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PREFACE

"Good things come in small packages."

Rapid Mathematics (**High Yield Facts** Book) is designed for people who have only enough time to glance at a book-literally. Our goal is to create an effective memory aid for those who wish to review mathematics.

The book covers complete syllabus in points form. The quality of the writing leaves the reader with the potential to achieve a good understanding of a given topic within a short period of learning. It gives a concise overview of the main concepts and formulaes of mathematics, for students studying mathematics and related courses at undergraduate level. Based on the highly successful and student friendly "at a glance" approach, the material developed in this book has been chosen to help the students grasp the essence of mathematics, ensuring that they can confidently use that knowledge when required.

The books has been crafted extremely well for a very specific purpose: review. A person who has been away from mathematics (but who understood it very well at the time) can use this book effectively for a rapid review of any basic topic. The book is so highly compressed that every page is like a food with a rich sauce that needs to be slowly savored and slowly digested for maximum benefit. You may only glance into the book, but you can think about the mathematics for much longer.

All the best!

contents

1

CHAPTER

sets, relations, functions & binary operations

Set

- Set is a well defined collection of objects.
- If a is an element of a set A, then we write $a \in A$ and say a belongs to A or a is in A or a is a member of A. If a does not belong to A, we write $a \notin A$.
- **Some standard notations**

  N : For the set of natural numbers.
  Z : For the set of integers.
  Z^+ : For the set of all positive integers.
  Q : For the set of all rational numbers.
  Q^+ : For the set of all positive rational numbers.
  R : For the set of all real numbers.
  R^+ : For the set of all positive real numbers.
  C : For the set of all complex numbers.

  Set is defined in two ways :

  (i) Roster method (ii) Set-builder method
- **Roster method :** In this method a set is described by listing elements, separated by commas, within braces {}.
- **Set-builder method :** In this method a set is described by characterizing property $P(x)$ of its element x. In such case a set is described by $\{x : P(x) \text{ holds}\}$ or $\{x|P(x) \text{ holds}\}$, which is read as 'the set of all x such that $P(x)$ holds'. The symbol '|' or ':' is read as 'such that'.

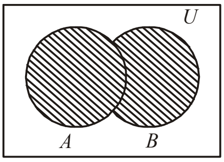
Types of Set

- **Empty set :** A set is said to be empty or null or void set if it has no element and it is denoted by ϕ.
- In Roster method, ϕ is denoted by {}.
- If A and B are any two empty sets, then $x \in A$ iff $x \in B$ is satisfied because there is no element x in either A or B to which the condition may be applied. Thus, $A = B$. Hence, there is only one empty set and we denote it by ϕ.
- **Singleton set :** A set consisting of a single element is called a singleton set.
- **Finite set :** A set is called a finite set if it is either void set or its elements can be listed (counted, labelled) by natural numbers 1, 2, 3, ... and the process of listing terminates at a certain natural number n (say).

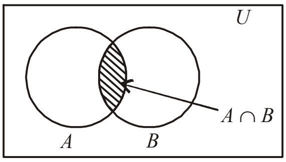
- **Cardinal number of a finite set :** The number n in the above definition is called the cardinal number or order of a finite set A and is denoted by $n(A)$.
- **Infinite set :** A set whose elements cannot be listed by the natural numbers 1, 2, 3,.........., n, for any natural number n is called an infinite set.
- **Equivalent sets :** Two sets A and B are equivalent if their cardinal numbers are same *i.e.* $n(A) = n(B)$.
- **Equal sets :** Two sets A and B are said to be equal if every element of A is a member of B, and every element of B is a member of A.
- **Subsets :** Let A and B be two sets. If every element of A is an element of B, then A is called a subset of B.
- A is subset of B, written as $A \subseteq B$.
- Every set is a subset of itself.
- The empty set is a subset of every set.
- The total number of subsets of a finite set containing n elements is 2^n.
- **Universal set :** A set that contains all sets in a given context is called the universal set.
- **Power set :** Let A be a set. Then the collection or family of all subsets of A is called the power set of A and is denoted by $P(A)$.
- **Venn diagram :** A diagram used to illustrate relationships between sets or a rectangle represents the universal set and circle within it represents the sets.

Operations on Sets

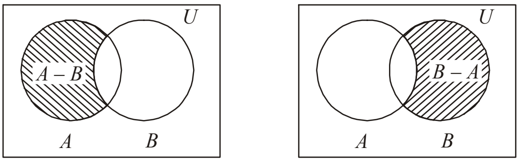
- **Union of sets :** Let A and B be two sets. The union of A and B is the set of all those elements which belong either to A or to B or both A and B.



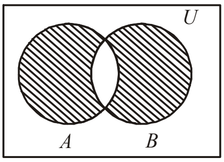
- **Intersection of sets :** Let A and B be two sets. The intersection of A and B is the set of all those elements that belong to both A and B.



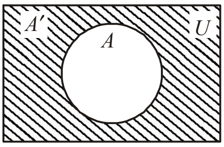
- If $A_1, A_2, ..., A_n$ is a finite family of sets, then their intersection is denoted by $\bigcap_{i=1}^{n} A_i$ or $A_1 \cap A_2 \cap ...... \cap A_n$.
- **Disjoint sets :** Two sets A and B are said to be disjoint, if $A \cap B = \phi$.
  If $A \cap B \neq \phi$, then A and B are said to be intersecting or overlapping sets.
- **Difference of sets :** Let A and B be two sets. The difference of A and B, written as $A - B$, is the set of all those elements of A which do not belong to B.



- **Symmetric difference of two sets :** Let A and B be two sets. The symmetric difference of sets A and B is the set $(A - B) \cup (B - A)$ and is denoted by $A \Delta B$.
- $A \Delta B = (A - B) \cup (B - A) = \{x : x \notin A \cap B\}$ and $x \in A \cup B$.



- **Complement of a set :** Let U be the universal set and let A be a set such that $A \subset U$. Then, the complement of A with respect to U is denoted by A' or A^c or $U - A$ and is defined the set of all those elements of U which are not in A.



- **Properties of complement of a set**

(i) $U' = \{x \in U : x \notin U\} = \phi$

(ii) $\phi' = \{x \in U : x \notin \phi\} = U$

(iii) $(A')' = \{x \in U : x \notin A'\} = \{x \in U : x \in A\} = A$

(iv) $A \cup A' = \{x \in U : x \in A\} \cup \{x \in U : x \notin A\} = U$

(v) $A \cap A' = \{x \in U : x \in A\} \cap \{x \in U : x \notin A\} = \phi$

- **Laws of Algebra of Sets**

I. Idempotent law : For any set A, we have
(i) $A \cup A = A$ (ii) $A \cap A = A$.

II. Identity law : For any set A, we have
(i) $A \cup \phi = A$ (ii) $A \cap U = A$.

III. Commutative law : For any two sets A and B, we have
(i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$.

IV. Associative law : If A, B and C are any three sets, then
(i) $(A \cup B) \cup C = A \cup (B \cup C)$
(ii) $A \cap (B \cap C) = (A \cap B) \cap C$.

V. Distributive law : If A, B and C are any three sets, then
(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

VI. De-Morgan's laws : If A and B are any two sets, then
(i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$.

More results on operations on sets

- If A and B are any two sets, then

(i) $A - B = A \cap B'$
(ii) $B - A = B \cap A'$
(iii) $A - B = A \Leftrightarrow A \cap B = \phi$
(iv) $(A - B) \cup B = A \cup B$
(v) $(A - B) \cap B = \phi$
(vi) $A \subseteq B \Leftrightarrow B' \subseteq A'$
(vii) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

- If A, B and C are any three sets, then

(i) $A - (B \cap C) = (A - B) \cup (A - C)$
(ii) $A - (B \cup C) = (A - B) \cap (A - C)$
(iii) $A \cap (B - C) = (A \cap B) - (A \cap C)$
(iv) $A \cap (B \,\Delta\, C) = (A \cap B) \,\Delta\, (A \cap C)$.

Some important results on number of elements in sets

- If A, B and C are finite sets and U be the finite universal set, then

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.
(ii) $n(A \cup B) = n(A) + n(B) \Leftrightarrow A, B$ are disjoint non-void sets.
(iii) $n(A - B) = n(A) - n(A \cap B)$ *i.e.*, $n(A - B) + n(A \cap B) = n(A)$.
(iv) $n(A \,\Delta\, B)$ = Number of elements which belong to exactly one of A or B.
$= n((A - B) \cup (B - A))$
$= n(A - B) + n(B - A)$ [$\because$ $(A - B)$ and $(B - A)$ are disjoint]
$= n(A) - n(A \cap B) + n(B) - n(A \cap B)$
$= n(A) + n(B) - 2n(A \cap B)$.
(v) $n(A \cup B \cup C)$
$= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$.
(vi) Number of elements in exactly two of the sets A, B, C
$= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$
(vii) Number of elements in exactly one of the sets A, B, C
$= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$
(viii) $n(A' \cup B') = n((A \cap B)') = n(U) - n(A \cap B)$
(ix) $n(A' \cap B') = n((A \cup B)') = n(U) - n(A \cup B)$

- **Ordered pair :** An ordered pair consists of two objects or elements in a given fixed order.
- **Equality of ordered pairs :** Two ordered pairs (a_1, b_1) and (a_2, b_2) are equal iff $a_1 = a_2$ and $b_1 = b_2$ *i.e.* $(a_1, b_1) = (a_2, b_2) \Leftrightarrow a_1 = a_2$ and $b_1 = b_2$.

Cartesian Product of Sets

- Let A and B be any two non-empty sets. The set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called the cartesian product of the sets A and B and is denoted by $A \times B$.
- If A and B are finite sets, then $n(A \times B) = n(A) \cdot n(B)$.
- If either A or B is an infinite set, then $A \times B$ is an infinite set.
- $A \times B = \phi \Leftrightarrow A = \phi$ or $B = \phi$.
- For any three sets A, B, C

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

- For any three sets A, B, C
  $A \times (B - C) = (A \times B) - (A \times C)$.
- If A and B are any two non-empty sets
  $A \times B = B \times A \Leftrightarrow A = B$.
- If $A \subseteq B$, then $A \times A = (A \times B) \cap (B \times A)$.
- If $A \subseteq B$, then $A \times C \subseteq B \times C$ for any set C.
- If $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$.
- For any sets A, B, C, D
  $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.
- For any sets A and B,
  $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$.
- For any three sets A, B, C

(i) $A \times (B' \cup C')' = (A \times B) \cap (A \times C)$
(ii) $A \times (B' \cap C')' = (A \times B) \cup (A \times C)$.

- Let A and B be two non-empty sets having n elements in common, then $A \times B$ and $B \times A$ have n^2 elements in common.

Relation

- Let A and B be two sets. Then a relation R from A to B is a subset of $A \times B$.
- R is a relation from A to $B \Leftrightarrow R \subseteq A \times B$.

Total Number of Relations

- Let A and B be two non-empty finite sets consisting of m and n elements respectively. Then $A \times B$ consists of mn ordered pairs. So, total number of relations from A to B is 2^{nm}.

Domain and range of a relation

- Let R be a relation from a set A to a set B. Then the set of all first components or co-ordinates of the ordered pairs belonging to R is called the domain of R, while the set of all second components or co-ordinates of the ordered pairs in R is called the range of R.
  Thus, Dom $(R) = \{a : (a, b) \in R\}$ and Range $(R) = \{b : (a, b) \in R\}$.

Inverse relation

- Let A, B be two sets and let R be a relation from a set A to a set B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$.

Types of Relations

- **Void relation :** Let A be a set. Then $\phi \subseteq A \times A$ and so it is a relation on A. This relation is called the void or empty relation on A. It is the smallest relation on set A.
- **Universal relation :** Let A be a set. Then $A \times A \subseteq A \times A$ and so it is a relation on A. This relation is called the universal relation on A. It is the largest relation on set A.

- **Identity relation :** Let A be a set. Then the relation $I_A = \{(a, a) : a \in A\}$ on A is called the identity relation on A.
- **Reflexive Relation :** A relation R on a set A is said to be reflexive if every element of A is related to itself. Thus, R reflexive $\Leftrightarrow (a, a) \in R.\ \forall\ a \in A.$
- A relation R on a set A is not reflexive if there exists an element $a \in A$ such that $(a, a) \notin R.$
- **Symmetric relation :** A relation R on a set A is said to be a symmetric relation iff $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$.
  i.e. $aRb \Rightarrow bRa$ for all $a, b \in A$.
- A relation R on a set A is not a symmetric relation if there are atleast two elements $a, b \in A$ such that $(a, b) \in R$ but $(b, a) \notin R$.
- **Transitive relation :** A relation R on A is said to be a transitive relation iff $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.
  i.e. aRb and $bRc \Rightarrow aRc$ for all $a, b, c \in A$.
- **Anti-symmetric relation :** A relation R on set A is said to be an anti-symmetric relation iff $(a, b) \in R$ and $(b, a) \in R$
  $\Rightarrow\ a = b$ for all $a, b \in A$.
- **Equivalence relation :** A relation R on a set A is said to be an equivalence relation on A iff

(i) It is reflexive *i.e.* $(a, a) \in R$ for all $a \in A$.
(ii) It is symmetric *i.e.* $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$.
(iii) It is transitive *i.e.* $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

Congruence modulo *m*

- Let m be an arbitrary but fixed integer. Two integers a and b are said to be congruence modulo m if $a - b$ is divisible by m and we write $a \equiv b\ (\text{mod } m)$.
  Thus, $a \equiv b\ (\text{mod } m) \Leftrightarrow a - b$ is divisible by m.

Some Results on Relations

- If R and S are two equivalence relations on a set A, then $R \cap S$ is also an equivalence relation on A.
- The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.
- If R is an equivalence relation on a set A, then R^{-1} is also an equivalence relation on A.

Composition of relations

- Let R and S be two relations from sets A to B and B to C respectively. Then we can define a relation SoR from A to C such that
  $(a, c) \in SoR \Leftrightarrow \exists\ b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.
  This relation is called the composition of R and S.

Functions

- Let A and B be two-empty sets. Then a function 'f' from set A to set B is a rule or method or correspondence which associates elements of set A to elements of set B such that

(i) All elements of set A are associated to elements in set B.

(ii) An element of set A is associated to a unique element in set B.

- A function 'f' from a set A to a set B associates each element of set A to a unique element of set B.
- If an element $a \in A$ is associated to an element $b \in B$, then b is called 'the f-image of a or 'image of a under f' or 'the value of the function f at a'. Also, a is called the pre-image of b under the function f. We write it as : $b = f(a)$.

Domain, Co-Domain and Range of a function

- Let $f : A \to B$. Then, the set A is known as the domain of f and the set B is known as the co-domain of f. The set of all f-images of elements of A is known as the range of f or image set of A under f and is denoted by $f(A)$.
  Thus, $f(A) = \{f(x) : x \in A\}$ = Range of f.
  Clearly, $f(A) \subseteq B$.

Equal functions

- Two functions f and g are said to be equal iff

(i) The domain of f = domain of g
(ii) The co-domain of f = the co-domain of g, and
(iii) $f(x) = g(x)$ for every x belonging to their common domain.

- If two functions f and g are equal, then we write $f = g$.

Types of Functions

(i) One-one function (injection)

A function $f : A \to B$ is said to be a one-one function or an injection if different elements of A have different images in B.
Thus, $f : A \to B$ is one-one
$\Leftrightarrow \quad a \neq b \Rightarrow f(a) \neq f(b)$ for all $a, b \in A$
$\Leftrightarrow \quad f(a) = f(b) \Rightarrow a = b$ for all $a, b \in A$.

- **Algorithm to check the injectivity of a function**

Step I : Take two arbitrary elements x, y (say) in the domain of f.

Step II : Put $f(x) = f(y)$

Step III : Solve $f(x) = f(y)$. If $f(x) = f(y)$ gives $x = y$ only, then $f : A \to B$ is a one-one function (or an injection) otherwise not.

- Graphically, if any straight line parallel to x-axis intersects the curve $y = f(x)$ exactly at one point, then the function $f(x)$ is one-one or an injection. Otherwise it is not.
- If $f : R \to R$ is an injective map, then the graph of $y = f(x)$ is either a strictly increasing curve or a strictly decreasing curve. Consequently, $\frac{dy}{dx} > 0$ or $\frac{dy}{dx} < 0$ for all x.
- Number of one-one functions from A to $B = \begin{cases} {}^nP_m, & \text{if} \quad n \geq m \\ 0, & \text{if} \quad n < m \end{cases}$

  where $m = n$(Domain)
  $n = n$(Co-domain)

(ii) Onto-function (surjection)

A function $f: A \to B$ is said to be an onto function or a surjection if every element of B is the f-image of some element of A *i.e.*, if $f(A) = B$ or range of f is the co-domain of f.

Thus, $f: A \to B$ is a surjection iff for each $b \in B$, $\exists\ a \in A$ that $f(a) = b$.

Algorithm for Checking the Surjectivity of a Function

Let $f: A \to B$ be the given function.

Step I : Choose an arbitrary element y in B.

Step II : Put $f(x) = y$.

Step III : Solve the equation $f(x) = y$ for x and obtain x in terms of y. Let $x = g(y)$.

Step IV : If for all values of $y \in B$, for which x, given by $x = g(y)$ are in A, then f is onto.

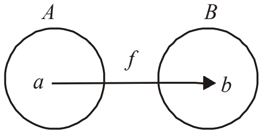
- If there are some $y \in B$ for which x, given by $x = g(y)$ is not in A. Then, f is not onto.
- **Number of onto functions :** If A and B are two sets having m and n elements respectively such that $1 \le n \le m$, then number of onto functions from A to B is

$$\sum_{r=1}^{n} (-1)^{n-r}\, {}^nC_r\, r^m.$$

(iii) Bijection (one-one onto function)

A function $f: A \to B$ is a bijection if it is one-one as well as onto.

In other words, a function $f: A \to B$ is a bijection if



f is one-one onto functioı

(i) It is one-one *i.e.* $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in A$.

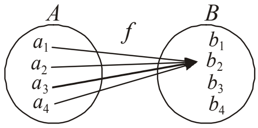
(ii) It is onto *i.e.* for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

- **Number of bijections :** If A and B are finite sets and $f: A \to B$ is a bijection, then A and B have the same number of elements. If A has n elements, then the number of bijections from A to B is the total number of arrangements of n items taken all at a time *i.e.* $n!$

(iv) Many-one function

A function $f: A \to B$ is said to be a many-one function if two or more elements of set A have the same image in B.

$\therefore\ f: A \to B$ is a many-one function if there exist x, $y \in A$ such that $x \ne y$ but $f(x) = f(y)$.



f is many-one and into function

(v) Into function

A function $f: A \to B$ is an into function if there exists an element in B having no pre-image in A.

In other words $f: A \to B$ is an into function if it is not an onto function.

(vi) Identity function

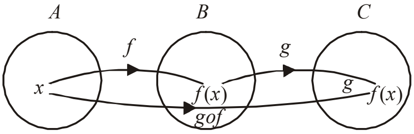
Let A be a non-empty set. A function $f : A \to A$ is said to be an identity function on set A if f associates every element of set A to the element itself.
Thus $f : A \to A$ is an identity function iff $f(x) = x$, for all $x \in A$.

(vii) Constant function

A function $f : A \to B$ is said to be a constant function if every element of A has the same image under function of B *i.e.* $f(x) = c$ for all $x \in A$, where $c \in B$.

Composition of functions

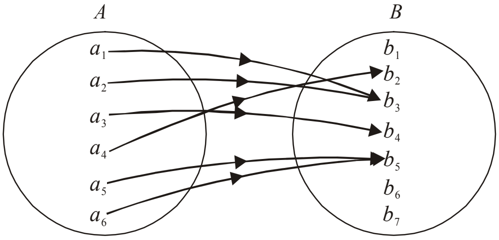
- Let A, B and C be three non-void sets and let $f : A \to B$, $g : B \to C$ be two functions. For each $x \in A$ there exists a unique element $g(f(x)) \in C$.



- The composition of functions is not commutative *i.e.* $fog \neq gof$.
- The composition of functions is associative *i.e.* if f, g, h are three functions such that $(fog)oh$ and $fo(goh)$ exist, then $(fog)oh = fo(goh)$.
- The composition of two bijections is a bijection *i.e.* if f and g are two bijections, then gof is also a bijection.
- Let $f : A \to B$. The $fo\, I_A = I_B\, of = f$ *i.e.* the composition of any function with the identity function is the function itself.

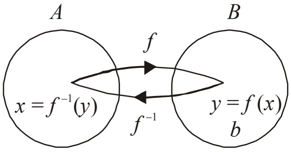
Inverse of an element

- Let A and B be two sets and let $f : A \to B$ be a mapping. If $a \in A$ is associated to $b \in B$ under the function f, then b is called the f image of a and we write it as $b = f(a)$.



Inverse of a function

- If $f : A \to B$ is a bijection, we can define a new function from B to A which associates each element $y \in B$ to its pre-image $f^{-1}(y) \in A$.



- **Algorithm to find the inverse of a bijection**

Let $f : A \to B$ be a bijection. To find the inverse of f we proceed as follows :

Step I : Put $f(x) = y$, where $y \in B$ and $x \in A$.

Step II : Solve $f(x) = y$ to obtain x in terms of y.

Step III : In the relation obtained in step II replace x by $f^{-1}(y)$ to obtain the inverse of f.

- **Properties of Inverse of a Function**

(i) The inverse of a bijection is unique.

(ii) The inverse of a bijection is also a bijection.

(iii) If $f : A \to B$ is a bijection and $g : B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A and I_B are the identity functions on the sets A and B respectively.

If in the above property, we have $B = A$. Then we find that for every bijection $f : A \to A$ there exists a bijection $g : A \to A$ such that $fog = gof = I_A$.

(iv) Let $f : A \to B$ and $g : B \to A$ be two functions such that $gof = I_A$ and $fog = I_B$. Then f and g are bijections and $g = f^{-1}$.

Binary Operation

- Let S be a non-void set. A function f from $S \times S$ to S is called a binary operation on S *i.e.* $f : S \times S \to S$ is a binary operation on set S.
- Generally binary operations are represented by the symbols *, ⊕ ,. etc. instead of letters f, g etc.
- Addition on the set N of all natural numbers is a binary operation.
- Subtraction is a binary operation on each of the sets Z, Q, R and C. But, it is a binary operation on N.
- Division is not a binary operation on any of the sets N, Z, Q, R and C. However, it is not a binary operation on the sets of all non-zero rational (real or complex) numbers.

Types of Binary Operations

(i) Commutative binary operation

- A binary operation * on a set S is said to be commutative if

$$a * b = b * a \text{ for all } a, b \in S$$

- Addition and multiplication are commutative binary operations on Z but subtraction is not a commutative binary operation, since $2 - 3 \neq 3 - 2$.
- Union and intersection are commutative binary operations on the power set $P(S)$ of all subsets of set S. But difference of sets is not a commutative binary operation on $P(S)$.

(ii) Associative binary operation

- A binary operation * on a set S is said to be associative if

$$(a * b) * c = a * (b * c) \text{ for all } a, b, c \in S.$$

(iii) Distributive binary operation

- Let * and o be two binary operations on a set S. Then * is said to be

(i) Left distributive over o if

$a*(b \; o \; c) = (a * b) \; o \; (a * c)$ for all $a, b, c \in S$

(ii) Right distributive over o if

$(b \; o \; c) * a = (b * a) \; o \; (c * a)$ for all $a, b, c \in S$.

(iv) Identity element

- Let * be a binary operation on a set S. An element $e \in S$ is said to be an identity element for the binary operation * if
$$a * e = a = e * a \text{ for all } a \in S.$$
- For addition on Z, 0 is the identity element, since
$$0 + a = a = a + 0 \text{ for all } a \in R.$$
- For multiplication on R, 1 is the identity element, since
$$1 \times a = a = a \times 1 \text{ for all } a \in R.$$
- For addition on N the identity element does not exist. But for multiplication on N the identity element is 1.

(v) Inverse of an element

- Let * be a binary operation on a set S and let e be the identity element in S for the binary operation *. An element $a' \in S$ is said to be an inverse of $a \in S$, if $a * a' = e = a' * a$.
- Addition on N has no identity element and accordingly N has no invertible element.
- Multiplication on N has 1 as the identity element and no element other than 1 is invertible.
- Let S be a finite set containing n elements. Then the total number of binary operations on S is n^{n^2}.
- Let S be a finite set containing n elements. Then the total number of commutative binary operation on S is $n\left[\frac{n(n+1)}{2}\right]$.

End

2

CHAPTER

complex numbers

- **$\sqrt{-1} = i \Rightarrow i^2 = -1$, i is called iota.**
- **Imaginary Quantities**
  The square root of a negative real number is called an imaginary quantity or an imaginary number.
- **A Useful Result :** If a, b are positive real numbers, then $\sqrt{-a} \times \sqrt{-b} = -\sqrt{ab}$.
- For any two real numbers $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is true only when at least one of a and b is either positive or zero. In order words, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is not valid if a and b both are negative.
- For any positive real number a, we have $\sqrt{-a} = \sqrt{-1 \times a} = \sqrt{-1} \times \sqrt{a} = i\sqrt{a}$.
- **Complex Numbers :** If a, b are two real numbers, then a number of the form $a + ib$ is called a complex number.
- **Real and Imaginary Parts of a Complex Number :** If $z = a + ib$ is a complex number, then 'a' is called the real part of z and 'b' is known as the imaginary part of z. The real part of z is denoted by Re(z) and the imaginary part by Im(z).
- **Purely Real and Purely Imaginary Complex Numbers**
  A complex number z is purely real if its imaginary part is zero *i.e.* Im(z) = 0 and purely imaginary if its real part is zero *i.e.*, Re(z) = 0.
- **Set of Complex Numbers :** The set of all complex numbers is denoted by C *i.e.* $C = \{a + ib |\ a, b \in R\}$.
- **Equality of Complex Numbers :** Two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are equal if $a_1 = a_2$ and $b_1 = b_2$ *i.e.* $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$.
- **Addition of Complex Numbers :** Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers. Then their sum $z_1 + z_2$ is defined as the complex number $(a_1 + a_2) + i(b_1 + b_2)$.

Properties of Addition of Complex Numbers

I. Addition is Commutative : For any two complex numbers z_1 and z_2, we have $z_1 + z_2 = z_2 + z_1$.

II. Addition is Associative : For any three complex numbers z_1, z_2, z_3, we have $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.

III. Existence of Additive Identity : The complex number $0 = 0 + i0$ is the identity element for addition *i.e.* $z + 0 = z = 0 + z$ for all $z \in C$.

- **Subtraction of Complex Numbers :** Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers. Then the subtraction of z_2 from z_1 is denoted by $z_1 - z_2$ and is defined as the addition of z_1 and $-z_2$.
- **Multiplication of Complex Numbers :** Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers. Then the multiplication of z_1 with z_2 is denoted by $z_1 z_2$ and is defined as the complex number $(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$.

Properties of Multiplication

I. Multiplication is Commutative : For any two complex numbers z_1 and z_2, we have $z_1 z_2 = z_2 z_1$.

II. Multiplication is Associative : For any three complex numbers z_1, z_2, z_3, we have $(z_1 z_2)z_3 = z_1(z_2 z_3)$.

III. Existence of Identity Element for Multiplication : The complex number $1 = 1 + i0$ is the identity element for multiplication *i.e.* for every complex number z, we have $z \cdot 1 = z = 1 \cdot z$.

IV. Existence of Multiplicative Inverse : Corresponding to every non-zero complex number $z = a + ib$ there exists a complex number $z_1 = x + iy$ such that

$$z \cdot z_1 = 1 = z_1 \cdot z.$$

V. Multiplication of Complex Number is Distributive Over Addition of Complex Numbers : For any three complex numbers z_1, z_2, z_3, we have

(i) $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ (Left distributivity)

(ii) $(z_2 + z_3)z_1 = z_2 z_1 + z_3 z_1$ (Right distributivity)

- **Division of Complex Numbers**

  The division of a complex number z_1 by a non-zero complex number z_2 is defined as the multiplication of z_1 by the multiplicative inverse of z_2 and is denoted by $\frac{z_1}{z_2}$.

Conjugate of a Complex Number

- Let $z = a + ib$ be a complex number. Then the conjugate of z is denoted by $\bar{z}$ and is equal to $a - ib$.

Properties of Conjugate

If z, z_1, z_2 are complex numbers, then

(i) $\overline{(\bar{z})} = z$ (ii) $z + \bar{z} = 2\,\text{Re}(z)$

(iii) $z - \bar{z} = 2i\,\text{Im}(z)$ (iv) $z = \bar{z} \Leftrightarrow z$ is purely real

(v) $z + \bar{z} = 0 \Rightarrow z$ is purely imaginary

(vi) $z\bar{z} = \{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2$

(vii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (viii) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(ix) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ (x) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0.$

Modulus of a Complex Number

- The modulus of a complex number $z = a + ib$ is denoted by $|z|$ and is defined as

$$|z| = \sqrt{a^2 + b^2} = \sqrt{\{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2}.$$

- In the set C of all complex numbers, the order relation is not defined *i.e.* $z_1 > z_2$ or $z_1 < z_2$ has no meaning but $|z_1| > |z_2|$ or $|z_1| < |z_2|$ has got its meaning since $|z_1|$ and $|z_2|$ are real numbers.

Properties of Modulus

If $z, z_1, z_2 \in C$, then

(i) $|z| = 0 \Leftrightarrow z = 0$ *i.e.* $\text{Re}(z) = \text{Im}(z) = 0$

(ii) $|z| = |\bar{z}| = |-z|$.

(iii) $-|z| \le \text{Re}(z) \le |z|$; $-|z| \le \text{Im}(z) \le |z|$

(iv) $z\bar{z} = |z|^2$

(v) $|z_1 z_2| = |z_1|\,|z_2|$

(vi) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$; $z_2 \ne 0$

(vii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\text{Re}(z_1\,\bar{z}_2)$

(viii) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\text{Re}(z_1\,\bar{z}_2)$

(ix) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

(x) $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$, where $a, b \in R$.

Multiplicative Inverse

- The multiplicative inverse of a non-zero complex number z is same as its reciprocal and is given by $\frac{\bar{z}}{|z|^2}$.

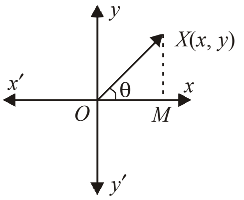
Square Roots of a Complex Number

- Let $a + ib$ be a complex number such that $\sqrt{a+ib} = x + iy$, where x and y are real numbers. Then,

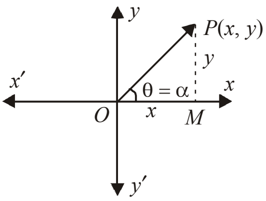
$$\sqrt{a+ib} = \pm\left\{\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}+a\right\}} \pm i\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}-a\right\}}\right\}.$$

To find the square root of $a - ib$, replace i by $-i$ in the above result.

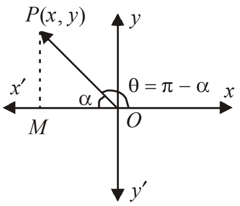
- A complex number $z = x + iy$ can be represented by a point (x, y) on the plane which is known as the **Argand plane.**
- A purely real number is represented by a point on x-axis.
- A purely imaginary complex number is represented by a point on y-axis.
- x-axis is known as the real axis and y-axis, as the imaginary axis.
- $P(x, y)$ is a point in the plane, then the point $P(x, y)$ represents a complex number $z = x + iy$.
- There exists a one-one correspondence between the points of the plane and the members (elements) of the set C of all complex numbers.
- The plane in which we represent a complex number geometrically is known as the **complex plane** or **Argand plane** or the **Gaussian plane.**
- The length of the line segment OX is called the **modulus** of z and is denoted by $|z|$.



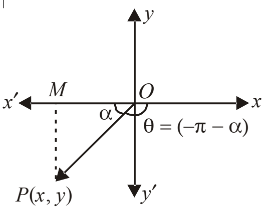
- The angle θ which OX makes with positive direction of x-axis in anti-clockwise sense is called the **argument or amplitude of** z and is denoted by $\arg(z)$ or $\text{amp}(z)$.
- The unique value of θ such that $-\pi < \theta \le \pi$ is called the **principal value of the amplitude or principal argument.**
- If x and y both are positive, then the argument of $z = x + iy$ is the acute angle given $\tan^{-1}\left|\frac{y}{x}\right|$.



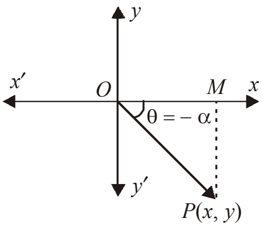
- If $x < 0$ and $y > 0$, then the argument of $z = x + iy$ is $\pi - \alpha$, where α is the acute angle given by $\tan^{-1}\left|\frac{y}{x}\right|$.



- If $x < 0$ and $y < 0$ then the argument of $z = x + iy$ is $\alpha - \pi$, where α is the acute angle given by $\tan\alpha = \left|\frac{y}{x}\right|$.



- If $x > 0$ and $y < 0$, then the argument of $z = x + iy$ is $-\alpha$, where α is the acute angle given by $\tan\alpha = \left|\frac{y}{x}\right|$.

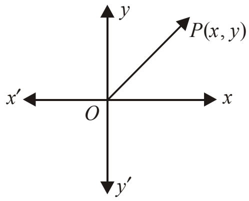


- **Steps for Finding Argument of $z = x + iy$**

  Step I : Find the value of $\tan^{-1}\left|\frac{y}{x}\right|$ lying between 0 and $\frac{\pi}{2}$. Let it be α.

  Step II : Determine in which quadrant the point $P(x, y)$ belongs. If $P(x, y)$ belongs to the first quadrant, then $\arg(z) = \alpha$.
  If $P(x, y)$ belongs to the second quadrant, then $\arg(z) = \pi - \alpha$.
  If $P(x, y)$ belongs to the third quadrant, then $\arg(z) = \alpha - \pi$.
  If $P(x, y)$ belongs to the fourth quadrant, then $\arg(z) = -\alpha$.

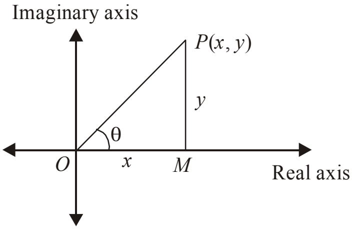
Vectorial Representation of a Complex Number



- The complex number $z = x + iy$ is represented by the vector OP and in such a case $|z|$ is the length OP and $\arg(z)$ is the angle which the directed line OP makes with the positive direction of x-axis.

Geometrical representation

- Let $z = x + iy$ be a complex number represented by a point $P(x, y)$ in the Argand plane.



  Length of $OP = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Polar Representation

- $z = r(\cos\theta + i\sin\theta)$, where $r = |z|$ and $\theta = \arg(z)$, is the polar form of z.
  If the general value of the argument is θ, then the polar form of z is $z = r[\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)]$, where $r = |z|$, $\theta = \arg(z)$ and n is an integer.
- **Polar form of $z = x + y$ for different signs of x and y**

  Let $|z| = r$ and α be the acute angle given by $\tan^{-1}\left(\frac{y}{x}\right)$. Let θ be the argument of z.

(i) Polar form of $z = x + iy$ when $x > 0$ and $y > 0$: In this case $\theta = \alpha$. So, the polar form of $z = x + iy$ is $r(\cos\alpha + i\sin\alpha)$.

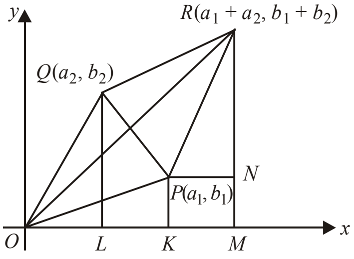
(ii) Polar form of $z = x + iy$ when $x < 0$ and $y > 0$: In this case $\theta = \pi - \alpha$. So, the polar form of $z = x + iy$ is $r[\cos(\pi - \alpha) + i\sin(\pi - \alpha)$ or $r(-\cos\alpha + i\sin\alpha)$.

(iii) Polar form of $z = x + iy$ when $x < 0$ and $y < 0$: In this case $\theta = -(\pi - \alpha)$. So, the polar form of z is $r[\cos(\pi - \alpha) + i\sin(-(\pi - \alpha))]$ or $r(-\cos\alpha - i\sin\alpha)$

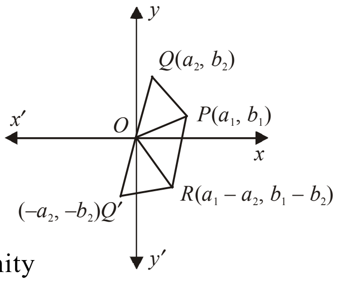
(iv) Polar form of $z = x + iy$ when $x > 0$ and $y < 0$: In this case $\theta = -\alpha$. So, the polar form of z is $r[\cos(-\alpha) + i\sin(-\alpha)]$ or $r(\cos\alpha - i\sin\alpha)$.

Eulerian Form of A Complex Number

- $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$. These two are called Euler's notations.
- **Geometrical Representation of Addition :** If two points P and Q represent complex numbers z_1 and z_2 respectively in the Argand plane, then the sum $z_1 + z_2$ is represented by the extremity R of the diagonal OR of parallelogram $OPRQ$ having OP and OQ as two adjacent sides.



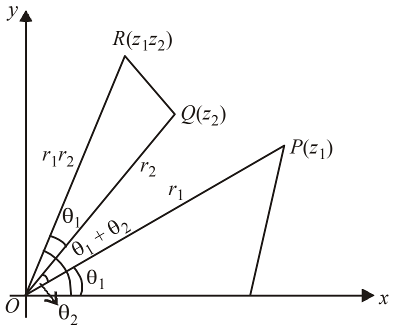
- **Geometrical Representation of Subtraction :** Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers represented by points $P(a_1, b_1)$ and $Q(a_2, b_2)$ in the Argand plane. Q' represents the complex number $(-z_2)$. Complete the paralleogram $OPRQ'$ by taking OP and OQ' as two adjacent sides.



  The sum of z_1 and $-z_2$ is represented by the extremity R of the diagonal OR of parallelogram $OPRQ'$. R represents the complex number $z_1 - z_2$.

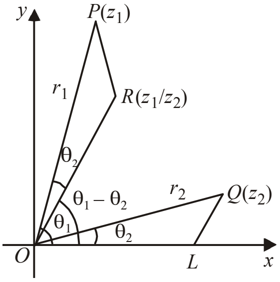
Modulus and Argument of Multiplication of Complex Numbers

- For any two complex numbers z_1, z_2, we have
  (i) $|z_1 z_2| = |z_1|\,|z_2|$ (ii) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.
- If $z_1, z_2, z_3, \ldots\ldots, z_n$ are complex numbers, then
  (i) $|z_1 z_2 z_3 \ldots\ldots z_n| = |z_1|\,|z_2|\,|z_3| \ldots\ldots |z_n|$
  (ii) $\arg(z_1 z_2 z_3 \ldots\ldots z_n) = \arg(z_1) + \arg(z_2) + \ldots\ldots + \arg(z_n)$.
- **Geometrical Representation of Multiplication of Complex Numbers**



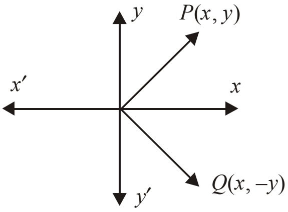
R has the polar co-ordinates $(r_1 r_2, \theta_1 + \theta_2)$ and it represents the complex numbers $z_1 z_2$.

- **Modulus and Argument of Division of two Complex Numbers**
  If z_1 and $z_2 (\neq 0)$ are two complex numbers, then
  (i) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ (ii) $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$.
- **Geometrical Representation of the Division of Complex Numbers**



R has the polar co-ordinates $\left(\dfrac{r_1}{r_2}, \theta_1 - \theta_2\right)$ and it represents the complex number z_1/z_2.

- $|z| = |\bar{z}|$ and $\arg(\bar{z}) = -\arg(z)$. The general value of $\arg(\bar{z})$ is $2n\pi - \arg(z)$.
- If a point P represents a complex number z, then its conjugate $\bar{z}$ is represented by the image of P in the real axis.



- **Some Important Results on Modulus and Argument**
  If z, z_1 and z_2 are complex numbers, then
  (i) $\arg(\bar{z}) = -\arg(z)$.
  (ii) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.
  (iii) $\arg(z_1 \bar{z}_2) = \arg(z_1) - \arg(z_2)$.
  (iv) $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$.
  (v) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1|\,|z_2|\cos(\theta_1 - \theta_2)$, where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$

  or

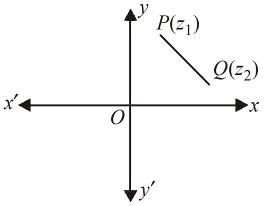
  $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\text{Re}(z_1 \bar{z}_2)$.
  (vi) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,|z_1|\,|z_2|\cos(\theta_1 - \theta_2)$, where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$

  or

  $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\text{Re}(z_1 \bar{z}_2)$.
  (vii) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$.

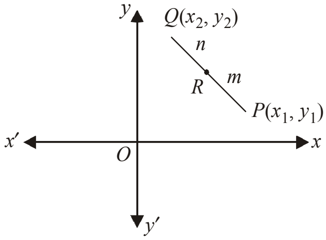
(viii) $|z_1 + z_2| = |z_1 - z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = \pi/2$.
(ix) $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow \arg(z_1) = \arg(z_2)$.
(x) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \frac{z_1}{z_2}$ is purely imaginary.
If $|z_1| \le 1, |z_2| \le 1$, then
(xi) $|z_1 - z_2|^2 \le (|z_1| - |z_2|)^2 + (\arg(z_1) - \arg(z_2))^2$.
(xii) $|z_1 + z_2|^2 \ge (|z_1| + |z_2|)^2 - (\arg(z_1) - \arg(z_2))^2$.
(xiii) $|z_1 + z_2| \le |z_1| + |z_2|$.
(xiv) $|z_1 - z_2| \le |z_1| + |z_2|$.
(xv) $|z_1 + z_2| \ge |z_1| - |z_2|$.
(xi) $|z_1 - z_2| \ge |z_1| - |z_2|$.

- **Distance Between Two Points**
  Let P and Q be two points in the Argand plane having affixes z_1 and z_2 respectively.



Then, $PQ = |z_2 - z_1| = |\text{affix of } Q - \text{affix of } P|$.

- **Division of a Line Segment in the ratio *m* : *n***



If two points P and Q have affixes z_1 and z_2 respectively in the Argand plane, then the affix of a point R dividing PQ internally in the ratio $m : n$ is $\frac{mz_2 + nz_1}{m+n}$.

- If R is the mid-point of PQ, then affix of R is $\frac{z_1 + z_2}{2}$.
- If R divides PQ externally in the ratio $m : n$, then affix of R is $\frac{mz_2 - nz_1}{m-n}$.
- If z_1, z_2, z_3 are affixes of the vertices of a triangle, then the affix of its centroid is $\frac{z_1 + z_2 + z_3}{3}$.

Equation of the Perpendicular Bisector

- The equation of the perpendicular bisector of the line segment joining points having affixes z_1 and z_2 is

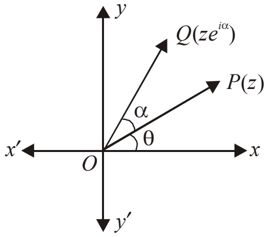
$$z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_1 - z_2) = |z_1|^2 - |z_2|^2.$$

Equation of a Circle

- The equation of a circle whose centre is at point having affix z_0 and radius R is
$$|z - z_0| = R.$$
- If the centre of the circle is at the origin and radius R, then its equation is $|z| = R$.
- $|z - z_0| < R$ represents the interior of the circle and $|z - z_0| \leq R$ is the set of a points lying inside and on the circle $|z - z_0| = R$. Similarly, $|z - z_0| > R$ is the set of all points lying outside the circle and $|z - z_0| \geq R$ is the set of all points lying outside and on the circle $|z - z_0| = R$.

Complex Number as a Rotating Arrow in the Argand Plane

- $ze^{i\alpha}$ is the complex number whose modulus is r and argument $\theta + \alpha$.
- Multiplication by $e^{-i\alpha}$ to z rotates the vector OP in clockwise sense through an angle α.



I. If z_1, z_2, z_3 are the affixes of the points A, B and C in the Argand plane, then

(i) $\angle BAC = \arg\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right)$.

(ii) $\dfrac{z_3 - z_1}{z_2 - z_1} = \dfrac{|z_3 - z_1|}{|z_2 - z_1|}(\cos\alpha + i\sin\alpha)$, where $\alpha = \angle BAC$.

(iii) $\angle BAC$ can also be remembered by using the following technique:

$$\angle BAC = \arg\left(\frac{\text{affix of } C - \text{affix of } A}{\text{affix of } B - \text{affix of } A}\right).$$

II. If z_1, z_2, z_3 and z_4 are the affixes of the points A, B, C and D respectively in the Argand plane.

(i) AB is inclined to CD at the angle $\arg\left(\dfrac{z_2 - z_1}{z_4 - z_3}\right)$

(ii) If CD is inclined at 90° to AB, then $\arg\left(\dfrac{z_2 - z_1}{z_4 - z_3}\right) = \pm\dfrac{\pi}{2}$.

(iii) If z_1 and z_2 are fixed complex numbers, then the locus of a point z satisfying

$\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \pm\dfrac{\pi}{2}$ is circle with 2, and z_2 as ends of diameter.

- **Some Standard Locii in the Argand Plane**

(I) If z is a variable point in the Argand plane such that $\arg(z) = \theta$, then locus of z is a straight line (excluding origin) through the origin inclined at an angle θ with x-axis.

(II) If z is a variable point and z_1 is a fixed point in the Argand plane such that $\arg(z - z_1) = \theta$, then locus of z is a straight line passing through the point representing z_1 and inclined at an angle θ with x-axis. Note that the point z_1 is excluded from the locus.

(III) If z is a variable point and z_1, z_2 are two fixed point in the Argand plane, then

(a) $|z - z_1| = |z - z_2|$.

$\Rightarrow$ Locus of z is the perpendicular bisector of the line segment joining z_1 and z_2.

(b) $|z - z_1| + |z - z_2| = \text{const}(\neq |z_1 - z_2|)$

$\Rightarrow$ Locus of z is an ellipse.

(c) $|z - z_1| + |z - z_2| = |z_1 - z_2|$

$\Rightarrow$ Locus of z is the line segment joining z_1 and z_2.

(d) $|z - z_1| - |z - z_2| = |z_1 - z_2|$

$\Rightarrow$ Locus of z is a straight line joining z_1 and z_2 but z does not lie between z_1 and z_2.

(e) $|z - z_1| - |z - z_2| = \text{constant}(\neq |z_1 - z_2|)$

$\Rightarrow$ Locus of z is a hyperbola.

(f) $|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$

$\Rightarrow$ Locus of z is a circle with z_1 and z_2 as the extremities of diameter.

(g) $|z - z_1| = k|z - z_2|,\ k \neq 1$

$\Rightarrow$ Locus of z is a circle.

(h) $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \alpha(\text{fixed})$

$\Rightarrow$ Locus of z is a segment of circle.

(i) $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \pm\pi/2$

$\Rightarrow$ Locus of z is a circle with z_1 and z_2 as the vertices of diameter.

(j) $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = 0 \text{ or } \pi$

$\Rightarrow$ Locus z is a straight line passing through z_1 and z_2.

Equation of Straight Line

- **Parametric Form :** Equation of a straight line joining the points having affixes z_1 and z_2 is $z = tz_1 + (1 - t)z_2$, where $t \in R$
- **Non-Parametric Form :** Equation of a straight line joining the points having affixes z_1 and z_2 is

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0 \text{ or } z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - \bar{z}_1 z_2 = 0$$

- **General Equation of a Straight Line**

The general equation of a straight line is of the form

$$a\bar{z} + \bar{a}z + b = 0$$

where a is a complex number and b is a real number.

Slope of the Line Segment Joining Two Points

- If A and B represent complex numbers z_1 and z_2 in the Argand plane, then the complex slope of AB is defined to be
$$\frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}.$$
- If ω_1 and ω_2 are the complex slope of two lines on the Argand plane, then the lines are :
  (i) Perpendicular, if $\omega_1 + \omega_2 = 0$
  (ii) Parallel, if $\omega_1 = \omega_2$.
- **Length of the Perpendicular From a Point on a Line**
  The length of the perpendicular from a point z_1 to the line $a\bar{z} + \bar{a}z + b = 0$ is given by
$$\frac{a\bar{z}_1 + \bar{a}z_1 + b}{2|a|}.$$
- **De-Moivre's Theorem**
  (i) If $n \in Z$ (the set of integers), then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
  (ii) If $n \in Q$ (the set of rational numbers), then $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.
- $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$
- $\dfrac{1}{\cos\theta + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta$
- $(\sin\theta \pm i\cos\theta)^n \neq \sin n\theta \pm i\cos n\theta$
- $(\sin\theta + i\cos)^n = [\cos(\pi/2 - \theta) + \sin(\pi/2 - \theta)]^n$
  $= \cos(n\,\pi/2 - n\theta) + i\sin(n\,\pi/2 - n\theta)$
- $(\cos\theta + i\sin\phi)^n \neq \cos n\theta + i\sin n\phi$.
- **Algorithm for finding n^{th} roots of a given complex number**
  Step 1 : Write the given complex number in polar form.
  Step 2 : Add $2m\pi$ to the argument.
  Step 3 : Apply De Moivre's theorem.
  Step 4 : Put $m = 0, 1, 2, ..., (n-1)$ *i.e.* one less than the number in the denominator of the given index in the lowest form.
- n^{th} roots of unity are : $\alpha^0 = 1, \alpha, \alpha^2, \alpha^3, ..., \alpha^{n-1}$, where $\alpha = e^{i2\pi/n} = \cos 2\pi/n + i\sin 2\pi/n$.

Properties of n^{th} Roots of Unity

I. n^{th} roots of unity form a G.P. with common ratio $e^{i2\pi/n}$.
II. Sum of n^{th} roots of unity is always zero.
III. Sum of p^{th} powers of n^{th} roots of unity is zero, if p is not a multiple of n.
IV. Sum of p^{th} powers of n^{th} roots of unity is n, if p is a multiple of n.
V. Product of n^{th} roots of unity is $(-1)^{n-1}$.
VI. n^{th} roots of unity lie on the unit circle $|z| = 1$ and divide its circumference into n equal parts.
VII. n^{th} roots of 1 are the solutions of the equation $z = 1^{1/n}$

i.e. $z^n = 1$ or $z^n - 1 = 0$

$\therefore \quad z^n - 1 = (z - 1)(z - \alpha)(z - \alpha^2).....(z - \alpha^{n-1})$

VIII. n^{th} roots of -1 are the solutions of the equation $z = (-1)^{1/n}$

i.e. $z^n = -1$ or $z^n + 1 = 0$.

Cube Roots of Unity

- Cube roots of unity are $1, \dfrac{-1+i\sqrt{3}}{2}$ and $\dfrac{-1-i\sqrt{3}}{2}$.
- Cube roots of unity are $1, \omega, \omega^2$, where $\omega = e^{i2\pi/3}$.

Properties of Cube Roots of Unity and Some Useful Results Related to Them

I. Cube roots of unity are $1, \omega, \omega^2$, where $\omega = -\dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2} = e^{i2\pi/3}$.

II. $\arg(\omega) = 2\pi/3$ and $\arg(\omega^2) = 4\pi/3$.

III. $z^3 - 1 = (z - 1)(z - \omega)(z - \omega^2)$.

IV. ω and ω^2 are roots of the equation $z^2 + z + 1 = 0$.

V. Cube roots of unity lie on the unit circle $|z| = 1$ and divide its circumference into three equal parts.

VI. $1 + \omega^n + \omega^{2n} = \begin{cases} 0, \text{ if } n \text{ is not a multiple of } 3 \\ 3, \text{ if } n \text{ is a multiple of } 3 \end{cases}$.

VII. Cube roots of -1 are $-1, -\omega, -\omega^2$.

VIII. $z^3 + 1 = (z + 1)(z + \omega)(z + \omega^2)$

IX. $-\omega$ and $-\omega^2$ are roots of $z^2 - z + 1 = 0$.

X. $a^3 + b^3 = (a + b)(a + b\omega)(a + b\omega^2)$, $a^3 - b^3 = (a - b)(a - b\omega)(a - b\omega^2)$

$a^2 + b^2 + c^2 - ab - bc - ca = (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$.

Logarithm of a Complex Number

- Let $x + iy$ and $a + ib$ be two complex numbers such that $a + ib = e^{x+iy}$, then $x + iy$ is called logarithm of $a + ib$ to the base e and we write

$$x + iy = \log_e (a + ib).$$

- $\log(z) = \log |z| + i \operatorname{amp}(z)$, where $z = a + ib$.

End

3

CHAPTER

sequences and series

- **Sequence :** A sequence is a function whose domain is the set N of natural numbers.
- **Real Sequence :** A sequence whose range is a subset of R is called a real sequence.
- **Series :** If $a_1, a_2, a_3, a_4, ..., a_n, ...$ is a sequence, then the expression $a_1 + a_2 + a_3 + a_4 + a_5 + ... + a_n + ...$ is a series.
- **Progressions :** Those sequences whose terms follow certain patterns are called progressions.
- **Arithmetic Progression (A.P.) :** A sequence is called an arithmetic progression if the difference of a term and the previous term is always same *i.e.*, $a_{n+1} - a_n = \text{constant } (= d)$ for all $n \in N$.
- **General Term of an A.P. :** Let a be the first term and d be the common difference of an A.P., then its n^{th} term is $a + (n - 1)\, d$ *i.e.* $a_n = a + (n - 1)d$.
- **n^{th} Term of an A.P. From the End :** Let a be the first term and d be the common difference of an A.P. having m terms. Then n^{th} term from the end is $(m - n + 1)^{th}$ term from the beginning.
- **Selection of Terms in an A.P. :**

No. of terms	Terms	Common difference
3	$a - d, a, a + d$	d
4	$a - 3d, a - d, a + d, a + 3d$	$2d$
5	$a - 2d, a - d, a, a + d, a + 2d$	d
6	$a - 5d, a - 3d, a - d, a + d, a + 3d, a + 5d$	$2d$

- Sum of n term of an Arithmetic Progression where a = first term, d = common difference is $S_n = \frac{n}{2}[2a + (n-1)d]$
- If the sum S_n of n terms of a sequence is given, then n^{th} term a_n of the sequence can be determined by the formula $a_n = S_n - S_{n-1}$
- A sequence is an A.P. iff the sum of its n terms is of the form $An^2 + Bn$ *i.e.* a quadratic expression in n and in such a case the common difference is twice the coefficient of n^2.

Properties of Arithmetic Progressions

- If a constant is added or subtracted from each term of an A.P., then the resulting sequence is also an A.P. with the same common difference.
- If each term of a given A.P. is multiplied or divided by a non-zero constant k, then the resulting sequence is also an A.P. with common difference kd or d/k, where d is the common difference of the given A.P.

- In a finite A.P. the sum of the terms equidistant from the beginning and end is always same and is equal to the sum of first and last term *i.e.*, $a_k + a_{n-(k-1)} = a_1 + a_n$ for all $k = 1, 2, 3, ..., n - 1$.
- Three numbers a, b, c are in A.P. iff $2b = a + c$.
- A sequence is an A.P. iff its n^{th} term is a linear expression in n *i.e.* $a_n = An + B$, where A, B are constants. In such a case the coefficient of n in a_n is the common difference of the A.P.
- A sequence is an A.P. iff the sum of its first n terms is of the form $An^2 + Bn$, where A, B are constants independent of n. In such a case the common difference is $2A$ *i.e.* 2 times the coefficient of n^2.
- If the terms of an A.P. are chosen at regular intervals, then they form an A.P.
- If a_n, a_{n+1} and a_{n+2} are three consecutive terms of an A.P., then $2a_{n+1} = a_n + a_{n+2}$.
- **Insertion of n Arithmetic Means**
  If between two given quantities a and b we have to insert n quantities $A_1, A_2, ..., A_n$ such that $a, A_1, A_2, ..., A_n, b$ form an A.P., then $A_1, A_2 ..., A_n$ are arithmetic means between a and b.
- **Insertion of a Single Arithmetic Mean Between a and b**
  Let A be the arithmetic mean of a and b. Then a, A, b are in A.P.
  $\Rightarrow \quad A = \frac{a+b}{2}$.
- If $a_1, a_2, a_3, ..., a_n$ are n numbers, then their A.M. is given by
  $$A = \frac{a_1 + a_2 + a_3 + ... + a_n}{n}.$$
- **Geometric Progression**
  A sequence of non-zero numbers is called a geometric progression (G.P.) if the ratio of a term and the term preceding to it is always a constant quantity.
- The constant ratio is called the common ratio of the G.P.
- In other words, a sequence, $a_1, a_2, a_3, ... a_n, ...$ is called a geometric progression if $\frac{a_{n+1}}{a_n}$ = constant for all $n \in N$.
- **Geometric Series**
  If $a_1, a_2, a_3, ..., a_n, ...$ is a G.P., then the expression $a_1 + a_2 + a_3 + ... + a_n + ...$ is called a geometric series.

The n^{th} or General Term of a G.P.

- The n^{th} term of a G.P. with first term a and common ratio r is given by $a_n = ar^{n-1}$.
- If a is the first term and r is the common ratio of a G.P., then the G.P. can be written as $a, ar, ar^2, ar^3, ar^4, ..., ar^{n-1}, ...$

The n^{th} Term from the End of a Finite G.P.

- The n^{th} term from the end of a finite G.P. consisting of m terms is ar^{m-n}, where a is the first term and r is the common ratio of the G.P.

- **Selection of Terms in G.P.**

No. of terms	Terms	Common ratio
3	$\frac{a}{r}, a, ar$	r
4	$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$	r^2
5	$\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$	r

Sum of n Terms of a G.P.

- The sum of n terms of a G.P. with first term 'a' and common ratio 'r' given by

$$S_n = a\left(\frac{r^n - 1}{r - 1}\right) \text{ or } S_n = a\left(\frac{1 - r^n}{1 - r}\right), r \neq 1.$$

- If l is the last term of the G.P., then $l = ar^{n-1}$.

$$S_n = \frac{a - lr}{1 - r} \text{ or } \frac{lr - a}{r - 1}; r \neq 1.$$

Sum of an Infinite G.P.

- The sum of an infinite G.P. with first term a and common ratio $r(-1 < r < 1$ *i.e.* $|r| < 1)$ is $S = \frac{a}{1-r}$.
- If $r \geq 1$, then the sum of an infinite G.P. tends to infinity.

Properties of Geometric Progression

- If all the terms of a G.P. be multiplied or divided by the same non-zero constant, then it remains a G.P. with the same common ratio.
- The reciprocals of the terms of a given G.P. form a G.P.
- If each term of a G.P. be raised to the same power, the resulting sequence also forms a G.P.
- In a finite G.P. the product of the terms equidistant from the beginning and the end is always same and is equal to the product of the first and the last term.
- Three non-zero numbers a, b, c are in G.P. iff $b^2 = ac$.
- If the terms of a given G.P. are chosen at regular intervals, then the new sequence so formed also forms a G.P.
- If $a_1, a_2, a_3, \ldots\ldots, a_n, \ldots\ldots$ is a G.P. of non-zero non-negative terms, then $\log a_1, \log a_2, \ldots\ldots, \log a_n, \ldots\ldots.$ is an A.P. and vice-versa.

Insertion of n Geometric Means Between Two Given Numbers

- Let a and b be two given numbers. If n numbers $G_1, G_2, \ldots, G_n$ are inserted between a and b such that the sequence $a, G_1, G_2, \ldots, G_n, b$ is a G.P., then the numbers $G_1, G_2, \ldots, G_n$ are known as n geometric means (G.M.'s) between a and b.

$$\Rightarrow \quad G_n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Insertion of Single Geometric Mean between Two Numbers

- If a single geometric mean G is inserted between two given numbers a and b, then G is known as the geometric mean between a and b.
  $\Rightarrow \quad G = \sqrt{ab}$.

An Important Property of Geometric Means

- If n geometric means are inserted between two quantities, then the product of n geometric means is the n^{th} power of the single geometric mean between the two quantities.
- If A and G are respectively arithmetic and geometric means between two positive numbers a and b, then $A > G$.
- If A and G are respectively arithmetic and geometric means between two positive quantities a and b, then the quadratic equation having a, b as its roots is $x^2 - 2Ax + G^2 = 0$.
- If A and G be the A.M. and G.M. between two positive numbers, then the numbers are $A \pm \sqrt{A^2 - G^2}$.

