generalisation of the Boltzmann–Gibbs–Shannon entropy, we could introduce the two-parameter generalisation of the Kullback–Leibler divergence as

$$D_{q,q'}(p(x)||q(x)) = \sum_{x\in X} p^q(x)\left(\ln\frac{p(x)}{q(x)}\right)^{q'} .$$

It is not difficult to see that the standard Kullback–Leibler divergence is recovered when one considers the limit $\lim_{q\to 1}\lim_{q'\to 1} D_{q,q'}(p||q)) = D_{KL}(p||q)$.

We will proceed as what we did in Sect. 3.3 for obtaining the metric. In the continuous case, the two-parameter generalisation of the Kullback–Leibler divergence reads

$$D_{q,q'}(p(x)||q(x)) = \int dx p^q(x)\left(\ln\frac{p(x)}{q(x)}\right)^{q'} . \tag{59}$$

In the case that $q(x) = p(x) + dp(x)$ (60) becomes

$$D_{q,q'}(p(x)||p(x)+dp(x)) = \int dx p^q(x)\left(\ln\frac{p(x)}{p(x)+dp(x)}\right)^{q'} . \tag{60}$$

Expanding (60) with respect to dp, the first order gives

$$D_{q,q'} \approx I_{q,q'}(p) \equiv \int dx (p(x))^{q-q'-1}\left(\frac{dp(x)}{dx}\right)^{q'+1} , \tag{61}$$

which is the two-parameter generalisation Fisher information.

4 Fisher Information and Its Generalisation

In this section, we will explain another type of the information measure called the Fisher information. What we have learnt so far is that the entropy can be used to predict the probabilistic behaviour of the system. Here, the Fisher information infers how much we know about the internal structure, what the system are made of and how the system compose, i.e. capacity to estimate the parameters that define the system. Then with this two pieces of information, we would have a complete description of the system: their behaviour through the entropy and their architecture through the Fisher information.

4.1 Likelihood Function

Recalling the random variable X, one defines

$$L(\theta|X) = f(x_1, x_2, \ldots, x_W|\theta) \ , \tag{62}$$

where θ is arbitrary parameter in probability models. Equation (62) is known as the likelihood function which measures how good the statistical model is comparing to the sample of data x for a given value of unknown parameter θ.

Let us provide a concrete example. We will consider here a tossing coin experiment to estimate the probability of the head-outcome p_H. This implies that $\theta = p_H$. The act of tossing coin n times gives 2^n possible outcomes forming the sample space. Basically, we can define a function known as a random variable, i.e. the number of times heads appears. What we need for the likelihood estimation are three main elements: (i) the data, (ii) a model describing the probability of measured data and (iii) a criterion to estimate the parameter from data and the model provided.

Data: We are interested to toss the coin 10 times: $n = 10$. We observe a sequence of head-outcomes and tail-outcomes which is supposedly $H, H, H, T, H, T, T, H, T, H$. If we label head-outcomes with 1 and tail-outcomes with 0, the data can be encoded as $X = \{1, 1, 1, 0, 1, 0, 0, 1, 0, 1, 0\}$.

Model: We see that outcomes are independent to each other. Then the Bernoulli distribution is an appropriate model to describe the probability of observing heads for any single flip $f(x_i|p_H) = p_H^{x_i}(1 - p_H)^{1-x_i}$.

Criterion: The condition that will be applied to estimate the probability p_H is the extremising likelihood function. The likelihood function is given by

$$L(p_H|x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i|p_H) = p_H^{\sum_{i=1}^{n} x_i}(1 - p_H)^{n-\sum_{i=1}^{n} x_i} \ .$$

Then we compute

$$\frac{d}{dp_H} L(p_H|x_1, x_2, \ldots, x_n) = 0 \ .$$

Before we proceed further, it is more convenient to express the likelihood function in terms of the logarithm function. This change will not affect the critical point of the likelihood function since the logarithm function is concave. Then we write

$$\log L(p_H|x_1, x_2, \ldots, x_n) = \left(\sum_{i=1}^{n}\right) \log p_H + \left(n - \sum_{i=1}^{n} x_i\right) \log(1 - p_H)$$

and the condition

$$\frac{d}{dp_H} \log L(p_H|x_1, x_2, \ldots, x_n) = 0$$

gives a critical value

$$p_H^* = \frac{\sum_{i=1}^n x_i}{n}$$

which implies that the probability p_H is just proportion of number of the head-outcome. Then with the above setup, the probability $p_H = 6/10 = 0.6$. The remain question is that how good accurate the estimate is. This issue can be settled according to the law of large number, a big bunch of data X.