Arithmetico-Geometric Sequence

- If $a_1, a_2, a_3, \ldots, a_n, \ldots$ is an A.P. and $b_1, b_2, \ldots, b_n, \ldots$ is a G.P., then the sequence $a_1b_1, a_2b_2, a_3b_3, \ldots, a_nb_n, \ldots$ is said to be an arithmetico-geometric sequence. n^{th} term of this sequence is $[a + (n-1)d]r^{n-1}$.
- Let $a, (a+d)r, (a+2d)r^2, (a+3d)r^3, \ldots$ be an arithmetico-geometric sequence. Then, $a + (a+d)r + (a+2d)r^2 + (a+3d)r^3 + \ldots$ is an arithmetico-geometric series.
- **Sum of n Terms of an Arithmetico-Geometric Sequence**
  The sum of n terms of an arithmetico-geometric sequence $a, (a+d)r, (a+2d)r^2, (a+3d)r^3, \ldots$ is given by
  $$S_n = \begin{cases} \dfrac{a}{1-r} + dr\dfrac{(1-r^{n-1})}{(1-r)^2} - \dfrac{\{a+(n-1)d\}r^n}{1-r}, & \text{when } r \neq 1 \\ \dfrac{n}{2}[2a+(n-1)]d, & \text{when } r = 1. \end{cases} \quad \ldots\text{(i)}$$
- **Sum of an Infinite Arithmetico-Geometric Sequence**
  Let $|r| < 1$. Then $r^n, r^{n-1} \to 0$ as $n \to \infty$ and it can also be shown that $n \cdot r^n \to 0$ as $n \to \infty$. So, from (i) we obtain that
  $$S_n \to \frac{a}{1-r} + \frac{dr}{(1-r)^2}, \text{ as } n \to \infty.$$
  In other words, when $|r| < 1$ the sum to infinity of an arithmetico-geometric series is $S_\infty = \dfrac{a}{1-r} + \dfrac{dr}{(1-r)^2}$.

Sum to n Terms of Some Special Sequences

- **Sum of first n natural numbers**
  Consider the sequence $1, 2, 3, \ldots, n$ of natural numbers. Let S_n denote the sum

of its first n terms *i.e.* $S_n = 1 + 2 + 3 + ... + n = \sum_{k=1}^{n} k$

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}.$$

- **Sum of the Squares of First *n* Natural Numbers :**

$$S_n = \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

- **Sum of the Cubes of First *n* Natural Numbers :**
Consider the sequence of cubes of the natural number *viz.*
$1^3, 2^3, 3^3, ..., n^3, (n+1)^3, ...$

$$1^3 + 2^3 + ... + n^3 = \left(\frac{n(n+1)}{2}\right)^2 = \left(\sum_{k=1}^{n} k\right)^2.$$

- **Harmonic Progression**
A sequence $a_1, a_2, a_3, ..., a_n, ...$ of non-zero number is called a harmonic progression (H.P.) if the sequence $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, ..., \frac{1}{a_n}, ...$ is an A.P.

- **Insertion of *n* Harmonic Means Between Two Given Numbers**
Let a, b be two given numbers. If n numbers $H_1, H_2, ..., H_n$ are inserted between a and b such that the sequence $a, H_1, H_2, H_3, ..., H_n, b$ is an H.P., then $H_1, H_2, ..., H_n$ are called n harmonic means between a and b.
Now, $a, H_1, H_2, ..., H_n, b$ are in H.P.

- **Harmonic Mean of Two Given Numbers**
If a and b are two non-zero numbers, then the harmonic mean of a and b is a number H such that the sequence a, H, b is a H.P.

$$H = \frac{2ab}{a+b}.$$

Properties of Harmonic Progression

- If $a_1, a_2, a_3, ..., a_n$ are n non-zero numbers, then the harmonic mean H of these numbers is given by $\frac{1}{H} = \frac{\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}}{n}$.
- Let A, G and H be arithmetic, geometric and harmonic means of two numbers a and b. Then, $A = \frac{a+b}{2}$, $G = \sqrt{ab}$ and $H = \frac{2ab}{a+b}$.
- $A \geq G \geq H$.
- A, G, H form a G.P., *i.e.* $G^2 = AH$ for only 2 numbers a, b.
- The equation having a and b as its roots is $x^2 - 2Ax + G^2 = 0$.
- If A, G, H are arithmetic, geometric and harmonic means between three given numbers a, b and c, then the equation having a, b, c as its roots is

$$x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0.$$

End

4

CHAPTER

quadratic equations & expressions

- **Real Polynomial**
  Let a_0, a_1, a_2, ..., a_n be real numbers and x is a real variable. Then, $f(x) = a_0 + a_1 x + a_2x^2 + ... + a_n x^n$ is called a real polynomial of real variable x with real coefficients.
- **Complex Polynomial**
  If $a_0, a_1, a_2, ... a_n$ be complex numbers and x is a varying complex number, then $f(x) = a_0 + a_1x + a_2x^2 +....+ a_{n-1}x^{n-1} + a_nx^n$ is called a complex polynomial or a polynomial of complex variable with complex coefficients.
- **Degree of a Polynomial**
  A polynomial $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n$, real or complex, is a polynomial of degree n, if $a_n \neq 0$.
- **Polynomial Equation**

  If $f(x)$ is a polynomial, real or complex, then $f(x) = 0$ is called a polynomial equation.

  If $f(x)$ is a polynomial of second degree, then $f(x) = 0$ is called a quadratic equation. The general form of a quadratic equation is $ax^2 + bx + c = 0$, where $a, b, c \in C$, set of all complex numbers, and $a \neq 0$.
- **Roots of an Equation**
  The values of the variable satisfying the given equation are called its roots.
- **Some Results on Roots of an Equation**

I. An equation of degree n has n roots, real or imaginary.

II. Surd and imaginary roots always occur in pairs *i.e.* if $2 - 3i$ is a root of an equation, then $2 + 3i$ is also its root. Similarly, if $2+\sqrt{3}$ is a root of a given equation, then $2-\sqrt{3}$ is also its root.

III. An odd degree equation has atleast one real, whose sign is opposite to that of its last term, provided that the coefficient of highest degree term is positive.

IV. Every equation of an even degree whose constant term is negative and the coefficient of highest degree term is positive, has atleast two real roots, one positive and one negative.

Position of Roots of a Polynomial Equation

- If $f(x) = 0$ is an equation and a, b are two real numbers such that $f(a)f(b) < 0$, then the equation $f(x) = 0$ has atleast one real root or an odd number of real roots between a and b. $f(a)$ and $f(b)$ are of the same sign, then either no real root or an even number of real roots of $f(x) = 0$ lie between a and b.

- **Deductions**

1. Every equation of an odd degree has atleast one real root, whose sign is opposite to that of its last term, provided that the coefficient of first term is +ve.
2. Every equation of an even degree whose last term is negative and the coefficient of first term positive, has atleast two real roots, one positive and one negative.
3. If an equation has only one change of sign, it has one +ve root and no more.
4. If all the terms of an equation are +ve and the equation involves no odd powers of x, then all its roots are complex.

Descartes Rule of Signs

- The maximum number of positive real roots of a polynomial equation $f(x) = 0$ is the number of changes of signs from positive to negative and negative to positive in $f(x)$.

Relations Between Roots and Coefficients

- If α, β are roots of the quadratic equation $ax^2 + bx + c = 0$, then

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}.$$

- **Cubic Equation**

If α, β, γ are roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$, then

$$\alpha + \beta + \gamma = -b/a,\ \alpha\beta + \beta\gamma + \gamma\alpha = (-1)^2\frac{c}{a} = \frac{c}{a} \text{ and } \alpha\beta\gamma = (-1)^3\frac{d}{1} = -\frac{d}{a}.$$

- **Biquadratic Equation**

If α, β, γ, δ are roots of biquadratic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then

$$S_1 = \alpha + \beta + \gamma + \delta = -\frac{b}{a},\ S_2 = \alpha\beta + \beta\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = (-1)^2\frac{c}{a} = \frac{c}{a}$$

$$S_3 = \alpha\beta\gamma + \beta\gamma\delta + \gamma\beta\alpha + \alpha\beta\delta = (-1)^3\frac{d}{a} = -\frac{d}{a}, \text{ and } S_4 = \alpha\beta\gamma\delta = (-1)^4\frac{e}{a} = \frac{e}{a}.$$

Formation of Roots of a Polynomial Equation from given Roots

- **Quadratic Equation**

If α, β are the roots of a quadratic equation, then the equation is $x^2 - S_1x + S_2 = 0$ *i.e.* $x^2 - (\alpha + \beta)x + \alpha\beta = 0$.

- **Cubic Equation**

If α, β, γ are the roots of a cubic equation, then the equation is $x^3 - S_1x^2 + S_2x - S_3 = 0$ *i.e.* $x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma = 0$.

- **Biquadratic Equation**

If α, β, γ, δ are the roots of a biquadratic equation, then the equation is

$$x^4 - S_1x^3 + S_2x^2 - S_3x + S_4 = 0$$

i.e. $x^4 - (\alpha + \beta + \gamma + \delta)x^3 + (\alpha\beta + \beta\gamma + \gamma\delta + \alpha\delta + \beta\delta + \alpha\gamma)x^2 - (\alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta + \gamma\delta\alpha)x + \alpha\beta\gamma\delta = 0$.

- To obtain an equation whose roots are reciprocals of the roots of a given equation is obtained by replacing x by $1/x$ in the given equation.
- Let x be a root of the given equation and y be that of the transformed equation. Then, $y = x^2 \Rightarrow x = \sqrt{y}$.

- Let x be a root of the given equation and y be that of the transformed equation. Then, $y = x^3 \Rightarrow x = y^{1/3}$.

Roots of a Quadratic Equation with Real Coefficients

- An equation of the form $ax^2 + bx + c = 0$ ....(i)
  where $a \neq 0$, $a, b, c \in R$ is called a quadratic equation with real coefficients.
- The quantity $D = b^2 - 4ac$ is known as the discriminant of the quadratic equation in (i) whose roots are given by

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

The nature of the roots is as given below :

1. The roots are real and distinct iff $D > 0$.
2. The roots are real and equal iff $D = 0$.
3. The roots are complex with non-zero imaginary part iff $D < 0$.
4. The roots are rational iff a, b, c are rational and D is a perfect square.
5. The roots are of the form $p + \sqrt{q}\ (p, q \in Q)$ iff a, b, c are rational and D is not a perfect square.
6. If $a = 1$, $b, c \in I$ and the roots are rational numbers, then these roots must be integers.
7. If a quadratic equation in x has more than two roots, then it is an identity in x that is $a = b = c = 0$.

Quadratic Equation and its Graphs

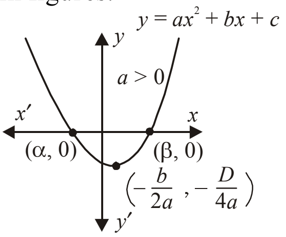
- Let $y = ax^2 + bx + c$, where $a \neq 0$. Then, $y = ax^2 + bx + c$.

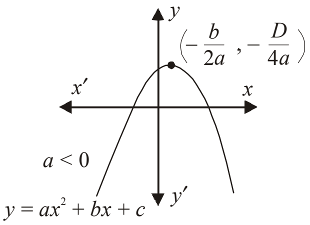
  This equation can be written as $\left(y + \frac{D}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2$

  which reduces to equation of parabola, where $Y = y + \frac{D}{4a}$ and $X = \left[x + \frac{b}{2a}\right]$.

$$Y = aX^2.$$

The parabola opens upwards or downwards according as $a > 0$ or $a < 0$ as shown in figures.

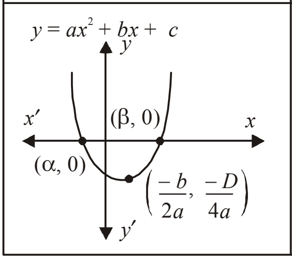




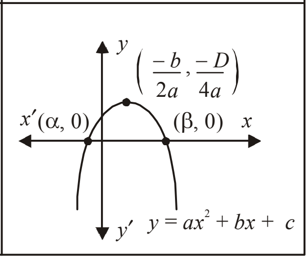
- For $a > 0$: The curve $y = ax^2 + bx + c$ is a parabola opening upwards such that:
  $y_{\min} = -\frac{D}{4a}$ at $x = -\frac{b}{2a}$ and $y_{\max} \to \infty$.
- For $a < 0$: The curve $y = ax^2 + bx + c$ is a parabola opening downwards such that
  $y_{\max} = \frac{-D}{4a}$ at $x = \frac{-b}{2a}$ and $y_{\min} \to -\infty$.

- The parabola will intersect the x-axis in two distinct points iff $D > 0$.

For $a > 0$

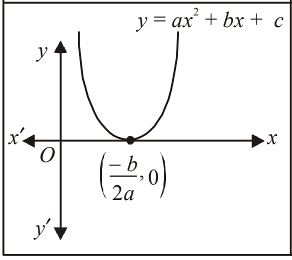


For $a < 0$

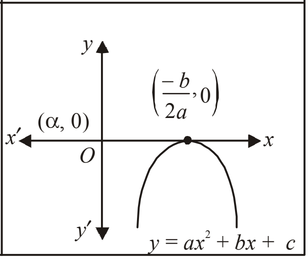


- The parabola will just touch the x-axis at one point iff $D = 0$

For $a > 0$

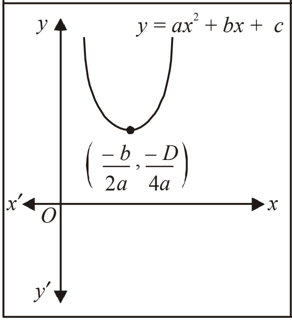


For $a < 0$

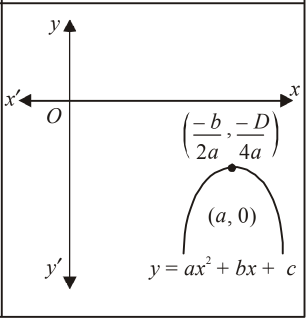


- The parabola will not intersect and will not touch x-axis iff $D < 0$.

For $a > 0$



For $a < 0$



Case I : When $D = b^2 - 4ac < 0$

If $D < 0$, $f(x) > 0$ iff $a > 0$ and $f(x) < 0$ iff $a < 0$.

Case II : When $D = b^2 - 4ac = 0$

When $D = 0$, we have

$f(x) > 0$ iff $a > 0$ and $f(x) < 0$ iff $a < 0$

Case III : When $D = b^2 - 4ac > 0$

(i) If $D = b^2 - 4ac > 0$and $a > 0$, then

$$f(x)\begin{cases} > 0 & \text{for} \quad x < \alpha \text{ or } x > \beta \\ < 0 & \text{for} \quad \alpha < x < \beta \\ = 0 & \text{for} \quad x = \alpha, \beta \end{cases}$$

(ii) If $D = b^2 - 4ac > 0$ and $a < 0$, then

$$f(x)\begin{cases} < 0 & \text{for} \quad x < \alpha \text{ or } x > \beta \\ > 0 & \text{for} \quad \alpha < x < \beta \\ 0 & \text{for} \quad x = \alpha, \beta \end{cases}$$

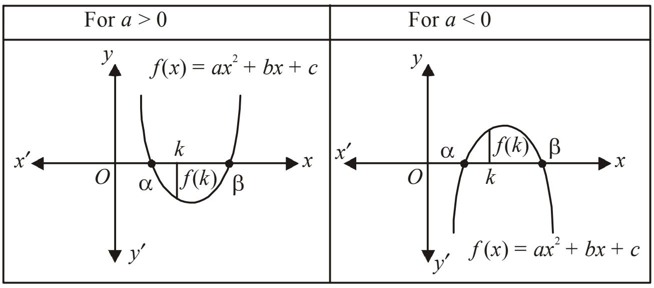
Therefore, for real values of x, the sign of the quadratic expression $f(x) = ax^2 + bx + c$ is same as that of 'a', except when the roots of the equation $ax^2 + bx + c = 0$ are real and distinct and x lies between them.

It also follows that :

- $ax^2 + bx + c > 0$ for all $x \in R$ iff $a > 0$ and $D < 0$, and
- $ax^2 + bx + c < 0$ for all $x \in R$ iff $a < 0$ and $D < 0$.

Position of Roots of a Quadratic Equation

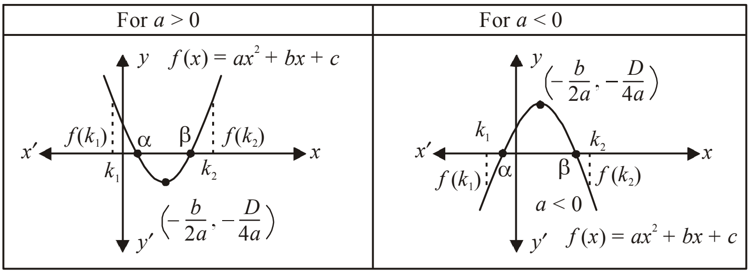
- If a number k lies between the roots of a quadratic equation $f(x) = ax^2 + bx + c = 0$, then the equation must have real roots and the sign of $f(k)$ is opposite to the sign of 'a'.



Thus, a number k lies between the roots of a quadratic equation $f(x) = ax^2 + bx + c = 0$, if

(i) $D > 0$ (ii) $af(k) < 0$.

- If both the roots α and β of a quadratic equation lie between number k_1 and k_2 then (i) $D > 0$ (ii) $af(k_1) > 0$, $af(k_2) > 0$ (iii) $k_1 < \dfrac{-b}{2a} < k_2$.

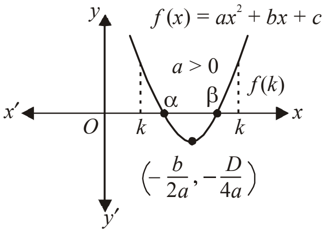


A number k is smaller than the roots of a quadratic equation $ax^2 + bx + c = 0$ if

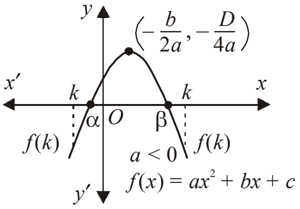
(i) $D > 0$

(ii) $af(k) > 0$

(iii) $k < \dfrac{-b}{2a}$.



- If a number k is more than the roots of a quadratic equation $ax^2 + bx + c$, then

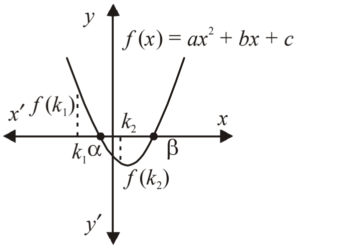
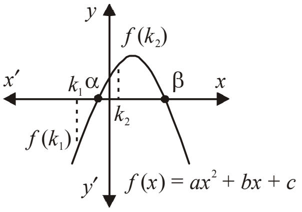


(i) $D > 0$

(ii) $af(k) > 0$

(iii) $k > -\dfrac{b}{2a}$.

- If exactly one root of the equation $ax^2 + bx + c = 0$ lies in the interval (k_1, k_2), then the equation $ax^2 + bx + c = 0$ must have real roots and $f(k_1), f(k_2)$ must be of opposite signs.

For $a > 0$	For $a < 0$
	

Thus exactly one root of the equation $ax^2 + bx + c = 0$ lies in the interval (k_1, k_2) if (i) $D > 0$ (ii) $f(k_1)f(k_2) < 0$.

Common Roots

The common roots of the quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ is given by $\alpha = \dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$ or $\alpha = \dfrac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1}$.

Note : To find the common root of two equations, make the coefficient of second degree terms in two equations equal and subtract. The value of x so obtained is the required common root.

- If the two equations have both roots common, then $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$.

- The quadratic function $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is resolvable into linear rational factors iff $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

i.e. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$

- Between any two roots of a polynomial $f(x)$, there is always a root of its derivative *i.e.* $f'(x) = 0$.

End

5

CHAPTER

permutations and combinations

Factorial

- The continued product of first n natural numbers is called the "n factorial" and is denoted by $n!$ or $\lfloor n$.
- Factorials of proper fractions or negative integers are not defined. Factorial n is defined only for whole numbers.
- $n! = n[(n-1)!]$.
- **Exponent of Prime p in $n!$**
  Let p be a prime number and n be a positive integer. Then the last integer amongst $1, 2, 3, \ldots, (n-1), n$ which is divisible by p is $\left[\frac{n}{p}\right]p$, where $\left[\frac{n}{p}\right]$ denotes the greatest integer less than or equal to $\frac{n}{p}\cdot$
- Let $E_p(n)$ denote the exponent of the prime p in the positive integer n. Then,
  $E_p(n!) = E_p(1 \cdot 2 \cdot 3 \ldots (n-1)n)$
  $$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots\ldots + \left[\frac{n}{p^s}\right]$$
  where s is the largest positive integer such that $p^s \le n < p^{s+1}$.
- **Fundamental Principle of Multiplication**
  If there are two jobs such that one of them can be completed in m ways, and when it has been completed in any one of these m ways, second job can be completed in n ways, then the two jobs in succession can be completed in $m \times n$ ways.
- This principle can be extended for any finite number of jobs as stated below : If there are n jobs $J_1, J_2, \ldots, J_n$ such that job J_i can be performed independently in m_i ways, where $i = 1, 2, \ldots, n$. Then the total number of ways in which all the jobs can be performed is $m_1 \times m_2 \times m_3 \times \ldots \times m_n$.
- **Fundamental Principle of Addition**
  If there are two jobs such that they can be performed independently in m and n ways respectively, then either of the two jobs can be performed in $(m + n)$ ways.

Permutations

- Each of the arrangements which can be made by taking some or all of a number of things is called a permutation.
- It should be noted that in permutations the order of arrangement is taken into account, when the order is changed, a different permutation is obtained.

- If n and r are positive integers such that $1 \le r \le n$, then the number of all permutations of n distinct things, taken r at a time is denoted by the symbol $P(n, r)$ or ${}^nP_r$.
- Let r and n be positive integers such that $1 \le r \le n$. Then the number of all permutations of n distinct things taken r at a time is given by
$n(n-1)(n-2)(n-3)$ .... $(n-(r-1))$.
i.e. $P(n, r) = {}^nP_r = n(n-1)(n-2)$ ..... $(n-(r-1))$.
- $P(n,r) = {}^nP_r = \dfrac{n!}{(n-r)!}$.
- The number of all permutations of n distinct things, taken all at a time is $n!$.
- $P(n, n) = 2P(n, n-2)$
- $P(n, n) = P(n, n-1)$
- $P(n, r) = P(n-1, r) + r \cdot P(n-1, r-1)$
$P(n, r) = n \cdot P(n-1, r-1)$.
- Sum of the number formed by n non-zero digits

$$= (\text{Sum of the digits})(n-1)!\left(\frac{10^n-1}{10-1}\right).$$

- The number of all permutations of n different objects taken r at a time, when a particular object is to be always included in each arrangement, is $r \cdot {}^{n-1}P_{r-1}$.
- The number of permutations of n distinct objects taken r at a time, when a particular object is never taken in each arrangement, is ${}^{n-1}P_r$.
- Number of permutations of n different objects taken r at a time in which two specified objects always occur together is $2!(r-1){}^{n-2}P_{r-2}$.
- The number of mutually distinguishable permutations of n things, taken all at a time, of which p are alike of one kind, q alike of second such that $p + q = n$, is

$$\frac{n!}{p!q!}.$$

- The number of permutations of n things, of which p_1 are alike of one kind, p_2 are alike of second kind, p_3 are alike of third kind,....., p_r are like of r^{th} kind such that $p_1 + p_2 + \ldots + p_r = n$, is $\dfrac{n!}{p_1!p_2!p_3!....p_r!}$.
- The number of permutations of n things, of which p are alike of one kind, q are alike of second kind and remaining all are distinct, is $\dfrac{n!}{p!q!}$.
- Suppose there are r things to be arranged, allowing repetitions. Let further p_1, p_2,...., p_r be the integers such that the first object occurs exactly p_1 times, the second occurs exactly p_2 times, etc. Then the total number of permutations of these r objects to the above condition is $\dfrac{(p_1+p_2+......+p_r)!}{p_1!p_2!p_3!......p_r!}$.
- The number of permutations of n different things, taken r at a time, when each may be repeated any number of times in each arrangement, is n^r.
- The number of circular permutations of n distinct objects is $(n-1)!$

- Anti-clockwise and clockwise order of arrangements are considered as distinct permutations.
- If anti-clockwise and clockwise order of arrangements are not distinct then number of circular permutations of n distinct items is $\frac{1}{2}\{(n-1)!\}$.

Combinations

- Each of the different selections made by taking some or all of a number of objects, irrespective of their arrangements is called a combination.
- The word 'arrangements' is used for permutations and 'selections' for combinations.
- The number of all combinations of n distinct objects, taken r at a time is generally denoted by $C(n, r)$ or ${}^nC_r$ or $\binom{n}{r}$.
- The number of all combinations of n distinct objects, taken r at a time is given by ${}^nC_r = \frac{n!}{(n-r)!r!}$
- ${}^nC_r = \frac{n!}{(n-r)!r!} = \frac{1}{r!}\left(\frac{n!}{(n-r)!}\right) = \frac{{}^nP_r}{r!}$.
- ${}^nC_r = {}^nC_{n-r}$ for $0 \le r \le n$.
- If x and y are non-negative integers such that $x + y = n$, then ${}^nC_x = {}^nC_y$.
- Let n and r be non-negative integers such that $r \le n$.
  Then, ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$.
- Let n and r be non-negative integers such that $1 \le r \le n$.
  Then, ${}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}$
- If $1 \le r \le n$, then $n\,{}^{n-1}C_{r-1} = (n - r + 1)\,{}^nC_{r-1}$
- ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$.
- If ${}^nC_x = {}^nC_y$ and $x \ne y$, then $x + y = n$.
- If n is even, then the greatest value of ${}^nC_r\,(0 \le r \le n)$ is ${}^nC_{n/2}$.
- If n is odd, then the greatest value of ${}^nC_r\ (0 \le r \le n)$ is ${}^nC_{\frac{n+1}{2}}$ or ${}^nC_{\frac{n-1}{2}}$.
- The number of ways of selecting one or more items from a group of n distinct items is $2^n - 1$.
- The number of ways of selecting r items out of n identical items is 1.
- The total number of ways of selecting zero or more from a group of n identical items is $(n + 1)$.
- The total number of selections of some or all out of $p + q + r$ items where p are alike of one kind, q are alike of second kind and rest are alike of third kind is
$$[(p + 1)(q + 1)(r + 1)]-1.$$
- The total number of ways of selecting one or more items from p identical items of one kind, q identical items of second kind, identical items of third kind and

n different items is $(p + 1)(q + 1)(r + 1)2^n - 1$.

- Number of ways in which $(m + n)$ items can be divided into two unequal groups containing m and n items is $\dfrac{(m+n)!}{m!n!}$.
- The number of ways in which $(m + n + p)$ items can be divided into unequal groups containing m, n, p items is ${}^{m+n+p}C_m \cdot {}^{n+p}C_n = \dfrac{(m+n+p)!}{m!\,n!\,p!}$.
- The number of ways to distribute $(m + n + p)$ items among 3 persons in the groups containing m, n and p items is

  $$= \text{(No. of ways to divide)} \times \text{(No. of groups)!} = \frac{(m+n+p)!}{m!\,n!\,p!} \times 3!.$$
- The number of ways in which mn different items can be divided equally into m groups, each containing n objects and the order of the groups is not important, is $\left(\dfrac{(mn)!}{(n!)^m}\right)\dfrac{1}{m!}$.
- The total number of ways of dividing n identical items among r persons, each one of whom, can receive 0, 1, 2 or more items ($\leq n$) is ${}^{n+r-1}C_{r-1}$
- The total number of ways of dividing n identical objects into r groups, if blank groups are allowed, is ${}^{n+r-1}C_{r-1}$
- The total number of ways of dividing n identical items among r persons, each one of whom, receives atleast one item is ${}^{n-1}C_{r-1}$
- The number of ways in which n identical items can be divided into r groups so that group contains more than or equal to m items and less than or equal to k($m < k$) is

  Coefficient of x^n in the expansion of $(x^m + x^{m+1} + ... + x^k)^r$.
- If n items are arranged in a row, then the number of ways in which they can be rearranged so that no one of them occupies the place assigned to it, is

  $$n!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+........+(-1)^n\frac{1}{n!}\right].$$
- If there are m items of one kind, n items of another kind and so on, then the number of ways of choosing r items out of these items is

  = Coefficients of x^r in $(1 + x + x^2 + ...+ x^m)(1 + x + x^2 + ... + x^n)$...
- If there are m items of one kind, n items of another kind and so, on then the number of ways of choosing r items out of these items such that atleast one item of each kind is included in every selections, is

  = Coefficient of x^r in $(x + x^2 + ... + x^m)(x + x^2 + ... + x^n)$....
- The number of ways of selecting r items from a group of n items in which p are identical items is $\begin{cases} {}^{n-p}C_r + {}^{n-p}C_{r-1} + {}^{n-p}C_{r-2} + \ldots + {}^{n-p}C_0, & \text{if } r \leq p \\ {}^{n-p}C_r + {}^{n-p}C_{r-1} + \ldots. + {}^{n-p}C_{r-p} & \text{, if } r > p \end{cases}$

End

6

CHAPTER

binomial theorem

Binomial Expression

An algebraic expression containing two terms is called a binomial expression.

Binomial Theorem for Positive Integral Index

- If x and a are real numbers, then for all $n \in N$,
  $(x + a)^n = {}^nC_0x^na^0 + {}^nC_1x^{n-1}a^1 + {}^nC_2x^{n-2}a^2$
  $+....+ {}^nC_rx^{n-r}a^r +...+ {}^nC_{n-1}x^1a^{n-1} + {}^nC_nx^0a^n$

  i.e. $(x+a)^n = \sum_{r=0}^{n} {}^nC_rx^{n-r}a^r.$

Some Important Conclusions from the Binomial Theorem

- ${}^nC_r = {}^nC_{n-r}$, where $r = 0, 1, 2, ..., n.$
- ${}^nC_0 = {}^nC_n$, ${}^nC_1 = {}^nC_{n-1}$, ${}^nC_2 = {}^nC_{n-2}$ and so on.
- $(x-a)^n = \sum_{r=0}^{n} (-1)^r \, {}^nC_rx^{n-r}a^r.$
- $(1+x)^n = \sum_{r=0}^{n} {}^nC_rx^r.$
- $(1+x)^n = \sum_{r=0}^{n} {}^nC_rx^{n-r}.$
- $(1-x)^n = \sum_{r=0}^{n} (-1)^r \, {}^nC_rx^r.$
- The coefficient of $(r + 1)^{\text{th}}$ term in the expansion of $(1 + x)^n$ is ${}^nC_r$.
- $(x + a)^n + (x - a)^n = 2[{}^nC_0\, x^n\, a^0 + {}^nC_2\, x^{n-2}\, a^2 + {}^nC_4\, x^{n-4}\, a^4 + ...].$
- $(x + a)^n - (x - a)^n = 2[{}^nC_1\, x^{n-1}\, a^1 + {}^nC_3\, x^{n-3}\, a^3 + ...].$
- If n is odd then $\{(x + a)^n + (x - a)^n\}$ and $\{(x + a)^n - (x - a)^n\}$ both have the same number of terms equal to $\left(\frac{n+1}{2}\right)$ whereas if n is even, then $\{(x + a)^n + (x - a)^n\}$ has $\left(\frac{n}{2}+1\right)$ terms and $\{(x + a)^n - (x - a)^n\}$ has $\left(\frac{n}{2}\right)$ terms.

General Term in a Binomial Expansion

- $(x + a)^n = {}^nC_0\, x^n\, a^0 + {}^nC_1\, x^{n-1}\, a^1 + {}^nC_2\, x^{n-2}\, a^2 + ... + {}^nC_r\, x^{n-r} + ... + {}^nC_n\, x^0\, a^n.$
  The first term $= {}^nC_0\, x^n\, a^0$
  The second term $= {}^nC_1\, x^{n-1}\, a^1$

The third term = $^nC_2\, x^{n-2}\, a^2$

The fourth term = $^nC_3\, x^{n-3}\, a^3$ and so on.

- Similarly in $(x - a)^n$, the general term, $T_{r+1} = (-1)^r\, {}^nC_r\, x^{n-r}\, a^r$.
- In expansion of $(x + a)^n$, the r^{th} term from the end is $(n - r + 2)^{\text{th}}$ term from starting.

Another Form of Binomial Expansion

- We have, $(x+a)^n = \sum_{s=0}^{n} {}^nC_s x^{n-s} a^s = \sum_{s=0}^{n} \frac{n!}{(n-s)!s!} x^{n-s} a^s$.
- Number of non-negative integral solutions of $x_1 + x_2 + \dots + x_r = n$ is $^{n+r-1}C_{r-1}$.
- **Multinomial Theorem**

We have $(x_1 + x_2)^n = \sum_{r_1 + r_2 = n} \frac{n!}{r_1! r_2!} x_1^{r_1} \cdot x_2^{r_2}$

$$\Rightarrow \quad (x_1 + x_2 + x_3 + \dots + x_k)^n = \sum_{r_1 + r_2 + \dots + r_k = n} \frac{n!}{r_1! r_2! r_3! \dots r_k!} \times x_1^{r_1} x_2^{r_2} x_3^{r_3} \dots x_k^{r_k}$$

General term is $\frac{n!}{r_1! r_2! r_3! \dots r_k!} x_1^{r_1} x_2^{r_2} \dots x_k^{r_k}$.

- $(x + a)^n$ contains $(n + 1)$ terms, therefore

I. If n is even, then $\left(\frac{n}{2}+1\right)^{\text{th}}$ term is the middle term.

II. If n is odd, then $\left(\frac{n+1}{2}\right)^{\text{th}}$ and $\left(\frac{n+3}{2}\right)^{\text{th}}$ terms are two middle terms.

Some Important Results

I. Coefficient of $(r + 1)^{\text{th}}$ term in the binomial expression of $(1 + x)^n$ is ^nC_r.

II. Coefficient of x^r in the binomial expansion of $(1 + x)^n$ is ^nC_r.

III. Coefficient of x^r in the expansion of $(1 - x)^n$ is $(-1)^r\, {}^nC_r$.

IV. Coefficient of $(r + 1)^{\text{th}}$ term in the expansion of $(1 - x)^n$ is $(-1)^r\, {}^nC_r$.

Greatest Term in the Expansion of $(x + a)^n$

- When $\dfrac{n+1}{1+\frac{x}{a}}$ is an integer.

$T_1 < T_2 < \dots < T_{m-1} < T_m = T_{m+1} > T_{m+2} > \dots > T_n$

$\therefore$ m^{th} and $(m + 1)^{\text{th}}$ terms are greatest terms.

- When $\dfrac{n+1}{1+\frac{x}{a}}$ is not an integer.

$T_1 < T_2 < T_3 < \dots < T_m = T_{m+1} > T_{m+2} > T_{m+3} \dots > T_{n+1}$

$(m + 1)^{\text{th}}$ term is the greatest term.

Algorithm for finding the Greatest Term

- Write T_{r+1} and T_r from the given expansion.
- Find $\frac{T_{r+1}}{T_r}$.
- Put $\frac{T_{r+1}}{T_r} > 1$.
- Solve the inequality in above step for r to get an inequality of the form $r < m$ or $r > m$.
- If m is an integer, then m^{th} and $(m + 1)^{th}$ terms are equal in magnitude and these two are the greatest terms. If m is not an integer, then obtain the integral part of m, say k. In this case, $(k + 1)^{th}$ term is the greatest term.

Properties of the Binomial Coefficients

- In the expansion of $(1+x)^n$ the coefficient of terms equidistant from the beginning and the end are equal.
- The sum of the binomial coefficients in the expansion of $(1+x)^n$ is $2^n \Rightarrow \sum_{r=0}^{n} {}^nC_r = 2^n$.
- $\sum_{r=0}^{n} {}^nC_r = 2^n$, $\sum_{r=1}^{n} {}^{n-1}C_{r-1} = 2^{n-1}$ etc.
- The sum of the coefficients of the odd terms in the expansion of $(1+x)^n$ is equal to the sum of the coefficient of the even terms and each is equal to 2^{n-1}.
- ${}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1} = \frac{n}{r} \cdot \frac{n-1}{r-1} \cdot {}^{n-2}C_{r-2}$ and so on.
- $C_0 - C_1 + C_2 - C_3 + C_4 - \ldots\ldots + (-1)^n C_n = 0$ *i.e.* $\sum_{r=0}^{n} (-1)^r \, {}^nC_r = 0$.
- $C_0 + \frac{C_1}{2} + \frac{C_2}{2^2} + \frac{C_3}{2^3} + \ldots\ldots + \frac{C_n}{2^n} = \left(\frac{3}{2}\right)^n$.
- $\sum_{r=0}^{n} (-1)^r \, {}^nC_r \left\{ \frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} + \ldots\ldots \text{ upto } m \text{ terms} \right\} = \frac{2^{mn} - 1}{2^{mn}(2^n - 1)}$

Binomial Theorem for any Index

- Let n be a rational number and x be a real number such that $|x| < 1$, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \ldots\ldots\infty.$$

Note :(i) The condition $|x| < 1$ is un-necessary, if n is a whole number while the same condition is essential if n is a rational number other than a whole number.
(ii) There are infinite number of terms in the expansion of $(1 + x)^n$, when n is a negative integer or a fraction.

(iii) If n is a positive integer the above expansion contains $(n + 1)$ terms and coincides with
$(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots\ldots + {}^nC_n x^n$, because ${}^nC_0 = 1$, ${}^nC_1 = n$,

$$^nC_2 = \frac{n(n-1)}{2!},\ ^nC_3 = \frac{n(n-1)(n-2)}{3!}, \ldots\ldots .$$

General Term in the Expansion of (1 + x)^n

The general term in the expansion of $(1 + x)^n$ is given by

$$T_{r+1} = \frac{n(n-1)(n-2)\ldots\ldots(n-(r-1))}{r!}x^r.$$

Expansion of (x + a)^n for any rational index

- **Case I :** When $x > a$ *i.e.* $\frac{a}{x} < 1$.

  In this case $(x+a)^n = \left\{x\left(1+\frac{a}{x}\right)\right\}^n = x^n\left(1+\frac{a}{x}\right)^n$

  $$= x^n\left\{1+n\cdot\frac{a}{x}+\frac{n(n-1)}{2!}\left(\frac{a}{x}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{a}{x}\right)^3+\ldots\ldots\right\}$$

- **Case II :** When $x < a$ *i.e.* $\frac{x}{a} < 1$.

  In this case $(x+a)^n = \left\{a\left(1+\frac{x}{a}\right)\right\}^n = a^n\left(1+\frac{x}{a}\right)^n$

  $$= a^n\left\{1+n\cdot\frac{x}{a}+\frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{x}{a}\right)^3+\ldots\ldots\right\}$$

- In the expansion of $(1 + x)^{-n}$, $T_{r+1} = \frac{(-1)^r n(n+1)(n+2)\ldots.(n+r-1)}{r!}x^r$.
- In the expansion of $(1 - x)^{-n}$, $T_{r+1} = \frac{n(n+1)(n+2)\ldots.(n+r-1)}{r!}x^r$.
- In the expansion of $(1 - x)^{n}$, $T_{r+1} = \frac{(-1)^r n(n-1)(n-2)\ldots.(n-r+1)}{r!}x^r$.

Difference Between the Binomial Theorem for Positive Integer exponent and for General Exponent

- If n is a positive integer, then $(1 + x)^n$ contains $(n + 1)$ terms *i.e.* a finite number of terms. When n is general exponent, then the expansion of $(1 + x)^n$ contains infinitely many terms.
- When n is a positive integer, the expansion of $(1 + x)^n$ is valid for all values of x. If n is general exponent, the expansion of $(1+ x)^n$ is valid for the values of x satisfying the condition $|x| < 1$.

End

7

CHAPTER

exponential and logarithmic series

- The sum of the series $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+$ ....∞ is denoted by the number e.

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

$$\therefore \quad e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+ ....= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

- e lies between 2 and 3.
- e is an irrational (incommensurable) number.
- $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+ ..... \infty$.

Exponential Theorem

- Let $a > 0$, then for all real values of x,

$$a^x = 1 + x(\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + \frac{x^3}{3!}(\log_e a)^3 + ...\infty.$$

- $\frac{e+e^{-1}}{2} = 1+\frac{1}{2!}+\frac{1}{4!}+ ......\infty$ or $\frac{e+e^{-1}}{2} = \sum_{n=0}^{\infty}\frac{1}{(2n)!}$
- $\frac{e-e^{-1}}{2} = 1+\frac{1}{3!}+\frac{1}{5!}+ ......\infty$ or $\frac{e-e^{-1}}{2} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}$

Some Important Formulae

(i) $\sum_{n=0}^{\infty}\frac{1}{n!} = e = \sum_{n=0}^{\infty}\frac{1}{(n-1)!} = \sum_{n=0}^{\infty}\frac{1}{(n-k)!} = e.$

(ii) $\sum_{n=1}^{\infty}\frac{1}{n!} = \frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+ .....\infty = e-1.$

(iii) $\sum_{n=2}^{\infty}\frac{1}{n!} = \frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+ .....\infty = e-2.$

(iv) $\sum_{n=0}^{\infty}\frac{1}{(n+1)!} = \frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+ .....\infty = e-1.$

(v) $\sum_{n=0}^{\infty}\frac{1}{(n+2)!} = \frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!} .....\infty = e-2.$

(vi) $\sum_{n=1}^{\infty}\frac{1}{(n+1)!} = \frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!} .....\infty = e-2.$

(vii) $\sum_{n=0}^{\infty}\frac{1}{(2n)!} = 1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+ ..... = \frac{e+e^{-1}}{2} = \sum_{n=1}^{\infty}\frac{1}{(2n-2)!}.$

(viii) $\sum_{n=1}^{\infty}\frac{1}{(2n-1)!}=\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+.....=\frac{e-e^{-1}}{2}=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}$.

(ix) $e^{ax}=1+\frac{(ax)}{1!}+\frac{(ax)^2}{2!}+\frac{(ax)^3}{3!}+.....+\frac{(ax)^n}{n!}+.....\infty$.

(x) $\sum_{n=0}^{\infty}\frac{n}{n!}=e=\sum_{n=0}^{\infty}\frac{n}{n!}$.

(xi) $\sum_{n=0}^{\infty}\frac{n^2}{n!}=2e=\sum_{n=1}^{\infty}\frac{n^2}{n!}$.

(xii) $\sum_{n=0}^{\infty}\frac{n^3}{n!}=5e=\sum_{n=1}^{\infty}\frac{n^3}{n!}$.

(xiii) $\sum_{n=0}^{\infty}\frac{n^4}{n!}=15e=\sum_{n=1}^{\infty}\frac{n^4}{n!}$.

Logarithmic Series

- $\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+......\infty$.

  $\therefore\quad \log_e(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$.

  $\Rightarrow\quad \log_e(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}......\infty$.

  $\Rightarrow\quad -\log_e(1-x)=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}......\infty$.

  $\Rightarrow\quad \log_e\left(\frac{1+x}{1-x}\right)=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+......\infty\right)$.

  $\Rightarrow\quad \log_e 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}......\infty$.

- In the exponential series $e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+.....\infty$, all terms carry positive signs whereas in the logarithmic series $\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+.....$, the terms are alternatively positive and negative.
- In the exponential series the denominators of the terms involve factorials of natural numbers. But in the logarithmic series the terms do not contain factorials.
- The exponential series is valid for all the values of x. The log series is valid when $|x|<1$.

End

8

CHAPTER

matrices

- **Matrix**
  A set of mn numbers (real or imaginary) arranged in the form of a rectangular array of m rows and n columns is called an $m \times n$ matrix (to be read as 'm by n' matrix).
- In compact form the above matrix is represented by $[a_{ij}]_{m \times n}$ or $A = [a_{ij}]$.

Types of Matrices

- **Row Matrix :** A matrix having only one row is called a row-matrix or a row-vector.
- **Column Matrix :** A matrix having only one column is called a column matrix or a column-vector.
- **Square Matrix :** A matrix in which the number of rows is equal to the number of columns, say n, is called a square matrix of order n.
- A square matrix of order n is also called a n-rowed square matrix.
- The elements $a_{11}, a_{22}, ..., a_{nn}$ are called the diagonal elements and the line along which they lie is called the principal diagonal or leading diagonal of the matrix.
- **Diagonal Matrix :** A square matrix $A = [a_{ij}]_{n\times n}$ is called a diagonal matrix if all the elements, except those in the leading diagonal, are zero *i.e.* $a_{ij} = 0$ for all $i \neq j$.
- **Scalar Matrix :** A square matrix $A = [a_{ij}]_{n\times n}$ is called a scalar matrix if
  (i) $a_{ij} = 0$ for all $i \neq j$, and (ii) $a_{ij} = C$ for all $i = j$, where $C \neq 0$.
- **Identity or Unit Matrix :** A square matrix $A = [a_{ij}]_{n\times n}$ is called identity or unit matrix if
  (i) $a_{ij} = 0$ for all $i \neq j$ and (ii) $a_{ii} = 1$ for all i.
- **Null Matrix :** A matrix whose all elements are zero is called a null matrix or a zero matrix.
- **Upper Triangular Matrix :** A square matrix $A = [a_{ij}]$ is called an upper triangular matrix if $a_{ij} = 0$ for all $i > j$.
- **Lower Triangular Matrix :** A square matrix $A = [a_{ij}]$ is called a lower triangular matrix if $a_{ij} = 0$ for all $i < j$.
  A triangular matrix $A = [a_{ij}]_{n \times n}$ is called a strictly triangular if $a_{ii} = 0$ for all $i = 1, 2, ..., n$.

Equality of Matrices

- Two matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{r \times s}$ are equal if
  (i) $m = r$, *i.e.*, the number of rows in A equals the number of rows in B.

(ii) $n = s$, *i.e.*, the number of columns in A equals the number of columns in B.
(iii) $a_{ij} = b_{ij}$ for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$.

- If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are two matrices of the same order, their sum $A + B$ is defined to be the matrix of order $m \times n$ such that
$(A + B)_{ij} = a_{ij} + b_{ij}$ for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$.
- The sum of two matrices is defined only when they are of the same order.

Properties of Matrix Addition

- **Matrix Addition is Commutative :** If A and B are two $m \times n$ matrices, then $A + B = B + A$.
- **Matrix Addition is Associative :** If A, B, C are three matrices of the same order, then $(A + B) + C = A + (B + C)$.
- **Existence of Identity :** The null matrix is the identity element for matrix addition, *i.e.*, $A + O = A = O + A$.
- **Existence of Inverse :** For every matrix $A = [a_{ij}]_{m\times n}$ there exists a matrix $[-a_{ij}]_{m\times n}$ denoted by $-A$, such that $A + (-A) = O = (-A) + A$.

Cancellation Laws Hold Good in Case of Addition of Matrices

- If A, B, C are matrices of the same order, then
$A + B = A + C \Rightarrow B = C$ (left cancellation law)
and $B + A = C + A \Rightarrow B = C$ (right cancellation law)
- **Multiplication of a Matrix by a Scalar (Scalar Multiplication)**
Let $A = [a_{ij}]$ be an $m \times n$ matrix and k be any number called a scalar. Then the matrix obtained by multiplying every element of A by k is called the scalar multiple of A by k and is denoted by kA. Thus $kA = [ka_{ij}]_{m\times n}$.

Properties of Scalar Multiplication

- If $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$ are two matrices and k, l are scalars, then
(i) $k(A + B) = kA + kB$
(ii) $(k + l)A = kA + lA$
(iii) $(kl)A = k(lA) = l(kA)$
(iv) $(-k)A = -(kA) = k(-A)$
(v) $IA = A$
(vi) $(-I)A = -A$.

Subtraction of Matrices

- For two matrices A and B of the same order, we define $A - B = A + (-B)$.
- If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ are two matrices of order $m \times n$ and $n \times p$ respectively, then their product AB is of order $m \times p$ and is defined as

$$(AB)_{ij} = \sum_{r=1}^{n} a_{ir}\, b_{rj} = a_{i1} b_{1j} + a_{i2} \cdot b_{2j} + ... + a_{in} b_{nj}.$$

- If A and B are two matrices such that AB exists, then BA may or not exist.

Properties of Matrix Multiplication

- Matrix multiplication is not commutative in general *i.e.* $AB \neq BA$.

- Matrix multiplication is associative *i.e.,* $(AB)C = A(BC)$, whenever both sides are defined.
- Matrix multiplication is distributive over matrix addition
  i.e., (i) $A(B + C) = AB + AC$,
  (ii) $(A + B)C = AB + AC$, whenever both sides of equality are defined.
- If A is an $m \times n$ matrix, then $I_m A = A = A I_n$.
- The product of two matrices can be the null matrix while neither of them is the null matrix.
- If A is $m \times n$ matrix and O is a null matrix, then
  (i) $A_{m \times n} O_{n \times p} = O_{m \times p}$ (ii) $O_{p \times m} A_{m \times n} = O_{p \times n}$.
- In the case of matrix multiplication if $AB = O$, then it does not necessarily imply that $BA = O$.
- **Positive integral powers of a square matrix :** Let A be a square matrix. Then we define
  (i) $A^1 = A$ and
  (ii) $A^{n+1} = A^n \cdot A$, where $n \in N$.

Matrix polynomial

- Let $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots + a_{n-1} x + a_n$ be a polynomial and let A be a square matrix of order n. Then
  $f(A) = a_0 A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots\ldots + a_{n-1} A + a_n I_n$ is called a matrix polynomial.
- **Transpose of a matrix**

  Let $A = [a_{ij}]$ be an $m \times n$ matrix. Then the transpose of A, denoted by A^T or A', is an $n \times m$ matrix such that

  $(A^T)_{ji} = a_{ij}$ for all $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

  Thus, A^T is obtained from A by changing its rows into columns and its columns into rows.

Properties of Transpose

- Let A and B be two matrices. Then
  (i) $(A^T)^T = A$
  (ii) $(A + B)^T = A^T + B^T$, where A and B being of the same order.
  (iii) $(kA)^T = kA^T$, k be any scalar (real or complex)
  (iv) $(AB)^T = B^T A^T$, A and B being conformable for the product AB. (Reversal law).
  (v) $(ABC)^T = C^T B^T A^T$.
- **Symmetric Matrix :** A square matrix $A = [a_{ij}]$ is called a symmetric matrix if $a_{ij} = a_{ji}$ for all i, j.
- A square matrix A is a symmetric matrix iff $A^T = A$.
- **Skew-Symmetric Matrix.** A square matrix $A = [a_{ij}]$ is a skew-symmetric matrix if $a_{ij} = -a_{ji}$ for all i, j.
  A square matrix A is a skew-symmetric matrix iff $A^T = -A$.

- **Determinants :** Every square matrix can be associated to an expression or a number which is known as its determinant. If $A = [a_{ij}]$ is a square matrix of order n, then the determinant of A is denoted by det A or $|A|$ or

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & & & \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}$$

- **Determinant of a Square Matrix of Order 1 :** If $A = [a_{11}]$ is a square matrix of order 1, then the determinant of A is defined as $|A| = a_{11}$ or $|a_{11}| = a_{11}$.
- **Determinant of a Square Matrix of Order 2 :** If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a square matrix of order 2, then the expression $a_{11}\,a_{22} - a_{12}\,a_{21}$ is defined as the determinant of A.
- Only square matrices have determinants. The matrices which are not square do not have determinants.
- The determinant of a square matrix of order 3 can be expanded along any row or column.
- If a row or a column of a determinant consists of all zeroes, then the value of the determinant is zero.
- **Sarrus Diagram Method :** The determinant of a square matrix of order 3 can be evaluated by the following procedure:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\,a_{22}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{13}\,a_{21}\,a_{32} - a_{13}\,a_{22}a_{31} - a_{11}\,a_{23}\,a_{32} - a_{12}\,a_{21}\,a_{33}.$$

This method does not work for determinant of order greater than 3.
- **Determinant of a square matrix of order 4 or more.**
To evaluate the determinant of a square matrix of order 4 or more than 4 we follow the same procedure as discussed in evaluating the determinant of a square matrix of order 3.
- A square matrix is a singular matrix if its determinant is zero. Otherwise, it is a non-singular matrix.
- **Adjoint of a square matrix**
Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be co-factor of a_{ij} in A. Then the transpose of the matrix of co-factor of elements of A is called the adjoint of A and is denoted by adj A.
- adj $A = [C_{ij}]^T \Rightarrow (\text{adj } A)_{ij} = C_{ji}$ = cofactor of a_{ji} in A.
- Let A be a square matrix of order n. Then $A(\text{adj } A) = |A|\, I_n = (\text{adj } A)A$.

Inverse of a matrix

- A square marix of order n is invertible if there exists a square matrix, B of the same order such that $AB = I_n = BA$
  In such a case, we say that the inverse of A is B and we write, $A^{-1} = B$.
- Every invertible matrix possesses a unique inverse.
- A square matrix is invertible iff it is non-singular.
- The inverse of A is given by $A^{-1} = \frac{1}{|A|} \cdot \text{adj} A$.
- (Cancellation Laws) : Let A, B, C be square matrices of the same order n. If A is a non-singular matrix, then
  (i) $AB = AC \Rightarrow B = C$ [Left cancellation law].
  (ii) $BA = CA \Rightarrow B = C$ [Right cancellation law].
- $AB = AC \Rightarrow B = C$ is true only when $|A| \neq 0$. Otherwise we can find matrices such that $AB = AC$ but $B \neq C$.
- (Reversal law) : If A and B are invertible matrices of the same order, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
- If A, B, C are invertible matrices, then $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.
- If A is an invertible square matrix, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$.
- Let A be a non-singular square matrix of order n. Then $|\text{adj } A| = |A|^{n-1}$.
- If A and B are non-singular square matrices of the same order, then adj (AB) = (adj B)(adj A).
- If A is an invertible square matirx, then adj A^T = (adj $A)^T$.
- If A is a non-singular square matrix, then adj (adj A) = $|A|^{n-2}A$.

Elementary Transformation

- The following three operations applied on the rows (columns) of a matrix are called elementary row (column) transformations.

(i) Interchange of any two rows (columns)
If i^{th} row (column) of a matrix is interchanged with the j^{th} row (column), it will be denoted by $R_i \leftrightarrow R_j (C_i \leftrightarrow C_j)$.

(ii) Multiplying all elements of a row (column) of a matrix by a non-zero scalar : $R_i \to kR_i$ or $C_i \to C_i(k)$.

(iii) If the elements of i^{th} row (column) are multiplied by a non-zero scalar k, it will be denoted by $R_i \to R_i (k)$ $[C_i \to C_i(k)]$ or $R_i \to kR_i[C_i \to kC_i]$.

- Adding to the elements of a row (column), the corresponding elements of any other row (column) multiplied by any scalar k.
  If k times the elements of j^{th} row (column) are added to the corresponding elements of the i^{th} row (column), it will be denoted by $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$.
- If a matrix B is obtained from a matrix A by one or more elementary transformations, then A and B are equivalent matrices and we write $A \sim B$.
- An elementary transformation is called a row transformation or a column transformation according as it is applied to rows or columns.

- **Elementary Matrix :** A matrix obtained from an identity matrix by a single elementary operation (transformation) is called an elementary matrix.
- Every elementary row (column) transformation of an $m \times n$ matrix (not identity matrix) can be obtained by pre-multiplication (post-multiplication) with the corresponding elementary matrix obtained from the identity matrix $I_m(I_n)$ by subjecting it to the same elementary row (column) transformation.
- Let $C = AB$ be a product of two matrices. Any elementary row (column) transformation of AB can be obtained by subjecting the pre-factor A (post-factor B) to the same elementary row (column) transformation.
- **Algorithm for finding the inverse of a non-singular matrix by elementary row transformations**
  Let A be a non-singular matrix of order n.
  Step I : Write $A = I_n A$.
  Step II : Perform a sequence of elementary row operations successively on A on the LHS and the pre-factor I_n on the RHS till we obtain the result $I_n = BA$.
  Step III : Write $A^{-1} = B$.
- **Orthogonal Matrix :** A square matrix A is called an orthogonal matrix if $AA^T = A^T A = I$.
- **Trace of a Matrix :** Let $A = [a_{ij}]_{n \times n}$ be a square matrix. Then the sum of all diagonal elements of A is called the trace of A and is denoted by $tr(A)$. Thus,

$$tr(A) = \sum_{i=1}^{n} a_{ii}.$$

- **Idempotent Matrix :** A square matrix A is called an idempotent matrix if $A^2 = A$.
- **Submatrix :** Let A be an $m \times n$ matrix. Then a matrix obtained by leaving some rows or columns or both of A is called a submatrix of A.
- **Rank of a Matrix :** A number r is said to be the rank of an $m \times n$ matrix A if
  (i) every square submatrix of order $(r + 1)$ or more is singular, and
  (ii) there exists atleast one square submatrix of order r which is non-singular.
  Thus, the rank of a matrix is the order of the highest order non-singular square submatrix.
- If A is a non-singular square matrix of order n, then its rank is n. Consequently rank of the identity matrix of order n is also n.
- The rank of the null matrix is not defined and the rank of every non-null matrix is greater than or equal to 1.
- The rank of a matrix is same as the rank of its transpose *i.e.* $r(A) = r(A^T)$.
- Elementary transformation do not alter the rank of matrix.
- **Equivalent Matrices :** Two matrices A and B are equivalent if one can be obtained from the other by a sequence of elementary row transformations.
- **Echelon Form of Matrix :** A matrix A is said to be in Echelon form if either A is the null matrix or A satisfies the following conditions :
  (i) Every non-zero row in A precedes every zero row.
  (ii) The number of zeroes before the first non-zero element in a row is less than the number of such zeroes in the next row.

It can be easily proved that the rank of a matrix in Echelon form is equal to the number of non-zero rows of the matrix.

- **Algorithm for Finding the Rank of a Matrix :** Let $A = [a_{ij}]$ be an $m \times n$ matrix.
  Step I : Using elementary row transformations make $a_{11} = 1$.
  Step II : Make $a_{21}, a_{31}, \ldots, a_{m1}$ all zeroes by using elementary transformation $R_2 \to R_2 - a_{21} R_1, R_3 \to R_3 - a_{31}R_1, \ldots, R_m \to R_m - a_{m1}R_1$.
  Step III : Make $a_{22} = 1$ by using elementary row transformations.
  Step IV : Make $a_{32}, a_{42}, \ldots, a_{m2}$ all zeroes by using $R_3 \to R_3 - a_{32} R_1$, $R_4 \to R_4 - a_{42} R_2, \ldots, R_m \to R_m - a_{m2} R_2$.
  The process used in steps III and IV is repeated upto $(m-1)^{\text{th}}$ row. Finally we obtain a matrix in echelon form which is equivalent to the matrix A. The rank of A will be equal to the number of non-zeroes rows in it.
- Consider the following system of m linear equations in n unknowns :

$$\begin{array}{l} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1 \\ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n = b_2 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_n. \end{array} \quad \ldots\text{(i)}$$

  This system of equations can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

  or $AX = B$,

$$\text{where } A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}_{m \times n} ; X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

  The $m \times n$ matrix A is called the coefficient matrix of the system of linear equations.
- A set of values of the variables $x_1, x_2, \ldots, x_n$ which simultaneously satisfy all the equations is called a solution of the system of equations.
- **Consistent System :** If the system of equations has one or more solutions, then it is said to be a consistent system of equations, otherwise it is an inconsistent system of equations.
- **Homogeneous and Non-Homogeneous Systems of Linear Equations :** A system of equations $AX = B$ is called a homogeneous system if $B = 0$. Otherwise, it is called a non-homogeneous system of equations.
- Let $AX = B$ be a system of n linear equations with n unknowns. If A is non-singular, then A^{-1} exists.
- Thus, the system of equations, $AX = B$ has a solution given by $X = A^{-1}B$.
- If A is a non-singular matrix, then the system of equations given by $AX = B$ has a unique solution given by $X = A^{-1}B$.

- If A is a singular matrix, then the system of equations given by $AX = B$ may be consistent with infinitely many solutions or it may be inconsistent also.
- Let $AX = B$ be a system of n-linear equations in n unknowns.
  (i) If $|A| \neq 0$, then the system is consistent and has a unique solution given by $X = A^{-1}B$.
  (ii) If $|A| = 0$ and (adj A) $B = 0$, then the system is consistent and has infinitely many solutions.
  (iii) If $|A| = 0$ and (adj A) $B \neq 0$, then the system is inconsistent.

Algorithm for solving a non-homogeneous system of linear equations

- Write the given system of equations in matrix form $AX = B$ and obtain A, B.
- Find $|A|$
- If $|A| \neq 0$, then write "the system is consistent with unique solution". Obtain the unique solution by the following procedure :

  Find A^{-1} by using $= \frac{1}{|A|} \text{adj} A$.

  Obtain the unique solution given by $X = A^{-1}B$.
- If $|A| = 0$, then write "the system is either consistent with infinitely many solutions or it is consistent.
  To distinguish these two proceed as follows :
  Find (adj A)B.
  If (adj A)$B \neq 0$, then write "the system is inconsistent".
  If (adj A)$B = 0$, then the system is consistent with infinitely many solutions. To find these solutions proceed as follows.
  Put $z = k$ (any real number) and take any two equations out of three equations. Solve these equations for x and y. Let the values of x and y be λ and μ respectively. Then, $x = \lambda$, $y = \mu$, $z = k$ is the required solution.
- The system of linear equations $AX = B$ is consistent iff the rank of the augmented matrix $[A : B]$ is equal to the rank of the coefficient matrix A.
- Let $AX = B$ be a system of m simultaneous linear equations in n unknowns.

  Case I : If $m > n$, then

  (i) If $r(A) = r(A : B) = n$, then system of linear equations has a unique solution.

  (ii) If $r(A) = r(A : B) = r < n$, then system of linear equations is consistent and has infinite number of solutions. In fact, in this case $(n - r)$ variables can be assigned arbitrary values.

  (iii) If $r(A) \neq r(A : B)$, then the system of linear equations is inconsistent *i.e.* it has no solution.

  Case II : If $m < n$ and $r(A) = r(A : B) = r$, then $r \leq m < n$ and so from (ii) in case I, there are infinite number of solutions.
- **Algorithm for solving a non-homogeneous system $AX = B$, of linear equations by rank method**
  Step I : Obtain A, B.
  Step II : Write the Augmented matrix $[A: B]$.
  Step III : Reduce the augmented matrix to echelon form by applying a sequence

of elementary row-operations.

Step IV : Determine the number of non-zero rows in A and $[A : B]$ to determine the ranks of A and $[A : B]$ respectively.

Step V : If $r(A) \neq r(A : B)$ then write "the system is inconsistent" STOP else write "the system is consistent" go to step VI.

Step VI : If $r(A) = r(A : B)$ = number of unknowns, then the system has a unique solution which can be obtained by back substitution.

- If $r(A) = r(A : B) <$ number of unknowns, then the system has an infinite number of solutions which can also be obtained by back substitution.

Solution of a Homogeneous System of Linear Equations

- **Matrix Method :** Let $AX = O$ be a homogeneous system of n-linear equations with a unknowns. If A is a non-singular matrix, then A^{-1} exists.
- A system $AX = 0$ of n homogeneous linear equations in n unknowns has non-trivial solution iff the coefficient matrix A is singular.

Algorithm for Solving a Homogeneous System of Linear Equations

- Let $AX = 0$ be a homogeneous system of 3 linear equations in 3 unknowns. The solve this system of equations we proceed as follows :

  Step I : Write the given system of equations in matrix form $AX = O$ and obtain A.

  Step II : Find $|A|$.

  Step III : If $|A| \neq 0$, the write "the system is consistent with unique solution $x = y = z = 0$." STOP else go to step IV.

  Step IV : The system of equations has infinitely many solutions. To find these solutions proceed as follows :

  Put $z = k$ (any real number) an solve any two equations for x and y in terms of k. The values of x and y so obtained with $z = k$ gives a solution of the given system of equations.

  Note : It should be noted here that a homogeneous system of equations is never inconsistent.
- **Rank Method :** In case of a homogeneous system of linear equations the rank of the augmented matrix is always same as that of the coefficient matrix. So a homogeneous system of linear equations is always consistent.
- If $r(A) = n$ = number of variables, then $AX = O$ has a unique solution $X = 0$ *i.e.* $x_1 = x_2 = \ldots\ldots = x_n = 0$.
- If $r(A) = r < n$ (= number of variables), then the system of equations has an infinite number of solutions.

End

9

CHAPTER

determinants

- Every square matrix can be associated to an expression or a number which is known as its determinant. If $A = [a_{ij}]$ is a square matrix of order n, then the determinant of A is denoted by det A or $|A|$ or

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & & & & \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & & \vdots & & \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix}$$

- **Determinant of a Square Matrix of Order 1 :** If $A = [a_{11}]$ is a square matrix of order 1, then the determinant of A is defined as $|A| = a_{11}$ or $|a_{11}| = a_{11}$.
- **Determinant of a Square Matrix of Order 2 :** If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a square matrix of order 2, then the expression $a_{11}\,a_{22} - a_{12}\,a_{21}$ is defined as the determinant of A *i.e.* $|A| = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}$ ....(i)

  Thus, the determinant of a square matrix of order 2 is equal to the product of the diagonal elements minus the product of off-diagonal elements.
- Only square matrices have determinants. The matrices which are not square do not have determinants.
- The determinant of a square matrix of order 3 can be expanded along any row or column.
- If a row or a column of a determinant consists of all zeroes, the value of the determinant is zero.
- **Singular matrix :** A square matrix is a singular matrix if its determinant is zero. Otherwise, it is a non-singular matrix.
- **Minor :** Let $A = [a_{ij}]$ be a square matrix of order n. Then the minor M_{ij} of a_{ij} in A is the determinant of the square sub-matrix of order $(n - 1)$ obtained by leaving i^{th} row and j^{th} column of A.
- **Cofactor :** Let $A = [a_{ij}]$ be a square matrix of order n. Then the co-factor C_{ij} of a_{ij} in A is equal to $(-1)^{i+j}$ times the determinant of the sub-matrix of order $(n - 1)$ obtained by leaving i^{th} row and j^{th} column of A.

Properties of Determinants

- Let $A = [a_{ij}]$ be a square matrix of order n, then the sum of the product of elements of any row (column) with their co-factors is always equal to $|A|$ or det(A) *i.e.*

$$\sum_{j=1}^{n} a_{ij} C_{ij} = |A| \quad \text{and} \quad \sum_{i=1}^{n} a_{ij} C_{ij} = |A|.$$

- Let $A = [a_{ij}]$ be a square matrix of order n, then the sum of the product of elements of any row (column) with the co-factors of the corresponding elements of some other row (column) is zero *i.e.* $\sum_{j=1}^{n} a_{ij} C_{kj} = 0$ and $\sum_{i=1}^{n} a_{ij} C_{ik} = 0$.
- Let $A = [a_{ij}]$ be a square matrix of order n, then $|A| = |AT|$.
- The value of a determinant remains unchanged if its rows and columns are interchanged.
- Let $A = [a_{ij}]$ be a square matrix of order n (≥ 2) and let B a matrix obtained from A by interchanging any two rows (columns) of A, then $|B| = -|A|$.
- If any two rows (columns) of a determinant are interchanged, then the value of the determinant changes by minus sign only.
- If any two rows (columns) of a square matrix $A = [a_{ij}]$ of order n (≥ 2) are identical, then its determinant is zero *i.e.* $|A| = 0$. If any two rows or columns of a determinant are identical, then its value is zero.
- Let $A = [a_{ij}]$ be a square matrix of order n, and let B be the matrix obtained from A by multiplying each element of a row (column) of A by a scalar k, then $|B| = k\,|A|$.
- If each element of a row (column) of a determinant is multiplied by a constant k, then the value of the new determinant is k times the value of the original determinant. Let $A = [a_{ij}]$ be a square matrix of order n, then $|kA| = k^n|A|$.
- Let A be a square matrix such that each element of a row (column) of A is expressed as the sum of two or more terms. Then the determinant of A can be expressed as the sum of the determinants of two or more matrices of the same order.
- If each element of a row (column) of a determinant is expressed as a sum of two or more terms, then the determinant can be expressed as the sum of two or more determinants.
- Let A be a square matrix and B be a matrix obtained from A by adding to a row (column) of A a scalar multiple of another row (column) of A, then $|B| = |A|$.
- If each element of a row (column) of a determinant is multiplied by the same constant and then added to the corresponding elements of some other row (column), then the value of the determinant remains same.
- Let A be a square matrix of order $n(\geq 2)$ such that each element in a row (column) of A is zero, then $|A| = 0$. If each element of a row (column) of a determinant is zero, then its value is zero.
- If $A = [a_{ij}]$ is a diagonal matrix of order n (≥ 2), then $|A| = a_{11} \cdot a_{22} \cdot a_{33} \ldots\ldots a_{nn}$.
- If A and B are square matrices of the same order, then $|AB| = |A||B|$.

- If $A = [a_{ij}]$ is a triangular matrix of order n, then $|A| = a_{11} \cdot a_{22} \cdot a_{33} \ldots\ldots a_{nn}$.

Differentiation of Determinants

- Let $\Delta(x) = \begin{vmatrix} f_1(x) & g_1(x) \\ f_2(x) & g_2(x) \end{vmatrix}$, where $f_1(x), f_2(x), g_1(x)$ and $g_2(x)$ are functions of x.

  Then, $\Delta'(x) = \begin{vmatrix} f_1'(x) & g_1'(x) \\ f_2(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(g) & g_1(x) \\ f_2'(x) & g_2'(x) \end{vmatrix}$

  Also, $\Delta'(x) = \begin{vmatrix} f_1'(x) & g_1(x) \\ f_2'(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(g) & g_1'(x) \\ f_2(x) & g_2'(x) \end{vmatrix}$

  Thus, to differentiate a determinant, we differentiate one row (or column) at a time, keeping others unchanged.

Application of Determinants to Co-ordinate Geometry

- **Area of a Triangle :** We know that the area of triangle whose vertices are $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is given by the expression:

  $\Delta = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$.

- The area of a triangle having vertices at $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is given by

  $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

- **Condition of Collinearity of Three Points**

  Let $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ be three points. Then A, B, C are collinear $\Leftrightarrow$ area of triangle $ABC = 0$

  $\Leftrightarrow \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$

Equation of a Line Passing Through Two given Points

- Let the two points be $A(x_1, y_1)$ and $B(x_2, y_2)$. Let $P(x, y)$ be any point on the line joining A and B. The points P, A and B are collinear.

  $\therefore \quad \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

- The equation of the line joining points (x_1, y_1) and (x_2, y_2) is given by

  $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$

Applications of Determinants in Solving a System of Linear Equations

- Consider a system of simultaneous linear equations given by

$$\left.\begin{aligned} a_1x+b_1y+c_1z &= d_1 \\ a_2x+b_2y+c_2z &= d_2 \\ a_3x+b_3y+c_3z &= d_3 \end{aligned}\right\} \quad \text{.....(i)}$$

A set of values of the variables x, y, z which simultaneously satisfy these three equations is called a solution.

- A system of linear equations may have a unique solution, or many solutions, or no solution at all. If it has a solution (whether unique or not) the system is said to be a **consistent.** If it has no solution, it is called an **inconsistent** system.
- If $d_1 = d_2 = d_3 = 0$ in (i), then the system of equations is said to be a **homogeneous system.** Otherwise it is called a **non-homogeneous system** of equations.
- **(Cramer's rule) :** The solution of the system of simultaneous linear equations

$$a_1x + b_1y = c_1 \quad \text{.......(i)}$$
$$a_2x + b_2y = c_2 \quad \text{...... (ii)}$$

is given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$, where $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ $D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$

and $D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ provided that $D \neq 0$.

- $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is the determinant of the coefficient matrix $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.
- The determinant D_1 is obtained by replacing first column in D by the column of the right hand side of the given equations.
- The determinant D_2 is obtained by replacing the second column in D by the right most column in the given system of equations.

Solution of system of linear equation in three variables by Cramer's Rule

- **(Cramer's Rule) :** The solution of the system of linear equations

$a_1x + b_1y + c_1z = d_1$ ....(i)

$a_2x + b_2y + c_2z = d_2$ ....(ii)

$a_3x + b_3y + c_3z = d_3$ ....(iii)

is given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$, where

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

provided that $D \neq 0$.

- **(Cramer's Rule) :** Let there be a system of n simultaneous linear equations n unknowns given by

$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$

$a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n$

Let $D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$ and let D_j be the determinant obtained from D after replacing the j^{th} column by $\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$. Then, $x_1 = \dfrac{D_1}{D}, x_2 = \dfrac{D_2}{D}, \ldots, x_n = \dfrac{D_n}{D}$, provided that $D \neq 0$.

Conditions for Consistency

- **Case I.** For a system of 2 simultaneous linear equations with 2 unknowns :

  (i) If $D \neq 0$, then the given system of equations is consistent and has a unique solution given by $x = \dfrac{D_1}{D}, y = \dfrac{D_2}{D}$.

  (ii) If $D = 0$ and $D_1 = D_2 = 0$, then the system is consistent and has infinitely many solutions.

  (iii) If $D = 0$ and one of D_1 and D_2 is non-zero, then the system is inconsistent.

- **Case II.** For a system of 3 simultaneous linear equations in three unknowns :

  (i) If $D \neq 0$, then the given system of equations is consistent and has a unique solution given by $x = \dfrac{D_1}{D}, y = \dfrac{D_2}{D}$ and $z = \dfrac{D_3}{D}$.

  (ii) If $D = 0$ and $D_1 = D_2 = D_3 = 0$, then the given system of equations is consistent with infinitely many solutions.

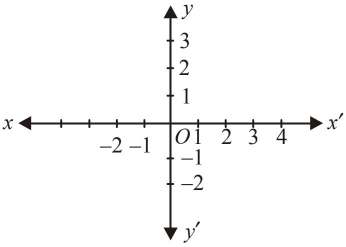
  (iii) If $D = 0$ and atleast one of the determinants D_1, D_2, D_3 is non-zero, then the given system of equations is inconsistent.

End

10
CHAPTER

cartesian system of rectangular co-ordinates and straight lines

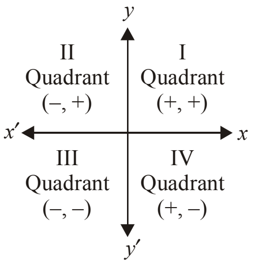
- Let XOX' and YOY' be two mutually perpendicular lines through any point O, where is point O is the origin, the line XOX' is x-axis and YOY' is y-axis.



For a given point, the abscissa and ordinate are the distances of the given point from y-axis and x-axis respectively.

System of co-ordinating an ordered pair (x, y) with every point in a plane is called the *Rectangular Cartesian Co-ordinates System.*

- There is a one-to-one correspondence between the set of all ordered pairs (x, y) of real numbers and points in a plane. The set of all points (x, y) of real numbers is called the Cartesian plane and is denoted by R^2.
- **Quadrants :** The four regions made by the two axes are called quadrants.



I quadrant : $x > 0, y > 0$

II quadrant : $x < 0, y > 0$

III quadrant : $x < 0, y < 0$

IV quadrant : $x > 0, y < 0$.

- The co-ordinates of the origin are taken as (0, 0). The co-ordinates of any point on x-axis are of the form $(x, 0)$ and the co-ordinates of any point on y-axis are of the form $(0, y)$. Thus, if the abscissa of a point is zero, it would lie somewhere on the y-axis and if its ordinate is zero it would lie on x-axis.
- If the co-ordinates of a point P are (x, y) we write it as $P(x, y)$.
- The distance between any two points in the plane is the length of the line segment joining them.
- The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

i.e., $PQ = \sqrt{(\text{difference of abscissae})^2 + (\text{difference of ordinates})^2}$

- If O is the origin and $P(x, y)$ is any point, then

$$OP = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}.$$

Some Important Tips !!!

I. In order to prove that a given figure is a

(i) square, prove that the four sides are equal and the diagonals are also equal.

(ii) rhombus, prove that the four sides are equal.

(iii) rectangle, prove that opposite sides are equal and diagonals are also equal.

(iv) a parallelogram, prove that the opposite sides are equal.

(v) parallelogram but not a rectangle, prove that its opposite sides are equal but the diagonals are not equal.

(vi) a rhombus but not a square, prove that its all sides are equal but the diagonals are not equal.

II. For three points to be collinear, prove that the sum of the distances between two pairs of points is equal to the third pair of points.

Area of a Triangle

- The area of a triangle, the co-ordinates of whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is $= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$.
- If the points A, B, C are plotted in the two dimensional plane and three points are taken in the anti-clockwise sense then the area calculated of the triangle ABC will be positive while if the points are taken in clockwise sense then the area calculated will be negative.
- To find the area of a polygon we divide it in triangles and take numerical value of the area of each of the triangles.

Condition of Collinearity of Three Points

- Three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear iff

(i) Area of $\Delta ABC = 0$, *i.e.* $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$ or $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

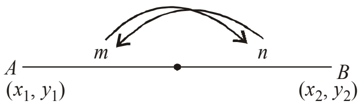
OR

(ii) $AB + BC = AC$ or $AC + BC = AB$ or $AC + AB = BC$.

Section Formulae

- **Formula for internal division :** The co-ordinates of the point which divides the line segment joining the points (x_1, y_1) and (x_2, y_2) internally in the ratio $m : n$ are given by

$$x = \frac{mx_2 + nx_1}{m + n}, \; y = \frac{my_2 + ny_1}{m + n}.$$



Mid-point formula : If P is the mid-point of AB, then it divides AB in the ratio 1 : 1, so its co-ordinates are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Formula for External Division

The co-ordinates of the point which divides the line segment joining the points (x_1, y_1) and (x_2, y_2) externally in the ratio $m : n$ are given by

$$x = \frac{mx_2 - nx_1}{m-n},\ y = \frac{my_2 - ny_1}{m-n}.$$

Co-ordinates of the centroid, in-centre and ex-centres of a triangle

- The co-ordinates of the centroid of the triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$.

  The medians of a triangle are concurrent meeting at centroid.
- The co-ordinates of the in-centre of a triangle whose vertices are (x_1, y_1), (x_2, y_2) (x_3, y_3) are $\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$.

  The internal bisectors of the angles of a triangle are concurrent meeting at in-centre.
- The co-ordinates of I_1, I_2 and I_3 (centres of escribed circles opposite to the angles A, B and C respectively) are given by

  $$I_1\left(\frac{-ax_1+bx_2+cx_3}{-a+b+c}, \frac{-ay_1+by_2+cy_3}{-a+b+c}\right), I_2\left(\frac{ax_1-bx_2+cx_3}{a-b+c}, \frac{ay_1-by_2+cy_3}{a-b+c}\right)$$

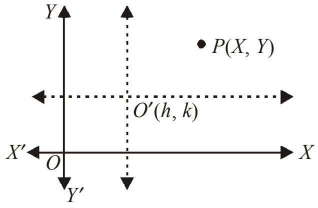
  and $I_3\left(\frac{ax_1+bx_2-cx_3}{a+b-c}, \frac{ay_1+by_2-cy_3}{a+b-c}\right)$ respectively.
- **Locus :** The curve described by a point which moves under given condition or conditions is called its locus.
- **Equation of the locus of a point**

  The equation of the locus of a point is the relation which is satisfied by the co-ordinates of every point on the locus of the point.

Shifting of Origin

- Let O be the origin and $X'OX$ and $Y'OY$ be the axis of x and y respectively. Let $O'(h, k)$ and $P(X, Y)$ be two points referred to the given axes. If the origin is transferred to O' and $X'O'X$ and $Y'O'Y$ the new axes, then the co-ordinates of $P(X, Y)$ referred to new axes as co-ordinates axes is given by

  $X = x - h$, $Y = y - k$ or $x = X + h$ and $y = Y + k$



The coordinates of the old origin referred to the new axes are $(-h, -k)$.

Rotation of Axes

- In any equation to turn the axes through an angle θ, we must substitute $X \cos \theta - Y \sin \theta$ and $X \sin \theta + Y \cos \theta$ for x and y respectively. Solving the equations for X and Y, we get $X = x \cos \theta + y \sin \theta$, $Y = y \cos \theta - x \sin \theta$.

Definition of a Straight Line

- A straight line is a curve such that every point on the line segment joining any two points on it lies on it.
- Every first degree equation in x, y represents a straight line.
- A first degree equation in x, y *i.e.*, $ax + by + c = 0$ represents a line, it means that all points (x, y) satisfying $ax + by + c = 0$ lie along a line. Thus, a line is also defined as the locus of a point satisfying the conditions $ax + by + c = 0$, where a, b, c are constants.

Slope (Gradient) of a Line

- The trigonometrical tangent of the angle that a line makes with the positive direction of the x-axis in anti-clockwise sense is called the slope or gradient of the line.
- The slope of a line is generally denoted by m. Thus $m = \tan \theta$.
- The angle of inclination of a line with the positive direction of x-axis in anti-clockwise sense always lies between 0° and 180°.

Angle Between Two Lines

- The angle θ between the lines having slopes m_1 and m_2 is given by

$$\tan \theta = \pm \frac{m_2 - m_1}{1 + m_1 m_2}.$$

- **Condition of Parallelism of Lines**
  It two lines of slopes m_1 and m_2 are parallel, then the angle θ between them is of 0°. $\therefore \quad \tan \theta = \tan 0° = 0 \quad \Rightarrow \quad m_2 = m_1$.

  Thus, when two lines are parallel, their slopes are equal.
- When the two lines are perpendicular, the product of their slopes is -1. If m is the slope of a line, then the slope of a line perpendicular to it is $-(1/m)$.

Intercepts of a Line on the Axes

- If a straight line cuts x-axis at A and the y-axis at B then OA and OB are known as the intercepts of the line on x-axis and y-axis respectively.

Equation of a Line Parallel to *x*-axis

- The equation of a line parallel to x-axis at a distance b from it is $y = b$.
  If a line is parallel to x-axis at a distance b and below x-axis, then its equation is $y = -b$.

Equation of a Line Parallel to *y*-axis

- The equation of a line parallel to y-axis at a distance a from it is $x = a$.

  If a line is parallel to y-axis at a distance a and to the left of y-axis, then its equation is $x = -a$.

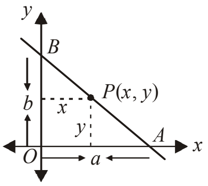
Different Forms of the Equation of a Straight Line

- **The Slope Intercept form of a Line :** The equation of a line with slope m and making an intercept c on y-axis is $y = mx + c$.

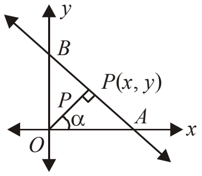
(i) If the line passes through the origin, then $0 = m0 + c \Rightarrow c = 0$. Therefore, the equation of a line passing through the origin is $y = mx$, where m is the slope of the line.

(ii) If the line is parallel to x-axis, then $m = 0$, therefore the equation of a line parallel to x-axis is $y = c$.

- **The Point-Slope Form of a Line :** The equation of a line which passes through the point (x_1, y_1) and has the slope 'm' is $(y - y_1) = m(x - x_1)$.
- **The Two-Point Form of a Line :** The equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is $(y - y_1) = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$.
- **The Intercept Form of a Line :** The equation of a line which cuts off intercepts a and b respectively from the x-axis and y-axis is $\frac{x}{a} + \frac{y}{b} = 1$.



- **The Normal Form or Perpendicular Form of a Line :**
The equation of the straight line upon which the length of the perpendicular from the origin is p and this perpendicular makes an angle α with x-axis is $x\cos\alpha + y\sin\alpha = p$.



- **The Distance Form of a Line :** The equation of the straight line passing through (x_1, y_1) and making an angle θ with the positive direction of x-axis is
$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r,$$
where r is the distance of the point (x, y) on the line from the point (x_1, y_1) and where $(x - x_1) = r\cos\theta$ and $(y - y_1) = r\sin\theta$.
- At a given distance r from the point (x_1, y_1) on the line $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta}$ there are two points *viz.* $(x_1 + r\cos\theta, y_1 + r\sin\theta)$ and $(x_1 - r\cos\theta, y_1 - r\sin\theta)$.

Transformation of General Equation in Different Standard Forms

- **Transformation of $Ax + By + C = 0$ in the slope intercept form ($y = mx + c$):**

We have $Ax + By + C = 0 \Rightarrow y = \left(\frac{-A}{B}\right)x + \left(\frac{-C}{B}\right)$, which is of the form $y = mx + c$

m = slope $= -\dfrac{A}{B} = -\dfrac{\text{Coeff. of } x}{\text{Coeff. of } y}$,

and Intercept on y-axis $= -\dfrac{C}{B} = -\dfrac{\text{Const. term}}{\text{Coeff. of } y}$.

- To determine the slope of a line by the formula $m = -\dfrac{\text{Coeff. of } x}{\text{Coeff. of } y}$ first transfer all terms in the equation on one side.

- **Transformation of $Ax + By + C = 0$ in intercept form $\left(\dfrac{x}{a} + \dfrac{y}{b} = 1\right)$:**

  We have $Ax + By + C = 0 \Rightarrow \dfrac{x}{\frac{-C}{A}} + \dfrac{y}{\frac{-C}{B}} = 1$

- Intercept on x-axis $= -\dfrac{C}{A} = -\dfrac{\text{Const. term}}{\text{Coeff. of } x}$

  Intercept on y-axis $= -\dfrac{C}{B} = -\dfrac{\text{Const. term}}{\text{Coeff. of } y}$.

- **Transformation of $Ax + By + C = 0$ in the normal form ($x\cos\alpha + y\sin\alpha = p$):**

  $-\dfrac{A}{\sqrt{A^2+B^2}}x - \dfrac{B}{\sqrt{A^2+B^2}}y = \dfrac{C}{\sqrt{A^2+B^2}}$ is the required normal form of the line $Ax + By + C = 0$.

- **Algorithm to transform the general equation to normal form**

  Step I : Shift the constant term on the RHS and make it positive.

  Step II : Divide both sides by $\sqrt{(\text{Coeff. of } x)^2 + (\text{Coeff. of } y)^2}$. The equation so obtained is in the normal form.

Point of intersection of two lines

- Let the equations of two lines be

  $a_1x + b_1y + c_1 = 0$ ....(i)

  $a_2x + b_2y + c_2 = 0$ ....(ii)

  The co-ordinates of the point of intersection of (i) and (ii) are

  $$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right).$$

- To find the co-ordinates of the point of intersection of two non-parallel lines, we solve the given equations simultaneously and the values of x and y so obtained determine the coordinates of the point of intersection.

Condition of Concurrency of Three Lines

- Three lines are said to be concurrent if they pass through a common point. *i.e.*, they meet at a point.

- Three lines $a_1x + b_1y = c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are concurrent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.
- Three lines $L_1 \equiv a_1x + b_1y + c_1 = 0$; $L_2 \equiv a_2x + b_2y + c_2 = 0$; $L_3 \equiv a_3x + b_3y + c_3 = 0$ are concurrent iff there exist constants $\lambda_1, \lambda_2, \lambda_3$ not all zero such that

  $\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 = 0$

  i.e. $\lambda_1(a_1x + b_1y + c_1) + \lambda_2(a_2x + b_2y + c_2) + \lambda_3(a_3x + b_3y + c_3) = 0$.

Equation of a Line Parallel to a given Line

- The equation of a line parallel to a given line $ax + by + c = 0$ is $ax + by + \lambda = 0$, where λ is a constant.

Equation of a Line Perpendicular to a given Line

- The equation of a line perpendicular to a given line $ax + by + c = 0$ is $bx - ay + \lambda = 0$, where λ is a constant.

Angle between two straight lines when their equations are given

- The angle θ between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is given by $\tan\theta = \left|\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}\right|$.
- **Condition for the lines to be parallel :** If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are parallel, then

$$m_1 = m_2 \Rightarrow -\frac{a_1}{b_1} = \frac{-a_2}{b_2} \Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2}.$$

- **Condition for the lines to be perpendicular :** If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are perpendicular, then

$$m_1m_2 = -1 \Rightarrow -\frac{a_1}{b_1} \times -\frac{a_2}{b_2} = -1 \Rightarrow a_1a_2 + b_1b_2 = 0.$$

Conditions for two lines to be coincident, parallel, perpendicular or intersecting

- Two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

(i) Coincident, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ (ii) Parallel, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

(iii) Intersecting, if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ (iv) Perpendicular, if $a_1a_2 + b_1b_2 = 0$.

- **Distance of a Point From a Line :** The length of the perpendicular from a point (x_1, y_1) to a line $ax + by + c = 0$ is $\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$.
- The two points (x_1, y_1) and (x_2, y_2) are on the same (or opposite) sides of the straight line $ax + by + c = 0$ according as the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same (or opposite) signs.

- **Equations of straight lines passing through a given point and making a given angle with a given line**
The equations of the straight lines which pass through a given point (x_1, y_1) and make a given angle α with the given straight line $y = mx + c$ are

$$(y - y_1) = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1).$$

- **Equations of the bisectors of the angles between two straight lines**
The equation of the bisector of the angles between the lines
$a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ are given by
$\dfrac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\dfrac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$.

- **Algorithm to find the bisector of the angle containing the origin**
Let the equation of the two lines be $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$. To find the bisector of the angle containing the origin, we proceed as follows :
Step I : See whether the constant terms c_1 and c_2 in the equations of two lines are positive or not. If not, then multiply both the sides of the equations by –1 to make the constant term positive.
Step II : Now obtain the bisector corresponding to the positive sign

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = +\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}.$$

This is the required bisector of the angle containing the origin.

- **Note :** The bisector of the angle containing the origin means the bisector of that angle between the lines which contains the origin within it.

- **Algorithm to find the bisectors of acute and obtuse angles between two lines**
Let the equation of the two lines be $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$. To separate the bisectors of the obtuse and acute angles between the lines we proceed as follows :
Step I : See whether the constant terms c_1 and c_2 in the two equations are positive or not. If not, then multiply both sides of the given equations by –1 to make the constant terms positive.
Step II : Determine the sign of the expression $a_1 a_2 + b_1 b_2$.
Step III : If $a_1 a_2 + b_1 b_2 > 0$, then the bisector corresponding to "+" sign gives the obtuse angle bisector and the bisector corresponding to "–" sign is the bisector of acute angle between the lines *i.e.*

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = +\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \text{ and } \frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

are the bisectors of obtuse and acute angles respectively.

- If $a_1 a_2 + b_1 b_2 < 0$, then the bisector corresponding to "+" and "–" sign give the acute and obtuse angle bisector respectively *i.e.*

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = +\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \text{ and } \frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

are the bisectors of acute and obtuse angles respectively.

- **Algorithm to determine whether the origin lies in the obtuse angle or acute angle between the lines**

  Let the equations of the two lines be $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$. To determine whether the origin lies in the acute angle or obtuse angle between the lines we proceed as follows :

  Step I : See whether the constant terms c_1 and c_2 in the equations of the two lines are positive or not. If not, make them positive by multiplying both sides of the equations by minus sign.

  Step II : Determine the sign of $a_1 a_2 + b_1 b_2$.

  Step III : If $a_1 a_2 + b_1 b_2 > 0$, then the origin lies in the obtuse angle and the "+"sign gives the bisector of the obtuse angle. If $a_1 a_2 + b_1 b_2 < 0$, then the origin lies in the acute angle and "+" sign gives the bisector of acute angle.

- **Some important points of a triangle**

  Centroid : The point of intersection of the medians of a triangle is called its centroid. It divides the medians in the ratio 2 : 1. If (x_1, y_1), (x_2, y_2) and (x_3, y_3) are the vertices of a triangle, then the co-ordinates of its centroid are

  $$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right).$$

  In-Centre : The point of intersection of the internal bisectors of the angles of a triangle is called its in-centre. If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of a triangle ABC such that $BC = a$, $CA = b$ and $AB = c$, then the co-ordinates of its in-centre are $\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$.

  Circum-Centre : The point of intersection of the perpendicular bisectors of the sides of a triangle is called its circum-centre. It is equidistant from the vertices of a triangle.

  Ortho-Centre : The point of intersection of the altitudes of a triangle is called its orthocentre.

  Circumcentre O, centroid G and orthocentre O' of a triangle ABC are collinear such that G divides $O'O$ in the ratio 2 : 1 *i.e.*, $O'G : OG = 2 : 1$.

End

11

CHAPTER

family of lines

Joint Equation of a Pair of Straight Lines

- The joint equation of the straight lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is $(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0$.
  $\Rightarrow \quad ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$,
  where $a = a_1a_2$, $b = b_1b_2$, $2h = a_1b_2 + a_2b_1$, $2g = a_1c_2 + a_2c_1$, $2f = b_1c_2 + b_2c_1$ and $c = c_1c_2$.
- The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is known as the general equation of second degree.
- **Homogeneous Equation of Second Degree :** The joint equation of a pair of straight lines $a_1x + b_1y = 0$ and $a_2x + b_2y = 0$ is $(a_1x + b_1y)(a_2x + b_2y) = 0$ or $a_1a_2x^2 + b_1b_2y^2 + xy\,(a_1b_2 + a_2b_1) = 0$. This equation can also be written as $ax^2 + 2hxy + by^2 = 0$, where $a = a_1a_2$, $b = b_1b_2$ and $2h = a_1b_2 + a_2b_1$. The equation $ax^2 + 2hxy + by^2 = 0$ is known as general form of the homogeneous equation of second degree where a, b, h are constants.

Pair of Straight Lines Through the Origin

- A rational, integral, algebraic equation in two variables x and y is said to be a homogeneous equation of the second degree, if the sum of the indices (exponents) of x and y in each term is equal to 2.
- The homogeneous equation of second degree $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines passing through the origin, if $h^2 \geq ab$.
- These lines are real and distinct, if $h^2 > ab$ and coincident if $h^2 = ab$, and the lines are imaginary *i.e.,* they do not exist if $h^2 < ab$.

Important points to remember

- If $y = m_1x$ and $y = m_2x$ are the lines represented by a homogeneous equation $ax^2 + 2hxy + by^2 = 0$, then
- The equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines given by

  $y = m_1x$ and $y = m_2x$, where $m_1 = \dfrac{-h+\sqrt{h^2-ab}}{b}$ and $m_2 = \dfrac{-h-\sqrt{h^2-ab}}{b}$.

  $$\therefore \quad m_1 + m_2 = \frac{-h+\sqrt{h^2-ab}}{b} + \frac{-h-\sqrt{h^2-ab}}{b} = -\frac{2h}{b} = \frac{-\text{coeff. of } xy}{\text{coeff. of } y^2}$$

  $$\text{and} \quad m_1m_2 = \left(\frac{-h+\sqrt{h^2-ab}}{b}\right)\left(\frac{-h-\sqrt{h^2-ab}}{b}\right)$$

$$= \frac{h^2 - (h^2 - ab)}{b^2} = \frac{a}{b} = \frac{\text{coeff. of } x^2}{\text{coeff. of } y^2}.$$

$y - m_1x$ and $y - m_2x$ are factors of $ax^2 + 2hxy + by^2 = 0$.

- The equation of the pair of lines through the origin and perpendicular to the pair of lines given by $ax^2 + 2hxy + by^2 = 0$ is $bx^2 - 2hxy + ay^2 = 0$.

Angle Between the Pair of Lines

- The angle θ between the pair of lines represented by $ax^2 + 2hxy + by^2 = 0$ is given by $\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$.

Condition for the Lines to be Coincident

- The angle between the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is given by $\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$.
- The lines are coincident if the angle between them is zero *i.e.*, Lines are coincident $\Leftrightarrow \theta = 0 \Leftrightarrow \tan\theta = 0 \Leftrightarrow h^2 = ab$.

  $\therefore$ The lines represented by $ax^2 + 2hxy + by^2 = 0$ are coincident iff $h^2 = ab$.

Condition for the Lines to be Perpendicular

- The lines are perpendicular if the angle between them is $\pi/2$.
  The lines represented by $ax^2 + 2hxy + by^2 = 0$ are perpendicular iff $a + b = 0$ *i.e.* Coeff. of x^2 + Coeff. of $y^2 = 0$.

Condition for the Lines to be Parallel

- The lines are parallel if the angle between them is zero. *i.e.*, if $\theta = 0$

$$\Rightarrow \quad \tan\theta = 0 \Rightarrow \frac{2\sqrt{h^2 - ab}}{a+b} = 0 \Rightarrow h^2 = ab.$$

Bisectors of the Angle Between the Lines Given by a Homogeneous Equation

- The equations of the bisectors of the angles between the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are given by

$$\frac{(x - x_1)^2 - (y - y_1)^2}{a - b} = \frac{(x - x_1)(y - y_1)}{h}$$

  where (x_1, y_1) is the point of intersection of the lines represented by the given equation.

- The joint equation of the bisector of the angles between the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$ is $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$.

$$\Rightarrow \quad h(x^2 - y^2) = (a - b)xy$$

$$\Rightarrow \quad hx^2 - hy^2 - (a - b)xy = 0.$$

- Here, coefficient of x^2 + coefficient of $y^2 = 0$. Hence, the bisectors of the angle between the lines are perpendicular to each other.

- The necessary and sufficient condition for $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent a pair of straight lines is that $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

  or $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$

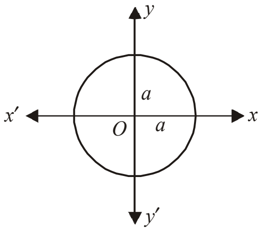
End

12

CHAPTER

circles

- A circle is defined as the locus of a point which moves in a plane such that its distance from a fixed point in that plane is always constant.
- If the centre of the circle is at the origin and radius is a, then equation of the circle is $x^2 + y^2 = a^2$.



- The equation of a circle with centre at (h, k) and radius equal to a, is $(x - h)^2 + (y - k)^2 = a^2$.

Some Particular Cases of the Central Form of the Equation of a Circle

- When the circle passes through the origin, centre (h, k) then equation of the circle is $x^2 + y^2 - 2hx - 2ky = 0$.
- When the circle touches x-axis equation becomes $x^2 + y^2 - 2hx - 2ay + h^2 = 0$.
- When the circle touches y-axis equation becomes $x^2 + y^2 - 2ax - 2ky + k^2 = 0$.
- When the circle touches both the axes is $x^2 + y^2 - 2ax - 2ay + a^2 = 0$.
- When the circle passes through the origin and centre lies on x-axis is $x^2 + y^2 - 2ax = 0$.
- When the circle passes through the origin and centre lies on y-axis, then equation is $x^2 + y^2 - 2ay = 0$.

General Equation of a Circle

- The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ always represents a circle whose centre is $(-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$.
- If $g^2 + f^2 - c > 0$ then the radius of the circle is real and hence the circle is also real.
- If $g^2 + f^2 - c = 0$ then the radius of the circle is zero. Such a circle is known as a point circle.
- If $g^2 + f^2 - c < 0$, then the radius $\sqrt{g^2 + f^2 - c}$ of the circle is imaginary but the centre is real. Such a circle is called an imaginary circle as it is not possible to draw such a circle.

- Special features of the general equation $x^2 + y^2 + 2gx + 2fy + c = 0$ of the circle are :
  (i) it is quadratic in both x and y
  (ii) coefficient of x^2 = coefficient of y^2
  (iii) there is no term containing xy *i.e.*, the coefficient of xy is zero
  (iv) it contains three arbitrary constants *viz.* g, f and c.
- The equation $ax^2 + ay^2 + 2gx + 2fy + c = 0, a \neq 0$ also represents a circle. This equation can also be written as $x^2 + y^2 + \frac{2g}{a}x + \frac{2f}{a}y + \frac{c}{a} = 0$.
  The co-ordinates of the centre are $\left(-\frac{g}{a}, -\frac{f}{a}\right)$ and radius $= \sqrt{\frac{g^2}{a^2} + \frac{f^2}{a^2} - \frac{c}{a}}$.
- On comparing the general equation $x^2 + y^2 + 2gx + 2fy + c = 0$ of a circle with the general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ we find that it represents a circle if $a = b$ *i.e.*, coefficient of x^2 = coefficient of y^2 and $h = 0$ *i.e.*, coefficient of $xy = 0$.
- The equation of the circle drawn on the straight line joining two given points (x_1, y_1) and (x_2, y_2) as diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.
- If the co-ordinates of the end points of a diameter of a circle are given, we can also find the equation of the circle by finding the co-ordinates of the centre and radius. The centre is the mid-point of the diameter and radius is half of the length of the diameter.

Intercepts on the Axes

- The lengths of intercepts made by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ with X and Y axes are $2\sqrt{g^2 - c}$ and $2\sqrt{f^2 - c}$ respectively.
- If $g^2 > c$, then the roots of the equation $x^2 + 2gx + c = 0$ are real and distinct, so the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ meets the x-axis in two real and distinct points and the length of the intercept on x-axis is $2\sqrt{g^2 - c}$.
- If $g^2 = c$, then the roots of the equation $x^2 + 2gx + c = 0$ are real and equal, so the circle touches x-axis and the intercept on x-axis is zero.
- If $g^2 < c$, then the roots of the equation $x^2 + 2gx + c = 0$ are imaginary, so the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ does not meet x-axis in real points.
- The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts the y-axis in real and distinct points, touches or does not meet in real points according as $f^2 >, =$ or $< c$.

Position of a Point with Respect to a Circle

- A point (x_1, y_1) lies outside, on or inside a circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ according as $S_1 >, =$ or < 0, where $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$.

Equation of a Circle in Parametric Form

- Parametric equations of $x^2 + y^2 = r^2$ are $x = r\cos\theta, y = r\sin\theta$, where θ is a parameter.
- Parameteric Equations of $(x - a)^2 + (y - b)^2 = r^2$ are $x = a + r\cos\theta; y = b + r\sin\theta$

Intersection of a Straight Line and a Circle

Case I : When points of intersection are real and distinct : A line intersects a given circle at two distinct points if the length of the perpendicular from the centre is less than the radius of the circle.

Case II : When the points of intersection are coincident : A line touches a circle if the length of the perpendicular from the centre is equal to the radius of the circle.

Case III : When points of intersection are imaginary : A line does not intersect a circle if the length of the perpendicular from the centre is greater than the radius of the circle.

The Length of the Intercept Cut off From a Line by a Circle

- The length of the intercept cut off from the line $y = mx + c$ by the circle $x^2 + y^2 = a^2$ is $2\sqrt{\dfrac{a^2(1+m^2)-c^2}{(1+m^2)}}$.
- The line $y = mx + c$ touches the circle $x^2 + y^2 = a^2$ if the length of the intercept is zero *i.e.*, $c = \pm a\sqrt{1+m^2}$.
- In case of a circle the above result can also be used as, if a line touches a circle, then the length of the perpendicular from the centre upon the line is equal to the radius of the circle.
- The tangent at a point P is the limiting position of a secant PQ when Q tends to P along the circle. The point P is called the point of contact of the tangent.

Different forms of the Equations of Tangents

- **Slope Form :** The equation of a tangent of slope m to the circle $x^2 + y^2 = a^2$ is $y = mx \pm a\sqrt{1 + m^2}$. The co-ordinates of the point of contact are $\left(\pm \dfrac{am}{\sqrt{1+m^2}}, \mp \dfrac{a}{\sqrt{1 + m^2}}\right)$.
- **Point Form :** The equation of the tangent at the point $P(x_1, y_1)$ to a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.
- **Algorithm to write the equation of tangent to a circle at a given point**

  Step I : In the equation of the circle substitute xx_1 for x^2, yy_1 for y^2, $\dfrac{x + x_1}{2}$ for x, $\dfrac{y + y_1}{2}$ for y and keep the constant as such.

  Step II : Simplify the equation obtained in Step I.
  The equation so obtained is the desired equation of the tangent.
- **Algorithm to obtain the equation of tangents drawn from a given point to a given circle**

  Step I : Obtain the point, say (x_1, y_1).
  Step II : Write a line passing through (x_1, y_1) having slope m
  i.e. $(y - y_1) = m(x - x_1)$.

Step III : Equate the length of the perpendicular from the centre of the circle to the line in Step II to the radius of the circle.
Step IV : Obtain the value of m from the equation in Step III.
Step V : Substitute m in the equation in Step II.

Normal to a circle at a given point

- The normal at any point on a curve is a straight line which is perpendicular to the tangent to the curve at the point.
- **Algorithm to find the normal to a circle at a given point (x_1, y_1)**
  Step I : Write the equation of the tangent to the given circle at the given point (x_1, y_1).
  Step II : Write the equation of a line perpendicular to the tangent in Step I and passing through (x_1, y_1). The equation obtained in Step II is the required equation of the normal at (x_1, y_1).

Length of the tangent from a point to a circle

- The length of the tangent from the point $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is equal to $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$.
- If PT is the length of the tangent from a point P to a given circle, then PT^2 is called the power of the point with respect to the given circle.
- If we write $S = x^2 + y^2 + 2gx + 2fy + c$ and $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$, then the equation of the circle is $S = 0$ and the length of the tangent from $P(x_1, y_1)$ is $\sqrt{S_1}$. The power of the point $P(x_1, y_1)$ is S_1.
- **Algorithm to find the length of the tangent from a point (x_1, y_1) to a given circle**
  Step I : Obtain the equation of the circle and the given point (x_1, y_1).
  Step II : Shift all terms on one side and introduce zero on the right hand side of the given equation.
  Step III : Make coefficients of x^2 and y^2 unity if they are not so by dividing both sides of the equation in Step II by the common value.
  Step IV : Replace x by x_1 and y by y_1 in the left hand side of the equation in Step III.
  Step V : Take the square root of the number obtained in Step IV. The number so obtained is the required length of the tangent.

Pair of tangents drawn from a point to a given circle

- Let (x_1, y_1) be a given point and let the equation of the circle be $x^2 + y^2 = a^2$. Then the equation of any tangent to the circle is of the form $y = mx + a\sqrt{1+m^2}$, where m is the slope of the tangent.
- If the tangent passes through the given point (x_1, y_1), we must have
  $m^2(x_1^2 - a^2) - 2m\, x_1y_1 + y_1^2 - a^2 = 0.$ ....(i)
- The tangents are real, coincident or imaginary according as the values of m obtained from (i) are real, coincident or imaginary.
  or, according as Discriminant >, = or < 0
  $\Rightarrow \quad 4x_1^2\, y_1^2 - 4(x_1^2 - a^2)(y_1^2 - a^2) \geq$ or < 0

$\Rightarrow \quad x_1^2 + y_1^2 - a^2 >, = \text{ or } < 0$

or, according as the point $P(x_1, y_1)$ lies outside on, or, inside the circle $x^2 + y^2 = a^2$.

Combined Equation of Pair of Tangents

- The combined equation of the pair of tangents drawn from a point (x_1, y_1) to a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is given by

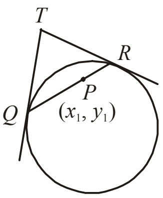
  $$(x^2 + y^2 + 2gx + 2fy + c)(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c) = \{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c\}^2$$

  or $\quad SS' = T^2$.
- **Director Circle :** The locus of the point of intersection of two perpendicular tangents to a given conic is known as its director circle.

  The equation of the director circle of the circle $x^2 + y^2 = a^2$ is $x^2 + y^2 = 2a^2$.

Chord of Contact of Tangents

- The chord joining the points of contact of the two tangents to a conic drawn from a given point, outside it, is called the chord of contact of tangents.
- The equation of the chord of contact of tangents drawn from a point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.
- The equation of the chord of contact is of the same form as the equation of the tangent, if the point (x_1, y_1) lies on the conic.
- The equation of the chord of contact or tangents drawn from (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.
- **Polar of a Point with respect to a Circle :** If through a point $P(x_1, y_1)$ (within or without a circle) there be drawn any straight line to meet the given circle at Q and R, the locus of the point of intersection of the tangents at Q and R is called the polar of P and P is called the pole of the polar.



- The polar of (x_1, y_1) with respect to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.
- The polar of (x_1, y_1) with respect to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ which is same as the equation of the tangent at (x_1, y_1), if (x_1, y_1) lies on the circle.
- **Conjugate Points :** Two points A and B are conjugate points with respect to a given circle if each lies on the polar of the other with respect to the circle.
- **Conjugate Lines :** If two lines be such that the pole of one lies on the other, then they are called conjugate lines with respect to the given circle.

Equation of the Chord Bisected at a Given Point

- The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ bisected at the point (x_1, y_1) is given by $T = S'$

  i.e. $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$.

Diameter of a Circle

- The locus of the middle points of a system of parallel chords of a circle is called a diameter of the circle.
- The equation of the diameter bisecting parallel chords $y = mx + c$ (c is a parameter) of the circle $x^2 + y^2 = a^2$ is $x + my = 0$.
- The diameter corresponding to a system of parallel chords of a circle always passes through the centre of the circle and is perpendicular to the parallel chords.

Common Tangents to Two Circles

- Let the equations of the two circles be

  $(x - h_1)^2 + (y - k_1)^2 = a_1^2$ ....(i)

  $(x - h_2)^2 + (y - k_2)^2 = a_2^2$ ....(ii)

  with centres $C_1(h_1, k_1)$ and $C_2(h_2, k_2)$ and radii a_1 and a_2 respectively. Then the following cases of intersection of these two circles may arise.

  Case I : When $C_1 C_2 > a_1 + a_2$, *i.e.*, the distance between the centres is greater than the sum of the radii. In this case the two circles do not intersect, two direct common and two transverse common tangents can be drawn to the circles.

  Case II : When $C_1 C_2 = a_1 + a_2$, *i.e.*, the distance between the centres is equal to the sum of the radii. In this case two direct tangents are real and distinct while the transverse tangents are coincident.

  Case III : When $C_1 > a_1 - a_2$ $C_2 < a_1 + a_2$, *i.e.*, the distance between the centres is less than the sum of the radii. In this case the two direct common tangents are real while the transverse tangents are imaginary.

  Case IV : When $C_1 C_2 = a_1 - a_2$, *i.e.*, the distance between the centres is equal to the difference of the radii. In this case two tangents are real and coincident while the other two tangents are imaginary.

  Case V : When $C_1 C_2 < a_1 - a_2$, *i.e.*, the distance between the centres is less than the difference of the radii. In this case all the four common tangents are imaginary.

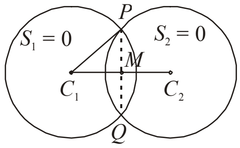
Common Chord of Two Circles

- The chord joining the points of intersection of two given circles is called their common chord.
- The equation of the common chord of two circles

  $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ ....(i)

  and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ ....(ii)

  is $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$ *i.e.* $S_1 - S_2 = 0$.
- The length of the common chord is



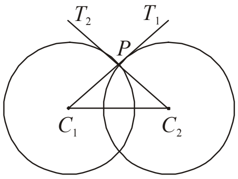
$$PQ = 2(PM) = 2\sqrt{C_1P^2 - C_1M^2}\ .$$

where C_1P = radius of the circle $S = 0$ and C_1M = length of the perpendicular from the centre C_1 to the common chord PQ.

- If the length of the common chord is zero, then the two circles touch each other and the common chord becomes the common tangent to the two circles at the common point of contact.

Angle of Intersection of Two Curves

- If two curves C_1 and C_2 intersect at a point P and PT_1 and PT_2 be the tangents to the two curves at P. Then the angle between the tangents at P is called the angle of intersection of the two curves at the point of intersection.



- **Orthogonal Curves :** Two curves are said to intersect orthogonally when the tangents at the common point are at right angles.

Condition for Two Intersecting Circles to be Orthogonal

- The condition for the circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ to intersect orthogonally is given by $2(g_1g_2 + f_1f_2) = c_1 + c_2$.

Radical Axis and Radical Centre

- The radical axis of two circles is the locus of a point which moves in such a way that the lengths of the tangents drawn from it to the two circles are equal.
- **Properties of Radical Axis**
  (i) The radical axis of two circles is always perpendicular to the line joining the centres.
  (ii) The radical axes of three circles whose centres are non-collinear, taken in pairs, meet in a point.
- **Radical Centre :** The point of concurrence of the radical axes of three circles whose centres are non-collinear, taken in pairs, is called the radical centre of the circles.
- The circle with centre at the radical centre and radius equal to the length of the tangent from it to any one of the circles intersects all the three circles orthogonally.
- **Equation of a Circle Through the Intersection of a Circle and Line :** The equation of a circle passing through the intersection of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ and the line $L \equiv lx + my + n = 0$ is $x^2 + y^2 + 2gx + 2fy + c + \lambda(lx + my + n) = 0$ or $S + \lambda L = 0$, where λ is a constant determined by an additional condition.
- **Circle Through the Intersection of Two Circles**

  The equation of a family of circles passing through the intersection of the circles
  $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and
  $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ is $S_1 + \lambda S_2 = 0$

i.e., $(x^2 + y^2 + 2g_1x + 2f_1y + c_1) + \lambda(x^2 + y^2 + 2g_2x + 2f_2y + c_2) = 0$,
where λ ($\neq -1$) is an arbitrary real number.

Coaxial System of Circles

- A system of circles, every pair of which has the same radical axis is called a coaxial system of circles.
- The equation $x^2 + y^2 + 2gx + c = 0$, where g is a variable and c is a constant, is the simplest equation of a coaxial system of circles. The common radical axis of this system of circles is y-axis.
- Let $S = x^2 + y^2 + 2gx + 2fy + c = 0$ be the circle and $P = lx + my + n = 0$ be the radical axis. Then $S + \lambda P = 0$ (λ is an arbitrary constant) represents the coaxial system of circles.
- Let $S_1 = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $S_2 = x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ be two circles of the coaxial system. Then $S_1 + \lambda S_2 = 0$ ($\lambda \neq -1$) represents the coaxial system.
- The simplest form of the equation of a coaxial system of circles can be put in the form $x^2 + y^2 + 2gx + c = 0$, where g is a variable and c is a constant.

End

13

CHAPTER

parabola

- Consider a double right circular cone of semi vertical angle α and let it be cut by a plane inclined at an angle θ to the axis of the cone. We will get different sections (curves) as follows :

  Case I : If the plane passes through the vertex *O*.

  The curve of intersection is a pair of straight lines passing through the vertex which are

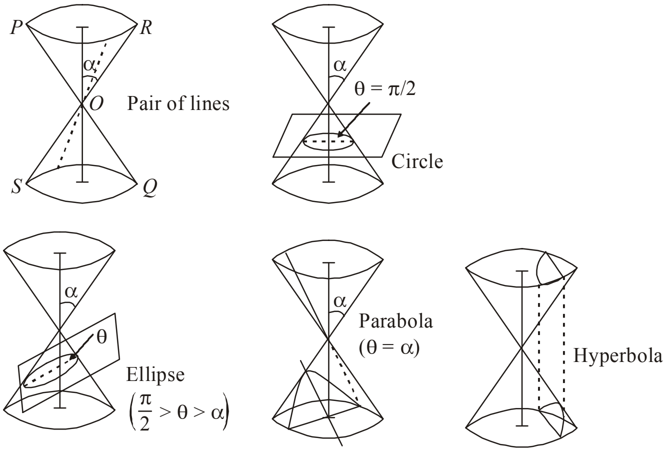
(i) Real and distinct for $\theta < \alpha$

(ii) Coincident for $\theta = \alpha$ *i.e* the plane touches the cone

(iii) Imaginary for $\theta > \alpha$

Case II : If the plane does not pass through the vertex *O*.

The curve of intersection is called



(i) A circle if $\theta = \dfrac{\pi}{2}$

(ii) A parabola for $\theta = \alpha$ *i.e.,* if the plane is parallel to the generator *PQ*

(iii) An ellipse for $\theta > \alpha$ $(\theta \neq \pi / 2)$ *i.e.*, if the plane cuts both the generating lines *PQ* and *RS*.

(iv) A hyperbola for $\theta < \alpha$ *i.e.*, if the plane cuts both the cones.

Thus we may get the section either as a pair of straight lines, a circle, a parabola, an ellipse or a hyperbola depending upon the different positions of the cutting plane. These curves of intersection are called the conic sections.

Conic Section

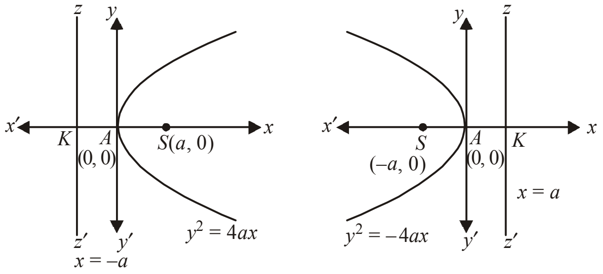
- A conic section or conic is the locus of a point *P* which moves in such a way that its distance from a fixed point *S* always bears a constant ratio to its distance from a fixed line, all being in the same plane.

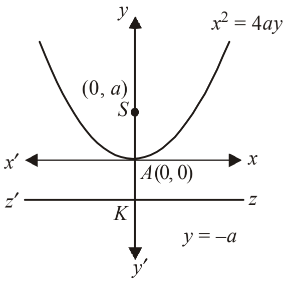
Important Terms

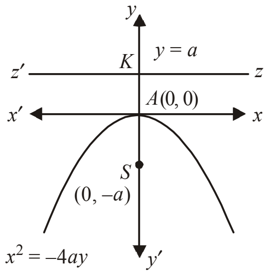
- **Focus :** The fixed point is called the focus of the conic section.
- **Directrix :** The fixed straight line is called the directrix of the conic section.
- **Eccentricity :** The constant ratio is called the eccentricity of the conic section and is denoted by e.
- **Axis :** The straight line passing through the focus and perpendicular to the directrix is called the axis of the conic section.
- **Vertex :** The points of intersection of the conic section and the axis are called vertices of the conic.
- **Centre :** The point which bisects every chord of the conic passing through it, is called the centre of the conic.
- **Latus-Rectum :** The latus-rectum of a conic is the chord passing through the focus and perpendicular to the axis.
- (i) For $e < 1$, the conic obtained is an ellipse.
  (ii) For $e = 1$, the conic obtained is a parabola.
  (iii) For $e > 1$, the conic is a hyperbola.
  (iv) For $e = 0$, the conic is a circle.
- The general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ always represents
  (i) A pair of straight lines if $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$
  (ii) A circle if $\Delta \neq 0$, $a = b$ and $h = 0$
  (iii) A parabola if $\Delta \neq 0$ and $h^2 = ab$
  (iv) An ellipse if $\Delta \neq 0$ and $h^2 < ab$
  (v) A hyperbola if $\Delta \neq 0$ and $h^2 > ab$
  (v) A rectangular hyperbola if $\Delta \neq 0$, $h^2 > ab$ and $a + b = 0$.

Parabola

- A parabola is the locus of a point which moves in a plane such that its distance from a fixed point in the plane is always equal to its distance from a fixed straight line in the plane.
- A double ordinate through the focus is called the latusrectum. *i.e.,* the latusrectum of a parabola is a chord passing through the focus perpendicular to the axes.
- The distance of $P(x, y)$ from the focus is called the focal distance of the point P. $x + a$ is the focal distance of the point $P(x, y)$.
- **Focal Chord :** A chord of the parabola is a focal chord if it passes through the focus.







Parabola at a Glance				
	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Co-ordinates of vertex	(0, 0)	(0, 0)	(0, 0)	(0, 0)
Co-ordinates of focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Equation of the directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Equation of the axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Length of the latusrectum	$4a$	$4a$	$4a$	$4a$
Focal distance of a point $P(x, y)$	$x + a$	$x - a$	$y + a$	$y - a$

- **Position of a Point With Respect to a Parabola**

  The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as $y_1^2 - 4ax_1 >, =$ or < 0.
- The point (x_1, y_1) lies inside, on, or outside $y^2 = -4ax$ according as $y_1^2 + 4ax_1 <, =$ or > 0.
- The point (x_1, y_1) lies inside, on, or outside $x^2 = 4ay$ according as $x_1^2 - 4ay_1 <, =,$ or > 0.
- The point (x_1, y_1) lies inside, on or outside $x^2 = -4ay$ according as $x_1^2 + 4ay_1 <, =$ or > 0.

Parametric Equations of a Parabola

Parabola	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Parametric co-ordinates	$(at^2, 2at)$	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$
Parametric equations	$x = at^2$, $y = 2at$	$x = -at^2$, $y = 2at$	$x = 2at$, $y = at^2$	$x = 2at$, $y = -at^2$

- The parametric equations of $(y - k)^2 = 4a(x - h)$ are $x = h + at^2$ and $y = k + 2at$.

Equation of The Chord Joning any Two Points on the Parabola

- Let $P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$ be any two points on the parabola $y^2 > 4ax$. Then, the equation of the chord joining these points is $y(t_1 + t_2) = 2x + 2at_1t_2$.

Condition for The Chord Joining Points having Parameters t_1 and t_2 to be a Focal Chord

- If the chord joining points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ on the parabola passes through its focus, then $(a, 0)$ satisfies the equation $t_1t_2 = -1$
- The length of a focal chord having parameters t_1 and t_2 for its end points is $a(t_2 - t_1)^2$.
- If l_1 and l_2 are the lengths of a focal chord of a parabola, then its latusrectum is $\frac{4l_1l_2}{l_1+l_2}$.

Intersection of a Line and a Parabola

- The equation of the line and the parabola are $y = mx + c$ ...(i) and $y^2 = 4ax$ ...(ii) respectively. The points of intersection of the line and the parabola are obtained by solving (i) and (ii), which gives $m^2x^2 + 2x(mc - 2a) + c^2 = 0$. This equation gives two values of x, then we can obtain the corresponding value of y.
- **Nature of The Points of Intersection :** The points of intersection of (i) and (ii) are real and distinct, coincident or imaginary, according as the roots of the equation (iii) are real and distinct, coincident or imaginary, *i.e.* according as $4(mc - 2a)^2 - 4m^2c^2 >, =, \text{or} < 0$

  $\Rightarrow \quad a^2 - amc >, =, \text{or} < 0$

  $\Rightarrow \quad a >, =, \text{or} < mc$

  $\Rightarrow \quad c <, =, \text{or} > \frac{a}{m}$.
- **Condition of Tangency :** The line $y = mx + c$ touches the parabola $y^2 = 4ax$ if $c = \frac{a}{m}$ and the co-ordinates of the points of contact are $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

Equation of Tangent in Different Forms

- **Point Form :** The equation of the tangent to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is $yy_1 = 2a(x + x_1)$.