Now we introduce the score function

$$\text{Score} = \frac{\partial}{\partial p_H} \log L(p_H|X) = \frac{x}{p_H} - \frac{n-x}{1-p_H} \;,$$

where $x = \sum_{i=1}^n x_i$. We find that

$$\begin{aligned}
\left\langle \frac{\partial}{\partial p_H} \log L(p_H|X) \right\rangle &= \sum_{x=0}^{n} \frac{\partial}{\partial p_H} \log L(p_H|\theta) \binom{n}{x} p_H^x (1-p_H)^{1-x} \\
&= \sum_{x=0}^{n} \left(\frac{x}{p_H} - \frac{n-x}{1-p_H} \right) \binom{n}{x} p_H^x (1-p_H)^{1-x} \\
&= \frac{np_H}{p_H} - \frac{n(1-p_H)}{1-p_H} = 0 \;.
\end{aligned}$$

Furthermore, we find that

$$\begin{aligned}
\left\langle \left(\frac{\partial}{\partial p_H} \log L(p_H|X) \right)^2 \right\rangle &= \sum_{x=0}^{n} \left(\frac{\partial}{\partial p_H} \log L(p_H|\theta) \right)^2 \binom{n}{x} p_H^x (1-p_H)^{1-x} \\
&= \sum_{x=0}^{n} \left(\frac{x}{p_H} - \frac{n-x}{1-p_H} \right)^2 \binom{n}{x} p_H^x (1-p_H)^{1-x} \\
&= \frac{n}{p_H(1-p_H)} = -\left\langle \frac{\partial^2}{\partial p_H^2} \log L(p_H|X) \right\rangle \\
&= \text{Var} \left(\frac{\partial}{\partial p_H} \log L(p_H|X) \right) \;,
\end{aligned}$$

which is the variance of the score function. What we have here the variance is proportional to the number n of tossing. This means that, for large n, the variance is negatively large.

4.2 Fisher Information

We now define

$$I(\theta) = \left\langle \left(\frac{\partial}{\partial \theta} \log L(\theta|X) \right)^2 \right\rangle$$
$$= -\left\langle \frac{\partial^2}{\partial \theta^2} \log L(\theta|X) \right\rangle$$
$$= \int \left(\frac{\partial}{\partial \theta} \log L(\theta|X) \right)^2 L(\theta|X) dX$$

as the Fisher information which measures amount of information expecting within the data X on the parameter θ(continuous). One can try to interpret the meaning of the Fisher information through the uncertainty of a parameter θ around the critical value θ^*. To see this, we consider

$$\log L(\theta) = \log L(\theta^*) + \frac{\partial}{\partial \theta} \log L(\theta)\Big|_{\theta=\theta^*} (\theta - \theta^*)$$
$$+ \frac{1}{2} \frac{\partial^2}{\partial \theta^2} \log L(\theta)\Big|_{\theta=\theta^*} (\theta - \theta^*)^2 + \cdots \ .$$

The second term vanishes according to the critical condition. We obtain

$$\log L(\theta) \approx \log L(\theta^*) + \frac{1}{2} \frac{\partial^2}{\partial \theta^2} \log L(\theta)\Big|_{\theta=\theta^*} (\theta - \theta^*)^2 \ .$$

To look for the error range of the parameter, we demand

$$\frac{\partial^2}{\partial \theta^2} \log L(\theta)\Big|_{\theta=\theta^*} (\theta - \theta^*)^2 = -1 \ ,$$

resulting in

$$\mathrm{Var}(\theta) = \langle (\theta - \theta^*)^2 \rangle = \frac{1}{I(\theta)} \ .$$