  The equation of tangent at (x_1, y_1) can also be obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{x+x_1}{2}$, y by $\frac{y+y_1}{2}$ etc.
- **Parametric Form :** The equation of the tangent to the parabola $y^2 = 4ax$ at $(at^2, 2at)$ is $ty = x + at^2$.
- **Slope Form :** The equation of a tangent to the parabola $y^2 = 4ax$ is $y = mx + \frac{a}{m}$. The co-ordinates of the point of contact are $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

Point of intersection of tangent at any two points on the parabola

- The point of intersection of tangents at two points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is $(at_1 t_2, a(t_1 + t_2))$.
- x co-ordinate of the points of intersection of tangents at P and Q on the parabola is the G.M. of the x co-ordinates of P and Q ,and y co-ordinates is the A.M. of y co-ordinates.
- **Equation of Normal in Different Forms**
- **Point form :** The equation of the normal to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is $(y - y_1) = -\frac{y_1}{2a}(x - x_1)$.
- **Parametric form :** The equation of the normal to the parabola $y^2 = 4ax$ at $(at^2, 2at)$ is $y + tx = 2at + at^3$
- **Slope form :** The equation of normal of slope m to the parabola $y^2 = 4ax$ is $y = mx - 2am - am^3$ at the point $(am^2, -2am)$.
- The equation of normals of slope m to $y^2 = -4ax$, $x^2 = 4ay$ and $x^2 = -4ay$, are $y = mx + 2am + am^3$, $x = \frac{y}{m} - \frac{2a}{m} - \frac{a}{m^3}$ and $x = \frac{y}{m} + \frac{2a}{m} + \frac{a}{m^3}$ respectively.
- Three normals can be drawn from a point to a parabola.
- These normals are real, coincident or imaginary according as the values of m are real coincident or imaginary. Out of the three normals atleast one will be real, because a cubic equation has atleast one real root.

Co-normal points

- The points on the curve at which the normals pass through a common point are called co-normal points.
- The sum of the slopes of the normals at co-normal points is zero.
- The sum of the ordinates of the co-normal points is zero.
- The centroid of the triangle formed by the co-normal points lies on the axis of the parabola.
- Two tangents can be drawn from a point to a parabola. The two tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the parabola.

Director Circle

- The locus of the point of intersection of perpendicular tangents to a conic is known as its director circle. The director circle of a parabola is its directrix.

Equation of the pair of tangents from a point to a parabola

- The combined equation of the pair of tangents drawn from a point to a parabola is $SS' = T^2$, where $S = y^2 - 4ax$, $S' = y_1^2 - 4ax_1$ and $T = yy_1 - 2a(x + x_1)$.
- $S = 0$ is the equation of the curve, S' is obtained from S by replacing x by x_1 and y by y_1 and $T = 0$ is the equation of the tangent.

Equation of the chord of contact of tangents to a parabola

- The chord of contact of tangents drawn from a point (x_1, y_1) to the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$.

- The area of the triangle formed by the tangents drawn from a point (x_1, y_1) to the parabola $y^2 = 4ax$ and their chord of contact is $\frac{1}{2a}(y_1^2 - 4ax_1)^{3/2}$.

Equation of the chord of the parabola which is bisected at a given point

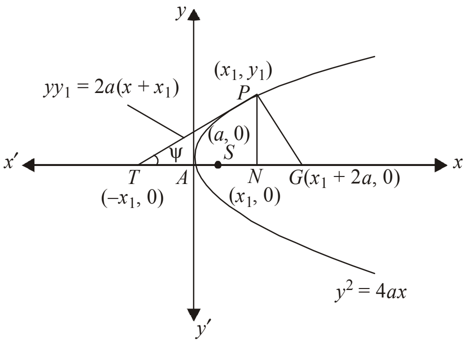
- The equation of the chord of the parabola $y^2 = 4ax$ which is bisected at (x_1, y_1), is $yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$ or $T = S'$, where $S' = y_1^2 - 4ax_1$ and $T = yy_1 - 2a(x + x_1)$.

Equation of diameter of a parabola

- The locus of the mid-points of a system of parallel chords of a conic is known as its diameter. The equation of the diameter bisecting chords of slope m of the parabola $y^2 - 4ax$ is $y = \frac{2a}{m}$.

Sub-tangent and Sub-normal

- Let the tangent and normal at any point $P(x_1, y_1)$ on a parabola $y^2 = 4ax$ meet the axis in T and G respectively. Then PT is called the length of the tangent at P and PG is called the length of the normal at P. NT is called the sub-tangent and NG the sub-normal at P.



- Sub-tangent = NT = twice the abscissa of P and sub-normal = $NG = 2a$ = semi-latusrectum. From right angled triangle NPT and PNG, we have that

(i) The length of the tangent = $PT = PN \operatorname{cosec}\psi = y_1 \operatorname{cosec}\psi$.

(ii) The length of the normal = $PG = PN \sec\psi = y_1 \sec\psi$

(iii) The length of the sub-tangent = $NT = PN \cot\psi = y_1 \cot\psi$

(iv) The length of the sub-normal = $NG = PN \tan\psi = y_1 \tan\psi$, where $\tan\psi = \frac{2a}{y_1} = m$.

Pole and Polar

- Let P be a point lying within or outside a given parabola. Suppose any straight line drawn through P intersects the parabola at Q and R. Then the locus of the point of intersection of the tangents to the parabola at Q and R is called the polar of the given point P with respect to the parabola and the point P is called the pole of the polar.
- The polar of the point (x_1, y_1) with respect to the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$.

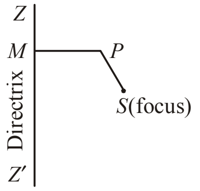
End

14

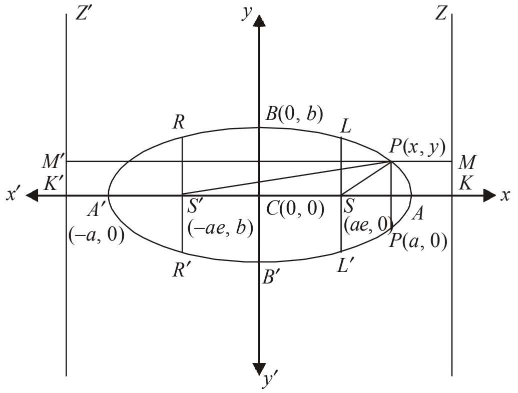
CHAPTER

ellipse

- **Definition :** An ellipse is the locus of a point in a plane which moves in the plane in such a way that the ratio of its distance from a fixed point (called focus) in the same plane to its distance from a fixed straight line (called directrix) is always constant which is always less than unity.



- The constant ratio is generally denoted by e and is known as the eccentricity of the ellipse.
- If S is the focus, ZZ' is the directrix and P is any point on the ellipse, then by definition $\frac{SP}{PM} = e \Rightarrow SP = e \cdot PM$.
- Equation of the ellipse in standard form is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $b^2 = a^2(1 - e^2)$.
- Since $e < 1$, therefore $a^2(1 - e^2) < a^2 \Rightarrow b^2 < a^2$.



- **Vertices :** The points A and A', in figure, where the curve meets the line joining the foci S and S', are called the vertices of the ellipse. The co-ordinates of A and A' are $(a, 0)$ and $(-a, 0)$ respectively.
- **Major and Minor Axes :** In figure the distance $AA' = 2a$ and $BB' = 2b$ are called the major and minor axes of the ellipse.

  Since $e < 1$ and $b^2 = a^2(1 - e^2)$, therefore $a > b \Rightarrow AA' > BB'$.
- **Focii :** In figure the points $S(ae, 0)$ and S' $(-ae, 0)$ are the focii of the ellipse.

- **Directrices :** ZK and $Z'K'$ are two directrices of the ellipse and their equations are $x = a/e$ and $x = -a/e$ respectively.
- **Centre :** Since the centre of a conic section is a point which bisects every chord passing through it. In case of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ every chord is bisected at $C(0, 0)$. Therefore C is the centre of the ellipse and is the mid point of AA'.
- $e = \sqrt{1 - \left(\frac{\text{Minor axis}}{\text{Major axis}}\right)^2}$.
- **Ordinate and Double Ordinate :** Let P be a point on the ellipse and let PN be perpendicular to the major axis AA' such that PN produced meets the ellipse at P'. Then PN is called the ordinate of P and PNP' is the double ordinate of P.
- **Latus Rectum :** It is a double ordinate passing through the focus, LSL' is the latus-rectum and LS is called the semi-latus-rectum. $RS'R'$ is also a latus-rectum. The co-ordinates of L are (ae, SL).
- Length of the latus-rectum $= 2\,(SL) = \frac{2b^2}{a} = 2a(1-e^2)$.

Focal Distances of a Point on the Ellipse

- The sum of the focal distances of any point on an ellipse is constant and equal to the length of the major axis of the ellipse.

	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a<b$
Co-ordinates of the centre	$(0, 0)$	$(0, 0)$
Co-ordinates of the vertices	$(a, 0)$ and $(-a, 0)$	$(0, -b)$ and $(0, -b)$
Co-ordinate of foci	$(ae, 0)$ and $(-ae, 0)$	$(0, be)$ and $(0, -be)$
Length of the major axis	$2a$	$2b$
Length of the minor axis	$2b$	$2a$
Equation of the major axis	$y = 0$	$x = 0$
Equation of the minor axis	$x = 0$	$y = 0$
Equations of the directrices	$x = \frac{a}{e}$ and $x = -\frac{a}{e}$	$y = \frac{b}{e}$ and $y = -\frac{b}{e}$
Eccentricity	$e = \sqrt{1 - \frac{b^2}{a^2}}$	$e = \sqrt{1 - \frac{a^2}{b^2}}$
Length of latusrectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Focal distances of a point (x, y)	$a \pm ex$	$b \pm ey$

- If the centre of the ellipse is at point (h, k) and the directions of the axes are parallel to the co-ordinate axes, then its equation is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$.
- We shift the origin at (h, k) without rotating the co-ordinate axes, then

$$x = X + h \text{ and } y = Y + k.$$

- The equation of the ellipse w.r.t. new origin, becomes $\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$.
- The point $P(x_1, y_1)$ lies outside, on or inside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, according as $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1>0, =$ or <0.
- **Auxiliary Circle :** The circle described on the major axis of an ellipse as diameter is called an auxiliary circle of the ellipse.
- If $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is an ellipse, then its auxiliary circle is $x^2 + y^2 = a^2$.
- The co-ordinates of point P having eccentric angle ϕ can be written as $(a\cos\phi, b\sin\phi)$ and are known as the parametric co-ordinates.
- The equations $x = a\cos\phi$, $y = b\sin\phi$ taken together are called the parametric equations of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where ϕ is the parameter.
- The eccentric angles of the extremities of the latus-rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are given by $\theta = \tan^{-1}(\pm b/ae)$.
- The equation of the chord joining two points having eccentric angles θ and ϕ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta-\phi}{2}\right)$.
- The condition for the line $y = mx + c$ to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is that $c^2 = a^2m^2 + b^2$ or $c = \pm\sqrt{a^2m^2+b^2}$.
- The equations of tangents of slope m to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are $y = mx \pm \sqrt{a^2m^2+b^2}$ and the co-ordinates of the points of contact are $\left(\pm\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$.

Equation of the Tangent At a Point

- The equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.
- The equation of tangent and normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $(a\cos\theta, b\sin\theta)$

are $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ and $ax\sec\theta - by\,\text{cosec}\,\theta = a^2 - b^2$ respectively.

- The equation of the tangent at (x_1, y_1) to an ellipse can be obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{x + x_1}{2}$, y by $\frac{y + y_1}{2}$ and keeping the constant unchanged.

Number of Tangents Drawn from a Point to an Ellipse

- Two tangents can be drawn from a point to an ellipse. The two tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the ellipse.
- Two tangents can be drawn from a point $P(h, k)$ to an ellipse.

Director Circle

- The director circle is the locus of points from which perpendicular tangents are drawn to the ellipse.
- The equation of the director circle is $x^2 + y^2 = a^2 + b^2$.
- It is a circle whose centre coincides with the centre of the ellipse and radius equal to $\sqrt{a^2 + b^2}$.

Equation of Normal in Different Forms

- **Point Form :** The equation of the normal at (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$.
- **Parametric Form :** The equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(a\cos\theta, b\sin\theta)$ is $ax\sec\theta - by\,\text{cosec}\,\theta = a^2 - b^2$.
- **Slope Form :** If m is the slope of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then the equation of the normal is $y = mx - \frac{m(a^2 - b^2)}{\sqrt{a^2 + b^2m^2}}$.
- Four normals can be drawn from a point to an ellipse.

Properties of Eccentric Angles of the Co-normal Points

- The sum of the eccentric angles of the co-normal points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to odd multiple of π.
- If α, β, γ are the eccentric angles of three points on the ellipse, the normals at which are concurrent, then $\sin(\alpha + \beta) + \sin(\beta + \gamma) + \sin(\gamma + \alpha) = 0$.

Equation of The Chord of Contact of Tangents Drawn from a Point to an Ellipse

- The chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.

Equation of The Chord of The Ellipse which is Bisected at a given Point

- The equation of a chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ bisected at a given point (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$ or $T = S'$, where S' and T have their usual meanings.

Diameter

- The locus of the mid-point of a system of parallel chords of an ellipse is called a diameter and the chords are called its double ordinates.
- The equation of the diameter bisecting the chords of slope m of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y=-\frac{b^2}{a^2m}x$.
- Two straight lines $y = m_1 x$ and $y = m_2 x$ are conjugate diameters of the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $m_1m_2=\frac{-b^2}{a^2}$.
- In an ellipse, the major axis bisects all chords parallel to the minor axis and vice-versa, therefore major and minor axes of an ellipse are conjugate diameters of the ellipse but they do not satisfy the condition $m_1 m_2 = -b^2/a^2$ and are the only perpendicular conjugate diameters.

Properties of Conjugate Diameters

(i) The eccentric angles of the ends of a pair of conjugate diameters of an ellipse differ by a right angle.

(ii) The sum of the squares of any two conjugate semi-diameters of an ellipse is constant and equal to the sum of the squares of the semi-axes of the ellipse.

(iii) The product of the focal distances of a point on an ellipse is equal to the square of the semi-diameter which is conjugate to the diameter through the point.

(iv) The tangents at the ends of a pair of conjugate diameters of an ellipse form a parallelogram.

(v) The area of the parallelogram formed by the tangents at the ends of conjugate diameters of an ellipse is constant and is equal to the product of the axes.

Some Properties of Ellipse

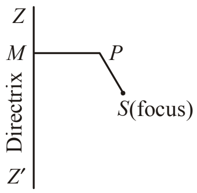
- The tangent and normal at point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meet the x-axis in two points, then the length of the sub-tangent is equal to $\frac{a^2}{x_1}-x_1$ and length of the sub-normal is equal to $(1 - e^2)x_1$.
- The normal and tangent at a point of an ellipse bisect the internal and external angles between the focal distances of the point.
- The locus of the feet of the perpendiculars from the foci on any tangent to an ellipse is the auxiliary circle.

End

15

CHAPTER

hyperbola



- **Definition :** A hyperbola is the locus of a point in a plane which moves in the plane in such a way that the ratio of its distance from a fixed point (called focus) in the same plane to its distances from a fixed line (called directrix) is always constant which is always greater than unity.
- The constant ratio is generally denoted by e and is known as the eccentricity of the hyperbola.
- If S is the focus, ZZ' is the directrix and P is any point on the hyperbola, then by definition $\frac{SP}{PM} = e \Rightarrow SP = e \cdot PM$.

Focal Distances of a Point

- The difference of the focal distances of any point on a hyperbola is constant and equal to the length of the transverse axis of the hyperbola.

	Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	Conjugate hyperbola $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
Co-ordinates of the centre	(0, 0)	(0, 0)
Co-ordinates of the vertices	$(a, 0)$ and $(-a, 0)$	$(0, b)$ and $(0, -b)$
Co-ordinates of foci	$(\pm ae, 0)$	$(0, \pm be)$
Length of the transverse axis	$2a$	$2b$
Length of the conjugate axis	$2b$	$2a$
Equations of the directrices	$x = \pm \frac{a}{e}$	$y = \pm \frac{b}{e}$
Eccentricity	$e = \sqrt{\frac{a^2 + b^2}{a^2}}$ or $b^2 = a^2(e^2 - 1)$	$e = \sqrt{\frac{b^2 + a^2}{b^2}}$ or $a^2 = b^2(e^2 - 1)$
Length of the latusrectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Equation of the transverse axis	$y = 0$	$x = 0$
Equation of the conjugate axis	$x = 0$	$y = 0$

- The equation $x = a\ \sec\theta$ and $y = b\ \tan\theta$ are known as the parametric equations of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
- The equations $x = a\ \cosh\theta$ and $y = b\sinh\theta$ are also known as the parametric equations of the hyperbola and the co-ordinates of any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are expressible as $(a\cosh\theta,\ b\sinh\theta)$, where $\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$ and $\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$.
- The equation of the chord joining two points $P(a\sec\theta_1,\ b\tan\theta_1)$ and $Q(a\sec\theta_2,\ b\tan\theta_2)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

  $$\frac{x}{a}\cos\left(\frac{\theta_1 - \theta_2}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta_1 + \theta_2}{2}\right) = \cos\left(\frac{\theta_1 + \theta_2}{2}\right)$$

- The condition for the line $y = mx + c$ to be tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is that $c^2 = a^2m^2 - b^2$ *i.e.* $c = \pm\sqrt{a^2m^2 - b^2}$.

Different Forms of the Equation of Tangent

- **Slope Form :** The equation of tangents of slope m to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $y = mx \pm \sqrt{a^2m^2 - b^2}$ and the co-ordinates of the points of contact are

  $$\left(\pm\frac{a^2m}{\sqrt{a^2m^2 - b^2}},\ \pm\frac{b^2}{\sqrt{a^2m^2 - b^2}}\right).$$

- **Point Form :** The equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.
- **Parametric Form :** The equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a\sec\theta,\ b\tan\theta)$ is $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$.
- The point of intersection of tangents at $(a\sec\theta_1,\ b\tan\theta_1)$ and $(a\sec\theta_2,\ b\tan\theta_2)$ is

  $$\left[\frac{a\cos\left(\frac{\theta_1 - \theta_2}{2}\right)}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)}\cdot\frac{b\sin\left(\frac{\theta_1 + \theta_2}{2}\right)}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)}\right].$$

- Two tangents can be drawn from a point to a hyperbola.

Director Circle

- The director circle is the locus of points from which perpendicular tangents are drawn to the given hyperbola.

- The equation of the director circle of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $x^2 + y^2 = a^2 - b^2$.

Equation of Pair of Tangents Drawn From a Point to a Hyperbola

- The combined equation of the pair of tangents drawn from point (x_1, y_1) to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is given by $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1\right)^2$
i.e., $S S_1 = T^2$, where S, S_1 and T have their usual meanings.

Equation of Chord of Contact of Tangents Drawn from a Point to a Hyperbola

- The chord of contact of tangents drawn from a point (x_1, y_1) to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

Different Forms of The Equation of Normal

- **Point Form :** The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at (x_1, y_1) is $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$.

- **Parametric Form :** The equation of the normal at $(a\sec\theta, b\tan\theta)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $ax\cos\theta + by\cot\theta = a^2 + b^2$.

- **Slope Form :** The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in terms of the slope m of the normal is given by $y=mx-\frac{m(a^2+b^2)}{\sqrt{a^2-b^2m^2}}$.

Number of Normals from a Point to a Hyperbola

- In general four normals can be drawn from a point (x_1, y_1) to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

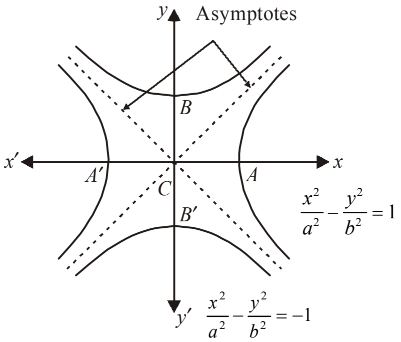
- The line $lx + my + n = 0$ will be a normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ if $\frac{a^2}{l^2}-\frac{b^2}{m^2}=\frac{(a^2+b^2)^2}{n^2}$.

- Equation of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, bisected at the given point (x_1, y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$.

Asymptotes of a Hyperbola

- An asymptote to a curve is a straight line, at a finite distance from the origin, to which the tangent to a curve tends as the point of contact goes to infinity.

- The equations of two asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $y=\pm\frac{b}{a}x$ or $\frac{x}{a}\pm\frac{y}{b}=0$.
- The combined equation of the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$.
- When $b = a$ *i.e.* the asymptotes of rectangular hyperbola $x^2 - y^2 = a^2$ are $y = \pm x$, which are at right angles.



- A hyperbola and its conjugate hyperbola have the same asymptotes.
- The equation of the pair of asymptotes differ the hyperbola and the conjugate hyperbola by the same constant only *i.e.*
  Hyperbola – Asymptotes = Asymptotes – Conjugate hyperbola
  or $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)-\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)=\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)-\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}+1\right)$
- The asymptotes pass through the centre of the hyperbola. The bisectors of the angles between the asymptotes are the co-ordinate axes.
- The product of the perpendiculars from any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ to its asymptotes is equal to $\frac{a^2b^2}{a^2+b^2}$.

Rectangular Hyperbola

- A hyperbola whose asymptotes are at right angles to each other is called a rectangular hyperbola.
- The transverse and conjugate axes of a rectangular hyperbola are equal and the equation is $x^2 - y^2 = a^2$.

  The equation of the asymptotes of the rectangular hyperbola are $y = \pm x$ *i.e.* $y = x$ and $y = -x$. Clearly, each of these two asymptotes is inclined at 45° to the transverse axis.
- Parametric co-ordinates of a point on the hyperbola $xy = c^2$ is $(ct, c/t)$.

- **Equation of The Chord Joining Points t_1 and t_2 :** The equation of the chord joining two points $\left(ct_1, \frac{c}{t_1}\right)$ and $\left(ct_2, \frac{c}{t_2}\right)$ on the hyperbola $xy = c^2$ is $x + yt_1t_2 = c(t_1 + t_2)$.

Equation of Tangent in Different Forms

- **Point Form :** The equation of tangent at (x_1, y_1) to the hyperbola $xy = c^2$ is $xy_1 + yx_1 = 2c^2$ or $\frac{x}{x_1} + \frac{y}{y_1} = 2$.
- **Parametric Form :** The equation of the tangent at $\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy = c^2$ is $\frac{x}{t} + yt = 2c$.

Equation of The Normal in Different Forms

- **Point Form :** The equation of the normal at (x_1, y_1) to the hyperbola $xy = c^2$ is $xx_1 - yy_1 = x_1^2 - y_1^2$.
- **Parametric Form :** The equation of the normal at $\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy = c^2$ is $xt^3 - yt - ct^4 + c = 0$.
- The equation of the chord of the hyperbola $xy = c^2$ whose middle is (x_1, y_1) is
$$xy_1 + yx_1 = 2x_1y_1 \text{ or } T = S'.$$
- The equations of tangents at t_1 and t_2 to $xy = c^2$ are
$\frac{x}{t_1} + yt_1 = 2c$ and $\frac{x}{t_2} + yt_2 = 2c$.
The point of intersection of these two lines is $\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right)$.

End

16

CHAPTER

real functions

- **Real Functions :** If the domain and co-domain of a function are subsets of R (set of all real numbers), it is called a real valued function or a real function.
- **Range of a Function :** The range of a function $f : A \rightarrow B$ is the set of all values taken by f.

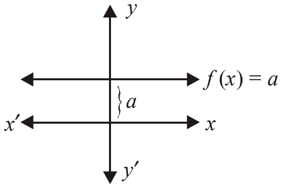
  $\therefore$ Range $(f) = \{f(x) | x \in A\}$. Also, Range $(f) \subset B$.
- **Closed Interval :** Let a and b be two given real numbers such that $a < b$. Then the set of all real numbers x such that $a \leq x \leq b$ is called a closed interval and is denoted by $[a, b]$. *i.e.* $[a, b] = \{x \in R \mid a \leq x \leq b\}$.
- **Open Interval :** Let a and b be two given real numbers such that $a < b$. The set of all real numbers x such that $a < x < b$ is called a open interval and is denoted by (a, b).*i.e.* $(a, b) = \{x \in R | \ a < x < b\}$.
- **Semi-Closed or Semi-Open Interval :** If a, b are two given real numbers such that $a < b$, then the sets $(a, b] = \{x \in R| \ a < x \leq b\}$ and $[a, b) = \{x \in R| \ a \leq x < b\}$ are known as semi-closed or semi-open intervals.
- **Domain of a Real Function :** When real functions in calculus are described by some formula and their domains are not explicitly stated, then the domain is the set of all real numbers x for which $f(x)$ is a real number.
- **Algorithm for Finding The Range of a Real Function**

  Step I : Put $f(x) = y$.

  Step II : Solve the equation in step I for x to obtain $x = \phi(y)$.

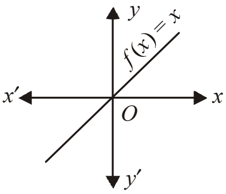
  Step III : Find the values of y for which the values of x, obtained from $x = \phi(y)$ are in the domain of f.

  Step IV : The set of values of y obtained in Step III is the range of f.
- **Constant Function :** Let a be a fixed real number. Then a function $f(x)$ given by $f(x) = a$ for all $x \in R$ is called a constant function.



Domain = R
Range = a

- **Identity Function :** The function defined by $I(x) = x$ for all $x \in R$, is called the identity function on R.

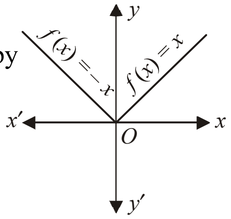


Domain = R
Range = R

- **Modulus Function :** The function defined by

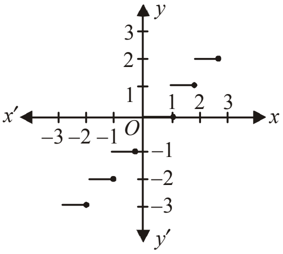
$$f(x)=|x|=\begin{cases} x, & \text{when } x \geq 0 \\ -x, & \text{when } x < 0 \end{cases}$$

is called the modulus function.



Domain = R
Range = R^+

- **The Greatest Integer Function :** For any real number x, we denote $[x]$, the greatest integer less than or equal to x. The function f defined by $f(x) = [x]$ for all $x \in R$, is called the greatest integer function.

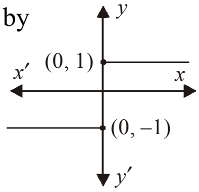


Domain = R
Range = I (Set of integers)

- **Signum Function :** The function defined by

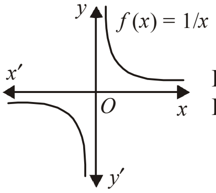
$$f(x)=\begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \text{ or } f(x)=\begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

is called the signum function.



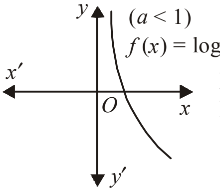
Domain = R
Range = $\{-1, 0, 1\}$

- **Reciprocal Function :** The function that associates each non-zero real number x to its reciprocal $\dfrac{1}{x}$ is called the reciprocal function.

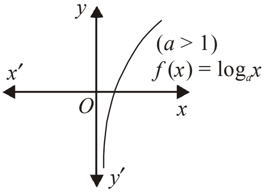


Domain = $R - \{0\}$
Range = $R - \{0\}$

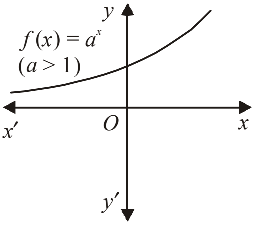
- **Logarithmic Function :** If a is a positive real number, then the function that associates every positive real number to $\log_a x$ *i.e.* $f(x) = \log_a x$ is called the logarithmic function.

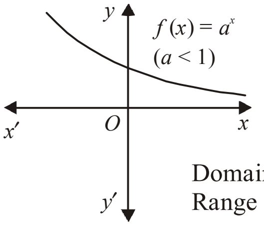


Domain = R^+
Range = R



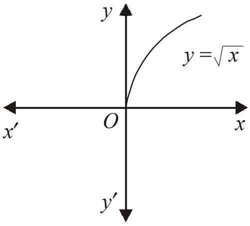
- **Exponential Function :** If a is positive real number, then the function which associates every real number x to a^x *i.e.* $f(x) = a^x$ is called the exponential function.





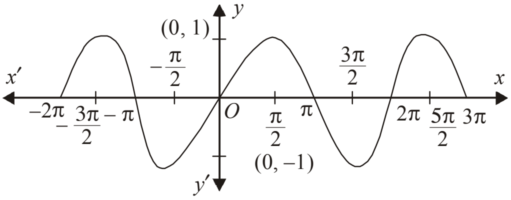
Domain = R
Range = R^+

- **Square Root Function :** The function that associates every positive real number x to $+\sqrt{x}$ is called the square root function, *i.e.*, $f(x) = +\sqrt{x}$.



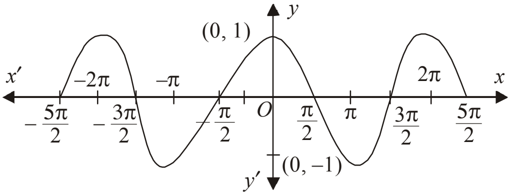
Domain = $[0, \infty)$
Range = $[0, \infty)$

- **Polynomial Function :** A function of the form $f(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$, where $a_0, a_1, a_2, ..., a_n$ are real numbers, $a_0 \neq 0$ and $n \in N$, is called a polynomial function of degree n.
  The domain of a polynomial function is always R.
- **Rational Function :** A function of the form $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial and $q(x) \neq 0$, is called a rational function. Domain of a rational function $\dfrac{p(x)}{q(x)}$ is the set of all real numbers except points where $q(x) = 0$.
- **Sine Function :** The function that associates each real number x to $\sin x$ is called the sine function, where x is the radian measure of the angle.



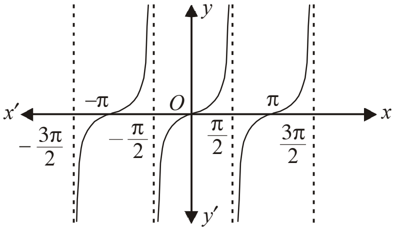
Domain = R, Range = $[-1, 1]$

- **Cosine Function :** The function that associates each real number x to $\cos x$ is called the cosine function, where x is the radian measure of the angle.



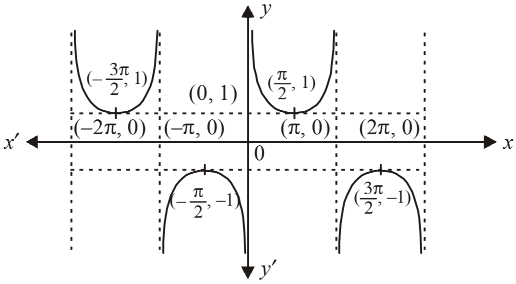
Domain = R, Range = $[-1, 1]$

- **Tangent Function :** The function that associates a real number x to tan x is called the tangent function.



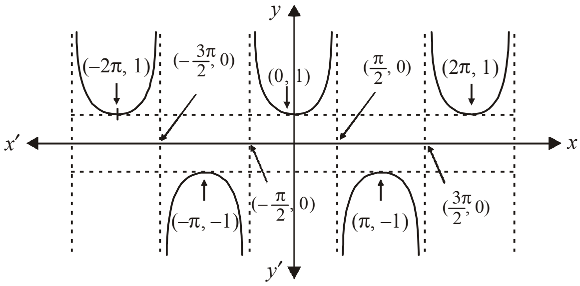
Domain = $R - \left\{(2n+1)\frac{\pi}{2} \mid n \in Z\right\}$, Range = R

- **Cosecant Function :** The function that associates a real number x to cosec x is called the cosecant function.



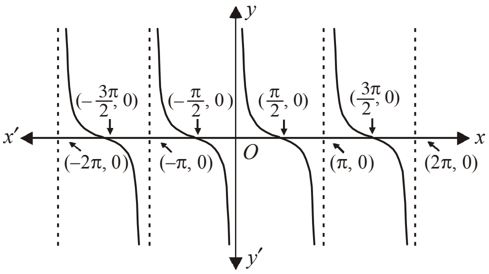
Domain = $R - \{n\pi \mid n \in Z\}$, Range = $(-\infty, -1] \cup [1, \infty)$

- **Secant Function :** The function that associates a real number x to sec x is called the secant function.



Domain = $R - \left\{(2n+1)\frac{\pi}{2} \mid n \in Z\right\}$, Range = $(-\infty, -1] \cup [1, \infty)$

- **Cotangent Function :** The function that associates a real number x to cot x is called the cotangent function.



Domain = $R - \{n\pi | n \in Z\}$, Range = R

- **Inverse Trigonometrical Functions :** The inverse trigonometrical functions $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc., are defined as the inverse of the corresponding trigonometric functions.

Function	Domain	Range	Definition of the function
$\sin^{-1}x$	$[-1, 1]$	$[-\pi/2, \pi/2]$	$y = \sin^{-1}x$ $\Leftrightarrow x = \sin y$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$	$y = \cos^{-1}x$ $\Leftrightarrow x = \cos y$
$\tan^{-1}x$	$(-\infty, \infty)$ or R	$(-\pi/2, \pi/2)$	$y = \tan^{-1}x$ $\Leftrightarrow x = \tan y$
$\cot^{-1}x$	$(-\infty, \infty)$ or R	$(0, \pi)$	$y = \cot^{-1} x$ $\Leftrightarrow x = \cot y$
$\text{cosec}^{-1}x$	$R-(-1, 1)$	$[-\pi/2, \pi/2]-\{0\}$	$y = \text{cosec}^{-1}x$ $\Leftrightarrow x = \text{cosec}\, y$
$\sec^{-1}x$	$R-(-1, 1)$	$[0, \pi] -\{\pi/2\}$	$y = \sec^{-1}x$ $\Leftrightarrow x = \sec y$

Operations on Real Functions

- **Sum :** Let f and g be two real functions with domain D_1 and D_2 respectively. Sum $f + g$ is a function from $D_1 \cap D_2$ to R which associates each $x \in D_1 \cap D_2$ to the number $f(x) + g(x)$.
- **Difference :** $f - g : D_1 \cap D_2 \to R$ such that $(f - g)(x) = f(x) - g(x)$ for all $x \in D_1 \cap D_2$.
- **Product :** $fg : D_1 \cap D_2 \to R$ such that $(fg)(x) = f(x)g(x)$ for all $x \in D_1 \cap D_2$.
- **Quotient :** $\dfrac{f}{g} : D_1 \cap D_2 - \{x \mid g(x) = 0\} \to R$ such that

  $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ for all $x \in D_1 \cap D_2 - \{x|g(x) = 0\}$.
- **Scalar Multiple :** For any real number c, the function cf is defined by $(cf)(x) = c{\cdot}f(x)$ for all $x \in D_1$.
- **Composition of Functions :** Let f and g be two functions with domain D_1 and D_2 respectively. If range (f) = domain (g), then $(gof)(x) = g(f(x)\ \forall\ x \in D_1$ and if range (g) = domain (f), then $(fog)(x) = f(g(x))\ \forall\ x \in D_2$.
- **Even Functions :** A function $f(x)$ is said to be an even function if $f(-x) = f(x)$ for all x.
- **Odd Function :** A function $f(x)$ is said to be an odd function if $f(-x) = -f(x)$ for all x.
- Every real valued function can be expressed uniquely as the sum of an even function and an odd function.

- If $f(x)$ is an even differentiable function on R, then $f'(x)$ is an odd function.
- If $f(x)$ is an odd function differentiable on R, then $f'(x)$ is an even function.
- **Periodic Function :** A function $f(x)$ is said to be a periodic function if there exists a positive real numbers T such that $f(x + T) = f(x)$ for all $x \in R$.

Some Standard Periodic Functions and their Periods	
Function	**Period**
$\sin x$	2π
$\cos x$	2π
$\tan x$	π
$\operatorname{cosec} x$	2π
$\sec x$	2π
$\cot x$	π
$x - [x]$	1
$\sin^2 x$, $\cos^2 x$	π
$\|\sin x\|$, $\|\cos x\|$	π
$\sin^4 x + \cos^4 x$	$\pi/2$

- If $f(x)$ is a periodic function with period T and $a, b \in R$ such that $a > 0$, then $f(ax + b)$ is periodic with period T/a.
- If $f_1(x), f_2(x)$ and $f_3(x)$ are periodic functions with periods T_1, T_2 and T_3 respectively, then $a_1 f_1(x) + a_2 f_2(x) + a_3 f_3(x)$ is a periodic function with period equal to L.C.M. of T_1, T_2 and T_3 where a_1, a_2, a_3 are non-zero real numbers.
- If $f(x)$ is a periodic function with period T and $g(x)$ is any function such that domain of f is a proper subset of domain of g, then gof is periodic with period T.

End

17

CHAPTER

limits

- **Neighbourhood (nbd.) of a Point :** Let a be a real number and let δ be a positive real number. Then the set of all real numbers lying between $a - \delta$ and $a + \delta$ is called the neighbourhood of a of radius 'δ' and is denoted by $N_\delta(a)$.
- If l is the limit of $f(x)$ as x tends to a, then we write $\lim_{x \to a} f(x) = l$.
- **Algorithm for Finding Left Hand Limit :**

  Step I : Write $\lim_{x \to a^-} f(x)$.

  Step II : Put $x = a - h$ and replace $x \to a^-$ by $h \to 0$ to obtain $\lim_{h \to 0} f(a-h)$.

  Step III: Simplify $\lim_{h \to 0} f(a-h)$ by using the formula for the given function.

  Step IV: The value obtain in Step III is the L.H.L. of $f(x)$ at $x = a$.
- **Algorithm for Finding Right Hand Limit :**

  Step I : Write $\lim_{x \to a^+} f(x)$.

  Step II : Put $x = a + h$ and replace $x \to a^+$ by $h \to 0$ to obtain $\lim_{h \to 0} f(a+h)$.

  Step III : Simplify $\lim_{h \to 0} f(a+h)$ by using the formula for the given function.

  Step IV : The value obtain in Step III is the R.H.L. of $f(x)$ at $x = a$.
- $\lim_{x \to a} f(x)$ exists if L.H.L. = R.H.L. at $x = a$

Algebra of Limits

- Let $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$. If l and m exist, then

(i) $\lim_{x \to a}(f \pm g)(x) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = l \pm m$.

(ii) $\lim_{x \to a}(fg)(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = l\,m$.

(iii) $\lim_{x \to a}\left(\frac{f}{g}\right)(x) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{l}{m}$, provided $m \neq 0$.

(iv) $\lim_{x \to a} Kf(x) = K \cdot \lim_{x \to a} f(x)$, where K is constant.

(v) $\lim_{x \to a} |f(x)| = |\lim_{x \to a} f(x)| = |l|$.

(vi) $\lim_{x \to a}(f(x))^{g(x)} = l^m$.

(vii) If $f(x) \leq g(x)$ for every x in the deleted nbd. of a (not including a) and $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$.

(viii) If $f(x) \le g(x) \le h(x)$ for every x in the deleted nbd. of a (not including a) and $\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x) = l$.

(ix) $\lim_{x\to a} fog(x) = f\left(\lim_{x\to a} g(x)\right) = f(m)$

i.e., $\lim_{x\to a} \log f(x) = \log\left(\lim_{x\to a} f(x)\right) = \log l$ & $\lim_{x\to a} e^{f(x)} = e^{\lim_{x\to a} f(x)} = e^l$.

(x) If $\lim_{x\to a} f(x) = +\infty$ or $-\infty$, then $\lim_{x\to a} \frac{1}{f(x)} = 0$.

- **Evaluation of Algebraic Limits :** Let $f(x)$ be an algebraic function and 'a' be a real number. Then $\lim_{x\to a} f(x)$ is known as an algebraic limit.
- **Methods of Evaluation of Limits**

(i) Direct Substitution Method : If by direct substitution of the point in the given expression we get a finite number, then the number obtained is the limit of the given expression.

(ii) Factorisation method : Consider $\lim_{x\to a} \frac{f(x)}{g(x)}$. If by puting $x = a$ the rational function $\frac{f(x)}{g(x)}$ takes the form $\frac{0}{0}, \frac{\infty}{\infty}$ etc., then $(x - a)$ is a factor of both $f(x)$ and $g(x)$. In such case cancel out the common factor and find the limit.

(iii) If $n \in Q$, then $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$.

(iv) Method of evaluating algebraic limit when $x \to \infty$: To evaluate this type of limit we follow the following procedure :

Step I : Write down the given expression in form of a rational function, *i.e.*, $\frac{f(x)}{g(x)}$, if it is not so.

Step II : If k is the highest power of x in numerator and denominator both, then divide each term in numerator and denominator by x^k.

Step III : Use the result $\lim_{x\to\infty} \frac{1}{x^n} = 0$, where $n > 0$.

(v) If m, n are positive integers and a_0, $b_0 \ne 0$ are non-zero real numbers, then

$$\lim_{x\to\infty} \frac{a_0x^m + a_1x^{m-1} + \dots + a_{m-1}x + a_m}{b_0x^n + b_1x^{n-1} + \dots + b_{n-1}x + b_n} = \begin{cases} \frac{a_0}{b_0}, & \text{if } m = n \\ 0, & \text{if } m < n. \\ \infty, & \text{if } m > n \end{cases}$$

- **Evaluation of Trigonometric Limits**

(i) $\lim_{x\to 0} \frac{\sin x}{x} = 1$ **(ii)** $\lim_{x\to 0} \frac{\tan x}{x} = 1$

(iii) $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1$ **(iv)** $\lim_{x\to 0} \frac{\tan^{-1} x}{x} = 1$

(v) $\lim_{x\to 0} \frac{\sin x^\circ}{x} = \frac{\pi}{180}$ **(vi)** $\lim_{x\to 0} \cos x = 1$

(vii) $\lim_{x\to a} \frac{\sin(x-a)}{(x-a)} = 1$ **(viii)** $\lim_{x\to a} \frac{\tan(x-a)}{(x-a)} = 1$.

- **Evaluation of Exponential Limits**

(i) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a$ **(ii)** $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$ **(iii)** $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$.

(iv) $\lim_{x\to 0}(1+x)^{1/x} = e$ **(v)** $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

(vi) $\lim_{x\to 0}(1+\lambda x)^{1/x} = e^{\lambda}$ **(vii)** $\lim_{x\to\infty}\left(1+\frac{\lambda}{x}\right)^x = e^{\lambda}$.

- If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then $\lim_{x\to a}[1+f(x)]^{1/g(x)} = e^{\lim_{x\to a}\frac{f(x)}{g(x)}}$.

- If $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a}[f(x)]^{g(x)} = \lim_{x\to a}[1+(f(x)-1)]^{g(x)}$

$= e^{\lim_{x\to a}(f(x)-1)g(x)}$.

- **Evaluation of Limits by using The L' Hospital's Rule**

If $f(x)$ and $g(x)$ be two functions of x such that

(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$.

(ii) Both are continuous at $x = a$.

(iii) Both are differentiable at $x = a$.

(iv) $f'(x)$ and $g'(x)$ are continuous at the point $x = a$, then

$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$, provided that $g(a) \neq 0$.

- The above rule is also applicable if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$.

- $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)}$.

End

18

CHAPTER

continuity and differentiability

- **Continuity at a Point :** A function $f(x)$ is said to be continuous at a point $x = a$ of its domain, iff $\lim_{x \to a} f(x) = f(a)$.
- **Continuity on an open interval :** A function $f(x)$ is said to be continuous on an open interval (a, b) iff it is continuous at every point on the interval (a, b).
- **Continuity on a closed interval :** A function $f(x)$ is said to be continuous on a closed interval $[a, b]$ iff

  (i) f is continuous on the open interval (a, b).

  (ii) $\lim_{x \to a^+} f(x) = f(a)$, and

  (iii) $\lim_{x \to b^-} f(x) = f(b)$

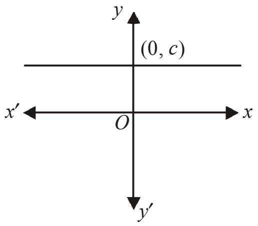
  In other words, $f(x)$ is continuous on $[a, b]$ iff it is continuous on (a, b) and it is continuous at a from the right and at b from the left.
- **Continuous Function :** A function $f(x)$ is said to be continuous, if it is continuous at each point of its domain.
- **Everywhere Continuous Function :** A function $f(x)$ is said to be everywhere continuous if it is continuous on the entire real line $(-\infty, \infty)$.
- Let $f(x)$ and $g(x)$ be two continuous functions on their common domain D and let c be a real number. Then

  (i) cf is continuous (ii) $f + g$ is continuous

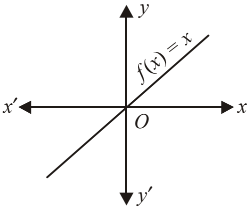
  (iii) $f - g$ is continuous (iv) fg is continuous

  (v) f/g is continuous (vi) $\frac{1}{g}$ is continuous

  (vii) f^n, $n \in N$ is continuous.
- **Constant Function :** Every constant function is everywhere continuous.

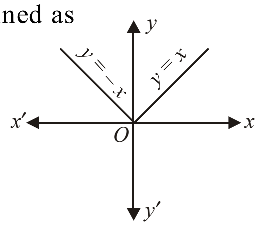


- **Identity Function :** The identity function $I(x)$ is defined by $I(x) = x$ for all $x \in R$. This function is everywhere continuous as evident from its graph shown in figure.

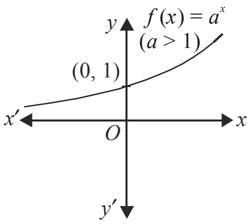


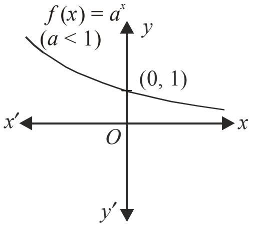
- **Modulus Function :** The modulus function $f(x)$ is defined as

$$f(x) = |x| = \begin{cases} x, & \text{if} \quad x \geq 0 \\ -x, & \text{if} \quad x < 0 \end{cases}$$

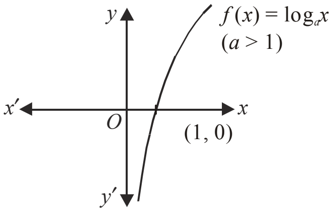


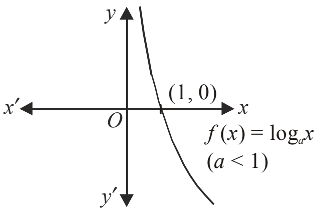
- **Exponential Function :** If a is a positive real number, other than 1, then the function $f(x)$ defined by $f(x) = a^x$ for all $x \in R$ is called the exponential function.





- **Logarithmic Function :** If a is positive real number other than unity, then a function defined by $f(x) = \log_a x$ is called the logarithmic function.





- **Polynomial Function :** A function of the form $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$, where $a_0, a_1, a_2, ..., a_n \in R$ is called a polynomial function.
  Polynomial function is everywhere continuous.
- **Rational function :** If $p(x)$ and $q(x)$ are two polynomials, then a function, $f(x)$ of the form $f(x) = \dfrac{p(x)}{q(x)}, q(x) \neq 0$ is called a rational function.
  Rational function is continuous on its domain, *i.e.*, it is everywhere continuous except at points where $q(x) = 0$.
- **Trigonometric Functions :** All trigonometrical functions viz. $\sin x$, $\cos x$, $\tan x$, $\sec x$ $\operatorname{cosec} x$, $\cot x$ are continuous at each point of their respective domains. Each of inverse trigonometric functions is continuous in its domain.

- The composition of two continuous functions is a continuous function *i.e.* if f and g are two and functions such that g is continuous at a point a and f is continuous at $g(a)$, then fog is continuous at a.

Cauchy's Definition of Continuity

- A function f is said to be continuous at a point a of its domain D iff for every $\varepsilon > 0$ there exists a $\delta > 0$ (dependent on ε) such that $|x - a| < \delta$ $\Rightarrow |f(x) - f(a)| < \varepsilon$
- $f(x)$ is continuous at $x = 0$ iff $\lim_{x\to a} f(x)$ exists and is equal to $f(a)$ *i.e.* iff $\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^+} f(x)$.
- **Continuity From the Left :** A function f defined on a left nbd. of a point a of its domain D is said to be continuous from the left at a iff $\lim_{x\to a^-} f(x) = f(a)$.
- f is continuous from the right at a iff $\lim_{x\to a^+} f(x) = f(a)$.
- **Continuity of a Function in an Open Interval (a, b) :** A function f is said to be continuous on (a, b) if f is continuous at each point of (a, b).
- **Continuity of a function in a closed interval $[a, b]$:** A function f is said to be continuous on a closed interval $[a, b]$ if

  (i) f is continuous on (a, b),

  (ii) f is continuous from the right at a *i.e.* $\lim_{x\to a^+} f(x) = f(a)$, and

  (iii) f is continuous from the left at b *i.e.* $\lim_{x\to b^-} f(x) = f(b)$.

Heine's Definition of Continuity

- A function f is said to be continuous at a point a of its domain D, if for every sequence $< a_n >$ of the points in D converging to a, the sequence $< f(a_n) >$ converges to $f(a)$ *i.e.* $\lim a_n = a \Rightarrow \lim f(a_n) = f(a)$.

Discontinuous Functions

- A function f is said to be discontinuous at a point a of its domain D if it is not continuous at a. The point a is then called a point of discontinuity of the function. The discontinuity may arise due to any of the following situations:

  (i) $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x)$ or both may not exist.

  (ii) $\lim_{x\to a^+} f(x)$ as well as $\lim_{x\to a^-} f(x)$ may exist, but are unequal.

  (iii) $\lim_{x\to a^+} f(x)$ as well as $\lim_{x\to a^-} f(x)$ both may exist, but either of the two or both may not be equal to $f(a)$.
- **Removable Discontinuity :** A function f is said to have removable discontinuity at $x = a$ if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$ but their common value is not equal to $f(a)$.
- **Discontinuity of the First Kind :** A function f is said to have a discontinuity of the first kind at $x = a$ if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist but are not equal.

- **Discontinuity of Second Kind :** A function f is said to have a discontinuity of the second kind at $x = a$ if neither $\lim_{x \to a^-} f(x)$ nor $\lim_{x \to a^+} f(x)$ exists.

Properties of Continuous Functions

(i) If f, g are two continuous functions at a point a of their common domain D, then $f \pm g$, fg are continuous at a and if $g(a) \neq 0$, then f/g is also continuous at a.

(ii) If f is continuous at a and $f(a) \neq 0$, then there exists an open interval $(a - \delta, a + \delta)$ such that for all $x \in (a - \delta, a + \delta)$, $f(x)$ has the same sign as $f(a)$.

(iii) If a function f is continuous on a closed interval $[a, b]$, then it is bounded on $[a, b]$ *i.e.* there exist real numbers k and K such that $k \leq f(x) \leq K$ for all $x \in [a, b]$.

(iv) If f is a continuous function defined on $[a, b]$ such that $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one solution of the equation $f(x) = 0$ in the open interval (a, b).

(v) If f is continuous on $[a, b]$, then f assumes atleast once, every value between its minimum and maximum values *i.e.* if K is any real number between minimum and maximum values of $f(x)$, then there exists at least one solution of the equation $f(x) = K$ in the open interval (a, b).

(vi) If f is continuous on $[a, b]$ and maps $[a, b]$ into $[a, b]$, then for some $x \in [a, b]$ we have $f(x) = x$.

(vii) If g is continuous at a and f is continuous at $g(a)$, then fog is continuous at a.

Differentiability at a Point

- Let $f(x)$ be a real valued function defined on an open interval (a, b) and let $c \in (a, b)$. Then $f(x)$ is said to be differentiable or derivable at $x = c$, iff $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists finitely.
- $f(x)$ is differentiable at $x = c \Leftrightarrow Lf'(c) = Rf'(c)$. If $Lf'(c) \neq Rf'(c)$, we say that $f(x)$ is not differentiable at $x = c$.
- $f(x)$ is differentiable at point P, iff there exists a unique tangent at point P. In other words, $f(x)$ is differentiable at a point P iff the curve does not have P as a corner point.
- $f(x)$ is differentiable at $x = c \Rightarrow f(x)$ is continuous at $x = c$.

Differentiability in a Set

- A function $f(x)$ defined on an open interval (a, b) is said to be differentiable or derivable in open interval (a, b) if it is differentiable at each point of (a, b).

Some Standard Results on Differentiability

I. Every polynomial function is differentiable at each $x \in R$.

II. The exponential function a^x, $a > 0$ is differentiable at each $x \in R$.

III. Every constant function is differentiable at each $x \in R$.

IV. The logarithmic function is differentiable at each point in its domain.

V. Trigonometric and inverse-trigonometric functions are differentiable in their domains.

VI. The sum, difference, product and quotient of two differentiable functions is differentiable.

VII. The composition of differentiable functions is a differentiable function.

End

19
CHAPTER

differentiation

- **Derivative :** The rate of change of a function with respect to the independent variable. For the function $y = f(x)$ it is denoted by $\frac{dy}{dx}$.
- **Differentiation :** The process of obtaining the derivative of a function by considering small changes in the function and independent variable, and finding the limiting value of the ratio of such changes.
- $\frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, $\frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x-h)-f(x)}{-h}$.

Geometrically Meaning of Derivative at a Point

- Geometrically derivative of a function at a point $x = c$ is the slope of the tangent to the curve $y = f(x)$ at the point $(c, f(c))$.
- Slope of tangent at $P = \lim_{x\to c}\frac{f(x)-f(c)}{x-c} = \left(\frac{df(x)}{dx}\right)_{x=c}$ or $f'(c)$.

Differentiation of Some Standard Functions

(i) $\frac{d}{dx}(x^n) = nx^{n-1}$

(ii) $\frac{d}{dx}(e^x) = e^x$

(iii) $\frac{d}{dx}(a^x) = a^x \log_e a$

(iv) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(v) $\frac{d}{dx}(\log_a x) = \frac{1}{x\log_e a}$

(vi) $\frac{d}{dx}(\sin x) = \cos x$

(vii) $\frac{d}{dx}(\cos x) = -\sin x$

(viii) $\frac{d}{dx}(\tan x) = \sec^2 x$

(ix) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(x) $\frac{d}{dx}(\sec x) = \sec x\tan x$

(xi) $\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x\cot x$

(xii) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, -1<x<1$

(xiii) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, -1<x<1$

(xiv) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, -\infty < x < \infty$

(xv) $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

(xvi) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$

(xvii) $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}, |x| > 1$.

- Differentiation of a constant function is zero *i.e.* $\frac{d}{dx}(c) = 0$.
- Let $f(x)$ be a differentiable function and let c be a constant. Then $c \cdot f(x)$ is also differentiable such that $\frac{d}{dx}\{c \cdot f(x)\} = c \cdot \frac{d}{dx}(f(x))$

  That is the derivative of a constant times a function is the constant times the derivatives of the function.
- If $f(x)$ and $g(x)$ are differentiable functions, then $f(x) \pm g(x)$ are also differentiable such that $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$.

  That is the derivative of the sum or difference of two functions is the sum or difference of their derivatives.
- **Product Rule :** If $f(x)$ and $g(x)$ are two differentiable functions, then $f(x) \cdot g(x)$ is also differentiable such that

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x)\frac{d}{dx}\{g(x)\} + g(x)\frac{d}{dx}\{f(x)\}.$$

  That is, derivative of the product of two functions = [(First function) × (derivative of second function) + (second function) × (derivative of first function)].
- **Quotient Rule :** If $f(x)$ and $g(x)$ are two differentiable functions and $g(x) \neq 0$, then $\frac{f(x)}{g(x)}$ is also differentiable such that

$$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x) \cdot \frac{d}{dx}\{f(x)\} - f(x) \cdot \frac{d}{dx}\{g(x)\}}{[g(x)]^2}.$$

- Relation between $\frac{dy}{dx}$ and $\frac{dx}{dy}$: $\frac{dy}{dx} = \frac{1}{dx/dy}$

Differentiation of Implicit Functions

- When it is not possible to express y as a function of x in the form of $y = \phi(x)$, then y is said to be an implicit function of x. To find the derivative in such case we differentiate both sides of the given relation with respect of x.

Differentiation of Logarithmic Functions

- $y = f(x)^{g(x)} = e^{g(x).\log\{f(x)\}}$ and then differentiating with respect to x, we may get

$$\frac{dy}{dx} = e^{g(x)\log\{f(x)\}}\left[g(x) \cdot \frac{1}{f(x)} \frac{df(x)}{dx} + \log\{f(x)\} \cdot \frac{dg(x)}{dx}\right]$$

$$= [f(x)]^{g(x)}\left[\frac{g(x)}{f(x)} \cdot \frac{df(x)}{dx} + \log\{f(x)\} \cdot \frac{dg(x)}{dx}\right].$$

Differentiation of Parametric Functions

- When x and y are given as functions of a single variable, *i.e.*, $x = \phi(t)$, $y = \psi(t)$ are two functions and t is a variable. Then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

Differentiation of a Function with respect to Another Function

- Let $u = f(x)$ and $v = g(x)$ be two functions of x. Then to find the derivative of $f(x)$ w.r.t. $g(x)$ *i.e.*, to find $\frac{du}{dv}$ we use the formula $\frac{du}{dv} = \frac{du/dx}{dv/dx}$.

  Thus, to find the derivative of $f(x)$ w.r.t. $g(x)$, we first differentiate both w.r.t. x and then divide the derivative of $f(x)$ w.r.t. x by the derivative of $g(x)$ w.r.t. x.

End

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CHAPTER

tangents, normals and other applications of derivatives

Slope of Tangent

- If the tangent at P is perpendicular to x-axis, or parallel to y-axis, then
$$\psi = \frac{\pi}{2} \Rightarrow \cot\psi = 0 \Rightarrow \frac{1}{\tan\psi} = 0 \Rightarrow \left(\frac{dx}{dy}\right)_P = 0.$$

- If $\left(\frac{dy}{dx}\right)_P = \infty$, then the tangent at (x_1, y_1) is parallel to y-axis and its equation is $x = x_1$.

Slope of Normal

- Slope of the normal at $P = \dfrac{-1}{\text{slope of the tangent at } P}$
$$= -\frac{1}{\left(\frac{dy}{dx}\right)_P} = -\left(\frac{dx}{dy}\right)_P.$$

- If $\left(\frac{dy}{dx}\right)_P = 0$, then the normal at (x_1, y_1) is parallel to y-axis and its equation is $x = x_1$.

- **Algorithm for Finding The Equation of Tangent and Normal to The Curve $y = f(x)$ at the given Point (x_1, y_1)**

  Step I : Find $\frac{dy}{dx}$ from the given equation $y = f(x)$.

  Step II : Find the value of $\frac{dy}{dx}$ at the given point $P(x_1, y_1)$.

  Step III : If $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ is a non-zero finite number, then obtain the equations of tangent and normal at (x_1, y_1) by using the formulae
$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x - x_1) \text{ and } (y - y_1) = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}(x - x_1)$$
  respectively. Otherwise go to Step IV or Step V.

  Step IV : If $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$, then the equations of the tangent and normal at (x_1, y_1) are $(y - y_1) = 0$ and $(x - x_1) = 0$ respectively.

Step V : If $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \pm\infty$, then the equation of the tangent and normal at (x_1, y_1) are $(x - x_1) = 0$ and $(y - y_1) = 0$ respectively.

- **Angle of Intersection of Two Curves :** The angle of intersection of two curves is defined to be the angle between the tangents to the two curves at their point of intersection.
- The other angle between the tangents is $180° - \phi$. Generally the smaller of these two angles is taken to be the angle of intersection.
- **Orthogonal Curves :** If the angle of intersection of two curves is a right angle, the two curves are said to intersect orthogonally and the curves are called orthogonal curves.
- **Lengths of Tangent, Normal, Sub-tangent and Sub-normal :**

  Sub-tangent $= \frac{y}{(dy/dx)}$, Sub-normal $= y\frac{dy}{dx}$

  Length of the Tangent $= y\sqrt{1+\cot^2\psi} = \dfrac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}}$

  Length of the Normal $= y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$.

Rolle's Theorem

- **Statement :** Let f be a real valued function defined on the closed interval $[a, b]$ such that

  (i) It is continuous on the closed interval $[a, b]$

  (ii) It is differentiable on the open interval (a, b)

  (iii) $f(a) = f(b)$.

  Then there exists a real number $c \in (a, b)$ such that $f'(c) = 0$.
- **Algebraic Interpretation of Rolle's Theorem :** Between any two roots of a polynomial $f(x)$, there is always a root of its derivative $f'(x)$.

Lagrange's Mean Value Theorem

- **Statement :** Let $f(x)$ be a function defined on $[a, b]$ such that

  (i) It is continuous on $[a, b]$.

  (ii) It is differentiable on (a, b).

  (iii) $f(a) \neq f(b)$

  Then there exists a real number $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.
- **Geometrical Interpretation of Lagrange's Mean Value Theorem :** Let $f(x)$ be a function defined on $[a, b]$ such that the curve $y = f(x)$ is a continuous curve between points $A(a, f(a))$ and $B(b, f(b))$ and at every point on the curve, except at the end points, it is possible to draw a unique tangent. Then there exists a point on the curve such that the tangent at this is parallel to the chord joining the end points of the curve.