The error on a parameter can be obtained by varying its value around the maximum until $\log L(\theta)$ decreases by a factor of $1/2$. This implies that small error range gives big value of the Fisher information and vice versa. In general, we could deal with a set of estimating parameters $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$. The Fisher information becomes

$$\boldsymbol{I}(\theta) = [I_{ij}(\theta)] \ ,$$

where

$$I_{ij}(\theta) = \left\langle \left(\frac{\partial}{\partial \theta_i} \log L(\theta|X) \right) \left(\frac{\partial}{\partial \theta_j} \log L(\theta|X) \right) \right\rangle = -\left\langle \frac{\partial^2}{\partial \theta_i \partial \theta_j} \log L(\theta|X) \right\rangle$$

which is called the Fisher–Rao matrix. The interesting point is that the Fisher matrix can also be obtained by considering the relative entropy or Kullback–Leibler divergence between two distributions $p(X)$ and $q(X)$ on the probability manifold

$$KL(p||q) = \int \ldots \int p(X) \log \left(\frac{p(X)}{q(X)} \right) dX \ .$$

Then the Kullback–Leibler divergence between two probability distributions $L(\theta|X)$ and $L(\theta'|X)$, parametrised by θ, is given by

$$D(\theta, \theta') \equiv KL(L(\theta|X)||L(\theta'|X)) = \int \ldots \int L(\theta|X) \log \left(\frac{L(\theta|X)}{L(\theta'|X)} \right) dX \ .$$

For θ being fixed, the Kullback–Leibler divergence can be expanded around θ as

$$D(\theta, \theta') = \frac{1}{2} \left(\theta' - \theta\right)^{\mathrm{T}} \left(\frac{\partial^2}{\partial \theta'_i \partial \theta'_j} D(\theta, \theta') \right) \Big|_{\theta=\theta'} \left(\theta' - \theta\right) + \mathcal{O}\left(\left(\theta' - \theta\right)^2 \right) \ ,$$

where the second-order derivative is

$$\begin{aligned} \left(\frac{\partial^2}{\partial \theta'_i \partial \theta'_j} D(\theta, \theta') \right) \Big|_{\theta=\theta'} &= -\int \ldots \int \left(\frac{\partial^2}{\partial \theta'_i \partial \theta'_j} L(\theta'|X) \right) \Big|_{\theta'=\theta} L(\theta|X) dX \\ &= [I_{ij}(\theta)] \ . \end{aligned}$$

With this connection, one may intuitively interpret the Fisher information as the metric between two points on the probability manifold. However, the Kullback–Leibler divergence is not symmetric and does not follow the triangle inequality [54]. Then the Fisher information cannot be treated as a true metric.

4.3 Cramér–Rao Inequality and Additivity

Fisher information also provides an information lower bound on the variance of an unbiased estimator for a parameter. This relation is known as the Cramér–Rao inequality. To obtain such relation, one can start to consider the unbiased estimator

$$B(\hat{\Theta}) \equiv \left\langle \hat{\Theta} - \theta \right\rangle = \int_{\Omega} \ldots \int_{\Omega} (\hat{\Theta} - \theta) L(\theta \mid X) dX = 0 \ , \qquad (63)$$

where $\hat{\Theta} = h(x_1, x_2, \ldots, x_N)$ is a point estimator. For now on, we might neglect subscription Ω on integrating for our convenience. Next, we consider the derivative of (63) with respect to the parameter θ resulting in

$$\frac{\partial}{\partial\theta}\left\langle\hat{\Theta} - \theta\right\rangle = -\int \ldots \int L(\theta \mid X)dX + \int \ldots \int (\hat{\Theta} - \theta)\frac{\partial}{\partial\theta}L(\theta \mid X)dX \ .$$

Using the fact that $\int \ldots \int L(X \mid \theta)dX = 1$, we obtain

$$\int \ldots \int \left[(\hat{\Theta} - \theta) \cdot L^{1/2}(\theta \mid X)\right]\left[\left(\frac{\partial}{\partial\theta}\log L(\theta \mid X)\right) \cdot L^{1/2}(\theta \mid X)\right]dX = 1 \ .$$

Applying Cauchy–Schwarz inequality, we obtain

$$\frac{1}{\int \ldots \int (\hat{\Theta} - \theta)^2 L(\theta \mid X)dX} \leq \int \ldots \int \left(\frac{\partial}{\partial\theta}\log L(\mathbf{x} \mid \theta)\right)^2 L(\theta \mid X)dX$$
$$\frac{1}{\mathrm{Var}(\hat{\Theta})} \leq \boldsymbol{I}(\theta) \ . \tag{64}$$

What we have in (64) is that the variance of any such estimator is at least as much as the inverse of the Fisher information.

Furthermore, there is also one more important feature of the Fisher information known as the additive property. From the right-hand side of above inequality (64), the Fisher information is given by

$$\boldsymbol{I}(\theta) = \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \int\int \frac{\partial p(x_i \mid \theta)}{\partial\theta}\frac{\partial p(x_j \mid \theta)}{\partial\theta}dx_i dx_j + \sum_{j=1}^{N}\int \frac{1}{p(x_j \mid \theta)}\left(\frac{\partial p(x_j \mid \theta)}{\partial\theta}\right)^2 dx_j \ . \tag{65}$$

Here we are dealing with identical and independent random variables. Then the Fisher information (65) can be simplified as

$$\boldsymbol{I}(\theta) = \sum_{j=1}^{N}\int \left(\frac{\partial}{\partial\theta}\log p(x_j \mid \theta)\right)^2 p(x_j \mid \theta)dx_j = \sum_{j=1}^{N} I_i(\theta) \ . \tag{66}$$

With many, independent data, random variables, the Fisher information can be splitted as the summation of all Fisher information of each random variable. Therefore, Fisher information possesses the additive property.

4.4 Connection with the Action Functional

In the case of one random variable, the likelihood function becomes $p(\theta|x)$ which is a probability distribution over x with respect to θ. The Fisher information reads

$$I[p] = \int p(x|\theta) \left(\frac{\partial}{\partial \theta} \log p(x|\theta) \right)^2 dx \ .$$

Next we define $y = \theta - x$ and then $p(x|\theta) = p(x - \theta) = p(y)$. The Fisher information is simplified to

$$I[p(y)] = \int p(y) \left(\frac{\partial}{\partial y} \log p(y) \right)^2 dy = \int \frac{(\partial p(y)/\partial y)^2}{p(y)} dy \ .$$

We further define $q(y) = \sqrt{p(y)}$, the Fisher information becomes

$$I[q(y)] = 4 \int dy q'^2(y) \ , \quad q'(y) = \frac{dq(y)}{dy} \ . \tag{67}$$

At this present form of the Fisher information, the $I[q(y)]$ can be treated as an action functional $S[q(y)] = I[q(y)]$ for a free particle with the Lagrangian $L(q'(y), q(y); y) = 4q'^2(y)$ and $q(y)$ is a solution of the second-order differential equation $-8q''(y) = 0$ and, of course, in the absence of the interaction, the equation of motion in physics is a direct result of extremising the Fisher information: $\delta I[q] = 0$.

4.5 One-Parameter Generalisation of the Fisher Information

In this section, we will employ the connection between the action functional and the Fisher information to derive a one-parameter generalisation of the Fisher information [55]. Without the interaction, we now propose one-parameter generalisation of the Lagrangian $L(q'(y), q(y); y) = 4q'^2(y)$, resulting in action functional

$$I_\lambda[q(y)] = \frac{4}{\lambda} \int dy \left[e^{\lambda q'^2(y)} - 1 \right] \ , \ \text{where} \ L_\lambda(q'(y), q(y); y) = \frac{4}{\lambda} \left(e^{\lambda q'^2(y)} - 1 \right) \ , \tag{68}$$

where λ is a parameter. One can see immediately that the critical value of the action functional would give the same equation of motion of the free particle $-8q''(y) = 0$. By considering the limit $\lambda \to 0$, one find that $\lim_{\lambda \to 0} L_\lambda(q'(y), q(y); y) = L(q'(y), q(y); y) = 4q'^2(y)$. Of course, this new Lagrangian is a direct result from the extension of the standard Lagrangian in Sect. 2.2.

We shall call (68) as a one-parameter extended Fisher information. The reason can be seen as follows. If we expand the functional (68) with respect to the parameter

λ, we obtain

$$I_\lambda[q(y)] = 4\int q'^2(y)dy + 4\frac{\lambda}{2!}\int q'^4(y)dy + 4\frac{\lambda^2}{3!}\int q'^6(y)dy + \cdots$$
$$= I_1[q(y)] + \frac{\lambda}{2!}I_2[q(y)] + \frac{\lambda^2}{3!}I_3[q(y)] + \cdots \ . \tag{69}$$

What we have in (69) is a hierarchy $\{I_1, I_2, I_3, \ldots\}$. The first term is nothing but the standard Fisher information $I_1[q] = I[q]$ coinciding with the limit $\lim_{\lambda\to 0} I_\lambda[q(y)] = I[q(y)]$. Next, we consider the second function in the hierarchy

$$I_2[q(y)] = 4\int q'^4(y)dy \ \rightarrow \ I_2[p(y)] = 4\int \left(\frac{p'(y)}{2q(y)}\right)^4 dy$$
$$= \frac{4}{2^4}\int \frac{p'^4(y)}{p^2(y)}dy \ . \tag{70}$$