End

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CHAPTER

increasing and decreasing functions

- **Strictly Increasing Function :** A function $f(x)$ is said to be a strictly increasing function on (a, b) if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in (a, b)$.
- **Strictly Decreasing Function :** A function $f(x)$ is said to be a strictly decreasing function on (a, b) if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in (a, b)$.
- **Monotonic Function :** A function $f(x)$ is said to be monotonic on an interval (a, b) if it is either increasing or decreasing on (a, b).
- A function $f(x)$ is said to be increasing (decreasing) at point x_0 if there is an interval $(x_0 - h, x_0 + h)$ containing x_0 such that $f(x)$ is increasing (decreasing) on $(x_0 - h, x_0 + h)$.
- A function $f(x)$ is said to be increasing on $[a, b]$ if it is increasing (decreasing) on (a, b) and it is also increasing at $x = a$ and $x = b$.
- If $f(x)$ is increasing function on (a, b), then tangent at every point on the curve $y = f(x)$ makes an acute angle θ with the positive direction of x-axis.

  $\therefore \tan\theta > 0 \Rightarrow \frac{dy}{dx} > 0$ or $f'(x) > 0$ for all $x \in (a, b)$.
- Let f be a differentiable real function defined on an open interval (a, b).

(a) If $f'(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is increasing on (a, b).

(b) If $f'(x) < 0$ for all $x \in (a, b)$, then $f(x)$ is decreasing on (a, b).

- Let f be a function defined on (a, b).

(a) If $f'(x) > 0$ for all $x \in (a, b)$ except for a finite number of points, where $f'(x) = 0$, then $f(x)$ is increasing on (a, b).

(b) If $f'(x) < 0$ for all $x \in (a, b)$ except for a finite number of points, where $f'(x) = 0$, then $f(x)$ is decreasing on (a, b).

- **Properties of Monotonically Increasing and Decreasing Functions**

I If $f(x)$ is strictly increasing function on an interval $[a, b]$, then f^{-1} exists and it is also a strictly increasing function.

II If $f(x)$ is strictly increasing function on an interval $[a, b]$, such that it is continuous, then f^{-1} is continuous on $[f(a), f(b)]$.

III If $f(x)$ is continuous on $[a, b]$ such that $f'(c) \geq 0$ ($f'(c) > 0$) for each $c \in (a, b)$, then $f(x)$ is monotonically (strictly) increasing function on $[a, b]$.

IV If $f(x)$ is continuous on $[a, b]$ such that $f'(c) \leq 0$ ($f'(c) < 0$) for each $c \in (a, b)$, then $f(x)$ is monotonically (strictly) decreasing function on $[a, b]$.

V If $f(x)$ and $g(x)$ are monotonically (or strictly) decreasing function on $[a, b]$, then $gof(x)$ is a monotonically (or strictly) increasing function on $[a, b]$.

VI If one of the functions $f(x)$ and $g(x)$ is strictly (or monotonically) increasing and other a strictly (monotonically) decreasing, then $gof(x)$ is strictly (monotonically) decreasing on $[a, b]$.

End

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CHAPTER

maxima and minima

- **Maximum :** Let $f(x)$ be a function with domain $D \subset R$. Then $f(x)$ is said to attain the maximum value at a point $a \in D$, if $f(x) \le f(a)$ for all $x \in D$.

  In such a case, a is called point of maxima and $f(a)$ is known as the maximum value or the greatest value or the absolute maximum value of $f(x)$.
- **Minimum :** Let $f(x)$ be a function with domain $D \subset R$. Then $f(x)$ is said to attain the minimum value at a point $a \in D$, if $f(x) \ge f(a)$ for all $x \in D$.

  In such a case, a is called point of minima and $f(a)$ is known as the minimum value or the least value or the absolute minimum value of $f(x)$.
- **Local Maximum :** A function $f(x)$ is said to attain a local maximum at $x = a$ if there exists a neighbourhood $(a - \delta, a + \delta)$ of a such that,

  $f(x) < f(a)$ for all $x \in (a - \delta, a + \delta), x \neq a$

  or, $f(x) - f(a) < 0$ for all $x \in (a - \delta, a + \delta), x \neq a$.

  In such a case $f(a)$ is called to attain a local maximum value of $f(x)$ at $x = a$.
- **Local Minimum :** $f(x) > f(a)$ for all $x \in (a - \delta, a + \delta), x \neq a$

  or $f(x) - f(a) > 0$ for all $x \in (a - \delta, a + \delta), x \neq a$.

  In such a case $f(a)$ is called the local minimum value of $f(x)$ at $x = a$.
- **Some Important Points**

(i) The points at which a function attains either the local maximum value or local minimum values are known as the extreme points or turning points and both local maximum and local minimum values are called the extreme values of $f(x)$. Thus, a function attains an extreme value at $x = a$ if $f(a)$ is either a local maximum value or a local minimum value. Consequently at an extreme point 'a', $f(x) - f(a)$ keeps the same sign for all values of x in a deleted nbd. of a.

(ii) A necessary condition for $f(a)$ to be an extreme value of a function $f(x)$ is that $f'(a) = 0$ in case it exists.

(iii) This condition is only a necessary condition for the point $x = a$ to be an extreme point. It is not sufficient *i.e.*, $f'(a) = 0$ does not necessarily imply that $x = a$ is an extreme point. There are functions for which the derivatives vanish at a point but do not have an extreme value. For example, the function $f(x) = x^3$, $f'(0) = 0$ but at $x = 0$ the function does not attain an extreme value.

(iv) Geometrically the above condition means that the tangent to the curve $y = f(x)$ at a point where the ordinate is maximum or minimum is parallel to the x-axis.

(v) All x, for which $f'(x) = 0$, do not give us the extreme values. The values of x for which $f'(x) = 0$ are called stationary values or critical values of x and the corresponding values of $f(x)$ are called stationary or turning values of $f(x)$.

- **First Derivative Test for Local Maxima and Minima**

(a) $x = a$ is a point of local maximum of $f(x)$ if

(i) $f'(a) = 0$ and

(ii) $f'(x)$ changes sign from positive to negative as x passes through a *i.e.*, $f'(x) > 0$ at every points in the left *nbd* $(a - \delta, a)$ of a and $f'(x) < 0$ at every point in the right nbd. $(a, a + \delta)$ of a.

(b) $x = a$ is a point of local minimum of $f(x)$ if

(i) $f'(a) = 0$ and

(ii) $f'(x)$ changes sign from negative to positive as x passes through a *i.e.*, $f'(x) < 0$ at every points in the left nbd. $(a - \delta, a)$ of a and $f'(x) > 0$ at every point in the right nbd. $(a, a + \delta)$ of a.

(c) If $f'(a) = 0$, but $f'(x)$ does not change sign, that is, $f'(a)$ has the same sign in the complete nbd. of a, then a is neither a point of local maximum nor a point of local minimum.

- **Algorithm for Determining Extreme Values of a Function by using First Derivative Test**

Step I : Put $y = f(x)$

Step II : Find $\frac{dy}{dx}$.

Step III : Put $\frac{dy}{dx} = 0$ and solve this equation for x. Let $c_1, c_2, c_3, ..., c_n$ be the roots of the equation. $c_1, c_2, c_3, ..., c_n$ are stationary values of x and these are the possible points where the function can attain a local maximum or a local minimum. So we test the function at each of these points.

Step IV : Consider $x = c_1$, if $\frac{dy}{dx}$ changes its sign from positive to negative as x increases through c_1, then the function attains a local maximum at $x = c_1$.

- If $\frac{dy}{dx}$ changes its sign from negative to positive as x increases through c_1, then the function attains a local minimum at $x = c_1$.

- If $\frac{dy}{dx}$ does not change sign as x increase through c_1, then $x = c_1$ is neither a points of local maximum nor a point of local minimum. In this case $x = c_1$ is a point inflexion.

- **Higher Order Derivative Test :** Let f be a differentiable function on an interval I and let c be an interior point of I such that

(i) $f'(c) = f''(c) = f'''(c) = ... = f^{n-1}(c) = 0$, and

(ii) $f^n(c)$ exists and is non-zero. Then,
if n is even and $f^n(c) < 0 \Rightarrow x = c$ is a point of local maximum
if n is even and $f^n(c) > 0 \Rightarrow x = c$ is a point of local minimum
if n is odd $\Rightarrow x = c$ is neither a point of local maximum nor a point of local minimum.

- **Algorithm for Determining Values of a Function by Using Second Derivative Test :**
From the Ist derivative test criteria we obtain the following rule for determining maximum and minimum of $f(x)$:
Step I : Find $f'(x)$
Step II : Put $f'(x) = 0$ and solve this equation for x. Let $c_1, c_2, c_3, \ldots, c_n$ be the roots of this equation $c_1, c_2, c_3 \ldots, c_n$ are stationary values of x and these are the possible points where the function can attain a local maximum or a local minimum value. So we test the function at each one of these points.
Step III : Find $f'(x)$. Consider $x = c_1$
if $f''(c_1) < 0$, then $x = c_1$ is point of local maximum.
if $f''(c_1) > 0$, then $x = c_1$ is point of local minimum.
if $f''(c_1) = 0$, we must find $f'''(x)$ and substitute in it c_1 for x.
- If $f'''(c_1) \neq 0$, then $x = c_1$ is neither a point of local maximum nor a point of local minimum and is called point of inflection.
- If $f'''(c_1) = 0$, we must find $f^{IV}(x)$ and substitute in it c_1 for x.
- If $f^{IV}(c_1) < 0$, then $x = c_1$ is a point of local maximum and if $f^{IV}(c_1) > 0$, then $x = c_1$ is a point of local minimum.
- If $f^{IV}(c_1) = 0$, we must find $f^{V}(x)$, and so on. Similarly the values of $c_2, c_3, \ldots$, may be tested.
- A point of inflection is a point at which a curve is changing concave upward to concave downward, or vice versa.
- $x = c$ is a point of inflection if $f''(c) = 0$ and $f'''(c) \neq 0$.
- **Properties of Maxima and Minima**

I If $f(x)$ is continuous function in its domain, then at least one maxima and one minima must lie between two equal values of x .

II Maxima and minima occur alternately, that is, between two maxima there is one minima and vice versa.

III If $f(x) \to \infty$ as $x \to a$ or b and $f'(x) = 0$ only for one value of x (say c) between a and b, then $f(c)$ is necessarily the minimum and the least value.

IV If $f(x) \to -\infty$ as $x \to a$ or b and $f'(x) = 0$ only for one value of x (say c) between a and b, then $f(c)$ is necessarily the maximum and the greatest value.

- **Maximum and Minimum Values in a Closed Interval :** Let $y = f(x)$ be a function defined on $[a, b]$. By a local maximum (or local minimum) value of a function at a point $c \in [a, b]$ we mean the greatest (or the least) value in the immediate neighbourhood of $x = c$. It does not mean the greatest or absolute maximum (or the least or absolute minimum) of $f(x)$ in the interval $[a, b]$. A function may have a number of local maxima or local minima in a given interval and even a local minimum may be greater than a relative maximum.

- **Algorithm for Finding The Maximum and The Minimum Values of a Function in a Closed Interval**
- Let $y = f(x)$ be function defined on $[a, b]$.

  Step I : Find $\frac{dy}{dx} = f'(x)$

  Step II : Put $f'(x) = 0$ and find values of x.

  Let $c_1, c_2, c_3, \ldots, c_n$ be the values of x.

  Step III : Take the maximum and minimum values out of the values $f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)$.
- The maximum and minimum values obtained in Step III are respectively the largest or absolute maximum and the smallest or absolute minimum values of the function.

End

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CHAPTER

indefinite integrals

- **Primitive or Anti-derivative :** A function $\phi(x)$ is called a primitive or an anti-derivative of a function $f(x)$ if $\phi'(x) = f(x)$.

Indefinite Integral

- Let $f(x)$ be a function. Then the collection of all its primitives is called the indefinite integral of $f(x)$ and is denoted by $\int f(x)\, dx$.
- $\frac{d}{dx}(\phi(x) + C) = f(x) \Leftrightarrow \int f(x)\, dx = \phi(x) + C$, where $\phi(x)$ is primitive of $f(x)$ and C is an arbitrary constant known as the **constant of integration.**
- Integration of a function $f(x)$ means finding a function $\phi(x)$ such that $\frac{d}{dx}(\phi(x)) = f(x)$.
- **Fundamental Integration Formulas**

(i) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, n \neq -1 \quad \Rightarrow \int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1.$

(ii) $\frac{d}{dx}(\log x) = \frac{1}{x} \quad \Rightarrow \int \frac{1}{x} dx = \log|x| + C$

(iii) $\frac{d}{dx}(e^x) = e^x \quad \Rightarrow \int e^x dx = e^x + C$

(iv) $\frac{d}{dx}\left(\frac{a^x}{\log_e a}\right) = a^x,\ a > 0, a \neq 1 \quad \Rightarrow \int a^x dx = \frac{a^x}{\log a} + C$

(v) $\frac{d}{dx}(-\cos x) = \sin x \quad \Rightarrow \int \sin x dx = -\cos x + C$

(vi) $\frac{d}{dx}(\sin x) = \cos x \quad \Rightarrow \int \cos x dx = \sin x + C$

(vii) $\frac{d}{dx}(\tan x) = \sec^2 x \quad \Rightarrow \int \sec^2 x dx = \tan x + C$

(viii) $\frac{d}{dx}(-\cot x) = \text{cosec}^2 x \quad \Rightarrow \int \text{cosec}^2 x dx = -\cot x + C$

(ix) $\frac{d}{dx}(\sec x) = \sec x \tan x \quad \Rightarrow \int \sec x \tan x dx = \sec x + C$

(x) $\frac{d}{dx}(-\text{cosec}\, x) = \text{cosec}\, x \cot x \quad \Rightarrow \int \text{cosec}\, x \cot x dx = -\text{cosec}\, x + C$

(xi) $\frac{d}{dx}(\log \sin x) = \cot x \quad \Rightarrow \int \cot x dx = \log|\sin x| + C$

(xii) $\frac{d}{dx}(-\log \cos x) = \tan x \quad \Rightarrow \int \tan x dx = -\log|\cos x| + C$

(xiii) $\frac{d}{dx}(\log(\sec x + \tan x)) = \sec x \quad \Rightarrow \int \sec x dx = \log|\sec x + \tan x| + C$

(xiv) $\frac{d}{dx}(\log(\operatorname{cosec} x - \cot x)) = \operatorname{cosec} x \quad \Rightarrow \int \operatorname{cosec} x dx = \log|\operatorname{cosec} x - \cot x| + C$

(xv) $\frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}} \quad \Rightarrow \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

(xvi) $\frac{d}{dx}\left(\cos^{-1}\frac{x}{a}\right) = \frac{-1}{\sqrt{a^2 - x^2}} \quad \Rightarrow \int -\frac{1}{\sqrt{a^2 - x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + C$

(xvii) $\frac{d}{dx}\left(\frac{1}{a}\tan^{-1}\frac{x}{a}\right) = \frac{1}{a^2 + x^2} \quad \Rightarrow \int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

(xviii) $\frac{d}{dx}\left(\frac{1}{a}\cot^{-1}\frac{x}{a}\right) = -\frac{1}{a^2 + x^2} \quad \Rightarrow \int -\frac{1}{a^2 + x^2} dx = \frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right) + C$

(xix) $\frac{d}{dx}\left(\frac{1}{a}\sec^{-1}\frac{x}{a}\right) = \frac{1}{x\sqrt{x^2 - a^2}} \quad \Rightarrow \int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$

(xx) $\frac{d}{dx}\left(\frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a}\right) = -\frac{1}{x\sqrt{x^2 - a^2}} \quad \Rightarrow \int -\frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a}\operatorname{cosec}^{-1}\left(\frac{x}{a}\right) + C$

Some Standard Results on Integration

(i) $\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$, *i.e.* the differentiation of an integral is the integrand itself or differentiation and integration are inverse operations.

(ii) $\int kf(x)dx = k\int f(x)\,dx$, where k is a constant *i.e.*, the integral of the product of a constant and a function = the constant × integral of the function.

(iii) $\int \{f(x) \pm g(x)\}dx = \int f(x)\,dx \pm \int g(x)dx$, *i.e.*, the integral of the sum or difference of a finite number of functions is equal to the sum or difference of the integrals of the various functions.

$$\int \{k_1 \cdot f_1(x) \pm k_2\, f_2\,(x) \pm k_3\, f_3\,(x) \pm ... \pm k_n\, f_n\,(x)\}\,dx = k_1 \cdot \int f_1\,(x)\,dx \pm k_2 \cdot \int f_2\,(x)\,dx \pm ... \pm k_n \int f_n\,(x)\,dx$$

i.e., the integration of the linear combination of a finite number of functions is equal to the linear combination of their integrals.

- If $\int f(x)\,dx = \phi\,(x)$, then $\int f(ax + b)\,dx = \frac{1}{a}\,\phi\,(ax + b)$.
- $\int (ax + b)^n\,dx = \frac{(ax + b)^{n+1}}{a\,(n+1)} + C,\ n \neq -1$.

- $\int \frac{1}{ax+b}dx = \frac{1}{a}\log|ax+b|+C.$
- $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b}+C.$
- $\int a^{bx+c}dx = \frac{1}{b}\cdot\frac{a^{bx+c}}{\log a}+C$, $a>0$ and $a\neq 1$.
- $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b)+C$.
- $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b)+C$.
- $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b)+C$.
- $\int \operatorname{cosec}^2(ax+b)dx = -\frac{1}{a}\cot(ax+b)+C$.
- $\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b)+C.$
- $\int \operatorname{cosec}(ax+b)\cot(ax+b)\,dx = \frac{-1}{a}\operatorname{cosec}(ax+b)+C.$
- $\int \tan(ax+b)dx = \frac{-1}{a}\log|\cos(ax+b)|+C.$
- $\int \cot(ax+b)\,dx = \frac{1}{a}\log|\sin(ax+b)|+C.$
- $\int \sec(ax+b)dx = \frac{1}{a}\log|\sec(ax+b)+\tan(ax+b)|+C.$
- $\int \operatorname{cosec}(ax+b)\,dx = \frac{1}{a}\log|\operatorname{cosec}(ax+b)-\cot(ax+b)|+C.$
- In rational algebraic functions if the degree of numerator is greater than or equal to the degree of denominator, then always divide the numerator by denominator and use the result.

$$\frac{\text{Numerator}}{\text{Denominator}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Denominator}}$$

- To evaluate integrals of the form $\int \sin mx \cos nx\, dx$, $\int \sin mx \sin nx\, dx$, $\int \cos mx \cos nx\, dx$ and $\int \cos mx \sin nx\, dx$, we use the following trigonometrical identities :
  $2\sin A\cos B = \sin(A+B)+\sin(A-B)$.
  $2\cos A\sin B = \sin(A+B)-\sin(A-B)$.
  $2\cos A\cos B = \cos(A+B)+\cos(A-B)$.
  $2\sin A\sin B = \cos(A-B)-\cos(A+B)$.
- $\int \frac{f'(x)}{f(x)}dx = \log\{f(x)\}+C.$
- $\int \tan x\,dx = \log|\sec x| + C.$
- $\int \cot x\,dx = \log|\sin x| + C.$

- $\int \sec x\, dx = \log|\sec x + \tan x| + C.$
- $\int \text{cosec}\, x\, dx = \log|\text{cosec}\, x - \cot x| + C.$
- $\int \text{cosec}\, x dx = \log\left|\tan \frac{x}{2}\right| + C.$
- $\int \sec x dx = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C.$
- $\int \{f(x)\}^n f'(x) dx = \frac{\{f(x)\}^{n+1}}{n+1} + C, n \neq -1.$

Some Special Integrals

(i) $\int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C.$

(ii) $\int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a} \log\left|\frac{x - a}{x + a}\right| + C.$

(iii) $\int \frac{1}{a^2 - x^2}\, dx = \frac{1}{2a} \log\left|\frac{a + x}{a - x}\right| + C.$

(iv) $\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + C.$

(v) $\int \frac{1}{\sqrt{a^2 + x^2}}\, dx = \log\left|x + \sqrt{a^2 + x^2}\right| + C.$

(vi) $\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \log\left|x + \sqrt{x^2 - a^2}\right| + C.$

Some Important Substitutions

- Following are some substitutions useful in evaluating integrals.

Expression	Substitution
$a^2 + x^2$	$x = a\tan\theta$ or $a\cot\theta$
$a^2 - x^2$	$x = a\sin\theta$ or $a\cos\theta$
$x^2 - a^2$	$x = a\sec\theta$ or $a\,\text{cosec}\,\theta$
$\sqrt{\frac{a - x}{a + x}}$ or $\sqrt{\frac{a + x}{a - x}}$	$x = a\cos 2\theta$
$\sqrt{\frac{x - \alpha}{\beta - x}}$ or $\sqrt{(x - \alpha)(x - \beta)}$	$x = \alpha\cos^2\theta + \beta\sin^2\theta$

Integrals of The Type $\int \frac{1}{ax^2 + bx + c} dx$

- To evaluate this type of integrals we express $ax^2 + bx + c$ as the sum or difference of two squares by using the following steps :

  Step I : Make the coefficient of x^2 unity by taking it common.
  Step II : Add and subtract the square of half of the coefficient of x and then integrate.

Integrals of The Type $\int \frac{1}{\sqrt{ax^2 + bx + c}} dx$

- To evaluate this type of integrals, we express $ax^2 + bx + c$ as the sum or difference of two squares by using the following steps :

  Step I : Make the coefficient of x^2 unity by taking it common.

  Step II : Add and subtract the square of half of the coefficient of x, then integrate.

Integrals of The Form $\int \frac{px + q}{ax^2 + bx + c} dx$

- To evaluate this type of integrals we express the numerator as follows:
  $px + q = \lambda$ (diff. of denominator) $+ \mu = \lambda (2ax + b) + \mu$

Integrals of the form $\int \frac{P(x)}{ax^2 + bx + c} dx$**, where** $P(x)$ **is a polynomial of degree greater than or equal to 2**

- $\int \frac{P(x)}{ax^2 + bx + c} dx = \int Q(x)\, dx + \int \frac{R(x)}{ax^2 + bx + c} dx.$

Integrals of the form $\int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx$

- To evaluate this type of integrals we express the numerator as follows:
  $Px + q = \lambda$ (diff. of denominator) $+ \mu = \lambda (2ax + b) + \mu$

Integrals of the form $\int \frac{1}{a\sin^2 x + b\cos^2 x} dx$, $\int \frac{1}{a + b\sin^2 x} dx$,

$\int \frac{1}{a + b\cos^2 x} dx$, $\int \frac{1}{(a\sin x + b\cos x)^2} dx$, $\int \frac{1}{a + b\sin^2 x + c\cos^2 x} dx$

- To evaluate this type of integrals we proceed as follows:

  Step I : Divide numerator and denominator both by $\cos^2 x$.

  Step II : Replace $\sec^2 x$, if any, in denominator by $1 + \tan^2 x$.

  Step III : Put $\tan x = t$ so that $\sec^2 x\, dx = dt$.

  After employing these three steps the integral will reduce to the form $\int \frac{1}{at^2 + bt + c} dt$ which can be evaluated by the method discussed earlier.

Integrals of The Form $\int \frac{1}{a\sin x + b\cos x} dx$, $\int \frac{1}{a + b\sin x} dx$, $\int \frac{1}{a + b\cos x} dx$,

$\int \frac{1}{a\sin x + b\cos x + c} dx$

- To evaluate this type of integrals we procced as follows:

  Step I : Put $\sin x = \frac{2\tan(x/2)}{1 + \tan^2(x/2)}$, $\cos x = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}$.

  Step II : Replace $1 + \tan^2 \frac{x}{2}$ in the numerator by $\sec^2 \frac{x}{2}$.

Step III : Put $\tan\frac{x}{2} = t$ so that $\frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$.

- After performing these three steps the integral reduces to the form $\int \frac{1}{at^2 + bt + c}dt$ which can be evaluated by the method discussed earlier.

Alternative Method to Evaluate Intergrals of The Form $\int \frac{1}{a\sin x + b\cos x}dx$

- To evaluate this type of integrals we substitute $a = r\cos\theta$, $b = r\sin\theta$ and so $r = \sqrt{a^2 + b^2}$, $\theta = \tan^{-1}\left(\frac{b}{a}\right)$.

Integrals of The Form $\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}dx$

- To evaluate this type of integrals we express the numerator as follows:
  Numerator = λ (diff. of denominator) + μ (denominator)

Integrals of The Form $\int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}dx$

- To evaluate this type of integrals we express the numerator as follows:
  Numerator = λ (denominator) + μ (diff. of denominator) + v.

Integration by Parts

- If u and v are two functions of x, then

  $$\int uv\,dx = u\left(\int v\,dx\right) - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

  i.e., the integral of the product of two functions = (First function) × (integral of second function) – integral of {(diff. of first function) × (integral of second function)}.

  Choose the first function as the function which comes first in the word **ILATE**, where

  I – stands for the inverse trignometrical functions ($\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc.)

  L – stands for the logarithmic functions

  A – stands for the algebraic functions

  T – stands for the trigonometrical functions

  E – stands for the exponential functions.

- $\int e^x\{f(x) + f'(x)\}dx = e^x f(x) + C.$
- $\int e^{kx}\{f(x) + f'(x)\}dx = e^{kx} f(x) + C.$
- $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx) + C.$
- $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx) + C.$

- $\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right) + C$
- $\int \sqrt{a^2 + x^2}\,dx = \frac{1}{2}x\sqrt{a^2 + x^2} + \frac{1}{2}a^2 \log|x + \sqrt{a^2 + x^2}|9 + C$
- $\int \sqrt{x^2 - a^2}\,dx = \frac{1}{2}x\sqrt{x^2 - a^2} - \frac{1}{2}a^2 \log|x + \sqrt{x^2 - a^2}| + C$
- $\int (px + q)\sqrt{ax^2 + bx + c}\,dx = \frac{p}{2a}\int (2ax + b)\sqrt{ax^2 + bx + c}\,dx + \left(\frac{q - pb}{2a}\right)\int \sqrt{ax^2 + bx + c}\,dx.$

Integration of Rational Algebraic Functions by using Partial Fractions

- **Partial Functions :** If $f(x)$ and $g(x)$ are two polynomials, then $\frac{f(x)}{g(x)}$ defines a rational algebraic function or a rational function of x.
- If degree of $f(x) <$ degree of $g(x)$, then $\frac{f(x)}{g(x)}$ is called a proper rational function.
- If degree of $f(x) \geq$ degree of $g(x)$, then $\frac{f(x)}{g(x)}$ is called improper rational function.
- If $\frac{f(x)}{g(x)}$ is an improper rational function, we divide $f(x)$ by $g(x)$ so that the rational function $\frac{f(x)}{g(x)}$ is expressed in the form $\phi(x) + \frac{\psi(x)}{g(x)}$ where $\phi(x)$ and $\psi(x)$ are polynomials such that the degree of $\psi(x)$ is less than that of $g(x)$. Thus, $\frac{f(x)}{g(x)}$ is expressible as the sum of a polynomial and a proper rational function.

  Any proper rational function $\frac{f(x)}{g(x)}$ can be expressed as the sum of rational functions, each having a simple factor of $g(x)$. Each such fraction is called a partial fraction and the process of obtaining them is called the resolution or decomposition of $\frac{f(x)}{g(x)}$ into partial fraction.
- When denominator is expressible as the product of non-repeating linear factors.

  Let $g(x) = (x - a_1)(x - a_2) \ldots\ldots (x - a_n)$.

  Then we assume that $\frac{f(x)}{g(x)} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \ldots + \frac{A_n}{x - a_n}$.

  where $A_1, A_2, \ldots\ldots, A_n$ are constants and can be determined by equating the numerator on RHS to the numerator on LHS and then substituting $x = a_1, a_2, \ldots\ldots, a_n$.
- When the denominator $g(x)$ is expressible as the product of the linear factor such that some of them are repeating.
- Let $g(x) = (x - a)^k (x - a_1)(x - a_2) \ldots\ldots (x - a_r)$.

Then we assume $\dfrac{f(x)}{g(x)}=\dfrac{A_1}{x-a}+\dfrac{A_2}{(x-a)^2}+\dfrac{A_3}{(x-a)^3}+....+\dfrac{A_k}{(x-a)^k}$

$$+\frac{B_1}{x-a}+\frac{B_2}{x-a_2}+....+\frac{B_r}{(x-a_r)}.$$

- When some of the factors of denominator $g(x)$ are quadratic but non-repeating. Corresponding to each quadratic factor ax^2+bx+c, we assume partial fraction of the type $\dfrac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be determined by comparing coefficients of similar powers of x in the numerator of both sides. It is advisable to assume partial fractions of the type $\dfrac{A(2ax+b)}{ax^2+bx+c}+\dfrac{B}{ax^2+bx+c}$.
- When some of the factors of the denominator $g(x)$ are quadratic and repeating. For every quadratic repeating factor of the type $(ax^2+bx+c)^k$, we assume $2k$ partial fractions

$$\left\{\frac{A_0(2ax+b)}{ax^2+bx+c}+\frac{A_1}{ax^2+bx+c}\right\}+\left\{\frac{A_1(2ax+b)}{(ax^2+bx+c)^2}+\frac{A_2}{(ax^2+bx+c)^2}\right\}$$
$$+\left\{\frac{A_{2k-1}(2ax+b)}{(ax^2+bx+c)^k}+\frac{A_{2k}}{(ax^2+bx+c)^k}\right\}.$$

Integrals of the forms $\int\dfrac{x^2+1}{x^4+\lambda x^2+1}dx$, $\int\dfrac{x_2-1}{x^4+\lambda x^2+1}dx$, $\int\dfrac{1}{x^4+\lambda x^2+1}dx$, where λ is constant.

- To evaluate this type of integrals, divide the numerator and denominator by x^2 and put $x+\dfrac{1}{x}=t$ or $x-\dfrac{1}{x}=t$, whichever on differentiation gives the numerator of the resulting integrand.

Integrals of the form $\int\dfrac{\phi(x)}{P\sqrt{Q}}dx$, where P and Q both are linear function of x

- To evaluate this type of integrals we put $Q=t^2$ i.e. to evaluate integrals of the form $\int\dfrac{1}{(ax+b)(\sqrt{cx+d})}dx$, put $cx+d=t^2$.

Integrals of the form $\int\dfrac{\phi(x)}{P\sqrt{Q}}dx$, where P is a quadratic expression and Q is a linear expression

- To evaluate this type of integrals we put $Q=t^2$ *i.e.* to evaluate integrals of the form $\int\dfrac{1}{(ax^2+bx+c)\sqrt{px+q}}dx$, put $px+q=t^2$.

Integrals of the form $\int \frac{\phi(x)}{P\sqrt{Q}}\,dx$, where P is a linear expression and Q is a quadratic expression

- To evaluate this type of integrals we put $p = 1/t$ *i.e.*, to evaluate integrals of the form $\int \frac{1}{(ax+b)\sqrt{px^2+qx+r}}\,dx$, put $ax + b = \frac{1}{t}$.

Integrals of the form $\int \frac{\phi(x)}{P\sqrt{Q}}\,dx$, where P and Q both are pure quadratic expression in x *i.e.* $P = ax^2 + b$ and $Q = cx^2 + d$

- To evaluate this type of integrals we put $x = \frac{1}{t}$ and then $c + dt^2 = u^2$ *i.e.*, to evaluate integrals of the form $\int \frac{1}{(ax^2+b)\sqrt{cx^2+d}}\,dx$, we put $x = \frac{1}{t}$ to obtain $\int \frac{-t\,dt}{(a+bt^2)\sqrt{c+dt^2}}$ and then $c + dt^2 = u^2$.

End

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CHAPTER

definite integrals

Definite Integral

- $\int_a^b f(x)dx = [\phi(x)]_a^b = (\phi(x) \text{ at } x = b) - (\phi(x) \text{ at } x = a)$
- $\int_a^b f(x)dx = \int_a^b f(t)\,dt$

  $\therefore$ Integration is independent of the change of variable.
- $\int_a^b f(x)dx = -\int_b^a f(x)\,dx$ *i.e.*, if the limits of a definite integral are interchanged then its value changes by minus sign only.
- $\int_a^b f(x)dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$ where $a < c < b$.

  $\therefore \int_a^b f(x)dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + ... + \int_{c_n}^{b} f(x)dx$

  where $a < c_1 < c_2 < c_3 < ...... < c_{n-1} < c_n < b$.
- $\int_0^a f(x)\,dx = \int_0^a f(a-x)dx$
- $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is an even function} \\ 0 & \text{, if } f(x) \text{ is an odd function} \end{cases}$
- $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0 & \text{, if } f(2a-x) = -f(x) \end{cases}$
- $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx.$
- If $f(x)$ is a periodic function with period T, then $\int_0^{nT} f(x)\,dx = n\int_0^T f(x)\,dx.$
- If $f(x)$ is a periodic function with period T and $a \in R^+$, then

  $\int_{nT}^{a+nT} f(x)dx = \int_0^a f(x)\,dx.$

Leibnitz's Rule for The Differentiation Under The Integral Sign

- If the function $\phi(x)$ and $\psi(x)$ are defined on $[a, b]$ and differentiable at a point $x \in (a, b)$ and $f(x, t)$ is continuous, then

$$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(x,t)\,dt\right]=\left[\int_{\phi(x)}^{\psi(x)} \frac{\partial}{\partial x} f(x,t)\,dt+\frac{d\psi(x)}{dx} f(x,\psi(x))-\frac{d}{dx}(\phi(x) f(x,\phi(x)))\right].$$

- If the functions $\phi(x)$ and $\psi(x)$ are defined on $[a, b]$ and differentiable at a point $x \in (a, b)$, and $f(t)$ is continuous on $[\phi(a), \phi(b)]$, then

$$\frac{d}{dx}\left(\int_{\phi(x)}^{\psi(x)} f(t)\,dt\right)=\frac{d}{dx}\{\psi(x)\} f(\psi(x))-\frac{d}{dx}\{\phi(x)\} f(\phi(x)).$$

- If $f(x) \geq 0$ on the interval $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$.
- If $f(x) \leq g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.
- If m and M are the smallest and greatest values of a function $f(x)$ defined on an interval $[a, b]$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.
- If $f(x)$ is a periodic function with period T, then $\int_a^{a+T} f(x)\,dx$ is independent of a.
- If $f(x)$ is defined on $[a, b]$, then $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$.
- If $f^2(x)$ and $g^2(x)$ are integrable on $[a, b]$, then

$$\left|\int_a^b f(x)g(x)\,dx\right| \leq \left(\int_a^b f^2(x)dx\right)^{1/2}\left(\int_a^b g^2(x)\,dx\right)^{1/2}.$$

- Let a function $f(x, \alpha)$ be continuous for $a \leq x \leq b$ and $c \leq \alpha \leq d$. Then for any $\alpha \in [c, d]$, if $I(\alpha)=\int_a^b f(x,\alpha)\,dx$, then $\frac{dI(\alpha)}{d\alpha}=\int_a^b \frac{\partial f(x,\alpha)}{\partial \alpha}dx$.
- If $f(t)$ is an odd function, then $\phi(x)=\int_0^x f(t)\,dt$ is an even function.
- If $f(t)$ is an even function, then $\phi(x)=\int_0^x f(t)\,dt$ is an odd function.
- If $f(t)$ is an even function, then for non-zero a, $\int_a^x f(t)\,dt$ is not necessarily an odd function. It will be an odd function if $\int_0^a f(t)\,dt = 0$.
- If $f(x)$ is continuous on $[a, \infty]$, then $\int_a^\infty f(x)\,dx$ is called an improper integeral and is defined as $\int_a^\infty f(x)\,dx=\lim_{b\to\infty}\int_a^b f(x)\,dx$.

- $\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_{a}^{b} f(x)\,dx$ and $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$.
- Geometrically, for $f(x) > 0$ the improper integral $\int_{a}^{\infty} f(x)\,dx$ gives area of the figure bounded by the curve $y = f(x)$, the x-axis and the straight line $x = a$.
- If $f(x)$ is a continuous function on $[a, b]$, then there exists a point $c \in (a, b)$ such that $\int_{a}^{b} f(x)\,dx = f(c)(b-a)$. The number $f(c) = \frac{1}{b-a}\int_{a}^{b} f(x)\,dx$ is called the mean value of the function $f(x)$ on $[a, b]$.

Integral Function

- Let $f(x)$ be a continuous function defined on $[a, b]$, then a function $\phi(x)$ defined by $\phi(x) = \int_{a}^{x} f(t)\,dt$, $x \in [a, b]$ is called the integral function of the function f.

Properties of Integral Function

- The integral function of an integrable function is continuous.
- If $\phi(x)$ is the integral function of continuous function, then $\phi(x)$ is derivable and $\phi'(x) = f(x)$ for all $x \in [a, b]$.

Summation of series using definite integral

- $\int_{0}^{1} f(x)dx = \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)$.
- **Algorithm**

  Step I : Obtain the given series.

  Step II : Express the series in the form $\lim_{n\to\infty}\left[\frac{1}{n}\Sigma f\left(\frac{r}{n}\right)\right]$

  Step III : Replace Σ by $\int$, $\frac{r}{n}$ by x and $\frac{1}{n}$ by dx.

  Step IV : Obtain lower and upper limits by computing $\lim_{n\to\infty}\left(\frac{r}{n}\right)$ for the least and greatest values of r respectively.

  Step V : Evaluate the integral obtained in previous step. The value so obtained is the required sum of the given series.

Gamma Function

- If n is a positive rational number, then the improper integral $\int_{0}^{\infty} e^{-x}x^{n-1}dx$ is defined as Gamma function and is denoted by Γn *i.e.* $\Gamma n = \int_{0}^{\infty} e^{-x}x^{n-1}dx$, where $x \in Q^{+}$.

- **Properties of Gamma Function**

(i) $\Gamma 1 = 1 \cdot \Gamma 0 = \infty$ and $\Gamma n+1 = n\Gamma n$.

(ii) If $n \in N$, then $\Gamma n+1 = n!$.

(iii) $\Gamma_{1/2} = \sqrt{\pi}$.

(iv) $\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \dfrac{\Gamma \frac{m+1}{2} \Gamma \frac{n+1}{2}}{2\Gamma \frac{m+n+2}{2}}$.

End

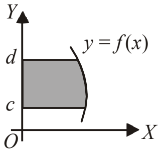
25
CHAPTER

areas of bounded regions

Applications of the Integrals

- Area bounded by the curve $y = f(x)$, the x-axis and between the abscissae at $x = a$ and $x = b$ is given by $\text{Area} = \int_a^b y\,dx = \int_a^b f(x)dx.$
- Area bounded by the curve $y = f(x)$, the y-axis and between ordinates at $y = c$ and $y = d$ is given by

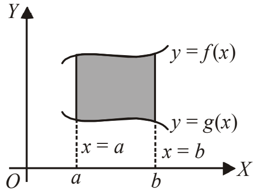
$$\text{Area} = \int_c^d x\,dy = \int_c^d g(y)dy$$



where $y = f(x) \Rightarrow x = g(y)$

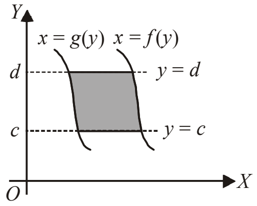
- Area bounded by the two curves $y = f(x)$ and $y = g(x)$, such that $0 \le g(x) \le f(x)$ for all $x \in [a, b]$ and between the abscissae at $x = a$, $x = b$ is given by

$$\text{Area} = \int_a^b \{f(x) - g(x)\}dx$$



- Area bounded by the two curves $x = f(y)$ and $x = g(y)$ such that $0 \le g(y) \le f(y)$ for all $y \in [c, d]$ and between the ordinates at $y = c$ and $y = d$ is given by

$$\text{Area} = \int_c^d \{f(y) - g(y)\}dy$$



- If $f(x) \ge g(x)$ in $[a, c]$ and $f(x) \le g(x)$ in $[c, b]$, where $a < c < b$, then area of the region bounded by the curves is given as

$$\text{Area} = \int_a^c \{f(x) - g(x)\}dx + \int_c^b \{g(x) - f(x)\}dx$$

End

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CHAPTER

differential equations

- **Differential Equations :** An equation containing an independent variable, dependent variable and differential coefficients of dependent variable with respect to independent variable is called a differential equation.
- **Order of a Differential Equation :** The order of a differential equation is the order of the highest order derivative appearing in the equation.
- The order of a differential equation is a positive integer.
- **Degree of a Differential Equation :** The degree of a differential equation is the degree of the highest order derivative, when differential coefficients are made free from radicals and fractions.
- **Linear and Non-Linear Differential Equations :** A differential equation is a linear differential equation if it is expressible in the form

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + ... + P_{n-1} \frac{dy}{dx} + P_n y = Q$$

where $P_0, P_1, P_2, ..., P_{n-1}, P_n$ and Q are either constants or functions of independent variable x.

- If a differential equation, when expressed in the form of a polynomial involves the derivatives and dependent variable in the first power and there are no product of these, and also the coefficient of the various terms are either constants or functions of the independent variable, then it is said to be linear differential equation. Otherwise, is a non linear differential equation.
- **Solution of a Differential Equation :** The solution of a differential equation is a relation between the variables involved which satisfies the differential equation. Such a relation and the derivatives obtained from there, when substituted in the differential equation, makes left hand, and right hand sides identically equal.
- **General Solution :** The solution which contains as many as arbitrary constants as the order of the differential equation is called the general solution of the differential equation.
- **Particular Solution :** Solution obtained by giving particular values to the arbitrary constants in the general solution of a differential equation is called a particular solution.
- Formulating a differential equation from a given equation representing a family of curves means finding a differential equation whose solution is the given equation. If an equation representing a family of curves, contains n arbitrary constants, then we differentiate the given equation n times to obtain n more equations. Using

all these equations, we eliminate the constants. The equation so obtained is the differential equation of order n for the family of given curves.

- **Differential Equations of First Order and First Degree :** A differential equation of first order and first degree involves x, y and $\frac{dy}{dx}$. So it can be put in any one of the following forms :

  $\frac{dy}{dx} = f(x, y)$ or $f\left(x, y\frac{dy}{dx}\right) = 0$ or $f(x, y)\, dx + g(x, y)dy = 0$

  where $f(x, y)$ and $g(x, y)$ are obviously the function of x and y.
- Differential equations of the type $\frac{dy}{dx} = f(x)$ has solution of the form $\int dy = \int f(x)dx + C$ or $y = \int f(x)dx + C$, where C is an arbitrary constant.
- Differential equations of the type $\frac{dy}{dx} = f(y)$ has solution of the form $\int dx = \int \frac{1}{f(y)}dy + C$ or $x = \int \frac{1}{f(y)}dy + C$.

Variable Separable Form

- If the differential equation can be put in the form $f(x)\, dx = g(y)\, dy$ we say that the variables are separable and such equation can be solved by integrating on both sides, *i.e.* $\int f(x)dx = \int g(y)dy + C$.

Equation Reducible to Variable Separable Form

- Differential equations of the form $\frac{dy}{dx} = f(ax + by + c)$ can be reduced to variable separable form by the substitution $ax + by + c = v$

Homogeneous Form

- A function $f(x, y)$ is called a homogeneous function of degree n if $f(\lambda x, \lambda y) = \lambda^n f(x, y)$.

  A homogeneous function $f(x, y)$ of degree n can always be written as

  $f(x, y) = x^n f\left(\frac{y}{x}\right)$ or $f(x, y) = y^n f\left(\frac{x}{y}\right)$.
- If a first-order first degree differential equation is expressible in the form $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$, where $f(x, y)$ and $g(x, y)$ are homogeneous functions of the same degree, then it is called a homogeneous differential equation.
- Such type equations can be reduced to variable separable form by the substitution $y = vx$.
- **Algorithm for Solving Homogeneous Differential Equations**

  Step I : Put the differential equation in the form $\frac{dy}{dx} = \frac{\phi(x, y)}{\psi(x, y)}$.

  Step II : Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in the equation in the Step I and cancel out x from the right hand side.

The equation reduces to the form $v + x\frac{dv}{dx} = F(v)$.

Step III : Shift v on RHS and separate the variable v and x.

Step IV : Integrate both sides to obtain the solution in terms of v and x.

Step V : Replace v by $\frac{y}{x}$ in the solution obtained in Step IV to obtain the solution in terms of x and y.

- **Equations Reducible to Homogeneous Form :** If a first order, first degree differential equation is of the form $\frac{dy}{dx} = \frac{a_1x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$ ....(i)

  Then it can be reduced to homogeneous form puting $x = X + h$, $y = Y + k$ in (i), where h and k are constants, which are to be determined.

Linear Differential Equations

- A differential equation is linear if the dependent variable (y) and its derivative appear only in first degree. The general form of a linear differential equation of first order is $\frac{dy}{dx} + Py = Q$, ...(i)

  where P and Q are functions of x (or constants) $ye^{\int Pdx} = \int Qe^{\int Pdx}dx + C$ is the required solution *i.e.*, $e^{\int Pdx}$ is called the integrating factor of (i).

  The solution of (i) is $y(\text{I.F.}) = \int Q \cdot (\text{I.F.})dx + c$

- **Algorithm for Solving a Linear Differential Equation**

  Step I : Write the differential equation in the form $\frac{dy}{dx} + Py = Q$ and obtain P and Q.

  Step II : Find integrating factor (I.F.) given by I.F. = $e^{\int Pdx}$

  Step III : Multiply both sides of equation in Step I by I.F.

  Step IV : Integrate both sides of the equation obtained in Step III w.r.t x to obtain y (I.F.) = $\int Q$ (I.F.) $dx + C$.

  This gives the required solution.

- **Algorithm for Solving a Linear Differential Equations of The Form $\frac{dx}{dy} + Rx = S$**

  Step I : Write the differential equation in the form $\frac{dx}{dy} + Rx = S$ and obtain R and S.

  Step II : Find I.F. by using I.F. = $e^{\int Rdy}$.

  Step III : Multiply both sides of equation in Step I by I.F.

  Step IV : Integrate both sides of the equation obtained in Step III w.r.t. y to obtain the solution given by $x(\text{I.F.}) = \int S(\text{I.F.})dx + C$, where C is the constant of integration.

Equations Reducible to Linear Form (Bernoulli's Differential Equation)

- The differential equation of type $\frac{dy}{dx} + Py = Qy^n$, where P and Q are constants or functions of x alone and n is a constant other than zero of unity, can be reduced to the linear form by dividing with y^n and then putting $y^{-n+1} = v$ gives $\frac{dv}{dx} + (1-n)Pv = (1-n)Q$ which is a linear differential equation.

  Note : If $n = 1$, then we find that the variables in Bernoulli's differential equation are separable and it can be easily integrated by the method discussed in variable separable form.

End

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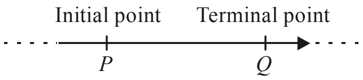
CHAPTER

vectors

- Those quantities which have only magnitude and which are not related to any fixed direction in space are called **scalar quantities** or **scalars.**
  Examples of scalars are mass, volume, density, work, temperature, etc.
- Those quantities which have both magnitude and direction, are called **vectors.** Displacement, velocity, acceleration, momentum, weight, force, etc. are examples of vector quantities.
- **Representation of Vectors :** A vector, denoted by $\overrightarrow{PQ}$, is determined by two points P, Q such that the magnitude of the vector is the length of the straight line PQ and its direction is that from P to Q. The point P is called the **initial point** of vector $\overrightarrow{PQ}$ and Q is called the **terminal point** or **tip.** Vectors are generally denoted by $\vec{a}, \vec{b}, \vec{c}$ etc.



- Every vector $\overrightarrow{PQ}$ has the following three characteristics:

  Length : The length of $\overrightarrow{PQ}$ will be denoted by $|\overrightarrow{PQ}|$ or PQ.

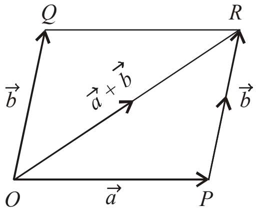
  Support : The line of unlimited length of which PQ is segment is called the support of the vector $\overrightarrow{PQ}$.

  Sense : The sense of $\overrightarrow{PQ}$ is from P to Q and that of $\overrightarrow{QP}$ is from Q to P. Thus, the sense of a directed line segment is from its initial point to the terminal point.
- **Equality of Vectors :** Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal, *i.e.*, $\vec{a} = \vec{b}$, if they have (i) the same length (ii) the same or parallel support, and (iii) the same sense.

Types of Vectors

- **Zero or Null Vector :** A vector whose initial and terminal points are coincident is called the zero or null vector.
- Vectors other than the null vector are called **proper vectors.**
- **Unit Vector :** A vector whose modulus is unity, is called a unit vector. The unit vector in the direction of a vector $\vec{a}$ is denoted by $\hat{a}$. Thus $|\hat{a}| = 1$.
- **Like and Unlike Vectors :** Vectors are said to be like when they have the same sense of direction and unlike when they have opposite directions.
- **Collinear or Parallel Vectors :** Vectors having the same or parallel support are called collinear vectors.

- **Co-Initial Vectors :** Vectors having the same initial point are called co-initial vectors.
- **Co-Planar Vectors :** A system of vectors is said to be co-planar, if their supports are parallel to the same plane.
- **Co-terminous Vectors :** Vectors having the same terminal point are called co-terminous vectors.
- **Negative of a Vector :** The vector which has the same magnitude as the vector but opposite direction, is called the negative of $\vec{a}$ and is denoted by $-\vec{a}$. Thus, if $\overrightarrow{PQ} = \vec{a}$, then $\overrightarrow{QP} = -\vec{a}$.
- **Reciprocal of a Vector :** A vector having the same direction as that of a given vector but magnitude equal to the reciprocal of the given vector is known as the reciprocal of $\vec{a}$ and is denoted by $\vec{a}^{-1}$. Thus, if $|\vec{a}| = a, |\vec{a}^{-1}| = \frac{1}{a}$.
- **Localized Vectors :** A vector which is drawn parallel to a given vector through a specified point in space is called a localized vector.
- **Free Vector :** If the value of a vector depends only on its length and direction and is independent of its position in the space, it is called a free vector.
- Two vectors are represented in magnitude and direction by the two sides of a triangle taken in the same order, then their sum is represented by the third side taken in the reverse order. This is called the triangle law of addition of vectors.



$|\vec{a} + \vec{b}| \neq |\vec{a}| + |\vec{b}|$.

Properties of Addition of Vectors

(i) Vector addition is commutative *i.e.*, $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ for any two vectors $\vec{a}$ and $\vec{b}$.

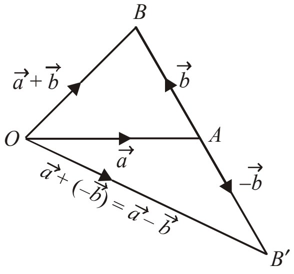
(ii) Vector addition is associative *i.e.*, $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ for any three vectors $\vec{a}, \vec{b}, \vec{c}$.

(iii) Existence of additive identity: For every vector $\vec{a}$, we have, $\vec{a} + \vec{0} = \vec{a} = \vec{0} + \vec{a}$, where $\vec{0}$ is the null vector.

(iv) Existence of additive inverse : For every vector $\vec{a}$, there corresponds a vector $-\vec{a}$ such that $\vec{a} + (-\vec{a}) = \vec{0} = (-a) + \vec{a}$.

- **Subtraction of Vectors :** To subtract $\vec{b}$ from $\vec{a}$, reverse the direction of $\vec{b}$ and add to $\vec{a}$.

- **Geometrical Representation of Addition and Subtraction :**



- **Multiplication of a Vector by a Scalar :** Let m be a scalar and $\vec{a}$ be a vector, then $m\vec{a}$ is defined as a vector having the same support as that of $\vec{a}$ such that its magnitude is $|m|$ times the magnitude of $\vec{a}$ and its direction is same as or opposite to the direction of $\vec{a}$ according as m is positive or negative.

Properties of Multiplication of Vectors by a Scalar

- For vectors $\vec{a}, \vec{b}$ and scalars m, n, we have

(i) $m(-\vec{a}) = (-m)\vec{a} = -(m\vec{a})$

(ii) $(-m)(-\vec{a}) = m\vec{a}$

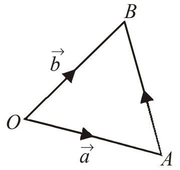
(iii) $m(n\vec{a}) = (mn)\vec{a} = n(m\vec{a})$

(iv) $(m+n)\vec{a} = m\vec{a} + n\vec{a}$

(v) $m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$.

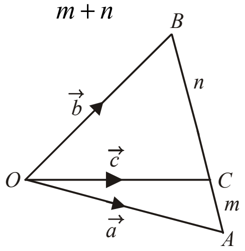
- **Position Vector :** If a point O is fixed as the origin in space (or plane) and P is any point, then $\overrightarrow{OP}$ is called the position vector of P with respect to O.

- $\overrightarrow{AB}$ = (Position vector of head) – (Position vector of tail) $= \vec{b} - \vec{a}$.



Section Formula

- **Internal Division :** Position vector of a point C dividing a vector $\overrightarrow{AB}$ internally in the ratio of $m : n$ is $\overrightarrow{OC} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$.



- If C is the mid-point of $\overrightarrow{AB}$, then $\overrightarrow{OC}$ divides $\overrightarrow{AB}$ in the ratio 1 : 1. Therefore, position vector of C is

$$\frac{1\cdot\vec{a} + 1\cdot\vec{b}}{1+1} = \frac{\vec{a} + \vec{b}}{2}.$$

- The position vector of the mid-point of $\overrightarrow{AB}$ is $\frac{1}{2}(\vec{a} + \vec{b})$.
- Position vector of any point C on $\overrightarrow{AB}$ can always be taken as $\vec{c} = \lambda\vec{a} + \mu\vec{b}$, where $\lambda + \mu = 1$.
- $n\cdot\overrightarrow{OA} + m\cdot\overrightarrow{OB} = (n + m)\,\overrightarrow{OC}$, where C is a point on $\overrightarrow{AB}$ dividing it in the ratio $m : n$.
- **External Division :** Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$ respectively and let C be a point dividing $\overrightarrow{AB}$ externally in the ratio $m : n$. Then the position vector of C is given by

$$\overrightarrow{OC} = \frac{m\vec{b} - n\vec{a}}{m - n}.$$

Linear Combination of Vectors

- A vector $\vec{r}$ is said to be a linear combination of vectors $\vec{a}, \vec{b}, \vec{c}, \ldots$.. etc. if there exist scalars x, y, z etc., such that $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + \ldots\ldots$

Collinear and Non-Collinear Vectors

- If $\vec{a}, \vec{b}$ are collinear vectors, then $\vec{a} = \lambda\vec{b}$ or $\vec{b} = \lambda\vec{a}$ for some scalar λ.
- If $\vec{a}, \vec{b}$ are any two non-zero non-collinear vectors and x, y are scalars such that $x\vec{a} + y\vec{b} = \vec{0}$, then $x = y = 0$.
- A, B, C are collinear $\Leftrightarrow$ $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are collinear $\Leftrightarrow$ $\overrightarrow{AB} = \lambda\overrightarrow{BC}$ for some scalar λ.
- **Components of a Vector :** If a point P in a plane has co-ordinates (x, y), then

  (i) $\overrightarrow{OP} = x\hat{i} + y\hat{j}$

  (ii) $|\overrightarrow{OP}| = \sqrt{x^2 + y^2}$

  (iii) The components of $\overrightarrow{OP}$ along x-axis is a vector $x\hat{i}$, whose magnitude is $|x|$ and whose direction is along OX or OX' according as x is positive or negative.
- **Components of a Vector in Terms of Co-ordinates of A and B :** If $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two points in XOY plane, then component of $\overrightarrow{AB}$ along x-axis $= (x_2 - x_1)\hat{i}$. Component of $\overrightarrow{AB}$ along y-axis $= (y_2 - y_1)\hat{j}$ and

  $|\overrightarrow{AB}| = AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Addition, Subtraction, Multiplication of a Vector by a Scalar and Equality in Terms of Components

- For any two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j}$, we define

  (i) $\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j}$

(ii) $\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j}$

(iii) $m\vec{a} = (ma_1)\hat{i} + (ma_2)\hat{j}$, where m is a scalar

(iv) $\vec{a} = \vec{b} \Leftrightarrow a_1 = b_1$ and $a_2 = b_2$.

- **Position Vector of a Point in Space :** If a point P in space has co-ordinates (x, y, z), then its position vector $\vec{r}$ is $x\hat{i} + y\hat{j} + z\hat{k}$ and $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$. The vectors $x\hat{i}$, $y\hat{j}$ and $z\hat{k}$ are known as the component vectors of $\vec{r}$ along x, y and z axes respectively.
- **Addition, Subtraction and Multiplication of a Vector by a Scalar and Equality in Terms of Components**

  For any vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ we define

  (i) $\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$

  (ii) $\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$

  (iii) $m\vec{a} = (ma_1)\hat{i} + (ma_2)\hat{j} + (ma_3)\hat{k}$, where m is a scalar

  (iv) $\vec{a} = \vec{b} \Leftrightarrow a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$.
- Three points with position vectors $\vec{a}, \vec{b}, \vec{c}$ are collinear iff there exist scalars x, y, z not all zero such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$, where $x + y + z = 0$.
- Let $\vec{a}$ and $\vec{b}$ be two given non-zero and non-collinear vectors. Then any vector $\vec{r}$ co-planar with $\vec{a}$ and $\vec{b}$ can be uniquely expressed as $\vec{r} = x\vec{a} + y\vec{b}$, for some scalars x and y.
- Three vectors are co-planar if one of them is expressible as a linear combination of the other two.
- Four points with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are co-planar iff there exist scalars x, y, z, u not all zero such that $x\vec{a} + y\vec{b} + z\vec{c} + u\vec{d} = \vec{0}$, where $x + y + z + u = 0$.
- If $\vec{a}, \vec{b}, \vec{c}$ are three given non-coplanar vectors, then every vector $\vec{r}$ in space can be uniquely expressed as $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$ for some scalars x, y and z.
- Between any four non-coplanar vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ in space there exists a unique relation connecting them which may be expressed as, $x\vec{a} + y\vec{b} + z\vec{c} + u\vec{d} = \vec{0}$.
- **Linearly Independent Vectors :** A set of non-zero vectors $\vec{a}_1, \vec{a}_2, \ldots\ldots, \vec{a}_n$ is said to be linearly independent, if $x_1\vec{a}_1 + x_2\vec{a}_2 + \ldots. + x_n\vec{a}_n = 0 \Rightarrow x_1 = x_2 = \ldots. = x_n = 0$.
- **Linearly Dependent Vectors :** A set of vectors $\vec{a}_1, \vec{a}_2, \ldots\ldots, \vec{a}_n$ is said to be linearly dependent if there exist scalars $x_1, x_2, \ldots\ldots, x_n$ not all zero such that $x_1\vec{a}_1 + x_2\vec{a}_2 + \ldots\ldots + x_n\vec{a}_n = \vec{0}$.
- **Angle Between Two Vectors :** θ is the measure of the angle between two vectors, then $0 \le \theta \le \pi$. For $\theta = \pi/2$, the vectors are said to be perpendicular or orthogonal and for $\theta = 0$ or π, the vectors are said to be parallel.
- **Scalar or Dot Product :** Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors inclined at an angle θ.

Then, $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$.

- If $\vec{a}$ or $\vec{b}$ or both is a zero vector, then θ is not defined as $\vec{0}$ has no direction. In this case their dot product $\vec{a}\cdot\vec{b}$ is defined as the scalar zero.
- **Geometrically,** the scalar product of two vectors is the product of modulus of either vector and the projection of the other in its direction.
- Projection of $\vec{b}$ on $\vec{a} = \hat{a}\cdot\vec{b}$.
- Projection of $\vec{a}$ on $\vec{b} = \vec{a}\cdot\hat{b}$.

Properties of Scalar Product

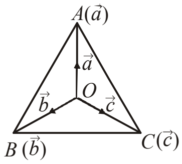
- The scalar product of two vectors is commutative *i.e.*, $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$.
- The scalar product of vectors is distributive over vector addition *i.e.*
  (i) $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ (Left distributivity)
  (ii) $(\vec{b}+\vec{c})\cdot\vec{a} = \vec{b}\cdot\vec{a} + \vec{c}\cdot\vec{a}$ (Right distributivity).
- Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors. Then $\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \vec{a}\perp\vec{b}$.
- Since $\hat{i}, \hat{j}, \hat{k}$ are mutually perpendicular unit vectors along the co-ordinate axes, therefore $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{i} = 0$; $\hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{j} = 0$; $\hat{k}\cdot\hat{i} = \hat{i}\cdot\hat{k} = 0$.
- For any vector $\vec{a}$, $\vec{a}\cdot\vec{a} = |\vec{a}|^2$.
- Since $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along the co-ordinate axes, therefore $\hat{i}\cdot\hat{i} = |\hat{i}|^2 = 1$, $\hat{j}\cdot\hat{j} = |\hat{j}|^2 = 1$ and $\hat{k}\cdot\hat{k} = |\hat{k}|^2 = 1$.
- If m is a scalar and $\vec{a}, \vec{b}$ be any two vectors, then $(m\vec{a})\cdot\vec{b} = m(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(m\vec{b})$.
- If m, n are scalars and $\vec{a}, \vec{b}$ be two vectors, then $m\vec{a}\cdot n\vec{b} = mn(\vec{a}\cdot\vec{b})$
  $= (mn\vec{a})\cdot\vec{b} = \vec{a}\cdot(mn\vec{b})$.
- For any two vectors $\vec{a}, \vec{b}$, we have
  (i) $\vec{a}\cdot(-\vec{b}) = -(\vec{a}\cdot\vec{b}) = (-\vec{a})\cdot\vec{b}$
  (ii) $(-\vec{a})\cdot(-\vec{b}) = \vec{a}\cdot\vec{b}$.
- For any two vectors $\vec{a}$ and $\vec{b}$, we have
  (i) $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b}$
  (ii) $|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}\cdot\vec{b}$
  (iii) $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$.
- **Scalar Product in Terms of Components :** Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. Then by using the distributivity of dot product over vector addition we get $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.
- **Angle Between Two Vectors :** Let $\vec{a}, \vec{b}$ be two vectors inclined at an angle θ.