Now we introduce a new variable $\mathrm{p}_1 \equiv p^2$ such that

$$I_2[\mathrm{p}_1] = \frac{4}{4^4}\int \frac{\mathrm{p}_1'^4(y)}{\mathrm{p}_1^4(y)}\mathrm{p}_1(y)dy$$

or

$$I_2[\theta] = \frac{4}{4^4}\int \left[\frac{\partial}{\partial\theta}\ln \mathrm{p}_1(x|\theta)\right]^4 \mathrm{p}_1(x|\theta)dx \ . \tag{71}$$

We shall call (71) as the second-order Fisher information. We can proceed the same technique of transformation and obtain the nth-order Fisher information as

$$I_n[\theta] = \frac{4}{(2n)^{2n}}\int \left[\frac{\partial}{\partial\theta}\ln \mathrm{p}_{n-1}(x|\theta)\right]^{2n} \mathrm{p}_{n-1}(x|\theta)dx, \tag{72}$$

where $\mathrm{p}_{n-1}(y) = p^n(y)$ and the extended Fisher information (69) can be expressed in terms of infinite series as

$$I_\lambda[\theta] = \sum_{j=1}^{\infty} I_j[\theta] \ .$$

At this point, we may treat (69) as the generating function for the entire hierarchy of the Fisher information by expanding with respect to the parameter λ. However, recalling the two-parameter Fisher information in (61)

$$I_{a,b}[p] = \int p^a(y)\left(\frac{dp(y)}{dy}\right)^b dy \ , \tag{73}$$

Table 1 Extended Fisher informations are contained in generalised Fisher information Eq. (73)

Extended Fisher information	Parameter a	Parameter b
First Fisher information: I_1	1	2
Second Fisher information: I_2	2	4
Third Fisher information: I_3	3	6
Fourth Fisher information: I_4	4	8

where $a = q - q' - 1$ and $b = q' + 1$, we find that, with a suitable choice of parameters, our whole hierarchy of Fisher information can be identified as shown in Table 1.

4.6 Generalisation of the Cramer–Rao Inequality and Additive Property

In the previous subsection, we introduce the one-parameter generalisation Fisher information. Here, in this section, we would like to establish the lower bound of it through the Cramer–Rao inequality. We first consider

$$\left\langle \hat{\Theta} - \theta \right\rangle = \int (\hat{\Theta} - \theta) p^q(x \mid \theta) dx = 0 \ ,$$

which is known as the q-expectation value, [56] . Taking the 1st derivative, we obtain

$$\begin{aligned} \frac{\partial}{\partial\theta}\left\langle \hat{\Theta} - \theta \right\rangle &= \int \frac{\partial}{\partial\theta}(\hat{\Theta} - \theta) p^q(x \mid \theta) dx + \int (\hat{\Theta} - \theta) \frac{\partial}{\partial\theta} p^q(x \mid \theta) dx \\ &= -\int p^q(x \mid \theta) dx + q \int (\hat{\Theta} - \theta) p^{q-1}(x \mid \theta) \frac{\partial p(x \mid \theta)}{\partial\theta} dx \\ &= -\int p^q(x \mid \theta) dx + q \int (\hat{\Theta} - \theta) p^{q-1}(x \mid \theta) p(x \mid \theta) \frac{\partial \ln p(x \mid \theta)}{\partial\theta} dx \\ &= -Q_q + qJ = 0 \ , \end{aligned} \tag{74}$$

where

$$\begin{aligned} Q_q &= \int p^q(x \mid \theta) dx \ , \\ J &= \int (\hat{\Theta} - \theta) p^{q-1}(x \mid \theta) p(x \mid \theta) \frac{\partial \ln p(x \mid \theta)}{\partial\theta} dx \ . \end{aligned}$$

Normally, the term J is well known as information generating function [57], it is also called incomplete normalisation [58], and p^q is called effective probability. Next, we rewrite the J in the form