  Then $\theta = \cos^{-1}\left(\dfrac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}\right)$.

- The components of $\vec{b}$ along and perpendicular to $\vec{a}$ are $\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$ and $\vec{b}-\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$ respectively.
- For any two vectors $\vec{a}$ and $\vec{b}$
  (i) $|\vec{a}+\vec{b}|^2+|\vec{a}-\vec{b}|^2=2(|\vec{a}|^2+|\vec{b}|^2)$.
  (ii) $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|\Leftrightarrow\vec{a}\perp\vec{b}$.
  (iii) $|\vec{a}+\vec{b}|=|\vec{a}|+|\vec{b}|\Leftrightarrow\vec{a}$ is parallel to $\vec{b}$.
  (iv) $|\vec{a}+\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2\Leftrightarrow\vec{a},\vec{b}$ are orthogonal.

Tetrahedron

- A tetradedron is a three-dimensional figure formed by four triangles.
- A tetrahedron in which all edges are equal, is called a **regular tetrahedron.**



- If two pairs of opposite edges of a tetrahedron are perpendicular, then the opposite edges of the third pair are also perpendicular to each other.
- In a tetrahedron, the sum of the squares of two opposite edges is the same for each pair.
- Any two opposite edges in a regular tetrahedron are perpendicular.
- Work done = (Force)·(Displacement).
  If a number of forces are acting on a particle, then the sum of the works done by the separate forces is equal to the work done by the resultant force.
- **Vector (Cross) Product :** Let $\vec{a},\vec{b}$ be two non-zero, non-parallel vectors. Then the vector product $\vec{a}\times\vec{b}$ in that order, is defined as a vector whose magnitude is $|\vec{a}||\vec{b}|\sin\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$ and whose direction is perpendicular to the plane of $\vec{a}$ and $\vec{b}$ in such a way that $\vec{a},\vec{b}$ and this direction constitute a right handed system.
- If one of $\vec{a}$ or $\vec{b}$ or both is $\vec{0}$ then, θ is not defined as $\vec{0}$ has no direction and so $\hat{\eta}$ is not defined. In this case we define $\vec{a}\times\vec{b}=\vec{0}$.
- If $\vec{a}$ and $\vec{b}$ are collinear *i.e.,* if $\theta = 0$ or π, then the direction of $\hat{\eta}$ is not well defined and so in this case also we define $\vec{a}\times\vec{b}=\vec{0}$.
- $\vec{a}\times\vec{b}$ is read as $\vec{a}$ cross $\vec{b}$. The resulting quantity is vector quantity so it is also known as the vector product.
- $\vec{a}\times\vec{b}$ is a vector whose magnitude is equal to the area of the parallelogram having $\vec{a}$ and $\vec{b}$ as its adjacent sides and whose direction $\hat{\eta}$ is $\perp$ to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a},\vec{b},\hat{\eta}$ form a right handed system.

Properties of Vector Product

- Vector product is not commutative *i.e.*, if $\vec{a}$ and $\vec{b}$ are any two vectors, then $\vec{a}\times\vec{b} \neq \vec{b}\times\vec{a}$, however, $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$.
- $(m\vec{a})\times\vec{b} = m(\vec{a}\times\vec{b}) = \vec{a}\times(m\vec{b})$, where m is a scalar.
- $m\vec{a}\times n\vec{b} = mn(\vec{a}\times\vec{b}) = m(\vec{a}\times n\vec{b}), n(m\vec{a}+\vec{b})$, where m and n are scalars.

  (i) $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$. (Left distributivity)

  (ii) $(\vec{b}+\vec{c})\times\vec{a} = \vec{b}\times\vec{a}+\vec{c}\times\vec{a}$. (Righ distributivity)
- $\vec{a}\times(\vec{b}-\vec{c}) = \vec{a}\times\vec{b}-\vec{a}\times\vec{c}$.
- The vector product of two non-zero vectors is zero vector iff they are parallel (collinear) *i.e.* $\vec{a}\times\vec{b}=\vec{0} \Leftrightarrow \vec{a}\,||\,\vec{b}$, where $\vec{a}$ and $\vec{b}$ are non-zero vectors.
- $\vec{a}\times\vec{a}=\vec{0}$ for every non-zero vector $\vec{a}$ *i.e.* $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=\vec{0}$.

Vector Product in Terms of Components

- Let $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ be two vectors.

  Then, $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$.
- $\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$ is a unit vector $\perp$ to the plane of $\vec{a}$ and $\vec{b}$.
- $-\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$ is also a unit vector $\perp$ to the plane of $\vec{a}$ and $\vec{b}$.
- A vector of magnitude 'λ' normal to the plane $\vec{a}$ and $\vec{b}$ is given by $\pm\dfrac{\lambda(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$.

Some Important Results

- The area of a parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$ is $|\vec{a}\times\vec{b}|$.
- The area of a triangle with adjacent sides $\vec{a}$ and $\vec{b}$ is $\dfrac{1}{2}|\vec{a}\times\vec{b}|$.
- The area of a triangle ABC is $\dfrac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$ or $\dfrac{1}{2}|\overrightarrow{BC}\times\overrightarrow{BA}|$ or $\dfrac{1}{2}|\overrightarrow{CB}\times\overrightarrow{CA}|$.
- The area of a parallelogram with diagonals $\vec{a}$ and $\vec{b}$ is $\dfrac{1}{2}|\vec{a}\times\vec{b}|$.
- The area of a plane quadrilateral $ABCD$ is $\dfrac{1}{2}|\overrightarrow{AC}\times\overrightarrow{BD}|$, where $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are its diagonals.

Lagrange's Identity

- If $\vec{a}, \vec{b}$ are any two vectors, then

  $|\vec{a}\times\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2$ or $|\vec{a}\times\vec{b}|^2+(\vec{a}\cdot\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2$.

- **Applications of Vector Product in Mechanics to Find the Moment of a Force About a Point :** Let a force $\vec{F}$ be applied at point P of a rigid body. Then the moment of $\vec{F}$ about a point O measures the tendency (amount) of $\vec{F}$ to turn the body about point O. If this tendency of rotation about O is in anti-clockwise direction the moment is positive, otherwise it is negative.
- **About a Line :** The moment of a force $\vec{F}$ acting at a point P about a line L is a scalar given by $(\vec{r} \times \vec{F})\cdot\hat{a}$, where $\hat{a}$ is a unit vector in the direction of the line, and $\overrightarrow{OP} = \vec{r}$, where O is any point on the line.
- The moment of a force $\vec{F}$ about a line is the resolve part (component) along this line of the moment of $\vec{F}$ about any point on the line.
- The moment of a force about a point is a vector while the moment about a straight line is a scalar quantity.
- The moment of a couple is a vector $\perp$ to the plane of the couple and its magnitude is the product of the magnitude of either force with the $\perp$ distance between the lines of the force.
- **Scalar Triple Product :** Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors. Then the scalar $(\vec{a} \times \vec{b})\cdot\vec{c}$ is called the scalar triple product of $\vec{a}, \vec{b}$ and $\vec{c}$ and is denoted by $[\vec{a}\ \vec{b}\ \vec{c}]$. Thus $[\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a} \times \vec{b})\cdot\vec{c}$.
- The scalar triple product $[\vec{a}\ \vec{b}\ \vec{c}]$ represents the volume of the parallelopiped whose coterminous edges $\vec{a}, \vec{b}, \vec{c}$ form a right handed system of vectors.
- If $\vec{a}, \vec{b}, \vec{c}$ are cyclically permuted the value of scalar triple product remains same. or $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$.
- The change of cyclic order of vectors in scalar triple product changes the sign of the scalar triple product but not the magnitude. *i.e.*, $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{b}\ \vec{a}\ \vec{c}] = -[\vec{c}\ \vec{b}\ \vec{a}] = -[\vec{a}\ \vec{c}\ \vec{b}]$.
- In scalar triple product, the position of dot and cross can be interchanged provided that the cyclic order of the vectors remains same *i.e.*, $(\vec{a} \times \vec{b})\cdot\vec{c} = \vec{a}\cdot(\vec{b} \times \vec{c})$.
- The scalar triple product of three vectors is zero if any two of them are equal.
- For any three vectors $\vec{a}, \vec{b}, \vec{c}$ and scalar λ, $[\lambda\vec{a}\ \vec{b}\ \vec{c}] = \lambda[\vec{a}\ \vec{b}\ \vec{c}]$.
- The scalar triple product of three vectors is zero if any two of them are parallel or collinear.
- If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are four vectors, then $[\vec{a} + \vec{b}\ \vec{c}\ \vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]$.
- The necessary and sufficient condition for three non-zero and non-collinear vectors $\vec{a}, \vec{b}, \vec{c}$ to be co-planar is that $[\vec{a}\ \vec{b}\ \vec{c}] = 0$. *i.e.* $\vec{a}, \vec{b}, \vec{c}$ are co-planar $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$.
- Four points with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ will be co-planar if $[\vec{d}\ \vec{b}\ \vec{c}] + [\vec{d}\ \vec{c}\ \vec{a}] + [\vec{d}\ \vec{a}\ \vec{b}] = [\vec{a}\ \vec{b}\ \vec{c}]$.

- **Volume of a Tetrahedron :** The volume of a tetrahedron, whose three coterminous edges in the right-handed system are $\vec{a}, \vec{b}, \vec{c}$ is $\frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$.
- If A, B, C, D are the vertices of a tetrahedron $ABCD$, then its volume is numerically equal to $\frac{1}{6}[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}]$ or $\frac{1}{6}[\overrightarrow{BC}\ \overrightarrow{BD}\ \overrightarrow{BA}]$ or $\frac{1}{6}[\overrightarrow{CD}\ \overrightarrow{CB}\ \overrightarrow{CA}]$.
- **Vector Triple Product :** Let $\vec{a}, \vec{b}, \vec{c}$ be any three vectors, then the vectors $\vec{a}\times(\vec{b}\times\vec{c})$ and $(\vec{a}\times\vec{b})\times\vec{c}$ are called vector triple products of $\vec{a}, \vec{b}, \vec{c}$.
- For any three vectors $\vec{a}, \vec{b}, \vec{c}$ we have $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$.
- The vector triple product $\vec{a}\times(\vec{b}\times\vec{c})$ is a linear combination of those two vectors which are within brackets.
- The vector $\vec{r} = \vec{a}\times(\vec{b}\times\vec{c})$ is perpendicular to $\vec{a}$ and lies in the plane of $\vec{b}$ and $\vec{c}$.
- The formula $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$ is true only when the vector outside the bracket is on the left-most side. If it is not, then we first shift on left by using the properties of cross-product and then apply the same formula.
- **Reciprocal System of Vectors :** Let $\vec{a}, \vec{b}, \vec{c}$ be three non-coplanar vectors, and let $\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}, \vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}, \vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

  $\vec{a}, \vec{b}, \vec{c}$ are said to form a reciprocal system of vectors for the vectors $\vec{a}, \vec{b}, \vec{c}$.

Properties of Reciprocal System of Vectors

- If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ form a reciprocal system of vectors, then

(i) $\vec{a}\cdot\vec{a}' = \vec{b}\cdot\vec{b}' = \vec{c}\cdot\vec{c}' = 1$.

(ii) $\vec{a}\cdot\vec{b}' = \vec{a}\cdot\vec{c}' = 0;\ \vec{b}\cdot\vec{c}' = \vec{b}\cdot\vec{a}' = 0;\ \vec{c}\cdot\vec{a}' = \vec{c}\cdot\vec{b}' = 0$

(iii) $[\vec{a}'\ \vec{b}'\ \vec{c}'] = \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}$.

(iv) $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar iff so are $\vec{a}', \vec{b}', \vec{c}'$.

Geometrical Applications of Vectors

- **Bisectors of an Angle :** If $\hat{a}$ and $\hat{b}$ are unit vectors along the sides of an angle, then $\hat{a}+\hat{b}$ and $\hat{a}-\hat{b}$ are the vectors along the internal and external bisectors of the angle, respectively.
- **Vector Equation of a Line :** The vector equation of a line passing through a point having position vector $\vec{a}$ and parallel to vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$, where λ is a scalar.
- **Vector equation of a plane.** The vector equation of a plane passing through a point having position vector $\vec{a}$ and containing vectors $\vec{b}$ and $\vec{c}$ is $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$, where $\lambda, \mu \in R$.

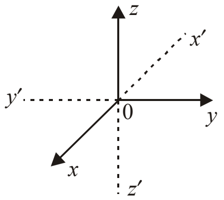
- **Bisector of The Angles Between Two Lines :** The bisectors of the angle between the lines $\vec{r} = \lambda\vec{a}$ and $\vec{r} = \mu\vec{b}$ are given by $\vec{r} = \lambda\left(\frac{\vec{a}}{|\vec{a}|} \pm \frac{\vec{b}}{|\vec{b}|}\right)$.
- The bisectors of the angle between the lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{a} + \mu\vec{c}$ are given by $\vec{r} = \vec{a} + \lambda\left(\frac{\vec{b}}{|b|} \pm \frac{\vec{c}}{|c|}\right)$.

End

28
CHAPTER

three dimensional geometry

- The co-ordinates of the point P are the perpendicular distances from P on the three mutually perpendicular rectangular co-ordinate planes YOZ, ZOX and XOY respectively.



The co-ordinates of a point are the distances from the origin of the feet of the perpendiculars from the point on the respective co-ordinate axes.

- The signs of co-ordinates of points in various octants:

Octant Coor-dinate	$OXYZ$	$OX'YZ$	$OXY'Z$	$OX'Y'Z$	$OXYZ'$	$OX'YZ'$	$OXY'Z'$	$OX'Y'Z'$
x	+	–	+	–	+	–	+	–
y	+	+	–	–	+	+	–	–
z	+	+	+	+	–	–	–	–

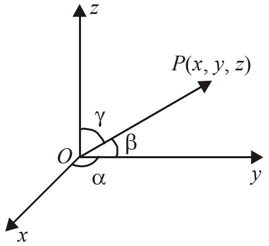
- If a point P lies in xy-plane, then by the definition of co-ordinates of a point, z-coordinate of P is zero. Therefore, the co-ordinates of a point on xy-plane are of the form $(x, y, 0)$ and we may take the equation of xy-plane as $z = 0$. Similarly, the co-ordinates of any point in yz and zx-planes are of the form $(0, y, z)$ and $(x, 0, z)$ respectively and their equations may be taken as $x = 0$ and $y = 0$ respectively.
- If a point lies on the x-axis, then its y and z-coordinates are both zero. Therefore, the co-ordinates of a point on x-axis are of the form $(x, 0, 0)$ and we may take the equation of x-axis as $y = 0$, $z = 0$. Similarly, the co-ordinates of a point on y and z-axes are of the form $(0, y, 0)$ and $(0, 0, z)$ respectively and their equations may be taken as $x = 0$, $z = 0$ and $x = 0$, $y = 0$ respectively.
- **Distance Formula :** The distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

- **Section Formula for Internal Division :** Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points. Let R be a point on the line segment joining P and Q such that its divides the join of P and Q internally in the ratio $m_1 : m_2$. Then the co-ordinates of R are $\left(\frac{m_1x_2 + m_2x_1}{m_1+m_2}, \frac{m_1y_2+m_2y_1}{m_1+m_2}, \frac{m_1z_2+m_2z_1}{m_1+m_2}\right)$.

If R is the mid-point of the segment joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, then $m_1 = m_2 = 1$ and the co-ordinates of R are given by $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$.

- **Section Formula for External Division :** Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points, and let R be a point on PQ produced dividing it externally in the ratio $m_1 : m_2 (m_1 \neq m_2)$. Then the co-ordinates of R are
$\left(\frac{m_1x_2 - m_2x_1}{m_1-m_2}, \frac{m_1y_2-m_2y_1}{m_1-m_2}, \frac{m_1z_2-m_2z_1}{m_1-m_2}\right)$
- **Direction Cosines :** If α, β, γ are the angles which a vector $\overrightarrow{OP}$ makes with the positive directions of the co-ordinate axes Ox, Oy, Oz respectively, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are known as the direction cosines of $\overrightarrow{OP}$ and are generally denoted by the letters l, m, n respectively *i.e.* $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.



- Let $P(x, y, z)$ be a point in space such that $\vec{r} = \overrightarrow{OP}$ has direction cosines l, m, n. Then
  (i) $l|\vec{r}|, m|\vec{r}|, n|\vec{r}|$ are projections of $\vec{r}$ on Ox, Oy, Oz respectively.
  (ii) $x = l|\vec{r}|, y = m|\vec{r}|, z = n|\vec{r}|$.
  (iii) $\vec{r} = |\vec{r}|(l\hat{i} + m\hat{j} + n\hat{k})\hat{r}$ and $\hat{r} = l\hat{i} + m\hat{j} + n\hat{k}$.
  (iv) $l^2 + m^2 + n^2 = 1$.
- **Direction Ratios :** Let l, m, n be direction cosines of a vector $\vec{r}$ and a, b, c, be three numbers such that $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$.
  Then a, b, c are known as direction ratios or direction numbers of vector $\vec{r}$.

Important Results on Direction Ratios and Directions Cosines

- If a, b, c are direction ratios of a vector, then its direction cosines are given by
$\pm\frac{a}{\sqrt{a^2+b^2+c^2}}, \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$
where the signs should be taken all positive or all negative.
- Direction cosines of $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ are $\frac{a}{|\vec{r}|}, \frac{b}{|\vec{r}|}, \frac{c}{|\vec{r}|}$.
- Direction ratios of $\overrightarrow{PQ}$ are $x_2 - x_1, y_2 - y_1, z_2 - z_1$ and its direction cosines are

$\frac{x_2 - x_1}{|\overrightarrow{PQ}|}, \frac{y_2 - y_1}{|\overrightarrow{PQ}|}, \frac{z_2 - z_1}{|\overrightarrow{PQ}|}$.

- If $\vec{a}$ and $\vec{b}$ are two parallel vectors, then $\vec{b} = \lambda\vec{a}$ for some scalar λ.
- If a vector $\vec{r}$ has direction ratios a, b, c then $\vec{r} = \frac{|\vec{r}|}{\sqrt{a^2+b^2+c^2}}(a\hat{i}+b\hat{j}+c\hat{k})$.
- Projection of $\vec{r}$ on the co-ordinate axes are $l|\vec{r}|, m|\vec{r}|, n|\vec{r}|$.
- The direction cosines of a line are defined as the direction cosines of any vector whose support is the given line.
- If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are two points on a line L, then its direction cosines are $\frac{x_2 - x_1}{AB}, \frac{y_2 - y_1}{AB}, \frac{z_2 - z_1}{AB}$ or $\frac{x_1 - x_2}{AB}, \frac{y_1 - y_2}{AB}, \frac{z_1 - z_2}{AB}$.
- The direction ratios of a line are the direction ratios of any vector whose support is the given line.
- If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are two points on a line, then its direction ratios are $x_2 - x_1, y_2 - y_1, z_2 - z_1$.
- The projection of segment joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on a line with D.C.'s l, m, n is $(x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n$.
- If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two concurrent lines, then the direction cosines of the lines bisecting the angles between them are proportional to $l_1 \pm l_2, m_1 \pm m_2, n_1 \pm n_2$.
- **Angle Between Two Vectors in Terms of Their Direction Cosines :** Let $\vec{a}$ and $\vec{b}$ be two vectors with direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 respectively. Let θ be the angle between $\vec{a}$ and $\vec{b}$, then $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$.
- **Condition for Perpendicularity :** $\vec{a}$ and $\vec{b}$ are perpendicular $\Leftrightarrow \hat{a} \perp \hat{b}$ $\Leftrightarrow \hat{a}\cdot\hat{b} = 0.\ l_1l_2 + m_1m_2 + n_1n_2 = 0$.
- **Condition for Parallelism :** $\vec{a}$ and $\vec{b}$ are parallel $\Leftrightarrow \hat{a}$ and $\hat{b}$ are parallel.
  $\Leftrightarrow \hat{a} = \lambda\hat{b}$, for some scalar λ.
  $\Leftrightarrow \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$.
- **Angle Between Two Vectors in Terms of Their Direction Ratios :** Let $\vec{a}$ and $\vec{b}$ be two vectors with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 respectively. Let θ be the angle between $\vec{a}$ and $\vec{b}$, then $\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$
- **Condition for Perpendicularity :** $\vec{a}$ and $\vec{b}$ are perpendicular $\Leftrightarrow \vec{A} \perp \vec{B}$ $\Leftrightarrow \vec{A}\cdot\vec{B} = 0 \Leftrightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$.
- **Condition for Parallelism :** $\vec{a}$ and $\vec{b}$ are parallel $\Leftrightarrow \vec{A}$ and $\vec{B}$ are parallel.
  $\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

- **Algorithm for Finding Angle Between Two Vectors in Terms of Their Direction Cosines or Directions Ratios**
  Step I : Obtain direction ratios or direction cosines of two vectors. Let the direction ratios of two vectors be a_1, b_1, c_1 and a_2, b_2, c_2 respectively.
  Step II : Write vectors parallel to the given vectors. Let $\vec{a}$ be vector parallel to the vector having direction ratios $a_1, b_1, c_1 = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\vec{b}$ be a vector parallel to the vector having direction ratios $a_2, b_2, c_2 = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$.
  Step III : Use the formula $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$.

Straight Line in Space

- **Vector Equation of a Line Passing Through a given Point and Parallel to a given Vector :** Vector equation of a straight line passing through a fixed point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is , $\vec{r} = \vec{a} + \lambda\vec{b}$, where λ is scalar.

  If $\vec{r}$ is the position vector of any point $P(x, y, z)$ on the line $\vec{r} = \vec{a} + \lambda\vec{b}$, then $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.
- **Cartesian Equation of a Line Passing Through a given Point and given Direction Ratios :** Cartesian equation of a straight line passing through a fixed point (x_1, y_1, z_1) and having direction ratios a, b, c is $\dfrac{x-x_1}{a} = \dfrac{y-y_1}{b} = \dfrac{z-z_1}{c}$.
- The parametric equation of the line $\dfrac{x-x_1}{a} = \dfrac{y-y_1}{b} = \dfrac{z-z_1}{c}$ are $x = x_1 + a\lambda$, $y = y_1 + b\lambda$, $z = z_1 + c\lambda$, where λ is the parameter.
- The co-ordinates of any point on the line are $\dfrac{x-x_1}{a} = \dfrac{y-y_1}{b} = \dfrac{z-z_1}{c}$ are $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$, and having direction cosines l, m, n is
  $\dfrac{x-x_1}{l} = \dfrac{y-y_1}{m} = \dfrac{z-z_1}{n}$.
- Since, the direction cosines of a line are also direction ratios, therefore equation of a line passing through (x_1, y_1, z_1) and having direction cosines l, m, n is
  $\dfrac{x-x_1}{l} = \dfrac{y-y_1}{m} = \dfrac{z-z_1}{n}$.
- Since x, y and z-axes pass through the origin and have direction cosines (1,0,0); (0,1,0) and (0,0,1) respectively. Therefore, their equations are

  x-axis : $\dfrac{x-0}{1} = \dfrac{y-0}{0} = \dfrac{z-0}{0}$ or $y = 0$ and $z = 0$

  y-axis : $\dfrac{x-0}{0} = \dfrac{y-0}{1} = \dfrac{z-0}{0}$ or $x = 0$ and $z = 0$

  z-axis : $\dfrac{x-0}{0} = \dfrac{y-0}{0} = \dfrac{z-0}{1}$ or $x = 0$ and $y = 0$.

- **Vector Equation of a Line Passing Through Two given Points :** The vector equation of a line passing through two points with position vector $\vec{a}$ and $\vec{b}$ is

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}).$$

- **Cartesian Equation of a Line Passing Through Two given Points :** The Cartesian equation of a line passing through two given points $(x_1, y_1\ z_1)$ and (x_2, y_2, z_2) is given by $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_2}{z_2 - z_1}$.

- **Cartesian to Vector :** The Cartesian equation of a line be $\dfrac{x-x_1}{a} = \dfrac{y-y_1}{b} = \dfrac{z-z_1}{c}$.

Angle Between Two Lines

- **Vector form :** Let the vector equations of the two lines be $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$. If θ is the angle between the given lines, then $\cos\theta = \dfrac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}$.

- **Condition of Perpendicularity :** If the lines are perpendicular, then $\vec{b}_1 \cdot \vec{b}_2 = 0$.

- **Condition of Parallelism :** If the lines are parallel, then $\vec{b}_1$ and $\vec{b}_2$ are parallel, therefore $\vec{b}_1 = \lambda\,\vec{b}_2$ for some scalar λ.

- **Cartesian Form :** $\cos\theta = \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$.

- If the lines are perpendicular, then $\vec{m}_1 \cdot \vec{m}_2 = 0 \Rightarrow a_1\,a_2 + b_1\,b_2 + c_1\,c_2 = 0$.

- If the lines are parallel, then $\vec{m}_1$ and $\vec{m}_2$ are parallel. Therefore

  $\vec{m}_1 = \lambda\,\vec{m}_2$, for some scalar $\lambda \Rightarrow \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$.

- Let $\vec{\alpha}'$ be the position vector of Q. Since L is the mid-point of PQ, therefore

  $\dfrac{\vec{\alpha} + \vec{\alpha}'}{2} = \vec{a} - \left(\dfrac{(\vec{a} - \vec{\alpha}) \cdot \vec{b}'}{|\vec{b}|^2}\right)\vec{b}$, $\vec{\alpha}' = 2\vec{a} - \left(\dfrac{(\vec{a} - \vec{\alpha}) \cdot \vec{b}}{|\vec{b}|^2}\right)\vec{b} - \alpha.$

  This gives the position vector of Q which is the image of P in the given line.

- **Skew Lines :** Two straight lines in space which are neither parallel nor intersecting are called skew lines.

- **Line of Shortest Distance :** If l_1 and l_2 are two skew-lines, then there is one and only one line perpendicular to each of lines l_1 and l_2 which is known as the line of shortest distance.

- **Shortest Distance :** The shortest distance between two lines l_1 and l_2 is the distance PQ between the points P and Q where the lines of shortest distance intersects the two given lines.

- **Shortest Distance Between Two Skew Lines (Vector Form) :** Let l_1 and l_2 be two lines whose equations are $l_1 : \vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $l_2 : \vec{r} = \vec{a}_2 + \mu\vec{b}_2$ respectively. Let $\overrightarrow{PQ}$ be the shortest distance vector between l_1 and l_2.

$$PQ = \left|\frac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right| = \frac{|[\vec{b}_1 \vec{b}_2 (\vec{a}_2 - \vec{a}_1)]|}{|\vec{b}_1 \times \vec{b}_2|}.$$

- **Condition for Two given Lines to Intersect :** If the lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ intersect, then the shortest distance between them is zero.

$$\left|\frac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right| = 0$$

$(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1) = 0 \Rightarrow [\vec{b}_1 \vec{b}_2 (\vec{a}_2 - \vec{a}_1)] = 0.$

- **Shortest Distance Between Two Skew Lines (Cartesian Form) :** Let the two skew lines be

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} \text{ and } \frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}.$$

The shortest distance between the lines is given by

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{(m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - l_1 n_2)^2 + (l_1 m_2 - l_2 m_1)^2}}.$$

- **Condition for Two given Lines to Intersect :** If the lines $\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$ and $\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$ intersect, then the shortest distance between them is zero.

$$\Rightarrow \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_1 & n_2 \end{vmatrix} = 0.$$

- The shortest distance between the parallel lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}$ is given by $d = \frac{|(\vec{a}_2 - \vec{a}_1) \times \vec{b})|}{|\vec{b}|}$.

Plane

- A plane is a surface such that if any two points are taken on it, the line segment joining them lies completely on the surface. In other words, every point on the line segment joining any two points lies on the plane.
- Every first degree equation in x, y and z represents a plane *i.e.*, $ax + by + cz + d = 0$ is the general equation of a plane.
- The general equation of a plane passing through a point (x_1, y_1, z_1) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$, where a, b and c are constants.

Intercept Form of a Plane

- The equation of a plane intercepting lengths a, b and c with x-axis, y-axis and z-axis respectively is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.
- The above equation is known as the intercept form of the plane, because the plane intercepts lengths a, b and c with x, y and z-axis respectively.
- **For x-intercept :** Put $y = 0$, $z = 0$ in the equation of the plane and obtain the value of x. The value of x is the intercept on x-axis.
- **For y-intercept :** Put $x = 0$, $z = 0$ in the equation of the plane and obtain the value of y. The value of y is the intercept on y-axis.
- **For z-intercept :** Put $x = 0$, $y = 0$ in the equations of the plane and obtain the value of z. The value of z is the intercept on z-axis.

Vector Equation of a Plane Passing Through a Given Point and Normal to a Given Vector

- The vector equation of a plane passing through a point having position vector $\vec{a}$ and normal to vector $\vec{n}$ is $(\vec{r}-a)\cdot\vec{n}=0$ or $\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$.
- Vector equation of a plane means a relation involving the position vector $\vec{r}$ of an arbitrary point on the plane.
- The above equation can also be written as $\vec{r}\cdot\vec{n}=d$, where $d=\vec{a}\cdot\vec{n}$.
  This is known as the scalar product form of a plane.
- The coefficient of x, y and z in the Cartesian equation of a plane are the direction ratios of normal to the plane.

Equation of a Plane in Normal Form

- **Vector Form :** The vector equation of a plane normal to unit vector $\hat{n}$ and at a distance d from the origin is $\vec{r}\cdot\hat{n}=d$.
- **Cartesian Form :** If l, m, n are direction cosines of the normal to a given plane which is at a distance p from the origin, then the equation of the plane is $lx + my + nz = p$.
- The equation $\vec{r}\cdot\vec{n}=d$ is in normal form if $\vec{n}$ is a unit vector and in such a case d denotes the distance of the plane from the origin. If $\vec{n}$ is not a unit vector, then to reduce the equation $\vec{r}\cdot\vec{n}=d$ to normal form we divide both sides by $|\vec{n}|$ to obtain $\vec{r}\cdot\frac{\vec{n}}{|\vec{n}|}=\frac{d}{|\vec{n}|}$ or $\vec{r}\cdot\hat{n}=\frac{d}{|\vec{n}|}$.
- The angle between two planes is defined as the angle between their normals.
- **Angles Between Two Planes in Vector Form :** The angle θ between the planes $\vec{r}\cdot\vec{n}_1=d_1$ and $\vec{r}\cdot\vec{n}_2=d_2$ is given by $\cos\theta=\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$.
- **Condition of Perpendicularity :** If the planes $\vec{r}\cdot\vec{n}_1=d_1$ and $\vec{r}\cdot\vec{n}_2=d_2$ are perpendicular, then $\vec{n}_1$ and $\vec{n}_2$ are perpendicular. Therefore $\vec{n}_1\cdot\vec{n}_2=0$.

- **Condition of Parallelism :** If the planes $\vec{r}\cdot\vec{n}_1 = d_1$ and $\vec{r}\cdot\vec{n}_2 = d_2$ are parallel, then $\vec{n}_1$ and $\vec{n}_2$ are parallel. Therefore, there exists a scalar λ such that $\vec{n}_1 = \lambda\vec{n}_2$.
- **Angle Between Two Planes in Cartesian Form :** The angle θ between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by
$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$
- The equation of the plane passing through a point having position vector $\vec{a}$ and parallel to $\vec{b}$ and $\vec{c}$ is $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$, where λ and μ are scalars.
- The equation of the plane passing through a point having position vector $\vec{a}$ and parallel to vectors $\vec{b}$ and $\vec{c}$ is $[\vec{r}\ \vec{b}\ \vec{c}] = [\vec{a}\ \vec{b}\ \vec{c}]$.

Equation of a Plane Parallel to a given Plane

- **Vector Form :** Since parallel planes have the common normal, therefore equation of a plane parallel to the plane $\vec{r}\cdot\vec{n} = d_1$ is $\vec{r}\cdot\vec{n} = d_2$, where d_2 is a constant determined by the given condition.
- **Cartesian Form :** Let $ax + by + cz + d = 0$ be a plane. Then direction ratios of its normal are a, b, c. Since, parallel planes have common normal, therefore the direction ratios of the normal to the parallel plane are also a, b, c. Thus, the equation of a plane parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz + k = 0$, where k is an arbitrary constant and is determined by the given condition.

Equation of a Plane Passing Through The Intersection of Two Planes

- **Vector Form :** The equation of a plane passing through the intersection of the planes $\vec{r}\cdot\vec{n} = d_1$ and $\vec{r}\cdot\vec{n} = d_2$ is $(\vec{r}\cdot\vec{n}_1 - d_1) + \lambda(\vec{r}\cdot\vec{n}_2 - d_2) = 0$.
- **Cartesian Form :** The equation of a plane passing through the intersection of $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is $(a_1x + b_1y + c_1z + d_1) + \lambda(a_2x + b_2y + c_2z + d_2) = 0$ where, λ is a constant.

Distance of a Point From a Plane

- **Vector Form :** The length of the perpendicular from a point having position vector $\vec{a}$ to the plane $\vec{r}\cdot\vec{n} = d$ is given by $p = \dfrac{|\vec{a}\cdot\vec{n} - d|}{|\vec{n}|}$.
- **Cartesian Form :** The length of the perpendicular from a point $P(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is given by $\dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$.

Equations of Planes Bisecting The Angles Between Two given Planes

- **Cartesian Form :** The equation of the planes bisecting the angles between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are
$$\frac{(a_1x + b_1y + c_1z + d_1)}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm\frac{(a_2x + b_2y + c_2z + d_2)}{\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$
- **Vector Form :** The equation of the planes bisecting the angles between the planes

$\vec{r}\cdot\vec{n}_1 = d_1$ and $\vec{r}\cdot\vec{n}_2 = d_2$ are $\dfrac{|\vec{r}\cdot\vec{n}_1 - d_1|}{|\vec{n}_1|} = \dfrac{|\vec{r}\cdot\vec{n}_2 - d_2|}{|\vec{n}_2|}$, $\vec{r}\cdot(\hat{n}_1 \pm \hat{n}_2) = \dfrac{d_1}{|\vec{n}_1|} \pm \dfrac{d_2}{|\vec{n}_2|}$.

Bisector of the Angle Containing the Origin

- If the given planes are $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, such that d_1 and d_2 are of the same sign, then the bisecting plane
  $$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$
  bisects the angle between the planes that contains the origin.

Angle Between a Line and a Plane

- **Vector Form :** If θ is the angle between a line $\vec{r} = (a + \lambda\vec{b})$ and the plane $\vec{r}\cdot\vec{n} = d$, then $\sin\theta = \dfrac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}$.
- **Condition of Perpendicularity :** $\vec{b}\times\vec{n} = 0$ or $\vec{b} = \lambda\vec{n}$, for some scalar λ.
- **Condition of Parallelism :** $\vec{b}$ and $\vec{n}$ are perpendicular. So, $\vec{b}\cdot\vec{n} = 0$.
- **Cartesian Form :** If θ is the angle between the line $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}$ and the plane $ax + by + cz + d = 0$, then $\sin\theta = \dfrac{al + bm + cn}{\sqrt{a^2 + b^2 + c^2}\sqrt{l^2 + m^2 + n^2}}$.
- **Condition of Perpendicularity :** $\vec{b} = l\hat{i} + m\hat{j} + n\hat{k}$ and $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$ are parallel, then $\dfrac{l}{a} = \dfrac{m}{b} = \dfrac{n}{c}$.
- **Condition of Parallelism :** $\vec{b} = l\hat{i} + m\hat{j} + n\hat{k}$ and $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$ are perpendicular. So, $\vec{b}\cdot\vec{n} = 0 \Rightarrow al + bm + cn = 0$.

Condition for a Line to Lie in a Plane

- **Vector form :** If the line $\vec{r} = \vec{a} + \lambda\vec{b}$ lies in the plane $\vec{r}\cdot\vec{n} = d$, then (i) $\vec{b}\cdot\vec{n} = 0$ and (ii) $\vec{a}\cdot\vec{n} = d$.
- **Cartesian Form :** If the line $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}$ lies in the plane $ax + by + cz + d = 0$ then (i) $ax_1 + by_1 + cz_1 + d = 0$ and (ii) $al + bm + cn = 0$.

Condition of co-planar of two Lines and Equation of the Plane containing them

- **Vector Form :** If the lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ are co-planar, then $[\vec{a}_1\ \vec{b}_1\ \vec{b}_2] = [\vec{a}_2\ \vec{b}_1\ \vec{b}_2]$ and the equation of the plane containing them is $[\vec{r}\ \vec{b}_1\ \vec{b}_2] = [\vec{a}_1\ \vec{b}_1\ \vec{b}_2]$ or $[\vec{r}\ \vec{b}_1\ \vec{b}_2] = [\vec{a}_2\ \vec{b}_1\ \vec{b}_2]$.
- **Cartesian Form :** If the line $\dfrac{x - x_1}{l_1} = \dfrac{y - y_1}{m_1} = \dfrac{z - z_1}{n_1}$ and $\dfrac{x - x_2}{l_2} = \dfrac{y - y_2}{m_2} = \dfrac{z - z_2}{n_2}$ are co-planar, then $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$.

Image of a Point in a Plane

- Let P and Q be two points and let π be a plane such that
  (i) line PQ is perpendicular to the plane π, and
  (ii) mid-point of PQ lies on the plane π.
  Then either of the point is the image of the other in the plane π.

Sphere

- A sphere is the locus of a point which moves in space in such a way that its distance from a fixed point always remains constant.
- The cartesian equation of a sphere with centre (a, b, c) and radius R is
  $(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$. ....(i)
- The equation is called the central form of a sphere. If the centre is at the origin, then equation (i) takes the form $x^2 + y^2 + z^2 = R^2$
  which is known as the standard form of the equation of the sphere.
- Equation (i) can be written as
  $x^2 + y^2 + z^2 - 2ax - 2by - 2cz + (a^2 + b^2 + c^2 - R^2) = 0$
  From this equation, we note the following characteristics of the equation of a sphere :
  (i) It is a second degree equation in x, y, z;
  (ii) The coefficients of x^2, y^2, z^2 are all equal;
  (iii) The terms containing the products xy, yz and zx are absent.
- The equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ represents a sphere with centre $(-u, -v, -w)$ *i.e.*
  (–(1/2) coefficient of x, –(1/2) coefficient of y, –(1/2) coefficient of z) and radius $= \sqrt{u^2 + v^2 + w^2 - d}$.
- The equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ represents a real sphere if $u^2 + v^2 + w^2 - d > 0$.
  If $u^2 + v^2 + w^2 - d = 0$, then it represents a point sphere. The sphere is imaginary if $u^2 + v^2 + w^2 - d < 0$.

Diameter Form of The Equation of Equation of Sphere Passing Through a Sphere of Four Non-Coplanar Points

- **Vector Form :** If the position vectors of the extremities of a diameter of a sphere are $\vec{a}$ and $\vec{b}$, then its equation $(\vec{r} - \vec{a})\cdot(\vec{r} - \vec{b}) = 0 \Rightarrow |\vec{r}|^2 - \vec{r}\cdot(\vec{a} - \vec{b}) + \vec{a}\cdot\vec{b} = 0$.
  i.e., if the position vectors of the extremities of a diameter of a sphere are $\vec{a}$ and $\vec{b}$, then its equation is $|\vec{r} - \vec{a}|^2 + |\vec{r} - \vec{b}|^2 = |\vec{a} - \vec{b}|^2$.
- **Cartesian Form :** If (x_1, y_1, z_1) and (x_2, y_2, z_2) are the co-ordinates of the extremities of a diameter of a sphere, then its equation is
  $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$.
- The plane $\vec{r}\cdot\vec{n} = d$ touches the sphere $|\vec{r} - \vec{a}| = R$, if $\dfrac{|\vec{a}\cdot\vec{n} - d|}{|\vec{n}|} = R$
  The plane $lx + my + nz = p$ touches the sphere
  $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, if
  $(ul + vm + wn + p)^2 = (l^2 + m^2 + n^2)(u^2 + v^2 + w^2 - d)$.

End

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CHAPTER

probability

- **Trial and Elementary Events :** Let a random experiment be repeated under identical conditions. Then the experiment is called a trial and the possible outcomes of the experiment are known as elementary events or cases.
- Elementary events are also known as indecomposable events.
- **Decomposable Events/Compound Events :** Events obtained by combining together two or more elementary events are known as the compound events or decomposable events.
- **Exhaustive Number of Cases :** The total number of possible outcomes of a random experiment in a trial is known as the exhaustive number of cases.
- The total number of elementary events of a random experiment is called the exhaustive number of cases.
- **Mutually Exclusive Events :** Events are said to be mutually exclusive or incompatible if the occurrence of anyone of them prevents the occurrence of all the others, *i.e.*, if no two or more of them can occur simultaneously in the same trial.
- **Equally Likely Events :** Events are equally likely if there is no reason for an event to occur in preference to any other events.
- The number of cases favourable to an events in a trial is the total number of elementary events such that the occurrence of any one of them ensures the happening of the event.
- **Independent Events :** Events are said to be independent if the happening (or non-happening)of one event is not affected by the happening (or non-happening) of others.
- **Classical Definition of Probability of An Event :** If there are n-elementary events associated with a random experiment and m of them are favourable to an event A, then the probability of happening of A is denoted by $P(A)$ and is defined as the ratio m/n.

  $P(A) = \frac{m}{n}$.

  $0 \le P(A) \le 1$. $\bar{A}$ denotes not happening of A
- $P(\bar{A}) = 1 - P(A)$
- $P(A) + P(\bar{A}) = 1$.
- If $P(A) = 1$, then A is called certain event and A is called an impossible event if $P(A) = 0$.

- The odds in favour of occurrence of the event A are defined by $m : n - m$ *i.e.*, $P(A) : P(\bar{A})$ and the odds againts the occurrence of A are defined by $n - m : m$, *i.e.*, $P(\bar{A}) : P(A)$.
- **Sample Space :** The set of all possible outcomes of a random experiment is called the sample space associated with it and it is generally denoted by S.
- If $E_1, E_2, ..., E_n$ are the possible outcomes of a random experiment, then $S = \{E_1, E_2, ..., E_n\}$. Each element of S is called a sample point.
- **Event :** A subset of the sample space associated with a random experiment is called an event.
- **Elementary Events :** Single element subsets of the sample space associated with a random experiment are known as the elementary events or indecomposable events.
- **Compound Events :** Those subsets of the sample space S associated to an experiment which are disjoint union of single element subsets of the sample space S are known as the compound or decomposable events.
- **Impossible and Certain Event :** Let S be the sample space associated with a random experiment.Then ϕ and S, being subsets of S are events.The event ϕ is called an impossible event and the event S is known as a certain event.
- **Occurence or Happening of An Event :** Let S be the sample space associated with a random experiment and let A be an event. If w is an outcome of a trial such that $w \in A$, then we say that the event A has occurred. If $w \notin A$, we say that the event A has not occured.

Algebra of Events

Verbal description of event	Equivalent set theoretic notation
Not A	$\bar{A}$
A or B (at least one of A or B)	$A \cup B$
A and B	$A \cap B$
A but not B	$A \cap \bar{B}$
Neither A nor B	$\bar{A} \cap \bar{B} = \overline{(A \cup B)}$
At least one of A, B or C	$A \cup B \cup C$
Exactly one of A and B	$(A \cap \bar{B}) \cup (\bar{A} \cap B)$
All three of A, B and C	$A \cap B \cap C$
Exactly two of A, B and C	$(A \cap B \cap \bar{C}) \cup (A \cap \bar{B} \cap C) \cap (\bar{A} \cap B \cap C)$.

- **Mutually Exclusive Events :** Let S be the sample space associated with a random experiment and let A_1 and A_2 be two events.Then A_1 and A_2 are mutually exclusive events if $A_1 \cap A_2 = \phi$.
- **Mutually Exclusive and Exhaustive System of Events :** Let S be the sample space associated with a random experiment. Let $A_1, A_2, ..., A_n$ be subsets of S such that

  (i) $A_i \cap A_j = \phi$ for $i \neq j$, and (ii) $A_1 \cup A_2 \cup ... \cup A_n = S$.

- If $E_1, E_2, ..., E_n$ are elementary events associated with a random experiment. Then (i) $E_i \cap E_j = \phi$ for $i \neq j$ and (ii) $E_1 \cup E_2 \cup ... \cup E_n = S$.
- **Favourable Events :** Let S be the sample space associated with a random experiment and let $A \subset S$. Then the elementary events belonging to A are known as the favourable events to A.
- **Axiomatic Definition of Probability :** Let S be the sample space associated with a random experiment, and let A be a subset of S representing an event. Then the probability of the event A is defined as

  $$P(A) = \frac{\text{Number of elements in } A}{\text{Number of elements in } S} = \frac{n(A)}{n(S)}$$

  $P(\phi) = 0,\ P(S) = 1.$
- **Addition theorem for two events :** If A and B are two events associated with a random experiment, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- If A and B are mutually exclusive events, then
  $P(A \cap B) = 0$, therefore $P(A \cup B) = P(A) + P(B)$.
  This is the addition theorem for mutually exclusive events.
- **Addition theorem for three events :** If A, B, C are three events associated with a random experiment then,

  $$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C).$$
- If A, B, C are mutually exclusive events , then
  $P(A \cap B) = P(B \cap C) = P(A \cap C) = P(A \cap B \cap C) = 0.$

  $\therefore P(A \cup B \cup C) = P(A) + P(B) + P(C)$
- Let A and B be two events associated with a random experiment. Then
  (i) $P(\bar{A} \cap B) = P(B) - P(A \cap B)$ (ii) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$
- $P(\bar{A} \cap B)$ is known as the probability of occurence of B only.
- $P(A \cap \bar{B})$ is known as the probability of occurence of A only.
- If $B \subset A$, then (i) $P(A \cap \bar{B}) = P(A) - P(B)$ (ii) $P(B) \leq P(A)$.
- **Generalization of the Addition Theorem :** If $A_1, A_2, ..., A_n$ are n events associated with a random experiment, then

  $$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{\substack{i,\ j=1 \\ i \neq j}}^{n} P(A_i \cap A_j) + \sum_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{n} P(A_i \cap A_j \cap A_k) + ... + (-1)^{n-1} P(A_1 \cap A_2 \cap ... \cap A_n)$$
- If all the events A_i $(i = 1, 2..., n)$ are mutually exclusive then

  $$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i).$$
- **Boole's Inequality :** If $A_1, A_2 ..., A_n$ are n events associated with a random experiment, then

  (i) $P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i)$ (ii) $P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$

Conditional Probability

- $P(A/B)$ = Probability of occurrence of A given that B has already happened.
- $P(B/A)$ = Probability of occurrence of B given that A has already happened.
- **Multiplication Theorem :** If A and B are two events, then
  $P(A \cap B) = P(A)P(B/A)$, if $P(A) \neq 0$
  $P(A \cap B) = P(B)P(A/B)$, if $P(B) \neq 0$.
  $P(B/A) = \dfrac{P(A \cap B)}{P(A)}$ and $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$.
- **Extension of multiplication theorem :** If $A_1, A_2, \ldots, A_n$ are n events related to a random experiment, then
  $$P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) = P(A_1)P\left(\frac{A_2}{A_1}\right)P\left(\frac{A_3}{A_1 \cap A_2}\right)\ldots P\left(\frac{A_n}{A_1 \cap A_2 \cap \ldots \cap A_{n-1}}\right)$$
  where $P(A_i/A_1 \cap A_2 \cap \ldots \cap A_{i-1})$ represents the conditional probability of the event A_i, given that the events $A_1, A_2, \ldots, A_{i-1}$ have already happened.
- **Independent Events :** Event are said to be independent, if the occurrence or non-occurrence of one does not affect the probability of the occurrence or non-occurrence of the other.
  If A and B are two independent events associated with a random experiment then, $P(A/B) = P(A)$ and $P(B/A) = P(B)$ and vice-versa.
- If A and B are independent events associated with a random experiment, then $P(A \cap B) = P(A)P(B)$ *i.e.*, the probability of simultaneous occurrence of two independent events is equal to the product of their probabilities.
- If $A_1, A_2, \ldots, A_n$ are independent events associated with a random experiment, then $P(A_1 \cap A_2 \cap A_3 \ldots \cap A_n) = P(A_1)\,P(A_2)\ldots P(A_n)$
- If $A_1, A_2, \ldots, A_n$ are n independent events associated with a random experiment, then $P(A_1 \cup A_2 \cup \ldots \cup A_n) = 1 - P(\overline{A_1})P(\overline{A_2})\ldots P(\overline{A_n})$.
- Events $A_1, A_2, \ldots, A_n$ are independent or mutually independent if the probability of the simultaneous occurence of (any) finite number of them is equal to the product of their separate probabilities while these events are pairwise independent if $P(A_i \cap A_j) = P(A_j)P(A_i)$ for all $i \neq j$.
- **The Law of Total Probability :** Let S be the sample space and let $E_1, E_2, \ldots, E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with E_1 or E_2 or ... or E_n, then
  $$P(A) = P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right) + \ldots + P(E_n)P\left(\frac{A}{E_n}\right)$$
- **Baye's Rule :** Let S be the sample space and let $E_1, E_2, \ldots, E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with E_1 or E_2 or ... or E_n, then

$$P\left(\frac{E_i}{A}\right)=\frac{P(E_i)P(A/E_i)}{\sum_{i=1}^{n}P(E_i)P(A/E_i)},\ i=1,2,...,n$$

- The events $E_1, E_2, ..., E_n$ are usually referred to as 'hypothesis' and the probabilities $P(E_1), P(E_2), ..., P(E_n)$ are known as the 'priori' probabilities as they exist before we obtain any information from the experiment.
- The probabilities $P(A/E_i)$; $i = 1, 2, ..., n$ are called the likelihood probabilities as they tell us how likely the event A under consideration occur, given each and every priori probabilities.
- The probabilities $P(E_i/A)$; $i = 1, 2, ..., n$ are called the posterior probabilities as they are determined after the result of the experiment are known.
- **Random Variable :** A random variable is a real valued function having domain as the sample space associated with a given random experiment.
- A random variable associated with a given random experiment associates every event to a unique real number.
- **Probability Distribution :** If a random variable X takes values $x_1, x_2, ..., x_n$ with respective, probabilities $p_1, p_2, ..., p_n$, then

X :	x_1	x_2	...	x_n
$P(X)$:	p_1	p_2	...	p_n

  is called the probability distribution of X.
- **Binomial Distribution :** A random variable X which takes values 0, 1, 2, ..., n is said to follow binomial distribution if its probability distribution function is given by $P(X = r) = {}^nC_r p^r q^{n-r}$, $r = 0, 1, 2, ..., n$ where $p, q > 0$ such that $p + q = 1$.
- If n trials constitute an experiment and the experiment is repeated N times, then the frequencies of 0, 1, 2, ..., n successes are given by
  $N{\cdot}P(X = 0), N{\cdot}P(X=1), N{\cdot}P(X=2), ..., N{\cdot}P(X=n)$.
- If X is a random variable with probability distribution

x :	x_1	x_2	x_3	...	x_n
$P(X)$:	p_1	p_2	p_3	...	p_n

  then the mean or expectation of X is defined as $\overline{X}=E(X)=\sum_{i=1}^{n}p_i x_i$ and the variance of X is defined as

  $$\text{Var }(X)=\sum_{i=1}^{n}p_i(x_i-E(X))^2=\sum_{i=1}^{n}p_i x_i^2-[(E(X)^2].$$
- The mean of the binomial variate $X \sim B(n, p)$ is $n\,p$.
- The variance of the binomial variate $X \sim B(n, p)$ is npq, where $p + q = 1$
- The standard deviation of a binomial variate $X \sim B(n, p)$ is $\sqrt{Var(X)}=\sqrt{npq}$.
  mean > variance.

Maximum Value of $P(X = r)$ for given Values of n and p for a Binomial Variate X

- If $(n + 1)p$ is an integer, say m, then $P(X = r) = {}^nC_r\, p^r (1 - p)^{n-r}$ is maximum when $r = m$ or $r = m - 1$.
- If $(n + 1)p$ is not an integer, then $P(X = r)$ is maximum when $r = [(n + 1)p]$.

End

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CHAPTER

trigonometry

- An angle is the amount of rotation of a revolving line with respect to a fixed line.
  1 right angle = 90 degrees (90°)
  1° = 60 minutes (60′)

$$1 \text{ radian} = \frac{180}{\pi} \text{ degree}, \ 1 \text{ degree} = \frac{\pi}{180} \text{ radian}$$

Some Basic Formulae

1. $\sin^2 A + \cos^2 A = 1$ or $1 - \cos^2 A = \sin^2 A$ or $1 - \sin^2 A = \cos^2 A$

2. $1 + \tan^2 A = \sec^2 A$ or $\sec^2 A - \tan^2 A = 1$

or $\sec A + \tan A = \dfrac{1}{\sec A - \tan A}$, where $A \neq n\pi + \dfrac{\pi}{2'}$ $n \in Z$. $n\pi + \dfrac{\pi}{2'}$ $n \in Z$.

3. $1 + \cot^2 A = \operatorname{cosec}^2 A$ or $\operatorname{cosec}^2 A - \cot^2 A = 1$

$\operatorname{cosec} A + \cot A = \dfrac{1}{\operatorname{cosec} A - \cot A}$, where $A \neq n\pi$, $n \in Z$

Domain and Range of Trigonometrical Functions

	Domain	**Range**
$\sin A$	R	$[-1, 1]$
$\cos A$	R	$[-1, 1]$
$\tan A$	$R - \{(2n+1)\pi/2 \mid n \in Z\}$	$(-\infty, \infty) = R$
$\operatorname{cosec} A$	$R - \{n\pi \mid n \in Z\}$	$(-\infty, -1] \cup [1, \infty)$
$\sec A$	$R - \{(2n+1)\pi/2 \mid n \in Z\}$	$(-\infty, -1] \cup [1, \infty)$
$\cot A$	$R - \{n\pi \mid n \in Z\}$	$(-\infty, \infty) = R$.

Thus, $|\sin A| \leq 1$, $|\cos A| \leq 1$, $\sec A \geq 1$ or $\sec A \leq -1$ and $\operatorname{cosec} A \geq 1$ or $\operatorname{cosec} A \leq -1$.

Sum and Difference Formulae

1. $\sin(A + B) = \sin A \cos B + \cos A \sin B$

2. $\sin(A - B) = \sin A \cos B - \cos A \sin B$

3. $\cos(A + B) = \cos A \cos B - \sin A \sin B$

4. $\cos(A - B) = \cos A \cos B + \sin A \sin B$

5. $\left.\begin{array}{l} \tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B} \\ \tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B} \end{array}\right]$, where $A \neq n\pi + \dfrac{\pi}{2}, B \neq n\pi + \dfrac{\pi}{2}$ and $A \pm B \neq m\pi + \dfrac{\pi}{2}$

6. $\left.\begin{array}{l}\cot(A+B)=\dfrac{\cot A\cot B-1}{\cot A+\cot B}\\[2ex] \cot(A-B)=\dfrac{\cot A\cot B+1}{\cot A-\cot B}\end{array}\right]$, where $A \neq n\pi$, $B \neq n\pi$ and $A + B \neq n\pi$.

7. $\sin(A+B)\sin(A-B)=\sin^2A-\sin^2B=\cos^2B-\cos^2A$

8. $\cos(A+B)\cos(A-B)=\cos^2A-\sin^2B=\cos^2B-\sin^2A$

9. $\sin2\theta=2\sin\theta\cos\theta=\dfrac{2\tan\theta}{1+\tan^2\theta}$

10. $\cos2\theta=\cos^2\theta-\sin^2\theta=2\cos^2\theta-1=1-2\sin^2\theta=\dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

11. $1+\cos2\theta=2\cos^2\theta$, $1-\cos2\theta=2\sin^2\theta$

12. $\tan2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}$, where $\theta\neq(2n+1)\dfrac{\pi}{4}$

13. $\dfrac{1-\cos\theta}{\sin\theta}=\tan\dfrac{\theta}{2}$, where $\theta\neq2n\pi$

14. $\dfrac{1+\cos\theta}{\sin\theta}=\cot\dfrac{\theta}{2}$, where $\theta\neq(2n+1)\pi$

15. $\dfrac{1-\cos\theta}{1+\cos\theta}=\tan^2\dfrac{\theta}{2}$, where $\theta\neq(2n+1)\pi$

16. $\dfrac{1+\cos\theta}{1-\cos\theta}=\cot^2\dfrac{\theta}{2}$, where $\theta\neq2n\pi$

17. $\sin3\theta=3\sin\theta-4\sin^3\theta$

18. $\cos3\theta=4\cos^3\theta-3\cos\theta$

19. $\tan3\theta=\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$

20. $\cos A\cos2A\cos2^2A\ \ldots\ \cos2^{n-1}A=\dfrac{\sin2^nA}{2^n\sin A}$

21. $\sin\left(\dfrac{A}{2}\right)+\cos\left(\dfrac{A}{2}\right)=\pm\sqrt{1+\sin A}$

22. $\sin\left(\dfrac{A}{2}\right)-\cos\left(\dfrac{A}{2}\right)=\pm\sqrt{1-\sin A}$

23. $2\sin\left(\dfrac{A}{2}\right)=\pm\sqrt{1+\sin A}\pm\sqrt{1-\sin A}$

24. $2\cos\left(\dfrac{A}{2}\right)=\pm\sqrt{1+\sin A}\mp\sqrt{1-\sin A}$

Sum and Difference Into Products

1. $\sin A+\sin B=2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$

2. $\sin A-\sin B=2\sin\left(\dfrac{A-B}{2}\right)\cos\left(\dfrac{A+B}{2}\right)$

3. $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

4. $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

5. $\tan A + \tan B = \frac{\sin(A+B)}{\cos A \cos B}$

6. $\tan A - \tan B = \frac{\sin(A-B)}{\cos A \cos B}$, where $A, B \neq n\pi + \frac{\pi}{2}$ and $n \in Z$

7. $\cot A + \cot B = \frac{\sin(A+B)}{\sin A \sin B}$

8. $\cot A - \cot B = \frac{\sin(B-A)}{\sin A \sin B}$, where $A, B \neq n\pi$, $n \in Z$

Product Into Sum or Difference

1. $2\sin A \cos B = \sin(A+B) + \sin(A-B)$.

2. $2\cos A \sin B = \sin(A+B) - \sin(A-B)$.

3. $2\cos A \cos B = \cos(A+B) + \cos(A-B)$.

4. $2\sin A \sin B = \cos(A-B) - \cos(A+B)$.

T-Ratios of The Sum of Three or More Angles

1. $\sin(A+B+C) = \sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C$

or $\sin(A+B+C) = \cos A\cos B\cos C(\tan A + \tan B + \tan C - \tan A\tan B\tan C)$

2. $\cos(A+B+C) = \cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C - \cos A\sin B\sin C$

or $\cos(A+B+C) = \cos A\cos B\cos C(1 - \tan A\tan B - \tan B\tan C - \tan C\tan A)$

3. $\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}$

4. $\sin(A_1 + A_2 + \ldots + A_n) = \cos A_1 \cos A_2 \ldots \cos A_n (S_1 - S_3 + S_5 - S_7 + \ldots)$

5. $\cos(A_1 + A_2 + \ldots + A_n) = \cos A_1 \cos A_2 \ldots \cos A_n (1 - S_2 + S_4 - S_6 + \ldots)$

6. $\tan(A_1 + A_2 + \ldots + A_n) = \frac{S_1 - S_3 + S_5 - S_7 + \ldots}{1 - S_2 + S_4 - S_6 + \ldots}$,

where $S_1 = \tan A_1 + \tan A_2 + \ldots + \tan A_n$
= the sum of the tangents of the separate angles,

$S_2 = \tan A_1 \tan A_2 + \tan A_1 \tan A_3 + \ldots$ = the sum of the tangents taken two at a time

$S_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \tan A_3 \tan A_4 + \ldots$
= the sum of the tangents taken two at a time, and so on.