Table 2 Extended Cramer–Rao inequalities are identified with three parameters

Type of Cramer–Rao inequality	Parameter q	Parameter β	Parameter α
First Fisher information	1	2	2
Second Fisher information	5/4	4/3	4
Third Fisher information	4/3	6/5	6
Fourth Fisher information	11/8	8/7	8

$$J = \int \left[(\hat{\Theta} - \theta)\right]\left[\frac{\partial \ln p(x \mid \theta)}{\partial \theta} p^{q-1}(x \mid \theta)\right] p(x \mid \theta) dx \ , \tag{75}$$

and applying the Hölder inequality to (75), we obtain

$$\begin{aligned} J &\le \left[\int (\hat{\Theta} - \theta)^{\beta} p(x \mid \theta) dx\right]^{1/\beta} \left[\int \left(\frac{\partial \ln p(x \mid \theta)}{\partial \theta}\right)^{\alpha} (p^{q-1}(x \mid \theta))^{\alpha} p(x \mid \theta) dx\right]^{1/\alpha} \\ &= \left[\int (\hat{\Theta} - \theta)^{\beta} p(x \mid \theta) dx\right]^{1/\beta} \left[\int \left(\frac{\partial \ln p(x \mid \theta)}{\partial \theta}\right)^{\alpha} p^{\alpha(q-1)+1}(x \mid \theta) dx\right]^{1/\alpha} , \end{aligned} \tag{76}$$

where $q \ge 1$ is the Tsallis entropy index [52], α and β are Hölder conjugates: $1/\alpha + 1/\beta = 1$ for $\alpha, \beta = [1, \infty]$. Finally, employing (74), inequality (76) becomes

$$\begin{aligned} \frac{Q_q}{q} = \frac{\int p^q(x \mid \theta) dx}{q} &\le \left[\int (\hat{\Theta} - \theta)^{\beta} p(x \mid \theta) dx\right]^{1/\beta} \\ &\times \left[\int \left(\frac{\partial \ln p(x \mid \theta)}{\partial \theta}\right)^{\alpha} p^{\alpha(q-1)+1}(x \mid \theta) dx\right]^{1/\alpha} , \end{aligned} \tag{77}$$

which is our extended Cramer–Rao inequality. It is not difficult to see that if one takes $q = 1$, $\beta = 2$ and $\alpha = 2$, the standard Cramer–Rao inequality can be recovered. For $\alpha = 4$, $\beta = 4/3$ and $q = 5/4$, we obtain

$$\frac{(4/5)^4 \, Q_{5/4}}{4^3 \left\langle (\hat{\Theta} - \theta)^{4/3} \right\rangle^3} \le I_2 \ ,$$

which is the Cramer–Rao inequality for the second extended Fisher information. Basically, inequality (77) provides the Cramer–Rao bound for the whole Fisher information hierarchy (Table 2).

Next, we will investigate the additive property of the higher order Fisher information. To make simple, we shall start with the second-order Fisher information.

Consider a system composed of two independent identically subsystems which is defined the random variable $X = (x_1, x_2)$, where superscription denotes for subsystems. The joint probability of the two subsystems is given by $p_{12} \equiv p(x_1, x_2|\theta) = p(x_1|\theta)p(x_2|\theta) \equiv p_1 p_2$. What we have for the second-order Fisher information is

$$\begin{aligned}
I_2[p_{12}] &= \frac{4}{2^4}\int\int\left(\frac{\partial}{\partial\theta}\ln(p_1p_2)\right)^4 p_1^2p_2^2dx_1dx_2\\
&= \frac{4}{2^4}\Bigg[\int\left(\frac{\partial}{\partial\theta}\ln p_1\right)^4 p_1^2dx_1\int p_2^2dx_2\\
&+4\int\left(\frac{\partial}{\partial\theta}\ln p_1\right)^3 p_1^2dx_1\int\left(\frac{\partial}{\partial\theta}\ln p_2\right)p_2^2dx_2\\
&+6\int\left(\frac{\partial}{\partial\theta}\ln p_1\right)^2 p_1^2dx^1\int\left(\frac{\partial}{\partial\theta}\ln p_2\right)^2 p_2^2dx_2\\
&+4\int\left(\frac{\partial}{\partial\theta}\ln p_1\right) p_1^2dx_1\int\left(\frac{\partial}{\partial\theta}\ln p_2\right)^3 p_2^2dx_2\\
&+\int p_1^2dx_1\int\left(\frac{\partial}{\partial\theta}\ln p_2\right)^4 p_2^2dx_2\Bigg]\\
&= \frac{4}{2^4}\Bigg[Q_2(p_2)\int\left(\frac{\partial}{\partial\theta}\ln p_1\right)^4 p_1^2dx_1\\
&+6\int\left(\frac{\partial}{\partial\theta}\ln p_1\right)^2 p_1^2dx_1\int\left(\frac{\partial}{\partial\theta}\ln p_2\right)^2 p_1^2dx_2\\
&+Q_2(p_1)\int\left(\frac{\partial}{\partial\theta}\ln p_2\right)^4 p_2^2dx_2\Bigg]\\
&= \frac{1}{4}\Bigg[Q_2(p_2)I_2(p_1)+Q_2(p_1)I_2(p_2)+6I(p_1)I(p_2)\Bigg]. \qquad (78)
\end{aligned}$$

Here see that the second-order Fisher information does not follow the additive rule. With the result in (78), it is not difficult now to see that the nth-order Fisher information could give

$$\begin{aligned}
I_n[p_{12}] =&\frac{4}{2^{2n}}\Bigg[\binom{2n}{0}Q_n(p(x_2|\theta))\int\left(\frac{\partial}{\partial\theta}\ln p(x_1|\theta)\right)^{2n} p^n(x_1|\theta)dx_1\\
&+\sum_{k=2}^{n-2}\binom{2n}{k}\int\left(\frac{\partial}{\partial\theta}\ln p(x_1|\theta)\right)^{n-k} p^n(x_1|\theta)dx_1\\
&\times\int\left(\frac{\partial}{\partial\theta}\ln p(x_2|\theta)\right)^{k} p^n(x_2|\theta)dx_2\\
&+\binom{2n}{2n}Q_n(p(x_1|\theta))\int\left(\frac{\partial}{\partial\theta}\ln p(x_2|\theta)\right)^{2n} p^n(x_2|\theta)dx_2\Bigg],
\end{aligned}$$

where the first and last terms refer to the Fisher information for each subsystem and the middle one is the crossing term. Therefore, our Fisher information hierarchy does not follow the additive property, except for $n = 1$ the standard Fisher information.

4.7 Two-Parameter Generalised Fisher Information

In this section, we would like to give a preliminary result on two-parameter generalised Fisher information. With all the ingredients we do have in the previous subsections, it is quite natural to extend the Fisher information to the case of two-parameter generalisation. Before doing that, we would like to rewrite the one-parameter Fisher information (68) as

$$I_q[q(y)] = \frac{4}{1-q}\int dy \left[e^{(1-q)q'^2(y)} - 1\right] ,$$

where the parameter λ is replaced by $1 - q$. With this form, the limit one could recover the standard Fisher information is that $q \to 1$.

With the definition of the q-exponential given in (1), the two-parameter generalised Fisher information reads

$$I_{q,r}[q(y)] = \frac{4}{1-q}\int dy \left[e_r^{(1-q)q'^2(y)} - 1\right] .$$

Here we see that the limit $r \to 1$, the one-parameter Fisher information is recovered and, of course, if further limit on the parameter q is considered, the standard Fisher information is obtained. We note here that what we obtain in this section for the two-parameter Fisher information is totally different from (61).

5 Concluding Summary

In this review, we begin to ask a simple question: for a given equation of motion, “Does there exist a corresponding Hamiltonian?” Degasperis and Ruijsenaars [40] have shown that Eq. (4), in the case of one degree of freedom, admits a solution attached with a parameter and an appropriate limit on the parameter, the standard Hamiltonian is recovered. Here, we have shown that there exist other kinds of solution attached with one and more parameters. The interesting point is that in the case of one-parameter generalised Hamiltonian, the form of the Hamiltonian (19) is accidentally in the same form with the Tsallis entropy. This invites us to treat this Hamiltonian as the q-deformation version of the standard Hamiltonian. At this point, it is hard to express the direct connection in terms of the physical meaning of these two quantities and we shall leave it as an open question. In the Lagrangian case, we impose the same question: for a given equation of motion, "Does there exist a corresponding Lagrangian?" Sonin provided the proof that every function $\mathcal{F}$ and every ordinary second-order differential equation

$$g(\mathcal{F} - \ddot{x}) = \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \ , \quad \text{where} \quad g = \frac{\partial^2 L}{\partial \dot{x}^2} \neq 0 \tag{1}$$

admits a Lagrangian [41–43]. However, what we present here is that (1) admits a special solution equipped with a parameter (or parameters). With this parameter, one could generate infinite type of Lagrangians producing the same equation of motion. Of course, this would point out that indeed the Lagrangian is not unique, but with a different perspective from what we had in the standard context, i.e. adding/multiplying a constant or adding the total time derivative of the function $F(x, t)$.