If $A_1 = A_2 = \ldots = A_n = A$, then

$S_1 = n\tan A$, $S_2 = {}^nC_2 \tan^2 A$, $S_3 = {}^nC_3 \tan^3 A$, … . Therefore,

7. $\sin nA = \cos^n A\,({}^nC_1 \tan A - {}^nC_3 \tan^3 A + {}^nC_5 \tan^5 A - \ldots)$

8. $\cos nA = \cos^n A\,(1 - {}^nC_2 \tan^2 A + {}^nC_4 \tan^4 A + \ldots)$

9. $\tan nA = \frac{{}^nC_1 \tan A - {}^nC_3 \tan^3 A + {}^nC_5 \tan^5 A \ldots}{1 - {}^nC_2 \tan^2 A + {}^nC_4 \tan^4 A - {}^nC_6 \tan^6 A + \ldots}$

10. $\sin nA + \cos nA = \cos^n A(1 + {}^nC_1 \tan A - {}^nC_2 \tan^2 A - {}^nC_3 \tan^3 A + {}^nC_4 \tan^4 A + {}^nC_5 \tan^5 A - {}^nC_6 \tan^6 A - {}^nC_7 \tan^7 A + \ldots)$

11. $\sin nA - \cos nA = \cos^n A(-1 + {}^nC_1 \tan A + {}^nC_2 \tan^2 A - {}^nC_3 \tan^3 A - {}^nC_4 \tan^4 A + {}^nC_5 \tan^5 A + {}^nC_6 \tan^6 A \ldots)$

12. $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \ldots + \sin(\alpha + (n-1)\beta)$

$$= \frac{\sin\left(\alpha + (n-1)\frac{\beta}{2}\right)}{\sin(\beta/2)} \sin\left(\frac{n\beta}{2}\right)$$

13. $\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots + \cos(\alpha + (n-1)\beta)$

$$= \frac{\cos\left(\alpha + (n-1)\frac{\beta}{2}\right)\sin\left(\frac{n\beta}{2}\right)}{\sin(\beta/2)}$$

Values of Trigonometrical Ratios of Some Important Angles

1. $\sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}}$

2. $\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$

3. $\tan 15° = 2 - \sqrt{3} = \cot 75°$

4. $\cot 15° = 2 + \sqrt{3} = \tan 75°$

5. $\sin 22\frac{1°}{2} = \frac{1}{2}(\sqrt{2-\sqrt{2}})$

6. $\cos 22\frac{1°}{2} = \frac{1}{2}(\sqrt{2+\sqrt{2}})$

7. $\tan 22\frac{1°}{2} = \sqrt{2} - 1$

8. $\cot 22\frac{1°}{2} = \sqrt{2} + 1$

9. $\sin 18° = \frac{\sqrt{5}-1}{4} = \cos 72°$

10. $\cos 18° = \frac{\sqrt{10+2\sqrt{5}}}{4} = \sin 72°$

11. $\sin 36° = \frac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54°$

12. $\cos 36° = \frac{\sqrt{5}+1}{4} = \sin 54°$

13. $\sin 9° = \frac{\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}}}{4} = \cos 81°$

14. $\cos 9° = \frac{\sqrt{3+\sqrt{5}} + \sqrt{5-\sqrt{5}}}{4} = \sin 81°$

15. $\sin\theta\sin(60°-\theta)\sin(60°+\theta)=\frac{1}{4}\sin 3\theta$

16. $\cos\theta\cos(60°-\theta)\cos(60°+\theta)=\frac{1}{4}\cos 3\theta$

17. $\tan\theta\tan(60°-\theta)\tan(60°+\theta)=\tan 3\theta$

18. $\cos 36°-\cos 72°=\frac{1}{2}$

19. $\cos 36°\cos 72°=\frac{1}{4}$.

Maximum and Minimum Values of Trigonometrical Functions

- The maximum and minimum values of a trigonometrical function of the form $a\sin x+b\cos x$ are $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$ respectively.

Properties of Triangle and Circles connected with them

- **Sine Rule :** In any ΔABC, $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$ *i.e.* the sines of the angles are proportional to the lengths of the opposite sides.

- **Cosine Formulae :** In any $\Delta\ ABC$, $\cos A=\frac{b^2+c^2-a^2}{2bc}$, $\cos B=\frac{c^2+a^2-b^2}{2ac}$, $\cos C=\frac{a^2+b^2-c^2}{2ab}$.

- **Projection Formulae :** In any $\Delta\ ABC$, $a=b\cos C+c\cos B$, $b=c\cos A+a\cos C$, $c=a\cos B+b\cos A$.

Trigonometrical Ratios of Half of the Angles of a Triangle

- In any $\Delta\ ABC$, we have

(i) $\sin\left(\frac{A}{2}\right)=\sqrt{\frac{(s-b)(s-c)}{bc}}$, $\sin\left(\frac{B}{2}\right)=\sqrt{\frac{(s-c)(s-a)}{ac}}$, $\sin\left(\frac{C}{2}\right)=\sqrt{\frac{(s-a)(s-b)}{ab}}$

(ii) $\cos\left(\frac{A}{2}\right)=\sqrt{\frac{s(s-a)}{bc}}$, $\cos\left(\frac{B}{2}\right)=\sqrt{\frac{s(s-b)}{ac}}$ $\cos\left(\frac{C}{2}\right)=\sqrt{\frac{s(s-c)}{ab}}$.

(iii) $\tan\left(\frac{A}{2}\right)=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$, $\tan\left(\frac{B}{2}\right)=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$, $\tan\left(\frac{C}{2}\right)=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

Area of a Triangle

- $\Delta=\frac{1}{2}ab\sin C$, $\Delta=\frac{1}{2}bc\sin A$, $\Delta=\frac{1}{2}ac\sin B$.

  Also, $\Delta=\frac{c^2\sin A\sin B}{2\sin C}$, $\Delta=\frac{a^2\sin B\sin C}{2\sin A}$ and $\Delta=\frac{b^2\sin C\sin A}{2\sin B}$.

- **Napier's Analogy :** In any triangle ABC, $\tan\left(\frac{B-C}{2}\right)=\left(\frac{b-c}{b+c}\right)\cot\frac{A}{2}$,

$$\tan\left(\frac{C-A}{2}\right)=\left(\frac{c-a}{c+a}\right)\cot\frac{B}{2},\quad \tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}\cot\frac{C}{2}.$$

- **Circum-Radius :** The radius of the circumcircle of a triangle ABC is called the circum-radius given by

  (i) $R=\dfrac{a}{2\sin A}=\dfrac{b}{2\sin B}=\dfrac{c}{2\sin C}$ (ii) $R=\dfrac{abc}{4\Delta}$.

Inscribed Circle or Incircle of a Triangle

- The circle which can be incribed within the triangle so as to touch each of its sides is called its inscribed circle or incircle.
- The centre of this circle is the point of intersection of bisectors of the angles of the triangle.
- The radius of this circle is always denoted by r and is equal to the length of the perpendicular from its centre to any one of the sides of triangle.
- **In-Radius :** The radius of the inscribed circle of a triangle is called the in-radius. It is denoted by r and is given by

(i) $r=\dfrac{\Delta}{s}$

(ii) $r=(s-a)\tan\dfrac{A}{2},\ r=(s-b)\tan\dfrac{B}{2},\ r=(s-c)\tan\dfrac{C}{2}$

(iii) $r=\dfrac{a\sin(B/2)\sin(C/2)}{\cos(A/2)}\quad r=\dfrac{b\sin(A/2)\sin(C/2)}{\cos(B/2)},\quad r=\dfrac{c\sin(B/2)\sin(A/2)}{\cos(C/2)},$

(iv) $r=4R\sin\left(\dfrac{A}{2}\right)\sin\left(\dfrac{B}{2}\right)\sin\left(\dfrac{C}{2}\right).$

Escribed Circles of a Triangle

- The circle which touches the sides BC and two sides AB and AC produced of a triangle ABC is called the escribed circle opposite to the angle A. Its radius is denoted by r_1. Similarly, r_2 and r_3 denotes the radius of the escribed circles opposite to the angles B and C respectively.
- The centres of the escribed circles are called the ex-centres. The centres of the escribed circle opposite to the angle A is the point of intersection of the external bisectors of angles B and C. This centre is generally denoted by I_1.
- **Formulae for r_1, r_2, r_3 :** In any $\Delta\,ABC$, we have

(i) $r_1=\dfrac{\Delta}{s-a},\ r_2=\dfrac{\Delta}{s-b},\ r_3=\dfrac{\Delta}{s-c}$

(ii) $r_1=s\tan\left(\dfrac{A}{2}\right),\ r_2=s\tan\left(\dfrac{B}{2}\right),\ r_3=s\tan\left(\dfrac{C}{2}\right)$

(iii) $r_1=\dfrac{a\cos(B/2)\cos(C/2)}{\cos(A/2)},\ r_2=\dfrac{b\cos(C/2)\cos(A/2)}{\cos(B/2)},\ r_3=\dfrac{c\cos(A/2)\cos(B/2)}{\cos(C/2)}$

(iv) $r_1=4R\sin\left(\dfrac{A}{2}\right)\cos\left(\dfrac{B}{2}\right)\cos\left(\dfrac{C}{2}\right),\quad r_2=4R\cos\left(\dfrac{A}{2}\right)\sin\left(\dfrac{B}{2}\right)\cos\left(\dfrac{C}{2}\right)$

$r_3 = 4R\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right).$

- The distances of the orthocentre of the triangle from the angular points are $2R\cos A$, $2R\cos B$, $2R\cos C$ and its distances from the sides are $2R\cos B\cos C$, $2R\cos C\cos A$, $2R\cos A\cos B$.
- The lengths of the medians AD, BE and CF of a triangle ABC are given by

$$AD = \frac{1}{2}\sqrt{b^2 + c^2 + 2bc\cos A} = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$$

$$BE = \frac{1}{2}\sqrt{c^2 + a^2 + 2ac\cos B} = \frac{1}{2}\sqrt{2c^2 + 2a^2 - b^2}$$

$$CF = \frac{1}{2}\sqrt{a^2 + b^2 + 2ab\cos C} = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$$

- The distance between the circumcentre O and orthocentre O' is given by :

$$OO' = R\sqrt{1 - 8\cos A\cos B\cos C}.$$

- The distance between the circumcentre O and the incentre I of $\Delta\, ABC$ is

$$OI = R\sqrt{1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

- If I_1 is the centre of the escribed circle opposite to the angle A, then

$$OI_1 = R\sqrt{1 + 8\sin\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)}.$$

Similarly, $OI_2 = R\sqrt{1 + 8\cos\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)}$

$$OI_3 = R\sqrt{1 + 8\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}.$$

- A regular polygon is a polygon which has all its sides equal and all its angles equal.
- Each interior angle of a regular polygon of n sides is $\left(\frac{2n-4}{n}\right)$ right angles $= \left(\frac{2n-4}{n}\right)\times\frac{\pi}{2}$ radians.

Area of a Cyclic Quadrilateral

- A quadrilateral is a cyclic quadrilateral if its vertices lie on a circle.
- Let $ABCD$ be a cyclic quadrilateral such that $AB = a$, $BC = b$, $CD = c$ and $DA = d$. Then $\Delta = \frac{1}{2}(ab + cd)\sin B$, $\cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}$
- **Ptolemy's Theorem :** In a cyclic quadrilateral $ABCD$, $AC\cdot BD = AB\cdot CD + BC\cdot AD$ *i.e.* in a cyclic quadrilateral the product of diagonals is equal to the sum of the products of the lengths of the opposite sides.

Circum-Radius of a Cyclic Quadrilateral

- Let $ABCD$ be a cyclic quadrilateral. Then the circumcircle of the quadrilateral $ABCD$ is also the circumcircle of $\Delta\, ABC$.

  If R = circum-radius of $\Delta\, ABC$

  $$R = \frac{1}{4}\sqrt{\frac{(ac+bd)(ad+bc)(ab+cd)}{(s-a)(s-b)(s-c)(s-d)}}.$$

- **Periodic Function :** A function $f(x)$ is said to be periodic if there exists $T > 0$ such that $f(x + T) = f(x)$ for all x in the domain of definition of $f(x)$.

- The general solution of the trigonometrical equations are
  If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n \alpha,\ n \in Z.$
  $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha,\ n \in Z$
  $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha,\ n \in Z$
  $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha,\ n \in Z$
  $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha,\ n \in Z$
  $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha,\ n \in Z$
  $\cos\theta = \cos\alpha$ and $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + \alpha,\ n \in Z.$

- **Inverse Trigonometrical Functions :** A function $f : A \to B$ is invertible if it is a bijection. The inverse of f is denoted by f^{-1} and is defined as $f^{-1}(y) = x \Leftrightarrow f(x) = y.$

  Clearly, domain of f^{-1} = range of f and range of f^{-1} = domain of f.

- The inverse of sine function is defined as $\sin^{-1}x = \theta \Leftrightarrow \sin\theta = x$, where $\theta \in [-\pi/2, \pi/2]$ and $x \in [-1, 1]$.
- Thus, $\sin^{-1}x$ has infinitely many values for given $x \in [-1, 1]$.
- There is one value among these values which lies in the interval $[-\pi/2, \pi/2]$. This value is called the principal value.
- **Domain and Range of Inverse Trigonometrical Functions**

Function	Domain	Range
$\sin^{-1}x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}x$	$(-\infty, \infty)$	$(-\pi/2, \pi/2)$
$\cot^{-1}x$	$(-\infty, \infty)$	$(0, \pi)$
$\sec^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$
$\text{cosec}^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, 0) \cup (0, \pi/2]$

Properties of Inverse Trigonometrical Functions

- $\sin^{-1}(\sin\theta) = \theta$ and $\sin(\sin^{-1}x) = x$, provided that $-1 \le x \le 1$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.
- $\cos^{-1}(\cos\theta) = \theta$ and $\cos(\cos^{-1} x) = x$, provided that $-1 \le x \le 1$ and $0 \le \theta \le \pi$.
- $\tan^{-1}(\tan\theta) = \theta$ and $\tan(\tan^{-1}x) = x$, provided that $-\infty < x < \infty$ and $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$.

- $\cot^{-1}(\cot\theta) = \theta$ and $\cot(\cot^{-1} x) = x$, provided that $-\infty < x < \infty$ and $0 < \theta < \pi$.
- $\sec^{-1}(\sec\theta) = \theta$ and $\sec(\sec^{-1} x) = x$
- $\text{cosec}^{-1}(\text{cosec}\theta) = \theta$ and $\text{cosec}(\text{cosec}^{-1}x) = x$,
- $\sin^{-1}x = \text{cosec}^{-1}\left(\frac{1}{x}\right)$ or $\text{cosec}^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$
- $\cos^{-1}x = \sec^{-1}\left(\frac{1}{x}\right)$ or $\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)$
- $\tan^{-1}x = \cot^{-1}\left(\frac{1}{x}\right)$ or $\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)$
- $$\sin^{-1} x = \cos^{-1} \sqrt{1-x^2} = \tan^{-1} \frac{x}{\sqrt{1-x^2}} = \cot^{-1} \frac{\sqrt{1-x^2}}{x}$$
  $$= \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \text{cosec}^{-1}\left(\frac{1}{x}\right)$$
- $$\cos^{-1} x = \sin^{-1} \sqrt{1-x^2} = \tan^{-1} \frac{\sqrt{1-x^2}}{x} = \sec^{-1} \frac{1}{x}$$
  $$= \text{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$
- $$\tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1} \frac{1}{\sqrt{1+x^2}} = \cot^{-1} \frac{1}{x}$$
  $$= \sec^{-1} \sqrt{1+x^2} = \text{cosec}^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$
- $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, where $-1 \le x \le 1$

  $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$, where $-\infty < x < \infty$

  $\sec^{-1} x + \text{cosec}^{-1} x = \frac{\pi}{2}$, where $x \le -1$ or $x \ge 1$.
- $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, if $xy < 1$

  $\tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, if $xy > 1$

  $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

  $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}(x\sqrt{1-y^2} \pm y\sqrt{1-x^2})$ if $x, y \ge 0$, $x^2 + y^2 \le 1$

  $\sin^{-1}x \pm \sin^{-1}y = \pi - \sin^{-1}(x\sqrt{1-y^2} \pm y\sqrt{1-x^2})$ if $x, y \ge 0$, $x^2 + y^2 > 1$

  $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}(xy \mp \sqrt{1-x^2}\sqrt{1-y^2})$, if $x, y \ge 0$ and $x^2 + y^2 \le 1$

  $\cos^{-1}x \pm \cos^{-1}y = \pi - \cos^{-1}(xy \mp \sqrt{1-x^2}\sqrt{1-y^2})$ if $x, y \ge 0$ and $x^2 + y^2 > 1$

- $\sin^{-1}(-x) = -\sin^{-1}x$, $\cos^{-1}(-x) = \pi - \cos^{-1}x$
  $\tan^{-1}(-x) = -\tan^{-1}x$, $\cot^{-1}(-x) = \pi - \cot^{-1}x$.
- $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$ $2\cos^{-1}x = \cos^{-1}(2x^2 - 1)$
  $$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \sin^{-1}\left(\frac{2x}{1-x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$
- $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$, $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$
  $$3\tan^{-1}x = \tan^{-1}\left(\frac{3x - x^3}{1-3x^2}\right).$$
- $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$
- $\tan^{-1}x_1 + \tan^{-1}x_2 + \ldots + \tan^{-1}x_n = \tan^{-1}\left[\frac{S_1 - S_3 + S_5 \ldots}{1 - S_2 + S_4 - S_6 + \ldots}\right]$, where S_k denotes the sum of the products of $x_1, x_2, \ldots, x_n$ taken k at a time.

Solution of Right Angled Triangle

- **Case I :** When two sides are given :
  Let the triangle be right angled at C. Then we can determine the remaining elements as given in the following table :

Given	Required
(i) a, b	$\tan A = \frac{a}{b}$, $B = 90° - A$, $C = \frac{a}{\sin A}$
(ii) a, c	$\sin A = \frac{a}{c}$, $b = c\cos A$, $B = 90° - A$

- **Case II :** When a side and an acute angle are given:
  In this table we can determine the remaining elements as given in the following table :

Given	Required
(i) a, A	$B = 90° - A$, $b = a\cot A$, $c = \frac{a}{\sin A}$
(ii) c, A	$B = 90° - A$, $a = c\sin A$, $b = c\cos A$

Solution of a Triangle in General

- **Case I :** When three sides a, b and c are given. In this case the remaining elements are determined by using the following formulae :
  $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, where $2s = a + b + c$
  $$\sin A = \frac{2\Delta}{bc}, \sin B = \frac{2\Delta}{ac}, \sin C = \frac{2\Delta}{ab}$$
  $$\tan\frac{A}{2} = \frac{\Delta}{s(s-a)}, \tan\frac{B}{2} = \frac{\Delta}{s(s-b)}, \tan\frac{C}{2} = \frac{\Delta}{s(s-c)}, A + B + C = 180°.$$

- **Case II :** When two sides a, b and the included angle C are given.

  In this case we use the following formulae:

  $$\Delta = \frac{1}{2} ab \sin C,\ \tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b} \cot\frac{C}{2},\ \frac{A+B}{2} = 90^\circ - \frac{C}{2},\ c = \frac{a \sin C}{\sin A}.$$

- **Case III :** When one side a and two angle A and B are given:

  In this case we use the following formulae to determine the remaining elements:

  $$A + B + C = 180^\circ \Rightarrow C = 180^\circ - A - B$$

  $$b = \frac{a \sin B}{\sin A} \text{ and } c = \frac{a \sin C}{\sin A},\ \Delta = \frac{1}{2} ca \sin B.$$

- **Case IV :** When two sides a, b and the angle A opposite to one side is given :

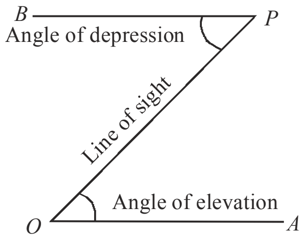
  In this case we use the following formulae:

  $$\sin B = \frac{b}{a}\sin A \text{ ....(i)},\ C = 180^\circ - (A+B),\ c = \frac{a \sin C}{\sin A}.$$

- From (i), the following possibilities will arise : When A is an acute angle and $a < b \sin A$. In this case the relation $\sin B = \frac{b}{\alpha} \sin A$ gives that $\sin B > 1$, which is impossible. Hence, no triangle is possible.
- When A is an acute angle and $a = b \sin A$. In this case only one triangle is possible which is right angle at B.
- When A is an acute angle and $a > b \sin A$.

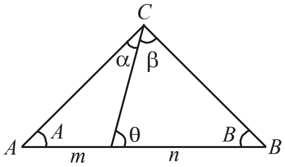
  In this case there are two values of B given by $\sin B = \frac{b \sin A}{a}$, say B_1 and B_2 such that $B_1 + B_2 = 180^\circ$ and side c can be obtained by using $c = \frac{a \sin C}{\sin A}$.
- In any right angle triangle, the orthocentre coincides with the vertex containing the right angle.
- The mid-point of the hypotenuse of a right angle triangle is equidistant from the three vertices of the triangle.
- The mid-point of the hypotenuse of a right angle triangle is the circum-centre of the triangle.

Height and Distance



- Any line perpendicular to a plane is perpendicular to every line lying in the plane.
- In a triangle the internal bisector of an angle divides the opposite side in the ratio of the arms of the angle.

- In an isosceles triangle the median is perpendicular to the base.
- Angles in the same segment of a circle are equal.



- ***m-n* Theorem :** In figure, we have

  (a) $(m + n)\cot\theta = m\cot\alpha - n\cot\beta$

  (b) $(m + n)\cot\theta = n\cot A - m\cot B$.

End

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CHAPTER

statistics

- **Arithmetic Mean :** If $x_1, x_2, ..., x_n$ are n values of variable X, then the arithmetic mean of these values is given by

$$\overline{X} = \frac{1}{n}(x_1 + x_2 + ... + x_n) = \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)$$

- In case of a frequency distribution x_i/f_i, $i = 1, 2, ..., n$ where f_i is the frequency of the variable x_i,

$$\overline{X} = \frac{f_1x_1 + f_2x_2 + ... + f_nx_n}{f_1 + f_2 + ... + f_n} = \frac{1}{N}\left(\sum_{i=i}^{n} f_ix_i\right), \quad ...(i)$$

where $N = f_1 + f_2 + ... + f_n = \sum_{i=1}^{n} f_i$

- If $w_1, w_2, w_3, ..., w_n$ be the weights assigned to the n values $x_1, x_2, ..., x_n$ respectively of a variable X, then the weighted A.M. is given by

$$\overline{X} = \frac{w_1x_1 + w_2x_2 + ... + w_nx_n}{w_1 + w_2 + ... + w_n} = \frac{\sum_{i=1}^{n} w_ix_i}{\sum_{i=1}^{n} w_i} \text{ or } \overline{X} = \frac{\Sigma wx}{\Sigma w}$$

Properties of Arithmetic Mean

(i) Algebraic sum of the deviations of a set of values from their arithmetic mean is zero.

(ii) The sum of the squares of the deviation of a set of values is minimum when taken about mean.

(iii) Mean of the composite series: If $\overline{X}_1, \overline{X}_2, ..., \overline{X}_k$ are the means of k series of sizes $n_1, n_2, ..., n_k$ respectively then the mean $\overline{X}$ of the composite series is given by $\overline{X} = \dfrac{n_1\overline{X}_1 + n_2\overline{X}_2 + ... + n_k\overline{X}_k}{n_1 + n_2 + ... + n_k}$

- **Geometric Mean :** Geometric mean of a set of n observations is the n^{th} root of their product. Thus, the geometric mean G of n non-zero observations $x_1, x_2, ..., x_n$ is $G = (x_1 \cdot x_2 \cdot x_3 \cdot \; ... \; \cdot x_n)^{1/n}$ $\Rightarrow G = \text{antilog}\left(\dfrac{1}{n}\sum_{i=1}^{n} \log x_i\right)$
- In case of a frequency distribution x_i/f_i; $i = 1, 2, ..., n$, geometric mean G is given by $G = \text{antilog}\left(\dfrac{1}{N}\sum_{i=1}^{n} f_i \log x_i\right)$.

- **Properties of Geometric Mean :** If G_1 and G_2 are the geometric means of two series of sizes n_1 and n_2 respectively, then the geometric mean G of the combined series is given by $\log G = \dfrac{n_1 \log G_1 + n_2 \log G_2}{n_1 + n_2}$.
- **Harmonic Mean :** Harmonic mean of a number of non-zero observations is the reciprocal of the A.M. of the reciprocals of the given values. Thus, harmonic mean H of n non-zero observations $x_1, x_2, ..., x_n$ is

$$H = \frac{1}{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)}$$

- In case of a frequency distribution x_i/f_i; $i = 1, 2, ..., n$

$$H = \frac{1}{\frac{1}{N}\sum_{i=1}^{n}\left(\frac{f_i}{x_i}\right)}$$

Median

- In case of individual observations $x_1, x_2, ..., x_n$ if the number of observations is odd, then median is the value of $\left(\frac{n+1}{2}\right)^{\text{th}}$ observation after the observations have been arranged in ascending or descending order of magnitude.
  In case of even number of observations median is the A.M. of the values of $\left(\frac{n}{2}\right)^{\text{th}}$ and $\left(\frac{n}{2}+1\right)^{\text{th}}$ observations.
- In case of discrete frequency distribution $\frac{x_i}{f_i}$; $i = 1, 2, ..., n$, the median is calculated by following algorithm :
- **Algorithm for finding out Median**
  Step I : Find cumulative frequencies (*c.f.*)
  Step II : Find $\frac{N}{2}$ where $N = \sum_{i=1}^{n} f_i$
  Step III : See the cumulative frequency (*c.f.*) just greater than $\frac{N}{2}$
  Step IV : The corresponding value of x is the median.
- In case of a grouped or continuous frequency distribution, the class corresponding to the *c.f.* just greater than $\frac{N}{2}$ is called the median class and the value of the median is calculated by the formula:

$$\text{Median} = l + \left(\frac{\frac{N}{2} - F}{f}\right)h$$

where l = lower limit of the median class
f = frequency of the median class

h = size (width) of the median class
F = $c.f.$ of the class preceeding the median class

Quartile

- As median divides a distribution into two equal parts, similarly the quartiles, quantiles, deciles and percentiles divide the distribution respectively into 4, 5, 10 and 100 equal parts. The i^{th} quartile is given by

$$Q_i = l + \left(\frac{\frac{iN}{4} - F}{f}\right)h;\ i = 1,\ 2,\ 3.$$

Q_1 is the lower quartile, Q_2 is the median and Q_3 is called the upper quartile.

- The i^{th} decile is given by $D_i = l + \left(\frac{\frac{iN}{10} - F}{f}\right)h; i = 1, 2, ..., 9.$ D_5 is the median.

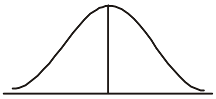
Mode

- The mode or modal value of a distribution is that value of the variable for which the frequency is maximum.
- In case of a discrete frequency distribution is that value of the mode determined by the method of grouping.
- In case of a grouped or continuous frequency distribution, mode is given by the formula

Mode = $l + \frac{f - f_1}{2f - f_1 - f_2} \times h$

where l = lower limit of the modal class
h = width of the modal class
f_1 = frequency of the class preceding the modal class
f_2 = frequency of the class following the modal class
f = frequency of the modal class.

- **Symmetric Distribution :** A distribution is a symmetric distribution if the values of mean, mode and median coincide. In a symmetric distribution frequencies are symmetrically distributed on both sides of the centre point of the frequency curve.



Mean = Median = Mode

- A distribution which is not symmetric is called a skewed- distribution.
Mean – Mode = 3 (Mean – Median)
Mode = 3 Median – 2 Mean.
- Dispersion is the measure of the variations. It measures the degree of scatteredness of the observations in a distribution around the central value.
- **Range :** The range is the difference between the greatest and the least observations of the distribution. Thus, if A and B are the greatest and the smallest observations respectively in a distribution, then its range = $A - B$.

The coefficient of range (or scatter) $= \frac{A-B}{A+B}$.

- **Quartile Deviation or Semi-Interquartile Range (Q.D.) :** If Q_1 and Q_3 be the lower and upper quartiles. Then quartile deviation or semi-interquartile range Q is given by $Q = \frac{1}{2}(Q_3 - Q_1)$

  Coefficient of quartile deviation $= \frac{(Q_3 - Q_1)/2}{(Q_3 + Q_1)/2} = \frac{Q_3 - Q_1}{Q_3 + Q_1}$

- **Mean Deviation (M.D.) :** If x_i/f_i ; $i = 1, 2, ..., n$ is the frequency distribution, then mean deviation from an average A (median, mode or A.M.) is given by

  M.D. $= \frac{1}{N}\sum_{i=1}^{n} f_i \ | x_i - A |$, where $\sum_{i=1}^{n} f_i = N$.

  Mean coefficient of dispersion $= \frac{\text{Mean deviation from the mean}}{\text{Mean}}$

  Median coefficient of dispersion $= \frac{\text{Mean deviation from the median}}{\text{Median}}$

  Mode coefficient of dispersion $= \frac{\text{Mean deviation from the mode}}{\text{Mode}}$

- **Standard Deviation (S.D.) and Variance :** If x_1/f_i; $i = 1, 2, ..., n$ is a frequency distribution, then $\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{n} f_1 (x_i - \overline{X})^2}$ ...(i)

  where $\overline{X}$ is the A.M. of the distribution and $N = \sum_{i=1}^{n} f_i$

- The square of the standard deviation is called the variance and is given by

  $\sigma^2 = \frac{1}{N}\sum_{i=1}^{n} f_i \, (x_i - \overline{X})^2$ ...(ii)

  Coefficient of dispersion $= \frac{\sigma}{\overline{X}}$

  Coefficient of variation $= \frac{\sigma}{\overline{X}} \times 100$

- If deviations of X are measured from an assumed mean A, then root-mean square deviation of X is denoted by s, and is given by

  $s = \sqrt{\frac{1}{N}\sum_{i=1}^{n} f_i (x_i - A)^2}$, s^2 is called mean square deviation. ...(iii)

- Mean square deviation and consequently root mean square deviation is least if the deviation are taken from the mean.

- **Variance of The Combined Series :** If n_1, n_2 are the sizes, $\overline{X}_1$, $\overline{X}_2$ the means and σ_1, σ_2 the standard deviations of two series, then the standard deviation σ of the combined series is given by

$$\sigma^2 = \frac{1}{n_1+n_2}[n_1(\sigma_1^2+d_1^2)+n_2(\sigma_2^2+d_2^2)]$$

where $d_1 = \overline{X}_1 - \overline{X}, d_2 = \overline{X}_2 - \overline{X}$ and $\overline{X} = \dfrac{n_1\overline{X}_1 + n_2\overline{X}_2}{n_1+n_2}$.

- A distribution is positively skewed if the value of mean is maximum and that of mode is least – the median in between the two. In a negatively skewed distribution the value of mode is maximum and that of mean is least – the median lies in beetween the two.
- **Absolute Measure of Skewness :** Various measures of skewness are

  (i) $S_k = M - M_d$ (ii) $S_k = M - M_0$

  (iii) $S_k = Q_3 + Q_1 - 2\,M_d$, where M_d = Median, M_0 = Mean, M = Mean
- **Karl Pearson's Coefficient of Skewness :** $S_k = \dfrac{M - M_0}{\sigma}$ or $S_k = \dfrac{3\,(M - M_d)}{\sigma}$
- **Bowley's Coefficient of Skewness :** $S_k = \dfrac{Q_3 + Q_1 - 2\,M_d}{Q_3 - Q_1}$
- **Kelly's Coefficient of Skewness :** $S_k = \dfrac{P_{10}+P_{90}-2M_d}{P_{90}-P_{10}}$ Also, $S_k = \dfrac{D_1+D_9-2M_d}{D_9-D_1}$.
- For a symmetrical distribution, the following area relationship holds good:

  $\overline{X} \pm \sigma$ covers 68.27% items

  $\overline{X} \pm 2\,\sigma$ covers 95.45% items

  $\overline{X} \pm 3\,\sigma$ covers 99.74% items

Relationship between different measures of dispersion

(a) Q.D. $= \frac{2}{3}\sigma$ (approx.) **(b)** M.D. $= \frac{4}{5}\sigma$ (approx.)

(c) S.D. $= \frac{3}{2}$Q.D.(approx.) **(d)** Q.D. $= \frac{5}{6}$M.D.(approx.)

(e) M.D. $= \frac{6}{5}$Q.D.(approx.)

- **Bivariate Distribution :** If the change in one variable is accompanied by a change in the other variable in such a way that an increase in one variable results in an increase or decrease in the other and also greater change in one variable result in a corresponding greater change in the other, then the two variables are said to be correlated.
- If the increase (or decrease) in one variable results in a corresponding increase (or decrease) in the other, correlation is said to be direct or positive.

  But, if the increase (or decrease) in one result in a corresponding decrease (or increase) in other, correlation is said to be negative.
- If the change in one variable is followed by a corresponding and proportional change in the other variable, correlation is said to be perfect.

- **Covariance :** Let (x_i, y_i); i = 1, 2, ..., n be a bivariate distribution, where $x_1, x_2, ..., x_n$ are the values of variable X and $y_1, y_2, ..., y_n$ those of Y. Then the covariance Cov. (X, Y) beetween X and Y is given by

$$\text{Cov}(X, Y) = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{X})(y_i - \overline{Y})$$

where, $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ are means of variables X and Y respectively.

- **Karl Pearson's Coefficient of Correlation :** The correlation coefficient $r(X, Y)$, between two variables X and Y is given by

$$r(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)}\sqrt{\text{Var}(Y)}} \text{ or } \frac{\text{Cov}(X, Y)}{\sigma_x \sigma_y}.$$

- **Coefficient of Correlation always lies between –1 and 1 :** If r = 1, the correlation is perfect and positive and if r = –1, correlation is perfect and negative. If r = 0, then there is no correlation between the variables
- The coefficient of correlation is independent of the change of origin and scale.
- **Probable Error and Standard Error :** If r is the correlation coefficient in a sample of n pairs of observations, then its standard error is given by

$$\text{S.E.}(r) = \frac{1 - r^2}{\sqrt{n}}$$

Probable error of correlation coefficient is given by

$$\text{P.E. }(r) = (0.6745)\ (\text{S.E.}) = (0.6745)\left(\frac{1 - r^2}{\sqrt{n}}\right)$$

- If r < P.E. (r) there is no evidence of correlation.
- If r > 6 P.E. (r) the existence of correlation is certain. The square of the coefficient of correlation for a bivariate distribution is known as coefficient of determination.
- **Rank Correlation :** $r(X, Y) = 1 - \dfrac{6\sum_{i=1}^{n} d_i^2}{n(n^2 - 1)}$

which is Spearman's formula for the rank correlation coefficient.

- The rank correlation coefficient lies between – 1 and 1.
- If two variables are correlated, then points in the scatter diagram generally cluster around a curve which we call the curve of regression and we say that there is curvilinear regression between the variables.

Line of Regression

- Line of regression of Y on X or regression line of Y on X :

$$(y - \overline{Y}) = r\frac{\sigma_Y}{\sigma_X}(x - \overline{X})$$

- Line of regression of X on Y or regression line of X on Y

$$(x - \overline{X}) = r\frac{\sigma_X}{\sigma_Y}(y - \overline{Y})$$

- The regression coefficient of Y on X is denoted by b_{YX} and is given by

  $b_{YX} = r\frac{\sigma_Y}{\sigma_X}$. .

- The coefficient of regression of X and Y is denoted by b_{XY} and is given by

  $b_{XY} = r\frac{\sigma_X}{\sigma_Y}$.

Properties of Regression Coefficients

- The sign of the correlation coefficient is the same as that of regression coefficients.
- Correlation coefficient is the geometric mean between the regression coefficients.
- If one of the regression coefficients is greater than unity, the other must be less than unity.
- Arithmetic mean of the regression coefficients is greater than the correlation coefficient.
- Regression coefficients are independent of change of origin but not of scale.

Angle Between Two Lines of Regression

- Equations of the two lines of regression are
  $(y - \overline{Y}) = b_{YX}(x - \overline{X})$ and $(x - \overline{X}) = b_{XY}(y - \overline{Y})$

  m_1 = slope of the line of regression of Y on $X = b_{YX} = r\frac{\sigma_Y}{\sigma_X}$.

  m_2 = slope of the line of regression of X on $Y = \frac{1}{b_{XY}} = \frac{1}{r}\frac{\sigma_Y}{\sigma_X}$

  Let θ be the angle between two lines of regression. Then,

  $$\therefore \quad \theta = \tan^{-1}\left\{\frac{1-r^2}{r}\left(\frac{\sigma_X \sigma_Y}{\sigma_X^2 + \sigma_Y^2}\right)\right\}.$$

- If $r = 0$, then $\tan\theta = \infty \Rightarrow \theta = \frac{\pi}{2}$.
- If two variables are uncorrelated, the line of r regression are perpendicular to each other.
- If $r = \pm 1$, then $\tan\theta = 0 \Rightarrow \theta = 0$ or π

End

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CHAPTER

numerical methods

- **Significant Digits**
  The significant digits in a number are determined by the following rules :
  (i) All non-zero digits in a number are significant.
  (ii) All zeroes between two non-zero digits are significant.
  (iii) If a number having embedded decimal point ends with a non-zero or a sequences of zeroes, then all these zeroes are significant digits.
  (iv) All zeroes preceding a non zero digits are non-significant.
- **Rounding off of Numbers :** If a number is to be rounded off to n significant digits, then we follow the following rules :
  (i) Discard all digits to the right of the n^{th} digit .
  (ii) If the $(n + 1)^{th}$ digit is greater than 5 or it is 5 followed by a non-zero digit, then n^{th} digit is increased by 1. If the $(n + 1)^{th}$ digit is less than 5 , then n^{th} digit remains unchanged.
  (iii) If the $(n + 1)^{th}$ digit is 5 and followed by zero, or zeroes, then n^{th} digit is increased by 1 if it is odd and it remains unchanged if it is even.
- If a number is rounded off to n decimal places, then $|E_R| < 0.5 \times 10^{-n+1}$.
- If a number is truncated to n decimal places, then $|E_R| < 10^{-n+1}$.
- **Fundamental Theorem of Algebra :** Every algebraic equation has atleast one root.
- If $f(x)$ is a polynomial of degree n, then $f(x) = 0$ has exactly n roots.
- In any algebraic equation $f(x) = 0$
  (i) The number of positive roots cannot exceed the number of changes of sign of the coefficients in the polynomial $f(x)$.
  (ii) The number of negative roots cannot exceed the number of changes of sign of the coefficients in $f(-x)$.
- If $f(x)$ is continuous on $[a, b]$ and $f(a), f(b)$ are of opposite signs, then there is atleast one number $\alpha \in (a, b)$ such that $f(\alpha) = 0$.
- If $f(x)$ is continuous on $[a, b]$ and a, b are two real numbers such that:
  (i) $f(a)$ and $f(b)$ are of opposite signs, then there are odd number of real roots of the equation $f(x) = 0$ in (a, b)
  (ii) $f(a)$ and $f(b)$ are of the same sign then either there is no real root or there are even number of roots of the equation $f(x) = 0$ in (a, b).
- Every polynomial equation of even degree whose last term is negative and the coefficient of its first term being positive, has atleast two real roots one positive and other negative.

- Every polynomial equation of an odd degree must have atleast one real root whose sign is opposite to that of its last term, the coefficient of first term being positive.
- Let $f(x) = a_0 x^n + a_1 x^{n-1} + ... + a_{n-1} x + a_n$. Then the largest root of polynomial $f(x) = 0$ may be approximated by the root of the linear equation $a_0 x + a_1 = 0$, or by the root of the quadratic equation $a_0 x^2 + a_1 x + a_2 = 0$, larger in absolute value.
- If $f(x) = 0$ be a polynomial equation and $x_1, x_2, ..., x_k$ are the consecutive real roots of $f(x) = 0$ then positive or negative sign of the values of $f(-\infty), f(x_1), ..., f(x_k), f(\infty)$ will determine the intervals in which the roots of $f(x) = 0$ will lie. Whenever there is a change of sign from $f(x_r)$ to $f(x_{r+1})$ the roots lies in the interval $[x_r, x_{r+1}]$.
- **Successive Bisection Method :** Let $f(x) = 0$ be an equation. Following are the steps involved to solve the equation $f(x) = 0$ by successive bisection method :
- **Step I :** Find an interval $[x_0, x_1]$ such that $f(x_0)$ and $f(x_1)$ are of opposite signs. Then the root of the equation $f(x) = 0$ lies between x_0 and x_1. Let $f(x_0) < 0$ and $f(x_1) > 0$.
- **Step II :** Let $x_2 = \frac{x_0 + x_1}{2}$. Then $f(x_2)$ may be positive, negative or zero. If $f(x_2) = 0$, then x_2 is the required root. If $f(x_2) > 0$, then the root lies between x_0 and x_2. If $f(x_2) < 0$, then the root lies between x_2 and x_1.
  Let us assume that $f(x_2) > 0$. Denote this x_2 by x_1.
- **Step III :** We have $f(x_0) < 0$. Therefore, root of the equation $f(x) = 0$ lies between x_0 and x_1. Let $x_2 = \frac{x_0 + x_2}{2}$, then $f(x_2)$ may be positive, negative or zero.If $f(x_2) = 0$, then $x = x_2$ is the required root.
- If we want to find the root(s) correct upto n places of decimal, then in intermediate steps we retain the digits upto $n + 2$ places of decimals and in the final answer we retain the digits upto n places of decimal and chop off the other digits.
- **Regula Falsi Method of False Position or Method of Interpolation :**
  $x_{n+1} = x_n - \frac{(x_n - x_{n-1}) f(x_n)}{f(x_n) - f(x_{n-1})}$ provided that at each step $f(x_{n-1}) f(x_n) < 0$.
- **Newton Raphson Method :** The iterative formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, f'(x_n) \neq 0$.

Rate of Convergence

- A positive real number k is called the order of convergence of a method iff $|x_{n+1} - \alpha| \leq C |x_n - \alpha|^k$, $n \geq 0$ and $C > 0$ is a finite constant.
  or $\varepsilon_{n+1} \leq C \varepsilon_n^k$, where ε_n is the error in n^{th} iteration.

Approximation of Functions

- Some of the commonly used power series expansions of some standard functions are given below :

(i) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$

$$+\frac{n(n-1)(n-2)\,...\,(n-(r-1))}{r!}x^r + ..., \text{ where } |x| < 1.$$

(ii) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ...$ for all x

(iii) $a^x = 1 + (x\log a) + \frac{(x\log a)^2}{2!} + \frac{(x\log a)^3}{3!} + ... + \frac{(x\log a)^n}{n!} + ...$

(iv) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$ for $|x| < 1$

(v) $\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + ..\right)$ for $|x| < 1$

(vi) $\log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + ...\right)$ for $|x| < 1$

(vii) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$ for all x

(viii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$ for all x

(ix) $\tan x = x - \frac{x^3}{3!} + \frac{2}{15!}x^5 + ...$ for all $x \neq (2n+1)\frac{\pi}{2}$, $n \in Z$.

- **The Trapezoidal Rule :** Let $y = f(x)$ be a function defined on $[a, b]$ which is divided into n equal sub-intervals each of width h so that $b - a = nh$.
  Let the values of $f(x)$ for $(n + 1)$ equidistant arguments $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, ..., $x_n = x_0 + nh = b$ be $y_0, y_1, y_2, ..., y_n$ respectively. Then

  $$\int_a^b f(x)dx = \int_{x_0}^{x_0+nh} y\,dx = h\left[\frac{1}{2}(y_0 + y_n) + (y_1 + y_2 + y_3 + ... + y_{n-1})\right]$$

  This rule is known as the Trapezoidal rule.

End

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CHAPTER

linear programming

- A half-plane in the xy-plane is called a **closed half-plane** if the points on the line separating the half-plane are also included in the half-plane.

  The graph of a linear inequation involving sign '≤' or '≥' is a closed half-plane.

  A half-plane in the xy-plane is called an **open half-plane** if the points on the line separating the half-plane are not included in the half-plane.

  The graph of linear inequation involving sign '<' or '>' is an open half-plane.

- Two or more linear inequations are said to constitute a **system of linear inequations.**

  The solution set of a system of linear inequations is defined as the intersection of solution sets of linear inequations in the system.

  A linear inequation is also called a **linear-constraint** as it restricts the freedom of choice of the values x and y.

- **Convex Set :** A subset S of xy-plane is called a convex set if corresponding to any two points P and Q in S, every point on the segment PQ lies in S *i.e.*, S is convex iff (x_1, y_1) and $(x_2, y_2) \in S$
  $\Rightarrow (\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \in S$, for $0 \le \lambda \le 1$.

  One can see that intersection of two convex sets is a convex set. The set of all feasible solutions of a L.P.P. is a convex set.

- In **linear programming** we deal with the optimization (maximization or minimization) of a linear function of a number of variables subject to a number of restrictions (or constraints) on variables, in the form of linear inequations in the variable of the optimization function.

- A function of certain variables, which is to be optimized, subject to given conditions (constraints) on the variables of the function is called the **objective function** of the problem under consideration.

- The common region determined by all constraints of an L.P.P. is called a **feasible region** and every point in this region is called a feasible solution of L.P.P. A feasible solution which maximizes the objective function is called an **optimal solution** of L.P.P.

Fundamental Extreme Point Theorem

The maximum/minimum value of a linear objective function over a convex polygon bounded by a number of lines, occurs at some vertex (corner) or the other.

- In order to solve a linear programming problem (L.P.P.), the following steps are followed :

I. Identify the variables in the given L.P.P. and denote these by x and y.

II. Translate all constraints in the form of linear inequations. No constraint is left out.

III. Formulate the objective function in terms of x and y, and decide whether the objective function is to be maximized or minimized.

IV. Draw the half-plane of each constraint and mark the common area by shaded lines. The shaded portion is called the **feasible region.**

V. Find the co-ordinates of all the vertices of the feasible region.

VI. Find the value of the objective function at each vertex of the feasible region.

VII. Find the values of x and y which maximize/minimize the values of the objective function.

End

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CHAPTER

computing

- Computer is a general purpose electronic machine which can process a large amount of information at a high speed using arithmetic and logical operations. A computer has five major components :
  (i) Input Unit
  (ii) Memory Unit
  (iii) Control Unit
  (iv) Arithmetic Logic Unit (ALU)
  (v) Output Unit
- **Input Unit :** Data are entered into the computer system by means of input devices. Examples of Input devices are keyboard, Punched cards, paper tape, magnetic tape, magnetic disk, light pen, optical scanner and the mouse.
- **Memory Unit :** A computer system has storage areas, often referred to as memory. The memory can receive, hold and deliver data when instructed to do so.
- **Control Unit :** This controls and co-ordinates the activities of all other units of a computer system. It performs the following function :
  (a) It can get instructions out of the memory unit.
  (b) It can decode the instructions
  (c) It can set up the routing, through the internal wiring of data to the correct place at the correct time.
- **Arithmetic Logic Unit (ALU) :** ALU performs all the arithmetic calculations and takes logical decisions. This unit can do additions, subtractions, etc., as a calculator and in addition, it can also perform some logical functions.
- **Output Unit :** The output unit receives the stored result from the memory unit, converts it into a form the user can understand and produces it in the desired format.

Software

- Software consists of the instructions to the computer that specify which functions the CPU is to perform. A list of sequenced instructions that direct the computer system to perform a particular task is called a **program.** Preparing these sequenced instructions is called **programming.**

Assignment and Comparison Operators

- While writing programs, many times we are required to assign a value to a variable. For example, if we assign the value 3 to the variable X, we write this instruction as $X \leftarrow 3$
  Similarly, consider the sequence

$A \leftarrow 8$
$B \leftarrow 7$
$C \leftarrow A + B$

C has the value 15 after three assignment operations.

- To make decisions, a computer often makes comparison between two values. For instance, a program may have a step where a determination must be made whether the current value of a variable is greater than the current value of other variable. Some binary relation symbols '>', '≥', '<', '≤', and '=' are used to write computer programs. These symbols are called **Comparison Operators.**

  $A < B$ *i.e.*, A is less than B
  $A \leq B$ *i.e.*, A is less than or equal to B
  $A > B$ *i.e.*, A is greater than B
  $A \geq B$ *i.e.*, A is greater than or equal to B
  $A = B$ *i.e.*, A is equal to B.

Input-Output Instructions

- While writing computer programmes, we shall use the word 'get' to have an instruction to fetch the values of the variables and the word 'output' to have an instruction to obtain the computer results. For example get A, B

  This means that the value of variables A and B are obtained and are available for the remaining part of the program.

  Output A

  This means that the value of A is obtained as the result.

Writing Programs

- We use various constructs to organize the program on paper. This technique is known as Pseudocode. Pseudocode expresses a program in outline form using English words and phrases to list procedures to be carried out in the program.

Pseudo Language Constructs

(i) If-then-else Construct : This construct is used to provide selection of actions

```
IF statement THEN
Procedure - 1
Procedure - 2
Procedure - 3
ELSE
Procedure - 2
Procedure - 1
Procedure - 4 ;
```

This statement means, "If the statement is true when the program tests it, the following procedures are performed : Procedure -1, Procedure -2, then Procedure-3. If the statement is false when the program tests it, then the following procedures are performed : Procedure - 2, Procedure -1, then Procedure - 4".

(ii) While-do construct : This construct is used to provide repetition of instructions and the statement is tested before each repetition of loop.

```
Loop initialization ;
WHILE statement Do
```

Procedure - 1
Procedure - 2
Procedure - 3
Procedure - 4
loop update ;

This statement performs the initialization for the loop. The program encounters the WHILE statement, and determines whether the statement is true. If it is, the program carries out procedures Procedure-1 to Procedure-4 and loop update. The program then go back to the WHILE statement and repeat it. If the statement is false. Then the program terminates the loop and goes to the next statement.

(iii) Repeat-Until construct : This is similar to the while-do construct.

Loop initialization ;
REPEAT
Procedure -1
Procedure-2
Procedure -3
loop update
UNTIL statement ;

In the Repeat-Until loop the statement is tested after each repetition of the loop.

- **For construct :** When we know in advance how many times a part of the program is to be executed we use for construct.

For identifies = initial value TO test value BY increment DO
Procedure - 1
Procedure - 2
Procedure - 3

Number System

- In computer, data can be stored in the form of **Bits.** Bit is the abbreviation of **Binary digits.** As the word binary suggests there two types of digits *i.e.* 0 and 1 only. Therefore, whole data processing in computers is done in the form of bits. Arithmetic applied an bits to represent number is called **Binary Arithmetic.** Like this, there are various number systems :

(i) Binary Number system
(ii) Decimal number system
(iii) Octal Number system
(iv) Hexadecimal Number system.

All the four systems, with first 16 numbers in the decimal, binary, octal and hexadecimal systems are listed in the following table :

Decimal (Base 10)	**Binary (Base 2)**	**Octal (Base 8)**	**Hexadecimal (Base 16)**
0	0000	00	0
1	0001	01	1
2	0010	02	2
3	0011	03	3
4	0100	04	4

5	0101	05	5
6	0110	06	6
7	0111	07	7
8	1000	10	8
9	1001	11	9
10	1010	12	A
11	1011	13	B
12	1100	14	C
13	1101	15	D
14	1110	16	E
15	1111	17	F

Numbers with Different Bases

- **Decimal Number System :** It has base 10. **0 to 9** are the digits represented in this system. All the arithmetic operations are applied on a number which is represented by these 10 digits and get the result in a number represented by these 10 digits.
- **Binary Number System :** It has base 2. Therefore, it has only two digits **0 and 1.** All the data manipulation is done by these two digits only. This system is used in computers.
- **Octal Number System :** It has base 8 and **0 to 7** are the 8 digits which is represented by this system.
- **Hexadecimal Number System :** It has base 16 and **0 to 15** are the 16 digits which is represented by this system.

Numbers Base Conversions

- **Conversion from any Base to Decimal :**A binary number can be converted to decimal by forming the sum of the powers of 2 of these coefficients whose value is 1.
- **Conversion from decimal to any base :** The conversion from decimal to binary as to any other base *r* system is more convenient if the number is separated into an integer part and a fraction part and conversion of each part done separately.
- **Conversion of Decimal Number Into Octal and Vice Versa :** For the conversion of a decimal number to an octal number the technique of division by 8 can be used.
- **Conversion of Decimal Number to Hexadecimal Number and Vice Versa :** For the conversion of a decimal to hecadecimal number, the technique of division by 16 can be used.

End

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CHAPTER

partial differentiation

- **Partial Differentiation :** Consider a function f which depends on more than one input variable, say x_1, x_2, ..., x_n. The symbol $\frac{\partial f}{\partial x_1}$ represents differentiation of f w.r.t x_1 only. In this differentiation all other variables are treated as constants.

An Important Result

- It α is m times repeated root of the equation $f(x) = 0$, then $f(x)$ can be written as $f(x) = (x - \alpha)^m g(x)$, where $g(\alpha) \neq 0$.
  From the above equation, we can see that
  $$f(\alpha) = 0, f'(\alpha) = 0, f''(\alpha) = 0, \ldots., f^{(m-1)}(\alpha) = 0$$
  Hence, we have the following proposition :
  $$f(\alpha) = 0, f'(\alpha) = 0, f''(\alpha) = 0, \ldots., f^{(m-1)}(\alpha) = 0$$
  $\Rightarrow \alpha$ is m times repeated root of the equation $f(x) = 0$.
- To find $\frac{dy}{dx}$ with the help of partial differentiation. If $f(x, y) = c$, then we can find $\frac{dy}{dx}$ with the help of partial differentiation as :
  $\frac{dy}{dx} = -\frac{f_x}{f_y}$, when f_x is the differential coefficient of $f(x, y)$ w.r.t. x treating y as constant and f_y is the differential coefficient of $f(x, y)$ w. r. t. y treating x constant.

End

36
CHAPTER

partial fraction

Integration by Partial Fraction

(a) (i) $\int \frac{dx}{(x \mp a)(x \mp b)(x \mp c)} = \int \left(\frac{A}{x \mp a} + \frac{B}{x \mp b} + \frac{K}{x \mp c} \right) dx$

$$A = \pm \frac{1}{(a-b)(a-c)}, B = \pm \frac{1}{(b-c)(b-a)}, K = \pm \frac{1}{(c-a)(c-b)}$$

(ii) $\int \frac{dx}{(1 \mp ax)(1 \mp bx)(1 \mp c)} = \int \left(\frac{A}{1 \mp ax} + \frac{B}{1 \mp bx} + \frac{K}{1 \mp cx} \right) dx$ values of A, B, K are same in both cases

$$A = \pm \frac{a^2}{(a-b)(a-c)}, B = \pm \frac{b^2}{(b-c)(b-a)}, K = \pm \frac{c^2}{(c-a)(c-b)}.$$

(iii) $\int \frac{x^2}{(x \pm a)(x \pm b)(x \pm c)} = \int \left(\frac{A}{x \pm a} + \frac{B}{x \pm b} + \frac{K}{x \pm c} \right) dx$ for both cases values of A, B, K are same given as $A = \frac{a^2}{(a-b)(a-c)}$.

(iv) $\int \frac{x^3 dx}{(x \pm a)(x \pm b)(x \pm c)} = \int 1 + \left(\frac{A}{x \pm a} + \frac{B}{x \pm b} + \frac{K}{x \pm c} \right) dx$

where $A = \mp \frac{a^3}{(a-b)(a-c)}, B = \mp \frac{b^3}{(b-c)(b-a)}, K = \mp \frac{c^3}{(c-a)(c-b)}$.

(v) $\int \frac{ax+b}{(cx+d)(ex+f)} dx = \int \left(\frac{A}{cx+d} + \frac{B}{ex+f} \right) dx$, $A = \frac{ad-bc}{de-cf}$, $B = -\frac{(af-be)}{de-cf}$

(vi) $\int \frac{1}{x^2(x-1)} dx = \int \left(\frac{A}{x-1} + \frac{B}{x} + \frac{C}{x^2} \right) dx$, $A = 1$, $B = C = -1$

(vii) $\int \frac{dx}{x(x+1)^2} dx = \int \left[\frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \right] dx$, $A = 1, B = -1 = C$

(viii) $\int \frac{ax+b}{cx+d} dx = \int \frac{a}{c} + \frac{bc-ad}{c} . \frac{1}{cx+d} dx$

(ix) $\int \frac{(x-a)(x-b)(x-c)}{(x-p)(x-q)(x-r)} dx = \int \left(1 + \frac{A}{x-p} + \frac{B}{x-q} + \frac{K}{x-r} \right) dx$

$$A = \frac{(p-a)(p-b)(p-c)}{(p-q)(p-r)} \text{ (when } x=p)$$

$$B = \frac{(q-a)(q-b)(q-c)}{(q-p)(q-r)} \text{ (when } x=q)$$

$$K = \frac{(r-a)(r-b)(r-c)}{(r-p)(r-q)} \text{ (when } x=r)$$

(b) $\int \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} = \int \left(\frac{Ax+B}{x^2+a^2} + \frac{Cx+D}{x^2+b^2} \right) dx$

(i) $\frac{1}{a^2-b^2} \int \left[\frac{a^2}{x^2+a^2} - \frac{b^2}{x^2+b^2} \right] dx \quad \because \quad B = \frac{a^2}{a^2-b^2}$ and $A = C = 0,\ D = \frac{-b^2}{a^2-b^2}$

(ii) $\int \frac{dx}{(x^2+a^2)(x^2+b^2)} = \int \left(\frac{Ax+B}{x^2+a^2} + \frac{Cx+D}{x^2+b^2} \right) dx$

$= \frac{1}{a^2-b^2} \int \left[\frac{1}{x^2+b^2} - \frac{1}{x^2+a^2} \right] dx$, as $A = C = 0,\ B = \frac{1}{a^2-b^2},\ D = \frac{-1}{a^2-b^2}$

(iii) $\int \frac{dx}{(x^2-a^2)(x^2-b^2)} = \int \left(\frac{Ax+B}{x^2-a^2} + \frac{Cx+D}{x^2-b^2} \right) dx,\ A = C = 0,\ B = \frac{1}{a^2-b^2},$

$D = \frac{-1}{a^2-b^2}$

(iv) $\int \frac{x^2 dx}{(x^2 \pm a^2)(x^2 \pm b^2)(x^2 \pm c^2)} = \int \frac{y\, dx}{(y \pm a^2)(y \pm b^2)(y \pm c^2)},\ y = x^2$

$$= \int \left(\frac{A}{y \pm a^2} + \frac{B}{y \pm b^2} + \frac{K}{y \pm c^2} \right) dx$$

where $A = \mp \frac{a^2}{(a^2-b^2)(a^2-c^2)},\ B = \mp \frac{b^2}{(b^2-c^2)(b^2-a^2)},\ K = \pm \frac{c^2}{(c^2-a^2)(c^2-b^2)}$

(v) $\int \frac{(x^2+a^2)(x^2+b^2)}{(x^2+c^2)(x^2+d^2)} dx = \int \left(1 + \frac{f(y)}{(y+c^2)(y+d^2)} \right) dx$, y is a function x.

$$= \int \left(1 + \frac{A}{y+c^2} + \frac{B}{y+d^2} \right) dx$$

$$= \int \left(1 + \frac{(a^2-c^2)(b^2-c^2)}{(d^2-c^2)(x^2+c^2)} + \frac{(a^2-d^2)(b^2-d^2)}{(c^2-d^2)(x^2+d^2)} \right) dx$$

(vi) $\int \frac{lx^2+mx+n}{(x-a)(x-b)(x-c)} dx = \int \left(\frac{A}{x-a} + \frac{B}{x-b} + \frac{K}{x-c} \right) dx$

$$\text{where} = \begin{cases} A = \dfrac{la^2 + ma + n}{(a-b)\,(a-c)} \text{ (when } x=a) \\ B = \dfrac{lb^2 + mb + n}{(b-a)\,(b-c)} \text{ (when } x=b) \\ K = \dfrac{lc^2 + mc + n}{(c-a)\,(c-b)} \text{ (when } x=c) \end{cases}$$

(vii) $$\int \frac{n!}{x(x+1)(x+2).....(x+n)}dx = \int\left(\frac{A_0}{x}+\frac{A_1}{x+1}+\frac{A_2}{(x+2)}+\frac{A_3}{x+3}+......+\frac{A_n}{x+n}\right)dx$$

$$=\int\left(\frac{C_0}{x}-\frac{C_1}{x+1}+\frac{C_2}{(x+2)}-\frac{C_3}{x+3}+....+\frac{(-1)^n C_n}{x+n}\right)dx$$

where $C_0, C_1, C_2,....., C_n$ are Binomial coefficients in the expansion $(1 + x)^n$ and $A_0 = C_0, A_1 = (-1)C_1, A_2 = (-1)^2 C_2, A_3 = (-1)^3 C_3, ... A_n = (-1)^n C_n.$

End

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CHAPTER

group theory

- **Algebraic Structure :** A non-empty set with a binary operation defined on it is called an algebraic structure if the binary operation satisfies certain axioms.
- **Semi group :** An algebraic structure $(G,*)$ consisting of a non-void set G and a binary operation * defined on G is called a semi-group if it satisfies the following axiom :

  Associative Law : The binary operation * is associative on G, *i.e.* $a * (b * c) = (a * b) * c \ \forall a, b, c \in G$.