Later, the standard entropies and generalised entropies are presented. The surprisal quantity is introduced and the linear average of the surprisal is called the Shannon entropy. The second derivative of the Shannon entropy with respect to the probabilities yields the Fisher–Rao metric. Furthermore, with the generating function $F_\alpha(p) = \sum_{i=1}^{W} p_i^\alpha$, one can obtain the Shannon entropy from the negative derivative of the generating function with respect to the parameter α and set $\alpha = 1$. In case that there are two probability distributions, one can distinguish them by means of the Kullback–Leibler divergence or relative entropy. In case the two probability distributions are very close to each other, one can approximate the Kullback–Leibler divergence and obtain Fisher metric. This invites us to consider the Kullback–Leibler divergence as the distance measure between two distributions. However, the Kullback–Leibler divergence is not symmetric under interchanging the argument. Then Kullback–Leibler divergence cannot be treated as a true distance measure. If we relax the linear average of the surprisal, the one-parameter extended version of the Shannon entropy called the Renyi entropy is obtained as the exponential average of the surprisal. Under an appropriate limit, the Shannon entropy can be recovered. In the perspective of the statistical mechanics, the Boltzmann–Gibbs entropy is a measure of "how much we do know on the distribution of the available microstates". The one-parameter extended version of the Boltzmann–Gibbs entropy is known as the Tsallis entropy. This entropy comes with many interesting features. First, with two independent subsystems, the Tsallis entropy provides an extra term apart from the standard addition of the entropy for individual subsystems. This feature is known as the non-additive property. Second, one can connect with the Tsallis entropy and Renyi entropy and, under an appropriate limit on the parameter, the standard Boltzmann–Gibbs entropy is recovered. Third, with the generating function $F_\alpha(p)$, if we replace the standard derivative with the q-derivative, the Tsallis entropy can be obtained. Fourth, the Tsallis entropy can be viewed as the linear average of the q-logarithm version of the surprisal. The two-parameter extended version of the Boltzmann–Gibbs entropy can be directly obtained by considering the expectation value of the q-exponential of the q-logarithm version of the surprisal. This two-parameter entropy is known as the Sharma–Mittal entropy. The two-parameter Kullback–Leibler divergence is also discussed and the two-parameter Fisher information is direct result from considering two close probability distributions. In the past few years, the black hole thermodynamics gains an enormous attention for physicists. An early interesting feature of the classical black hole is that, according to Bekenstein's view, the horizon area is nothing but (proportional to) the entropy of the black hole itself [62]. Later, Hawking came up with the idea that the static black hole can radiate and therefore there must be temperature (later known as Hawking temperature) [63]. At this point, the proportional constant is determined and the Bekenstein–Hawking entropy takes a form $S = A/4$. This structure of the black hole entropy is not quite common in the thermodynamical sense as it depends on the area rather than the volume. This means that the black hole entropy is not extensive quantity. Then, this allows new ideas on studying black hole statistic by replacing the Bekenstein–Hawking entropy with Renyi and Tsallis entropies, please see [64] and references therein.

The Fisher information is properly discussed and together with the entropy, one possesses a complete description of the system. The main features of the Fisher information are the following. First, Fisher information must satisfy the Cramer–Rao inequality. Second, Fisher information is additive with respect to two independent subsystems. Third, with the one-parameter case, one can consider the Fisher information as the action functional of the free particle. This suggests that, in the absence of the interaction, all Lagrangians are schematically identical to the Fisher information Lagrangian, and moreover the fundamental equations of motion in physics are the direct result of extremising the Fisher information. With an available connection between the Fisher information and action functional, one can immediately apply the result on one-parameter extended Lagrangian provided in the first part of this work to obtain the one-parameter extended Fisher information. We find that, under the appropriate limit on the parameter, the standard Fisher information is recovered. However, expanding the one-parameter Fisher information with respect to the parameter, the Fisher information hierarchy is obtained and the standard Fisher information is the first one in the family. With appropriate choices on parameters, every single Fisher information in the hierarchy can be identified with the two-parameter Fisher information obtained from the two-parameter Kullback–Leibler divergence. At this stage, the application of the Fisher information hierarchy is not so obvious and we shall leave this as an open problem for further investigation.

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