  A semi group $(G, *)$ is non-void set G together with an associativity binary operation * on G.

  $(N, +)$, $(Z, +)$, $(Q, +)$, $(R, +)$, $(C, +)$ are all semi groups.

  $(Z, -)$, $(Q, -)$, $(R, -)$ and $(C, -)$ are not semi groups.
- **Monoid :** An algebraic structure $(G, *)$, consisting of a non-void set G and a binary operation * defined on G is called monoid, if it satisfies the following axioms:

  Associative Law : The binary operation * is associative on G,

  i.e. $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$.

  Existence of identity : There exists an element $e \in G$ such that $a*e = a = e*a$ for all $a \in G$. The element e is called the identity element of G for the binary operation *.

Group

- An algebraic structure $(G,*)$ consisting of a non-void set G and a binary operation * defined on G is called a group if it satisfies the following axioms:

(i) **Associative Law :** The binary operation * is associative on G,

i.e. $a*(b*c) = (a*b)*c$ for all $a, b, c \in G$.

(ii) **Existence of Identity :** There exists an element $e \in G$ such that $a*e = a = e*a$ for all $a \in G$. The element e is called the identity element.

(iii) **Existence of Inverse :** For each $a \in G$ there exist an element $a' \in G$ such that $a * a' = e = a' * a$. The element a' is called the inverse of a and is denoted by a^{-1}.

- **Abelian Group or Commutative Group :** A group $(G, *)$ is called an abelian group or a commutative group if the binary operation * is commutative on G. *i.e.* $a * b = b * a$ for all $a, b \in G$

  To define a group we should have a set consisting of atleast one element *viz*, identity element and a binary operation defined on it.

- **Finite Group :** A group $(G, *)$ is called a finite group if the set G is a finite set, otherwise it is called an infinite group.
- **Order of a Group :** Let $(G, *)$ be *a* finite group. Then the number of elements in the set G is called the order of the group $(G,*)$. The order of a group G is generally denoted by $O(G)$.
- **Examples of Groups**

1. The set Z of integers is an infinite abelian group under the addition of integers as a binary operation.

2. The set Q, R, C are also infinite abelian group under addition.

3. Let m be a fixed integer. Then the set $mZ = \{mx \mid x \in Z\}$ of all multiples of m is an infinite abelian group under addition of integers.

4. The set of all even integers is an infinite additive abelian group but the set of all odd integers is not a group under addition of integers because it is not closed under addition.

5. The set Z of all integers is not a group under multiplication of integers as a binary operation because except 1 and –1 no integer possesses its multiplicative inverse.

6. The sets of all non-zero rational, real and complex numbers form multiplicative infinite abelian groups.

7. For any fixed positive integer n, the set of all n n^{th} roots of unity forms a finite abelian group under multiplication of complex numbers as a binary operation.

8. The set Q^+ of all positive rational numbers is an infinite abelian group under the operation defined as $a * b = \dfrac{ab}{2}$ for all $a, b \in Q^+$.

9. The set $Q - \{1\}$ of all rational numbers other than unity forms an infinite abelian group under the binary operation * defined as $a * b = a + b - ab$ for all $a, b \in Q - \{1\}$.

10. The set of all $m \times n$ matrices over integers is an infinite abelian group under matrix addition. But it is not a group under matrix multiplication.The set of all $m \times n$ matrices over rational, real and complex numbers also form infinite abelian groups under matrix addition.

11. The set of all $n \times n$ non-singular square matrices over rational and real numbers form an infinite non-abelian group under matrix multiplication. The set M of all $n \times n$ non-singular matrices over integers is not a group under matrix multiplication, because all $n \times n$ non singular matrices over integers are not invertible. For example, the non singular matrix $A = \begin{bmatrix} 3 & -1 \\ 7 & 2 \end{bmatrix}$ is not invertible over Z, because $A^{-1} = \begin{bmatrix} 2/13 & 1/13 \\ -7/13 & 3/13 \end{bmatrix}$ is not matrix over Z.

12. The set of matrices $A\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix}$, where $\alpha \in R$ is a group under matrix multiplication.

13. The set $G = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \middle| 0 \neq a \in R \right\}$ is an abelian group under matrix multiplication.
14. $G = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} \middle| 0 \neq a \in R \right\}$ is an abelian group under of singular matrices under matrix multiplication.
15. The set V of all vectors in a plane forms an infinite abelian group under vector addition.
16. The set of all vectors in a plane does not form a group under vector product as a binary operation, because vector product is not associative.
17. Let X be a non-void set. Then the set of all bijections from X to itself is a group under the composition of functions as a binary operation.
18. The set of all real valued continuous functions defined on the closed interval [0, 1] is an abelian group under the point wise sum defined as $(f + g)(x) = f(x) + g(x)$ for all $x \in [0, 1]$.

- **Addition Modulo m :** Addition modulo m is denoted by $\oplus_m$ and it is defined as $a \oplus_m b = r$, where r is the least non-negative remainder when $a + b$ is divided by m. Clearly $0 \leq r < m$.
- **Multiplication Modulo m :** Multiplication modulo m is denoted by $\otimes_m$ and is defined as $a \otimes_m b = r$ where r is least non-negative remainder when ab is divided by m.
- **Additive Group of Integers Modulo m :** Let m be an arbitrary but fixed positive integer. Then the set $G = \{0, 1, 2, ..., (m-1)\}$ is a finite abelian group of order m, under addition modulo m as a binary operation.

Multiplicative Group of Integers Modulo a Prime Integer p

- Let p be a prime integer. Then the set $G = \{1, 2, ..., (p-1)\}$ is a finite abelian group of order $(p-1)$, with respect to multiplication modulo p as a binary operation.
- **Properties of Groups**

1. The identity element in a group $(G, *)$ is unique. If the binary operation is additive, then 0 denotes the identity element for multiplicative binary operation 1 is used to denote the identity element.
2. The inverse of every element of a group $(G, *)$ is unique. The inverse of a is denoted by a^{-1}. If the binary operation is additive, then $-a$ denotes the inverse of a.
3. The inverse of the identity element in a group is identity element itself.
4. **Reversal Law :** Let $(G, *)$ be a group. Then $(a * b)^{-1} = b^{-1} * a^{-1}$ for all $a, b \in G$.
5. **Cancellation Laws :** Let $(G, *)$ be a group, then for all $a, b, c \in G$
   $a * b = a * c \Rightarrow b = c$ (Left cancellation law)
   and $b * a = c * a \Rightarrow b = c$ (Right cancellation law)
6. Let $(G, *)$ be a group and $a \in G$ Then $(a^{-1})^{-1} = a$.
7. Let $(G, *)$ be a group. Then for any $a, b \in G$, the equations $a * x = b$ and $y * a = b$ have unique solutions in G.

8. A semi-group $(G, *)$ is a group iff for all $a, b \in G$, the equation $a * x = b$ and $y * a = b$ have unique solutions in G.

9. A finite semi-group is a group iff cancellation laws hold in it.

10. A semi-group $(G, *)$ is a group iff
(i) there exists $e \in G$ such that $a * e = a$ for all $a \in G$.
(ii) for each $a \in G$, there exists $b \in G$ such that $a * b = e$.

11. A semi-group $(G, *)$ is a group iff
(i) there exists $e \in G$ such that $e * a = a$ for all $a \in G$.
(ii) for each $a \in G$, there exists $b \in G$ *s.t.* $b * a = e$.

12. If $(G, *)$ is a group such that $(a * a) = a$ for all $a \in G$, then G is abelian.

13. If $(G, *)$ is a group such that $(a * b)^2 = (a * a) * (b * b)$ for all $a, b \in G$, then G is abelian.

14. Every group of order less than or equal to 5 is abelian.

15. The order of every element of a finite group is finite and less than or equal to the order of the group.

16. $O(a) = O(a^{-1})$.

17. If a is an element of a group $(G, *)$ then $O(x^{-1} * a * x) = O(a)$.

18. If an element a of a group $(G, *)$ is of order n and e is the identity element in G, then $a^m = e \Leftrightarrow n$ is a divisor of m.

19. If the order of an element of a group G is n and p is an integer relatively prime to n, then $O(a^p) = n$.

20. If the order of an element of a group G is n, then $O(a^s) = n/d$, where d is the greatest common divisor of n and s.

Order of An Element of a Group

- Let $(G, *)$ be a group, and let $a \in G$. Then a is said to be of finite order if there exists a positive integer n such that $a^n = e$.
- If n is the smallest positive integer such that $a^n = e$, then we say that order of a is n and we write $O(a) = n$.
- Identity element in a group is the the only element whose order is 1.

Cyclic Groups

- A group $(G, *)$ is said to be cyclic if there exists an element $a \in G$, such that every element of G is expressible as some integral power of a.
- The element a is called the generator of G and write $G = [a]$.

Properties of Cyclic Groups

1. Every cyclic group is abelian but an abelian group need not be cyclic.

2. If a is a generator of a cyclic group G, then a^{-1} is also a generator of G.

3. The order of a cyclic group is the same as the order of its generator.

4. A finite group of order n containing an element of order n is cyclic.

5. Every infinite cyclic group has two and only two generators.

6. If G is a finite cyclic group of order n generated by an element a, then other generators of G are of the form a^s, where s is relative prime n and less than n.

End

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CHAPTER

hyperbolic functions

- For any x (real or complex) the expressions $\frac{e^x - e^{-x}}{2}$ and $\frac{e^x + e^{-x}}{2}$ are respectively defined as **hyperbolic sine x** and **hyperbolic cosine x** and are written as $\sinh x$ and $\cosh x$ respectively.

  Thus, $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$.

- $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, $\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

  $\text{cosech}\, x = \frac{2}{e^x - e^{-x}}$, $\text{sech}\ x = \frac{2}{e^x + e^{-x}}$

  $\tanh x = \frac{\sinh x}{\cosh x}$, $\coth x = \frac{\cosh x}{\sinh x}$, $\text{cosech}\ x = \frac{1}{\sinh x}$ and $\text{sech}\ x = \frac{1}{\cosh x}$.

-

Hyperbolic function	Domain	Range
$\sinh x$	R	R
$\cosh x$	R	$(1, \infty)$
$\tanh x$	R	$(-1, 1)$
$\coth x$	$R_0 = R - \{0\}$	$R - [-1, 1]$
$\text{cosech}\, x$	R_0	R_0
$\text{sech}\, x$	R	$(0, 1)$

Important Facts

-

$\sin ix = i \sinh x$	$\sinh x = -i \sin(ix)$
$\cos ix = \cosh x$	$\cosh x = \cos(ix)$
$\tan ix = i \tanh x$	$\tanh x = -i \tan(ix)$
$\cot ix = -i \coth x$	$\coth x = i \cot(ix)$
$\sec ix = \text{sech}\, x$	$\text{sech}\, x = \sec(ix)$
$\text{cosec}\, ix = -i\, \text{cosech}\, x$	$\text{cosech}\, x = i\ \text{cosec}\ (ix)$

(i) $\cosh^2 x - \sinh^2 x = 1$
(ii) $\cosh^2 x + \sinh^2 x = \cosh 2x$
(iii) $1 - \tanh^2 x = \text{sech}^2\, x$
(iv) $1 - \coth^2 x = -\,\text{cosech}^2\, x$

Formulae for The Sum and Difference

(i) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
(ii) $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

(iii) $\tan(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$

Formulae for Multiples of *x*

(i) $\sinh 2x = 2\sinh x \cosh x$

(ii) $\cosh 2x = 2\cosh^2 x - 1$ or $1 + \cosh 2x = 2\cosh^2 x$

(iii) $\cosh 2x = 1 + 2\sinh^2 x$ or $1 - \cosh 2x = -2\sinh^2 x$

(iv) $\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$

(v) $\sinh 2x = \dfrac{2\tanh x}{1 - \tanh^2 x}$

(vi) $\cosh 2x = \dfrac{1 + \tanh^2 x}{1 - \tanh^2 x}$

(vii) $\sinh 3x = 3\sinh x + 4\sinh^3 x$

(viii) $\cosh 3x = 4\cosh^3 x - 3\cosh x$

(ix) $\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$

Formulae to Convert The Sum into Product

(i) $\sinh x + \sinh y = 2\sinh\left(\dfrac{x+y}{2}\right)\cosh\left(\dfrac{x-y}{2}\right)$

(ii) $\sinh x - \sinh y = 2\cosh\left(\dfrac{x+y}{2}\right)\sinh\left(\dfrac{x-y}{2}\right)$

(iii) $\cosh x + \cosh y = 2\cosh\left(\dfrac{x+y}{2}\right)\cosh\left(\dfrac{x-y}{2}\right)$

(iv) $\cosh x - \cosh y = 2\sinh\left(\dfrac{x+y}{2}\right)\sinh\left(\dfrac{x-y}{2}\right)$.

Formulae to Convert The Product Into Sum

(i) $2\sinh x \cosh y = \sinh(x + y) + \sinh(x - y)$

(ii) $2\cosh x \sinh y = \sinh(x + y) - \sinh(x - y)$

(iii) $2\cosh x \cosh y = \cosh(x + y) + \cosh(x - y)$

(iv) $2\sinh x \sinh y = \cosh(x + y) - \cosh(x - y)$.

Some Important Formulaes

(i) $\sinh^2 x - \sinh^2 y = \sinh(x + y)\sinh(x - y)$

(ii) $\cosh^2 x + \sinh^2 y = \cosh(x + y)\cosh(x - y)$

(iii) $\cosh^2 x - \cosh^2 y = \sinh(x + y)\sinh(x - y)$

(iv) (a) $(\cosh x + \sinh x)^n = e^{nx} = \cosh nx + \sinh nx$

(b) $(\cosh x - \sinh x)^n = e^{-nx} = \cosh nx - \sinh nx$

(v) $\sinh^{-1} x = \log_e(x + \sqrt{x^2 + 1})$

(vi) $\cosh^{-1} x = \log_e(x+\sqrt{x^2-1})$

(vii) $\tanh^{-1} x = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)$

(viii) $\coth^{-1} x = \frac{1}{2}\log_e\left(\frac{x+1}{x-1}\right)$, for $|x|>1$

(ix) $\text{sech}^{-1} x = \log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right)$, for $x \in (0, 1]$

(x) $\text{cosech}^{-1} x = \log_e\left(\frac{1+\sqrt{1+x^2}}{x}\right)$, if $x > 0$

$= \log_e\left[\frac{1-\sqrt{1+x^2}}{x}\right]$, if $x < 0$

Relations Between Inverse Circular Functions and Inverse Hyperbolic Functions

(i) $\sinh^{-1}x = -i\sin^{-1}(ix)$

(ii) $\cosh^{-1}x = -i\cos^{-1}x$

(iii) $\tanh^{-1}x = -i\tan^{-1}(ix)$

(iv) $\sinh^{-1} x = \text{cosech}^{-1}\left(\frac{1}{x}\right)$

(v) $\sinh^{-1} x = \cosh^{-1}\sqrt{1+x^2}$

(vi) $\sinh^{-1} x = \tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

(vii) $\sinh^{-1} x = \coth^{-1}\left(\frac{\sqrt{1+x^2}}{2}\right)$

Logarithm of a Complex Quantity

- $\log z = \log|z| + i\tan^{-1}(\arg(z)) + 2in\pi$.

End

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CHAPTER

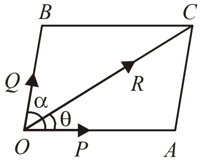
statics

- If a numbers of forces P_1, P_2, P_3 ..... act upon a rigid body or a particle and if a single force R can be found whose effect upon the body is same as that of all the forces P_1, P_2, P_3, ... then this single force R is called the resultant of the given forces. Also, the given forces P_1, P_2, ... are called components of R.
- **Law of Parallelogram of Forces :** If two forces acting at a point are represented in magnitude and direction by two sides of a parallelogram drawn through that point, then their resultant is represented in magnitude and direction by the diagonal of the parallelogram passing through that point.
- **Resultant of Two Forces :** If two forces P and Q are acting at a point at an angle α, then the magnitude R of their resultant is given by

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}.$$

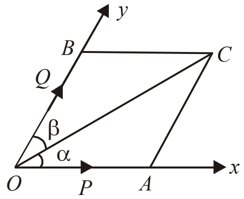
- Angle θ which the direction of resultant R makes with the direction of the force P is given by

$$\theta = \tan^{-1}\left(\frac{Q\sin\alpha}{P+Q\cos\alpha}\right).$$



- The angle θ_1 which the resultant R makes with the direction of the forces Q is given by $\theta_1 = \tan^{-1}\left(\frac{P\sin\alpha}{Q+P\cos\alpha}\right)$
- The resultant of two forces is closer to the larger force.
- The resultant two equal forces of magnitude P acting at an angle α is $2P\cos(\alpha/2)$ and it bisects the angle between the forces.
- The resultant of two perpendicular forces P and Q is given by $R = \sqrt{P^2 + Q^2}$.
- The greatest value of the resultant of the forces P and Q is equal to their sum $P + Q$ which happens only when the forces act in the same direction .
- The least value of the resultant of two forces P and Q is equal to the absolute value of their difference and it happens only when the forces act in opposite directions.
- If the resultant R of two forces P and Q acting at an angle α makes an angle θ with the direction of P, then $\sin\theta = \frac{Q\sin\alpha}{R}$ and $\cos\theta = \frac{P + Q\cos\alpha}{R}$.

- If the resultant R of the forces P and Q acting at an angle α makes an angle θ with the direction of the force Q then $\sin\theta = \dfrac{P\sin\alpha}{R}$ and $\cos\theta = \dfrac{Q + P\sin\alpha}{R}$
- The components of a forces R in two given directions making angle α and β with the line of action of R on an opposite sides of it are



$$P = \frac{R\sin\beta}{\sin(\alpha+\beta)} \text{ and } Q = \frac{R\sin\alpha}{\sin(\alpha+\beta)} \text{ respectively.}$$

- **Resolved Parts of a Force :** The resolved part of a force F in a direction making an angle θ with the direction of the force F is $F\cos\theta$.
- The necessary and sufficient conditions for a system of coplanar forces acting at a point be in equilibrium is that the algebraic sums of the resolved parts of the forces in any two mutually perpendicular directions in the place of the forces vanish separately.

Triangle Law of Forces

- If two forces acting at a point be represented in magnitude and direction by the two sides of a triangle taken in the same order, then the resultant will be represented in magnitude and direction by the third side taken in the opposite order.

Converse of the Triangle Law of Forces

- If three forces acting at a point be in equilibrium, then any triangle drawn with its sides parallel to the lines of action of the forces shall have its sides proportional to the forces.
- If three forces acting at a point are in equilibrium, the sum of any two cannot be less than the third.

Extension of Triangle Law of Forces

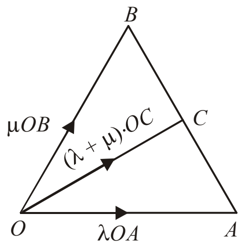
- If three forces acting at a point can be represented in magnitude by the sides of a triangle and their directions are perpendicular to the sides, all inwards or all outwards, then the forces shall be in equilibrium.

Polygon Law of Forces

- If any number of forces acting at a point be such that they can be represented in magnitude and direction by the sides of a closed polygon taken in order shall be in equilibrium.

$\lambda - \mu$ Theorem

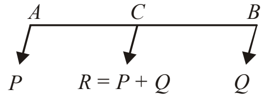
- The resultant of two forces, acting at a point O along OA and OB and represented in magnitude by $\lambda \cdot OA$ and $\mu \cdot OB$, is represented by a force $(\lambda + \mu)\cdot OC$, where C is a point in AB such that $\lambda\, CA = \mu\, CB$ *i.e.* C divides AB in the ratio $\mu : \lambda$.



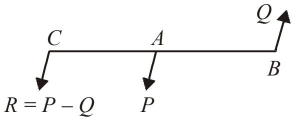
- If $\lambda = \mu$, then C becomes the mid-point of AB and $\lambda \cdot \overrightarrow{OA} + \mu \cdot \overrightarrow{OB} = 2\lambda \cdot \overrightarrow{OC}$ or $\overrightarrow{OA} + \overrightarrow{OB} = 2 \cdot \overrightarrow{OC}$.
- **Lami's Theorem :** If three forces acting at a point be in equilibrium then each force is proportional to the sine of the angle between the other two .
- **Converse of Lami's Theorem :** If three forces acting at a point be such that each is proportional to the sine of the angle between the other two, then the three forces are in equilibrium.
- **Conditions of Equilibrium of Concurrent Forces :** The necessary and sufficient conditions that a system of coplanar forces acting at a point may be in equilibrium are that the algebraic sum of the resolved parts of the forces along two mutually perpendicular directions should be separately zero.

Parallel Forces

- If two parallel forces P and Q ($P > Q$) act at the points A and B of a rigid body, then we have the following results :
  (i) The magnitude of the resultant forces is $P + Q$ or $P - Q$ according as the forces are like or unlike.



  (ii) The direction of the resultant force is same as the direction of either of the component forces if the forces are like or that of the greater force, if the forces are unlike.



  (iii) The point of application of the resultant forces divides AB internally in the inverse ratio of the forces if the forces are like. If the forces are unlike, then the point of application of the resultant force divides BA produced externally in the inverse ratio of the forces.

  i.e. $\dfrac{AC}{BC} = \dfrac{Q}{P}$ or $P \cdot AC = Q \cdot BC$

  (iv) The line of action of the resultant force is nearer to the greater force.

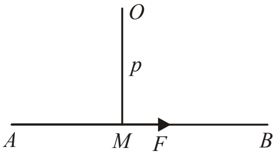
  (v) If the forces are like, then the resultant acts at C such that

$$AC = \left(\frac{AB}{P+Q}\right)Q \text{ and } BC = \left(\frac{AB}{P+Q}\right)P$$

- If the forces are unlike then the resultant force acts at a point C on BC produced such that $AC = \left(\frac{AB}{P-Q}\right)Q$ and $BC = \left(\frac{AB}{P-Q}\right)P$.

Moment of a Force

- The moment of a force about a point is the product of the force and the perpendicular distance of the point from the line of action of the force.



- The moment of a force about any point on its line of action is zero, because in such a case $p = 0$.
- If the moment of a force F about a point O is zero. Then, either $F = 0$ or $p = 0$.
- The moment of a force about a point O is numerically equal to twice the area the triangle formed with the point O as vertex and the measure of the line segment representing the force as base.
- The moment of a force about a point measures the tendency of rotation of the force to turn the body about that point.
- When the tendency of the force is to rotate the body about the given point in the anti-clockwise direction, then the moment is taken as positive whereas for its tendency to rotate the body in the clockwise direction, the moment is taken as negative.

Units of Moment

- The moment of a unit force about a point at a unit perpendicular distance from the line of action of the force is defined as the unit of moment.
- In F.P.S. system the unit of moment is foot-poundal. In C.G.S. system the unit of moment is centimetre-dyne.
- In S. I. unit, the unit of moment is Newton-metre. In symbol, this unit is written as Nm.
- **Varignon's Theorem :**The algebraic sum of the moments of two forces about any point in their plane is equal to the moment of their resultant about the same point.
- **Generalization of Varignon's Theorem :** If any number of coplanar forces acting on a rigid body have a resultant; the algebraic sum of their moments about any point in their plane is equal to the moment of their resultant about the same point.

Some Useful Results Related to Centroid, Circumcentre, Incentre and Orthocentre of a Triangle

- **Centroid :** Let ABC be a triangle. Then the point of intersection of its medians is the centroid. The centroid divides medians in the ratio 2 : 1.

- If p_1, p_2, p_3 denote the lengths of perpendiculars from the vertices A, B, C on the opposite sides, then

  (i) $p_1 = c\sin B = b\sin C,\ p_2 = a\sin C = c\sin A,$
  $p_3 = b\sin A = a\sin B$

  (ii) $\Delta = \frac{1}{2}ap_1, \Delta = \frac{1}{2}bp_2, \Delta = \frac{1}{2}cp_3.$

  (iii) Distance of the centroid from $BC = \frac{1}{3}p_1 = \frac{2\Delta}{3a}$

  Distance of the centroid from $CA = \frac{1}{3}p_2 = \frac{2\Delta}{3b}$

  Distance of the centroid from $AB = \frac{1}{3}p_3 = \frac{2\Delta}{3c}.$

- **Circumcentre :** Let ABC be a triangle. The point of intersection of the perpendicular bisectors of the sides of $\Delta\, ABC$ is known as its circumcentre, it is equidistant from the vertices. The distance of the circumcentre from the vertex is circumradius and it is denoted by R. We have :

  (i) $R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}$

  (ii) Distance of the circumcentre from $BC = R\cos A$
  (iii) Distance of the circumcentre from $CA = R\cos B$
  (iv) Distance of the circumcentre from $AB = R\cos C$

- **Incentre :** It is the point of intersection of the bisectors of the angles of a triangle. It is generally denoted by I. A circle with centre at I and radius equal to the length of perpendicular from I on the sides of a triangle is called the incircle. Its radius is denoted by r. The distances of I from the sides of a triangle are all equal to r.

- **Orthocentre :** The point of intersection of altitudes of a triangle is known as its orthocentre. The distances of the orthocentre of a $\Delta\, ABC$ from the sides BC, CA, and AB are $2R\cos B\cos C$, $2R\cos C\cos A$ and $2R\cos A\cos B$ respectively.

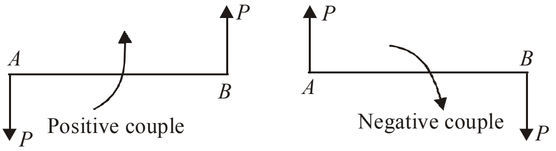
- **Couple :** Two equal and unlike parallel forces not having the same line of action are said to form a couple.

- **Arm of The Couple :** The perpendicular distance between the lines of action of the forces forming the couple is known as the arm of the couple.

- **Moment of a Couple :** The moment of a couple is defined as the product of the magnitude of each force framing the couple and the arm of the couple.

  Thus the moment of the couple (P, p) is $P \cdot p$
  i.e. Moment of a couple = Force × Arm of the couple.

- **Sign of the Moment of a Couple :** If the forces forming the couple tend to produce rotation in the body in anti-clockwise direction, the moment of the couple is positive. Also, then the couple is regarded as a positive couple. If the forces forming the couple tend to produce rotation in clockwise direction, we say that its moment is negative and the couple is regarded as a negative couple.



- The algebraic sum of the moments of the two forces forming a couple about any point in their plane is constant and equal to the moment of the couple.
- The effect of a couple acting on a rigid body which can turn freely about any point is independent of the position of the point and depends only on its moments.
- Couples of equal moments acting in the same plane are equivalent and consequently couples of equal but opposite moments in the plane balance each other.
- **Physical Significance of a Couple :** A couple acting on a rigid body does not produce any translatory motion *i.e.* the motion in a straight line. In fact it produces rotation in the body. Thus, physically the effect of a couple acting upon a rigid body is to turn the body about a point in the plane of the couple.
- If a system of forces acting on a body reduces to a couple, then there is no single resultant force. Consequently, the algebraic sum of the resolved parts of the forces along two perpendicular lines must vanish separately.
  (i) If a system of forces acting on a body is reducible to a couple, then the algebraic sum of the moments of the forces about any point in their plane is equal to the moment of the couple.
  (ii) The sum of the resolved parts of the forces forming a couple in any direction is zero.
  (iii) A couple cannot be balanced by a single force, but can be balanced by a couple of opposite sign.
- **Triangle Theorem of a Couple :** If three forces acting on a rigid body be represented in magnitude, direction and line of action by the sides of a triangle taken in order, then they are equivalent to a couple whose moment is represented by twice the area of the triangle.

Equilibrium of Coplanar Forces

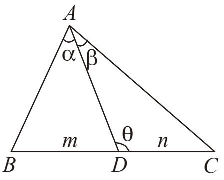
I. If three forces keep a body in equilibrium, then they must be coplanar.

II. If three forces acting in one plane upon a rigid body, keep it in equilibrium, then they must either meet at a point or be parallel.

III. A system of coplanar forces acting upon a rigid body will be in equilibrium if the algebraic sum of their resolved parts in any two mutually perpendicular directions vanish separately, and the algebraic sum of their moments about any point in their plane is zero.

IV. A system of coplanar forces acting upon a rigid body will be in equilibrium if the algebraic sum of the moments of the forces about each of three non-collinear points is zero.

V. Trigonometrical Theorems : If D is any point on the base BC of ΔABC such that $BD : CD = m : n$. Then,



(i) $(m + n)\cot\theta = m\cot\alpha - n\cot\beta$

(ii) $(m + n)\cot\theta = n\cot B - m\cot C$.

Friction

- **Statical Friction :** When two bodies in contact with each other are in equilibrium, the friction in such a position exerted at their point of contact is called statical friction.
- **Limiting Friction :** When one body is just about to slide over another body (*i.e.* in the position of limiting equilibrium) the force of friction exerted at their point of contact attains its maximum value and then it is called limiting friction.
- **Dynamical Friction :** When one body slides over another body, the force of friction exerted at their point of contact is called dynamical friction.
- λ is the limiting value of α when the force of friction F attains its maximum value.

  $$\therefore \quad \tan\lambda = \frac{\text{Maximum force of friction}}{\text{Normal reaction}}.$$

- **Coefficient of Friction :** When one body is in limiting equilibrium in contact with another body, the constant ratio which the limiting force of friction bears to normal reaction at their point of contact, is called the coefficient of friction and it is generally denoted by μ.

  $$\mu = \tan\lambda = \frac{\text{Maximum force of friction}}{\text{Normal reaction}}.$$

- The value of μ depends on the substance of which the bodies are made and so it differs from one body to the other. Also, the value of μ always lie between 0 and 1. Its value is zero for a perfectly smooth body.
- **Cone of Friction :** A cone whose vertex is at the point of contact of two rough bodies and whose axis lies along the common normal and whose semi-vertical angle is equal to the angle of friction λ, is called the cone of friction.

Laws of Friction

I. When two bodies are in contact, the force of friction at their point of contact acts in the direction opposite to what would be the direction of relative motion.

II. In the equilibrium position, the magnitude of the force of friction is just sufficient to prevent the relative motion. Thus friction is a self-adjusting force.

III. Limiting friction between two bodies bears a constant ratio to the normal reaction between them. This constant ratio is denoted by μ and it depends only on the nature of the substance of which the bodies are made.

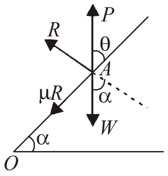
IV. Limiting friction is independent of the shape of the surfaces in contact provided that the normal reaction is unaltered.

V. When motion takes place, the above laws of limiting friction are still true, but the coefficient of friction μ is slightly less for bodies in motion than for the same bodies at rest. Moreover, the friction is independent of the velocity of the body in motion.

- If a body be on the point of sliding down an inclined plane under its own weight, the inclination of the plane is equal to the angle of the friction.

Least Force Required to Pull a Body up an Inclined Rough Plane

- Let a body of weight W be at the point A on the plane inclined at an angle α to the horizon. Let R be the normal reaction and μ be the coefficient of friction. Let P be the force, acting at an angle θ with the plane, required just to move the body up the plane. Since the body is on the point of moving up the plane, so the limiting friction μR acts down the plane.



Then $P = W\dfrac{\sin(\alpha + \lambda)}{\cos(\theta - \lambda)}$

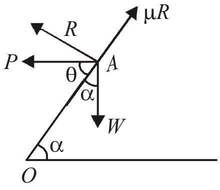
Clearly, the force P is least when $\cos(\theta - \lambda)$ is maximum

i.e. when $\cos(\theta - \lambda) = 1$ *i.e.* $\theta - \lambda = 0$ or $\theta = \lambda$.

The least value of P is $W\sin(\alpha + \lambda)$.

Least Force Required to Pull a Body Down an Inclined Rough Plane

- Let a body of weight W be at the point A on the plane inclined at an angle α to the horizon. Let R be the normal reaction and μ be the coefficient of friction. Let P be the force acting at an angle θ with the plane, required just to move the body down the plane. Since the body is on the point of moving down the plane, so the limiting friction μR acts up the plane.



Then, $P = \dfrac{W\sin(\lambda - \alpha)}{\cos(\theta - \lambda)}$

Clearly, P is least when $\cos(\theta - \lambda)$ is maximum

i.e. when $\theta - \lambda = 0$ or $\theta = \lambda$.

The least value of P is $W\sin(\lambda - \alpha)$.

- If a body be placed on a rough inclined plane and if it be just on the point of moving under its own weight and reaction of the plane, then

$$P = 0 \Rightarrow \frac{W\sin(\lambda - \alpha)}{\cos(\theta - \lambda)} = 0 \Rightarrow \sin(\lambda - \alpha) = 0 \Rightarrow \lambda = \alpha$$

Thus, the inclination of the plane to the horizontal is equal to the angle of friction.

- If $\alpha > \lambda$, then the body cannot rest on the plane under its own weight and reaction of the plane. So, the least force does not exist.
- If $\alpha < \lambda$, then the least force required to move the body down the plane is $W\sin(\lambda - \alpha)$.
- If $\alpha = 0$ *i.e.* the plane is horizontal, then the least force to pull the body on a horizontal plane is $W\sin\lambda$.
- If $\alpha = \lambda$, the body is in limiting equilibrium and is just on the point of sliding.
- If $\alpha > \lambda$, the body will move down the plane under the action of its weight and normal reaction.

Centre of Gravity

- **Centre of Gravity of Particles Lying on a Straight Line :** If n particles of weights $w_1, w_2, w_3, \ldots, w_n$ be placed at points $A_1, A_2, A_3, \ldots, A_n$ on the straight line OA_n such that the distances of these points from O are $x_1, x_2, \ldots, x_n$ respectively. Then the distance of their centre of gravity G(say) from O is given by

$$OG = \frac{w_1x_1 + w_2x_2 + \ldots + w_nx_n}{w_1 + w_2 + \ldots + w_n} = \frac{\Sigma w_i x_i}{\Sigma w_i}.$$

- The centre of gravity G of these particles has the co-ordinates $(\bar{x}, \bar{y})$ given by

$$\bar{x} = \frac{\Sigma w_i x_i}{\Sigma w_i} \text{ and } \bar{y} = \frac{\Sigma w_i y_i}{\Sigma w_i}.$$

- **Position of Centre of Gravity of Some Standard Bodies**

Body	**Position of C.G.**
Uniform rod	Mid-point
Triangular Lamina	Centroid
Parallelogram, Rectangle or Square	Intersection of diagonals
Circular Lamina	Centre
Circular arc subtending an angle 2α at the centre of the circle	At a distance $\frac{a\sin\alpha}{\alpha}$ from the centre, where a is the radius of the circle.
Sector of a circle subtending an angle 2α at the centre.	At a distance $2a/3 \sin\alpha/\alpha$ from the centre on the symmetrical radius, where a is the radius of the circle.
Semi-circular arc	At a distance $2a/\pi$ from the centre on the symmetrical radius, a being radius of the circle.
Semi-circular area	At a distance $4a/3\pi$ from the centre on symmetrical radius, where a is the radius.
Hemisphere	At a distance $3a/8$ from the centre on symmetrical radius, where a is the radius.
Hemispherical shell	At a distance $a/2$ from the centre on the symmetrical radius.
Solid cone	At a distance $h/4$ from the base on the axis, where h is the height of the cone.
Conical shell	At a distance $h/3$ from the base on the axis, where h is the height of the cone.
Semi-ellipse cut off by minor axis	At a distance $4a/3\pi$ on major axis from the centre, a being the length of semi-major axis.
Semi-ellipse cut off by major axis	At a distance $4b/3\pi$ on minor axis from the centre, b being the length of semi-minor axis.
Three particles placed at the vertices of a triangle	Centroid

- Let there be a body of weight W and x be its C.G. If a portion of weight W_1 is removed from it and x_1 be the C.G. of the removed portion. Then, the C.G. of the remaining portion of body given by $x_2 = \frac{Wx - W_1x_1}{W - W_1}$.
- Let x be the C.G. of a body of weight W. If x_1, x_2, x_3 are the C.G. of portions of weights W_1, W_2, W_3 respectively which are removed from the body, then the C.G. of the remaining body is given by $x_4 = \frac{Wx - W_1x_1 - W_2x_2 - W_3x_3}{W - W_1 - W_2 - W_3}$.

End

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CHAPTER

dynamics

- The displacement of a moving particle or a body during any interval of time is the change of its position, which is indicated by the line segment joining the initial position and final position.
- **Average Speed :** The average speed of a moving particle in a time interval is defined as the distance travelled by the particle divided by the time interval.
- If a particle travels a distance s in time interval t, then

  Average speed = $\frac{s}{t}$
- **Instantaneous Speed :** The instantaneous speed or simply speed of a moving particle is defined as the rate of change of distance along its path, straight or curved.

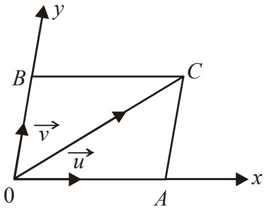
  Instantaneous speed at time $t = \frac{ds}{dt}$ Or, Speed at time $t = \frac{ds}{dt}$
- The average speed is defined for a time interval and the instantaneous speed is defined at a particular instant.

  In M.K.S. or S.I. system, the unit of speed is metre per second (m/s).
- **Uniform Speed :** The speed of a moving particle is uniform if it describes equal lengths of its path in any equal intervals, however small.

  $\frac{ds}{dt} = \lambda$ (constant).
- **Velocity :** The rate of change of displacement of a moving particle is called its velocity.
- **Average Velocity :** The average velocity of a particle in a given interval of time is defined as its displacement divided by the time interval.
- **Uniform Velocity :** If a particle moves in a constant direction and covers equal distances in equal intervals of time, then we say that it is moving with uniform velocity.
- **Acceleration :** The rate of change of velocity of a moving particle is called its acceleration.
- If $\vec{v}$ is the velocity of a moving particle at any time, then by definition the acceleration vector $\vec{a}$ is given by $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{x}}{dt^2}$, where $\vec{x}$ is the displacement.
- In M.K.S. or S.I. system, the unit of acceleration is m/sec^2.

  $a > 0$, if v increases with time and $a < 0$, if v decreases with time. A negative acceleration is called retardation.

- **Uniform Acceleration :** A particle is said to be moving with uniform acceleration if equal changes in velocity take place in equal intervals of time, however small these intervals may be.
- **Resultant and Components of Velocities :** If a moving particle possesses several simultaneous velocities in different directions such that the joint effect of all the velocities is same as if the particle moves with a single velocity in a definite direction, then this single velocity is called the resultant of the given velocities and the given velocities are known as the components of the single resultant.
- **Parallelogram Law of Velocities :** If a moving particle has two simultaneous velocities represented in magnitude and direction by the two sides of a parallelogram drawn from an angular point, then their resultant is represented in magnitude and direction by the diagonal of the parallelogram passing through that point.



- If a particle possesses two simultaneous velocities $\vec{u}$ and $\vec{v}$ inclined at an angle α such that $|\vec{u}| = u$ and $|\vec{v}| = v$, then the magnitude of their resultant velocity is given by $w = \sqrt{u^2 + v^2 + 2uv\cos\alpha}$
- The direction of this resultant velocity makes an angle θ with the direction of $\vec{u}$ such that $\tan\theta = \dfrac{v\sin\alpha}{u + v\cos\alpha}$.
- The angle made by the direction of the resultant velocity with the direction of $\vec{v}$ is given by $\tan^{-1}\left(\dfrac{u\sin\alpha}{v + u\cos\alpha}\right)$.
- If the direction of resultant velocity w makes an angle θ with the direction of $\vec{u}$, then $\sin\theta = \dfrac{v\sin\alpha}{w}$ and $\cos\theta = \dfrac{u + v\cos\alpha}{w}$.
- The resultant of two simultaneous velocities along the same line in the same direction is their sum. Also, the resultant of two velocities is maximum only when they act along the same line and in the same direction.
- The resultant of two simultaneous velocities acting along the same line but in opposite directions is the difference of the component velocities and acts in the direction of the greater velocity. Also, the resultant of two velocities is minimum only when they act along the same line but in opposite directions.
- The resultant of two simultaneous velocities u and v acting at right angle to each other is of magnitude $\sqrt{u^2 + v^2}$ and its direction makes an angle $\tan^{-1}\left(\dfrac{v}{u}\right)$ with the direction of u.

- The resultant of two velocities each of magnitude u acting at an angle α with each other is $2u\cos\alpha/2$ and acts in a direction bisecting the angle between them.
- The components of velocity u in the directions making angle α and β are $\dfrac{u\sin\beta}{\sin(\alpha+\beta)}$ and $\dfrac{u\sin\alpha}{\sin(\alpha+\beta)}$ respectively.
- The algebraic sum of the resolved parts of a number of velocities in a given direction is equal to the resolved parts of their resultant in that direction.
- **Relative Velocity :** If P and Q are two moving bodies, then the relative velocity of P with respect to Q is defined as the rate of change of the position of P as seen from Q.
- The relative velocity of P with respect to Q is also called the velocity of P relative to Q and is denoted by $\vec{V}_{PQ}$.
  In other words, the velocity of P relative to Q is the velocity with which P appears to move as viewed from Q.
- If the observer is one of the two moving bodies, then the relative velocity is also called the apparent velocity.
- The relative velocity of a moving body P with respect to another moving body Q is the resultant of the velocity of P and the reversed velocity of Q *i.e.* $\vec{V}_{PQ}=\vec{V}_P-\vec{V}_Q$.
- If two bodies P and Q are moving with the velocities of magnitudes V_P and V_Q respectively inclined at an angle α with each other, then

$$V_{PQ}=\sqrt{V_P^2+V_Q^2-2V_PV_Q\cos\alpha}.$$

  And the angle θ made by the direction of the relative velocity with the direction of velocity of P is given by $\tan\theta=\dfrac{V_Q\sin\alpha}{V_P-V_Q\cos\alpha}$.
- If the direction of the relative velocity of P with respect to Q makes an angle θ with the direction of velocity of Q, then $\tan\theta=\dfrac{V_P\sin\alpha}{V_Q-V_P\cos\alpha}$.
- When the particles P and Q move parallel to each other in the same direction with velocities u and v respectively, then $V_{PQ}=(u-v)$ in the direction of the velocity of P
  $V_{QP}=(v-u)$ in the direction of the velocity of Q.
- When the particles P and Q move parallel to each other in opposite directions with velocities u and v respectively, then $V_{PQ}=(u+v)$ in the direction of the velocity of P
  $V_{QP}=(v+u)$ in the direction of velocity of Q.
- When the relative velocity of particle Q with respect to a particle P and the true velocity of P are given, then the true velocity of Q is obtained by compounding these velocities by the parallelogram law of velocities.
- True velocity of Q = Resultant of the relative velocity of Q with respect to P and the true velocity of P.

- True velocity of P = Resultant of the relative velocity of P with respect to Q and the true velocity of Q.

Equations of Motion

(i) $v = u + ft$

(ii) $s = ut + \frac{1}{2} ft^2$

(iii) $v^2 = u^2 + 2fs$

(iv) If s_n is the distance travelled by the particle in n^{th} second, then

$$s_n = u + \frac{1}{2} f(2n - 1).$$

- In case of retardation

(i) $v = u - ft$

(ii) $s = ut - \frac{1}{2} ft^2$

(iii) $v^2 = u^2 - 2fs$

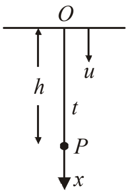
(iv) $S_n = u - \frac{1}{2} f(2n - 1)$.

Average Velocity

- A particle moves in a straight line with initial velocity u m/sec and uniform acceleration f m/sec^2. If v is the velocity of the particle at time t, then its average velocity during the interval of time t is given by $\frac{u + v}{2}$.
- If a particle moves in a straight line with initial velocity u m/sec and constant acceleration f m/sec^2. Then distance travelled in t second is given by = Average velocity × time.
- The average velocity of a particle moving along a straight line with a constant acceleration during time t is the velocity at time $\frac{t}{2}$.

Equations of Motion of a Body Projected Vertically Downwards

- If a body is projected vertically downwards from a point O with velocity u and at time t it is at a point P such that $OP = h$ and v is its velocity at P. Then, the equations describing the motion are:



(i) $v = u + gt$

(ii) $h = ut + \frac{1}{2} gt^2$

(iii) $v^2 = u^2 + 2gh$

(iv) h_n = distance covered in n^{th} second = $u + \frac{1}{2} g(2n - 1)$.

- If the body falls freely, then the initial velocity u will be zero and the above equations describing the motion reduce to:

(i) $v = gt$

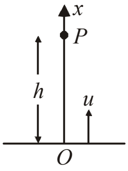
(ii) $h = \frac{1}{2} g t^2$

(iii) $v^2 = 2\ gh$

(iv) $h_n = \frac{1}{2} g(2n - 1).$

Equations of Motion of a Body Projected Vertically Upwards

- If a body is projected vertically upwards from a point O with a velocity u and at time t it is at a point P such that $OP = h$ and the velocity at P is v. Then the equations describing the motion are:



(i) $v = u - gt$

(ii) $h = ut - \frac{1}{2} g t^2$

(iii) $v^2 = u^2 - 2gh$

(iv) h_n = Distance covered in n^{th} second $= u - \frac{1}{2} g(2n-1).$

(v) Greatest height attained $= \frac{u^2}{2g}$

(vi) Time of the greatest height $= \frac{u}{g}$

(vii) Time to a given height $h = \frac{u}{g} \pm \frac{\sqrt{u^2 - 2gh}}{g}$

(viii) Time from any point on the path to the highest point is same as the time from the highest point to the given point when the body is returning.

(ix) Velocity at a given height $v = \pm\sqrt{u^2 - 2gh}.$

The particle at the same point possesses same magnitude of velocity whether going upward or coming downward.

- For a body projected vertically upwards the time to the highest point is same as the time of fall.
- The velocity on reaching the point of projection is equal to the velocity of projection.

Total Time of Flight

- Since time of rise is equal to the time of fall. Therefore,

Time of fight $= \frac{u}{g} + \frac{u}{g} = \frac{2u}{g}.$

Some Important Results

I When a lift is ascending with uniform acceleration of f m/sec^2 and after t seconds a body is dropped from it. Then at the time when the body is dropped:

(i) Initial velocity of the body is same as that of the lift and is in the same direction. So, the velocity of the body is ft m/sec.

(ii) Initial velocity of the body relative to the lift
$=$ Velocity of the body – velocity of the lift $= ft - ft = 0$.

(iii) Acceleration of the body $= g$ m/sec^2 in downward direction.

(iv) Acceleration of the lift $= f$ m/sec^2 in upward direction

(v) Acceleration of the body relative to the lift
$=$ Acceleration of the body – Acceleration of the lift in downward direction
$= g - (-f) = f + g$.

II When a lift is ascending with uniform acceleration of f m/sec^2 and after t seconds a body is thrown vertically upwards with velocity v m/sec. Then at that time, we have the following:

(i) Initial velocity of the body $= v +$ velocity of lift in upward direction $= v + ft$.

(ii) Initial velocity of the body related to the lift = velocity of the body – velocity of lift $= (v + ft) - ft = v$ m/sec.

(iii) Acceleration of the body relative to the lift in vertically upward direction is $(f + g)$ m/sec^2.

III When a lift is descending with uniform acceleration f m/sec^2 and after time t a body is dropped from it. Then at that time, we have the following:

(i) Velocity of the body = Velocity of the lift $= ft$ m/sec in downward direction.

(ii) Initial velocity of the body related to the lift = 0.

(iii) Acceleration of the body relative to the lift in downward direction = Acceleration of the body – Acceleration of lift $= (g - f)$ m/sec^2.

Laws of Motion

- **First law :** Every body continues in its state of rest or of uniform motion in a straight line except in so far as it is compelled by some external impressed force to change that state.
- **Second Law :** The rate of change of momentum of a body is proportional to the impressed force, and takes place in the direction in which the force acts.
- **Third Law :** To every action, there is an equal and opposite reaction.
- **Momentum :** The momentum of a moving particle or body at any time is defined as the product of its mass and its velocity at that instant.
- Impressed force in any direction is equal to the effective force in that direction.
- **Weight of a Body :** The weight of a body is the force with which it is attracted by the earth towards its centre.
- The weight of a body is given by the product of its mass in kg and the acceleration g due to gravity in m/sec^2.

Units of Force

- In M.K.S. system the unit of force is Newton. A Newton is that amount of force which produces an acceleration of 1 m/sec^2 in a mass of 1 kg.

- In C.G.S. system the unit of force is dyne. A dyne is that amount of force which produces an acceleration of 1 cm/sec^2 in a mass of 1 gm.
- In S.I. system the unit of force is Newton.

Gravitational Units of Force

- In the M.K.S. system the gravitational unit of force is kilogram weight or kg.wt. Clearly, 1 kg.wt. = $(1 \times g)$ Newtons = 9.8 Newtons.
- In C.G.S. system the gravitational unit of force is gram weight or gm.wt. Clearly, 1 gm.wt = 981 dynes.

Pressure of a Body Resting on a Moving Horizontal Plane or a Lift

- **When the Horizontal Plane is Rising Vertically Upwards :** If a man of mass m is standing on a lift which is moving vertically upwards with an acceleration f, then the pressure or thrust R on the floor of the lift is given by
  $R = m(g + f)$ ....(i)
  This pressure is also known as the apparent weight of a man.
  Apparent weight of the man is $\frac{f}{g}$ times more than this actual weight.
- **When the Horizontal Plane is Descending Vertically Downwards :** If a man of mass m is standing on a lift which is moving vertically downwards with an acceleration f, then the pressure or thrust R on the floor of the lift is given by
  $R = m(g - f)$ ....(ii)
  This pressure is also known as the apparent weight of a man.
  The apparent weight of the man is $\frac{f}{g}$ times less than this actual weight.
- If the apparent weight of a body resting on a moving horizontal plane is more than its actual weight, then the plane moves in vertically upward direction.
- If the apparent weight of a body resting on a moving horizontal plane is less than its actual weight, then the plane moves in vertically downward direction.
- If the plane be at rest or if it moves up or down with constant velocity, then $f = 0$. Then from (i) and (ii), we get $R = mg$.
  So, the pressure of the body on the plane is equal to the weight mg of the body.
- If the plane moves vertically upwards with same retardation *i.e.* if its velocity gradually diminishes, then from (i), the pressure of the body on the plane will be less than its weight.
  If the plane moves vertically downwards with some retardation, *i.e.* with diminishing velocity, then from (ii), the pressure of the body on the plane will be greater than its actual weight.
- If the plane moves vertically upwards with retardation equal to g *i.e.* $f = -g$ then from (i), $R = 0$. Thus, there is no pressure of the body on the plane when the plane rises vertically with retardation equal to g.
- If the plane moves down freely under gravity *i.e.* with acceleration equal to g, then from (ii) $R = 0$. Thus, there is no pressure of the body on the plane, when it moves vertically downward with an acceleration equal to g.

Motion of Two Particles or Bodies Connected by a String

- If two particles or bodies of masses m_1 and m_2 ($m_2 < m_1$) are connected by a light inextensible string which passes over a small smooth fixed pulley then,

  (i) f = Common acceleration of the mass $= \left(\dfrac{m_1 - m_2}{m_1 + m_2}\right) g$.

  (ii) T = Tension in the string $= \left(\dfrac{2m_1 \, m_2}{m_1 + m_2}\right) g$.

  (iii) P = Pressure on the pulley $= 2T = \left(\dfrac{4m_1 \, m_2}{m_1 + m_2}\right) g$.

  The pressure on the pulley is twice the harmonic mean between the two given weights.

Motion on a Smooth Horizontal Plane

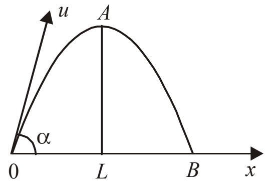
- If two particles of masses m_1 and m_2 ($m_1 < m_2$) are connected by a light inextensible string, passing over a small smooth fixed pulley at the edge of a smooth horizontal plane, m_2 is placed on the plane and m_1 hangs freely, then

  (i) f = Common acceleration $= \left(\dfrac{m_1}{m_1 + m_2}\right) g$

  (ii) T = Tension in the string $= \left(\dfrac{m_1 \, m_2}{m_1 + m_2}\right) g$

  (iii) P = Pressure on the pulley $= \sqrt{2}\,T = \left(\dfrac{\sqrt{2}\, m_1 m_2}{m_1 + m_2}\right) g$

- **Projectile :** If a particle or a body is projected or thrown in any direction into vacuum, then the particle or body is called a projectile.



- **Point of Projection :** The point from which the particle is projected is called the point of projection.
- **Velocity of Projection :** The initial velocity with which the projectile is projected is called the velocity of projection.
- **Angle of Projection :** The angle which the direction of projection makes with the horizon is called the angle of projection *i.e.* the inclination of the direction of velocity of projection with the horizon is called the angle of projection.
- **Trajectory :** The curved path described by particle or body is called its trajectory.
- **Range :** The distance between the point of projection and the point where the projectile strikes a given plane through the point of projection is called its range on the plane. When the given plane is horizontal, we call it the horizontal range.

- **Time of Flight :** The time interval between the instant of projection and the instant when the projectile meets a fixed plane through the point of projection, *i.e.* the time for which the particle remains in air is called the time of flight.
- **Greatest Height :** The maximum height reached by a projectile during its motion is called greatest height.
- **Equations of Motion of a Projectile :** If a particle of mass m is projected into the space with velocity u, in a direction making an angle α with the horizontal direction. Then the equations describing its motion are $v\cos\theta = u\cos\alpha$, $x = (u\cos\alpha)\cdot t$, $v\sin\theta = u\sin\alpha - gt$, $y = (u\sin\alpha)t - \frac{1}{2}gt^2$.
- If a particle (or body) is projected in space with velocity u in a direction making angle α with the horizontal line through the point of projection. Then the equation of the path described by it with reference to the horizontal and vertical lines, lying in the plane of projection, through the point of projection as the co-ordinate axes is a parabola whose equation is $y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$

  The vertex, focus, directrix and latusrectum are :

  Vertex : $\left(\frac{u^2}{2g}\sin 2\alpha, \frac{u^2}{2g}\sin^2\alpha\right)$ Focus : $\left(\frac{u^2}{2g}\sin 2\alpha, -\frac{u^2}{2g}\cos 2\alpha\right)$

  Directrix : $y = \frac{u^2}{2g}$ Latusrectum : $\frac{2u^2}{g}\cos^2\alpha$.
- The velocity of the projectile at a given point on the trajectory is same as the velocity acquired by a particle, projected vertically upwards with initial velocity equal to the initial velocity of the projectile, to arrive at the given point.
- The velocity of the projectile at any point on its path is same as the velocity acquired by a particle in falling freely from rest from the level of the directrix upto the given point.

  Maximum height $= \frac{u^2\sin^2\alpha}{2g} = \frac{(\text{vertical component of initial velocity})^2}{2g}$

  Time to the maximum height $= \frac{u\sin\alpha}{g} = \frac{\text{Vertical component of initial velocity}}{g}$.

Maximum Horizontal Range

- If a particle is projected with velocity u and angle of projection α, then the horizontal range R is given by $R = \frac{u^2}{g}\sin 2\alpha$.
- **Directions of Projection for a Given Range :** For a given velocity of projection and a given horizontal range there are in general two directions of projection which are equally inclined to the direction of the maximum range.

Time for a given Height

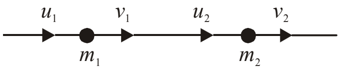
- The two values of t are $t_1 = \frac{u\sin\alpha - \sqrt{u^2\sin^2\alpha - 2gh}}{g}$

and $t_2 = \dfrac{u\sin\alpha + \sqrt{u^2\sin^2\alpha - 2gh}}{g}$.

- The projectile attains the same height at two instants of time.
- The minimum velocity of projection for the projectile to hit the point $P(a, b)$ is $\sqrt{g\,\{b + \sqrt{a^2 + b^2}\}}$.

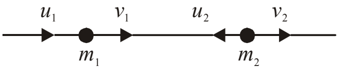
Impact

- **Impact :** When two bodies strike against each other, the state of striking is known as their impact. It is of two kinds : Direct and Oblique.
- **Direct Impact :** Impact between two bodies is said to be direct when the direction of motion of each of them before impact is along the common normal at their point of contact.
- **Oblique Impact :** Impact between two bodies is said to be oblique when the direction of velocities of one or both of them, before impact is not along the common normal at their point of contact.
- **Line of Impact :** The direction of the common normal at the point of contact of the two bodies is called the line of impact.
- **Newton's Experimental Law of Impact :** It states that when two elastic bodies collide, their velocity along the common normal after impact bears a constant ratio of their relative velocity before impact and is in opposite direction.
- The constant ratio is called the coefficient of elasticity or coefficient of restitution and is denoted by e.
- If v_1 and v_2 be the velocities of two bodies after impact in the same direction and u_1 and u_2 are their initial velocities along the common normal at their point of contact, then, $v_1 - v_2 = -e\,(u_1 - u_2)$.
- The value of e depends upon the substances of the bodies and is independent of the masses of the bodies. If the bodies are perfectly elastic, then $e = 1$ and for inelastic bodies, $e = 0$.
- **Law of Conservation of Momentum :** When two bodies impinge, the sum of their moments before impact along the line of impact remains the same.
- If u_1, u_2 are the velocities before impact and v_1, v_2 are the velocities after impact of two bodies of masses m_1 and m_2 then, $m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$
- If the two bodies move in directions as shown,



then by the laws of direct impact, we have
(i) $v_1 - v_2 = -e\,(u_1 - u_2)$ (ii) $m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2$

- If the direction of motion of two bodies before and after the impact are as shown,



then by the laws of direct impact, we have
(i) $v_1 - v_2 = -e(u_1 - (-u_2))$ or $v_1 - v_2 = -e(u_1 + u_2)$

and, (ii) $m_1v_1 + m_2v_2 = m_1u_1 + m_2(-u_2)$

- If the spheres are perfectly elastic *i.e.* $e = 1$, we have loss in K.E. = 0. If the spheres are perfectly inelastic *i.e.* $e = 0$, then we have

  Loss in K.E. $= \dfrac{m_1 m_2}{2(m_1 + m_2)}(u_1 - u_2)^2$.

  = K.E. of a body whose mass is half the H.M. between the mass of the two spheres and whose velocity equal their relative velocity before impact.

End

41
CHAPTER

work, power and energy

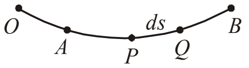
- **Work :** A force is said to do work when a body undergoes some displacement.
- **Work Done by a Constant Force :** It is defined as the product of the force and the displacement of its point of application in the direction of the force.
- Work done by the force F

  = Resolved part of F along the distance × Actual displacement.
- The work done will be positive or negative according as the force and displacement are in the same direction or opposite direction.

Work done by a Variable Force

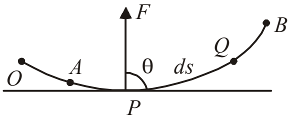
Case I : When the force F varies in magnitude only.

Work done by the force $= \int_a^b F\, ds.$



Case II : When the force F varies in magnitude as well as direction.

- Then work done by the force F in moving from A to $B = \int_a^b F \cos\theta\, ds.$



Units of Work

- In F.P.S. system, the unit of work is foot poundal which is defined as the work done by a force of one poundal in moving its point of application through one foot in the direction of the force.
- In C.G.S. system, the unit of work is erg which is defined as the work done by a force of one dyne in moving its point of application through one centimetre in the direction of the force.
- In M.K.S. system, the unit of work is a joule is defined as the work done by a force of one Newton in moving its point of application through one metre in the direction of the force.
- 1 Joule $=10^7$ Ergs.

Power

- Power is the rate of doing work *i.e.* the amount of the work done in unit time.
- Power = Force × Velocity.
- In F.P.S. system, the unit of power is Horse power. If an agent is doing work at the rate of 550 foot poundals per second *i.e* 33000 foot poundals per minute, it is said to work with one Horse power.

- 1 H.P. = 550 foot poundals per second.
- In M.K.S system, the unit of power is Watt. If an agent is doing work at the rate of 1 Joule per second ,its power is said to be one watt.
- 1 Watt = 1 Joule of work per second.
- 1 Kilowatt = 1000 Joules of work per second =1000 Watts.
- 1 Metric Horse Power = 735 Joules of work per second.
- 1 H.P. = 746 watts.
- The capacity of an agent to do work is defined as its energy.
- Kinetic energy of an agent is its capacity of doing work by virtue of its motion.

$$\text{K.E.} = \frac{1}{2}mv^2$$

- The capacity of a body to do work by virtue of its position is defined as its potential energy.
  Potential energy (P.E.) = mgh.
- **Principle of Conservation of Energy :** If a particle or a body is acted on by a conservative system of force and be in motion, then the sum of the kinetic energy and potential energy of the particle or body remains constant *i.e.*
  K.E. + P.E. = Constant.
- **Principle of Energy :** Change in K.E. = Work done by the force in moving the particle from one position to the other.
- **Impulse of a Force :** The impulse of a constant force F in a given time is defined as the product of the force F and time t during which it acts.
- Impulse of a force = Change in momentum produced.

End